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Etude des sous-variétés dans les variétés kählériennes, presque kählériennes et les variétés produit

JURY

Président du jury

Opozda, Barbara. Professeur. Jagiellonian University, Krakow, Poland.

Rapporteurs

Leschke, Katrin. Professeur. University of Leicester, Leicester, UK.

Voisin, Claire. Professeur. Collège de France, Paris.

Examinateurs

Birembaux, Olivier. Professeur. Université de Valenciennes.

El Kacimi, Aziz. Professeur. Université de Valenciennes.

Munteanu, Marian Ioan. Professeur. UAIC University of Iași, Roumanie.

Van der Veken, Joeri. Professor. KU Leuven, Belgium.

Directeur de thèse

Vrancken, Luc. Professeur. Université de Valenciennes.

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Résumé

Cette thèse est constituée de quatre chapitres. Le premier contient les notions de base qui permettent d'aborder les divers thèmes qui y sont étudiés. Le second est consacré à l'étude des sous-variétés lagrangiennes d'une variété presque kählérienne $\mathbb{S}^3 \times \mathbb{S}^3$.
Une variété presque kählérienne est une variété presque hermitienne munie d'une structure presque complexe J pour laquelle le tenseur $(\tilde{\nabla} J)$ est anti-symétrique, où $\tilde{\nabla}$ est la connexion de Levi-Civita. Je m'intéresse à des sous-variétés lagrangiennes de $\mathbb{S}^3 \times \mathbb{S}^3$ non totalement geodésiques dont la projection sur le premier facteur $\mathbb{S}^3$ n'est nulle part de rang maximal. J'exprime cette propriété à l'aide des fonctions d'angle et j'étudie plus particulièrement la relation entre ces sous-variétés et les surfaces minimales dans $\mathbb{S}^3$. Dans ce chapitre, je démontre que les sous-variétés lagrangiennes de $\mathbb{S}^3 \times \mathbb{S}^3$ dont les fonctions d'angle sont constantes sont totalement géodésiques ou ont leur courbure sectionnelle constante. Puis je donne une classification complète de ces sous-variétés lagrangiennes. Les résultats présentés ici ont été obtenus en collaboration avec Burcu Bektaş, Joeri Van der Veken et Luc Vrancken (voir [3] et [4]).
Dans le troisième chapitre, je m'intéresse à un problème de géométrie différentielle affine et donne une classification des hypersphères affines qui sont isotropiques. Ce résultat a été obtenu en collaboration avec Luc Vrancken (voir [41]).
Et enfin dans le dernier, je présente quelques résultats sur les surfaces de translation et les surfaces homothétiques. Ce travail a été réalisé avec Rafael López ([35]).

Courte présentation des résultats obtenus

§ L'étude des sous-variétés lagrangiennes de la variété presque kählérienne $\mathbb{S}^3 \times \mathbb{S}^3$

L'étude des variétés presque kählériennes débute dans les années 1970 avec A. Gray. Des théorèmes de structure, traitant le cas très spécial de la dimension 6, ont été obtenus par P.-A. Nagy dans les années 2000. Plus récemment, il a été démontré par J.-B. Butruille que les seuls espaces presque kählériens homogènes de dimension 6 sont la sphère $\mathbb{S}^6$ (avec la structure presque complexe introduite par le produit vectoriel sur $\mathbb{R}^7$), l'espace $\mathbb{S}^3 \times \mathbb{S}^3$ (mais pas équipé de la métrique canonique et muni d'une structure presque complexe introduite en utilisant les quaternions), l'espace projectif $\mathbb{C}P^3$ (avec une métrique et une structure presque complexe non canoniques) et la variété des drapeaux $SU(3)/U(1) \times U(1)$. Tous ces espaces sont compacts et 3-symétriques. En 2014 V. Cortés and J. J. Vásquez ont découvert les premiéres structures non homogènes (mais locallement homogènes) presque Kähler dans [14].

Les premières structures non homogènes complètes des espaces presque kählériens ont été découvertes seulement en 2015 par L. Foscolo et M. Haskins (voir [23]). Il existe deux types de sous-variétés des espaces presque kählériens (ou plus généralement, presque hermitiens), à savoir les sous-variétés presque complexes et les sous-variétés totalement réelles. Les sous-variétés presque complexes sont des sous-variétés dont les espaces tangents sont invariants par l'opérateur J. Pour les sous-variétés totalement réelles, les vecteurs tangents sont envoyés sur les vecteurs normaux par la structure presque complexe J. Dans ce dernier cas, si en plus la dimension de la sous-variété est la moitié de la dimension de l'espace ambiant, alors la sous-variété est appelée lagrangienne. On notera que les sous-variétés lagrangiennes des variétés presque kählériennes sont particulièrement intéressantes car elles sont toujours minimales et orientables (voir [21] pour $\mathbb{S}^6$ et [27] ou [52] pour le cas général).

Les sous-variétés lagrangiennes ont été étudiées par de nombreux auteurs dans le passé. Par contre pour les autres espaces, jusqu'à présent, il existe très peu de résultats. Dans le cas de $\mathbb{S}^3 \times \mathbb{S}^3$ les premiers exemples ont été obtenus respectivement par Schäfer et Smoczyk (2010) et par Moroianu et Semmelmann (2014). Un exemple de tore plat et la classification de tous les exemples totalement géodésiques ou avec courbure sectionelle constante ont été obtenus en 2014 par Dioos, Vrancken et Wang.

Une phase fondamentale dans cette étude est l'utilisation d'une structure presque produit P sur $\mathbb{S}^3 \times \mathbb{S}^3$, qui est liée, mais est différente de la structure produit canonique de $\mathbb{S}^3 \times \mathbb{S}^3$. La décomposition de P en une partie tangentielle et une partie normale le long d'une sous-variété lagrangienne permet alors d'introduire trois directions principales, E_1, E_2, E_3, avec des fonctions angulaires correspondantes θ_1, θ_2, θ_3.
Les résultats obtenus dans ce domaine sont présentés dans la suite.

Theorem 1. *Soit*

$$\begin{aligned} f : M &\to \mathbb{S}^3 \times \mathbb{S}^3 \\ x &\mapsto f(x) = (p(x), q(x)) \end{aligned}$$

une immersion Lagrangienne telle que la première projection $p : M \to \mathbb{S}^3$ est nulle part une immersion (cela veut dire que $p(M)$ est une surface dans $\mathbb{R}^3$). Alors une des fonctions angulaires est constante et égale à $\frac{\pi}{3}$. La réciproque est vraie aussi.

Theorem 2. *Soit*

$$\begin{aligned} f : M &\to \mathbb{S}^3 \times \mathbb{S}^3 \\ x &\mapsto f(x) = (p(x), q(x)), \end{aligned}$$

une immersion lagrangienne telle que la première projection $p : M \to \mathbb{S}^3$ est nulle part une immersion. Alors $p(M)$ est une surface minimale dans $\mathbb{S}^3$.

En distinguant plusieurs cas, nous avons aussi réussi à montrer la réciproque, c'est-à-dire comment construire, à partir d'une surface minimale dans $\mathbb{S}^3$ (qui est totalement géodésique ou qui correspond à une solution de l'équation de sinh-Gordon), une sous-variété lagrangienne de dimension 3 de $S^3 \times S^3$. Ici il faut remarquer que cette variété lagrangienne n'est pas unique. En effet, pour chaque solution d'une équation différentielle supplémentaire il existe une telle variété lagrangienne.

Theorem 3. *Soit ω et μ des solutions de l'équation différentielle de respectivement, Sinh-Gordon ($\Delta\omega = -8\sinh\omega$) et Liouville ($\Delta\mu = -e^{\mu}$) définies sur un ouvert simplement connexe $U \subseteq \mathbb{C}$ et soit $p : U \to \mathbb{S}^3$ la surface minimale associée.*
Soit $V = \{(z,t) \mid z \in U, t \in \mathbb{R}, e^{\omega+\mu} - 2 - 2\cos(4t) > 0\}$ et soit Λ une solution de

$$\left(\frac{2\sqrt{3}e^{\omega}}{\tan\Lambda} - 2\sin(2t)\right)^2 = e^{\omega+\mu} - 2 - 2\cos(4t)$$

on V. Alors, il existe une immersion lagrangienne $f : V \to S^3 \times S^3 : x \mapsto (p(x), q(x))$, où q est déterminée par

$$\begin{aligned}
\frac{\partial q}{\partial t} =& -\frac{\sqrt{3}}{2\sqrt{3}e^{\omega} - 2\sin(2t)\tan\Lambda}\ q\ \alpha_2 \times \alpha_3,\\
\frac{\partial q}{\partial u} =& \frac{1}{8}\Big(e^{-\omega}\left(\mu_v + \omega_v - \frac{(\mu_u + \omega_u)\cos(2t)\tan\Lambda}{\sqrt{3}e^{\omega} - \sin(2t)\tan\Lambda}\right) q\ \alpha_2 \times \alpha_3 - 4(\sqrt{3}\cot\Lambda\cos(2t) + 1)\ q\ \alpha_2 -\\
&-4\sqrt{3}\sin(2t)\cot\Lambda\ q\ \alpha_3\Big),\\
\frac{\partial q}{\partial v} =& \frac{1}{8}\Big(-e^{-\omega}\left(\mu_u + \omega_u + \frac{(\mu_v + \omega_v)\cos(2t)\tan\Lambda}{\sqrt{3}e^{\omega} - \sin(2t)\tan\Lambda}\right)\ q\ \alpha_2 \times \alpha_3 - 4\sqrt{3}\cot\Lambda\sin(2t)\ q\ \alpha_2 +\\
&+4(1 + \sqrt{3}\cos(2t)\cot\Lambda)\ q\ \alpha_3\Big),
\end{aligned}$$

avec $\alpha_2 = \bar{p}p_u$ et $\alpha_3 = \bar{p}p_v$.

Theorem 4. *Soit X_1, X_2, X_3 les champs de vecteurs canoniques sur $\mathbb{S}^3$. Soit β une solution de l'équation différentielle*

$$\begin{aligned}
&X_1(\beta) = 0,\\
&X_2(X_2(\beta)) + X_3(X_3(\beta)) = \frac{2(3 - e^{4\beta})}{e^{4\beta}},
\end{aligned}$$

sur un domaine connexe, simplement connexe U de $\mathbb{S}^3$.
Alors il existe une immersion lagrangienne $f : U \to \mathbb{S}^3 \times \mathbb{S}^3 : x \mapsto (p(x), q(x))$, où $p(x) = xix^{-1}$ et q est déterminée par

$$\begin{aligned}
X_1(q) &= -2qhxix^{-1}h^{-1},\\
X_2(q) &= q\left(-X_3(\beta)hxix^{-1}h^{-1} - (1 - \sqrt{3}e^{-2\beta})\ hxjx^{-1}h^{-1}\right),\\
X_3(q) &= q\left(X_2(\beta)\ hxix^{-1}h^{-1} - (1 + \sqrt{3}e^{-2\beta})\ hxkx^{-1}h^{-1}\right).
\end{aligned}$$

Dans le théorème précédent, l'image de p est une surface totalement géodésique dans $\mathbb{S}^3$. Dans la construction réciproque, il y a un cas exceptionnel à considérer.

Theorem 5. *Soit ω une solution l'équation de Sinh-Gordon $\Delta\omega = -8\sinh\omega$ définie sur un domaine ouvert et simplement connexe U de $\mathbb{C}$ et soit $p : U \to \mathbb{S}^3$ la surface minimale associée. Alors il existe une immersion lagrangienne $f : U \times \mathbb{R} \to \mathbb{S}^3 \times \mathbb{S}^3 : x \mapsto (p(x), q(x))$, où q est déterminée par*

$$\frac{\partial q}{\partial t} = -\frac{\sqrt{3}e^{-\omega}}{4} q\ \alpha_2 \times \alpha_3,$$

$$\frac{\partial q}{\partial u} = \frac{e^{-\omega}}{8}(4e^{\omega}q\alpha_2 - 4q\alpha_3 + \omega_v q\ \alpha_2 \times \alpha_3),$$
$$\frac{\partial q}{\partial v} = -\frac{e^{-\omega}}{8}(4q\alpha_2 - 4e^{\omega}q\alpha_3 + \omega_u q\ \alpha_2 \times \alpha_3).$$

où $\alpha_2 = \bar{p}p_u$ *and* $\alpha_3 = \bar{p}p_v$*.*

Finalement nous avons montré que localement, une immersion lagrangienne, pour laquelle p est nulle part une immersion, est obtenue comme décrit dans un des théorèmes précédents.

Les fonctions angulaires θ_1, θ_2, θ_3, définies à partir des directions principales de P, jouent un rôle très important dans l'étude des sous-variétés lagrangiennes de $\mathbb{S}^3 \times \mathbb{S}^3$. Dans ce sens, nous avons un premier résultat:

Theorem 6. *Soit*

$$f : M \to \mathbb{S}^3 \times \mathbb{S}^3$$
$$x \mapsto f(x) = (p(x), q(x)),$$

une immersion lagrangienne. Si toutes les fonctions angulaires sont constantes alors M est totalement géodésique ou M est un espace à courbure sectionnelle constante.

Cela signifie qu'en appliquant le résultat de Dioos, Vrancken et Wang ([20]), nous obtenons une classification complète de ces sous-variétés. Il est connu que la somme des fonctions angulaires est toujours un multiple de π. Donc si deux de ces fonctions sont constantes, la troisième doit aussi être constante. Pour cette raison, nous avons étudié le cas des sous-variétés Lagrangiennes pour lesquelles seulement une de ces fonctions est constante. Ainsi, nous avons montré :

Theorem 7. *Soit*

$$f : M \to \mathbb{S}^3 \times \mathbb{S}^3$$
$$x \mapsto f(x) = (p(x), q(x)),$$

une immersion lagrangienne. Si exactement une fonction angulaire θ est constante, alors $\theta = 0$ ou $\theta = \frac{\pi}{3}$ ou $\theta = \frac{2\pi}{3}$.

Géométriquement, l'angle de $\frac{\pi}{3}$ correspond au cas où $p(M)$ est une surface minimale dans $\mathbb{S}^3$ (et donc une classification complète est obtenue en appliquant les théorèmes précédents). Les autres cas suivent des deux constructions remarquables obtenues dans les résultats qui suivent.

Theorem 8. *Soit*

$$f : M \to \mathbb{S}^3 \times \mathbb{S}^3$$
$$x \mapsto f(x) = (p(x), q(x)),$$

une immersion lagrangienne. Notons les fonctions angulaires $\theta_1, \theta_2, \theta_3$ et les vecteurs propres correspondants E_1, E_2, E_3. Alors $\tilde{f} : M \longrightarrow \mathbb{S}^3 \times \mathbb{S}^3$ donnée par $\tilde{f} = (q, p)$ a les propriétés suivantes :

(i) $\tilde{f}$ est une immersion Lagrangienne,

(ii) *la métrique induite par f et $\tilde{f}$ sur M est la même,*

(iii) *les fonctions angulaires sont liées par $\tilde{\theta}_i = \pi - \theta_i$, où $i = 1, 2, 3$ et les vecteurs propres correspondants sont les mêmes.*

et

Theorem 9. *Soit*

$$\begin{aligned} f &: M \to \mathbb{S}^3 \times \mathbb{S}^3 \\ x &\mapsto f(x) = (p(x), q(x)), \end{aligned}$$

une immersion lagrangienne. Notons les fonctions angulaires $\theta_1, \theta_2, \theta_3$ et les vecteurs propres correspondants E_1, E_2, E_3. Alors $f^* : M \longrightarrow \mathbb{S}^3 \times \mathbb{S}^3$ donné par $f^* = (\bar{p}, q\bar{p})$ a les propriétés suivantes:

(i) *f^* est une immersion lagrangienne,*

(ii) *la métrique induite par f et f^* sur M est la même,*

(iii) *les fonctions angulaires sont liées par $\theta_i^* = \frac{2\pi}{3} - \theta_i$, où $i = 1, 2, 3$ et les vecteurs propres correspondants sont les mêmes.*

§ La géométrie différentielle affine

Dans ce domaine, nous étudions les sous-variétés M de $\mathbb{R}^{n+1}$. Cette étude fait partie du programme de Felix Klein, c'est-à-dire la géométrie est l'étude des propriétés qui restent invariantes sous l'action d'un groupe donné de transformations. L'étude de la géométrie différentielle affine commence par le travail de Blaschke et de ses collègues au début du siècle précédent. Les 30 dernières années, il y a eu une reprise d'intérêt pour ce domaine et beaucoup de géomètres célèbres tels que Bobenko, Calabi, Chern, Nomizu, Pinkall, Sasaki, Simon, Terng, Trudinger et Yau ont étudié cette géométrie. Le premier problème fondamental rencontré dans le domaine de la géométrie différentielle affine est comment on peut, à partir de la structure équiaffine donnée sur $\mathbb{R}^{n+1}$, introduire une structure équiaffine sur la sous-variété M. Pour les hypersurfaces, la solution à ce problème est bien connue. Dans le cas où l'hypersurface est nondégénérée, il est possible de déterminer un champ de vecteurs transversal canonique et une forme bilinéaire symétrique, qu'on appelle respectivement *le normal affine* et *la métrique affine h*.

Une hypersurface affine est appelée *sphère affine* si soit tous les normaux affines passent à travers un point fixe (*les sphères affines propres*), soit tous les normaux affines sont parallèles (*les sphères affines impropres*). Cette classe est, sans aucun doute, la plus étudiée, voir par exemple les résultats de classification obtenus par Chern, Li et Yau dans le cas où la métrique est définie positive et complète. Cependant, dans tous les autres cas, il reste de nombreux problèmes non résolus. Malgré ce que l'on pourrait imaginer, il existe localement beaucoup de sphères affines, propres et impropres. En effet, par exemple, l'étude des sphères impropres est équivalente à l'étude de l'équation différentielle de Monge Ampère.

Donc si nous voulons obtenir plus de résultats, nous avons besoin de conditions supplémentaires.

Pour cela le tenseur le plus adapté à utiliser est le tenseur de différence K. Ce tenseur donne la différence entre la connection induite et la connection de Levi Civita de la métrique affine. Un théorème classique, dû à Berwald, montre que le tenseur K est nul si et seulement si l'hypersurface est une quadrique non dégénérée. Une condition naturelle sur K est la notion d'isotropie. On dit que l'hypersurface est λ-*isotrope* si et seulement si

$$h(K(v,v),K(v,v)) = \lambda(p)h(v,v)h(v,v),$$

pour tout vecteur v tangent à un point p.
Dans le cas où la métrique est définie positive, une classification a été obtenue par O. Birembaux et M. Djoric. Ils ont montré en 2012 que si $K \neq 0$, la dimension ne peut être que 2, 5, 8, 14 ou 26. De plus, en dimension 2 toute sphère affine est λ-isotrope. Dans les quatre autres dimensions, il existe un seul exemple canonique qui est, respectivement, l'immersion standard de l'espace symétrique $SL(3,\mathbb{R})/SO(3)$, $SL(3,\mathbb{C})/SU(3)$, $SU^*\big(6\big)/Sp(3)$ ou $E_{6(-26)}/F_4$. Ici il faut remarquer que l'ingrédient crucial dans la preuve est que l'espace tangent unitaire en un point p est un espace compact et qu' une fonction continue sur un espace compact admet un maximum. C'est un argument qui, bien sûr, ne peut pas du tout être adapté pour traiter le cas où la métrique n'est pas définie positive. Néanmoins, nous avons réussi à démontrer que le théorème qui donne les dimensions possibles reste vrai dans le cas où la métrique n'est pas définie positive. De plus nous avons montré que dans le cas où la métrique n'est pas définie positive :

1. en dimension 5, nous avons précisément un exemple supplémentaire qui est l'immersion canonique de l'espace symétrique $SL(3,\mathbb{R})/SO(2,1)$,
2. en dimension 8, nous avons précisément deux exemples supplémentaires que sont l'immersion canonique de l'espace symétrique $SL(3,\mathbb{C})/SU(2,1)$ et l'immersion canonique de $SL(3,\mathbb{R})$,
3. en dimension 14, nous avons précisément deux exemples supplémentaires que sont l'immersion canonique de l'espace symétrique $SU^*\big(6\big)/Sp(1,2)$ et l'immersion canonique de $SL\big(6,\mathbb{R}\big)/Sp(6)$,
4. en dimension 26, nous avons précisément deux exemples supplémentaires construits, respectivement, à partir des nombres octonions et des nombres split-octonions.

§Surfaces dans $\mathbb{E}^3$ et $\mathbb{L}^3$

Rappelons que $\mathbb{L}^3$ est l'espace affine euclidien muni de la métrique de signature $+,+,-$. Dans ce chapitre, je présente les résultats qui ont été obtenus en collaboration avec Rafael López ([35]), sur les surfaces de translation et surfaces homothétiques dans les espaces $\mathbb{R}^3$ et $\mathbb{L}^3$.
On dit qu'une surface S est une *surface de translation* si elle peut être exprimée comme la somme de deux courbes $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ et $\beta : J \subset \mathbb{R} \to \mathbb{R}^3$. Elle peut être parametrisée comme $X(s,t) = \alpha(s) + \beta(t)$, $s \in I, t \in J$ (voir [15, p. 138]). La classification des surfaces de translation ayant une courbure moyenne constante (CMC) ou une courbure gaussienne constante (CGC) est un problème ouvert. Un premier exemple de surface de translation est la surface de Scherk (voir figure n^o 1) donnée par

$$z(x,y) = \frac{1}{a}\log\left(\left|\frac{\cos(ay)}{\cos(ax)}\right|\right),\ a > 0. \tag{1}$$

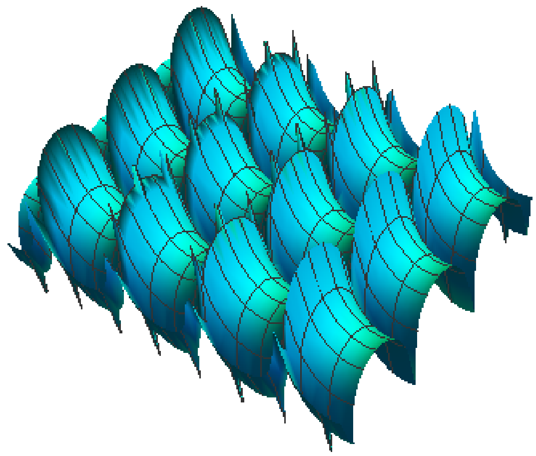


Figure 1: La surface de Scherk donnée par la paramétrisation dans (1).

Cette surface est minimale ($H = 0$) et appartient à une famille plus large de surfaces de Scherk ([45, pp. 67-73]). Les courbes génératrices α et β se trouvent dans des plans orthogonaux et, après un changement de coordonnées, la surface peut être representée localement comme le graphe de la somme de deux fonctions $z = f(x)+g(y)$. Les résultats connus jusqu'à présent comportent des conditions supplémentaires sur les deux courbes génératrices. Je les énumère ci-dessous. Si α et β sont dans des plans orthogonaux, alors les surfaces de translation de type CMC sont le plan, la surface minimale de Scherk et le cylindre circulaire ([33], [50]), tandis que celles de type CGC ont leur courbure gaussienne K nulle et s'identifient aux surfaces cylindriques ([33]). Dans le cas où les courbes α et β sont toutes les deux planes, alors les seuls surfaces minimales de translation sont le plan et une surface de la famille des surfaces de Scherk ([16]). Finalement, si seulement une des deux courbes α et β est plane, alors il n'existe pas de surfaces de translation minimales ([16]).
Nous avons obtenu un premier résultat sur les surfaces de translation qui ont leur courbure gaussienne K constante ([35]). Sans aucune condition supplémentaire, nous montrons que les seules surfaces de translation plates ($K = 0$) sont les surfaces cylindriques (voir figure n° 2). Ici, une *surface cylindrique* est une surface réglée, engendrée par une droite parallèle à une position fixe dans $\mathbb{R}^3$ et dont la directrice est plane. Nous avons montré le théoreme suivant:

Theorem 10. *1. Les seules surfaces de translation à courbure gaussienne nulle sont les surfaces cylindriques.*

2. Si une des courbes génératrices est plane, alors il n'existe pas de surfaces de translation ayant leur courbure gaussienne constante $K \neq 0$.

Dans le cas oú $K = 0$, nous donnons une classification complète des surfaces de type CGC et, pour $K \neq 0$, nous étendons le résultat donné en [16] pour les surfaces de type CMC.
Un deuxième type de surfaces étudiées dans cette partie est celui des *surfaces homothéthiques*. Elles se définissent comme les surfaces de translation, sauf qu'à la place du signe + dans la paramétrisation $z = f(x) + g(y)$, nous avons la multiplication $z = f(x) \cdot g(y)$. Le premier résultat que nous obtenons se réfère aux surfaces minimales. Précédemment, Van de Woestyne

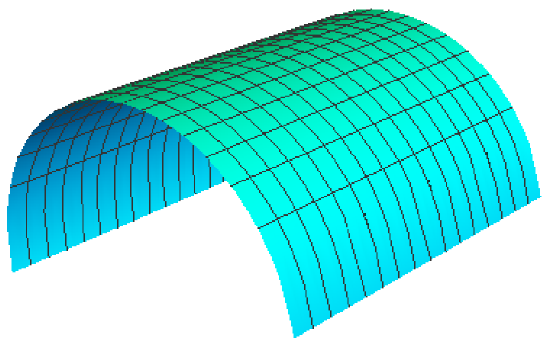


Figure 2: Une surface cylindrique dont la directrice est un demi-cercle.

a démontré (cf.[54]) que les seules surfaces minimales homothétiques non-dégénérées dans $\mathbb{L}^3$ sont les plans et les hélycoïdes. A la fin de son article, l'auteur affirme qu'un résultat similaire peut être obtenu dans l'espace euclidean $\mathbb{E}^3$. Dans [35], nous avons donné une démonstration différente pour le cas euclidien. Plus précisemment, nous avons prouvé les théorèmes suivants:

Theorem 11. *Les plans et les hélycoïdes sont les seules surfaces minimales homothétiques en* $\mathbb{E}^3$.

La paramétrisation de l'hélycoïde (voir figure n^o 3) n'est pas celle usuelle d'une surface réglée, ayant l'hélice comme base, mais

$$z(x,y) = (x+b)\tan(cy+d), \tag{2}$$

où $b, c, d \in \mathbb{R}$, $c \neq 0$ ([45, p. 20]).

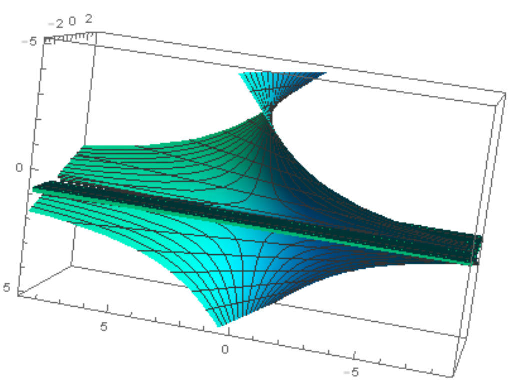


Figure 3: L'hélycoïde donné par la paramétrisation dans (2).

Un dernier résultat représente une classification complète des surfaces homothétiques dans $\mathbb{E}^3$ ayant une courbure gausienne constante:

Theorem 12. *Soit S une surface homothétique dans $\mathbb{E}^3$ de courbure gaussienne K constante. Alors, $K = 0$. De plus, la surface est soit un plan, une surface cylindrique ou une surface dont la paramétrisation est :*

(i)

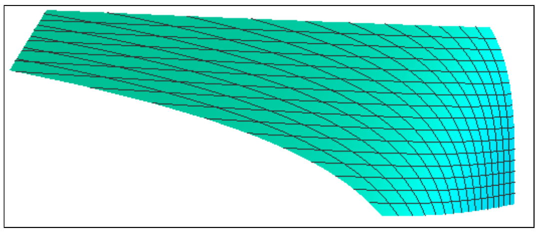
$$z(x,y) = ae^{bx+cy}, \tag{3}$$

où $a, b, c > 0$ (voir figure n^o 4a) ou

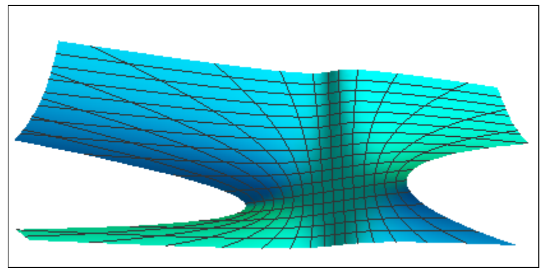
(ii)

$$z(x,y) = \left(\frac{bx}{m} + d\right)^m \left(\frac{cy}{m-1} + e\right)^{1-m}, \tag{4}$$

où $b, c, d, e, m \in \mathbb{R}$, $b, c \neq 0$, $m \neq 0, 1$ *(voir figure n^o 4b).*



(a) Une surface homothétique donnée par la paramétrisation dans le théorème 12.(i).



(b) Une surface homothétique donnée par la paramétrisation dans le théorème 12.(ii).

Figure 4: Des surfaces homothétiques de courbure gaussienne constante et, donc, nulle.

Summary

This work is structured in four chapters. In the first one, there is a brief presentation of the basic notions on which the studied problems rely. The second chapter develops around the study of Lagrangian submanifolds of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$.
The nearly Kähler manifolds are almost Hermitian manifolds with almost complex structure J for which the tensor field $\tilde{\nabla} J$ is skew-symmetric, where $\tilde{\nabla}$ is the Levi-Civita connection. I study non-totally geodesic Lagrangian submanifolds of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$ for which the projection on the first component is nowhere of maximal rank. I show that this property can be expressed in terms of the so called angle functions and that such Lagrangian submanifolds are closely related to minimal surfaces in $\mathbb{S}^3$.
Moreover, I study as well Lagrangian submanifolds of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$ for which all angle functions are constant. In this case the submanifold is either totally geodesic or has constant sectional curvature. Finally, if precisely one angle function is constant, I obtain a classification of such Lagrangian submanifolds. The results in this chapter are based on two articles written in collaboration with Burcu Bektaş, Joeri Van der Veken and Luc Vrancken (see [3], [4]).
The third chapter presents the results obtained together with Luc Vrancken on a problem from affine differential geometry treated in [41], where I gave a classification of isotropic affine hyperspheres. Finally, the last chapter presents some results on the study of translation and homothetical surfaces in $\mathbb{E}^3$ and $\mathbb{L}^3$. They may be found in [35], as they are based on a joint work with Rafael López, which has been finished during the first year of my PhD.

Short presentation of the results obtained

§ Lagrangian submanifolds of the nearly Kähler manifold $\mathbb{S}^3 \times \mathbb{S}^3$

The nearly Kähler manifolds have been studied intensively in the 1970's by Gray ([24]). Nagy ([43], [44]) made further contribution to the classification of nearly Kähler manifolds and more recently it has been shown by Butruille ([9]) that the only homogeneous 6-dimensional nearly Kähler manifolds are the nearly Kähler 6-sphere $\mathbb{S}^6$, $\mathbb{S}^3 \times \mathbb{S}^3$, the projective space $\mathbb{C}P^3$ and the flag manifold $SU(3)/U(1) \times U(1)$, where the last three are not endowed with the standard metric. All these spaces are compact 3-symmetric spaces. Note that in 2014 V. Cortés and J. J. Vásquez have discovered the first non homogeneous (but locally homogeneous) nearly Kähler structures in [14], while more recently, the first complete non homogeneous nearly Kähler structures were discovered on $\mathbb{S}^6$ and $\mathbb{S}^3 \times \mathbb{S}^3$ in [23].
A natural question for the above mentioned four homogeneous nearly Kähler manifolds is

to study their submanifolds. There are two natural types of submanifolds of nearly Kähler (or more generally, almost Hermitian) manifolds, namely *almost complex* and *totally real* submanifolds. Almost complex submanifolds are submanifolds whose tangent spaces are invariant under J. For a totally real submanifold, a tangent vector is mapped by the almost complex structure J into a normal vector. In this case, if additionally, the dimension of the submanifold is half the dimension of the ambient manifold, then the submanifold is called Lagrangian.

Note that the Lagrangian submanifolds of nearly strict Kähler manifolds are especially interesting as they are always minimal and orientable (see [22] for $\mathbb{S}^6$ or [52], [27] for the general case). Lagrangian submanifolds of $\mathbb{S}^6$ have been studied by many authors (see, amongst others, [19], [18], [21], [22],[57], [58], [36] and [49]), whereas the study of Lagrangian submanifolds of $\mathbb{S}^3 \times \mathbb{S}^3$ only started recently. The first examples of those were given in [52] and [40]. Moreover, in [59] and [20], the authors obtained a classification of the Lagrangian submanifolds, which are either totally geodesic or have constant sectional curvature. An important tool in the study in [20] and [59] is the use of an almost product structure P on $\mathbb{S}^3 \times \mathbb{S}^3$, which was introduced in [7]. The decomposition of P into a tangential part and a normal part along a Lagrangian submanifold allows us to introduce three principal directions, E_1, E_2, E_3, with corresponding angle functions $\theta_1, \theta_2, \theta_3$.

We study non-totally geodesic Lagrangian submanifolds of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$ for which the projection on the first factor is nowhere of maximal rank. We show that this property can be expressed in terms of the angle functions and that the Lagrangian submanifolds are closely related to minimal surfaces, in the sense of the following two results.

Theorem 13. *Let*

$$\begin{aligned} f : M &\to \mathbb{S}^3 \times \mathbb{S}^3 \\ x &\mapsto f(x) = (p(x), q(x)) \end{aligned}$$

be a Lagrangian immersion such that $p : M \to \mathbb{S}^3$ *has nowhere maximal rank. Then* $\frac{\pi}{3}$ *is an angle function up to a multiple of* π*. The converse is also true.*

Theorem 14. *Let*

$$\begin{aligned} f : M &\to \mathbb{S}^3 \times \mathbb{S}^3 \\ x &\mapsto f(x) = (p(x), q(x)) \end{aligned}$$

be a Lagrangian immersion such that $p : M \to \mathbb{S}^3$ *has nowhere maximal rank. Assume that* M *is not totally geodesic. Then* $p(M)$ *is a (branched) minimal surface in* $\mathbb{S}^3$.

For the next part, the study of the submanifold is separated into three cases and we manage to prove the reverse problem. That is, starting from a minimal surface in $\mathbb{S}^3$ (which is totally geodesic or corresponds to a solution of the sinh-Gordon equation), we can construct a Lagrangian submanifold of $\mathbb{S}^3 \times \mathbb{S}^3$. One should remark that the Lagrangian submanifold thus obtained is not unique, as for each solution of the sinh-Gordon equation there is locally more than one corresponding Lagrangian submanifold. The following theorems comprise these results.

Theorem 15. *Let* ω *and* μ *be solutions of, respectively, the Sinh-Gordon equation* $\Delta\omega = -8\sinh\omega$ *and the Liouville equation* $\Delta\mu = -e^{\mu}$ *on an open simply connected domain* $U \subseteq \mathbb{C}$

and let $p : U \to \mathbb{S}^3$ be the associated minimal surface with complex coordinate z such that $\sigma(\partial z, \partial z) = -1$.
Let $V = \{(z,t) \mid z \in U, t \in \mathbb{R}, e^{\omega+\mu} - 2 - 2\cos(4t) > 0\}$ and let Λ be a solution of

$$\left(\frac{2\sqrt{3}e^{\omega}}{\tan\Lambda} - 2\sin(2t)\right)^2 = e^{\omega+\mu} - 2 - 2\cos(4t)$$

on V. Then, there exists a Lagrangian immersion $f : V \to \mathbb{S}^3 \times \mathbb{S}^3 : x \mapsto (p(x), q(x))$, where q is determined by

$$\begin{aligned}
\frac{\partial q}{\partial t} =& -\frac{\sqrt{3}}{2\sqrt{3}e^{\omega} - 2\sin(2t)\tan\Lambda}\ q\ \alpha_2 \times \alpha_3,\\
\frac{\partial q}{\partial u} =& \frac{1}{8}\Big(e^{-\omega}\left(\mu_v + \omega_v - \frac{(\mu_u+\omega_u)\cos(2t)\tan\Lambda}{\sqrt{3}e^{\omega} - \sin(2t)\tan\Lambda}\right) q\ \alpha_2 \times \alpha_3 - 4(\sqrt{3}\cot\Lambda\cos(2t) + 1)\ q\ \alpha_2 -\\
& -4\sqrt{3}\sin(2t)\cot\Lambda\ q\ \alpha_3\Big),\\
\frac{\partial q}{\partial v} =& \frac{1}{8}\Big(-e^{-\omega}\left(\mu_u + \omega_u + \frac{(\mu_v+\omega_v)\cos(2t)\tan\Lambda}{\sqrt{3}e^{\omega} - \sin(2t)\tan\Lambda}\right)\ q\ \alpha_2 \times \alpha_3 - 4\sqrt{3}\cot\Lambda\sin(2t)\ q\ \alpha_2 +\\
& +4(1 + \sqrt{3}\cos(2t)\cot\Lambda)\ q\ \alpha_3\Big),
\end{aligned}$$

where $\alpha_2 = \bar{p}p_u$ and $\alpha_3 = \bar{p}p_v$.

Theorem 16. *Let X_1, X_2, X_3 be the standard vector fields on $\mathbb{S}^3$. Let β be a solution of the differential equations*

$$\begin{aligned}
&X_1(\beta) = 0,\\
&X_2(X_2(\beta)) + X_3(X_3(\beta)) = \frac{2(3 - e^{4\beta})}{e^{4\beta}},
\end{aligned}$$

on a connected, simply connected open subset U of $\mathbb{S}^3$.
Then there exist a Lagrangian immersion $f : U \to \mathbb{S}^3 \times \mathbb{S}^3 : x \mapsto (p(x), q(x))$, where $p(x) = xix^{-1}$ and q is determined by

$$\begin{aligned}
&X_1(q) = -2qhxix^{-1}h^{-1},\\
&X_2(q) = q\left(-X_3(\beta)hxix^{-1}h^{-1} - (1 - \sqrt{3}e^{-2\beta})\ hxjx^{-1}h^{-1}\right),\\
&X_3(q) = q\left(X_2(\beta)\ hxix^{-1}h^{-1} - (1 + \sqrt{3}e^{-2\beta})\ hxkx^{-1}h^{-1}\right).
\end{aligned}$$

Note that in the previous theorem the image of p is a totally geodesic surface in $\mathbb{S}^3$.

Theorem 17. *Let ω be a solution of the Sinh-Gordon equation $\Delta\omega = -8\sinh\omega$ on an open connected domain of U in $\mathbb{C}$ and let $p : U \to \mathbb{S}^3$ be the associated minimal surface with complex coordinate z such that $\sigma(\partial z, \partial z) = -1$. Then, there exists a Lagrangian immersion $f : U \times \mathbb{R} \to \mathbb{S}^3 \times \mathbb{S}^3 : x \mapsto (p(x), q(x))$, where q is determined by*

$$\begin{aligned}
\frac{\partial q}{\partial t} &= -\frac{\sqrt{3}e^{-\omega}}{4} q\ \alpha_2 \times \alpha_3,\\
\frac{\partial q}{\partial u} &= \frac{e^{-\omega}}{8}(4e^{\omega}q\alpha_2 - 4q\alpha_3 + \omega_v q\ \alpha_2 \times \alpha_3),\\
\frac{\partial q}{\partial v} &= -\frac{e^{-\omega}}{8}(4q\alpha_2 - 4e^{\omega}q\alpha_3 + \omega_u q\ \alpha_2 \times \alpha_3),
\end{aligned}$$

for $\alpha_2 = \bar{p}p_u$ and $\alpha_3 = \bar{p}p_v$.

Finally, we indicate that a Lagrangian immersion for which p has nowhere maximal rank is always obtained in the way indecated by the latter three theorems:

Theorem 18. *Let $f : M \to \mathbb{S}^3 \times \mathbb{S}^3 : x \mapsto (p(x), q(x))$ be a Lagrangian immersion such that p has nowhere maximal rank. Then every point x of an open dense subset of M has a neighborhood U such that $f|_U$ is obtained as described in Theorem 15, 16 or 17.*

As already seen so far, the angle functions play an important role in the study of the Lagrangian submanifolds of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$. They provide important information about the submanifold, as it may be further seen in the results obtained in section 2.2.
In case that all the angle functions are constant, we have the following theorem.

Theorem 19. *A Lagrangian submanifold of the nearly Kähler manifold $\mathbb{S}^3 \times \mathbb{S}^3$ given by*

$$\begin{aligned} f &: M \to \mathbb{S}^3 \times \mathbb{S}^3 \\ x &\mapsto f(x) = (p(x), q(x)), \end{aligned}$$

for which all angle functions are constant, is either totally geodesic or has constant sectional curvature in $\mathbb{S}^3 \times \mathbb{S}^3$.

This means that, by applying the result of Dioos, Vrancken and Wang ([20]), we obtain a complete classification for such Lagrangian submanifolds. One should remark that the sum of the angle functions is always a multiple of π. Hence, if two of the angles are constant, so is the third one too. Therefore, one of the results obtained concerns the case when exactly on angle function is constant:

Theorem 20. *Let M be a Lagrangian submanifold in the nearly Kähler manifold $\mathbb{S}^3 \times \mathbb{S}^3$ given by*

$$\begin{aligned} f &: M \to \mathbb{S}^3 \times \mathbb{S}^3 \\ x &\mapsto f(x) = (p(x), q(x)), \end{aligned}$$

with angle functions $\theta_1, \theta_2, \theta_3$. If precisely one of the angle functions is constant, then up to a multiple of π, it can be either $0, \frac{\pi}{3}$ or $\frac{2\pi}{3}$.

From a geometrical point of view, the angle $\frac{\pi}{3}$ corresponds to the case when $p(M)$ is a minimal surface in $\mathbb{S}^3$ and therefore, the Lagrangian immersion is determined in the sense of theorem 18. The other two cases corresponding to the remaining values of θ (0 and $\frac{2\pi}{3}$) follow easily from the case when $\theta = \frac{\pi}{3}$, by using the two constructions given in the following two theorems.

Theorem 21. *Let $f : M \longrightarrow \mathbb{S}^3 \times \mathbb{S}^3$ be a Lagrangian immersion into the nearly Kähler manifold $\mathbb{S}^3 \times \mathbb{S}^3$, given by $f = (p, q)$ with angle functions $\theta_1, \theta_2, \theta_3$ and eigenvectors E_1, E_2, E_3. Then $\tilde{f} : M \longrightarrow \mathbb{S}^3 \times \mathbb{S}^3$ given by $\tilde{f} = (q, p)$ satisfies:*

(i) $\tilde{f}$ is a Lagrangian immersion,

(ii) f and $\tilde{f}$ induce the same metric on M,

(iii) E_1, E_2, E_3 are also eigendirections of the operators $\tilde{A}, \tilde{B}$ corresponding to the immersion $\tilde{f}$ and the angle functions $\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3$ are given by $\tilde{\theta}_i = \pi - \theta_i$, for $i = 1, 2, 3$.

Theorem 22. *Let $f : M \longrightarrow \mathbb{S}^3 \times \mathbb{S}^3$ be a Lagrangian immersion into the nearly Kähler manifold $\mathbb{S}^3 \times \mathbb{S}^3$ given by $f = (p, q)$ with angle functions $\theta_1, \theta_2, \theta_3$ and eigenvectors E_1, E_2, E_3. Then, $f^* : M \longrightarrow \mathbb{S}^3 \times \mathbb{S}^3$ given by $f^* = (\bar{p}, q\bar{p})$ satisfies:*

(i) f^* is a Lagrangian immersion,

(ii) f and f^* induce the same metric on M,

(iii) E_1, E_2, E_3 are also eigendirections of the operators A^*, B^* corresponding to the immersion f^* and the angle functions $\theta_1^*, \theta_2^*, \theta_3^*$ are given by $\theta_i^* = \frac{2\pi}{3} - \theta_i$, for $i = 1, 2, 3$.

§Affine differential geometry

In this domain, the submanifolds M in $\mathbb{R}^{n+1}$ are the main subject of interest. Their study is part of the Felix Klein program, which sees geometry as the study of properties which stay invariant under the action of some group of transformations. The study of affine differential geometry starts at the beginning of the previous century with the work of Blaschke and his collegues. In the last 30 years there was a rise of interest in this domain and many famous geometers such as Bobenko, Calabi, Chern, Nomizu, Pinkall, Sasaki, Simon, Terng, Trudinger and Yau have studied this geometry. The first fundamental problem that one finds in affine geometry refers to finding a way to introduce an equiaffine structure on the submanifold M, starting from the equiaffine structure given from $\mathbb{R}^{n+1}$. As far as the hypersurfaces are concerned, the solution for this problem is well known. In this case, it is possible to find a canonical transversal vector field and a biliniar symmetric form, which we call, respectively, *the affine normal* and *the affine metric h*. An affine hypersurface is called *affine sphere* if all the affine normals pass through the same fixed point (*proper hyperspheres*) or, if all the affine normals are parallel (*improper spheres*). This class is definetely the most studied one (see, for example, the results obtained by Chern, Li and Yau for the case when the metric is positive definite and complete). Nevertheless, in all the other cases, there are still many other unsolved problems. Despite what one could think, locally, there are many proper or improper affine spheres in the sense that, in fact, the study of improper spheres is equivalent to the study of the Monge Ampère differential equation. Therefore, in order to obtain some results, one actually needs extra conditions and this is where *the difference tensor K* intervenes. This tensor gives the difference between the induced connection and the Levi-Civita connection of the affine metric h. A classical theorem shows that K is zero if and only if the affine hypersurface is a non-degenerate quadric. A natural condition on K that one could look at leads to the notion of *isotropy*. An hypersurface M is called *λ-isotropic* if and only if there exist a function λ on M such that

$$h(K(v, v), K(v, v)) = \lambda(p)h(v, v)h(v, v),$$

for all tangent vector v at $p \in M$. In the case when the metric is positive definite, one classification was obtained by Birembaux and Djoric. They proved in 2012 that if $K \neq 0$, the dimension can be $2, 5, 8, 14$ or 26. Moreover, in dimension 2, any affine sphere is λ-isotropic. For the other remaining dimensions, there exists one sole canonical example, that is, respectively, the standard immersion of the symmetric space $SL(3, \mathbb{R})/SO(3)$, $SL(3, \mathbb{C})/SU(3)$, $SU^*(6)/Sp(3)$ and $E_{6(-26)}/F_4$. It is important to remark that the essential argument in the proof in the positive definite case is the fact that the unit tangent bundle at a point p is a compact space, and, a continuous function on a compact space attains a maximum. This does not hold anymore when the metric is not positive definite. Nevertheless, as one may see it is proven in chapter 3, the same theorem which gives the possible dimensions of the affine hyperspheres in the positive definite case, holds as well for the indefinite case. Therefore, the results of Djoric and Birembaux from the definite case are completed by the ones obtained in chapter 3, concerning the indefinite case, as follows:

1. in dimension 5, there is precisely one additional example for the studied affine hyperspheres: the standard immersion of $\frac{SL(3,\mathbb{R})}{SO(2,1)}$ in $\mathbb{R}^6$,
2. in dimension 8, there are precisely two additional examples,that is the cannonical immersion of $\frac{SL(3,\mathbb{C})}{SU(2,1)}$ and $SL(3,\mathbb{R})$, respectively,
3. in dimension 14 there are precisely two additional examples, namely the cannonical immersion of $\frac{SU^*(6)}{Sp(1,2)}$ and $\frac{SL(6,\mathbb{R})}{Sp(6)}$,
4. in dimension 26, there are exactly two examples which are constructed by use of octonions and split-octonions.

§Surfaces in $\mathbb{E}^3$ and $\mathbb{L}^3$

Chapter 4 presents the results obtained in [35], together with professor Rafael López on translation and on homothetical surfaces, respectively, in the Euclidean space $\mathbb{R}^3$ and $\mathbb{L}^3$. A *translation surface* S is a surface that can be expressed as the sum of two curves $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$, $\beta : J \subset \mathbb{R} \to \mathbb{R}^3$. In a parametric form, the surface S writes as $X(s,t) = \alpha(s) + \beta(t)$, $s \in I, t \in J$. See [15, p. 138]. It is an open problem to classify all translation surfaces with constant mean curvature (CMC) or constant Gauss curvature (CGC). A first example of a CMC translation surface is the Scherk surface (see figure no.5) given by

$$z(x,y) = \frac{1}{a}\log\left(\left|\frac{\cos(ay)}{\cos(ax)}\right|\right), \ a > 0. \tag{5}$$

This surface is minimal ($H = 0$) and belongs to a more general family of Scherk surfaces ([45, pp. 67-73]). In this case, the curves α and β lie in two orthogonal planes and after a change of coordinates, the surface is locally described as the graph of $z = f(x) + g(y)$. The progress made so far on this problem always asked for extra conditions on the two generating curves, in the following sense. If α and β lie in orthogonal planes, the only CMC translation surfaces are the plane, the Scherk surface and the circular cylinder ([33], [50]) and the only CGC translation surfaces have in fact $K = 0$ and are cylindrical surfaces ([33]). Another case is that when both curves α and β are planar: then the only minimal translation surfaces are the plane or a surface which belongs to the family of Scherk surfaces ([16]). Finally, if one of the curves α or β is planar and the other one is not, there are no minimal translation surfaces ([16]).

The first result obtained in [35] and, therefore, presented in this chapter, concerns the case when the Gauss curvature K is constant. Without making any assumption on the curves α and β, we prove that the only flat ($K = 0$) translation surfaces are cylindrical surfaces. Notice

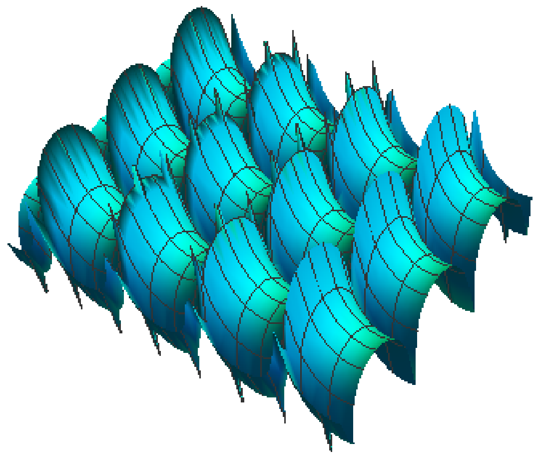


Figure 5: The surface of Scherk given in (5).

that by *a cylindrical surface* we mean a ruled surface whose directrix is contained in a plane and the rulings are parallel to a fixed direction in $\mathbb{R}^3$ (see figure no.6). The corresponding theorem is the following:

Theorem 23. *1. The only translation surfaces with zero Gauss curvature are cylindrical surfaces.*

2. There are no translation surfaces with constant Gauss curvature $K \neq 0$ if one of the generating curves is planar.

For the case when $K = 0$ we give a complete classification of the CGC translation surfaces and, for $K \neq 0$, we extend the result given in [16] for CMC translation surfaces.

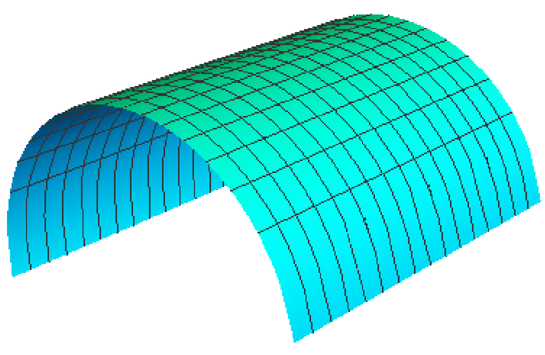


Figure 6: A cylindrical surface whose directrix is a semi-circle.

A second kind of surfaces studied in this chapter are *the homothetical surfaces*. Roughly speaking, we replace the plus sign $+$ in the definition of a translation surface $z = f(x) + g(y)$ by the multiplication operation $z = f(x)g(y)$.

Our first result on this problem concerns minimal surfaces. Van de Woestyne proved in [54] that the only minimal homothetical non-degenerate surfaces in $\mathbb{L}^3$ are planes and helicoids.

At the end of [54] the author asserted that, up to small changes in the proof, a similar result can be obtained in the Euclidean space $\mathbb{R}^3$. In the present paper we do a different proof of the Euclidean version and in section 4.2 we prove:

Theorem 24. *Planes and helicoids are the only minimal homothetical surfaces in Euclidean space.*

The parametrization of the helicoid (see figure no.7) is not the usual one as for a ruled surface which has a helix as base, but

$$z(x, y) = (x + b) \tan(cy + d), \tag{6}$$

where $b, c, d \in \mathbb{R}$, $c \neq 0$ ([45, p. 20]).

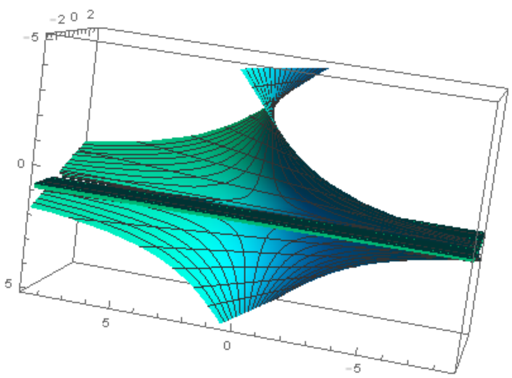


Figure 7: A helicoid given by the parametrization in (6).

The third result considers homothetical surfaces in the Euclidean space with constant Gauss curvature, for which we obtained a complete classification.

Theorem 25. *Let S be a homothetical surface in Euclidean space $\mathbb{R}^3$ with constant Gauss curvature K. Then $K = 0$. Furthermore, the surface is either a plane, a cylindrical surface or a surface whose parametrization is:*

(i)

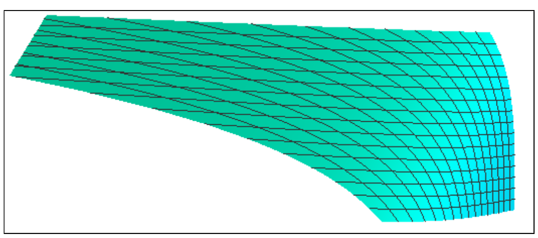
$$z(x, y) = ae^{bx+cy}, \tag{7}$$

with $a, b, c > 0$ (see figure no.8a), or

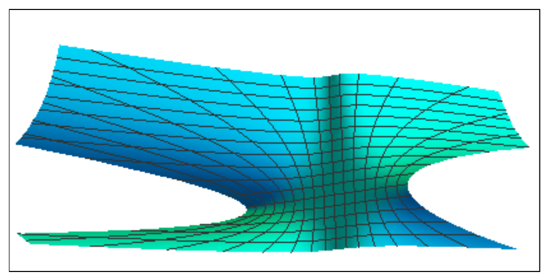
(ii)

$$z(x, y) = \left(\frac{bx}{m} + d\right)^m \left(\frac{cy}{m-1} + e\right)^{1-m}, \tag{8}$$

with $b, c, d, e, m \in \mathbb{R}$, $b, c \neq 0$, $m \neq 0, 1$ (see figure no.8b).



(a) A homothetical surface given by the parametrization in Theorem 25.(i)



(b) A homothetical surface given by the parametrization in Theorem 25.(ii).

Figure 8: Homothetical surfaces with constant Gauss curvature (K=0).

Chapter 1

Preliminaries

1.1 Affine, Riemannian and semi-Riemannian manifolds

In this section we briefly recalle the basic definitions on the geometry of affine manifolds, Riemannian manifolds and semi-Riemannian manifolds. The summary is based on [8], [11], [48] and [56], where one may find more details.

We say that M is a *topological manifold of dimension* n if it is a Hausdorff space with a countable basis of open sets and with the property that each point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$. Each pair (U, ϕ), where U is an open set of M and ϕ is a homeomorphism of U to an open subset of $\mathbb{R}^n$, is called a *coordinate neighborhood*: to $q \in U$ we assign the n-coordinates $x^1(q), \ldots, x^n(q)$ of its image $\phi(q)$ in $\mathbb{R}$- each x^i is a real-valued function on U, the ith *coordinate function*. We shall say that the charts (U, ϕ) and (V, ψ) are C^∞-compatible if $U \cap V$ nonempty implies that $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism of class C^∞ .

