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Stochastic control applied in the theory of decision in a discrete time non-dominated multiple-priors framework

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Contents

Table des matières

Introduction et résumé

Cette dissertation s'articule autour de trois thématiques importantes : la notion d'incertitude, la notion de fonction d'utilité et enfin le concept d'absence d'opportunité d'arbitrage. Dans cette introduction, nous nous proposons de faire un bref survol de ces sujets et d'une partie de la littérature mathématique correspondante. Nous nous efforcerons en particulier de donner un aperçu des outils mathématiques innovants qui ont été mis en place pour traiter les problèmes qui apparaissent lorsqu'il y a de l'incertitude sur les lois de probabilités des phénomènes étudiés. Nous précisons que le but n'est pas, à ce stade, de donner une présentation formelle et rigoureuse. Au contraire nous espérons que cette introduction pourra intéresser des lecteurs non-spécialistes. Enfin nous insistons sur le fait que les problèmes que nous allons aborder ne sont pas uniquement d'un interêt purement mathématique. Non seulement ils sont liés à des problèmes concrets qui apparaissent sur les marchés financiers (évaluation de produits dérivés, gestion des risques, régulations,..) mais ils interviennent aussi dans beaucoup d'autres disciplines telles que les sciences économiques, les politiques monétaires et budgétaires, la psychologie,...

Aléas, risque et incertitude

Quelques exemples

L'aléa est omniprésent dans notre vie quotidienne et apparait à travers de multiples phénomènes et dans de nombreuses disciplines. Au cours du vingtième siècle, les mathématiciens ont développé à travers la théorie des probabilités et de la statistique des outils très puissants pour étudier et comprendre ces situations. Ces outils ont été un élément essentiel de la compréhension et la modélisation de phénomènes complexes dans des domaines aussi variés que la mécanique quantique, de la génétique jusqu'au dévelopement récent du "big data". De façon similaire, le monde de la finance et plus particulièrement la finance quantitative, a suscité un grand nombre de recherches et ainsi également contribué au développement d'outils mathématiques adaptés et innovants et cela essentiellement depuis 1970. De façon réflexive ces outils ont eux-mêmes profondément modifié la façon dont les marchés financiers fonctionnent.
Des recherches récentes dans le domaine de la psychologie rendues accessibles à un grand public dans [87] par exemple ainsi que dans [125] pour une audience peut-être plus spécifique, ont radicalement changé la façon dont est modelisée la réaction d'un agent économique qui fait face à une situation incertaine. En quelques

mots, ces études montrent que notre esprit n'est pas toujours apte à appréhender ce qui est aléatoire. Même lorsque nous pensons que nous nous comportons de façon rationnelle, nous sommes en réalité souvent victimes de multiples biais conscients ou inconscients. D'une certaine façon notre cerveau n'est pas, d'un point de vue biologique, adapté pour comprendre et traiter les probabilités et la statistique : il suffit de voir par exemple comment notre intuition (et même celle d'esprits experts ou entrainés) est souvent fausse face à des situations impliquant des espérances conditionnelles et le Théorème de Bayes.
Citons un exemple extrait de [87] : *"Linda a 31 ans, est célibataire, ouverte et brillante. Elle a un diplôme en philosophie. Quand elle était étudiante, elle se sentait très concernée par les problèmes de discrimination et d'injustice sociale et elle a participé à des manifestations anti-nucléaires." Classez les deux asserstions suivantes en fonction de la probabilité de la situation qu'elles décrivent a) Linda est une employée de banque , b) Linda est une employée de banque qui est active dans des mouvements féministes.* De façon surprenante, des expériences ont montré que la réponse b) est souvent vue comme plus probable que a) bien qu'elle décrive une situation moins générale. Il y a bien sûr beaucoup d'explications à cela : la formulation de la question n'est pas étrangère aux réponses obtenues mais cela reste profondément surprenant et montre à quel point notre esprit peut nous jouer des tours. Pour plus d'exemples tout aussi surprenants et révélateurs de certaines de nos faiblesses, nous invitons le lecteur curieux à se plonger dans la lecture de [87].
Il est bien sûr en dehors du cadre de cette dissertation de faire une liste exhaustive de toutes ces problématiques. Toutefois dans les lignes qui suivent, nous voudrions mettre l'accent sur quelques exemples typiques et montrer aussi comment ils sont en relation avec les problèmes mathématiques que nous traiterons par la suite.
Pour commencer nous nous intéressons au concept d'aversion au risque qui est tout particulièrement important lorsque l'on modélise le comportement d'un agent économique. Nous verrons par ailleurs qu'il est lié au concept d'utilité que nous introduirons plus loin et tout particulièrement à la concavité des fonctions d'utilité.
Historiquement, on a souvent considéré un agent économique "rationnel" comme étant averse au risque : un agent préfère en général une situation dont le résultat est connu à une situation au résultat inconnu même si son espérance de gain dans la seconde est un peu supérieure à celle de la première. Cependant si un agent économique préfèrera recevoir 50 plutôt que de jouer à pile ou face et gagner 100 (si pile) ou 0 (si face) , son comportement n'est pas forcement le même si l'on parle de pertes potentielles. Ainsi un agent économique préfère en général prendre le risque de jouer à un jeu ou il peut perdre 100 (si pile) ou 0 (si face) (encore avec une chance sur deux) plutôt que de perdre de façon certaine 50 : la possibilité de ne rien perdre l'incite à prendre le risque alors que dans le cas précédent c'était l'éventualité de ne rien gagner qui le poussait à ne pas jouer. En terme de gestion des risques c'est bien évidemment une attitude problématique (car contraire à une

gestion raisonnable dans laquelle on essaie de limiter les pertes et on ne prend des risques que lorsque l'on peut se le permettre) et qui doit être prise en compte par exemple lorsque l'on considère la régulation des établissement financiers. D'un point de vue mathématique c'est l'une des motivations du Chapitre 2 ou nous étudierons des fonctions d'utilité qui ne sont pas concaves.
Un second exemple concerne la notion de distorsion de la loi de probabilité. Il y a dans la littérature économique et mathématique une longue historique de débats sur la nature des lois de probabilités. En résumant et simplifiant, est-ce qu'une loi de probabilité est un élément purement objectif ou bien est-elle subjective, c'est à dire dépendante de la personne qui prend la décision ? Nous verrons comment cette distinction entre des lois de probabilités objectives et subjectives intervient dans la contexte d'espérance d'utilité. Toutefois la notion de distorsion de probabilité va plus loin : non seulement l'agent économique utilise sa propre loi de probabilité subjective pour appréhender une situation incertaine mais en plus il modifie mentalement (consciemment ou pas) cette loi : en général la probabilité des évènements rares est sur-estimée et celle des évènements fréquents sous-estimée. Cette notion a été introduite par exemple dans [88] et on peut trouver des exemples dans [87]. D'un point de vue mathématique, pour modéliser ce comportement, on introduit une fonction croissante $f : [0,1] \to [0,1]$ telle que $f(0) = 0$ and $f(1) = 1$ et on remplace la probabilité $P(A)$d'un évenement A par $c(A) = f(P(A))$. Il est important de noter que par cette transformation , il n'y a aucune raison que c demeure une loi de probabilité. On parle alors de capacité et ces problématiques sont liées également à la notion d'intégrale de Choquet et (voir [41]) d'espérance non linéaire que nous présentons brièvement plus bas.

Risque vs incertitude, le risque de modèle et le dévelopement d'outils mathématiques adaptés

Dans cette section, nous abordons une distinction fondamentale au sujet de l'aléatoire et qui est au coeur de notre étude. Il s'agit de la distinction entre le risque et l'incertitude. Le risque est "l'inconnu connu" alors que l'incertitude représente "l'inconnu inconnu". On parle d'incertitude knigthienne, en référence à F. Knight à qui on attribue la parenté de ce concept (voir [90]). Pour illustrer ce concept prenons l'exemple suivant inspiré du paradoxe d'Ellsberg (nous le reprendrons par la suite). On vous propose de jouer à un jeu dans lequel vous devez choisir entre recevoir 20 ou bien tirer une boule dans une urne qui contient 100 boules. Si vous tirez une boule rouge vous recevrez 100 et sinon 0. Cependant, vous ne savez pas quelle est la quantité exacte de boules rouges présente dans l'urne. On vous dit simplement qu'il y a entre 20 et 80 boules rouges. Si on vous avait indiqué la quantité précise de boules rouges, alors c'est votre aversion au risque qui aurait déterminé

votre choix. Si il y a 20 boules rouges, la plupart d'entre nous choisirons de ne pas tirer de boule et de recevoir 20. Si par contre il y a 50 ou 80 boules rouges alors un plus grand nombre décidera de tirer une boule dans l'urne (et donc de prendre le risque de ne rien recevoir). Mais le jeu présenté est différent : il ne s'agit pas seulement d'aversion au risque. Comment peut on calculer l'espérance de gain de ce jeu ? Doit-on prendre le cas le plus défavorable (20 boules), le cas moyen (50 boules) ou bien le cas le plus favorable (80 boules) ? En général, face à ce genre de situation, un agent économique fait preuve d'une aversion à l'incertitude au sens oú il préfère une situation oú l'incertitude est réduite. Par exemple, si dans le cas oú il y a exactement 50 boules rouges, un agent économique était content de tirer une boule, ce ne sera pas forcement le cas dans la situation avec incertitude sur le nombre de boules rouges même si en moyenne on peut dire que l'espérance de gain est la même. Il est important de remarquer qu'il est finalement très facile de trouver des exemples quotidiens similaires où nous faisons face à des situations d'incertitude. Il s'agit donc d'un concept particulièrement pertinent et nous verrons plus bas comment il intervient lorsque l'on modélise de façon plus formelle le comportement d'un agent économique.
Ce n'est pas un concept récent mais au cours des 15 à 20 dernières années, il est réapparu dans le contexte des marchés financiers. En effet c'est un concept bien adapté à l''etude des problèmes de risque de modèles. De façon un peu schématique on peut distinguer deux formes d'incertitude. La première forme est liée aux questions de stabilité d'un modèle. Plutôt que fixer un modèle précis, on considère un ensemble de modèles que l'on interprête comme de petites pertubations autour du modèle initial. C'est finalement une forme modérée d'incertitude dans le sens où les perturbations sont censées être limitées. L'incertitude de modèle peut prendre une forme plus extréme. Prenons l'exemple oú les hypothèses sous-jacentes du modèle ne décrivent pas suffisamment précisément le phénomène que l'on veut étudier (typiquement le prix d'un actif financier). La meilleure illustration de ces problématiques est probablement celles relatives à la modélisation de la volatilité d'un actif financier. C'est un sujet qui a une longue histoire et qui a suscité de nombreuses recherches tant du côté académique que du côté des salles de marchés. Dans les travaux originaux de [18], la volatilité de l'actif sous-jacent est supposée constante. Très vite il est apparu évident que ce modèle ne correspondait pas à la réalite, ne serait-ce que parce qu'en pratique la volatilité observée à travers les prix des options observés sur les marchés dépend du strike et de la maturité de l'option. De nouveaux modèles ont alors été construits : d'abord des modèles à volatilité locale ([56]) ou la volatilité est une fonction de la valeur de l'actif, puis des modèles à volatilité stochastiques (le modèle de Hull and White [83], le modèle de Heston [77], et enfin le modèle SABR [71]) où la volatilité est elle-même aléatoire jusqu'aux travaux récents sur la "rough" volatilité (voir par exemple [65]). Quel que soit le modéle choisi, on peut considérer pour chacun de petites variations de

ses paramètres. C'est d'ailleurs ce que font souvent les opérateurs de marché en pratique en calculant des bid-offer et/ou des réserves en fonction de la variation du prix de modèle lorsque l'on fait bouger certains des paramètres. Mais chacun de ces modèles repose sur des hypothèses complexes, parfois cachées et ne décrivent le comportement de l'actif sous-jacent que dans des situations particulières. Si l'on veut comprendre (mesurer) vraiment le risque d'une position financière, peut-être faut-il considérer l'ensemble de ces modèles. Une fois encore c'est souvent ce qui se passe en pratique où le gestionnaire de risque peut utiliser un modèle ou un autre en fonction des caractéristiques du produit et des qualités respectives de chacun des modèles. Mais d'un point de vue mathématique cette approche a besoin d'être formalisée. Comment s'assurer par exemple que l'utilisation de ces différentes méthodes d'évaluation n'amène pas à des arbitrages ? L'approche adoptée dans le modèle à volatilité incertaine (voir [5], [93]) s'appuie aussi sur cette idée : le processus de la volatilité n'est pas directement modélisé mais on suppose seulement que la volatilité de l'actif sous-jacent se trouve entre deux bornes. Une situation finalement assez similaire à l'exemple décrit ci-dessus de l'urne où la proportion exacte de boules rouges était entre deux bornes.
L'évolution importante des marchés financiers dont le comportement et la structure semblent par ailleurs de plus en plus déconnectés des réalites économiques sous-jacentes a motivé plus encore ce type de questionnements. Les épisodes récents de volatilité extrême autour d'évènements politiques ou encore les épisodes de "flash-crash" observés ces dernières années soulèvent des questions et des inquiétudes légitimes. Dans ce contexte, la notion d'incertitude est un cadre puissant permettant aussi de mieux décrire et modéliser le comportement et les interactions des acteurs économiques. Cela est particulièrement important dans des périodes de tensions importantes comme ce fut le cas lors de la crise de 2008 où l'interaction entre les agents économiques dans des situations d'incertitude extrême a joué un role essentiel dans la propagation de la crise. Il s'agit là bien entendu d'une forme beaucoup plus profonde d'incertitude et qui dépasse le problème de risque de modèle.
D'un point de vue mathématique, ces questions ont amené au dévelopement d'outils mathématiques innovants afin de mieux formaliser et modéliser le problème d'incertitude des lois de probabilités. Nous présentons brièvement la notion d'espérance non-linéaire qui est un des outils sous-jacents aux problèmes que nous étudierons. Le but n'est bien entendu pas de donner une présentation rigoureuse et complète et nous renvoyons le lecteur à [103] et [104] pour plus de détails et d'autres références. On considère un espace mesurable $(\Omega, \mathcal{S})$ qui représente l'ensemble des évènements possibles et un espace linéaire $\mathcal{H}$ de fonctions réelles et mesurables définies sur Ω (et qui contient les fonctions constantes). Chacune des ces fonctions correspond au résultat financier d'une décision ou d'un investissement. On introduit une espérance sous-lineaire, c'est à dire une fonctionnelle $\mathcal{E} : \mathcal{H} \to \mathbb{R}$

qui

- est monotone : si $X \geq Y$ alors $\mathcal{E}(X) \geq \mathcal{E}(Y)$
- préserve les constantes : pour $c \in \mathbb{R}$, $\mathcal{E}(c) = c$
- est sous-additive : pour $X, Y \in \mathcal{H}$, $\mathcal{E}(X+Y) \leq \mathcal{E}(X) + \mathcal{E}(Y)$
- est positivement homogène : pour $\lambda \geq 0$, $X \in \mathcal{H}$, $\mathcal{E}(\lambda X) = \lambda \mathcal{E}(X)$.

La motivation sous-jacente est la suivante : une façon de représenter l'incertitude est d'introduire non pas une loi de probabilité P mais un ensemble de lois de probabilité $\mathcal{P}$. Ainsi pour une variable aléatoire $X \in \mathcal{H}$, il semble naturel de remplacer l'espérance $\mathcal{E}(X) = \sup_{P \in \mathcal{P}} E_P X$ [1]. La moyenne de la variable aléatoire est ainsi remplacée par l'intervale $[-\mathcal{E}(-X), \mathcal{E}(X)]$: celui-ci représente l'incertitude sur la moyenne (et de même sur la variance). Il est alors possible de généraliser dans ce contexte les notions de variables indentiquement distribuées, de variables aléatoires indépendantes, de convergence en loi (au sens de l'espérance sous-linéaire). On peut aussi obtenir dans le contexte d'incertitude un équivalent de la loi des grands nombres, du théorème central limite et de bien d'autres résultats et concepts classiques.

