





École Doctorale Carnot-Pasteur

Thèse de doctorat

Présentée par :

Tianxiang Gou

en vue de l'obtention du grade de

Docteur de l'Université de Bourgogne Franche-Comté
Spécialité Mathématiques et Applications

Existence and orbital stability of normalized solutions for nonlinear Schrödinger equations

Soutenue publiquement le 29 september 2017 devant le jury composé de :

Mathieu Colin	Université de Bordeaux	Rapporteur
Alberto Farina	Université de Picardie Jules Verne	Examinateur
Louis Jeanjean	Université de Bourgogne Franche-Comté	Directeur de thèse
Simona Rota Nodari	Université de Bourgogne	Examinateur
Tobias Weth	Goethe-Universität Frankfurt	Rapporteur

Laboratoire de Mathématiques
UMR CNRS 6623
Université de Franche-Comté
16 route de Gray
25030 Besançon Cedex, France.

École doctorale Carnot-Pasteur

To my dear parents and brother

Acknowledgements

First of all, I should express my deep gratitude to the China Scholarship Council for supporting my study in France during these two years.

I thank Louis Jeanjean for supervising and guiding my Ph.D. thesis. I also want to thank members in my defence jury. It is my honor to have Mathieu Colin and Tobias Weth review my thesis. Many thanks to them for their careful reading and useful suggestions to further polish my thesis. Simultaneously, I would like to thank Alberto Farina and Simona Rota Nodari for accepting pleasantly as members of my defence jury.

In addition, I have to thank Denis Bonheure and Jean-Baptiste Casteras, who are collaborators of some parts of my thesis. This thesis would not be completed without their contributions.

My gratitude should go to the Laboratoire de Mathématiques de Besançon, Université de Bourgogne Franche-Comté for providing an enjoyable research environment to carry out my thesis. In particular, I want to express my gratitude to my colleagues, Johann Cuenin, Olivier Ho, Othman Kadmiri, Aline Moufleh, Quentin Richard, Simeng Wang, Xumin Wang, Runlian Xia and Haonan Zhang who never hesitate to lend me a hand when I ask for help.

I would like to thank Hongrui Sun for her persistent encouragement and help at the beginning of my research career. As advisor of mine during master period, she offered me several vital opportunities to stimulate my research interest and expand my knowledge, which does lay a solid foundation for my following study. Besides, I should really thank my friend Quanguo Zhang for his constant assistance and valuable suggestions when I ever trapped in dilemma.

In these years, it is quite glad and lucky to encounter so many intimate friends. Especially, I am grateful to Yunchuan Chen, Long Li, Kuan Li, Wenjun Pei as well as Yantao Wang who aided me a lot in my daily life when I got trouble.

Finally, some people deserve a huge thank without any explanation. I owe many thanks to my dear parents Qinbao Gou, Lihua Yang and little brother Yuxiang Gou who always support and encourage me to pursuit my dream. I end the acknowledgements by expressing my gratitude to those important ones who ever touched my life through one way or another, however their names have not been mentioned.

Résumé

Dans cette thèse nous étudions l'existence et la stabilité orbitale de solutions ayant une norme L^2 prescrite, pour deux types d'équations Schrödinger non linéaires dans $\mathbb{R}^N$, à savoir, une classe de systèmes non linéaires couplés de Schrödinger dans $\mathbb{R}^N$ et une classe d'équations nonlinéaires de Schrödinger du quatrième ordre dans $\mathbb{R}^N$. Ces deux types d'équations nonlinéaires de Schrödinger surviennent dans de nombreuses applications en mathématiques et physique, et sont devenus une grande attention dans les années récentes. D'un point de vue physique, de telles solutions sont souvent référées comme des solutions normalisées, qui sont obtenues comme points critiques d'energie fonctionnelle associée sous contrainte avec une norme L^2. Les éléments clés de nos preuves sont les méthodes variationnelles.

La thèse est divisée en 5 chapitres. Le chapitre 1 est une introduction de la thèse, qui contient une brève présentation des problèmes traités et résultats correspondants obtenus dans cette thèse. Dans les chapitres 2 et 3, nous sommes intéressés par l'existence et la stabilité orbitale de solutions normalisées pour une classe de systèmes nonlinéaires couplés de Schrödinger dans $\mathbb{R}^N$. Plus précisément, dans le chapitre 2, nous considérons solutions normalisées dans un cas où la fonctionnelle d'énergie associée est minorée sous contrainte. Par conséquent nous présentons un problème de minimisation de l'énergie fonctionnelle associée sous contrainte. Dans ce cas, les solutions normalisées sont en effet obtenues comme minimiseurs globaux. Notre but est d'être établir la compacité de toute suite minimisante en utilisant la technique de réarrangement couplé, qui est une alternative du principe de concentration-compacité de Lions, et n'exige pas la vérification de l'inégalité stricte de la subadditivité associée. En corollaire de la compacité de toute suite minimisante, la stabilité orbitale de minimiseurs globaux est prouvée. Au chapitre 3, nous nous concentrons sur l'existence de solutions normalisées dans deux autres cas, dans lesquels l'énergie fonctionnelle associée n'est pas minorée sous contrainte. En conséquence le minimiseur global pour l'énergie fonctionnelle associée sous contrainte n'existe plus. L'existence de deux solutions normalisées strictement positives est établi par méthodes du minimax. La première solution est un minimiseur local dont l'existence est assurée par l'étude de compacité de toute suite minimisante à un problème de minimisation localisée, et la deuxième, est respectivement de type point col ou de type linking. En particulier nous relâchons le hypothèse sur la dimension induites par les résultats de type de Liouville. En outre, nous obtenons la stabilité orbitale de minimiseurs locaux. Dans le chapitre 4, nous étudions des solutions normalisées pour une classe d'équations non linéaires du quatrième ordre de Schrödinger dans le cas de masse critique et dans le cas supercritique. Dans les deux cas, la fonctionnelle d'énergie associée n'est pas minorée sous contrainte. En utilisant une approche par contrainte naturelle, nous établissons l'existence de solutions d'états fondamentaux et la multiplicité de solutions radiales. De plus, nous discutons l'instabilité orbitale par explosion en temps fini d'états fondamentaux radiaux. Pour finir, dans le chapitre 5, nous mettons quelques remarques relatives à cette thèse, et proposons également quelques problèmes intéressants.

Mots-clefs

Équations de Schrödinger non linéaires, norme L^2 prescrite, solutions normalisées, états fondamentaux, stabilité orbitale, minimiseurs, explosion, réarrangement, méthodes variationnelles, principe de concentration-compacité, identité de type Pohozaev, variété.

Existence and orbital stability of normalized solutions for nonlinear Schrödinger equations

Abstract

In this thesis, we are concerned with the existence and orbital stability of solutions having prescribed L^2-norm for two types of nonlinear Schrödinger equations in $\mathbb{R}^N$, namely a class of coupled nonlinear Schrödinger systems in $\mathbb{R}^N$ and a class of fourth-order nonlinear Schrödinger equations in $\mathbb{R}^N$. These two types of nonlinear Schrödinger equations arise in a variety of mathematical and physical models, and have drawn wide attention in recent years. From a physical point of view, such solutions are often referred as normalized solutions, which correspond to critical points of the underlying energy functional restricted to the L^2-norm constraint. The main ingredients of our proofs are variational methods.

The thesis is divided into five chapters. Chapter 1 is an introduction to this thesis, which contains a brief presentation of issues treated and corresponding results attained in the thesis. In Chapter 2 and Chapter 3, we are interested in the existence and orbital stability of normalized solutions for a class of coupled nonlinear Schrödinger systems in $\mathbb{R}^N$. More precisely, Chapter 2 is devoted to investigating normalized solutions in a case where the associated energy functional is bounded from below on constraint. Accordingly, we introduce a global minimization problem as the energy functional subject to constraint. In this situation, normalized solutions are indeed achieved as global minimizers to the minimization problem. Our purpose consists in establishing the compactness of any minimizing sequence by means of the coupled rearrangement arguments, which is alternative to the Lions' concentration compactness principle and does not require the verification of related strict subadditivity inequality. As a corollary of the compactness of any minimizing sequence, the orbital stability of global minimizers is proved. In Chapter 3, we focus on the existence of normalized solutions in another two cases, in which the energy functional becomes unbounded from below on constraint. Thus global minimizer to the energy functional restricted to constraint does not exist. The existence of two normalized solutions is established in each case with the aid of minimax methods. The first solution is a local minimizer, whose existence is insured through the study of the compactness of any minimizing sequence to a localized minimization problem, and the second one is a mountain pass type and a linking type, respectively. In particular, we relax the limitation on dimension induced by the Liouville's type results. Furthermore, we obtain the orbital stability of local minimizers. In Chapter 4, we study normalized solutions for a class of fourth-order nonlinear Schrödinger equations in the mass critical and supercritical regime. In both cases, the associated energy functional is unbounded from below on constraint. Using a natural constraint approach, we establish the existence of ground state solutions and multiplicity of radial solutions. In addition, we discuss the orbital instability by blowup in finite time of radial ground state solutions. Finally, in Chapter 5, we present some remarks related to this thesis and also put forward some interesting issues.

Keywords

Nonlinear Schrödinger equations, prescribed L^2-norm, normalized solutions, ground states, orbital stability, minimizers, blowup, rearrangement, variational methods, concentration compactness principle, Pohozaev type identity, manifold.

Contents

1 Introduction **13**
1.1 Normalized solutions for coupled nonlinear Schrödinger system 13
1.2 Normalized solutions for fourth-order nonlinear Schrödinger equation 20

2 Existence and orbital stability of normalized solutions for coupled nonlinear Schrödinger system **27**
2.1 Introduction 27
2.2 Preliminary results 30
2.3 Proofs of the main results 31

3 Multiple normalized solutions for coupled nonlinear Schrödinger system **39**
3.1 Introduction 39
3.2 Preliminary results 44
3.3 Existence of local minimizers 47
3.4 Existence of minimax solutions 52
3.5 Appendix 60

4 Normalized solutions for fourth-order nonlinear Schrödinger equation in the mass critical and supercritical regime **63**
4.1 Introduction 63
4.2 Preliminary results 69
4.3 Some properties of the constraint $\mathcal{M}(c)$ 72
4.4 Existence of ground state solutions 77
4.5 Multiplicity of radial solutions 79
4.6 Properties of the function $c \mapsto \gamma(c)$ 80
4.7 A concentration phenomenon 86
4.8 Positive and sign-changing solutions 88
4.9 Dynamical behaviors 89
4.10 Appendix 92

5 Remarks and Perspectives **99**
5.1 Remarks 99
5.2 Perspectives 100

Bibliography **103**

Chapter 1

Introduction

The thesis which collects some works obtained during my Ph.D. in these two years is devoted to the study of normalized solutions for a class of coupled nonlinear Schrödinger systems in $\mathbb{R}^N$ and a class of fourth-order nonlinear Schrödinger equations in $\mathbb{R}^N$. Chapter 2 and Chapter 3 correspond to works with L. Jeanjean. Chapter 4 is an collaboration with D. Bonheure, J.-B. Casteras and L. Jeanjean.

1.1 Normalized solutions for coupled nonlinear Schrödinger system

An important feature in quantum physics is played by the following time-dependent coupled nonlinear Schrödinger system in $\mathbb{R} \times \mathbb{R}^N$,

$$\begin{cases} -i\partial_t \Psi_1 = \Delta \Psi_1 + \mu_1 |\Psi_1|^{p_1-2}\Psi_1 + \beta r_1 |\Psi_1|^{r_1-2}\Psi_1 |\Psi_2|^{r_2}, \\ -i\partial_t \Psi_2 = \Delta \Psi_2 + \mu_2 |\Psi_2|^{p_2-2}\Psi_2 + \beta r_2 |\Psi_1|^{r_1} |\Psi_2|^{r_2-2}\Psi_2. \end{cases} \tag{1.1.1}$$

This system governs various physical phenomena, such as the Bose-Einstein condensates with multiple states, or propagation of mutually incoherent waves packets in nonlinear optics, see for instance [2, 48, 50, 59, 80, 84, 109]. In the system (1.1.1), the functions Ψ_1, Ψ_2 are corresponding condensate amplitudes, μ_i and β are intraspecies and interspecies scattering length, describing interaction of the same state and different states, respectively. The positive sign of μ_i (and β) represents attractive interaction, the negative one represents repulsive interaction.

One of the most fundamental research regarding (1.1.1) in mathematical and physical field consists in standing waves, namely solutions with the form of

$$\Psi_1(t,x) = e^{-i\lambda_1 t} u_1(x), \quad \Psi_2(t,x) = e^{-i\lambda_2 t} u_2(x)$$

for $(\lambda_1, \lambda_2) \in \mathbb{R}^2$. This ansatz then gives rise to the following elliptic system satisfied by u_1 and u_2,

$$\begin{cases} -\Delta u_1 = \lambda_1 u_1 + \mu_1 |u_1|^{p_1-2}u_1 + \beta r_1 |u_1|^{r_1-2}u_1 |u_2|^{r_2}, \\ -\Delta u_2 = \lambda_2 u_2 + \mu_2 |u_2|^{p_2-2}u_2 + \beta r_2 |u_1|^{r_1} |u_2|^{r_2-2}u_2. \end{cases} \tag{1.1.2}$$

In order to study solutions to (1.1.2), two possible options arise. The first one is to consider (1.1.2) with the given parameters $(\lambda_1, \lambda_2) \in \mathbb{R}^2$. In this situation, a solution (u_1, u_2) to (1.1.2) corresponds to a critical point of energy functional $F : H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$F(u_1, u_2) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_1|^2 + \lambda_1 |u_1|^2 \, dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_2|^2 + \lambda_2 |u_2|^2 \, dx$$

$$-\sum_{i=1}^{2}\frac{\mu_i}{p_i}\int_{\mathbb{R}^N}|u_i|^{p_i}\,dx-\beta\int_{\mathbb{R}^N}|u_1|^{r_1}|u_2|^{r_2}\,dx.$$

Recently, considerable literature has been devoted to this subject concerning the existence and phase separation of solutions, see for instance [5, 6, 15, 37, 39, 41, 42, 43, 44, 46, 60, 71, 79, 81, 98, 102, 107, 108, 110, 112] and references therein.

The second one is motivated by the fact that the L^2-norm of solution to the Cauchy problem of (1.1.1) is conserved along time, i.e. for any $t>0$,

$$\int_{\mathbb{R}^N}|\Psi_i(t,x)|^2\,dx=\int_{\mathbb{R}^N}|\Psi_i(0,x)|^2\,dx\quad\text{for}\quad i=1,2.$$

Thus it is of particular interest to search for solutions to (1.1.2) having prescribed L^2-norm, namely, for given $a_1,a_2>0$, to find $(\lambda_1,\lambda_2)\in\mathbb{R}^2$ and $(u_1,u_2)\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ satisfying (1.1.2), together with normalized condition

$$\int_{\mathbb{R}^N}|u_1|^2\,dx=a_1,\quad \int_{\mathbb{R}^N}|u_2|^2\,dx=a_2. \tag{1.1.3}$$

Physically, such solutions are often referred as normalized solutions. In this case, we emphasize that (λ_1,λ_2) are unknown and appear as Lagrange multipliers. For convenience of terminology, we shall identify a solution $(\lambda_1,\lambda_2,u_1,u_2)$ to (1.1.2)-(1.1.3) with (u_1,u_2), where (u_1,u_2) is obtained as a critical point of energy functional $J:H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)\to\mathbb{R}$ defined by

$$J(u_1,u_2):=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u_1|^2+|\nabla u_2|^2\,dx-\sum_{i=1}^{2}\frac{\mu_i}{p_i}\int_{\mathbb{R}^N}|u_i|^{p_i}\,dx-\beta\int_{\mathbb{R}^N}|u_1|^{r_1}|u_2|^{r_2}\,dx$$

on the constraint $S(a_1,a_2):=S(a_1)\times S(a_2)$, here

$$S(a):=\{u\in H^1(\mathbb{R}^N):\int_{\mathbb{R}^N}|u|^2\,dx=a>0\}.$$

The purpose of Chapter 2 and Chapter 3 is to investigate the existence and orbital stability of solutions to (1.1.2)-(1.1.3). In Chapter 2, we deal with the existence and orbital stability of normalized solutions in a case where the energy functional J is bounded from below on $S(a_1,a_2)$. In Chapter 3, we consider the multiple existence of normalized solutions in another two cases, in which the energy functional J becomes unbounded from below on $S(a_1,a_2)$.

1.1.1 Compactness of any minimizing sequence

In Chapter 2, we study the existence and orbital stability of solutions to (1.1.2)-(1.1.3) under the assumption

(H_0) $\mu_1,\mu_2,\beta>0, 2<p_1,p_2<2+\frac{4}{N}, r_1,r_2>1, r_1+r_2<2+\frac{4}{N}$.

Note that under the assumption (H_0) the energy functional J is bounded from below on $S(a_1,a_2)$. Thus it is natural to introduce the following minimization problem

$$M(a_1,a_2):=\inf_{(u_1,u_2)\in S(a_1,a_2)}J(u_1,u_2)<0. \tag{1.1.4}$$

Clearly, minimizers to (1.1.4) correspond to critical points of energy functional J restricted to $S(a_1,a_2)$, then solutions to (1.1.2)-(1.1.3). Hence our aim is to look for minimizers to (1.1.4), whose existence is a straightforward consequence of the following statement.

Theorem 1.1.1. *Let $N \geq 1$. Assume that (H_0) holds. Then any minimizing sequence to* (1.1.4) *is compact, up to translation, in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. In particular, there exists a solution to* (1.1.2)-(1.1.3) *as a minimizer to* (1.1.4).

Remark 1.1.2. *When $N = 1$, $\mu_1, \mu_2, \beta > 0$, $p_1 = p_2 = 4, r_1 = r_2 = 2$, the authors [91] studied the compactness of any minimizing sequence to* (1.1.4)*, which is indeed based on the Lions' concentration compactness principle [73, 74]. When $N \geq 1$, we mention the paper [100], where the compactness of any minimizing sequence was discussed by taking advantage of the coupled rearrangement arguments, which is alterantive to the Lions' concentration compactness principle. However, embedding the minimization problem* (1.1.4) *into the one as presented in [100], the compactness result is only valid under condition (H_0) with $r_1, r_2 \geq 2$. Our Theorem 1.1.1 provides a fairly complete result concerning the compactness of any minimizing sequence to* (1.1.4) *under more general assumption (H_0) in any dimension.*

Let $\{(u_1^n, u_2^n)\} \subset S(a_1, a_2)$ be an arbitrary minimizing sequence to (1.1.4). To see the compactness of $\{(u_1^n, u_2^n)\}$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, if employing the Lions' concentration compactness principle [73, 74], one has to rule out the possibilities of vanishing and dichotomy. Notice that $M(a_1, a_2) < 0$ and the energy functional J is invariant under translations, then vanishing can be excluded easily as a result of the Lions' concentration compactness Lemma [74, Lemma I.1]. Next in order to prevent dichotomy from occurring, the heuristic argument is to establish the following strict subadditivity inequality

$$M(a_1, a_2) < M(b_1, b_2) + M(a_1 - b_1, a_2 - b_2), \tag{1.1.5}$$

where $0 \leq b_i < a_i$ for $i = 1, 2$, $(b_1, b_2) \neq (0, 0)$ and $(b_1, b_2) \neq (a_1, a_2)$.

To deal with only one constraint problem, several techniques have been developed to establish strict subadditivity inequality. Most are based on some homogeneity type properties, such as in autonomous case, one can make use of scaling technique to check related strict subadditivity inequality, we refer the readers to [19, 40, 101]. However, when it comes to multiple constraints problem, this technique is generally not applicable, thus how to achieve strict subadditivity inequality in this situation is much less understood in addition to some special cases, where constraints cannot be chosen independently, see for instance [89, 90, 93]. In addition, when $N = 1$, we mention the papers [23, 24, 91], where the authors established strict subadditivity inequality by means of crucially applying [3, Lemma 2.10], which depends on the original idea as introduced in [33]. The readers can also refer to [51] for an application of [3, Lemma 2.10] to a minimization problem in the case of dimension $N \geq 1$. This result is however available under the condition that one can identify a radially symmetric minimizing sequence to associated minimization problem.

Coming back to the minimization problem (1.1.4), it seems hard to check (1.1.5). For this reason, as inspired by Ikoma [61], we propose the coupled rearrangement arguments to discuss the compactness of minimizing sequence $\{(u_1^n, u_2^n)\}$, whose original spirit however comes from Shibata [100].

We now sketch the virtue to prove Theorem 1.1.1. Firstly, observe that under the assumption (H_0), the minimizing sequence $\{(u_1^n, u_2^n)\}$ is bounded in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. By using the Lions' concentration compactness Lemma [74, Lemma I.1], we then denote by $(u_1, u_2) \neq (0, 0)$ the weak limit of $\{(u_1^n, u_2^n)\}$, up to translation, in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Next in light of the coupled rearrangement arguments, we are able to prove that $(u_1^n, u_2^n) \to (u_1, u_2)$, up to translation, in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ for $2 < p < 2^*$. This joints with the

weakly lower semicontinuous of norm, we see that $J(u_1,u_2)\leq M(a_1,a_2)$. At this point, to obtain the compactness of minimizing sequence $\{(u_1^n,u_2^n)\}$, it remains to prove that $(u_1,u_2)\in S(a_1,a_2)$. This is guaranteed by the property that if $0\leq \bar{a}_i<a_i$ for $i=1,2$ and $(\bar{a}_1,\bar{a}_2)\neq(a_1,a_2)$, then

$$M(a_1,a_2)<M(\bar{a}_1,\bar{a}_2). \tag{1.1.6}$$

Remark 1.1.3. *When $N\geq 2$, if one is only interested in the existence of minimizers to* (1.1.4)*, the paper [8] should be mentioned. It was assumed that (H_0) holds, in addition $p_1,p_2<2+\frac{2}{N-2}$ if $N\geq 5$, the authors [8] successfully proved the existence of minimizers to* (1.1.4) *by essentially making use of the Liouville's type results. We now extend this result under (H_0). Indeed, this can be done by considering a radially symmetric minimizing sequence $\{(u_1^n,u_2^n)\}\subset S(a_1,a_2)$ to* (1.1.4)*. Such minimizing sequence is obtained by the Schwarz's rearrangement of a minimizing sequence. Recall that the embedding $H^1_{rad}(\mathbb{R}^N)\hookrightarrow L^p(\mathbb{R}^N)$ is compact for $N\geq 2$, $2<p<2^*$, where $H^1_{rad}(\mathbb{R}^N)$ stands for a subspace of $H^1(\mathbb{R}^N)$, which consists of radially symmetric functions in $H^1(\mathbb{R}^N)$. Noticing first the assumption (H_0) and the Lions' concentration compactness Lemma, we then denote by $(u_1,u_2)\neq(0,0)$ the weak limit of $\{(u_1^n,u_2^n)\}$, up to translation, in $H^1_{rad}(\mathbb{R}^N)\times H^1_{rad}(\mathbb{R}^N)$. Thus by using the compact embedding and the weakly lower semicontinuous of norm, it then readily follows that $J(u_1,u_2)\leq M(a_1,a_2)$. At this point, the fact that $(u_1,u_2)\in S(a_1,a_2)$ comes from the property* (1.1.6)*. Hence the claim follows.*

Alternatively, it is possible to establish the existence of minimizers to (1.1.4) *by working directly in $H^1_{rad}(\mathbb{R}^N)\times H^1_{rad}(\mathbb{R}^N)$. For more details, see Remark 2.3.4.*

Defining the set

$$G_M(a_1,a_2):=\{(u_1,u_2)\in S(a_1,a_2):J(u_1,u_2)=M(a_1,a_2)\},$$

we now show the orbital stability of minimizers to (1.1.4) in the following sense.

Definition 1.1.4. *We say the set $G(a_1,a_2)$ is orbitally stable, i.e. for any $\epsilon>0$, there exists $\delta>0$ so that if $(\Psi_{1,0},\Psi_{2,0})\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ satisfies*

$$\inf_{(u_1,u_2)\in G(a_1,a_2)}\|(\Psi_{1,0},\Psi_{2,0})-(u_1,u_2)\|\leq\delta,$$

then

$$\sup_{t\in[0,T)}\inf_{(u_1,u_2)\in G(a_1,a_2)}\|(\Psi_1(t),\Psi_2(t))-(u_1,u_2)\|\leq\epsilon,$$

where $(\Psi_1(t),\Psi_2(t))$ is a solution to the Cauchy problem of (1.1.1) *with initial datum $(\Psi_{1,0},\Psi_{2,0})$, T denotes the maximum existence time of the solution, and $\|\cdot\|$ stands for the standard norm in the Sobolev space $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$.*

Based upon Theorem 1.1.1, making use of the elements in Cazenave and Lions [36], we are able to prove the following result.

Theorem 1.1.5. *Let $N\geq 1$. Assume that (H_0) and the local existence and uniqueness of the Cauchy problem to* (1.1.1) *hold. Then the set $G_M(a_1,a_2)$ is orbitally stable.*

Remark 1.1.6. *Note that under the assumption (H_0), the local existence and uniqueness to the Cauchy problem of* (1.1.1) *are unknown. The point being that when $1<r_1,r_2<2$, the interaction parts are not Lipschitz continuous. Thus the orbital stability of minimizers to* (1.1.4) *is under the condition. However, let us point out that when $N=1$, $2\leq r_1=r_2<3$, the local existence to the Cauchy problem of* (1.1.1) *holds, see [88].*

1.1.2 Existence of multiple normalized solutions

In Chapter 3, we consider the existence of multiple solutions to (1.1.2)-(1.1.3) in the following two cases,

(H_1) $\mu_1, \mu_2, \beta > 0$, $2 < p_1, p_2 < 2 + \frac{4}{N}$, $r_1, r_2 > 1$, $2 + \frac{4}{N} < r_1 + r_2 < 2^*$;

(H_2) $\mu_1, \mu_2, \beta > 0$, $2 + \frac{4}{N} < p_1, p_2 < 2^*$, $r_1, r_2 > 1$, $r_1 + r_2 < 2 + \frac{4}{N}$.

Recall that under the assumption (H_0) the energy functional J is bounded from below on $S(a_1, a_2)$, then one can obtain a solution to (1.1.2)-(1.1.3) as a global minimizer to (1.1.4) through studying the compactness of any minimizing sequence to (1.1.4), see [58]. In contrast, under the assumption (H_1) or (H_2), the energy functional J is not bounded from below on $S(a_1, a_2)$ anymore. Indeed, to see this, for any $t > 0$ let us introduce the scaling of $u \in S(a)$ as

$$u^t(x) := t^{\frac{N}{2}} u(tx).$$

Clearly, $\|u^t\|_2 = \|u\|_2 = a$. For any $(u_1, u_2) \in S(a_1, a_2)$, a straightforward calculation leads to,

$$\begin{aligned} J(u_1^t, u_2^t) = \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla u_1|^2 + |\nabla u_2|^2 \, dx - \sum_{i=1}^{2} t^{(\frac{p_i}{2} - 1)N} \frac{\mu_i}{p_i} \int_{\mathbb{R}^N} |u_i|^{p_i} \, dx \\ - \beta t^{(\frac{r_1 + r_2}{2} - 1)N} \int_{\mathbb{R}^N} |u_1|^{r_1} |u_2|^{r_2} \, dx. \end{aligned} \tag{1.1.7}$$

By consequence, if $\mu_1, \mu_2, \beta > 0$, when either $p_i > 2 + \frac{4}{N}$ for some $i = 1, 2$ or $r_1 + r_2 > 2 + \frac{4}{N}$, it then follows from (1.1.7) that J becomes unbounded from below on $S(a_1, a_2)$. As a result, under the assumption (H_1) or (H_2), it is no more possible to look for a solution to (1.1.2)-(1.1.3) as a global minimizer to (1.1.4).

When global minimizer to (1.1.4) fails to exist, finding a solution to (1.1.2)-(1.1.3) is more delicate and involved. In this situation, minimax methods come into play. We now point out some literature in this direction. When $2 \le N \le 4$, $\mu_1, \mu_2, \beta > 0$, if either $2 < p_1 < 2 + \frac{4}{N} < p_2 < 2^*$, $2 + \frac{4}{N} < r_1 + r_2 < 2^*, r_2 > 2$ or $2 + \frac{4}{N} < p_1, p_2, r_1 + r_2 < 2^*$, the authors [8] studied the existence of solution to (1.1.2)-(1.1.3) with the aid of the mountain pass arguments, see also [12]. When $N = 3$, $\mu_1, \mu_2 > 0, \beta < 0$, $p_1 = p_2 = 4, r_1 = r_2 = 2$, by using a natural constraint approach, the existence of solution to (1.1.2)-(1.1.3) was established in [13]. In addition, concerning a multiplicity result to (1.1.2)-(1.1.3), we refer the reader to [14]. Let us also mention the papers [86, 87, 94], where the authors considered the existence of normalized solutions to problems confined on a bounded domain in $\mathbb{R}^N$ or with a trapping potential. While a periodic potential is included to problem, the existence of normalized solutions was discussed in [1].

As mainly motivated by [8, 12], we investigate the existence of multiple solutions to (1.1.2)-(1.1.3) under two new assumptions (H_1) and (H_2). Our aim is to prove that (1.1.2)-(1.1.3) admits two positive solutions when $N \ge 1$ and (H_1) or (H_2) holds. Up to our knowledge, it is the first time that a multiplicity result to (1.1.2)-(1.1.3) is obtained when $N \ge 1$ and $\beta > 0$.

In order to address our results, for any $\rho > 0$ let us introduce the notation,

$$\mathcal{B}(\rho) := \{(u_1, u_2) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla u_1|^2 + |\nabla u_2|^2 \, dx < \rho\}.$$

Firstly, on account of (1.1.7), under (H_1) or (H_2) there holds

$$\inf J(u_1,u_2)<0 \quad for \quad (u_1,u_2)\in S(a_1,a_2)\cap\mathcal{B}(\rho), \tag{1.1.8}$$

see Lemma 3.2.4. Furthermore, there exist $\beta_0=\beta_0(a_1,a_2)>0$ and $\rho_0=\rho_0(a_1,a_2)>0$ such that

$$\inf J(u_1,u_2)>0 \quad for \quad (u_1,u_2)\in S(a_1,a_2)\cap\partial\mathcal{B}(\rho_0) \tag{1.1.9}$$

holds for any $0<\beta\le\beta_0$, see Lemma 3.3.1. Together (1.1.8) with (1.1.9), then there may exist a local minimizer for the energy functional J restricted to $S(a_1,a_2)\cap\mathcal{B}(\rho_0)$. Hence, for $0<\beta\le\beta_0$ we introduce the following localized minimization problem

$$m(a_1,a_2):=\inf_{(u_1,u_2)\in S(a_1,a_2)\cap\mathcal{B}(\rho_0)} J(u_1,u_2). \tag{1.1.10}$$

Obviously, minimizers to (1.1.10) are critical points for the energy functional J restricted to $S(a_1,a_2)$, i.e. solutions to (1.1.2)-(1.1.3). Thus our first solution to (1.1.2)-(1.1.3) is obtained as a local minimizer to (1.1.10), whose existence is insured by the study the compactness of any minimizing sequence to (1.1.10) in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$.

In addition, for any $(u_1,u_2)\in S(a_1,a_2)$, it follows from (1.1.7) that $J(u_1^t,u_2^t)\to-\infty$ as $t\to\infty$ when (H_1) or (H_2) holds, and note also that $(u_1^t,u_2^t)\notin\mathcal{B}(\rho_0)$ for $t>0$ large enough. This property along with (1.1.9) reveal that there may exist other critical points for the energy functional J restricted to $S(a_1,a_2)$. In fact, under the assumption (H_1), the second critical point is obtained through the mountain pass arguments, while under the assumption (H_2), the second one is achieved by means of a linking type procedure. Let us now state our main results.

Theorem 1.1.7. *Let $a_1,a_2>0$ be given and assume that (H_1) holds. Then there exist $\beta_0=\beta_0(a_1,a_2)>0$ and $\rho_0=\rho_0(a_1,a_2)>0$ such that for any $0<\beta\le\beta_0$,*

(i) if $N\ge1$, any minimizing sequence to (1.1.10) is compact, up to translation, in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. In particular, there exists a positive solution (v_1,v_2) to (1.1.2)-(1.1.3) with $(v_1,v_2)\in\mathcal{B}(\rho_0)$ and $J(v_1,v_2)<0$;

(ii) If either $2\le N\le4$ or $N\ge5$ with $p_1,p_2\le r_1+r_2-\frac{2}{N}$ or $|p_1-p_2|\le\frac{2}{N}$, there exists a second positive solution (u_1,u_2) to (1.1.2)-(1.1.3) with $J(u_1,u_2)>0$.

Theorem 1.1.8. *Let $a_1,a_2>0$ be given and assume that (H_2) holds. Then there exist $\beta_0=\beta_0(a_1,a_2)>0$ and $\rho_0=\rho_0(a_1,a_2)>0$ such that for any $0<\beta\le\beta_0$,*

(i) if either $1\le N\le4$ or $N\ge5$, $r_i>(\frac{r_1+r_2}{2}-1)N$ for $i=1,2$, any minimizing sequence to (1.1.10) is compact, up to translation, in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. In particular, there exists a positive solution (v_1,v_2) to (1.1.2)-(1.1.3) with $(v_1,v_2)\in\mathcal{B}(\rho_0)$ and $J(v_1,v_2)<0$;

(ii) If $2\le N\le4$, there exists a second positive solution (u_1,u_2) to (1.1.2)-(1.1.3) with $J(u_1,u_2)>0$.

Remark 1.1.9. *i) The value of β_0 in Theorem 1.1.7 and Theorem 1.1.8 can be explicitly computed in terms of N,p_i,a_i,r_i for $i=1,2$, instead of being obtained through a limit process. Additionally, for any given $\beta>0$, we can assume that $\beta\le\beta_0$ at the expense of taking $a_1,a_2>0$ sufficiently small, because $\beta_0(a_1,a_2)\to\infty$ as $a_1,a_2\to0$, to see this property, we refer Lemma 3.3.1. This indeed implies that for any given $\beta>0$, there are two positive solutions to (1.1.2)-(1.1.3) under the assumptions of Theorem 1.1.7 or*

Theorem 1.1.8 for $a_1, a_2 > 0$ sufficient small. Finally, let us also point out that our results are not a consequence of perturbation arguments.

ii) Notice that the existence of the second solution to (1.1.2)-(1.1.3) *in Theorem 1.1.7 (ii) and Theorem 1.1.8 (ii) is under the condition $N \geq 2$, this is because the second one is established in the framework of radially symmetric functions space $H^1_{rad}(\mathbb{R}^N) \times H^1_{rad}(\mathbb{R}^N)$, and the compact embedding $H^1_{rad}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for $2 < p < 2^*$ holds for $N \geq 2$.*

iii) When $N \geq 2$, we conjecture that the existence of the second solution to (1.1.2)-(1.1.3) *remains valid by only assuming (H_1), we refer to Remark 3.4.5 for a discussion concerning this subject.*

To establish the compactness of any minimizing sequence to (1.1.10) under the assumption (H_1) or (H_2), we essentially make use of the coupled rearrangement arguments due to Shibata [100] as developed by Ikoma [61]. Assume $\{(v_1^n, v_2^n)\}$ be an arbitrary minimizing sequence to (1.1.10). Note that $m(a_1, a_2) < 0$, from the Lions' concentration compactness Lemma [74, Lemma I.1], we then denote by $(v_1, v_2) \neq (0, 0)$ the weak limit of $\{(v_1^n, v_2^n)\}$, up to translation, in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. In the following, using the coupled rearrangement arguments,one can show that $(v_1^n, v_2^n) \to (v_1, v_2)$, up to translation, in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ for $2 < p < 2^*$. Nevertheless, unlike the global minimization problem (1.1.4), to prove this, one faces a difficulty arising from the fact that the sum of two elements in $\mathcal{B}(\rho_0)$ may not belong to $\mathcal{B}(\rho_0)$. This causes more technique to discuss the possibility of dichotomy. To overcome this difficulty, one needs to analyze carefully some properties of the energy functional J restricted to $S(a_1, a_2) \cap \mathcal{B}(\rho_0)$. Finally, to see the compactness of minimizing sequence $\{(v_1^n, v_2^n)\}$, it remains to assert that $(v_1, v_2) \in S(a_1, a_2)$. Reasoning as the proof of Theorem 1.1.1, under the assumption (H_1), this is insured by the fact that $m(a_1, a_2)$ satisfies the property (1.1.6). However, under the assumption (H_2), it is unknown if $m(a_1, a_2)$ satisfies (1.1.6), thus, in this situation we apply the Liouville's type results, see Lemma 3.2.2, which is however available when $N \leq 4$, and in order to deal with the case $N \geq 5$, a restriction is eventually imposed on the range of r_1, r_2.

The proofs of Theorem 1.1.7 (ii) and Theorem 1.1.8 (ii) depend on the virtue as presented in [8, 12]. Roughly speaking, the proofs can be divided into three steps. Firstly, one requires to identify a suspected critical level. This can be done by introducing a minimax structure of mountain pass type under the assumption (H_1), and linking one under the assumption (H_2). Secondly, one needs to find a bounded Palais-Smale sequence $\{(u_1^n, u_2^n)\} \subset S(a_1, a_2)$ for the energy functional J restricted to $S(a_1, a_2)$ at the energy level. To this end, the classical methods developed to derive the boundedness of any Palais-Smale sequence for unconstrained problem collapse. Actually, this step benefits from the presence of a Pohozaev type constraint, on which the energy functional J is coercive. Thus taking advantage of this constraint and adapting the approach introduced in [63] which consists in adding an artificial variable within the variational procedure, one can end this step. Having obtained a bounded Palais-Smale sequence $\{(u_1^n, u_2^n)\}$ for the energy functional J restricted to $S(a_1, a_2)$, we denote by (u_1, u_2) its weak limit in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, and we immediately find that (u_1, u_2) solves (1.1.2) with some $(\lambda_1, \lambda_2) \in \mathbb{R}^2$, see Lemma 3.2.7. At this point, the last step is to show that $(u_1, u_2) \in S(a_1, a_2)$. It is this step where the limitation on dimension was imposed in [8, 12, 13]. Because in this step the authors of the literature took into account the Liouville's type results and also used the property that the scalar problem

$$-\Delta w - \lambda w = \mu |w|^{p-2} w, \quad w \in S(a) \quad \text{for} \quad \mu > 0 \tag{1.1.11}$$

has a unique positive radial solution for $2 < p < 2^*$. We now relax these two restrictions under the assumption (H_1), thus Theorem 1.1.7 (ii) allows to consider the case $N \geq 5$. This is essentially based on the fact that when $2 < p < 2 + \frac{4}{N}$, $\mu > 0$,

$$-\infty < \inf_{u \in S(a)} I(u) < 0, \tag{1.1.12}$$

where $I(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{\mu}{p} \int_{\mathbb{R}^N} |u|^p \, dx$.

We now continue the proof of the last step. Under the assumption (H_1), when $2 \leq N \leq 4$, the fact that $(u_1, u_2) \in S(a_1, a_2)$ is a direct consequence of the Liouville's type results. When $N \geq 5$, assuming by contradiction that $(u_1, u_2) \in S(\bar{a}_1, \bar{a}_2)$ for $0 \leq \bar{a}_i \leq a_i$ for $i = 1, 2$ and $(\bar{a}_1, \bar{a}_2) \neq (a_1, a_2)$. Thus one can crucially apply the property (1.1.12) and Lemma 3.4.4 to construct a path, on which the maximum of J is strictly below mountain pass level. We then reach a contradiction. Here the path is constructed by "adding some masses" technique somehow in the spirit of [62], but using the coupled rearrangement arguments.

On the contrary, when $p > 2 + \frac{4}{N}$, $\mu > 0$, the property (1.1.12) is violated, hence to prove that $(u_1, u_2) \in S(a_1, a_2)$ under the assumption (H_2), it indeed depends on the Liouville's type results, which induces a restriction on dimension $N \leq 4$.

We now define the set

$$G_m(a_1, a_2) := \{(u_1, u_2) \in S(a_1, a_2) \cap \mathcal{B}(\rho_0) : J(u_1, u_2) = m(a_1, a_2)\}.$$

In view of Remark 2.1.5, as a counterpart one to Theorem 1.1.1, we have the orbital stability of minimizers to (1.1.10).

Theorem 1.1.10. *Let $N \geq 1$. Assume that (H_1) or (H_2) with either $1 \leq N \leq 4$ or $N \geq 5$, $r_i > (\frac{r_1+r_2}{2} - 1)N$ for $i = 1, 2$, and the local existence and uniqueness of the Cauchy problem to* (1.1.1) *hold. Then the set $G_m(a_1, a_2)$ is orbitally stable.*

1.2 Normalized solutions for fourth-order nonlinear Schrödinger equation

In Chapter 4, we deal with a class of fourth-order nonlinear Schrödinger equations in $\mathbb{R} \times \mathbb{R}^N$,

$$i\partial_t \psi - \gamma \Delta^2 \psi + \Delta \psi + |\psi|^{2\sigma} \psi = 0, \tag{1.2.1}$$

where $\gamma > 0$.

The classical nonlinear Schrödinger equation with pure power nonlinearity in $\mathbb{R} \times \mathbb{R}^N$ is given by

$$i\partial_t \psi + \Delta \psi + |\psi|^{2\sigma} \psi = 0.$$

It is well known that when $0 < \sigma N < 2$, any solution to the Cauchy problem of (1.2) with initial datum in $H^1(\mathbb{R}^N)$ exists globally in time, and standing waves are orbitally stable. While $\sigma N \geq 2$, blowup in finite time may occur, then standing waves become singular, see for instance [35].

In order to regularize and stabilize solution to the Cauchy problem of (1.2), Karpman and Shagalov [70] introduced a small fourth-order dispersion term to (1.2), i.e. they considered the fourth-order nonlinear Schrödinger equation (1.2.1), see also [64]. Using

a combination of stability analysis and numerical simulations, they showed that standing waves are orbitally stable for any $\gamma > 0$, when $0 < \sigma N < 2$, and for $\gamma > 0$ small, when $2 \leq \sigma N < 4$. Whereas $\sigma N \geq 4$, they observed an unstable phenomenon. This result indicates that adding a small fourth-order dispersion term to (1.2) helps to stabilize standing waves.

In nonlinear optics, the classical nonlinear Schrödinger equation (for example (1.2)) is traditionally derived from the scalar nonlinear Helmhotz equation through so-called paraxial approximation. The fact that the solution to the Cauchy problem of (1.2) with initial datum in $H^1(\mathbb{R}^N)$ may blow up in finite time suggests that some small terms neglected by the paraxial approximation which play an important role to prevent this phenomenon. Therefore a small fourth-order dispersion term was proposed in [49] as a nonparaxial correction, see also [9, 10, 11], which eventually gives rise to the fourth-order nonlinear Schrödinger equations (1.2.1). Applying the arguments as developed in [111], when $0 < \sigma N < 4$ the authors [49] proved that any solution to the Cauchy problem of (1.2.1) with initial datum in $H^2(\mathbb{R}^N)$ exists globally in time.

Nevertheless, despite of these physical relevance, the dispersion equation (1.2.1) is far from being well understood. There are only few papers studying (1.2.1), for instance [20, 27, 28, 30, 85, 95, 96, 97].

From a physical and mathematical point of view, a center issue to study (1.2.1) consists in standing waves, namely solutions with the form of $\psi(t, x) = e^{i\alpha t}u(x)$ for $\alpha \in \mathbb{R}$. Then u satisfies the following elliptic equation

$$\gamma\Delta^2 u - \Delta u + \alpha u = |u|^{2\sigma}u. \tag{1.2.2}$$

In order to study solutions to (1.2.2), two possible options have been developed. The first one is to investigate solutions to (1.2.2) with the given parameter $\alpha \in \mathbb{R}$. In this case, a solution to (1.2.2) is obtained as a critical point of energy functional $F : H^2(\mathbb{R}^N) \to \mathbb{R}$ given by

$$F(u) := \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{\alpha}{2}\int_{\mathbb{R}^N} |u|^2\,dx - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2}\,dx,$$

and of particular interest is to investigate least energy solutions, i.e. solutions to (1.2.2) minimize the energy functional F among all solutions. Concerning this subject, we refer to [31].

Note that the L^2-norm of the solution to the Cauchy problem of (1.2.1) is conserved along time, i.e. for any $t > 0$,

$$\int_{\mathbb{R}^N} |\psi(t, x)|^2\,dx = \int_{\mathbb{R}^N} |\psi(0, x)|^2\,dx.$$

As motivated by this physical fact, the second one is to research solutions to (1.2.2) having prescribed L^2-norm, namely, for given $c > 0$, to find $\alpha \in \mathbb{R}$ and $u \in H^2(\mathbb{R}^N)$ satisfying (1.2.2), together with normalized condition

$$\int_{\mathbb{R}^N} |u|^2\,dx = c. \tag{1.2.3}$$

Conventionally, the solutions are referred as normalized solutions, In this situation, the parameter α is unknown and determined as Lagrange multiplier. For the sake of simplicity,

we identify a solution (α, u) to (1.2.2)-(1.2.3) with u, where u is obtained as a critical point of energy functional $E : H^2(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$E(u) := \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{1}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2}\,dx$$

on the constraint

$$S(c) := \{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2\,dx = c\}.$$

From now on, we are concerned with normalized solutions to (1.2.2), i.e. solutions to (1.2.2)-(1.2.3). Observe that when $0 < \sigma N < 4$, the energy functional E is bounded from below on $S(c)$, the authors [28] then studied the following minimization problem

$$m(c) := \inf_{u\in S(c)} E(u). \tag{1.2.4}$$

In this case, it is possible to find a solution to (1.2.2)-(1.2.3) as a minimizer to (1.2.4). We mention the following result as obtained in [28].

Theorem 1.2.1. *If $0 < \sigma N < 2$, then $m(c)$ is achieved for any $c > 0$. If $2 \le \sigma N < 4$, then there exists a critical mass $\tilde{c} = \tilde{c}(\sigma, N)$ such that*

(i) $m(c)$ is not achieved if $c < \tilde{c}$;

(ii) $m(c)$ is achieved if $c > \tilde{c}$ and $\sigma = 2/N$;

(iii) $m(c)$ is achieved if $c \ge \tilde{c}$ and $\sigma \ne 2/N$.

Moreover, if $\sigma \in \mathbb{N}^+$ and $m(c)$ is achieved, then there exists at least one radially symmetric minimizer to (1.2.4).

Remark 1.2.2. *The appearance of a critical mass when $2 \le \sigma N < 4$ is linked to the fact that every term of the energy functional E behaves differently with respect to dilations.*

In Chapter 4, as inspired by [28], our aim is to study solutions to (1.2.2)-(1.2.3) under the mass critical case $\sigma N = 4$ and the mass supercritical case $4 < \sigma N < 4^*$, where $4^* := \frac{4N}{(N-4)^+}$. Firstly, we note that, in these two cases it is no more possible to look for a solution to (1.2.2)-(1.2.3) as a minimizer to (1.2.4). Indeed, to see this, for any $u \in S(c)$, $\lambda > 0$, let us define the scaling of u as

$$u_\lambda(x) := \lambda^{\frac{N}{4}} u(\sqrt{\lambda}x).$$

By direct calculations, one can check that $\|u_\lambda\|_2 = \|u\|_2$ and

$$E(u_\lambda) = \frac{\gamma\lambda^2}{2}\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \frac{\lambda}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{\lambda^{\sigma N/2}}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2}\,dx. \tag{1.2.5}$$

Thus, when $4 < \sigma N < 4^*$, we find that $E(u_\lambda) \to -\infty$ as $\lambda \to \infty$, then $m(c) = -\infty$ for any $c > 0$.

We now turn to the case $\sigma N = 4$. To prove the claim, we first recall the Gagliardo-Nirenberg's inequality (see [92]) for $u \in H^2(\mathbb{R}^N)$

$$\|u\|_{2\sigma+2}^{2\sigma+2} \le B_N(\sigma)\|\Delta u\|_2^{\frac{\sigma N}{2}}\|u\|_2^{2+2\sigma-\frac{\sigma N}{2}}, \tag{1.2.6}$$

where

$$\begin{cases} 0 \le \sigma, & \text{if } N \le 4, \\ 0 \le \sigma < \dfrac{4}{N-4}, & \text{if } N \ge 5, \end{cases}$$

and $B_N(\sigma)$ is a constant depending on σ and N. We thus obtain the following result.