Definition 1. *A differentiable or C^∞ (or smooth) structure on a topological manifold M is a family $\mathcal{U} = \{(U_\alpha, \phi_\alpha)_\alpha\}$ of coordinate neighborhoods such that:*

1. *the set $\{U_\alpha\}$ covers M,*
2. *for any α, β the coordinate neighborhoods (U_α, ϕ_α) and (U_β, ϕ_β) are C^∞-compatible,*
3. *any coordinate neighborhood (V, ψ) compatible with every $(U_\alpha, \phi_\alpha) \in \mathcal{U}$ is itself in $\mathcal{U}$.*

A C^∞*-manifold* is a topological manifold together with a C^∞-differentiable structure. Let M be a differentiable manifold and let $p \in M$. We denote *the tangent space* of M at the point p by T_pM. Let X, Y be vector fields on M. Then we define a new vector field $[X, Y]$ by

$$[X, Y]f = X(Y(f)) - Y(X(f)),$$

which is called *the bracket* of X and Y. Let $\mathcal{X}(M)$ be the set of all vector fields of class C^∞ on M and $D(M)$ the ring of real-valued functions of class C^∞ defined on M.

Definition 2. *An* affine connection ∇ *on M is a mapping*

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$
$$(X, Y) \mapsto \nabla_X Y,$$

which satisfies:

(i) $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$,

(ii) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$,

(iii) $\nabla_X(fY) = f\nabla_X Y + X(f)Y$,

in which $X, Y, Z \in \mathcal{X}(M)$ *and* $f, g \in D(M)$.

We say that an affine connection ∇ on a smooth manifold M is *symmetric* when

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

for all $X, Y \in \mathcal{X}(M)$.

1.1.1 Affine manifolds

Definition 3. *A differentiable manifold equipped with a symmetric affine connection is called an* ***affine manifold****.*

The curvature tensor of an affine manifold is defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{1.1}$$

where X, Y, Z are tangent vector fields on M. It can be proved that $R(X,Y)Z$ at a point p of M is completely determined by the values of X, Y, Z at the point p. We call an affine manifold *flat* if R vanishes identically.
An affine manifold (M, ∇) is said to be *equiaffine* if there exists a volume form ω, i.e. a non-vanishing n-form, on M which is parallel with respect to ∇, that is

$$(\nabla_X\omega)(X_1 \ldots, X_n) = X(\omega(X_1, \ldots, X_n)) - \omega(\nabla_X X_1, \ldots, X_n) - \ldots - \omega(X_1, \ldots, \nabla_X X_n).$$

In this case, we say that (∇, ω) determine an *equiaffine structure* on M.

Example 1. Take $M = \mathbb{R}^n$. Let us denote by D the standard connection on M. Then, for vector fields $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$, we have that

$$D_X Y = (X(Y_1), X(Y_2), \ldots, X(Y_n)).$$

If we take on $\mathbb{R}^n$ the volume form Ω given by the determinant, a straightforward computation shows that (D, Ω) detrmines an equiaffine structure on $\mathbb{R}^n$. We also find that $\mathbb{R}^n$ is flat.

1.1.2 Riemannian manifolds

Definition 4. *A* Riemannian metric (or Riemannian structure) *on a differentiable manifold M is a correspondance which associates to each point $p \in M$ an inner product $\langle\cdot,\cdot\rangle_p$ (that is, a symmetric, bilinear, positive-definite form) which varies differentiably, i.e. for any pair of vector fields X, Y in a neighborhood V of M, the function $\langle X, Y\rangle$ is differentiable on V.*

Definition 5. *A differentiable manifold with a given Riemannian metric is called* ***a Riemannian manifold****.*

A connection ∇ on a Riemannian manifold is *compatible* with the metric if and only if

$$X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

Theorem 26. *(Levi-Civita). Given a Riemannian manifold M, there exists a unique affine connection ∇ on M, satisfying the conditions:*

a) ∇ is symmetric,

b) ∇ is compatible with the Riemannian metric.

∇ is called the *Levi-Civita connection* or *Riemannian connection* and is characterized by *the Koszul formula*

$$2\langle \nabla_V W, X\rangle = V\langle W,X\rangle + W\langle X,V\rangle - X\langle V,W\rangle - \langle V[W,X]\rangle + \langle W,[X,V]\rangle + \langle X,[V,W]\rangle. \quad (1.2)$$

It is clear that (M,∇) is an affine manifold. Thus, for the Levi-Civita connection, we define the Riemannian curvature tensor R of M by (1.1). Next, given the metric, we can define a $(0,4)$ curvature tensor $\tilde{R}$ associated with the curvature R by:

$$\tilde{R}(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle.$$

We have the following properties:

$$\begin{aligned}
&\tilde{R}(X,Y,Z,W) = -\tilde{R}(X,Y,W,Z),\\
&\tilde{R}(X,Y,Z,W) = \tilde{R}(Z,W,X,Y),\\
&R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0,\\
&(\nabla_X R)(Y,Z)W + (\nabla_Y R)(Z,X)W + (\nabla_Z R)(X,Y)W = 0.
\end{aligned} \quad (1.3)$$

The last two identities are called *the first* and *the second identity of Bianchi*, respectively. Let $y,z \in T_pM$. We define the *Ricci curvature* as

$$Ric(y,z) = \frac{1}{n-1} trace\{x \mapsto R(x,y)z\},$$

for all $x \in T_pM$. Let $\{z_1, z_2, \ldots, z_n\}$ be an orthonormal basis of T_pM. Then, we rewrite the Ricci curvature and define *the scalar curvature* for M as the following averages, respectively:

$$Ric_p(x,y) = \frac{1}{n-1}\sum_i \langle R(z_i,x)y, z_i\rangle,\ i = 1,2,\ldots,n-1, \quad (1.4)$$

$$K(p) = \frac{1}{n}\sum_j Ric_p(z_j) = \frac{1}{n(n-1)}\sum_{i,j}\langle R(z_j,z_i)z_i, z_j\rangle,\ j = 1,2,\ldots,n. \quad (1.5)$$

Definition 6. *Furthermore, given a point $p \in M$ and a two dimensional subspace $\sigma \subset T_pM$, the real number*

$$K(\sigma) = \frac{\tilde{R}(x,y,y,x)}{\langle x,x\rangle\langle y,y\rangle - \langle x,y\rangle^2},$$

where $\{x,y\}$ is any basis of σ, is called the sectional curvature *of σ at p.*

If the sectional curvature is independent of the tangent plane σ and of the point $p \in M$, then M is *a space of constant curvature.* A complete connected manifold of constant curvature is called *a real space form.* The curvature tensor R of a space of constant curvature c is given by

$$R(X,Y)Z = c\{\langle Y, Z\rangle X - \langle X, Z\rangle Y\}.$$

Example 2. Examples of real space forms:

(i) the Euclidean space $\mathbb{R}^n$ is a flat space (i.e. has constant zero sectional curvature),

(ii) the n-dimensional sphere $\mathbb{S}^n(r)$ of radius r has constant sectional curvature equal to $\frac{1}{r^2}$,

(iii) the hyperbolic space $\mathbb{H}^n(r)$ has constant negative sectional curvature equal to $-\frac{1}{r^2}$.

Next, if the Riemannian manifold M, admits an endomorphism J of the tangent space such that $J^2 = -Id$ and such that J maps differentiable vector fields into differentiable vector field, we say that M is *a Riemannian almost complex manifold.* M must have even real dimension, $2n$, that is, complex dimension n.
We say that J is *a complex structure* if the Nijenhuis tensor N defined by

$$N(X,Y) = [JX, JY] - [X,Y] - J[X, JY] - J[JX, Y],$$

vanishes identically (that is, J is *integrable*). By the theorem of $Newlander - Nirenberg$, we know that if J is a complex structure on M, then we can choose charts on M such that the coordinate changes are holomorphic functions from $\mathbb{C}^n$ to $\mathbb{C}^n$. Thus M becomes a complex manifold of dimension n. Moreover, if for a (an almost) complex manifold, the complex structure J is compatible with the metric, that is

$$\langle JX, JY\rangle = \langle X, Y\rangle,$$

then we call M an *(almost) Hermitian manifold.*
An almost Hermitian manifold is called *a nearly Kähler manifold* if the complex structure satisfies

$$(\nabla_X J)X = 0,$$

for all vector fields X on M. A Hermitian manifold is called *a Kähler manifold* if the complex structure satisfies

$$(\nabla_X J)Y = 0,$$

for all vector fields X and Y.

Example 3. $\mathbb{S}^3 \times \mathbb{S}^3$ is an example of a nearly Kähler manifold which is not a Kähler manifold. Its nearly Kähler structure is described in Chapter 2.

1.1.3 Semi-Riemannian manifolds

Definition 7. *A symmetric bilinear form b on V is*

(i) positive [negative] definite *provided $v \neq 0$ implies $b(v,v) > 0[< 0]$,*

(ii) positive [negative] semidefinite *provided $b(v,v) \geq 0, [\leq 0]$ for all $v \in V$,*

(iii) nondegenerate provided $b(v, w) = 0$ *for all* $w \in V$ *implies* $v = 0$.

Also, b is *definite* provided either alternative in (i) [(ii)] holds.

Definition 8. *The* index ν *of a symmetric bilinear form* b *on* V *is the largest integer that is the dimension of a subspace* $W \subset V$ *on which* $b_{|_W}$ *is negative definite.*

Notice that we call g a *scalar product* on a vector space V if it is a nondegenerate symmetric bilinear form on V, whereas an *inner product* is a *positive definite scalar product.* The results presented in the following part concern the case when the positive definiteness of the inner product is weakened to nondegeneracy.
A symmetric nondegenerate $(0, 2)$ tensor field g on a smooth manifold M of constant index is called *a metric tensor* on M. That is, $g \in \mathcal{T}(M)$ smoothly assigns to each point $p \in M$ a scalar product g_p on the tangent space T_pM and the index of g_p is the same for all p.

Definition 9. *Thus, a smooth manifold* M *furnished with a metric tensor* g *is called* ***a semi-Riemannian manifold****.*

Notice that the semi-Riemannian manifolds are often called *pseudo-Riemannian manifolds.* We will use $\langle \cdot, \cdot \rangle$ as an alternative notation for g.
The value ν of the index of g_p on a semi-Riemannian manifold M is called *the index of* M*:* $0 < \nu \leq dim M$. If $\nu = 0$, M is a Riemannian manifold; each g_p is then a (positive definite) inner product on T_pM. If $\nu = 1$ and $n > 2$, M is a *Lorentz manifold.*
The dot product on $\mathbb{R}^n$ gives rise to a metric tensor on R^n with

$$\langle v_p, w_p \rangle = v \cdot w = \sum v_i w_i,$$

where $v = \sum_{i=1}^{n} v_i \partial_i$, $v = \sum_{i=1}^{n} v_i \partial_i$ in the basis $\{\partial_i\}$ of T_pM. In any geometric context R^n will denote the resulting Riemannian manifold, called the *Euclidean n-space.* For $n \geq 2$, R^n_1 is called the *Minkowski n-space.*
If we denote by

$$\varepsilon_i = \begin{cases} -1 & for \quad 1 \leq i \leq \nu, \\ +1 & for \quad \nu + 1 \leq i \leq n, \end{cases}$$

then the metric tensor of $\mathbb{R}^n_\nu$ can be written $g = \sum \varepsilon du_i \otimes du_i$. The geometric significance of the index of a semi-Riemannian manifold derives from the following trichotomy:
We say that a tangent vector $v \in M$ is

$$\begin{aligned} &\textit{spacelike} && \text{if } \langle v, v \rangle > 0 \text{ or } v = 0, \\ &\textit{null} && \text{if } \langle v, v \rangle = 0 \text{ and } v \neq 0, \\ &\textit{timelike} && \text{if } \langle v, v \rangle < 0. \end{aligned}$$

The set of all null vectors in T_pM is called *the nullcone* at $p \in M$. The category into which a given tangent vector falls is called *its causal character.* This terminology derives from relativity theory, and particularly in the Lorentz case, null vectors are also said to be *lightlike.*
As for a Riemannian manifold, on a semi-Riemannian manifold M there is also a unique connection ∇ such that

(i) $[V, W] = \nabla_V W - \nabla_W V$,

(ii) $X\langle V, W\rangle = \langle \nabla_X V, W\rangle + \langle V, \nabla_X W\rangle$

for all X, V, W tangent vector fields on M. ∇ is called *the Levi-Civita connection of* M, and is characterized by *the Koszul formula* in (1.2).
Let M be a semi-Riemannian manifold with the Levi-Civita connection ∇. Then relation (1.1) defines the *Riemannian curvature tensor of* M. Given the metric $\langle \cdot, \cdot \rangle$ on T_pM, we associate the *curvature tensor* $\tilde{R}$ to the curvature R, defined by

$$\tilde{R}(X, Y, Z, W) = \langle R(X, Y)Z, W\rangle,$$

which satisfy the properties from (1.3).
Let $x, y \in T_pM$. Then we define the *Ricci curvature* as

$$Ric(x, y) = \frac{1}{n-1} trace\{z \mapsto R(x, z)y\},$$

for all $z \in T_pM$. Let $\{z_1, z_2, \ldots, z_n\}$ be an orthonormal basis of T_pM, that is $\langle z_i, z_j\rangle = 0$, for $i \neq j$ and $\langle z_i, z_i\rangle = \varepsilon_i$, where $\varepsilon_i = \pm 1$. Then, we rewrite the Ricci curvature and define *the scalar curvature* for M as the following averages, respectively:

$$Ric_p(x, y) = \frac{1}{n-1} \sum_i \varepsilon_i \langle R(z_i, x)y, z_i\rangle, \; i = 1, 2, \ldots, n-1, \tag{1.6}$$

$$K(p) = \frac{1}{n} \sum_j Ric_p(z_j) = \frac{1}{n(n-1)} \sum_{i,j} \varepsilon_j \varepsilon_i \langle R(z_j, z_i)z_i, z_j\rangle, \; j = 1, 2, \ldots, n, \tag{1.7}$$

where $\langle z_i, z_i\rangle = \varepsilon_i$.
A two-dimensional subspace σ of the tangent space T_pM is called *a tangent plane* to M at p. For tangent vectors v, w, define

$$Q(v, w) = \langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle^2.$$

A tangent plane σ is nondegenerate if and only if $Q(v, w) \neq 0$ for one- hence for every- basis v, w for σ. $Q(v, w)$ is positive if $g|_\sigma$ is definite, negative if it is indefinite. Let $\sigma \subset T_pM$ be a nondegenerate tangent plane to M at p. The number

$$K(v, w) = \frac{\langle R(w, v)v, w\rangle}{Q(v, w)},$$

is independent of the choice of basis v, w for σ and is called *the sectional curvature* $K(\sigma)$ of σ at p.

1.2 Submanifolds

Definition 10. *Let M^m and N^n be differentiable manifolds.*

- *A differentiable mapping $\phi : M \to N$ is said to be an* immersion *if $d\phi_p : T_pM \to T_{\phi(p)}N$ is injective for all $p \in M$.*

- *If, in addition, ϕ is a homeomorphism onto $\phi(M) \subset N$, where $\phi(M)$ has the subset topology induced from N, we say that ϕ is an* embedding.
- *If $M \subset N$ and the inclusion $i : M \to N$ is an embedding, we say that M is a* submanifold *of N.*

It can be seen that if $\phi : M \to N$ is an immersion, then $m \leq n$; the difference $n - m$ is called the codimension of the immersion ϕ.
A mapping which associates to each point $p \in M$ a tangent vector to N at $\phi(p)$ is called *a vector field* along ϕ. Then $d\phi$ maps the tangent space of M at $p \in M$ to a subspace of the tangent space of N at $\phi(p)$. Since ϕ is injective, this subspace has dimension m. Usually, this subspace is identified with the tangent space at the point p. Tangent vectors to N which do not belong to this subspace are called *transversal vectors to M*.
Affine immersions. Let M^m and N^n be affine manifolds. We denote the connection of M by ∇ and the connection on N by D. Then we call an immersion f an *affine immersion* if there exist $(n - m)$-transversal vector fields ξ_i such that for every $X, Y \in \mathcal{X}(M)$ we have

$$D_{\tilde{X}}\tilde{Y} \circ \phi = d\phi(\nabla_X Y) + \sum_{i=1}^{n-m} h^i(X,Y)\xi_i,$$

where $\tilde{X}(\phi(q)) = d\phi(X(q))$, $\tilde{Y}(\phi(q)) = d\phi(Y(q))$ and $h^i(X,Y)$ are symmetric bilinear forms on M. It can be shown that this definition is independent of the extension of $d\phi(X)$ and $d\phi(Y)$.
Isometric immersions. Suppose now that (M, g) and $(N, \tilde{g})$ are Riemannian manifolds. Then, we call an immersion $\phi : M \to N$ an *isometric immersion* if $\phi^*\tilde{g} = g$. We will make similar identifications as for affine manifolds and we will identify the metric g with $\phi^*\tilde{g}$, and denote both by $\langle \cdot, \cdot \rangle$.
Gauss and Weingarten formulas. A vector field ξ is called a *normal* vector field if, after making the necessary identifications, we have

$$\langle \xi, X \rangle = 0$$

for all tangent vector fields X to M. The normal space at a point $p \in M$ will be denoted by $T_p^\perp M$. Thus, it is easy to see that every vector field on N can be decomposed into a tangent vector field to M and a normal vector field to M. Therefore, if we denote by D the Levi-Civita connection on N, the formula of Gauss gives the decomposition of $D_X Y$, for $X, Y \in \mathcal{X}(M)$ as

$$D_X Y = \nabla_X Y + h(X,Y).$$

It is easy to see that ∇ is actually the Levi-Civita connection on M and $h(X,Y)$ is a bilinear and symmetric normal vector field on M. Then, h is called *the second fundamental form* of the immersion ϕ.
For $X \in \mathcal{X}(M)$ and ξ a normal vector field, we have the following decomposition along a tangent and the normal direction ξ, given by the formula of Weingarten:

$$D_X \xi = -A_\xi X + \nabla_X^\perp \xi.$$

A is called the *Weingarten endomorphism (or the shape operator)* and $\nabla^\perp$ is called *the normal connection* on M. They satisfy the following properties:

$$A_{f_1\xi_1 + f_2\xi_2} = f_1 A_{\xi_1} + f_2 A_{\xi_2}, \qquad \langle h(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle,$$

where $f_1, f_2 \in C^\infty(M)$ and ξ_1, ξ_2 normal vector fields.
We call M *a totally geodesic* manifold if $h = 0$. This implies that the geodesics in M are also geodesics in N.
Minimal submanifolds. For a point $p \in H$ we define *the mean curvature vector H* by

$$H(p) = \frac{1}{m} \sum_{i=1}^{m} h(e_i, e_i),$$

where $\{e_1, e_2, \ldots, e_m\}$ is an orthonormal basis of T_pM. We call M a *minimal submanifold* if H is identically zero.
Curvature. For normal vector fields ξ and η and tangent vector fields X and Y we define

$$[A_\xi, A_\eta]X = A_\xi A_\eta X - A_\eta A_\xi X,$$
$$R^\perp(X,Y)\xi = \nabla^\perp_X \nabla^\perp Y \xi - \nabla^\perp_Y \nabla^\perp X \xi - \nabla^\perp_{[X,Y]}\xi.$$

We call $R^\perp$ *the normal curvature tensor*, which is linear in each argument.

Theorem 27. *(the equations of Gauss,Codzzi and Ricci) Let $\tilde{R}$ denote the curvature tensor of N and let ^{t} (resp.) ^{n} denote the tangent (resp. the normal component) of a vector field, we get that*

$$R(X,Y)Z = (\tilde{R}(X,Y)Z)^t + A_{h(Y,Z)}X - A_{h(X,Z)}Y,$$
$$(\nabla h)(X,Y,Z) - (\nabla h)(Y,X,Z) = (\tilde{R}(X,Y)Z)^n,$$
$$\langle \tilde{R}(X,Y)\xi, \eta\rangle = \langle R^\perp(X,Y)\xi, \eta\rangle - \langle [A_\xi, A_\eta]X, Y\rangle$$

Notice that for submanifolds of real space forms these equations reduce to

$$R(X,Y)Z = c(\langle Y,Z\rangle X - \langle X,Z\rangle Y) + A_{h(Y,Z)}X - A_{h(X,Z)}Y,$$
$$(\nabla h)(X,Y,Z) = (\nabla h)(Y,X,Z),$$
$$\langle R^\perp(X,Y)\xi, \eta\rangle = \langle [A_\xi, A_\eta]X, Y\rangle.$$

Let ξ be a normal vector field on M. We say that ξ is *parallel at a point p*, if $\nabla^\perp_v \xi = 0$, for all $v \in T_pM$. We call ξ *parallel on M* if ξ is parallel at every point p of M.
The derivatives of h are called *higher order fundamental forms.* The first two are defined by

$$(\nabla h)(X,Y,Z) = \nabla^\perp_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),$$
$$(\nabla^2 h)(X,Y,Z,W) = \nabla^\perp_X (\nabla h)(Y,Z,W) - (\nabla h)(\nabla_X Y, Z, W) - (\nabla h)(Y, \nabla_X Z, W) - (\nabla h)(Y,Z,\nabla_X W),$$

where $X, Y, Z, W \in \mathcal{X}(M)$. A straightforward computation shows that the following formula, called *the Ricci identity*, holds:

$$(\nabla^2 h)(X,Y,Z,W) - (\nabla^2 h)(Y,X,Z,W) = R^\perp(X,Y)h(Z,W) - h(R(X,Y)Z,W) - h(X, R(Y,Z)W).$$

1.2.1 On affine hypersurfaces

Let $f : M \longrightarrow \mathbb{R}^{n+1}$ be a nondegenerate affine hypersurface immersion. Let D be the covariant derivative on $\mathbb{R}^{n+1}$ and Ω the volume form given by $\Omega(u_1, \ldots, u_{n+1}) = \det(u_1, \ldots, u_{n+1})$, such that $\mathbb{R}^{n+1}$ is endowed with its standard equiaffine structure (D, Ω). In a general setting, an affine manifold (M^n, ∇) is said to be equiaffine if there exists a volume form ω, i.e. a non-vanishing n-form, on M which is parallel with respect to ∇:

$$(\nabla_X \omega)(X_1 \dots, X_n) = X(\omega(X_1, \dots, X_n)) - \omega(\nabla_X X_1, \dots, X_n) - \dots \\ - \omega(X_1, \dots, \nabla_X X_n). \quad (1.8)$$

In this case we may also say that (∇, ω) is *an equiaffine structure* on M^n. In what follows, we briefly recall the construction of an equiaffine structure on an affine hypersurface M^n in $\mathbb{R}^{n+1}$. For more details we refer to [46].

Blaschke approach. First, let $p \in M$ and $X, Y \in T_pM$. If we choose an arbitrary transversal vector field η we can decompose $D_X Y$ into a tangent and a normal part, denoted as:

$$D_X Y = \nabla^\eta_X Y + h^\eta(X, Y)\eta.$$

It is easy to see that ∇^η is a connection on M and h^η is a symmetric bilinear form. Note that the fact whether this bilinear form is degenerate or not is independent of the choice of transversal vector field η. Notice that M is called *nondegenerate* if and only if this bilinear form is nondegenerate. Hence, locally there exists a volume form on M associated to h^η, given by

$$\omega_{h^\eta}(X_1, \dots, X_n) = \sqrt{|\det h^\eta(X_i, X_j)|}.$$

Next, we want to introduce a canonical transversal vector field ξ. In order to make a good choice, we define $\omega_\eta(X_1, \dots, X_n) := \Omega(X_1, \dots, X_n, \eta)$, for $X_1, \dots, X_n$ vector fields on M^n and we ask that the volume forms ω_ξ and ω_{h^ξ} coincide and that (∇^ξ, ω_ξ) is an equiaffine structure on M^n. Notice that these conditions guarantee the existence of a unique (up to sign) transversal vector field ξ, see [46]. It is called *the affine normal vector field*, or the *Blaschke normal vector field*. For convenience, we will denote from now on $\nabla := \nabla^\xi$.

Finally, in terms of this transversal vector field we get for M the formulas of Gauss and Weingarten, respectively, as follows:

$$D_X Y = \nabla_X Y + h(X, Y)\xi, \quad \text{(G)}$$

$$D_X \xi = -SX, \quad \text{(W)}$$

where we call ∇ the *induced affine connection*, h the *affine metric*, ξ the *affine normal field* or *Blaschke normal field* and S the affine *shape operator*. An affine hypersurface is called *a (proper) affine sphere* if S is a (non zero) multiple of the identity.

Moreover, let R denote the curvature tensor of M^n. Then, the following fundamental equations hold with respect to the induced affine connection:

Gauss equation:	$R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY;$
Codazzi equation for h :	$(\nabla h)(X, Y, Z) = (\nabla h)(Y, X, Z);$
Codazzi equation for S :	$(\nabla_X S)Y = (\nabla_Y S)X;$
Ricci equation:	$h(SX, Y) = h(X, SY).$

The Codazzi equation implies that for a proper affine sphere, the multiple of the identity is constant, in which case, by applying a homothety of the ambient space, we may assume that $S = \varepsilon I$, where $\varepsilon = \pm 1$. Moreover we have that $\xi + \varepsilon f$, where f denotes the position vector, is a constant vector which is called *the center* of the proper affine hypersphere. By applying

a translation in the ambient space we may, of course, always assume that the center is the origin.
As ∇ is not necessarily compatible with the affine metric h, it is interesting to look at the difference between the affine connection and the Levi-Civita connection ($\hat{\nabla}$). Thus, we obtain the *difference tensor* K, a $(1,2)$-type vector field defined as:

$$K(X,Y) = \nabla_X Y - \hat{\nabla}_X Y.$$

By convention, one may also write $K_X Y$ instead of $K(X,Y)$. The classical *Berwald theorem* states that K *vanishes identically if and only if* M *is congruent to a nondegenerate quadric.*

Proposition 1. *We have the following properties for* K*:*

1. $K(X,Y) = K(Y,X)$*;*
2. *for any* X *we have that* $Y \mapsto K_X Y$ *is a symmetric linear map and* $\mathrm{trace} K_X = 0$ *(the apolarity condition);*
3. $h(K(X,Y),Z) = h(K(X,Z),Y)$.

Moreover, it is easy to prove that ∇h is related to K by:

$$\nabla h(X,Y,Z) = -2h(Z,K(X,Y)).$$

We denote by $[K_X, K_Y]$ and $\hat{\nabla}K$ the following:

$$\begin{aligned} [K_X, K_Y]Z &= K_X K_Y Z - K_Y K_X Z, \\ \hat{\nabla}K(X,Y,Z) &= \hat{\nabla}_X K(Y,Z) - K(\hat{\nabla}_X Y, Z) - K(Y, \hat{\nabla}_X Z). \end{aligned}$$

Then, the equations of Gauss, Ricci and Codazzi, respectively, may also be written out with respect to the Levi-Civita connection as follows:

$$\hat{R}(X,Y)Z = \frac{1}{2}\{h(Y,Z)SX - h(X,Z)SY + h(SY,Z)X - h(SX,Z)Y\} - [K_X, K_Y]Z,$$
$$\hat{\nabla}K(X,Y,Z) - \hat{\nabla}K(Y,X,Z) = \frac{1}{2}\{h(Y,Z)SX - h(X,Z)SY - h(SY,Z)X + h(SX,Z)Y\},$$

$$\begin{aligned} &(\hat{\nabla}_X S)Y - (\hat{\nabla}_Y S)X = K(Y,SX) - K(X,SY), \\ &(\nabla h)(X,Y,Z) = (\nabla h)(Y,X,Z) \end{aligned}$$

and

$$h(X,SY) = h(SX,Y).$$

We have the following *Ricci identity*:

$$\hat{\nabla}^2 K(X,Y,Z,W) - \hat{\nabla}^2 K(Y,X,Z,W) = \hat{R}(X,Y)K(Z,W) - K(\hat{R}(X,Y)Z,W) - h(Z,\hat{R}(X,Y)W). \quad (1.9)$$

Homogeneity. A nondegenerate hypersurface M of the equiaffine space $\mathbb{R}^{n+1}$ is called *locally homogeneous* if for all points p and q of M, there exists a neighborhood U_p of p in

M, and an equiaffine transformation A of $\mathbb{R}^{n+1}$, i.e. $A \in \mathrm{SL}(n+1,\mathbb{R}) \ltimes \mathbb{R}^{n+1}$, such that $A(p) = q$ and $A(U_p) \subset M$. If $U_p = M$ for all p, then M is called *homogeneous.* Let G be the pseudogroup defined by

$$G = \{A \in \mathrm{SL}(n+1,\mathbb{R}) \ltimes \mathbb{R}^{n+1} \mid \exists U, \text{ open in } M : A(U) \subset M\},$$

then M is locally homogeneous if and only if G "acts" transitively on M. If M is homogeneous, then G is a group and every element of G maps the whole of M into M. The following proposition is probably well known, however, because of the lack of an explicit reference, a small proof is included.

Proposition 2. *Let M^n be a nondegenerate homogeneous affine hypersurface. Assume that $G \subset SL(n+1,\mathbb{R})$. Then M is an affine sphere centered at the origin.*

Proof. We denote the immersion by f. Let p and q be in M and let g be the affine transformation which maps p to q. We have that

$$\xi(g(p)) = dg(\xi(p)),$$

and

$$dg(f(p)) = g(f(p)) = f(q)$$

Moreover as M is homogeneous we know that the position vector can not be a tangent vector at one point (and therefore at every point). Indeed if that were the case, we would habe a tangent vector field X such that $X(p) = f(p)$. This would imply that $D_Y X = Y$, and therefore $h(X,Y) = 0$ for any vector field Y. This implies that the immersion f would be degenerate.
Therefore we may write $\xi = \rho f + Z$, where Z is a tangent vector field and ρ a function. As M is locally homogeneous and g belongs to $\mathrm{SL}(n+1,\mathbb{R})$ it follows that ρ is constant. The construction of the affine normal of [46] then implies that M is an affine sphere centered at the origin. □

1.2.2 On surfaces in $\mathbb{E}^3$ and Lorentzian space

Definition 11. *A subset $S \subset \mathbb{R}$ is a regular surface if, for every $p \in S$, there exist a neighborhood $V \subset \mathbb{R}^3$ and a map $r : U \to V \cap S$, $U \subset \mathbb{R}^2$ an open set, $V \cap S \subset \mathbb{R}^3$ such that*

(i) r is differentiable: for $r(u,v) = (x(u,v), y(u,v), z(u,v)), (u,v) \in U$, the functions x, y, z have continuous partial derivatives of all orders in U,

(ii) r is a homeomorphism: since r is continuous by condition (i), this means that r has an inverse $r^{-1} : V \cap S \to U$ which is continuous,

(iii) (the regularity condition) $r_u \times r_v \neq 0$.

Tangent plane. By a tangent vector to S at a point $p \in S$, we mean the tangent vector $\alpha'(0)$ of a differentiable parametrized cure $\alpha : (-\varepsilon, \varepsilon) \to S$ with $\alpha(0) = p$, $\varepsilon > 0$. The set of all tangent vectors on S at a point $p \in S$ forms the *tangent space* at $p \in S$.

Proposition 3. *Let $r : U \to S$ be a parametrization of a regular surface S and let $q \in U$. The vector space of dimension 2, $dr_q(\mathbb{R}^2) \subset \mathbb{R}^3$, coincides with the set of tangent vectors to S at $r(q)$.*

The first fundamental form. The natural inner product of $\mathbb{R}^3 \supset S$ induces on each tangent plane T_pS of the regular surface S an inner product, to be denoted by $\langle \cdot, \cdot \rangle_p$. If $w_1, w_2 \in T_pS \subset \mathbb{R}^3$ then $\langle w_1, w_2 \rangle_p$ is equal to the inner product of w_1 and w_2 as vectors in $\mathbb{R}^3$. To this inner product, which is a symmetric bilinear form, there corresponds a quadratic form $I_p : T_pS \to \mathbb{R}$, called the first fundamental form of the surface S at the point p, and which is given by

$$I_p(w) = \langle w, w \rangle_p = \mid w \mid^2 \geq 0.$$

The first fundamental form is the expression of how the surface S inherits the natural inner product of $\mathbb{R}^3$. Geometrically, the first fundamental form allows us to make measurements on the surface (lengths of curves, angles of tangent vectors, areas of regions) without referring back to the ambient space $\mathbb{R}^3$ where the surface lies. It can be expressed in local coordinates as well. Let $r(u, v)$ be a parametrisation of S at p and let $\alpha(t) = r(u(t), v(t))$ be a parametrized curve on S, with $\alpha(0) = p$. Then for the basis $\{r_u, r_v\}$ of T_pS, we have

$$I_p(\alpha'(0)) = E(u')^2 + 2Fu'v' + G(v')^2,$$

where

$$E = \langle r_u, r_u \rangle, \quad F = \langle r_u, r_v \rangle, \quad G = \langle r_v, r_v \rangle \tag{1.10}$$

are *the coefficients of the first fundamental form.* Notice that we define the *area element* of the surface as $dM = \sqrt{EG - F^2} du \wedge dv$.

Let $r : U \subset \mathbb{R}^2 \to S$ be a parametrization of a regular surface S around a point $p \in S$. We can choose a unit normal vector at each point of $r(U)$ by

$$N(q) = \frac{r_u \times r_v}{\mid r_u \times r_v \mid}(q), \; q \in r(U).$$

Thus, we have a differentiable map $N : r(U) \to \mathbb{R}^3$ that associates to each $q \in r(U)$ a unit normal vector $N(q)$. More generally, if $V \subset S$ is an open set in S and $N : V \to \mathbb{R}^3$ is a differentiable map which associates to each $q \in V$ a unit normal vector at q, we say that N is *a differentiable field of unit normal vectors on V.*

Orientable surface. We shall say that a regular surface is *orientable* if it admits a differentiable field of unit normal vectors defined on the whole surface; the choice of such a field N is called an *orientation* of S.

The Gauss map. A large number of local properties of S at p can be derived from the study of the so called *Gauss map*, which, in a general sense, measures the rate of change at p of the unit normal vector field N on a neighborhood of p.

Definition 12. *Let $S \subset \mathbb{R}^3$ be a surface with an orientation N. The map $N : S \to \mathbb{R}^3$ takes its values in the unit sphere $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$. The map $N : S \to \mathbb{S}^2$ thus defined is called* the Gauss map *of S.*

The second fundamental form. Notice that the Gauss map is differentiable and that the differential dN_p of N at $p \in S$ is a linear map from T_pS to $T_{N(p)}\mathbb{S}^2$. Since T_pS and $T_{N(p)}\mathbb{S}^2$ are parallel planes, dN_p can be looked upon as a linear map on T_pS. The Gauss map $dN_p : T_pS \to T_pS$ is a self-adjoint linear map and this fact allows us to associate to dN_p a quadratic form in T_pS, $I\!I$, as follows.

$$I\!I_p(v) = -\langle dN_p(v), v \rangle, \; v \in T_pS.$$

The quadratic form $I\!I$ is called *the second fundamental form of S at p.*
Curvature. Let C be a regular curve in S passing through $p \in S$, k the curvature of C at p, and $\cos\theta = \langle n, N\rangle$, where n is the normal vector to C and N is the normal vector to S at p. The number $k_n = k\cos\theta$ is then called *the normal curvature of $C \subset S$ at p.*
We know that for the self-adjoint operator dN_p there exist an orthonormal basis $\{e_1, e_2\} \in T_pS$ such that

$$dN_p(e_1) = -k_1 e_1, \quad dN_p(e_2) = -k_1 e_2. \tag{1.11}$$

Moreover, k_1 and k_2 ($k_1 > k_2$) are the maximum and minimum of the second fundamental form $I\!I_p$ restricted to the unit circle of T_pS; that is, they are the extreme values of the normal curvature at p.

Definition 13. *Let $p \in S$ and let $dN_p : T_pS \to T_pS$ be the differential of the Gauss map. The determinant of dN_p is* the Gaussian curvature K *of S at p. The negative of half of the trace of dN_p is called* the mean curvature H *of S at p. In terms of the principal curvatures we can write*

$$K = k_1 k_2, \quad H = \frac{k_1 + k_2}{2}. \tag{1.12}$$

We may express the previous invariants using local coordinates as well. Let $r(u,v)$ be a parametrisation of S at p and let $\alpha(t) = r(u(t), v(t))$ be a parametrized curve on S, with $\alpha(0) = p$. Then for the basis $\{r_u, r_v\}$ of T_pS, let $dN = (a_{ij})$, $i, j = 1, 2$. The following hold.

$$I\!I_p(\alpha'(0)) = e(u')^2 + 2fu'v' + g(v')^2,$$

for

$$\begin{aligned} e &= -\langle N_u, r_u\rangle = \langle N, r_{uu}\rangle, \\ f &= -\langle N_v, r_u\rangle = \langle N, r_{uv}\rangle, \\ g &= -\langle N_v, r_v\rangle = \langle N, r_{vv}\rangle, \end{aligned}$$

and, moreover,

$$K = det(a_{ij}) = \frac{eg - f^2}{EG - F^2} \tag{1.13}$$

and

$$H = \frac{1}{2}(k_1 + k_2) = -\frac{1}{2}(a_{11} + a_{22}) = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}. \tag{1.14}$$

Lorentzian case. We consider the Lorentzian-Minkowski space $\mathbb{L}^3$, that is, $\mathbb{R}^3$ endowed with the metric $(dx)^2 + (dy)^2 - (dz)^2$. A surface immersed in $\mathbb{L}^3$ is said non-degenerate if the induced metric on S is not degenerate. The induced metric can only be of two types: positive definite and the surface is called spacelike, or a Lorentzian metric, and the surface is called timelike. For both types of surfaces, the mean curvature H and the Gauss curvature K are defined and they have the following expressions in local coordinates $X = X(s,t)$:

$$H = \varepsilon\,\frac{1}{2}\frac{lG - 2mF + nE}{EG - F^2}, \quad K = \varepsilon\,\frac{ln - m^2}{EG - F^2},$$

where $\varepsilon = -1$ if S is spacelike and $\varepsilon = 1$ if S is timelike. Here $\{E, F, G\}$ and $\{l, m, n\}$ are the coefficients of the first and second fundamental forms with respect to X, respectively. See [34] for more details.

Chapter 2

On the Nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$

In this chapter we first present the homogeneous nearly Kähler structure of $\mathbb{S}^3 \times \mathbb{S}^3$ and we mention some of the known results from [20] and [59]. Next, we first explain how the metric, the almost complex structure and the almost product structure of the homogeneous nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$ can be recovered from the submersion $\pi : \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3$, together with some further properties of $\mathbb{S}^3 \times \mathbb{S}^3$. Then, in the two following subsections, we present the results obtained in [3] and [4] , respectively, as joint work between B. Bektaş, M. Moruz, J. Van der Veken and L. Vrancken.

The nearly Kähler structure of $\mathbb{S}^3 \times \mathbb{S}^3$. By the natural identification $T_{(p,q)}(\mathbb{S}^3 \times \mathbb{S}^3) \cong T_p\mathbb{S}^3 \oplus T_q\mathbb{S}^3$, we may write a tangent vector at (p, q) as $Z(p, q) = (U(p, q), V(p, q))$ or simply $Z = (U, V)$. We regard the 3-sphere as the set of all unit quaternions in $\mathbb{H}$ and we use the notations i, j, k to denote the imaginary units of $\mathbb{H}$. In computations it is often useful to write a tangent vector $Z(p, q)$ at (p, q) on $\mathbb{S}^3 \times \mathbb{S}^3$ as $(p\alpha, q\beta)$, with α and β imaginary quaternions. This is possible as for $v \in T_p\mathbb{S}^3$ we know that $\langle v, p\rangle = 0$ and, in addition, for $p \in \mathbb{S}^3$ we can always find $\tilde{v} \in \mathbb{H}$ such that $v = p\tilde{v}$. Moreover, $Re(\tilde{v}) = 0$ as $0 = \langle p, v\rangle = Re(\bar{p}v) = Re(\bar{p}p\tilde{v}) = Re(\tilde{v})$. Hence, we will work with tangent vectors at $(p, q) \in \mathbb{S}^3 \times \mathbb{S}^3$ of the form $Z(p, q) = (pU, qV)$, for U, V imaginary quaternions.
We define the vector fields

$$\begin{aligned} \tilde{E}_1(p,q) &= (pi, 0), & \tilde{F}_1(p,q) &= (0, qi),\\ \tilde{E}_2(p,q) &= (pj, 0), & \tilde{F}_2(p,q) &= (0, qj),\\ \tilde{E}_3(p,q) &= -(pk, 0), & \tilde{F}_3(p,q) &= -(0, qk), \end{aligned} \tag{2.1}$$

which are mutually orthogonal with respect to the usual Euclidean product metric on $\mathbb{S}^3 \times \mathbb{S}^3$. The Lie brackets are $[\tilde{E}_i, \tilde{E}_j] = -2\varepsilon_{ijk}\tilde{E}_k$, $[\tilde{F}_i, \tilde{F}_j] = -2\varepsilon_{ijk}\tilde{F}_k$ and $[\tilde{E}_i, \tilde{F}_j] = 0$, where

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{if } (ijk) \text{ is an even permutation of } (123),\\ -1, & \text{if } (ijk) \text{ is an odd permutation of } (123),\\ 0, & \text{otherwise.} \end{cases}$$

The almost complex structure J on the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$ is defined by

$$J(pU, Vq)_{(p,q)} = \frac{1}{\sqrt{3}}(p(2V - U), q(-2U + V)), \tag{2.2}$$

for U, V imaginary quaternions and therefore $(pU, qV) \in T_{(p,q)}(\mathbb{S}^3 \times \mathbb{S}^3)$. The nearly Kähler metric on $\mathbb{S}^3 \times \mathbb{S}^3$ is the Hermitian metric associated to the usual Euclidean product metric on $\mathbb{S}^3 \times \mathbb{S}^3$:

$$\begin{aligned} g(Z, Z') =& \frac{1}{2}(\langle Z, Z'\rangle + \langle JZ, JZ'\rangle) \\ =& \frac{4}{3}(\langle U, U'\rangle + \langle V, V'\rangle) - \frac{2}{3}(\langle U, V'\rangle + \langle U', V\rangle), \end{aligned} \tag{2.3}$$

where $Z = (pU, qV)$ and $Z' = (pU', qV')$. In the first line $\langle\cdot,\cdot\rangle$ stands for the usual Euclidean product metric on $\mathbb{S}^3 \times \mathbb{S}^3$ and in the second line $\langle\cdot,\cdot\rangle$ stands for the usual Euclidean metric on $\mathbb{S}^3$. By definition, the almost complex structure is compatible with the metric g.
From [7] we have the following lemma.

Lemma 1. *The Levi-Civita connection $\tilde{\nabla}$ on $\mathbb{S}^3 \times \mathbb{S}^3$ with respect to the metric g is given by*

$$\begin{array}{ll} \tilde{\nabla}_{\tilde{E}_i}\tilde{E}_j = -\varepsilon_{ijk}\tilde{E}_k & \tilde{\nabla}_{\tilde{E}_i}\tilde{F}_j = \frac{\varepsilon_{ijk}}{3}(\tilde{E}_k - \tilde{F}_k) \\ \tilde{\nabla}_{\tilde{F}_i}\tilde{E}_j = \frac{\varepsilon_{ijk}}{3}(\tilde{F}_k - \tilde{E}_k) & \tilde{\nabla}_{\tilde{F}_i}\tilde{F}_j = -\varepsilon_{ijk}\tilde{F}_k. \end{array}$$

Then we have that

$$\begin{array}{ll} (\tilde{\nabla}_{\tilde{E}_i}J)\tilde{E}_j = -\frac{2}{3\sqrt{3}}\varepsilon_{ijk}(\tilde{E}_k + 2\tilde{F}_k), & (\tilde{\nabla}_{\tilde{E}_i}J)\tilde{F}_j = -\frac{2}{3\sqrt{3}}\varepsilon_{ijk}(\tilde{E}_k - \tilde{F}_k), \\ (\tilde{\nabla}_{\tilde{F}_i}J)\tilde{E}_j = -\frac{2}{3\sqrt{3}}\varepsilon_{ijk}(\tilde{E}_k - \tilde{F}_k), & (\tilde{\nabla}_{\tilde{F}_i}J)\tilde{F}_j = -\frac{2}{3\sqrt{3}}\varepsilon_{ijk}(2\tilde{E}_k + \tilde{F}_k). \end{array} \tag{2.4}$$

Let $G := \tilde{\nabla}J$. Then G is skew-symmetric and satisfies that

$$G(X, JY) = -JG(X, Y), \quad g(G(X, Y), Z) + g(G(X, Z), Y) = 0, \tag{2.5}$$

for any vectors fields X, Y, Z tangent to $\mathbb{S}^3 \times \mathbb{S}^3$. Therefore, $\mathbb{S}^3 \times \mathbb{S}^3$ equipped with g and J, becomes a nearly Kähler manifold.
The almost product structure P introduced in [7] and defined as

$$P(pU, qV) = (pV, qU), \quad \forall Z = (pU, qV) \in T_{(p,q)}(\mathbb{S}^3 \times \mathbb{S}^3), \tag{2.6}$$

plays an important role in the study of the Lagrangian submanifolds of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$. It has the following properties:

$$P^2 = Id, \text{ i.e. } P \text{ is involutive,} \tag{2.7}$$

$$PJ = -JP, \text{ i.e. } P \text{ and } J \text{ anti-commute,} \tag{2.8}$$

$$g(PZ, PZ') = g(Z, Z'), \text{ i.e. } P \text{ is compatible with } g, \tag{2.9}$$

$$g(PZ, Z') = g(Z, PZ'), \text{ i.e. } P \text{ is symmetric.} \tag{2.10}$$

Moreover, the almost product structure P can be expressed in terms of the usual product structure $QZ = Q(pU, pV) = (-pU, qV)$ and vice versa:

$$\begin{aligned} QZ &= \frac{1}{\sqrt{3}}(2PJZ - JZ), \\ PZ &= \frac{1}{2}(Z - \sqrt{3}QJZ) \end{aligned}$$

and we know from [20] that

$$PG(Z,Z') + G(PZ,PZ') = 0, \tag{2.11}$$

$$(\tilde{\nabla}_Z P)Z' = \frac{1}{2}J(G(Z,PZ') + PG(Z,Z')). \tag{2.12}$$

In addition, the Riemannian curvature tensor $\tilde{R}$ on $(\mathbb{S}^3 \times \mathbb{S}^3, g, J)$ is given by

$$\begin{aligned}\tilde{R}(U,V)W = & \frac{5}{12}\big(g(V,W)U - g(U,W)V\big)\\ & + \frac{1}{12}\big(g(JV,W)JU - g(JU,W)JV - 2g(JU,V)JW\big)\\ & + \frac{1}{3}\big(g(PV,W)PU - g(PU,W)PV\\ & \qquad + g(JPV,W)JPU - g(JPU,W)JPV\big).\end{aligned} \tag{2.13}$$

Next, we recall the relation between the Levi-Civita connections $\tilde{\nabla}$ of g and ∇^E of the Euclidean product metric $\langle \cdot, \cdot \rangle$.

Lemma 2. *[20] The relation between the nearly Kähler connection $\tilde{\nabla}$ and the Euclidean connection $\nabla^E$is*

$$\nabla^E_X Y = \tilde{\nabla}_X Y + \frac{1}{2}(JG(X,PY) + JG(Y,PX)).$$

We recall here a useful formula, already known in [20].
Let D be the Euclidean connection on $\mathbb{R}^8$. For vector fields $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ on $\mathbb{S}^3 \times \mathbb{S}^3$, we may decompose $D_X Y$ along the tangent and the normal directions as follows:

$$D_X Y = \nabla^E_X Y + \frac{1}{2}\langle D_X Y, (p,q)\rangle (p,q) + \frac{1}{2}\langle D_X Y, (-p,q)\rangle(-p,q). \tag{2.14}$$

Here, notice the factor $\frac{1}{2}$ due to the fact that (p,q) and $(-p,q)$ have length $\sqrt{2}$. Moreover, as $\langle Y, (p,q)\rangle = 0$, (2.14) is equivalent with

$$D_X Y = \nabla^E_X Y - \frac{1}{2}\langle Y, X\rangle (p,q) - \frac{1}{2}\langle Y, (-X_1, X_2)\rangle(-p,q).$$

In the special case that $Y_2 = 0$, the previous formula reduces to

$$D_X(Y_1, 0) = \nabla^E_X(Y_1,0) - \langle X_1, Y_1\rangle (p,0). \tag{2.15}$$

We find it appropriate here to prove an additional important formula not explicitly mentioned in [7], that allows us to evaluate G for any tangent vector fields.

Proposition 4. *Let $X = (p\alpha, q\beta), Y = (p\gamma, q\delta) \in T_{(p,q)}\mathbb{S}^3 \times \mathbb{S}^3$. Then*

$$G(X,Y) = \frac{2}{3\sqrt{3}}(p(\beta\times\gamma + \alpha\times\delta + \alpha\times\gamma - 2\beta\times\delta), q(-\alpha\times\delta - \beta\times\gamma + 2\alpha\times\gamma - \beta\times\delta)). \tag{2.16}$$

Proof. As α is an imaginary unit quaternion, we may write $\alpha = \alpha_1 \cdot i + \alpha_2 \cdot j + \alpha_3 \cdot k$ and similarly for β, γ, δ. Then, using (2.1), we write for more convenience in computations $X = U_\alpha + V_\beta$, where $U_\alpha = \alpha_1 \tilde{E}_1 + \alpha_2 \tilde{E}_2 - \alpha_3 \tilde{E}_3$ and $V_\beta = \beta_1 \tilde{F}_1 + \beta_2 \tilde{F}_2 - \beta_3 \tilde{F}_3$. Similarly, $Y = U_\gamma + V_\delta$. We now use the relations in (2.4) and compute

$$G(U_\alpha, V_\beta) = \frac{2}{3\sqrt{3}}(U_{\alpha\times\beta} - V_{\alpha\times\beta}), \quad G(U_\alpha, U_\beta) = \frac{2}{3\sqrt{3}}(U_{\alpha\times\beta} + 2V_{\alpha\times\beta}).$$

As $PU_\alpha = V_\alpha$, we obtain that

$$G(V_\alpha, V_\beta) = -\frac{2}{3\sqrt{3}}(V_{\alpha\times\beta} + 2U_{\alpha\times\beta}).$$

Finally, by linearity we get the relation in (2.16). □

From now on we will restrict ourselves to 3-dimensional Lagrangian submanifolds M of $\mathbb{S}^3 \times \mathbb{S}^3$. It is known from [20] and [59] that, as the pull-back of $T(\mathbb{S}^3 \times \mathbb{S}^3)$ to M splits into $TM \oplus JTM$, there are two endomorphisms $A, B : TM \to TM$ such that the restriction $P|_{TM}$ of P to the submanifold equals $A + JB$, that is $PX = AX + JBX$, for all $X \in TM$. Note that the previous formula, together with the fact that P and J anti-commute, also determines P on the normal space by $PJX = -JPX = BX - JAX$. In addition, from the properties of J and P it follows that A and B are symmetric operators which commute and satisfy moreover that $A^2 + B^2 = Id$ (see [20]). Hence A and B can be diagonalised simultaneously at a point p in M and there is an orthonormal basis $e_1, e_2, e_3 \in T_pM$ such that

$$Pe_i = \cos(2\theta_i)e_i + \sin(2\theta_i)Je_i. \tag{2.17}$$

The functions θ_i are called the angle functions of the immersion. Next, for a point p belonging to an open dense subset of M on which the multiplicities of the eigenvalues of A and B are constant (see [53]), we may extend the orthonormal basis e_1, e_2, e_3 to a frame on a neighborhood in the Lagrangian submanifold. Finally, taking into account the properties of G we know that there exists a local orthonormal frame $\{E_1, E_2, E_3\}$ on an open subset of M such that

$$AE_i = \cos(2\theta_i)E_i, \quad BE_i = \sin(2\theta_i)E_i \tag{2.18}$$

and

$$JG(E_i, E_j) = \frac{1}{\sqrt{3}}\varepsilon_{ijk}E_k. \tag{2.19}$$

Notice that, in a general sense, for an immersion $f : M \to \mathbb{S}^3 \times \mathbb{S}^3$ there exist $A, B : TM \to TM$ with eigenvectors E_i and corresponding angle functions θ_i such that, on the image of M we may write by (2.18) and (2.17):

$$Pdf(E_i) = df(AE_i) + Jdf(BE_i) \quad \Leftrightarrow \quad Pdf(E_i) = \cos(2\theta_i)df(E_i) + \sin(2\theta_i)Jdf(E_i), \tag{2.20}$$

for $i = 1, 2, 3$.
The equations of Gauss and Codazzi, respectively, state that

$$\begin{aligned} R(X,Y)Z = {} & \frac{5}{12}\left(g(Y,Z)X - g(X,Z)Y\right) \\ & + \frac{1}{3}\left(g(AY,Z)AX - g(AX,Z)AY + g(BY,Z)BX - g(BX,Z)BY\right) \\ & + [S_{JX}, S_{JY}]Z \end{aligned} \tag{2.21}$$

and

$$\nabla h(X,Y,Z) - \nabla h(Y,X,Z) = \frac{1}{3}\left(g(AY,Z)JBX - g(AX,Z)JBY - g(BY,Z)JAX + g(BX,Z)JAY\right). \tag{2.22}$$

For the Levi-Civita connection ∇ on M we introduce (see[20]) the functions ω_{ij}^k satisfying

$$\nabla_{E_i}E_j = \sum_{k=1}^{3}\omega_{ij}^k E_k \quad \text{and} \quad \omega_{ij}^k = -\omega_{ik}^j.$$

As usual, we write:

$$\begin{aligned}
\tilde{\nabla}_X Y &= \nabla_X Y + h(X,Y),\\
\tilde{\nabla}_X JY &= -S_{JY}X + \nabla_X^{\perp} JY,
\end{aligned}$$

where h is the second fundamental form on M and S_{JY} is the shape operator in the direction of JY. As for the Lagrangian manifolds of a strict 6-dimensional nearly Kähler manifold we have that $G(X,Y)$ is normal (see [27], [52]), it follows that

$$\begin{aligned}
\nabla_X^{\perp} JY &= J\nabla_X Y + G(X,Y),\\
Jh(X,Y) &= -S_{JY}X.
\end{aligned}$$

The latter equation implies in particular that the cubic form $g(h(X,Y),JZ)$ is totally symmetric. We denote by h_{ij}^k the components of this cubic form on M:

$$h_{ij}^k = g(h(E_i,E_j),JE_k). \tag{2.23}$$

We recall the following lemmas.

Lemma 3. *[20] The sum of the angles $\theta_1+\theta_2+\theta_3$ is zero modulo π.*

Lemma 4. *[20] The derivatives of the angles θ_i give the components of the second fundamental form*

$$E_i(\theta_j) = -h_{jj}^i, \tag{2.24}$$

except h_{12}^3. The second fundamental form and covariant derivative are related by

$$h_{ij}^k\cos(\theta_j-\theta_k) = \left(\frac{\sqrt{3}}{6}\varepsilon_{ij}^k - \omega_{ij}^k\right)\sin(\theta_j-\theta_k). \tag{2.25}$$

Lemma 5. *[20] If two of the angles are equal modulo π, then the Lagrangian submanifold is totally geodesic.*

Remark 1. By Lemma 5, we may see that if the Lagrangian submanifold is not totally geodesic, then $\sin(\theta_i-\theta_j)\neq 0$, for $i\neq j$.

2.1 Properties of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$ as a Riemannian submersion

In this section we show how the nearly Kähler metric g and the almost complex structure J of $\mathbb{S}^3 \times \mathbb{S}^3$ can be recovered in a natural way by looking at the submersion $\pi : \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3$. The results may be found in [42] and one should note that the approach used here is also more or less implicitly present in [25], [26], [40] and [37].

We also show how the almost product structure P defined in [7] can be introduced using the submersion π from a structure on $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$. This way we actually obtain three different almost product structures

$$P_\ell = \cos(\tfrac{2\pi\ell}{3})P - \sin(\tfrac{2\pi\ell}{3})JP.$$

We show in the final subsection that these are precisely the three possible almost product structures which preserve the basic equations for $\mathbb{S}^3 \times \mathbb{S}^3$ derived in [7].

We also show how the maps which interchange the components of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ give rise to isometries of $\mathbb{S}^3 \times \mathbb{S}^3$. We call $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) the isometry corresponding to interchanging the first two coordinates (respectively interchanging the first and third coordinate). We show that both these isometries preserve up to sign the almost complex structure. And even though they do not preserve the almost product structures individually, they do preserve the set of almost product structures $\{P_1, P_2, P_3\}$. This is of course the reason why in several classification theorems for Lagrangian submanifolds, see for example [3], [4], [59], one often has 3 isometric examples with slightly different properties of the almost product structure P. These examples are precisely obtained one from another by applying the isometries $\mathcal{F}_1$ and $\mathcal{F}_2$. The only exception so far to this is the classification of non totally geodesic Lagrangian submanifolds with constant sectional curvature in [20]. This is due to the special property of the angle functions (which determine P) of these last examples.