Nous finissons cette section en considérant le problème suivant dont les conséquences mathématiques sont importantes. Lorsque l'on fixe un ensemble de lois de probabilité $\mathcal{P}$ (celui-ci pouvant être réduit à une seule probabilité le cas échéant) la distinction entre les évènements pouvant se produire (c'est à dire de probabilité strictement positive pour au moins l'un des éléments de $\mathcal{P}$) et ceux qui ne peuvent pas arriver est essentielle et tout aussi importante que la quantification précise des probabilités respectives de chacun des événements. A titre d'exemple : est-ce qu'un modèle permet ou non que les taux d'interêts soient négatifs ? Il est bien évident que l'attention et la réponse apportées à cette question de modélisation ne sont plus aujourd'hui les mêmes qu'il y a 10 ou 15 ans. Sur le plan mathématique, on distingue deux situations : soit il existe une probabilité P^* telle que pour tout $A \in \mathcal{S}$, $P^*(A) = 0$ implique que $P(A) = 0$ pour tout $P \in \mathcal{P}$. On dit alors que $\mathcal{P}$ est dominée par P^* et c'est P^* qui détermine les événements qui peuvent arriver ou non. D'un point de vue mathématique c'est le cas le plus simple car on pourra utiliser les outils classiques de la théorie des probabilités. Mais si par malchance, l'ensemble $\mathcal{P}$ n'est pas dominé la situation devient beaucoup plus délicate. En effet les outils classiques comme l'esperance conditionelle ou le supremum essentiel sont définis P-presque sûrement (pour un P donné) c'est à dire seulement sur l'événements visibles par P. Ces outils ne sont donc a priori pas adaptés à l'étude d'un ensemble de probabilité non-dominé puisque leurs définitions posent problème. Cela amène à la

[1] Notons que $\inf_{P \in \mathcal{P}} E_P X = -\sup_{P \in \mathcal{P}} E_P(-X)$

problématique d'agrégation (voir par exemple [124] or [43]) et qui sera également au coeur du Chapitre 4. Il est important de noter qu'il ne s'agit pas seulement d'une question théorique. Ainsi, dans le modèle à volatilité incertaine que nous avons mentionné plus tôt, on obtient des ensembles non-dominés de probabilites dans lesquelles les lois de probabilités sont deux à deux mutuellement singulières.

Fonctions d'utilité et la notion d'espérance d'utilité

Nous présentons dans cette section le concept d'espérance d'utilité qui sera lui aussi, central dans cette dissertation. Nous donnerons brièvement l'idée générale sous-jacente. Le lecteur interessé pourra se reporter au [62, Chapter 2] par exemple pour une présentation plus complète et détaillée. La formalisation de la théorie de von Neumann et Morgenstern a été initialement développée dans [126]. Supposons que l'ensemble des scenarios possibles soit représenté par un espace mesurable $(\Omega, \mathcal{F})$ et que chaque décision (un investissement par exemple) soit representée par une variable aléatoire $X : \Omega \to \mathbb{R}$ correspondant au résultat financier de cette décision. Supposons par ailleurs qu'il existe une loi de probabilité P connue sur $(\Omega, \mathcal{F})$ décrivant la distribution de chacune de ces variables aléatoires. En d'autres termes, l'agent économique est en situation de risque (et non d'incertitude). Alors sous l'axiomatique développée par [126], pour un agent donné chaque décison peut être representée par

$$u(X) = E_P U(X)$$

ou $U : \mathbb{R} \to \mathbb{R}$ est une fonction concave et croissante que l'on appelle fonction d'utilité et qui est propre à chaque agent. Ainsi $u(x)$ représente l'espérance d'utilité de X et une décision X sera preférée à Y si et seulement si $u(X) = E_P U(X) \geq u(Y) = E_P U(Y)$. Un agent essaie toujours de maximiser son espérance d'utilité en choisissant parmi toutes les actions X disponibles. On distingue souvent deux types de fonctions d'utilité : soit U est définie sur tout $\mathbb{R}$, soit uniquement sur (a, ∞) pour un réel a (et $U = -\infty$ en dessous de a). Dans ce cas, a correspond au capital maximum que l'agent peut perdre. Parmi les fonctions d'utilité usuelles on trouve : les fonctions logarithmes, puissances ou exponentielles (cette dernière correspond à une aversion au risque constante quelque soit la richesse). La concavité de la fonction d'utilité est liée à l'aversion au risque comme nous l'avons déjà mentionné. Pour une décision X, notons $m(X) = E_P X$ la moyenne de X sous P. L'aversion au risque signifie que l'agent économique préfère recevoir $m(X)$ qui est sûr plutôt que X (sauf si bien sur X n'est pas aléatoire). C'est une hypothèse importante et comme nous l'avons vu qui en pratique n'est pas toujours vérifiée. C'est ce qui nous poussera à étudier dans le Chapitre 2 des fonctions non-concaves.
Au-delà de ce problème d'aversion au risque, la modélisation ci-dessous présente

d'autres faiblesses importantes : en particulier peut-on vraiment supposer qu'il existe une probablité P objective, connue et partagée par tous les agents ? Le paradoxe d'Allais (voir [62, Example 2.32]) montre par ailleurs que les axiomes sous-jacents de la théorie de von Neumann et Morgenstern ne sont pas toujours vérifiés. Pour répondre à ces critiques, L.J. Savage ([120]) proposa une approche modifiée. Dans le cadre de Savage, on ne fait plus l'hypothèse qu'il existe une probabilité P objective et connue. En ajoutant des hypothèses supplémentaires sur les préférences des agents, on peut obtenir une nouvelle représentation numérique des préférences des agents sous la forme suivante :

$$u(X) = E_Q U(X)$$

où Q est une loi de probabilité subjective on $(\Omega, \mathcal{F})$ et qui dépend donc de chaque agent (U est toujours une fonction concave et croissante). Dans ce cadre un agent économique cherche toujours à maximiser son espérance d'utilité mais en fonction de sa propre vision du monde.

Malheureusement cette représentation n'est toujours pas complètement satisfaisante. En effet, le paradoxe d'Ellsberg [62, Exemple 2.32] montre que les hypothèses sous-jacentes de la représentation de Savage ne sont, elles aussi, pas toujours vérifiées expérimentalement. Le concept d'aversion à l'incertitude décrit plus haut a alors été introduit. C'est en quelques sorte l'analogue de l'aversion au risque pour l'incertitude. Cette approche, utilisée dans [69], permet de proposer une nouvelle théorie d'espérance d'utilité. Cette fois les préférences des agents peuvent s'écrire sous la forme

$$u(X) = \inf_{P \in \mathcal{P}} E_P U(X)$$

où $\mathcal{P}$ est un ensemble de loi de probabilités subjective (et donc dépendant de chaque agent) et qui représente toutes les croyances d'un agent économique. La fonction $U : \mathbb{R} \to \mathbb{R}$ est toujours concave et croissante. Dans ce cadre, un agent économique va donc maximiser éspérances d'utilités calculées sous la croyance la plus défavorable. Le cadre introduit par [69] a ensuite été étendu dans [94] où une fonction de pénalité $c(P)$ est introduite dans la fonctionelle précédente. Finalement, dans [39], les préférences des agents sont représentées par une fonctionnelle de la forme $\inf_{P \in \mathcal{Q}} G(E_P U(X), P)$ ou G est appellé l'indice d'incertitude et représente l'attitude de l'agent économique face à l'incertain. Il existe bien entendu d'autres réponses aux critiques formulées à l'encontre du paradigme de l'espérance d'utilité de von Neumann et Morgenstern : on peut se rapporter par exemple [42] pour un exposé de ces différentes idées.

Dans le contexte des marchés financiers, les fonctions d'utilité n'ont pas toujours été utilisées autant que d'autres techniques essentiellement car il est difficile de savoir comment les estimer. Toutefois elles ont gagné en popularité récemment. Tout d'abord parce qu'elles permettent justement d'introduire des distinctions entre les agents economiques. Par ailleurs elles sont aussi utilisée dans des situations où les couvertures (hedge) parfaits sont impossibles et ou l'on considère des couvertures partielles. C'est en particulier le point de vue utilisé dans l'évaluation par indifférence d'utilité dont certains aspects en présence d'incertitude seront étudiés dans le Chapitre 5.

Notons qu'une revue de la littérature mathématique récente sur le problème de maximisation d'espérance d'utilité est proposée plus loin en Section 1.2.2.

La notion d'arbitrage en mathématiques financières

Nous concluons cette introduction en présentant brièvement la notion d'arbitrage en mathématiques financières qui est un concept essentiel. En quelques mots l'absence d'opportunité d'arbitrage veut simplement dire qu'un investisseur ne peut pas faire de profit certain sans prendre un risque : c'est à dire sans s'exposer à une perte potentielle. ll s'agit bien entendu d'une vision idealisée des marchés. En pratique, des arbitrages existent et pour de multiples raisons d'ailleurs. Ceci étant dit, il semble raisonnable de supposer qu'une fois qu'une opportunité d'arbitrage est detectée, un ou plusieurs "arbitragiste" interviendront sur les marchés pour en profiter et que cette opportunité va donc rapidement disparaitre.
Dans le cadre classique où il existe une probabilité objective (historique), on dit de façon informelle, qu'une stratégie est un arbitrage si sur un horizon de temps T, elle délivre toujours un résultat positif ou nul et qu'il existe des situations dans lesquelles elle délivre un profit. Attention pour qu'il y ait arbitrage, il faut que les situations dans lesquelles la stratégie délivre un profit soient visibles pour le modèle sous-jacent, c'est à dire qu'elles aient une probablité strictement positive dans ce modèle. Ainsi si une stratégie amène un profit dans une situation où les taux d'interêts sont négatifs et qu'elle délivre toujours un résultat positif dans tous les autres situations mais que le modèle en question ne permet pas d'avoir des taux d'interêts négatifs, il n'y a pas d'arbitrage. La notion d'arbitrage et ses conséquences en terme de loi de probabilité d'évaluation sont au coeur du développement de la finance moderne. Cette notion a véritablement révolutionné la façon dont beaucoup de marchés financiers fonctionnent et a contribué à leur croissance exponentielle. Cette croissance est allée de pair avec le dévelopement d'outils mathématiques complexes qui ont eux-même incité beaucoup d'intervenants à construire et élaborer des instruments financiers toujours plus sophistiqués. Cette croyance parfois aveugle en la toute-puissance et l'infaillibilité des outils mathématiques malgré des accidents et crises récurrentes sont sans aucun doute des sources d'inquiétudes. Mais ces questions sortent largement du cadre de cette dissertation.

Pour revenir sur la notion d'arbitrage, les bases de la théorie ont été développées et formalisées par [74], [75] and [92]. Un résultat essentiel de cette théorie est le théorème fondamental de l'évaluation des actifs qui fait le lien entre la notion de non-arbitrage et celle de probabilité risque-neutre (ou probabilité martingale) : un modèle est sans opportunité d'arbitrage si et seulement si il existe une loi de probabilité risque-neutre. De façon informelle, probabilité risque neutre est probabilité sous laquelle le prix actualisé à la date t d'un actif est exactement l'espérance conditionnellement à l'information disponible à la date t, du prix de l'actif actualisé à la date $t+1$. En d'autres termes que l'on achète ou vende cet actif à la date t, l'espérance de gain en date $t+1$ est nulle. Notons qu'il s'agit du prix actualisé de l'actif : c'est à dire son prix exprimé pas rapport à un actif de référence (souvent appelé, à tort, actif sans risque). Par ailleurs dans ce cas, le prix actualisé d'un actif contingent (c'est à dire un actif dont le prix dépend du prix d'un ou plusieurs actifs sous-jacents) est exactement son espérance mais calculée sous cette probabilité risque neutre (et non en utilisant une loi de probabilité historique). Ces résultats sont désormais bien connus et bien ancrés dans les pratiques des marchés. Mais répétons qu'ils ont véritablement revolutionné la façon dont les marchés fonctionnent.
Rappelons que la notion d'arbitrage est aussi liée à la notion de sur-réplication. Pour un actif contingent, le prix de sur-réplication correspond au prix minimum que demande un agent économique vendant cet actif contingent afin de pouvoir le sur-répliquer en achetant/vendant dynamiquement les actifs sous-jacents dans le marché : en d'autre terme la stratégie de sur-réplication doit permettre à l'agent de délivrer l'actif contingent dans tous les situations possibles sans aucun risque. La relation entre ce prix et le(s) prix obtenus par espérance risque neutre est donnée par le théorème de sur-réplication : le prix de sur-réplication d'un actif contigent est égal au supremum des prix obtenus par espérance (parmi toutes les probabilités risque-neutre).
Pour conclure cette introduction, nous évoquons brièvement comment le concept de non-arbitrage est impacté en présence d'incertitude. Ces questions ont suscité en effet un regain d'intérêt récemment, en particulier pour répondre aux problèmes liés au risque de modèle comme nous l'avons déjà evoqué. Dans ce cadre, il existe différentes approches et nous proposons une discussion plus detaillée sur ces différentes approches et une revue de la litteŕature mathématique récente plus bas (voir section 1.3.2). Dans cette dissertation, nous suivrons essentiellement la modélisation proposée par [25] où l'incertitude est modélisée en introduisant un ensemble non-dominé de lois de probabilité (qui représentent donc les différentes croyances des agents). Dans ce cadre, une généralisation assez naturelle de la notion classique (avec une seule loi de probabilité) d'arbitrage est proposée. Cette définition permet en particulier d'étendre les résultas classiques sur l'existence de lois de probabilité risque neutre et de sur-réplication. Nous reviendrons plus en détails sur

ces points en particulier dans le Chapitre 3.

Résumé de la dissertation

Le travail présenté dans cette dissertation est le résultat de différents articles (voir [20], [19], [21], [22]) dont certains ont été soumis pour publication. Le contenu de chacun des chapitres correspond donc à une version détaillée et complète. En particulier, nous avons volontairement laissé les répétitions entre les différents chapitres. Par exemples nous répétons dans la Section 3.2 du Chapitre 3, la Section 4.2 du Chapitre 4 et la Section 5.2.1 du Chapitre 5 tout ce qui concerne les notations utilisées, la façon dont l'incertitude est modélisée et les outils de théorie de la mesure utilisés. Notons aussi qu'un certain nombres de preuves dans les différents chapitres utilisent des arguments similaires. De même, les introductions de chacune des parties sont parfois redondantes et reprennent certains éléments de cette introduction générale. Nous avons fait ce choix afin que chacun de ces chapitres puisse être lu de façon (presque) indépendante du reste de la dissertation.

Nous proposons ci-dessous un résumé des résultats et outils utilisés dans chacun de ces chapitres.

Chapitre 1

Le Chapitre 1 reprend en anglais l'introduction et propose aussi une revue de la littérature sur les problématiques d'utilité et d'arbitrage.

Chapitre 2

Dans le Chapitre 2 nous étudions dans un cadre classique (c'est-à-dire avec une seule probabilité), le problème de maximisation d'espérance d'utilité pour une fonction d'utilité qui n'est ni concave, ni continue et qui est définie sur l'axe des réels positifs. Dans ce cadre, nous établissons le résultat qui est le plus complet à notre connaissance, garantissant l'existence d'une stratégie d'investissement optimale. Une des raisons qui nous a poussée à étudier des fonctions non-concaves a déjà été abordée précédemment : un agent économique n'est pas forcement averse au risque dans toutes les situations. Par ailleurs, le fait d'avoir des fonctions d'utilité discontinues est aussi intéressant sur le plan pratique : une fois que l'on dépasse un certain seuil de richesse, l'utilité d'un agent peut sauter d'un niveau par exemple. Enfin, notre résultat permet aussi de considérer des fonctions d'utilité qui ne sont

pas finies en 0 (le logarithme par exemple) ce qui n'était pas le cas dans les résultats précédemment obtenus. Notre preuve utilise des outils de théorie de la mesure et de sélection mesurable et repose sur le principe de programmation dynamique. Nous donnons les preuves de tous les résultats utilisés (certains étant des résultats classiques dont nous rappelons les démonstrations en les adaptant à notre cadre). En particulier, nous démontrons tous les résultas nécessaires relatifs à la condition de non-arbitrage.
Les deux résultats principaux sont les suivants

Theorem 0.0.1 *Supposons que la condition (NA) et les hyptothèses 2.4.7, 2.4.8 and 2.4.10 soient verifiées. Soit $x \geq 0$. Alors, $u(x) < \infty$ et il existe une stratégie optimale $\phi^* \in \Phi(U, x)$ telle que*

$$u(x) = EU(\cdot, V_T^{x,\phi^*}(\cdot)).$$

De plus, $\phi_t^*(\cdot) \in D^t(\cdot)$ p.s. pour tout $0 \leq t \leq T$.