Theorem 1.2.3. *Let $N \geq 1, \sigma N = 4$. There exists $c_N^* > 0$ such that*

$$m(c) = \inf_{u \in S(c)} E(u) = \begin{cases} 0, & 0 < c \leq c_N^*, \\ -\infty, & c > c_N^*. \end{cases}$$

For $c \in (0, c_N^*)$, (1.2.2)-(1.2.3) has no solution, and in particular $m(c)$ is not achieved. In addition, $c_N^* = (\gamma C(N))^{\frac{N}{4}}$ where

$$C(N) := \frac{N+4}{NB_N(\frac{4}{N})}, \tag{1.2.7}$$

and $B_N(\sigma)$ is the constant in (1.2.6).

In view of Theorem 1.2.3, when $\sigma N = 4$, and $c > c_N*$, it is also unlikely to find a solution to (1.2.2)-(1.2.3) as a minimizer to (1.2.4).

From previous observations, since minimizer to (1.2.4) fails to exist under the mass critical and supercritical case, one will see that it is more delicate to seek for solutions to (1.2.2)-(1.2.3) in these two cases. In comparison with unconstrained problem, when facing similar issue, one can search for a solution as a minimizer to associated energy functional restricted to the Nehari manifold. However, in our situation, no Nehari manifold is available because α is unknown. Thus to overcome this difficulty, we introduce a natural constraint $\mathcal{M}(c)$ given by

$$\mathcal{M}(c) := \{u \in S(c) : Q(u) = 0\},$$

where

$$Q(u) := \gamma \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{\sigma N}{2(2\sigma+2)} \int_{\mathbb{R}^N} |u|^{2\sigma+2} \, dx.$$

Using (1.2.5), we immediately see that

$$Q(u) = \frac{\partial E(u_\lambda)}{\partial \lambda}|_{\lambda=1}. \tag{1.2.8}$$

thus, heuristically, $\mathcal{M}(c)$ contains all critical points for E restricted to $S(c)$, i.e. all solutions to (1.2.2)-(1.2.3). This fact is to be rigourously proved in Lemma 4.10.1. Actually, the condition $Q(u) = 0$ corresponds to a Pohozaev type identity related to (1.2.2)-(1.2.3), and $\mathcal{M}(c)$ is regarded as the Pohozaev manifold. Furthermore, borrowing the key spirit from Bartsch and Soave [13], we are able to prove that a critical point for E restricted to $\mathcal{M}(c)$ is a critical point for E restricted to $S(c)$, then a solution to (1.2.2)-(1.2.3), see Lemma 4.3.5. In addition, there holds that $E(u) \geq 0$ for $u \in \mathcal{M}(c)$. For these reasons, we now introduce the following minimization problem

$$\gamma(c) := \inf_{u \in \mathcal{M}(c)} E(u). \tag{1.2.9}$$

We shall look for a minimizer to (1.2.9). Note that, if it exists, it then corresponds to a ground state solution to (1.2.2)-(1.2.3) in the sense that it minimizes the energy functional E among all solutions to (1.2.2)-(1.2.3) with same L^2-norm.

For convenience, we define $c_0 \in \mathbb{R}$ as

$$c_0 := \begin{cases} 0, & \text{if } 4 < \sigma N < 4^*, \\ c_N^*, & \text{if } \sigma N = 4, \end{cases}$$

where c_N^* is given in Theorem 1.2.3.

Theorem 1.2.4. *Let $N \geq 1$, $4 \leq \sigma N < 4^*$. Then there exists $c_{\sigma,N} > c_0$ such that for any $c \in (c_0, c_{\sigma,N})$, (1.2.2)-(1.2.3) has a ground state solution u_c satisfying $E(u_c) = \gamma(c)$, and the associated Lagrange parameter α_c is strictly positive. Moreover*

(i) $c_{\sigma,1} = c_{\sigma,2} = \infty$, and $c_{\sigma,3} = \infty$ if $4/3 \leq \sigma < 2$;

(ii) When $\sigma N = 4$, then $c_{\sigma,4} = \infty$, and $c_{\sigma,N} \geq \left(\frac{N}{N-4}\right)^{\frac{N}{4}} c_N^*$ if $N \geq 5$.

The proof of Theorem 1.2.4 crucially relies on a key element Lemma 4.3.5. Using this result and the Ekeland variational principle [47], we then obtain a Palais-Smale sequence $\{u_n\} \subset \mathcal{M}(c)$ for E restricted to $S(c)$ at level $\gamma(c)$ as a minimizing sequence to (1.2.9). Our aim is to prove that $\{u_n\}$ is compact, up to translation, in $H^2(\mathbb{R}^N)$. Firstly, notice that E is coercive on $\mathcal{M}(c)$, see Lemma 4.3.1, thus $\{u_n\}$ is bounded in $H^2(\mathbb{R}^N)$, and it readily follows that there is $u_c \in H^2(\mathbb{R}^N)$ such that $u_n \rightharpoonup u_c$, up to a subsequence and translation, in $H^2(\mathbb{R}^N)$. Furthermore, there exists $\alpha_c \in \mathbb{R}$ such that u_c satisfies

$$\gamma \Delta^2 u_c - \Delta u_c + \alpha_c u_c = |u_c|^{2\sigma} u_c. \tag{1.2.10}$$

At this point, proving the compactness of $\{u_n\}$ then reduces to show that the strong convergence of $\{u_n\}$ in $L^{2\sigma+2}(\mathbb{R}^N)$ and the Lagrange parameter $\alpha_c > 0$, see Lemma 4.3.6. The strong convergence of $\{u_n\}$ in $L^{2\sigma+2}(\mathbb{R}^N)$ is indeed beneficial from the fact that the function $c \mapsto \gamma(c)$ is nonincreasing on (c_0, ∞), see Lemma 4.4.1. The restriction on the size of c is to insure that $\alpha_c > 0$, see Lemma 4.2.1.

Taking advantage of the genus theory, we obtain the existence of multiple radial solutions to (1.2.2)-(1.2.3).

Theorem 1.2.5. *Assume $N \geq 2$.*

(i) If $4 < \sigma N < 4^*$, then for any $c \in (0, c_{\sigma,N})$, where $c_{\sigma,N}$ is defined in Theorem 1.2.4, (1.2.2)-(1.2.3) admits infinitely many radial solutions;

(ii) If $2 \leq N \leq 4$, $\sigma N = 4$, then for any $k \in \mathbb{N}^+$, there exists a $c_k > c_N^*$ such that, for any $c \geq c_k$, (1.2.2)-(1.2.3) admits at least k radial solutions.

To establish Theorem 1.2.5, we work in the subspace $H^2_{rad}(\mathbb{R}^N)$ of $H^2(\mathbb{R}^N)$, which consists of radially symmetric functions in $H^2(\mathbb{R}^N)$. Accordingly, we define $\mathcal{M}_{rad}(c) := \mathcal{M}(c) \cap H^2_{rad}(\mathbb{R}^N)$.

The proof of Theorem 1.2.5 is based on the Kranosel'skii genus theory. The key step is to prove that E restricted to $\mathcal{M}_{rad}(c)$ satisfies the Palais-Smale condition. To this end, let us consider an arbitrary Palais-Smale sequence $\{u_n\} \subset \mathcal{M}_{rad}(c)$ for E restricted to $\mathcal{M}_{rad}(c)$. Our purpose is to prove that $\{u_n\}$ is compact in $H^2(\mathbb{R}^N)$. Noting the coerciveness of E on $\mathcal{M}_{rad}(c)$, we then denote by u_c its weak limit in $H^2(\mathbb{R}^N)$. Moreover, there exists a $\alpha_c \in \mathbb{R}$ such that u_c satisfies (1.2.10). The fact that the strong convergence of $\{u_n\}$ in $L^{2\sigma+2}(\mathbb{R}^N)$ is given here for free, because the embedding $H^2_{rad}(\mathbb{R}^N) \hookrightarrow L^{2\sigma+2}(\mathbb{R}^N)$ is compact for $N \geq 2$. Thus reasoning as Theorem 1.2.4, to show the compactness it remains to check that the Lagrange parameter α_c is strictly positive, which is indeed guaranteed by Lemma 4.2.1. The second step is to show that the set $\mathcal{M}(c)$ is sufficiently large. This is always the case when $4 < \sigma N < 4^*$ for any $c > 0$. However, when $\sigma N = 4$, the set $\mathcal{M}_{rad}(c)$ may be too small. In particular, it shrinks to the empty set as $c \to c_N^*$. To obtain a given number of solutions, we require that $c > c_N^*$ is sufficiently large.

The monotonicity of the function $c \mapsto \gamma(c)$ on (c_0, ∞) is crucially used in the proof of Theorem 1.2.4. We now present additional properties of this function, its behaviors depend in an essential way on the couple (σ, N).

Theorem 1.2.6. *Assume $N \geq 1$. Let $4 \leq \sigma N < 4^*$. The function $c \mapsto \gamma(c)$ is continuous for any $c > c_0$, is decreasing on (c_0, ∞), and $\lim_{c \to c_0^+} \gamma(c) = \infty$. In addition,*

(i) if $N = 1, 2$, $N = 3$ with $\frac{4}{3} \leq \sigma < 2$ or $N = 4$ with $\sigma = 1$, then $c \mapsto \gamma(c)$ is strictly decreasing and $\lim_{c \to \infty} \gamma(c) = 0$;

(ii) If $N = 3$ with $\sigma \geq 2$ or $N = 4$ with $\sigma > 1$, then $\lim_{c \to \infty} \gamma(c) := \gamma(\infty) > 0$ and $\gamma(c) > \gamma(\infty)$ for all $c > c_0$;

(iii) If $N \geq 5$, then $\lim_{c \to \infty} \gamma(c) := \gamma(\infty) > 0$, and there exists a $c_\infty > c_0$ such that $\gamma(c) = \gamma(\infty)$ for all $c \geq c_\infty$.

Note that Theorem 1.2.6, the difference of behavior of $\gamma(c)$ as $c \to \infty$ between $N \leq 4$ and $N \geq 5$ arises from the fact that the equation

$$\gamma \Delta^2 u - \Delta u = |u|^{2\sigma} u \tag{1.2.11}$$

does not admit least energy solution in $H^2(\mathbb{R}^N)$ when $N \leq 4$, but it does when $N \geq 5$, see Proposition 4.6.5 for more details.

Next when $\sigma N = 4$, we show a concentration behavior of ground state solutions to (1.2.2)-(1.2.3) as c approaches to c_N^* from above.

Theorem 1.2.7. *Let $N \geq 1, \sigma N = 4$, and $\{c_n\} \subset \mathbb{R}$ be a sequence satisfying for any $n \in \mathbb{N}$, $c_n > c_N^*$ with $c_n \to c_N^*$ as $n \to \infty$, and u_n be a ground state solution to (1.2.2)-(1.2.3) for $c = c_n$ at level $\gamma(c_n)$. Then there exist a sequence $\{y_n\} \subset \mathbb{R}^N$ and a least energy solution u to the equation*

$$\gamma \Delta^2 u + u = |u|^{\frac{8}{N}} u,$$

such that up to a subsequence,

$$\left(\frac{\epsilon_n^4 c_N^* N}{4}\right)^{\frac{N}{8}} u_n \left(\left(\frac{\epsilon_n^4 c_N^* N}{4}\right)^{\frac{1}{4}} x + \epsilon_n y_n\right) \to u \text{ in } L^q(\mathbb{R}^N) \text{ as } n \to \infty$$

for $2 \leq q < \frac{2N}{(N-4)^+}$, where $\epsilon_n \to 0$ as $n \to \infty$.

Proposition 1.2.7 gives a description of ground state solution to (1.2.2)-(1.2.3) as c_n approaches to c_N^* from above. Roughly speaking, it shows for $n \in \mathbb{N}$ large enough,

$$u_n(x) \approx \left(\frac{4}{\epsilon_n^4 c_N^* N}\right)^{\frac{N}{8}} u\left(\left(\frac{4}{\epsilon_n^4 c_N^* N}\right)^{\frac{1}{4}} (x - \epsilon_n y_n)\right).$$

In the following we consider the sign and radially symmetric property of solutions to (1.2.2)-(1.2.3). Concerning this subject, we first mention the case that $\alpha \in \mathbb{R}^+$ is given in (4.1.2). In this case, it is known that when $\alpha \in \mathbb{R}^+$ is sufficiently small, all least energy solutions have a sign and are radial. On the contrary, when $\alpha \in \mathbb{R}^+$ is large, radial solutions are necessarily sign-changing. In addition, when $\sigma \in \mathbb{N}^+$, at least one least energy solution is radial. For more details, see [31, Theorem 4]. When $0 < \sigma N < 4$, regarding the sign and radially symmetric property of minimizers to (1.2.4), we refer to [28]. However, when $4 \leq \sigma N < 4^*$, it seems more complex to derive these information for ground state solutions to (1.2.2)-(1.2.3). In this direction, we only present the following result.

Theorem 1.2.8. *Let $N \geq 1$, $4 \leq \sigma N < 4^*$, and $\sigma \in \mathbb{N}^+$. Then there exists a $c_r > c_0$ such that, for any $c \in (c_0, c_r)$,* (1.2.2)-(1.2.3) *admits a ground state solution, which is radial and sign-changing.*

In our next result, we prove that positive radial solutions to (1.2.2)-(1.2.3) do exist.

Theorem 1.2.9. *Let $1 \leq N \leq 4$, $4 \leq \sigma N < 4^*$. Then there exists a $\bar{c}_{\sigma,N} > c_0$ such that* (1.2.2)-(1.2.3) *admits a positive and radial solution for any $c \geq \bar{c}_{\sigma,N}$.*

We now turn our attention to investigate dynamical behaviors of solution to the Cauchy problem of the dispersion equation (1.2.1). From [95], when $0 < \sigma N < 4^*$ the local well-posedness to the Cauchy problem of (1.2.1) is known. Moreover, in the mass subcritical case $0 < \sigma N < 4$, any solution to the Cauchy problem of (1.2.1) with initial datum in $H^2(\mathbb{R}^N)$ exists globally in time, see [49, 95]. While in the mass critical and supercritical case $4 \leq \sigma N < 4^*$, blowup in finite time may happen, but it is also likely to show that the solution to the Cauchy problem of (1.2.1) with some initial datums exists globally in time.

Theorem 1.2.10. *Let $N \geq 1$, $4 \leq \sigma N < 4^*$. For any $c > c_0$, the solution $\psi \in C([0,T); H^2(\mathbb{R}^N))$ to* (1.2.1) *with initial datum $\psi_0 \in \mathcal{O}_c$ with*

$$\mathcal{O}_c := \{u \in S(c) : E(u) < \gamma(c), Q(u) > 0\}.$$

exists globally in time.

When $0 < \sigma N < 4$, it was proved in [28] that minimizers to (1.2.4) are orbitally stable, see also [85]. While $4 \leq \sigma N < 4^*$, we show that radial ground state solutions to (1.2.2)-(1.2.3) are unstable by blowup in finite time.

Definition 1.2.11. *We say that $u \in H^2(\mathbb{R}^N)$ is unstable by blowup in finite time, if for any $\varepsilon > 0$, there exists $v \in H^2(\mathbb{R}^N)$ such that $\|v - u\|_{H^2} < \varepsilon$ and the solution $\psi(t) \in C([0,T); H^2(\mathbb{R}^N))$ to* (1.2.1) *with initial datum $\psi(0) = v$ blows up in finite time in H^2-norm.*

Making use of a key element in Boulenger and Lenzmann [30], we have

Theorem 1.2.12. *Let $4 \leq \sigma N < 4^*$, $N \geq 2$ and $\sigma \leq 4$. Then the standing waves associated to radial ground state solutions to* (1.2.2)-(1.2.3) *are unstable by blowup in finite time.*

In the case where $\alpha \in \mathbb{R}^+$ is given in (1.2.2), the fact that radial least energy solutions are unstable by blowup in finite time was recently established, see our paper [27]. It should be noted that the results of [27] are also strongly based on the arguments from Boulenger and Lenzmann [30]

Chapter 2

Existence and orbital stability of normalized solutions for coupled nonlinear Schrödinger system

2.1 Introduction

In this chapter, we consider the existence of solutions having prescribed L^2-norm to a class of coupled nonlinear Schrödinger systems in $\mathbb{R}^N$. More precisely, for given $a_1, a_2 > 0$, we look for $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ and $(u_1, u_2) \in H^1(\mathbb{R}^N) \times H^1(R^N)$ satisfying

$$\begin{cases} -\Delta u_1 = \lambda_1 u_1 + \mu_1 |u_1|^{p_1-2} u_1 + r_1 \beta |u_1|^{r_1-2} u_1 |u_2|^{r_2}, \\ -\Delta u_2 = \lambda_2 u_2 + \mu_2 |u_2|^{p_2-2} u_2 + r_2 \beta |u_1|^{r_1} |u_2|^{r_2-2} u_2, \end{cases} \tag{2.1.1}$$

and

$$\int_{\mathbb{R}^N} |u_1|^2 \, dx = a_1, \quad \int_{\mathbb{R}^N} |u_2|^2 \, dx = a_2. \tag{2.1.2}$$

Physically, such solutions are often referred as normalized solutions.

The problem under consideration is associated to the research of standing waves to the following nonlinear Schrödinger system in $\mathbb{R} \times \mathbb{R}^N$,

$$\begin{cases} -i\partial_t \Psi_1 = \Delta \Psi_1 + \mu_1 |\Psi_1|^{p_1-2} \Psi_1 + \beta |\Psi_1|^{r_1-2} \Psi_1 |\Psi_2|^{r_2}, \\ -i\partial_t \Psi_2 = \Delta \Psi_2 + \mu_2 |\Psi_2|^{p_2-2} \Psi_2 + \beta |\Psi_1|^{r_1} |\Psi_2|^{r_2-2} \Psi_2. \end{cases} \tag{2.1.3}$$

Here by standing waves, we mean solutions to (2.1.3) with the form of

$$\Psi_1(t, x) = e^{-i\lambda_1 t} u_1(x), \quad \Psi_2(t, x) = e^{-i\lambda_2 t} u_2(x),$$

for $(\lambda_1, \lambda_2) \in \mathbb{R}^2$. Thus (u_1, u_2) satisfies the elliptic system (2.1.1).

Note that the L^2-norm of solution to the Cauchy problem of (2.1.3) is conserved along time, i.e.

$$\int_{\mathbb{R}^N} |\Psi_i(t, x)|^2 \, dx = \int_{\mathbb{R}^N} |\Psi_i(0, x)|^2 \, dx \quad \text{for} \quad i = 1, 2,$$

which leads to the study of normalized solutions quite interesting. For simplicity, in the following we shall regard a solution $(\lambda_1, \lambda_2, u_1, u_2)$ to (2.1.1)-(2.1.2) as (u_1, u_2), where

(u_1, u_2) is obtained as a critical point of energy functional $J : H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$J(u_1, u_2) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_1|^2 + |\nabla u_2|^2 \, dx - \sum_{i=1}^{2} \frac{\mu_i}{p_i} \int_{\mathbb{R}^N} |u_i|^{p_i} \, dx - \beta \int_{\mathbb{R}^N} |u_1|^{r_1} |u_2|^{r_2} \, dx$$

on the constraint $S(a_1, a_2) := S(a_1) \times S(a_2)$ with

$$S(a) := \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 \, dx = a > 0\},$$

and (λ_1, λ_2) is determined as Lagrange multipliers.

In this chapter, we are interested in the existence of solutions to (2.1.1)-(2.1.2) under the following assumption

(H_0) $\mu_1, \mu_2, \beta > 0, 2 < p_1, p_2 < 2 + \frac{4}{N}$, $r_1, r_2 > 1$, $r_1 + r_2 < 2 + \frac{4}{N}$.

Note that under the assumption (H_0), the energy functional J is bounded from below on $S(a_1, a_2)$, we then consider the following minimization problem

$$M(a_1, a_2) := \inf_{(u_1, u_2) \in S(a_1, a_2)} J(u_1, u_2). \tag{2.1.4}$$

It is standard that minimizers to (2.1.4) are critical points for the energy functional J restricted to $S(a_1, a_2)$, then solutions to (2.1.1)-(2.1.2). Hence, we look for minimizers to (2.1.4), and whose existence is a consequence of the following statement.

Theorem 2.1.1. *Let $N \geq 1$. Assume that (H_0) holds. Then any minimizing sequence to (2.1.4) is compact, up to translations, in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.*

Remark 2.1.2. *If one only concerns the existence of minimizers to (2.1.4), we mention paper [8]. When $N \geq 2$, assume that (H_0) holds, in addition $2 < p_1, p_2 < 2 + \frac{2}{N-2}$ if $N \geq 5$, the authors [8] obtained the existence of minimizers to (2.1.4). In this related direction, we also refer to [34, 100].*

Following some initial works [105, 106], from the last thirty years, the Lions' concentration compactness principle [73, 74] had a deep influence on solving minimization problem under constraint. Regarding our problem (2.1.4), if employing the concentration compactness principle, then the heuristic arguments readily convince that the compactness of any minimizing sequence holds if the following strict subadditivity inequality are satisfied,

$$M(a_1, a_2) < M(b_1, b_2) + M(a_1 - b_1, a_2 - b_2), \tag{2.1.5}$$

where $0 \leq b_i < a_i$ for $i = 1, 2$, and $(b_1, b_2) \neq (0, 0)$ and $(b_1, b_2) \neq (a_1, a_2)$.

To deal with only one constraint problem, several techniques have been developed to prove strict subadditivity inequality. Most are based on some homogeneity type properties. In autonomous case, then it is possible to use scaling techniques, see for example [19, 40, 101]. In the case of multiple constraints problem, how to establish strict subadditivity inequality is much less understood. As a matter of fact, in this situation few papers addressed the issue of compactness of any minimizing sequence. Moreover, among most of them, constraints cannot be chosen independently, for instance [89, 90, 93]. Concerning minimization problem (2.1.4), when $N = 1$, a more complete result seems to be due to

[91], where the compactness of any minimizing sequence was obtained by checking (2.1.5). To estblish (2.1.5), the authors [91] crucially applied [3, Lemma 2.10], which depends in turn on original idea introduced in [33], see also [51]. We also refer to [75] for similar arguments on related problem.

As inspired by Ikoma [61], we propose an alternative approach to verify the compactness of any minimizing sequences to (2.1.4). Let $\{(u_1^n, u_2^n)\} \subset S(a_1, a_2)$ be a minimizing sequence to (2.1.4). Firstly, under (H_0), we see that $\{(u_1^n, u_2^n)\}$ is bounded in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, we then denote by (u_1, u_2) the weak limit of $\{(u_1^n, u_2^n)\}$. At this point, to demonstrate the compactness, we first prove that $(u_1^n, u_2^n) \to (u_1, u_2)$, up to translations, in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ for $2 < p < 2^*$. To prove this, we make use of a nice result of Shibata [100] as developed in [61, Lemma A.1]. This result can somehow be considered as an extension of [3, Lemma 2.10] to any dimension.

With this strong convergence in hand, then using weakly lower semicontinuous of norm, we find that $J(u_1, u_2) \leq M(a_1, a_2)$. Namely the energy functional J is weakly lower semicontinuous on minimizing sequence. If $\|u_1\|_2^2 = a_1$ and $\|u_2\|_2^2 = a_2$, the compactness immediately follows. Suppose not and assume that $\|u_1\|_2^2 := \bar{a}_1 < a_1$ or $\|u_2\|_2^2 := \bar{a}_2 < a_2$. Since $J(u_1, u_2) \leq M(a_1, a_2)$, it follows that $M(\bar{a}_1, \bar{a}_2) \leq M(a_1, a_2)$. We then reach a contradiction via observing the weak version (2.1.5) where an equality is allowed, which implies that the function $(a_1, a_2) \mapsto M(a_1, a_2)$ is strictly decreasing in both variables.

Remark 2.1.3. *Note that when $N \geq 2$ and (H_0) holds, if one is interested in the existence of minimizers to* (2.1.4)*, a shorter proof can be given. Choosing a radially symmetric minimizing sequence $\{(u_1^n, u_2^n)\} \subset S(a_1, a_2)$ to* (2.1.4)*. Such minimizing sequence can be obtained as the Schwartz's reaarangement of a minimizing sequence. Recall that the embedding $H^1_{rad}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is compact for $N \geq 2$, and $2 < p < 2^*$, where $H^1_{rad}(\mathbb{R}^N)$ stands for a subspace of $H^1(\mathbb{R}^N)$, which consists of radially symmetric functions in $H^1(\mathbb{R}^N)$. By means of the assumption (H_0) and the Lions' concentration Lemma, we then denote by (u_1, u_2) the weak limit of $\{(u_1^n, u_2^n)\}$, up to transaltion, in $H^1_{rad}(\mathbb{R}^N) \times H^1_{rad}(\mathbb{R}^N)$. By using the fact that the energy functional J is weakly lower semicontinuous on minimizing sequence, it readily follows that $J(u_1, u_2) \leq M(a_1, a_2)$. At this point, the remaining proof is identical to the one of Theorem 2.1.1.*

Alternatively, it is possible to obtain the existence of minimizers to (2.1.4) *by working directly in $H^1_{rad}(\mathbb{R}^N) \times H^1_{rad}(\mathbb{R}^N)$. In this direction, we refer to Remark 2.3.4.*

Defining the set

$$G_M(a_1, a_2) := \{(u_1, u_2) \in S(a_1, a_2) : J(u_1, u_2) = M(a_1, a_2)\},$$

we show the orbital stability of minimizers to (2.1.4) in the following sense.

Definition 2.1.4. *We say a set $G(a_1, a_2)$ is orbitally stable, i.e. for any $\epsilon > 0$, there exists $\delta > 0$ so that if $(\Psi_{1,0}, \Psi_{2,0}) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ satisfies*

$$\inf_{(u_1,u_2)\in G(a_1,a_2)} \|(\Psi_{1,0}, \Psi_{2,0}) - (u_1, u_2)\| \leq \delta,$$

then

$$\sup_{t\in[0,T)} \inf_{(u_1,u_2)\in G(a_1,a_2)} \|(\Psi_1(t), \Psi_2(t)) - (u_1, u_2)\| \leq \epsilon,$$

where $(\Psi_1(t), \Psi_2(t))$ *is solution to the Cauchy problem of* (2.1.3) *with initial datum* $(\Psi_{1,0}, \Psi_{2,0})$, T *denotes the maximum existence time of solution, and* $\|\cdot\|$ *stands for the standard norm in the Sobolev space* $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

Remark 2.1.5. *Note that under assumption* (H_0), *the local well-posedness to the Cauchy problem of* (2.1.3) *is unknown. The point being that when* $1 < r_1, r_2 < 2$, *the interaction parts are not Lipchitz continuous, in particular the uniqueness might fail. For this reason, the orbital stability of minimizers is under condition. However, let us point out that when* $N = 1$, $2 \le r_1 = r_2 < 3$, *the local existence and uniqueness to the Cauchy problem of* (2.1.3) *holds, see for instance [88].*

Based upon Theorem 2.1.1, as a dierct consequence of the elements in Cazenave and Lions [36] we are able to prove the following result.

Theorem 2.1.6. *Let* $N \ge 1$. *Assume that* (H_0) *and the local existence and uniqueness to the Cauchy problem of* (1.1.1) *hold. Then the set* $G_M(a_1, a_2)$ *is orbitally stable.*

This chapter is organized as follows. In Section 2.2, we display some preliminary results. Theorem 2.1.1 and Theorem 2.1.6 will be established in Section 2.3.

Notation 2.1.7. *In this chapter, we write* $L^p(\mathbb{R}^N)$ *the usual Lebesgue space endowed with the norm*

$$\|u\|_p^p := \int_{\mathbb{R}^N} |u|^p \, dx,$$

and $H^1(\mathbb{R}^N)$ *the usual Sobolev space endowed with the norm*

$$\|u\|^2 := \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 \, dx.$$

We denote by $' \to '$ *and* $' \rightharpoonup '$ *strong convergence and weak convergence in corresponding space, respectively, and denote by* $B(x, R)$ *a ball in* $\mathbb{R}^N$ *of center* x *and radius* $R > 0$.

2.2 Preliminary results

Firstly, let us observe that the energy functional J is well-defined in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Indeed, for $r_1, r_2 > 1, r_1 + r_2 < 2^*$, there is $q > 1$ satisfying $2 < r_1 q, r_2 q' \le 2^*$, $q' := \frac{q}{q-1}$. Hence

$$\int_{\mathbb{R}^N} |u_1|^{r_1} |u_2|^{r_2} \, dx \le \|u_1\|_{r_1 q}^{r_1} \|u_2\|_{r_2 q'}^{r_2} < \infty.$$

The Gagliardo-Nirenberg's inequality for $u \in H^1(\mathbb{R}^N)$ and $2 \le p \le 2^*$,

$$\|u\|_p \le C(N, p) \|\nabla u\|_2^{\alpha} \|u\|_2^{1-\alpha}, \quad \text{where} \alpha := \frac{N(p-2)}{2p},$$

this implies for $(u_1, u_2) \in S(a_1, a_2)$:

$$\begin{aligned} \int_{\mathbb{R}^N} |u_1|^{p_1} \, dx &\le C(N, p_1, a_1) \|\nabla u_1\|_2^{\frac{N(p_1-2)}{2}}, \\ \int_{\mathbb{R}^N} |u_2|^{p_2} \, dx &\le C(N, p_2, a_2) \|\nabla u_2\|_2^{\frac{N(p_2-2)}{2}}, \end{aligned} \tag{2.2.1}$$

and

$$\int_{\mathbb{R}^N} |u_1|^{r_1}|u_2|^{r_2}\,dx \le \|u_1\|_{r_1q}^{r_1}\|u_2\|_{r_2q'}^{r_2} \le C\|\nabla u_1\|_2^{\frac{N(r_1q-2)}{2q}}\|\nabla u_2\|_2^{\frac{N(r_2q'-2)}{2q'}} \tag{2.2.2}$$

with $C = C(N, r_1, r_2, a_1, a_2, q)$.

Now recall the rearrangement results of Shibata [100] as presented in [61]. Let u be a Borel measurable function on $\mathbb{R}^N$. It is said to vanish at infinity if $|\{x \in \mathbb{R}^N : |u(x)| > t\}| < \infty$ for every $t > 0$. Here $|A|$ stands for the N-dimensional Lebesgue measure of a Lebesgue mesurable set $A \subset \mathbb{R}^N$. Considering two Borel mesurable functions u, v which vanish at infinity in $\mathbb{R}^N$, we define for $t > 0$, $A^\star(u, v; t) := \{x \in \mathbb{R}^N : |x| < r\}$ where $r > 0$ is chosen so that

$$|B(0, r)| = |\{x \in \mathbb{R}^N : |u(x)| > t\}| + |\{x \in \mathbb{R}^N : |v(x)| > t\}|,$$

and $\{u, v\}^\star$ by

$$\{u, v\}^\star(x) := \int_0^\infty \chi_{A^\star(u,v;t)}(x)\,dt,$$

where $\chi_A(x)$ is a characteristic function of the set $A \subset \mathbb{R}^N$.

Lemma 2.2.1. *[61, Lemma A.1]*

(i) The function $\{u, v\}^\star$ is radially symmetric, non-increasing and lower semi-continuous. Moreover, for each $t > 0$, there holds $\{x \in \mathbb{R}^N : \{u, v\}^\star > t\} = A^\star(u, v; t)$.

(ii) Let $\Phi : [0, \infty) \to [0, \infty)$ be increasing, lower semicontinuous, continuous at 0 and $\Phi(0) = 0$. Then $\{\Phi(u), \Phi(v)\}^\star = \Phi(\{u, v\}^\star)$.

(iii) $\|\{u, v\}^\star\|_p^p = \|u\|_p^p + \|v\|_p^p$ for $1 \le p < \infty$.

(iv) If $u, v \in H^1(\mathbb{R}^N)$, then $\{u, v\}^\star \in H^1(\mathbb{R}^N)$ and $\|\nabla\{u, v\}^\star\|_2^2 \le \|\nabla u\|_2^2 + \|\nabla v\|_2^2$. In addition, if $u, v \in (H^1(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)) \setminus \{0\}$ are radially symmetric, positive and decreasing, then

$$\int_{\mathbb{R}^N} |\nabla\{u, v\}^\star|^2\,dx < \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} |\nabla v|^2\,dx.$$

(v) Let $u_1, u_2, v_1, v_2 \ge 0$ be Borel measurable functions which vanish at infinity, then

$$\int_{\mathbb{R}^N} (u_1u_2 + v_1v_2)\,dx \le \int_{\mathbb{R}^N} \{u_1, v_1\}^\star\{u_2, v_2\}^\star\,dx.$$

2.3 Proofs of the main results

In this scetion, we are aim at proving Theorem 2.1.1 - 2.1.6. Hereafter, we use the same notation $M(a_1, a_2)$ for $a_1, a_2 \ge 0$, namely, one component of (a_1, a_2) may be zero.

In what follows, we collect some basic properties of $M(a_1, a_2)$.

Lemma 2.3.1. *(i) If $a_1, a_2 \ge 0$ with either $a_1 > 0$ or $a_2 > 0$, then $-\infty < M(a_1, a_2) < 0$.*

(ii) $M(a_1, a_2)$ is continuous with respect to $a_1, a_2 \ge 0$.

(iii) If $a_1 \ge b_1 \ge 0, a_2 \ge b_2 \ge 0$, then $M(a_1, a_2) \le M(b_1, b_2) + M(a_1 - b_1, a_2 - b_2)$.

Proof. (i) Observe that $\frac{N(p_i-2)}{2} < 2$ by $p_i < 2 + \frac{4}{N}$ for $i = 1, 2$ and that

$$\frac{N(r_1 q - 2)}{2q} + \frac{N(r_2 q' - 2)}{2q'} < 2,$$

owing to $r_1 + r_2 < 2 + \frac{4}{N}$. Thus, it follows from (2.2.1)-(2.2.2) that J is coercive and in particular, $M(a_1, a_2) > -\infty$. Now taking into account that $\beta > 0$, one has

$$M(a_1, a_2) \leq M(a_1, 0) + M(0, a_2).$$

Since $2 < p_1, p_2 < 2 + \frac{4}{N}$, it is standard to show that $M(a_1, 0) < 0$ if $a_1 > 0$ and $M(0, a_2) < 0$ if $a_2 > 0$. Thus $M(a_1, a_2) < 0$.

(ii) We assume $(a_1^n, a_2^n) = (a_1, a_2) + o_n(1)$, where $o_n(1) \to 0$ as $n \to 0$. From the definition of $M(a_1^n, a_2^n)$, for any $\epsilon > 0$, there exists $(u_1^n, u_2^n) \in S(a_1^n, a_2^n)$ such that

$$J(u_1^n, u_2^n) \leq M(a_1^n, a_2^n) + \epsilon. \tag{2.3.1}$$

Setting

$$v_i^n := \frac{u_i^n}{\|u_i^n\|_2} a_i^{\frac{1}{2}}$$

for $i = 1, 2$, we have that $(v_1^n, v_2^n) \in S(a_1, a_2)$ and

$$M(a_1, a_2) \leq J(v_1^n, v_2^n) = J(u_1^n, u_2^n) + o_n(1). \tag{2.3.2}$$

Combining (2.3.1) and (2.3.2) we obtain

$$M(a_1, a_2) \leq M(a_1^n, a_2^n) + \epsilon + o_n(1).$$

Reversing the arguments, we obtain similarly that

$$M(a_1^n, a_2^n) \leq M(a_1, a_2) + \epsilon + o(1).$$

Therefore, since $\epsilon > 0$ is arbitrary, we deduce that $M(a_1^n, a_2^n) = M(a_1, a_2) + o_n(1)$.

(iii) By density of $C_0^\infty(\mathbb{R}^N)$ in $H^1(\mathbb{R}^N)$, for any $\epsilon > 0$, there exist $(\bar{\varphi}_1, \bar{\varphi}_2)$, $(\hat{\varphi}_1, \hat{\varphi}_2) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$ with $\|\bar{\varphi}_i\|_2^2 = b_i$, $\|\hat{\varphi}_i\|_2^2 = a_i - b_i$ for $i = 1, 2$ such that

$$J(\bar{\varphi}_1, \bar{\varphi}_2) \leq M(b_1, b_2) + \frac{\epsilon}{2},$$
$$J(\hat{\varphi}_1, \hat{\varphi}_2) \leq M(a_1 - b_1, a_2 - b_2) + \frac{\epsilon}{2}.$$

Since J is invariant by translations, without loss of generality, we may assume that $supp\, \bar{\varphi}_i \cap supp\, \hat{\varphi}_i = \emptyset$, and then $\|\bar{\varphi}_i + \hat{\varphi}_i\|_2^2 = \|\bar{\varphi}_i\|_2^2 + \|\hat{\varphi}_i\|_2^2 = a_i$ for $i = 1, 2$, as well as

$$M(a_1, a_2) \leq J(\bar{\varphi}_1 + \hat{\varphi}_1, \bar{\varphi}_2 + \hat{\varphi}_2) \leq M(b_1, b_2) + M(a_1 - b_1, a_2 - b_2) + \epsilon.$$

Thus

$$M(a_1, a_2) \leq M(b_1, b_2) + M(a_1 - b_1, a_2 - b_2).$$

□

Lemma 2.3.2. *Assume $r_1, r_2 > 1, r_1 + r_2 < 2 + \frac{4}{N}$. If $(u_1^n, u_2^n) \rightharpoonup (u_1, u_2)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} |u_1^n|^{r_1} |u_2^n|^{r_2} - |u_1^n - u_1|^{r_1} |u_2^n - u_2|^{r_2}\, dx = \int_{\mathbb{R}^N} |u_1|^{r_1} |u_2|^{r_2}\, dx + o_n(1).$$

Proof. Since this lemma can be proved following closely the approach of [38, Lemma 2.3], we only provide the outline of the proof. For any $b_1, b_2, c_1, c_2 \in \mathbb{R}$ and $\epsilon > 0$, set $r := r_1 + r_2$. The mean value theorem and Young's inequality lead to

$$\begin{aligned}&\big||b_1+b_2|^{r_1}|c_1+c_2|^{r_2} - |b_1|^{r_1}|c_1|^{r_2}\big| \\ &\le C\epsilon\big(|b_1|^r + |c_1|^r + |b_2|^r + |c_2|^r\big) + C_\epsilon\big(|b_2|^r + |c_2|^r\big).\end{aligned}$$

Denote $b_1 := u_1^n - u_1, c_1 := u_2^n - u_2, b_2 := u_1, c_2 := u_2$. Then

$$\begin{aligned}f_n^\epsilon := \big[&\big||u_1^n|^{r_1}|u_2^n|^{r_2} - |u_1^n - u_1|^{r_1}|u_2^n - u_2|^{r_2} - |u_1|^{r_1}|u_2|^{r_2}\big| \\ &- C\epsilon\big(|u_1^n - u_1|^r + |u_2^n - u_2|^r + |u_1|^r + |u_2|^r\big)\big]^+ \\ &\le |u_1|^{r_1}|u_2|^{r_2} + C_\epsilon\left(|u_1|^r + |u_2|^r\right),\end{aligned}$$

where $u^+(x) := \max\{u(x), 0\}$, so the dominated convergence theorem implies that

$$\int_{\mathbb{R}^N} f_n^\epsilon \, dx \to 0 \quad \text{as } n \to \infty. \tag{2.3.3}$$

Since

$$\begin{aligned}&\big||u_1^n|^{r_1}|u_2^n|^{r_2} - |u_1^n - u_1|^{r_1}|u_2^n - u_2|^{r_2} - |u_1|^{r_1}|u_2|^{r_2}\big| \\ &\le f_n^\epsilon + C\epsilon\big(|u_1^n - u_1|^r + |u_2^n - u_2|^r + |u_1|^r + |u_2|^r\big),\end{aligned}$$

by the boundedness of $\{(u_1^n, u_2^n)\}$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and (2.3.3), it follows that

$$\int_{\mathbb{R}^N} |u_1^n|^{r_1}|u_2^n|^{r_2} - |u_1^n - u_1|^{r_1}|u_2^n - u_2|^{r_2} \, dx = \int_{\mathbb{R}^N} |u_1|^{r_1}|u_2|^{r_2} \, dx + o_n(1).$$

□

Lemma 2.3.3. *Any minimizing sequence to* (2.1.4) *is, up to translations, strongly convergent in* $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ *for* $2 < p < 2^*$.

Proof. Assume that $\{(u_1^n, u_2^n)\} \subset S(a_1, a_2)$ is a minimizing sequence to (2.1.4). By the coerciveness of the energy functional J on $S(a_1, a_2)$, $\{(u_1^n, u_2^n)\}$ is bounded in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. If

$$\sup_{y \in \mathbb{R}^N} \int_{B(y,R)} |u_1^n|^2 + |u_2^n|^2 \, dx = o_n(1),$$

for some $R > 0$, then $u_i \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*, i = 1, 2$, see [74, Lemma I.1]. This is incompatible with the fact that $M(a_1, a_2) < 0$, see Lemma 2.3.1 (i). Thus, there exist a $\beta_0 > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^N$ such that

$$\int_{B(y_n,R)} |u_1^n|^2 + |u_2^n|^2 \, dx \ge \beta_0,$$

and we deduce from the weak convergence in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and the local compactness in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ that $(u_1^n(x - y_n), u_2^n(x - y_n)) \rightharpoonup (u_1, u_2) \neq (0, 0)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Our aim is to prove that $w_i^n(x) := u_i^n(x) - u_i(x + y_n) \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*, i = 1, 2$. To do this, we suppose by contradiction that there exists a $2 < q < 2^*$ such that $(w_1^n, w_2^n) \nrightarrow (0, 0)$ in $L^q(\mathbb{R}^N) \times L^q(\mathbb{R}^N)$. Note that under this assumption there exists a sequence $\{z_n\} \subset \mathbb{R}^N$ such that

$$(w_1^n(x - z_n), w_2^n(x - z_n)) \rightharpoonup (w_1, w_2) \neq (0, 0)$$

in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^N)$. Indeed, otherwise

$$\sup_{y\in\mathbb{R}^N} \int_{B(y,R)} |w_1^n|^2 + |w_2^n|^2\, dx = o_n(1),$$

which leads to $(w_1^n, w_2^n) \to (0,0)$ in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ for $2 < p < 2^*$.

Now, combining the Brezis-Lieb Lemma, Lemma 2.3.2 and the translational invariance we conclude

$$\begin{aligned}
&J(u_1^n, u_2^n) = J(u_1^n(x-y_n), u_2^n(x-y_n))\\
&= J(u_1^n(x-y_n) - u_1 + u_1, u_2^n(x-y_n) - u_2 + u_2)\\
&= J(u_1^n(x-y_n) - u_1, u_2^n(x-y_n) - u_2) + J(u_1,u_2) + o_n(1)\\
&= J(w_1^n(x-y_n), w_2^n(x-y_n)) + J(u_1,u_2) + o_n(1)\\
&= J(w_1^n(x-z_n), w_2^n(x-z_n)) + J(u_1,u_2) + o_n(1)\\
&= J(w_1^n(x-z_n) - w_1 + w_1, w_2^n(x-z_n) - w_2 + w_2) + J(u_1,u_2) + o_n(1)\\
&= J(w_1^n(x-z_n) - w_1, w_2^n(x-z_n) - w_2) + J(w_1,w_2) + J(u_1,u_2) + o_n(1),
\end{aligned} \tag{2.3.4}$$

and

$$\begin{aligned}
\|u_i^n(x-y_n)\|_2^2 &= \|u_i^n(x-y_n) - u_i + u_i\|_2^2\\
&= \|u_i^n(x-y_n) - u_i\|_2^2 + \|u_i\|_2^2 + o_n(1)\\
&= \|w_i^n(x-z_n) - w_i + w_i\|_2^2 + \|u_i\|_2^2 + o_n(1)\\
&= \|w_i^n(x-z_n) - w_i\|_2^2 + \|w_i\|_2^2 + \|u_i\|_2^2 + o_n(1).
\end{aligned}$$

Thus

$$\begin{aligned}
\|w_i^n(x-z_n) - w_i\|_2^2 &= \|u_i^n(x-y_n)\|_2^2 - \|w_i\|_2^2 - \|u_i\|_2^2 + o_n(1)\\
&= a_i - \|w_i\|_2^2 - \|u_i\|_2^2 + o_n(1)\\
&= b_i + o_n(1),
\end{aligned} \tag{2.3.5}$$

where $b_i := a_i - \|w_i\|_2^2 - \|u_i\|_2^2$. Noting that

$$\begin{aligned}
\|w_i\|_2^2 &\le \liminf_{n\to\infty} \|w_i^n(x-z_n)\|_2^2 = \liminf_{n\to\infty} \|u_i^n(x-y_n) - u_i\|_2^2\\
&= a_i - \|u_i\|_2^2,
\end{aligned}$$

then $b_i \ge 0$ for $i = 1, 2$. Recording that $J(u_1^n, u_2^n) \to M(a_1, a_2)$, in view of (2.3.5), Lemma 2.3.1 (ii) and (2.3.4), we get

$$M(a_1,a_2) \ge M(b_1,b_2) + J(w_1,w_2) + J(u_1,u_2). \tag{2.3.6}$$

If $J(w_1,w_2) > M(\|w_1\|_2^2, \|w_2\|_2^2)$ or $J(u_1,u_2) > M(\|u_1\|_2^2, \|u_2\|_2^2)$, then, from (2.3.6) and Lemma 2.3.1 (iii), it follows

$$M(a_1,a_2) > M(b_1,b_2) + M(\|w_1\|_2^2, \|w_2\|_2^2) + M(\|u_1\|_2^2, \|u_2\|_2^2) \ge M(a_1,a_2),$$

which is impossible. Hence

$$J(w_1,w_2) = M(\|w_1\|_2^2, \|w_2\|_2^2), \quad J(u_1,u_2) = M(\|u_1\|_2^2, \|u_2\|_2^2).$$

We denote by u_i^*, w_i^* the classical Schwartz's rearrangement of u_i, w_i for $i = 1, 2,$. Since

$$\|u_i^*\|_2^2 = \|u_i\|_2^2, \quad \|w_i^*\|_2^2 = \|w_i\|_2^2,$$

$$J(u_1^*, u_2^*) \leq J(u_1, u_2), \quad J(w_1^*, w_2^*) \leq J(w_1, w_2)$$

see for example [68], we deduce that

$$J(u_1^*, u_2^*) = M(\|u_1\|_2^2, \|u_2\|_2^2), \quad J(w_1^*, w_2^*) = M(\|w_1\|_2^2, \|w_2\|_2^2).$$

Therefore, $(u_1^*, u_2^*), (w_1^*, w_2^*)$ are solutions of the system (2.1.1) and from standard regularity results we have that $u_i^*, w_i^* \in C^2(\mathbb{R}^N)$ for $i = 1, 2$.

At this point, Lemma 2.2.1 comes into play. Without restriction, we may assume $u_1 \neq 0$. We divide into two cases.