2.1.1 The structure on $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$

We consider $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ with its usual induced structure. For tangent vectors (g_1V_1, g_2V_2, g_3V_3) and (g_1W_1, g_2W_2, g_3W_3) at the point (g_1, g_2, g_3) we have that $V_1, V_2, V_3, W_1, W_2, W_3$ are imaginary quaternions and that the induced metric is given by

$$\begin{aligned}
\langle (g_1V_1, g_2V_2, g_3V_3), (g_1W_1, g_2W_2, g_3W_3)\rangle &= \sum_{\ell=1}^{3} Re(g_\ell V_\ell \bar{W}_\ell \bar{g}_\ell) \\
&= -\sum_{\ell=1}^{3} Re(g_\ell V_\ell W_\ell \bar{g}_\ell) \\
&= \sum_{\ell=1}^{3} Re(g_\ell(< V_\ell, W_\ell > - V_\ell \times W_\ell)\bar{g}_\ell) \\
&= \sum_{\ell=1}^{3} < V_\ell, W_\ell > .
\end{aligned}$$

We define the following vector fields on $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ as

$$E_1(g_1, g_2, g_3) = (g_1\bar{g}_3 i g_3, 0, 0), \quad F_1(g_1, g_2, g_3) = (0, g_2\bar{g}_3 i g_3, 0), \quad G_1(g_1, g_2, g_3) = (0, 0, i g_3),$$

$$E_2(g_1,g_2,g_3)=(g_1\bar g_3 j g_3,0,0),\quad F_2(g_1,g_2,g_3)=(0,g_2\bar g_3 j g_3,0),\quad G_2(g_1,g_2,g_3)=(0,0,jg_3),$$
$$E_3(g_1,g_2,g_3)=-(g_1\bar g_3 k g_3,0,0), F_3(g_1,g_2,g_3)=-(0,g_2\bar g_3 k g_3,0), G_3(g_1,g_2,g_3)=-(0,0,kg_3).$$

Note that using the induced metric, it immediately follows that E_1, E_2, E_3, F_1, F_2, F_3, G_1, G_2, G_3 form an orthonormal basis of the tangent space.
We also have that for any (g_1,g_2,g_3), γ_1 given by

$$\gamma_1(t)=(g_1\bar g_3 e^{it} g_3,g_2,g_3)$$

is a curve in $\mathbb{S}^3\times\mathbb{S}^3\times\mathbb{S}^3$ with initial conditions $\gamma_1(0)=(g_1,g_2,g_3)$ and $\gamma_1'(0)=E_1(g_1,g_2,g_3)$. Similarly we have that the curves $\gamma_2(t)=(g_1,g_2\bar g_3 e^{it}g_3,g_3)$ and $\gamma_3(t)=(g_1,g_2,e^{it}g_3)$ are curves in $\mathbb{S}^3\times\mathbb{S}^3\times\mathbb{S}^3$ with initial conditions respectively

$$\begin{aligned}\gamma_2(0)&=(g_1,g_2,g_3), &\quad \gamma_2'(0)&=F_1(g_1,g_2,g_3),\\ \gamma_3(0)&=(g_1,g_2,g_3), &\quad \gamma_3'(0)&=G_1(g_1,g_2,g_3).\end{aligned}$$

By replacing i with j and $-k$ in the expressions of the curves γ_1,γ_2 and γ_3, we define similarly the corresponding curves for the other vectors in the basis .

We also have on each tangent space the natural linear applications:

$$\begin{aligned}\tilde\tau(g_1V_1,g_2V_2,g_3V_3)&=(g_1V_2,g_2V_3,g_3V_1),\\ \tilde P_1(g_1V_1,g_2V_2,g_3V_3)&=(g_1V_2,g_2V_1,g_3V_3),\\ \tilde P_2(g_1V_1,g_2V_2,g_3V_3)&=(g_1V_3,g_2V_2,g_3V_1,)\\ \tilde P_3(g_1V_1,g_2V_2,g_3V_3)&=(g_1V_1,g_2V_3,g_3V_2).\end{aligned}$$

Note that these applications all preserve the induced metric. Moreover we have that $\tilde\tau^3=I=\tilde P_1^2=\tilde P_2^2=\tilde P_3^2$, $\tilde P_3\tilde P_1=\tilde\tau$. In terms of the previously induced vector fields, we have that

$$\begin{aligned}\tilde\tau E_\ell&=G_\ell, &\quad \tilde\tau F_\ell&=E_\ell, &\quad \tilde\tau G_\ell&=F_\ell,\\ \tilde P_1E_\ell&=F_\ell, & \tilde P_1F_\ell&=E_\ell, & \tilde P_1G_\ell&=G_\ell,\\ \tilde P_2E_\ell&=G_\ell, & \tilde P_2F_\ell&=F_\ell, & \tilde P_2G_\ell&=E_\ell,\\ \tilde P_3E_\ell&=E_\ell, & \tilde P_3F_\ell&=G_\ell, & \tilde P_3G_\ell&=F_\ell.\end{aligned}$$

2.1.2 The nearly Kähler $\mathbb{S}^3\times\mathbb{S}^3$ as a Riemannian submersion

We look at the map

$$\pi:\mathbb{S}^3\times\mathbb{S}^3\times\mathbb{S}^3\to\mathbb{S}^3\times\mathbb{S}^3:(g_1,g_2,g_3)\mapsto(g_1\bar g_3,g_2\bar g_3).$$

It follows immediately that

$$\pi(g_1,g_2,g_3)=\pi(g_1',g_2',g_3')\iff(g_1',g_2',g_3')=(g_1a,g_2a,g_3a),$$

where $a\in\mathbb{S}^3$ is a unit quaternion. We have that

$$\begin{aligned}d\pi(E_1(g_1,g_2,g_3))&=\tfrac{d}{dt}\pi(g_1\bar g_3e^{it}g_3,g_2,g_3)|_{t=0}\\ &=\tfrac{d}{dt}(g_1\bar g_3e^{it},g_2\bar g_3)|_{t=0}\end{aligned}$$

$$= (g_1\bar{g}_3 i, 0)$$
$$= \tilde{E}_1(g_1\bar{g}_3, g_2\bar{g}_3)$$
$$= \tilde{E}_1(\pi(g_1, g_2, g_3)).$$

By similar computations we obtain that

$$d\pi(E_\ell(g_1, g_2, g_3)) = \tilde{E}_\ell(\pi(g_1, g_2, g_3)),$$
$$d\pi(F_\ell(g_1, g_2, g_3)) = \tilde{F}_\ell(\pi(g_1, g_2, g_3)),$$
$$d\pi(G_\ell(g_1, g_2, g_3)) = -\tilde{E}_\ell(\pi(g_1, g_2, g_3)) - \tilde{F}_\ell(\pi(g_1, g_2, g_3)).$$

This implies that $d\pi$ is surjective (and hence π is a submersion). We also see that the space of vertical vectors $\mathcal{V}$ is given by $\mathcal{V} = span\{E_1+F_1+G_1, E_2+F_2+G_2, E_3+F_3+G_3\}$. Therefore, we have the space of horizontal vector fields $\mathcal{H}$ spanned by $\{\frac{1}{3}(2E_\ell - F_\ell - G_\ell), \frac{1}{3}(-E_\ell + 2F_\ell - G_\ell)\}$. It also follows that

$$d\pi(\tfrac{1}{3}(2E_\ell - F_\ell - G_\ell))(g_1, g_2, g_3) = \tilde{E}_\ell(\pi(g_1, g_2, g_3),$$
$$d\pi(\tfrac{1}{3}(-E_\ell + 2F_\ell - G_\ell))(g_1, g_2, g_3) = \tilde{F}_\ell(\pi(g_1, g_2, g_3).$$

Note that

$$< \tfrac{1}{3}(2E_\ell - F_\ell - G_\ell), \tfrac{1}{3}(2E'_\ell - F'_\ell - G'_\ell) > = \tfrac{2}{3}\delta_{\ell\ell'}$$
$$< \tfrac{1}{3}(2E_\ell - F_\ell - G_\ell), \tfrac{1}{3}(-E'_\ell + 2F'_\ell - G'_\ell) > = -\tfrac{1}{3}\delta_{\ell\ell'}$$
$$< \tfrac{1}{3}(-E_\ell + 2F_\ell - G_\ell), \tfrac{1}{3}(-E'_\ell + 2F'_\ell - G'_\ell) > = \tfrac{2}{3}\delta_{\ell\ell'}$$

Moreover, as the right-hand sides are independent of the point (g_1, g_2, g_3) for which $\pi(g_1, g_2, g_3) = (p, q)$, we see from the above formulas that we can define the canonical metric, g_s on $\mathbb{S}^3 \times \mathbb{S}^3$, of the submersion π by

$$g_s(\tilde{E}_\ell, \tilde{E}_\ell) = g_s(\tilde{F}_\ell, \tilde{F}_\ell) = -2g_s(\tilde{E}_\ell, \tilde{F}_\ell) = \tfrac{2}{3},$$

and such that all other components vanish. Note that $g = 2g_s$ and therefore the nearly Kähler metric is twice the metric induced by the submersion.

Theorem 28. *The map*

$$\pi : \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : (g_1, g_2, g_3) \mapsto (g_1\bar{g}_3, g_2\bar{g}_3)$$

is a submersion. Moreover there exists a canonical metric g_s on $\mathbb{S}^3 \times \mathbb{S}^3$ such that the submersion becomes a Riemannian submersion. This metric is related to the nearly Kähler metric by $g = 2g_s$.

Next we look at what happens with the applications $\tilde{\tau}$, $\tilde{P}_1$, $\tilde{P}_2$ and $\tilde{P}_3$. In order to do so we will use the following lemma.

Lemma 6. *Let $\tilde{A}$ be a linear application on the tangent space of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$. Suppose that*

1. *$\tilde{A}$ maps vertical vector fields to vertical vector fields*
2. *$\tilde{A}$ preserves the metric*

3. If v, w are horizontal tangent vectors at resp. (g_1, g_2, g_3) and $(g_1 a, g_2 a, g_3 a)$ such that $d\pi(v) = d\pi(w)$, then we also have that $d\pi(\tilde{A}v) = d\pi(\tilde{A}w)$.

Then there exists a linear isometry A of the tangent space of $\mathbb{S}^3 \times \mathbb{S}^3$, such that

$$AZ(p, q) = d\pi(\tilde{A}\tilde{Z}(g_1, g_2, g_3)),$$

where (g_1, g_2, g_3) is any point such that $\pi(g_1, g_2, g_3) = (p, q)$ and $\tilde{Z}(g_1, g_2, g_3)$ is the unique horizontal tangent vector such that $d\pi(\tilde{Z}(g_1, g_2, g_3)) = Z(p, q)$.

Proof. As $\tilde{A}$ maps vertical vector fields to vertical vector fields and preserves the metric, $\tilde{A}$ also maps horizontal vector fields to horizontal vector fields. The third condition then implies that the map

$$AZ(p, q) = d\pi(\tilde{A}\tilde{Z}(g_1, g_2, g_3)),$$

is well defined and is an isometry. □

Note that the maps $\tilde{\tau}$, $\tilde{P}_1$, $\tilde{P}_2$, $\tilde{P}_3$ satisfy the conditions of the above lemma. Therefore we get the corresponding maps at the tangent space of a point $(p, q) \in \mathbb{S}^3 \times \mathbb{S}^3$ given by τ, P_1, P_2 and P_3. In terms of the vector fields $\tilde{E}_\ell$ and $\tilde{F}_\ell$, the map τ can be described by

$$\begin{aligned}
\tau(\tilde{E}_\ell) &= d\pi(\tilde{\tau}(\tfrac{1}{3}(2E_\ell - F_\ell - G_\ell)) \\
&= d\pi(\tfrac{1}{3}(2G_\ell - E_\ell - F_\ell)) \\
&= -d\pi(\tfrac{1}{3}(2E_\ell - F_\ell - G_\ell) - d\pi(\tfrac{1}{3}(-E_\ell + F_\ell - G_\ell)) \\
&= -\tilde{E}_\ell - \tilde{F}_\ell, \\
\tau(\tilde{F}_\ell) &= d\pi(\tilde{\tau}(\tfrac{1}{3}(-E_\ell + 2F_\ell - G_\ell)) \\
&= d\pi(\tfrac{1}{3}(2E_\ell - F_\ell - G_\ell)) \\
&= \tilde{E}_\ell.
\end{aligned}$$

It now follows by straightforward computations that

$$(\tfrac{2}{\sqrt{3}}(\tau + \tfrac{1}{2}I))^2 = -I$$

and that the nearly Kähler structure is given by

$$J = \tfrac{2}{\sqrt{3}}(\tau + \tfrac{1}{2}I).$$

In particular

$$\begin{aligned}
J\tilde{E}_\ell &= \tfrac{2}{\sqrt{3}}(-\tfrac{1}{2}\tilde{E}_\ell - \tilde{F}_\ell) = \tfrac{1}{\sqrt{3}}(-\tilde{E}_\ell - 2\tilde{F}_\ell, \\
J\tilde{F}_\ell &= \tfrac{2}{\sqrt{3}}(\tilde{E}_\ell + \tfrac{1}{2}\tilde{F}_\ell) = \tfrac{1}{\sqrt{3}}(2\tilde{E}_\ell + \tilde{F}_\ell).
\end{aligned}$$

Using similar computations, for the maps P_1, P_2 and P_3 we obtain the following lemma.

Lemma 7. *We have that*

$$\begin{aligned}
P_1 &= P \\
P_2 &= -\tfrac{1}{2}P - \tfrac{\sqrt{3}}{2}JP \\
P_3 &= -\tfrac{1}{2}P + \tfrac{\sqrt{3}}{2}JP
\end{aligned}$$

Note that in a subsequent section, we will show that these are precisely the three possible almost product structures on $\mathbb{S}^3 \times \mathbb{S}^3$ which preserve the basic equations. We will also see that even though the maps $\mathcal{F}_1$ and $\mathcal{F}_2$ are isometries of $\mathbb{S}^3 \times \mathbb{S}^3$ which do not necessarily preserve the almost product structure P, they do preserve the triple of almost product structures $\{P_1, P_2, P_3\}$.

2.1.3 Properties of the application $\mathcal{F}_1$, $\mathcal{F}_2$

We look at the maps $\widetilde{\mathcal{F}_{abc}}$, $\widetilde{\mathcal{F}_1}$ and $\widetilde{\mathcal{F}_2}$ of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ defined respectively by

$$\begin{aligned}\widetilde{\mathcal{F}_{abc}}(g_1, g_2, g_3) &= (ag_1, bg_2, cg_3),\\ \widetilde{\mathcal{F}_1}(g_1, g_2, g_3) &= (g_2, g_1, g_3),\\ \widetilde{\mathcal{F}_2}(g_1, g_2, g_3) &= (g_3, g_2, g_1),\end{aligned}$$

where a, b, c are unitary quaternions. An elementary computation shows that $\widetilde{\mathcal{F}_{abc}}$, $\widetilde{\mathcal{F}_1}$ and $\widetilde{\mathcal{F}_2}$ are isometries of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$. Note that these isometries all have the property that for any unit quaternion d we have that

$$\begin{aligned}&\pi\widetilde{\mathcal{F}_{abc}}(g_1 d, g_2 d, g_3 d) = \pi(ag_1 d, bg_2 d, cg_3 d) = (ag_1\bar{g}_3\bar{c}, bg_2\bar{g}_3\bar{c}),\\ &\pi\widetilde{\mathcal{F}_1}(g_1 d, g_2 d, g_3 d) = \pi(g_2 d, g_1 d, g_3 d) = (g_2\bar{g}_3, g_1\bar{g}_3),\\ &\pi\widetilde{\mathcal{F}_2}(g_1 d, g_2 d, g_3 d) = \pi(g_3 d, g_2 d, g_1 d) = (g_3\bar{g}_1, g_2\bar{g}_1)\end{aligned}$$

are independent of the unit quaternion d. Therefore we can define the applications $\mathcal{F}_{abc}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathbb{S}^3 \times \mathbb{S}^3$ such that

$$\begin{aligned}\pi \circ \widetilde{\mathcal{F}_{abc}} &= \mathcal{F}_{abc} \circ \pi,\\ \pi \circ \widetilde{\mathcal{F}_1} &= \mathcal{F}_1 \circ \pi,\\ \pi \circ \widetilde{\mathcal{F}_2} &= \mathcal{F}_2 \circ \pi.\end{aligned}$$

As $\widetilde{\mathcal{F}_{abc}}$, $\widetilde{\mathcal{F}_1}$ and $\widetilde{\mathcal{F}_2}$ are isometries of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ and the nearly Kähler metric is a constant multiple of the metric of the Riemannian submersion, it follows that $\mathcal{F}_{abc}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ are isometries of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$. The same remains of course valid for all compositions of these applications. Note that these applications are given by

$$\mathcal{F}_{abc}(p, q) = (ap\bar{c}, bq\bar{c}), \tag{2.26}$$
$$\mathcal{F}_1(p, q) = (q, p), \tag{2.27}$$
$$\mathcal{F}_2(p, q) = (\bar{p}, q\bar{p}). \tag{2.28}$$

As indicated in [7], the isometries $\mathcal{F}_{abc}$ also preserve both the almost complex structure J and the almost product structure P. As we will see in the next lemmas, this is no longer true for the isometries $\mathcal{F}_1$ and $\mathcal{F}_2$.

In order to investigate the behaviour of J, P_1, P_2 and P_3 under the maps $\mathcal{F}_1$ and $\mathcal{F}_2$ we write an arbitrary tangent vector at a point (p, q) by

$$X(p, q) = (p\alpha, q\beta),$$

where α and β are imaginary quaternions. This is a tangent vector to a curve $\delta(t) = (\delta_1(t), \delta_2(t))$ in $\mathbb{S}^3 \times \mathbb{S}^3$ with initial conditions:

$$\delta_1(0) = p, \qquad \delta_2(0) = q, \qquad \delta_1'(0) = p\alpha, \qquad \delta_2'(0) = q\beta.$$

It now follows that

$$\begin{aligned} d\mathcal{F}_1(p\alpha, q\beta) &= d\mathcal{F}_1(X(p,q)) \\ &= \tfrac{d}{dt}\mathcal{F}_1(\delta(t))|_{t=0} \\ &= \tfrac{d}{dt}(\delta_2(t), \delta_1(t))|_{t=0} \\ &= (q\beta, p\alpha) \end{aligned}$$

and

$$\begin{aligned} d\mathcal{F}_2(p\alpha, q\beta) &= d\mathcal{F}_2(X(p,q)) \\ &= \tfrac{d}{dt}\mathcal{F}_2(\delta(t))|_{t=0} \\ &= \tfrac{d}{dt}(\bar{\delta}_1(t), \delta_2(t)\bar{\delta}_1(t))|_{t=0} \\ &= (\bar{\alpha}\bar{p}, q(\beta + \bar{\alpha})\bar{p}) \\ &= (\bar{p}(p(-\alpha)\bar{p}), q\bar{p}(p(\beta - \alpha)\bar{p})). \end{aligned}$$

On the other hand we recall that

$$\begin{aligned} J(p\alpha, q\beta) &= \tfrac{1}{\sqrt{3}}(p(2\beta - \alpha), q(-2\alpha + \beta)), \\ P_1(p\alpha, q\beta) &= P(p\alpha, q\beta) = (p\beta, q\alpha), \end{aligned}$$

from which we deduce that

$$\begin{aligned} P_2(p\alpha, q\beta) &= -\tfrac{1}{2}P(p\alpha, q\beta) - \tfrac{\sqrt{3}}{2}JP((p\alpha, q\beta)) \\ &= -\tfrac{1}{2}((p\beta, q\alpha)) - \tfrac{1}{2}(p(2\alpha - \beta), q(-2\beta + \alpha)) \\ &= (-p\alpha, q(\beta - \alpha)) \end{aligned}$$

and

$$\begin{aligned} P_3(p\alpha, q\beta) &= -\tfrac{1}{2}P(p\alpha, q\beta) + \tfrac{\sqrt{3}}{2}JP((p\alpha, q\beta)) \\ &= -\tfrac{1}{2}((p\beta, q\alpha)) + \tfrac{1}{2}(p(2\alpha - \beta), q(-2\beta + \alpha)) \\ &= (p(\alpha - \beta), q(-\beta)). \end{aligned}$$

Using the above formulas, if necessary at different points and for different tangent vectors, we now can prove:

Theorem 29. *The differential of the isometry $\mathcal{F}_1$ anticommutes with J, i.e. $d\mathcal{F}_1 \circ J = -J \circ d\mathcal{F}_1$. For the almost product structures P_1, P_2 and P_3 we have*

$$\begin{aligned} d\mathcal{F}_1 \circ P_1 &= P_1 \circ d\mathcal{F}_1, \\ d\mathcal{F}_1 \circ P_2 &= P_3 \circ d\mathcal{F}_1, \\ d\mathcal{F}_1 \circ P_3 &= P_2 \circ d\mathcal{F}_1. \end{aligned}$$

Proof. We have

$$
\begin{aligned}
d\mathcal{F}_1(JX) &= \tfrac{1}{\sqrt{3}}(q(-2\alpha+\beta), p(2\beta-\alpha)),\\
Jd\mathcal{F}_1(X) &= J_{(q,p)}(q\beta, p\alpha) = \tfrac{1}{\sqrt{3}}(q(2\alpha-\beta), p(-2\beta+\alpha)),
\end{aligned}
$$

from which the first claim follows. The other claims follow from comparing

$$
\begin{aligned}
d\mathcal{F}_1(P_1X) &= (q\alpha, p\beta),\\
d\mathcal{F}_1(P_2X) &= (q(\beta-\alpha), p(-\alpha)),\\
d\mathcal{F}_1(P_3X) &= (q(-\beta), p(\alpha-\beta)),\\
P_1d\mathcal{F}_1(X) &= (q\alpha, p\beta),\\
P_2d\mathcal{F}_1(X) &= (q(-\beta), p(\alpha-\beta)),\\
P_3d\mathcal{F}_1(X) &= (q(\beta-\alpha), p(-\alpha)).
\end{aligned}
$$

□

Theorem 30. *The differential of the isometry $\mathcal{F}_2$ anticommutes with J, i.e. $d\mathcal{F}_2 \circ J = -J \circ d\mathcal{F}_2$. For the almost product structures P_1, P_2 and P_3 we have*

$$
\begin{aligned}
d\mathcal{F}_2 \circ P_1 &= P_3 \circ d\mathcal{F}_2,\\
d\mathcal{F}_2 \circ P_2 &= P_2 \circ d\mathcal{F}_2,\\
d\mathcal{F}_2 \circ P_3 &= P_1 \circ d\mathcal{F}_2.
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
d\mathcal{F}_2(JX) &= d\mathcal{F}_2(\tfrac{1}{\sqrt{3}}(p(2\beta-\alpha), q(-2\alpha+\beta)))\\
&= \tfrac{1}{\sqrt{3}}(\bar{p}(-p(2\beta-\alpha)\bar{p}), q\bar{p}(p(-\alpha-\beta)\bar{p})),\\
Jd\mathcal{F}_2(X) &= J_{(\bar{p},q\bar{p})}\tfrac{1}{\sqrt{3}}(\bar{p}(p(-\alpha)\bar{p}), q\bar{p}(p(\beta-\alpha)\bar{p}))\\
&= \tfrac{1}{\sqrt{3}}(\bar{p}(p(2\beta-\alpha)\bar{p}), q\bar{p}(p(\beta+\alpha)\bar{p})),
\end{aligned}
$$

from which the first claim follows. The other claims follow from comparing

$$
\begin{aligned}
d\mathcal{F}_2(P_1X) &= (\bar{p}(p(-\beta)\bar{p}), q\bar{p}(p(\alpha-\beta)\bar{p})),\\
d\mathcal{F}_2(P_2X) &= (\bar{p}(p(\alpha)\bar{p}), q\bar{p}(\beta)\bar{p})),\\
d\mathcal{F}_2(P_3X) &= (\bar{p}(p(\beta-\alpha)\bar{p}), q\bar{p}(p(-\alpha)\bar{p})),\\
P_1d\mathcal{F}_1(X) &= (\bar{p}(p(\beta-\alpha)\bar{p}), q\bar{p}(p(-\alpha)\bar{p})),\\
P_2d\mathcal{F}_2(X) &= (\bar{p}(p(\alpha)\bar{p}), q\bar{p}(p(\beta)\bar{p})),\\
P_3d\mathcal{F}_2(X) &= (\bar{p}(p(-\beta)\bar{p}), q\bar{p}(p(\alpha-\beta)\bar{p})).
\end{aligned}
$$

□

From the above two theorems we see that J is preserved up to sign by $\mathcal{F}_1$ and $\mathcal{F}_2$ (and therefore preserved by the composition of the two). On the other hand, by a suitable composition of $\mathcal{F}_1$ and $\mathcal{F}_2$, we see that we can switch between $P = P_1$, P_2 and P_3.

2.1.4 The role of the almost product structure P

The tensor P appears in the basic equations of $\mathbb{S}^3 \times \mathbb{S}^3$ in (2.7), (2.8), (2.9), (2.10), (2.11), (2.12) and (2.13).

We call a tensor P^* satisfying the above conditions a nearly productlike structure on $\mathbb{S}^3 \times \mathbb{S}^3$. In order to determine all nearly productlike structures on $\mathbb{S}^3 \times \mathbb{S}^3$, we have the following lemmas.

Lemma 8. *Let P^* be a structure which satisfies* (2.7)*,* (2.8)*,* (2.9)*,* (2.10)*, and* (2.13)*. Then there exists an angle θ such that*

$$P^* = \cos\theta P + \sin\theta JP.$$

The converse is also true.

Proof. We use (2.13) and we take an arbitrary vector $U = X$. We take $V = Y$ orthogonal to X, JX, P^*PX and JP^*PX. We take $W = P^*Y$. Then we have that

$$\begin{aligned}\big(g(P^*Y, P^*Y)P^*X - g(P^*X, P^*Y)P^*V \\ +g(JP^*Y, P^*Y)JP^*X - g(JP^*X, P^*Y)JP^*V\big) \\ = \big(g(PY, P^*Y)PX - g(PX, P^*Y)PY \\ +g(JPY, P^*Y)JPX - g(JPX, P^*Y)JPY\big).\end{aligned} \tag{2.29}$$

Using the properties of P and P^*, we see that the left hand side of (2.29) reduces to $g(Y,Y)P^*X$, whereas the right hand side reduces to $g(PY, P^*Y)PX + g(JPY, P^*Y)JPX$. Hence for any X there exists an angle $\theta(X)$ such that

$$P^*X = \cos(\theta(X))PX + \sin(\theta(X))JPX.$$

Using the properties of P and P^* we deduce that

$$P^*JX = -JP^*X = \cos(\theta(X))PJX + \sin(\theta(X))JPJX.$$

Hence $\theta(JX) = \theta(X)$. By linearity the same is now true for any linear combination of X and JX. Take now a vector field Y, orthogonal to X and JX, such that $\|Y\| = \|X\|$. For any angle α, we can now compute $\psi_\alpha = \theta(\cos\alpha X + \sin\alpha Y)$. On the one hand we have that

$$\begin{aligned}P^*(\cos\alpha X + \sin\alpha Y) = \cos\alpha(\cos(\theta(X))PX + \sin(\theta(X))JPX) \\ + \sin\alpha(\cos(\theta(Y))PY + \sin(\theta(Y))JPY),\end{aligned}$$

while on the other hand

$$\begin{aligned}P^*(\cos\alpha X + \sin\alpha Y) = \cos\psi_\alpha(\cos\alpha PX + \sin\alpha PY) \\ + \sin\psi_\alpha(\cos\alpha JPX + \sin\alpha JPY)).\end{aligned}$$

As the above formula is valid for any angle α and the vector fields X,JX, Y and JY are mutually orthogonal (and therefore independent) we deduce that $\theta(Y) = \theta(X) = \psi_\alpha$. Hence $\theta(X) = \theta$ is constant. The converse can be verified by a straightforward computation. $\square$

Lemma 9. *P^* satisfies moreover* (2.11) *if and only if θ is a multiple of $\frac{2\pi}{3}$, i.e. if and only if P^* is either P_1, P_2 or P_3. Moreover, in that case* (2.12) *is trivially satisfied.*

Proof. We write
$$P^*(X) = \cos\theta PX + \sin\theta JPX.$$
It then follows that
$$\begin{aligned}
G(P^*X, P^*Y) &= \cos^2\theta G(PX, PY) + \sin^2\theta G(JPX, JPY) \\
&\qquad + \cos\theta\sin\theta(G(PX, JPY) + G(JPX, PY)) \\
&= \cos 2\theta G(PX, PY) - \sin 2\theta JG(PX, PY) \\
&= -\cos 2\theta PG(X, Y) + \sin 2\theta JPG(X, Y).
\end{aligned}$$
On the other hand, we have that
$$-P^*G(X, Y) = -\cos\theta PG(X, Y) - \sin\theta JPG(X, Y).$$
As $PG(X, Y)$ and $JPG(X, Y)$ are mutually orthogonal, we see that equality holds if and only if $\cos 2\theta = \cos\theta = \cos(-\theta)$ and $\sin 2\theta = -\sin\theta = \sin(-\theta)$. Hence, if and only if, 3θ is a multiple of 2π.
In order to show that $P^\star$ now satisfies also (2.12) it is sufficient to consider the case that $P^* = -\frac{1}{2}P + \frac{\sqrt{3}}{2}\varepsilon JP$ where $\varepsilon = \pm 1$. On the one hand we get that
$$\begin{aligned}
(\tilde\nabla_X P^*)Y &= \tilde\nabla_X P^*Y - P^*\tilde\nabla_X Y \\
&= -\frac{1}{2}(\tilde\nabla_X P)Y + \varepsilon\frac{\sqrt{3}}{2}\tilde\nabla_X JPY - \varepsilon\tfrac{\sqrt{3}}{2}JP\tilde\nabla_X Y \\
&= -\frac{1}{2}(\tilde\nabla_X P)Y + \varepsilon\frac{\sqrt{3}}{2}(G(X, PY) + J\tilde\nabla_X PY - JP\tilde\nabla_X Y) \\
&= -\frac{1}{2}(\tilde\nabla_X P)Y + \varepsilon\frac{\sqrt{3}}{2}(G(X, PY) + J(\tilde\nabla_X P)Y) \\
&= -\frac{1}{4}J(G(X, PY) + PG(X, Y)) + \varepsilon\frac{\sqrt{3}}{4}(2G(X, PY) - G(X, PY) - PG(X, Y)) \\
&= -\frac{1}{4}J(G(X, PY) + PG(X, Y)) + \varepsilon\frac{\sqrt{3}}{4}(G(X, PY) - PG(X, Y)).
\end{aligned}$$
On the other hand we get that
$$\begin{aligned}
\frac{1}{2}J(G(X, P^*Y) + P^*G(X, Y)) &= -\frac{1}{4}(JG(X, PY) + JPG(X, Y)) \\
&\qquad + \varepsilon\frac{\sqrt{3}}{2}(JG(X, JPY) - PG(X, Y)) \\
&= -\frac{1}{4}(JG(X, PY) + JPG(X, Y)) \\
&\qquad + \varepsilon\frac{\sqrt{3}}{4}(G(X, PY) - PG(X, Y)).
\end{aligned}$$
Comparing now both right-hand sides completes the proof of the lemma. □

Combining the previous lemmas, we deduce that the only nearly productlike structures on $\mathbb{S}^3 \times \mathbb{S}^3$ are $P_1 = P$, P_2 and P_3. Of course applying the isometries $\mathcal{F}_1$ and $\mathcal{F}_2$ allows us to switch between these structures and therefore from an isometrical point of view these can not be distinguished. As a consequence, in many classification theorems of submanifolds, there will appear 3 isometrical examples which slightly different tensors P.

2.2 Lagrangian submanifolds with constant angle functions in the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$

As already seen from the results of Dioos, Vrancken and Wang in [20], the angle functions provide important information in the characterization of the Lagrangian submanifolds of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$. Therefore, we continue the study of these submanifolds from the point of view of the angle functions.
In this section we show that if all angle functions are constant, then the submanifold is either totally geodesic or has constant sectional curvature and there is a classification theorem that follows from [20]. Moreover, we show that if precisely one angle function is constant, then it must be equal to $0, \frac{\pi}{3}$ or $\frac{2\pi}{3}$. Using then two remarkable constructions together with the classification of Lagrangian submanifolds of which the first component has nowhere maximal rank from [3] (see the next section), we obtain a classification of such Lagrangian submanifolds.
From now on, we identify the tangent vector X with $df(X)$.

Theorem 31. *Let $f : M \longrightarrow \mathbb{S}^3 \times \mathbb{S}^3$ be a Lagrangian immersion into the nearly Kähler manifold $\mathbb{S}^3 \times \mathbb{S}^3$, given by $f = (p, q)$ with angle functions $\theta_1, \theta_2, \theta_3$ and eigenvectors E_1, E_2, E_3. Then $\tilde{f} : M \longrightarrow \mathbb{S}^3 \times \mathbb{S}^3$ given by $\tilde{f} = (q, p)$ satisfies:*

(i) $\tilde{f}$ is a Lagrangian immersion,

(ii) f and $\tilde{f}$ induce the same metric on M,

(iii) E_1, E_2, E_3 are also eigendirections of the operators $\tilde{A}, \tilde{B}$ corresponding to the immersion $\tilde{f}$ and the angle functions $\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3$ are given by $\tilde{\theta}_i = \pi - \theta_i$, for $i = 1, 2, 3$.

Proof. Let $f : M \longrightarrow \mathbb{S}^3 \times \mathbb{S}^3$ given by $f = (p, q)$ be a Lagrangian immersion with the angle functions $\theta_1, \theta_2, \theta_3$. Then, for any point on M, we have a differentiable frame $\{E_1, E_2, E_3\}$ along M satisfying (2.20) such that

$$df(E_i) = (p\alpha_i, q\beta_i)_{(p,q)},\ i = 1, 2, 3, \tag{2.30}$$

where α_i, β_i are imaginary quaternions. Moreover, for $\tilde{f}$ we have as well

$$d\tilde{f}(E_i) = (q\beta_i, p\alpha_i)_{(q,p)},\ i = 1, 2, 3.$$

From equations (2.2) and (2.6) a direct calculation gives that

$$Pdf(E_i) = (p\beta_i, q\alpha_i)_{(p,q)}, \tag{2.31}$$

$$Jdf(E_i) = \frac{1}{\sqrt{3}} \left(p(2\beta_i - \alpha_i), q(-2\alpha_i + \beta_i)\right)_{(p,q)}, \tag{2.32}$$

and

$$Pd\tilde{f}(E_i) = (q\alpha_i, p\beta_i)_{(q,p)}, \tag{2.33}$$

$$Jd\tilde{f}(E_i) = \frac{1}{\sqrt{3}} \left(q(2\alpha_i - \beta_i), p(-2\beta_i + \alpha_i)\right)_{(q,p)}, \tag{2.34}$$

for $i = 1, 2, 3$. The conditions for f and $\tilde{f}$ to be Lagrangian immersions write out, respectively, as

$$g(df(E_i), Jdf(E_j)) = 0 \text{ for } i \neq j,$$

$$g(d\tilde{f}(E_i), Jd\tilde{f}(E_j)) = 0 \text{ for } i \neq j.$$

By (2.3) and by the previous relations, these conditions become

$$\frac{4}{3}(\langle \alpha_i, 2\beta_j - \alpha_j\rangle + \langle \beta_i, -2\alpha_j + \beta_j\rangle) - \frac{2}{3}(\langle \alpha_i, -2\alpha_j + \beta_j\rangle + \langle 2\beta_j - \alpha_j, \beta_i\rangle) = 0,$$
$$\frac{4}{3}(\langle \beta_i, 2\alpha_j - \beta_j\rangle + \langle \alpha_i, -2\beta_j + \alpha_j\rangle) - \frac{2}{3}(\langle \beta_i, -2\beta_j + \alpha_j\rangle + \langle 2\alpha_j - \beta_j, \alpha_i\rangle) = 0,$$

respectively. Since both are equivalent to $\langle \alpha_i, \beta_j\rangle - \langle \beta_i, \alpha_j\rangle = 0$, we conclude that $\tilde{f}$ is a Lagrangian immersion if and only if f is a Lagrangian immersion. Therefore, one may notice that this also implies that $\tilde{f}$ is an isometry.
In order to prove (ii), we must show that $g(df(E_i), df(E_j)) = g(d\tilde{f}(E_i), d\tilde{f}(E_j))$. By straightforward computations, using (2.3), we have

$$\begin{aligned} g(df(E_i), df(E_j)) =& \frac{1}{2}\Big(\langle df(E_i), df(E_j)\rangle + \langle Jdf(E_i), Jdf(E_j)\rangle\Big) \\ =& \frac{1}{2}\Big(\langle (p\alpha_i, q\beta_i), (p\alpha_j, q\beta_j)\rangle + \\ &+ \frac{1}{3}\langle (p(2\beta_i - \alpha_i), q(-2\alpha_i + \beta_i)), (p(2\beta_j - \alpha_j), q(-2\alpha_j + \beta_j))\rangle\Big) \\ =& \frac{2}{3}\Big(2\langle \alpha_i, \alpha_j\rangle + 2\langle \beta_i, \beta_j\rangle - \langle \beta_i, \alpha_j\rangle - \langle \alpha_i, \beta_j\rangle\Big). \end{aligned} \tag{2.35}$$

Similarly, we have

$$\begin{aligned} g(d\tilde{f}(E_i), d\tilde{f}(E_j)) =& \frac{1}{2}\Big(\langle d\tilde{f}(E_i), d\tilde{f}(E_j)\rangle + \langle Jd\tilde{f}(E_i), Jd\tilde{f}(E_j)\rangle\Big) \\ =& \frac{1}{2}\Big(\langle (q\beta_i, p\alpha_i), (q\beta_j, p\alpha_j)\rangle + \\ &+ \frac{1}{3}\langle (q(2\alpha_i - \beta_i), p(-2\beta_i + \alpha_i)), (q(2\alpha_j - \beta_j), p(-2\beta_j + \alpha_j))\rangle\Big) \\ =& \frac{2}{3}\Big(2\langle \alpha_i, \alpha_j\rangle + 2\langle \beta_i, \beta_j\rangle - \langle \beta_i, \alpha_j\rangle - \langle \alpha_i, \beta_j\rangle\Big) \end{aligned}$$

and we can easily notice that the metric is preserved under the transformation $\tilde{f}$.
In order to prove (iii), we see from (2.20) that

$$Pdf(E_i) = \cos(2\theta_i)df(E_i) + \sin(2\theta_i)Jdf(E_i), \tag{2.36}$$

and there exist $\tilde{A}, \tilde{B} : TM \to TM$ with eigenvectors $\tilde{E}_i$ and angle functions $\tilde{\theta}_i$ such that

$$Pd\tilde{f}(\tilde{E}_i) = \cos(2\tilde{\theta}_i)d\tilde{f}(\tilde{E}_i) + \sin(2\tilde{\theta}_i)Jd\tilde{f}(\tilde{E}_i). \tag{2.37}$$

From (2.30) and (2.32), we replace $df(E_i)$ and $Jdf(E_i)$ in (2.36) and get:

$$Pdf(E_i) = \Big(p\Big(\cos(2\theta_i)\alpha_i + \frac{1}{\sqrt{3}}\sin(2\theta_i)(2\beta_i - \alpha_i)\Big), q\Big(\cos(2\theta_i)\beta_i + \frac{1}{\sqrt{3}}\sin(2\theta_i)(-2\alpha_i + \beta_i)\Big)\Big). \tag{2.38}$$

Considering now equation (2.31) as well, we obtain

$$\alpha_i = \cos(2\theta_i)\beta_i + \frac{1}{\sqrt{3}}\sin(2\theta_i)(-2\alpha_i + \beta_i),$$

$$\beta_i = \cos(2\theta_i)\alpha_i + \frac{1}{\sqrt{3}}\sin(2\theta_i)(2\beta_i - \alpha_i).$$

Replacing α_i and β_i in (2.33) with the latter expressions gives

$$Pd\tilde{f}(E_i) = \cos(-2\theta_i)d\tilde{f}(E_i) + \sin(-2\theta_i)Jd\tilde{f}(E_i).$$

Comparing this with (2.37), we see that E_i are the eigenvectors of $\tilde{A}$ and $\tilde{B}$ with angle functions

$$\tilde{\theta}_i = \pi - \theta_i.$$

□

Remark 2. One should notice that by making use of the fact that $\mathcal{F}_1$ defined in (2.26) is an isometry, conditions $(i), (ii)$ in the theorem become trivial to prove:

$$g(d\tilde{f}(E_i), d\tilde{f}(E_j)) = g(d\mathcal{F}_1(d\tilde{f}(E_i)), d\mathcal{F}_1(d\tilde{f}(E_j))) = g(df(E_i), df(E_j)).$$

This implies that f and $\tilde{f}$ induce the same metric on M. Similarly, using the $\mathcal{F}_1$ anticommutes with J, we obtain $g(d\tilde{f}(E_i), Jd\tilde{f}(E_j)) = g(df(E_i), Jdf(E_j))$.

Theorem 32. *Let $f : M \longrightarrow \mathbb{S}^3 \times \mathbb{S}^3$ be a Lagrangian immersion into the nearly Kähler manifold $\mathbb{S}^3 \times \mathbb{S}^3$ given by $f = (p, q)$ with angle functions $\theta_1, \theta_2, \theta_3$ and eigenvectors E_1, E_2, E_3. Then, $f^* : M \longrightarrow \mathbb{S}^3 \times \mathbb{S}^3$ given by $f^* = (\bar{p}, q\bar{p})$ satisfies:*

(i) f^* is a Lagrangian immersion,

(ii) f and f^* induce the same metric on M,

(iii) E_1, E_2, E_3 are also eigendirections of the operators A^*, B^* corresponding to the immersion f^* and the angle functions $\theta_1^*, \theta_2^*, \theta_3^*$ are given by $\theta_i^* = \frac{2\pi}{3} - \theta_i$, for $i = 1, 2, 3$.

Proof. Let $f : M \longrightarrow \mathbb{S}^3 \times \mathbb{S}^3$ given by $f = (p, q)$ be a Lagrangian immersion with the angle functions $\theta_1, \theta_2, \theta_3$. Then, for any point on M, we have a differentiable frame $\{E_i\}$ along M satisfying (2.20) and we may write

$$df(E_i) = (p\alpha_i, q\beta_i)_{(p,q)}, \tag{2.39}$$
$$df^*(E_i) = (\bar{p}\alpha_i^*, q\bar{p}\beta_i^*)_{(\bar{p},q\bar{p})}, \tag{2.40}$$

for $i = 1, 2, 3$ and $\alpha_i, \beta_i, \alpha_i^*, \beta_i^*$ imaginary quaternions. Moreover we have that

$$\alpha_i^* = -p\alpha_i\bar{p},$$
$$\beta_i^* = p(\beta_i - \alpha_i)\bar{p},$$

where we have used

$$df^*(E_i) = D_{E_i}f^* = (D_{E_i}\bar{p}, D_{E_i}(q\bar{p})) = (\overline{D_{E_i}p}, (D_{E_i}q)\bar{p} + q(\overline{D_{E_i}p})) \overset{(2.39)}{=} (-\alpha_i\bar{p}, q(\beta_i - \alpha_i)\bar{p})$$

for the Euclidean connection D. Furthermore, by (2.2) and (2.6), we obtain again (2.31) and (2.32) as well as

$$Pdf^*(E_i) = ((\beta_i - \alpha_i)\bar{p}, -q\alpha_i\bar{p})_{(\bar{p},q\bar{p})}, \tag{2.41}$$

$$Jdf^*(E_i) = \frac{1}{\sqrt{3}}\left((2\beta_i - \alpha_i)\bar{p}, q(\alpha_i + \beta_i)\bar{p}\right)_{(\bar{p}, q\bar{p})} \tag{2.42}$$

for $i = 1, 2, 3$. A straightforward computation gives, for $i \neq j$, that

$$g(df^*(E_i), Jdf^*(E_j)) = \frac{2}{\sqrt{3}}(\langle \beta_i, \alpha_j \rangle - \langle \alpha_i, \beta_j \rangle),$$

which, as in the proof of Theorem 31, shows that f^* is a Lagrangian immersion if and only if f is a Lagrangian immersion. Therefore, one may notice that this also implies that f^* is an isometry.

To prove (ii), we must show that $g(df(E_i), df(E_j)) = g(df^*(E_i), df^*(E_j))$. By straightforward computations, using (2.3), we have

$$\begin{aligned} g(df^*(E_i), df^*(E_j)) &= \frac{1}{2}\Big(\langle df^*(E_i), df^*(E_j)\rangle + \langle Jdf^*(E_i), Jdf^*(E_j)\rangle\Big) \\ &= \frac{1}{2}\Big(\langle(-\alpha_i\bar{p}, q(\beta_i - \alpha_i)\bar{p}), (-\alpha_j\bar{p}, q(\beta_j - \alpha_j)\bar{p})\rangle + \\ &+ \frac{1}{3}\langle((2\beta_i - \alpha_i)\bar{p}, q(\alpha_i + \beta_i)\bar{p}), ((2\beta_j - \alpha_j)\bar{p}, q(\alpha_j + \beta_j)\bar{p})\rangle\Big) \\ &= \frac{2}{3}\Big(2\langle \alpha_i, \alpha_j\rangle + 2\langle \beta_i, \beta_j\rangle - \langle \beta_i, \alpha_j\rangle - \langle \alpha_i, \beta_j\rangle\Big), \end{aligned}$$

and, comparing it to (2.35), we can easily notice that the metric is preserved under the transformation f^*.
In order to prove (iii), we see from (2.20) that

$$Pdf(E_i) = \cos(2\theta_i)df(E_i) + \sin(2\theta_i)Jdf(E_i), \tag{2.43}$$

and, associated with the second immersion f^*, there exist $A^*, B^* : TM \to TM$ with eigenvectors E_i^* and angle functions θ_i^* such that

$$Pdf^*(E_i^*) = \cos(2\theta_i^*)df^*(E_i^*) + \sin(2\theta_i^*)Jdf^*(E_i^*). \tag{2.44}$$

As in the proof of the previous theorem, we have

$$\begin{aligned} \alpha_i &= \cos(2\theta_i)\beta_i + \frac{1}{\sqrt{3}}\sin(2\theta_i)(-2\alpha_i + \beta_i), \\ \beta_i &= \cos(2\theta_i)\alpha_i + \frac{1}{\sqrt{3}}\sin(2\theta_i)(2\beta_i - \alpha_i). \end{aligned}$$

On the one hand, replacing α_i and β_i in (2.41) with the latter expressions, we see that

$$Pdf^*(E_i) = \Big([\cos(2\theta_i)(\alpha_i - \beta_i) + \frac{1}{\sqrt{3}}\sin(2\theta_i)(\beta_i + \alpha_i)]\bar{p}, -q[\cos(2\theta_i)\beta_i + \frac{1}{\sqrt{3}}\sin(2\theta_i)(-2\alpha_i + \beta_i)]\bar{p}\Big).$$

On the other hand, we see that for $\theta_i^* = \frac{2\pi}{3} - \theta_i$, the following holds:

$$\begin{aligned} \cos(2\theta_i^*)df^*(E_i) + \sin(2\theta_i^*)Jdf^*(E_i) &= \cos(\frac{4\pi}{3} - 2\theta_i)df^*(E_i) + \sin(\frac{4\pi}{3} - 2\theta_i)Jdf^*(E_i) \\ &= \frac{1}{2}[(-\cos(2\theta_i - \sqrt{3}\sin(2\theta_i)))df^*(E_i) + (-\sqrt{3}\cos(2\theta_i) + \sin(2\theta_i)Jdf^*(E_i))] \end{aligned}$$

$$\overset{(2.40),(2.42)}{=}([\cos(2\theta_i)(\alpha_i-\beta_i)+\frac{\sin(2\theta_i)}{\sqrt{3}}(\alpha_i+\beta_i)]\bar{p}, q[\cos(2\theta_i)(-\beta_i)+\frac{\sin(2\theta_i)}{\sqrt{3}}(2\alpha_i-\beta_i)]\bar{p}).$$

Therefore, (2.44) holds for $E_i^*=E_i$ and $\theta_i^*=\frac{2\pi}{3}-\theta_i$. This concludes point (iii) of the theorem.

□

Remark 3. One should notice that by making use of the fact that $\mathcal{F}_2$ defined in (2.26) is an isometry, conditions $(i),(ii)$ of the theorem become trivial to prove:

$$g(df^*(E_i),df^*(E_j))=g(d\mathcal{F}_2(df^*(E_i)),d\mathcal{F}_2(df^*(E_j)))=g(df(E_i),df(E_j)).$$

This implies that f and f^* induce the same metric on M. Similarly, using the fact that $\mathcal{F}_2$ anticommutes with J, we obtain $g(df^*(E_i),Jdf^*(E_j))=g(df(E_i),Jdf(E_j))$.

Lemma 10. *Let M be a Lagrangian submanifold of the nearly Kähler manifold $\mathbb{S}^3\times\mathbb{S}^3$ with constant angle functions $\theta_1,\theta_2,\theta_3$.*

i. If M is a non-totally geodesic submanifold, then the nonzero components of ω_{ij}^k are given by

$$\omega_{12}^3=\frac{\sqrt{3}}{6}-\frac{\cos(\theta_2-\theta_3)}{\sin(\theta_2-\theta_3)}h_{12}^3, \tag{2.45}$$

$$\omega_{23}^1=\frac{\sqrt{3}}{6}+\frac{\cos(\theta_1-\theta_3)}{\sin(\theta_1-\theta_3)}h_{12}^3, \tag{2.46}$$

$$\omega_{31}^2=\frac{\sqrt{3}}{6}-\frac{\cos(\theta_1-\theta_2)}{\sin(\theta_1-\theta_2)}h_{12}^3. \tag{2.47}$$

ii. The Codazzi equations of the submanifold M are as followings:

$$E_i(h_{12}^3)=0,\quad i=1,2,3, \tag{2.48}$$

$$h_{12}^3\left(2(\omega_{13}^2+\omega_{21}^3)+\frac{1}{\sqrt{3}}\right)=\frac{1}{3}\sin(2(\theta_1-\theta_2)), \tag{2.49}$$

$$h_{12}^3\left(2(\omega_{12}^3+\omega_{31}^2)-\frac{1}{\sqrt{3}}\right)=\frac{1}{3}\sin(2(\theta_1-\theta_3)), \tag{2.50}$$

$$h_{12}^3\left(2(\omega_{21}^3+\omega_{32}^1)+\frac{1}{\sqrt{3}}\right)=\frac{1}{3}\sin(2(\theta_2-\theta_3)). \tag{2.51}$$

iii. The Gauss equations of the submanifold M are given by

$$\frac{5}{12}+\frac{1}{3}\cos(2(\theta_1-\theta_2))-(h_{12}^3)^2=-\omega_{21}^3\omega_{13}^2+\omega_{12}^3\omega_{31}^2-\omega_{21}^3\omega_{31}^2, \tag{2.52}$$

$$\frac{5}{12}+\frac{1}{3}\cos(2(\theta_1-\theta_3))-(h_{12}^3)^2=-\omega_{31}^2\omega_{12}^3+\omega_{13}^2\omega_{21}^3-\omega_{31}^2\omega_{21}^3, \tag{2.53}$$

$$\frac{5}{12}+\frac{1}{3}\cos(2(\theta_2-\theta_3))-(h_{12}^3)^2=-\omega_{32}^1\omega_{21}^3+\omega_{23}^1\omega_{12}^3-\omega_{32}^1\omega_{12}^3. \tag{2.54}$$

Proof. Suppose that M is a Lagrangian submanifold of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$ for which the angle functions $\theta_1, \theta_2, \theta_3$ are constant. Thus, equation (2.24) immediately implies that all coefficients of the second fundamental form are zero except h_{12}^3. Using (2.25), we see that $\omega_{ii}^j = 0, i \neq j$. As $\omega_{ij}^k = -\omega_{ik}^j$, it follows that $\omega_{12}^3, \omega_{23}^1, \omega_{31}^2$ are the only non-zero components out of ω_{ij}^k. From (2.25) and by Remark 1, we calculate the nonzero connection forms as in (2.45)-(2.47). Taking E_1, E_2, E_3 and E_3, E_1, E_2 for the vector fields X, Y, Z in the Codazzi equation (2.22), we get (2.48), and for $E_1, E_2, E_2; E_1, E_3, E_3; E_2, E_3, E_3$ we obtain (2.48)-(2.51). Moreover, we evaluate the Gauss equation (2.21) successively for E_1, E_2, E_2; E_1, E_3, E_3; E_3, E_2, E_2 and then we obtain the given equations, respectively. □

Theorem 33. *A Lagrangian submanifold of the nearly Kähler manifold $\mathbb{S}^3 \times \mathbb{S}^3$ for which all angle functions are constant is either totally geodesic or has constant sectional curvature in $\mathbb{S}^3 \times \mathbb{S}^3$.*

Proof. Suppose that M is a Lagrangian submanifold in the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$ with constant angle functions $\theta_1, \theta_2, \theta_3$. From equation (2.24) and the fact that h_{ij}^k are totally symmetric, all coefficients are zero except h_{12}^3. Also, the Codazzi equations given by (2.48)-(2.51) are valid for M. Equation (2.48) implies that h_{12}^3 is constant and thus, there are two cases that may occur:

Case 1. $h_{12}^3 = 0$, that is, M is a totally geodesic Lagrangian submanifold in the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$.

Case 2. h_{12}^3 is a nonzero constant, that is, M is non-totally geodesic. In this case, the nonzero components of ω_{ij}^k for the submanifold M are given by (2.45)-(2.47). Replacing the coresponding ω_{ij}^k in the Codazzi equations given by (2.49)-(2.51), we obtain the following system of equations:

$$2(h_{12}^3)^2 - \frac{1}{\sqrt{3}} \frac{\sin(\theta_1 - \theta_3)\sin(\theta_2 - \theta_3)}{\sin(\theta_1 - \theta_2)} h_{12}^3 - \frac{2}{3}\cos(\theta_1 - \theta_2)\sin(\theta_1 - \theta_3)\sin(\theta_2 - \theta_3) = 0, \tag{2.55}$$

$$2(h_{12}^3)^2 - \frac{1}{\sqrt{3}} \frac{\sin(\theta_1 - \theta_2)\sin(\theta_1 - \theta_3)}{\sin(\theta_2 - \theta_3)} h_{12}^3 - \frac{2}{3}\cos(\theta_2 - \theta_3)\sin(\theta_1 - \theta_2)\sin(\theta_1 - \theta_3) = 0, \tag{2.56}$$

$$2(h_{12}^3)^2 - \frac{1}{\sqrt{3}} \frac{\sin(\theta_2 - \theta_3)\sin(\theta_1 - \theta_2)}{\sin(\theta_1 - \theta_3)} h_{12}^3 + \frac{2}{3}\cos(\theta_1 - \theta_3)\sin(\theta_2 - \theta_3)\sin(\theta_1 - \theta_2) = 0. \tag{2.57}$$

Notice that by Remark 1, we have $\sin(\theta_i - \theta_j) \neq 0$. Considering the above system of equations as a linear system in $2(h_{12}^3)^2, h_{12}^3, 1$ and since h_{12}^3 is a nonzero constant, we see that the matrix of the system must have determinant zero. By a direct calculation, we find

$$\sin(\theta_1 + \theta_2 - 2\theta_3)\sin(\theta_2 + \theta_3 - 2\theta_1)\sin(\theta_1 + \theta_3 - 2\theta_2) = 0.$$

Given the symmetry in $\theta_1, \theta_2, \theta_3$, it is sufficient to assume that $\sin(\theta_1 + \theta_3 - 2\theta_2) = 0$. Thus, considering also Lemma 3, we may write

$$\theta_1 + \theta_3 - 2\theta_2 = k_1\pi, \tag{2.58}$$

$$\theta_1 + \theta_2 + \theta_3 = k_2\pi \tag{2.59}$$

for $k_1, k_2 \in \mathbb{Z}$. As each angle function is determined modulo π, there exist $l_1, l_2, l_3 \in \mathbb{Z}$ such that the above equations are satisfied by new angle functions $\theta_1 + l_1\pi$, $\theta_2 + l_2\pi$ and $\theta_3 + l_3\pi$:

$$
\begin{aligned}
(\theta_1 + l_1\pi) + (\theta_3 + l_3\pi) - 2(\theta_2 + l_2\pi) &= k_1^*\pi, \\
(\theta_1 + l_1\pi) + (\theta_2 + l_2\pi) + (\theta_3 + l_3\pi) &= k_2^*\pi.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
k_1 &= k_1^* - (l_1 + l_3 - 2l_2), \\
k_2 &= k_2^* - (l_1 + l_2 + l_3).
\end{aligned}
$$

Hence, we may assume $k_2 = 1$. Allowing now only changes of angles which preserve this property, we must have that $l_2 = -l_1 - l_3$ and $k_1 = k_1^* - 3(l_1 + l_3)$. So we may additionally assume that $k_1 \in \{-1, 0, 1\}$. Therefore, we have three cases:

(i) $\theta_1 + \theta_3 - 2\theta_2 = -\pi$ and $\theta_1 + \theta_2 + \theta_3 = \pi$,

(ii) $\theta_1 + \theta_3 - 2\theta_2 = 0$ and $\theta_1 + \theta_2 + \theta_3 = \pi$,

(iii) $\theta_1 + \theta_3 - 2\theta_2 = \pi$ and $\theta_1 + \theta_2 + \theta_3 = \pi$.