Theorem 0.0.2 *Supposons que la condition (NA) et l'hyptothèses2.4.10 soit verifiées. Suppons par ailleurs que $EU^+(\cdot, 1) < +\infty$ et que pour tout $0 \leq t \leq T$ $|\Delta S_t|$, $\frac{1}{\alpha_t} \in \mathcal{W}_t$. Soit $x \geq 0$. Alors, pour tout $\phi \in \Phi(x)$ et tout $0 \leq t \leq T$, $V_t^{x,\phi} \in \mathcal{W}_t$. De plus, il existe une stratégie optimale $\phi^* \in \Phi(U, x)$ telle que*

$$u(x) = EU(\cdot, V_T^{x,\phi^*}(\cdot)) < \infty$$

Plus précisement, le Chapitre 2 est structuré de la façon suivante. Après une partie introductive qui replace le chapitre par rapport à la recherche existante, nous présentons le cadre et les notations utilisées : nous décrivons l'espace probabilisé et présentons les notions de noyau stochastique et d'intégrale généralisée qui seront utilisées par la suite. Nous présentons ensuite la condition de non-arbitrage : après avoir rappelé des propriétés de mesurabilité importantes, nous établissons la Proposition 2.3.7 qui sera essentielle par la suite. Dans la section suivante, nous introduisons la définition de fonction d'utilité et posons précisement le problème qui nous intéresse. Nous présentons et discutons ensuite les différentes hypothèses nécessaires à la démonstration de notre théorème principal (Théorème 2.4.17) et en particulier les nouvelles conditions d'intégrabilité et d'élasticité asymptotique.
La démonstration du théorème s'effectue en deux étapes : nous considérons le cas d'un modèle à une période avec données initiales déterministes. Il s'agit alors d'un problème relativement simple d'optimisation dans $\mathbb{R}^d$. Ensuite, dans le cas multipériode (qui est la partie techniquement difficile), il s'agit d'utiliser le résultat obtenu dans le cas une période. Notre démonstration repose sur deux idées essentielles : il s'agit d'utiliser des outils de sélection mesurable ainsi que le principe de

programmation dynamique pour construire pour chaque étape une solution optimale en recollant bout à bout les solutions obtenues dans le cas une période avec conditions initales déterministes. Le résultat essentiel est obtenu dans la Proposition 2.6.10 qui est le principal outils utilisé dans la démonstration du théorème. Cette démonstration s'effectue elle aussi en deux étapes : nous contruisons grâce à la Proposition 2.6.10 une stratégie qui sera notre candidate pour être la solution optimale. Ensuite, nous vérifions qu'elle est effectivement une solution optimale.
Nous proposons ensuite dans le Théorème 2.4.17 une application dans un cadre général de notre résultat. Finalement, en appendice, nous reprenons précisement un certain nombres de détails techniques utilisés dans le chapitre. Nous proposons également un certain nombres de rappels sur les ensembles aléatoires ainsi que les problèmes de mesurabilité sous-jacents. Ces notions seront d'ailleurs utilisées à travers toute la dissertation et ce chapitre qui peut aussi être vu comme une préparation en vue du Chapitre 4.
En conclusion de ce chapitre, il semble naturel de se demander si et comment l'on peut étendre les résultats obtenus à des fonctions d'utilités définies sur l'ensemble des réels. Une partie de la réponse a déjà été apportée dans [33]. Toutefois la condition d'intégrabilité proposée ([33, Assumption 2.9]) n'est pas totalement satisfaisante car elle n'est pas facile à vérifier en pratique. De plus d'un point de vue technique (et aussi esthétique), cela rend la preuve délicate et complexe car la condition d'intégrabilié doit être vérifiée par récurence ascendante (contrairement à ce que nous faisons dans ce chapitre) alors que les autres conditions nécessaires pour appliquer le principe de programmation dynamique sont, elles, vérifiées par une récurence descendante qui est plus naturelle. Malheureusement il n'est pas évident de remplacer cette hypothèse par des hypothèses similaires à celles introduites dans ce chapitre (voir Assumptions 2.4.7 and 2.4.8) qui ne soient pas trop restrictives. En effet dans le cas de fonctions définies sur l'ensemble des réels, l'argument de compacité (c'est à dire l'equivalent du Lemma 2.5.10) demande plus de travail et la borne obtenue dans le modèle une période dépend de la fonction d'utliité. Il n'est donc pas évident de trouver une condition d'intégrabilité qui soit préservée dans la programmation dynamique. Dit autrement et de façon plus intuitive : dans le cas d'une fonction définie sur les réels positifs il n'est finalement pas vraiment restrictif d'imposer des conditions d'integrabilité puisque cette condition est d'une certaine façon déjà imposé par la contrainte d'admissibilité (à savoir que l'agent ne veut pas perdre d'argent et donc que la valeur du portefeuille doit rester postive ou nulle). Dans le cas d'une fonction définie sur tous les réels, cette contraintes n'existe pas (l'agent n'a pas de limite de perte) et les conditions d'intégrabilités sont une vraie contrainte. En effet rien n'empêche un investisseur d'avoir une stratégie qui lui procure une utilité très importante dans certaines situations. Bien sur, la condition d'arbitrage implique que ce type de stratégie conduira à des utilités très négatives pour certains évènements et donc que de telles strategies ne

sont vraisemblablement pas optimales. Mais on voit pourquoi une conditions d'intégrabilité sur toutes les stratégies est, sur le plan mathématique, trop contraignant. La généralisation de notre résultat et de notre preuve à des fonctions définies sur $\mathbb{R}$ est donc un problème intéressant mais dont la résolution n'est pas une adaptation tout à fait évidente du cas $(0,\infty)$. Ce sera donc le sujet de recherches futures.

Chapitre 3

Le Chapitre 3 peut également être considéré comme un chapitre préparatif au Chapitre 4. Nous nous intéressons à un marché financier en temps discret avec un horizon de temps fini mais en présence cette fois d'incertitude. Les deux résultats principaux sont les suivants

Theorem 0.0.3 *Supposons que les hypothèses 3.2.1 et 3.2.2 soient verifiéds. Alors le non- arbitrage quantitatif (voir Définition 3.4.4), le non-arbitrage géometirque (voir Définition 3.4.6) et la condition* $NA(\mathcal{Q}^T)$ *(voir Définition 3.4.1) sont équivalent et* $\Omega^t_{NA}=\Omega^t_{qNA}=\Omega^t_{gNA}$ *for all* $0\leq t\leq T$*. De plus , pour tout* $\omega^t\in\Omega^t_{NA}$ *on peut choisir* $\alpha_t(\omega^t)=\varepsilon(\omega^t)$ *tels que (3.5) et (3.6) soient vraies.*

Theorem 0.0.4 *Supposons que les hypothèses 3.2.1 et 3.2.2 soient vraies . Alors les affirmations suivantes sont équivalentes*
1. $sNA(\mathcal{Q}^T)$ *est vraie.*
2. Pour tout $0\leq t\leq T-1$*,* $\Omega^t_{sNA}\in\mathcal{CA}(\Omega^t)$ *est un ensemble de pleine mesure pour* $\mathcal{Q}^t$*.*

La proposition suivante est également importante.

Proposition 0.0.5 *Supposons que la condition* $sNA(\mathcal{Q}^T)$ *et que les hypothèses 3.2.1 et 3.2.2 soient vérifiées. Soit* $0\leq t\leq T-1$*. Fixons* $P=Q_1\otimes q_2\otimes\cdots\otimes q_T\in\mathcal{Q}^T$*. Alors il existe* $\Omega^t_P\in\mathcal{B}(\Omega^t)$ *tel que* $P_t(\Omega^t_P)=1$ *et pour tout* $\omega^t\in\Omega^t_P$*,* $D^{t+1}_P(\omega^t)$ *est un sous-espace vectoriel et il existe* $\alpha^P_t(\omega^t)\in(0,1]$ *tel que pour tout* $h\in D^{t+1}_P({}^t)$*,* $h\neq 0$

$$q_{t+1}\big(h\Delta S_{t+1}(\omega^t,\cdot)\leq-\alpha^P_t(\omega^t)|h|,\omega^t\big)\geq\alpha^P_t(\omega^t).$$

Enfin, $\omega^t\to\alpha^P_t(\omega^t)$ *est* $\mathcal{B}(\Omega^t)$*-mesurable.*

Pour cela nous nous plaçons dans le cadre introduit dans [25] que nous présentons en détails après une partie introductive. Ce cadre utilise des outils de théorie de la mesure et, en particulier, la notion d'ensemble analytique qui est au coeur de difficultés techniques du cadre à croyances multiples non-dominées. Nous introduisons ensuite la notion de support (dans le cadre de croyance multiples et non-dominées) du processus de prix des actifs. C'est un outil important à travers

toute la dissertation qui permet de se débarasser des actifs "redondants" et dont nous établissons des propriétés de mesurabilités importantes. Ces propriétés seront utilisées de façon récurente à travers la dissertation. Leur preuve repose sur des théorèmes de projection ce qui justifie l'introduction des ensembles analytiques introduit plus haut.
Nous rappelons ensuite la définition de la condition de non-arbitrage quasi-sûre, ainsi que sa caractérisation locale. Cette notion sera essentielle par la suite. Nous proposons ensuite des définitions alternatives, mais équivalentes, de cette condition de non-arbitrage. Plus précisement, il s'agit d'une caractérisation dite quantitative du non-arbitrage et d'une caractérisation dite géométrique et nous prouvons dans le Théorème 3.4.7 que ces définitions sont bien équivalentes ce qui généralise un résultat bien connu dans le cadre classique (sans croyances multiples). La preuve du Théorème 3.4.7 s'effectue en deux étapes : en Proposotion 3.4.14 nous établissons l'équivalence des différentes définitions dans un modèle une période avec données initiales deterministes. Pour cela nous utilisons des résultats classiques de séparation d'ensemble convexe dans $\mathbb{R}^d$. La preuve du Théorème 3.4.7 repose ensuite sur la caracterisation locale du non-arbitrage quasi-sûre établie dans [25, Theorem 4.5]. Ce résultat utilise des techniques de selection mesurable et les ensembles analytiques. A l'aide du Théorème 3.4.7, nous établissons aussi la proposition 3.4.9 dans laquelle nous obtenons des propriétes de mesurabilités utiles par la suite.
Dans la section suivante et afin d'illustrer l'intérêt de la caractérisation quantitative de la notion de non-arbitrage, nous étudions le problème de la maximisation d'espérance d'utilité la plus défavorable pour une fonction d'utilité définie sur $\mathbb{R}$, non bornée et toujours dans le cadre de croyance multiples et non-dominées. Le Lemme 3.5.12 est la clé de voûte de la preuve du Théorème 3.5.13 : la caracterisation quantitative du non-arbitrage permet d'utiliser un argument de compacité, en obtenant une borne sur la norme des éventuelles stratégies optimales. Toutefois, nous nous limitons au cas une période et nous reverrons dans le Chapitre 4 pourquoi l'extension au cas multi-période est délicate.
Enfin, dans une dernière section nous introduisons la condition dite de non-arbitrage fort. C'est une condition plus contraignante que la condition de non-arbitrage quasi-sûre introduite précédemment. Elle sera utilisée dans le Chapitre 4 dans un théorème d'application car elle simplifie certaines questions techniques. Nous illustrons ces aspects techniques à travers quelques résultats de mesurabilité liés à la caractérisation locale de cette condition. C'est l'occasion de manipuler les ensembles analytiques (et coanalytiques) et de se familiariser avec certaines des difficultés techniques sous-jacentes qui apparaitront dans le Chapitre 4. Puis, nous proposons à nouveau une caractérisation quantitative de cet arbitrage qui sera aussi utilisée dans le Chapitre 4. Finalement, nous établissons en Proposition 3.6.12, un théorème fondamental pour la condition de non-arbitrage fort.

Pour conclure ce chapitre citons deux axes éventuels pour des recherches futures. Le premier consiste à creuser plus encore la relation entre la condition de non-arbitrage fort et la condition de non-arbitrage quasi-sure. On a vu en effet dans ce chapitre que la condition de non-arbitrage fort est d'un point de vue de technique mathématique plus facile à manipuler mais qu'elle est plus forte que la condition de non-arbitrage quasi-sure. Il serait donc intéressant de trouver sous quelles conditions on peut par exemple espérer obtenir l'implication réciproque.
Par ailleurs il est bien connu que dans le cadre classique si la preuve initiale du Théorème Fondamental de l'évaluation des actifs financiers proposée dans citepdmw repose sur des outils puissants de théorie de la mesure et de selection mesurable d'autres preuves, reposant par exemples sur des outils d'analyse fonctionnelle, sur une version aléatoire du Lemme de Bolzano-Weistrass ou les fonctions d'utilités, furent ensuite proposées (voir par exemple [86], [117] et [49]). Il serait alors intéressant de voir si des approches similaires peuvent aboutir à une preuve alternative du Théorème Fondamental de l'évaluation des actifs financiers obtenu par [25] dans le cadre de croyances multiples non-dominées.

Chapitre 4

Dans le Chapitre 4, nous nous intéressons cette fois au problème de maximisation de la plus défavorable des espérances d'utilité pour une fonction concave et non bornée, définie sur l'axe des réels positif. Les deux résultats principaux sont les suivants

Theorem 0.0.6 *Supposons que la condition $NA(\mathcal{Q}^T)$ et les hyptothèses 4.2.1, 4.2.2, 4.2.4, 4.4.2 et 4.4.12 soient verifiées. Soit $x \geq 0$. Alors, il existe une stratégie optimale $\phi^* \in \Phi(x, U, \mathcal{Q}^T)$ telle que*

$$u(x) = \inf_{P \in \mathcal{Q}^T} E_P U(\cdot, V_T^{x,\phi^*}(\cdot)) < \infty.$$

De plus, $\phi_t^*(\cdot) \in D^t(\cdot)$ $\mathcal{Q}^t$-q.s. pour tout $0 \leq t \leq T$.

Theorem 0.0.7 *Supposons que la conditions$NA(\mathcal{Q}^T)$ et les hyptothèses 4.2.1, 4.2.2, 4.2.4 et 4.4.2 soient verifiées. Supposons en plus que $U^+(\cdot, 1), U^-(\cdot, \frac{1}{4}) \in \mathcal{W}_T$ et que pour tout $1 \leq t \leq T$, $P \in \mathcal{Q}^t$, $\Delta S_t, \frac{1}{\alpha_t^P} \in \mathcal{W}_t$ (voir Proposition 4.3.6 pour la définition de α_t^P). Soit $x \geq 0$. Alors, pour tout $P \in \mathcal{Q}^T$, $\phi \in \Phi(x, P)$ et $0 \leq t \leq T$, $V_t^{x,\phi} \in \widehat{\mathcal{W}_t}$. De plus, il existe une stratégie optimale $\phi^* \in \Phi(x, U, \mathcal{Q}^T)$ telle que*

$$u(x) = \inf_{P \in \mathcal{Q}^T} E_P U(\cdot, V_T^{x,\phi^*}(\cdot)) < \infty.$$