Case 1: $u_1 \neq 0$ *and* $w_1 \neq 0$.
By virtue of Lemma 2.2.1 $(ii), (iv), (v)$,

$$\int_{\mathbb{R}^N} |\nabla\{u_1^*, w_1^*\}^\star|\, dx < \int_{\mathbb{R}^N} |\nabla u_1^*|^2 + |\nabla w_1^*|^2\, dx \leq \int_{\mathbb{R}^N} |\nabla u_1|^2 + |\nabla w_1|^2\, dx,$$

$$\begin{aligned}
\int_{\mathbb{R}^N} |\{u_1^*, w_1^*\}^\star|^{r_1} |\{\tilde{u}_2, w_2^*\}^\star|^{r_2}\, dx &= \int_{\mathbb{R}^N} \{|u_1^*|^{r_1}, |w_1^*|^{r_1}\}^\star \{|u_2^*|^{r_2}, |w_2^*|^{r_2}\}^\star\, dx, \\
&\geq \int_{\mathbb{R}^N} |u_1^*|^{r_1} |u_2^*|^{r_2} + |\tilde{w}_1|^{r_1} |w_2^*|^{r_2}\, dx \\
&= \int_{\mathbb{R}^N} (|u_1|^{r_1})^* (|u_2|^{r_2})^* + (|w_1|^{r_1})^* (|w_2|^{r_2})^*\, dx, \\
&\geq \int_{\mathbb{R}^N} |u_1|^{r_1} |u_2|^{r_2} + |w_1|^{r_1} |w_2|^{r_2}\, dx,
\end{aligned}$$

and thus

$$J(u_1, u_2) + J(w_1, w_2) > J(\{u_1^*, w_1^*\}^\star, \{u_2^*, w_2^*\}^\star). \tag{2.3.7}$$

Also from Lemma 2.2.1 (iii), for $i = 1, 2$,

$$\int_{\mathbb{R}^N} |\{u_i^*, w_i^*\}^\star|^2\, dx = \int_{\mathbb{R}^N} |u_i^*|^2 + |w_i^*|^2\, dx = \int_{\mathbb{R}^N} |u_i|^2 + |w_i|^2\, dx, \tag{2.3.8}$$

and taking (2.3.6)-(2.3.8) and Lemma 2.3.1 (iii) into consideration, one obtains the contradiction

$$M(a_1, a_2) > M(b_1, b_2) + M(a_1 - b_1, a_2 - b_2) \geq M(a_1, a_2).$$

Case 2: $u_1 \neq 0$, $w_1 = 0$ *and* $w_2 \neq 0$.
If $u_2 \neq 0$, we can reverse the role of u_1, w_1 and u_2, w_2 in *Case 1* to get a contradiction. Thus, we suppose that $u_2 = 0$. Due to Lemma 2.2.1 (ii)-(v),

$$\begin{aligned}
J(\{u_1^*, 0\}^\star, \{w_2^*, 0\}^\star) &\leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_1^*|^2 + |\nabla w_2^*|^2\, dx - \frac{\mu_1}{p_1} \int_{\mathbb{R}^N} |u_1^*|^{p_1}\, dx \\
&\quad - \frac{\mu_2}{p_2} \int_{\mathbb{R}^N} |w_2^*|^{p_2}\, dx - \beta \int_{\mathbb{R}^N} |u_1^*|^{r_1} |w_2^*|^{r_2} \\
&< J(u_1^*, 0) + J(0, w_2^*) \\
&\leq J(u_1, 0) + J(0, w_2),
\end{aligned} \tag{2.3.9}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^N} |\{u_1^*, 0\}^\star|^2\, dx &= \int_{\mathbb{R}^N} |u_1^*|^2\, dx = \int_{\mathbb{R}^N} |u_1|^2\, dx, \\
\int_{\mathbb{R}^N} |\{w_2^*, 0\}^\star|^2\, dx &= \int_{\mathbb{R}^N} |w_2^*|^2\, dx = \int_{\mathbb{R}^N} |w_2|^2\, dx.
\end{aligned} \tag{2.3.10}$$

Thus using (2.3.6), (2.3.9), (2.3.10) and Lemma 2.3.1, we also have that

$$M(a_1, a_2) > M(b_1, b_2) + M(a_1 - b_1, a_2 - b_2) \geq M(a_1, a_2).$$

The contradictions obtained in *Cases* 1-2 indicate that $w_i^n(x) = u_i^n(x) - u_i(x + y_n) \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*, i = 1, 2$. □

Proof of Theorem 2.1.1. Let $\{(u_1^n, u_2^n)\} \subset S(a_1, a_2)$ be a minimizing sequence to (2.1.4). In light of Lemma 2.3.3, $(u_1^n, u_2^n) \to (u_1, u_2)$, up to translations, in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ for $2 < p < 2^*$. Hence by the weakly lower semicontinuous of norm,

$$J(u_1, u_2) \leq M(a_1, a_2). \tag{2.3.11}$$

Note that if $||u_1||_2^2 = a_1$ and $||u_2||_2^2 = a_2$, we have done. Indeed, the compactness of $\{(u_1^n, u_2^n)\}$ then directly follows. To show that $||u_1||_2^2 = a_1$ and $||u_2||_2^2 = a_2$, we assume by contradiction that $\|u_1\|_2^2 := \bar{a}_1 < a_1$ or $\|u_2\|_2^2 := \bar{a}_2 < a_2$. By definition, $J(u_1, u_2) \geq M(\bar{a}_1, \bar{a}_2)$ and thus it results from (2.3.11) that

$$M(\bar{a}_1, \bar{a}_2) \leq M(a_1, a_2). \tag{2.3.12}$$

At this point, from Lemma 2.3.1 (iii), $M(a_1, a_2) \leq M(\bar{a}_1, \bar{a}_2) + M(a_1 - \bar{a}_1, a_2 - \bar{a}_2)$ and Lemma 2.3.1 (i), $M(a_1 - \bar{a}_1, a_2 - \bar{a}_2) < 0$, we have reached a contradiction from (2.3.12), then Theorem 2.1.1 follows. □

Remark 2.3.4. *As indicated in Remark 2.1.3, a proof for the existence of minimizers to* (2.1.4) *can be given by working directly in* $H_{rad}^1(\mathbb{R}^N) \times H_{rad}^1(\mathbb{R}^N)$. *In such space, the strong convergence in* $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ *for* $2 < p < 2^*$, *and* $N \geq 2$, *is given for free. Now define*

$$M_r(a_1, a_2) := \inf_{(u_1, u_2) \in S_r(a_1, a_2)} J(u_1, u_2), \tag{2.3.13}$$

where

$$S_r(a_1, a_2) := \{(u_1, u_2) \in H_{rad}^1(\mathbb{R}^N) \times H_{rad}^1(\mathbb{R}^N) : \|u_1\|_2^2 = a_1, \|u_2\|_2^2 = a_2\}.$$

We observe that

$$M_r(a_1, a_2) \leq M_r(b_1, b_2) + M_r(a_1 - b_1, a_2 - b_2), \tag{2.3.14}$$

where $0 \leq b_i \leq a_i$ *for* $i = 1, 2$. *Indeed, since for any minimizing sequence to* (2.1.4), *one can find a radially symmetric minimizing sequence by the Schwartz's rearrangement, thus it results that* $M_r(a_1, a_2) = M(a_1, a_2)$ *for any* $a_1 \geq 0, a_2 \geq 0$, *and* (2.3.14) *then follows from Lemma 2.3.1 (iii). Thus we can end the proof as previously.*

We now turn to the proof of Theorem 2.1.6, whose proof relies on the classical arguments of Cazenave and Lions [36], hence we only give a sketch.

Proof of Theorem 2.1.6. By contradiction, we assume that there is a $\epsilon_0 > 0$, $(\Psi_1^n(0), \Psi_2^n(0)) \subset H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, and $\{t_n\} \subset \mathbb{R}^+$ such that

$$\inf_{(u_1, u_2) \in G(a_1, a_2)} \|(\Psi_1^n(0), \Psi_2^n(0)) - (u_1, u_2)\| \to 0,$$

and

$$\inf_{(u_1,u_2)\in G(a_1,a_2)} \|(\Psi_1^n(t_n),\Psi_2^n(t_n))-(u_1,u_2)\| \geq \epsilon_0, \tag{2.3.15}$$

where $(\Psi_1^n(t),\Psi_2^n(t))$ is solution to the Cauchy problem of (2.1.3) with initial datum $(\Psi_1^n(0),\Psi_2^n(0))$. By the conservation laws,

$$\|\Psi_i^n(t_n)\|_2^2 = \|\Psi_i^n(0)\|_2^2, \quad \text{for } i=1,2,$$

also

$$J(\Psi_1^n(t_n),\Psi_2^n(t_n)) = J(\Psi_1^n(0),\Psi_2^n(0)).$$

Define

$$\hat{\Psi}_i^n = \frac{\Psi_i^n(t_n)}{\|\Psi_i^n(t_n)\|_2^2} a_i^{\frac{1}{2}}, \quad \text{for } i=1,2,$$

we get that

$$\|\hat{\Psi}_i^n\|_2^2 = a_i, \quad J(\hat{\Psi}_1^n,\hat{\Psi}_2^n) = M(a_1,a_2)+o_n(1).$$

Namely, $\{(\hat{\Psi}_1^n,\hat{\Psi}_2^n)\}$ is a minimizing sequence to (2.1.1). From Theorem 2.1.1, it follows that it is compact up to translation in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, thus (2.3.15) fails. We reach a contradiciton. □

Chapter 3

Multiple normalized solutions for coupled nonlinear Schrödinger system

3.1 Introduction

In this chapter, we are concerned with standing waves to the following coupled nonlinear Schrödinger system in $\mathbb{R} \times \mathbb{R}^N$,

$$\begin{cases} -i\partial_t \Psi_1 = \Delta \Psi_1 + \mu_1 |\Psi_1|^{p_1-2}\Psi_1 + \beta |\Psi_1|^{r_1-2}\Psi_1 |\Psi_2|^{r_2}, \\ -i\partial_t \Psi_2 = \Delta \Psi_2 + \mu_2 |\Psi_2|^{p_2-2}\Psi_2 + \beta |\Psi_1|^{r_1} |\Psi_2|^{r_2-2}\Psi_2. \end{cases} \tag{3.1.1}$$

Here by standing waves to (3.1.1), we mean solutions with the form of

$$\Psi_1(t,x) = e^{-i\lambda_1 t}u_1(x), \quad \Psi_2(t,x) = e^{-i\lambda_2 t}u_2(x)$$

for $(\lambda_1, \lambda_2) \in \mathbb{R}^2$. This then gives rise to the following elliptic system satisfied by u_1 and u_2,

$$\begin{cases} -\Delta u_1 = \lambda_1 u_1 + \mu_1 |u_1|^{p_1-2}u_1 + \beta r_1 |u_1|^{r_1-2}u_1 |u_2|^{r_2}, \\ -\Delta u_2 = \lambda_2 u_2 + \mu_2 |u_2|^{p_2-2}u_2 + \beta r_2 |u_1|^{r_1} |u_2|^{r_2-2}u_2. \end{cases} \tag{3.1.2}$$

Notice that the L^2-norm of solution to the Cauchy problem of (3.1.1) is conserved along time, i.e. for any $t > 0$,

$$\int_{\mathbb{R}^N} |\Psi_i(t,x)|^2 \, dx = \int_{\mathbb{R}^N} |\Psi_i(0,x)|^2 \, dx \quad \text{for} \quad i = 1, 2.$$

Thus it is of particular interest to study solutions to (3.1.2) having prescribed L^2-norm. More precisely, for given $a_1, a_2 > 0$, to search for $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ and $(u_1, u_2) \in H^1(\mathbb{R}^N) \times H^1(R^N)$ satisfying (3.1.2), together with normalized condition

$$\int_{\mathbb{R}^N} |u_1|^2 \, dx = a_1, \quad \int_{\mathbb{R}^N} |u_2|^2 \, dx = a_2. \tag{3.1.3}$$

Such solutions are often referred as normalized solutions. In what follows, for the sake of convenience, we identify a solution $(\lambda_1, \lambda_2, u_1, u_2)$ to (3.1.2)-(3.1.3) with (u_1, u_2), where

(u_1, u_2) is obtained as a critical point of energy functional $J : H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$J(u_1, u_2) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_1|^2 + |\nabla u_2|^2\, dx - \sum_{i=1}^{2} \frac{\mu_i}{p_i}\int_{\mathbb{R}^N} |u_i|^{p_i}\, dx - \beta\int_{\mathbb{R}^N} |u_1|^{r_1}|u_2|^{r_2}\, dx$$

on the constraint $S(a_1, a_2) := S(a_1) \times S(a_2)$ with

$$S(a) := \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2\, dx = a > 0\},$$

and (λ_1, λ_2) is determined as Lagrange multipliers.

When $2 < p_1, p_2 < 2 + \frac{4}{N}$, $r_1, r_2 > 1, r_1 + r_2 < 2 + \frac{4}{N}$, the energy functional J is bounded from below on $S(a_1, a_2)$. Then one may search for a critical point for J restricted to $S(a_1, a_2)$ as a global minimizer for J subject to $S(a_1, a_2)$ through studying the compactness of any minimizing sequence. In this direction, a more complete result was recently obtained in [58]. On the contrary, if $\mu_1, \mu_2, \beta > 0$, when either $p_i > 2 + \frac{4}{N}$ for some $i = 1, 2$ or $r_1 + r_2 > 2 + \frac{4}{N}$, then the energy functional J becomes unbounded from below on $S(a_1, a_2)$. To see this, for $t > 0$, let us introduce the scaling of $u \in H^1(\mathbb{R}^N)$ as

$$u^t(x) := t^{\frac{N}{2}} u(tx).$$

Clearly, $\|u^t\|_2 = \|u\|_2$. A direct calculation then shows that for any $(u_1, u_2) \in S(a_1, a_2)$,

$$\begin{aligned} J(u_1^t, u_2^t) = {} & \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla u_1|^2 + |\nabla u_2|^2\, dx - \sum_{i=1}^{2} t^{(\frac{p_i}{2}-1)N}\frac{\mu_i}{p_i}\int_{\mathbb{R}^N} |u_i|^{p_i}\, dx \\ & - \beta t^{(\frac{r_1+r_2}{2}-1)N}\int_{\mathbb{R}^N} |u_1|^{r_1}|u_2|^{r_2}\, dx, \end{aligned} \tag{3.1.4}$$

from which the claim immediately follows.

When global minimizer fail to exist, finding a critical point for J restricted to $S(a_1, a_2)$ is more delicate and involved. In such situation, minimax methods come into play. When $2 \le N \le 4$, $\mu_1, \mu_2, \beta > 0$, if either $2 < p_1 < 2 + \frac{4}{N} < p_2 < 2^*$, $2 + \frac{4}{N} < r_1 + r_2 < 2^*, r_2 > 2$ or $2 + \frac{4}{N} < p_1, p_2, r_1 + r_2 < 2^*$, the authors [8] studied the existence of positive solution to (3.1.2)-(3.1.3), see also [12]. When $N = 3$, $\mu_1, \mu_2 > 0, \beta < 0$, $p_1 = p_2 = 4, r_1 = r_2 = 2$, the existence of positive solution to (3.1.2)-(3.1.3) was also established in [13], concerning a multiplicity result, we refer to [14]. Let us also mention the papers [86, 87, 94], where the authors considered the existence of normalized solutions to problem confined on a bounded domain in $\mathbb{R}^N$ or with a trapping potential. Although more compactness is available in these cases, but it is unlikely to take advantage of the dilations, which play an essential role in [7, 8, 12, 13, 17, 18, 63]. When a periodic potential is included in equation, the existence of normalized solutions was discussed in [1].

In this chapter, as mainly inspired by [8, 12], we consider the existence of multiple solutions to (3.1.2)-(3.1.3) under the following two new assumptions,

(H_1) $\mu_1, \mu_2, \beta > 0$, $2 < p_1, p_2 < 2 + \frac{4}{N}$, $r_1, r_2 > 1$, $2 + \frac{4}{N} < r_1 + r_2 < 2^*$;

(H_2) $\mu_1, \mu_2, \beta > 0$, $2 + \frac{4}{N} < p_1, p_2 < 2^*$, $r_1, r_2 > 1$, $r_1 + r_2 < 2 + \frac{4}{N}$.

From above observations, the energy functional J is not bounded from below on $S(a_1, a_2)$ under (H_1) or (H_2). Thus in order to find a critical point for J restricted to $S(a_1, a_2)$, we are indeed based on the minimax methods. Our aim is to prove that(3.1.2)-(3.1.3) admits two positive solutions when $N \geq 1$ and (H_1) or (H_2) holds. Up to our knowledge, it is the first time that a multiplicity result to (3.1.2)-(3.1.3) is obtained when $N \geq 1, \beta > 0$.

In order to address our results, for $\rho > 0$, let us introduce

$$\mathcal{B}(\rho) := \{(u_1, u_2) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla u_1|^2 + |\nabla u_2|^2 \, dx < \rho\}.$$

On account of (3.1.4), under either (H_1) or (H_2), for any $\rho > 0$ there holds

$$\inf J(u_1, u_2) < 0 \quad for \quad (u_1, u_2) \in S(a_1, a_2) \cap \mathcal{B}(\rho), \tag{3.1.5}$$

see Lemma 3.2.4. Furthermore, we will prove that there exist $\beta_0 = \beta_0(a_1, a_2) > 0$ and $\rho_0 = \rho_0(a_1, a_2) > 0$ such that

$$\inf J(u_1, u_2) > 0 \quad for \quad (u_1, u_2) \in S(a_1, a_2) \cap \partial\mathcal{B}(\rho_0), \tag{3.1.6}$$

for any $0 < \beta \leq \beta_0$, see Lemma 3.3.1.

Together (3.1.5) with (3.1.6), then there may admit a local minimizer for J restricted to $S(a_1, a_2) \cap \mathcal{B}(\rho)$. Thus for $0 < \beta \leq \beta_0$, it is natural to introduce the following minimization problem

$$m(a_1, a_2) := \inf_{(u_1,u_2) \in S(a_1,a_2) \cap \mathcal{B}(\rho_0)} J(u_1, u_2). \tag{3.1.7}$$

Obviously, minimizers to (3.1.7) are critical points for J restricted to $S(a_1, a_2)$, i.e. solutions to (3.1.2)-(3.1.3). We shall prove that any minimizing sequence to (3.1.7) is compact, up to translations, in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, and in particular this implies the existence of a critical point for J restricted to $S(a_1, a_2)$ as a minimizer to (3.1.7).

As observed from (3.1.4), for any $(u_1, u_2) \in S(a_1, a_2)$, we have that $J(u_1^t, u_2^t) \to -\infty$ as $t \to \infty$ when (H_1) or (H_2) holds, and note also that $(u_1^t, u_2^t) \notin \mathcal{B}(\rho_0)$ for $t > 0$ large enough. This property along with (3.1.6) suggest that there may exist other critical points for the energy functional J restricted to $S(a_1, a_2)$. Actually, under (H_1), the second critical ponit is obtained by mountain pass arguments. Under (H_2), inspired by [12], the second one is achieved by a linking type procedure. Let us now state our main results.

Theorem 3.1.1. *Let $a_1, a_2 > 0$ be given and assume that (H_1) holds. Then there exist $\beta_0 = \beta_0(a_1, a_2) > 0$ and $\rho_0 = \rho_0(a_1, a_2) > 0$ such that for any $0 < \beta \leq \beta_0$,*

(i) if $N \geq 1$, any minimizing sequence to (3.1.7) is compact, up to translation, in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. In particular, there exists a positive solution (v_1, v_2) to (3.1.2)-(3.1.3) with $(v_1, v_2) \in \mathcal{B}(\rho_0)$ and $J(v_1, v_2) < 0$;

(ii) If either $2 \leq N \leq 4$ or $N \geq 5$, $p_1, p_2 \leq r_1 + r_2 - \frac{2}{N}$ or $|p_1 - p_2| \leq \frac{2}{N}$, there exists a second positive solution (u_1, u_2) to (3.1.2)-(3.1.3) with $J(u_1, u_2) > 0$.

Theorem 3.1.2. *Let $a_1, a_2 > 0$ be given and assume that (H_2) holds. Then there exist $\beta_0 = \beta_0(a_1, a_2) > 0$ and $\rho_0 = \rho_0(a_1, a_2) > 0$ such that for any $0 < \beta \leq \beta_0$,*

(i) if either $1 \leq N \leq 4$ or $N \geq 5$, $r_i > (\frac{r_1+r_2}{2} - 1)N$ for $i = 1, 2$, any minimizing sequence to (3.1.7) is compact, up to translation, in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. In particular, there exists a positive solution (v_1, v_2) to (3.1.2)-(3.1.3) with $(v_1, v_2) \in \mathcal{B}(\rho_0)$ and $J(v_1, v_2) < 0$;

(ii) If $2 \leq N \leq 4$, there exists a second positive solution (u_1, u_2) to (3.1.2)-(3.1.3) with $J(u_1, u_2) > 0$.

Remark 3.1.3. *i) The value of β_0 in Theorem 3.1.1 and Theorem 3.1.2 can be explicitly computed in terms of N, p_i, a_i, r_i for $i = 1, 2$, instead of being obtained through a limit process. In addition, for any given $\beta > 0$, we can assume that $\beta \leq \beta_0$ at the expense of taking $a_1 > 0$ and $a_2 > 0$ sufficiently small, because $\beta_0(a_1, a_2) \to \infty$ as $a_1, a_2 \to 0$, for this property, see Lemma 3.3.1. Finally we point out that our results are not perturbative.*

ii) The existence of second solution in Theorem 3.1.1 (ii) and Theorem 3.1.2 (ii) is under the condition $N \geq 2$. This is because we search for solutions in the radially symmetric functions space $H^1_{rad}(\mathbb{R}^N) \times H^1_{rad}(\mathbb{R}^N)$, and the compact embedding $H^1_{rad}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for $2 < p < 2^*$ holds when $N \geq 2$.

iii) When $N \geq 2$, we conjecture that Theorem 3.1.1 (ii) is true by only assuming (H_1), we refer to Remark 3.4.5 for a discussion in this direction.

Proving the compactness of any minimizing sequence to (3.1.7) under the assumption (H_1) or (H_2), we make use of the coupled rearrangement arguments due to Shibata [100] as developed by Ikoma [61], instead of directly employing the Lions' compactness concentration principle [73, 74]. This is already the approach as presented in [58], but here we need to adapt it to a local minimization problem (3.1.7). In this case, a new difficulty arises from the fact that in general the sum of two elements in $\mathcal{B}(\rho_0)$ does not belong to $\mathcal{B}(\rho_0)$, and this makes more technical to discuss dichotomy. To overcome this difficulty, we need to analyze carefully some properties of the energy functional J restricted to $S(a_1, a_2) \cap \mathcal{B}(\rho_0)$.

The proofs of Theorem 3.1.1 (ii) and Theorem 3.1.2 (ii) follow the virtue in the papers [8, 12]. Our proofs can be divided into three steps. Firstly, one needs to identify a possible critical level. This is done by introducing a minimax structure of mountain pass type when (H_1) holds, and of linking one when (H_2) holds. Secondly, one has to show that there exists a bounded Palais-Smale sequence $\{(u_1^n, u_2^n)\} \subset S(a_1, a_2)$ for the energy functional J restricted to $S(a_1, a_2)$ at this energy level. This step relies on the presence of a natural constraint of Pohozaev type, on which the energy functional J is coercive. Taking advantage of this constraint and making use of the approach introduced in [63] which consists in adding an artificial variable within the variational procedure, one can end this step. Having obtained a bounded Palais-Smale sequence $\{(u_1^n, u_2^n)\}$ for J restricted to $S(a_1, a_2)$, we denote by (u_1, u_2) its weak limit in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, then (u_1, u_2) solves (1.1.2) with some $(\lambda_1, \lambda_2) \in R^2$, see Lemma 3.2.7. At this point, the last step is to show that $(u_1, u_2) \in S(a_1, a_2)$. It is this step where the limitation on dimension was imposed in [8, 12, 13]. Because the authors applied the Liouville's type results, see Lemma 3.2.2, which is only available when $N \leq 4$, and also used the property that the scalar problem

$$-\Delta w - \lambda w = \mu |w|^{p-2} w, \quad u \in S(a). \tag{3.1.8}$$

has a unique positive radial solution for $\mu > 0$, and $2 < p < 2^*$.

We start to relax these two restrictions. Thus Theorem 3.1.1 (ii) allows to consider the case $N \geq 5$. Indeed, under the assumption (H_1), the second critical point for J restricted to $S(a_1, a_2)$ is found through the mountain pass arguments. More precisely, we first prove

tha there exist $\beta_0 = \beta_0(a_1, a_2) > 0$, $\rho_0 = \rho_0(a_1, a_2) > 0$ and $0 < \bar{\rho} = \bar{\rho}(a_1, a_2) < \rho_0$ such that for any $0 < \beta \leq \beta_0$,

$$\gamma(a_1, a_2) := \inf_{g\in\Gamma} \max_{t\in[0,1]} J(g(t)) > \max\{J(g(0)), J(g(1))\},$$

where

$$\Gamma := \{g \in C([0,1], S(a_1, a_2)) : g(0) \in \mathcal{B}(\bar{\rho}), g(1) \notin \overline{B(\rho_0)} \text{ with } J(g(1)) < 0\}.$$

Having obtained a bounded Palais-Smale sequence for J restricted to $S(a_1, a_2)$ at the level $\gamma(a_1, a_2)$, we denote by (u_1, u_2) its weak limit. Furthermore, (u_1, u_2) solves (3.1.2) with some $(\lambda_1, \lambda_2) \in \mathbb{R}^2$. An appropriate choice of the Palais-Smale sequence insures that

$$J(u_1, u_2) \leq \gamma(a_1, a_2). \tag{3.1.9}$$

When $2 \leq N \leq 4$, the fact that $(u_1, u_2) \in S(a_1, a_2)$ is obtained directly by the Liouville's type results. When $N \geq 5$, we argue by contradiction. If $\bar{a}_1 := ||u_1||_2^2 < a_1$ or $\bar{a}_2 := ||u_2||_2^2 < a_2$, we manage to construct a path $g \in \Gamma$, on which the maximum of J is strictly below $J(u_1, u_2)$. By the characterization of $\gamma(a_1, a_2)$, we thus get

$$\gamma(a_1, a_2) \leq \max_{0\leq t\leq 1} J(g(t)) < J(u_1, u_2),$$

in contradiction with (3.1.9). The construction of this path $g \in \Gamma$ relies on the property that when $2 < p < 2 + \frac{4}{N}$, $\mu > 0$,

$$-\infty < \inf_{u\in S(a)} I(u) < 0, \tag{3.1.10}$$

where $I(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - \frac{\mu}{p}\int_{\mathbb{R}^N} |u|^p\, dx$, and using "adding some masses" technique somehow in the spirit of [62], but here again the coupled rearrangement arguments come into play.

In Theorem 3.1.2 (ii), to look for the second critical point, we establish a linking structure for J restricted to $S(a_1, a_2)$. Since $p > 2 + \frac{4}{N}$, $\mu > 0$, (3.1.10) does not hold, then our proof benefits from the Liouville's type results, which eventually induces the restriction on dimension $N \leq 4$.

We now set

$$G_m(a_1, a_2) := \{(u_1, u_2) \in S(a_1, a_2) \cap \mathcal{B}(\rho_0) : J(u_1, u_2) = m(a_1, a_2)\}.$$

Note that under assumption (H_1) or (H_2), the local well-posedness to the Cauchy problem of (3.1.1) is unknown. The point being that when $1 < r_1, r_2 < 2$, the interaction parts are not Lipschitz continuous, and in particular the uniqueness might fail. As a consequence, our result which states the orbital stability of the set $G_m(a_1, a_2)$ is valid under condition. Having the compactness of any minimizing sequence to (3.1.7) in hand, the proof is a direct adaption of the classical arguments in Cazenave and Lions [36], thus we do not provide it.

Theorem 3.1.4. *Assume that (H_1) or (H_2) with either $1 \leq N \leq 4$ or $N \geq 5$, $r_i > (\frac{r_1+r_2}{2} - 1)N$ for $i = 1, 2$, and the local existence and uniqueness of the Cauchy problem to (3.1.1) hold. Then the set $G_m(a_1, a_2)$ is orbitally stable, i.e. for any $\epsilon > 0$, there exists $\delta > 0$ so that if $(\Psi_{1,0}, \Psi_{2,0}) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ satisfies*

$$\inf_{(u_1,u_2)\in G_m(a_1,a_2)} \|(\Psi_{1,0}, \Psi_{2,0}) - (u_1, u_2)\| \leq \delta,$$

then

$$\sup_{t\geq[0,T)}\inf_{(u_1,u_2)\in G_m(a_1,a_2)}\|(\Psi_1(t),\Psi_2(t))-(u_1,u_2)\|\leq\epsilon,$$

where $(\Psi_1(t),\Psi_2(t))$ *is solution to the Cauchy problem of* (3.1.1) *with initial datum* $(\Psi_{1,0},\Psi_{2,0})$*, T denotes the maximum existence time of solution, and* $\|\cdot\|$ *stands for the standard norm in the Sobolev space* $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$*.*

This chapter is organized as follows. In Section 3.2, we establish some preliminary results. Section 3.3 is devoted to the proofs of Theorem 3.1.1 (i) and Theorem 3.1.2 (i). In Section 3.4, we give the proofs of Theorem 3.1.1 (ii) and Theorem 3.1.2 (ii). Finally, in Appendix we establish a key technical result, Lemma 3.4.4.

Notation 3.1.5. *In this chapter, for any* $1\leq p<\infty$*, we write* $L^p(\mathbb{R}^N)$ *the usual Lebesgue space endowed with the norm*

$$\|u\|_p^p:=\int_{\mathbb{R}^N}|u|^p\,dx,$$

and $H^1(\mathbb{R}^N)$ *the usual Sobolev space endowed with the norm*

$$\|u\|^2:=\int_{\mathbb{R}^N}|\nabla u|^2+|u|^2\,dx.$$

We denote by $'\to'$ *and* $'\rightharpoonup'$ *strong convergence and weak convergence in corresponding space, respectively, and denote by* $B(x,R)$ *a ball in* $\mathbb{R}^N$ *of center* x *and radius* $R>0$*.*

3.2 Preliminary results

First of all, observe that the energy functional J is well-defined in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, thanks to the Hölder inequality,

$$\int_{\mathbb{R}^N}|u_1|^{r_1}|u_2|^{r_2}\,dx\leq\|u_1\|_{r_1q}^{r_1}\|u_2\|_{r_2q'}^{r_2}<\infty,$$

for some $1<q<2^*, q'=\frac{q}{q-1}$ with $2\leq r_1q, r_2q'\leq 2^*$. Recalling the Gagliardo-Nirenberg's inequality, for $u\in H^1(\mathbb{R}^N)$, $2\leq p\leq 2^*$,

$$\|u\|_p\leq C(N,p)\|\nabla u\|_2^{\alpha(p)}\|u\|_2^{1-\alpha(p)},\quad\text{where }\alpha(p)=\frac{N(p-2)}{2p},\tag{3.2.1}$$

then we get for $(u_1,u_2)\in S(a_1)\times S(a_2)$,

$$\begin{aligned}&\int_{\mathbb{R}^N}|u_1|^{r_1}|u_2|^{r_2}\,dx\leq\|u_1\|_{r_1q}^{r_1}\|u_2\|_{r_2q'}^{r_2}\\&\leq Ca_1^{\frac{(1-\alpha(r_1q))r_1}{2}}a_2^{\frac{(1-\alpha(r_2q))r_2}{2}}\|\nabla u_1\|_2^{\frac{N(r_1q-2)}{2q}}\|\nabla u_2\|_2^{\frac{N(r_2q'-2)}{2q'}}\end{aligned}\tag{3.2.2}$$

with $C=C(N,r_1,r_2,q)$.

We now introduce the coupled rearrangement results of Shibata [100] as developed in [61]. Let u be a Borel measurable function on $\mathbb{R}^N$. It is said to vanish at infinity if the level set $|\{x\in\mathbb{R}^N:|u(x)|>t\}|<\infty$ for every $t>0$. Here $|A|$ stands for the N-dimensional Lebesgue measure of a Lebesgue measurable set $A\subset\mathbb{R}^N$. Considering

two Borel mesurable functions u, v which vanish at infinity in $\mathbb{R}^N$, we define for $t > 0$, $A^\star(u, v; t) := \{x \in \mathbb{R}^N : |x| < r\}$ where $r > 0$ is chosen so that

$$|B(0, r)| = |\{x \in \mathbb{R}^N : |u(x)| > t\}| + |\{x \in \mathbb{R}^N : |v(x)| > t\}|,$$

and $\{u, v\}^\star$ by

$$\{u, v\}^\star(x) := \int_0^\infty \chi_{A^\star(u,v;t)}(x)\, dt, \tag{3.2.3}$$

where $\chi_A(x)$ is a characteristic function of the set $A \subset \mathbb{R}^N$.

Lemma 3.2.1. *[61, Lemma A.1]*

(i) The function $\{u, v\}^\star$ is radially symmetric, decreasing and lower semicontinuous. Moreover, for each $t > 0$ there holds $\{x \in \mathbb{R}^N : \{u, v\}^\star > t\} = A^\star(u, v; t)$.

(ii) Let $\Phi : [0, \infty) \to [0, \infty)$ be increasing, lower semicontinuous, continuous at 0 and $\Phi(0) = 0$. Then $\{\Phi(u), \Phi(v)\}^\star = \Phi(\{u, v\}^\star)$.

(iii) $\|\{u, v\}^\star\|_p^p = \|u\|_p^p + \|v\|_p^p$ for $1 \le p < \infty$.

(iv) If $u, v \in H^1(\mathbb{R}^N)$, then $\{u, v\}^\star \in H^1(\mathbb{R}^N)$ and $\|\nabla\{u, v\}^\star\|_2^2 \le \|\nabla u\|_2^2 + \|\nabla v\|_2^2$. In addition, if $u, v \in (H^1(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)) \setminus \{0\}$ are radially symmetric, positive and non-increasing, then

$$\int_{\mathbb{R}^N} |\nabla\{u, v\}^\star|^2\, dx < \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} |\nabla v|^2\, dx.$$

(v) Let $u_1, u_2, v_1, v_2 \ge 0$ be Borel measurable functions which vanish at infinity, then

$$\int_{\mathbb{R}^N} (u_1 u_2 + v_1 v_2)\, dx \le \int_{\mathbb{R}^N} \{u_1, v_1\}^\star \{u_2, v_2\}^\star\, dx.$$

Lemma 3.2.2. *[61, Lemma A.2] Suppose $p \in (1, \frac{N}{N-2}]$ when $N \ge 3$, and $p \in (1, \infty)$ when $N = 1, 2$. Let $u \in L^p(\mathbb{R}^N)$ be a smooth nonnegative function satisfying $-\Delta u \ge 0$ in $\mathbb{R}^N$. Then $u \equiv 0$.*

Lemma 3.2.3. *Assume $r_1, r_2 > 1, r_1 + r_2 \le 2^*$. If $(u_1^n, u_2^n) \rightharpoonup (u_1, u_2)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} |u_1^n|^{r_1} |u_2^n|^{r_2} - |u_1^n - u_1|^{r_1} |u_2^n - u_2|^{r_2}\, dx = \int_{\mathbb{R}^N} |u_1|^{r_1} |u_2|^{r_2}\, dx + o(1).$$

Proof. In [58], see also [38, Lemma 2.3], this result was proved under the assumption $r_1, r_2 > 1, r_1 + r_2 < 2 + \frac{4}{N}$, but the proof can extend to the case $r_1, r_2 > 1, r_1 + r_2 \le 2^*$. □

Lemma 3.2.4. *Assume that (H_1) or (H_2) holds. Then for any $b_1, b_2 \ge 0$ with $(b_1, b_2) \ne (0, 0)$ if (H_1) holds, and $b_1 \ne 0$, $b_2 \ne 0$ if (H_2) holds,*

$$\inf_{(u_1,u_2) \in S(b_1,b_2) \cap \mathcal{B}(\rho)} J(u_1, u_2) < 0, \quad \text{for any } \rho > 0.$$

Proof. Observing that $(\frac{p_i}{2} - 1)N < 2$, $i = 1, 2$ if (H_1) holds, and $(\frac{r_1 + r_2}{2} - 1)N < 2$ if (H_2) holds. In light of (3.1.4), the lemma follows directly by taking $t > 0$ small enough. □

Our next result, which is borrowed from [61, Lemma 2.2], shows that when considering a minimizing sequence to (3.1.7), it is not restrictive to assume that two components are nonnegative.

Lemma 3.2.5. *Assume that* $\{(v_1^n, v_2^n)\}$ *is a minimizing sequence to* (3.1.7). *If* $\{(|v_1^n|, |v_2^n|)\}$ *is compact in* $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, *so is* $\{(v_1^n, v_2^n)\}$.

Proof. First note that there exists $(w_1, w_2) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ such that, up to a subsequence, $(|v_1^n|, |v_2^n|) \to (w_1, w_2)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, and $(|v_1^n(x)|, |v_2^n(x)|) \to (w_1(x), w_2(x))$ for a.e. $x \in \mathbb{R}^N$. Since $\{(v_1^n, v_2^n)\}$ is a bounded sequence, theen there exists $(v_1, v_2) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ such that, up to a subsequence, $(v_1^n, v_2^n) \rightharpoonup (v_1, v_2)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and $(v_1^n(x), v_2^n(x)) \to (v_1(x), v_2(x))$ for a.e. $x \in \mathbb{R}^N$. By the uniqueness of the limit, $w_i = |v_i|$, then $(v_1^n, v_2^n) \to (v_1, v_2)$ in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. Now since $(v_1^n, v_2^n) \to (v_1, v_2)$ in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ for $2 < p < 2^*$, it follows that

$$m(a_1, a_2) = J(v_1^n, v_2^n) + o_n(1) \geq J(v_1, v_2) \geq m(a_1, a_2),$$

and thus $(v_1^n, v_2^n) \to (v_1, v_2)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. □

Next recalling (3.1.4), we define for $(u_1, u_2) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$,

$$\begin{aligned} Q(u_1, u_2) :&= \frac{d}{dt} J(u_1^t, u_2^t)|_{t=1} = \int_{\mathbb{R}^N} |\nabla u_1|^2 + |\nabla u_2|^2 \, dx \\ &- \sum_{i=1}^{2} \frac{\mu_i}{p_i} \left(\frac{p_i}{2} - 1 \right) N \int_{\mathbb{R}^N} |u_i|^{p_i} \, dx - \beta \left(\frac{r_1 + r_2}{2} - 1 \right) N \int_{\mathbb{R}^N} |u_1|^{r_1} |u_2|^{r_2} \, dx. \end{aligned} \tag{3.2.4}$$

It is standard that any solution (u_1, u_2) to (3.1.2) for some $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ must satisfy $Q(u_1, u_2) = 0$.

Lemma 3.2.6. *Assume* $2 < p_1, p_2, r_1 + r_2 < 2^*$. *If* $(u_1, u_2) \neq (0, 0)$ *solves* (3.1.2) *for some* $(\lambda_1, \lambda_2) \in \mathbb{R}^2$, *then* $\lambda_1 < 0$ *or* $\lambda_2 < 0$.

Proof. Testing (3.1.2) by (u_1, u_2) and integrating in $\mathbb{R}^N$, one has

$$\lambda_1 a_1 + \lambda_2 a_2 = \int_{\mathbb{R}^N} |\nabla u_1|^2 + |\nabla u_2|^2 \, dx - \sum_{i=1}^{2} \mu_i \int_{\mathbb{R}^N} |u_i|^{p_i} \, dx + \beta(r_1 + r_2) \int_{\mathbb{R}^N} |u_1|^{r_1} |u_2|^{r_2} \, dx.$$

Since (u_1, u_2) satisfies (1.1.2), then $Q(u_1, u_2) = 0$, which implies

$$\begin{aligned} \lambda_1 a_1 + \lambda_2 a_2 = &\sum_{i=1}^{2} \left(\frac{\mu_i}{p_i} \left(\frac{p_i}{2} - 1 \right) N - \mu_i \right) \int_{\mathbb{R}^N} |u_i|^{p_i} \, dx \\ &+ \beta \left(\left(\frac{r_1 + r_2}{2} - 1 \right) N - (r_1 + r_2) \right) \int_{\mathbb{R}^N} |u_1|^{r_1} |u_2|^{r_2} \, dx < 0. \end{aligned}$$

Then the lemma follows. □

We recall that a sequence $\{(u_1^n, u_2^n)\} \subset S(a_1, a_2)$ is a Palais-Smale sequence for J restricted to $S(a_1, a_2)$ at the level c, if $J(u_1^n, u_2^n) \to c$ and $(J_{|S(a_1,a_2)})'(u_1^n, u_2^n) \to 0$ in $H^{-1}(\mathbb{R}^N) \times H^{-1}(\mathbb{R}^N)$ as $n \to \infty$. The proof of our next lemma can be found in [8, Lemma 3.2].

Lemma 3.2.7. *Assume* $2 < p_1, p_2, r_1 + r_2 < 2^*$. *For any bounded Palais-Smale sequence* $\{(u_1^n, u_2^n)\}$ *for* J *restricted to* $S(a_1, a_2)$, *there exist* $(u_1, u_2) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ *and a sequence* $\{(\lambda_1^n, \lambda_2^n)\} \subset \mathbb{R}^2$ *such that, up to a subsequence,*

(i) $(u_1^n, u_2^n) \rightharpoonup (u_1, u_2)$ *in* $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$*;*

(ii) $(\lambda_1^n, \lambda_2^n) \to (\lambda_1, \lambda_2)$ *in* $\mathbb{R}^2$*;*

(iii) $J'(u_1^n, u_2^n) - \lambda_1^n(u_1^n, 0) - \lambda_2^n(0, u_2^n) \to 0$ *in* $H^{-1}(\mathbb{R}^N) \times H^{-1}(\mathbb{R}^N)$*;*

(iv) (u_1, u_2) *is solution to the system* (3.1.2) *where* (λ_1, λ_2) *is given in (ii).*

In addition, if $(u_1^n, u_2^n) \to (u_1, u_2)$ *in* $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ *for* $2 < p < 2^*$*, then* $u_1^n \to u_1$ *in* $H^1(\mathbb{R}^N)$ *if* $\lambda_1 < 0$*. Similarly,* $u_2^n \to u_2$ *in* $H^1(\mathbb{R}^N)$ *if* $\lambda_2 < 0$*.*

3.3 Existence of local minimizers

In this section, we establish Theorem 3.1.1 (i) and Theorem 3.1.2 (i).

Lemma 3.3.1. *Assume that* (H_1) *or* (H_2) *holds. There exist a* $\beta_0 = \beta_0(a_1, a_2) > 0$ *and a* $\rho_0 = \rho_0(a_1, a_2) > 0$*, such that*

$$J(u_1, u_2) \geq 0 \quad on \quad S(a_1, a_2) \cap [\mathcal{B}(2\rho_0) \backslash \mathcal{B}(\rho_0)] \tag{3.3.1}$$

for any $0 < \beta \leq \beta_0$*. Moreover, if* $0 \leq d_1 \leq a_1$*,* $0 \leq d_2 \leq a_2$ *with* $(d_1, d_2) \neq (0, 0)$*, then*

$$J(u_1, u_2) \geq 0 \quad on \quad S(d_1, d_2) \cap [\mathcal{B}(2\rho_0) \backslash \mathcal{B}(\rho_0)] \tag{3.3.2}$$

for any $0 < \beta \leq \beta_0$*. In addition,* $\beta_0(a_1, a_2) \to \infty$ *as* $a_1, a_2 \to 0$*.*

Proof. For any $(u_1, u_2) \in S(a_1, a_2)$, let $\rho := \int_{\mathbb{R}^N} |\nabla u_1|^2 + |\nabla u_2|^2 dx$. Using (3.2.1)-(3.2.2), we have

$$\begin{aligned}
J(u_1, u_2) &= \frac{1}{2}\rho - \sum_{i=1}^{2} \frac{\mu_i}{p_i} \int_{\mathbb{R}^N} |u_i|^{p_i}\, dx - \beta \int_{\mathbb{R}^N} |u_1|^{r_1} |u_2|^{r_2}\, dx \\
&\geq \frac{1}{2}\rho - \sum_{i=1}^{2} K_i \left(\int_{\mathbb{R}^N} |\nabla u_i|^2\, dx \right)^{\frac{N(p_i-2)}{4}} \\
&\quad - \beta K_3 \left(\int_{\mathbb{R}^N} |\nabla u_1|^2\, dx \right)^{\frac{N(r_1 q - 2)}{4q}} \left(\int_{\mathbb{R}^N} |\nabla u_1|^2\, dx \right)^{\frac{N(r_2 q' - 2)}{4q'}} \\
&\geq \frac{1}{2}\rho - \sum_{i=1}^{2} K_i \left(\int_{\mathbb{R}^N} |\nabla u_1|^2 + |\nabla u_2|^2\, dx \right)^{\frac{N(p_i-2)}{4}} \\
&\quad - \beta K_3 \left(\int_{\mathbb{R}^N} |\nabla u_1|^2 + |\nabla u_2|^2\, dx \right)^{\frac{N(r_1 q - 2)}{4q}} \left(\int_{\mathbb{R}^N} |\nabla u_1|^2 + |\nabla u_2|^2\, dx \right)^{\frac{N(r_2 q' - 2)}{4q'}}, \\
&\geq \frac{1}{2}\rho - K_1 \rho^{\frac{N(p_1-2)}{4}} - K_2 \rho^{\frac{N(p_2-2)}{4}} - \beta K_3 \rho^{\frac{N(r_1+r_2-2)}{4}},
\end{aligned}$$

where

$$K_i := \frac{\mu_i}{p_i} C_i(N, p_i) a_i^{\frac{(1-\alpha(p_i))p_i}{2}} \qquad K_3 := C(N, r_1, r_2) a_1^{\frac{(1-\alpha(r_1 q))r_1}{2}} a_2^{\frac{(1-\alpha(r_2 q'))r_2}{2}}. \tag{3.3.3}$$

Now if (H_1) holds, then $\frac{N(p_i-2)}{4} < 1$ for $i = 1, 2$, and $\frac{N(r_1+r_2-2)}{4} > 1$. We fix a $\rho = \rho_0 > 0$ sufficiently large so that

$$K_1 \rho_0^{\frac{N(p_1-2)}{4}-1} + K_2 \rho_0^{\frac{N(p_2-2)}{4}-1} \leq \frac{1}{8}, \tag{3.3.4}$$

and then we fix a $\beta_0 > 0$ small enough, satisfying

$$\beta_0 K_3 (2\rho_0)^{\frac{N(r_1+r_2-2)}{4}-1} \leq \frac{1}{8}. \tag{3.3.5}$$

Observe that the left hand side of (3.3.4) and of (3.3.5) is decreasing and increasing with respect to ρ_0, respectively. Thus we deduce that

$$J(u_1, u_2) \geq \frac{1}{4}\rho_0 \quad \text{for } (u_1, u_2) \in \mathcal{B}(2\rho_0)\backslash\mathcal{B}(\rho_0). \tag{3.3.6}$$

If we assume that (H_2) holds, then $\frac{N(p_i-2)}{4} > 1$ for $i = 1, 2$, and $\frac{N(r_1+r_2-2)}{4} < 1$. Thus we fix a $\rho = \rho_0 > 0$ sufficiently small so that

$$K_1 (2\rho_0)^{\frac{N(p_1-2)}{4}-1} + K_2 (2\rho_0)^{\frac{N(p_2-2)}{4}-1} \leq \frac{1}{8}, \tag{3.3.7}$$

and then we fix a $\beta_0 > 0$ small enough, satisfying

$$\beta_0 K_3 \rho_0^{\frac{N(r_1+r_2-2)}{4}-1} \leq \frac{1}{8}. \tag{3.3.8}$$

Here again one can readily check that (3.3.6) holds. Now to establish (3.3.2) it suffices to observe that the choices of $\beta_0 > 0$ and $\rho_0 > 0$ done with (a_1, a_2) in (3.3.1) can be retain for (d_1, d_2). This follows directly from the observation that the K_j, $j = 1, 2, 3$ are increasing functions with respect to a_1 and a_2. Finally, we prove that $\beta_0(a_1, a_2) \to \infty$ as $a_1, a_2 \to 0$. Indeed, when (H_0) holds, since $K_j \to 0$, $j = 1, 2, 3$ as $a_i \to 0$, $i = 1, 2$, then $\rho_0 > 0$ in (3.3.4) can be taken arbitrarily small, thus in (3.3.5), $\beta_0 > 0$ can be taken large if $\rho_0 > 0$ is small. When (H_1) holds we reach the same conclusion by similar arguments. □

From now on, for $a_1, a_2 \geq 0$ given, we fix a $\rho_0 > 0$ and a $\beta_0 > 0$ as determined in Lemma 3.3.1. For any $0 \leq d_1 \leq a_1$, $0 \leq d_2 \leq a_2$ we define

$$m(d_1, d_2) := \inf_{(u_1,u_2)\in S(d_1,d_2)\cap\mathcal{B}(\rho_0)} J(u_1, u_2). \tag{3.3.9}$$

Lemma 3.3.2. *Assume that (H_0) or (H_1) holds. Then for $0 < \beta \leq \beta_0$,*

(i) if $(d_1, d_2) \neq (0, 0)$ when (H_1) holds or $d_1 \neq 0$ and $d_2 \neq 0$ when (H_2) holds, we have $m(d_1, d_2) < 0$.

(ii) If (d_1^n, d_2^n) is such that $(d_1^n, d_2^n) \to (d_1, d_2)$ as $n \to \infty$ with $0 \leq d_i^n \leq a_i$ for $i = 1, 2$, we have $m(d_1^n, d_2^n) \to m(d_1, d_2)$ as $n \to \infty$.

(iii) For any $0 \leq d_i \leq a_i$, $i = 1, 2$ if $m(d_1, d_2) < 0$ and $m(a_1 - d_1, a_2 - d_2) < 0$, we have $m(a_1, a_2) \leq m(d_1, d_2) + m(a_1 - d_1, a_2 - d_2)$.