Finally, this reduces to

(i) $\theta_2 = 0$ and $\theta_1 + \theta_3 = \pi$,

(ii) $\theta_2 = \frac{\pi}{3}$ and $\theta_1 + \theta_3 = \frac{2\pi}{3}$,

(iii) $\theta_2 = \frac{2\pi}{3}$ and $\theta_1 + \theta_3 = \frac{\pi}{3}$.

Using the relations between the angles θ_i and $\tilde{\theta}_i$, θ_i^* of the Lagrangian immersions $\tilde{f}$ and f^* in Theorem 31 and Theorem 32 respectively, these three cases can be reduced to a single case, as shown below:

$$
\begin{matrix} \theta_2 = \frac{\pi}{3} \\ \theta_1 + \theta_3 = \frac{2\pi}{3} \end{matrix} \quad \overset{\tilde{f}}{\longleftrightarrow} \quad \begin{matrix} \tilde{\theta}_2 = \frac{2\pi}{3} \\ \tilde{\theta}_1 + \tilde{\theta}_3 = \frac{\pi}{3} \end{matrix} \quad \overset{f^*}{\longleftrightarrow} \quad \begin{matrix} \theta_2^* = 0 \\ \theta_1^* + \tilde{\theta}_3^* = \pi. \end{matrix} \tag{2.60}
$$

Remark that according to Theorems 31 and 32, the metric g given by (2.3) is preserved under transformations $\tilde{f}, f^*$ from which we deduce that the sectional curvature of M is the same in each case. Therefore, it is sufficient to consider the case that $\theta_2 = \frac{\pi}{3}$ and $\theta_1 + \theta_3 = \frac{2\pi}{3}$. By straightforward computations, equations (2.49)-(2.54) reduce to

$$
2(h_{12}^3)^2 - \frac{1}{\sqrt{3}} \sin(2\alpha)\; h_{12}^3 - \frac{1}{3} \sin^2(2\alpha) = 0, \tag{2.61}
$$

$$
2(h_{12}^3)^2 - \frac{1}{\sqrt{3}} \frac{\sin^2\alpha}{\sin(2\alpha)}\; h_{12}^3 + \frac{2}{3} \sin^2\alpha \cos(2\alpha) = 0, \tag{2.62}
$$

where $\alpha := \theta_1 - \frac{\pi}{3}$. Solving this system of equations, we see that there are four cases that we must discuss:

(a) $h_{12}^3 = -\frac{1}{2}$ and $\alpha = -\frac{\pi}{3} + k\pi$,

(b) $h_{12}^3 = -\frac{1}{4}$ and $\alpha = \frac{\pi}{3} + k\pi$,

(c) $h_{12}^3 = \frac{1}{4}$ and $\alpha = -\frac{\pi}{3} + k\pi$,

(d) $h_{12}^3 = \frac{1}{2}$ and $\alpha = \frac{\pi}{3} + k\pi$

for some $k \in \mathbb{Z}$.
Remark that cases (c) and (d) reduce to cases (a) and (b), respectively, by changing the basis $\{E_1, E_2, E_3\}$ with $\{E_3, E_2, -E_1\}$. Therefore, we will only consider cases (a) and (b).

Case (a): $h_{12}^3 = -\frac{1}{2}$ and $\theta_1 = 0, \theta_2 = \frac{\pi}{3}, \theta_3 = \frac{2\pi}{3}$. From (2.45)-(2.47), we find that all connection forms are zero. Thus, M is a flat Lagrangian submanifold in the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$.

Case (b): $h_{12}^3 = -\frac{1}{4}$ and $\theta_1 = \frac{2\pi}{3}, \theta_2 = \frac{\pi}{3}, \theta_3 = 0$, In this case, we have that $\omega_{12}^3 = \omega_{23}^1 = \omega_{31}^2 = \frac{\sqrt{3}}{4}$. By a straightforward computation, we find that M has constant sectional curvature which is equal to $\frac{3}{16}$. As a result, the Lagrangian submanifold M of the nearly Kähler manifold $\mathbb{S}^3 \times \mathbb{S}^3$ with constant angle functions $\theta_1, \theta_2, \theta_3$, which is not totally geodesic, has constant sectional curvature. □

Combining the classification theorems in [20] and [59] and Theorem 33, we state the following:

Corollary 1. *A Lagrangian submanifold in the nearly Kähler manifold $\mathbb{S}^3 \times \mathbb{S}^3$ whose all angle functions are constant is locally congruent to one of the following immersions:*

1. $f\colon \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (u, 1)$,
2. $f\colon \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (1, u)$,
3. $f\colon \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (u, u)$,
4. $f\colon \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (u, ui)$,
5. $f\colon \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (u^{-1}, uiu^{-1})$,
6. $f\colon \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (uiu^{-1}, u^{-1})$,
7. $f\colon \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (uiu^{-1}, uju^{-1})$,
8. $f : \mathbb{R}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : (u, v, w) \mapsto (p(u, w), q(u, v))$, *where p and q are constant mean curvature tori in $\mathbb{S}^3$*

$$\begin{aligned}
p(u, w) &= (\cos u \cos w, \cos u \sin w, \sin u \cos w, \sin u \sin w)\,,\\
q(u, v) &= \frac{1}{\sqrt{2}} (\cos v\,(\sin u + \cos u)\,, \sin v\,(\sin u + \cos u)\,,\\
&\qquad \cos v\,(\sin u - \cos u)\,, \sin v\,(\sin u - \cos u))\,.
\end{aligned}$$

Theorem 34. *Let M be a Lagrangian submanifold in the nearly Kähler manifold $\mathbb{S}^3 \times \mathbb{S}^3$ with angle functions $\theta_1, \theta_2, \theta_3$. If precisely one of the angle functions is constant, then up to a multiple of π, it can be either $0, \frac{\pi}{3}$ or $\frac{2\pi}{3}$.*

Proof. First, we may denote the three angle functions by

$$
\begin{aligned}
2\theta_1 &= 2c, \\
2\theta_2 &= 2\Lambda - c, \\
2\theta_3 &= -2\Lambda - c,
\end{aligned}
$$

where $c \in \mathbb{R}$ and Λ is some non constant function. Then, we may write the conditions following from the minimality of the Lagrangian immersion:

$$
\begin{aligned}
h_{11}^1 + h_{12}^2 + h_{13}^3 &= 0, \\
h_{11}^2 + h_{22}^2 + h_{23}^3 &= 0, \\
h_{11}^3 + h_{22}^3 + h_{33}^3 &= 0.
\end{aligned} \tag{2.63}
$$

We are going to use the definitions of ∇A and ∇B in (2.92) and (2.93) and then evaluate these relations for different vectors in the basis in order to get information about the functions ω_{ij}^k and h_{ij}^k. For $X = Y = E_1$ in (2.92) we obtain that

$$
\begin{aligned}
h_{12}^2 &= -h_{13}^3, \\
\omega_{11}^2 &= -\frac{h_{11}^2(\cos(c-2\Lambda) + \cos(2c))}{\sin(c-2\Lambda) + \sin(2c)}, \\
\omega_{11}^3 &= -\frac{h_{11}^3(\sin(2c) - \sin(2\Lambda + c))}{\cos(2c) - \cos(2\Lambda + c)}.
\end{aligned} \tag{2.64}
$$

If we take $X = E_1$ and $Y = E_2$ in (2.92) and (2.93), we see that

$$
E_1(\Lambda) = h_{13}^3, \tag{2.65}
$$

$$
\omega_{12}^3 = \frac{\sqrt{3}}{6} - h_{12}^3 \cot 2\Lambda \tag{2.66}
$$

and, for $X = E_2$ and $Y = E_1$ in (2.92), we obtain

$$
h_{11}^2 = 0, \tag{2.67}
$$

$$
\omega_{21}^2 = -\frac{h_{13}^3(\sin(2c) - \sin(c - 2\Lambda))}{\cos(2c) - \cos(c - 2\Lambda)} \tag{2.68}
$$

$$
\omega_{21}^3 = -h_{12}^3 \cot\left(\Lambda + \frac{3c}{2}\right) - \frac{\sqrt{3}}{6}. \tag{2.69}
$$

Then we choose successively $X = E_3, Y = E_1$, $X = E_2, Y = E_3$ and $X = E_3, Y = E_2$ in relations (2.92) and (2.93) and obtain

$$
h_{11}^3 = 0, \tag{2.70}
$$

$$
\omega_{31}^2 = \frac{\sqrt{3}}{6} - h_{12}^3 \cot\left(\frac{3c}{2} - \Lambda\right), \tag{2.71}
$$

$$
\omega_{31}^3 = \frac{h_{13}^3(\sin(2c) - \sin(2\Lambda + c))}{\cos(2c) - \cos(2\Lambda + c)}, \tag{2.72}
$$

$$
\omega_{22}^3 = -\cot 2\Lambda h_{22}^3, \tag{2.73}
$$

$$
\omega_{32}^3 = -\cot 2\Lambda h_{23}^3 \tag{2.74}
$$

$$E_2(\Lambda) = h^3_{23}, \tag{2.75}$$
$$E_3(\Lambda) = -h^3_{22}. \tag{2.76}$$

We can easily see from (2.64), (2.67) and (2.70) that

$$\omega^2_{11} = 0 \quad \text{and} \quad \omega^3_{11} = 0,$$

and if we consider, as well, the relations in (2.91), we have that

$$h^3_{33} = -h^3_{22}, \quad h^1_{11} = 0 \quad \text{and} \quad h^2_{22} = -h^3_{23}.$$

Next, we are going to use the definition for ∇h in (2.22) and take different values for the vectors X, Y and Z. Thus, we evaluate it for E_1, E_2, E_1 and E_1, E_3, E_1. Looking at the component in E_3 of the resulting two vectors, we obtain the following relations, respectively:

$$E_1(h^3_{12}) = \frac{h^3_{13}}{\sqrt{3}} - 2h^3_{12}h^3_{13}\left(\cot(2\Lambda) + \frac{2\sin(2\Lambda)}{\cos(2\Lambda) - \cos(3c)}\right),$$
$$E_1(h^3_{13}) = \frac{1}{3}\Big(\cot\left(\Lambda + \frac{3c}{2}\right)\left(1 - \cos(2\Lambda + 3c) + 6(h^3_{12})^2 + 6(h^3_{13})^2\right) + 6(h^3_{12})^2\cot(2\Lambda) - \sqrt{3}h^3_{12}\Big).$$

Taking again $X = E_1, Y = E_2, Z = E_1$ in (2.22) as just done previously, we look at the component of E_2 this time, after replacing $E_1(h^3_{13})$ from the above equations, and we get that

$$\sin(3c)\csc\left(\frac{3c}{2} - \Lambda\right)\csc\left(\Lambda + \frac{3c}{2}\right)\left(\cos(4\Lambda) - 2\cos(2\Lambda)\cos(3c) + 12(h^3_{12})^2 + 12(h^3_{13})^2 + 1\right) = 0.$$

As Λ is not constant, this implies that $\cos(4\Lambda) - 2\cos(2\Lambda)\cos(3c) + 12(h^3_{12})^2 + 12(h^3_{13})^2 + 1 = 0$ or $\sin(3c) = 0$.

Case 1. $\sin(3c) = 0$. In this case, considering that $\theta_1 \in [0, \pi]$, it is straightforward to see that $c \in \{0, \frac{\pi}{3}, \frac{2\pi}{3}\}$.

In the following we will show that the other case cannot occur.

Case 2. $\sin(3c) \neq 0$. It follows that

$$\cos(4\Lambda) - 2\cos(2\Lambda)\cos(3c) + 12(h^3_{12})^2 + 12(h^3_{13})^2 + 1 = 0, \tag{2.77}$$

and, therefore, its derivative with respect to E_1 vanishes too:

$$h^3_{13}\left(\sin(4\Lambda) - 2\cos(2\Lambda)\sin(3c) - 3\sin(2\Lambda)\cos(3c) - 12\left((h^3_{12})^2 + (h^3_{13})^2\right)\cot\left(\Lambda + \frac{3c}{2}\right)\right) = 0.$$

We must split again into two cases.

Case 2.1. $h^3_{13} \neq 0$. We have, of course, that

$$\sin(4\Lambda) - 2\cos(2\Lambda)\sin(3c) - 3\sin(2\Lambda)\cos(3c) - 12\left((h^3_{12})^2 + (h^3_{13})^2\right)\cot\left(\Lambda + \frac{3c}{2}\right) = 0$$

and by (2.77), we may write

$$\left(\cos(4\Lambda) - 2\cos(2\Lambda)\cos(3c) + 12(h^3_{12})^2 + 12(h^3_{13})^2 + 1\right)\cot\left(\frac{3c}{2} - \Lambda\right) -$$

$$-\left(\sin(4\Lambda) - 2\cos(2\Lambda)\sin(3c) - 3\sin(2\Lambda)\cos(3c) - 12\left((h_{12}^3)^2 + (h_{13}^3)^2\right)\cot\left(\Lambda + \frac{3c}{2}\right)\right) = 0. \tag{2.78}$$

The latter equation reduces to $-3\cos(3c)\sin(2\Lambda) = 0$, which implies $\cos(3c) = 0$. With this information, we evaluate (2.22) for E_1, E_2, E_1 and, looking at the component of E_2 of the resulting vector gives

$$\sin(3c)\csc\left(\frac{3c}{2} - \Lambda\right)\csc\left(\Lambda + \frac{3c}{2}\right)\left(\cos(4\Lambda) - 2\cos(2\Lambda)\cos(3c) + 12(h_{12}^3)^2 + 12(h_{13}^3)^2 + 1\right) = 0.$$

This yields $\cos(4\Lambda) + 12(h_{12}^3)^2 + 12(h_{13}^3)^2 + 1 = 0$, which is a contradiction, as, given that Λ is not constant, the expression is actually strictly greater than 0.

Case 2.2 $h_{13}^3 = 0$. From (2.22) evaluated for E_1, E_2, E_2; E_1, E_3, E_3; E_2, E_3, E_3; E_3, E_2, E_2, by looking at the components of $E_2, E_3; E_3, E_2; E_3; E_3$, we obtain, respectively:

$$\begin{aligned}
E_1(h_{23}^3) =& -h_{12}^3 h_{22}^3 \cot\left(\Lambda + \frac{3c}{2}\right) + h_{12}^3 h_{22}^3 \cot(2\Lambda) - \frac{h_{22}^3}{\sqrt{3}}, \\
E_2(h_{12}^3) =& -h_{12}^3 h_{23}^3 \left(-2\cot(2\Lambda) + \cot\left(\frac{3c}{2} - \Lambda\right) + \cot\left(\Lambda + \frac{3c}{2}\right)\right), \\
0 =& -\frac{1}{3}\sin(2\Lambda + 3c) - 2(h_{12}^3)^2\left(\cot(2\Lambda) + \cot\left(\frac{3c}{2} - \Lambda\right)\right) + \frac{h_{12}^3}{\sqrt{3}}, \\
E_1(h_{22}^3) =& \frac{1}{3}h_{23}^3\left(\sqrt{3} - 3h_{12}^3\left(\cot(2\Lambda) + \cot\left(\frac{3c}{2} - \Lambda\right)\right)\right), \\
E_3(h_{12}^3) =& -h_{12}^3 h_{22}^3\left(2\cot(2\Lambda) + \cot\left(\frac{3c}{2} - \Lambda\right) + \cot\left(\Lambda + \frac{3c}{2}\right)\right), \\
E_2(h_{22}^3) =& -E_3(h_{23}^3), \\
E_3(h_{22}^3) =& \frac{1}{3}\Big(-\sin(4\Lambda) - 6(h_{12}^3)^2\left(\cot\left(\Lambda + \frac{3c}{2}\right) - \cot\left(\frac{3c}{2} - \Lambda\right)\right) + 3E_2(h_{23}^3) - \sqrt{3}h_{12}^3 - \\
& - 9\cot(2\Lambda)\left((h_{22}^3)^2 + (h_{23}^3)^2\right)\Big).
\end{aligned} \tag{2.79}$$

Next, for the vector fields E_1, E_2, E_1, E_2, we may evaluate the sectional curvature once using the definition for the curvature tensor, once using (2.21), and subtract the results. This gives

$$\begin{aligned}
&-\sin(2c)\sin(c - 2\Lambda) + \cos(2c)\cos(c - 2\Lambda) - \\
&\quad - 6(h_{12}^3)^2\csc(2\Lambda)\cos\left(\frac{3c}{2} - \Lambda\right)\csc\left(\Lambda + \frac{3c}{2}\right) + \sqrt{3}h_{12}^3\cot\left(\frac{3c}{2} - \Lambda\right) + 1 = 0.
\end{aligned} \tag{2.80}$$

From (2.79), we obtain

$$(h_{12}^3)^2 = \frac{\sqrt{3}h_{12}^3 - \sin(2\Lambda + 3c)}{6\left(\cot(2\Lambda) + \cot\left(\frac{3c}{2} - \Lambda\right)\right)}, \tag{2.81}$$

so that we may replace $(h_{12}^3)^2$ in (2.80) and solve for h_{12}^3:

$$h_{12}^3 = \frac{(\cos(3c) - \cos(2\Lambda))\csc(2\Lambda)}{\sqrt{3}}. \tag{2.82}$$

Nevertheless, $(h^3_{12})^2$ from (2.82) does not coincide with (2.81), as it would imply

$$\csc^2(2\Lambda)(\cos(3c)-\cos(2\Lambda))(-9\cos(2\Lambda)+\cos(6\Lambda)+8\cos(3c))=0,$$

i.e. Λ should be constant, which is a contradiction. $\square$

A complete classification of the Lagrangian submanifolds with $\theta_1=\frac{\pi}{3}$ is given in [3] and therefore, presented in the next section. Similarly, for those with angle functions $\theta_1=0$ or $\theta_1=\frac{2\pi}{3}$, we obtain the same result by constructions $\tilde{f}$ and f^*, respectively.

2.3 Lagrangian submanifolds of the nearly Kähler $\mathbb{S}^3\times\mathbb{S}^3$ from minimal surfaces in $\mathbb{S}^3$

In this section we study non-totally geodesic Lagrangian submanifolds of the nearly Kähler $\mathbb{S}^3\times\mathbb{S}^3$ for which the projection on the first component is nowhere of maximal rank. We show that this property can be expressed in terms of the angle functions and that such Lagrangian submanifolds are closely related to minimal surfaces in $\mathbb{S}^3$. Indeed, starting from an arbitrary minimal surface, we can construct locally a large family of such Lagrangian immersions, including one exceptional example. We also show that locally all such Lagrangian submanifolds can be obtained in this way.

2.3.1 Elementary properties of orientable minimal surfaces in $\mathbb{S}^3$.

Let $p:S\to\mathbb{S}^3\subset\mathbb{R}^4$ be an oriented minimal surface. We are going to check that, away from isolated points, the immersion either admits local isothermal coordinates for which the conformal factor satisfies the Sinh-Gordon equation or is totally geodesic. First, we take isothermal coordinates u,v such that $\partial u,\partial v$ is positively oriented, $\langle\partial u,\partial u\rangle=\langle\partial v,\partial v\rangle=2e^{\omega}$ and $\langle\partial u,\partial v\rangle=0$ in a neighborhood of a point of S. As it is often more useful to use complex notation we write $z=u+Iv$ and consider $\partial z=\frac{1}{2}(\partial u-I\partial v)$ and $\partial\bar{z}=\frac{1}{2}(\partial u+I\partial v)$. Note that we use I here in order to distinguish between the i,j,k introduced in the quaternions. We also extend everything in a linear way in I. This means that $\langle\partial z,\partial z\rangle=\langle\partial\bar{z},\partial\bar{z}\rangle=0$ and $\langle\partial z,\partial\bar{z}\rangle=e^{\omega}$. If we write $\partial u=p\alpha$ and $\partial v=p\beta$, the unit normal is given by $N=p\frac{\alpha\times\beta}{2e^{\omega}}$. It is elementary to check that this is independent of the choice of complex coordinate and that the matrix $\left(p\ \frac{\partial u}{|\partial u|}\ \frac{\partial v}{|\partial v|}\ N\right)$ belongs to $SO(4)$. We denote by σ the component of the second fundamental form in the direction of N. Remark that with this choice, the minimality of the surface implies $\sigma(\partial z,\partial\bar{z})=0$ and we may determine the components of the connection ∇ on the surface:

$$\nabla_{\partial z}\partial z=\omega_z\partial z,\quad \nabla_{\partial z}\partial\bar{z}=\nabla_{\partial\bar{z}}\partial z=0\quad\text{and}\quad \nabla_{\partial\bar{z}}=\omega_{\bar{z}}\partial\bar{z}. \tag{2.83}$$

The Codazzi equation of a surface in $\mathbb{S}^3$ states that

$$\nabla\sigma(\partial z,\partial\bar{z},\partial z)=\nabla\sigma(\partial\bar{z},\partial z,\partial z).$$

So it follows that $\partial\bar{z}(\sigma(\partial z,\partial z))=0$. Hence $\sigma(\partial z,\partial z)$ is a holomorphic function. Then we have two cases:

Case 1. If $\sigma(\partial z,\partial z)=0$ on an open set, then by conjugation $\sigma(\partial\bar{z},\partial\bar{z})=0$ and therefore, using the analyticity of a minimal surface, $\sigma=0$ everywhere.

Case 2. If $\sigma(\partial z, \partial z) \neq 0$, then there exists a function $g(z)$ such that $\sigma(\partial z, \partial z) = g(z)$. Away from isolated points we can always make a change of coordinates if necessary such that $\sigma(\partial z, \partial z) = -1$. Notice that by conjugation we get also $\sigma(\partial \bar{z}, \partial \bar{z}) = -1$. Such a change of coordinates is unique up to translations and replacing z by $-z$.

Next, given the immersions $p : S \to \mathbb{S}^3(1) \xhookrightarrow{i} \mathbb{R}^4$, from the Gauss formula we obtain:

$$\begin{aligned}
p_{zz} &= \omega_z p_z - N, \\
p_{z\bar{z}} &= -e^{\omega} p, \\
p_{\bar{z}\bar{z}} &= \omega_{\bar{z}} p_{\bar{z}} - N,
\end{aligned} \tag{2.84}$$

where N is the normal on $\mathbb{S}^3$ and $N_z = e^{-\omega} p_{\bar{z}}$, $N_{\bar{z}} = e^{-\omega} p_z$. Therefore

$$p_{zz\bar{z}} = (\omega_{z\bar{z}} - e^{-\omega}) p_z - \omega_z e^{\omega} p, \quad p_{z\bar{z}z} = -e^{\omega} \omega_z p - e^{\omega} p_z,$$

which shows that ω satisfies

$$\begin{aligned}
\omega_{z\bar{z}} &= -2 \sinh \omega \quad \Leftrightarrow \\
\Delta \omega &= -8 \sinh \omega \quad \text{(Sinh-Gordon equation).}
\end{aligned} \tag{2.85}$$

Notice that by $\Delta\omega$ we denote the Euclidean Laplacian of ω in $\mathbb{R}^2 = \mathbb{C}$.

Let $\mathcal{P}$ be the lift of the minimal immersion to the immersion of the frame bundle in $SO(4)$, i.e.

$$\mathcal{P} : US \to SO(4) : w \mapsto (p\ w\ \tilde{J}w\ N),$$

where US denotes the unit tangent bundle of S and $\tilde{J}$ denotes the natural complex structure on an orientable surface. In terms of our chosen isothermal coordinate this map can be parametrised by

$$\mathcal{P}(u, v, t) = \left(p(u, v), \cos t \frac{p_u}{\mid p_u \mid} + \sin t \frac{p_v}{\mid p_v \mid}, -\sin t \frac{p_u}{\mid p_u \mid} + \cos t \frac{p_v}{\mid p_v \mid}, N(u, v) \right),$$

for some real parameter t. Note that we have the frame equations which state that

$$d\mathcal{P} = \mathcal{P}\Omega^t = -\mathcal{P}\Omega,$$

where in terms of the coordinates u, v and t the matrix Ω is given by

$$\begin{pmatrix}
0 & \sqrt{2}e^{\frac{\omega}{2}}(\cos(t)du + \sin(t)dv) & \sqrt{2}e^{\frac{\omega}{2}}(\cos(t)dv - \sin(t)du) & 0 \\
-\sqrt{2}e^{\frac{\omega}{2}}(\cos(t)du + \sin(t)dv) & 0 & \frac{1}{2}(\omega_u dv - \omega_v du) + dt & -\sqrt{2}e^{-\frac{\omega}{2}}(\cos(t)du - \sin(t)dv) \\
-\sqrt{2}e^{\frac{\omega}{2}}(\cos(t)dv - \sin(t)du) & -\frac{1}{2}(\omega_u dv - \omega_v du) - dt & 0 & \sqrt{2}e^{-\frac{\omega}{2}}(\sin(t)du + \cos(t)dv) \\
0 & \sqrt{2}e^{-\frac{\omega}{2}}(\cos(t)du - \sin(t)dv) & -\sqrt{2}e^{-\frac{\omega}{2}}(\sin(t)du + \cos(t)dv) & 0
\end{pmatrix}.$$

2.3.2 From the Lagrangian immersion to the minimal surface

Now we will consider Lagrangian submanifolds in the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$. We write the Lagrangian submanifold M as

$$\begin{aligned} &f : M \to \mathbb{S}^3 \times \mathbb{S}^3 \\ &x \mapsto f(x) = (p(x), q(x)), \end{aligned}$$

and we assume that the first component has nowhere maximal rank. We have the following:

Theorem 35. *Let*

$$\begin{aligned} &f : M \to \mathbb{S}^3 \times \mathbb{S}^3 \\ &x \mapsto f(x) = (p(x), q(x)), \end{aligned}$$

be a Lagrangian immersion such that $p : M \to \mathbb{S}^3$ has nowhere maximal rank. Then $\frac{\pi}{3}$ is an angle function up to a multiple of π. The converse is also true.

Proof. It is clear that p has nowhere maximal rank if and only if there exists a non zero vector field X such that $dp(X) = 0$. As usual we identify $df(X)$ with X, so we have that $X = df(X) = (dp(X), dq(X))$ and $QX = (-dp(X), dq(X))$. Therefore p has nowhere maximal rank if and only if

$$\begin{aligned} X &= QX \\ &= \tfrac{1}{\sqrt{3}}(2PJX - JX) \\ &= \tfrac{1}{\sqrt{3}}(2BX - 2JAX - JX). \end{aligned}$$

Comparing tangent and normal components we see that this is the case if and only if

$$AX = -\tfrac{1}{2}X \qquad\qquad BX = \frac{\sqrt{3}}{2}X.$$

So we see that X is an eigenvector of both A and B and that the corresponding angle function is $\frac{\pi}{3}$ (up to a multiple of π). $\square$

For the remainder of the paper we will consider Lagrangian immersions for which the map p has nowhere maximal rank. In view of the previous lemma this means that one of the angle functions is constant, namely $\theta_1 = \frac{\pi}{3}$. Then using that the angles are only determined up to a multiple of π and given that $2\theta_1 + 2\theta_2 + 2\theta_3$ is a multiple of 2π, we may write

$$\begin{aligned} 2\theta_1 &= \tfrac{2\pi}{3}, \\ 2\theta_2 &= 2\Lambda + \tfrac{2\pi}{3}, \\ 2\theta_3 &= -2\Lambda + \tfrac{2\pi}{3}, \end{aligned} \tag{2.86}$$

for Λ an arbitrary function which takes values in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. If necessary, by interchanging E_2, E_3 with $-E_3, E_2$, we may assume that $\Lambda \geq 0$ and therefore Λ takes values only in $[0, \frac{\pi}{2}]$. Similarly, if necessary, interchanging E_1, E_3 by $-E_1, -E_3$, we may also assume that $h_{13}^3 \leq 0$ (see equation (2.23)).
Note however that at the points where Λ is 0 or $\frac{\pi}{2}$ modulo π, we have that two of the angle functions coincide. If this is true on an open set, it follows from [59] that the Lagrangian submanifold is totally geodesic and is congruent either with $f\colon \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (1, u)$ or $f\colon \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (uiu^{-1}, u^{-1})$. So by restricting to an open dense subset of M which

we denote by M^*, we may actually assume that $\Lambda \in (0, \frac{\pi}{2})$, in which case the function Λ, as well as the vector fields E_1, E_2, E_3 are differentiable.
Notice that the case when Λ is constant is treated in [4], where such Lagrangian submanifolds are determined to be either totally geodesic or of constant sectional curvature. As we consider here $\Lambda \in (0, \frac{\pi}{2})$, the only possibility is $\Lambda = \frac{\pi}{3}$, in which case the Lagrangian submanifold is not totally geodesic, but of constant sectional curvature.

Theorem 36. *Let M be a Lagrangian submanifold of constant sectional curvature in the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$. If M is not totally geodesic, then up to an isometry of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$, M is locally congruent with one of the following immersions:*

1. $f\colon \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : u \mapsto (uiu^{-1}, uju^{-1})$,
2. $f : \mathbb{R}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 : (u, v, w) \mapsto (p(u, w), q(u, v))$, *where p and q are constant mean curvature tori in $\mathbb{S}^3$ given by*

$$\begin{aligned}
p(u, w) &= (\cos u \cos w, \cos u \sin w, \sin u \cos w, \sin u \sin w)\,,\\
q(u, v) &= \frac{1}{\sqrt{2}} \left(\cos v \left(\sin u + \cos u\right), \sin v \left(\sin u + \cos u\right)\,,\right.\\
&\qquad \left.\cos v \left(\sin u - \cos u\right), \sin v \left(\sin u - \cos u\right)\right).
\end{aligned}$$

Note that these are precisely the two Lagrangian immersions with constant sectional curvature obtained in [20]. These two examples will appear as special solutions in respectively Case 2 and Case 3. However we will mainly focus on the case that Λ is not constant.
In the following, we will identify a tangent vector X in T_xM with its image through df in $T_{(p,q)}\mathbb{S}^3 \times \mathbb{S}^3$, that is $X \equiv df(X) = (dp(X), dq(X))$, and we can write $QX \equiv Q(df(X)) = (-dp(X), dq(X))$. Therefore, if we see $dp(X)$ projected on the first factor of $\mathbb{S}^3 \times \mathbb{S}^3$, that is $dp(X) \equiv (dp(X), 0)$, we can write

$$dp(X) = \frac{1}{2}(X - QX). \tag{2.87}$$

We use relations (2.17) and (2.86) to compute $PE_1 = -\frac{1}{2}E_1 + \frac{\sqrt{3}}{2}JE_1$. As mentioned before this is equivalent with stating that $dp(E_1) = 0$ and that p has nowhere maximal rank. By straightforward computations we obtain

$$\begin{aligned}
(dp(E_2), 0) &= \left(\tfrac{1}{2} - \tfrac{1}{\sqrt{3}} \sin(2\Lambda + \tfrac{2\pi}{3})\right) E_2 + \tfrac{1}{\sqrt{3}} \left(\tfrac{1}{2} + \cos(2\Lambda + \tfrac{2\pi}{3})\right) JE_2,\\
(dp(E_3), 0) &= \left(\tfrac{1}{2} - \tfrac{1}{\sqrt{3}} \sin(-2\Lambda + \tfrac{2\pi}{3})\right) E_3 + \tfrac{1}{\sqrt{3}} \left(\tfrac{1}{2} + \cos(-2\Lambda + \tfrac{2\pi}{3})\right) JE_3
\end{aligned} \tag{2.88}$$

and

$$\begin{aligned}
\langle dp(E_2), dp(E_2)\rangle &= \sin^2 \Lambda,\\
\langle dp(E_3), dp(E_3)\rangle &= \sin^2 \Lambda,\\
\langle dp(E_2), dp(E_3)\rangle &= 0.
\end{aligned} \tag{2.89}$$

We denote

$$v_2 := dp(E_2) \equiv (dp(E_2), 0),$$

$$v_3 := dp(E_3) \equiv (dp(E_3), 0), \tag{2.90}$$
$$\xi = \frac{1}{\sqrt{3}} E_1 - JE_1$$

and we may easily see that $Q\xi = -\xi$, i.e. ξ lies entirely on the first factor of $\mathbb{S}^3 \times \mathbb{S}^3$. Moreover, $\langle v_i, v_j \rangle = \delta_{ij} \sin \Lambda$, $\langle \xi, v_2 \rangle = \langle \xi, v_3 \rangle = 0$ and $\langle \xi, \xi \rangle = 1$. Therefore, $p(M)$ is a surface in $\mathbb{S}^3$ and ξ can be seen as a unit normal to the surface.
As far as the Lagrangian immersion itself is concerned we also have due to the minimality that

$$\begin{aligned} h_{11}^1 + h_{12}^2 + h_{13}^3 &= 0, \\ h_{11}^2 + h_{22}^2 + h_{23}^3 &= 0, \\ h_{11}^3 + h_{22}^3 + h_{33}^3 &= 0. \end{aligned} \tag{2.91}$$

From [20] we know that the covariant derivatives of the endomorphisms A and B are

$$(\nabla_X A)Y = BS_{JX}Y - Jh(X, BY) + \frac{1}{2}(JG(X, AY) - AJG(X, Y)), \tag{2.92}$$
$$(\nabla_X B)Y = -AS_{JX}Y + Jh(X, AY) + \frac{1}{2}(JG(X, AY) - AJG(X, Y)). \tag{2.93}$$

We are going to use the definition of ∇A and ∇B in the previous expressions and then evaluate them for different vectors in the basis in order to get information about the functions ω_{ij}^k and h_{ij}^k. For $X = Y = E_1$ in (2.92) we obtain that

$$\begin{aligned} h_{12}^2 &= -h_{13}^3, \\ \omega_{11}^2 &= h_{11}^2 \cot \Lambda, \\ \omega_{11}^3 &= -h_{11}^3 \cot \Lambda. \end{aligned} \tag{2.94}$$

If we take $X = E_1$ and $Y = E_2$ in (2.92) and (2.93), we see that

$$E_1(\Lambda) = h_{13}^3, \tag{2.95}$$
$$\omega_{12}^3 = \frac{\sqrt{3}}{6} - h_{12}^3 \cot 2\Lambda \tag{2.96}$$

and, for $X = E_2$ and $Y = E_1$ in (2.92), we obtain

$$h_{11}^2 = 0, \tag{2.97}$$
$$\omega_{21}^2 = -\cot \Lambda h_{13}^3, \tag{2.98}$$
$$\omega_{21}^3 = -\frac{\sqrt{3}}{6} - h_{12}^3 \cot \Lambda. \tag{2.99}$$

Then we choose successively $X = E_3, Y = E_1$, $X = E_2, Y = E_3$ and $X = E_3, Y = E_2$ in relations (2.92) and (2.93) and obtain

$$h_{11}^3 = 0, \tag{2.100}$$
$$\omega_{31}^2 = \frac{\sqrt{3}}{6} + \cot \Lambda h_{12}^3, \tag{2.101}$$
$$\omega_{31}^3 = -\cot \Lambda h_{13}^3, \tag{2.102}$$

$$\omega_{22}^3 = -\cot 2\Lambda h_{22}^3, \tag{2.103}$$
$$\omega_{32}^3 = -\cot 2\Lambda h_{23}^3, \tag{2.104}$$
$$E_2(\Lambda) = h_{23}^3, \tag{2.105}$$
$$E_3(\Lambda) = -h_{22}^3. \tag{2.106}$$

We can easily see from (2.94), (2.97) and (2.100) that

$$\omega_{11}^2 = 0 \quad \text{and} \quad \omega_{11}^3 = 0$$

and, if we consider as well the relations in (2.91), we have that

$$h_{33}^3 = -h_{22}^3, \quad h_{11}^1 = 0 \quad \text{and} \quad h_{22}^2 = -h_{23}^3.$$

Later on we will also need to study the Codazzi equations for M. From [20] we know their general form:

$$\begin{aligned}\nabla h(X,Y,Z) - \nabla h(Y,X,Z) =& \frac{1}{3}(g(AY,Z)JBX - g(AX,Z)JBY \\ &- g(BY,Z)JAX + g(BX,Z)JAY).\end{aligned} \tag{2.107}$$

We are going to use the definition for ∇h in the previous relation and take different values for the vectors X, Y and Z. Thus, we evaluate it successively for E_1, E_2, E_1; E_1, E_2, E_2; E_1, E_3, E_3; E_1, E_3, E_2 and E_2, E_3, E_3 and we obtain the following relations, respectively:

$$\begin{aligned}E_1(h_{13}^3) &= \tfrac{1}{3}(-\sqrt{3}h_{12}^3 + 6(h_{13}^3)^2 \cot\Lambda - 6(h_{12}^3)^2 \csc(2\Lambda) + \sin(2\Lambda)),\\ E_1(h_{12}^3) &= \tfrac{1}{3}h_{13}^3(\sqrt{3} + 9h_{12}^3 \cot\Lambda + 3h_{12}^3 \tan\Lambda),\end{aligned} \tag{2.108}$$

$$\begin{aligned}E_2(h_{13}^3) - E_1(h_{23}^3) =& \frac{1}{\sqrt{3}}h_{22}^3 + h_{12}^3h_{22}^3 \cot\Lambda - h_{13}^3h_{23}^3 \cot\Lambda - h_{12}^3h_{22}^3 \cot(2\Lambda),\\ E_1(h_{22}^3) - E_2(h_{12}^3) =& h_{13}^3h_{22}^3(2\cot\Lambda - \tan\Lambda) + \frac{1}{6}h_{23}^3(2\sqrt{3} - 3h_{12}^3 \cot\Lambda + 9h_{12}^3 \tan\Lambda),\\ E_3(h_{12}^3) - E_1(h_{23}^3) =& \frac{1}{\sqrt{3}}h_{22}^3 + (h_{12}^3h_{22}^3 - h_{13}^3h_{23}^3)\cot\Lambda - (3h_{12}^3h_{22}^3 + 2h_{13}^3h_{23}^3)\cot(2\Lambda),\\ E_3(h_{13}^3) + E_1(h_{22}^3) =& \frac{1}{\sqrt{3}}h_{23}^3 + h_{13}^3h_{22}^3 \cot\Lambda + h_{12}^3h_{23}^3 \cot\Lambda - h_{12}^3h_{23}^3 \cot(2\Lambda),\\ E_2(h_{13}^3) - E_3(h_{12}^3) =& 2(h_{12}^3h_{22}^3 + h_{13}^3h_{23}^3)\cot(2\Lambda),\\ E_3(h_{22}^3) - E_2(h_{23}^3) =& -\frac{1}{2}(8(h_{12}^3)^2 + 4(h_{13}^3)^2 + 3((h_{22}^3)^2 + (h_{23}^3)^2))\cot\Lambda -\\ & \frac{1}{3}(\sqrt{3}h_{12}^3 + \sin 4\Lambda) + \frac{3}{2}((h_{22}^3)^2 + (h_{23}^3)^2)\tan\Lambda,\\ E_2(h_{22}^3) + E_3(h_{23}^3) =& -\frac{1}{3}h_{13}^3(\sqrt{3} + 6h_{12}^3 \cot\Lambda).\end{aligned} \tag{2.109}$$

Theorem 37. *Let*

$$\begin{aligned}&f : M \to \mathbb{S}^3 \times \mathbb{S}^3\\ &x \mapsto f(x) = (p(x), q(x)),\end{aligned}$$

be a Lagrangian immersion such that $p : M \to \mathbb{S}^3$ has nowhere maximal rank. Assume that M is not totally geodesic. Then $p(M)$ is a (branched) minimal surface in $\mathbb{S}^3$. Moreover

$$\tilde{P} : M^* \to SO(4) : \ x \mapsto \begin{pmatrix} p(x) & \dfrac{v_2}{\sin\Lambda} & \dfrac{v_3}{\sin\Lambda} & \xi \end{pmatrix},$$

where v_2, v_3 and ξ are defined by (2.90), *is a map which is contained into the frame bundle over the minimal surface p.*

Proof. Recall that $dp(E_1) = 0$, hence $p(M)$ is a surface. Denoting the second fundamental form of the surface in the direction of ξ by σ, a straightforward computation yields that

$$\begin{aligned} &\sigma(E_2, E_2) = h^3_{13}, \\ &\sigma(E_2, E_3) = \sigma(E_3, E_2) = \tfrac{1}{\sqrt{3}} \cos\Lambda \sin\Lambda - h^3_{12}, \\ &\sigma(E_3, E_3) = -h^3_{13}. \end{aligned} \tag{2.110}$$

As $dp(E_2)$ and $dp(E_3)$ are orthogonal and have the same length, the above formulas indeed imply that the surface is minimal.
Moreover we also see that the surface is totally geodesic if and only if $h^3_{13} = 0$ and $h^3_{12} = \frac{1}{\sqrt{3}} \cos\Lambda \sin\Lambda$. Note also that if we write $(dp(E_2), 0) = (p\alpha, 0)$ and $(dp(E_3), 0) = (p\gamma, 0)$, we have that

$$\begin{aligned} G((dp(E_2), 0), (dp(E_3), 0)) &= G((p\alpha, 0), (p\gamma, 0)) \\ &= \tfrac{2}{3\sqrt{3}} (p(\alpha \times \gamma), 2q(\alpha \times \gamma)). \end{aligned}$$

Therefore,

$$(p(\alpha \times \gamma), 0) = \tfrac{3\sqrt{3}}{4} (G((dp(E_2), 0), (dp(E_3), 0)) - Q(G((dp(E_2), 0), (dp(E_3), 0)))).$$

A straightforward computation, using (2.88) and (2.19), shows that this gives

$$(p(\alpha \times \gamma), 0) = (\sin\Lambda)^2 \xi.$$

Therefore ξ corresponds with the normal N on the surface. $\square$

2.3.3 The reverse construction

In the following, we will separate the study of the submanifold into three cases, according to whether the surface is totally geodesic or not and whether the map to the frame bundle is an immersion or not.

Case 1. $p(M)$ is not a totally geodesic surface and the map $\tilde{\mathcal{P}}$ is an immersion into the frame bundle

In that case, in view of the dimension, we can locally identify M with the frame bundle on the minimal surface induced earlier. Recall that

$$\tilde{\mathcal{P}} : x \in M^* \mapsto \begin{pmatrix} p & \dfrac{v_2}{\sin\Lambda} & \dfrac{v_3}{\sin\Lambda} & \xi \end{pmatrix}.$$

Writing again $d\tilde{\mathcal{P}} = -\tilde{\mathcal{P}}\tilde{\Omega}$, we can express the matrix $\tilde{\Omega}$ in terms of $\{E_1, E_2, E_3\}$ by

$$\begin{pmatrix} 0 & \sin(\Lambda)\omega_2 & \sin(\Lambda)\omega_3 & 0 \\ -\sin(\Lambda)\omega_2 & 0 & \begin{matrix}\left(\frac{1}{\sqrt{3}} + h_{12}^3\csc(2\Lambda)\right)\omega_1 + \\ h_{22}^3\csc(2\Lambda)\omega_2 + \\ h_{23}^3\csc(2\Lambda)\omega_3\end{matrix} & \begin{matrix} h_{13}^3\csc(\Lambda)\omega_2 + \\ \left(\frac{\cos\Lambda}{\sqrt{3}} - h_{12}^3\csc\Lambda\right)\omega_3\end{matrix} \\ -\sin(\Lambda)\omega_3 & \begin{matrix} -(\frac{1}{\sqrt{3}} + h_{12}^3\csc(2\Lambda))\omega_1 - \\ h_{22}^3\csc(2\Lambda)\omega_2 - h_{23}^3\csc(2\Lambda)\omega_3\end{matrix} & 0 & \begin{matrix}\left(\frac{\cos\Lambda}{\sqrt{3}} - h_{12}^3\csc(\Lambda)\right)\omega_2 - \\ h_{13}^3\csc(\Lambda)\omega_3\end{matrix} \\ 0 & \begin{matrix}\left(-\frac{\cos\Lambda}{\sqrt{3}} + h_{12}^3\csc\Lambda\right)\omega_3 - \\ h_{13}^3\csc(\Lambda)\omega_2\end{matrix} & \begin{matrix}\left(-\frac{\cos\Lambda}{\sqrt{3}} - h_{12}^3\csc(\Lambda)\right)\omega_2 - \\ h_{13}^3\csc(\Lambda)\omega_3\end{matrix} & 0 \end{pmatrix},$$

where $\omega_i(E_j) = \delta_{ij}$. The above matrix implies that the map $\tilde{\mathcal{P}}$ into $SO(4) \subset \mathbb{R}^{16}$ is an immersion if and only if

$$\tfrac{1}{\sqrt{3}} + h_{12}^3\csc(2\Lambda) \neq 0.$$

As it is an immersion, in view of the dimensions, its image is an open part of the frame bundle and we can identify M with an open part of the frame bundle on the minimal surface. Moreover we can write

$$\frac{v_2}{\sin\Lambda} = \cos(t + \gamma(t,u,v))\frac{p_u}{\mid p_u \mid} + \sin(t + \gamma(t,u,v))\frac{p_v}{\mid p_v \mid},$$

where γ is some function. As $\tilde{\mathcal{P}}$ is an immersion, we have that $t + \gamma(t,u,v)$ depends on t and can be taken as the new variable t on the frame bundle. Doing so, we have that $\tilde{\mathcal{P}} = \mathcal{P}$ and $\tilde{\Omega} = \Omega$ (for $\mathcal{P}, \Omega$ as in subsection 2.3.1). Comparing both expressions for the matrix Ω we deduce

$$\begin{aligned}
\omega_1 =& \frac{1}{\frac{1}{\sqrt{3}} + h_{12}^3\csc 2\Lambda}\left(-\left(\sqrt{2}\frac{\csc 2\Lambda}{\sin\Lambda}e^{\omega/2}(h_{22}^3\cos t - h_{23}^3\sin t) + \frac{1}{2}\omega_v\right)du - \right. \\
&\left.\left(\sqrt{2}\frac{\csc 2\Lambda}{\sin\Lambda}e^{\omega/2}(h_{22}^3\sin t + h_{23}^3\cos t) - \frac{1}{2}\omega_u\right)dv + dt\right), \\
\omega_2 =& \frac{1}{\sin\Lambda}\sqrt{2}e^{\omega/2}(\cos(t)du + \sin(t)dv), \\
\omega_3 =& \frac{1}{\sin\Lambda}\sqrt{2}e^{\omega/2}(\cos(t)dv - \sin(t)du),
\end{aligned}$$

as well as

$$\begin{cases} e^{-\omega}\cos(2t) + h_{13}^3\frac{1}{\sin^2\Lambda} = 0, \\ e^{-\omega}\sin(2t) + \left(h_{12}^3\csc\Lambda - \frac{\cos\Lambda}{\sqrt{3}}\right)\frac{1}{\sin\Lambda} = 0, \end{cases} \tag{2.111}$$

which implies that

$$\begin{cases} h_{13}^3 &= -e^{-\omega}\cos(2t)\sin^2\Lambda, \\ h_{12}^3 &= \left(-e^{-\omega}\sin(2t)\sin\Lambda + \frac{\cos\Lambda}{\sqrt{3}}\right)\sin\Lambda. \end{cases} \tag{2.112}$$

We may express E_1, E_2, E_3 with respect to the basis $\{\partial t, \partial u, \partial v\}$ as follows. For $E_i = a_i \partial t + b_i \partial u + c_i \partial v$, we use the previously obtained expressions of ω_j in $\omega_j(E_i) = \delta_{ij}$ and by straightforward computations we get

$$\begin{aligned}
E_1 =& \left(\frac{1}{\sqrt{3}} + h_{12}^3 \csc(2\Lambda)\right) \partial t, \\
E_2 =& (\csc(2\Lambda) h_{22}^3 + \frac{1}{2\sqrt{2}} \sin \Lambda e^{-\omega/2} (\cos(t)\omega_v - \sin(t)\omega_u))\partial t + \\
& \frac{e^{-\omega/2} \cos t \sin \Lambda}{\sqrt{2}} \partial u + \frac{e^{-\omega/2} \sin t \sin \Lambda}{\sqrt{2}} \partial v, \\
E_3 =& \left(\csc(2\Lambda) h_{23}^3 - \frac{1}{2\sqrt{2}} \sin \Lambda \; e^{-\omega/2} (\cos(t)\omega_u + \sin(t)\omega_v)\right) \partial t - \\
& \frac{e^{-\omega/2} \sin t \sin \Lambda}{\sqrt{2}} \partial u + \frac{e^{-\omega/2} \cos t \sin \Lambda}{\sqrt{2}} \partial v.
\end{aligned} \tag{2.113}$$

In order to be able to proceed with the reverse construction, i.e. in order to be able to construct a Lagrangian immersion starting from the minimal surface we need to express Λ, h_{22}^3 and h_{23}^3 in terms of the variables t, u, v. Remark that, as $E_1(\Lambda) = h_{13}^3$, we may use (2.112) and the expression of E_1 in (2.113) to determine how Λ depends on the variable t. We get

$$\Lambda_t = -\frac{2\cos(2t)\sin^2 \Lambda}{\sqrt{3}e^{\omega} - 2\cos t \sin t \tan \Lambda}. \tag{2.114}$$

In order to solve the above differential equation, we use (2.114) to compute the derivative of the expression $\frac{\sqrt{3}e^{\omega}}{\tan \Lambda} - \sin(2t)$:

$$\partial t \left(\frac{\sqrt{3}e^{\omega}}{\tan \Lambda} - \sin(2t)\right)^2 = 2\sin(4t),$$

which, by integration, gives $\left(\frac{\sqrt{3}e^{\omega}}{\tan \Lambda} - \sin(2t)\right)^2 = -\frac{1}{2}\cos(4t) + \frac{c_1}{4}$, where c_1 does not depend on t. Notice that this implies

$$\tan \Lambda = \frac{2\sqrt{3}e^{\omega}}{\varepsilon_1 \sqrt{c_1 - 2\cos(4t)} + 2\sin(2t)}, \tag{2.115}$$

where $\varepsilon_1 = \pm 1$ and, at the same time, the surface is defined on an open set where $c_1 - 2\cos(4t) \geq 0$. Note that as the above expression contains a square root which would complicate simplifications later on, we will avoid its use as much as possible. For later use, remark that we can write

$$\left(\frac{2\sqrt{3}e^{\omega}}{\tan \Lambda} - 2\sin(2t)\right)^2 = c_1 - 2\cos(4t). \tag{2.116}$$

As we can rewrite the above equation as

$$\left(\frac{2\sqrt{3}e^{\omega}}{\tan \Lambda} - 2\sin(2t)\right)^2 + 2\cos(4t) + 2 = c_1 + 2,$$

we see that $c_1 \geq -2$ and equality can hold if $t \in \{\pm\frac{\pi}{4}, \pm\frac{5\pi}{4}\}$ and $\frac{2\sqrt{3}e^{\omega}}{\tan\Lambda} \pm 2 = 0$. So on an open dense subset we can write

$$c_1 = e^{\omega+\mu} - 2.$$

Combining this with the previous expression of c_1 and taking the derivative with respect to u and v, we can compute

$$\Lambda_u = -\frac{\sin^2\Lambda\left(\mu_u + e^{\omega}\cot\Lambda\left(3e^{\omega}\cot\Lambda(\mu_u-\omega_u) - 2\sqrt{3}\mu_u\sin(2t)\right) + \omega_u\right)}{6e^{2\omega}\cot\Lambda - 2\sqrt{3}e^{\omega}\sin(2t)}$$
$$\Lambda_v = -\frac{\sin^2\Lambda\left(\mu_v + e^{\omega}\cot\Lambda\left(3e^{\omega}\cot\Lambda(\mu_v-\omega_v) - 2\sqrt{3}\mu_v\sin(2t)\right) + \omega_v\right)}{6e^{2\omega}\cot\Lambda - 2\sqrt{3}e^{\omega}\sin(2t)}.$$

Using this, together with (2.113), we can solve in (2.105) and (2.106), for h^3_{22} and h^3_{23}. This gives us

$$\begin{aligned} h^3_{22} = {} & \frac{e^{-3\omega/2}\sin^2\Lambda}{6\sqrt{2}}\big(3e^{\omega}\cos\Lambda((\omega_u-\mu_u)\sin t + (\mu_v-\omega_v)\cos t) - \\ & \sqrt{3}\sin\Lambda((\mu_u+\omega_u)\cos(3t) + (\mu_v+\omega_v)\sin(3t))\big), \\ h^3_{23} = {} & \frac{e^{-3\omega/2}\sin^2\alpha}{6\sqrt{2}}\big(\sqrt{3}\sin\Lambda((\mu_u+\omega_u)\sin(3t) + (-\mu_v-\omega_v)\cos(3t)) - \\ & 3e^{\omega}\cos\Lambda(\mu_u-\omega_u)\cos t - 3e^{\omega}\cos\Lambda(\mu_v-\omega_v)\sin t\big). \end{aligned}$$

In order to determine a differential equation for the function μ we now apply the previously obtained Codazzi equations for M. By (2.113), it turns out that (2.108) and the first 5 equations of (2.109) are trivially satisfied. Recall from (2.85) that $\Delta\omega = -8\sinh\omega$. The seventh equation of (2.109) reduces to

$$\Delta\mu = -4e^{\omega}(\cos(2\Lambda)+2)\csc^2\Lambda + 8\sqrt{3}\cot\Lambda\sin(2t) + 8\sinh\omega. \tag{2.117}$$

A straightforward computation, using the definition of μ and (2.116), shows that this reduces to

$$\Delta\mu = -e^{\mu}. \tag{2.118}$$

Further on, with these new notations, we may see by straightforward computations that the sixth equation of (2.109) is now trivially satisfied.