Nous utilisons les résultats obtenus dans [25] ainsi que dans le Chapitre 3. Nous sommes donc encore donc dans un cadre de croyances multiples dans lequel nous ne supposons pas que l'ensemble de lois de probablité est dominé. Comme nous l'avons déjà indiqué dans cette introduction, ce cadre augmente la difficulté mathématique du problème mais semble tout à fait justifié d'un point de vue pratique. Nous établissons le premier(à notre connaissance) résultat déxistence dans un cadre de croyances multiples et non-dominées pour des fonctions d'utilité non-bornées. Nous généralisons ainsi le résultat obtenu dans [99] pour des fonctions bornées par dessus. Nous introduisons deux conditions d'intégrabilité : les hypotheses 4.4.2 et 4.4.12. La preuve repose, comme dans le Chapitre 2, sur le principe de programmation dynamique ainsi que sur des outils de théorie de la mesure et de sélection mesurable. Cependant, comme dans le Chapitre 3 déjà, le cadre de croyances multiples non-dominées complique les problématiques de mesurabilités. Nous utilisons à nouveau les ensembles analytiques et des théorèmes de selection mesurable relatifs aux ensembles analytiques. Rappelons que les ensembles analytiques interviennent car ils sont stables par projection contrairement aux ensembles boréliens par exemple. Ils sont aussi stables par unions ou intersections dénombrables mais pas par passage au complémentaire. Et c'est la raison pour laquelle beaucoup de problèmes de mesurabilité ne peuvent pas être résolus aussi facilement que dans le cas du Chapitre 2. Les problématiques liées aux conditions d'integrabilités ajoutent des difficultés techniques supplémentaires.
Le chapitre est structuré de façon similaire au Chapitre 2. Après une partie introductive, nous présentons de façon détaillée le cadre, les notations, ainsi que les outils de théorie de la mesure qui seront utilisés dans le chapitre. Nous rapellons ensuite les résultats sur le support conditionnel des variations des prix des actifs sous-jacent et sur la condition de non-arbitrage quasi-sûre et sa caractérisation quantitative. Tout comme dans la Section 3.5 du Chapitre 3, cette caractérisation sera utilisée pour obtenir de la compacité. Ensuite, après avoir formulé explicitement le problème que nous cherchons à résoudre, les hypothèses utilisées et les résultats obtenus, nous nous attaquons à la preuve, celle-ci s'articulant en deux temps. Dans un premier temps, nous traitons le cas une période avec données initiales déterministes. Le résultat principal de cette section est le Théorème 4.5.23, il donne des conditions sous lesquelles une strategie optimale existe dans ce cadre et il sera utilisé dans le cadre multi-periode. Nous établissons aussi dans cette partie des résultats techniques (Lemmes 4.5.17, 4.5.18, 4.5.22). Ils seront utilisés pour résoudre des questions de mesurabilité et d'intégrabilité dans le cadre multi-periode. Ces problèmes sont propres au cadre de croyances multiples non-dominées de ce chapitre. Enfin dans un second temps en utilisant à la fois des outils de sélection mesurable et programmation dynamique nous prouvons notre théorème. La structure de la preuve est tout à fait similaire à celle du Chapitre 2, mais les arguments sont beaucoup plus délicats pour les raisons de mesurabilité mentionnées

plus haut. La Proposition 4.6.12 permet de construire pour chaque période une solution optimale en recollant de façon mesurable les solutions obtenues dans le cas une période. Puis, nous contruisons grâce à la Proposition 4.6.12 une stratégie qui sera notre candidate pour être la solution optimale et vérifions qu'elle est en effet une solution optimale. Finalement, en utilisant la condition de non-arbitrage fort qui se révèle plus adaptée à la condition d'intégrabilité que nous avons introduite, nous appliquons notre résultat dans un cadre relativement large. Enfin l'appendice reprend les détails de certains résultats techniques et revient aussi sur un problème de mesurabilité identifié dans [25].
En conclusion de ce chapitre, nous proposons des pistes pour des recherches futures afin d'améliorer ce résultat. Tout d'abord il serait interessant d'etudier le généralisation suivante de notre problème d'óptimisation

$$\sup_{\phi \in \Phi_G(x,U,\mathcal{Q}^T)} \inf_{P \in \mathcal{Q}^T} E_P U(\cdot, V_T^{x,\phi}(\cdot) - G(\cdot))$$

où $G : \Omega^T \to [0, \infty)$ est une variable aléatoire et

$$\phi_G(x, U, \mathcal{Q}^T) = \{\phi \in \phi(x, U, \mathcal{Q}^T),\ V_T^{x,\phi}(\cdot) - G(\cdot) \geq 0\ \mathcal{Q}^T\text{-q.s}\}.$$

En effet cela pourrait permettre dans le Chapitre 5, d'obtenir une strategie optimale pour realiser le prix d'indifférence d'utilité. En s'inspirant de [112], en supposant G bornée et vérifiant de bonnes conditions de mesurabilité et en utilisant [25, SuperHedging Theorem, Lemma 4.10], notre preuve pourrait être adaptée en introduisant $G_t(^t) = \sup_{P \in \mathcal{Q}^{t+1}(^t)} E_P G_{t+1}(^t, \cdot)$ et en replacant $\mathcal{H}^{t+1}(^t)$ avec $\mathcal{H}^{t+1}(^t) := \left\{h \in \mathbb{R}^d,\ x + h\Delta S_{t+1}(^t, \cdot) \geq G_t(^t)\ \mathcal{Q}_{t+1}(^t)\text{-q.s.}\right\}$. La vérification de ce travail minutieux est toutefois laissé pour des recherches futures.
Une autre amélioration importante serait de supprimer l'hypothèse de concavité imposée dans la Définition 4.4.1. En effet on a vu que cette hypothèse n'est pas requise dans le Chapitre 2 et comme nous l'avons déjà mentionné qu'elle n'est pas toujours justifié d'un point de vue pratique. Malheureusement dans un cadre à croyances multiples non-dominés, même si la concavité n'est pas indispensable pour le problème d'optimisation en lui-même, elle est essentielle pour obtenir les bonnes proprietés de mesurabilités (comme c'est illustré dans l'Example 4.8.2 et le Lemme 4.8.3). Nous avons exploré différentes alternatives mais aucune ne s'est révélée satisfaisante.
Enfin, la question de la condition d'intégrabilité reste ouverte : en effet initialement nous espérions pouvoir utiliser la condition suivante

$$\sup_{P \in \mathcal{Q}^T} \sup_{\phi \in \Phi(1,\mathcal{Q}^T)} E_P U^+(\cdot, V_T^{1,\phi}(\cdot)) < \infty.$$

qui est moins contraignante et plus esthétique que l'hypothèse 4.4.12 puisque $\Phi(1, \mathcal{Q}^T) \subset \Phi(1, P)$ for all $P \in \mathcal{Q}^T$. Malheureusement l'utilisation de cette condition soulève des

problémes de mesurabilité pour la fonction I_t (dont la définition doit être adaptée) que nous n'avons pas réussi à résoudre.
Pour des fonctions d'utilités définies sur l'ensemble des réels, [98] ont obtenus des résultats pour des fonctions bornées par au-dessus. Le cas de fonctions non-bornées (par au-dessus) reste donc ouvert. Comme cela a été évoqué plus haut, déjà dans le cas classique il n'est pas évident de remplacer l'hypothése traditionnelle $u(x) < \infty$ par une hypothése du type $E_P U(V_T^{x,\phi})$ qui ne soit pas trop restrictive. Dans le cadre à croyances multiples non-dominées cette difficulté est amplifiée par deux facteurs : d'abord parce que même la condition $u(x) < \infty$ ne peut pas fonctionner (pour des raisons similaires à celles qui sont exposées dans la Remarque 4.5.15) et ensuite parce que des problèmes techniques de mesurabilité apparaissent.

Chapitre 5

Finalement, dans le Chapitre 5 qui conclut cette dissertation, nous nous intéressons à la notion d'évaluation par indifférence d'utilité.
Les résultats principaux obtenus sont les suivants

Theorem 0.0.8 *Soit $G \in \mathcal{W}_T^{0,+}$ et $G \neq 0$ $\mathcal{Q}^T$-q.s. Supposons que les hypothèses 5.2.1, 5.2.2, 5.2.3 et 5.4.1 soient vraies ainsi que les hypothèses 5.4.2 and 5.4.4 pour un $x_0 > 0$.*
Alors pour tout $n \geq 1$, $p_n(G, x_0)$ est bien défini et $\lim_{n\to+\infty} p_n(G, x_0) = \pi(G)$.

Theorem 0.0.9 *Soit $G \in \mathcal{W}_T^{0,+}$ et $G \neq 0$ $\mathcal{Q}^T$-q.s. Supposons que les hypothèses 5.2.1, 5.2.2, 5.2.3 et 5.4.1 soient vraies. Supposons de plus que U_n est une fonction d'utilité non aléatoire pour tout $n \geq 1$ et que $\lim_{n\to\infty} r_n(x) = +\infty$ pour tout $x > 0$.*
Alors, $\lim_{n\to+\infty} p_n(G, x) = \pi(G)$ pour tout $x > 0$.

Les trois propositions suivantes sont également des résultats importants. La première est une conséquence des deux résultats précédents.

Proposition 0.0.10 *Soit $G \in \mathcal{W}_T^{0,+}$ et $G \neq 0$ $\mathcal{Q}^T$-q.s. Supposons que les hypothèses 5.2.1, 5.2.2, 5.2.3 et 5.4.1 soient vraies ainsi que les hypothèses 5.4.2 et 5.4.4 pour un $x_0 > 0$. Alors, pour tout $n \geq 1$, $p_n^B(G, x_0)$ est bien défini et $\lim_{n\to+\infty} p_n^B(G, x_0) = \pi^{sub}(G)$.*

Proposition 0.0.11 *Soit $x \geq 0$. Supposons que $\mathcal{A}(U, G, x) = \mathcal{A}(G, x)$ pour tout $G \in \mathcal{W}_T^0$.*
1. *Alors la fonction ρ_x est une mesure de risque monétaire sur $\mathcal{W}_T^0$.*
2. *Si les hypothèses 5.2.2 et 5.2.3 sont vérifiées et $u(0, x) > -\infty$ alors ρ_x est une mesure de risque monétaire convexe sur $\{G \in \mathcal{W}_T^0,\ u(-G, z) < \infty, \forall z \in \mathbb{R}\}$.*
3. *Si de plus nous supposons que $u(0, x) > -\infty$, $u(0, x - \delta) < u(0, x)$ pour tout $\delta > 0$, alors ρ_x est normalisée.*

Proposition 0.0.12 *Soient U_A, U_B des fonctions d'utilités non aléatoires définies sur $(0, \infty)$ et qui vérifient la Définition 5.2.11. Soit*

$$\mathcal{W}_T^+(U) := \mathcal{W}_T^{0,+} \cap \left\{ G,\ P(G(\cdot) < +\infty) = 1,\ E_P U^+(G(\cdot)) < \infty, \forall P \in \mathcal{Q}^T,\ \sup_{P \in \mathcal{Q}^T} E_P U^-(G(\cdot)) < \infty \right\}.$$

1. *Si pour tout $x > 0$, $r_A(x) \geq r_B(x)$, alors $e_A(G) \leq e_B(G)$ pour tout $G \in \mathcal{W}_T^+(U)$.*
2. *Si pour tout $G \in \mathcal{W}_T^+(U)$ $e_A(G) < e_B(G)$ alors $r_A(x) \geq r_B(x)$ pour tout $x > 0$.*

Rappelons tout d'abord la définition de la notion de prix de sur-réplication (dans un cadre de croyances multiples non-dominés) que nous avons déjà mentionné plus haut dans l'introduction : il correspond au prix minimum que demande un agent économique vendant un actif contingent afin de pouvoir le sur-répliquer en achetant et vendant dynamiquement dans le marché. En d'autres termes, la stratégie de sur-réplication doit permettre à l'agent de délivrer l'actif contingent dans toutes les situations possibles : c'est à dire tous les évènement qui ont une probablité strictement positive pour au moins une des lois de probabilité de l'ensemble considéré. Le prix d'indifference d'utilité, introduit initialement par [82], correspond à la quantité d'argent minimum que demande un agent économique vendant un actif contingent pour que, ajouté à sa richesse initiale, son utilité en vendant l'actif contingent et en traitant dynamiquement sur le marché soit supérieure ou égale à son utilité si elle n'avait rien fait. On voit donc qu'il s'agit d'un prix qui fait intervenir les préférences de l'agent économique et qui donc semble plus réaliste d'un point de vue pratique.
Le résultat principal obtenu montre que sous de bonnes conditions, le prix d'indifférence d'utilité converge vers le prix de sur-réplication dans le cadre de croyances multiples et non-dominées. Dans le cas de fonctions d'utilité non-aleatoires si l'aversion au risque de l'agent économique tends vers $+\infty$ alors les conditions d'application du théorème sont vérifées. Cela généralise un résultat bien connu dans le cas classique et correspond aussi à l'intuition initiale selon laquelle le prix de sur-réplication correspond à un agent ayant une aversion au risque infinie.
Nous commençons le chapitre par un rappel du cadre dans lequel nous travaillons. Le cadre est similaire à celui des Chapitres 3 et 4 toutefois les difficultés techniques seront moindres. Nous rappelons ensuite les définitions précises que nous utiliserons : le prix de sur-réplication (et le prix de sous-réplication qui correspond à la situation d'achat) le prix d'indifférence d'utilité pour l'acheteur et le vendeur. Nous faisons le lien entre prix d'utilité et mesure de risque, une notion importante en finance puisqu'elle permet de quantifier le risque d'une position donnée. Dans la Proposition 5.2.19, nous montrons que sous de bonnes hypothèses, le prix d'indifférence d'utilité est effectivement une mesure convexe de risque (dont la définition est rapellée en Définition 5.2.17).

Dans la Section 5.3, nous revisitons aussi la notion d'équivalent certain introduite dans [108]. La Proposition 5.3.2 que nous rappelons ci-dessous généralise le concept d'équivalent certain dans notre cadre à croyances multiples pour des fonctions d'utilités éventuellement aléatoires et en présence de croyances multiples et non-dominées. Nous obtenons en particulier en Proposition 5.3.4 une généralisation du résultat classique qui relie l'équivalent certain et le coéfficient d'aversion au risque. La Section 5.4 présente et demontre les differents résultats de convergence obtenus, ainsi que les hypothèses utilisées. En particulier, nous considérons des fonctions d'utilité éventuellement aléatoires ce qui complique la démonstration, mais une fois encore, est justifié sur le plan pratique. Le Théorème 5.4.8 constitue le résultat principal. L'hypothèse 5.33 remplace dans le cas de fonctions aléatoires la convergence vers l'infini du coefficient d'aversion au risque, ce qui est illustré dans le Théorème 5.4.10. La preuve du théorème de convergence est ensuite découpée en plusieurs étapes. En dehors des difficultés techniques liées à des problèmes d'integrabilité, le Lemma 5.2.6 est la clé de voûte de la preuve : il repose sur [25, Theorem 2.2]. Comme précisé plus haut, la preuve n'est pas aussi technique que celles du Chapitre 4. La caractérisation quantitative de l'arbitrage introduite au Chapitre 3 est à nouveau utilisée car elle permet de contrôler le comportement de la valeur du portefeuille de couverture et donc de résoudre les problèmes d'integrabilité.
L'approche adoptée dans ce chapitre est volontairement théorique. L'évaluation par indifférence d'utilité ayany un interêt pratique important nous nous efforcerons dans le cadre de future recherches nous nous efforcerons d'étudier et de proposer des modélisations qui permettent des applications concrètes des résultats obtenus.

Conclusion

En conclusion, rappelons tout d'abord que les difficultés techniques liées aux questions de mesurabilité sont bien entendu au coeur de cette dissertation. Beaucoup de preuves utilisent des outils fins de théorie de la mesure. Cela nécessite une approche minutieuse rendant les preuves souvent assez fastidieuses. Dans un souci de précision et de minutie, mais au risque de paraitre parfois trop prudent et sans aucun doute de rendre la lecture des preuves difficiles, nous avons détaillé la plupart de ces preuves.
Par ailleurs, si dans le Chapitre 2, c'est à dire dans un cadre classique sans croyances multiples, les problèmes de mesurabilité peuvent tous être (relativement) facilement réglés (ce qui n'est pas une surprise puisque l'on sait que dans ce cas les outils classiques tels que espérances conditionnelles et supremum essentiel sont adaptés), ce n'est pas toujours le cas dans un cadre à croyances multiples non-dominées. C'est bien sur particulièrement évident dans le Chapitre 4 où ces dif-

ficultés nous obligent souvent à des "gymnastiques" mathématiques délicates. La raison sous-jacente principale de ces difficultés est simple : les ensembles boreliens (respectivement universellement mesurables, voir la Section 3.2 dans le Chapitre 3 pour un rappel de leurs définitions) ne sont pas stables par projection. C'est à dire, si X, Y sont des espaces Polonais et si $A \in \mathcal{B}(X \times Y)$ (respectivement $X \in \mathcal{B}_c(X \times Y)$), on ne sait pas à priori si $Proj_X(A) \in \mathcal{B}(X)$ (respectivement $Proj_X(A) \in \mathcal{B}_c(X)$) et c'est la raison pour laquelle on doit introduire les ensembles analytiques qui sont eux stables pr projection. Toutefois le prix à payer est que l'ensemble des ensembles analytiques n'est pas stable par passage au complementaire.
Est-ce qu'il existe une alternative aux ensembles analytiques et à une utilisation intensive d'outils puissants de théorie de la mesure pour résoudre ces problémes tout en gardant le même degres de généralité ? La question reste ouverte. Nous ne sommes en tout cas pas parvenus à en trouver. Notons toutefois que si $O \subset X \times Y$ est un ensemble ouvert de $X \times Y$ (pour la topologie produit) alors $Proj_{X \times Y}(O)$ est encore ouvert dans X. Ainsi en introduisant des conditions de continuité on peut espérer remplacer les ensembles analytiques par des ouverts et ainsi simplifier considérablement toutes les questions techniques de mesurabilités. L'étude de cette question est bien entendu un axe naturel de recherches futures. Parmis les questions intéressantes et stimulantes sur cette thématique, citons à nouveau celle de l'obtention d'une preuve du Théorème Fondamental de l'évaluation des actifs financiers dans le cadre de croyances multiples non-dominées obtenus par [25] qui ne repose pas (autant) sur des outils de théorie de la mesure.
Dans le prolongement du travail effectué dans cette dissertation, l'étude du cas de fonctions d'utilités non-concaves ainsi que celui des fonctions d'utilités non-bornées définies sur l'ensemble des réels (et éventuellement non-concaves) sont bien entendus des axes naturels de recherches futures qui devraient être plus accessibles. L'introduction de distorsion est également (dans un cadre de croyance multiples non-dominées) une généralisation interéssante non seulement sur le plan mathématiques mais aussi en terme d'applications.
Finalement, il nous semble important et essentiel à plus moyen terme d'étudier et tester des applications peut-être plus concrètes et pratiques de ces résultats dans des domaines tels que la modélisation economique qui reste en plus de l'aspect purement mathématiques, l'une des principales motivations de cette dissertation.