Proof. (i) It follows directly from Lemma 3.2.4. (ii) By definition of $m(d_1^n, d_2^n)$, for any $\epsilon > 0$, there exists $(u_1^n, u_2^n) \in S(d_1^n, d_2^n) \cap \mathcal{B}(\rho_0)$ such that

$$J(u_1^n, u_2^n) \leq m(d_1^n, d_2^n) + \epsilon.$$

Setting $w_i^n := \dfrac{u_i^n}{\|u_i^n\|_2} a_i^{\frac{1}{2}}$ for $i = 1, 2$, we have $(w_1^n, w_2^n) \in S(a_1, a_2)$ and

$$\|\nabla w_1^n\|_2^2 + \|\nabla w_2^n\|_2^2 = \|\nabla u_1^n\|_2^2 + \|\nabla u_2^n\|_2^2 + o_n(1) < 2\rho_0.$$

Consequently from the definition (3.3.9) and using (3.3.2), we get

$$m(d_1,d_2) \le J(w_1^n,w_2^n) = J(u_1^n,u_2^n) + o(1) \le m(d_1^n,d_2^n) + \epsilon + o_n(1),$$

and thus $m(d_1,d_2) \le m(d_1^n,d_2^n) + o_n(1)$. Similarly, reversing the argument, it follows that $m(d_1^n,d_2^n) \le m(d_1,d_2) + o_n(1)$. Now we deal with (iii). For any $\epsilon > 0$, there exist $(\varphi_1,\varphi_2) \in S(d_1,d_2) \cap \mathcal{B}(\rho_0)$ and $(\psi_1,\psi_2) \in S(a_1-d_1,a_2-d_2) \cap \mathcal{B}(\rho_0)$ such that

$$J(\varphi_1,\varphi_2) \le m(d_1,d_2) + \frac{\epsilon}{2} \quad J(\psi_1,\psi_2) \le m(a_1-d_1,a_2-d_2) + \frac{\epsilon}{2}. \tag{3.3.10}$$

Setting $w_i = \{\varphi_i,\psi_i\}^\star$ for $i=1,2$, it follows from, Lemma 3.2.1 (iii)-(iv), that $(w_1,w_2) \in S(a_1,a_2)$ and

$$\|\nabla w_1\|_2^2 + \|\nabla w_2\|_2^2 \le \sum_{i=1}^{2} \|\nabla\varphi_i\|_2^2 + \|\nabla\psi_i\|_2^2.$$

If $\|\nabla w_1\|_2^2 + \|\nabla w_2\|_2^2 < \rho_0$, using Lemma 3.2.1 and (3.3.10), we have

$$\begin{aligned} m(a_1,a_2) &\le J(w_1,w_2) \le J(\varphi_1,\varphi_2) + J(\psi_1,\psi_2) \\ &\le m(d_1,d_2) + m(a_1-d_1,a_2-d_2) + \epsilon, \end{aligned}$$

from which it follows that $m(a_1,a_2) \le m(d_1,d_2) + m(a_1-d_1,a_2-d_2)$. Otherwise, $\rho_0 \le \|\nabla w_1\|_2^2 + \|\nabla w_2\|_2^2 < 2\rho_0$ and in view of (3.3.2), we get

$$0 \le J(w_1,w_2) \le J(\varphi_1,\varphi_2) + J(\psi_1,\psi_2) \le m(d_1,d_2) + m(a_1-d_1,a_2-d_2) + \epsilon,$$

which is impossible since $m(d_1,d_2) < 0$ and $m(a_1-d_1,a_2-d_2) < 0$. □

Lemma 3.3.3. *Assume that (H_1) or (H_2) holds. Any minimizing sequence to (3.1.7) is, up to translations, strongly convergent in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ for any $2 < p < 2^*$ as $0 < \beta \le \beta_0$.*

Proof. The proof follows closely the one of [58, Lemma 3.3]. Let $\{(v_1^n,v_2^n)\}$ be a minimizing sequence to (3.1.7). If

$$\sup_{y\in\mathbb{R}^N} \int_{B(y,R)} |v_1^n|^2 + |v_2^n|^2\,dx = o_n(1),$$

for some $R > 0$, then $v_i \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*, i = 1,2$, see [74, Lemma I.1]. This contradicts the property $m(a_1,a_2) < 0$, obtained in Lemma 3.3.2 (i). Thus, there exist a $\gamma_0 > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^N$ such that

$$\int_{B(y_n,R)} |v_1^n|^2 + |v_2^n|^2\,dx \ge \gamma_0,$$

and we deduce that $(v_1^n(x-y_n), v_2^n(x-y_n)) \rightharpoonup (v_1,v_2) \neq (0,0)$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. Our aim is to prove that $w_i^n(x) := v_i^n(x) - v_i(x+y_n) \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*, i = 1,2$ and so we suppose by contradiction that there exists a $2 < q < 2^*$ such that $(w_1^n,w_2^n) \nrightarrow (0,0)$ in $L^q(\mathbb{R}^N) \times L^q(\mathbb{R}^N)$. Still using [74, Lemma I.1] it follows that there exists a sequence $\{z_n\} \subset \mathbb{R}^N$ such that $(w_1^n(x-z_n), w_2^n(x-z_n)) \rightharpoonup (w_1,w_2) \neq (0,0)$ in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^N)$.

Now, combining Lemma 3.2.3, the Brezis-Lieb Lemma and the translational invariance, we see that

$$\begin{aligned} &J(v_1^n,v_2^n) = J(v_1^n(x-y_n), v_2^n(x-y_n)) \\ &= J(v_1^n(x-y_n) - v_1, v_2^n(x-y_n) - v_2) + J(v_1,v_2) + o_n(1) \\ &= J(w_1^n(x-y_n), w_2^n(x+-y_n)) + J(v_1,v_2) + o(1) \\ &= J(w_1^n(x-z_n) - w_1, w_2^n(x-z_n) - w_2) + J(w_1,w_2) + J(v_1,v_2) + o_n(1), \end{aligned} \tag{3.3.11}$$

and

$$\begin{aligned} a_i = \|v_i^n(x-y_n)\|_2^2 &= \|v_i^n(x-y_n)-v_i\|_2^2 + \|v_i\|_2^2 + o_n(1) \\ &= \|w_i^n(x-z_n)-w_i+w_i\|_2^2 + \|v_i\|_2^2 + o_n(1) \\ &= \|w_i^n(x-z_n)-w_i\|_2^2 + \|w_i\|_2^2 + \|v_i\|_2^2 + o_n(1). \end{aligned}$$

Setting for $i=1,2$, $b_i := a_i - \|w_i\|_2^2 - \|v_i\|_2^2$ we then have $\|w_i^n(x-z_n)-w_i\|_2^2 = b_i + o(1)$. Thus recording that $J(v_1^n, v_2^n) \to m(a_1,a_2)$, in view of (3.3.11) and Lemma 3.3.2 (ii) we get

$$m(a_1,a_2) \geq J(w_1,w_2) + J(v_1,v_2) + m(b_1,b_2). \tag{3.3.12}$$

If $J(w_1,w_2) > m(\|w_1\|_2^2, \|w_2\|_2^2)$ or $J(v_1,v_2) > m(\|v_1\|_2^2, \|v_1\|_2^2)$, then, from (3.3.12) and Lemma 3.3.2 (iii) , it follows

$$m(a_1,a_2) > m(\|w_1\|_2^2, \|w_2\|_2^2) + m(\|v_1\|_2^2, \|v_2\|_2^2) + m(b_1,b_2) \geq m(a_1,a_2)$$

which is impossible. Hence $J(w_1,w_2) = m(\|w_1\|_2^2, \|w_2\|_2^2)$ and $J(v_1,v_2) = m(\|v_1\|_2^2, \|v_2\|_2^2)$. We denote by v_i^*, w_i^* the classical Schwartz's rearrangement of v_i, w_i for $i=1,2,$. Since

$$\|v_i^*\|_2^2 = \|v_i\|_2^2, \quad \|w_i^*\|_2^2 = \|w_i\|_2^2,$$

$$J(v_1^*, v_2^*) \leq J(v_1,v_2), \quad J(w_1^*, w_2^*) \leq J(w_1,w_2),$$

see for example [68], we deduce that

$$J(v_1^*, v_2^*) = m(\|u_1\|_2^2, \|u_2\|_2^2), \quad J(w_1^*, w_2^*) = m(\|w_1\|_2^2, \|w_2\|_2^2).$$

Therefore, $(v_1^*, v_2^*), (w_1^*, w_2^*)$ are solutions to (3.1.2) for some $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ and from the standard regularity results we have that $v_i^*, w_i^* \in C^2(\mathbb{R}^N)$ for $i=1,2$.

We distinguish two cases to preceed the proof. Without loss of generality, we may assume $v_1 \neq 0$.

Case 1: $v_1 \neq 0$ and $w_1 \neq 0$.
By virtue of Lemma 3.2.1 $(ii),(iv),(v)$,

$$\int_{\mathbb{R}^N} |\nabla\{v_1^*, w_1^*\}^\star|\,dx < \int_{\mathbb{R}^N} |\nabla v_1^*|^2 + |\nabla w_1^*|^2\,dx \leq \int_{\mathbb{R}^N} |\nabla v_1|^2 + |\nabla w_1|^2\,dx,$$

$$\begin{aligned} \int_{\mathbb{R}^N} |\{v_1^*, w_1^*\}^\star|^{r_1} |\{v_2^*, w_2^*\}^\star|^{r_2}\,dx &= \int_{\mathbb{R}^N} \{|v_1^*|^{r_1}, |w_1^*|^{r_1}\}^\star \{|v_2^*|^{r_2}, |w_2^*|^{r_2}\}^\star\,dx, \\ &\geq \int_{\mathbb{R}^N} |v_1^*|^{r_1}|v_2^*|^{r_2} + |w_1^*|^{r_1}|w_2^*|^{r_2}\,dx \\ &= \int_{\mathbb{R}^N} (|v_1|^{r_1})^*(|v_2|^{r_2})^* + (|w_1|^{r_1})^*(|w_2|^{r_2})^*\,dx, \\ &\geq \int_{\mathbb{R}^N} |v_1|^{r_1}|v_2|^{r_2} + |w_1|^{r_1}|w_2|^{r_2}\,dx, \end{aligned} \tag{3.3.13}$$

and thus

$$J(v_1,v_2) + J(w_1,w_2) > J(\{v_1^*, w_1^*\}^\star, \{v_2^*, w_2^*\}^\star). \tag{3.3.14}$$

Also from Lemma 3.2.1 (iii), for $i = 1, 2$,

$$\int_{\mathbb{R}^N} |\{v_i^*, w_i^*\}^\star|^2 \, dx = \int_{\mathbb{R}^N} |v_i^*|^2 + |w_i^*|^2 \, dx = \int_{\mathbb{R}^N} |v_i|^2 + |w_i|^2 \, dx, \tag{3.3.15}$$

and hence taking (3.3.12)-(3.3.15) and Lemma 3.3.2 (iii) into consideration, one obtains the contradiction

$$m(a_1, a_2) > m(b_1, b_2) + m(a_1 - b_1, a_2 - b_2) \geq m(a_1, a_2).$$

Case 2: $v_1 \neq 0$, $w_1 = 0$ *and* $w_2 \neq 0$.
If $v_2 \neq 0$, we can reverse the role of v_1, w_1 and v_2, w_2 in *Case 1* to get a contradiction. Thus, we suppose that $v_2 = 0$. Due to Lemma 3.2.1 (ii)-(v),

$$\begin{aligned} J(\{v_1^*, 0\}^\star, \{w_2^*, 0\}^\star) &\leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_1^*|^2 + |\nabla w_2^*|^2 \, dx - \frac{\mu_1}{p_1} \int_{\mathbb{R}^N} |v_1^*|^{p_1} \, dx \\ &\quad - \frac{\mu_2}{p_2} \int_{\mathbb{R}^N} |w_2^*|^{p_2} \, dx - \beta \int_{\mathbb{R}^N} |v_1^*|^{r_1} |w_2^*|^{r_2} \\ &< J(v_1^*, 0) + J(0, w_2^*) \leq J(v_1, 0) + J(0, w_2), \end{aligned} \tag{3.3.16}$$

with $||\{v_1^*, 0\}^\star||_2^2 = ||v_1^*||_2^2 = ||v_1||_2^2$ and $||\{w_2^*, 0\}^\star||_2^2 = ||w_2^*||_2^2 = ||w_2||_2^2$. Thus using (3.3.12), (3.3.16) and Lemma 3.3.2, we also have

$$m(a_1, a_2) > m(b_1, b_2) + m(a_1 - b_1, a_2 - b_2) \geq m(a_1, a_2).$$

The contradictions obtained in *Cases* 1 - 2 indicate that $w_i^n(x) = v_i^n(x) - v_i(x + y_n) \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*, i = 1, 2$. □

Proof of Theorem 3.1.1 (i). Let $\{(v_1^n, v_2^n)\}$ be an arbitrary minimizing sequence to (3.1.7). In view of Lemma 4.3.14, there exists $(v_1, v_2) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ such that, up to a subsequence and translations, $(v_1^n, v_2^n) \rightharpoonup (v_1, v_2)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and $(v_1^n, v_2^n) \to (v_1, v_2)$ in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ for $2 < p < 2^*$. Hence, by the weak lower semi-continuity of the norm, $\|\nabla v_1\|_2^2 + \|\nabla v_2\|_2^2 < \rho_0$, namely, $(v_1, v_2) \in \mathcal{B}(\rho_0)$, and $J(v_1, v_2) \leq m(a_1, a_2) < 0$, from which we deduce that $(v_1, v_2) \neq (0, 0)$. To show the compactness of $\{(v_1^n, v_2^n)\}$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, it suffices to prove that $(v_1, v_2) \in S(a_1, a_2)$. Assume by contradiction that $\|v_1\|_2^2 := \bar{a}_1 < a_1$ or $\|v_2\|_2^2 := \bar{a}_2 < a_2$. Then by the definition (3.3.9), $m(\bar{a}_1, \bar{a}_2) \leq J(v_1, v_2)$. At this point, in light of Lemma 3.3.2 (i) and (iii), we get

$$J(v_1, v_2) \leq m(a_1, a_2) \leq m(\bar{a}_1, \bar{a}_2) + m(a_1 - \bar{a}_1, a_2 - \bar{a}_2) < m(\bar{a}_1, \bar{a}_2) \leq J(v_1, v_2).$$

This contradiction proves that $(v_1, v_2) \in S(a_1, a_2)$. To end the proof, we note that without restriction we can choose a minimizer (v_1, v_2) of $m(a_1, a_2)$ with $v_1 \geq 0$ and $v_2 \geq 0$. From the classical regularity theory, and using the strong maximum principle we then deduce that $v_1, v_2 > 0$. □

Proof of Theorem 3.1.2 (i). Let $\{(v_1^n, v_2^n)\}$ be a minimizing sequence to (3.1.7) whose two components are nonnegative. We know by Lemma 3.2.5 that it is not a restriction. Now it is classical, see for example [55], that there exists another minimizing sequence $\{(\tilde{v}_1^n, \tilde{v}_2^n)\} \subset S(a_1, a_2)$ which is a Palais-Smale sequence for J restricted to $S(a_1, a_2)$, and such that $||(\tilde{v}_1^n, \tilde{v}_2^n) - (v_1^n, v_2^n)|| \to 0$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Because of this convergence, we have in particular that $(\tilde{v}_1^n)^- \to 0$ and $(\tilde{v}_2^n)^- \to 0$ as $n \to \infty$ and we obtain that $(\tilde{v}_1^n, \tilde{v}_2^n) \rightharpoonup (v_1, v_2)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ with $v_1 \geq 0$ and $v_2 \geq 0$. Furthermore, it results from Lemma 3.2.7

that (v_1, v_2) satisfies (3.1.2)-(3.1.3) with some $(\lambda_1, \lambda_2) \in \mathbb{R}^2$, from which we infer that $Q(v_1, v_2) = 0$. From Lemma 4.3.14, we also get that $J(v_1, v_2) \leq m(a_1, a_2) < 0$. It remains to show that $(v_1, v_2) \in S(a_1, a_2)$. By Lemma 3.2.6, we can assume without restriction that $\lambda_1 < 0$ and then Lemma 3.2.7 gives $v_1 \in S(a_1)$. If $\lambda_2 < 0$ we also have that $v_2 \in S(a_2)$. Let us thus assume by contradiction that $\lambda_2 \geq 0$. In the case $1 \leq N \leq 4$, since

$$-\Delta v_2 = \lambda_2 v_2 + \mu_2 v_2^{p_2-1} + \beta r_2 v_1^{r_1} v_2^{r_2-1} \geq 0,$$

by the Liouville's results recalled in Lemma 3.2.2, we obtain that $v_2 = 0$. It then follows that $J(v_1, v_2) = J(v_1, 0)$ with $v_1 \in S(a_1)$ and satisfying $-\Delta v_1 = \lambda_1 v_1 + \mu_1 v_1^{p_1-1}$. Since $p_1 > 2 + \frac{4}{N}$, we necessarily have $J(v_1, 0) > 0$, and this provides the contradiction. If we now assume that $N \geq 5$, testing the second equation of (3.1.2) with v_2, and integrating in $\mathbb{R}^N$, because $\lambda_2 \geq 0$, we get that

$$\int_{\mathbb{R}^N} |\nabla v_2|^2 \, dx - \mu_2 \int_{\mathbb{R}^N} |v_2|^{p_2} \, dx - \beta r_2 \int_{\mathbb{R}^N} |v_1|^{r_1} |v_2|^{r_2} \, dx \geq 0. \tag{3.3.17}$$

Now jointing (3.3.17) with $Q(v_1, v_2) = 0$, we obtain that

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla v_1|^2 - \frac{\mu_1}{p_1} \left(\frac{p_1}{2} - 1 \right) N \int_{\mathbb{R}^N} |v_1|^{p_2} \, dx + \left(\mu_2 - \frac{\mu_2}{p_2} \left(\frac{p_2}{2} - 1 \right) N \right) \int_{\mathbb{R}^N} |v_2|^{p_2} \, dx \\ &+ \beta \left(r_2 - \left(\frac{r_1 + r_2}{2} - 1 \right) N \right) \int_{\mathbb{R}^N} |v_1|^{r_1} |v_2|^{r_2} \, dx \leq 0. \end{aligned}$$

Note that the coefficient of $\int_{\mathbb{R}^N} |v_2|^{p_2} \, dx$ is positive. From the Gagliardo-Nirenberg's inequality (3.2.1), we can assume without restriction that

$$\int_{\mathbb{R}^N} |\nabla v_1|^2 - \frac{\mu_1}{p_1} \Big(\frac{p_1}{2} - 1 \Big) N \int_{\mathbb{R}^N} |v_1|^{p_1} \, dx \geq 0$$

by taking, if necessary, $\rho_0 > 0$ (and thus $\beta_0 > 0$) smaller in Lemma 3.3.1. Thus we also obtain a contradiction, since we have assumed that $r_2 > \big(\frac{r_1 + r_2}{2} - 1 \big) N$. Knowing that $\lambda_2 < 0$, we deduce that $v_2 \in S(a_2)$ and then we conclude as before that $v_1 > 0$ and $v_2 > 0$. □

3.4 Existence of minimax solutions

This section is devoted to the proofs of Theorem 3.1.1 (ii) and Theorem 3.1.2 (ii). To obtain our second solution and in order to benefit from additional compactness, we replace $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ by $H^1_{rad}(\mathbb{R}^N) \times H^1_{rad}(\mathbb{R}^N)$. It is well-known that the subspace $H^1_{rad}(\mathbb{R}^N)$ of $H^1(\mathbb{R}^N)$ consisting of radially symmetric functions is compactly embedded into $L^q(\mathbb{R}^N)$ for $2 < q < 2^*$ and $N \geq 2$. Also it is classical that a constrained critical point of J defined on $H^1_{rad}(\mathbb{R}^N) \times H^1_{rad}(\mathbb{R}^N)$ is a constrained critical point of J defined on $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Accordingly, we define $S_{rad}(a_1, a_2) := S(a_1, a_2) \cap H^1_{rad}(\mathbb{R}^N) \times H^1_{rad}(\mathbb{R}^N)$.

We first deal with the case where (H_1) holds. By Lemma 3.2.4 and 3.3.1, we know that there exists a $0 < \bar{\rho} = \bar{\rho}(a_1, a_2) < \rho_0$ such that, for any $0 < \beta \leq \beta_0$,

$$\gamma(a_1, a_2) := \inf_{g \in \Gamma} \max_{t \in [0,1]} J(g(t)) > \max\{J(g(0)), J(g(1))\},$$

where

$$\Gamma := \{ g \in C([0,1], S(a_1, a_2)) : g(0) \in B(\bar{\rho}), g(1) \notin \overline{B(\rho_0)} \text{ with } J(g(1)) < 0 \}.$$

Lemma 3.4.1. *Assume that (H_1) holds. Then, for any $0 < \beta \leq \beta_0$, there exists a Palais-Smale sequence $\{(u_1^n, u_2^n)\} \subset S(a_1, a_2)$ for J restricted to $S_{rad}(a_1, a_2)$ at the level $\gamma(a_1, a_2)$, which satisfies $(u_1^n)^- \to 0$, $(u_2^n)^- \to 0$ in $H^1(\mathbb{R}^N)$ and the property $Q(u_1^n, u_2^n) \to 0$ as $n \to \infty$.*

Proof. The proof of such result is now standard, similar statements appear in [63, 8, 12]. □

Lemma 3.4.2. *Assume that (H_1) holds and that $0 < \beta \leq \beta_0$. Then there exists $(u_1, u_2) \in H^1_{rad}(\mathbb{R}^N) \times H^1_{rad}(\mathbb{R}^N)$ solving to (3.1.2) for some $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ such that $J(u_1, u_2) = \gamma(a_1, a_2)$. Moreover $u_1 \geq 0$ and $u_2 \geq 0$.*

Proof. The couple (u_1, u_2) will be obtained as a weak limit of the Palais-Smale sequence whose existence is provided by Lemma 3.4.1. To this aim, we first show that $\{(u_1^n, u_2^n)\}$ is bounded in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. As we shall see this property follows from the fact that the functional J restricted to the set where $Q = 0$ is coercive. Indeed, we can write, for any $\varepsilon > 0$,

$$J(u_1, u_2) = \frac{\varepsilon}{2}||\nabla u_1^n||_2^2 + \frac{\varepsilon}{2}||\nabla u_2^n||_2^2 + a_1(\varepsilon)||u_1^n||_{p_1}^{p_1} + a_2(\varepsilon)||u_2||_{p_2}^{p_2} + \beta b(\varepsilon)\int_{\mathbb{R}^N} |u_1^n|^{r_1}|u_2^n|^{r_2}\,dx + \frac{1-\varepsilon}{2}Q(u_1^n, u_2^n).$$

where

$$a_1(\varepsilon) = \frac{(1-\varepsilon)\mu_1 N}{2p_1}\left(\frac{p_1}{2} - 1\right) - \frac{\mu_1}{p_1}, \quad a_2(\varepsilon) = \frac{(1-\varepsilon)\mu_2 N}{2p_2}\left(\frac{p_2}{2} - 1\right) - \frac{\mu_2}{p_2}$$

and

$$b(\varepsilon) = \frac{(1-\varepsilon)N}{2}\left(\frac{r_1 + r_2}{2} - 1\right) - 1.$$

The coefficients $a_i(\varepsilon)$, $i = 1, 2$ are strictly negative, but the corresponding terms can be controlled by $\varepsilon||\nabla u_i^n||_2^2$, using the Gagliardo-Nirenberg's inequality (3.2.1) because $p_1, p_2 < 2 + \frac{4}{N}$. Now since $r_1 + r_2 > 2 + \frac{4}{N}$, we also have that $b(\varepsilon) > 0$ for $\varepsilon > 0$ small enough. Recalling that $Q(u_1^n, u_2^n) \to 0$, the boundedness of our Palais-Smale sequence follows.

At this point, using Lemma 3.2.7, we can assume that $u_i^n \rightharpoonup u_i$, $i = 1, 2$ in $H^1(\mathbb{R}^N)$ and that $u_i^n \to u_i$, $i = 1, 2$ in $L^q(\mathbb{R}^N)$ with $2 < q < 2^*$. Lemma 3.2.7 also insures that (u_1, u_2) is a solution to (3.1.2) for some $(\lambda_1, \lambda_2) \in \mathbb{R}^2$, and thus $Q(u_1, u_2) = 0$. Clearly, the property $u_1 \geq 0$ and $u_2 \geq 0$ follows from $(u_1^n)^- \to 0$, $(u_2^n)^- \to 0$ in $H^1(\mathbb{R}^N)$. It remains to show that $J(u_1, u_2) = \gamma(a_1, a_2)$. Since $Q(u_1^n, u_2^n) \to 0$ we have,

$$\int_{\mathbb{R}^N} |\nabla u_1^n|^2 + |\nabla u_2^n|^2\,dx \to \sum_{i=1}^2 \frac{\mu_i}{p_i}\left(\frac{p_i}{2} - 1\right) N\int_{\mathbb{R}^N} |u_i^n|^{p_i}\,dx + \beta\left(\frac{r_1 + r_2}{2} - 1\right) N\int_{\mathbb{R}^N} |u_1^n|^{r_1}|u_2^n|^{r_2}\,dx.$$

From the strong convergence in $L^q(\mathbb{R}^N)$, the right hand side converges to

$$\sum_{i=1}^2 \frac{\mu_i}{p_i}\left(\frac{p_i}{2} - 1\right) N\int_{\mathbb{R}^N} |u_i|^{p_i}\,dx + \beta\left(\frac{r_1 + r_2}{2} - 1\right) N\int_{\mathbb{R}^N} |u_1|^{r_1}|u_2|^{r_2}\,dx.$$

Thanks to $Q(u_1, u_2) = 0$, this gives that $\int_{\mathbb{R}^N} |\nabla u_1^n|^2 + |\nabla u_2^n|^2\,dx \to \int_{\mathbb{R}^N} |\nabla u_1|^2 + |\nabla u_2|^2\,dx$. As a consequence, we deduce that $J(u_1^n, u_2^n) \to J(u_1, u_2)$. Thus recalling that $J(u_1^n, u_2^n) \to \gamma(a_1, a_2)$, we get $J(u_1, u_2) = \gamma(a_1, a_2)$. □

Proof of Theorem 3.1.1 (ii). First we consider the case $2 \leq N \leq 4$. In view of Lemma 3.4.2, it remains to prove that $(u_1, u_2) \in S(a_1, a_2)$. Recall that here we work in the radially symmetric space $H^1_{rad}(\mathbb{R}^N) \times H^1_{rad}(\mathbb{R}^N)$, thus in view of Lemma 3.2.7, we only need to prove that $\lambda_1, \lambda_2 < 0$. At this point, as in the proof of Theorem 3.1.2 (i), reasoning by contradiction if necessary, we assume that $\lambda_2 \geq 0$, we obtain that $J(u_1, u_2) = J(u_1, 0)$ with $u_1 \in S(a_1)$ satisfying $-\Delta u_1 = \lambda_1 u_1 + \mu_1 u_1^{p_1 - 1}$. Since $p_1 < 2 + \frac{4}{N}$, we necessarily have that $J(u_1, 0) < 0$, this provides the contradiction $J(u_1, 0) = \gamma(a_1, a_2) > 0$. We then conclude as before. □

Let us now consider the case $N \geq 5$, where the Liouville's type results cannot be applied.

Lemma 3.4.3. *Assume that (H_1) holds and that either $p_i \leq r_1 + r_2 - \frac{2}{N}$, $i = 1, 2$ or $|p_1 - p_2| \leq \frac{2}{N}$. If $Q(u_1, u_2) = 0$, and $J(u_1, u_2) > 0$, then $u_1 \neq 0$, $u_2 \neq 0$ and*

$$J(u_1, u_2) = \max_{t>0} J(u_1^t, u_2^t). \tag{3.4.1}$$

The proof of Lemma 3.4.3 relies on the following technical result whose proof will be postponed until the Appendix.

Lemma 3.4.4. *Assume that (H_1) holds and that either $p_1, p_2 \leq r_1 + r_2 - \frac{2}{N}$ or $|p_1 - p_2| \leq \frac{2}{N}$. Let $(v_1, v_2) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ be arbitrary. Then the function $t \mapsto J(v_1^t, v_2^t)$ admits at most two stationary points for $t > 0$.*

Remark 3.4.5. *It is only in the proof of Lemma 3.4.4 that we need the assumption $p_1, p_2 \leq r_1 + r_2 - \frac{2}{N}$, or alternatively $|p_2 - p_1| \leq \frac{2}{N}$. These conditions are used to establish the key property, on which our proof of Theorem 3.1.1(ii) relies, namely that if $(v_1, v_2) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ is such that $Q(v_1, v_2) = 0$ and $J(v_1, v_2) \geq 0$, then $J(v_1, v_2) = \max_{t>0} J(v_1^t, v_2^t)$.*

Proof of Lemma 3.4.3. We first assert that $u_1 \neq 0$ and $u_2 \neq 0$. If we assume that $u_1 = 0$, then by using $Q(0, u_2) = 0$ and $2 < p_2 < 2 + \frac{4}{N}$,

$$J(0, u_2) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_2|^2 \, dx - \frac{\mu_2}{p_2} \int_{\mathbb{R}^N} |u_2|^{p_2} \, dx = \frac{\mu_2}{p_2} \left(\frac{1}{2} - \left(\frac{p_2}{2} - 1 \right) N \right) \int_{\mathbb{R}^N} |u_2|^{p_2} \, dx \leq 0,$$

this is impossible, which proves that $u_1 \neq 0$. Similarly we get that $u_2 \neq 0$. Next we are going to prove that $\max_{t>0} \theta(t) := \max_{t>0} J(u_1^t, u_2^t) = J(u_1, u_2)$. Since $Q(u_1, u_2) = 0$, it follows from that $t = 1$ is a stationary point of θ. Note that $\lim_{t \to 0^+} \theta(t) = 0^-$, $\lim_{t \to \infty} \theta(t) = -\infty$. Due to $\theta(1) > 0$, we then deduce from Lemma 3.4.4 that (3.4.1) necessarily holds. □

End of the proof of Theorem 3.1.1 (ii). We now deal with the case $N \geq 5$. In view of Lemma 3.4.2, it remains to prove that $(u_1, u_2) \in S(a_1, a_2)$. Let $\bar{a}_1 := \|u_1\|_2^2 \leq a_1$ and $\bar{a}_2 := \|u_2\|_2^2 \leq a_2$. Assuming by contradiction that either $\bar{a}_1 < a_1$ or $\bar{a}_2 < a_2$, we shall obtain a contradiction by constructing a path $g \in \Gamma$ such that

$$\max_{t \in [0,1]} J(g(t)) < \gamma(a_1, a_2).$$

Let $0 < t_1 < 1 < t_2$ be such that $(u_1^{t_1}, u_2^{t_1}) \in B(\bar{\rho}/2)$ and $J(u_1^{t_2}, u_2^{t_2}) < m(a_1, a_2) < 0$. The existence of $0 < t_1 < 1$ is insured by Lemma 3.2.4 and the one of $t_2 > 1$ by the property that $J(u_1^t, u_2^t) \to -\infty$ as $t \to \infty$. Now because of (3.1.10), if $\bar{a}_1 < a_1$, there

exists a $w_1 \in S(a_1 - \bar{a}_1)$ such that $w_1^{t_1} \in \mathcal{B}(\bar{\rho}/2)$, and $J(w_1^t, 0) < 0$ for $t \in [t_1, t_2]$. Here $w^t(x) := t^{\frac{N}{2}} w(tx)$ and without restriction we can assume that $w_1 \in S(a_1 - \bar{a}_1)$ is radially symmetric. Similarly, if $\bar{a}_2 < a_2$, we can choose a radially symmetric $w_2 \in S(a_2 - \bar{a}_2)$ such that $w_2^{t_1} \in \mathcal{B}(\bar{\rho}/2)$, and $J(0, w_2^t) < 0$ for $t \in [t_1, t_2]$. Note that we just take $w_1 = 0$ if $\bar{a}_1 = a_1$, and $w_2 = 0$ if $\bar{a}_2 = a_2$.

We now set

$$v_i := \{u_i, w_i\}^*, \quad \text{for} \quad i = 1, 2,$$

where $\{u, v\}^*$ is the coupled rearrangement of u, v defined by (3.2.3). Then we consider a path $[t_1, t_2] \mapsto (v_1^t, v_2^t)$. From Lemma 3.2.1 (iii)-(iv), for all $t \in [t_1, t_2]$, we see that $(v_1^t, v_2^t) \in S(a_1, a_2)$, and

$$\begin{aligned} ||\nabla v_1^t||_2^2 + ||\nabla v_1^t||_2^2 &= t^2 \left(||\nabla v_1||_2^2 + ||\nabla v_1||_2^2 \right) \leq t^2 \sum_{i=1}^{2} ||\nabla u_i||_2^2 + ||\nabla w_i||_2^2 \\ &= \sum_{i=1}^{2} ||\nabla u_i^t||_2^2 + ||\nabla w_i^t||_2^2. \end{aligned}$$

Thus $(v_1^{t_1}, v_2^{t_2}) \in \mathcal{B}(\bar{\rho})$, due to $(u_1^{t_1}, u_2^{t_2}), (w_1^{t_1}, w_2^{t_1}) \in \mathcal{B}(\bar{\rho}/2)$. Also

$$\begin{aligned} J(v_1^t, v_2^t) =& \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla v_1|^2 + |\nabla v_2|^2 dx - \sum_{i=1}^{2} \frac{\mu_i}{p_i} t^{(\frac{p_i}{2} - 1)N} \int_{\mathbb{R}^N} |v_i|^{p_i} dx \\ &- \beta t^{(\frac{r_1 + r_2}{2} - 1)N} \int_{\mathbb{R}^N} |v_1|^{r_1} |v_2|^{r_2} \, dx \\ &\leq \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla u_1|^2 + |\nabla u_2|^2 dx + \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla w_1|^2 + |\nabla w_2|^2 \, dx \\ &- \sum_{i=1}^{2} \frac{\mu_i}{p_i} t^{(\frac{p_i}{2} - 1)N} \int_{\mathbb{R}^N} |u_i|^{p_i} \, dx - \sum_{i=1}^{2} \frac{\mu_i}{p_i} t^{(\frac{p_i}{2} - 1)N} \int_{\mathbb{R}^N} |w_i|^{p_i} \, dx \\ &- \beta t^{(\frac{r_1 + r_2}{2} - 1)N} \int_{\mathbb{R}^N} |u_1|^{r_1} |u_2|^{r_2} dx \end{aligned}$$

where we have used the property, see (3.3.13), that

$$\int_{\mathbb{R}^N} |v_1|^{r_1} |v_2|^{r_2} \, dx \geq \int_{\mathbb{R}^N} |u_1|^{r_1} |u_2|^{r_2} + |w_1|^{r_1} |w_2|^{r_2} \, dx.$$

As a consequence, for $t \in [t_1, t_2]$,

$$J(v_1^t, v_2^t) \leq J(u_1^t, u_2^t) + J(w_1^t, 0) + J(0, w_2^t). \tag{3.4.2}$$

In particular, since $J(w_1^{t_2}, 0) \leq 0$ and $J(0, w_2^{t_2}) \leq 0$, we get from (3.4.2) that $J(v_1^{t_2}, v_2^{t_2}) \leq J(u_1^{t_2}, u_2^{t_2}) < m(a_1, a_2)$. Thus we both have that $(v_1^{t_2}, v_2^{t_2}) \notin \overline{\mathcal{B}(\rho_0)}$ and $J(v_1^{t_2}, v_2^{t_2}) < 0$. Now from (3.4.2) and using Lemma 3.4.2, we also have that

$$\begin{aligned} \max_{t \in [t_1, t_2]} J(v_1^t, v_2^t) &\leq \max_{t \in [t_1, t_2]} \left[J(u_1^t, u_2^t) + J(w_1^t, 0) + J(0, w_2^t) \right] \\ &\leq \max_{t \in [t_1, t_2]} J(u_1^t, u_2^t) + \max_{t \in [t_1, t_2]} J(w_1^t, 0) + \max_{t \in [t_1, t_2]} J(0, w_2^t) \\ &= J(u_1, u_2) + \max_{t \in [t_1, t_2]} J(w_1^t, 0) + \max_{t \in [t_1, t_2]} J(0, w_2^t) < \gamma(a_1, a_2), \end{aligned}$$

because $\max_{t \in [t_1, t_2]} J(w_1^t, 0) < 0$ if $w_1 \neq 0$ and $\max_{t \in [t_1, t_2]} J(0, w_2^t) < 0$ if $w_2 \neq 0$. Thus, after a renormalization $[t_1, t_2] \to [0, 1]$, we obtain a path g lying in Γ such that $\max_{t \in [0,1]} J(g(t)) < \gamma(a_1, a_2)$ and this ends the proof. $\square$

We now turn to the existence of the second solution of Theorem 3.1.2 (ii). Our proof borrows several key ingredients from [12]. First we recall some properties of the scalar nonlinear Schrödinger equation. Let $w_{a,\mu,p} > 0, w_{a,\mu,p} \in S(a)$ be radially symmetric and satisfy

$$-\Delta w_{a,\mu,p} - \lambda w_{a,\mu,p} = \mu |w_{a,\mu,p}|^{p-2} w_{a,\mu,p}, \tag{3.4.3}$$

for $2 + \frac{4}{N} < p < 2^*$ and $\lambda < 0$. It is well known that $w_{a,\mu,p}$ is unique and is given by

$$w_{a,\mu,p}(x) = \left(-\frac{\lambda}{\mu}\right)^{\frac{1}{p-2}} w_0((-\lambda)^{\frac{1}{2}} x), \tag{3.4.4}$$

where w_0 is the unique positive radial solution of the equation $-\Delta w + w = |w|^{p-2}w$. In what follows, we set

$$C_0(N,p) = \int_{\mathbb{R}^N} |\nabla w_0|^2 \, dx, \quad C_1(N,p) = \int_{\mathbb{R}^N} |w_0|^p \, dx. \tag{3.4.5}$$

Let us now introduce a Pohozaev type manifold

$$\mathcal{P}(N,a,\mu,p) := \{u \in S(a) : \int_{\mathbb{R}^N} |\nabla u|^2 \, dx = \frac{\mu}{p}\left(\frac{p}{2} - 1\right) N \int_{\mathbb{R}^N} |u|^p \, dx\}$$

and the functional $I_{\mu,p} : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$I_{\mu,p}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{\mu}{p}\int_{\mathbb{R}^N} |u|^p \, dx.$$

Lemma 3.4.6. *The solution $w_{a,\mu,p}$ to (3.4.3) belongs to $\mathcal{P}(N,a,\mu,p)$, and it minimizes the functional $I_{\mu,p}$ on the manifold $\mathcal{P}(N,a,\mu,p)$.*

Proof. The proof of such results can be directly deduced from [63, Lemma 2.7 and 2.10]. □

From (3.4.4)-(4.2.15), it is not difficult to check that

$$\begin{aligned} \|\nabla w_{a,\mu,p}\|_2^2 &= \left(\frac{a}{C_0(N,p)}\right)^{\frac{2p-N(p-2)}{4-N(p-2)}} \mu^{\frac{4}{4-N(p-2)}} C_0(N,p), \\ \|w_{a,\mu,p}\|_p^p &= \left(\frac{a}{C_0(N,p)}\right)^{\frac{2p-N(p-2)}{4-N(p-2)}} \mu^{\frac{N(p-2)}{4-N(p-2)}} C_1(N,p), \end{aligned} \tag{3.4.6}$$

and then the least energy level of $I_{\mu,p}$ on $\mathcal{P}(N,a,\mu,p)$ is given by

$$\begin{aligned} l(N,a,\mu,p) &:= \inf_{u\in\mathcal{P}(N,a,\mu,p)} I_{\mu,p}(u) = I_{\mu,p}(w_{a,\mu,p}) \\ &= \frac{\mu}{p}\left(\left(\frac{p}{2}-1\right)\frac{N}{2} - 1\right)\int_{\mathbb{R}^N} |w_{a,\mu,p}|^p \, dx \\ &= \frac{1}{p}\left(\left(\frac{p}{2}-1\right)\frac{N}{2} - 1\right)\left(\frac{a}{C_0(N,p)}\right)^{\frac{2p-N(p-2)}{4-N(p-2)}} \mu^{\frac{4}{4-N(p-2)}} C_1(N,p). \end{aligned} \tag{3.4.7}$$

We now define, for $s \in \mathbb{R}$ and $w \in H^1(\mathbb{R}^N)$, the dilation $(s * w)(x) := e^{\frac{Ns}{2}} w(e^s x)$.

Lemma 3.4.7. *For any $w \in H^1(\mathbb{R}^N)$, there holds*

$$I_{\mu,p}(s*w) = \frac{e^{2s}}{2}\int_{\mathbb{R}^N}|\nabla w|^2\,dx - \frac{\mu}{p}e^{s(\frac{p}{2}-1)N}\int_{\mathbb{R}^N}|w|^p\,dx,$$
$$\frac{\partial}{\partial s}I_{\mu,p}(s*w) = e^{2s}\int_{\mathbb{R}^N}|\nabla w|^2\,dx - \frac{\mu}{p}\left(\frac{p}{2}-1\right)Ne^{s(\frac{p}{2}-1)N}\int_{\mathbb{R}^N}|w|^p\,dx.$$

In particular, if $w = w_{a,\mu,p}$, then

$$\frac{\partial}{\partial s}I_{\mu,p}(s*w_{a,\mu,p}) = 0 \text{ if } s = 0,$$
$$\frac{\partial}{\partial s}I_{\mu,p}(s*w_{a,\mu,p}) > 0(<0) \text{ if } s < 0(>0).$$

Proof. We refer to [12, Lemma 3.1] for a very similar proof. □

Now define, for $i = 1, 2$,

$$\begin{aligned} c_i := c_i(r_1+r_2, p_i) &:= \frac{p_i-(r_1+r_2)}{p_i}\left(\frac{p_i(r_1+r_2)}{p_i-2}\right)^{\frac{r_1+r_2-2}{p_i-(r_1+r_2)}} \\ &= \max_{t\geq 0}\Big[t^{r_1+r_2-2} - \frac{1}{p_i}t^{p_i-2}\Big]. \end{aligned} \tag{3.4.8}$$

In view of (3.4.7), since $p_1, p_2 > 2 + \frac{4}{N}$, then there exists a $\beta_1 = \beta_1(a_1, a_2) > 0$ such that

$$\begin{aligned} &l(N, a_1, \mu_1+\beta_1, p_1) + l(N, a_2, \mu_2+\beta_1, p_2) - \beta_1 c_1 a_1 - \beta_1 c_2 a_2 \\ &= \max\{l(N, a_1, \mu_1, p_1), l(N, a_2, \mu_2, p_2)\} > 0 \end{aligned}$$

and

$$\begin{aligned} &l(N, a_1, \mu_1+\beta_1, p_1) + l(N, a_2, \mu_2+\beta_1, p_2) - \beta_1 c_1 a_1 - \beta_1 c_2 a_2 \\ &> \max\{l(N, a_1, \mu_1, p_1), l(N, a_2, \mu_2, p_2)\} > 0 \end{aligned}$$

for any $0 < \beta < \beta_1$. Note that $\beta_1(a_1, a_2) \to \infty$ as $a_1, a_2 \to 0$. Choosing if necessary $\beta_0 > 0$ smaller in Lemma 3.3.1, we can assume that $\beta_1 = \beta_0$.

Lemma 3.4.8. *For any $0 < \beta < \beta_0$,*

$$\begin{aligned} &\inf\{J(u_1,u_2) : (u_1,u_2) \in \mathcal{P}(N, a_1, \mu_1+\beta, p_1) \times \mathcal{P}(N, a_2, \mu_2+\beta, p_2)\} \\ &> \max\{l(N, a_1, \mu_1, p_1), l(N, a_2, \mu_2, p_2)\}. \end{aligned}$$

Proof. For any $(u_1, u_2) \in \mathcal{P}(N, a_1, \mu_1+\beta, p_1) \times \mathcal{P}(N, a_2, \mu_2+\beta, p_2)$, we have

$$\begin{aligned} J(u_1,u_2) &= I_{\mu_1,p_1}(u_1) + I_{\mu_2,p_2}(u_2) - \beta\int_{\mathbb{R}^N}|u_1|^{r_1}|u_2|^{r_2}\,dx \\ &\geq I_{\mu_1,p_1}(u_1) + I_{\mu_2,p_2}(u_2) - \beta\sum_{i=1}^{2}\int_{\mathbb{R}^N}|u_i|^{r_1+r_2}\,dx \\ &\geq I_{\mu_1,p_1}(u_1) + I_{\mu_2,p_2}(u_2) - \beta\sum_{i=1}^{2}\int_{\mathbb{R}^N}c_i|u_i|^2 + \frac{1}{p_i}|u_i|^{p_i}\,dx \\ &= I_{\mu_1+\beta,p_1}(u_1) + I_{\mu_2+\beta,p_2}(u_2) - \beta c_1 a_1 - \beta c_2 a_2 \\ &\geq l(N, a_1, \mu_1+\beta, p_1) + l(N, a_2, \mu_2+\beta, p_2) - \beta c_1 a_1 - \beta c_2 a_, \end{aligned}$$

where c_i for $i = 1, 2$ are defined by (3.4.8). □

Now for any given $\beta \in (0, \beta_0)$, according to Lemma 4.3.2, we can fix a $\epsilon > 0$ such that

$$\begin{aligned} &\inf\{J(u_1,u_2) : (u_1,u_2) \in \mathcal{P}(N,a_1,\mu_1+\beta,p_1) \times \mathcal{P}(N,a_2,\mu_2+\beta,p_2)\} \\ &> \max\{l(N,a_1,\mu_1,p_1), l(N,a_2,\mu_2,p_2)\} + \epsilon. \end{aligned} \tag{3.4.9}$$

We set

$$w_1 := w_{a_1,\mu_1+\beta,p_1},\ w_2 := w_{a_2,\mu_2+\beta,p_2},$$

and

$$\phi_i(s) := I_{\mu_i,p_i}(s * w_i),\ \psi_i(s) := \frac{\partial}{\partial s} I_{\mu_i,p_i}(s * w_i).$$

From these definitions and as in [12, Lemma 3.3], one obtains the following result.

Lemma 3.4.9. *For $i = 1, 2$, there exists $\rho_i < 0$ and $R_i > 0$ such that*

(i) $0 < \phi_i(\rho_i) < \epsilon$ and $\phi_i(R_i) \le 0$, where $\epsilon > 0$ is determined in (3.4.9);

(ii) $\psi_i(s) > 0$ for $s < 0$ and $\psi_i(s) < 0$ for $s > 0$. In particular $\psi_i(\rho_i) > 0, \psi_i(R_i) < 0$.