Reverse construction

We denote by $p : S \to \mathbb{S}^3 \subset \mathbb{R}^4$ a given minimal surface S which is not totally geodesic, on which we take suitable isothermal coordinates as introduced before. Hence we have a solution ω of $\Delta\omega = -8\sinh\omega$. Additionally, we take a solution of

$$\Delta\mu = -e^{\mu} \tag{2.119}$$

and we take the open part of the frame bundle such that

$$\left(\frac{2\sqrt{3}e^{\omega}}{\tan\Lambda} - 2\sin(2t)\right)^2 = e^{\omega+\mu} - 2 - 2\cos(4t) \tag{2.120}$$

has a solution for the function Λ on an open domain. We define

$$
\begin{aligned}
h_{13}^3 &= -e^{\omega}\cos(2t)\sin^2\Lambda,\\
h_{12}^3 &= (-e^{-\omega}\sin(2t)\sin\Lambda + \frac{\cos\Lambda}{\sqrt{3}})\sin\Lambda,\\
h_{22}^3 &= \frac{e^{-3\omega/2}\sin^2\Lambda}{6\sqrt{2}}\big(3e^{\omega}\cos\Lambda((\omega_u-\mu_u)\sin t + (\mu_v-\omega_v)\cos t) - \\
&\qquad \sqrt{3}\sin\Lambda((\mu_u+\omega_u)\cos(3t) + (\mu_v+\omega_v)\sin(3t))\big),\\
h_{23}^3 &= \frac{e^{-3\omega/2}\sin^2\Lambda}{6\sqrt{2}}\big(\sqrt{3}\sin\Lambda((\mu_u+\omega_u)\sin(3t) + (-\mu_v-\omega_v)\cos(3t)) - \\
&\qquad 3e^{\omega}\cos\Lambda(\mu_u-\omega_u)\cos t - 3e^{\omega}\cos\Lambda(\mu_v-\omega_v)\sin t\big)
\end{aligned}
$$

and we define as well a metric on the open part of the frame bundle, by assuming that the vectors

$$
\begin{aligned}
E_1 =& \frac{1}{2}\left(\sqrt{3} - 2e^{-\omega}\tan\Lambda\sin t\cos t\right)\partial t,\\
E_2 =& -\frac{e^{-3\omega/2}\sin\Lambda}{12\sqrt{2}}\Big(\sqrt{3}\tan\Lambda((\mu_u+\omega_u)\cos(3t) + (\mu_v+\omega_v)\sin(3t)) + 3e^{\omega}((\mu_u+\omega_u)\sin t + \\
&(-\mu_v-\omega_v)\cos\ t)\Big)\partial t + \frac{e^{-\frac{\omega}{2}}\cos t\sin\Lambda}{\sqrt{2}}\partial u + \frac{e^{-\frac{\omega}{2}}\sin t\sin\Lambda}{\sqrt{2}}\partial v, \qquad (2.121)\\
E_3 =& \frac{e^{-3\omega/2}\sin\Lambda}{12\sqrt{2}}\Big(\sqrt{3}\tan\Lambda((\mu_u+\omega_u)\sin(3t) + (-\mu_v-\omega_v)\cos(3t)) - 3e^{\omega}((\mu_u+\omega_u)\cos t + \\
&(\mu_v+\omega_v)\sin t)\Big)\partial t - \frac{e^{-\frac{\omega}{2}}\sin t\sin\Lambda}{\sqrt{2}}\partial u + \frac{e^{-\frac{\omega}{2}}\cos t\sin\Lambda}{\sqrt{2}}\partial v
\end{aligned}
$$

form an orthonormal basis.
We now want to determine the Lagrangian immersion

$$
\begin{aligned}
&f : S \times I \to \mathbb{S}^3 \times \mathbb{S}^3\\
&(u, v, t) \mapsto f(u, v, t) = (p(u, v, t), q(u, v, t)).
\end{aligned}
$$

We already know that the first component is the given minimal surface p. We write for both bases

$$
\begin{array}{llcll}
\frac{\partial}{\partial t}(q) = q\beta_1, & \frac{\partial}{\partial t}(p) = p\alpha_1, & & E_1(q) = q\tilde{\beta}_1, & E_1(p) = p\tilde{\alpha}_1,\\
\frac{\partial}{\partial u}(q) = q\beta_2, & \frac{\partial}{\partial u}(p) = p\alpha_2, & \text{and} & E_2(q) = q\tilde{\beta}_2, & E_2(p) = p\tilde{\alpha}_2,\\
\frac{\partial}{\partial v}(q) = q\beta_3, & \frac{\partial}{\partial v}(p) = p\alpha_3, & & E_3(q) = q\tilde{\beta}_3, & E_3(p) = p\tilde{\alpha}_3.
\end{array}
$$

Note that $\alpha_1 = 0$ and α_2 and α_3 are determined by the minimal surface. In particular α_2 and α_3 are mutually orthogonal imaginary quaternions with length squared $2e^{\omega}$. From (2.121) we then get that

$$
\begin{aligned}
\tilde{\alpha}_1 &= 0,\\
\tilde{\alpha}_2 &= \frac{e^{-\frac{\omega}{2}}\cos t\sin\Lambda}{\sqrt{2}}\alpha_2 + \frac{e^{-\frac{\omega}{2}}\sin t\sin\Lambda}{\sqrt{2}}\alpha_3,\\
\tilde{\alpha}_3 &= -\frac{e^{-\frac{\omega}{2}}\sin t\sin\Lambda}{\sqrt{2}}\alpha_2 + \frac{e^{-\frac{\omega}{2}}\cos t\sin\Lambda}{\sqrt{2}}\alpha_3
\end{aligned}
$$

and from the properties of the minimal surface we obtain

$$\begin{aligned}
\tfrac{\partial \alpha_2}{\partial u} &= -\tfrac{\partial \alpha_3}{\partial v} = \tfrac{1}{2}\omega_u \alpha_2 - \tfrac{1}{2}\omega_v \alpha_3 - e^{\omega}\alpha_2 \times \alpha_3,\\
\tfrac{\partial \alpha_2}{\partial v} &= \tfrac{1}{2}\omega_v \alpha_2 + \tfrac{1}{2}\omega_u \alpha_3 + \alpha_2 \times \alpha_3,\\
\tfrac{\partial \alpha_3}{\partial u} &= \tfrac{1}{2}\omega_v \alpha_2 + \tfrac{1}{2}\omega_u \alpha_3 - \alpha_2 \times \alpha_3.
\end{aligned}$$

Using the properties of the vector cross product, this also implies that

$$\begin{aligned}
\tfrac{\partial \alpha_2 \times \alpha_3}{\partial u} &= 2\alpha_2 + 2e^{\omega}\alpha_3 + \omega_u \alpha_2 \times \alpha_3,\\
\tfrac{\partial \alpha_2 \times \alpha_3}{\partial v} &= -2e^{\omega}\alpha_2 - 2\alpha_3 + \omega_v \alpha_2 \times \alpha_3.
\end{aligned}$$

Now, in order to find $\tilde{\beta}_i$, we remark that the vectors E_1, E_2 and E_3 need to correspond with eigenvectors of the operators A and B with suitable eigenfunctions. We have

$$\begin{aligned}
E_1 &= (0, q\tilde{\beta}_1),\\
E_2 &= (p\tilde{\alpha}_2, q\tilde{\beta}_2),\\
E_3 &= (p\tilde{\alpha}_3, q\tilde{\beta}_3).
\end{aligned} \tag{2.122}$$

The angle functions are $\theta_1 = \frac{2\pi}{3}, \theta_2 = 2\Lambda + \frac{2\pi}{3}, \theta_3 = -2\Lambda + \frac{2\pi}{3}$ and

$$PE_i = \cos(2\theta_i)E_i + \sin(2\theta_i)JE_i, \tag{2.123}$$

for $i = 1, 2, 3$. At the same time, by the definition of P in (2.6) and by (2.122) we have

$$PE_1 = (p\tilde{\beta}_1, 0), \quad PE_2 = (p\tilde{\beta}_2, q\tilde{\alpha}_2), \quad PE_3 = (p\tilde{\beta}_3, q\tilde{\alpha}_3). \tag{2.124}$$

Now we use the definition of J to write out JE_i:

$$\begin{aligned}
JE_1 &= \frac{1}{\sqrt{3}}(2p\tilde{\beta}_1, q\tilde{\beta}_1),\\
JE_2 &= \frac{1}{\sqrt{3}}(p(2\tilde{\beta}_2 - \tilde{\alpha}_2), q(-2\tilde{\alpha}_2 + \tilde{\beta}_2)),\\
JE_3 &= \frac{1}{\sqrt{3}}(p(2\tilde{\beta}_3 - \tilde{\alpha}_3), q(-2\tilde{\alpha}_3 + \tilde{\beta}_3)).
\end{aligned} \tag{2.125}$$

Then, by using (2.125), (2.122) and the values of θ_i in (2.86), we rewrite equation (2.123) and, by comparing it to (2.124), we obtain

$$\begin{aligned}
\tilde{\beta}_2 &= \frac{\cos(2\Lambda + \frac{2\pi}{3}) - \frac{1}{\sqrt{3}}\sin(2\Lambda + \frac{2\pi}{3})}{1 - \frac{2}{\sqrt{3}}\sin(2\Lambda + \frac{2\pi}{3})}\tilde{\alpha}_2 = \tfrac{1}{2}(1 - \sqrt{3}\cot\Lambda)\tilde{\alpha}_2,\\
\tilde{\beta}_3 &= \frac{\cos(-2\Lambda + \frac{2\pi}{3}) - \frac{1}{\sqrt{3}}\sin(-2\Lambda + \frac{2\pi}{3})}{1 - \frac{2}{\sqrt{3}}\sin(-2\Lambda + \frac{2\pi}{3})}\tilde{\alpha}_3 = \tfrac{1}{2}(1 + \sqrt{3}\cot\Lambda)\tilde{\alpha}_3.
\end{aligned}$$

Next we continue the computations in order to determine $\tilde{\beta}_1$. For this, we compute $G(E_2, E_3)$ in two different ways, once using (2.19) and once using (2.16). We obtain, respectively

$$G(E_2, E_3) = -\frac{1}{\sqrt{3}}JE_1 = -\frac{1}{3}(p2\tilde{\beta}_1, q\tilde{\beta}_1),$$

and

$$
\begin{aligned}
G(E_2,E_3) &= G((p\tilde{\alpha}_2, q\tilde{\beta}_2),(p\tilde{\alpha}_3, q\tilde{\beta}_3))\\
&= \tfrac{2}{3\sqrt{3}}(p(\tilde{\beta}_2\times\alpha_3+\tilde{\alpha}_2\times\tilde{\beta}_3+\tilde{\alpha}_2\times\tilde{\alpha}_3-2\tilde{\beta}_2\times\tilde{\beta}_3,\\
&\qquad\qquad q(-\tilde{\beta}_2\times\alpha_3-\tilde{\alpha}_2\times\tilde{\beta}_3+2\tilde{\alpha}_2\times\tilde{\alpha}_3-\tilde{\beta}_2\times\tilde{\beta}_3)\\
&= \tfrac{2}{3\sqrt{3}}(p(2-\tfrac{1}{2}(1-3\cot^2\Lambda))\tilde{\alpha}_2\times\tilde{\alpha}_3, q(1-\tfrac{1}{3}(1-3\cot^2\Lambda))\tilde{\alpha}_2\times\tilde{\alpha}_3)\\
&= \tfrac{1}{2\sqrt{3}}(1+\cot^2\Lambda)(2p\tilde{\alpha}_2\times\tilde{\alpha}_3, q\tilde{\alpha}_2\times\tilde{\alpha}_3).
\end{aligned}
$$

Hence, comparing both expressions we get that

$$\tilde{\beta}_1=-\tfrac{\sqrt{3}}{2}\csc^2\Lambda\ \tilde{\alpha}_2\times\tilde{\alpha}_3=-\tfrac{\sqrt{3}}{4}e^{-\omega}\alpha_2\times\alpha_3.$$

Moreover, we also obtain that

$$
\begin{aligned}
\tilde{\beta}_2 =& \tfrac{1}{2\sqrt{2}}(1-\sqrt{3}\cot\Lambda)e^{-\frac{\omega}{2}}\sin\Lambda(\cos t\alpha_2+\sin t\alpha_3),\\
\tilde{\beta}_3 =& \tfrac{1}{2\sqrt{2}}(1+\sqrt{3}\cot\Lambda)e^{-\frac{\omega}{2}}\sin\Lambda(-\sin t\alpha_2+\cos t\alpha_3).
\end{aligned}
$$

We then take the inverse of (2.121) and deduce that

$$
\begin{aligned}
\beta_1 =& -\frac{\sqrt{3}\alpha_2\times\alpha_3}{2\sqrt{3}e^{\omega}-2\sin(2t)\tan(\Lambda)},\\
\beta_2 =& \frac{1}{8}\left(e^{-\omega}\left(\mu_v+\omega_v-\frac{(\mu_u+\omega_u)\cos(2t)\tan(\Lambda)}{\sqrt{3}e^{\omega}-\sin(2t)\tan(\Lambda)}\right)\alpha_2\times\alpha_3-4(\sqrt{3}\cot(\Lambda)\cos(2t)+1)\alpha_2\right.\\
&\left.-4\sqrt{3}\sin(2t)\cot\Lambda\alpha_3\right),\\
\beta_3 =& \frac{1}{8}\left(-e^{-\omega}\left(\mu_u+\omega_u+\frac{(\mu_v+\omega_v)\cos(2t)\tan(\Lambda)}{\sqrt{3}e^{\omega}-\sin(2t)\tan(\Lambda)}\right)\alpha_2\times\alpha_3-4\sqrt{3}\cot(\Lambda)\sin(2t)\alpha_2\right.\\
&\left.+4(1+\sqrt{3}\cos(2t)\cot\Lambda)\alpha_3\right).
\end{aligned}
$$

By straightforward computations, it now follows that

$$
\begin{aligned}
\frac{\partial\beta_1}{\partial u}-\frac{\partial\beta_2}{\partial t}-2\beta_1\times\beta_2=0,\\
\frac{\partial\beta_1}{\partial v}-\frac{\partial\beta_3}{\partial t}-2\beta_1\times\beta_3=0,\\
\frac{\partial\beta_3}{\partial u}-\frac{\partial\beta_2}{\partial v}-2\beta_3\times\beta_2=0,
\end{aligned}
$$

from which we deduce that the integrability conditions for the immersion q are satisfied.

Case 2. The minimal surface $p(M)$ is totally geodesic, i.e. $\sigma = 0$

As mentioned before this means that $h^3_{13}=0$, $h^3_{12}=\frac{\cos\Lambda\sin\Lambda}{\sqrt{3}}$. The equations following from (2.92) and (2.93), just like in the first case, give

$$
\begin{aligned}
&h^2_{12}=0, && \omega^2_{11}=0, && \omega^3_{21}=-\tfrac{2+\cos(2\Lambda)}{2\sqrt{3}},\\
&h^2_{11}=0, && \omega^3_{11}=0, && \omega^3_{22}=-h^3_{22}\cot(2\Lambda),\\
&h^3_{11}=0, && \omega^3_{12}=\tfrac{\sin^2\Lambda}{\sqrt{3}}, && \omega^2_{31}=\tfrac{2+\cos(2\Lambda)}{2\sqrt{3}},\\
&\omega^2_{21}=0, && \omega^3_{31}=0, && \omega^3_{32}=-h^3_{23}\cot(2\Lambda)
\end{aligned}
\tag{2.126}
$$

and

$$\begin{aligned} E_1(\Lambda) &= 0, \\ E_2(\Lambda) &= h^3_{23}, \\ E_3(\Lambda) &= -h^3_{22}. \end{aligned} \tag{2.127}$$

In this case, the equations of Codazzi become

$$E_1(h^3_{23}) = -\frac{\sqrt{3}}{2}h^3_{22}, \quad E_1(h^3_{22}) = \frac{\sqrt{3}}{2}h^3_{23}, \quad E_2(h^3_{22}) = -E_3(h^3_{23}) \tag{2.128}$$

and

$$-1-(1+12(h^3_{22})^2+12(h^3_{23})^2)\cos(2\Lambda)+\cos(4\Lambda)+\cos(6\Lambda)+4(E_2(h^3_{23})-E_3(h^3_{22}))\sin(2\Lambda) = 0. \tag{2.129}$$

In what follows we are going to introduce new vector fields on M by:

$$\begin{aligned} X_1 &= \frac{4}{\sqrt{3}}E_1, \\ X_2 &= -\frac{2h^3_{22}\csc^2\Lambda\sec\Lambda}{\sqrt{3}}E_1 + 2\csc\Lambda\ E_2, \\ X_3 &= -\frac{2h^3_{23}\csc^2\Lambda\sec\Lambda}{\sqrt{3}}E_1 + 2\csc\Lambda\ E_3. \end{aligned} \tag{2.130}$$

We can easily check that

$$\begin{aligned} [X_1, X_2] &= 2X_3, \\ [X_2, X_3] &= 2X_1, \\ [X_3, X_1] &= 2X_2. \end{aligned} \tag{2.131}$$

Taking a canonical metric on M such that X_1, X_2 and X_3 have unit length and are mutually orthogonal, it follows from the Koszul formula that all connection components are determined. From (4.1), Proposition 5.2 and its preceeding paragraph in [20] it follows that we can locally identify M with $\mathbb{S}^3$ and we can consider X_1, X_2 and X_3 as the standard vector fields on $\mathbb{S}^3$ with

$$\begin{aligned} X_1(x) &= xi, \\ X_2(x) &= xj, \\ X_3(x) &= xk. \end{aligned} \tag{2.132}$$

Using the above formulas, the component p of the map can now be determined explicitly. First, we write

$$D_{X_i}p = p\alpha_i, \tag{2.133}$$

for $i = 1, 2, 3$, where D denotes the Euclidean covariant derivative. Of course, by Theorem 35, $D_{X_1}p = 0$. Then, we may compute by (2.87)

$$\begin{aligned} (dp(X_2), 0) &= (\frac{2\cos\Lambda}{\sqrt{3}} + 2\sin\Lambda)E_2 + (-2\cos\Lambda + \frac{\sin\Lambda}{\sqrt{3}})JE_2, \\ (dp(X_3), 0) &= (-\frac{2\cos\Lambda}{\sqrt{3}} + 2\sin\Lambda)E_3 + (2\cos\Lambda + \frac{\sin\Lambda}{\sqrt{3}})JE_3 \end{aligned}$$

and we see that

$$\begin{aligned} &\nabla^E_{X_1}(dp(X_2),0) = (2dp(X_3),0), && \nabla^E_{X_2}(dp(X_3),0) = (0,0),\\ &\nabla^E_{X_1}(dp(X_3),0) = (-2dp(X_2),0), && \nabla^E_{X_3}(dp(X_2),0) = (0,0),\\ &\nabla^E_{X_2}(dp(X_2),0) = (0,0), && \nabla^E_{X_3}(dp(X_3),0) = (0,0). \end{aligned} \tag{2.134}$$

Moreover, it is straightforward to get

$$\langle dp(X_2), dp(X_2)\rangle = \langle dp(X_3), dp(X_3)\rangle = 4, \quad \langle dp(X_2), dp(X_3)\rangle = 0. \tag{2.135}$$

Next, we want to determine a system of differential equations satisfied by α_2 and α_3. For this, we consider $\mathbb{S}^3\times\mathbb{S}^3 \in \mathbb{R}^4\times\mathbb{R}^4$. On the one hand, we use (2.133) together with $D_X(dp(Y),0) = (D_X dp(Y),0)$. On the other hand, we use (2.15) and, therefore, we obtain

$$\begin{aligned} &X_1(\alpha_2) = 2\alpha_3, && X_1(\alpha_3) = -2\alpha_2,\\ &X_2(\alpha_2) = 0, && X_2(\alpha_3) = -\alpha_2\times\alpha_3,\\ &X_3(\alpha_2) = -\alpha_3\times\alpha_2, && X_3(\alpha_3) = 0. \end{aligned} \tag{2.136}$$

We choose a unit quaternion h such that at the point $p(x) = 1$ we have

$$\begin{aligned} &\alpha_2(1) = -2hjh^{-1},\\ &\alpha_3(1) = -2hkh^{-1},\\ &\alpha_2\times\alpha_3(1) = 4hih^{-1}. \end{aligned}$$

Using (2.132), we can check that $\alpha_2 = -2hxjx^{-1}h^{-1}$, $\alpha_3 = -2hxkx^{-1}h^{-1}$ and $\alpha_2\times\alpha_3 = 4hxix^{-1}h^{-1}$ are the unique solutions for the system of differential equations in (2.136):

$$\begin{aligned} X_1(\alpha_2) = X_1(-2hxjx^{-1}h^{-1}) &= -2(hX_1(x)jx^{-1}h^{-1} + hxjX_1(x^{-1})h^{-1})\\ &= -4hxkx^{-1}h^{-1}\\ &= 2\alpha_3, \end{aligned}$$

$$\begin{aligned} X_1(\alpha_3) = X_1(-2hxkx^{-1}h^{-1}) &= -2(hX_1(x)kx^{-1}h^{-1} + hxkX_1(g^{-1})h^{-1})\\ &= 4hxjx^{-1}h^{-1}\\ &= -2\alpha_2, \end{aligned}$$

$$\begin{aligned} X_2(\alpha_3) = X_2(-2hxkx^{-1}h^{-1}) &= -2(hxjkx^{-1}h^{-1} + hxk(-j)x^{-1}h^{-1})\\ &= -4hxix^{-1}h^{-1}\\ &= -\alpha_2\times\alpha_3, \end{aligned}$$

$$\begin{aligned} X_2(\alpha_2) = X_2(-2hxjx^{-1}h^{-1}) &= -2(hxjjx^{-1}h^{-1} + hxj(-j)x^{-1}h^{-1})\\ &= 0, \end{aligned}$$

$$\begin{aligned} X_3(\alpha_3) = X_3(-2hxkx^{-1}h^{-1}) &= -2(hxkkx^{-1}h^{-1} + hxk(-k)x^{-1}h^{-1})\\ &= 0, \end{aligned}$$

$$
\begin{aligned}
X_3(\alpha_2) = X_3(-2hxjx^{-1}h^{-1}) &= -2(hxkjx^{-1}h^{-1} + hxj(-k)x^{-1}h^{-1}) \\
&= 4hxix^{-1}h^{-1} \\
&= \alpha_2 \times \alpha_3.
\end{aligned}
$$

This in its turn implies that

$$
p(x) = -hixix^{-1}h^{-1} \tag{2.137}
$$

is the unique solution of $X_i(p) = p\alpha_i$ with initial conditions $p(1) = 1$. Indeed we have

$$
\begin{aligned}
&X_1(p) = X_1(-hixix^{-1}h^{-1}) = 0 = p\alpha_1, \\
&X_2(p) = X_2(-hixix^{-1}h^{-1}) = 2hixkx^{-1}h^{-1} = (-hixix^{-1}h^{-1})(-2hxjx^{-1}h^{-1}) = p\alpha_2, \\
&X_3(p) = X_3(-hixix^{-1}h^{-1}) = -2hixjx^{-1}h^{-1} = (-hixix^{-1}h^{-1})(-2hxkx^{-1}h^{-1}) = p\alpha_3.
\end{aligned}
$$

Before we can determine the second component q of the Lagrangian immersion, we need to explore the Codazzi equations further. First we look at the system of differential equations for the function Λ in (2.128) and (2.129). Notice that by using the relations in (2.130) we have that

$$
\begin{aligned}
&X_1(\Lambda) = 0, \\
&X_2(\Lambda) = 2h_{23}^3 \csc\Lambda, \\
&X_3(\Lambda) = -2h_{22}^3 \csc\Lambda,
\end{aligned} \tag{2.138}
$$

where the last two equations can be seen as the definition for the functions h_{23}^3 and h_{22}^3. The first one is, of course, a condition for the unknown function of Λ. Three out of the four Codazzi equations then can be seen as integrability conditions for the existence of a solution of this system, whereas the last one reduces to

$$
X_2(X_2(\Lambda)) + X_3(X_3(\Lambda)) = (\cot(\Lambda) - \tan(\Lambda))((X_2(\Lambda))^2 + (X_3(\Lambda))^2) + 4(1 + 2\cos(2\Lambda))\cot(\Lambda).
$$

Under the change of variable $\Lambda = \arctan(e^{2\beta})$, this equation simplifies to

$$
X_2(X_2(\beta)) + X_3(X_3(\beta)) = \frac{2(3 - e^{4\beta})}{e^{4\beta}}. \tag{2.139}
$$

Note also that for $\Lambda = \frac{\pi}{3}$, we get the solution corresponding to example (1) in Theorem 36, as it follows. From (2.130) and (2.138) we see that

$$
\begin{aligned}
X_1 &= \frac{4}{\sqrt{3}} E_1, \\
X_2 &= \frac{4}{\sqrt{3}} E_2, \\
X_3 &= \frac{4}{\sqrt{3}} E_3.
\end{aligned}
$$

This implies that M has constant sectional curvature $\frac{\sqrt{3}}{4}$. Hence this corresponds to example (1) in Theorem 36.

Remark 4. Note that there exist at least locally many solutions of the system

$$
X_1(\beta) = 0,
$$

$$X_2(X_2(\beta)) + X_3(X_3(\beta)) = \frac{2(3 - e^{4\beta})}{e^{4\beta}}.$$

This can be seen by choosing special coordinates on the usual $\mathbb{S}^3$. We take

$$\begin{aligned}
x_1 &= \cos v \cos(t + u), \\
x_2 &= \cos v \sin(t + u), \\
x_3 &= \sin v \cos(u - t), \\
x_4 &= \sin v \sin(u - t).
\end{aligned}$$

As, given (2.132), at the point $x = (x_1, x_2, x_3, x_4)$ the vectors in the basis are

$$\begin{aligned}
X_1(x) &= (-x_2, x_1, x_4, -x_3), \\
X_2(x) &= (-x_3, -x_4, x_1, x_2), \\
X_3(x) &= (-x_4, x_3, -x_2, x_1),
\end{aligned}$$

it is straightforward to see that

$$\begin{aligned}
&\partial t = X_1, \\
&\partial u = \cos(2v)X_1 + \sin(2t)\sin(2v)X_2 + \cos(2t)\sin(2v)X_3, \\
&\partial v = \cos(2t)X_2 - \sin(2t)X_3,
\end{aligned}$$

and conversely,

$$\begin{aligned}
X_1 &= \partial t, \\
X_2 &= \frac{\sin(2t)}{\sin(2v)}\partial u - \sin(2t)\frac{\cos(2v)}{\sin(2v)}\partial t + \cos(2t)\partial v, \\
X_3 &= \frac{\cos(2t)}{\sin(2v)}\partial u - \cos(2t)\frac{\cos(2v)}{\sin(2v)}\partial t - \sin(2t)\partial v.
\end{aligned}$$

At last, the equations in (2.139) become $\frac{\partial}{\partial t}\beta = 0$ and

$$\csc^2(2v)\frac{\partial^2\beta}{\partial u^2} + \frac{\partial^2\beta}{\partial v^2} + 2\cot(2v)\frac{\partial\beta}{\partial v} = 2(3e^{-4\beta} - 1). \tag{2.140}$$

The above differential equation is an elliptic quasilinear second order PDE. Hence, we can apply the Cauchy-Kowalevskaya theorem (see [30]). Therefore, if we start with an analytic regular curve without self intersections and analytic Cauchy data along the curve, we locally have a unique (analytic) solution. Given that we can choose arbitrarily both the curve and the Cauchy data along the curve, locally there exist many solutions for the system in (2.140).

In the following part we are going to determine the second part of the immersion. We start with an arbitrary solution of

$$\begin{aligned}
&X_1(\beta) = 0, \\
&X_2(X_2(\beta)) + X_3(X_3(\beta)) = \frac{2(3 - e^{4\beta})}{e^{4\beta}}
\end{aligned}$$

and we are going to find a system of differential equations determining the immersion q. We define h_{22}^3 and h_{23}^3 as in (2.138) and such that $\Lambda = \arctan(e^{2\beta})$. First, we can write for each of the bases that we took, $\{E_i\}$ and $\{X_i\}$, the following:

$$\begin{array}{ll} X_1(q) = q\beta_1, \quad X_1(p) = p\alpha_1, & E_1(q) = q\tilde{\beta}_1, \quad E_1(p) = p\tilde{\alpha}_1, \\ X_2(q) = q\beta_2, \quad X_2(p) = p\alpha_2, \text{ and } & E_2(q) = q\tilde{\beta}_2, \quad E_2(p) = p\tilde{\alpha}_2, \\ X_3(q) = q\beta_3, \quad X_3(p) = p\alpha_3, & E_3(q) = q\tilde{\beta}_3, \quad E_3(p) = p\tilde{\alpha}_3, \end{array}$$

where $\alpha_1 = 0$ and α_2 and α_3 are as determined previously. Then, we prove as before that

$$\begin{aligned} \tilde{\beta}_1 &= -\frac{\sqrt{3}}{2\sin^2\Lambda}\tilde{\alpha}_2 \times \tilde{\alpha}_3, \\ \tilde{\beta}_2 &= \frac{\cos(2\Lambda + \frac{2\pi}{3}) - \frac{1}{\sqrt{3}}\sin(2\Lambda + \frac{2\pi}{3})}{1 - \frac{2}{\sqrt{3}}\sin(2\Lambda + \frac{2\pi}{3})}\tilde{\alpha}_2 = \tfrac{1}{2}(1 - \sqrt{3}\cot\Lambda)\tilde{\alpha}_2, \\ \tilde{\beta}_3 &= \frac{\cos(-2\Lambda + \frac{2\pi}{3}) - \frac{1}{\sqrt{3}}\sin(-2\Lambda + \frac{2\pi}{3})}{1 - \frac{2}{\sqrt{3}}\sin(-2\Lambda + \frac{2\pi}{3})}\tilde{\alpha}_3 = \tfrac{1}{2}(1 + \sqrt{3}\cot\Lambda)\tilde{\alpha}_3 \end{aligned} \tag{2.141}$$

and we continue the computations in order to find the system of differential equations for the immersion q in terms of the basis $\{X_i\}$. As we identify $df(X_1) \equiv X_1$, we have

$$D_{X_1}f = (X_1(p), X_1(q)) = (0, q\beta_1) \equiv X_1 \overset{(2.130)}{=} \frac{4}{\sqrt{3}}E_1 = \frac{4}{\sqrt{3}}(p\tilde{\alpha}_1, q\tilde{\beta}_1).$$

Therefore, $\beta_1 = \frac{4}{\sqrt{3}}\tilde{\beta}_1$. We may compute similarly for $D_{X_2}f$ and $D_{X_3}f$ and find

$$\begin{cases} \beta_2 = 2\csc\Lambda\ \tilde{\beta}_2 - \frac{2}{\sqrt{3}}h_{22}^3\csc^2\Lambda\sec\Lambda\tilde{\beta}_1, \\ \beta_3 = 2\csc\Lambda\ \tilde{\beta}_3 - \frac{2}{\sqrt{3}}h_{23}^3\csc^2\Lambda\sec\Lambda\tilde{\beta}_1, \end{cases} \quad \begin{cases} \tilde{\alpha}_2 = -\frac{1}{\csc\Lambda}hxjx^{-1}h^{-1}, \\ \tilde{\alpha}_3 = -\frac{1}{\csc\Lambda}hxkx^{-1}h^{-1} \end{cases}$$

and

$$\tilde{\beta}_1 = -\frac{\sqrt{3}}{2}hxix^{-1}h^{-1}.$$

Using now relations (2.141) we may express

$$\begin{aligned} \beta_2 &= -(1 - \sqrt{3}\cot\Lambda)\ hxjx^{-1}h^{-1} + h_{22}^3\csc^2\Lambda\sec\Lambda\ hxix^{-1}h^{-1}, \\ \beta_3 &= -(1 + \sqrt{3}\cot\Lambda)\ hxkx^{-1}h^{-1} + h_{23}^3\csc^2\Lambda\sec\Lambda\ hxix^{-1}h^{-1}. \end{aligned}$$

Finally, as $X_i(q) = q\beta_i$, we find

$$\begin{cases} X_1(q) = -2qhxix^{-1}h^{-1}, \\ X_2(q) = q(h_{22}^3\csc^2\Lambda\sec\Lambda\ hxix^{-1}h^{-1} - (1 - \sqrt{3}\cot\Lambda)\ hxjx^{-1}h^{-1}), \\ X_3(q) = q(h_{23}^3\csc^2\Lambda\sec\Lambda\ hxix^{-1}h^{-1} - (1 + \sqrt{3}\cot\Lambda)\ hxkx^{-1}h^{-1}), \end{cases}$$

which, given (2.138) and $\Lambda = \arctan(e^{2\beta})$, is equivalent to

$$\begin{cases} X_1(q) = -2qhxix^{-1}h^{-1}, \\ X_2(q) = q\left(-X_3(\beta)hxix^{-1}h^{-1} - (1 - \sqrt{3}e^{-2\beta})\ hxjx^{-1}h^{-1}\right), \\ X_3(q) = q\left(X_2(\beta)\ hxix^{-1}h^{-1} - (1 + \sqrt{3}e^{-2\beta})\ hxkx^{-1}h^{-1}\right). \end{cases} \tag{2.142}$$

By straightforward computations, one may see that $X_i(X_j(q)) - X_j(X_i(q)) = [X_i, X_j](q)$ hold for $i, j = 1, 2, 3$. Therefore, the immersion f is completely determined by (2.137) and (2.142).

Case 3. The minimal surface $p(M)$ is not totally geodesic, but the map $\tilde{\mathcal{P}}$ is not an immersion

As mentioned before this means that

$$h^3_{12} = -\frac{\sin(2\Lambda)}{\sqrt{3}}. \tag{2.143}$$

Therefore, the equations in subsection 2.3.2 which follow from (2.92) and (2.93) become

$$\begin{array}{lll} h^2_{12} = -h^3_{13}, & \omega^3_{12} = \frac{1+2\cos(2\Lambda)}{2\sqrt{3}}, & \omega^2_{21} = \omega^3_{31} = -h^3_{13}\cot\Lambda, \\ h^2_{11} = h^3_{11} = 0, & \omega^3_{21} = \frac{1+2\cos(2\Lambda)}{2\sqrt{3}}, & \omega^2_{31} = -\frac{1+2\cos(2\Lambda)}{2\sqrt{3}}, \\ \omega^2_{11} = \omega^3_{11} = 0, & \omega^3_{22} = -h^3_{22}\cot(2\Lambda), & \omega^3_{32} = -h^3_{23}\cot(2\Lambda) \end{array}$$

and

$$E_1(\Lambda) = h^3_{13}, \quad E_2(\Lambda) = h^3_{23}, \quad E_3(\Lambda) = -h^3_{22}. \tag{2.144}$$

Moreover, the equations of Codazzi in (2.108) yield $h^3_{13} = 0$ and, therefore, $\omega^2_{21} = \omega^3_{31} = 0$. The first two equations in (2.109) imply that

$$E_1(h^3_{23}) = 0 \quad \text{and} \quad E_1(h^3_{22}) = 0,$$

while the next three ones vanish identically. The last two equations in (2.109) become

$$E_2(h^3_{22}) = -E_3(h^3_{23}) \tag{2.145}$$

and

$$-1-[1+6(h^3_{22})^2+6(h^3_{23})^2]\cos 2\Lambda+\cos 4\Lambda+\cos 6\Lambda+2[-E_3(h^3_{22})+E_2(h^3_{23})]\sin 2\Lambda = 0, \tag{2.146}$$

respectively. The Lie brackets of the vector fields E_1, E_2, E_3 give

$$\begin{aligned} &[E_1, E_2] = 0, \\ &[E_1, E_3] = 0, \\ &[E_2, E_3] = -\frac{1+2\cos(2\Lambda)}{\sqrt{3}}E_1 + h^3_{22}\cot(2\Lambda)E_2 + h^3_{23}\cot(2\Lambda)E_3. \end{aligned}$$

Next, we take new vector fields X_1, X_2, X_3 of the form

$$\begin{aligned} X_1 &= E_1, \\ X_2 &= \frac{\sqrt{2}(h^3_{22} - h^3_{23})}{3^{\frac{3}{4}}(\sin(2\Lambda))^{\frac{3}{2}}}E_1 + \frac{\sqrt{2}}{3^{\frac{1}{4}}\sqrt{\sin(2\Lambda)}}E_2 - \frac{\sqrt{2}}{3^{\frac{1}{4}}\sqrt{\sin(2\Lambda)}}E_3, \\ X_3 &= \frac{\sqrt{2}(h^3_{22} + h^3_{23})}{3^{\frac{3}{4}}(\sin(2\Lambda))^{\frac{3}{2}}}E_1 + \frac{\sqrt{2}}{3^{\frac{1}{4}}\sqrt{\sin(2\Lambda)}}E_2 + \frac{\sqrt{2}}{3^{\frac{1}{4}}\sqrt{\sin(2\Lambda)}}E_3. \end{aligned} \tag{2.147}$$

We can easily check that $[X_1, X_2] = 0$, $[X_1, X_3] = 0$ and $[X_2, X_3] = 0$, therefore, by the lemma on page 155 in [8], we know that there exist coordinates $\{t, u, v\}$ on M such that

$$\begin{aligned} X_1 &= \partial t, \\ X_2 &= \partial u, \end{aligned}$$

$$X_3 = \partial v.$$

Using (2.144) we obtain:

$$\begin{aligned}
\Lambda_t &= 0,\\
\Lambda_u &= \frac{h^3_{22} + h^3_{23}}{3^{1/4}\sqrt{\cos\Lambda\sin\Lambda}},\\
\Lambda_v &= \frac{-h^3_{22} + h^3_{23}}{3^{1/4}\sqrt{\cos\Lambda\sin\Lambda}}.
\end{aligned}$$

Furthermore, we express h^3_{22} and h^3_{23} from the previous relations as

$$\begin{aligned}
h^3_{22} &= \frac{1}{2}3^{1/4}(\Lambda_u - \Lambda_v)\sqrt{\cos\Lambda\sin\Lambda},\\
h^3_{23} &= \frac{1}{2}3^{1/4}(\Lambda_u + \Lambda_v)\sqrt{\cos\Lambda\sin\Lambda}
\end{aligned}$$

and therefore, the expression of (2.147) becomes

$$\begin{aligned}
X_1 &= E_1,\\
X_2 &= -\frac{\Lambda_v\csc(2\Lambda)}{\sqrt{3}}E_1 + \frac{1}{3^{\frac{1}{4}}\sqrt{\cos\Lambda\sin\Lambda}}E_2 - \frac{1}{3^{\frac{1}{4}}\sqrt{\cos\Lambda\sin\Lambda}}E_3,\\
X_3 &= \frac{\Lambda_u\csc(2\Lambda)}{\sqrt{3}}E_1 + \frac{1}{3^{\frac{1}{4}}\sqrt{\cos\Lambda\sin\Lambda}}E_2 + \frac{1}{3^{\frac{1}{4}}\sqrt{\cos\Lambda\sin\Lambda}}E_3.
\end{aligned} \tag{2.148}$$

Finally, by straightforward computations, one may see that equation (2.146) becomes

$$-\sqrt{3}(\Lambda_u^2 + \Lambda_v^2)\cos(2\Lambda) + 3^{\frac{1}{4}}(E_2(\Lambda_u) - E_3(\Lambda_u) + E_2(\Lambda_v) + E_3(\Lambda_v))\sqrt{\cos\Lambda\sin\Lambda} - {} - 2(\sin(2\Lambda) + \sin(4\Lambda)) = 0. \tag{2.149}$$

We compute $dp(\partial u)$ and $dp(\partial v)$:

$$dp\left(\partial u\right) = \frac{\sqrt{3} - 2\cos\left(2\Lambda + \frac{\pi}{6}\right)}{3^{3/4}\sqrt{2}\sqrt{\sin(2\Lambda)}}E_2 + \frac{2\sin\left(2\Lambda + \frac{\pi}{3}\right) - \sqrt{3}}{3^{3/4}\sqrt{2}\sqrt{\sin(2\Lambda)}}E_3 + \frac{2\cos\left(2\Lambda + \frac{2\pi}{3}\right) + 1}{3^{3/4}\sqrt{2}\sqrt{\sin(2\Lambda)}}JE_2 + \frac{2\cos\left(2\Lambda + \frac{\pi}{3}\right) - 1}{3^{3/4}\sqrt{2}\sqrt{\sin(2\Lambda)}}JE_3,$$

$$dp(\partial v) = \frac{\sqrt{3} - 2\cos\left(2\Lambda + \frac{\pi}{6}\right)}{3^{3/4}\sqrt{2}\sqrt{\sin(2\Lambda)}}E_2 - \frac{2\sin\left(2\Lambda + \frac{\pi}{3}\right) - \sqrt{3}}{3^{3/4}\sqrt{2}\sqrt{\sin(2\Lambda)}}E_3 + \frac{2\cos\left(2\Lambda + \frac{2\pi}{3}\right) + 1}{3^{3/4}\sqrt{2}\sqrt{\sin(2\Lambda)}}JE_2 - \frac{2\cos\left(2\Lambda + \frac{\pi}{3}\right) - 1}{3^{3/4}\sqrt{2}\sqrt{\sin(2\Lambda)}}JE_3,$$

and we remark that they are mutually orthogonal and that their length is $\frac{2\tan\Lambda}{\sqrt{3}}$. So, as u, v are isothermal coordinates on the surface, for which $\langle\partial u, \partial u\rangle = \langle\partial v, \partial v\rangle = 2e^\omega$, we obtain that

$$e^\omega = \frac{\tan\Lambda}{\sqrt{3}}. \tag{2.150}$$

On the one hand, for $z = x + Iy$ as in subsection 2.3.1, we may compute $dp(\partial z)$:

$$\begin{aligned}
dp(\partial z) &= \frac{1}{2}\left[dp\left(\partial u\right) - I\cdot dp\left(\partial v\right)\right]\\
&= \frac{1}{2\sqrt{2}\ 3^{3/4}\sqrt{\sin(2\Lambda)}}\Big[(1-I)\left(\sqrt{3} - 2\cos\left(2\Lambda + \frac{\pi}{6}\right)\right)E_2 - \\
&(1+I)\left(\sqrt{3} - 2\sin\left(2\Lambda + \frac{\pi}{3}\right)\right)E_3 + (1-I)\left(2\cos\left(2\Lambda + \frac{2\pi}{3}\right) + 1\right)JE_2 + \\
&(1+I)\left(2\cos\left(2\Lambda + \frac{\pi}{3}\right) - 1\right)JE_3\Big].
\end{aligned}$$

As

$$\begin{aligned}
&\sqrt{3} - 2\cos\left(2\Lambda + \frac{\pi}{6}\right) = 2\sin\Lambda(\sqrt{3}\sin\Lambda + \cos\Lambda), \quad 2\sin\left(2\Lambda + \frac{\pi}{3}\right) - \sqrt{3} = 2\sin\Lambda(\cos\Lambda - \sqrt{3}\sin\Lambda),\\
&2\cos\left(2\Lambda + \frac{2\pi}{3}\right) + 1 = 2\sin\Lambda(\sin\Lambda - \sqrt{3}\cos\Lambda), \quad 2\cos\left(2\Lambda + \frac{\pi}{3}\right) - 1 = -2\sin\Lambda(\sin\Lambda + \sqrt{3}\cos\Lambda),
\end{aligned}$$

we finally have

$$\begin{aligned}
dp(\partial z) = &\frac{\sin\Lambda}{\sqrt{2}\ 3^{3/4}\sqrt{\sin(2\Lambda)}}\Big[(1-I)\left(\sqrt{3}\sin\Lambda + \cos\Lambda\right)E_2 + (1+I)\left(\cos\Lambda - \sqrt{3}\sin\Lambda\right)E_3 + \\
&(1-I)\left(\sin\Lambda - \sqrt{3}\cos\Lambda\right)JE_2 - (1+I)\left(\sin\Lambda + \sqrt{3}\cos\Lambda\right)JE_3\Big].
\end{aligned}$$

Moreover, from (2.150), it follows that $\omega_z = \frac{1}{\sin(2\Lambda)}\left(\Lambda_u - i\Lambda_v\right)$.
On the other hand, we may compute $\nabla^E_{\partial z} dp(\partial z)$ using the Euclidean connection ∇^E :

$$\begin{aligned}
\nabla^E_{\partial z} dp(\partial z) = &-\frac{1}{\sqrt{3}}E_1 + \frac{e^{-\frac{i\pi}{4}}\sin^2\Lambda\left(\sqrt{3}\cot\Lambda + 3\right)\left(\Lambda_u - i\Lambda_v\right)}{3\sqrt[4]{3}\sin^{\frac{3}{2}}(2\Lambda)}E_2 + \\
&\frac{e^{-\frac{i\pi}{4}}\sin\Lambda(\Lambda_v + i\Lambda_u)\left(\sqrt{3}\cos\Lambda - 3\sin\Lambda\right)}{3\sqrt[4]{3}\sin^{\frac{3}{2}}(2\Lambda)}E_3 + JE_1 + \\
&\frac{e^{-\frac{i\pi}{4}}\sin\Lambda(\Lambda_u - i\Lambda_v)\left(\sqrt{3}\sin\Lambda - 3\cos\Lambda\right)}{3\sqrt[4]{3}\sin^{\frac{3}{2}}(2\Lambda)}JE_2 - \\
&\frac{\left(\frac{1}{3} + \frac{i}{3}\right)\sin\Lambda(\Lambda_u - i\Lambda_v)\left(\sqrt{3}\sin\Lambda + 3\cos\Lambda\right)}{\sqrt{2}\sqrt[4]{3}\sin^{\frac{3}{2}}(2\Lambda)}JE_3.
\end{aligned}$$

From the previous computations we see, indeed, that

$$\nabla^E_{\partial z} dp(\partial z) = -N + \omega_z dp(\partial z),$$

which corresponds to (2.84). From here, we remark the component in the direction of the normal $N = \xi$ (see subsection 2.3.2) and we see that the choice of coordinates $\{t, u, v\}$ following from (2.147) is the right one, as we have indeed $\sigma(\partial z, \partial z) = -1$, as in subsection 2.3.1. Using (2.150) together with the fact that, by taking the inverse in (2.148), we have

$$E_1 = \partial t,$$

$$
\begin{aligned}
E_2 &= \frac{3^{\frac{1}{4}}\sqrt{\sin(2\Lambda)}}{2\sqrt{2}}\left(\frac{\Lambda_v-\Lambda_u}{\sqrt{3}\sin(2\Lambda)}\partial t+\partial u+\partial v\right),\\
E_3 &= -\frac{3^{\frac{1}{4}}\sqrt{\sin(2\Lambda)}}{2\sqrt{2}}\left(\frac{\Lambda_v+\Lambda_u}{\sqrt{3}\sin(2\Lambda)}\partial t+\partial u-\partial v\right),
\end{aligned}
$$

we may prove that equation (2.149) is equivalent to the Sinh-Gordon equation in (2.85), which characterizes the minimal surface.

Reverse construction

Let S be a minimal surface given by $p : S \to \mathbb{S}^3 \subset \mathbb{R}^4$, on which we take isothermal coordinates u and v as in subsection 2.3.1. Hence, we have a solution ω of the Sinh-Gordon equation $\Delta\omega = -8\sinh\omega$. Next, we define a function $\Lambda \in (0, \frac{\pi}{2})$ such that

$$
e^{\omega} = \frac{\tan\Lambda}{\sqrt{3}}.
$$

Remark 5. If $\omega = 0$, then $\Lambda = \frac{\pi}{3}$, which corresponds to example (2) in Theorem 36.

We then define a metric on an open part of the unit frame bundle of the surface by assuming that the vectors

$$
\begin{aligned}
E_1 &= \partial t,\\
E_2 &= \frac{\sqrt{3}e^{\omega/2}}{2\sqrt{1+3e^{2\omega}}}\left(\frac{\omega_v-\omega_u}{2\sqrt{3}}\partial t+\partial u+\partial v\right),\\
E_3 &= -\frac{\sqrt{3}e^{\omega/2}}{2\sqrt{1+3e^{2\omega}}}\left(\frac{\omega_v+\omega_u}{2\sqrt{3}}\partial t+\partial u-\partial v\right)
\end{aligned}
\tag{2.151}
$$

form an orthonormal basis. Next, we want to determine the Lagrangian immersion

$$
\begin{aligned}
&f : S\times I \to \mathbb{S}^3\times\mathbb{S}^3\\
&(u,v,t)\mapsto f(u,v,t) = (p(u,v,t), q(u,v,t)),
\end{aligned}
$$

for which we already know that the first component is the given minimal surface p. We write for both bases

$$
\begin{array}{lll}
\frac{\partial}{\partial t}(q) = q\beta_1, \quad \frac{\partial}{\partial t}(p) = p\alpha_1, & & E_1(q) = q\tilde{\beta}_1, \quad E_1(p) = p\tilde{\alpha}_1,\\
\frac{\partial}{\partial u}(q) = q\beta_2, \quad \frac{\partial}{\partial u}(p) = p\alpha_2, & \text{and} & E_2(q) = q\tilde{\beta}_2, \quad E_2(p) = p\tilde{\alpha}_2,\\
\frac{\partial}{\partial v}(q) = q\beta_3, \quad \frac{\partial}{\partial v}(p) = p\alpha_3 & & E_3(q) = q\tilde{\beta}_3, \quad E_3(p) = p\tilde{\alpha}_3.
\end{array}
$$

Note that $\alpha_1 = 0$ and α_2 and α_3 are determined by the minimal surface. In particular α_2 and α_3 are mutually orthogonal imaginary quaternions with length squared $2e^{\omega}$. From the derivates of p in the latter relations together with (2.151), we obtain

$$
\begin{aligned}
\tilde{\alpha}_1 &= 0,\\
\tilde{\alpha}_2 &= \frac{\sqrt{3}e^{\omega/2}}{2\sqrt{1+3e^{2\omega}}}(\alpha_2+\alpha_3),
\end{aligned}
\tag{2.152}
$$

$$\tilde{\alpha}_2 = -\frac{\sqrt{3}e^{\omega/2}}{2\sqrt{1+3e^{2\omega}}}(\alpha_2 - \alpha_3).$$

We then follow the same steps as in *Case 1* and obtain

$$\begin{aligned}
\tilde{\beta}_1 &= -\frac{\sqrt{3}e^{-\omega}}{4}\alpha_2 \times \alpha_3, \\
\tilde{\beta}_2 &= \frac{\sqrt{3}(e^{\omega/2} - e^{-\omega/2})}{4\sqrt{1+3e^{2\omega}}}(\alpha_2 + \alpha_3), \\
\tilde{\beta}_3 &= -\frac{\sqrt{3}(e^{\omega/2} - e^{-\omega/2})}{4\sqrt{1+3e^{2\omega}}}(\alpha_2 - \alpha_3).
\end{aligned} \tag{2.153}$$

Finally, we take the inverse of the matrix which give $\{E_i\}$ in the basis $\{\partial t, \partial u, \partial v\}$ in (2.151) and obtain

$$\begin{aligned}
\beta_1 &= -\frac{\sqrt{3}e^{-\omega}}{4}\alpha_2 \times \alpha_3, \\
\beta_2 &= \frac{e^{-\omega}}{8}(4e^{\omega}\alpha_2 - 4\alpha_3 + \omega_v \alpha_2 \times \alpha_3), \\
\beta_3 &= -\frac{e^{-\omega}}{8}(4\alpha_2 - 4e^{\omega}\alpha_3 + \omega_u \alpha_2 \times \alpha_3).
\end{aligned}$$

By straightforward computations, it now follows that

$$\begin{aligned}
&\frac{\partial \beta_1}{\partial u} - \frac{\partial \beta_2}{\partial t} - 2\beta_1 \times \beta_2 = 0, \\
&\frac{\partial \beta_1}{\partial v} - \frac{\partial \beta_3}{\partial t} - 2\beta_1 \times \beta_3 = 0, \\
&\frac{\partial \beta_3}{\partial u} - \frac{\partial \beta_2}{\partial v} - 2\beta_3 \times \beta_2 = 0,
\end{aligned}$$

from which we deduce that the integrability conditions for the immersion q are satisfied.

2.4 Conclusion

The results in Section 2.3.3 can now be summarized in the following theorems.

Theorem 38. *Let ω and μ be solutions of, respectively, the Sinh-Gordon equation $\Delta\omega = -8\sinh\omega$ and the Liouville equation $\Delta\mu = -e^{\mu}$ on an open simply connected domain $U \subseteq \mathbb{C}$ and let $p : U \to \mathbb{S}^3$ be the associated minimal surface with complex coordinate z such that $\sigma(\partial z, \partial z) = -1$.*
Let $V = \{(z,t) \mid z \in U, t \in \mathbb{R}, e^{\omega+\mu} - 2 - 2\cos(4t) > 0\}$ and let Λ be a solution of

$$\left(\frac{2\sqrt{3}e^{\omega}}{\tan\Lambda} - 2\sin(2t)\right)^2 = e^{\omega+\mu} - 2 - 2\cos(4t)$$

on V. Then, there exists a Lagrangian immersion $f : V \to \mathbb{S}^3 \times \mathbb{S}^3 : x \mapsto (p(x), q(x))$, where q is determined by

$$\frac{\partial q}{\partial t} = -\frac{\sqrt{3}}{2\sqrt{3}e^{\omega} - 2\sin(2t)\tan\Lambda}\ q\ \alpha_2 \times \alpha_3,$$

$$\frac{\partial q}{\partial u} =\frac{1}{8}\Big(e^{-\omega}\left(\mu_v+\omega_v-\frac{(\mu_u+\omega_u)\cos(2t)\tan\Lambda}{\sqrt{3}e^{\omega}-\sin(2t)\tan\Lambda}\right) q\ \alpha_2\times\alpha_3-4(\sqrt{3}\cot\Lambda\cos(2t)+1)\ q\ \alpha_2-$$
$$4\sqrt{3}\sin(2t)\cot\Lambda\ q\ \alpha_3\Big),$$
$$\frac{\partial q}{\partial v} =\frac{1}{8}\Big(-e^{-\omega}\left(\mu_u+\omega_u+\frac{(\mu_v+\omega_v)\cos(2t)\tan\Lambda}{\sqrt{3}e^{\omega}-\sin(2t)\tan\Lambda}\right)\ q\ \alpha_2\times\alpha_3-4\sqrt{3}\cot\Lambda\sin(2t)\ q\ \alpha_2+$$
$$4(1+\sqrt{3}\cos(2t)\cot\Lambda)\ q\ \alpha_3\Big),$$

where $\alpha_2=\bar{p}p_u$ *and* $\alpha_3=\bar{p}p_v$.

Theorem 39. *Let* X_1, X_2, X_3 *be the standard vector fields on* $\mathbb{S}^3$. *Let* β *be a solution of the differential equations*

$$\begin{aligned} &X_1(\beta)=0,\\ &X_2(X_2(\beta))+X_3(X_3(\beta))=\frac{2(3-e^{4\beta})}{e^{4\beta}}, \end{aligned}$$

on a connected, simply connected open subset U *of* $\mathbb{S}^3$.
Then there exist a Lagrangian immersion $f: U\to\mathbb{S}^3\times\mathbb{S}^3 : x\mapsto(p(x),q(x))$, *where* $p(x)=xix^{-1}$ *and* q *is determined by*

$$\begin{aligned} X_1(q)&=-2qhxix^{-1}h^{-1},\\ X_2(q)&=q\left(-X_3(\beta)hxix^{-1}h^{-1}-(1-\sqrt{3}e^{-2\beta})\ hxjx^{-1}h^{-1}\right),\\ X_3(q)&=q\left(X_2(\beta)\ hxix^{-1}h^{-1}-(1+\sqrt{3}e^{-2\beta})\ hxkx^{-1}h^{-1}\right). \end{aligned}$$

Note that in the previous theorem the image of p is a totally geodesic surface in $\mathbb{S}^3$.

Theorem 40. *Let* ω *be a solution of the Sinh-Gordon equation* $\Delta\omega=-8\sinh\omega$ *on an open connected domain of* U *in* $\mathbb{C}$ *and let* $p: U\to\mathbb{S}^3$ *be the associated minimal surface with complex coordinate* z *such that* $\sigma(\partial z,\partial z)=-1$. *Then, there exist a Lagrangian immersion* $f: U\times I\to\mathbb{S}^3\times\mathbb{S}^3 : x\mapsto(p(x),q(x))$, *where* q *is determined by*

$$\begin{aligned} \frac{\partial q}{\partial t}&=-\frac{\sqrt{3}e^{-\omega}}{4}q\ \alpha_2\times\alpha_3,\\ \frac{\partial q}{\partial u}&=\frac{e^{-\omega}}{8}(4e^{\omega}q\alpha_2-4q\alpha_3+\omega_v q\ \alpha_2\times\alpha_3),\\ \frac{\partial q}{\partial v}&=-\frac{e^{-\omega}}{8}(4q\alpha_2-4e^{\omega}q\alpha_3+\omega_u q\ \alpha_2\times\alpha_3). \end{aligned}$$

where $\alpha_2=\bar{p}p_u$ *and* $\alpha_3=\bar{p}p_v$.

Theorem 41. *Let* $f: M\to\mathbb{S}^3\times\mathbb{S}^3 : x\mapsto(p(x),q(x))$ *be a Lagrangian immersion such that* p *has nowhere maximal rank. Then every point* x *of an open dense subset of* M *has a neighborhood* U *such that* $f|_U$ *is obtained as described in Theorem 38, 39 or 40.*

Chapter 3

A classification of isotropic affine hyperspheres

The notion of a submanifold with isotropic second fundamental form was first introduced in [47] by O'Neill for immersions in Riemannian manifolds and recently extended by Cabrerizo et al. in [10] for pseudo-Riemannian manifolds. We say that M has isotropic second fundamental form h if and only if for any tangent vector X at a point p we have that

$$< h(X(p), X(p)), h(X(p), X(p)) >= \lambda(p) < X(p), X(p) >^2 .$$

If λ is independent of the point p, the submanifold is called constant isotropic. Given the similarities between the basic equations that charactherise the manifolds and the important role played by the difference tensor it is natural to introduce the equivalent notion of isotropy in affine geometry. That is, a hypersurface M has isotropic difference tensor K if and only if for any tangent vector X at a point p we have that

$$h(K(X(p), X(p)), K(X(p), X(p))) = \lambda(p)h(X(p), X(p))^2,$$

where h is the affine metric on the hypersurface. Note that a 2-dimensional affine surface is always isotropic. In case that the affine metric is positive definite such submanifolds have been previously studied in [5] and [6]. In [5], beside a restriction on the dimension, a complete classification was obtained in case that the affine hypersurface is an affine sphere. In [6] a complete classification was given of 5 dimensional positive definite affine hypersurfaces.
Here we will always deal with the case that $\lambda \neq 0$. Therefore, if necessary, by replacing ξ with $-\xi$, we may assume that λ is positive and therefore there exists a positive function μ such that $\lambda = \mu^2$.

In the present study we deal with the case that the induced affine metric has arbitrary signature. We will first show that the restriction of the dimension remains valid in the indefinite case. Even though the proof remains based on the Hurwitz theorem it is essentially different from the proof in the definite case. This is because unlike in the definite case, the unit tangent bundle at a point p is no longer a compact manifold. Instead of this null vectors will play an important role in the proof of the restriction of the dimension.