CHAPTER 1

Introduction

This dissertation evolves around the following three main general thematics: uncertainty, utility and non-arbitrage. In this introduction, we propose a brief and general overview of these subjects and related literature. In particular, we will try to give a flavour of some of the innovative mathematical tools developed to treat problems of risk and randomness under the uncertainty of probability measures. Note that at this stage, however, we do not intend to give a mathematically rigorous presentation. On the contrary we hope that it could be accessible and of some interest for non-specialists. We will also emphasise the fact that these issues are not only of mathematical interest but are deeply enshrined in reality: not only do they relate to very concrete questions arising in financial markets concerning issues such as pricing, risk management and regulations but they can also be applied to a large range of other fields such as economics, theory of decision under risk, policy making and psychology amongst many others.

1.1 Randomness, risk and uncertainty

1.1.1 Some motivating examples

Randomness is a constant part of our life that appears in many domains and disciplines. During the 20th century, mathematicians have developed a set of very powerful tools to study and analyse these situations, namely probability and statistic. These tools have been an essential element to understand and model complex phenomenon from quantum mechanics to genetics or to the recent development of big data. In a similar spirit, finance and quantitative finance, mostly since the seventies, have triggered a tremendous amount of research and progress in these domains, which in turn have profoundly modified the way finance and financial market works and operates.

Recent researchs and developments in the domain of psychology, as made popular and accessible to a wide audience in [87] for instance or [125] for a more specific public, have led to a profound rethinking of how to model the way economic agents behave when facing randomness. Very roughly speaking, these studies show that our minds are ill-suited to properly deal with random events: we are easily "fooled by randomness" to quote again Nassim Taleb. Even when we think that we behave

rationally (whatever this means!), we are very often subject to various conscious or unconscious biases (confirmation bias, conjunction fallacy, illusion of understanding, illusion of validity,...). Somehow our brains are biologically not equipped to properly handle probability and statistics: think of how easily our intuition is wrong when it comes to issues involving conditional probabilities and Bayes's Theorem. The following famous excerpt from [p158][87] is truly puzzling and not only for mathematicians. *"Linda is 31 years old, single, outspoken and very bright. She majored in philosophy. As a student, she was deeply concerned with the issues of discrimination and social justice, and she participated in antinuclear demonstrations" Rank the following two descriptions in terms of the probability that they describe: a) Linda is a bank teller, b) Linda is a bank teller who is active in a feminist movement.* Surprisingly, some experiences have shown that very often b) is seen as more likely than a) despite the fact that a) describes a more general situation than b). There are many reasons for this and clearly the way the question is formulated is not neutral. We invite the interested and curious reader to dive into the whole book for further insight. As a second illustration of how “poorly" we perform when faced with randomness, we mention the results from [72]. Some students were invited to play a game where they are given upfront some money and can bet as much and as many times as they want within the next 30 minutes on coin toss. They are also told that the coin is biased and has a 0.6 probability of coming up with heads. The aim is obviously to maximise their gain at the end of the period (in the experience their maximum gain is capped). Note that the students involved were supposedly already familiar with concepts related to financial markets, asset management and underlying mathematical techniques and ideas. Still, the results of the experience were very disappointing with for instance almost one third of the students ending up the game with less money than they started with.
It is obviously out of our scope to go through an exhaustive review of these type of issues. But, in the next paragraph, we would like to illustrate further the kind of irrational mistakes typical "rational agents" do through a few simples examples that are also connected to the mathematical problems we will study. An interesting concept, particularly relevant when modelling the behaviour of agents, is the notion of risk aversion. We will see in Section 1.2 that in the expected utility paradigm it is related to the concavity property of the utility function and "rational agents" have for a long time been deemed to be risk adverse: they tend to prefer a sure thing rather than a risky bet. However, it was shown that if agents are generally risk adverse when it comes to gain (an agent prefers receiving 50 rather than tossing a fair coin where he will receive 100 or 0 with probability $1/2$), this is not the case when losses are involved. In this case, agents may actually be risk seeking. The agent tends to prefer playing a game where he can lose 100 or nothing each with probability $1/2$ rather than losing surely 50. Note that from a risk-management perspective this is a rather unfortunate and worrying feature

that needs to be seriously taken into account when looking at regulation issues. From a mathematical point of view, this is one of the motivation behind Chapter 2, where we study utility functions that are not assumed to be concave.
Moving to another concept, there is, both in economics and mathematics, a long history of debates on the distinction between objective and subjective probability: is probability an objective feature of the phenomena under study, or merely a subjective judgment of the decision maker. We will see in Section 1.2 how this plays out in the context of expected utility. The issue of probability distortion introduced by [88] in the context of cumulative prospect theory goes on step further: it has been shown indeed that not only can an agent have her own subjective probability but she can also mentally modify and distort it. In particular the agent tends to overweight the probability of rare events. From a mathematical point of view, it is modelled by introducing a non-decreasing function $f : [0,1] \to [0,1]$ such that $f(0) = 0$ and $f(1) = 1$ and replacing the probability $P(A)$ of an event with $c(A) = f(P(A))$. Note that by doing so, there is no reason for the set function c to still be a probability measure. This approach leads to the notion of capacity, Choquet Integrals (see [41]) and non-linear expectation that we will briefly present below.

1.1.2 Risk vs uncertainty, model risk and related mathematical developments

We now move to a fundamental distinction regarding random events that is at the heart of this PhD dissertation: namely the distinction between risk and uncertainty. More precisely, by risk we refer to the “known unknown", and by uncertainty to the “unknown unknown". The fatherhood of the concept is attributed to F. Knight ([90]) and often called Knightian uncertainty. Consider the following situation inspired by the Ellsberg Paradox that we will revisit later. You are asked to play a game where you have to choose between receiving surely 20 or drawing a ball from a box and receiving 100 if you pick a red ball and nothing otherwise. You are told that there 100 balls in the box in total and between 20 and 80 red balls. If you knew exactly how many red balls there are, your risk aversion would determine your choice: if there are only 20 red balls, most people would take the cash while if there are 50 or 80 red balls a larger proportion is likely to take the risk. But here the situation slightly differs. It is not only about risk aversion: how do you evaluate your expected gain from the bet? Do you consider the worst case (20 red balls), the best case (80 red balls) or some kind of average case (50 red balls)? In general faced with this kind of situation, an agent will demonstrate uncertainty aversion: she will tend to choose a situation where the uncertainty is reduced. For instance, if in the case where there are exactly 50 red balls she is happy to take the bet, this will not be the case in the situation described above (even if one could say that in average there is also 50 red balls in the box). We will come back in Section 1.2.1 on some of the consequences uncertainty aversion has when modelling agents preferences and behaviour. Note that it is easy to come up with concrete and real life examples of similar situations, illustrating how relevant the concept of uncertainty is.

This is not a new concept but it has re-emerged in the context of financial markets over the last 15 to 20 years. Indeed, it is a very appropriate framework to study model risk. Roughly speaking, we can somehow distinguish between two forms of uncertainty. The first one is related to robust questions: rather than fixing a given model, we take a set of models resulting from small perturbations of the initial reference model. This is a "mild" form of uncertainty where the deviation from the initial model is understood to be relatively limited. This can be the case for instance because we can only partially estimate the parameters of the model. But the issue of model risk can be of another and more serious kind: simply put the underlying assumption of the model do not describe accurately enough the phenomenon modelled (namely the price of a given asset). These questions of model risk are best illustrated with the issue of the volatility of financial assets (stocks, bonds, ...) where actually a lot of these problems were initiated. There is a very long history on the subject starting with the seminal work of [18] where the volatility of the underlying asset is assumed to be a constant. Very quickly, it was clear that the constant volatility assumption was wrong and that the model needed to be improved: indeed each option (*i.e* for a given strike and a maturity) on a given underlying was quoted with a specific volatility. New approaches were considered: from local volatility models (see [56]) where the volatility is a function of the value of the asset, to stochastic volatility models (such as the Hull and White model [83], the Heston model [77], and the SABR model [71]) where the volatility is itself random and to the recent work on "rough" volatility (see [65]), academics and practitioners alike have developed a wide range of techniques and model to deal with the issue. In all of these frameworks a robust approach consists of taking small variations of some of the parameters used in the model. This is actually applied in practice where risk managers and traders will use bid-offer or reserves (when the parameters are not directly tradable). But each of the above models relies on complex (sometimes hidden) assumptions and provides an accurate representation of the reality only under certain conditions. If one wants to fully understand (*i.e.* measure) the risk taken, somehow all of the models have to be taken into account. This is also what happen very often in practice, where a blend of all the models is used especially to account for period of acute stress where the dynamic of markets can become very surprising. But this needs to be formalised: for instance if the risk manager of a complex derivatives book uses different models to price and risk-manage positions depending on the characteristic of each product, how can he be sure that this is not leading to arbitrage opportunity? In a similar spirit but with

a rigorous analysis, the uncertain volatility approach (see [5], [93], on which we will come back later) is not directly modelling the dynamic of the volatility process but only assumes that the volatility of the underlying asset lies between certain bounds (this is similar to the previous example where we only knew that there was between 20 and 80 red balls).
These kinds of questions have followed the profound evolution of the way financial markets operates, becoming even more endogenous and seemingly disconnected from the underlying economics. Some of the recent extreme behaviour observed in markets such as so-called flash crashes or huge bout of volatility surrounding certain political events are without doubt also a source of concern at least from a regulation and stability point of view. In this context, the concept of uncertainty is also a very powerful framework to describe and model the behaviour of investors and their interactions, especially in periods of acute stress and uncertainty such as financial crisis. This was evidentely the case after the 2008 Great Financial Crisis since the investors's behaviours played a crucial role in its development (see for instance [29]). This is obviously a much deeper form of uncertainty that goes beyond the issue of model misspecification.
From a mathematical point of view, these questions have triggered interesting development to formalise and model the uncertainty of the underlying distribution. We would like to illustrate this through a brief overview of non-linear expectations and stochastic calculus under uncertainty which are important topics underlying the problems we will study. We obviously do not aim to give a mathematically rigorous presentation here. For more details we refer the reader to [103] and [104] and the references therein, we will also revisit more precisely some of these aspects later. Given a measurable space $(\Omega, \mathcal{S})$ representing the set of possible scenarii and a linear space $\mathcal{H}$ of real valued measurable functions defined on Ω (containing the constant functions) that represent the monetary outcome of a decision, we consider a sublinear expectation: $i.e$ a functional $\mathcal{E} : \mathcal{H} \to \mathbb{R}$ satisfying

- Monotonicity: if $X \geq Y$ then $\mathcal{E}(X) \geq \mathcal{E}(Y)$
- Constant preserving: for $c \in \mathbb{R}$, $\mathcal{E}(c) = c$
- Sub-additivity: for $X, Y \in \mathcal{H}$, $\mathcal{E}(X+Y) \leq \mathcal{E}(X) + \mathcal{E}(Y)$
- Positive homogeneity: for $\lambda \geq 0$, $X \in \mathcal{H}$, $\mathcal{E}(\lambda X) = \lambda \mathbb{E}(X)$.

The underlying motivation and link with uncertainty is as follow. One way to model uncertainty is indeed to introduce, rather than a single probability P measure defined on $(\Omega, \mathcal{S})$, a set of probability measures $\mathcal{P}$. Thus, for a real valued random variable X it seems natural to replace the expectation $E_P X$ with $\mathcal{E} = \sup_{P \in \mathcal{P}} E_P X$ [1].

[1]Note that $\inf_{P \in \mathcal{P}} E_P X = -\sup_{P \in \mathcal{P}} E_P(-X)$

Hence, the mean of a random variable $X \in \mathcal{H}$ is replaced by the mean-uncertainty interval $[-\mathcal{E}(-X), \mathcal{E}(X)]$, and its variance is replaced by the variance-uncertainty $[-\mathcal{E}(-X^2), \mathcal{E}(X^2)]$. Interestingly, it is shown in [103] how in this framework, most of the notions and tools of probability spaces can be extended. To name a few, the notions of identically distributed and independent random variables and of convergence in law in the sense of sublinear expectations can be introduced and one can seek to obtain a law of large numbers and a central limit theorem in the context of uncertainty. Similarly, the pendant of the normal law in the context of multiple-priors gives rise to the G-normal law, G-brownian motion and a G-Ito-integral can then be constructed using similar ideas as in the classical case.
The notion of sub-linear expectation is also strongly related to the problem of coherent risk-measure (see [62, Chapter 4] for a general presentation), which corresponds to a very acute and concrete question in quantitative finance, namely measuring the risk of a given financial position.
From a more probabilistic point of view, non-linear expectations are also related to the notion of capacity [2], the non-additive pendant of a probability measure. For instance, from the set of probability measures introduced previously, we can set $c(A) = \sup_{P \in \mathcal{P}} P(A)$ for all $A \in \mathcal{S}$ and another sublinear expectation can be defined (using the Choquet Integral) which is actually different from the previous one, even if they agree on the function of the form 1_A for $A \in \mathcal{S}$. We refer for instance to [70, Section 0.2] for more details and a general overview. Using the notion of capacity, one can also develop a quasi-sure stochastic analysis which is the pendant of the P-almost sure stochastic analysis. We refer here to [53], [51] and again [103] for more details on this. The notion of capacity is also useful when the set of probability measures $\mathcal{P}$ is non-dominated. Recall that when there is just one probability measure, the null-set consists of all the events whose probability is zero (for simplicity we do not mention the issue of measurability of these sets). When considering different models (probability measures), the distinction between the events that can or cannot happen is as important as the quantification of the probability of each event. As a typical illustration: does a given model allow for negative interest rate? Today's answer to this modelling question is probably very different to the one we would have given 10 or 15 years ago. When we deal with a set of probabilities measure, this distinction is crucial. If we are lucky, there exists some probability measure P^* such that for all $A \in \mathcal{S}$, $P(A^*) = 0$ implies that $P(A) = 0$ for all $P \in \mathcal{P}$. We say that $\mathcal{P}$ is dominated by P^* and P^* determines the events that are possible or not [3]. But if we are not lucky and our set of probability measures is non-dominated, then from a mathematical point of view the situation is more challenging. Indeed, the classical tools of probability theory such as conditional

[2] A capacity is a set function $c : \mathcal{S} \to [0,1]$, normalised ($c(\emptyset) = 0$, $c(\Omega) = 1$) and monotone ($c(A) \leq c(B)$ if $A \subset B$).

[3] Note that if $\mathcal{P}$ is finite or countable one can always find such a P^*

expectations or essential supremum are defined P-almost-surely ($i.e$ they are valid only for all events visible for P) and are therefore ill-suited for non-dominated set of probabilities measures. This leads to the non-trivial aggregation issue (see for instance [124] or [43]). Importantly, the issue of a non-dominated set of probability measures is not only of theoretical interest but linked to real life problems: in the already mentioned context of volatility uncertainty certain sets of probability measures are mutually singular. These aggregation problems and related measurability issues will be at the heart of the problem we face in Chapter 4 when trying to find a version of the value function having the required properties.