Let $M := [\rho_1, R_1] \times [\rho_2, R_2]$, and for $(t_1, t_2) \in M$,

$$g_0(t_1,t_2) := (t_1 * w_1, t_2 * w_2) \in S(a_1,a_2).$$

We now introduce the min-max class

$$\Gamma := \{g \in C(M, S(a_1,a_2)) : g_{|\partial M} = g_0\}.$$

Lemma 3.4.10. *If $g \in \Gamma$, then there holds*

$$\sup_{\partial M} J(g) < \max\{l(N,a_1,\mu_1+\beta,p_1), l(N,a_2,\mu_2+\beta,p_2)\} + \epsilon.$$

Proof. In view of Lemma 3.4.9 and (3.4.6),

$$\begin{aligned} J(t_1 * w_1, \rho_2 * w_2) &\le I_{\mu_1,p_1}(t_1 * w_1) + I_{\mu_2,p_2}(\rho_2 * w_2) \\ &\le I_{\mu_1,p_1}(t_1 * w_1) + \epsilon \le \sup_{s\in\mathbb{R}} I_{\mu_1,p_1}(s * w_1) + \epsilon \\ &= \left(\frac{\mu_1+\beta}{\mu_1}\right)^{\frac{4}{4-N(p_1-2)}} \sup_{s\in\mathbb{R}} I_{\mu_1,p_1}(s * w_{a_1,\mu_1,p_1}) + \epsilon \\ &\le l(N,a_1,\mu_1,p_1) + \epsilon, \end{aligned}$$

because $p_1 > 2 + \frac{4}{N}$. Consequently, for $t_1 \in [\rho_1, R_1]$,

$$J(t_1 * w_1, \rho_2 * w_2) \le l(N,a_1,\mu_1,p_1) + \epsilon,$$

and in a similar way, for $t_2 \in [\rho_2, R_2]$,

$$J(\rho_1 * w_1, t_2 * w_2) \le l(N,a_2,\mu_2,p_2) + \epsilon.$$

On the other hand, using Lemma 3.4.9, one can show that for $t_1 \in [\rho_1, R_1]$,

$$\begin{aligned} J(t_1 * w_1, R_2 * w_2) &\le I_{\mu_1,p_1}(t_1 * w_1) + I_{\mu_2,p_2}(R_2 * w_2) \\ &\le \sup_{s\in\mathbb{R}} I_{\mu_1,p_1}(s * w_1) \le l(N,a_1,\mu_1,p_1). \end{aligned}$$

Analogously, for $t_2 \in [\rho_2, R_2]$, $J(R_1 * w_1, t_2 * w_2) \le l(N,a_2,\mu_2,p_2)$. Then the lemma follows. □

Lemma 3.4.11. *For every $g \in \Gamma$, there exists $(t_1, t_2) \in M$ such that $g(t_1, t_2) \in \mathcal{P}(N, a_1, \mu_1 + \beta, p_1) \times \mathcal{P}(N, a_2, \mu_2 + \beta, p_2)$.*

Proof. Let $g \in \Gamma$ be arbitrary, we write $g(t_1, t_2) := (g_1(t_1, t_2), g_2(t_1, t_2))$, and we introduce the map $F_g : M \to \mathbb{R}^2$ as,

$$F_g(t_1, t_2) := \left(\frac{\partial}{\partial s} I_{\mu_1+\beta,p_1}(s * g_1(t_1, t_2))|_{s=0}, \frac{\partial}{\partial s} I_{\mu_2+\beta,p_2}(s * g_2(t_1, t_2))|_{s=0} \right).$$

Since

$$\begin{aligned} &\frac{\partial}{\partial s} I_{\mu_i+\beta,p_i}(s * g_i(t_1, t_2))|_{s=0} \\ &= \int_{\mathbb{R}^N} |\nabla g_i(t_1, t_2)|^2 \, dx - \frac{\mu_i}{p_i} \left(\frac{p_i}{2} - 1 \right) N \int_{\mathbb{R}^N} |g_i(t_1, t_2)|^{p_i} \, dx, \end{aligned}$$

we deduce that $F_g(t_1, t_2) = (0, 0)$ if and only if $g(t_1, t_2) \in \mathcal{P}(N, a_1, \mu_1+\beta, p_1) \times \mathcal{P}(N, a_2, \mu_2+\beta, p_2)$. To show that $F_g(t_1, t_2) = 0$ has a solution we can follow the proof given in [12, Lemma 3.5]. □

At this point, we know from Lemma 4.3.2, 3.4.10 and 3.4.11, that there exists a Palais-Smale sequence for J restricted to $S(a_1, a_2)$ at the level

$$c(a_1, a_2) := \inf_{g \in \Gamma} \max_{(t_1,t_2) \in M} J(g(t_1, t_2)) > \max\{l(N, a_1, \mu_1, p_1), l(N, a_2, \mu_2, p_2)\}. \tag{3.4.10}$$

In addition, arguing as in the proof of Theorem 3.1.1 (ii), we obtain the following result.

Lemma 3.4.12. *For any $0 < \beta < \beta_0$, there exists a Palais-Smale sequence $\{(u_1^n, u_2^n)\} \subset S_{rad}(a_1, a_2)$ for J restricted to $S_{rad}(a_1, a_2)$ at the level $c(a_1, a_2)$, which satisfies $(u_1^n)^- \to 0$, $(u_2^n)^- \to 0$ in $H^1(\mathbb{R}^N)$ and the property $Q(u_1^n, u_2^n) \to 0$ as $n \to \infty$.*

Proof of Theorem 3.1.2 (ii). Let $\{(u_1^n, u_2^n)\} \subset S_{rad}(a_1, a_2)$ be given by Lemma 3.4.12. Then there exists $u_1, u_2 \geq 0$ such that, up to a subsequence, $(u_1^n, u_2^n) \rightharpoonup (u_1, u_2)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and $(u_1^n, u_2^n) \to (u_1, u_2)$ in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ for $2 < p < 2^*$. It follows as before that (u_1, u_2) is a weak solution to (1.1.2) for some $(\lambda_1, \lambda_2) \in \mathbb{R}^2$, thus $Q(u_1, u_2) = 0$. Since $Q(u_1^n, u_2^n) = o_n(1)$, we deduce that $\int_{\mathbb{R}^N} |\nabla u_1^n|^2 + |\nabla u_2^n|^2 \, dx \to \int_{\mathbb{R}^N} |\nabla u_1|^2 + |\nabla u_2|^2 \, dx$. This results that $J(u_1, u_2) = c(a_1, a_2) > 0$, and in particular $(u_1, u_2) \neq (0, 0)$. It remains to prove that $(u_1, u_2) \in S(a_1, a_2)$. From Lemma 3.2.6, we may suppose $\lambda_1 < 0$, and thus $u_1 \in S(a_1)$. If $\lambda_2 < 0$ we also have that $u_2 \in S(a_2)$. If we assume $\lambda_2 \geq 0$, then

$$-\Delta u_2 = \lambda_2 u_2 + \mu_2 u_2^{p_2-1} + \beta r_2 u_1^{r_1} u_2^{r_2-1} \geq 0,$$

and applying Lemma 3.2.2, it follows that $u_2 = 0$. Therefore $Q(u_1, 0) = 0$, namely, $u_1 \in \mathcal{P}(N, a_1, p_1, \mu_1)$, and this implies that

$$c(a_1, a_2) = J(u_1, 0) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_1|^2 \, dx - \frac{\mu_1}{p_1} \int_{\mathbb{R}^N} |u_1|^{p_1} \, dx = l(N, a_1, \mu_1, p_1),$$

in contradiction with (3.4.10). Knowing that $(u_1, u_2) \in S(a_1, a_2)$, we conclude as previously. □

3.5 Appendix

Proof of Lemma 3.4.4. To begin with, we set for $i = 1, 2$,

$$a := \int_{\mathbb{R}^N} |\nabla u_1|^2 + |\nabla u_2|^2 \, dx, \quad b_i := \frac{\mu_i}{p_i} \int_{\mathbb{R}^N} |u_i|^{p_i} \, dx, \quad c := \beta \int_{\mathbb{R}^N} |u_1|^{r_1} |u_2|^{r_2} \, dx.$$

Thus defining, for $t > 0$, $\theta(t) := J(u_1^t, u_2^t)$, we then have

$$\theta(t) := a \, \frac{t^2}{2} - \sum_{i=1}^{2} b_i t^{\tilde{p}_i} - c \, t^r, \tag{3.5.1}$$

where we have set, for $i = 1, 2$,

$$\tilde{p}_i := \Big(\frac{p_i}{2} - 1\Big)N, \quad r := \Big(\frac{r_1 + r_2}{2} - 1\Big)N.$$

Note that, under (H_1), $\tilde{p}_1, \tilde{p}_2 \in (0, 1)$ if $2 < p_i < 2 + \frac{2}{N}$, $\tilde{p}_1, \tilde{p}_2 \in (1, 2)$ if $p_i > 2 + \frac{2}{N}$, and $r > 2$.

To prove the lemma, it suffices to show that θ' admits at most two zeros on $(0, \infty)$. This is clearly equivalent to show that $g(t) := \frac{\theta'(t)}{t^\alpha}$ has at most two zeros for $t > 0$, and for a $\alpha \in \mathbb{R}$ to be chosen later, . Note that it is not restrictive to assume that $p_1 \leq p_2$. We have

$$g(t) = a t^{1-\alpha} - b_1 \, \tilde{p}_1 \, t^{\tilde{p}_1 - 1 - \alpha} - b_2 \, \tilde{p}_2 \, t^{\tilde{p}_2 - 1 - \alpha} - c \, r \, t^{r-1-\alpha}.$$

Thus

$$\begin{aligned} g'(t) = {} & a \, (1-\alpha) t^{-\alpha} - b_1 \, \tilde{p}_1 (\tilde{p}_1 - 1 - \alpha) t^{\tilde{p}_1 - 2 - \alpha} \\ & - b_2 \, \tilde{p}_2 (\tilde{p}_2 - 1 - \alpha) \, t^{\tilde{p}_2 - 2 - \alpha} - c \, r \, (r - 1 - \alpha) \, t^{r-2-\alpha}, \end{aligned}$$

and

$$\begin{aligned} g''(t) = {} & a \, (1-\alpha)(-\alpha) t^{-\alpha-1} - b_1 \, \tilde{p}_1 (\tilde{p}_1 - 1 - \alpha)(\tilde{p}_1 - 2 - \alpha) t^{\tilde{p}_1 - 3 - \alpha} \\ & - b_2 \, \tilde{p}_2 (\tilde{p}_2 - 1 - \alpha)(\tilde{p}_2 - 2 - \alpha) t^{\tilde{p}_2 - 3 - \alpha} - c \, r (r - 1 - \alpha)(r - 2 - \alpha) t^{r-3-\alpha}. \end{aligned}$$

For convenience, we write

$$g''(t) = \alpha_0 \, t^{-\alpha-1} - \alpha_1 \, t^{\tilde{p}_1 - 3 - \alpha} - \, \alpha_2 \, t^{\tilde{p}_2 - 3 - \alpha} - \alpha_3 \, t^{r-3-\alpha}, \tag{3.5.2}$$

where we have set $\alpha_0 := a \, (1-\alpha)(-\alpha)$, $\alpha_i := b_i \, \tilde{p}_i (\tilde{p}_i - 1 - \alpha)(\tilde{p}_i - 2 - \alpha)$ for $i = 1, 2$, and $\alpha_3 := c \, r (r - 1 - \alpha)(r - 2 - \alpha)$. We now consider the following two cases.

Case 1: $2 < p_1 \leq p_2 \leq r_1 + r_2 - \frac{2}{N}$. If we assume that $\tilde{p}_2 \leq 1$, namely, $p_2 \leq 2 + \frac{2}{N}$, then setting $\alpha = 0$, we get that $\alpha_0 = 0$, $\alpha_1 \leq 0$, $\alpha_2 \leq 0$, and $\alpha_3 > 0$. Thus $g''(t) < 0$ for any $t > 0$, we then deduce that g' is strictly decreasing on $(0, \infty)$. It follows that g cannot have more than two zeros. Now if we assume that $\tilde{p}_2 > 1$, we choose $\alpha = \tilde{p}_2 - 1 \in (0, 1)$. Then $g''(t)$ becomes

$$g''(t) = \alpha_0 \, t^{-\tilde{p}_2} - \alpha_1 \, t^{\tilde{p}_1 - \tilde{p}_2 - 2} - \alpha_3 \, t^{r - \tilde{p}_2 - 2}$$

with $\alpha_0 < 0$ and $\alpha_1 > 0$. Also under our assumption we have $r \geq \tilde{p}_2 + 1$ and we obtain that $\alpha_3 \geq 0$. Thus $g''(t) < 0$ for any $t > 0$, and we conclude as in the first case.

Case 2: $|p_1 - p_2| \leq \frac{2}{N}$. In view of the first case we can assume that $\tilde{p}_2 > 1$. We now write (3.5.2) as

$$g''(t) = t^{-\alpha-1}\Big[\alpha_0 - \alpha_1\, t^{\tilde{p}_1-2} - \alpha_2\, t^{\tilde{p}_2-2} - \alpha_3\, t^{r-2}\Big] := t^{-\alpha-1}\xi(t).$$

Let us prove that, for a convenient choice of $\alpha \leq 0$ we can insure that ξ is a strictly decreasing on $(0,\infty)$. Recall that we assume that $p_1 \leq p_2$. Since $|p_1 - p_2| \leq \frac{2}{N}$, it implies that $\tilde{p}_2 \leq \tilde{p}_1 + 1$, thus we can choose a $\alpha \leq 0$ satisfying $\tilde{p}_2 - 2 \leq \alpha \leq \tilde{p}_1 - 1$. With this choice $\alpha_1 \leq 0, \alpha_2 \leq 0$, and $\alpha_3 > 0$ because of $r > 2$. It follows that ξ is strictly decreasing on $(0,\infty)$.

Now having proved that ξ is strictly decreasing and since $\lim_{t\to 0^+} \xi(t) > 0$ and $\lim_{t\to\infty} \xi(t) = -\infty$, there exists exactly one $t_1 > 0$ satisfying $\xi(t_1) = 0$. Thus $g'(t)$ is strictly increasing on $(0, t_1)$, and strictly decreasing on $[t_1, \infty)$. Also we can check that $\lim_{t\to 0^+} g'(t) < 0$ and $\lim_{t\to\infty} g'(t) = -\infty$. At this point, we can assume without restriction that

$$\max_{t>0} g'(t) > 0. \tag{3.5.3}$$

Otherwise, since $\lim_{t\to 0^+} g(t) < 0$, then $g(t) < 0$ for $t > 0$, and g has no zero on $(0,\infty)$.

From (3.5.3) and the limits of $g'(t)$, we deduce that there are exactly two values $t_2 < t_3$ such that $g'(t_2) = g'(t_3) = 0$. In addition, $0 < t_2 < t_1 < t_3$. Clearly, g is strictly decreasing on $(0, t_2) \cup (t_3, \infty)$, and strictly increasing on $[t_2, t_3)$. Recording that $\lim_{t\to 0^+} g(t) = 0^-$, it implies that g may have at most two zeros. □

Chapter 4

Normalized solutions for fourth-order nonlinear Schrödinger equation in the mass critical and supercritical regime

4.1 Introduction

In this chapter, we deal with a class of time-dependent fourth-order nonlinear Schrödinger equations in $\mathbb{R} \times \mathbb{R}^N$,

$$i\partial_t \psi - \gamma \Delta^2 \psi + \Delta \psi + |\psi|^{2\sigma} \psi = 0, \tag{4.1.1}$$

where $\gamma > 0$. A fundamental step to study solutions of (4.1.1) consists in standing waves, namely solutions with the form of $\psi(t, x) = e^{i\alpha t} u(x)$ for $\alpha \in \mathbb{R}$. This then leads to the following elliptic equation satisfied by u,

$$\gamma \Delta^2 u - \Delta u + \alpha u = |u|^{2\sigma} u. \tag{4.1.2}$$

Observe that the L^2-norm of solution to the Cauchy problem of (4.1.1) is conserved along time, i.e. for any $t > 0$,

$$\int_{\mathbb{R}^N} |\psi(t, x)|^2 \, dx = \int_{\mathbb{R}^N} |\psi(0, x)|^2 \, dx.$$

Thus it is of great interest to research solutions to (4.1.2) having prescribed L^2-norm, namely, for given $c > 0$, we find $\alpha \in \mathbb{R}$ and $u \in H^2(\mathbb{R}^N)$ satisfying (4.1.2), together with normalized condition

$$\int_{\mathbb{R}^N} |u|^2 \, dx = c. \tag{4.1.3}$$

Such solutions are so-called normalized solutions. For the simplicity of terminology, in the following we shall refer a solution (α, u) to (4.1.2)-(4.1.3) as u, where u is obtained as a critical point of energy functional $E : H^2(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$E(u) := \frac{\gamma}{2} \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{1}{2\sigma + 2} \int_{\mathbb{R}^N} |u|^{2\sigma+2} \, dx$$

on the constraint

$$S(c) := \{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 \, dx = c\},$$

and α is then determined as Lagrange multiplier.

From now on, we are concerned with normalized solutions, i.e. solutions to (4.1.2)-(4.1.3). When $0 < \sigma N < 4$, the energy functional E is bounded from below on $S(c)$, then the authors [28] studied the following minimizing problem

$$m(c) := \inf_{u \in S(c)} E(u). \tag{4.1.4}$$

In this case, it is possible to find a solution to (4.1.2)-(4.1.3) as a minimizer to (4.1.4). We mention the following result obtained in [28].

Theorem 4.1.1. *Assume that $0 < \sigma N < 2$, then $m(c)$ is achieved for every $c > 0$. If $2 \leq \sigma N < 4$ then there exists a critical mass $\tilde{c} = \tilde{c}(\sigma, N)$ such that*

(i) $m(c)$ is not achieved if $c < \tilde{c}$;

(ii) $m(c)$ is achieved if $c > \tilde{c}$ and $\sigma = 2/N$;

(iii) $m(c)$ is achieved if $c \geq \tilde{c}$ and $\sigma \neq 2/N$.

Moreover, if $\sigma \in \mathbb{N}^+$ and $m(c)$ is achieved, then there exists at least one radially symmetric minimizer.

Remark 4.1.2. *The appearance of a critical mass when $2 \leq \sigma N < 4$ is linked to the fact that every term of the energy functional E behaves differently with respect to dilations.*

In this chapter, as inspired by [28], we study solutions to (4.1.2)-(4.1.3) under the mass critical case $\sigma N = 4$ and the mass supercritical case $4 < \sigma N < 4^*$, where $4^* := \frac{4N}{(N-4)^+}$. In this subject, our first result concerns the mass critical case $\sigma N = 4$. To show the statement, we recall the well known Gagliardo-Nirenberg's inequality (see [92]) for $u \in H^2(R^N)$,

$$\|u\|_{2\sigma+2}^{2\sigma+2} \leq B_N(\sigma) \|\Delta u\|_2^{\frac{\sigma N}{2}} \|u\|_2^{2+2\sigma-\frac{\sigma N}{2}}, \tag{4.1.5}$$

where

$$\begin{cases} 0 \leq \sigma, & \text{if } N \leq 4, \\ 0 \leq \sigma < \dfrac{4}{N-4}, & \text{if } N \geq 5, \end{cases}$$

and $B_N(\sigma)$ is a constant depending on σ and N.

Theorem 4.1.3. *Let $N \geq 1, \sigma N = 4$. There exists a $c_N^* > 0$ such that*

$$m(c) = \inf_{u \in S(c)} E(u) = \begin{cases} 0, & 0 < c \leq c_N^*, \\ -\infty, & c > c_N^*, \end{cases}$$

For $c \in (0, c_N^*)$, (4.1.2)-(4.1.3) has no solution, and in particular $m(c)$ is not achieved. In addition, $c_N^* = (\gamma C(N))^{\frac{N}{4}}$ where

$$C(N) := \frac{N+4}{N B_N(\frac{4}{N})}, \tag{4.1.6}$$

and $B_N(\sigma)$ is the constant in (4.1.5).

Theorem 4.1.3 shows that $m(c) = -\infty$ when $c > c_N^*$. Actually, when $4 < \sigma N < 4^*$, we will obtain that $m(c) = -\infty$ when $c > 0$. To see this, for any $u \in S(c)$, $\lambda > 0$, we define

$$u_\lambda(x) := \lambda^{\frac{N}{4}} u(\sqrt{\lambda}x).$$

By direct calculations one can check that $\|u_\lambda\|_2 = \|u\|_2$ and

$$E(u_\lambda) = \frac{\gamma\lambda^2}{2}\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \frac{\lambda}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{\lambda^{\sigma N/2}}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2}\,dx. \tag{4.1.7}$$

Thus $E(u_\lambda) \to -\infty$ as $\lambda \to \infty$ when $4 < \sigma N < 4^*$, then we deduce that $m(c) = -\infty$ for any $c > 0$. By consequence, in both cases, it is no more possible to obtain a solution to (4.1.2)-(4.1.3) as a minimizer to (4.1.4). To overcome this difficulty, we introduce a natural constraint $\mathcal{M}(c)$ given by

$$\mathcal{M}(c) := \{u \in S(c) : Q(u) = 0\},$$

where

$$Q(u) := \gamma\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{\sigma N}{2(2\sigma+2)}\int_{\mathbb{R}^N} |u|^{2\sigma+2}\,dx.$$

Using (4.1.7), we immediately see that

$$Q(u) = \frac{\partial E(u_\lambda)}{\partial\lambda}|_{\lambda=1}. \tag{4.1.8}$$

and thus, heuristically, $\mathcal{M}(c)$ contains all critical points for E restricted to $S(c)$, then solutions to (4.1.2)-(4.1.3). This fact will be rigourously proved in Lemma 4.10.1. Actually, the condition $Q(u) = 0$ corresponds to a Pohozaev type identity, and $\mathcal{M}(c)$ is called as the Pohozaev manifold elated to (4.1.2)-(4.1.3). Borrowing the key spirit from [13], we shall prove that a critical point of E restricted to $\mathcal{M}(c)$ is a critical point of E restricted to $S(c)$, see Lemma 4.3.5. For these reasons, we define the following minimization problem

$$\gamma(c) := \inf_{u\in\mathcal{M}(c)} E(u). \tag{4.1.9}$$

We now search for a minimizer to (4.1.9). Note that, if it exists, it then corresponds to a ground state solution to (4.1.2)-(4.1.3) in the sense that it minimizes the energy functional E among all solutions having the same L^2-norm.

For convenience, we define $c_0 \in \mathbb{R}$ as

$$c_0 := \begin{cases} 0, & \text{if } 4 < \sigma N < 4^*, \\ c_N^*, & \text{if } \sigma N = 4, \end{cases} \tag{4.1.10}$$

where c_N^* is given in Theorem 4.1.3.

Theorem 4.1.4. *Let $N \geq 1$, $4 \leq \sigma N < 4^*$. Then there exists a $c_{\sigma,N} > c_0$ such that for any $c \in (c_0, c_{\sigma,N})$, (4.1.2)-(4.1.3) has a ground state solution u_c satisfying $E(u_c) = \gamma(c)$, and the associated Lagrange parameter α_c is strictly positive. Moreover*

(i) $c_{\sigma,1} = c_{\sigma,2} = \infty$, and $c_{\sigma,3} = \infty$ if $4/3 \leq \sigma < 2$;

(ii) If $\sigma N = 4$, then $c_{\sigma,4} = \infty$, and $c_{\sigma,N} \geq \left(\frac{N}{N-4}\right)^{\frac{N}{4}} c_N^*$ if $N \geq 5$.

The proof of Theorem 4.1.4 crucially relies on a key ingredient Lemma 4.3.5. Using this result and the Ekeland variational principle [47], we then obtain a Palais-Smale sequence $\{u_n\} \subset \mathcal{M}(c)$ for E restricted to $S(c)$ at level $\gamma(c)$ as a minimizing sequence to (4.1.9). Our aim is to prove that $\{u_n\}$ is compact, up to translations, in $H^2(\mathbb{R}^N)$. Firstly, since E is coercive on $\mathcal{M}(c)$, see Lemma 4.3.1, thus $\{u_n\}$ is bounded in $H^2(\mathbb{R}^N)$, and it then follows that there exists $u_c \in H^2(\mathbb{R}^N)$ such that $u_n \rightharpoonup u_c$, up to translation, in $H^2(\mathbb{R}^N)$. Furthermore, there is $\alpha_c \in \mathbb{R}$ such that u_c satisfies

$$\gamma\Delta^2 u_c - \Delta u_c + \alpha_c u_c = |u_c|^{2\sigma} u_c. \tag{4.1.11}$$

At this point, proving the compactness of $\{u_n\}$ then reduces to show that the strong convergence of $\{u_n\}$ in $L^{2\sigma+2}(\mathbb{R}^N)$ and the Lagrange parameter $\alpha_c > 0$, see Lemma 4.3.6. The strong convergence of $\{u_n\}$ in $L^{2\sigma+2}(\mathbb{R}^N)$ benefits from the fact that $c \mapsto \gamma(c)$ is nonincreasing on (c_0, ∞), see Lemma 4.4.1. The restriction on the size of c is to insure that $\alpha_c > 0$, see Lemma 4.2.1.

Taking advantage of the genus theory, we obtain the existence of multiple radial solutions to (4.1.2)-(4.1.3).

Theorem 4.1.5. *Assume $N \geq 2$.*

(i) If $4 < \sigma N < 4^*$, then for any $c \in (0, c_{\sigma,N})$, where $c_{\sigma,N}$ is defined in Theorem 4.1.4, (4.1.2)-(4.1.3) *admits infinitely many radial solutions.*

(ii) If $2 \leq N \leq 4$, $\sigma N = 4$, then for any $k \in \mathbb{N}^+$, there exists a $c_k > c_N^*$ such that, for any $c \geq c_k$, (4.1.2)-(4.1.3) *admits at least k radial solutions.*

To establish Theorem 4.1.5, we shall work in the subspace $H^2_{rad}(\mathbb{R}^N)$ of $H^2(\mathbb{R}^N)$, which consists of radially symmetric functions in $H^2(\mathbb{R}^N)$. Accordingly, we define $\mathcal{M}_{rad}(c) := \mathcal{M}(c) \cap H^2_{rad}(\mathbb{R}^N)$.

The proof of Theorem 4.1.5 is based on the Kranosel'skii genus theory. The key step is to prove that E restricted to $\mathcal{M}_{rad}(c)$ satisfies the Palais-Smale condition. To this end, we consider an arbitrary Palais-Smale sequence $\{u_n\} \subset \mathcal{M}_{rad}(c)$ for E restricted to $\mathcal{M}_{rad}(c)$. Applying the coerciveness of E on $\mathcal{M}_{rad}(c)$, we then denote by u_c its weak limit in $H^2_{rad}(\mathbb{R}^N)$. Moreover, there exists $\alpha_c \in \mathbb{R}$ such that u_c satisfies (4.1.11). Note that the strong convergence of $\{u_n\}$ in $L^{2\sigma+2}(\mathbb{R}^N)$ is given here for free, because the embedding $H^2_{rad}(\mathbb{R}^N) \hookrightarrow L^{2\sigma+2}(\mathbb{R}^N)$ is compact for $N \geq 2$. Reasoning as the proof of Theorem 4.1.4, to show the compactness it remains to check that the Lagrange parameter α_c is strictly positive, which is insured by Lemma 4.2.1. The second step is to show that the set $\mathcal{M}(c)$ is sufficiently large. This is always the case when $4 < \sigma N < 4^*$ for any $c > 0$. However, when $\sigma N = 4$, the set $\mathcal{M}_{rad}(c)$ may be too small. In particular, it shrinks to the empty set as $c \to c_N^*$. To obtain a given number of solutions, we require that $c > c_N^*$ is sufficiently large.

The monotonicity of the function $c \mapsto \gamma(c)$ on (c_0, ∞) is crucially used in the proof of Theorem 4.1.4. We now present additional properties of this function, its behaviors depend in an essential way on the couple (σ, N).

Theorem 4.1.6. *Let $4 \leq \sigma N < 4^*$. The function $c \mapsto \gamma(c)$ is continuous for any $c > c_0$, is decreasing on (c_0, ∞) and $\lim_{c \to c_0^+} \gamma(c) = \infty$. In addition,*

(i) if $N = 1, 2$, $N = 3$ with $\frac{4}{3} \leq \sigma < 2$ or $N = 4$ with $\sigma = 1$, then $c \mapsto \gamma(c)$ is strictly decreasing and $\lim_{c \to \infty} \gamma(c) = 0$.

(ii) *If* $N = 3$ *with* $\sigma \geq 2$ *or* $N = 4$ *with* $\sigma > 1$, *then* $\lim_{c\to\infty} \gamma(c) := \gamma(\infty) > 0$ *and* $\gamma(c) > \gamma(\infty)$ *for all* $c > c_0$.

(iii) *If* $N \geq 5$, *then* $\lim_{c\to\infty} \gamma(c) := \gamma(\infty) > 0$ *and there exists a* $c_\infty > c_0$ *such that,* $\gamma(c) = \gamma(\infty)$ *for all* $c \geq c_\infty$.

Note that Theorem 4.1.6, the difference of behavior of $\gamma(c)$ as $c \to \infty$ between $N \leq 4$ and $N \geq 5$ arises from the fact that the equation

$$\gamma\Delta^2 u - \Delta u = |u|^{2\sigma} u \tag{4.1.12}$$

does not admit a least energy solution in $H^2(\mathbb{R}^N)$ when $N \leq 4$, but it does when $N \geq 5$, see Proposition 4.6.5 for more details.

Next, when $\sigma N = 4$ we show a concentration behavior of ground state solutions to (4.1.2)-(4.1.3) as c approaches to c_N^* from above.

Theorem 4.1.7. *Let* $N \geq 1, \sigma N = 4$, *and* $\{c_n\} \subset \mathbb{R}$ *be a sequence satisfying for any* $n \in \mathbb{N}$, $c_n > c_N^*$ *with* $c_n \to c_N^*$ *as* $n \to \infty$, *and* u_n *be a ground state solution to* (4.1.2)-(4.1.3) *for* $c = c_n$ *at level* $\gamma(c_n)$. *Then there exist a sequence* $\{y_n\} \subset \mathbb{R}^N$ *and a least energy solution* u *to the equation*

$$\gamma\Delta^2 u + u = |u|^{\frac{8}{N}} u, \tag{4.1.13}$$

such that up to a subsequence,

$$\left(\frac{\epsilon_n^4 c_N^* N}{4}\right)^{\frac{N}{8}} u_n\left(\left(\frac{\epsilon_n^4 c_N^* N}{4}\right)^{\frac{1}{4}} x + \epsilon_n y_n\right) \to u \text{ in } L^q(\mathbb{R}^N) \text{ as } n \to \infty$$

for $2 \leq q < \frac{2N}{(N-4)^+}$, *where* $\epsilon_n \to 0$ *as* $n \to \infty$.

Proposition 4.1.7 gives a description of ground state solution to (4.1.2)-(4.1.3) as c_n approaches to c_N^* from above. Roughly speaking, it shows for $n \in \mathbb{N}$ large enough,

$$u_n(x) \approx \left(\frac{4}{\epsilon_n^4 c_N^* N}\right)^{\frac{N}{8}} u\left(\left(\frac{4}{\epsilon_n^4 c_N^* N}\right)^{\frac{1}{4}} (x - \epsilon_n y_n)\right).$$

In the folowing we consider the sign and radially symmetric property of solutions to (4.1.2)-(4.1.3). Concerning this subject, we first mention the case that $\alpha \in \mathbb{R}^+$ is given in (4.1.2). In this case, it is known that when $\alpha \in \mathbb{R}^+$ is sufficiently small, all least energy solutions have a sign and are radial. On the contrary, when $\alpha \in \mathbb{R}^+$ is large, radial solutions are necessarily sign-changing. In addition, when $\sigma \in \mathbb{N}^+$, at least one least energy solution is radial. For more details, see [31, Theorem 4]. When $0 < \sigma N < 4$, regarding the sign and radially symmetric property of minimizers to (4.1.4), we refer to [28]. However, when $4 \leq \sigma N < 4^*$, it seems more complex to derive these information for ground state solutions to (1.2.2)-(1.2.3). In this direction, we only present the following result.

Theorem 4.1.8. *Let* $N \leq 1$, $4 \leq \sigma N < 4^*$ *and* $\sigma \in \mathbb{N}^+$. *Then there exists a* $c_r > c_0$ *such that, for any* $c \in (c_0, c_r)$, (4.1.2)-(4.1.3) *admits a ground state solution, which is radial and sign-changing.*

In our next result, we prove that positive radial solutions to (4.1.2)-(4.1.3) do exist.

Theorem 4.1.9. *Let $1 \leq N \leq 4$, $4 \leq \sigma N < 4^*$. There exists a $\bar{c}_{\sigma,N} > c_0$ such that* (4.1.2)-(4.1.3) *admits a positive and radial solution for any $c \geq \bar{c}_{\sigma,N}$.*

In the last section of this chapter, we investigate dynamical behaviors of solution to the Cauchy problem of (4.1.1). From [95], when $0 < \sigma N < 4^*$, the local well-posedness to the Cauchy problem of (4.1.1) holds. Futhermore, in the mass subcritical case $0 < \sigma N < 4$, any solution to the Cauchy problem of (4.1.1) with initial datum in $H^2(\mathbb{R}^N)$ exists globally in time, see [49, 95]. While in the mass critical and supercritical case $4 \leq \sigma N < 4^*$, blowup in finite time may happen, but it is also possible to prove that the solution to the Cauchy problem of (4.1.1) with some initial datums exists globally in time.

Theorem 4.1.10. *Let $4 \leq \sigma N < 4^*$. For any $c > c_0$, the solution $u \in C([0,T); H^2(\mathbb{R}^N))$ to* (4.1.1) *with initial datum $u_0 \in H^2(\mathbb{R}^N)$ in*

$$\mathcal{O}_c := \{u \in S(c) : E(u) < \gamma(c), \text{ and } Q(u) > 0\}.$$

exists globally in time.

When $0 < \sigma N < 4$, it was prove in [28] that minimizers to (4.1.4) are orbitally stable. When $4 \leq \sigma N < 4^*$, we now prove that radial ground state solutions are unstable by blowup in finite time.

Definition 4.1.11. *We say that $u \in H^2(\mathbb{R}^N)$ is unstable by blowup in finite time, if for any $\varepsilon > 0$, there exists $v \in H^2(\mathbb{R}^N)$ such that $\|v - u\|_{H^2} < \varepsilon$ and the solution $\psi(t)$ to* (4.1.1) *with initial datum $\psi(0) = v$ blows up in finite time in the H^2-norm.*

Making use of a key element in Boulenger and Lenzmann [30], we have

Theorem 4.1.12. *Let $4 \leq \sigma N < 4^*$, $N \geq 2$ and $\sigma \leq 4$. Then standing waves associated to radial ground states to* (4.1.2)-(4.1.3) *are unstable by blowup in finite time.*

In the case where $\alpha \in \mathbb{R}$ is fixed in (4.1.2), the fact that radial least energy solutions are unstable by blowup in finite time was recently established, see our paper [27]. It should be noted that the results of [27] are also strongly based on arguments due to Boulenger and Lenzmann [30].

This chapter is organized as follows. In Section 4.2, we establish some preliminary results and give the proof of Theorem 4.1.3. In Section 4.3, we reveal some properties of the constraint $\mathcal{M}(c)$, in particular we show that in order to find a critical point for E restricted to $S(c)$, we can work directly with a minimizing sequence to (4.1.9), see Lemma 4.3.5. The following Section 4.4 is devoted to the proof of Theorem 4.1.4, and Section 4.5 is devoted to the proof of the multiplicity result Theorem 4.1.5. The subject of Section 4.6 is to establish the properties of $c \mapsto \gamma(c)$ as presented in Theorem 4.1.6. In Section 4.7, we show the proof of the concentration result Theorem 4.1.7. In Section 4.8, Theorem 4.1.8 and Theorem 4.1.9 are established. Afterwards, in Section 4.9, we deal with the unstable issue and prove Theorem 4.1.10 and Theorem 4.1.12. Finally, in Appendix we prove that any solution $u \in H^2(\mathbb{R}^N)$ to (4.1.2) satisfies $Q(u) = 0$, and all solutions to equation (4.1.12) belong to $H^2(\mathbb{R}^N)$ when $N \geq 5$.

Notation 4.1.13. *For $1 \le p < \infty$, $L^p(\mathbb{R}^N)$ is the usual Lebesgue space with norm*

$$\|u\|_p^p := \int_{\mathbb{R}^N} |u|^p \, dx.$$

The Sobolev space $H^2(\mathbb{R}^N)$ is endowed with its standard norm

$$\|u\|^2 := \int_{\mathbb{R}^N} |\Delta u|^2 + |\nabla u|^2 + |u|^2 \, dx.$$

We denote by $' \to '$ reps. $' \rightharpoonup '$ strong convergence reps. weak convergence in corresponding space, and denote by $B_R(x)$ a ball in $\mathbb{R}^N$ of center x and radius $R > 0$. In the rest of this chapter, the constant c_0 is defined by (4.1.10), and we will assume that $N \ge 1$ unless stated the contrary.

4.2 Preliminary results

To begin with, we recall the following well known Gagliardo-Nirenberg's inequality for $u \in H^1(\mathbb{R}^N)$,

$$\|u\|_{2\sigma+2}^{2\sigma+2} \le C_N(\sigma) \|\nabla u\|_2^{\sigma N} \|u\|_2^{2+\sigma(2-N)}, \tag{4.2.1}$$

where

$$\begin{cases} 0 \le \sigma, & \text{if } N \le 2, \\ 0 \le \sigma < \dfrac{2}{N-2}, & \text{if } N \ge 3. \end{cases}$$

Using the Sobolev inequalities and interpolation inequalities in Lebesgue space, we obtain for $u \in H^2(\mathbb{R}^N)$,

$$\|u\|_{2\sigma+2}^{2\sigma+2} \le C_N(\sigma) \|\nabla u\|_2^{N-(\sigma+1)(N-4)} \|\Delta u\|_2^{(N-2)(\sigma+1)-N}, \tag{4.2.2}$$

where

$$\begin{cases} \dfrac{2}{N-2} \le \sigma, & \text{if } N = 3, 4, \\ \dfrac{2}{N-2} \le \sigma < \dfrac{4}{N-4}, & \text{if } N \ge 5. \end{cases}$$

Let us also recall the Cauchy-Schwarz's inequality for $u \in H^2(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \le \Big(\int_{\mathbb{R}^N} |\Delta u|^2 dx \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^N} |u|^2 dx \Big)^{\frac{1}{2}}. \tag{4.2.3}$$

Lemma 4.2.1. *Let $4 \le \sigma N < 4^*$. If $u_c \in S(c)$ is a solution to*

$$\gamma \Delta^2 u - \Delta u + \alpha_c u = |u|^{2\sigma} u \tag{4.2.4}$$

then there exists a $c_{N,\sigma} > 0$ such that $\alpha_c > 0$ for any $c \in (0, c_{N,\sigma})$. Moreover, we have

(i) $c_{1,\sigma} = c_{2,\sigma} = \infty$, and $c_{3,\sigma} = \infty$ if $4/3 \le \sigma \le 2$.

(ii) If $\sigma N = 4$, then $c_{4,\sigma} = \infty$, and $c_{N,\sigma} \ge \Big(\frac{N}{N-4}\Big)^{\frac{N}{4}} c_N^*$ for $N \ge 5$.

Proof. Since any solution to (4.2.4) satisfies $Q(u) = 0$, see Lemma 4.10.1, we have

$$\gamma \int_{\mathbb{R}^N} |\Delta u_c|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_c|^2 \, dx = \frac{\sigma N}{2(2\sigma+2)} \int_{\mathbb{R}^N} |u_c|^{2\sigma+2} \, dx. \tag{4.2.5}$$

Also multiplying (4.2.4) by u_c and integrating in $\mathbb{R}^N$, we get

$$\gamma \int_{\mathbb{R}^N} |\Delta u_c|^2 \, dx + \int_{\mathbb{R}^N} |\nabla u_c|^2 \, dx + \alpha_c \int_{\mathbb{R}^N} |u_c|^2 \, dx = \int_{\mathbb{R}^N} |u_c|^{2\sigma+2} \, dx. \tag{4.2.6}$$

Combining (4.2.5) and (4.2.6) gives

$$-\alpha_c \int_{\mathbb{R}^N} |u_c|^2 \, dx = \gamma \left(1 - \frac{4\sigma+4}{\sigma N}\right) \int_{\mathbb{R}^N} |\Delta u_c|^2 \, dx + \left(1 - \frac{2\sigma+2}{\sigma N}\right) \int_{\mathbb{R}^N} |\nabla u_c|^2 \, dx. \tag{4.2.7}$$

Since u_c is nontrivial, (4.2.7) implies that $\alpha_c > 0$ for any $c > 0$ provided that either $N = 1, 2$ or $N = 3$ with $4/3 \leq \sigma \leq 2$ or $N = 4$ with $\sigma N = 4$. Next we consider the remaining cases. Using the Gagliardo-Nirenberg's inequality (4.1.5), we get from (4.2.5) that

$$\gamma \int_{\mathbb{R}^N} |\Delta u_c|^2 \, dx \leq C c^{1+\sigma-\frac{\sigma N}{4}} \left(\int_{\mathbb{R}^N} |\Delta u_c|^2 \, dx \right)^{\frac{\sigma N}{4}},$$

which implies that

$$\left(\int_{\mathbb{R}^N} |\Delta u_c|^2 \, dx \right)^{1-\frac{\sigma N}{4}} \leq C c^{1+\sigma-\frac{\sigma N}{4}}. \tag{4.2.8}$$

Thus, when $4 < \sigma N < 4^*$, one obtains

$$\int_{\mathbb{R}^N} |\Delta u_c|^2 \, dx \to \infty \text{ as } c \to 0. \tag{4.2.9}$$

On the other hand, using (4.2.3) we get from (4.2.7) that

$$-\alpha_c \int_{\mathbb{R}^N} |u_c|^2 \, dx \leq \gamma \left(1 - \frac{4\sigma+4}{\sigma N}\right) \int_{\mathbb{R}^N} |\Delta u_c|^2 \, dx + C(N,\sigma) \left(\int_{\mathbb{R}^N} |\Delta u_c|^2 \, dx \right)^{\frac{1}{2}} c^{\frac{1}{2}}, \tag{4.2.10}$$

and taking (4.2.9) into account, it follows that $\alpha_c > 0$ provided that $c > 0$ is small enough. It remains to treat the case $\sigma N = 4$ with $N \geq 5$. Observe that from (4.2.5) and (4.2.6), we can obtain

$$-\alpha_c \int_{\mathbb{R}^N} |u_c|^2 \, dx = -\gamma \int_{\mathbb{R}^N} |\Delta u_c|^2 \, dx + \frac{N-4}{N+4} \int_{\mathbb{R}^N} |u_c|^{2+\frac{8}{N}} \, dx. \tag{4.2.11}$$

Now applying the Gagliardo-Nirenberg's inequality (4.1.5) to (4.2.11), it then gives

$$-\alpha_c \int_{\mathbb{R}^N} |u_c|^2 \, dx \leq \left(\frac{N-4}{N} \left(\frac{c}{c_N^*} \right)^{\frac{4}{N}} - 1 \right) \gamma \int_{\mathbb{R}^N} |\Delta u_c|^2 \, dx, \tag{4.2.12}$$

and we deduce that $\alpha_c > 0$ for $c < \tilde{c}_N := \left(\frac{N}{N-4}\right)^{\frac{N}{4}} c_N^*$. □

The last two results of this section concern the mass critical case $\sigma N = 4$. We start by proving the nonexistence result Theorem 4.1.3.

Proof of Theorem 4.1.3. First observe from (4.1.7) that, for any $u \in S(c)$, $E(u_\lambda) \to 0$ as $\lambda \to 0^+$. Thus $m(c) \leq 0$ for $c > 0$. Now using the Gagliardo-Nirenberg's inequality (4.1.5), we have for any $u \in S(c)$,

$$\begin{aligned} E(u) &= \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{N}{2N+8}\int_{\mathbb{R}^N} |u|^{2+\frac{8}{N}}\,dx \\ &\geq \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{c^{\frac{4}{N}}}{2C(N)}\int_{\mathbb{R}^N} |\Delta u|^2\,dx \\ &\geq \frac{1}{2}\left(\gamma - \frac{c^{\frac{4}{N}}}{C(N)}\right)\int_{\mathbb{R}^N} |\Delta u|^2\,dx, \end{aligned} \tag{4.2.13}$$

where $C(N)$ is defined by (4.1.6). Hence (4.2.13) implies that $m(c) \geq 0$ for $c \leq c_N^* := (\gamma C(N))^{\frac{N}{4}}$. Therefore we deduce that $m(c) = 0$ for $c \leq c_N^*$. Next we prove that there is no solution to (4.1.2)-(4.1.3) when $c \leq c_N^*$. Indeed, if u is a solution to (4.1.2)-(4.1.3), then $Q(u) = 0$ and applying (4.1.5), we get

$$\gamma\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx = \frac{N}{N+4}\int_{\mathbb{R}^N} |u|^{2+\frac{8}{N}}\,dx \leq \left(\frac{c}{c_N^*}\right)^{\frac{4}{N}} \gamma\int_{\mathbb{R}^N} |\Delta u|^2\,dx,$$

which implies that $u = 0$ because $c \leq c_N^*$. Finally, let us prove that $m(c) = -\infty$ for $c > c_N^*$. It follows from [30] that the constant $B_N(\frac{4}{N})$ in (4.1.5) is achieved, then there exists a $U \in H^2(\mathbb{R}^N)$ satisfying

$$\|U\|_{2+\frac{8}{N}}^{2+\frac{8}{N}} = B_N(\frac{4}{N})\|U\|_2^{\frac{8}{N}}\|\Delta U\|_2^2. \tag{4.2.14}$$

Setting

$$w := c^{\frac{1}{2}}\frac{U}{\|U\|_2} \in S(c), \tag{4.2.15}$$

and using (4.2.14), we obtain

$$\begin{aligned} E(w_\lambda) &= \frac{c}{2\|U\|_2^2}\lambda^2\gamma\int_{\mathbb{R}^N} |\Delta U|^2\,dx + \frac{c}{2\|U\|_2^2}\lambda\int_{\mathbb{R}^N} |\nabla U|^2\,dx \\ &\quad - \frac{N}{2N+8}\left(\frac{c^{\frac{1}{2}}}{\|U\|_2}\right)^{2+\frac{8}{N}}\lambda^2\int_{\mathbb{R}^N} |U|^{2+\frac{8}{N}}\,dx \\ &= \frac{c}{2\|U\|_2^2}\left(\gamma - \frac{c^{\frac{4}{N}}}{C(N)}\right)\lambda^2\int_{\mathbb{R}^N} |\Delta U|^2\,dx + \frac{c}{2\|U\|_2^2}\lambda\int_{\mathbb{R}^N} |\nabla U|^2\,dx, \end{aligned} \tag{4.2.16}$$

which implies that $E(w_\lambda) \to -\infty$ as $\lambda \to \infty$ for $c > c_N^*$. □

We now show that the two quadratic terms in E behave somehow in a similar manner. This observation will be only used to treat the case $\sigma N = 4$ but we state here under more general assumptions.

Lemma 4.2.2. *Assume that $\sigma N \geq 4$ if $N = 1, 2$ and $4 \leq \sigma N < \frac{2N}{N-2}$ if $N \geq 3$. Let $\{u_n\} \subset S(c_n)$ for every $n \in \mathbb{N}$, where $\{c_n\} \subset (0, a]$ for some $0 < a < \infty$, be such that $\{E(u_n)\} \subset \mathbb{R}$ is bounded. Then*

$$\left\{\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right\} \subset \mathbb{R} \text{ is bounded if and only if } \left\{\int_{\mathbb{R}^N} |\Delta u_n|^2 dx\right\} \subset \mathbb{R} \text{ is bounded.}$$

Proof. By the Cauchy-Schwarz's inequality (4.2.3), the reverse implication obviously holds. To prove the direct implication, we assume by contradiction that

$$\int_{\mathbb{R}^N} |\Delta u_n|^2 \, dx \to \infty \text{ as } n \to \infty. \tag{4.2.17}$$

Using the definition of E and the fact that $\{E(u_n)\}$ remains bounded, one obtains

$$\int_{\mathbb{R}^N} |\Delta u_n|^2 \, dx \leq \frac{1}{\gamma(\sigma+1)} \int_{\mathbb{R}^N} |u_n|^{2\sigma+2} \, dx + C \tag{4.2.18}$$

for some $C > 0$. Thus if $N = 1, 2$ with $\sigma N \geq 4$ or $4 \leq \sigma N < \frac{2N}{N-2}$ if $N \geq 3$, we obtain a contradiction with (4.2.1). If $N = 3, 4$ with $\frac{2N}{N-2} \leq \sigma N$ or $N \geq 5$ with $\frac{2N}{N-2} \leq \sigma N < 4^*$, using (4.2.2) we obtain from (4.2.18) that

$$\int_{\mathbb{R}^N} |\Delta u_n|^2 \, dx \leq C\Big(\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx\Big)^{\frac{N}{2} - \frac{\sigma+1}{2}(N-4)} \Big(\int_{\mathbb{R}^N} |\Delta u_n|^2 \, dx\Big)^{\frac{N-2}{2}(\sigma+1) - \frac{N}{2}}$$

and since, under our assumptions, $\frac{N-2}{2}(\sigma+1) - \frac{N}{2} < 1$ we also reach a contradiction in this case. □

4.3 Some properties of the constraint $\mathcal{M}(c)$

We say that E restricted to $\mathcal{M}(c)$ is coercive if for any $a \in \mathbb{R}$ the subset $\{u \in \mathcal{M}(c) : E(u) \leq a\}$ is bounded.

Lemma 4.3.1. *Let $4 \leq \sigma N < 4^*$ and $c > c_0$, then E restricted to $\mathcal{M}(c)$ is coercive and bounded from below by a positive constant.*

Proof. For any $u \in \mathcal{M}(c)$, we can write

$$E(u) = E(u) - \frac{2}{\sigma N} Q(u) = \gamma \frac{\sigma N - 4}{2\sigma N} \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \frac{\sigma N - 2}{2\sigma N} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx. \tag{4.3.1}$$

In view of (4.3.1), when $\sigma N > 4$ the coerciveness trivially holds. When $\sigma N = 4$, we obtain this from Lemma 4.2.2. Let us now prove that E is bounded from below by a positive constant. First we assume that $\sigma N > 4$. Then, from the Gagliardo-Nirenberg's inequality (4.1.5), for any $u \in \mathcal{M}(c)$,

$$\begin{aligned} \gamma \int_{\mathbb{R}^N} |\Delta u|^2 \, dx \leq \gamma \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx &= \frac{\sigma N}{2(2\sigma+2)} \int_{\mathbb{R}^N} |u|^{2\sigma+2} \, dx \\ &\leq \frac{\sigma N B_N(\sigma)}{2(2\sigma+2)} c^{1+\sigma-\sigma N/4} \left(\int_{\mathbb{R}^N} |\Delta u|^2 \, dx \right)^{\frac{\sigma N}{4}}, \end{aligned}$$

and thus

$$\int_{\mathbb{R}^N} |\Delta u|^2 \, dx \geq \left(\frac{4\sigma + 4}{\sigma N B_N(\sigma) c^{1+\sigma-\frac{\sigma N}{4}}} \right)^{\frac{4}{\sigma N - 4}}. \tag{4.3.2}$$

From (4.3.2), we see that there exists a $\delta > 0$ such that $\int_{\mathbb{R}^N} |\Delta u|^2 \, dx \geq \delta$ and then by (4.3.1) we obtain the lower bound. When $\sigma N = 4$, we first consider the case $1 \leq N \leq 4$.