In the second part of our study we will then restrict ourselves to the case that M is an affine hypersphere and we will deduce that in that case the immersion also has parallel difference tensor (and is a pseudo-Riemannian symmetric space). We then look at each of the possible dimensions and determine in each case explicitly by elementary means the form of the difference tensor and the possible examples. Note that for this second part also a more involved Lie group approach would be possible. We show the following theorems.

Theorem 42. *Let M^5 be a 5-dimensional affine hypersphere of $\mathbb{R}^6$. Assume that M is λ-isotropic with $\lambda \neq 0$. Then either*

1. *the metric is positive definite, M is isometric with $\frac{SL(3,\mathbb{R})}{SO(3)}$ and is affine congruent to an open part of the hypersurface $\{gg^T|g \in SL(3,\mathbb{R})\}$ of $\mathbb{R}^6 \equiv s(3) \subset \mathbb{R}^{3\times 3}$ (see [5]) , or*
2. *the metric has signature 2, M is isometric with $\frac{SL(3,\mathbb{R})}{SO(2,1)}$ and is affine congruent to an open part of the hypersurface $\{gAg^T|g \in SL(3,\mathbb{R})\}$ of $\mathbb{R}^6 \equiv s(3) \subset \mathbb{R}^{3\times 3}$, where $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.*

Theorem 43. *Let M^8 be an 8-dimensional affine hypersphere of $\mathbb{R}^9$. Assume that M is λ-isotropic with $\lambda \neq 0$. Then, either*

1. *the metric is positive definite, M is isometric with $\frac{SL(3,\mathbb{C})}{SU(3)}$ and M is affine congruent to an open part of the hypersurface $\{g\bar{g}^T|g \in SL(3,\mathbb{C})\}$ of $\mathbb{R}^9$, identified with the space of Hermitian symmetric matrices (see [5]), or*
2. *the metric has signature 4, M is isometric with $\frac{SL(3,\mathbb{C})}{SU(2,1)}$ and M is affine congruent to an open part of the hypersurface $\{gA\bar{g}^T|g \in SL(3,\mathbb{C})\}$ of $\mathbb{R}^9$ identified with the space of Hermitian symmetric matrices, where $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, or*
3. *the metric has signature 3, M is isometric with $SL(3,\mathbb{R})$ and M is affine congruent with $SL(3,\mathbb{R})$ considered as a hypersurface in $\mathbb{R}^9$ identified with $\mathbb{R}^{3\times 3}$.*

Theorem 44. *Let M^{14} be a 14-dimensional affine hypersphere of $\mathbb{R}^{15}$. Assume that M is λ-isotropic with $\lambda \neq 0$. Then, either*

1. *the metric is positive definite, M is locally isometric with $\frac{SU^*(6)}{Sp(3)}$ and is affine congruent with the connected component of the identity of the matrices with determinant 1 in $a = \left\{ \begin{pmatrix} E & F \\ -\bar{F} & \bar{E} \end{pmatrix}, E = \bar{E}^T, F = -F^T \right\} \subset \mathbb{C}^{6\times 6}$ (see [5]), or*
2. *the metric has signature 6 and we may identify $\mathbb{R}^{15}$ with the set of the skew symmetric matrices in $\mathbb{R}^{6\times 6}$ and therefore M is isometric with $\frac{SL(6,\mathbb{R})}{Sp(6)}$ and is affine congruent with the connected component of*
$$I_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
of skew symmetric matrices with determinant 1, or

3. *the metric has signature* 8 *and we may identify* $\mathbb{R}^{15}$ *with the set of matrices* $a = \left\{\begin{pmatrix} E & F \\ -\bar{F} & \bar{E}\end{pmatrix}, E = \bar{E}^T, F = -F^T\right\} \subset \mathbb{C}^{6\times 6}$, *such that* M *is locally isometric with* $\frac{SU^*(6)}{Sp(1,2)}$ *and is affine congruent with the connected component of* $I_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$ *of* a, *consisting of matrices with determinant* 1.

Theorem 45. *Let* M^{26} *be a 26-dimensional affine hypersphere of* $\mathbb{R}^{27}$. *Assume that* M *is* λ*-isotropic with* $\lambda \neq 0$. *Then, either*

1. *the metric is positive definite,* M *is locally isometric with* $E_6(-26)/F_4$ *and is affine congruent to the connected component of the identity of the hypersurface* $\{\bar{N}N^T | N \in \mathfrak{h}_3(\mathbb{O}), det(N) = 1\}$, *where* $\mathfrak{h}_3(\mathbb{O})$ *denotes the set of Hermitian matrices with entries in the space of octonions* $\mathbb{O}$ *(see [5]), or*

2. *the metric has signature* 16 *and we may identify* $\mathbb{R}^{27}$ *with* $\mathfrak{h}_3(\mathbb{O})$, *such that* M *is affine congruent with the connected component of* $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ *of the hypersurface* $\{\bar{N}AN^T | N \in \mathfrak{h}_3(\mathbb{O}), det(N) = 1\}$, *or*

3. *the metric has signature* 12 *and we may identify* $\mathbb{R}^{27}$ *with the set of Hermitian matrices with entries in the space of split-octonions, such that* M *is affine congruent with the connected component of* $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ *of the hypersurface* $\{\bar{N}AN^T | N \in \mathfrak{h}_3(\mathbb{O}), det(N) = 1\}$.

Before proceding with the presentation of the results obtained, we mention the following known lemmas and theorems:

Lemma 11. *([46]) Let* $F : M \to \mathbb{R}^{n+1}$ *be an equiaffine immersion. If the metric on* $\mathbb{R}^{n+1}$ *is indefinite, then the immersion is isotropic if and only if for any tangent vectors* $X_1, X_2, X_3, X_4 \in T_pM$, *we have that*

$$h(K(X_1, X_2), K(X_3, X_4)) + h(K(X_1, X_3), K(X_2, X_4)) + h(K(X_1, X_4), K(X_2, X_3)) = \\ \lambda(p)\{h(X_1, X_2)h(X_3, X_4) + h(X_1, X_3)h(X_2, X_4) + h(X_1, X_4)h(X_2, X_3)\}. \quad (3.1)$$

By using lemma 11 and property (3) in Proposition 2.137 we get that an affine submanifold M^n in $\mathbb{R}^{n+1}$ is isotropic if and only if for any tangent vectors X_1, X_2, $X_3 \in T_pM$ we have that

$$K_{X_1}K_{X_2}X_3 + K_{X_2}K_{X_1}X_3 + K_{X_3}K_{X_1}X_2 = \\ \lambda(p)\left(h(X_2, X_3)X_1 + h(X_1, X_3)X_2 + h(X_1, X_2)X_3\right). \quad (3.2)$$

Theorem 46. *([13]) Let (M^n_k, h) be an $n-$dimensional simply connected pseudo-Riemannian manifold with index k. Let $\hat{\nabla}$ denote the Levi Civita connection, $\hat{R}$ its curvature tensor and let TM denote the tangent bundle of M^n_k. If K is a TM-valued symmetric bilinear form on M^n_k satisfying that*
i) $h(K(X,Y),Z)$ is totally symmetric
ii) $(\hat{\nabla}K)(X,Y,Z) = \hat{\nabla}_X K(Y,Z) - K(\hat{\nabla}_X Y, Z) - K(Y, \hat{\nabla}_X Z)$ is totally symmetric,
iii) $\hat{R}(X,Y)Z = c(h(Y,Z)X - h(X,Z)Y) + K(K(Y,Z),X) - K(K(X,Z),Y)$,
then there exists an affine immersion $\phi : M^n_k \to \mathbb{R}^{n+1}$ as an affine sphere with induced difference tensor K and induced affine metric h.

Theorem 47. *([13]) Let $\phi^1, \phi^2 : M^n_k \longrightarrow \mathbb{R}^{n+1}$ be two affine immersions of an pseudo-Riemannian n-manifold (M^n_k, h) with difference tensors K^1, K^2, respectively. If*

$$h(K^1(X,Y), \phi^1_* Z) = h(K^2(X,Y), \phi^2_* Z)$$

for all tangent vectors fields $X, Y, Z \in T_p M^n_k$, then there exists an isometry ϕ of $\mathbb{R}^{n+1}$ such that $\phi^1 = \phi \circ \phi^2$.

3.1 Possible dimensions and choice of frame

From now on we will always assume that M^n_k is an affine isotropic hypersurface in $\mathbb{R}^{n+1}$. Here n denotes the dimension and k the index of the affine metric. In case that the metric is definite, a classification was obtained already in [5]. In view of this we will also assume thay M is neither positive nor negative definite, i.e. $1 \leq k < n$. Also recall that because of the properties of K any surface is isotropic. Therefore we will also assume that $n > 2$. First, we have the following lemma:

Lemma 12. *Let M^n_k be an n-dimensional isotropic affine hypersurface and let $p \in M^n_k$. If for any null vector $v \in T_pM$ we have that $K(v,v)$ is a null vector such that $h(K(v,v),v) = 0$, then the difference tensor K vanishes.*

As its proof is very similar to the proof of Lemma 3.1 in [32], we omit it here. From now on, we will assume that $\lambda \neq 0$. By Lemma 12, there exists a null vector v_0 such that v_0 and $K(v_0, v_0)$ are linearly independent and $h(v_0, K(v_0, v_0)) \neq 0$. Using Lemma 11, we have that for any null vector u

$$h(K(v_0,v_0), K(v_0,u)) = \lambda h(v_0,v_0) h(v_0,u) = 0. \tag{3.3}$$

As K_{v_0} is a symmetric operator with respect to the metric h, we get that $K_{v_0} K_{v_0} v_0 = 0$. Moreover, taking in particular $u = v_0$ in (3.3), we get that $K(v_0, v_0)$ is a null vector.
We can now take a null frame such that

$$e_1 = v_0, \quad e_2 = K_{v_0} v_0.$$

By rescaling v_0 if necessary, we may assume that $h(K(v_0,v_0), v_0) = -4\lambda^2$. Then we get

$$\begin{aligned} &h(e_1,e_1) = h(e_2,e_2) = 0 \quad h(e_1,e_2) = -4\lambda^2, \\ &K(e_1,e_1) = e_2, \quad K(e_1,e_2) = K_{v_0}K_{v_0}v_0 = 0. \end{aligned} \tag{3.4}$$

Using the isotropy condition in (3.2) for $X_1 = X_2 = e_1, X_3 = e_2$ we get that

$$K_{e_2}e_2 = -8\lambda^3 e_1.$$

From relation (3.4) we can see that the space $span\{e_1, e_2\}$ is invariant under the operator K_{e_1}. As the operator K_{e_1} is symmetric with respect to the metric, it follows that also the space $span\{e_1, e_2\}^\perp$ is invariant under K_{e_1}.

Now we follow precisely the computations of [32]. We get a basis $\{e_1, e_2, u_1, \ldots, u_r, \omega_1^1, \omega_2^1, \ldots, \omega_1^r, \omega_2^r\}$, which satisfies that $\{u_1, \ldots, u_r, \omega_1^1, \omega_2^1, \ldots, \omega_1^r, \omega_2^r\}$ is an orthogonal basis of $\{e_1, e_2\}^\perp$ and

$$\begin{cases} h(e_1, e_1) = h(e_2, e_2) = 0, \quad h(e_1, e_2) = -4\lambda^2, \\ h(u_i, u_j) = \varepsilon_i \delta_{ij}, \quad \varepsilon_i = \pm 1, \quad h(\omega_1^\alpha, \omega_1^\alpha) = 1, \quad h(\omega_2^\alpha, \omega_2^\alpha) = -1, \end{cases} \tag{3.5}$$

$$\begin{cases} K_{e_1}e_1 = e_2, \quad K_{e_1}e_2 = 0, \quad K_{e_1}u_i = \lambda u_i, \\ K_{e_1}\omega_1^\alpha = -\frac{1}{2}\lambda\omega_1^\alpha - \frac{\sqrt{3}}{2}\lambda\omega_2^\alpha, \quad K_{e_1}\omega_2^\alpha = -\frac{1}{2}\lambda\omega_2^\alpha + \frac{\sqrt{3}}{2}\lambda\omega_1^\alpha, \\ K_{e_2}e_2 = -8\lambda^3 e_1, \quad K_{e_2}u_i = -2\lambda^2 u_i, \\ K_{e_2}\omega_1^\alpha = \lambda^2\omega_1^\alpha - \sqrt{3}\lambda^2\omega_2^\alpha, \quad K_{e_2}\omega_2^\alpha = \sqrt{3}\lambda^2\omega_1^\alpha + \lambda^2\omega_2^\alpha, \\ K_{e_1}K_{u_i}u_j = \frac{\delta_{ij}\varepsilon_i}{4\lambda}(2\lambda e_1 - e_2), \quad K_{\omega_1^\alpha}\omega_1^\alpha = L(\omega_1^\alpha, \omega_1^\alpha) - \frac{1}{8\lambda}(2\lambda e_1 - e_2), \\ K_{\omega_2^\alpha}\omega_2^\alpha = L(\omega_1^\alpha, \omega_1^\alpha) + \frac{1}{8\lambda}(2\lambda e_1 - e_2), \quad K_{\omega_1^\alpha}\omega_2^\alpha = \frac{\sqrt{3}}{8\lambda}(2\lambda e_1 + e_2), \\ K_{\omega_k^\alpha}\omega_l^\beta = L(\omega_k^\alpha, \omega_l^\beta), k, l \in 1, 2, 1 \leq \alpha \neq \beta \leq r. \end{cases} \tag{3.6}$$

In the above formulas, U and W correspond to the invariant subspaces of K_{e_1} and the operator L is an operator on $W \times W$, defined by

$$L(\omega, \tilde{\omega}) = K_\omega\tilde{\omega} + \frac{1}{4\lambda^2}h(K_\omega\tilde{\omega}, e_2)e_1 + \frac{1}{4\lambda^2}h(K_\omega\tilde{\omega}, e_1)e_2, \quad \omega, \tilde{\omega} \in W, \tag{3.7}$$

which is a symmetric operator, satisfies $ImL \subset U = span\{u_1, \ldots, u_k\}$ and

$$\begin{matrix} L(\omega_1^\alpha, \omega_1^\alpha) = L(\omega_2^\alpha, \omega_2^\alpha), \quad L(\omega_1^\alpha, \omega_2^\alpha) = 0, \quad K_{\omega_1^\alpha}\omega_2^\alpha = \frac{\sqrt{3}}{4}\left(e_1 + \frac{1}{2\lambda}e_2\right), \\ L(\omega_1^\alpha, \omega_1^\beta) = L(\omega_2^\alpha, \omega_2^\beta), \quad L(\omega_1^\alpha, \omega_2^\beta) = -L(\omega_2^\alpha, \omega_1^\beta). \end{matrix} \tag{3.8}$$

As in [32], changing the frame by taking

$$f_1 = (2\lambda e_1 - e_2)/(4\mu^3), \quad f_2 = (2\lambda e_1 + e_2)/(4\mu^3), \tag{3.9}$$

we get that

$$h(f_1, f_1) = -h(f_2, f_2) = 1, \quad h(f_1, f_2) = 0 \tag{3.10}$$

and

$$\begin{cases} K_{f_1}f_1 = -\mu f_1, \quad K_{f_1}f_2 = \mu f_2, \quad K_{f_2}f_2 = -\mu f_1, \quad K_{f_1}u_i = \mu u_i, \quad K_{f_2}u_i = 0, \\ K_{f_1}\omega_1^\alpha = -\frac{\mu}{2}\omega_1^\alpha, \quad K_{f_1}\omega_2^\alpha = -\frac{\mu}{2}\omega_2^\alpha, \quad K_{f_2}\omega_1^\alpha = \frac{\sqrt{3}\mu}{2}\omega_2^\alpha, \quad K_{f_2}\omega_1^\alpha = -\frac{\sqrt{3}\mu}{2}\omega_1^\alpha, \\ K_{u_i}u_j = \mu\varepsilon_i\delta_{ij}f_1, \quad K_{\omega_1^\alpha}\omega_2^\alpha = \frac{\sqrt{3}}{2}\mu f_2, \quad K_{\omega_1^\alpha}\omega_1^\alpha = L(\omega_1^\alpha, \omega_1^\alpha) - \frac{\mu}{2}f_1, \\ K_{\omega_2^\alpha}\omega_2^\alpha = L(\omega_1^\alpha, \omega_1^\alpha) + \frac{\mu}{2}f_1, \quad K_{\omega_k^\alpha}\omega_l^\beta = L(\omega_k^\alpha, \omega_l^\beta), k, l \in 1, 2, 1 \leq \alpha \neq \beta \leq r. \end{cases} \tag{3.11}$$

Therefore, in order to determine the difference tensor explicitly, we only need to determine all the terms $L(\omega_k^\alpha, \omega_l^\beta), k, l \in \{1,2\}, 1 \leq \alpha, \beta \leq r$. In order to do so we will summarize the above properties in a more invariant way.

Let I be the identity map and define for any $w \in W$

$$Tw = \frac{2}{\sqrt{3}\lambda}\left(K_{e_1} + \frac{1}{2}\lambda I\right) w. \tag{3.12}$$

We can easily check that T satisfies

$$\begin{array}{llll} T\omega_1^\alpha = \omega_2^\alpha, & T\omega_2^\alpha = -\omega_1^\alpha, & T^2 w = -w, & h(Tv, w) = h(v, Tw), \\ T\omega_1^\alpha = \omega_2^\alpha, & T\omega_2^\alpha = -\omega_1^\alpha, & h(Tw, Tv) = -h(w, v), & h(Tv, w) = h(v, Tw), \end{array}$$

for $w, v \in W$. In addition, from (3.8) it follows that $L(w, Tv) = -L(v, Tw)$ and $L(Tw, Tv) = L(v, w)$. We also have that L satisfies an isotropy condition. Indeed, let $w = \sum_{\alpha=1}^{r} a_\alpha \omega_1^\alpha + \sum_{\beta=1}^{r} b_\beta \omega_2^\beta$. By using (6) in lemma 11 we have

$$\begin{aligned} h(K_w w, e_1) &= \sum_{\alpha,\beta=1}^{n} a_\alpha a_\beta h(K_{e_1}\omega_1^\alpha, \omega_1^\beta) + \sum_{\alpha,\beta=1}^{n} b_\alpha b_\beta h(K_{e_1}\omega_2^\alpha, \omega_2^\beta) \\ &\quad + \sum_{\alpha,\beta=1}^{n} a_\alpha b_\beta h(K_{e_1}\omega_1^\alpha, \omega_2^\beta) + \sum_{\alpha,\beta=1}^{n} b_\alpha a_\beta h(K_{e_1}\omega_2^\alpha, \omega_1^\beta) \\ &= -\frac{\lambda}{2}\sum_{\alpha,\beta=1}^{n}(a_\alpha a_\beta - b_\alpha b_\beta)\delta_{\alpha\beta} - \frac{\sqrt{3}\lambda}{2}\sum_{\alpha,\beta=1}^{n}(a_\alpha b_\beta + b_\alpha a_\beta)\delta_{\alpha\beta} \\ &= -\frac{\lambda}{2}h(w,w) + \frac{\sqrt{3}\lambda}{2}h(w, Tw). \end{aligned} \tag{3.13}$$

Similarly, we obtain

$$h(K_w w, e_2) = \lambda^2 h(w,w) + \sqrt{3}\lambda^2 h(w, Tw). \tag{3.14}$$

By combining (3.7), (3.13) and (3.14) we get

$$\begin{aligned} h(L(w,w), L(w,w)) &= h(K_w w, K_w w) + \frac{1}{2\lambda^2} h(K_w w, e_1) h(K_w w, e_2) \\ &= \lambda h(w,w)^2 + \frac{1}{2\lambda^2}\left(-\frac{1}{2}\lambda^3 h(w,w)^2 + \frac{3}{2}\lambda^3 h(w,Tw)^2\right) \\ &= \frac{3}{4}\lambda(h(w,w)^2 + h(w,Tw)^2). \end{aligned} \tag{3.15}$$

Linearizing the previous expression for arbitrary vectors $W_1, W_2, W_3, W_4 \in W$, we obtain:

$$\begin{aligned} &h(L(W_1,W_2), L(W_3,W_4)) + h(L(W_1,W_3), L(W_2,W_4)) + h(L(W_1,W_4), L(W_2,W_3)) \\ &\quad = \frac{3\lambda}{4}(h(W_1,W_2)h(W_3,W_4) + h(W_1,W_3)h(W_2,W_4) + h(W_1,W_4)h(W_2,W_3) \\ &\quad + h(W_1,TW_2)h(W_3,TW_4) + h(W_1,TW_3)h(W_2,TW_4) + h(W_1,TW_4)h(W_2,TW_3)). \end{aligned} \tag{3.16}$$

Note that given a metric of neutral signature on $\{f_1, f_2\}^\perp$ and operators T and L satisfying the previous conditions, we can define a frame such that (3.11) holds. We start with a vector

$u \in \{f_1, f_2\}^\perp$ with length 1. Then Tu has length -1. We now write $w = au + bTu$. The fact that w has length 1 and is orthogonal to Tw implies that

$$(a^2 - b^2) + 2ab < u, Tu >= 1,$$
$$(a^2 - b^2) < u, Tu > -2ab = 0,$$

which determines a and b uniquely up to sign. It is then sufficient to take $w_1^1 = w$ and $w_2^1 = Tw$ and to complete the construction is an inductive way.

In what follows we are going to determine the possible dimensions of the studied submanifold M^n. In order to do this, we will use a well known result from the theory of composition of quadratic forms, namely the '1,2,4,8 Theorem' proved by Hurwitz in 1898. One can find it for example in [51]. It states that there exists an n-square identity over the complex numbers of the form

$$(x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2) = z_1^2 + \ldots + z_n^2, \tag{3.17}$$

where $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ are systems of indeterminates and each $z_k = z_k(X, Y)$ is a bilinear form in X and Y, if and only if $n = 1, 2, 4$ or 8.
We are going to see how this result applies in our case and then determine the values of L on the components of the basis in order to determine the difference tensor of our immersion.

In order to apply the *1,2,4,8 Theorem*, we are going to find conveniently defined complex vector spaces and an operator which preserves lengths.
First, we denote by $U^{\mathbb{C}}$ the complex linear extension of U and by $W^{\mathbb{C}}$ the complex linear extension of W. We now take

$$\mathcal{W}_1 = \{v + iTv | v \in W\},$$
$$\mathcal{W}_2 = \{w - iTw | w \in W\}.$$

Note that these are indeed complex linear vector spaces as $i(v \pm iTv) = \mp(Tv \mp iv) = (\mp Tv \pm iT(\mp Tv))$ and we complexify the metric and the previously defined operator L. Note that L is symmetric and that from the properties of L and T it follows that the restriction of L to $\mathcal{W}_1 \times \mathcal{W}_1$ and $\mathcal{W}_2 \times \mathcal{W}_2$ vanishes identically. Therefore in order to determine L it is sufficient to study L on

$$\begin{cases} L : \mathcal{W}_1 \times \mathcal{W}_2 \to U^{\mathbb{C}} \\ L(\omega, \tilde{\omega}) = K_\omega \tilde{\omega} + \frac{1}{4\lambda^2} h(K_\omega \tilde{\omega}, e_2) e_1 + \frac{1}{4\lambda^2} h(K_\omega \tilde{\omega}, e_1) e_2, \end{cases} \tag{3.18}$$

where $U^{\mathbb{C}} := span\{u_1, \ldots, u_r\}$ over $\mathbb{C}$.

Proposition 5. *The operator L defined in* (3.18) *satisfies:*

1. *For any vectors $x \in \mathcal{W}_1$ and $y \in \mathcal{W}_2$ we have*

$$h(L(x, y), L(x, y)) = \frac{3\mu^2}{4} h(x, x) h(y, y); \tag{3.19}$$

2. *Given x_0 in $\mathcal{W}_1$ such that $h(x_0, x_0) = 1$, we have that $L(x_0, -)$ preserves norms in the sense that*

$$h(L(x_0, y), L(x_0, y)) = \frac{3}{4}\mu^2 h(y, y), \forall y \in \mathcal{W}_2;$$

3. *Given x_0 a non-null vector, we have that $L(x_0, -) : \mathcal{W}_2 \mapsto U^{\mathbb{C}}$ is a bijective operator;*

4. *For any $x, x' \in \mathcal{W}_1, y, z \in \mathcal{W}_2$ we have that*

$$h(L(x,y), L(x',z)) + h(L(x',y), L(x,z)) = \frac{3}{2}\mu^2 h(x,x')h(y,z). \tag{3.20}$$

Proof. 1. Take $W_1 = W_3 = \omega_1$ and $W_2 = W_4 = \omega_2$ in relation (3.16), where $\omega_1 := v + iTv \in \mathcal{W}_1$ and $\omega_2 := w - iTw \in \mathcal{W}_2$. Using the properties of T in (3.12) and the fact that ω_1 and ω_2 are orthogonal, we obtain $h(L(\omega_1,\omega_2), L(\omega_1,\omega_2)) = \frac{3\mu^2}{4}h(\omega_1,\omega_1)h(\omega_2,\omega_2)$.

2. This property follows directly from the previously proved one.

3. We linearize in the second argument in property (3.19), that is $y \rightsquigarrow y + z$, for $y, z \in \mathcal{W}_2$ and we get for arbitrary $x \in \mathcal{W}_1$

$$h(L(x,y), L(x,z)) = \frac{3}{4}\mu^2 h(x,x)h(y,z). \tag{3.21}$$

Fix $x = x_0$, for x_0 arbitrarily chosen in $\mathcal{W}_1$, and write equation (3.21) once for $y = y_1$ and once for $y = y_2$. Assuming $L(x_0, y_1) = L(x_0, y_2)$, as h is nondegenerate and x_0 is a non-null vector, we get that $L(x_0, -)$ is injective. This gives $\dim Im(L(x_0,-)) = \dim \mathcal{W}_2 = r$, but, as $\dim U^{\mathbb{C}} = r$, we obtain that L is also surjective.

4. The property in (3.20) follows immediately by liniarizing in (3.21) for $x \rightsquigarrow x + x', \forall x, x' \in \mathcal{W}_1$.

□

Theorem 48. *Let M^n_k be a λ-isotropic affine hypersurface. Assume that $\lambda \neq 0$. Then either $n = 2, 5, 8, 14$ or 26.*

Proof. We assume that $n > 2$. We can write out equation (3.19) for the elements of the bases. For more convenience, choose $\{e_i\}_{i=\{1,\ldots,r\}}, \{f_j\}_{j=\{1,\ldots,r\}}, \{g_k\}_{k=\{1,\ldots,r\}}$ bases for $\mathcal{W}_1, \mathcal{W}_2, U$ respectively, and let $u = \sum_{i=1}^{r} u_i e_i, v = \sum_{j=1}^{r} v_j f_j$. With this choice, relation (3.19) becomes

$$(u_1^2 + \ldots + u_r^2)(v_1^2 + \ldots + v_r) = z_1^2 + \ldots + z_r^2, \tag{3.22}$$

where $L(e_i, f_j) = l_{ij}^k g_k$ and $z_k = \sum_{i,j=1}^{n} u_i v_j l_{ij}^k$. Equation (3.22) yields an r-square quadratic equation. Thus, we may apply now the theorem of Hurwitz and obtain $r = 1, 2, 4, 8$, which implies that $n = 5, 8, 14, 26$. □

3.2 Isotropic affine hyperspheres

From now on, we will assume that M is a λ-isotropic affine hypersphere with $\lambda \neq 0$.

Proposition 6. *Let $n \geq 3$ and M^n be an n-dimensional affine λ-isotropic hypersphere in $\mathbb{R}^{n+1}$. Then M^n is constant isotropic.*

Proof. Let $e'_1 := f_1, e'_2 := f_2, e'_3 := u_1, \dots, e'_{r+2} := u_r, e'_{r+3} := \omega_1^1, \dots, e'_{2r+2} := \omega_1^r, e'_{2r+3} := \omega_2^r$. Then $\{e'_1, \dots, e'_n\}$ is an orthogonal basis with $h(e'_i, e'_j) = \varepsilon_i \delta_{ij}, \varepsilon_i = \pm 1$. We denote by Ric the Ricci tensor of M^n with respect to the affine metric h. As M^n is an affine sphere, we have that the shape operator is a multiple of the identity, say $S = \varepsilon I$. Using as well the Gauss equation, the apolarity condition in proposition 2.137 and the isotropy condition (3.1) we have

$$\begin{aligned} Ric(e'_j, e'_k) &= \sum_{i=1}^{n} \varepsilon_i h(\hat{R}(e'_i, e'_j)e'_k, e'_i) \\ &= h(\varepsilon(\varepsilon_j \delta_{jk} e'_i - \varepsilon_i \delta_{ik} e'_j) - [K_{e'_i}, K_{e'_j}]e'_k, e'_i) \\ &= n\varepsilon\varepsilon_j \delta_{jk} - \varepsilon\varepsilon_j \delta_{jk} - \sum_i \varepsilon_i h([K_{e_i}, K_{e_j}]e_k, e_i). \end{aligned} \tag{3.23}$$

For $k \neq j$ we obtain

$$\begin{aligned} Ric(e'_j, e'_k) &= -\sum_{i=1}^{n} \varepsilon_i h([K_{e'_i}, K_{e'_j}]e'_k, e'_i) \\ &= \sum_{i=1}^{n} \varepsilon_i h(K(e'_i, e'_j), K(e'_i, e'_k)) \\ &= -\frac{1}{2} \sum_{i=1}^{n} \varepsilon_i h(K(e'_i, e'_i), K(e'_j, e'_k)) \\ &= 0 \end{aligned} \tag{3.24}$$

and for $k = j$

$$\begin{aligned} Ric(e'_j, e'_j) &= \sum_{i=1}^{n} \varepsilon h(K(e'_i, e'_j), K(e'_i, e'_j)) \\ &= \sum_{i=1}^{n} \frac{1}{2}[-h(K(e'_j, e'_j), K(e'_i, e'_i)) + 2\lambda(p)\delta_{ij} + \lambda(p)\varepsilon_i\varepsilon_j] \\ &= \left(\frac{n}{2} + 1\right) \varepsilon_j \lambda(p). \end{aligned} \tag{3.25}$$

Since $n \geq 3$, by using the fact that the Levi-Civita connection on M^n is torsion free and using the second Bianchi identity, we get that λ is constant. $\square$

Similarly to [32], Proposition 3.6, we can prove the following:

Proposition 7. *Let M^n be an n-dimensional affine submanifold in $\mathbb{R}^{n+1}$. If M^n is constant isotropic with $\lambda \neq 0$, then M^n has parallel difference tensor.*

Proof. Since M^n is constant isotropic, we have $\lambda = h(K(v, v), K(v, v))$ and by taking the derivative, we obtain $h(\hat{\nabla}_\chi K(v, v), K(v, v)) = 0, \forall p \in M^n, \forall v, \chi \in T_pM^n, h(v, v) = 1$.
In the isotropy relation (3.1) we take $X_1 = \hat{\nabla}_\chi v$, $X_2 = X_3 = X_4 = v$ and obtain $h(K(\hat{\nabla}_\chi v, v), K(v, v)) = \lambda h(\hat{\nabla}_\chi v, v)h(v, v) = 0$, for $h(v, v) = 1$. This implies

$$h((\hat{\nabla} K)(\chi, v, v), K(v, v)) = 0 \tag{3.26}$$

for any $v, \chi \in T_pM^n$ such that $h(v,v) = 1$ and in particular, we have

$$h((\hat{\nabla}K)(v,v,v), K(v,v)) = 0. \tag{3.27}$$

Further on, we take the derivative with respect to some vector $w \in T_pM$ in equation (3.1) for $X_1 = X_2 = X_3 = v, X_4 = w$ and for $h(v,w) = 0$ and obtain

$$h((\hat{\nabla}K)(v,v,v), K(v,w)) - h((\hat{\nabla}K)(v,v,w), K(v,v)) = 0.$$

As $\hat{\nabla}K$ is totally symmetric, using also (3.26) we have

$$h((\hat{\nabla}K)(v,v,v,), K(v,w)) = 0, \tag{3.28}$$

for any $v, w \in T_pM$ such that $h(v,v) = 1$. We can write $K(v, K(v,v)) = av + bw$, for $v \in T_pM^n$, w an $(n-1)$-dimensional tangent vector , $h(v,w) = 0$. Since

$$\begin{cases} h(K(v,K(v,v)),v) = h(K(v,v),K(v,v)) = \lambda, \\ h(K(v,K(v,v)),w) = bh(w,w) = 0, \end{cases}$$

we get $a = \lambda, b = 0$ so that $K(v, K(v,v)) = \lambda v$. If we take $w = K(v,v)$ in equation (3.28) we get

$$\lambda h((\hat{\nabla}K)(v,v,v), v) = 0. \tag{3.29}$$

As $\lambda \neq 0$, using (3.29) and the symmetry of $\hat{\nabla}K$, we also have $\hat{\nabla}K = 0$. $\square$

Proposition 8. *Let $n \geq 3$ and M^n be an $n-$dimensional $\lambda-$isotropic affine hypersphere in $\mathbb{R}^{n+1}$, such that $S = \varepsilon I$, with ε constant. Assume that $\lambda \neq 0$. If $\mathbb{R}^{n+1}$ is endowed with an indefinite metric and M^n is not totally geodesic, then M^n is a locally symmetric space and $\lambda = -\frac{1}{2}\varepsilon$.*

Proof. From the previous propositions we conclude that $\hat{\nabla}K = 0$. Hence, by the Gauss equation we have $\hat{\nabla}R = 0$, which means that M^n is a locally symmetric space. Using the Ricci identity, from $\hat{\nabla}K = 0$ we also have $\hat{R}.h = 0$, that is

$$\hat{R}(X,Y)K(Z,W) - K(\hat{R}(X,Y)Z,W) - K(Z,\hat{R}(X,Y)W) = 0, \tag{3.30}$$

for X, Y, Z, W tangent vector fields. If we take $X = Z = W = f_1, Y = f_2$, it implies

$$\hat{R}(f_1,f_2)K(f_1,f_1) = 2K(\hat{R}(f_1,f_2)f_1,f_1) \tag{3.31}$$

and then from (3.11) and Gauss equation we have

$$\hat{R}(f_1,f_2)f_1 = -(\varepsilon + 2\lambda)f_2,$$

which together with (3.31) implies $\varepsilon + 2\lambda = 0$. $\square$

Proposition 9. *Let $n \geq 3$, $f_1 : M_1^n \to \mathbb{R}^{n+1}$ and $f_2 : M_2^n \to \mathbb{R}^{n+1}$ be $n-$dimensional $\lambda-$isotropic affine hypersphere in $\mathbb{R}^{n+1}$, such that $S_1 = S_2 = \varepsilon I$, with $\varepsilon = \pm 1$ constant. Let $p_1 \in M_1$ and $p_2 \in M_2$ and assume that there exists an isometry $A : T_{p_1}M_1 \to T_{p_2}M_2$ such that*

$$AK_1(v,w) = K_2(Av, Aw),$$

i.e. A preserves the difference tensor. Then there exists a local isometry $F : (M_1, h_1) \to (M_2, h_2)$ such that

$$dF(K_1(X,Y)) = K_2(dF(X), dF(Y)),$$

for any vector fields X, Y on M_1. Moreover the immersions f_1 and $f_2 \circ F$ are locally congruent.

Proof. From the previous propositions we know that λ is a constant, and that with respect to the Levi Civita connection, M_1 and M_2 are locally symmetric spaces whose difference tensor is parallel with respect to the Levi Civita connection.

We take $p_1 \in M_1$ and we take a basis $\{e'_1, \dots, e'_n\}$ of $T_{p_1}M_1$. As A is an isometry we take as basis of $T_{p_2}M_2$ the vectors $\{Ae'_1, \dots, Ae'_n\}$. By the initial conditions we have that

$$\begin{aligned} &h_1(e'_i, e'_j) = h_2(Ae'_i Ae'_j) \qquad \text{(isometry)} \\ &h_1(K_1(e'_i, e'_j), e'_k) = h_2(AK_1(e'_i, e'_j), Ae'_k) = h_2(K_2(Ae'_i, Ae'_j), Ae'_k). \end{aligned}$$

We now extend $\{e'_1, \dots, e'_n\}$ to a local differential basis $\{X_1, \dots, X_n\}$ by parallel translation along geodesics with respect to the Levi Civita connection of the affine metric. In the same way we extend $\{Ae'_1, \dots, Ae'_n\}$ to local vector fields $\{Y_1, \dots, Y_n\}$. As the difference tensors are parallel, we have that the components of the difference tensor stay constant along geodesics. Therefore by construction, we have that

$$\begin{aligned} &h_1(X_i, X_j) = h_2(Y_i, Y_j), \\ &h_1(K_1(X_i, X_j), X_k) = h_2(K_2(Y_i, Y_j), Y_k). \end{aligned}$$

Hence by the lemma of Cartan, see [12], we know that there exists a local isometry F such that $dF(X_i) = Y_i$. In order to complete the proof it is now sufficent to apply Theorem 47. $\square$

So in order to complete the classification it is now sufficient to determine, up to isometries, the possible forms of the difference tensor and for each of those forms obtained to determine an explicit example of an affine hypersphere with isotropic difference tensors. This is done explicitly for the 4 remaining dimensions 5, 8, 14 and 26 in the next sections.

3.3 Affine hyperspheres of dimension 5

3.3.1 The form of L, $\dim \mathcal{U} = 1$

We start with $\omega = v + iTv \in \mathcal{W}_1$, a vector of length 2. As the length of ω is 2, it follows that v has unit length and is orthogonal to Tv. So we can take $\omega_1^1 = v$ and $\omega_2^1 = Tv$. Note that by the properties of L we have that $L(v + iTv, v - iTv)$ is a real vector in $\mathcal{U}^{\mathbb{C}}$ whose square length is $3\mu^2$. Hence we can pick a unit vector u_1 in U such that

$$L(v + iTv, v - iTv) = \sqrt{3}\mu u_1.$$

By the properties of L this implies that

$$L(\omega_1^1, \omega_1^1) = \tfrac{\sqrt{3}}{2}\mu u_1.$$

From the properties of T we see $L(\omega_1^1, \omega_2^1) = 0$ and $L(\omega_1^1, \omega_1^1) = L(\omega_2^1, \omega_2^1)$, hence L is completely determined. Therefore L and also K are completely determined and the signature of the metric, if necessary after replacing ξ by $-\xi$ in order to make $\lambda > 0$, equals 2.

3.3.2 A canonical example

We consider $\mathbb{R}^6 = s(3)$ as the set of all symmetric 3×3 matrices and we take as hypersurface M those symmetric matrices with determinant 1. We define an action σ of $SL(3,\mathbb{R})$ on M as follows

$$\sigma: \; SL(3,\mathbb{R}) \times M \to M, \text{ such that } (g,p) \mapsto \sigma_g(p) = gpg^T.$$

Note that M has two connected components and that the action is transitive on each of the connected components. The connected component of I has been studied in [5], where it was shown that it gives a positive definite isotropic affine hypersurface. It also appears in [28]. Here we are interested in the component of the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, which we denote by M_1. So $M_1 = \{gAg^T | g \in SL(3,\mathbb{R})\}$. The isotropy group of A consists of the matrices g of determinant 1 such that $gAg^T = A$. This Lie group is congruent to $SO(2,1)$ and therefore, by Theorem 9.2 of [8], we know that M_1 is locally isometric with $\frac{SL(3,\mathbb{R})}{SO(2,1)}$.

Note that, of course, every element of $SL(3,\mathbb{R})$ acts at the same time also on $s(3)$ and that this action belongs to $SL(6,\mathbb{R})$, see [46]. This implies that M_1 is at the same time an homogeneous affine hypersurface and by Proposition 2 an equiaffine sphere centered at the origin.

In order to determine the tangent space at $p = gAg^T$, we look at the curves in M_1

$$\gamma(s) = g e^{sX} A e^{sX^T} g^T.$$

These are indeed curves in M_1, provided that $e^{sX} \in SL(3,\mathbb{R})$ or, equivalently, provided that $\operatorname{Tr} X = 0$. Note that $\gamma'(s) = g e^{sX}(XA + AX^T) e^{sX^T} g^T$, where $v = (XA + AX^T)$ is a symmetric matrix. So by using a dimension argument we see that the tangent space is given by

$$\{gvg^T | v = 2XA, XA = AX^T, \operatorname{Tr} X = 0, X \in \mathbb{R}^{3\times 3}\} = T_p M_1.$$

Working now at the point A, taking $g = I$ and $X \in so(2,1) = \{X \in \mathbb{R}^{3\times 3} | \operatorname{Tr} X = 0, XA = AX^T\}$ we see that

$$\begin{aligned}
\nabla_{\gamma'(s)}\gamma'(s) + h(\gamma',\gamma')\gamma &= \gamma''(s) \\
&= e^{sX}(4X^2 A) e^{sX^T} \\
&= e^{sX}((4X^2 - \tfrac{4}{3}\operatorname{Tr}(X^2) I)A) e^{sX^T} + \tfrac{4}{3}\operatorname{Tr}(X^2) e^{sX} A e^{sX^T} \\
&= e^{sX}((4X^2 - \tfrac{4}{3}\operatorname{Tr}(X^2) I)A) e^{sX^T} + \tfrac{4}{3}\operatorname{Tr}(X^2)\gamma(s).
\end{aligned}$$

As the matrix $(4X^2 - \frac{4}{3}\operatorname{Tr}(X^2)I)$ commutes with A, we can decompose the above expression into a tangent part and a part in the direction of the affine normal given by the position vector, and therefore we find that

$$h(\gamma'(s), \gamma'(s)) = \tfrac{4}{3}\operatorname{Tr}(X^2).$$

So we see that s is a constant length parametrisation of the curve γ and therefore we have that $h(\gamma', \hat{\nabla}_{\gamma'}\gamma') = 0$ and

$$h(\gamma', \nabla_{\gamma'}\gamma') = h(\gamma', K(\gamma', \gamma')).$$

In addition, we have

$$\begin{aligned}\gamma'''(s) &= \nabla_{\gamma'(s)}\nabla_{\gamma'(s)}\gamma'(s) + h(\gamma',\gamma')\gamma' + h(\gamma',K(\gamma',\gamma'))\gamma \\ &= e^{sX}(8X^3A)e^{sX^T} \\ &= e^{sX}((8X^3 - \tfrac{8}{3}\operatorname{Tr}(X^3)I)A)e^{sX^T} + \tfrac{8}{3}\operatorname{Tr}(X^3)\gamma(s).\end{aligned}$$

Therefore, working at $s=0$ and writing $v = 2XA$ as tangent vector, we obtain that

$$\begin{aligned}h(v,v) &= \tfrac{4}{3}\operatorname{Tr}(X^2), \\ h(v,K(v,v)) &= \tfrac{8}{3}\operatorname{Tr}X^3.\end{aligned}$$

Linearising the above expressions, i.e. writing $v = \alpha_1 v_1 + \alpha_2 v_2$, respectively $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$, for $v_i = 2X_iA, i = 1,2,3$, and looking at the coefficient of $\alpha_1\alpha_2$, respectively $\alpha_1\alpha_2\alpha_3$, we obtain that

$$\begin{aligned}h(v_1,v_2) &= \tfrac{4}{3}\operatorname{Tr}(X_1X_2) = \tfrac{4}{3}\operatorname{Tr}(X_2X_1), \\ 6h(K(v_1,v_2),v_3) &= \tfrac{8}{3}(\operatorname{Tr}X_1X_2X_3 + \operatorname{Tr}X_3X_1X_2 + \operatorname{Tr}X_2X_3X_1 + \\ &\quad + \operatorname{Tr}X_1X_3X_2 + \operatorname{Tr}X_3X_2X_1 + \operatorname{Tr}X_2X_1X_3) \\ &= 8(\operatorname{Tr}X_1X_2X_3 + \operatorname{Tr}X_2X_1X_3).\end{aligned}$$

So we see that

$$K(v_1,v_2) = 2(X_1X_2 + X_2X_1 - \tfrac{2}{3}\operatorname{Tr}(X_1X_2)I)A.$$

Indeed, we have that $(X_1X_2 + X_2X_1 - \frac{2}{3}\operatorname{Tr}(X_1X_2)I)$ has vanishing trace, commutes with A and therefore $K(v_1,v_2)$ is indeed the unique tangent vector such that

$$h(K(v_1,v_2),v_3) = \tfrac{4}{3}(\operatorname{Tr}(X_1X_2X_3) + \operatorname{Tr}(X_2X_1X_3)).$$

As by Cayley Hamilton, for a matrix X with vanishing trace, we have that $X^3 = 1/2\operatorname{Tr}(X^2)X + det(X)I$, we deduce that

$$\operatorname{Tr}X^4 = \tfrac{1}{2}(\operatorname{Tr}X^2)^2,$$

and therefore we have that

$$\begin{aligned}h(K(v,v),K(v,v)) &= \tfrac{4}{3}\operatorname{Tr}(2X^2 - \tfrac{2}{3}\operatorname{Tr}X^2I)^2 \\ &= \tfrac{4}{3}(4\operatorname{Tr}X^4 + \tfrac{4}{9}(TrX^2)^2TrI - \tfrac{8}{3}(\operatorname{Tr}X^2)^2) \\ &= \tfrac{8}{9}(TrX^2)^2 \\ &= \tfrac{1}{2}(h(v,v))^2.\end{aligned}$$

Hence M_1 is isotropic with positive λ. A straightforward computation also shows that the index of the metric is 2. Combining therefore the results in this section with Proposition 9 and the classification result of O. Birembaux and M. Djoric, see [5] in the positive definite case, we get theorem 42.

3.4 Affine hyperspheres of dimension 8

3.4.1 The form of L, $\dim \mathcal{U} = 2$

Let $W = span\{\omega_1^1, \omega_2^1, \omega_1^2, \omega_2^2\}$ and $\mathcal{W}_1 = span\{\omega_1^1 + i\omega_2^1, \omega_1^2 + i\omega_2^2\}$, $\mathcal{W}_2 = span\{\omega_1^1 - i\omega_2^1, \omega_1^2 - i\omega_2^2\}$. Remark that all the bases are orthogonal and in addition

$$\begin{aligned} h(\omega_1^1 + i\omega_2^1, \omega_1^1 + i\omega_2^1) &= h(\omega_1^2 + i\omega_2^2, \omega_1^2 + i\omega_2^2) = 2, \\ h(\omega_1^1 - i\omega_2^1, \omega_1^1 - i\omega_2^1) &= h(\omega_1^2 - i\omega_2^2, \omega_1^2 - i\omega_2^2) = -2. \end{aligned}$$

Then, straightforward computations lead to

$$\begin{cases} L(\omega_1^1 + i\omega_2^1, \omega_1^1 - i\omega_2^1) = 2L(\omega_1^1, \omega_1^1), \\ L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2) = 2L(\omega_1^1, \omega_1^2) - 2iL(\omega_1^1, \omega_2^2), \\ L(\omega_1^2 + i\omega_2^2, \omega_1^1 - i\omega_2^1) = 2L(\omega_1^1, \omega_1^2) + 2iL(\omega_1^1, \omega_2^2), \\ L(\omega_1^2 + i\omega_2^2, \omega_1^2 - i\omega_2^2) = 2L(\omega_1^2, \omega_1^2). \end{cases} \tag{3.32}$$

Notice that the vector $L(\omega_1^1 + i\omega_2^1, \omega_1^1 - i\omega_2^1)$ is a real vector of length $3\mu^2$. So we can pick $u_1 \in \mathcal{U}$, $h(u_1, u_1) = 1$ such that

$$L(\omega_1^1 + i\omega_2^1, \omega_1^1 - i\omega_2^1) = \sqrt{3}\mu u_1. \tag{3.33}$$

With this choice, from property (3.21) we obtain that $L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2)$ is orthogonal to u_1. Moreover as its length is a real number, we must have that $Re(L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2)$ and $Im(L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2)$ are orthogonal to each other. As they are also both orthogonal to u_1, one of them has to vanish. Therefore, we get two cases:

Case II–1. $Re(L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2)) = 0$

Now we obtain that $L(\omega_1^1, \omega_1^2) = 0$ and $L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2)$ is an imaginary vector of length $3\mu^2$, orthogonal to u_1. Thus, we can pick $u_2 \in \mathcal{U}$ in the direction of $L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2)$ such that $h(u_2, u_2) = -1$ and such that

$$L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2) = i\sqrt{3}\mu u_2. \tag{3.34}$$

Consider now $L(\omega_1^2 + i\omega_2^2, \omega_1^2 - i\omega_2^2)$. It is a real vector orthogonal to u_2, of length $3\mu^2$ and thus we can write $L(\omega_1^2 + i\omega_2^2, \omega_1^2 - i\omega_2^2) = \pm\sqrt{3}\mu u_1$.

Furthermore, from (3.32), (3.34) and Proposition 5- (4), we obtain

$$\begin{aligned} 3\mu^2 &= -h(L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2), L(\omega_1^2 + i\omega_2^2, \omega_1^1 - i\omega_2^1)) \\ &= h(L(\omega_1^1 + i\omega_2^1, \omega_1^1 - i\omega_2^1), L(\omega_1^2 + i\omega_2^2, \omega_1^2 - i\omega_2^2)) \\ &= \sqrt{3}\mu h(u_1, L(\omega_1^2 + i\omega_2^2, \omega_1^2 - i\omega_2^2)). \end{aligned}$$

So we get that $L(\omega_1^2 + i\omega_2^2, \omega_1^2 - i\omega_2^2) = \sqrt{3}\mu u_1$. In this case, the signature of the metric is 4.

Case II–2. $Im(L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2)) = 0$

Reasoning in a similar way, we choose $u_2 \in \mathcal{U}$ a real vector in the direction of $L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2)$, with $h(u_2, u_2) = 1$ such that $L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2) = \sqrt{3}\mu u_2$. We find $L(\omega_1^2 + i\omega_2^2, \omega_1^2 - i\omega_2^2) = -\sqrt{3}\mu u_1$ and in this case the signature of the metric is 3.

3.4.2 Two canonical examples

First we consider $\mathbb{R}^9$ as the set of Hermitian symmetric matrics $Y \in \mathbb{C}^{3\times 3}$. We take as hypersurface M those Hermitian symmetric matrices with determinant 1. We define an action σ of $SL(3,\mathbb{C})$ on M as follows

$$\sigma: \ SL(3,\mathbb{C}) \times M \to M, \text{ such that } (g,p) \mapsto \sigma_g(p) = gp\bar{g}^T.$$

Note that M has two connected components and that the action is transitive on each of the connected components. The connected component of I has been studied in [5], where it was shown that it gives a positive definite isotropic affine hypersurface. It also appears in [28]. Here we are interested in the component of the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, which we denote by M_1. So $M_1 = \{gA\bar{g}^T | g \in SL(3,\mathbb{C})\}$. The isotropy group consists of the matrices g of determinant 1 such that $gA\bar{g}^T = A$. This Lie group is congruent to $SU(2,1)$ and therefore, by Theorem 9.2 of [8], we know that M_1 is locally isometric with $\frac{SL(3,\mathbb{C})}{SU(2,1)}$.

Note that of course every element of $SL(3,\mathbb{C})$ acts at the same time also on $\mathbb{R}^9$ in a linear way and that, therefore, this action belongs to $GL(9,\mathbb{R})$. A straightforward computation shows that this action actually belongs to $SL(9,\mathbb{R})$. This implies that M_1 is at the same time an homogeneous affine hypersurface and, by Proposition 2, an equiaffine sphere centered at the origin. So, in order to determine the properties of M_1, it is sufficient to look at a single point.

In order to determine the tangent space at the point $p = gA\bar{g}^T$, we look at the curves in M_1

$$\gamma(s) = ge^{sX}Ae^{s\bar{X}^T}\bar{g}^T.$$

These are indeed curves in M_1 provided that $e^{sX} \in SL(3,\mathbb{C})$ or equivalently provided that $\operatorname{Tr} X = 0$. Note that $\gamma'(s) = ge^{sX}(XA + A\bar{X}^T)e^{s\bar{X}^T}\bar{g}^T$, where $v = (XA + A\bar{X}^T)$ is a Hermitian symmetric matrix. So by using a dimension argument we see that the tangent space is given by

$$\{gv\bar{g}^T | v = 2XA, XA = A\bar{X}^T, \operatorname{Tr} X = 0, X \in \mathbb{C}^{3\times 3}\} = T_pM_1.$$

Working now at the point A, taking $g = I$ and $X \in su(2,1) = \{X \in \mathbb{C}^{3\times 3} | \operatorname{Tr} X = 0, XA = A\bar{X}^T\}$ we see that

$$\begin{aligned}
\nabla_{\gamma'(s)}\gamma'(s) + h(\gamma',\gamma')\gamma &= \gamma''(s) \\
&= e^{sX}(4X^2A)e^{sX^T} \\
&= e^{sX}((4X^2 - \tfrac{4}{3}\operatorname{Tr}(X^2)I)A)e^{s\bar{X}^T} + \tfrac{4}{3}\operatorname{Tr}(X^2)e^{sX}Ae^{s\bar{X}^T} \\
&= e^{sX}((4X^2 - \tfrac{4}{3}\operatorname{Tr}(X^2)I)A)e^{s\bar{X}^T} + \tfrac{4}{3}\operatorname{Tr}(X^2)\gamma(s).
\end{aligned}$$

As the matrix $(4X^2 - \frac{4}{3}\operatorname{Tr}(X^2)I)$ commutes with A, we can decompose the above expression into a tangent part and a part in the direction of the affine normal given by the position vector, and therefore we find that

$$h(\gamma'(s),\gamma'(s)) = \tfrac{4}{3}\operatorname{Tr}(X^2).$$

So we see that s is a constant length parametrisation of the curve γ and therefore we have that $h(\gamma', \hat{\nabla}_{\gamma'}\gamma') = 0$ and

$$h(\gamma', \nabla_{\gamma'}\gamma') = h(\gamma', K(\gamma', \gamma')).$$

As

$$\begin{aligned}\gamma'''(s) &= \nabla_{\gamma'(s)}\nabla_{\gamma'(s)}\gamma'(s) + h(\gamma', \gamma')\gamma' + h(\gamma', K(\gamma', \gamma'))\gamma \\ &= e^{sX}(8X^3A)e^{s\bar{X}^T} \\ &= e^{sX}((8X^3 - \tfrac{8}{3}\operatorname{Tr}(X^3)I)A)e^{s\bar{X}^T} + \tfrac{8}{3}\operatorname{Tr}(X^3)\gamma(s),\end{aligned}$$

working at $s = 0$ and writing $v = 2XA$ as tangent vector, we have that

$$\begin{aligned}h(v, v) &= \tfrac{4}{3}\operatorname{Tr}(X^2), \\ h(v, K(v, v)) &= \tfrac{8}{3}\operatorname{Tr} X^3.\end{aligned}$$

Linearising the above expressions, i.e. writing $v = \alpha_1 v_1 + \alpha_2 v_2$, respectively $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$, for $v = 2X_iA, i = 1, 2, 3$, and looking at the coefficient of $\alpha_1\alpha_2$, respectively $\alpha_1\alpha_2\alpha_3$, we obtain that

$$\begin{aligned}h(v_1, v_2) &= \tfrac{4}{3}\operatorname{Tr}(X_1X_2) = \tfrac{4}{3}\operatorname{Tr}(X_2X_1), \\ 6h(K(v_1, v_2), v_3) &= \tfrac{8}{3}(\operatorname{Tr} X_1X_2X_3 + \operatorname{Tr} X_3X_1X_2 + \operatorname{Tr} X_2X_3X_1 + \\ &\quad + \operatorname{Tr} X_1X_3X_2 + \operatorname{Tr} X_3X_2X_1 + \operatorname{Tr} X_2X_1X_3) \\ &= 8(\operatorname{Tr} X_1X_2X_3 + \operatorname{Tr} X_2X_1X_3).\end{aligned}$$

So we see that

$$K(v_1, v_2) = 2(X_1X_2 + X_2X_1 - \tfrac{2}{3}\operatorname{Tr}(X_1X_2)I)A.$$

Indeed, we have that $(X_1X_2 + X_2X_1 - \frac{2}{3}\operatorname{Tr}(XY)I)$ has vanishing trace, commutes with A and therefore $K(v_1, v_2)$ is indeed the unique tangent vector such that

$$h(K(v_1, v_2), v_3) = \tfrac{4}{3}(\operatorname{Tr}(X_1X_2X_3) + \operatorname{Tr}(X_2X_1X_3)).$$

As by Cayley Hamilton for a matrix X with vanishing trace we have that $X^3 = 1/2\operatorname{Tr}(X^2)X + det(X)I$, we deduce that

$$\operatorname{Tr} X^4 = \tfrac{1}{2}\operatorname{Tr} X^2,$$

and therefore, we have that

$$\begin{aligned}h(K(v, v), K(v, v)) &= \tfrac{4}{3}\operatorname{Tr}(2X^2 - \tfrac{2}{3}\operatorname{Tr} X^2 I)^2 \\ &= \tfrac{4}{3}(4\operatorname{Tr} X^4 + \tfrac{4}{9}(TrX^2)^2 TrI - \tfrac{8}{3}(\operatorname{Tr} X^2)^2) \\ &= \tfrac{8}{9}(TrX^2)^2 \\ &= \tfrac{1}{2}(h(v, v))^2.\end{aligned}$$

Hence M_1 is isotropic with positive λ. A straightforward computation also shows that the index of the metric is 4.