1.2 Utility functions

1.2.1 The concept of expected utility

We now move to the second fundamental topic of this PhD dissertation: the expected utility paradigm. It would be too long to retrace the full history and all the evolutions and ramifications of the concept over the last 50 to 60 years. We refer to [62, Chapter 2] for a detailed presentation and further references. Very briefly, the concept of utility is related to the numerical representation of the preferences of agents. The underlying axiomatic of the theory was initiated by [126]. We assume that the set of possible scenarios is given by a measurable space $(\Omega, \mathcal{F})$ and that each decisions (an investment for instance) is represented by some random variable $X : \Omega \to \mathbb{R}$ that corresponds to its monetary outcome. Furthermore, we assume that there exists an *objective* known probability P on $(\Omega, \mathcal{F})$ that describes the distribution of the monetary outcome of each decision. In other words the agent is facing risk and not uncertainty. Under the axiomatic proposed in [126], for a given agent, each decision can be represented by

$$u(X) = E_P U(X) \tag{1.1}$$

where $U : \mathbb{R} \to \mathbb{R}$ is a non-decreasing and concave function called a utility function that depends on the agent. Thus, $u(X)$ represents the expected utility of X and decision X will be preferred over Y if and only if $u(X) = E_P U(X) \geq u(Y) = E_P U(Y)$. An agent will always try to maximise her expected utility over all the actions X available to her. We usually distinguish between two cases: either the utility function is defined (and finite) on all $\mathbb{R}$ or it is only finite on (a, ∞) for some real number a (and equals $-\infty$ below a). In this case, it means that the agent cannot lose more than some capital a. Typical examples of utility functions are given by: logarithm, power functions (know as Constant Relative Risk Aversion utility) and exponential (Constant Absolute Risk Aversion). The underlying intuition of the expected utility is very simple: for each possible outcome ω, $X(\omega)$ is measured using its utility

$U(X(\omega))$ (rather than $X(\omega)$) and the agent then takes the average of all outcomes. This formulation provides for instance an answer to the St. Petersburg Paradox (see for instance [62]) where an agent is asked how much she is willing to pay to play a game where she will receive 2^n where n is the number of successive tails she will obtain when playing a coin toss. The expected value of the bet is infinite, still in practice no-one will pay more than a few euros to play it. The non-decreasing property of the utility function is relatively natural: more is deemed to be always better. The concavity of the utility function is related to risk-aversion: for an outcome X, we denote by $m(X) = E_P X$ its expected value under P. Risk aversion means that the agent prefers to receive $m(X)$ rather than X (unless X is not random). This is a crucial assumption. But, as already eluded to, agents are not always risk adverse in real life situations. This is one of the reasons why in Chapter 2 we will study the case of non-concave utility functions.
Irrespective of the risk-aversion, the previous representation presents serious flaws: most importantly, can we really assume the existence of an objective probability P ? The well-known Allais Paradox (see for instance [62, Example 2.32]) illustrates that some of the underlying assumptions of the von Neumann and Morgenstern representation are not verified in real life situation. To answer this L.J. Savage ([120]) proposed an improved approach. In his framework the assumption that the distributions of the outcome are known at time 0 is dropped (hence we do not assume the existence of an "objective" probability P) but under additional assumptions, he shows that the numerical representation of the agent's preferences can be written as

$$u(X) = E_Q U(X) \tag{1.2}$$

where Q is a *subjective* probability on $(\Omega, \mathcal{F})$ (that depends on the agent) and $U : \mathbb{R} \to \mathbb{R}$ is still a concave and monotone function if the preference relation is monotone and risk-adverse. In this framework, the agent is still a utility maximiser but she is using her own belief.
Unfortunately, the representation in (1.2) is still not satisfying. Indeed, the Ellsberg Paradox [62, Exemple 2.32] reveals that Savage's representation are not satisfied in concrete situations. The concept underpinning the Ellsberg Paradox is called *ambiguity aversion* and is somehow the pendant in presence of uncertainty of the risk aversion in presence of risk. An answer to this challenge and a new axiomatic of expected utility was provided by [69]. Under their axiomatic, the preference of the agent can be written as

$$u(X) = \inf_{P \in \mathcal{P}} E_P U(X) \tag{1.3}$$

where $\mathcal{P}$ is a *subjective* set of probability on $(\Omega, \mathcal{F})$ representing all the possible beliefs of the agent and $U : \mathbb{R} \to \mathbb{R}$ is still a concave and monotone function. In other words, the agent tries to maximise her *wort-case expected utility* according to her own beliefs. Note that in this case, her beliefs are still assumed to be represented by a probability measure, in other words there is no distortion: introducing uncertainty and distortion together would be a nice extension of this approach. Note that if the functionals (1.1) and (1.2) are linear this is not the case for (1.3) which is in the form already mentioned in Section 1.1.2.

As already mentioned previously, this approach can also be seen as a form of robust approach: rather than fixing a given model, we take a set of models resulting from small perturbations of the initial reference model. This is related to the work of [73]. The framework introduced in [69] was extended in [94] who introduced a penalty function $c(P)$ to the utility functional. Finally, in [39] the preferences are represented by a more general functional $\inf_{P\in\mathcal{Q}} G(E_P U(X), P)$ where G is a so-called uncertainty index reflecting the decision maker's attitudes toward uncertainty. This is obviously not the only answer to the criticism of the expected utility paradigm and we refer for example to [42] for insight on the various related and non-linear theory of utility developed such as Choquet expected utility theory, Yaari's dual theory [127] and Rank-dependent expected utility theory [109].

In the context of financial markets utility functions have not been used as much as some other techniques for pricing or risk management: one of the reasons being that they depend on the preferences of the agents and thus cannot provide general results. Note that this could also be seen as an advantage as it allows to consider agent with different preferences. They have gained in popularity when perfect hedging is not feasible (or reasonable) and partial hedging is considered with some residual risk tolerated. We illustrate this with the notion of utility indifference. For a general overview of the subject we refer to the book [36] and we will also revisit the concept in more detail in Chapter 5. Utility indifference pricing was first introduced in [82] and represents the minimal amount of money to be paid to an agent selling a contingent claim G so that added to her initial capital, her utility when selling G and hedging it by trading dynamically in the market is greater than or equal to the one she would get without selling the product. It is particularly useful in the context of incomplete markets (for instance you have traded a derivative on an instrument that you cannot trade directly but only use an other instrument as a proxy) and provide prices that are cheaper than the price obtained by trying to superreplicate the claim. These prices are more acceptable from a marketing point of view while still controlling in a way the downside risks (by using different utility functions for instance). Utility indifference pricing and utility functions are also related to the concept of risk measure that we have already mentioned. We will only briefly touch upon issues related to risk measure but it is a subject very close to our problems.

1.2.2 Mathematic literature

The problem of maximisation of expected utility has a very long history and has been extensively studied. In mathematical terms, $EU(X)$ needs to be maximised in X where X runs over values of admissible portfolios. We propose here a few references from the literature. For more detailed presentations of the discrete time and the continuous time cases, we refer to [62, Section 3] and [89]. It was initiated by [96] in a continuous time setting with power utility function and a geometric brownian dynamic for the risky asset (Merton-Black and Scholes model). In continuous time setting for incomplete markets and in a mono-prior setting [91] and [121] used a duality approach for utility functions defined on the half-real line and the whole real line. In this context, the concept of "asymptotic elasticity" plays a key role (see [91, Section 6] for further insight) and if the utility function U is differentiable it also has a nice economic interpretation as the ratio of "marginal utility": U' and the "average utility" $x \to \frac{U(x)}{x}$. We will come back to this more precisely in Chapter 2. Always in continuous time, a direct approach through the primal problem was used in [122] for utility functions defined on the half-real line and recently in [110] for utility functions defined on the whole-real line. Orlicz spaces were also introduced (see for instance [15]) as an appropriate framework for utility maximisation. Now, in discrete time (and with a single probability measure), [111], [112] work directly on the primal problem and used dynamic programming: we will follow a similar approach both in Chapter 2 and 4. We make a small comment on dynamic programming which is a powerful technic to solve multi-period problems. It corresponds to a form of induction as the optimisation part of the problem is somehow reduced to a one-step optimisation. However, it crucially relies on the time-consistency of the problem which is a very well-known issue for risk measure. Very roughly speaking, it means that if in the future the agent prefers a decision over another one, it has also to be true seen from today. We refer to the surveys [2] and [16] for detailed overviews. This will be also an important issue to keep in mind in a multiple-priors setting: in [113, Appendix D] a simple example illustrates what can happen if one is not careful enough on the structure of the initial set of priors. Most importantly in this case one cannot hope to solve an optimisation problem such as maximising expected utility by dynamic programming.

Finally, note that in most of these maximisation problems, the existence of an optimal trading strategy relies generally on some compacity argument that is implied by some form of no-arbitrage condition (we will revisit this when we study the so-called quantitative characterisation of the no-arbitrage in Chapter 3). This is one of the reason why we also study no-arbitrage question.

We now focus on some of the more recent developments on the question of maximising expected utility. In [105] and [106], a general duality framework is introduced that covers a large range of stochastic optimisation and mathematical

finance problem (amongst other utility maximisation) in discrete time and offers a very insightful approach. They combine convex analysis and measure theory technics (namely normal-integrand and horizon function see for instance [116]). Interestingly, to obtain the existence of an optimal solution (in a general setting) a linearity condition is introduced that once translated in the context of mathematical finance corresponds exactly to the notion of no-arbitrage even in the presence of transaction costs. As mentioned previously, the maximisation of non-concave utility functions is not only of theoretical interest but is supported by empirical evidence and has seen renewed interest. We cite for instance [33], [34] where the case of utility function defined on $\mathbb{R}$ and $(0, +\infty)$ are studied and solved using again dynamic programming arguments. In [107] the techniques of [106] and [105] cited previously are generalised.
Now in the case of uncertainty the problem has also triggered a lot of interest. As already mentioned previously, the distinction between the dominated case and non-dominated case is essential. In the dominated case we refer to [63] for a general survey: the technics used are usually built on the mono-prior case solutions. However, the non-dominated case is more involved and requires to adapt the traditional approach and techniques. In [52] the existence of an optimal strategy, a worst case measure as well as some “minmax" results are obtained under some compacity assumption on the set of probability measures and with a bounded (from above and below) utility function. Techniques related to quasi-sure stochastic analysis, the theory of capacity and also some duality results are used. In the context of volatility uncertainty, [95] obtained some results for specific utility functions (namely exponential, power utility and logarithmic utility) using second-order backward stochastic differential equations (BSDE) techniques. Finally we mention [59] that used similar technique to study the issue of recursive utility.
In the discrete time case, [99], using the framework introduced in [25], was able to obtain the first result using a functional as in (1.3) for a non-dominated and non-compact set of probability measures and for a utility function bounded from above and defined on the half real line with unbounded endowments. The proof uses dynamic programming as in [111], [112] but in a specific measure-theory setting and making extensive use of measurable selection arguments. The same framework has been used in [7] where the case of an exponential utility function defined on $\mathbb{R}$ is studied and some duality result obtained. Finally, [98] managed to generalise the approach used in [106] based on normal-integrand together with the framework introduced in [99] and obtained results for a utility function defined on both the real line and the half real line with frictions but for utility functions bounded from above. We will revisit some of these aspects in Chapter 4, where in a similar framework we tackle the case of unbounded utility functions. Extending the result for functional where we introduce the uncertainty index G as presented above leads to difficult questions related to the time-consistency of the functional involved and

therefore on the ability to use dynamic programming. This was done in [8] in the case where G takes the form of a penalty function.

1.3 Arbitrage

1.3.1 The concept of arbitrage in mathematical finance

In this section we give a brief overview of our third main concept: the notion of arbitrage. Arbitrage, or more accurately the absence of arbitrage opportunity is a fundamental concept in mathematical finance. Roughly speaking, it means that an investor cannot make a certain profit without taking risk, *i.e.* without facing the possibility of losing money in some scenarios. It is obviously an idealised view of markets: in practice there may be arbitrage opportunities for many reasons. However it is reasonable to assume that once the opportunity is detected some "arbitrageur" will quickly make the opportunity disappear. It is also interesting to study the notion of arbitrage when there are constraints: the case of no-short sale is practically important as regulators introduce very often such constraints in periods of acute stress.

The mathematical notion of no-arbitrage and the consequence in terms of pricing measure is at the root of development of modern finance since the seventies. It has profoundly altered the way financial markets work and contributed to the exponential development of the derivatives markets in particular. It has happened in conjunction with the development of complex mathematics which in turn have pushed market participants to gain confidence, sometimes blindly and despite recurrent crisis, in trading more and more complex financial products. This is without any doubt a source of concern but probably out of the scope of this dissertation.

In a few words, the intuition underpinning the arbitrage theory for derivatives pricing was initiated by [18]. A rigorous theory was then formalised in [74], [75] and [92]. One of the main results is the so-called Fundamental Theorem of Asset Pricing (FTAP in short) that makes the link between the notion of non-arbitrage and the risk-neutral probability measure (also called martingale measures, pricing measures): a model is arbitrage free if and only if the set of martingale measure is not empty. Roughly speaking a martingale measure is a probability measure that makes the discounted underlying asset a fair game, *i.e* where the value of the asset at time t is exactly the expected value of the asset a $t+1$ given the information available at time t. These results are now well-known and well established in the finance community, however their importance and revolutionary nature should not be underestimated. They introduced a profound shift in the way markets and financial markets operate.

1.3.2 Mathematic literature

We now give a small and incomplete list of references on the FTAP. Most of the proofs rely on the use of some version of the Hahn-Banach separation argument. In discrete time [46] were the first to obtain the result in a general setting. Their proof makes use of measure theory tool and measurable selection arguments. Alternative proofs of the theorem were later offered without making us of measurable selection: we mention for instance [86] and also [117] for a proof using utility functions. Anecdotally, when dealing with no-arbitrage in the multiple-priors case, the approach of [25] is really in the same spirit as [46]. Finding an alternative proof of [25, Theorem 3.1] without making use of heavy measure theory tools remains an open and interesting mathematical question.
In the discrete time setting the mathematical formulation of no-arbitrage is relatively straightforward. However, in continuous time and already in the mono-prior case, it is a bit more subtle and mathematically more involved. These questions are not at the heart of our study as we will remain in a discrete-time setting. Briefly, one of the main issue is related to finding the "good" set of trading strategies to be used in order to define arbitrage. This raises topological questions as well as problems related to stochastic integration. We refer to [49] for a detailed presentation and overview. The concept of arbitrage is also related to the notion of superreplication price of contingent claims (or equivalently the subreplication). Roughly speaking, the superreplication price of a contingent claim G is the minimun amount of money required so that an agent can put in place a trading strategy that will always deliver at maturity at least G. Hence, this is somehow the minimum price at which an agent that replicates the strategy and doesn't want to incur any loss would be willing to sell the contingent claim G (and the subreplication is the symmetric version for buying, $i.e$ the maximum price at which an agent would be willing to buy the contingent claim G). This concept is particularly relevant in incomplete markets, where for a given claim, one cannot find in general a strategy that exactly replicates it. Indeed, in complete markets the superreplication price is just the replication price which can be computed as the expectation of the discounted value of G under the (unique) risk neutral probability measure. The relation between arbitrage and the superreplication price (or subreplication price) is the so-called Superhedging Theorem that makes the link between the pricing measure and the set of on-arbitrage prices for a given contingent claim. In this case, there is dual representation of the lower and upper arbitrage free prices and the infimum of prices of super-hedging strategies is equal to the supremum over the martingale measures equivalent to the historical probability measure.
We will now focus on the concept of no-arbitrage under uncertainty. As already eluded to, studying no-arbitrage in the presence of uncertainty has triggered renewed interest. The first attempt is not recent and was first developed in [78],