Then $2+\frac{8}{N} \leq 2^*$ and using the Gagliardo-Nirenberg's inequality (4.2.1), for any $u \in \mathcal{M}(c)$, we get

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx \leq \frac{N}{N+4}\int_{\mathbb{R}^N} |u|^{2+\frac{8}{N}}\,dx \leq C\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^2,$$

which gives the existence of a $\delta > 0$ such that $\int_{\mathbb{R}^N} |\nabla u|^2\,dx \geq \delta$ and we conclude as before. In the case $N \geq 5$, we have $2^* < 2 + \frac{8}{N} < 4^*$ and using the Sobolev inequalities and interpolation inequalities in Lebesgue space, it follows that

$$\begin{aligned}\int_{\mathbb{R}^N} |u|^{2+\frac{8}{N}}\,dx &\leq \left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}}\,dx\right)^{\lambda}\left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-4}}\,dx\right)^{1-\lambda}\\ &\leq C\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^{\frac{N\lambda}{N-2}}\left(\int_{\mathbb{R}^N} |\Delta u|^2\,dx\right)^{\frac{N(1-\lambda)}{N-4}},\end{aligned} \tag{4.3.3}$$

where $0 < \lambda < 1$ with $2 + \frac{8}{N} = \lambda\frac{2N}{N-2} + (1-\lambda)\frac{2N}{N-4}$. Thus for any $u \in \mathcal{M}(c)$,

$$\begin{aligned}\gamma\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx &\leq C\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^{\frac{N\lambda}{N-2}}\left(\int_{\mathbb{R}^N} |\Delta u|^2\,dx\right)^{\frac{N(1-\lambda)}{N-4}}\\ &\leq C\left(\int_{\mathbb{R}^N} |\Delta u|^2 + |\nabla u|^2\,dx\right)^{1+\frac{4}{N}},\end{aligned}$$

and there exists a $\delta > 0$ such that

$$\gamma\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx \geq \delta. \tag{4.3.4}$$

At this point, in view of (4.3.1), we assume by contradiction that there exists a sequence $\{u_n\} \subset \mathcal{M}(c)$ such that $\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx \to 0$. Since, by Lemma 4.2.2, $\{\int_{\mathbb{R}^N} |\Delta u_n|^2\,dx\}$ then remains bounded, it follows from (4.3.3) that $\int_{\mathbb{R}^N} |u_n|^{2+\frac{8}{N}}\,dx \to 0$. Recording that $u_n \in \mathcal{M}(c)$, we then obtain

$$\gamma\int_{\mathbb{R}^N} |\Delta u_n|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx \to 0,$$

which contradicts (4.3.4), and thus we end the proof of the lemma. □

Lemma 4.3.2. *Let $4 \leq \sigma N < 4^*$. For $u \in S(c)$ if $4 < \sigma N < 4^*$, and for $u \in S(c)$ such that $\sup_{\lambda>0} E(u_\lambda) < \infty$ if $\sigma N = 4$, there is a unique $\lambda_* > 0$ such that $u_{\lambda_*} \in \mathcal{M}(c)$. Moreover, $E(u_{\lambda_*}) = \max_{\lambda>0} E(u_\lambda)$ and the function $\lambda \mapsto E(u_\lambda)$ is concave on $[\lambda_*, \infty)$.*

Proof. For any $u \in S(c)$, differentiating (4.1.7) with respect to $\lambda > 0$, we obtain

$$\begin{aligned}\frac{d}{d\lambda}E(u_\lambda) &= \gamma\lambda\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{\sigma N\lambda^{\sigma N/2-1}}{2(2\sigma+2)}\int_{\mathbb{R}^N} |u|^{2\sigma+2}\,dx\\ &= \frac{1}{\lambda}Q(u_\lambda).\end{aligned}$$

When $\sigma N > 4$, it is easily seen that there exists a unique $\lambda_* > 0$ such that $Q(u_{\lambda^*}) = 0$ and also that

$$\frac{d}{d\lambda}E(u_\lambda) > 0 \text{ if } \lambda \in (0, \lambda_*) \quad \text{and} \quad \frac{d}{d\lambda}E(u_\lambda) < 0 \text{ if } \lambda \in (\lambda_*, \infty) \tag{4.3.5}$$

from which we deduce that $E(u_\lambda) < E(u_{\lambda_*})$, for any $\lambda > 0$, $\lambda \neq \lambda_*$. When $\sigma N = 4$, since we assume that $\sup_{\lambda>0} E(u_\lambda) < \infty$, then

$$\gamma \int_{\mathbb{R}^N} |\Delta u|^2 \, dx < \frac{\sigma N}{2(2\sigma+2)} \int_{\mathbb{R}^N} |u|^{2\sigma+2} \, dx, \tag{4.3.6}$$

and thus there also exists a unique $\lambda_* > 0$ such that $Q(u_{\lambda_*}) = 0$ and (4.3.5) holds. Now writing $\lambda = t\lambda_*$, we have

$$\begin{aligned}\frac{d^2}{d^2\lambda} E(u_\lambda) &= \gamma \int_{\mathbb{R}^N} |\Delta u|^2 \, dx - \frac{\sigma N(\sigma N - 2)}{4(2\sigma+2)} t^{\frac{\sigma N}{2}-2} \lambda_*^{\frac{\sigma N}{2}-2} \int_{\mathbb{R}^N} |u|^{2\sigma+2} \, dx \\ &= \frac{1}{\lambda_*^2}\Big[\gamma\lambda_*^2 \int_{\mathbb{R}^N} |\Delta u|^2 \, dx - \frac{\sigma N(\sigma N - 2)}{4(2\sigma+2)} t^{\frac{\sigma N}{2}-2} \lambda_*^{\frac{\sigma N}{2}} \int_{\mathbb{R}^N} |u|^{2\sigma+2} \, dx\Big].\end{aligned}$$

Thus using that

$$0 = Q(u_{\lambda_*}) = \gamma\lambda_*^2 \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \frac{1}{2}\lambda_* \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{\sigma N}{2(2\sigma+2)} \lambda_*^{\frac{\sigma N}{2}} \int_{\mathbb{R}^N} |u|^{2\sigma+2} \, dx$$

it follows that $\dfrac{d^2}{d^2\lambda} E(u_\lambda) < 0$ for any $t \geq 1$. This proves the lemma. □

Lemma 4.3.3. *Let $4 \leq \sigma N < 4^*$, then $\mathcal{M}(c)$ is a C^1 manifold of codimension 2 in $H^2(\mathbb{R}^N)$, hence a C^1 manifold of codimension 1 in $S(c)$.*

Proof. By definition, $u \in \mathcal{M}(c)$ if and only if $G(u) := \|u\|_2^2 - c = 0$ and $Q(u) = 0$. It is easy to check that G, Q are of C^1 class. Hence we only have to prove that for any $u \in \mathcal{M}(c)$,

$$(dG(u), dQ(u)) : H^2(\mathbb{R}^N) \to \mathbb{R}^2 \text{ is surjective.}$$

If this failed, we would have that $dG(u)$ and $dQ(u)$ are linearly dependent, which implies that there exists a $\nu \in \mathbb{R}$ such that for any $\varphi \in H^2(\mathbb{R}^N)$,

$$2\gamma \int_{\mathbb{R}^N} \Delta u \Delta\varphi \, dx + \int_{\mathbb{R}^N} \nabla u \cdot \nabla\varphi \, dx - \frac{\sigma N}{2} \int_{\mathbb{R}^N} |u|^{2\sigma} u\varphi \, dx = 2\nu \int_{\mathbb{R}^N} u\varphi \, dx,$$

namely, u solves

$$2\gamma\Delta^2 u - \Delta u = 2\nu u + \frac{\sigma N}{2}|u|^{2\sigma}u.$$

At this point from Lemma 4.10.1, we deduce

$$4\gamma \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \int_{\mathbb{R}^N} |\nabla u|^2 \, dx = \frac{(\sigma N)^2}{2(2\sigma+2)} \int_{\mathbb{R}^N} |u|^{2\sigma+2} \, dx,$$

and since $Q(u) = 0$ we then obtain

$$4\gamma \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \int_{\mathbb{R}^N} |\nabla u|^2 \, dx = \sigma N\gamma \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \frac{\sigma N}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx,$$

which is impossible since $\sigma N \geq 4$ and $u \in S(c)$. □

Lemma 4.3.4. *Let $4 \leq \sigma N < 4^*$, then for any $u \in C_0^\infty(\mathbb{R}^N) \cap \mathcal{M}(c)$, there holds*

$$T_u S(c) = T_u \mathcal{M}(c) \bigoplus \mathbb{R}\left(\frac{d}{d\lambda}(u_\lambda)|_{\lambda=1}\right), \tag{4.3.7}$$

and

$$dE(u)\left[\frac{d}{d\lambda}(u_\lambda)|_{\lambda=1}\right] = 0. \tag{4.3.8}$$

Proof. By Lemma 4.3.3, we know that $\mathcal{M}(c)$ has codimension 1 in $S(c)$, thus in order to prove (4.3.7), it suffices to show that

$$\frac{d}{d\lambda}(u_\lambda)|_{\lambda=1} \in T_u S(c) \backslash T_u \mathcal{M}(c).$$

For $u \in C_0^\infty(\mathbb{R}^N)$, one has

$$2\frac{d}{d\lambda}(u_\lambda)|_{\lambda=1}(x) = \frac{N}{2}u(x) + \nabla u(x) \cdot x \in C_0^\infty(\mathbb{R}^N) \tag{4.3.9}$$

It directly follows from the divergence theorem that

$$\int_{\mathbb{R}^N} (\nabla u \cdot x) u \, dx = -\frac{N}{2} \int_{\mathbb{R}^N} |u|^2 \, dx,$$

from which we deduce

$$\frac{d}{d\lambda}(u_\lambda)|_{\lambda=1} \in T_u S(c). \tag{4.3.10}$$

Now, using the divergence theorem again, we obtain

$$\begin{aligned} dQ(u)\left[\frac{d}{d\lambda}(u_\lambda)|_{\lambda=1}\right] &= \frac{N}{2}\gamma \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \gamma \int_{\mathbb{R}^N} \Delta u \Delta(\nabla u \cdot x) \, dx + \frac{N}{4} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \\ &+ \frac{1}{2} \int_{\mathbb{R}^N} \nabla u \cdot \nabla(\nabla u \cdot x) \, dx - \frac{\sigma N^2}{8} \int_{\mathbb{R}^N} |u|^{2\sigma+2} \, dx - \frac{\sigma N}{4} \int_{\mathbb{R}^N} |u|^{2\sigma} u(\nabla u \cdot x) \, dx \\ &= 2\gamma \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{(\sigma N)^2}{4(2\sigma+2)} \int_{\mathbb{R}^N} |u|^{2\sigma+2} \, dx. \end{aligned}$$

Since $Q(u) = 0$ and $\sigma N \geq 4$, we deduce

$$dQ(u)\left[\frac{d}{d\lambda}(u_\lambda)|_{\lambda=1}\right] = \left(2 - \frac{\sigma N}{2}\right)\gamma \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \left(\frac{1}{2} - \frac{\sigma N}{4}\right) \int_{\mathbb{R}^N} |\nabla u|^2 \, dx < 0.$$

This implies

$$\frac{d}{d\lambda}(u_\lambda)|_{\lambda=1} \notin T_u \mathcal{M}(c). \tag{4.3.11}$$

At this point, the proof of (4.3.7) follows directly from (4.3.10) and (4.3.11). Finally, recalling (4.1.8) and in view of (4.3.9), we can write

$$0 = Q(u) = \frac{\partial E(u_\lambda)}{\partial \lambda}|_{\lambda=1} = dE(u)\left[\frac{d}{d\lambda}(u_\lambda)|_{\lambda=1}\right] = 0,$$

then (4.3.8) holds. □

Our next result is directly inspired from [13], see also [14].

Lemma 4.3.5. *Let $4 \leq \sigma N < 4^*$. If $\{v_n\} \subset \mathcal{M}(c)$ is a Palais-Smale sequence for E restricted to $\mathcal{M}(c)$, then there exists a possible different Palais-Smale sequence $\{u_n\} \subset \mathcal{M}(c)$ for E restricted to $S(c)$ such that $\|u_n - v_n\| \to 0$ as $n \to \infty$. In particular, if $\{v_n\} \subset \mathcal{M}(c)$ is converging to a $v \in \mathcal{M}(c)$, then this limit is a critical point for E restricted to $S(c)$.*

Proof. Let us first prove that if $\{v_n\} \subset \mathcal{M}(c)$ is a Palais-Smale sequence for E restricted to $\mathcal{M}(c)$, then there exists a Palais-Smale sequence $\{u_n\} \subset C_0^\infty(\mathbb{R}^N) \cap \mathcal{M}(c)$ for E restricted to $\mathcal{M}(c)$ satisfying $\|u_n - v_n\| \to 0$ as $n \to \infty$. For this we just need to show that $C_0^\infty(\mathbb{R}^N) \cap \mathcal{M}(c)$ is dense in $\mathcal{M}(c)$. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^2(\mathbb{R}^N)$, for any $w \in \mathcal{M}(c)$ there exists a sequence $\{w_n\} \subset C_0^\infty(\mathbb{R}^N)$ such that $w_n \to w$ in $H^2(\mathbb{R}^N)$. From Lemma 4.3.2 without restriction we can assume that for any $n \in \mathbb{N}$, there exists a unique $\lambda_n^* \in \mathbb{R}$ such that $(w_n)_{\lambda_n^*} \in \mathcal{M}(c) \cap C_0^\infty(\mathbb{R}^N)$. Since $w \in \mathcal{M}(c)$, one can easily check that $\lambda_n^* \to 1$, which gives that $(w_n)_{\lambda_n^*} \to w$ in $H^2(\mathbb{R}^N)$.

Now let us prove that if $\{u_n\} \subset C_0^\infty(\mathbb{R}^N) \cap \mathcal{M}(c)$ is a Palais-Smale sequence for E restricted to $\mathcal{M}(c)$, then $\{u_n\}$ is a Palais-Smale sequence for E restricted to $S(c)$. We denote by $(T_uS(c))^*$ resp. $(T_u\mathcal{M}(c))^*$ the dual space to $T_uS(c)$ resp. $T_u\mathcal{M}(c)$. In view of Lemma 4.3.4, we have

$$\begin{aligned}
&\|dE(u_n)\|_{(T_uS(c))^*} = \sup\{dE(u_n)[\varphi] : \varphi \in T_uS(c), \|\varphi\| \le 1\} \\
&= \sup\left\{dE(u_n)[\varphi] : \varphi = \varphi_1 + \varphi_2, \|\varphi\| \le 1, \varphi_1 \in T_u\mathcal{M}(c), \varphi_2 \in \mathbb{R}\left(\frac{d}{d\lambda}(u_\lambda)|_{\lambda=1}\right)\right\} \\
&= \sup\{dE(u_n)[\varphi_1] : \|\varphi_1\| \le 1\} \\
&= \|dE(u_n)\|_{(T_u\mathcal{M}(c))^*},
\end{aligned}$$

from which it follows that $\{u_n\}$ is a Palais-Smale sequence for E restricted to $S(c)$. □

Lemma 4.3.6. *Let $4 \le \sigma N < 4^*$, and $\{u_n\} \subset \mathcal{M}(c)$ be a Palais-Smale sequence for E restricted to $\mathcal{M}(c)$. Then there exist a $u_c \in H^2(\mathbb{R}^N)$ and a sequence $\{\alpha_n\} \subset \mathbb{R}$ such that, up to a subsequence and translations,*

(i) $u_n \rightharpoonup u_c \neq 0$ *in* $H^2(\mathbb{R}^N)$ *as* $n \to \infty$;

(ii) $\alpha_n \to \alpha_c$ *in* $\mathbb{R}$ *as* $n \to \infty$;

(iii) $\gamma\Delta^2 u_n - \Delta u_n + \alpha_n u_n - |u_n|^{2\sigma}u_n \to 0$ *in* $H^{-2}(\mathbb{R}^N)$ *as* $n \to \infty$;

(iv) $\gamma\Delta^2 u_c - \Delta u_c + \alpha_c u_c = |u_c|^{2\sigma}u_c$.

In addition, if $\|u_n - u_c\|_{2\sigma+2} \to 0$ and $\alpha_c > 0$, then $\|u_n - u_c\| \to 0$ as $n \to \infty$. Here $H^{-2}(\mathbb{R}^N)$ denotes the dual space to $H^2(\mathbb{R}^N)$.

Proof. First observe that, because of Lemma 4.3.1 and Lemma 4.3.5, we can assume without restriction that $\{u_n\} \subset \mathcal{M}(c)$ is a bounded Palais-Smale sequence for E restricted to $S(c)$. After a suitable translation in $\mathbb{R}^N$, passing to a subsequence, we can assume that $u_n \rightharpoonup u_c \neq 0$. Indeed, if not this readily implies, see [74, Lemma I.1], that $\int_{\mathbb{R}^N} |u_n|^{2\sigma+2}\,dx = o_n(1)$, where $o_n(1) \to 0$ as $n \to \infty$. Thus, since $\{u_n\} \subset \mathcal{M}(c)$, it follows that $\int_{\mathbb{R}^N} |\Delta u_n|^2\,dx = o_n(1)$ and $\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx = o_n(1)$, which is turn implies that $E(u_n) = o_n(1)$. However, this contradicts the fact that E is bounded below by a positive constant on $\mathcal{M}(c)$ and thus (i) holds. Now since $\{u_n\}$ is bounded in $H^2(\mathbb{R}^N)$, we know from [22, Lemma 3] that $\|dE_{|_{S(c)}}(u_n)\|_{H^{-2}} = o_n(1)$ is equivalent to $\|dE(u_n) - dE(u_n)[u_n]u_n\|_{H^{-2}} = o_n(1)$. Therefore for any $\varphi \in H^2(\mathbb{R}^N)$, we have

$$\gamma\int_{\mathbb{R}^N} \Delta u_n \Delta\varphi\,dx + \int_{\mathbb{R}^N} \nabla u_n \nabla\varphi\,dx + \alpha_n\int_{\mathbb{R}^N} u_n\varphi\,dx - \int_{\mathbb{R}^N} |u_n|^{2\sigma}u_n\varphi\,dx = o_n(1), \tag{4.3.12}$$

where

$$-\alpha_n = \frac{1}{c}\left(\gamma\int_{\mathbb{R}^N} |\Delta u_n|^2\,dx + \int_{\mathbb{R}^N} |\nabla u_n|^2\,dx - \int_{\mathbb{R}^N} |u_n|^{2\sigma+2}\,dx\right). \tag{4.3.13}$$

From (4.3.12)-(4.3.13), we deduce that (ii)-(iii) hold and using that $u_n \rightharpoonup u_c$ in $H^2(\mathbb{R}^N)$ we obtain in a standard way from (ii)-(iii) that (iv) holds.

Finally, let us show that under our additional assumptions $\{u_n\}$ strongly converges to u_c in $H^2(\mathbb{R}^N)$. Recalling that $\{u_n\}$ is bounded in $H^2(\mathbb{R}^N)$ and using that $u_n \to u$ in $L^{2\sigma+2}(\mathbb{R}^N)$ as $n \to \infty$, it follows from (ii)-(iv) that

$$\begin{aligned} &\gamma \int_{\mathbb{R}^N} |\Delta u_n|^2 \, dx + \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx + \alpha_n \int_{\mathbb{R}^N} |u_n|^2 \, dx \\ &= \gamma \int_{\mathbb{R}^N} |\Delta u_c|^2 \, dx + \int_{\mathbb{R}^N} |\nabla u_c|^2 \, dx + \alpha_c \int_{\mathbb{R}^N} |u_c|^2 \, dx + o_n(1). \end{aligned} \tag{4.3.14}$$

But since $u_n \rightharpoonup u_c$ in $H^2(\mathbb{R}^N)$ as $n \to \infty$, by weak convergence

$$\gamma \int_{\mathbb{R}^N} |\Delta u_c|^2 \, dx + \int_{\mathbb{R}^N} |\nabla u_c|^2 \, dx \le \liminf_{n\to\infty} \gamma \int_{\mathbb{R}^N} |\Delta u_n|^2 \, dx + \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx,$$
$$\int_{\mathbb{R}^N} |u_c|^2 \, dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^2 \, dx.$$

At this point, using that $\alpha_n \to \alpha_c > 0$ as $n \to \infty$ and the previous inequalities we get from (4.3.14) that $u_n \to u$ in $H^2(\mathbb{R}^N)$ as $n \to \infty$. Thus the proof is complete. □

4.4 Existence of ground state solutions

In this section, we give the proof of Theorem 4.1.4.

Lemma 4.4.1. *Let $4 \le \sigma N < 4^*$ and $c > c_0$. Let $\{u_n\} \subset \mathcal{M}(c)$ be a Palais-Smale sequence for E restricted to $\mathcal{M}(c)$ at the level $\gamma(c)$, such that $u_n \rightharpoonup u_c \neq 0$ in $H^2(\mathbb{R}^N)$. If*

$$\gamma(c) \le \gamma(c_1) \text{ for any } c_1 \in (0, c], \tag{4.4.1}$$

then $\|u_n - u_c\|_{2\sigma+2} \to 0$ as $n \to \infty$. In particular $E(u_c) = \gamma(c)$.

Proof. By Lemma 4.3.6, we know that there exists a $\alpha_c \in \mathbb{R}$ such that u_c satisfies (4.1.2), and thus $Q(u_c) = 0$ by Lemma 4.10.1. Now we set $0 < \|u_c\|_2^2 =: c_1 \le c$, observing that in the case $\sigma N = 4$, we know from Theorem 4.1.3 that $c_1 > c_N^*$. Since $u_n \rightharpoonup u_c$ in $H^2(\mathbb{R}^N)$ as $n \to \infty$, we have from the Brezis-Lieb's Lemma,

$$\begin{aligned} &\|\Delta(u_n - u_c)\|_2^2 + \|\Delta u_c\|_2^2 = \|\Delta u_n\|_2^2 + o_n(1), \\ &\|\nabla(u_n - u_c)\|_2^2 + \|\nabla u_c\|_2^2 = \|\nabla u_n\|_2^2 + o_n(1), \\ &\|u_n - u_c\|_{2\sigma+2}^{2\sigma+2} + \|u_c\|_{2\sigma+2}^{2\sigma+2} = \|u_n\|_{2\sigma+2}^{2\sigma+2} + o_n(1). \end{aligned} \tag{4.4.2}$$

Since $Q(u_c) = 0$, and $Q(u_n) = 0$, it follows from (4.4.2) that $Q(u_n - u_c) = o_n(1)$, as well as

$$E(u_n - u_c) + E(u_c) = \gamma(c) + o_n(1). \tag{4.4.3}$$

Since $u_c \in \mathcal{M}(c_1)$, then (4.4.3) implies that

$$E(u_n - u_c) + \gamma(c_1) \le \gamma(c) + o_n(1)$$

and because of (4.4.1) it follows that $E(u_n - u_c) \leq o_n(1)$. We also have

$$\begin{aligned} &E(u_n - u_c) - \frac{2}{\sigma N} Q(u_n - u_c) \\ &= \gamma \frac{\sigma N - 4}{2\sigma N} \int_{\mathbb{R}^N} |\Delta(u_n - u_c)|^2 \, dx + \frac{\sigma N - 2}{2\sigma N} \int_{\mathbb{R}^N} |\nabla(u_n - u_c)|^2 \, dx, \end{aligned} \tag{4.4.4}$$

and since $Q(u_n - u_c) = o_n(1)$, this implies that $E(u_n - u_c) \geq o_n(1)$. Consequently $E(u_n - u_c) = o_n(1)$. When $\sigma N > 4$, we directly deduce from (4.4.4)

$$\|\Delta(u_n - u_c)\|_2 = o_n(1), \ \|\nabla(u_n - u_c)\|_2 = o_n(1),$$

and using again that $Q(u_n - u_c) = o_n(1)$, it follows that $\|u_n - u_c\|_{2\sigma+2} = o_n(1)$. When $\sigma N = 4$, then (4.4.4) only gives that $\|\nabla(u_n - u_c)\|_2 = o_n(1)$. Since, by Lemma 4.2.2, $\{\|\Delta(u_n - u_c)\|_2\}$ remains bounded we conclude using (4.2.1) if $N \leq 4$ and using (4.3.3) if $N \geq 5$ that $\|u_n - u_c\|_{2\sigma+2} = o_n(1)$. Now from (4.4.3) and using that $E(u_n - u_c) = o_n(1)$, it follows that $E(u_c) = \gamma(c)$. □

Lemma 4.4.2. *Let $4 \leq \sigma N < 4^*$, then the function $c \mapsto \gamma(c)$ is decreasing on (c_0, ∞).*

Proof. First we show that $\gamma(c)$ enjoys the variational characterization

$$\inf_{u \in \mathcal{M}(c)} E(u) = \inf_{u \in S(c)} \sup_{\lambda > 0} E(u_\lambda). \tag{4.4.5}$$

Indeed, on one hand, we observe that for any $u \in S(c)$ either $\sup_{\lambda>0} E(u_\lambda) = +\infty$ or there exists a $\lambda_* > 0$ such that $u_{\lambda_*} \in \mathcal{M}(c)$ and $E(u_{\lambda_*}) \leq \max_{\lambda>0} E(u_\lambda)$. It implies that

$$\inf_{u \in S(c)} \sup_{\lambda > 0} E(u_\lambda) \geq \inf_{u \in \mathcal{M}(c)} E(u).$$

On the other hand, for any $u \in \mathcal{M}(c)$, $E(u) \geq \max_{\lambda>0} E(u_\lambda)$ and then

$$\inf_{u \in \mathcal{M}(c)} E(u) \geq \inf_{u \in S(c)} \sup_{\lambda > 0} E(u_\lambda).$$

Thus (4.4.5) holds. To prove the lemma we have to demonstrate that if $0 < c_1 < c_2$, then $\gamma(c_2) \leq \gamma(c_1)$. Noting the definition of $\gamma(c)$ and (4.4.5), for any $\varepsilon > 0$ there exists a $u_1 \in \mathcal{M}(c_1)$ such that

$$E(u_1) \leq \gamma(c_1) + \frac{\varepsilon}{2} \quad \text{and} \quad \max_{\lambda > 0} E((u_1)_\lambda) = E(u_1) \tag{4.4.6}$$

where we recall that $(u_1)_\lambda(x) := \lambda^{\frac{N}{4}} u_1(\sqrt{\lambda} x)$. For $\delta > 0$, one can find $u_1^\delta \in H^2(\mathbb{R}^N)$ such that *supp* $u_1^\delta \subset B_{\frac{1}{\delta}}(0)$ and $||u_1 - u_1^\delta|| = o_\delta(1)$. Thus, as $\delta \to 0$

$$\int_{\mathbb{R}^N} |\Delta u_1^\delta|^2 \, dx \to \int_{\mathbb{R}^N} |\Delta u_1|^2 \, dx, \quad \int_{\mathbb{R}^N} |\nabla u_1^\delta|^2 \, dx \to \int_{\mathbb{R}^N} |\nabla u_1|^2 \, dx,$$

and

$$\int_{\mathbb{R}^N} |u_1^\delta|^{2\sigma+2} \, dx \to \int_{\mathbb{R}^N} |u_1|^{2\sigma+2} \, dx.$$

Let $v^\delta \in C_0^\infty(\mathbb{R}^N)$ be such that *supp* $v^\delta \subset B_{\frac{2}{\delta}+1}(0) \setminus B_{\frac{2}{\delta}}(0)$, and set

$$v_0^\delta := (c_2 - \|u_1^\delta\|_2^2)^{\frac{1}{2}} \frac{v^\delta}{\|v^\delta\|_2}.$$

We now define for $\lambda \in (0,1)$, $w_\lambda^\delta := u_1^\delta + (v_0^\delta)_\lambda$. Since

$$\operatorname{dist}(supp\ (v_0^\delta)_\lambda, supp\ u_1^\delta) \geq \frac{1}{\delta}\left(\frac{2}{\lambda} - 1\right) > 0,$$

we have that $\|w_\lambda^\delta\|_2^2 = c_2$. Also by standard scaling arguments we see that as $\lambda,\ \delta \to 0$,

$$\int_{\mathbb{R}^N} |\Delta w_\lambda^\delta|^2\, dx \to \int_{\mathbb{R}^N} |\Delta u_1|^2\, dx, \quad \int_{\mathbb{R}^N} |\nabla w_\lambda^\delta|^2\, dx \to \int_{\mathbb{R}^N} |\nabla u_1|^2\, dx,$$

and

$$\int_{\mathbb{R}^N} |w_\lambda^\delta|^{2\sigma+2}\, dx \to \int_{\mathbb{R}^N} |u_1|^{2\sigma+2}\, dx.$$

In [18, Lemma 5.2], it has been proved that the function $f : \mathbb{R}^+ \times (\mathbb{R}^+ \cup \{0\}) \times \mathbb{R}^+ \mapsto \mathbb{R}$ defined by $f(a,b,c) = \max_{t>0}(t^2 a + tb - ct^{\frac{\sigma N}{2}})$ is continuous. Setting $(w_\lambda^\delta)_t := t^{\frac{N}{4}} w_\lambda^\delta(\sqrt{t}x)$. If $\sigma N > 4$, using the above convergences and (4.4.5), we deduce that for $\lambda, \delta > 0$ small enough,

$$\gamma(c_2) \leq \max_{t>0} E((w_\lambda^\delta)_t) \leq \max_{t>0} E((u_1)_t) + \frac{\varepsilon}{2} = E(u_1) + \frac{\varepsilon}{2} \leq \gamma(c_1) + \varepsilon,$$

then this concludes the proof when $\sigma N > 4$. If $\sigma N = 4$, note that for $\lambda, \delta > 0$ small enough

$$\gamma \int_{\mathbb{R}^N} |\Delta w_\lambda^\delta|^2\, dx < \frac{N}{N+4} \int_{\mathbb{R}^N} |w_\lambda^\delta|^{2+\frac{8}{N}}\, dx,$$

thus $\sup_{t>0} E((w_\lambda^\delta)_t) < \infty$. Under this condition, [18, Lemma 5.2] can be easily extended and we conclude as in the case $\sigma N > 4$. □

We can now prove our result Theorem 4.1.4 concerning the existence of ground state solutions.

Proof of Theorem 4.1.4. For any $c > c_0$ fixed, by the Ekeland variational principle, there exists a Palais-Smale sequence $\{u_n\} \subset \mathcal{M}(c)$ for E restricted to $\mathcal{M}(c)$ at level $\gamma(c)$. By Lemma 4.3.6 we know that $u_n \rightharpoonup u_c$, where u_c is solution to

$$\gamma\Delta^2 u_c - \Delta u_c + \alpha_c u_c = |u_c|^{2\sigma} u_c$$

for some $\alpha_c \in \mathbb{R}$. We also know from Lemma 4.3.6 that this convergence is strong whenever $||u_n - u_c||_{2\sigma+2} \to 0$ and $\alpha_c > 0$. The first property is guaranteed by Lemma 4.4.1 and Lemma 4.4.2, and the second one comes from Lemma 4.2.1. □

4.5 Multiplicity of radial solutions

Next we turn to the proof of Theorem 4.1.5. First we recall the definition of genus of a set due to M.A. Krasnosel'skii.

Definition 4.5.1. *Let $\mathcal{A}$ be a family of sets $A \subset \mathcal{F}$ such that A is closed and symmetric ($u \in A$ if and only if $-u \in A$). For every $A \in \mathcal{A}$, the genus of A is defined by*

$$\gamma(A) := \min\{n \in \mathbb{N} : \exists\ \varphi : A \to \mathbb{R}^n \backslash \{0\}, \varphi \text{ is continuous and odd}\}.$$

When there is no φ as described above, we set $\gamma(A) = \infty$.

Now let $\mathcal{F} := \mathcal{M}_{rad}(c) := \mathcal{M}(c) \cap H^2_{rad}(\mathbb{R}^N)$. For any $k \in \mathbb{N}^+$, define

$$\Gamma_k := \{A \in \mathcal{A} : A \text{ is compact}, \gamma(A) \geq k\},$$

and

$$\beta_k := \inf_{A \in \Gamma_k} \sup_{u \in A} E(u).$$

Lemma 4.5.2.

(i) Let $4 < \sigma N < 4^*$, then for any $k \in \mathbb{N}^+$, $\Gamma_k \neq \emptyset$.

(ii) Let $\sigma N = 4$, then for any $k \in \mathbb{N}^*$ there exists a $c_k > c^*_N$ such that $\Gamma_k \neq \emptyset$ for all $c \geq c_k$.

Proof. First we consider the case $4 < \sigma N < 4^*$. Let $V \subset H^2_{rad}(\mathbb{R}^N)$ be such that $\dim V = k$. We set $SV(c) := V \cap S(c)$. By the basic property of the genus, see [4, Theorem 10.5], we have that $\gamma(SV(c)) = \dim V = k$. In view of Lemma 4.3.2, for any $u \in SV(c)$ there exists unique $\lambda^*_u > 0$ such that $u_{\lambda^*_u} \in \mathcal{M}(c)$. It is easy to check that the mapping $\varphi : SV(c) \to \mathcal{M}(c)$ defined by $\varphi(u) = u_{\lambda^*_u}$ is continuous and odd. Then [4, Lemma 10.4] leads to $\gamma(\varphi(SV(c))) \geq \gamma(SV(c)) = k$ and this shows that $\Gamma_k \neq \emptyset$. In the case $\sigma N = 4$, we shall prove that, for $k \in \mathbb{N}^+$ given, taking $c > c^*_N$ large enough $\Gamma_k \neq \emptyset$. Let $V \subset H^2_{rad}(\mathbb{R}^N)$ satisfies $\dim V = k$, and set $SV(c) := V \cap S(c)$. Using the fact that all norms are equivalent in a finite dimensional subspace, we get, for $c > c^*_N$ large enough and for any $u \in SV(c)$,

$$\gamma \int_{\mathbb{R}^N} |\Delta u|^2 \, dx < \frac{N}{N+4} \int_{\mathbb{R}^N} |u|^{2+\frac{8}{N}} \, dx.$$

This shows that $\sup_{\lambda>0} E(u_\lambda) < \infty$ and thus from Lemma 4.3.2 for any $u \in SV(c)$ that there exists unique $\lambda^*_u > 0$ such that $Q(u_{\lambda^*_u}) = 0$. At this point, we pursue as in the case $4 < \sigma N < 4^*$ to conclude the proof. □

Lemma 4.5.3. *Let $N \geq 2$ if $4 < \sigma N < 4^*$ or $2 \leq N \leq 4$ if $\sigma N = 4$. Then E restricted to $\mathcal{M}_{rad}(c)$ satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\} \subset \mathcal{M}_{rad}(c)$ be a Palais-Smale sequence for E restricted to $\mathcal{M}_{rad}(c)$. By Lemma 4.3.6 we know that, up to a subsequence, $\{u_n\}$ converges strongly in $H^2(\mathbb{R}^N)$ if $\{u_n\}$ converges strongly in $L^{2\sigma+2}(\mathbb{R}^N)$ and if the associated parameter $\alpha_c \in \mathbb{R}$ is strictly positive. The first property holds because the embedding $H^2_{rad}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is compact for $N \geq 2$, $2 < p < \frac{2N}{(N-4)^+}$, and the second one is guaranteed by Lemma 4.2.1. □

Proof of Theorem 4.1.5. In view of Lemma 4.5.2 and Lemma 4.5.3, Theorem 4.1.5 then follows directly from [4, Proposition 10.8]. □

4.6 Properties of the function $c \mapsto \gamma(c)$

In this section, we investigate further properties of the function $c \mapsto \gamma(c)$ and prove Theorem 4.1.6. We begin by showing its continuity.

Lemma 4.6.1. *Let $4 \leq \sigma N < 4^*$, then the function $c \mapsto \gamma(c)$ is continuous on (c_0, ∞).*

Proof. Let us prove that, for any $c > c_0$, if $\{c_n\} \subset (c_0, \infty)$ is such that $c_n \to c$, then $\lim_{n\to\infty} \gamma(c_n) = \gamma(c)$. From the definition of $\gamma(c)$, for any $\epsilon > 0$, there exists a $v \in \mathcal{M}(c)$ such that $E(v) \leq \gamma(c) + \frac{\varepsilon}{2}$. Now defining $v_n := \sqrt{\frac{c_n}{c}}\, v \in S(c_n)$, then as $n \to \infty$ we clearly have

$$\int_{\mathbb{R}^N} |\Delta v_n|^2\, dx \to \int_{\mathbb{R}^N} |\Delta v|^2\, dx,\ \int_{\mathbb{R}^N} |\nabla v_n|^2\, dx \to \int_{\mathbb{R}^N} |\nabla v|^2\, dx,$$

and

$$\int_{\mathbb{R}^N} |v_n|^{2\sigma+2}\, dx \to \int_{\mathbb{R}^N} |v|^{2\sigma+2}\, dx.$$

In particular, for $n \in \mathbb{N}$ large enough

$$\gamma \int_{\mathbb{R}^N} |\Delta v_n|^2\, dx < \frac{N}{N+4} \int_{\mathbb{R}^N} |v_n|^{2+\frac{8}{N}}\, dx$$

when $\sigma N = 4$. Now using [18, Lemma 5.2] and the above convergences, we deduce

$$\begin{aligned}
\gamma(c_n) &\leq \max_{\lambda>0} E((v_n)_\lambda) \\
&= \max_{\lambda>0} \left(\frac{\lambda^2}{2}\gamma \int_{\mathbb{R}^N} |\Delta v_n|^2\, dx + \frac{\lambda}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2\, dx - \frac{\lambda^{\sigma N/2}}{2(2\sigma+2)}\int_{\mathbb{R}^N} |v_n|^{2\sigma+2}\, dx \right) \\
&\leq \max_{\lambda>0} \left(\frac{\lambda^2}{2}\gamma \int_{\mathbb{R}^N} |\Delta v|^2\, dx + \frac{\lambda}{2}\int_{\mathbb{R}^N} |\nabla v|^2\, dx - \frac{\lambda^{\sigma N/2}}{2(2\sigma+2)}\int_{\mathbb{R}^N} |v|^{2\sigma+2}\, dx \right) + \frac{\varepsilon}{2} \\
&= \max_{\lambda>0} E((v)_\lambda) + \frac{\varepsilon}{2} = E(v) + \frac{\varepsilon}{2} \leq \gamma(c) + \varepsilon.
\end{aligned}$$

This shows that

$$\limsup_{n\to\infty} \gamma(c_n) \leq \gamma(c). \tag{4.6.1}$$

Now let $\{u_n\} \subset \mathcal{M}(c_n)$ be such that

$$E(u_n) \leq \gamma(c_n) + \frac{\varepsilon}{3}. \tag{4.6.2}$$

Since $Q(u_n) = 0$, using (4.6.1) and (4.6.2), we obtain that, for $n \in \mathbb{N}$ large enough

$$\gamma\frac{\sigma N - 4}{2\sigma N} \int_{\mathbb{R}^N} |\Delta u_n|^2\, dx + \frac{\sigma N - 2}{2\sigma N} \int_{\mathbb{R}^N} |\nabla u_n|^2\, dx = E(u_n) \leq \gamma(c_n) + \frac{\varepsilon}{3} \leq \gamma(c) + \frac{\varepsilon}{2},$$

thus when $\sigma N > 4$, we immediately get that $\{u_n\} \subset H^2(\mathbb{R}^N)$ is bounded. The same holds when $\sigma N = 4$ by Lemma 4.2.2. Thus we can assume without restriction that

$$\int_{\mathbb{R}^N} |\Delta u_n|^2\, dx \to A,\ \int_{\mathbb{R}^N} |\nabla u_n|^2\, dx \to B,\ \int_{\mathbb{R}^N} |u_n|^{2\sigma+2}\, dx \to C.$$

We claim that A and C are strictly positive constants. Indeed, when $4 < \sigma N < 4^*$, since $Q(u_n) = 0$, using the Gagliardo-Nirenberg's inequality (4.1.5), we get from (4.3.2) that $A > 0$. Using again that $Q(u_n) = 0$, we then obtain that $C > 0$. When $\sigma N = 4$, we can reach the same assertions by the virtue in Lemma 4.3.1.

Now we define $\tilde{u}_n := \sqrt{\frac{c}{c_n}}\, u_n \in S(c)$. Using [18, Lemma 5.2], we obtain

$$\gamma(c) \leq \max_{\lambda>0} E((\tilde{u}_n)_\lambda)$$

$$
\begin{aligned}
&= \max_{\lambda>0} \frac{c}{c_n}\left(\frac{\lambda^2}{2}\gamma\int_{\mathbb{R}^N}|\Delta u_n|^2\,dx + \frac{\lambda}{2}\int_{\mathbb{R}^N}|\nabla u_n|^2\,dx - \frac{\lambda^{\sigma N/2}}{2(2\sigma+2)}\left(\frac{c}{c_n}\right)^{\sigma}\int_{\mathbb{R}^N}|u_n|^{2\sigma+2}\,dx\right)\\
&\le \max_{\lambda>0}\left(\frac{\lambda^2}{2}\gamma A + \frac{\lambda}{2}B - \frac{\lambda^{\sigma N/2}}{2(2\sigma+2)}C\right) + \frac{\varepsilon}{3}\\
&\le \max_{\lambda>0}\left(\frac{\lambda^2}{2}\gamma\int_{\mathbb{R}^N}|\Delta u_n|^2\,dx + \frac{\lambda}{2}\int_{\mathbb{R}^N}|\nabla u_n|^2\,dx - \frac{\lambda^{\sigma N/2}}{2(2\sigma+2)}\int_{\mathbb{R}^N}|u_n|^{2\sigma+2}\,dx\right) + \frac{2\varepsilon}{3}\\
&= \max_{\lambda>0} E((u_n)_\lambda) + \frac{2\varepsilon}{3} = E(u_n) + \frac{2\varepsilon}{3} \le \gamma(c_n) + \varepsilon,
\end{aligned}
$$

from which we conclude that

$$\gamma(c) \le \limsup_{n\to\infty}\gamma(c_n). \tag{4.6.3}$$

From (4.6.1) and (4.6.3), we deduce that $\lim_{n\to\infty}\gamma(c_n) = \gamma(c)$. □

Lemma 4.6.2. *Let $4 \le \sigma N < 4^*$, then $\lim_{c\to c_0^+}\gamma(c) = +\infty$.*

Proof. When $4 < \sigma N < 4^*$, jointing (4.3.1) with (4.3.2), we immediately deduce that $\lim_{c\to 0}\gamma(c) = \infty$. When $\sigma N = 4$, to show that $\lim_{c\to {c_N^*}^+}\gamma(c) = \infty$, we first observe that for $u \in \mathcal{M}(c)$,

$$
\begin{aligned}
\gamma(c) \le E(u) &= \frac{\gamma}{2}\int_{\mathbb{R}^N}|\Delta u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx - \frac{N}{2N+8}\int_{\mathbb{R}^N}|u|^{2+\frac{8}{N}}\,dx\\
&= \frac{N}{2N+8}\int_{\mathbb{R}^N}|u|^{2+\frac{8}{N}}\,dx - \frac{\gamma}{2}\int_{\mathbb{R}^N}|\Delta u|^2\,dx\\
&\le \frac{1}{2}\left(\left(\frac{c}{c_N^*}\right)^{\frac{4}{N}} - 1\right)\gamma\int_{\mathbb{R}^N}|\Delta u|^2\,dx.
\end{aligned}
\tag{4.6.4}
$$

Thus combining the fact that E is bounded from below on $M(c)$ by a positive constant, see Lemma 4.3.1, and the property obtained in Lemma 4.4.2, that $c \mapsto \gamma(c)$ is decreasing, we deduce for any sequence $\{c_n\}$ with $c_n \to {c_N^*}^+$ and $\{u_{c_n}\} \subset \mathcal{M}(c_n)$ that

$$\int_{\mathbb{R}^N}|\Delta u_{c_n}|^2\,dx \to \infty \text{ as } n\to\infty. \tag{4.6.5}$$

If $E(u_{c_n}) \to \infty$ as $n\to\infty$, we have readily finished the proof. Otherwise, by Lemma 4.2.2, it then follows that $\int_{\mathbb{R}^N}|\nabla u_{c_n}|^2\,dx \to \infty$ as $n\to\infty$, and we conclude by using (4.3.1). □

Lemma 4.6.3. *Let $4 \le \sigma N < 4^*$ and $u_c \in S(c)$ be a solution to*

$$\gamma\Delta^2 u - \Delta u + \alpha_c u = |u|^{2\sigma}u$$

with $E(u_c) = \gamma(c)$. Then $\alpha_c \ge 0$, if $\alpha_c > 0$ the function $c \mapsto \gamma(c)$ is strictly decreasing in a right neighborhood of c.

Proof. In view of Lemma 4.4.2, to prove the lemma it suffices to show that if $\alpha_c > 0$ ($\alpha_c < 0$) the function $c \to \gamma(c)$ is strictly decreasing (increasing) in a right (left) neighbourhood of c. The strict monotonicity of the function $c \to \gamma(c)$ when $\alpha_c \ne 0$ is obtained as a consequence of the Implicit Function Theorem. Let $(u_c)_{t,\lambda}(x) := \lambda^{\frac{N}{4}}\sqrt{t}u_c(\sqrt{\lambda}x)$ for $t,\lambda > 0$. We define $\beta_E(t,\lambda) := E((u_c)_{t,\lambda})$, and $\beta_Q(t,\lambda) := Q((u_c)_{t,\lambda})$. By direct calculations, we obtain

$$\frac{\partial\beta_E}{\partial t}(1,1) = -\frac{1}{2}\alpha_c c, \quad \frac{\partial\beta_E}{\partial\lambda}(1,1) = 0, \quad \frac{\partial^2\beta_E}{\partial\lambda^2}(1,1) < 0,$$

which yields for sufficiently small $|\delta_\lambda|$, and $\delta_t > 0$,

$$\beta_E(1+\delta_t, 1+\delta_\lambda) < \beta_E(1,1) \text{ if } \alpha_c > 0 \tag{4.6.6}$$

$$\beta_E(1-\delta_t, 1-\delta_\lambda) < \beta_E(1,1) \text{ if } \alpha_c < 0. \tag{4.6.7}$$

Observe that $\beta_Q(1,1) = 0$, and $\dfrac{\partial \beta_Q}{\partial \lambda}(1,1) < 0$. Using Implicit Function Theorem, we obtain the existence of a $\varepsilon > 0$ small and of a continuous function $g : [1-\varepsilon, 1+\varepsilon] \to \mathbb{R}$ satisfying $g(1) = 1$ such that $\beta_Q(t, g(t)) = 0$ for $t \in [1-\varepsilon, 1+\varepsilon]$. Therefore we have from (4.6.6),

$$\gamma((1+\varepsilon)c) = \inf_{u \in \mathcal{M}((1+\varepsilon)c)} E(u) \le E((u_c)_{1+\varepsilon, g(1+\varepsilon)}) < E(u_c) = \gamma(c).$$

Similarly by (4.6.7), $\gamma((1-\varepsilon)c) < \gamma(c)$ when $\alpha_c < 0$. □

We now investigate the behaviors of the function $c \mapsto \gamma(c)$ as $c \to \infty$.