Next, we consider $\mathbb{R}^9 = \mathbb{R}^{3\times 3}$. We take as hypersurface M_2 those matrices with determinant 1. We define an action σ of $SL(3, \mathbb{R})$ on M_2 as follows

$$\sigma : SL(3, \mathbb{R}) \times M_2 \to M_2, \text{ such that } (g, p) \mapsto \sigma_g(p) = gp.$$

The isotropy group of the identity matrix consists only of the identity matrix. Therefore, by Theorem 9.2 of [8] we know that M_2 is locally isometric with $SL(3,\mathbb{R})$.

Note that, of course, every element of $SL(3,\mathbb{R})$ acts at the same time also on $\mathbb{R}^9$ in a linear way and that therefore this action belongs to $GL(9,\mathbb{R})$. A straightforward computation shows that this action actually belongs to $SL(9,\mathbb{R})$. This implies that M_2 is at the same time an homogeneous affine hypersurface and, by Proposition 2, an equiaffine sphere centered at the origin. So in order to determine the properties of M_2 it is sufficient to look at a single point.

In order to determine the tangent space at a point p, we look at the curves in M_2

$$\gamma(s) = e^{sX}p.$$

These are indeed curves in M_2, provided that $e^{sX} \in SL(3,\mathbb{R})$ or equivalently, provided that $\operatorname{Tr} X = 0$. Note that $\gamma'(s) = e^{sX}Xp$, so by using a dimension argument we see that the tangent space is given by

$$\{Xp| \operatorname{Tr} X = 0, X \in \mathbb{R}^{3\times 3}\} = T_pM_2.$$

Working now at the point I and $X \in sl(3,\mathbb{R}) = \{X \in \mathbb{R}^{3\times 3}| \operatorname{Tr} X = 0\}$, we see that

$$\begin{aligned}
\nabla_{\gamma'(s)}\gamma'(s) + h(\gamma',\gamma')\gamma &= \gamma''(s)\\
&= e^{sX}X^2\\
&= e^{sX}(X^2 - \tfrac{1}{3}\operatorname{Tr}(X^2)I) + \tfrac{1}{3}\operatorname{Tr}(X^2)e^{sX}\\
&= e^{sX}(X^2 - \tfrac{1}{3}\operatorname{Tr}(X^2)I) + \tfrac{1}{3}\operatorname{Tr}(X^2)\gamma(s).
\end{aligned}$$

As the matrix $(X^2 - \frac{1}{3}\operatorname{Tr}(X^2)I)$ commutes with e^{sX} and has vanishing trace, we can interprete $e^{sX}(X^2 - \frac{1}{3}\operatorname{Tr}(X^2)I)$ as a tangent vector at the point e^{sX}. By decomposing the above expression into a tangent part and a part in the direction of the affine normal given by the position vector, we deduce that

$$h(\gamma'(s),\gamma'(s)) = \tfrac{1}{3}\operatorname{Tr}(X^2).$$

So we see that s is a constant length parametrisation of the curve γ and therefore we have that $h(\gamma',\hat{\nabla}_{\gamma'}\gamma') = 0$ and

$$h(\gamma',\nabla_{\gamma'}\gamma') = h(\gamma',K(\gamma',\gamma')).$$

As

$$\begin{aligned}
\gamma'''(s) &= \nabla_{\gamma'(s)}\nabla_{\gamma'(s)}\gamma'(s) + h(\gamma',\gamma')\gamma' + h(\gamma',K(\gamma',\gamma'))\gamma\\
&= e^{sX}X^3\\
&= e^{sX}(X^3 - \tfrac{1}{3}\operatorname{Tr}(X^3)I)e^{sX^T} + \tfrac{1}{3}\operatorname{Tr}(X^3)\gamma(s),
\end{aligned}$$

working at $s = 0$ and writing $v = X$ as tangent vector, we have that

$$\begin{aligned}
h(v,v) &= \tfrac{1}{3}\operatorname{Tr}(X^2),\\
h(v,K(v,v)) &= \tfrac{1}{3}\operatorname{Tr} X^3.
\end{aligned}$$

Linearising the above expressions, i.e. writing $v = \alpha_1 v_1 + \alpha_2 v_2$, respectively $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$, for $v_i = X_i, i = 1,2,3$, and looking at the coefficient of $\alpha_1\alpha_2$, respectively $\alpha_1\alpha_2\alpha_3$ we obtain that

$$h(v_1,v_2) = \tfrac{1}{3}\operatorname{Tr}(X_1X_2) = \tfrac{1}{3}\operatorname{Tr}(X_2X_1),$$

$$\begin{aligned}6h(K(v_1,v_2),v_3) &= \tfrac{1}{3}(\operatorname{Tr} X_1X_2X_3 + \operatorname{Tr} X_3X_1X_2 + \operatorname{Tr} X_2X_3X_1 + \\ &\quad + \operatorname{Tr} X_1X_3X_2 + \operatorname{Tr} X_3X_2X_1 + \operatorname{Tr} X_2X_1X_3) \\ &= (\operatorname{Tr} X_1X_2X_3 + \operatorname{Tr} X_2X_1X_3).\end{aligned}$$

So we see that

$$K(v_1,v_2) = \tfrac{1}{2}(X_1X_2 + X_2X_1 - \tfrac{2}{3}\operatorname{Tr}(X_1X_2)I).$$

Indeed, we have that $(X_1X_2 + X_2X_1 - \frac{2}{3}\operatorname{Tr}(X_1,X_2)I)$ has vanishing trace and therefore $K(v_1,v_2)$ is indeed the unique tangent vector such that

$$h(K(v_1,v_2),v_3) = \tfrac{1}{6}(\operatorname{Tr}(X_1X_2X_3) + \operatorname{Tr}(X_2X_1X_3)).$$

As by Cayley Hamilton for a matrix X with vanishing trace we have that $X^3 = 1/2\operatorname{Tr}(X^2)X + det(X)I$, we deduce that

$$\operatorname{Tr} X^4 = \tfrac{1}{2}(\operatorname{Tr} X^2)^2,$$

and therefore we have that

$$\begin{aligned}h(K(v,v),K(v,v)) &= \tfrac{1}{3}\operatorname{Tr}(X^2 - \tfrac{1}{3}\operatorname{Tr} X^2 I)^2 \\ &= \tfrac{1}{3}(\operatorname{Tr} X^4 + \tfrac{1}{9}(TrX^2)^2 TrI - \tfrac{2}{3}(\operatorname{Tr} X^2)^2) \\ &= \tfrac{1}{18}(TrX^2)^2 \\ &= \tfrac{1}{2}(h(v,v))^2.\end{aligned}$$

Hence M_2 is isotropic with positive λ. A straightforward computation also shows that the index of the metric is 3. Combining therefore the results in this section with Proposition 9 and the classification result of O. Birembaux and M. Djoric, see [5] in the positive definite case, we get theorem 43.

3.5 Affine hyperspheres of dimension 14

3.5.1 The form of L, $\dim \mathcal{U} = 4$

We start with $w_1 \in \mathcal{W}$ a vector with length 1. As $L(w_1 + iTw_1, w_1 - iTw_1)$ is a real vector in U with length $3\mu^2$ there exists a real unit length vector u_1 in U such that

$$L(w_1 + iTw_1, w_1 - iTw_1) = \sqrt{3}\mu u_1.$$

We now complete u_1 to a basis of $\mathcal{U}$ by choosing orthogonal u_2, u_3, u_4 in $\{u_1\}^\perp$ such that $h(u_k,u_k) = \varepsilon_k$, where $\varepsilon_k = \pm 1$. We also introduce δ_k, for k=2,3,4, by

$$\delta_k = i, \qquad \text{if } \varepsilon_k = -1 \quad \text{and} \quad \delta_k = 1, \qquad \text{if } \varepsilon_k = 1.$$

Now we apply Proposition 5, which tells us that we can find vectors w_2, w_3, w_4 such that

$$\begin{aligned}L(w_1 + iTw_1, w_2 - iTw_2) &= \sqrt{3}\mu\delta_2 u_2, \\ L(w_1 + iTw_1, w_3 - iTw_3) &= \sqrt{3}\mu\delta_3 u_3, \\ L(w_1 + iTw_1, w_4 - iTw_4) &= \sqrt{3}\mu\delta_4 u_4.\end{aligned}$$

The first two properties of Proposition 5 then tells us that $\{w_1, Tw_1, \ldots, w_4, Tw_4\}$ is a basis of W, as in Lemma 7. Of course the previous equations also imply that

$$L(w_k + iTw_k, w_1 - iTw_1) = \sqrt{3}\mu\bar{\delta}_k u_k.$$

We now look at $L(w_2 + iTw_2, w_3 - iTw_3)$. From the last part of Proposition 5 it follows that this vector is orthogonal to $L(w_2 + iTw_2, w_1 - iTw_1)$, $L(w_1 + iTw_1, w_3 - iTw_3)$ and $L(w_1 + iTw_1, w_1 - iTw_1)$. So this implies that there exists a complex number b_4 such that

$$L(w_2 + iTw_2, w_3 - iTw_3) = b_4 u_4.$$

Similarly, we have that

$$\begin{aligned} L(w_2 + iTw_2, w_4 - iTw_4) &= b_3 u_3, \\ L(w_3 + iTw_3, w_4 - iTw_4) &= b_2 u_2. \end{aligned}$$

Using again Proposition 5 we see that there exists real numbers c_k such that

$$h(L(w_k + iTw_k, w_k - iTw_k), L(w_k + iTw_k, w_k - iTw_k)) = c_k u_k.$$

From

$$\begin{aligned} h(L(w_k + iTw_k, w_k - iTw_k), L(w_1 + iTw_1, w_1 - iTw_1)) = \\ - h(L(w_k + iTw_k, w_1 - iTw_1), L(w_1 + iTw_1, w_k - iTw_k)), \end{aligned} \quad (3.35)$$

it follows that $c_k = -\sqrt{3}\mu\varepsilon_k$. Next we use the fact that for different indices k and ℓ we have that

$$\begin{aligned} h(L(w_k + iTw_k, w_k - iTw_k), L(w_\ell + iTw_\ell, w_\ell - iTw_\ell)) = \\ - h(L(w_k + iTw_k, w_\ell - iTw_\ell), L(w_\ell + iTw_\ell, w_k - iTw_k)). \end{aligned} \quad (3.36)$$

Expressing this for the different possibilities for k and ℓ we find that

$$\begin{aligned} 3\mu^2\varepsilon_2\varepsilon_3 &= -|b_4|^2\varepsilon_4, \\ 3\mu^2\varepsilon_2\varepsilon_4 &= -|b_3|^2\varepsilon_3, \\ 3\mu^2\varepsilon_4\varepsilon_3 &= -|b_2|^2\varepsilon_4. \end{aligned}$$

Hence, up to permuting the vectors, we see that there are two possibilities. Either $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = -1$, in which case the index of the metric is 8 or $\varepsilon_2 = -1$ and $\varepsilon_3 = \varepsilon_4 = 1$, in which case the index of the metric is 6.

Computing the length of $L(w_2 + iTw_2, w_3 - iTw_3)$ we have in both cases that $b_4^2\varepsilon_4 = 3\mu^2$. So if necessary, by changing the sign of u_4 and w_4, we may assume that $b_4 = \sqrt{3}\mu$. We now complete the argument by looking at

$$\begin{aligned} h(L(w_2 + iTw_2, w_3 - iTw_3), L(w_1 + iTw_1, w_4 - iTw_4) = \\ - h(L(w_1 + iTw_1, w_3 - iTw_3), L(w_2 + iTw_2, w_4 - iTw_4). \end{aligned} \quad (3.37)$$

This yields that $b_3 = -\sqrt{3}\mu$. Interchanging the indices 2 and 3 in the formula above finally gives that $b_2 = -\sqrt{3}\mu$ in the first case, and $-\sqrt{3}\mu i$ in the second case.

3.5.2 Two canonical examples

First we look at the following example. We identify $\mathbb{R}^{15}$ with the set of all skew symmetric matrices in $\mathbb{R}^{6\times 6}$. So an element $p \in \mathbb{R}^{15}$ is of the form

$$p = \begin{pmatrix} 0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ -a_1 & 0 & a_6 & a_7 & a_8 & a_9 \\ -a_2 & -a_6 & 0 & a_{10} & a_{11} & a_{12} \\ -a_3 & -a_7 & -a_{10} & 0 & a_{13} & a_{14} \\ -a_4 & -a_8 & -a_{11} & -a_{13} & 0 & a_{15} \\ -a_5 & -a_9 & -a_{12} & -a_{14} & -a_{15} & 0 \end{pmatrix}.$$

We take as hypersurface M in $\mathbb{R}^{15}$ the skew symmetric matrices with determinant 1. Let $G = SL(6,\mathbb{R})$. Then, we have an action ρ of G on M by $\rho(g)(p) = gpg^T$ Here we are interested in the connected component of the matrix

$$I_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If necessary, we restrict now M to the orbit of I_0. Its isotropy group consists of the matrices g of determinant 1 such that $gI_0g^T = I_0$. This Lie group is congruent to $Sp(6)$ and therefore by Theorem 9.2 of [8] we know that M is locally isometric with $\frac{SL(6,\mathbb{R})}{Sp(6)}$.

Note that of course every element of $SL(6,\mathbb{R})$ acts at the same time also on $\mathbb{R}^{15}$ in a linear way and that therefore this action belongs to $GL(15,\mathbb{R})$. A straightforward computation shows that this action actually belongs to $SL(15,\mathbb{R})$. This implies that M is at the same time an homogeneous affine hypersurface and by Proposition 2 an equiaffine sphere centered at the origin. So in order to determine the properties of M it is sufficient to look at a single point.

In order to determine the tangent space at a point $p = gI_0g^T$, we look at the curves in M

$$\gamma(s) = ge^{sX} I_0 e^{sX^T} g^T.$$

These are indeed curves in M, provided that $e^{sX} \in SL(6,\mathbb{R})$ or equivalently, provided that $\operatorname{Tr} X = 0$. Note that $\gamma'(s) = ge^{sX}(XI_0 + I_0X^T)e^{sX^T}g^T$, where $v = (XI_0 + I_0X^T)$ is a symmetric matrix. So by using a dimension argument we see that the tangent space is given by

$$\{gvg^T | v = 2XI_0, XI_0 = I_0X^T, \operatorname{Tr} X = 0, X \in \mathbb{R}^{6\times 6}\} = T_pM.$$

In fact, such a matrix X is of the form

$$X = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & 0 & -a_5 \\ c_1 & c_2 & -a_1 - b_2 & 0 & -b_4 & -a_4 \\ d_1 & d_2 & 0 & -a_1 - b_2 & b_3 & a_3 \\ e_1 & 0 & -d_2 & c_2 & b_2 & a_2 \\ 0 & -e_1 & -d_1 & c_1 & b_1 & a_1 \end{pmatrix}.$$

Working now at the point I_0, taking $g = I$ and $X \in \{X \in \mathbb{R}^{6\times 6} | \operatorname{Tr} X = 0, XI_0 = I_0 X^T\}$ we see that

$$\begin{aligned}
\nabla_{\gamma'(s)}\gamma'(s) + h(\gamma', \gamma')\gamma &= \gamma''(s) \\
&= e^{sX}(4X^2 I_0)e^{sX^T} \\
&= e^{sX}((4X^2 - \tfrac{4}{6}\operatorname{Tr}(X^2)I)I_0)e^{sX^T} + \tfrac{4}{6}\operatorname{Tr}(X^2)e^{sX}I_0 e^{sX^T} \\
&= e^{sX}((4X^2 - \tfrac{4}{6}\operatorname{Tr}(X^2)I)I_0)e^{sX^T} + \tfrac{4}{6}\operatorname{Tr}(X^2)\gamma(s).
\end{aligned}$$

As the matrix $(4X^2 - \frac{4}{6}\operatorname{Tr}(X^2)I)$ commutes with I_0, we can decompose the above expression into a tangent and a part in the direction of the affine normal given by the position vector, and therefore we find that

$$h(\gamma'(s), \gamma'(s)) = \tfrac{4}{6}\operatorname{Tr}(X^2).$$

So we see that s is a constant length parametrisation of the curve γ and therefore we have that $h(\gamma', \hat{\nabla}_{\gamma'}\gamma') = 0$ and

$$h(\gamma', \nabla_{\gamma'}\gamma') = h(\gamma', K(\gamma', \gamma')).$$

As

$$\begin{aligned}
\gamma'''(s) &= \nabla_{\gamma'(s)}\nabla_{\gamma'(s)}\gamma'(s) + h(\gamma', \gamma')\gamma' + h(\gamma', K(\gamma', \gamma'))\gamma \\
&= e^{sX}(8X^3 I_0)e^{sX^T} \\
&= e^{sX}((8X^3 - \tfrac{8}{6}\operatorname{Tr}(X^3)I)I_0)e^{sX^T} + \tfrac{8}{6}\operatorname{Tr}(X^3)\gamma(s),
\end{aligned}$$

working at $s = 0$ and writing $v = 2XI_0$ as tangent vector, we have that

$$\begin{aligned}
h(v, v) &= \tfrac{4}{6}\operatorname{Tr}(X^2), \\
h(v, K(v, v)) &= \tfrac{8}{6}\operatorname{Tr} X^3.
\end{aligned}$$

Linearising the above expressions, i.e. writing $v = \alpha_1 v_1 + \alpha_2 v_2$, respectively $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$, for $v_i = 2X_i I_0, i = 1, 2, 3$, and looking at the coefficient of $\alpha_1\alpha_2$, respectively $\alpha_1\alpha_2\alpha_3$, we obtain that

$$\begin{aligned}
h(v_1, v_2) &= \tfrac{4}{6}\operatorname{Tr}(X_1 X_2) = \tfrac{4}{6}\operatorname{Tr}(X_2 X_1), \\
6h(K(v_1, v_2), v_3) &= \tfrac{8}{3}(\operatorname{Tr} X_1 X_2 X_3 + \operatorname{Tr} X_3 X_1 X_2 + \operatorname{Tr} X_2 X_3 X_1 + \\
&\quad + \operatorname{Tr} X_1 X_3 X_2 + \operatorname{Tr} X_3 X_2 X_1 + \operatorname{Tr} X_2 X_1 X_3) \\
&= 4(\operatorname{Tr} X_1 X_2 X_3 + \operatorname{Tr} X_2 X_1 X_3).
\end{aligned}$$

So we see that

$$K(v_1, v_2) = 2(X_1 X_2 + X_2 X_1 - \tfrac{2}{6}\operatorname{Tr}(X_1 X_2)I)I_0.$$

Indeed we have that $(X_1 X_2 + X_2 X_1 - \frac{2}{6}\operatorname{Tr}(X_1 X_2)I)$ has vanishing trace, commutes with I_0 and therefore $K(v_1, v_2)$ is indeed the unique tangent vector such that

$$h(K(v_1, v_2), v_3) = \tfrac{2}{3}(\operatorname{Tr}(X_1 X_2 X_3) + \operatorname{Tr}(X_2 X_1 X_3)).$$

By straightforward computations we deduce that

$$\operatorname{Tr} X^4 = \tfrac{1}{4}(\operatorname{Tr} X^2)^2,$$

and therefore we have that

$$\begin{aligned} h(K(v,v),K(v,v)) &= \tfrac{4}{6}\operatorname{Tr}(2X^2 - \tfrac{2}{3}\operatorname{Tr}X^2 I)^2 \\ &= \tfrac{4}{6}(4\operatorname{Tr}X^4 + \tfrac{4}{9}(TrX^2)^2 TrI - \tfrac{8}{3}(\operatorname{Tr}X^2)^2) \\ &= \tfrac{2}{9}(TrX^2)^2 \\ &= \tfrac{1}{2}(h(v,v))^2. \end{aligned}$$

Hence M is isotropic with positive λ. A straightforward computation also shows that the index of the metric is 6.

Next, the following example ilustrates the case when the signature of the indefinite metric on M^{14} is 8. First we identify $\mathbb{R}^{15}$ with the set of matrices $a = \left\{ \begin{pmatrix} E & F \\ -\bar{F} & \bar{E} \end{pmatrix}, E = \bar{E}^T, F = -F^T \right\} \subset \mathbb{C}^{6\times 6}$. An element in a is of the form

$$p := \begin{pmatrix} a_1 & a_2 + ia_3 & a_4 + ia_5 & 0 & a_6 + ia_7 & a_8 + ia_9 \\ a_2 - ia_3 & a_{10} & a_{11} + ia_{12} & -a_6 - ia_7 & 0 & a_{13} + ia_{14} \\ a_4 - ia_5 & a_{11} + ia_{12} & a_{15} & -a_8 - ia_9 & -a_{13} - ia_{14} & 0 \\ 0 & -a_6 + ia_7 & -a_8 + ia_9 & a_1 & a_2 - ia_3 & a_4 - ia_5 \\ a_6 - ia_7 & 0 & -a_{13} + ia_{14} & a_2 + ia_3 & a_{10} & a_{11} - ia_{12} \\ a_8 - ia_9 & a_{13} - ia_{14} & 0 & a_4 + ia_5 & a_{11} + ia_{12} & a_{15} \end{pmatrix}.$$

We take as hypersurface M in $\mathbb{R}^{15}$ all such matrices with determinant 1. Let $G = SU^*(6)$. Then, we have an action ρ of G on M by $\rho(g)(p) = gp\bar{g}^T$. Note that M has two connected components and that the action is transitive on each of the connected components. The connected component of I has been studied in [5], where it was shown that it gives a positive definite isotropic affine hypersurface. Here we are interested in the connected component M_1 containing the matrix

$$I_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Its isotropy group consists of the matrices g of determinant 1 such that $gI_0\bar{g}^T = I_0$. This Lie group is congruent to $Sp(1,2)$ and therefore, by Theorem 9.2 of [8], we know that M_1 is locally isometric with $\frac{SU^*(6)}{Sp(1,2)}$.

Note that of course every element of $SU^*(6)$ acts at the same time also on $\mathbb{R}^{15}$ in a linear way and that therefore this action belongs to $GL(15,\mathbb{R})$. A straightforward computation shows that this action actually belongs to $SL(15,\mathbb{R})$. This implies that M_1 is at the same time an homogeneous affine hypersurface and, by Proposition 2, an equiaffine sphere centered at the origin. So in order to determine the properties of M_1, it is sufficient to look at a single point.

In order to determine the tangent space at a point $p = gI_0\bar{g}^T$, we look at the curves in M_1

$$\gamma(s) = ge^{sX} I_0 e^{s\bar{X}^T} \bar{g}^T.$$

These are indeed curves in M_1, provided that $\operatorname{Tr} X = 0$ and $XJ = J\bar{X}$, for $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Note that $\gamma'(s) = ge^{sX}(XI_0 + I_0\bar{X}^T)e^{s\bar{X}^T}\bar{g}^T$. So by using a dimension argument, we see that the tangent space is given by

$$\{gv\bar{g}^T | v = 2XI_0, XI_0 = I_0\bar{X}^T, \operatorname{Tr} X = 0, XJ = J\bar{X}, X \in \mathbb{C}^{6\times 6}\} = T_pM_1.$$

In fact, such an X if of the form

$$\begin{pmatrix} -x-x_0 & x_1+iy_1 & x_2+iy_2 & 0 & x_3-iy_3 & x_4-iy_4 \\ x_1-iy_1 & x & x_5+iy_5 & -x_3+iy_3 & 0 & -x_6+iy_6 \\ -x_2+iy_2 & -x_5+iy_5 & x_0 & x_4-iy_4 & x_6-iy_6 & 0 \\ 0 & -x_3-iy_3 & -x_4-iy_4 & -x-x_0 & x_1-iy_1 & -x_2-iy_2 \\ x_3+iy_3 & 0 & -x_6-iy_6 & x_1+iy_1 & x & -x_5+iy_5 \\ x_4+iy_4 & x_6+iy_6 & 0 & x_2+iy_2 & x_5+iy_5 & x_0 \end{pmatrix}.$$

Working now at the point I_0, taking $g = I$ and $X \in \mathbb{C}^{6\times 6}$ satisfying $XI_0 = I_0\bar{X}^T, \operatorname{Tr} X = 0, XJ = J\bar{X}$, we see that

$$\begin{aligned} \nabla_{\gamma'(s)}\gamma'(s) + h(\gamma',\gamma')\gamma &= \gamma''(s) \\ &= e^{sX}(4X^2I_0)e^{s\bar{X}^T} \\ &= e^{sX}((4X^2 - \tfrac{4}{6}\operatorname{Tr}(X^2)I)I_0)e^{s\bar{X}^T} + \tfrac{4}{6}\operatorname{Tr}(X^2)e^{sX}I_0e^{s\bar{X}^T} \\ &= e^{sX}((4X^2 - \tfrac{4}{6}\operatorname{Tr}(X^2)I)I_0)e^{s\bar{X}^T} + \tfrac{4}{6}\operatorname{Tr}(X^2)\gamma(s). \end{aligned}$$

As the matrix $(4X^2 - \frac{4}{6}\operatorname{Tr}(X^2)I)$ has the same properties as X, we can decompose the above expression into a tangent part and a part in the direction of the affine normal given by the position vector, and therefore we find that

$$h(\gamma'(s),\gamma'(s)) = \tfrac{4}{6}\operatorname{Tr}(X^2).$$

So we see that s is a constant length parametrisation of the curve γ and therefore we have that $h(\gamma', \hat{\nabla}_{\gamma'}\gamma') = 0$ and

$$h(\gamma', \nabla_{\gamma'}\gamma') = h(\gamma', K(\gamma',\gamma')).$$

As

$$\begin{aligned} \gamma'''(s) &= \nabla_{\gamma'(s)}\nabla_{\gamma'(s)}\gamma'(s) + h(\gamma',\gamma')\gamma' + h(\gamma', K(\gamma',\gamma'))\gamma \\ &= e^{sX}(8X^3I_0)e^{s\bar{X}^T} \\ &= e^{sX}((8X^3 - \tfrac{8}{6}\operatorname{Tr}(X^3)I)I_0)e^{s\bar{X}^T} + \tfrac{8}{6}\operatorname{Tr}(X^3)\gamma(s), \end{aligned}$$

working at $s = 0$ and writing $v = 2XI_0$ as tangent vector, we have that

$$\begin{aligned} h(v,v) &= \tfrac{4}{6}\operatorname{Tr}(X^2), \\ h(v,K(v,v)) &= \tfrac{8}{6}\operatorname{Tr} X^3. \end{aligned}$$

Linearising the above expressions, i.e. writing $v = \alpha_1v_1 + \alpha_2v_2$, respectively $v = \alpha_1v_1 + \alpha_2v_2 + \alpha_3v_3$, for $v_i = 2X_iI_0, i = 1,2,3$, and looking at the coefficient of $\alpha_1\alpha_2$, respectively $\alpha_1\alpha_2\alpha_3$ we obtain that

$$h(v_1,v_2) = \tfrac{4}{6}\operatorname{Tr}(X_1X_2) = \tfrac{4}{6}\operatorname{Tr}(X_2X_1),$$

$$\begin{aligned} 6h(K(v_1,v_2),v_3) &= \tfrac{8}{3}(\operatorname{Tr} X_1X_2X_3 + \operatorname{Tr} X_3X_1X_2 + \operatorname{Tr} X_2X_3X_1 + \\ &\quad + \operatorname{Tr} X_1X_3X_2 + \operatorname{Tr} X_3X_2X_1 + \operatorname{Tr} X_2X_1X_3) \\ &= 4(\operatorname{Tr} X_1X_2X_3 + \operatorname{Tr} X_2X_1X_3). \end{aligned}$$

So we see that

$$K(v_1,v_2) = 2(X_1X_2 + X_2X_1 - \tfrac{2}{6}\operatorname{Tr}(X_1X_2)I)I_0.$$

Indeed we have that $(X_1X_2 + X_2X_1 - \frac{2}{6}\operatorname{Tr}(X_1X_2)I)$ has vanishing trace, commutes with I_0 and therefore $K(v_1,v_2)$ is indeed the unique tangent vector such that

$$h(K(v_1,v_2),v_3) = \tfrac{2}{3}(\operatorname{Tr}(X_1X_2X_3) + \operatorname{Tr}(X_2X_1X_3)).$$

By straightforward computations we deduce that

$$\operatorname{Tr} X^4 = \tfrac{1}{4}(\operatorname{Tr} X^2)^2,$$

and therefore we have that

$$\begin{aligned} h(K(v,v),K(v,v)) &= \tfrac{4}{6}\operatorname{Tr}(2X^2 - \tfrac{2}{3}\operatorname{Tr} X^2 I)^2 \\ &= \tfrac{4}{6}(4\operatorname{Tr} X^4 + \tfrac{4}{9}(TrX^2)^2 TrI - \tfrac{8}{3}(\operatorname{Tr} X^2)^2) \\ &= \tfrac{2}{9}(TrX^2)^2 \\ &= \tfrac{1}{2}(h(v,v))^2. \end{aligned}$$

Hence M_1 is isotropic with positive λ. A straightforward computation also shows that the index of the metric is 8.

3.6 Affine hyperspheres of dimension 26

3.6.1 The form of L, $\dim\mathcal{U} = 8$

Before treating each case of the signature for the metric, we first will give some lemmas which will be very useful in order to simplify the proof significantly. We start with an arbitrary vector $w + iTw \in \mathcal{W}_1$ with length 2 and define a real vector u_1 such that

$$\sqrt{3}\mu u = L(w + iTw, w - iTw). \tag{3.38}$$

We call $w_1^1 = w$ and $w_2^1 = Tw$. Next, we choose arbitrary orthogonal vectors $u_2,\ldots,u_8$ such that $u_1,u_2,\ldots,u_8$ forms an orthonormal (real) basis in $\mathcal{U}$, that is $h(u_j,u_k) = \varepsilon_j\delta_{jk}$, where $\varepsilon_j = \pm 1$ indicate the length of u_j. As the operator $L(\omega_1^1 + i\omega_2^1, -)$ is bijective, for every u_j we find ω_j^1, ω_j^2, such that

$$L(\omega_1^1 + i\omega_2^1, \omega_1^j - i\omega_2^j) = \sqrt{3}\mu\delta_j u_j, \text{ where } \delta_j = \begin{cases} 1, \text{ if } \varepsilon = 1 \\ i, \text{ if } \varepsilon = -1. \end{cases} \tag{3.39}$$

Lemma 13. *For the previously defined vectors, L satisfies*

$$L(\omega_1^k + i\omega_2^k, \omega_1^k - i\omega_2^k) = -\sqrt{3}\mu\varepsilon_k u_1.$$

Proof. The result is straightforward, by properties (3.20) and (3.21):

$$\begin{aligned} h(L(\omega_1^k + i\omega_2^k, \omega_1^k - i\omega_2^k), L(\omega_1^1 + i\omega_2^1, \omega_1^j - i\omega_2^j)) &= \\ &= -h(L(\omega_1^1 + i\omega_2^1, \omega_1^k - i\omega_2^k), L(\omega_1^k + i\omega_2^k, \omega_1^j - i\omega_2^j)) \\ &= -\frac{\delta}{\bar{\delta}} h(L(\omega_1^k + i\omega_2^k, \omega_1^1 - i\omega_2^1), L(\omega_1^k + i\omega_2^k, \omega_1^j - i\omega_2^j)) \\ &= -\frac{\delta}{\bar{\delta}}\frac{3\mu^2}{2} h(\omega_1^1 + i\omega_2^1, \omega_1^j - i\omega_2^j)) \\ &= \begin{cases} 0, & j \neq 1 \\ -3\mu^2\varepsilon_k, & j = 1. \end{cases} \end{aligned} \tag{3.40}$$

□

Lemma 14. *Let u_j and u_k determine ε_j and ε_k such that $\varepsilon_j = \varepsilon_k$, for $k, j > 1$. Then $L(\omega_1^k + i\omega_2^k, \omega_1^j - i\omega_2^j)$ is an imaginary vector.*

Proof. Let us define the orthonormal basis of $\mathcal{U}$ given by

$$\begin{cases} u_k^* = \cos(t)u_k + \sin(t)u_j, \\ u_j^* = -sin(t)u_k + cos(t)u_k, \\ u_l^* = u_l, l \neq k, j. \end{cases}$$

By relation (3.39), we compute

$$L(\omega_1^1 + i\omega_2^1, \cos(t)(\omega_1^k - i\omega_2^k) + \sin(t)(\omega_1^j - i\omega_2^j)) = \sqrt{3}\mu\delta_j(\cos(t)u_k + \sin(t)u_j)$$

and therefore we find $\omega_1^{*k} + i\omega_2^{*k} = \cos(t)(\omega_1^k + i\omega_2^k) + \sin(t)(\omega_1^j + i\omega_2^j)$ and $\omega_1^{*j} + i\omega_2^{*j} = -\sin(t)(\omega_1^k + i\omega_2^k) + \cos(t)(\omega_1^j + i\omega_2^j)$ such that

$$L(\omega_1^1 + i\omega_2^1, \omega_1^{*k} + i\omega_2^{*k}) = \sqrt{3}\delta_k u_k^*.$$

Next, by lemma (13) we may write

$$L(\omega_1^{*k} + i\omega_2^{*k}, \omega_1^{*k} - i\omega_2^{*k}) = -\sqrt{3}\mu\varepsilon_k u_1$$

and using the bilinearity of L, we get the conclusion. □

Lemma 15. *Let u_j and u_k determine ε_j and ε_k such that $\varepsilon_j = -1$ and $\varepsilon_k = 1$, for $k, j > 1$. Then $L(\omega_1^k + i\omega_2^k, \omega_1^j - i\omega_2^j)$ is a real vector.*

Proof. First, define an orthonormal basis of $\mathcal{U}$ given by

$$\begin{cases} u_k^* = \cosh(t)u_k + \sinh(t)u_j, \\ u_j^* = \sinh(t)u_k + \cosh(t)u_k, \\ u*_l = u_l, l \neq k, j \end{cases}$$

and notice that $L(\omega_1^1 + i\omega_2^1, \omega_1^k - i\omega_2^k) = \sqrt{3}\mu u_k$ and $L(\omega_1^1 + i\omega_2^1, \omega_1^j - i\omega_2^j)\sqrt{3}\mu u_j$. We take a, b, c, d complex functions and find $\omega_1^{*k} - i\omega_2^{*k} = a(\omega_1^k - i\omega_2^k) + b(\omega_1^j - i\omega_2^j)$ and $\omega_1^{*j} - i\omega_2^{*j} = c(\omega_1^k - i\omega_2^k) + d(\omega_1^j - i\omega_2^j)$ to be the unique vectors satisfying

$$L(\omega_1^1 + i\omega_2^1, \omega_1^{*k} - i\omega_2^{*k}) = \sqrt{3}\mu u_k^* \quad \text{and} \quad L(\omega_1^1 + i\omega_2^1, \omega_1^{*j} - i\omega_2^{*j})\sqrt{3}\mu u_j^*.$$

Therefore, we find

$$a = \cosh(t), \quad b = i\sinh(t), \quad c = -i\sinh(t) \quad \text{and} \quad d = \cosh(t).$$

Finally, using the bilinearity of L, the conclusion follows easily from

$$L(\omega_1^{*k} + i\omega_2^{*k}, \omega_1^{*k} - i\omega_2^{*k}) = -\sqrt{3}\mu u_1.$$

$\square$

In what follows, we will study different cases depending on the structure of the metric on $\mathcal{U}$. First we deal with the case that the signature of the metric is 4, 5 or 6. Let u_1 be defined as in the beginning of this section. Next, choose $u_2 \perp u_1$ such that $h(u_2, u_2) = -1$, w_1^2 and w_2^2 such that $L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2) = \sqrt{3}\mu u_2$ and $u_3 \perp u_1, u_2$ and w_1^3 and w_2^3 such that $h(u_3, u_3) = -1, L(\omega_1^1 + i\omega_2^1, \omega_1^3 - i\omega_2^3) = \sqrt{3}\mu u_3$. Then, we look at the vector $L(\omega_1^2 + i\omega_2^2, \omega_1^3 - i\omega_2^3)$ and see, by property (3.20), that it is orthogonal to u_1, u_2 and u_3 and has length $3\mu^2$ and by lemma (14), that it is an imaginary vector. Therefore, we define u_4 of length -1 such that

$$L(\omega_1^2 + i\omega_2^2, \omega_1^3 - i\omega_2^3) = \sqrt{3}\mu i u_4.$$

Next, by surjectivity of $L(\omega_1^1 + i\omega_2^1, -)$ and by (3.39) we can pick w_1^4 and w_2^4 such that $L(\omega_1^1 + i\omega_2^1, \omega_1^4 - i\omega_2^4) = \sqrt{3}\mu i u_4$. In the following, we pick $u_5 \perp u_1, u_2, u_3, u_4$ of length 1 and obtain ω_1^5, ω_2^5 such that $L(\omega_1^1 + i\omega_2^1, \omega_1^5 - i\omega_2^5) = \sqrt{3}\mu u_5$. Remark that the vectors $L(\omega_1^2 + i\omega_2^2, \omega_1^5 - i\omega_2^5)$, $L(\omega_1^3 + i\omega_2^3, \omega_1^5 - i\omega_2^5), L(\omega_1^4 + i\omega_2^4, \omega_1^5 - i\omega_2^5)$ are real, of positive length, mutually orthogonal and orthogonal to u_1, u_5. Therefore, the choice of $u_1, \ldots, u_5$ implies that the metric on $\{u_2, u_3, u_4\}^\perp$ is positive definite. Therefore, the cases when the metric has signature $4, 5$ or 6 cannot happen.

In case that the index of the metric is 0, we proceed as follows. Let u_1 be defined as before, choose $u_2 \perp u_1$ of length 1 and obtain the existence of ω_1^2, ω_2^2 such that $L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2) = \sqrt{3}\mu u_2$ and $L(\omega_1^2 + i\omega_2^2, \omega_1^2 - i\omega_2^2) = -\sqrt{3}\mu u_1$. Then, choose $u_3 \perp u_1, u_2$ of length 1 and obtain again $L(\omega_1^1 + i\omega_2^1, \omega_1^3 - i\omega_2^3) = \sqrt{3}\mu u_3$ and $L(\omega_1^3 + i\omega_2^3, \omega_1^3 - i\omega_2^3) = -\sqrt{3}\mu u_1$. Moreover, the vector $L(\omega_1^2 + i\omega_2^2, \omega_1^3 - i\omega_2^3)$ is an imaginary vector, orthogonal on u_1, u_2, u_3 (by relation (3.20)) and therefore, we get the existence of a unit vector of negative length, u_4, such that $L(\omega_1^2 + i\omega_2^2, \omega_1^3 - i\omega_2^3) = \sqrt{3}\mu i u_4$. This contradicts the fact that the index equals 0.

Next, we start anew, with different choices of vectors in order to eliminate the case when the signature of the metric is 1.

Let u_1 be defined as before, choose $u_2 \perp u_1$ of length -1 and obtain the existence of ω_1^2, ω_2^2 such that $L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2) = \sqrt{3}\mu i u_2$ and $L(\omega_1^2 + i\omega_2^2, \omega_1^2 - i\omega_2^2) = \sqrt{3}\mu u_1$. Then, choose $u_3 \perp u_1, u_2$ of length 1 and obtain again $L(\omega_1^1 + i\omega_2^1, \omega_1^3 - i\omega_2^3) = \sqrt{3}\mu u_3$ and $L(\omega_1^3 + i\omega_2^3, \omega_1^3 - i\omega_2^3) = -\sqrt{3}\mu u_1$. Moreover, the vector $L(\omega_1^2 + i\omega_2^2, \omega_1^3 - i\omega_2^3)$ is a real vector, orthogonal on u_1, u_2, u_3 (by relation (3.20)) and therefore, we get the existence of a unit vector of positive length, u_4, such that $L(\omega_1^2 + i\omega_2^2, \omega_1^3 - i\omega_2^3) = \sqrt{3}\mu u_4$. Consequently, $L(\omega_1^1 + i\omega_2^1, \omega_1^4 - i\omega_2^4) = \sqrt{3}\mu u_4$ and $L(\omega_1^4 + i\omega_2^4, \omega_1^4 - i\omega_2^4) = -\sqrt{3}\mu u_1$. Next, we pick $u_5 \perp u_1, u_2, u_3, u_4$ of length 1 and find

$$L(\omega_1^1 + i\omega_2^1, \omega_1^5 - i\omega_2^5) = \sqrt{3}\mu u_5 \quad \text{and} \quad L(\omega_1^5 + i\omega_2^5, \omega_1^5 - i\omega_2^5) = -\sqrt{3}\mu u_1.$$

Finally, by lemma (14) and the property in (3.20), we see that the vectors $L(\omega_1^3 + i\omega_2^3, \omega_1^5 - i\omega_2^5)$ and $L(\omega_1^3 + i\omega_2^3, \omega_1^4 - i\omega_2^4)$ are orthogonal imaginary vectors. This implies that the index of the metric is at least 2.

Now, we will prove that the metric on U cannot have signature 2. Let u_1 be defined as in (3.38), choose $u_2 \perp u_1$ of length -1 and obtain ω_1^2, ω_2^2 such that $L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2) = \sqrt{3}\mu i u_2$ and $L(\omega_1^2 + i\omega_2^2, \omega_1^2 - i\omega_2^2) = \sqrt{3}\mu u_1$. Then, choose $u_3 \perp u_1, u_2$ of length -1 and obtain again $L(\omega_1^1 + i\omega_2^1, \omega_1^3 - i\omega_2^3) = \sqrt{3}\mu i u_3$ and $L(\omega_1^3 + i\omega_2^3, \omega_1^3 - i\omega_2^3) = \sqrt{3}\mu u_1$. Remark now that the vector $L(\omega_1^2 + i\omega_2^2, \omega_1^3 - i\omega_2^3)$ is an imaginary vector, orthogonal on u_1, u_2, u_3 (by relation (3.20)). So we have that $L(\omega_1^2 + i\omega_2^2, \omega_1^3 - i\omega_2^3) = \sqrt{3}\mu i u_4$, where u_4 has negative length and belongs to $\{u_1, u_2, u_3\}^\perp$, where the metric is positive definite, which is a contradiction.

Next we deal with the case that the index of the metric equals 7. So on $\{u_1\}^\perp$ the metric is negative definite. We may take $u_2 \in \mathcal{U}$ such that $h(u_2, u_2) = -1$ and $h(u_1, u_2) = 0$.
As $L(\omega_1^1 + i\omega_2^1, -)$ is a surjective operator, we can pick w_1^1 and $w_2^1 = Tw_1^1$ such that

$$L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2) = \sqrt{3}\mu i u_2, \tag{3.41}$$

$$L(\omega_1^2 + i\omega_2^2, \omega_1^1 - i\omega_2^1) = -\sqrt{3}\mu i u_2. \tag{3.42}$$

By the lemma we have $L(\omega_1^2 + i\omega_2^2, \omega_1^2 - i\omega_2^2) = \sqrt{3}\mu u_1$. Next, we take $u_3 \in \mathcal{U}$ such that $h(u_3, u_3) = -1$. In a similar way as before, we define ω_1^3 and ω_2^3 and obtain

$$L(\omega_1^1 + i\omega_2^1, \omega_1^3 - i\omega_2^3) = \sqrt{3}\mu i u_3. \tag{3.43}$$

By the lemma this implies that $L(\omega_1^3 + i\omega_2^3, \omega_1^3 - i\omega_2^3) = \sqrt{3}\mu u_1$. Next, we find that $L(\omega_1^3 + i\omega_2^3, \omega_1^2 - i\omega_2^2)$ is an imaginary vector which is orthogonal to u_1, u_2 and u_3 such that we may write $L(\omega_1^3 + i\omega_2^3, \omega_1^2 - i\omega_2^2) = \sqrt{3}\mu i u_4$, for some $u_4 \in U$, $u_4 \perp u_1, u_2, u_3$. Given u_4, we define new ω_1^4 and ω_2^4 in $\mathcal{W}_2$ such that $L(\omega_1^1 + i\omega_2^1, \omega_1^4 - i\omega_2^4) = \sqrt{3}\mu i u_4$ and we have $L(\omega_1^4 + i\omega_2^4, \omega_1^4 - i\omega_2^4) = \sqrt{3}\mu u_1$. Next, we want to determine $L(\omega_1^2 + i\omega_2^2, \omega_1^4 - i\omega_2^4)$. We immediately obtain that it is an imaginary vector of length $3\mu^2$ which is orthogonal to u_1, u_2 and u_4. As

$$\begin{aligned} h(L(\omega_1^2 + i\omega_2^2, \omega_1^4 - i\omega_2^4), L(\omega_1^1 + i\omega_2^1, \omega_1^3 - i\omega_2^3)) = \\ - h(L(\omega_1^2 + i\omega_2^2, \omega_1^3 - i\omega_2^3), L(\omega_1^1 + i\omega_2^1, \omega_1^4 - i\omega_2^4)) = 3\mu^2, \end{aligned}$$

it follows from the Cauchy-Schwartz inequality on $\{u_1\}^\perp$ that $L(\omega_1^2 + i\omega_2^2, \omega_1^4 - i\omega_2^4) = \sqrt{3}\mu i u_3$. Similarly it follows that $L(\omega_1^4 + i\omega_2^4, \omega_1^3 - i\omega_2^3) = \sqrt{3}\mu i u_2$.
Remember that so far we have defined u_1, u_2, u_3 and $u_4 \in \mathcal{U}$ and ω_1^1, ω_2^1, ω_1^2, ω_2^2, ω_1^3, ω_2^3 ,$\omega_1^4, \omega_2^4 \in \mathcal{W}$. We take now some arbitrary $u_5 \in \{u_1, u_2, u_3, u_4\}^\perp$ such that $h(u_5, u_5) = -1$ and use again the surjectivity of $L(\omega_1^1 + i\omega_2^1, -)$ to define w_1^5 and $w_2^5 = Tw_1^5$ such that $L(\omega_1^1 + i\omega_2^1, \omega_1^5 - i\omega_2^5) = \sqrt{3}\mu i u_5$ and

$$L(\omega_1^5 + i\omega_2^5, \omega_1^5 - i\omega_2^5) = \sqrt{3}\mu u_1. \tag{3.44}$$

Next, we proceed with the computations as we did, for instance, for $L(\omega_1^3 + i\omega_2^3, \omega_1^2 - i\omega_2^2)$ and define $u_6, u_7, u_8 \in \mathcal{U}$ such that

$$\begin{aligned} L(\omega_1^5 + i\omega_2^5, \omega_1^2 - i\omega_2^2) = \sqrt{3}\mu i u_6, \\ L(\omega_1^5 + i\omega_2^5, \omega_1^3 - i\omega_2^3) = \sqrt{3}\mu i u_7, \\ L(\omega_1^5 + i\omega_2^5, \omega_1^4 - i\omega_2^4) = \sqrt{3}\mu i u_8. \end{aligned}$$

Given u_6, u_7, u_8, we use the surjectivity of $L(\omega_1^1 + i\omega_2^1, -)$ and just like previously done, we define $\omega_1^k, \omega_2^k \in \mathcal{U}$, for $k = 6, 7, 8$ and determine

$$L(\omega_1^1 + i\omega_2^1, \omega_1^k - i\omega_2^k) = \sqrt{3}\mu i u_k.$$

Next, we find $L(\omega_1^k + i\omega_2^k, \omega_1^k - i\omega_2^k) = \sqrt{3}\mu u_1$ for $k = 6, 7, 8$. Then, we compute similarly as for $L(\omega_1^2 + i\omega_2^2, \omega_1^4 - i\omega_2^4)$ in order to determine

$$L(\omega_1^6 + i\omega_2^6, \omega_1^2 - i\omega_2^2) = -\sqrt{3}\mu i u_5. \tag{3.45}$$

As for the vectors $L(\omega_1^3 + i\omega_2^3, \omega_1^6 - i\omega_2^6), L(\omega_1^4 + i\omega_2^4, \omega_1^6 - i\omega_2^6)$ and $L(\omega_1^2 + i\omega_2^2, \omega_1^7 - i\omega_2^7)$, by using property (3.21) and the determined vectors so far, we see they are in the directions of u_8, u_7 and u_8 respectively. We can easily determine their components by following the same procedure as for $L(\omega_1^3 + i\omega_2^3, \omega_1^2 - i\omega_2^2)$. Thus, we may write

$$L(\omega_1^3 + i\omega_2^3, \omega_1^6 - i\omega_2^6) = \varepsilon u_8,$$

$$\begin{aligned} L(\omega_1^4 + i\omega_2^4, \omega_1^6 - i\omega_2^6) &= \varepsilon_1 u_7, \\ L(\omega_1^2 + i\omega_2^2, \omega_1^7 - i\omega_2^7) &= \varepsilon_2 u_8, \end{aligned}$$

where $\varepsilon, \varepsilon_1, \varepsilon_2 = \pm\sqrt{3}\mu i$. Further on, in order to determine $L(\omega_1^3 + i\omega_2^3, \omega_1^7 - i\omega_2^7)$, we first see by property (3.21) that it is orthogonal to $\{u_1, u_2, u_3, u_4, u_7, u_8\}$. Next, as

$$\begin{multline} h(L(\omega_1^3 + i\omega_2^3, \omega_1^7 - i\omega_2^7), L(\omega_1^1 + i\omega_2^1, \omega_1^6 - i\omega_2^6)) + \\ h(L(\omega_1^1 + i\omega_2^1, \omega_1^7 - i\omega_2^7), L(\omega_1^3 + i\omega_2^3, \omega_1^6 - i\omega_2^6)) = 0 \end{multline} \tag{3.46}$$

and

$$\begin{multline} h(L(\omega_1^3 + i\omega_2^3, \omega_1^7 - i\omega_2^7), L(\omega_1^1 + i\omega_2^1, \omega_1^5 - i\omega_2^5)) + \\ h(L(\omega_1^1 + i\omega_2^1, \omega_1^7 - i\omega_2^7), L(\omega_1^3 + i\omega_2^3, \omega_1^5 - i\omega_2^5)) = 0 \end{multline} \tag{3.47}$$

we find

$$L(\omega_1^3 + i\omega_2^3, \omega_1^7 - i\omega_2^7) = \sqrt{3}\mu i u_5. \tag{3.48}$$

It is easy to see that $L(\omega_1^4 + i\omega_2^4, \omega_1^7 - i\omega_2^7)$ is colinear with u_6. From (3.20) we obtain

$$\begin{multline} h(L(\omega_1^1 + i\omega_2^1, \omega_1^6 - i\omega_2^6), L(\omega_1^4 + i\omega_2^4, \omega_1^7 - i\omega_2^7)) + \\ h(L(\omega_1^4 + i\omega_2^4, \omega_1^6 - i\omega_2^6), L(\omega_1^1 + i\omega_2^1, \omega_1^7 - i\omega_2^7)) = 0 \Leftrightarrow \end{multline} \tag{3.49}$$

$$h(u_6, L(\omega_1^4 + i\omega_2^4, \omega_1^7 - i\omega_2^7)) = \varepsilon_1,$$

so that $L(\omega_1^4 + i\omega_2^4, \omega_1^7 - i\omega_2^7) = -\varepsilon_1 u_6$.
Using similar methods we consecutively obtain that

$$\begin{aligned} L(\omega_1^2 + i\omega_2^2, \omega_1^8 - i\omega_2^8) &= -\sqrt{3}\varepsilon_2 \mu i u_7, \\ L(\omega_1^3 + i\omega_2^3, \omega_1^8 - i\omega_2^8) &= -\sqrt{3}\varepsilon \mu i u_7. \end{aligned}$$

Note that by applying (3.20) on

$$h(L(\omega_1^3 + i\omega_2^3, \omega_1^8 - i\omega_2^8), L(\omega_1^2 + i\omega_2^2, \omega_1^5 - i\omega_2^5)),$$

we see that $\varepsilon = -\varepsilon_2$. Using similar arguments, we proceed to find that

$$L(\omega_1^4 + i\omega_2^4, \omega_1^8 - i\omega_2^8) = \sqrt{3}\varepsilon_1\varepsilon_2 \mu i u_5,$$

$$\begin{aligned}
&\varepsilon_1 = \varepsilon_2,\\
&L(\omega_1^5 + i\omega_2^5, \omega_1^6 - i\omega_2^6) = -\sqrt{3}\mu i u_2,\\
&L(\omega_1^5 + i\omega_2^5, \omega_1^7 - i\omega_2^7) = -\sqrt{3}\mu i u_3,\\
&L(\omega_1^5 + i\omega_2^5, \omega_1^8 - i\omega_2^8) = -\sqrt{3}\mu i u_4,\\
&L(\omega_1^6 + i\omega_2^6, \omega_1^7 - i\omega_2^7) = \sqrt{3}\mu i\varepsilon_2 u_4,\\
&L(\omega_1^6 + i\omega_2^6, \omega_1^8 - i\omega_2^8) = -\sqrt{3}\varepsilon_2\mu i u_3,\\
&L(\omega_1^7 + i\omega_2^7, \omega_1^8 - i\omega_2^8) = \sqrt{3}\mu i\varepsilon_2 u_2.
\end{aligned}$$

Moreover it now immediately follows that $\epsilon_2 = 1$.