[79]. This corresponds to a model independent approach, $i.e$ where no specific underlying model or set of models is chosen. The initial approach corresponds to a very practical question: given a risky asset, assuming that the prices of a certain set of vanilla options (puts and calls) for different dates and strikes are observed, can we establish t if an arbitrage free model that matches the observed prices exists? If this is the case, the second question is to describe the set of models that can fit the observed prices and in particular to find some bounds for the prices of exotic options on the same underlying (say a digital option for instance). This problem is related to the Skorokhod Embedding Problem. This area has been very active and many results have been obtained for specific exotic options. We refer to [80] for a detailed presentation and further references.
Model independent arbitrage has been studied in other settings also. In [1], the authors obtain a FTAP considering static trading on options and under the additional assumption that there exists an option with a super-linearly growing payoff (used to obtained some compacity). In [26], [27] and [28] a scenario based approach is chosen. Rather than selecting a model or a set of possible models, the authors consider a general set of admissible scenarii $\mathcal{S}$ which represents the relevant events (corresponding for instance to the agent's belief). Note that from a practical point of view, this corresponds to a very natural behaviour where a risk-manager or a trader will in certain circumstances manages its position based on some scenarii. An "arbitrage de la classe $\mathcal{S}$" is a trading strategy that will always be non-negative on $\mathcal{S}$ and strictly positive for some event in $\mathcal{S}$. This approach encompasses a wide ranges of settings (and therefore different kinds of arbitrage) as $\mathcal{S}$ can be determined by a given probability measure, a set of probability measure or some topological properties (see [28, Section 4.1]). In this framework various versions of the FTAP and of the superhedging duality results are obtained. In a similar spirit, a model free approach is also used in [40] where the notion of acceptance set is introduced. In a one-period setting and under some continuity assumption on the asset's payoff [114] also proves a multiple-priors FTAP using the concept of full-measure support martingale measure.
In [25] the uncertainty is modelled by introducing a (non-dominated) set of probability measures $\mathcal{P}$ and a multiple-priors version of the no-arbitrage denoted $NA(\mathcal{P})$ which seems like a very natural generalisation of the classical no-arbitrage ($i.e$ when $\mathcal{P}$ is reduced to a singleton). The corresponding FTAP yields a family $\mathcal{R}$ of martingale measures having the same polar sets (in a sense) as $\mathcal{P}$. This formulation of no-arbitrage is reminiscent of Hypothesis (H) introduced in a continuous time framework (and without the compacity assumption) in [52]. The technics and difficulties arising from this approach are at the heart of some of the topics we have already mentioned such as quasi-sure stochastic analysis, non-linear expectations, aggregation and will also be at the heart of our results (we will also propose in Chapters 4 and 3 an alternative definition of no-arbitrage that simplifies some of

the measurability issues). Still in [25], an extension of the classical duality result is also obtained where the multiple-priors superreplication price for european payoffs is equal to the supremum of the model prices, $i.e.$ their expectations computed under the measures of $\mathcal{R}$: their is no duality gap. The superreplication results have been extended to the case of an american payoff in [9]. However, in the context of multiple priors, the concept of semi-static strategy as it is defined raises some issues (see Remark 1.3.1). Note that this framework has also been extended in the continuous time case, see [14], where another generalisation of the FTAP has been proved allowing the set of martingales to visit a cemetery state invisible from the initial set of models.
To conclude this introduction on arbitrage, we would also like to mention that the dual problem to superhedging can also be seen as a so-called martingale optimal transport problem. This approach has proven to be very fruitful in recent years: we refer for instance to [10] in the discrete time case and to [64] in the continuous time.

Remark 1.3.1 A semi-static strategy refers to a strategy where some assets are traded dynamically (typically the stocks prices) while some other assets (typically some options) are traded statically: $i.e$ bought at time 0 and held to maturity. However and specifically in the context of uncertainty, the distinction between two types of assets seems problematic. Indeed, by studying uncertainty, one of the goals is to forego any specific assumptions on the underlying dynamics and in particular not to prevent "a priori" a given scenario or situation to happen because it seems unlikely or unrealistic. By considering static instruments we just remove a large set of scenarii. Furthermore this is not a realistic description of the way markets works and behave: each single instrument (a simple stock but also a vanilla option or even a more complex derivative product) has its own life and thus a bid-offer price everyday.
The no-arbitrage condition allowing all instruments to be dynamically traded would also provide much more informations (and constraints) on the dynamic of each options and stocks. Moreover, taking for instance [1, Definition 1.2] or [25, Definition 1.1], some form of "time-consistency" is broken since the arbitrage is only defined for a given time horizon, while nothing can be said for smaller period and in particular for a one-period. When considering european payoffs this is not an issue as everything depends on what will happen at maturity T. However considering american payoffs and at the same time instruments that cannot be traded after time 0 and whose price dynamics over the period is not taken into account sounds inconsistent. Unsurprisingly, this raises serious mathematical difficulties (see for instance [9, Remark 2.1] and also [81]). In particular the set of equivalent martingales measures fails to be stable by pasting which is an issue when looking at time-consistency issues, see for instance the discussion in Chapter 5. Some of these

problems could probably be avoided by allowing the less liquid instruments to be traded. Thus, introducing transaction costs as in [100] seems to be a more realistic model.

1.4 Brief overview of the dissertation

The work presented in this dissertation is the result of different papers (see [20], [19], [21], [22]) some of them submitted for publications. The content of the following chapters corresponds to a detailed version of these papers.
There are often repetitions and redundancies between different chapters: for instance the notations and set-up are to a large extend repeated between Section 3.2 in Chapter 3, Section 4.2 in Chapter 4 and Section 5.2.1 in Chapter 5 or the proof of certain results, especially between Chapter 2 and 4 use similar arguments. Similarly, the introduction of each chapter often recall some elements of this general introduction. Overall, we have chosen to do so, so that each chapter can be read independently from others.

In Chapter 2 we study in a mono-prior framework a non-concave and non-smooth random utility functions with domain of definition equal to the non-negative half-line. In this setting, we provide to the best of our knowledge, the most complete result on the existence of an optimal strategy. Our proof relies on dynamic programming and measurable selection arguments. The paper is self-contained and in particular we prove in this specific framework all the required results related to the no-arbitrage condition. From an economic point of view, we have already mentioned some underlying motivation for introducing utility functions that are not concave: agents are not always consistently risk adverse. They can for instance exhibit risk aversion above a certain threshold but not below (where somehow they have nothing to loose so to speak). It is interesting also to allow the utility function to jump. From a mathematical point of view, we use an approach based on measure theory rather than a traditional probabilistic approach that can be seen as a preparation ahead of Chapters 3 and 4. Technical details are often tedious but work well which is not surprising as we know that the usual tools from probability theory such as conditional expectations and essential supremum would also work well.

Chapter 3 can be seen as a prelude to Chapter 4. We consider a discrete-time financial market model with a finite horizon but under non-dominated model uncertainty: in other words, we introduce a multiple-priors setting. We first recall the definition of the quasi-sure no-arbitrage introduced in [25] and its local characterisation. We then introduce alternative definitions of the no-arbitrage: namely a quantitative and a geometric version and establish the equivalence between these different definitions. The so-called quantitative version of the no-arbitrage will be useful in Chapter 4 and we prove in this chapter some related measurability properties. As an application we establish also the existence of an optimal strategy for an unbounded utility function defined on the whole real line but for a one-period model with deterministic initial data. Finally, we introduce a different notion of no-arbitrage: the so-called strong no-arbitrage. It is more restrictive than the previous definition, however we illustrate how some of the technical measurability difficulties can be simplified. This will used and further developed in Chapter 4.

Then, in Chapter 4 we investigate a problem of maximising worst-case expected terminal concave utility in the framework introduced in Chapter 3: $i.e$ in a discrete-time financial market model with a finite horizon but under non-dominated model uncertainty. Here again, we use a dynamic programming framework together with measurable selection arguments to prove that under mild integrability conditions, an optimal portfolio exists for an unbounded utility function defined on the half-real line. We use the notion of the quasi-sure no-arbitrage introduced in [25] and build on the results of Chapter 3 on its quantitative formulation. We hope that from the previous introduction the underlying motivation for this result is clear. At a technical level, the issues arising are more involved than in Chapter 2. They stem from the difficulties in carrying out integrability and measurability properties through the dynamic programming. Finally, we apply our result together with the strong no-arbitrage condition introduced in Chapter 3 in a large range of settings.

In Chapter 5 we move on to the concept of utility indifference pricing. The main result of the chapter is to prove a convergence theorem: namely that the multiple-priors utility indifference price of a contingent claim converges under appropriate conditions to its multiple-priors superreplication prices: for non-random utility function this is the case when the risk aversion of the agents tends to infinity. We also review briefly the connection between utility indifference pricing and risk measure and revisit some important concepts such as certainty equivalent that we extend for random utility functions under uncertainty. In this chapter the technical aspects are lighter than in the previous three chapters and rely mostly on simple elements of quasi-sure stochastic analysis.

As general a comment, a lot of our proofs rely heavily on measure theory tools and can be sometimes tedious. At the risk of sounding sometimes to cautious and burdening the text, we have purposely spelled out a lot of details to be as accurate as possible.

CHAPTER 2

Non-concave optimal investment and no-arbitrage: a measure theoretical approach

This chapter is an extended version of [22] that has been submitted for publication.

We consider non-concave and non-smooth random utility functions with domain of definition equal to the non-negative half-line. We use a dynamic programming framework together with measurable selection arguments to establish both the no-arbitrage condition characterisation and the existence of an optimal portfolio in a (generically incomplete) discrete-time financial market model with finite time horizon.

2.1 Introduction

We consider investors trading in a multi-asset and discrete-time financial market. We revisit two classical problems: the characterisation of no-arbitrage and the maximisation of the expected utility of the terminal wealth of an investor.

We consider a general random, possibly non-concave and non-smooth utility function U, defined on the non-negative half-line (that can be "S-shaped" but our results apply to a broader class of utility functions e.g. to piecewise concave ones) and we provide sufficient conditions which guarantee the existence of an optimal strategy. Similar optimization problems constitute an area of intensive study in recent years, see e.g. [12] , [76], [85] and [35].

We are working in the setting of [34] and remove certain restrictive hypothesis. Furthermore, we use methods that are different from the ones in [111], [112], [33] and [34] where similar multistep problems were treated. In contrast to the existing literature, we propose to consider a probability space which is not necessarily complete.

We extend the paper of [34] in several directions. First, we propose an alternative integrability condition (see Assumption 2.4.8 and Proposition 2.6.1) to the rather restrictive one of [34] stipulating that $E^{-}U(\cdot, 0) < \infty$. The property $U(0) = -\infty$ holds for a number of important (non-random and concave) utility functions (logarithm, $-x^{\alpha}$ for $\alpha < 0$). It is a rather natural requirement since it expresses the fear of investor for defaulting ($i.e$ reaching 0). We also introduce a new (weaker) version of the asymptotic elasticity assumption (see Assumption 2.4.10). In particular, Assumption 2.4.10 holds true for concave functions (see Remark 2.4.15) and therefore our result extends the one obtained in [112] to random utility function and incomplete probability spaces. Next, we do not require that the value function is finite for all initial wealth as it was postulated in [34]; instead we only assumed the less restrictive and more tractable Assumption 2.4.7. Finally, instead of using some Carathéodory utility function U as in [34] ($i.e$ function measurable in ω and continuous in x, see [3, Definition 4.50, Lemma 4.51] for instance), we consider function which is measurable in ω and upper semicontinuous (usc in the rest of the chapter) in x. As U is also non-decreasing, we point out that this implies that U is jointly measurable in (ω, x). Note that in the case of complete sigma-algebra -U is then a normal integrand (see Definition 2.8.23 and Remark 2.8.24). This will play an important role in the dynamic programming part to obtain certain measurability properties. Allowing non-continuous U is unusual in the financial mathematics literature (though it is common in optimization). We highlight that this generalisation has a potential to model investor's behaviour which can change suddenly after reaching a desired wealth level. Such a change can be expressed by a jump of U at the given level.

To solve our optimisation problem, we use dynamic programming as in [111], [112], [33] and [34] but here we propose a different approach which provides simpler proofs. As in [99], we consider first a one period case with strategy in $\mathbb{R}^d$. Then we use dynamic programming and measurable selection arguments, namely the Aumann Theorem (see, for example [119, Corollary 1]) to solve the multi-period problem. Our modelisation of $(\Omega, \mathcal{F}, \mathfrak{F}, P)$ is more general than in [99] and Chapter 4 as there is only one probability measure and we don't have to postulate Borel space or analytic sets. We also use the same methodology to reprove classical results on no-arbitrage characterisation (see [111] and [84]) in our context of possibly incomplete sigma-algebras.

The chapter is organised as follows: in section 2.2 we introduce our setup; section 2.3 contains the main results on no-arbitrage; section 2.4 presents the main theorem on terminal wealth expected utility maximisation; section 2.5 establishes the existence of an optimal strategy for the one period case; we prove our main theorem on utility maximisation in section 2.6. Finally, section 2.8 collects some elements about generalised integral, random sets measurability and normal integrand (that will actually be used throughout the disseration) and the proof of Lemma 2.2.2. We propose also some theoretical results that are not directly used in this chapter.

2.2 Set-up

Fix a time horizon $T \in \mathbb{N}$ and let $(\Omega_t)_{1 \leq t \leq T}$ be a sequence of spaces and $(\mathcal{G}_t)_{1 \leq t \leq T}$ be a sequence of sigma-algebra where $\mathcal{G}_t$ is a sigma-algebra on Ω_t for all $t = 1, \dots, T$. For $t = 1, \dots, T$, we denote by Ω^t the t-fold Cartesian product

$$\Omega^t = \Omega_1 \times \ldots \times \Omega_t.$$

An element of Ω^t will be denoted by $\omega^t = (\omega_1, \dots, \omega_t)$ for $(\omega_1, \dots, \omega_t) \in \Omega_1 \times \ldots \times \Omega_t$. We also denote by $\mathcal{F}_t$ the product sigma-algebra on Ω^t

$$\mathcal{F}_t = \mathcal{G}_1 \otimes \ldots \otimes \mathcal{G}_t.$$

For the sake of simplicity we consider that the state $t = 0$ is deterministic and set $\Omega^0 := \{\omega_0\}$ and $\mathcal{F}_0 = \mathcal{G}_0 = \{\emptyset, \Omega^0\}$. To avoid heavy notations we will omit the dependency in ω_0 in the rest of the chapter. We denote by $\mathfrak{F}$ the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$.

Let P_1 be a probability measure on $\mathcal{F}_1$ and q_{t+1} be a stochastic kernel on $\mathcal{G}_{t+1} \times \Omega^t$ for $t = 1, \dots, T-1$. Namely we assume that for all $\omega^t \in \Omega^t$, $B \in \mathcal{G}_{t+1} \to q_{t+1}(B|\omega^t)$ is a probability measure on $\mathcal{G}_{t+1}$ and for all $B \in \mathcal{G}_{t+1}$, $\omega^t \in \Omega^t \to q_{t+1}(B|\omega^t)$ is $\mathcal{F}_t$-measurable. Here we DO NOT assume that $\mathcal{G}_1$ contains the null sets of P_1 and that $\mathcal{G}_{t+1}$ contains the null sets of $q_{t+1}(.|\omega^t)$ for all $\omega^t \in \Omega^t$. Then we define for $A \in \mathcal{F}_t$ the probability P_t by Fubini's Theorem for stochastic kernel (see Lemma 2.8.1).

$$P_t(A) = \int_{\Omega_1} \int_{\Omega_2} \cdots \int_{\Omega_t} 1_A(\omega_1, \dots, \omega_t) q_t(d\omega_t|\omega^{t-1}) \cdots q_2(d\omega_2|\omega^1) P_1(d\omega_1). \tag{2.1}$$

Finally $(\Omega, \mathcal{F}, \mathfrak{F}, P) := (\Omega^T, \mathcal{F}_T, \mathfrak{F}, P_T)$ will be our basic measurable space. The expectation under P_t will be denoted by E_{P_t} ; when $t = T$, we simply write E.

Remark 2.2.1 If we choose for Ω some Polish space, then any probability measure P can be decomposed in the form of (2.1) (see the measure decomposition theorem in [50, III.70-7]).

From now on the positive (resp. negative) part of some number or random variable X is denoted by X^+ (resp. X^-). We will also write $f^{\pm}(X)$ for $(f(X))^{\pm}$ for any random variable X and (possibly random) function f.

In the rest of the chapter we will use generalised integral: for some $f_t : \Omega^t \to \mathbb{R} \cup \{\pm\infty\}$, $\mathcal{F}_t$-measurable, such that $\int_{\Omega^t} f_t^+(\omega^t) P_t(d\omega^t) < \infty$ or $\int_{\Omega^t} f_t^-(\omega^t) P_t(d\omega^t) < \infty$, we define

$$\int_{\Omega^t} f_t(\omega^t) P_t(d\omega^t) := \int_{\Omega^t} f_t^+(\omega^t) P_t(d\omega^t) - \int_{\Omega^t} f_t^-(\omega^t) P_t(d\omega^t),$$

where the equality holds in $\mathbb{R} \cup \{\pm\infty\}$. We refer to Lemma 2.8.1, Definition 2.8.2 and Proposition 2.8.4 of the Appendix for more details and properties. In particular, if f_t is non-negative or if f_t is such that $\int_{\Omega^t} f_t^+(\omega^t)P_t(d\omega^t) < \infty$ (this will be the two cases of interest in the chapter) we can apply Fubini's Theorem [1] and we have

$$\int_{\Omega^t} f_t(\omega^t)P_t(d\omega^t) = \int_{\Omega_1}\int_{\Omega_2}\cdots\int_{\Omega_t} f_t(\omega_1,\dots,\omega_t)q_t(d\omega_t|\omega^{t-1})\cdots q_2(d\omega_2|\omega^1)P_1(d\omega_1),$$

where the equality holds in $[0,\infty]$ if f_t is non-negative and in $[-\infty,\infty)$ if $\int_{\Omega^t} f_t^+(\omega^t)P_t(d\omega^t) < \infty$.