Proposition 4.6.4. *If $N = 1, 2$, $N = 3$ with $\frac{4}{3} \le \sigma < 2$ or $N = 4$ with $\sigma = 1$, then $c \mapsto \gamma(c)$ is strictly decreasing and $\lim_{c\to\infty} \gamma(c) = 0$.*

Proof. The fact that $c \mapsto \gamma(c)$ is strictly decreasing follows directly from Lemma 4.2.1 and Lemma 4.6.3. To show that $\lim_{c\to\infty} \gamma(c) = 0$, we first treat the case $\sigma N = 4$. Using (4.2.16) and (4.4.5) we obtain that

$$\gamma(c) \le \max_{\lambda>0} E(w_\lambda) = \frac{\frac{c}{\|U\|_2^2} \left(\int_{\mathbb{R}^N} |\nabla U|^2 \, dx\right)^2}{8 \left(\left(\frac{c}{c_N^*}\right)^{\frac{4}{N}} - 1 \right) \gamma \int_{\mathbb{R}^N} |\Delta U|^2 \, dx},$$

where w is defined by (4.2.15). This shows that $\gamma(c) \to 0$ as $c \to \infty$ when $1 \le N \le 4$. For the remaining cases we fix an arbitrary $u \in H^2(\mathbb{R}^N)$ satisfying $\|u\|_2 = 1$. For any $c > 0$, $\sqrt{c}u \in S(c)$, and from Lemma 4.3.2 we know that there exists a unique $\lambda_c > 0$ such that $Q((\sqrt{c}u)_{\lambda_c}) = 0$, i.e.

$$\lambda_c \gamma \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx = \frac{\sigma N}{2(2\sigma+2)} (c\lambda_c)^{\frac{\sigma N}{2}-1} c^{\sigma+1-\frac{\sigma N}{2}} \int_{\mathbb{R}^N} |u|^{2\sigma+2} \, dx.$$

Since $\sigma N > 1$, we deduce that $(c\lambda_c) \to 0$ as $c \to \infty$. Now using again (4.4.5), it follows that

$$\gamma(c) \le E(\sqrt{c}u)_{\lambda_c}) = c{\lambda_c}^2 \gamma \frac{\sigma N - 4}{2\sigma N} \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + c\lambda_c \frac{\sigma N - 2}{2\sigma N} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx,$$

and thus $\gamma(c) \to 0$ as $c \to \infty$. □

To treat the remaining cases, namely, $\sigma \ge 2$ if $N = 3$, $\sigma > 1$ if $N = 4$ or $4 \le \sigma N < 4^*$ if $N \ge 5$, we need to consider the following equation

$$\gamma \Delta^2 u - \Delta u = |u|^{2\sigma} u. \tag{4.6.8}$$

Let $X := \{u \in D^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\Delta u|^2 \, dx < \infty\}$ be equipped with the norm

$$\|u\|_X^2 := \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \int_{\mathbb{R}^N} |\nabla u|^2 \, dx.$$

Under our assumptions, we see from (4.2.2) that $X \hookrightarrow L^{2\sigma+2}(\mathbb{R}^N)$ and in particular E, Q are well-defined in X. Now let

$$\gamma(\infty) := \inf\{E(u) : u \in X \backslash \{0\}, E'(u) = 0\}.$$

Proposition 4.6.5. *Assume that $\sigma \geq 2$ if $N = 3$, $\sigma > 1$ if $N = 4$ and $4 \leq \sigma N < 4^*$ if $N \geq 5$. Then $\gamma(\infty)$ is reached and*

1. *When $N = 3, 4$, (4.6.8) does not admit nonnegative solution. In particular, $\gamma(\infty)$ is not reached by an element in $H^2(\mathbb{R}^N)$.*
2. *When $N \geq 5$, all minimizers of $\gamma(\infty)$ belongs to $H^2(\mathbb{R}^N)$.*

Remark 4.6.6. *If one considers the equation (4.6.8) assuming that $N = 1, 2$ or $N = 3$ with $\sigma \leq 2$ or $N = 4$ and $\sigma = 1$, we see directly from Lemma 4.2.1 that it has no solutions in $H^2(\mathbb{R}^N)$ nor in X.*

Proof of Proposition 4.6.5. It is classical to show that $\gamma(\infty)$ is reached if and only if the problem

$$m := \inf_{u \in M} J(u), \tag{4.6.9}$$

where

$$J(u) = \int_{\mathbb{R}^N} \gamma |\Delta u|^2 + |\nabla u|^2 \, dx \quad \text{and} \quad M := \{u \in X : ||u||_{2\sigma+2} = 1\}$$

admit a minimizer. To prove that m is reached we proceed as in [31, Remark 3.2]. Let $\{u_n\} \subset X$ be a minimizing sequence for m. Without restriction, since $H^2(\mathbb{R}^N)$ is dense in X, we can assume that $\{u_n\} \subset H^2(\mathbb{R}^N)$. Then we set $f_n := -\sqrt{\gamma}\Delta u_n + \frac{1}{2\sqrt{\gamma}} u_n 2$ and define $v_n \in H^2(\mathbb{R}^N)$ to be the strong solution of $-\sqrt{\gamma}\Delta v_n + \frac{1}{2\sqrt{\gamma}} v_n = |f_n|^*$ in $\mathbb{R}^N$, where $|f_n|^*$ denotes the Schwarz rearrangement of $|f_n|$. Thus for each $n \in \mathbb{N}$ we have $v_n \in H^2_{rad}(\mathbb{R}^N)$ and a particular case of [29, Lemma 3.4] implies that

$$\begin{aligned} J\Big(\frac{v_n}{||v_n||_{2\sigma+2}}\Big) &= \frac{\int_{\mathbb{R}^N} (-\sqrt{\gamma}\Delta v_n + \frac{1}{2\sqrt{\gamma}} v_n)^2 \, dx - \frac{1}{4\gamma} \int_{\mathbb{R}^N} v_n^2 \, dx}{||v_n||^2_{2\sigma+2}} \\ &\leq \frac{\int_{\mathbb{R}^N} (-\sqrt{\gamma}\Delta u_n + \frac{1}{2\sqrt{\gamma}} u_n)^2 \, dx - \frac{1}{4\gamma} \int_{\mathbb{R}^N} u_n^2 \, dx}{||u_n||^2_{2\sigma+2}} = J\Big(\frac{u_n}{||u_n||_{2\sigma+2}}\Big). \end{aligned}$$

Thus $\{\tilde{v}_n\} := \Big\{\frac{v_n}{||v_n||_{2\sigma+2}}\Big\}$ is a minimizing sequence for m. Now we claim that X_{rad}, the subset of radially symmetric functions in X, is compactly embedded into $L^{2\sigma+2}(\mathbb{R}^N)$. Indeed, applying [21, Radial Lemma AIII], if $u \in D^{1,2}(\mathbb{R}^N)$ is radially symmetric, we have

$$|u(x)| \leq C|x|^{-(N-2)/2} \|\nabla u\|_2.$$

Using this decay we get

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_R(0)} |u|^{2\sigma+2} dx &\leq \int_{\mathbb{R}^N \setminus B_R} |u|^{2\sigma+2-2N/(N-2)} |u|^{2N/(N-2)} dx \\ &\leq C R^{-\frac{N-2}{2}(2\sigma+2-\frac{2N}{N-2})} \|\nabla u\|_2^{1+(N-2)/2N}, \end{aligned}$$

from which the claim follows. Using this embedding, we get that $\{\tilde{v}_n\}$ weakly converges to some $v \in X$ with $||v||_{2\sigma+2} = 1$ and the remaining arguments are standard. We thus obtain a minimizer for J on M and $\gamma(\infty)$ is reached.

Let us now prove that $\gamma(\infty)$ does not have a minimizer in $H^2(\mathbb{R}^N)$ when $N = 3, 4$. Assuming by contradiction that u is such a minimizer we deduce from [31, Lemma 4.1] that u must have a sign and without restriction we can assume that $u \geq 0$. To conclude

it suffices to show that (4.6.8) has no nonnegative solutions in $H^2(\mathbb{R}^N)$. For this aim, we decompose (4.6.8) into the elliptic system

$$\begin{cases} -\gamma\Delta u = v, \\ -\Delta v + \dfrac{1}{\gamma} v = |u|^{2\sigma+2}u. \end{cases} \tag{4.6.10}$$

If u is a solution to (4.6.8), then by the standard elliptic regularity theory, $u \in C^4(\mathbb{R}^N)$. Hence applying the maximum principle to the second equation in (4.6.10), we deduce that $v \geq 0$ and thus any nontrivial nonnegative solution u to (4.6.8) has to satisfy $-\Delta u \geq 0$. Using the Liouville's type result [61, Lemma A.2], we conclude that $u \notin L^2(\mathbb{R}^N)$. Finally, when $N = 5$ one can show that any solution to (4.6.8) in X belongs to $H^2(\mathbb{R}^N)$. This is proved in Proposition 4.10.2 that can be found in Appendix. □

Since m is reached where m is defined by (4.6.9), then clearly $\gamma(\infty) > 0$ and by standard arguments, it can also be defined as

$$\gamma(\infty) := \inf\{E(u) : u \in X\backslash\{0\}, Q(u) = 0\}. \tag{4.6.11}$$

Proposition 4.6.7. *If $N = 3$ and $\sigma \geq 2$ or $N = 4$ and $\sigma > 1$, then $\lim_{c\to\infty} \gamma(c) = \gamma(\infty) > 0$.*

Proof. Using the definition (4.6.11), we directly obtain that $\gamma(c) \geq \gamma(\infty)$ for all $c > c_0$. Now still from (4.6.11) and taking Proposition 4.6.5 into account, we know that there exists a $u \in X$ such that $E(u) = \gamma(\infty)$ and $Q(u) = 0$. For $R > 0$ we define $u_R(x) := \eta(\frac{x}{R})u(x)$, where $\eta(x) = 1$ for $|x| \leq 1$, $\eta(x) = 0$ for $|x| \geq 2$, and $0 \leq \eta \leq 1$. Thus, as $R \to \infty$,

$$\|u_R\|_{2\sigma+2} \to \|u\|_{2\sigma+2}, \ \|\nabla u_R\|_2 \to \|\nabla u\|_2, \text{ and } \|\Delta u_R\|_2 \to \|\Delta u\|_2.$$

Now let $\lambda_R^* > 0$ be such that $Q((u_R)_{\lambda_R^*}) = 0$. By continuity we obtain that $\lambda_R^* \to 1$ as $R \to \infty$. Thereby

$$\gamma(\|u_R\|_2^2) \leq E((u_R)_{\lambda_R}) \leq E(u) + o_R(1) = \gamma(\infty) + o_R(1),$$

where $o_R(1) \to 0$ as $R \to \infty$, then $\gamma(c) \to \gamma(\infty)$ as $c \to \infty$. □

Proposition 4.6.8. *Let $N = 3$ and $\sigma \geq 2$ or $N = 4$ and $\sigma > 1$. Then $\gamma(c) > \gamma(\infty)$ for all $c > c_0$.*

Proof. When $N = 3, \sigma \geq 2$, it is a direct consequence of Lemma 4.2.1 and Lemma 4.6.3. In the other cases let us assume by contradiction that there exists $c > 0$ such that $\gamma(c) = \gamma(\infty)$. From Lemma 4.3.6 and Lemma 4.4.1, we obtain the existence of a $u_c \in H^2(\mathbb{R}^N)$ satisfying $0 < ||u_c||_2^2 \leq c$ and $\gamma(||u_c||_2^2) = \gamma(c) = \gamma(\infty)$. At this point, we have obtained that $u_c \in H^2(\mathbb{R}^N)$ is a solution of (4.6.8) at the energy level $\gamma(\infty)$, it is a ground state. But we know from Proposition 4.6.5 that such ground state does not exist. This contradiction ends the proof. □

Proposition 4.6.9. *If $N \geq 5$ there exists a $c_\infty > 0$ such that $\gamma(c) = \gamma(\infty)$ for all $c \geq c_\infty$.*

Proof. By Proposition 4.6.5 and using (4.6.11), we know that there exists a $u \in H^2(\mathbb{R}^N)$ such that $E(u) = \gamma(\infty)$ and $Q(u) = 0$. Setting $c_\infty := ||u||_2^2$, we obtain that $\gamma(c_\infty) \leq \gamma(\infty)$. Now recording that $\gamma(c) \geq \gamma(\infty)$ for any $c > c_0$ and that, by Lemma 4.4.2, $c \mapsto \gamma(c)$ is decreasing we conclude. □

Proof of Theorem 4.1.6. The proof follows directly from Proposition 4.6.4-Proposition4.6.9. □

4.7 A concentration phenomenon

In this section, when $\sigma N = 4$, we establish the concentration of solutions to (4.1.2)-(4.1.3) as c approaches to c_N^* from above, described in Theorem 4.1.7. As a preliminary result, we derive

Lemma 4.7.1. *Let $\sigma N = 4$ and $u \in H^2(\mathbb{R}^N)$ be a nontrivial solution to the equation*

$$\gamma \Delta^2 u + u = |u|^{\frac{8}{N}} u. \tag{4.7.1}$$

Then $\|u\|_2^2 \geq c_N^*$, furthermore, u is a least energy solution if $\|u\|_2^2 = c_N^*$,.

Proof. We define the energy functional associated to (4.7.1)in $H^2(\mathbb{R}^N)$ as

$$F(u) := \frac{\gamma}{2} \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} |u|^2 \, dx - \frac{N}{2N+8} \int_{\mathbb{R}^N} |u|^{2+\frac{8}{N}} \, dx.$$

If u is a solution to (4.7.1), then by Lemma 4.10.1, we get

$$\gamma \int_{\mathbb{R}^N} |\Delta u|^2 \, dx = \frac{N}{N+4} \int_{\mathbb{R}^N} |u|^{2+\frac{8}{N}}, \tag{4.7.2}$$

which implies that

$$F(u) = \frac{1}{2} \int_{\mathbb{R}^N} |u|^2 \, dx. \tag{4.7.3}$$

If u is a nontrivial solution to (4.7.1), then there holds $\|u\|_2^2 \geq c_N^*$. Indeed, using the Gagliardo-Nirenberg's inequality (4.1.5), we get from (4.7.2) that

$$\gamma \int_{\mathbb{R}^N} |\Delta u|^2 \, dx \leq \left(\frac{\|u\|_2^2}{c_N^*} \right)^{\frac{4}{N}} \gamma \int_{\mathbb{R}^N} |\Delta u|^2 \, dx.$$

Thus necessarily $\|u\|_2^2 \geq c_N^*$ and taking into account (4.7.3), this ends the proof. □

Proof of Theorem 4.1.7. By Theorem 4.1.4, there exist a sequence $\{c_n\}$ with $c_n \to c_N^*$ with $c_n > c_N^*$ and $\{u_n\} \subset \mathcal{M}(c_n)$ such that $E(u_n) = \gamma(c_n)$. From (4.6.5) in the proof of Lemma 4.6.2, we deduce that

$$\int_{\mathbb{R}^N} |\Delta u_n|^2 \, dx \to \infty \text{ as } n \to \infty, \tag{4.7.4}$$

and using Cauchy-Schwarz' inequality (4.2.3), it follows from (4.7.4) that

$$\frac{\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx}{\int_{\mathbb{R}^N} |\Delta u_n|^2 \, dx} \to 0 \text{ as } n \to \infty. \tag{4.7.5}$$

Since $Q(u_n) = 0$, we then obtain

$$\frac{\int_{\mathbb{R}^N} |u_n|^{2+\frac{8}{N}} \, dx}{\gamma \int_{\mathbb{R}^N} |\Delta u_n|^2 \, dx} \to \frac{N+4}{N} \text{ as } n \to \infty. \tag{4.7.6}$$

At this point, we introduce $\tilde{u}_n(x) := \epsilon_n^{\frac{N}{2}} u_n(\epsilon_n x)$, where

$$\epsilon_n^{-4} := \gamma \int_{\mathbb{R}^N} |\Delta u_n|^2 \, dx \to \infty \text{ as } n \to \infty. \tag{4.7.7}$$

It is easy to check that $\|\tilde{u}_n\|_2^2 = \|u_n\|_2^2 = c_n$, $\|\Delta \tilde{u}_n\|_2^2 = 1$, and

$$\int_{\mathbb{R}^N} |\tilde{u}_n|^{2+\frac{8}{N}} \, dx \to \frac{N+4}{N} \text{ as } n \to \infty, \tag{4.7.8}$$

Then, as in the proof of Lemma 4.3.1, necessarily there exist a $\delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^N$ such that for some $R > 0$,

$$\int_{B_R(y_n)} |\tilde{u}_n|^2 \, dx \geq \delta. \tag{4.7.9}$$

Thus defining

$$v_n(x) := \tilde{u}_n(x + y_n) = \epsilon_n^{\frac{N}{2}} u_n(\epsilon_n x + \epsilon_n y_n), \tag{4.7.10}$$

we get from (4.7.9) that there is a nontrivial v so that $v_n \rightharpoonup v$ in $H^2(\mathbb{R}^N)$. Since u_n satisfies the following equation

$$\gamma \Delta^2 u_n - \Delta u_n + \alpha_n u_n = |u_n|^{\frac{8}{N}} u_n,$$

where the Lagrange multiplier is given by

$$\alpha_n = \frac{1}{c_n} \left(-\gamma \int_{\mathbb{R}^N} |\Delta u_n|^2 \, dx - \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx + \int_{\mathbb{R}^N} |u_n|^{2+\frac{8}{N}} \, dx \right)$$

therefore v_n satisfies

$$\gamma \Delta^2 v_n - \epsilon_n^2 \Delta v_n = \epsilon_n^4 \alpha_n v_n + |v_n|^{\frac{8}{N}} v_n.$$

Combining (4.7.4) and (4.7.5)-(4.7.7), we get

$$\epsilon_n^4 \alpha_n \to \frac{4}{c_N^* N} \text{ as } n \to \infty. \tag{4.7.11}$$

Since $v_n \rightharpoonup v$ in $H^2(\mathbb{R}^N)$ as $n \to \infty$, then v solves

$$\gamma \Delta^2 v + \frac{4}{c_N^* N} v = |v|^{\frac{8}{N}} v. \tag{4.7.12}$$

Now setting

$$w_n(x) := \left(\frac{c_N^* N}{4} \right)^{\frac{N}{8}} v_n \left(\left(\frac{c_N^* N}{4} \right)^{\frac{1}{4}} x \right), \quad u(x) := \left(\frac{c_N^* N}{4} \right)^{\frac{N}{8}} v \left(\left(\frac{c_N^* N}{4} \right)^{\frac{1}{4}} x \right),$$

it is easily seen that $w_n \rightharpoonup u$ in $H^2(\mathbb{R}^N)$ as $n \to \infty$, and $\|w_n\|_2^2 = \|v_n\|_2^2 = c_n$. Moreover, it follows from (4.7.12) that u is solution to (4.7.1), and thus by Lemma 4.7.1, we have that $\|u\|_2^2 \geq c_N^*$. On the other hand, since $w_n \rightharpoonup u$ in $H^2(\mathbb{R}^N)$ as $n \to \infty$, we see that $\|u\|_2^2 \leq \liminf_{n\to\infty} \|w_n\|_2^2 = c_N^*$ and thus we obtain that $\|u\|_2^2 = c_N^*$. By Lemma 4.7.1 u is a least energy solution to (4.7.1). Since $\|u\|_2^2 = c_N^*$, $\|w_n\|_2^2 = c_n \to c_N^*$ as $n \to \infty$, and $w_n \rightharpoonup u$ in $H^2(\mathbb{R}^N)$ as $n \to \infty$, it follows that $w_n \to u$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$. Now from the definition (4.7.10), and by interpolation inequalities in Lebesgue space, there holds for $2 \leq q < \frac{2N}{(N-4)^+}$,

$$\left(\frac{\epsilon_n^4 c_N^* N}{4} \right)^{\frac{N}{8}} u_n \left(\left(\frac{\epsilon_n^4 c_N^* N}{4} \right)^{\frac{1}{4}} x + \epsilon_n y_n \right) \to u \text{ in } L^q(\mathbb{R}^N) \text{ as } n \to \infty.$$

This completes the proof. □

4.8 Positive and sign-changing solutions

In this section, we study the sign and radial symmetry property of ground states to (4.1.2)-(4.1.3).

Proof of Theorem 4.1.8. For any $c \in (c_0, c_{\sigma,N})$, the existence of a ground state is guaranteed by Theorem 4.1.4. To show that, when $\sigma \in \mathbb{N}$, one of them is radial we make use of the Fourier rearrangement arguments as presented in [30]. For $u \in L^2(\mathbb{R}^N)$, let $u^\sharp$ be the Fourier rearrangement to u defined by

$$u^\sharp := \mathcal{F}^{-1}((\mathcal{F}u)^*),$$

where $\mathcal{F}$ reps. $\mathcal{F}^{-1}$ denotes the Fourier transform reps. the Fourier inverse transform, and f^* stands for the Schwarz rearrangement of a measurable function f. Notice that $u^\sharp$ is radial, and $\|u^\sharp\|_2 = \|u\|_2$. Moreover, in view of [30, Lemma A.1],

$$\|\Delta u^\sharp\|_2 \le \|\Delta u\|_2, \quad \|\nabla u^\sharp\|_2 \le \|\nabla u\|_2, \quad \|u^\sharp\|_{2\sigma+2} \ge \|u\|_{2\sigma+2}. \tag{4.8.1}$$

Let u_c be a ground state associated to $\gamma(c)$, then $Q(u_c) = 0$. From (4.8.1), we obtain that $Q(u_c^\sharp) \le Q(u_c) = 0$. Hence by Lemma 4.3.2, there exists a $0 < \lambda \le 1$ such that $Q((u_c^\sharp)_\lambda) = 0$. Observe that

$$\begin{aligned}
\gamma(c) \le E((u_c^\sharp)_\lambda) &= E((u_c^\sharp)_\lambda) - \frac{2}{\sigma N} Q((u_c^\sharp)_\lambda) \\
&= \lambda^2 \frac{\sigma N - 4}{2\sigma N} \gamma \int_{\mathbb{R}^N} |\Delta u_c^\sharp|^2 \, dx + \lambda \frac{\sigma N - 2}{2\sigma N} \int_{\mathbb{R}^N} |\nabla u_c^\sharp|^2 \, dx \\
&\le E(u_c) - \frac{2}{\sigma N} Q(u_c) = \gamma(c),
\end{aligned}$$

and thus necessarily $\lambda = 1$, and $E(u_c^\sharp) = \gamma(c)$. Therefore, $u_c^\sharp$ is a ground state solution to (4.1.2)-(4.1.3). It remains to prove that $u_c^\sharp$ is sign-changing. Associated to $u_c^\sharp$ there exists a Lagrange multiplier $\alpha_c \in \mathbb{R}$ so that

$$\gamma \Delta^2 u_c^\sharp - \Delta u_c^\sharp + \alpha_c u_c^\sharp = |u_c^\sharp|^{2\sigma} u_c^\sharp.$$

Now, when $4 < \sigma N < 4^*$, we deduce from (4.2.8) and (4.2.10) that $\alpha_c \to +\infty$ as $c \to 0$. When $\sigma N = 4$, the same result can be established as $c \to c_N^*$ by combining (4.6.5) with (4.2.7) and (4.2.12). At this point, using [28, Theorem 3.2] we deduce that $u_c^\sharp$ is sign-changing. □

Proof of Theorem 4.1.9. We borrow here an idea from [28]. We consider the following modified minimization problem

$$\bar{\gamma}(c) := \inf_{u \in \bar{\mathcal{M}}(c)} \bar{E}(u), \tag{4.8.2}$$

where

$$\begin{aligned}
\bar{E}(u) :&= \frac{\gamma}{2} \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{1}{2\sigma + 2} \int_{\mathbb{R}^N} |u^+|^{2\sigma+2} \, dx, \\
\bar{Q}(u) :&= \gamma \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{\sigma N}{2(2\sigma + 2)} \int_{\mathbb{R}^N} |u^+|^{2\sigma+2} \, dx,
\end{aligned}$$

and

$$\bar{\mathcal{M}}(c) := \{u \in S(c) : \bar{Q}(u) = 0\}.$$

It is straightforward to check that the analysis done with E, Q, and $\mathcal{M}(c)$ remains unchanged if we now work with $\bar{E}$, $\bar{Q}$ and $\bar{\mathcal{M}}(c)$. Thus, in particular, for any $c > c_0$, if $\{\bar{u}_n\} \subset \bar{\mathcal{M}}(c)$ is a minimizing Palais-Smale sequence to (4.8.2), by the modified version of Lemma 4.3.6, there exists a $\bar{u}_c \in H^2(\mathbb{R}^N)\backslash\{0\}$, and a Lagrange multiplier $\bar{\alpha}_c \in \mathbb{R}$ such that

$$\gamma\Delta^2\bar{u}_c - \Delta\bar{u}_c + \bar{\alpha}_c\bar{u}_c = |\bar{u}_c^+|^{2\sigma}\bar{u}_c^+. \tag{4.8.3}$$

Also by the corresponding versions of Lemma 4.4.1, Lemma 4.4.2, and Lemma 4.6.3, we deduce that $0 < \|\bar{u}_c\|_2^2 \le c$, $\bar{u}_n \to \bar{u}_c$ in $L^{2\sigma+2}(\mathbb{R}^N)$, $\bar{E}(\bar{u}_c) = \bar{\gamma}(c) > 0$, and $\bar{\alpha}_c \ge 0$.

Next we show that $\bar{u}_c > 0$. To this aim, we first observe that $\bar{\alpha}_c \ge 0$ can be assumed arbitrarily small by taking $c > 0$ large enough. Indeed, $\bar{\alpha}_c$ satisfies

$$\bar{\alpha}_c = \frac{1}{c}\left(-2\bar{E}(\bar{u}_c) + \frac{\sigma}{\sigma+1}\int_{\mathbb{R}^N}|\bar{u}_c^+|^{2\sigma+2}\,dx\right) \le \frac{\sigma}{c(\sigma+1)}\int_{\mathbb{R}^N}|\bar{u}_c^+|^{2\sigma+2}\,dx. \tag{4.8.4}$$

Recording the fact that $\bar{\gamma}(c)$ remains bounded as $c \to \infty$. When $4 < \sigma N < 4^*$, then from (4.3.1) and $Q(\bar{u}_c) = 0$, we see that $\int_{\mathbb{R}^N}|\bar{u}_c^+|^{2\sigma+2}\,dx \le C$ for some $C > 0$ as $c \to \infty$. Thus, in view of (4.8.4) we deduce that $\bar{\alpha}_c \ge 0$ can be arbitrarily small by taking $c > 0$ large enough. When $\sigma N = 4$, it follows from (4.3.1) and (4.2.1) that $\int_{\mathbb{R}^N}|\bar{u}_c^+|^{2\sigma+2}\,dx \le C$ for some $C > 0$ as $c \to \infty$. Then we can reach the same argument from (4.8.4).

Since $\bar{\alpha}_c \ge 0$ is small when $c > c_0$ is sufficiently large, then we are able to write (4.8.3) into the following system

$$\begin{cases} -\gamma\Delta\bar{u}_c + \lambda_1\bar{u}_c = \bar{v}_c, \\ -\Delta\bar{v}_c + \dfrac{\lambda_2}{\gamma}\bar{v}_c = |\bar{u}_c^+|^{2\sigma}\bar{u}_c^+, \end{cases}$$

where $\lambda_1, \lambda_2 \ge 0$ satisfying $\lambda_1\lambda_2 = \gamma\bar{\alpha}_c$, and $\lambda_1 + \lambda_2 = 1$. It is then standard, by the strong maximum principle, to deduce that $\bar{u}_c > 0$ and in particular $\bar{u}_c$ satisfies (4.1.2). By Proposition 4.6.5 and Remark 4.6.6, then $\bar{\alpha}_c > 0$, thus Lemma 4.6.3 indicates that $\bar{\gamma}(c)$ is achieved by $\bar{u}_c$. Finally, let us show that $\bar{u}_c$ is radially symmetric around some point. Setting

$$f(u,v) := \left(\frac{1}{4\gamma} - \bar{\alpha}_c\right)u - \frac{1}{2\gamma}v + |u|^{2\sigma}u, \quad g(u,v) := v - \frac{1}{2}u$$

we see that (4.1.2) is equivalent to the elliptic system

$$\begin{cases} \gamma\Delta\bar{u}_c + g(\bar{u}_c, \bar{v}_c) = 0, \\ \Delta\bar{v}_c + f(\bar{u}_c, \bar{v}_c) = 0. \end{cases}$$

We are now in the setting of Busca and Sirakov [32] and from [32, Theorem 2], we readily deduce that $\bar{u}_c$ is radially symmetric. □

4.9 Dynamical behaviors

This section is devoted to the study of dynamic behaviors of the solution to the Cauchy problem of the dispersive equation (4.1.1). First we give a class of initial datums such that

the solutions to (4.1.1) exist globally in time. Next we discuss the instability of the standing waves associated to radial ground states to (4.1.2)-(4.1.3) in the sense of Definition 4.1.11.

We start by recalling the local well-posedness of the solutions to the Cauchy problem of (4.1.1) and a blow-up alternative due to [95].

Lemma 4.9.1. *([95, Proposition 4.1]) Let $\sigma N < 4^*$. For any $u_0 \in H^2(\mathbb{R}^N)$, there exist a $T > 0$ and a unique solution $u(t) \in C([0,T); H^2(\mathbb{R}^N))$ to (4.1.1) with initial datum u_0 so that the mass and the energy are conserved along time, that is for any $t \in [0,T)$,*

$$\|u(t)\|_2 = \|u_0\|_2, \text{ and } E(u(t)) = E(u_0).$$

Moreover, either $T = \infty$, or $\lim_{t\to T^-} \|\Delta u(t)\|_2 = \infty$.

Proof of Theorem 4.1.10. Let $c > c_0$ be arbitrary. First observe that $\mathcal{O}_c \neq \emptyset$. Indeed, for any $u \in \mathcal{M}(c)$ we know from Lemma 4.3.2 that $u_\lambda \in \mathcal{O}_c$ for $\lambda > 0$ small enough. Now let $u_0 \in \mathcal{O}_c$ and denote by $u \in C([0,T); H^2(\mathbb{R}^N))$ the solution to (1.2.1) with initial datum u_0. We shall prove that u exists globally in time, i.e. $T = \infty$. If we suppose by contradiction that $T < \infty$, it follows from Lemma 4.9.1 that,

$$\lim_{t\to T^-} \int_{\mathbb{R}^N} |\Delta u(t)|^2 \, dx = \infty. \tag{4.9.1}$$

Now we observe that $E(u(t)) = E(u_0)$ for $0 \leq t < T$, and

$$E(u(t)) - \frac{2}{\sigma N} Q(u(t)) = \gamma \frac{\sigma N - 4}{2\sigma N} \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \frac{\sigma N - 2}{2\sigma N} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx. \tag{4.9.2}$$

Since $E(u(t)) = E(u_0)$, thus when $4 < \sigma N < 4^*$, we deduce from (4.9.1) that

$$\lim_{t\to T^-} Q(u(t)) = -\infty. \tag{4.9.3}$$

When $\sigma N = 4$, using that the energy and the mass are conserved, then Lemma 4.2.2 applies to give that

$$\lim_{t\to T^-} \int_{\mathbb{R}^N} |\nabla u(t)|^2 \, dx = \infty.$$

and we also deduce from (4.9.2) that (4.9.3) holds.

By continuity, there exists a $t_0 \in (0,T)$ such that $Q(u(t_0)) = 0$. Since $\|u(t_0)\|_2 = \|u_0\|_2 = c$, by the definition of $\gamma(c)$ it follows that $E(u(t_0)) \geq \gamma(c)$. This contradicts the fact that $E(u(t_0)) = E(u_0) < \gamma(c)$. Then Theorem 4.1.10 follows. □

Let us now prove Theorem 4.1.12. For this aim we first recall the localized virial identity introduced in [30],

$$M_{\varphi_R}[u] := 2\mathrm{Im} \int_{\mathbb{R}^N} \bar{u} \nabla \varphi_R \nabla u \, dx,$$

where $u \in H^1(\mathbb{R}^N)$, $\varphi : \mathbb{R}^N \to \mathbb{R}$ is a radial function such that $\nabla^j \varphi \in L^\infty(\mathbb{R}^N)$, $1 \leq j \leq 6$ satisfying

$$\varphi(r) := \begin{cases} \dfrac{r^2}{2} & for\ r \leq 1 \\ const. & for\ r \geq 10 \end{cases}, \ \varphi''(r) \leq 1 \ for\ r \geq 0,$$

and $\varphi_R(r) := R^2\varphi(\frac{r}{R})$ for $R > 0$.

In [30, Lemma 3.1], it is proved that for $N \geq 2$, if $u(t) \in C([0,T); H^2(\mathbb{R}^N))$ is the radial solution to (1.2.1) with initial datum $u_0 \in H^2_{rad}(\mathbb{R}^N)$, it holds

$$\begin{aligned}\frac{d}{dt}M_{\varphi_R}[u(t)] &\leq 4N\sigma E(u_0) - (2N\sigma - 8)\gamma\|\Delta u(t)\|_2^2 - (2N\sigma - 4)\|\nabla u(t)\|_2^2 \\ &\quad + O\left(\frac{\|\nabla u(t)\|_2^2}{R^2} + \frac{\|\nabla u(t)\|_2^\sigma}{R^{\sigma(N-1)}} + \frac{1}{R^2} + \frac{1}{R^4}\right) \\ &= 8Q(u(t)) + O\left(\frac{\|\nabla u(t)\|_2^2}{R^2} + \frac{\|\nabla u(t)\|_2^\sigma}{R^{\sigma(N-1)}} + \frac{1}{R^2} + \frac{1}{R^4}\right).\end{aligned} \tag{4.9.4}$$

Proof of Theorem 4.1.12. Suppose that u_c is a radial ground state, and define

$$\Theta := \{v \in H^2_{rad}(\mathbb{R}^N)\backslash\{0\} : E(v) < E(u_c),\ \|v\|_2 = \|u_c\|_2,\ Q(v) < 0\}.$$

The set Θ contains elements arbitrarily close to u_c in $H^2(\mathbb{R}^N)$. Indeed, letting $v_0 := (u_c)_\lambda$, we see from Lemma 4.3.2 that $v_0 \in \Theta$ if $\lambda > 1$ and that $v_0 \to u_c$ in $H^2_{rad}(\mathbb{R}^N)$ as $\lambda \to 1^+$. Let $v \in C([0,T); H^2_{rad}(\mathbb{R}^N))$ be the solution to (1.2.1) with radial initial datum v_0, and $T \in (0,\infty]$ be the maximal existence time. To prove the theorem, we just need to show that $v(t)$ blows up in finite time. We divide the rest of the proof into three steps.

First step : We claim that there exists a $\beta > 0$ such that $Q(v(t)) \leq -\beta$ for any $t \in [0,T)$. Indeed, reasoning as the proof of Theorem 4.1.10, we easily check that $v(t) \in \Theta$ and in particular $Q(v(t)) < 0$ for any $t \in [0,T)$. Now setting $v := v(t)$, in view of Lemma 4.3.2, since $Q(v) < 0$ there exists a $\lambda^* < 1$ such that $Q(v_{\lambda^*}) = 0$. Moreover, the function $\lambda \mapsto E(v_\lambda)$ is concave for $\lambda \in [\lambda^*, 1]$, thus

$$E(v_{\lambda^*}) - E(v) \leq (\lambda^* - 1)\frac{\partial E(u_\lambda)}{\partial\lambda}|_{\lambda=1} = (\lambda^* - 1)Q(v).$$

Using that $Q(v) < 0$, $E(v) = E(v_0)$ and $v_{\lambda^*} \in \mathcal{M}_{rad}(c)$, we have

$$Q(v) \leq (1 - \lambda^*)Q(v) \leq E(v) - E(v_{\lambda^*}) \leq E(v_0) - E(u_c) =: -\beta. \tag{4.9.5}$$

Second step : We claim that there exists a constant $\delta > 0$ such that

$$\frac{d}{dt}M_{\varphi_R}[v(t)] \leq -\delta\|\nabla v(t)\|_2^2 \text{ for } t \in [0,T), \tag{4.9.6}$$

and a $t_1 \geq 0$ such that

$$M_{\varphi_R}[v(t)] < 0 \text{ for } t \geq t_1. \tag{4.9.7}$$

To prove (4.9.6) we need to distinguish two cases.

Case 1: Let

$$T_1 := \{t \in [0,\infty) : (\sigma N - 2)\|\nabla v(t)\|_2^2 \leq 4N\sigma E(v_0)\}.$$

In view of (4.9.4) and the First Step, taking $R > 0$ sufficiently large, we obtain

$$\frac{d}{dt}M_{\varphi_R}[v(t)] \leq -7\beta \leq -\delta||\nabla v(t)||_2^2 \text{ for } t \in T_1, \tag{4.9.8}$$

with some $\delta > 0$ sufficiently small.

Case 2: Set

$$T_2 := [0,\infty)\backslash T_1 = \{t \in [0,\infty) : (\sigma N - 2)\|\nabla v(t)\|_2^2 > 4N\sigma E(v_0)\}.$$

Using (4.2.3), we get from (4.9.4)

$$\frac{d}{dt}M_{\varphi_R}[v(t)] \leq -(N\sigma-2)\|\nabla v(t)\|_2^2 - \frac{(2N\sigma-8)\gamma}{||v_0||_2^2}\|\nabla v(t)\|_2^4 + O\left(\frac{1}{R^4} + \frac{\|\nabla v(t)\|_2^2}{R^2} + \frac{\|\nabla v(t)\|_2^\sigma}{R^{\sigma(N-1)}} + \frac{\mu}{R^2}\right).$$

Taking R large enough and noticing that under our assumptions, $\sigma \leq 2$ if $N\sigma = 4$ and $\sigma \leq 4$ if $\sigma N > 4$, we deduce

$$\frac{d}{dt}M_{\varphi_R}[v(t)] \leq -\frac{(N\sigma-2)}{2}\|\nabla v(t)\|_2^2. \tag{4.9.9}$$

Now combining (4.9.8) and (4.9.9), we see that there exists a $\delta > 0$ such that (4.9.6) holds. Finally since

$$M_{\varphi_R}[v(t_1)] = M_{\varphi_R}[v_0] + \int_0^{t_1} \frac{d}{ds}M_{\varphi_R}[v(s)]ds,$$

the inequality (4.9.7) follows from the estimate

$$|\frac{d}{dt}M_{\varphi_R}[v(t)]| \geq \min\left\{7\beta, \frac{(N\sigma-2)}{2}||\nabla v(t)||_2^2\right\}.$$

Third step : We now conclude that the solution $v(t)$ to (1.2.1) with initial datum v_0 blows up. Here we adapt another argument from [30]. Suppose by contradiction that $T = \infty$, then integrating (4.9.6) on $[t_1, t]$, and taking (4.9.7) into account, we have that

$$M_{\varphi_R}[v(t)] \leq -\delta\int_{t_1}^t \|\nabla v(s)\|_2^2 ds.$$

Now using the Cauchy-Schwarz's inequality (4.2.3), we get from the definition of $M_{\varphi_R}[v(t)]$ that

$$|M_{\varphi_R}[v(t)]| \leq 2\|\nabla\varphi_R\|_\infty\|v(t)\|_2\|\nabla v(t)\|_2 \leq C\|\nabla v(t)\|_2.$$

Thus for some $\tau > 0$,

$$M_{\varphi_R}[v(t)] \leq -\tau\int_{t_1}^t |M_{\varphi_R}[v(s)]|^2 ds. \tag{4.9.10}$$

Setting $z(t) := \int_{t_1}^t |M_{\varphi_R}[v(s)]|^2 ds$, we obtain from (4.9.10) that $z'(t) \geq \tau^2 z(t)^2$. Integrating this equation, we deduce that $M_{\varphi_R}[v(t)] \to -\infty$, when t tends to some finite time t^*. Therefore the solution $v(t)$ cannot exist for all $t > 0$. By the blow-up alternative recalled in Lemma 4.9.1, this ends the proof of the theorem. □

4.10 Appendix

Lemma 4.10.1. *Let $0 < \sigma N < 4^*$. If $v \in H^2(\mathbb{R}^N)$ is a weak solution to*

$$\gamma\Delta^2 v - \mu\Delta v + \omega v = d|v|^{2\sigma}v \tag{4.10.1}$$

with γ, μ, ω, d are constants, then v satisfies $I(v) = P(v) = Q(v) = 0$, where

$$I(u) = \gamma\|\Delta u\|_2^2 + \mu\|\nabla u\|_2^2 + \omega\|u\|_2^2 - d\|u\|_{2\sigma+2}^{2\sigma+2}.$$

$$P(u) = \frac{(N-4)\gamma}{2}\|\Delta u\|_2^2 + \frac{(N-2)\mu}{2}\|\nabla u\|_2^2 + \frac{N\omega}{2}\|u\|_2^2 - \frac{dN}{2\sigma+2}\|u\|_{2\sigma+2}^{2\sigma+2},$$

and

$$Q(u) = \gamma\|\Delta u\|_2^2 + \frac{\mu}{2}\|\nabla u\|_2^2 - \frac{d\sigma N}{2(2\sigma+2)}\|u\|_{2\sigma+2}^{2\sigma+2}.$$

Proof. Since $u \in H^2(\mathbb{R}^N)$ is a solution to (4.10.1), multiplying (4.10.1) by u and integrating in $\mathbb{R}^N$, we get that $I(u) = 0$. Next, we notice that $Q(u) = \frac{N}{4}I(u) - \frac{1}{2}P(u)$. Therefore to prove that $Q(u) = 0$, we only need to show that $P(u) = 0$. This last identity is usually referred to as a Derrick-Pohozaev identity. To establish it we closely follow the proof of [21, Proposition 1]. First multiplying (4.10.1) by $x \cdot \nabla u$ and integrating on $B_R(0)$ for some $R > 1$, we have

$$\int_{B_R(0)} \gamma(x\cdot\nabla u)\Delta^2 u - \mu(x\cdot\nabla u)\Delta u + \omega(x\cdot\nabla u)u\,dx = d\int_{B_R(0)}(x\cdot\nabla u)|u|^{2\sigma}u\,dx. \tag{4.10.2}$$

In a first time, we focus on the first left-hand side term of (4.10.2). Integration by parts, we find

$$\begin{aligned}\gamma\int_{B_R(0)}(x\cdot\nabla u)\Delta^2 u\,dx &= -\gamma\int_{B_R(0)}\nabla(x\cdot\nabla u)\cdot\nabla(\Delta u)\,dx\\ &\quad+\gamma\int_{\partial B_R(0)}(\nabla(\Delta u)\cdot\mathbf{n})(x\cdot\nabla u)\,dS\\ &= \gamma\int_{B_R(0)}\Delta(x\cdot\nabla u)\Delta u\,dx\\ &\quad-\gamma\int_{\partial B_R(0)}(\nabla(x\cdot\nabla u)\cdot\mathbf{n})\Delta u - (\nabla(\Delta u)\cdot\mathbf{n})(x\cdot\nabla u)\,dS,\end{aligned}$$

where $\mathbf{n} := \mathbf{n}_x = \frac{x}{R}$ denotes the unit outward normal at $x \in \partial B_R(0)$. Integrating by parts one more time, we have

$$\begin{aligned}\gamma\int_{B_R(0)}\Delta(x\cdot\nabla u)\Delta u\,dx &= 2\gamma\int_{B_R(0)}|\Delta u|^2\,dx + \gamma\int_{B_R(0)}(x\cdot\nabla(\Delta u))\Delta u\,dx\\ &= 2\gamma\int_{B_R(0)}|\Delta u|^2\,dx + \frac{\gamma}{2}\int_{B_R(0)}x\cdot\nabla(|\Delta u|^2)\,dx\\ &= \frac{(4-N)\gamma}{2}\int_{B_R(0)}|\Delta u|^2\,dx + \frac{\gamma}{2}\int_{\partial B_R(0)}(x\cdot\mathbf{n})|\Delta u|^2\,dS.\end{aligned}$$

Combining the previous two equalities, we obtain

$$\begin{aligned}\gamma\int_{B_R(0)}(x\cdot\nabla u)\Delta^2 u\,dx &= \frac{(4-N)\gamma}{2}\int_{B_R(0)}|\Delta u|^2\,dx + \frac{\gamma}{2}\int_{\partial B_R(0)}(x\cdot\mathbf{n})|\Delta u|^2\,dS\\ &\quad-\gamma\int_{\partial B_R(0)}(\nabla(x\cdot\nabla u)\cdot\mathbf{n})\Delta u - (\nabla(\Delta u)\cdot\mathbf{n})(x\cdot\nabla u)\,dS.\end{aligned}$$

Next, we deal with the second left-hand side term of (4.10.2). We have

$$-\mu\int_{B_R(0)}(x\cdot\nabla u)\Delta u\,dx = \mu\int_{B_R(0)}\nabla(x\cdot\nabla u)\cdot\nabla u\,dx - \mu\int_{\partial B_R(0)}(\nabla u\cdot\mathbf{n})(x\cdot\nabla u)\,dS$$

$$= \frac{(2-N)\mu}{2}\int_{B_R(0)}|\nabla u|^2\,dx + \frac{\mu}{2}\int_{\partial B_R(0)}(x\cdot\mathbf{n})|\nabla u|^2\,dS$$
$$-\mu\int_{\partial B_R(0)}(\nabla u\cdot\mathbf{n})(x\cdot\nabla u)\,dS.$$

Finally, for the last two terms of (4.10.2), we get

$$\omega\int_{B_R(0)}(x\cdot\nabla u)u\,dx = -\frac{\omega N}{2}\int_{B_R(0)}|u|^2\,dx + \frac{\omega}{2}\int_{\partial B_R(0)}(x\cdot\mathbf{n})|u|^2\,dS,$$

and

$$d\int_{B_R(0)}(x\cdot\nabla u)|u|^{2\sigma}u\,dx = -\frac{dN}{2\sigma+2}\int_{B_R(0)}|u|^{2\sigma+2}\,dx + \frac{d}{2\sigma+2}\int_{\partial B_R(0)}(x\cdot\mathbf{n})|u|^{2\sigma+2}\,dS.$$

Taking into account the above calculations, it follows from (4.10.2) that

$$\begin{aligned}&\frac{(N-4)\gamma}{2}\int_{B_R(0)}|\Delta u|^2\,dx + \frac{(N-2)\mu}{2}\int_{B_R(0)}|\nabla u|^2\,dx + \frac{N\omega}{2}\int_{B_R(0)}|u|^2\,dx\\ &= \frac{Nd}{2\sigma+2}\int_{B_R(0)}|u|^{2\sigma+2}\,dx + I_R(u),\end{aligned}\tag{4.10.3}$$

where

$$\begin{aligned}I_R(u) = &\frac{R}{2}\int_{\partial B_R(0)}\left(\gamma|\Delta u|^2 + \mu|\nabla u|^2 + \omega|u|^2 - d\frac{|u|^{2\sigma+2}}{\sigma+1}\right)dS\\ &+\frac{1}{R}\int_{\partial B_R(0)}\Big(\gamma(\nabla(\Delta u)\cdot x)(x\cdot\nabla u) - \gamma(\nabla(x\cdot\nabla u)\cdot x)\Delta u - \mu|x\cdot\nabla u|^2\Big)\,dS.\end{aligned}$$

We now show that $I_{R_n}(u)\to 0$ for a suitable sequence $(R_n)_n\subset\mathbb{R}$ with $R_n\to\infty$ as $n\to\infty$. First, using the Cauchy-Schwarz's inequality, we have, for any $x\in\partial B_R(0)$,

$$\begin{aligned}&|(\nabla(\Delta u)\cdot x)(x\cdot u)| \le R^2\left(|\nabla(\Delta u)|^2 + |u|^2\right)\\ &|(\nabla(x\cdot\nabla u)\cdot x)\Delta u| \le C_N R^2\left(|\Delta u|^2 + \sum_{i,j=1}^N|u_{i,j}|^2 + |\nabla u|^2\right),\end{aligned}\tag{4.10.4}$$

where $u_{i,j} := \frac{\partial^2 u}{\partial x_i\partial x_j}$. In view of the elliptic regularity theory, we have that $u\in H^4(\mathbb{R}^N)$, in particular $u\in H^3(\mathbb{R}^N)$. This yields to

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla(\Delta u)|^2 + |\Delta u|^2 + \sum_{i,j=1}^N|u_{i,j}|^2 + |\nabla u|^2 + |u|^2 + |u|^{2\sigma+2}\,dx\\ &= \int_0^\infty\Big(\int_{\partial B_R(0)}|\nabla(\Delta u)|^2 + |\Delta u|^2 + \sum_{i,j=1}^N|u_{i,j}|^2 + |\nabla u|^2 + |u|^2 + |u|^{2\sigma+2}\,dS\Big)\,dR < \infty.\end{aligned}\tag{4.10.5}$$

As a consequence, there exists a sequence $(R_n)_n\subset\mathbb{R}^N$ satisfying $R_n\to\infty$ as $n\to\infty$ so that

$$R_n\int_{\partial B_{R_n}(0)}|\nabla(\Delta u)|^2 + |\Delta u|^2 + \sum_{i,j=1}^N|u_{i,j}|^2 + |\nabla u|^2 + |u|^2 + |u|^{2\sigma+2}\,dS\to 0.$$

This implies that $I_{R_n}(u)\to 0$ as $n\to\infty$. Now substituting R by R_n in (4.10.3), we then obtain that $P(u)=0$. This completes the proof. □

Proposition 4.10.2. *Let $N \geq 5$ and $\frac{2}{N-2} < \sigma < \frac{4}{N-4}$. Then any solution $u \in X$ to (4.6.8) belongs to $L^2(\mathbb{R}^N)$.*

Proof. We can assume without loss of generality that $\gamma = 1$. The main idea of the proof consists in testing (4.6.8) with a function $\varphi^2 u$ where, roughly, $\varphi(x) = 1 + |x|$.