At last, we will study the solution given by the case when the metric on $\mathcal{U}$ has signature 3. Start with u_1 defined as in (3.39), choose $u_2 \perp u_1$ of length -1 and by surjectivity of $L(\omega_1^1 + i\omega_2^1, -)$ find ω_1^2, ω_2^2 such that $L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2) = \sqrt{3}\mu i u_2$. Similarly, choose $u_3 \perp u_1, u_2$ of length -1 and find $L(\omega_1^1+i\omega_2^1, \omega_1^3-i\omega_2^3) = \sqrt{3}\mu i u_3$. Then, by lemma (14) we can see that the vector $L(\omega_1^2 + i\omega_2^2, \omega_1^3 - i\omega_2^3)$ is imaginary, therefore, it defines a unit vector u_4, of length -1, such that $L(\omega_1^2+i\omega_2^2, \omega_1^3-i\omega_2^3) = \sqrt{3}\mu i u_4$. Moreover, we find the unique vectors ω_1^4 and ω_2^4 such that $L(\omega_1^1+i\omega_2^1, \omega_1^4-i\omega_2^4) = \sqrt{3}\mu i u_4$ and $L(\omega_1^4+i\omega_2^4, \omega_1^4-i\omega_2^4) = \sqrt{3}\mu u_1$. Further on, we see that $L(\omega_1^2+i\omega_2^2, \omega_1^4-i\omega_2^4)$ and $L(\omega_1^3+i\omega_2^3, \omega_1^4-i\omega_2^4)$ are orthogonal to u_1, u_2, u_4 and u_1, u_3, u_4. We compute by property (3.20) $h(L(\omega_1^2+i\omega_2^2, \omega_1^4-i\omega_2^4), L(\omega_1^1+i\omega_2^1, \omega_1^3-i\omega_2^3))$ and $h(L(\omega_1^3+i\omega_2^3, \omega_1^4-i\omega_2^4), L(\omega_1^1+i\omega_2^1, \omega_1^2-i\omega_2^2))$ and, as the metric on $\{u_2, u_3, u_4\}^\perp$ is positive definite, we find

$$L(\omega_1^2 + i\omega_2^2, \omega_1^4 - i\omega_2^4) = -\sqrt{3}\mu i u_3 \quad \text{and} \quad L(\omega_1^3 + i\omega_2^3, \omega_1^4 - i\omega_2^4) = \sqrt{3}\mu i u_2.$$

Next, we choose $u_5 \perp u_1, u_2, u_3, u_4$ of length 1 and find ω_1^5, ω_2^5 such that $L(\omega_1^1+i\omega_2^1, \omega_1^5-i\omega_2^5) = \sqrt{3}\mu u_5$. Then, we notice by property (3.20) that $L(\omega_1^2 + i\omega_2^2, \omega_1^5 - i\omega_2^5)$, $L(\omega_1^3 + i\omega_2^3, \omega_1^5 - i\omega_2^5)$ and $L(\omega_1^4+i\omega_2^4, \omega_1^5-i\omega_2^5)$ are real vectors and satisfy the orthogonality conditions which allow us to pick u_6, u_7, u_8 of length 1, in their directions respectively, and complete $\{u_1, u_2, u_3, u_4\}$ to an orthonormal basis, that is $L(\omega_1^2+i\omega_2^2, \omega_1^5-i\omega_2^5) = \sqrt{3}\mu u_6$, $L(\omega_1^3+i\omega_2^3, \omega_1^5-i\omega_2^5) = \sqrt{3}\mu u_7$ and $L(\omega_1^4 + i\omega_2^4, \omega_1^5 - i\omega_2^5) = \sqrt{3}\mu u_8$. Notice that, by lemmas (14) and property (3.39) we obtain

$$\begin{aligned}
&L(\omega_1^6 + i\omega_2^6, \omega_1^6 - i\omega_2^6) = -\sqrt{3}\mu u_1, \quad L(\omega_1^1 + i\omega_2^1, \omega_1^6 - i\omega_2^6) = \sqrt{3}\mu u_6,\\
&L(\omega_1^7 + i\omega_2^7, \omega_1^7 - i\omega_2^7) = -\sqrt{3}\mu u_1, \quad L(\omega_1^1 + i\omega_2^1, \omega_1^7 - i\omega_2^7) = \sqrt{3}\mu u_7,\\
&L(\omega_1^8 + i\omega_2^8, \omega_1^8 - i\omega_2^8) = -\sqrt{3}\mu u_1, \quad L(\omega_1^1 + i\omega_2^1, \omega_1^8 - i\omega_2^8) = \sqrt{3}\mu u_8.
\end{aligned}$$

In the following, we determine $L(\omega_1^2+i\omega_2^2, \omega_1^6-i\omega_2^6) = -\sqrt{3}\mu u_5$, as it is a real vector of length $3\mu^2$, orthogonal on u_1, u_2, u_3, u_4, u_6, and given that its component in the direction of u_5 is $-\sqrt{3}\mu$ (by property (3.20)). Furthermore, we find $L(\omega_1^2 + i\omega_2^2, \omega_1^7 - i\omega_2^7) = \varepsilon_1\sqrt{3}\mu u_8$, as it is orthogonal to $L(\omega_1^2 + i\omega_2^2, \omega_1^k - i\omega_2^k)$ and $L(\omega_1^1 + i\omega_2^1, \omega_1^7 - i\omega_2^7)$, for $k = 2, \ldots, 6$ and $\varepsilon_1 = \pm 1$. Similarly, we determine for $\varepsilon_j = \pm 1$, $j = 2, \ldots, 8$ the following vectors

$$\begin{aligned}
&L(\omega_1^2 + i\omega_2^2, \omega_1^8 - i\omega_2^8) = \varepsilon_2\sqrt{3}\mu u_7, \quad L(\omega_1^5 + i\omega_2^5, \omega_1^6 - i\omega_2^6) = -i\sqrt{3}\mu u_2\\
&L(\omega_1^3 + i\omega_2^3, \omega_1^6 - i\omega_2^6) = \varepsilon_3\sqrt{3}\mu u_8, \quad L(\omega_1^5 + i\omega_2^5, \omega_1^7 - i\omega_2^7) = -i\sqrt{3}\mu u_3,\\
&L(\omega_1^3 + i\omega_2^3, \omega_1^7 - i\omega_2^7) = -\sqrt{3}\mu u_5, \quad L(\omega_1^5 + i\omega_2^5, \omega_1^8 - i\omega_2^8) = \varepsilon_8\sqrt{3}\mu u_4,\\
&L(\omega_1^3 + i\omega_2^3, \omega_1^8 - i\omega_2^8) = \varepsilon_4\sqrt{3}\mu u_6, \quad L(\omega_1^6 + i\omega_2^6, \omega_1^7 - i\omega_2^7) = -\varepsilon_5 i\sqrt{3}\mu u_4,\\
&L(\omega_1^4 + i\omega_2^4, \omega_1^6 - i\omega_2^6) = \varepsilon_5\sqrt{3}\mu u_7, \quad L(\omega_1^6 + i\omega_2^6, \omega_1^8 - i\omega_2^8) = -\varepsilon_3 i\sqrt{3}\mu u_3,\\
&L(\omega_1^4 + i\omega_2^4, \omega_1^7 - i\omega_2^7) = \varepsilon_6\sqrt{3}\mu u_6, \quad L(\omega_1^7 + i\omega_2^7, \omega_1^8 - i\omega_2^8) = -\varepsilon_1 i\sqrt{3}\mu u_2.\\
&L(\omega_1^4 + i\omega_2^4, \omega_1^8 - i\omega_2^8) = \varepsilon_7\sqrt{3}\mu u_5,
\end{aligned}$$

Then, we can easily find the relations between the coefficients ε_j using property (3.20): $\varepsilon_2 = -\varepsilon_1$, $\varepsilon_4 = -\varepsilon_3$, $\varepsilon_6 = -\varepsilon_5$ and $\varepsilon_7 = -1, \varepsilon_8 = -i$. Moreover, we can find $\varepsilon_1 = -1, \varepsilon_3 = 1$ and $\varepsilon_5 = -1$ by applying property (3.20) successively to $L(\omega_1^6 + i\omega_2^6, \omega_1^7 - i\omega_2^7)$ and $L(\omega_1^2 + i\omega_2^2, \omega_1^3 - i\omega_2^3)$, $L(\omega_1^2 + i\omega_2^2, \omega_1^7 - i\omega_2^7)$ and $L(\omega_1^5 + i\omega_2^5, \omega_1^4 - i\omega_2^4)$, and finally, to $L(\omega_1^3 + i\omega_2^3, \omega_1^8 - i\omega_2^8)$ and $L(\omega_1^2 + i\omega_2^2, \omega_1^5 - i\omega_2^5)$.

3.6.2 Two canonical examples

When the indefinite signature on U is 7, we have the following example.
Let $\mathfrak{h}_3(\mathbb{O})$ denote the set of Hermitian matrices with entries in $\mathbb{O}$, the space of octonions endowed with the Jordan multiplication $\circ$:

$$\mathfrak{h}(\mathbb{O})_3 = \{N \in \mathcal{M}_3(\mathbb{O}) | \bar{N}^T = N\},$$
$$X \circ Y = \frac{1}{2}(XY + YX).$$

By definition, we have that the determinant of $N \in \mathfrak{h}_3(\mathbb{O})$ is given by

$$\det N = \frac{1}{3} Tr(N \circ N \circ N) - \frac{1}{2} Tr(N \circ N) + \frac{1}{6}(TrN)^3.$$

Remark that a matrix $N \in \mathfrak{h}_3(\mathbb{O})$ is of the form $N = \begin{pmatrix} \xi_1 & x_3 & \bar{x_2} \\ \bar{x_3} & \xi_2 & x_1 \\ x_2 & \bar{x_1} & \xi_3 \end{pmatrix}$, where $\xi_i \in \mathbb{R}, x_i \in \mathbb{O}$.
For more details for the space of octonions see [1]. Next, we define $G = \{N \in \mathfrak{h}(\mathbb{O})_3 | det(N) = 1\}$. We take as hypersurface $M_1 = \{\bar{N} A N^T | N \in G\}$ and we define an action of G on M_1 by

$$\rho : G \times M_1 \longrightarrow M_1$$
$$\rho(N) X = \bar{N} X N^T,$$

where $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. By construction, this action is transitive and therefore, by Theorem 9.2 of [8], M_1 is locally isometric with G/H, where $H = \{N \in G | A\bar{N} A N^T = I\}$. Note that $\rho(N)$ can be seen as a linear transformation acting on $\mathbb{R}^{27}$ and a straightforward computation shows that $\rho(N) \in SL(27, \mathbb{R})$. Therefore, M_1 is an homogeneous affine hypersphere in $\mathbb{R}^{27}$. It is now sufficient to work around a point. We introduce local coordinates around a point $p \in M_1$ by taking $y_1, \cdots, y_{26}$ such that $\xi_1 = 1$,

$$\xi_2 = y_1, \quad \xi_3 = y_2, \quad x_1 = \sum_{i=0}^{7} y_{3+i} e_i, \quad x_2 = \sum_{i=0}^{7} y_{11+i} e_i, \quad x_3 = \sum_{i=0}^{7} y_{19+i} e_i,$$

for $\{e_0, \cdots, e_7\}$ a basis of $\mathbb{O}$. Therefore, the parametrization for our hypersurface is given by

$$\begin{cases} F : \mathbb{R}^{26} \to \mathbb{R}^{27} \\ p \longmapsto g(p)^{-\frac{1}{3}}(1, p), \end{cases}$$

where $p = (y_1, \cdots, y_{26})$ and $g(p) := \det N$. By using the multiplication table for octonions, we can determine $g(p)$ and then, straightforward computations around the point $N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ allow us to find that the isotropy condition holds for $\lambda = \frac{1}{2}$. Thus, the signature of the metric on M is 16.

When the indefinite signature on U is 3, we have the following example.

Consider the set of Hermitian matrices with entries in the split-octonions space endowed with the Jordan multiplication $\circ$, as previously defined. Note that for $\{1, i, j, k, li, lj, lk\}$ an orthogonal basis of the split-octonion space, the length of a vector $x = x_0 + x_1 i + x_2 j + x_3 k + x_4 l + x_5 li + x_6 lj + x_7 lk$ is given by

$$h(x, x) = \bar{x}x = (x_0^2 + x_1^2 + x_2^2 + x_3^2) - (x_4^2 + x_5^2 + x_6^2 + x_7^2).$$

We define the manifold in a similar way as in the previous example and, by similar arguments, we get that M is an isotropic affine hypersphere of dimension 26 for which the signature of the metric is 12.

Chapter 4

Translation and homothetical surfaces in Euclidean space with constant curvature

This chapter is based on the results obtained in [35], and presents as well as some general considerations on minimal surfaces from [2], where one may find extra details.

Translation and homothetical surfaces. Two types of surfaces make the object of study of this chapter - translation surfaces and homothetical surfaces. *A translation surface* S is a surface that can be expressed as the sum of two curves $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$, $\beta : J \subset \mathbb{R} \to \mathbb{R}^3$. In a parametric form, the surface S writes as $X(s,t) = \alpha(s) + \beta(t)$, $s \in I, t \in J$. See [15, p. 138]. Similarly, the homothetical surfaces are defined by replacing the plus sign $+$ in the definition of a translation surface by the multiplication operation. That is, *a homothetical surface* S in Euclidean space $\mathbb{R}^3$ is a surface that is a graph of a function $z = f(x)g(y)$, where $f : I \subset \mathbb{R} \to \mathbb{R}$ and $g : J \subset \mathbb{R} \to \mathbb{R}$ are two smooth functions.
A translation surface S has the property that the translations of a parametric curve $s =$ *constant* by $\beta(t)$ remain in S (similarly for the parametric curves $t =$ *constant*).
It is an open problem to classify all translation surfaces with constant mean curvature (CMC) or constant Gauss curvature (CGC). An example of a CMC translation surface is the Scherk surface

$$z(x,y) = \frac{1}{a} \log \left(\left| \frac{\cos(ay)}{\cos(ax)} \right| \right), \ a > 0.$$

This surface is minimal ($H = 0$) and belongs to a more general family of Scherk surfaces ([45, pp. 67-73]). In this case, the curves α and β lie in two orthogonal planes and after a change of coordinates, the surface is locally described as the graph of $z = f(x) + g(y)$. Other examples of CMC or CGC translation surfaces given as a graph $z = f(x) + g(y)$ are: planes ($H = K = 0$), circular cylinders ($H = \mathit{constant} \neq 0$, $K = 0$) and cylindrical surfaces ($K = 0$).

Minimal surfaces. A surface with the property that the mean curvature H vanishes everywhere is called *minimal.* The study of minimal surfaces originates with the work of Lagrange in [31]. He considered surfaces in $\mathbb{R}^3$ that were graphs of C^2-differentiable functions $z = f(x,y)$.

For such surfaces the area element is given by

$$dM = (1 + f_x^2 + f_y^2)^{\frac{1}{2}} dx \wedge dy.$$

He studied the problem of determining a surface of this kind with the least possible area among all surfaces that assume given values on the boundary of an open set U of the plane (with compact closure and smooth boundary). Let $z = f(x, y)$ be a solution for this problem and consider a 1-parameter family of functions $z_t(x, y) = f(x, y) + t\eta(x, y)$, where η is a C^2-function that vanishes on the boundary of U, and we define

$$A(t) = \int_{\bar{U}} (1 + (z_t)_x^2 + (z_t)_y^2)^{1/2}\, dxdy. \tag{4.1}$$

It follows that

$$A(t) = \int_{\bar{U}} [(1 + f_x^2 + f_y^2) + 2t(f_x\eta_x + f_y\eta_y) + t^2(\eta_x^2 + \eta_y^2)]^{1/2}\, dxdy. \tag{4.2}$$

We set $p := f_x$, $q = f_y$ and $w = (1 + p^2 + q^2)^{1/2}$ and derive with respect to t in the above equation. We obtain

$$A'(0) = \int_{\bar{U}} \left(\frac{p}{w}\eta_x + \frac{q}{w}\eta_y\right) dxdy.$$

Next, we integrate by parts and recalling that $\eta_{|\partial\bar{U}} = 0$, we have

$$A'(0) = \int_{\bar{U}} \left[\frac{\partial}{\partial x}\left(\frac{p}{w}\right) + \frac{\partial}{\partial y}\left(\frac{q}{w}\right)\right] \eta dxdy. \tag{4.3}$$

Since $z = f(x, y)$ is a solution for the problem, then $A(0)$ is a minimum for the function $A(t)$ and hence $A'(0) = 0$. This occurs for any function η which vanishes on the boundary of U. It follows that

$$\frac{\partial}{\partial x}\left(\frac{p}{w}\right) + \frac{\partial}{\partial y}\left(\frac{q}{w}\right) = 0,$$

which implies that

$$f_{xx}(1 + f_y^2) - 2f_x f_y f_{xy} + f_{yy}(1 + f_x^2) = 0. \tag{4.4}$$

The solutions of the above equation were called *minimal surfaces*, and they are given by real analytic functions. It was only in 1776 that Meusnier gave a geometrical interpretation for (4.4) as meaning that

$$H = \frac{k_1 + k_2}{2} = 0, \tag{4.5}$$

where k_1 and k_2 stand for the principal curvatures. Meusnier also found *the catenoid* as the only minimal surface of revolution in $\mathbb{R}^3$. In 1835 Scherk discovered another example of minimal surface, by solving the equation (4.4) for functions of the type $f(x, y) = g(x) + h(y)$, which resulted in *the Scherk's minimal surface*. In 1842 Catalan proved that *the helicoid* is the only ruled minimal surface in $\mathbb{R}^3$.

Results obtained. The progress on the problem of classification of translation surfaces with constant mean curvature or constant Gauss curvature has been as follows.

1. If α and β lie in orthogonal planes, the only minimal translation surfaces are the plane and the Scherk surface [50].

2. If α and β lie in orthogonal planes, the only CMC translation surfaces are the plane, the Scherk surface and the circular cylinder [33].

3. If α and β lie in orthogonal planes, the only CGC translation surfaces have $K = 0$ and are cylindrical surfaces [33].

4. If both curves α and β are planar, the only minimal translation surfaces are the plane or a surface which belongs to the family of Scherk surfaces [16].

5. If one of the curves α or β is planar and the other one is not, there are no minimal translation surfaces [16].

The first result presented in this chapter, and therefore in [35] as well, concerns the case when the Gauss curvature K is constant. We prove that, without any assumption on the curves α and β, the only flat ($K = 0$) translation surfaces are cylindrical surfaces. By *a cylindrical surface* we mean a ruled surface whose directrix is contained in a plane and the rulings are parallel to a fixed direction in $\mathbb{R}^3$. The corresponding theorem is the following.

Theorem 49. *1. The only translation surfaces with zero Gauss curvature are cylindrical surfaces (see figure no.4.1).*

2. There are no translation surfaces with constant Gauss curvature $K \neq 0$ if one of the generating curves is planar.

For the case $K = 0$, we give a complete classification of CGC translation surfaces and for $K \neq 0$, we extend the result given in [16] for CMC translation surfaces.

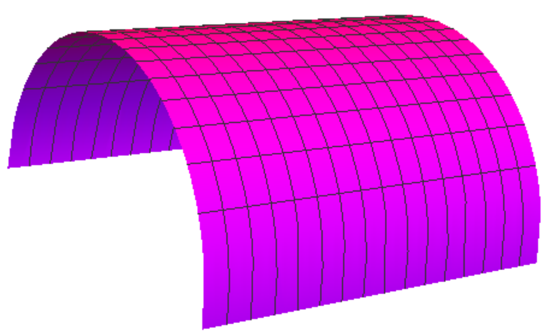


Figure 4.1: A cylindrical surface whose directrix is a semi-circle.

The first approach to this kind of surfaces appeared in [54], when studying minimal homothetical non-degenerate surfaces in Lorentz-Minkowski space $\mathbb{L}^3$ (see also [55]). Some authors have considered minimal homothetical hypersurfaces in Euclidean space and in semi-Euclidean spaces ([29, 55]. The first result concerns minimal surfaces. Van de Woestyne proved in [54] that the only minimal homothetical non-degenerate surfaces in $\mathbb{L}^3$ are planes and helicoids. At the end of [54] the author asserted that, up to small changes in the proof, a similar result can be obtained in Euclidean space $\mathbb{R}^3$. In the present paper we do a different proof of the Euclidean version and in section 4.2 we prove:

Theorem 50. *Planes and helicoids are the only minimal homothetical surfaces in Euclidean space.*

The parametrization of the helicoid is not the usual one as for a ruled surface which has a helix as base, but

$$z(x,y) = (x+b)\tan(cy+d), \tag{4.6}$$

where $b, c, d \in \mathbb{R}$, $c \neq 0$ ([45, p. 20]) (see figure no.4.2).

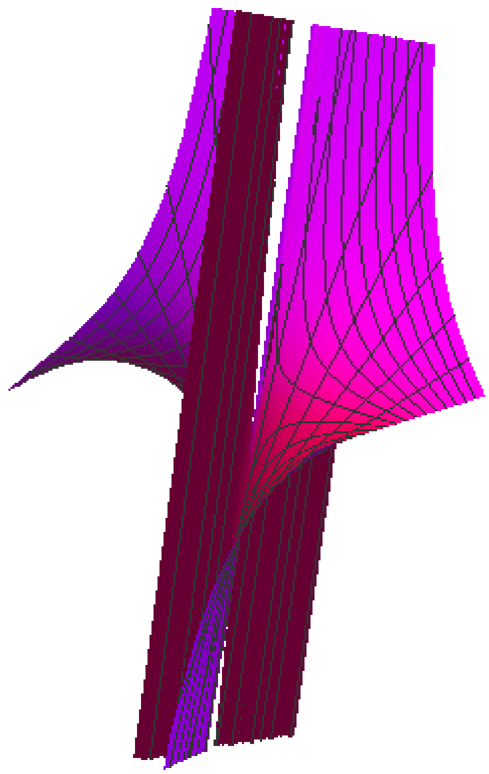


Figure 4.2: A helicoid given by the parametrization in (4.6).

The third result gives a complete classification of homothetical surfaces in Euclidean space with constant Gauss curvature.

Theorem 51. *Let S be a homothetical surface in Euclidean space $\mathbb{R}^3$ with constant Gauss curvature K. Then $K = 0$. Furthermore, the surface is either a plane, a cylindrical surface or a surface whose parametrization is:*

(i)

$$z(x,y) = ae^{bx+cy}, \tag{4.7}$$

where $a, b, c > 0$ (see figure no.4.3), or

(ii)

$$z(x,y) = \left(\frac{bx}{m} + d\right)^m \left(\frac{cy}{m-1} + e\right)^{1-m}, \tag{4.8}$$

with $b, c, d, e, m \in \mathbb{R}$, $b, c \neq 0$, $m \neq 0, 1$ (see figure no.4.4).

This theorem is proved in section 4.3. Finally, in section 4.4 we extend Theorems 49 and 51 in Lorentz-Minkowski space, obtaining similar results.

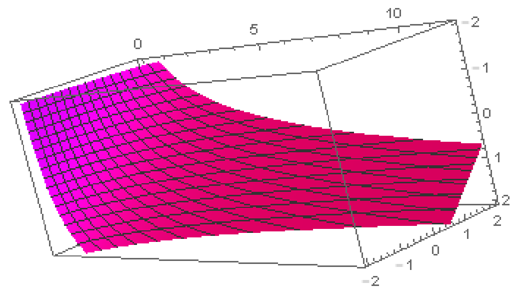


Figure 4.3: A homothetical surface given by the parametrization in Theorem 51.(i)

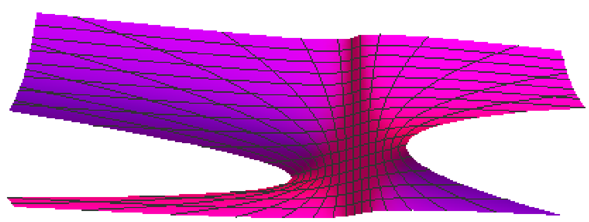


Figure 4.4: A homothetical surface given by the parametrization in Theorem51.(ii).

4.1 Proof of Theorem 49

Throughout this chapter, we consider the rectangular coordinates (x, y, z) of the Euclidean space $\mathbb{R}^3$. Assume S is the sum of the curves $\alpha(s)$ and $\beta(t)$. Locally, α and β are graphs on the axis coordinates of $\mathbb{R}^3$, so we may assume that $\alpha(s) = (s, f_1(s), f_2(s))$ and $\beta(t) = (g_1(t), t, g_2(t))$, $s \in I$, $t \in J$, for some functions f_1, f_2, g_1, g_2. Let us observe that if we replace the functions f_i or g_i by an additive constant, the surface changes by a translation of Euclidean space and thus, in what follows, we will take these functions up to additive constants. The Gauss curvature in local coordinates $X = X(s,t)$ writes as

$$K = \frac{ln - m^2}{EG - F^2},$$

where $\{E, F, G\}$ and $\{l, m, n\}$ are the coefficients of the first and second fundamental form with respect to X, respectively. In our case, the parametrization of S is $X(s,t) = \alpha(s) + \beta(t)$ and as $\partial^2_{st} X = 0$, then $m = 0$. The computation of K leads to

$$K = \frac{\left(f_2'' - f_1'' g_2' + g_1'(f_1'' f_2' - f_1' f_2'')\right)\left(g_2'' - f_2' g_1'' + f_1'(g_1'' g_2' - g_1' g_2'')\right)}{\left((1 + f_1'^2 + f_2'^2)(1 + g_1'^2 + g_2'^2) - (f_1' + g_1' + f_2' g_2')^2\right)^2}. \tag{4.9}$$

4.1.1 Case $K=0$

Then $l=0$ or $n=0$. Assume $l=0$ and the argument is similar if $n=0$. Thus

$$f_2''-f_1''g_2'+g_1'(f_1''f_2'-f_1'f_2'')=0 \tag{4.10}$$

We distinguish several cases.

1. Assume $f_1''=0$. Then $f_1(s)=as$, $a\in\mathbb{R}$, and (4.10) gives $f_2''(1-ag_1')=0$. If $f_2''=0$, then f_2 is linear, proving that the curve α is a straight-line and the surface is a cylindrical surface whose base curve is β (see figure no.4.1). If $f_2''\neq 0$, then $a\neq 0$. Solving for g_1, we obtain $g_1(t)=t/a$. Then $X(s,t)=(s+t/a, as+t, f_2(s)+g_2(t))$ and the surface is the plane of equation $ax-y=0$.

2. Assume $f_1''\neq 0$ and $g_1''=0$. Then $g_1(t)=at$, $a\in\mathbb{R}$, and (4.10) implies

$$\frac{f_2''+a(f_1''f_2'-f_1'f_2'')}{f_1''}=g_2'. \tag{4.11}$$

   As the left hand-side of this equation depends only on s, while the right hand-side only on t, we conclude that both functions in (4.11) must be equal to the same constant $b\in\mathbb{R}$. In particular, $g_2(t)=bt$. Now the curve β is a straight-line and the surface is a cylindrical surface with the curve α as base. Let us notice that under these conditions, equation (4.11) does not add further information on the curve α.

3. Assume $f_1''g_1''\neq 0$. Differentiating (4.10) with respect to t, we have $-f_1''g_2''+g_1''(f_1''f_2'-f_1'f_2'')=0$. With a similar argument as above, one proves that there exists $a\in\mathbb{R}$ such that
$$\frac{f_1''f_2'-f_1'f_2''}{f_1''}=a=\frac{g_2''}{g_1''}.$$
   The identity $g_2''=ag_1''$ implies that $\det(\beta',\beta'',\beta''')=0$ and this means that the torsion of β is 0 identically. This proves that β is a planar curve. Now we come back to the beginning of the proof assuming that β is included in the yz-plane (or equivalently, $g_1=0$). We compute K again obtaining

$$g_2''(f_2''-f_1''g_2')=0.$$

   If $g_2''=0$, then g_2 is linear and β is a straight-line, proving that S is a cylindrical surface with the curve α as base. If $g_2''\neq 0$, then $f_2''-f_1''g_2'=0$ and it follows that there exists $a\in\mathbb{R}$ such that
$$\frac{f_2''}{f_1''}=a=g_2',$$
   and so, $g_2''=0$, a contradiction.

4.1.2 Case $K\neq 0$

We will follow the same ideas as in [16] by distinguishing two cases: first, we suppose that both curves are planar, and second, we assume that only one is planar.

1. Case when α and β are planar curves. By the result of Liu in [33] , we only consider the case when the curves α and β cannot lie in planes mutually orthogonal. Let us notice that if the curves lie in parallel planes, the translation surface is (part) of a plane. Without loss of generality we can assume that α lies in the xz-plane and β in the plane of equation $x\cos\theta - y\sin\theta = 0$, with $\cos\theta, \sin\theta \neq 0$. Then α and β write as

$$\alpha(s) = (s, 0, f(s)),\ \ \beta(t) = (t\sin\theta, t\cos\theta, g(t))$$

with f and g smooth functions on s and t, respectively. The computation of K leads to

$$K = \frac{\cos^2\theta f''g''}{(f'^2 + g'^2 + \cos^2\theta - 2\sin\theta f'g')^2}.$$

Notice that $K \neq 0$ implies $f'', g'' \neq 0$. Differentiating with respect to s and with respect to t, we obtain respectively

$$\cos^2\theta f'''g'' = 4K(f'^2 + g'^2 + \cos^2\theta - 2\sin\theta f'g')(f'f'' - \sin\theta f''g')$$
$$\cos^2\theta f''g''' = 4K(f'^2 + g'^2 + \cos^2\theta - 2\sin\theta f'g')(g'g'' - \sin\theta f'g'').$$

Using $f''g'' \neq 0$, we have

$$\frac{f'''}{f''^2}(g' - \sin\theta f') = \frac{g'''}{g''^2}(f' - \sin\theta g'). \tag{4.12}$$

Differentiating now with respect to s and next with respect to t, we get

$$\left(\frac{f'''}{f''^2}\right)' g'' = f''\left(\frac{g'''}{g''^2}\right)'.$$

Dividing by $f''g''$, we have an identity of two functions, one depending on s and the other one depending on t. Then both functions are equal to the same constant $a \in \mathbb{R}$ and there exist $b, c \in \mathbb{R}$ such that

$$\frac{f'''}{f''^2} = af' + b,\ \ \frac{g'''}{g''^2} = ag' + c.$$

Substituting in (4.12) gives

$$a\sin\theta f'^2 + b\sin\theta f' + cf' = a\sin\theta g'^2 + c\sin\theta g' + bg'.$$

Again we have two functions, one depending on s and other one depending on t. Therefore both functions are constant and hence, f' and g' are constant, which is in contradiction with $f''g'' \neq 0$.

2. Assume that α is a planar curve and β does not lie in a plane. After a change of coordinates, we may suppose

$$\alpha(s) = (s, 0, f(s)),\ \ \beta(t) = (g_1(t), t, g_2(t)),$$

for smooth functions f, g_1 and g_2. The contradiction will arrive proving that β is a planar curve. For this reason, let us first observe that β is planar if and only if its torsion vanishes for all s, that is, $\det(\beta'(t), \beta''(t), \beta'''(t)) = 0$ for all t, or equivalently,

$$g_1'''g_2'' - g_1''g_2''' = 0. \tag{4.13}$$

We compute K obtaining

$$K = \frac{f''(g_2'' - f'g_1'')}{(1 + g_2'^2 + f'^2 + f'^2 g_1'^2 - 2f'g_1'g_2')^2}. \tag{4.14}$$

As $K \neq 0$, we have $f'' \neq 0$. We move f'' to the left hand-side of equation (4.14) and we obtain a function depending only on the variable s. Then the derivative of the right hand-side with respect to t is 0. This means

$$4(f'g_1' - g_2')(f'g_1'' - g_2'')^2 - (f'g_1''' - g_2''')(1 + f'^2(1 + g_1'^2) - 2f'g_1'g_2' + g_2'^2) = 0.$$

For each fixed t, we can view this expression as a polynomial equation on $f'(s)$ and thus, all coefficients vanish. The above equation writes precisely as $\sum_{n=0}^{3} A_n(t)f'(s)^n = 0$. The computations of A_n give:

$$\begin{array}{rcl}
A_0 & = & (1 + g_2'^2)g_2''' - 4g_2'g_2''^2 \\
A_1 & = & 8g_1''g_2'g_2'' + 4g_1'g_2''^2 - (1 + g_2'^2)g_1''' - 2g_1'g_2'g_2''' \\
A_2 & = & -8g_1'g_1''g_2'' - 4g_1''^2 g_2' + 2g_1'g_2'g_1''' + (1 + g_1'^2)g_2''' \\
A_3 & = & -(1 + g_1'^2)g_1''' + 4g_1'g_1''^2.
\end{array}$$

From $A_0 = 0$ and $A_3 = 0$ we get for $i = 1, 2$,

$$(1 + g_i'^2)g_i''' - 4g_i'g_i''^2 = 0, \tag{4.15}$$

or

$$\frac{g_i'''}{g_i''} = \frac{4g_i'g_i''}{1 + g_i'^2}.$$

A first integration leads to

$$g_i'' = \lambda_i(1 + g_i'^2)^2,\ \ \lambda_i > 0, i = 1, 2. \tag{4.16}$$

In particular, from (4.15),

$$g_i''' = 4\lambda_i^2 g_i'(1 + g_i'^2)^3.$$

Before continuing with the information obtained so far, we rewrite the condition (4.13) that β is a planar curve. In terms of g_1' and g_2', and using (4.16), the equation (4.13) is equivalent to

$$\lambda_1 g_1'(1 + g_1'^2) - \lambda_2 g_2'(1 + g_2'^2) = 0. \tag{4.17}$$

From the data obtained for g_i'' and g_i''', we now substitute into the coefficients A_1 and A_2. After some manipulations, the identity $A_1 g_2'(1 + g_1'^2) + A_2 g_1'(1 + g_2'^2) = 0$ simplifies into

$$\left[(\lambda_1 g_2'(1 + g_1'^2) + \lambda_2 g_1'(1 + g_2'^2)\right]\left[\lambda_1 g_1'(1 + g_1'^2) - \lambda_2 g_2'(1 + g_2'^2)\right] = 0.$$

If the right bracket is zero, then β is planar by (4.17), obtaining a contradiction. If the first bracket vanishes, then

$$1 + g_2'^2 = -\frac{\lambda_1}{\lambda_2}\frac{g_2'}{g_1'}(1 + g_1'^2).$$

We place this information together with (4.16) into the coefficient A_1, and we obtain that $A_1 = 0$ is equivalent to the identity

$$g_1'^4 + g_2'^4 + g_1'^2 + g_2'^2 + 2g_1'^2 g_2'^2 = 0.$$

Then $g_1' = g_2' = 0$, that is, the curve β is planar, obtaining a contradiction again. This finishes the proof of Theorem 49 for the case $K \neq 0$.

4.2 Proof of Theorem 50

Assume that S is a homothetical surface which is the graph of $z = f(x)g(y)$ and let $X(x, y) = (x, y, f(x)g(y))$ be a parametrization of S. The computation of $H = 0$ leads to

$$f''g(1 + f^2g'^2) - 2ff'^2gg'^2 + fg''(1 + f'^2g^2) = 0. \tag{4.18}$$

Since the roles of f and g in (4.18) are symmetric, we only discuss the cases according to the function f. We distinguish several cases.

1. Case $f' = 0$. Then $f(x) = \lambda$, $\lambda \in \mathbb{R}$ and (4.18) gives $fg'' = 0$. If $f = 0$, S is the horizontal plane of equation $z = 0$. If $g'' = 0$, then $g(y) = ay + b$, $a, b \in \mathbb{R}$ and $X(x, y)$ parametrizes the plane of equation $\lambda ay - z = \lambda b$.

2. Case $f'' = 0$, $f' \neq 0$, and by symmetry, $g' \neq 0$. Then $f(x) = ax + b$, for $a, b \in \mathbb{R}$, $a \neq 0$. Now (4.18) reduces into
$$-2a^2gg'^2 + g''(1 + a^2g^2) = 0.$$
Then
$$\frac{g''}{g'} = 2a^2\frac{gg'}{1 + a^2g^2}.$$
By integrating, we obtain that there exists a constant $k > 0$ such that
$$g' = k(1 + a^2g^2).$$
Solving this ODE, we get
$$g(y) = \frac{1}{a}\tan(aky + d),\ \ d \in \mathbb{R}.$$
It only remains to conclude that we obtain a helicoid. In such a case, the parametrization of S is
$$X(x, y) = (x, y, f(x)g(y)) = (0, y, bg(y)) + x(1, 0, ag(y)),$$
which indicates that the surface is ruled. But it is well known that the only ruled minimal surfaces in $\mathbb{R}^3$ are planes and helicoids ([2]) and since g is not a constant function, S must be a helicoid (see figure no.7).

3. Case $f'' \neq 0$. We will prove that this case is not possible. By symmetry in the discussion of the case, we also suppose $g'' \neq 0$. If we divide (4.18) by$ff'^2gg'^2$, we have
$$\frac{f''}{ff'^2g'^2} + \frac{f''f}{f'^2} - 2 + \frac{g''}{f'^2gg'^2} + \frac{gg''}{g'^2} = 0.$$
Let us differentiate with respect to x and then with respect to y, and we obtain
$$\left(\frac{f''}{ff'^2}\right)'\left(\frac{1}{g'^2}\right)' + \left(\frac{1}{f'^2}\right)'\left(\frac{g''}{gg'^2}\right)' = 0. \tag{4.19}$$
Since $f''g'' \neq 0$, we divide (4.19) by $(1/g'^2)'(1/f'^2)'$ and we conclude that there exists a constant $a \in \mathbb{R}$ such that
$$\left(\frac{f''}{ff'^2}\right)'\frac{1}{\left(\frac{1}{f'^2}\right)'} = a = -\left(\frac{g''}{gg'^2}\right)'\frac{1}{\left(\frac{1}{g'^2}\right)'}.$$

Hence there are constants $b, c \in \mathbb{R}$ such that

$$\frac{f''}{ff'^2} = a\frac{1}{f'^2} + b, \quad -\frac{g''}{gg'^2} = a\frac{1}{g'^2} + c,$$

or equivalently,

$$f'' = f(a + bf'^2), \quad g'' = -g(a + cg'^2). \tag{4.20}$$

Taking into account (4.20), we replace f'' and g'' in (4.18), obtaining

$$(a + bf'^2)(1 + f^2g'^2) - 2f'^2g'^2 - (a + cg'^2)(1 + f'^2g^2) = 0.$$

If we divide by $f'^2g'^2$, we get

$$\frac{c - af^2}{f'^2} + 2 - bf^2 = \frac{b - ag^2}{g'^2} - cg^2.$$

We use again the fact that each side of this equation depends on x and y respectively, hence there exists $\lambda \in \mathbb{R}$ such that

$$f'^2 = \frac{c - af^2}{\lambda + bf^2 - 2}, \quad g'^2 = \frac{b - ag^2}{\lambda + cg^2}. \tag{4.21}$$

Differentiating with respect to x and y, respectively, we have

$$f'' = -\frac{f(bc + a(\lambda - 2))}{(\lambda + bf^2 - 2)^2}, \quad g'' = -\frac{a\lambda + bc}{(\lambda + cg^2)^2}. \tag{4.22}$$

Let us compare these expressions of f'' and g'' with the ones that appeared in (4.20) and replace the value of f'^2 and g'^2 obtained in (4.21). After some manipulations, we get

$$(bc + a(\lambda - 2))\,(\lambda - 1 + bf^2) = 0.$$

$$(bc + a\lambda)\,(\lambda - 1 + cg^2) = 0.$$

We discuss all possibilities.

(a) If $bc + a(\lambda - 2) = bc + a\lambda = 0$, then $a = 0$ and $bc = 0$. Then (4.22) gives $f'' = 0$ or $g'' = 0$, a contradiction.

(b) If $bc + a(\lambda - 2) = 0$ and $c = \lambda - 1 = 0$, we obtain $a = 0$. From (4.22), we get $g'' = 0$, a contradiction.

(c) If $bc + a\lambda = 0$ and $b = \lambda - 1 = 0$, then $a = 0$ and (4.22) gives $f'' = 0$, a contradiction.

(d) If $b = c = 0$ and $\lambda = 1$, from the expressions of f'^2 and g'^2 in (4.21), we deduce $f'^2 = af^2$ and $g'^2 = -ag^2$, that is, $a = 0$. Then (4.22) gives $f' = g' = 0$, a contradiction again.

4.3 Proof of Theorem 51

The computation of K for the surface $X(x, y) = (x, y, f(x)g(y))$ gives

$$K = \frac{fgf''g'' - f'^2g'^2}{(1 + f'^2g^2 + f^2g'^2)^2}. \tag{4.23}$$

4.3.1 Case $K = 0$

If $K = 0$, then
$$ff''gg'' = f'^2g'^2. \tag{4.24}$$
Since the roles of the functions f and g are symmetric in (4.24), we discuss the different cases according to the function f.

1. Case $f' = 0$. Then f is a constant function $f(x) = x_0$ and the parametrization of the surface writes as $X(x, y) = (0, y, x_0g(y)) + x(1, 0, 0)$. This means that S is a cylindrical surface whose directrix lies in the yz-plane and the rulings are parallel to the x-axis.

2. Case $f'' = 0$ and $f', g' \neq 0$. Now $f(x) = ax + b$, $a, b \in \mathbb{R}$, $a \neq 0$. Moreover, (4.24) gives $g' = 0$ and $g(y) = y_0$ is a constant function. Now S is the plane of equation $z = x_0(ax + b)$.

3. Case $f'' \neq 0$. By the symmetry on the arguments, we also suppose $g'' \neq 0$. Equation (4.24) writes as
$$\frac{ff''}{f'^2} = \frac{g'^2}{gg''}.$$
As in each side of this equation we have a function depending on x and other depending on y, there exists $a \in \mathbb{R}$, $a \neq 0$, such that
$$\frac{ff''}{f'^2} = a = \frac{g'^2}{gg''}.$$
A direct integration implies that there exist $b, c > 0$ such that
$$f' = bf^a, \quad g' = cg^{1/a}.$$
   (a) Case $a = 1$. Then
   $$f(x) = pe^{bx}, g(y) = qe^{cy}, \quad p, q > 0.$$
   (b) Case $a \neq 1$. Then
   $$f(x) = ((1-a)bx + p)^{\frac{1}{1-a}}, \quad g(y) = \left(\frac{a-1}{a}cy + q\right)^{\frac{a}{a-1}},$$
   for $p, q \in \mathbb{R}$. This concludes the case $K = 0$.

4.3.2 Case $K \neq 0$

The proof is by contradiction. We assume the existence of a homothetical surface S with constant Gauss curvature $K \neq 0$. Let us observe the symmetry of the expression (4.23) on f and g. If $f = 0$ or $f' = 0$, then (4.23) implies $K = 0$, which is not our case. If $f'' = 0$, then $f(x) = ax + b$, for some constants a, b, $a \neq 0$. Then (4.23) writes as
$$K(1 + a^2g^2 + (ax + b)^2g'^2)^2 + a^2g'^2 = 0.$$
This is a polynomial equation on x of degree 4 because $K \neq 0$. Then the leading coefficient, namely $Ka^4g'^4$, must vanish. This means $g' = 0$ and (4.23) gives now $K = 0$: contradiction.

Thus $f'' \neq 0$. Interchanging the argument with g, we also suppose $g'' \neq 0$. In particular, $fgf'g' \neq 0$.

We write (4.23) as

$$K(1+f'^2g^2+f^2g'^2)^2 - fgf''g'' + f'^2g'^2 = 0. \tag{4.25}$$

Then

$$\log\left((1+f'^2g^2+f^2g'^2)^2 + \frac{1}{K}f'^2g'^2\right) = \log\left(\frac{1}{K}fgf''g''\right)$$

and so

$$\frac{\partial^2}{\partial x \partial y}\log\left((1+f'^2g^2+f^2g'^2)^2 + \frac{1}{K}f'^2g'^2\right) = 0.$$

This implies

$$\begin{aligned}(f'^2g'^2 + KD^2)\left(f''g'' + 2K\left(D(f''g+fg'') + (fg'^2+f''g^2)(f'^2g+f^2g'')\right)\right)\\ - \left(f''g'^2 + 2KD(fg'^2+f''g^2)\right)\left(f'^2g'' + 2KD(f'^2g+f^2g'')\right) = 0,\end{aligned} \tag{4.26}$$

where $D = 1+f'^2g^2+f^2g'^2$. On the other hand, we take the derivative in (4.25) with respect to x and obtain

$$4Kf'D(fg'^2+f''g^2) + 2f'f''g'^2 - (f'f''+ff''')gg'' = 0. \tag{4.27}$$

Next, from equation (4.25) we obtain g'' as

$$g'' = \frac{KD^2+f'^2g'^2}{ff''g}$$

and we replace it first in equation (4.26) and then in equation (4.27), obtaining two equations $P_1(f,f',f'',f''',g,g') = 0$ and $P_2(f,f',f'',f''',g,g') = 0$. We see both expressions as two polynomials in g'. As they have a common solution for g', then their resultant will vanish. The computation for their resultant gives a polynomial in g, with coefficients depending on f and its first, second and third derivatives. Taking the coefficients identically zero, we obtain a system of equations for f and its derivatives. We are only interested in the leading coefficient, namely, the one for g^{28}, which must vanish. This is equivalent to

$$K^{16}f^{16}f'^{20}(f'^2-ff'')^{14} = 0.$$

This implies $f'^2 - ff'' = 0$ and leads to $f(x) = ce^{dx}$, for c,d positive constants. Finally, we will prove that this gives a contradiction. For this value of f, we substitute f into (4.25), obtaining

$$K + c^2\left(2d^2Kg^2 + (d^2+2K)g'^2 - d^2gg''\right)e^{2dx} + c^4K(d^2g^2+g'^2)^2e^{4dx} = 0.$$

This expression is a polynomial equation on e^{dx} and so, the coefficients vanish. This implies $K=0$, a contradiction.

4.4 The Lorentzian case

Again we ask for those translation and homothetical surfaces in $\mathbb{L}^3$ with constant mean curvature and constant Gauss curvature. Recall that the property of a surface to be a translation surface or a homothetical surface is not metric but it is given by the affine structure of $\mathbb{R}^3$ and the multiplication of real functions of $\mathbb{R}$.

We generalize the results obtained in the previous sections for non-degenerate surfaces of $\mathbb{L}^3$. The proofs are similar, and we omit the details.

1. Extension of Theorem 49. Assume that S is a translation surface. The computation of K gives
$$K=-\frac{(f_2''-f_1''g_2'+g_1'(f_1''f_2'-f_1'f_2''))\,(g_2''-f_2'g_1''+f_1'(g_1''g_2'-g_1'g_2''))}{\left((1+f_1'^2-f_2'^2)(1+g_1'^2-g_2'^2)-(f_1'+g_1'-f_2'g_2')^2\right)^2}.$$
If $K=0$, then the numerator coincides with the one in (4.9) and the conclusion is that S is a cylindrical surface. In the case $K\neq 0$, the result asserts that, under the same hyposthesis, there are no further examples. We discuss the cases when α and β lies in two non-orthogonal planes and when one curve is planar. In the former case, the expression of K is
$$K=-\frac{\cos^2\theta f''g''}{(-f'^2-g'^2+\cos^2\theta+2\sin\theta f'g')^2}.$$
The proof works in the same way. In the second case,
$$K=-\frac{f''(g_2''-f'g_1'')}{(1-g_2'^2-f'^2-f'^2g_1'^2+2f'g_1'g_2')^2}.$$
Again, the proof is similar because we can move f'' to the left hand-side, differentiate with respect to t and observe that there appears an expression which is a polynomial on the function f'.

2. Extension of Theorem 50. As we have pointed out, this result was proved in [54].

3. Extension of Theorem 51. Assume now that S is a homothetical surface and we study those surfaces with constant Gauss curvature. If S is spacelike, then the surface is locally a graph on the xy-plane and S writes as $z=f(x)g(y)$. The expression of K is
$$K=-\frac{fgf''g''-f'^2g'^2}{(1-f'^2g^2-f^2g'^2)^2},\ \text{with } 1-f'^2g^2-f^2g'^2>0.$$
If S is timelike, then the surface is locally a graph on the xz-plane or on the yz-plane. Without loss of generality, we assume that the surface writes as $x=f(y)g(z)$. Now the Gauss curvature K is
$$K=-\frac{fgf''g''-f'^2g'^2}{(1+f^2g'^2-f'^2g^2)^2},\ \text{with } 1+f^2g'^2-f'^2g^2<0.$$
Because both expressions are the same as in (4.23) and the arguments are the same as in Euclidean space, we only give the statements. If $K\neq 0$, then there are no exist homothetical (spacelike or timelike) surfaces with constant Gauss curvature K. If $K=0$, then $fgf''g''-f'^2g'^2=0$, which is the same as (4.24). Then the conclusion is:

(a) The surface is a plane or a cylindrical surface whose directrix is contained in one of the three coordinates planes and the rulings are orthogonal to this plane.

(b) The function $z = f(x)g(y)$ agrees with Theorem 51, items 1) and 2).

Bibliography

[1] J. C. Baez, The octonions, Bull. Amer. Math. Soc., 39(23),145-205 (2001).

[2] J. L. M. Barbosa, A. Colares, A. G.: Minimal Surfaces in R^3, Lecture Notes in Mathematics, 1195, Springer-Verlag, Berlin, 1986.

[3] B. Bektaş, M. Moruz, J. Van der Veken, L. Vrancken, Lagrangian submanifolds of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$ from minimal surfaces in $\mathbb{S}^3$, Proceedings of the Royal Society of Edinburgh Section A, 2017.

[4] B. Bektaş, M. Moruz, J. Van der Veken, L. Vrancken, Lagrangian submanifolds with constant angle functions in the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$, preprint.

[5] O. Birembaux, M. Djoric, Isotropic Affine Spheres, Acta Mathematica Sinica, English Series, Oct., 2012, Vol. 28, No. 10, pp. 1955-1972.

[6] O. Birembaux, L. Vrancken, Isotropic affine hypersurfaces of dimension 5, J. Math. Anal. Appl. 417 (2014), no. 2, 918-962.

[7] J. Bolton, F. Dillen, B. Dioos and L. Vrancken, Almost complex surfaces in the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$, Tohoku Math. J. 67 (2015), 1–17.

[8] W. M. Boothby, An introduction to differentiable manifolds and Riemannian geometry, Revised second edition, Singapore: Elsevier, 2007.

[9] J.-B. Butruille, Homogeneous nearly Kähler manifolds, in: Handbook of Pseudo-Riemannian Geometry and Supersymmetry, 399–423, RMA Lect. Math. Theor. Phys. 16, Eur. Math. Soc., Zürich, 2010.

[10] J.L. Cabrerizo, M. Fernández, J.S. Gómez, Rigidity of pseudo-iotropic immersions, J. Geom. Phys. 59 (2009) 834-842.

[11] M.P. do Carmo, Differential Geometry of Curves and Surfaces, Prentice-Hall, Inc. Englewood Cliffs, New Jersey, 1976.

[12] M.P. do Carmo, Riemannian geometry, Birkhäuser, Boston, 1992.

[13] B.-Y. Chen, Complex extensors and Lagrangian submanifolds in indinite complex Euclidean spaces, Bull. Inst. Math. Acad. Sin. (NS) 31 (3) (2013) 151-179.

[14] V. Cortés, J. J. Vásquez, Locally homogeneous nearly Kähler manifolds, arXiv: 1410.6912.

[15] J. G. Darboux, Théorie Génerale des Surfaces, Livre I, Gauthier-Villars, Paris, 1914.

[16] F. Dillen, I. Van de Woestyne, L. Verstraelen, J. T. Walrave, The surface of Scherk in E^3: a special case in the class of minimal surfaces defined as the sum of two curves, Bull. Inst. Math. Acad. Sin. 26 (1998), 257–267.

[17] F. Dillen, L. Vrancken, Hypersurfaces with parallel difference tensor, Japan. J. Math., Vol. 24, No. 1, 1998.

[18] F. Dillen, L. Vrancken, Totally real submanifolds in $\mathbb{S}^6(1)$ satisfying Chen's equality, Trans. Amer. Math. Soc. 348 (1996), no. 4, 1633–1646.

[19] F. Dillen, L. Verstraelen, L. Vrancken, Classification of totally real 3-dimensional submanifolds of $\mathbb{S}^6(1)$ with $K \geq 1/16$, J. Math. Soc. Japan 42 (1990), no. 4, 565–584.

[20] B. Dioos, L. Vrancken and X. Wang, Lagrangian submanifolds in the nearly Kahler $\mathbb{S}^3 \times \mathbb{S}^3$, 2016, preprint, arXiv:1604.05060.

[21] N. Ejiri, Equivariant minimal immersions of $\mathbb{S}^2$ into $\mathbb{S}^{2m}(1)$, Trans. Amer. Math. Soc. 297 (1986), no. 1, 105–124.

[22] N. Ejiri, Totally real submanifolds in a 6-sphere, Proc. Amer. Math. Soc. 83 (1981), no. 4, 759–763.

[23] L. Foscolo, M.Haskins, New G2 holonomy cones and exotic nearly Kaehler structures on the 6-sphere and the product of a pair of 3-spheres, to appear in Ann. Math..

[24] A. Gray, Nearly Kähler manifolds, J. Diff. Geom. 4 (1970), 283–309.

[25] A. Gray, Riemannian manifolds with geodesic symmetries of order 3, J. Differential Geometry 7 (1972), 343–369.

[26] J.C. González Dávila, F. Martín Cabrera, Homogeneous nearly Kähler manifolds, Ann. Global Anal. Geom. **42** (2012), no. 2, 147–170

[27] J. Gutowski, S. Ivanov and G. Papadopoulos, Deformations of generalized calibrations and compact non-Kähler manifolds with vanishing first Chern class, Asian J. Math. 7 (2003), no. 1, 039–080.

[28] Z. Hu, H. Li, L.Vrancken, Locally strongly convex affine hypersurfaces with parallel cubic form, J. Differential Geom. 87 (2011), no.2, 239-307.

[29] L. Jiu, H. Sun, On minimal homothetical hypersurfaces, Colloq. Math. 109 (2007), 239–249.

[30] F. John, Partial Differential Equations, Volume 1, Springer US, 1978.

[31] J. L. Lagrange, Oeuvres, vol. I, (1760), 335.

[32] H. Li, L. Vrancken, X. Wang, Minimal Lagrangian isotropic immersions in indefinite complex space forms, Journal of Geometry and Physics, 62 (2012) 707-723.

[33] H. Liu, Translation surfaces with constant mean curvature in 3-dimensional spaces, J. Geom. 64 (1999), 141–149.

[34] R. López, Differential Geometry of curves and surfaces in Lorentz-Minkowski space, Internat. Electronic J. Geom. 7 (2014), 44–107.

[35] R. López, M. Moruz, Translation and homothetical surfaces in Euclidean space with constant curvature, J. Korean Math. Soc. **52** (2015), No. 3, pp. 523-535.

[36] J. D. Lotay, Ruled Lagrangian submanifolds of the 6-sphere, Trans. Amer. Math. Soc. 363 (2011), no. 5, 2305–2339.

[37] F. Lubbe, L. Schäfer, *Pseudo-holomorphic curves in nearly Kähler manifolds*, Differential Geom. Appl. **36** (2014), 24–43.

[38] M.A. Magid, Shape operator in Einstein hypersurfaces in indefinite space forms, Proc. Amer. Math. Soc. 84 (1982) 237-242.

[39] S. Montiel, F. Urbano, Isotropic totally real submanifolds, Math. Z. 199 (1988) 55-60.

[40] A. Moroianu, U. Semmelmann, Generalized Killing spinors and Lagrangian graphs, Diff. Geom. Appl. 37 (2014), 141–151.

[41] M. Moruz, L. Vrancken, A classification of isotropic affine hyperspheres, Int. J. Math. Vol 27, No. 9 (2016).

[42] M. Moruz, L. Vrancken, Properties of the nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$, preprint.

[43] P-A. Nagy, Nearly Kähler geometry and Riemannian foliations, Asian J. Math. 6 (2002), no. 3, 481–504.

[44] P-A. Nagy, On nearly- Kähler geometry, Ann. Global Anal. Geom. 22 (2002), no. 2, 167–178.

[45] J. C. C. Nitsche, Lectures on Minimal Surfaces, Cambridge University Press, Cambridge, 1989.

[46] K. Nomizu, T. Sasaki, Affine differential geometry, Cambridge University Press, 1994.

[47] B. O'Neill, Isotropic and Kaehler immersions, Canad. J. Math. 17 (1965) 907-915.

[48] B. O'Neill, Semi-Riemannian geometry with applications to relativity, Academic Press, 1983.

[49] B. Palmer, Calibrations and Lagrangian submanifolds in the six sphere (English summary), Tohoku Math. J. (2) 50 (1998), no. 2, 303–315.

[50] H. F. Scherk, Bemerkungen über die kleinste Fläche innerhalb gegebener Gren- zen, J. Reine Angew. Math. 13 (1835), 185–208.

[51] D.B. Shapiro, Compositions of Quadratic Forms, W. de Gruyter Verlag, 2000.

[52] L. Schäfer, K. Smoczyk, Decomposition and minimality of Lagrangian submanifolds in nearly Kähler manifolds, Ann. Global Anal. Geom. 37 (2010), no. 3, 221–240.

[53] Z. I. Szabo, Structure theorems on riemannian spaces satisfying $R(X,Y) \cdot R = 0$. I. the local version, J. Differential Geom. 17 (1982), no. 4, 531–582.

[54] I. Van de Woestyne, A new characterization of the helicoids, Geometry and topology of submanifolds, V (Leuven/Brussels, 1992), 267–273, World Sci. Publ., River Edge, NJ, 1993.

[55] I. Van de Woestyne, Minimal homothetical hypersurfaces of a semi-Euclidean space, Results Math. 27 (1995), 333–342.

[56] L. Vrancken, Differentialmeetkunde van speciale deelvarieteiten van sferen en affiene ruimten, 1989.

[57] L. Vrancken, Killing vector fields and Lagrangian submanifolds of the nearly Kaehler $\mathbb{S}^6$, J. Math. Pures Appl. (9) 77 (1998), no. 7, 631–645.

[58] L. Vrancken, Special Lagrangian submanifolds of the nearly Kaehler 6-sphere, Glasg. Math. J. 45 (2003), no. 3, 415–426.

[59] Y. Zhang, B. Dioos, Z. Hu, L. Vrancken and X. Wang, Lagrangian submanifolds in the 6-dimensional nearly Kähler manifolds with parallel second fundamental form, J. Geom. Phys. 108 (2016), 21–37.

Résumé

Cette thèse est constituée de quatre chapitres. Le premier contient les notions de base qui permettent d'aborder les divers thèmes qui y sont étudiés. Le second est consacré à l'étude des sous-variétés lagrangiennes d'une variété presque kàhlérienne. J'y présente les résultats obtenus en collaboration avec Burcu Bektaş, Joeri Van der Veken et Luc Vrancken. Dans le troisième, je m'intéresse à un problème de géométrie différentielle affine et je donne une classification des hypersphères affines qui sont isotropiques. Ce résultat a été obtenu en collaboration avec Luc Vrancken. Et enfin dans le dernier chapitre, je présente quelques résultats sur les surfaces de translation et les surfaces homothétiques, objet d'un travail en commun avec Rafael López.

Discipline

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Adresse du Laboratoire

LAMAV
Université de Valenciennes
ISTV2, Le Mont Houy
59313 Valenciennes Cedex 9
FRANCE