Let $\psi \in C^\infty(\mathbb{R}^N)$ with *supp* $\psi \subset \mathbb{R}^N \backslash B_R(0)$ be such that $\psi(x) = 1$ for $|x| \geq 2R$. Here $R > 0$ is a constant to be determined later. For $R_1 > 2R$, we define $\varphi := \psi h_{R_1}$, where $h_{R_1} \in C^2(\mathbb{R}^N)$ satisfies

$$h_{R_1}(x) = \begin{cases} |x| & 2R \leq |x| < R_1, \\ R_1 \left(1 + th\left(\dfrac{|x| - R_1}{R_1}\right)\right), & |x| \geq R_1. \end{cases}$$

Let

$$\lambda_1(R_1) := \sup_{|x| \geq 2R} \frac{|x||\nabla\varphi(x)|}{\varphi(x)}, \quad \lambda_2(R_1) := \sup_{|x| \geq 2R} \frac{|x||\Delta\varphi(x)|}{\varphi(x)}. \tag{4.10.6}$$

From the definition of φ it readily follow that $\lambda_1(R_1) = 1$, for all $R_1 > 0$ and that $\lambda_2 := \lambda_2(R_1) \to 0$ as $R_1 \to \infty$.

As a preliminary step we derive some pointwise identities. By simple calculations

$$\Delta(\varphi^2 u) = \varphi^2 \Delta u + 4\varphi \nabla u \nabla\varphi + u(2\varphi\Delta\varphi + 2|\nabla\varphi|^2),$$

and

$$\begin{aligned}(\Delta(\varphi u))^2 = {} & \varphi^2(\Delta u)^2 + 4|\nabla\varphi\nabla u|^2 + u^2(\Delta\varphi)^2 + 4\varphi\Delta u\nabla\varphi\nabla u \\ & + 2\varphi u\Delta u\Delta\varphi + 4u\Delta\varphi\nabla\varphi\nabla u.\end{aligned}$$

Using the two previous lines, we obtain

$$\begin{aligned}(\Delta(\varphi u))^2 = {} & \Delta u\Delta(\varphi u^2) + 4|\nabla\varphi\nabla u|^2 + u^2(\Delta\varphi)^2 \\ & + 4\nabla\varphi\nabla u u\Delta\varphi - 2u\Delta u|\nabla\varphi|^2.\end{aligned} \tag{4.10.7}$$

We also need that

$$|\nabla(\varphi u)|^2 = \nabla u \nabla(\varphi^2 u) + |\nabla\varphi|^2 u^2. \tag{4.10.8}$$

Now testing (4.6.8) with $\varphi^2 u$ and using (4.10.7)-(4.10.8), there holds

$$\begin{aligned}\int_{\mathbb{R}^N} |\Delta(\varphi u)|^2 + |\nabla(\varphi u)|^2 \, dx = {} & \int_{\mathbb{R}^N} |\varphi u|^2 |u|^{2\sigma} + \int_{\mathbb{R}^N} |\nabla\varphi|^2 |u|^2 \, dx \\ & + 4\int_{\mathbb{R}^N} |\nabla\varphi\nabla u|^2 \, dx + \int_{\mathbb{R}^N} |u\Delta\varphi|^2 \, dx \\ & + 4\int_{\mathbb{R}^N} u\Delta\varphi\nabla\varphi\nabla u \, dx - 2\int_{\mathbb{R}^N} u\Delta u|\nabla\varphi|^2 \, dx.\end{aligned} \tag{4.10.9}$$

Recalling Hölder inequality and taking into account (4.2.2), we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |\varphi u|^2 |u|^{2\sigma} \, dx &\leq \left(\int_{|x| \geq R} |u|^{2\sigma+2} \, dx\right)^{\frac{\sigma}{\sigma+1}} \left(\int_{\mathbb{R}^N} |\varphi u|^{2\sigma+2} \, dx\right)^{\frac{1}{\sigma+1}} \\ &\leq C \left(\int_{|x| \geq R} |u|^{2\sigma+2}\right)^{\frac{\sigma}{\sigma+1}} \int_{\mathbb{R}^N} |\Delta(\varphi u)|^2 + |\nabla(\varphi u)|^2 \, dx.\end{aligned}$$

Setting $\delta(R) := C\left(\int_{|x|\geq R}|u|^{2\sigma+2}\right)^{\frac{\sigma}{\sigma+1}}$ where we note that $\delta(R)\to 0$ as $R\to\infty$, it then follows from (4.10.9) that,

$$\begin{aligned}(1-\delta(R))\int_{\mathbb{R}^N}|\Delta(\varphi u)|^2+|\nabla(\varphi u)|^2\,dx &\leq \int_{\mathbb{R}^N}|\nabla\varphi|^2|u|^2\,dx+4\int_{\mathbb{R}^N}|\nabla\varphi\nabla u|^2\,dx\\ &\quad+\int_{\mathbb{R}^N}|u\Delta\varphi|^2\,dx+4\int_{\mathbb{R}^N}u\Delta\varphi\nabla\varphi\nabla u\,dx\\ &\quad-2\int_{\mathbb{R}^N}u\Delta u|\nabla\varphi|^2\,dx=:\sum_{i=1}^{5}I_i.\end{aligned}\tag{4.10.10}$$

From now on, we estimate I_i for $1\leq i\leq 5$. In view of (4.10.6), then

$$\begin{aligned}I_1=\int_{\mathbb{R}^N}|\nabla\varphi|^2|u|^2\,dx&=\int_{|x|<2R}|\nabla\varphi|^2|u|^2\,dx+\int_{|x|\geq 2R}|\nabla\varphi|^2|u|^2\,dx\\ &\leq C\int_{|x|<2R}|u|^2\,dx+\int_{|x|\geq 2R}\frac{|\varphi u|^2}{|x|^2}\,dx.\end{aligned}$$

Noting that $\nabla\varphi\nabla(\varphi u)=|\nabla\varphi|^2u+(\nabla\varphi\nabla u)\varphi$, it follows for $|x|\geq 2R$, that

$$|\nabla\varphi\nabla u|\leq\frac{|\nabla\varphi\nabla(\varphi u)|}{\varphi}+\frac{|\nabla\varphi|^2}{\varphi^2}|\varphi u|.$$

Combines this inequality and the Young's inequality, we obtain for any $\epsilon>0$,

$$\begin{aligned}\frac{I_2}{4}=\int_{\mathbb{R}^N}|\nabla\varphi\nabla u|^2\,dx&\leq\int_{|x|<2R}|\nabla\varphi\nabla u|^2\,dx+\int_{|x|\geq 2R}|\nabla\varphi\nabla u|^2\,dx\\ &\leq C\int_{|x|<2R}|\nabla u|^2\,dx+\int_{|x|\geq 2R}(1+\epsilon)\frac{|\nabla\varphi\nabla(\varphi u)|^2}{|\varphi|^2}+\left(1+\frac{1}{\epsilon}\right)\frac{|\nabla\varphi|^4}{|\varphi|^4}|\varphi u|^2\,dx\\ &\leq C\int_{|x|<2R}|\nabla u|^2\,dx+(1+\epsilon)\int_{|x|\geq 2R}\frac{|\nabla(\varphi u)|^2}{|x|^2}+\left(1+\frac{1}{\epsilon}\right)\int_{\mathbb{R}^N}\frac{|\varphi u|^2}{|x|^4}\,dx.\end{aligned}$$

Also using (4.10.6), we have

$$\begin{aligned}I_3=\int_{\mathbb{R}^N}|u\Delta\varphi|^2\,dx&\leq\int_{|x|<2R}|u|^2|\Delta\varphi|^2\,dx+\int_{|x|\geq 2R}|u|^2|\Delta\varphi|^2\,dx\\ &\leq C\int_{|x|<2R}|u|^2\,dx+\lambda_2^2\int_{|x|\geq 2R}\frac{|\varphi u|^2}{|x|^2}\,dx.\end{aligned}$$

Next we deal with I_4. Using the Young's inequality for $\epsilon>0$ again, leads to

$$\begin{aligned}I_4=4\int_{\mathbb{R}^N}u\Delta\varphi\nabla\varphi\nabla u\,dx&\leq 4\int_{|x|\geq 2R}|u\Delta\varphi||\nabla\varphi\nabla u|\,dx\\ &\leq 2\epsilon\int_{|x|\geq 2R}|\nabla\varphi\nabla u|^2\,dx+\frac{2}{\epsilon}\int_{|x|\geq 2R}|u\Delta\varphi|^2\,dx=\frac{\epsilon I_2}{2}+\frac{2I_3}{\epsilon}\\ &\leq C_\epsilon\int_{|x|<2R}|\nabla u|^2+|u|^2\,dx+2(\epsilon+\epsilon^2)\int_{|x|\geq 2R}\frac{|\nabla(\varphi u)|^2}{|x|^2}\,dx\\ &\quad+2\,(1+\epsilon)\int_{|x|\geq 2R}\frac{|\varphi u|^2}{|x|^4}\,dx+\frac{2\lambda_2^2}{\epsilon}\int_{|x|\geq 2R}\frac{|\varphi u|^2}{|x|^2}\,dx.\end{aligned}$$

Finally we estimate I_5. We have for $|x|\geq 2R$,

$$\Delta u=\frac{\Delta(u\varphi)}{\varphi}-2\frac{\nabla u\nabla\varphi}{\varphi}+\frac{u\Delta\varphi}{\varphi}.$$

This implies that

$$\begin{aligned}\frac{I_5}{2} &\le \int_{|x|<2R} |u\Delta u||\nabla\varphi|^2\,dx + \int_{|x|\ge 2R} |u\Delta u||\nabla\varphi|^2\,dx\\ &\le C\int_{|x|<2R} |u\Delta u|\,dx + \int_{|x|\ge 2R} |\Delta(u\varphi)|\frac{|u||\nabla\varphi|^2}{|\varphi|}\,dx\\ &+ 2\int_{|x|\ge 2R} |\nabla u\nabla\varphi|\frac{|u||\nabla\varphi|^2}{|\varphi|}\,dx + \int_{|x|\ge 2R} |u|^2\frac{|\nabla\varphi|^2|\Delta\varphi|}{|\varphi|}\,dx\\ &:= C\int_{|x|<2R} |u\Delta u|\,dx + \sum_{i=1}^{3} J_i.\end{aligned}$$

We now treat J_i for $i = 1, 2, 3$. By the Young's inequality for $\tau > 0$,

$$\begin{aligned}J_1 &\le \frac{\tau}{2}\int_{|x|\ge 2R} |\Delta(u\varphi)|^2\,dx + \frac{1}{2\tau}\int_{|x|\ge 2R} |u|^2\frac{|\nabla\varphi|^4}{|\varphi|^2}\,dx\\ &\le \frac{\tau}{2}\int_{|x|\ge 2R} |\Delta(u\varphi)|^2\,dx + \frac{1}{2\tau}\int_{|x|\ge 2R} \frac{|\varphi u|^2}{|x|^4}\,dx.\end{aligned}$$

We also get

$$\begin{aligned}J_2 &\le \int_{|x|\ge 2R} |\nabla u\nabla\varphi|^2\,dx + \int_{|x|\ge 2R} \frac{|u|^2|\nabla\varphi|^4}{|\varphi|^2}\,dx \le \frac{I_2}{4} + \int_{|x|\ge 2R} \frac{|\varphi u|^2}{|x|^4}\,dx\\ &\le C\int_{|x|<2R} |\nabla u|^2\,dx + \left(2+\frac{1}{\epsilon}\right)\int_{|x|\ge 2R} \frac{|\varphi u|^2}{|x|^4}\,dx\\ &+ (1+\epsilon)\int_{|x|\ge 2R} \frac{|\nabla(\varphi u)|^2}{|x|^2}\,dx\end{aligned}$$

and

$$J_3 = \int_{|x|\ge 2R} |u|^2\frac{|\Delta\varphi||\nabla\varphi|^2}{|\varphi|}\,dx \le \lambda_2\int_{|x|\ge 2R} \frac{|\varphi u|^2}{|x|^3}\,dx.$$

Thus combining the estimates to J_i for $i = 1, 2, 3$, we obtain

$$\begin{aligned}I_5 &\le C\int_{|x|<2R} |u|^2 + |\nabla u|^2\,dx + (2+2\epsilon)\int_{|x|\ge 2R} \frac{|\nabla(\varphi u)|^2}{|x|^2}\,dx\\ &+ 2\lambda_2\int_{|x|\ge 2R} \frac{|\varphi u|^2}{|x|^3}\,dx + \left(4+\frac{1}{\tau}+\frac{2}{\epsilon}\right)\int_{|x|\ge 2R} \frac{|\varphi u|^2}{|x|^4}\,dx\\ &+ \tau\int_{|x|\ge 2R} |\Delta(u\varphi)|^2\,dx.\end{aligned}$$

Now taking into account above estimates to I_i for $1 \le i \le 5$, there holds

$$\begin{aligned}&\sum_{i=1}^{5} I_i \le C(R) + \left(6+8\epsilon+2\epsilon^2\right)\int_{|x|\ge 2R} \frac{|\nabla(\varphi u)|^2}{|x|^2}\,dx + 2\lambda_2\int_{|x|\ge 2R} \frac{|\varphi u|^2}{|x|^3}\,dx\\ &+ \left(10+\frac{1}{\tau}+\frac{6}{\epsilon}+2\epsilon\right)\int_{|x|\ge 2R} \frac{|\varphi u|^2}{|x|^4}\,dx + \left(1+\lambda_2^2+\frac{2\lambda_2^2}{\epsilon}\right)\int_{|x|\ge 2R} \frac{|\varphi u|^2}{|x|^2}\,dx\\ &+ \tau\int_{|x|\ge 2R} |\Delta(u\varphi)|^2\,dx.\end{aligned}$$

Recalling the Hardy's inequalities

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla v|^2\,dx &\geq \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|v|^2}{|x|^2}\,dx,\\ \int_{\mathbb{R}^N} |\Delta v|^2\,dx &\geq \frac{N^2}{4}\int_{\mathbb{R}^N} \frac{|\nabla v|^2}{|x|^2}\,dx,\end{aligned} \tag{4.10.11}$$

from (4.10.10) and (4.10.11), we arrive at

$$\begin{aligned}(1-\delta(R))\int_{\mathbb{R}^N} |\Delta(\varphi u)|^2 + |\nabla(\varphi u)|^2\,dx &\leq C(R) + \left(\frac{4}{N^2}\left(6+8\epsilon+2\epsilon^2\right)+\tau\right)\int_{\mathbb{R}^N}|\Delta(\varphi u)|^2\,dx\\ &+\left(\frac{2}{N-2}\right)^2\left(\frac{1}{4R^2}\left(10+\frac{1}{\tau}+\frac{6}{\epsilon}+2\epsilon\right)+\left(1+\lambda_2^2+\frac{2\lambda_2^2}{\epsilon}\right)+\frac{\lambda_2}{R}\right)\int_{\mathbb{R}^N}|\nabla(\varphi u)|^2\,dx.\end{aligned} \tag{4.10.12}$$

Since $N \geq 5$, taking $\epsilon, \tau > 0$ small enough, $R > 0$ large enough and recording that $\delta(R) \to 0$ as $R \to \infty$, we can insure that

$$\left(\frac{4}{N^2}\left(6+8\epsilon+2\epsilon^2\right)+\tau\right) < 1-\delta,$$

and

$$\left(\frac{2}{N-2}\right)^2\left(\frac{1}{4R^2}\left(10+\frac{1}{\tau}+\frac{6}{\epsilon}+2\epsilon\right)+\left(1+\lambda_2^2+\frac{2\lambda_2^2}{\epsilon}\right)+\frac{\lambda_2}{R}\right) < 1-\delta.$$

Thereby there exists a constant $C > 0$ just depending on $R > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla(\varphi u)|^2\,dx \leq C.$$

It follow from (4.10.11) that

$$\int_{|x|\geq 2R} \frac{|\varphi u|^2}{|x|^2}\,dx \leq C$$

uniformly with respect to R_1. Finally, letting $R_1 \to \infty$, we observe that

$$\frac{|\varphi u|^2}{|x|^2} \to u^2 \; quad \text{a.e} \quad \text{for} \quad |x| \geq 2R$$

and using the Fatou's Lemma, it follows that $u \in L^2(\mathbb{R}^N \backslash B_{2R}(0))$. Thus obviously $u \in L^2(\mathbb{R}^N)$. □

Chapter 5

Remarks and Perspectives

To begin with, we shall present some remarks related to the problems treated in the thesis.

5.1 Remarks

In Chapter 2, we consider the existence and orbital stability of normalized solutions in a case where the energy functional J restricted to $S(a_1, a_2)$ is bounded from below. The main goal in this chapter consists in detecting the compactness of any minimizing sequence to (2.1.4), up to translation. To this aim, borrowing the spirit from the Lions' concentration compactness principle, one requires to exclude the possiblities of vanishing and dichotomy. Recall that the energy functional J is invariant under translations in $\mathbb{R}^N$, thus vanishing can be avoided as a simple consequence of the Lions' concentration compactness Lemma. Next to see the compactness, it remains to rule out dichotomy. In general, this can be done by checking the strict subadditivity inequality (2.1.5). However, we alternatively propose the coupled rearrangement arguments to remove dichotomy. More precisely, we crucially make use of the coupled rearrangement arguments to guarantee the strong convergence of any minimizing sequence in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$, up to translation, for $2 < p < 2^*$.

A natural question is that whether we are able to prevent dichotomy from happening by means of directly establishing the strict subadditivity inequality (2.1.5). At this moment, the answer is positive. In fact, in order to establish the strict subadditivity inequality (2.1.5), one can adopt the approach as introduced in [51, Proposition 4], which is based on [51, Lemma 2]. However, we remark that this lemma is applicable to establish related strict subadditivity inequality provided one can identify a radially symmtric minimizing sequence to corresponding minimization problem. From this point of view, the coupled rearrangement arguments are more flexible to deal with the compactness of any minimizing sequence, regarding this subject, we refer the readers to [57].

Furthermore, let us also point out a method to discuss the compactness of any minimizing sequence as proposed by Lopes [69], which is also alternative to the Lions' concentration compactness principle and does not need the verification of related strict subadditivity inequality. But this method is available under a stronger requirement that associated energy functional is of class C^2.

In contrast, Chapter 3 is devoted to looking for normalized solutions in another two cases where the energy functional J is unbounded from below on $S(a_1, a_2)$. Despite we manage to relax limitation on dimension inducing by the Liouville's type results, but we still fail to find two solutions to (3.1.2)-(3.1.3) under only assuming (H_1) or (H_2). This is because so far we are unable to prove the conjecture that if two nonnegative functions $u_1, u_2 \in H^1(\mathbb{R}^N)$ solve (3.1.2) with some $\lambda_1, \lambda_2 \in \mathbb{R}$ satisfying $\lambda_i \geq 0$ for some $i = 1, 2$, then $u_i = 0$.

In Chapter 4, we focus on the study of normalized solutions to a class of fourth-order nonlinear Schrödinger equations in the mass critical and supercritical regime, in which the energy functional E is unbounded from below on $S(c)$ for $c > c_0$, where c_0 is defined by (4.1.10). Using a natural constraint approach, we then introduce the minimization problem (4.1.9). In order to seek for ground state solutions to (4.1.2)-(4.1.3), our aim is to prove the existence of minimizers to (4.1.9). To this end, one of key steps is to show that the weak limit of a Palais-Smale sequence for the energy funcional E restricted to $S(c)$ stays in $S(c)$. This essentially relies on the fact that the associated Lagrange multiplier α_c is strictly positive. Actually, from Lemma 4.4.1 and Lemma 4.6.3, we know that $\alpha_c \geq 0$ is always the case for any $c > c_0$. Hence it is open that if minimizers to (4.1.9) exist when $\alpha_c = 0$ and $N \geq 5$.

Additionally, as we know that the Lions' concentration compactness principle is a useful means to handle various minimization problems under constraint, then we question whether it is possible to adapt directly the spirit of the Lions' concentration compactness principle to solve the minimization problem (4.1.9).

In Theorem 4.1.7, we obtain a concentration behavior of ground state solutions to (4.1.2)-(4.1.3) as c approaches to c_N^* from above in the mass critical case $\sigma N = 4$. Since the uniqueness of least energy solution to (4.1.13) is unknown, hence we cannot describe precisely the ground state solutions. At this point, a challenging question is that whether the uniqueness of least energy solution to (4.1.13) holds.

When $\sigma \in \mathbb{N}^+$, using the Fourier rearrangement technique we can prove that at least one of ground state solutions to (4.1.2)-(4.1.3) is radial, see Theorem 4.1.8. However, when $4 \leq \sigma N < 4^*$, radial symmetry of the ground state solutions is still open.

Finally, let us mention an issue concerning the orbital instability by blowup in finite time of radial ground state solutions to (4.1.2)-(4.1.3), see Theorem 4.1.12. As we have already seen, this result is valid under the restriction that $\sigma \leq 4$. This is because its proof strongly depends on an essential element coming from Boulenger and Lenzmann [30], which is only applicable when $\sigma \leq 4$. Thereby we would like to know if Theorem 4.1.12 remains true when $\sigma > 4$.

5.2 Perspectives

In the following, as a possible extension of this thesis we put forward some interesting issues to be exploited in forthcoming works.

5.2.1 Fractional minimization problem

We consider the existence of solutions to the following fractional nonlinear Schrödinger system in $\mathbb{R}^N$,

$$\begin{cases} (-\Delta)^s u_1 = \lambda_1 u_1 + \mu_1 |u_1|^{p_1-2}u_1 + \beta r_1 |u_1|^{r_1-2}u_1 |u_2|^{r_2}, \\ (-\Delta)^s u_2 = \lambda_2 u_2 + \mu_2 |u_2|^{p_2-2}u_2 + \beta r_2 |u_1|^{r_1} |u_2|^{r_2-2}u_2, \end{cases} \tag{5.2.1}$$

under the constraint

$$\int_{\mathbb{R}^N} |u_1|^2 \, dx = a_1 > 0, \quad \int_{\mathbb{R}^N} |u_2|^2 \, dx = a_2 > 0, \tag{5.2.2}$$

where $0 < s < 1, \mu_1, \mu_2, \beta > 0$, $2 < p_1, p_2, r_1 + r_2 < \frac{2N}{(N-2s)^+}$.

We denote by $H^s(\mathbb{R}^N)$ the fractional Sobolev space of order s with the norm

$$\|u\|_{H^s}^2 := \|u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} u\|_2^2,$$

where up to a multiplicative constant

$$\|(-\Delta)^{\frac{s}{2}} u\|_2^2 := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dxdy.$$

Clearly, a solution (u_1, u_2) to (5.2.1)-(5.2.2) corresponds to a critical points of energy functional $\tilde{J} : H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\tilde{J}(u_1, u_2) := \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_1|^2 + |(-\Delta)^{\frac{s}{2}} u_2|^2 \, dx - \sum_{i=1}^{2} \frac{\mu_i}{p_i} \int_{\mathbb{R}^N} |u_i|^{p_i} \, dx - \beta \int_{\mathbb{R}^N} |u_1|^{r_1} |u_2|^{r_2} \, dx,$$

on the constraint $\tilde{S}(a_1, a_2) := \tilde{S}(a_1) \times \tilde{S}(a_2)$, where

$$\tilde{S}(a) := \{u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 \, dx = a > 0\},$$

and the parameters λ_1, λ_2 are determined as Lagrange multipliers.

We are concerned with the existence of solutions to (5.2.1)-(5.2.2) under the assumption

$$(\tilde{H}_0) \quad N \geq 1, 0 < s < 1, \mu_1, \mu_2, \beta > 0, 2 < p_1, p_2 < \frac{4s}{N}, r_1, r_2 > 1, r_1 + r_2 < \frac{4s}{N}.$$

Observe that under the assumption $(\tilde{H}_0)$ the energy functional $\tilde{J}$ is bounded from below on $\tilde{S}(a_1, a_2)$. We then define the following minimization problem

$$\tilde{M}(a_1, a_2) := \inf_{(u_1,u_2) \in \tilde{S}(a_1,a_2)} \tilde{J}(u_1, u_2) < 0. \tag{5.2.3}$$

Indeed, minimizers to (5.2.7) are solutions to (5.2.1)-(5.2.2). Our aim is to prove that when $(\tilde{H}_0)$ holds, any minimizing sequence to (5.2.3) is compact, up to translation, in $H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$.

In this direction, we mention a related paper [25], where the author took advantage of the Lions' concentration compactness principle to obtain the compactness of any minimizing sequence where scaling technique is available. However, under the assumption $(\tilde{H}_0)$, it seems hard to establish the compactness of any minimizing sequence to (5.2.3) through the Lions' concentration compactness principle. For this reason, we employ the coupled rearrangement spirit. Thus the heuristic ingredient consists in showing assertion that

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \{u, v\}^\star|^2 \, dx < \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 \, dx. \tag{5.2.4}$$

5.2.2 Fourth-order minimization problem

We study the existence of solutions to the following fourth-order nonlinear Schrödinger system in $\mathbb{R}^N$,

$$\begin{cases} \Delta^2 u_1 = \lambda_1 u_1 + \mu_1 |u_1|^{p_1-2} u_1 + \beta r_1 |u_1|^{r_1-2} u_1 |u_2|^{r_2}, \\ \Delta^2 u_2 = \lambda_2 u_2 + \mu_2 |u_2|^{p_2-2} u_2 + \beta r_2 |u_1|^{r_1} |u_2|^{r_2-2} u_2. \end{cases} \tag{5.2.5}$$

under the constraint

$$\int_{\mathbb{R}^N} |u_1|^2 \, dx = a_1 > 0, \quad \int_{\mathbb{R}^N} |u_2|^2 \, dx = a_2 > 0, \tag{5.2.6}$$

where $\mu_1, \mu_2, \beta > 0$, $2 < p_1, p_2, r_1 + r_2 < \frac{2N}{(N-4)^+}$.

Apparently, a solution (u_1, u_2) to (5.2.1)-(5.2.2) is obtained as a critical point of energy functional $\hat{J} : H^2(\mathbb{R}^N) \times H^2(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\hat{J}(u_1, u_2) := \frac{1}{2} \int_{\mathbb{R}^N} |\Delta u_1|^2 + |\Delta u_2|^2 \, dx - \sum_{i=1}^{2} \frac{\mu_i}{p_i} \int_{\mathbb{R}^N} |u_i|^{p_i} \, dx - \beta \int_{\mathbb{R}^N} |u_1|^{r_1} |u_2|^{r_2} \, dx,$$

on the constraint $\hat{S}(a_1, a_2) := \hat{S}(a_1) \times \hat{S}(a_2)$, where

$$\hat{S}(a) := \{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 \, dx = a > 0\},$$

and the parameters λ_1, λ_2 are determined as Lagrange multipliers.

We are interested in the existence of solutions to (5.2.5)-(5.2.6) under the assumption

$$(\hat{H}_0) \quad N \geq 1, \mu_1, \mu_2, \beta > 0, 2 < p_1, p_2 < \frac{8}{N}, r_1, r_2 > 1, r_1 + r_2 < \frac{8}{N}.$$

On account of the fact that the energy functional $\hat{J}$ is bounded from below on $\hat{S}(a_1, a_2)$, we then introduce the following minimization problem

$$\hat{M}(a_1, a_2) := \inf_{(u_1, u_2) \in \hat{S}(a_1, a_2)} \hat{J}(u_1, u_2) < 0. \tag{5.2.7}$$

Indeed, any minimizer to (5.2.7) is a solution to (5.2.5)-(5.2.6). Our purpose is to detect the compactness of any minimizing sequence to (5.2.7), up to translation, in $H^2(\mathbb{R}^N) \times H^2(\mathbb{R}^N)$ under the assumption $(\hat{H}_0)$. Although (5.2.5) can be viewed as a replacement of $-\Delta$ by Δ^2 in (2.1.1), which however brings new challenges to discuss the compactness of any minimizing sequence to (5.2.7).

Bibliography

[1] N. Ackermann, T. Weth, *Existence and orbital instability of normalized multibump standing waves for nonlinear Schrödinger equations,* arXiv 1706.06950

[2] N. Akhmediev, A. Ankiewicz, *Partially coherent solitons on a finite background,* Phys. Rev. Lett. (1999) (82) 2661.

[3] J. Albert, S. Bhattarai, *Existence and stability of a two-parameter family of solitary waves for an NLS-KdV system*, Adv. Differential Equations 18 (2013) 1129-1164.

[4] A. Ambrosetti, A. Malchiodi, *Nonlinear analysis and semilinear elliptic problems,* Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2007.

[5] A. Ambrosetti, E. Colorado, *Standing waves of some coupled nonlinear Schrödinger equations,* J. London Math. Soc. 75 (2007) 67-82.

[6] A. Ambrosetti, E. Colorado, D. Ruiz *Multi-bump solitons to linearly coupled systems of nonlinear Schrödinger equations,* Calc. Var. Partial Differ. Equ. 30 (2007) 85-112

[7] T. Bartsch, S. De Valeriola, *Normalized solutions of nonlinear Schrödinger equations,* Arch. Math. 100 (2013) 75-83.

[8] T. Bartsch, L. Jeanjean, *Normalized solutions for nonlinear Schrödinger systems,* Proc. Roy. Soc. Edinburgh, to appear.

[9] G. Baruch, G. Fibich, *Singular solutions of the L^2-supercritical biharmonic nonlinear Schrödinger equation,* Nonlinearity 24 (6) (2011) 1843-1859.

[10] G. Baruch, G. Fibich, E. Mandelbaum, *Ring-type singular solutions of the biharmonic nonlinear Schrödinger equation,* Nonlinearity 23 (11) (2010) 2867-2887.

[11] G. Baruch, G. Fibich, E. Mandelbaum, *Singular solutions of the biharmonic nonlinear Schrödinger equation,* SIAM J. Appl. Math. 70 (8) (2010) 3319-3341.

[12] T. Bartsch, L. Jeanjean, N. Soave, *Normalized solutions for a system of coupled cubic Schrödinger equations on $\mathbb{R}^3$,* J. Math. Pures. Appl. 106 (4) (2016) 583-614.

[13] T. Bartsch, N. Soave, *A natural constraint approach to normalized solutions on nonlinear Schrödinger equations and systems,* J. Funct. Anal. 272 (12) (2017) 4998-5037.

[14] T. Bartsch, N. Soave, *Multiple normalized solutions for a competing system of Schrödinger equations,* arXiv 1703.02832.

[15] T. Bartsch, Z.-Q. Wang, J. Wei, *Bound states for a coupled Schrödinger system,* J. Fixed Point Theory Appl. 2 (2) (2007) 353-367.

[16] L. Battaglia, J. Van Schaftingen, *Existence of Groundstates for a Class of Nonlinear Choquard Equations in the Plane,* Adv. Nonlinear Stud. 17 (2017) 581-594.

[17] J. Bellazzini, L. Jeanjean, *On dipolar quantum gases in the unstable regime,* SIAM J. Math. Anal. 48 (3) (2016) 2028-2058.

[18] J. Bellazzini, L. Jeanjean, T. Luo, *Existence and instability of standing waves with prescribed norm for a class of Schrödinger-Poisson equations,* Proc. London Math. Soc. 107 (2013) 303-339.

[19] J. Bellazzini, G. Siciliano, *Stable standing waves for a class of nonlinear Schrödinger-Poisson equations,* Z. Angew. Math. Phys. 62 (2011) 267-280.

[20] M. Ben-Artzi, H. Koch, J.-C Saut, *Dispersion estimates for fourth order Schrödinger equations,* C. R. Acad. Sci. Paris Sér. I Math. 330 (2) (2000) 87-92.

[21] H. Berestycki, P.-L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state,* Arch. Rational Mech. Anal. 82 (4) (1983) 313-345.

[22] H. Berestycki, P.-L. Lions, *Nonlinear scalar field equations. II. Existence of a ground state,* Arch. Rational Mech. Anal. 82 (4) (1983) 347-375.

[23] S. Bhattarai, *Stability of solitary-wave solutions of coupled NLS equations with power-type nonlinearities,* Adv. Nonlinear Anal. 4 (2015) 73-90.

[24] S. Bhattarai, *Stability of normalized solitary waves for three coupled nonlinear Schrödinger equations,* Discrete Contin. Dyn. Syst- A, 36 (2016) 1789-1811.

[25] S. Bhattarai, *On fractional Schrödinger systems of Choquard type,* J. Differential Equations 263 (2017) 3197-3229.

[26] C. Bonanno, P. d'Avenia, M. Ghimenti, M. Squassina, *Soliton dynamics for the generalized Choquard equation,* J. Math. Anal. Appl. 417 (2014) 180-199.

[27] D. Bonheure, J.-B. Casteras, T. Gou, L. Jeanjean, *Strong instability of ground states to a fourth order Schrödinger equation,* arXiv 1703.07977.

[28] D. Bonheure, J.-B. Casteras, E. Moreira Dos Santos, R. Nascimento, *Orbitally stable standing waves of a mixed dispersion nonlinear Schrödinger equation,* submitted.

[29] D. Bonheure, E. Moreira dos Santos, M. Ramos, *Ground state and non ground state solutions of some strongly coupled elliptic systems,* Trans. Amer. math. Soc. 364 (1) (2012) 447-491.

[30] T. Boulenger, E. Lenzmann, *Blowup for Biharmonic NLS,* arXiv 1503.01741, to appear in Ann. Sci. ENS.

[31] D. Bonheure, R. Nascimento, *Waveguide solutions for a nonlinear Schrödinger equation with mixed dispersion*, Progr. Nonlinear Differential Equations Appl. 87 (2015) 31-53.

[32] J. Busca, B. Sirakov, *Symmetry results for semilinear elliptic systems in the whole space,* J. Differential Equations, 163 (1) (2000) 41-56.

[33] J. Byeon, *Effect of symmetry to the structure of positive solutions in nonlinear elliptic problems*, J. Differential Equations 163 (2000) 429-474.

[34] D. Cao, I-L. Chern, J. Wei, *On ground state of spinor Bose-Einstein condensates*, Nonlinear. Differ. Equ. Appl. 18 (2011) 427-445.

[35] T. Cazenave, *Semilinear Schrödinger equations,* Courant Lecture Notes in Mathematics, 10. New York University, Courant Institute of Mathematical Sciences, New York, American Mathematical Society, Providence, RI, 2003. xiv+323 pp.

[36] T. Cazenave, P.-L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys. 85 (1982) 549-561.

[37] Z. Chen, W. Zou, *On linearly coupled Schrödinger systems,* Proc. Amer. Math. Soc. 142 (2014) 323-333.

[38] Z. Chen, W. Zou, *Existence and symmetry of positive ground states for a doubly critical Schrödinger system*, Trans. Amer. Math. Soc. 367 (2015) 3599-3646.

[39] Z. Chen, C.-S. Lin, W. Zou, *Sign-changing solutions and phase separation for an elliptic system with critical exponent,* Comm. Partial Differential Equations 39 (2014) 1827-1859.

[40] M. Colin, L. Jeanjean, M. Squassina, *Stability and instability results for standing waves of quasi-linear Schrödinger equations*, Nonlinearity 23 (2010) 1353-1385.

[41] M. Colin, M. Ohta, *Bifurcation from semitrivial standing waves and ground states for a system of nonlinear Schrödinger equations,* SIAM J. Math. Anal. 44 (2012) 206-223.

[42] S. Correia, *Characterization of ground-states for a system of M coupled semilinear Schrödinger equations and applications,* J. Differential Equations 260 (2016) 3302-3326.

[43] S. Correia, *Ground-states for systems of M coupled semilinear Schrödinger equations with attraction-repulsion effects: characterization and perturbation results,* Nonlinear Anal. 140 (2016) 112-129.

[44] S. Correia, F. Oliveira, H. Tavares, *Semitrivial vs. fully nontrivial ground states in cooperative cubic Schrödinger systems with $d \geq 3$ equations,* J. Funct. Anal. 271 (2016) 2247-2273.

[45] P. d'Avenia, M. Squassina, *Soliton dynamics for the Schrödinger-Newton system,* Math. Models Methods Appl. Sci. 24 (2014) 553-572.

[46] D.G. de Figueiredo, O. Lopes, *Solitary waves for some nonlinear Schrödinger systems,* Ann. Inst. H. Poincaré Anal. Non Linéaire 25 (2008) 149-161.

[47] I. Ekeland, *On the variational principle,* J. Math. Anal. Appl. 47 (1974) 324-353.

[48] B. Esry, C. Greene, J. Burke Jr, J. Bohn, *Hartree-Fock theory for double condensates,* Phys. Rev. Lett. 78 (1997) 3594-3597.

[49] G. Fibich, B. Ilan, G. Papanicolaou, *Self-focusing with fourth-order dispersion,* SIAM J. Appl. Math. 62 (2002) 1437-1462.

[50] D. J. Frantzeskakis, *Dark solitons in atomic Bose-Einstein condensates from theory to experiments,* J. Phys. A, Math. Theor. 43 (2010).

[51] D. Garrisi, *On the orbital stability of standing-wave solutions to a coupled non-linear Klein-Gordon equation*, Adv. Nonlinear Stud. 12 (2012) 639-658.

[52] H. Genev, G. Venkov, *Soliton and blow-up solutions to the time-dependent Schrödinger-Hartree equation,* Discrete Contin. Dyn. Syst. (2012) 903-923.

[53] M. Ghimenti, V. Moroz, J. Van Schaftingen, *Least action nodal solutions for the quadratic Choquard equation,* Proc. Amer. Math. Soc. 145 (2017) 737-747.

[54] M. Ghimenti, J. Van Schaftingen, *Nodal solutions for the Choquard equation,* J. Funct. Anal. 271 (2016) 107-135.

[55] N. Ghoussoub, *Duality and perturbation methods in critical point theorey,* Cambridge University Press, (1993).

[56] J. Ginibre, G. Velo, *On a class of nonlinear Schrödinger equations with nonlocal interaction*, Math. Z. 170 (1980) 109-136.

[57] T. Gou, *Existence and orbital stability of standing waves to nonlinear Schrödinger system with partial confinement,* arXiv:1709.00217.

[58] T. Gou, L. Jeanjean, *Existence and orbital stability of standing waves for nonlinear Schrödinger systems,* Nonlinear Anal. 144 (2016) 10-22.

[59] D. Hall, R. Matthews, J. Ensher, C. Wieman, E. Cornell, *Dynamics of component separation in a binary mixture of Bose-Einstein condensates,* Phys. Rev. Lett. 81 (1998) 1539-1542.

[60] Q. He, S. Peng, *Synchronized vector solutions to an elliptic system,* Proc. Amer. Math. Soc. 144 (2016) 4055-4063.

[61] N. Ikoma, *Compactness of minimizing sequences in nonlinear Schrödinger systems under multiconstraint conditions*, Adv. Nonlinear Studies 14 (2014) 115-136.

[62] L. Jeanjean, M. Squassina, *An approach to minimization under a constraint, The added mass technique,* Calc. Var. Partial Differential Equations 41 (2011) 511-534.

[63] L. Jeanjean, *Existence of solutions with prescribed norm for semilinear elliptic equations,* Nonlinear Anal. 28 (1997) 1633-1659.

[64] V.I. Karpman, *Stabilization of soliton instabilities by higher-order dispersion fourth order nonlinear Schröinger-type equations,* Phys. Rev. E 53 (1996) 1336-1339.

[65] E. Laedke, K. Spatschek, *Stability properties of multidimensional finite-amplitude solitions,* Phys. Rev. A 30 (1984) 3279-3288.

[66] E. Laedke, K. Spatschek, L. Stenflo, *Evolution theorem for a class of perturbed envelope soliton solutions,* J. Math. Phys. 24 (12) (1983) 2764-2769.

[67] E. H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation,* Studies in Appl. Math. 57 (1976) 93-105.

[68] E.H. Lieb, M. Loss, *Analysis*, Second edition, Graduate studies in mathematics, vol. 14, American Mathematical Society, Providence, 2001.

[69] O. Lopes, *A constrained minimization problem with integrals on the entire space,* Bol. Soc. Brasil. Mat. (N.S.) 25 (1) (1994) 77-92.

[70] V.I. Karpman, A. Shagalov, *Stability of solitons described by nonlinear Schröinger-type equations with higher-order dispersion,* Phys. D 144 (2000) 194-210.

[71] T.-C. Lin, J. Wei, *Ground state of N coupled nonlinear Schrödinger equations in $\mathbb{R}^n, n \leq 3$,* Comm. Math. Phys. 255 (2005) 629-653.

[72] P.-L. Lions, *The Choquard equation and related questions,* Nonlinear Anal. 4 (1980) 1063-1072.

[73] P-L. Lions, *The concentration-compactness principle in the Calculus of Variations. The locally compact case, Part I*, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984) 109-145.

[74] P-L. Lions, *The concentration-compactness principle in the Calculus of Variations. The locally compact case, Part II*, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984) 223-283.

[75] C. Liu, N.V. Nguyen, Z-Q. Wang, *Existence and stability of solitary waves of an M-coupled nonlinear Schrödinger system*, J. Math. Study 49 (2016) 132-148.

[76] Z. Liu, Z.-Q. Wang, *Ground states and bound states of a nonlinear Schrd̈inger system,* Adv. Nonlinear Stud. 10 (1) (2010) 175-193.

[77] L. Ma, L. Zhao, *Uniqueness of ground states of some coupled nonlinear Schrödinger systems and their application,* J. Differential Equations 245 (2008) 2551-2565.

[78] L. Ma, L. Zhao, *Classification of positive solitary solutions of the nonlinear Choquard equation,* Arch. Ration. Mech. Anal. 195 (2010) 455-467.

[79] L.A. Maia, E. Montefusco, B. Pellacci, *Positive solutions for a weakly coupled nonlinear Schrödinger system,* J. Differential Equations 229(2) (2006) 743-767.

[80] B. Malomed, *Multi-component Bose-Einstein condensates, Theory. In P.G. Kevrekidis, D.J. Frantzeskakis, R. Carretero-Gonzalez (Eds.), Emergent Nonlinear Phenomena in BoseEinstein Condensation,* Springer-Verlag, Berlin, (2008) 287-305.

[81] R. Mandel, *Minimal energy solutions for cooperative nonlinear Schrödinger systems,* NoDEA Nonlinear Differential Equations Appl. 22 (2) (2015) 239-262.

[82] V. Moroz, J. Van Schaftingen, *Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics,* J. Funct. Anal. 265 (2013) 153-184.

[83] V. Moroz, J. Van Schaftingen, *A guide to the Choquard equation.,* J. Fixed Point Theory Appl. 19 (2017) 773-813.

[84] C. Myatt, E. Burt, R. Ghrist, E. Cornell, C. Wieman, *Production of two overlapping Bose-Einstein condensates by sympathetic cooling,* Phys. Rev. Lett. 78 (1997) 586-589.

[85] F. Natali, A. Pastor, *The fourth-order dispersive nonlinear Schrödinger equation, orbital stability of a standing wave,* SIAM J. Appl. Dyn. Syst. 14 (3) (2015) 1326-1347.

[86] B. Noris, H. Tavares, G. Verzini, *Existence and orbital stability of the ground states with prescribed mass for the L^2 critical and supercritical NLS on bounded domains*, Anal. PDE 7, no. 8, (2014), 1807-1838.

[87] B. Noris, H. Tavares, G. Verzini, *Stable solitary waves with prescribed L^2-mass for the cubic Schrödinger system with trapping potentials,* Discrete Contin. Dyn. Syst.-A 35 (2015) 6085-6112.

[88] N.V. Nguyen, R. Tian, B. Deconinck, N. Sheils, *Global existence of a coupled system of Schrödinger equations with power-type nonlinearities,* J. Math. Phys. 54 (2013) p. 011503.

[89] N.V. Nguyen, Z.-Q. Wang, *Orbital stability of solitary waves for a nonlinear Schröinger system*, Adv. Differential Equations 16 (2011) 977-1000.

[90] N.V. Nguyen, Z-Q. Wang, *Orbital stability of solitary waves of a 3-coupled nonlinear Schrödinger system*, Nonlinear Analysis 90 (2013) 1-26.

[91] N.V. Nguyen, Z-Q. Wang, *Existence and stability of a two-parameter family of solitary waves for a 2-couple nonlinear schrödinger system*, Discrete Contin. Dyn. Syst. 36 (2016) 1005-1021.

[92] L. Nirenberg, *On elliptic partial differential equations,* Ann. Scuola Norm. Sup. Pisa 13 (3) (1959) 115-162.

[93] M. Ohta, *Stability of solitary waves for coupled nonlinear Schrödinger equations*, Nonlinear Anal. Theory, Methods & Appl. 26 (1996) 933-939.

[94] D. Pierotti, G. Verzini, *Normalized bound states for the nonlinear Schrödinger equation in bounded domains,* arXiv 1607.04520.

[95] B. Pausader, *Global well-posedness for energy critical fourth-order Schrödinger equations in the radial case*, Dyn. Partial Differ. Equ. 4 (3) (2007) 197-225.

[96] B. Pausader, S. Shao, *The mass-critical fourth-order Schrödinger equation in high dimensions,* J. Hyperbolic Differ. Equ. 7 (4) (2000) 651-705.

[97] B. Pausader, S. Xia, *Scattering theory for the fourth-order Schrödinger equation in low dimensions,* Nonlinearity 26 (8) (2013) 2175-2191.

[98] P. Quittner, P. Souplet, *Symmetry of components for semilinear elliptic systems,* SIAM J. Math. Anal. 44 (4) (2012) 2545-2559.

[99] Y. Sato, Z.-Q. Wang, *Least energy solutions for nonlinear Schrödinger systems with mixed attractive and repulsive couplings,* Adv. Nonlinear Stud. 15 (1) (2015) 1-22.

[100] M. Shibata, *A new rearrangement inequality and its application for L^2-constraint minimizing problems*, Math. Z. (2016) doi 10.1007/s00209-016-1828-1.

[101] M. Shibata, *Stable standing waves of nonlinear Schrödinger equations with a general nonlinear term*, manuscripta math. 143 (2014) 221-237.

[102] B. Sirakov, *Least energy solitary waves for a system of nonlinear Schröodinger equations in $\mathbb{R}^n$,* Comm. Math. Phys. 271 (2007) 199-221.

[103] N. Soave, *On existence and phase separation of solitary waves for nonlinear Schrödinger systems modelling simultaneous cooperation and competition,* Calc. Var. Partial Differential Equations, 53 (3) (2015) 689-718.

[104] N. Soave, H. Tavares, *New existence and symmetry results for least energy positive solutions of Schrödinger systems with mixed competition and cooperation terms,* J. Differential Equations 261 (2016) 505-537.

[105] C.A. Stuart, *Bifurcation from the continuous spectrum in L^2-theory of elliptic equations on $\mathbb{R}^N$,* Recent Methods in Nonlinear Analysis and Applications, Liguori, Napoli, (1981).

[106] C.A. Stuart, *Bifurcation for Dirichlet problems without eigenvalues*, Proc. London Math. Soc. (3) 45 (1982) 169-192.

[107] H. Tavares, S. Terracini, G. Verzini, T. Weth, *Existence and nonexistence of entire solutions for non-cooperative cubic elliptic systems,* Comm. Partial Differential Equations 36 (2011) 1988-2010.

[108] S. Terracini, G. Verzini, *Multipulse phases in k-mixtures of Bose-Einstein condensates,* Arch. Ration. Mech. Anal. 194 (3) (2009) 717-741.

[109] E. Timmermans, *Phase separation of Bose-Einstein condensates,* Phys. Rev. Lett. 81 (1998) 5718-5721.

[110] J. Wei, W. Yao, *Uniqueness of positive solutions to some coupled nonlinear Schrödinger equations,* Commun. Pure Appl. Anal. 11 (3) (2012) 1003-1011.

[111] M.I. Weinstein: *Nonlinear Schrödinger Equations and Sharp Interpolation Estimates*, Commum. Math. Phys. 87 (1983) 67-576.

[112] J. Yang, *Classification of the solitary waves in coupled nonlinear Schrödinger equations,* Phys. D 108 (1997) 92-112.