Finally, we give some notations about completion of the probability space $(\Omega^t,\mathcal{F}_t,P_t)$ for some $t\in\{1,\dots,T\}$. We will denote by $\mathcal{N}_{P_t}$ the set of P_t negligible sets of Ω^t $i.e$

$$\mathcal{N}_{P_t} = \left\{N\subset\Omega^t,\ \exists M\in\mathcal{F}_t,\ N\subset M \text{ and } P_t(M)=0\right\}. \tag{2.2}$$

Let $\overline{\mathcal{F}}_t = \{A\cup N, A\in\mathcal{F}_t, N\in\mathcal{N}_{P_t}\}$ and $\overline{P}_t(A\cup N) = P_t(A)$ for $A\cup N\in\overline{\mathcal{F}}_t$. Then it is well known that $\overline{P}_t$ is a measure on $\overline{\mathcal{F}}_t$ which coincides with P_t on $\mathcal{F}_t$, that $(\Omega^t,\overline{\mathcal{F}}_t,\overline{P}_t)$ is a complete probability space and that $\overline{P}_t$ restricted to $\mathcal{N}_{P_t}$ is equal to zero.

For $t=0,\dots,T-1$, let Ξ_t be the set of $\mathcal{F}_t$-measurable random variables mapping Ω^t to $\mathbb{R}^d$.

The following lemma makes the link between conditional expectation and kernel. To do that, we introduce $\mathcal{F}_t^T$, the filtration on Ω^T associated to $\mathcal{F}_t$, defined by

$$\mathcal{F}_t^T = \mathcal{G}_1\otimes\ldots\otimes\mathcal{G}_t\otimes\{\emptyset,\Omega_{t+1}\}\ldots\otimes\{\emptyset,\Omega_T\}.$$

Let Ξ_t^T be the set of $\mathcal{F}_t^T$-measurable random variables from Ω^T to $\mathbb{R}^d$ and let also X_t $:\Omega^T\to\Omega_t$, $X_t(\omega_1,\dots,\omega_T)=\omega_t$ be the coordinate mapping corresponding to t. Then $\mathcal{F}_t^T = \sigma(X_1,\dots,X_t)$. So $h\in\Xi_t^T$ if and only if there exists some $g\in\Xi_t$ such that $h=g(X_1,\dots,X_t)$. This implies that $h(\omega^T)=g(\omega^t)$. For ease of notation we will identify h and g and also $\mathcal{F}_t$, $\mathcal{F}_t^T$, Ξ_t and Ξ_t^T.

Lemma 2.2.2 *Let $0\le s\le t\le T$. Let $h\in\Xi_t$ such that $\int_{\Omega^t} h^+dP_t<\infty$ then*

$$E(h|\mathcal{F}_s) = \varphi(X_1,\dots,X_s)\,P_s\ a.s.$$
$$\varphi(\omega_1,\dots,\omega_s) = \int_{\Omega_{s+1}\times\ldots\times\Omega_t} h(\omega_1,\dots,\omega_s,\omega_{s+1},\dots\omega_t)q_t(\omega_t|\omega^{t-1})\dots q_{s+1}(\omega_{s+1}|\omega^s).$$

Proof. For the sake of completeness, the proof is reported in Section 2.8.4 of the Appendix. □

[1] From now, we call Fubini's theorem the Fubini theorem for stochastic kernel (see eg Lemma 2.8.1, Proposition 2.8.4).

Let $\{S_t,\ 0 \le t \le T\}$ be a d-dimensional $\mathcal{F}_t$-adapted process representing the price of d risky securities in the financial market in consideration. There exists also a riskless asset for which we assume a constant price equal to 1, for the sake of simplicity. Without this assumption, all the developments below could be carried out using discounted prices. The notation $\Delta S_t := S_t - S_{t-1}$ will often be used. If $x, y \in \mathbb{R}^d$ then the concatenation xy stands for their scalar product. The symbol $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$ (or on $\mathbb{R}$).

Trading strategies are represented by d-dimensional predictable processes $(\phi_t)_{1\le t\le T}$, where ϕ_t^i denotes the investor's holdings in asset i at time t; predictability means that $\phi_t \in \Xi_{t-1}$. The family of all predictable trading strategies is denoted by Φ.

We assume that trading is self-financing. As the riskless asset's price is constant 1, the value at time t of a portfolio ϕ starting from initial capital $x \in \mathbb{R}$ is given by

$$V_t^{x,\phi} = x + \sum_{i=1}^{t} \phi_i \Delta S_i.$$

2.3 No-arbitrage condition

The following absence of arbitrage condition or NA condition is standard, it is equivalent to the existence of a risk-neutral measure in discrete-time markets with finite horizon, see e.g. [46].

(NA) *If* $V_T^{0,\phi} \ge 0$ *P-a.s. for some* $\phi \in \Phi$ *then* $V_T^{0,\phi} = 0$ *P-a.s.*

Remark 2.3.1 It is proved in [112, Proposition 1.1] that (NA) is equivalent to the no-arbitrage assumption which stipulates that no investor should be allowed to make a profit out of nothing and without risk, even with a budget constraint: for all $x_0 \ge 0$ if $\phi \in \Phi$ is such that with $V_T^{x_0,\phi} \ge x_0$ a.s., then $V_T^{x_0,\phi} = x_0$ a.s.

We now provide classical tools and results about the (NA) condition and its "concrete" local characterisation, see Proposition 2.3.7, that we will use in the rest of the chapter. We start by introducing a random set (see Definition 2.8.18) that is denoted D^{t+1} and is the smallest affine subspace of $\mathbb{R}^d$ containing the support of the distribution of $\Delta S_{t+1}(\omega^t, .)$ under $q_{t+1}(.|\omega^t)$ and If $D^{t+1}(\omega^t) = \mathbb{R}^d$ then, intuitively, there are no redundant assets. Otherwise, for $\phi_{t+1} \in \Xi_t$, one may always replace $\phi_{t+1}(\omega^t, \cdot)$ by its orthogonal projection $\phi_{t+1}^{\perp}(\omega^t, \cdot)$ on $D^{t+1}(\omega^t)$ without changing the portfolio value since $\phi_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t, \cdot) = \phi_{t+1}^{\perp}(\omega^t)\Delta S_{t+1}(\omega^t, \cdot)$, $q_{t+1}(\cdot|\omega^t)$ a.s., see Lemma 2.3.5 and Remark 2.5.3 below as well as [62, Remark 9.1].

Definition 2.3.2 Let $0 \le t \le T$ be fixed. We define the random set (recall Definition 2.8.18) $\widetilde{D}^{t+1} : \Omega^t \twoheadrightarrow \mathbb{R}^d$ by

$$\widetilde{D}^{t+1}(\omega^t) := \bigcap \left\{ A \subset \mathbb{R}^d,\ \text{closed},\ q_{t+1}\left(\Delta S_{t+1}(\omega^t, .) \in A|\omega^t) = 1\right) \right\}. \tag{2.3}$$

For $\omega^t \in \Omega^t$, $\widetilde{D}^{t+1}(\omega^t) \subset \mathbb{R}^d$ is the support of the distribution of $\Delta S_{t+1}(\omega^t, \cdot)$ under $q_{t+1}(\cdot|\omega^t)$. We also define the random set $D^{t+1} : \Omega^t \twoheadrightarrow \mathbb{R}^d$ by

$$D^{t+1}(\omega^t) := \text{Aff}\left(\widetilde{D}^{t+1}(\omega^t)\right), \tag{2.4}$$

where Aff denotes the affine hull of a set.

The following lemma establishes some important properties of $\widetilde{D}^{t+1}$ and D^{t+1} and in particular $Graph(D^{t+1}) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$. This result will be central in the proof of most of our results.

Lemma 2.3.3 *Let $0 \leq t \leq T$ be fixed. Let $\widetilde{D}^{t+1} : \Omega^t \twoheadrightarrow \mathbb{R}^d$ and $D^{t+1} : \Omega^t \twoheadrightarrow \mathbb{R}^d$ be the random sets defined in* (2.3) *and* (2.4) *of Definition 2.3.2. Then $\widetilde{D}^{t+1}$ and D^{t+1} are both non-empty, closed-valued and $\mathcal{F}_t$-measurable random sets (see Definition 2.8.19). In particular, $Graph(D^{t+1}) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$.*

Proof. We first prove that $\widetilde{D}^{t+1}$ is a non-empty, closed-valued and $\mathcal{F}_t$-measurable random set. It is clear from its definition (see (2.3)) that for all $\omega^t \in \Omega^t$, $\widetilde{D}^{t+1}(\omega^t)$ is a non-empty and closed subset of $\mathbb{R}^d$. We now show that $\widetilde{D}^{t+1}$ is measurable. Let O be a fixed open set in $\mathbb{R}^d$ and introduce

$$\begin{aligned}\mu_O : \omega^t \in \Omega^t \to \mu_O(\omega^t) &:= q_{t+1}\left(\Delta S_{t+1}(\omega^t, .) \in O|\omega^t\right) \\ &= \int_{\Omega_{t+1}} 1_{\Delta S_{t+1}(\cdot,\cdot)\in O}(\omega^t, \omega_{t+1}) q_{t+1}(d\omega_{t+1}|\omega^t).\end{aligned}$$

We prove that μ_O is $\mathcal{F}_t$-measurable. As $(\omega^t, \omega_{t+1}) \in \Omega^t \times \Omega_{t+1} \to \Delta S_{t+1}(\omega^t, \omega_{t+1})$ is $\mathcal{F}_t \otimes \mathcal{G}_{t+1}$-measurable and $O \in \mathcal{B}(\mathbb{R}^d)$, $(\omega^t, \omega_{t+1}) \to 1_{\Delta S_{t+1}(\cdot,\cdot)\in O}(\omega^t, \omega_{t+1})$ is $\mathcal{F}_t \otimes \mathcal{G}_{t+1}$-measurable and the result follows from Proposition 2.8.9.
By definition of $\widetilde{D}^{t+1}(\omega^t)$ we get that

$$\{\omega^t \in \Omega^t,\ \widetilde{D}^{t+1}(\omega^t) \cap O \neq \emptyset\} = \{\omega^t \in \Omega^t,\ \mu_O(\omega^t) > 0\} \in \mathcal{F}_t.$$

Next we prove that D^{t+1} is a non-empty, closed-valued and $\mathcal{F}_t$-measurable random set. Using (2.4), D^{t+1} is a non-empty and closed-valued random set. It remains to prove that D^{t+1} is $\mathcal{F}_t$-measurable. [2] As $\widetilde{D}^{t+1}$ is $\mathcal{F}_t$-measurable, applying the Castaing representation (see Proposition 2.8.20), we obtain a countable family of $\mathcal{F}_t$-measurable functions $(f_n)_{n\geq 1} : \Omega^t \to \mathbb{R}^d$ such that for all $\omega^t \in \Omega^t$, $\widetilde{D}^{t+1}(\omega^t) = \overline{\{f_n(\omega^t),\ n \geq 1\}}$ (where the closure is taken in $\mathbb{R}^d$ with respect to the usual topology). Let $\omega^t \in \Omega^t$ be fixed. It can be easily shown that

$$D^{t+1}(\omega^t) = \text{Aff}(\widetilde{D}^{t+1}(\omega^t)) = \overline{\left\{f_1(\omega^t) + \sum_{i=2}^{p} \lambda_i(f_i(\omega^t) - f_1(\omega^t)),\ (\lambda_2, \ldots, \lambda_p) \in \mathbb{Q}^{p-1},\ p \geq 2\right\}}.$$

[2] This is the proof of [116, Exercise 14.12] that we provide for sake of completeness as we will use it again.

So, using again the Castaing representation (see Proposition 2.8.20), we obtain that $D^{t+1}(\omega^t)$ is $\mathcal{F}_t$-measurable. From [116, Theorem 14.8] (see (2.72)), $Graph(D^{t+1}) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ (recall that D^{t+1} is closed-valued). □

In Lemma 2.3.4, which is used in the proof of Lemma 2.3.6 for projection purposes, we obtain a well-know result : for $\omega^t \in \Omega^t$ fixed and under a local version of (NA), $D^{t+1}(\omega^t)$ is a vector subspace of $\mathbb{R}^d$ (see for instance [62, Theorem 1.48] or [111, Proposition 3.2]). Then in Lemma 2.3.6 we prove that under the (NA) assumption, for P_t-almost all ω^t, $D^{t+1}(\omega^t)$ is a vector subspace of $\mathbb{R}^d$. We also provide a local version of the (NA) condition (see (2.8)). Note that Lemma 2.3.6 is a direct consequence of [111, Proposition 3.3] (which doesn't use measurable selection arguments and provides directly the $\mathcal{F}_t$ measurability of α_t) combined with Lemma 2.2.2. We propose alternative proofs of Lemmata 2.3.4, 2.3.5 and 2.3.6 which are coherent with our framework and our methodology. We will also revisit similar issues in Chapter 3 in the presence of uncertainty.

Lemma 2.3.4 *Let* $0 \leq t \leq T$ *and* $\omega^t \in \Omega^t$ *be fixed. Assume that for all* $h \in D^{t+1}(\omega^t)\backslash\{0\}$

$$q_{t+1}(h\Delta S_{t+1}(\omega^t, \cdot) \geq 0|\omega^t) < 1.$$

Then $0 \in D^{t+1}(\omega^t)$ *and the set* $D^{t+1}(\omega^t)$ *is actually a vector subspace of* $\mathbb{R}^d$.

Proof. Introduce $C^{t+1}(\omega^t) := \overline{\text{Conv}}(\widetilde{D}^{t+1}(\omega^t))$ the closed convex hull generated by $\widetilde{D}^{t+1}(\omega^t)$. As $C^{t+1}(\omega^t) \subset D^{t+1}(\omega^t)$ we will prove that $0 \in C^{t+1}(\omega^t)$. Since $C^{t+1}(\omega^t) \subset D^{t+1}(\omega^t)$, for all $h \in C^{t+1}(\omega^t)\backslash\{0\}$ we know by assumption that

$$q_{t+1}(h\Delta S_{t+1}(\omega^t, \cdot) \geq 0|\omega^t) < 1. \tag{2.5}$$

Thus if we find some $h_0 \in C^{t+1}(\omega^t)$ such that $q_{t+1}(h_0\Delta S_{t+1}(\omega^t, \cdot) \geq 0|\omega^t) = 1$ then $h_0 = 0$.
We distinguish two cases. First assume that for all $h \in \mathbb{R}^d$, $h \neq 0$, $q_{t+1}(h\Delta S_{t+1}(\omega^t, .) \geq 0|\omega^t) < 1$. Then the polar cone of $C^{t+1}(\omega^t)$, $i.e$ the set

$$\left(C^{t+1}(\omega^t)\right)^\circ := \{y \in \mathbb{R}^d,\ yx \leq 0, \forall\, x \in C^{t+1}(\omega^t)\}$$

is reduced to $\{0\}$. Indeed if this is not the case there exists $y_0 \in \mathbb{R}^d\backslash\{0\}$ such that $-y_0x \geq 0$ for all $x \in C^{t+1}(\omega^t)$. As $A := \{\omega_{t+1} \in \Omega_{t+1},\ \Delta S_{t+1}(\omega^t, \omega_{t+1}) \in \widetilde{D}^{t+1}(\omega^t)\} \subset \{\omega_{t+1} \in \Omega_{t+1},\ -y_0\Delta S_{t+1}(\omega^t, \omega_{t+1}) \geq 0\}$ and $q_{t+1}(A|\omega^t) = 1$ we obtain that $q_{t+1}(-y_0\Delta S_{t+1}(\omega^t, \cdot) \geq 0|\omega^t) = 1$ a contradiction. As $\left((C^{t+1}(\omega^t))^\circ\right)^\circ = \text{cone}\,(C^{t+1}(\omega^t))$ where cone $(C^{t+1}(\omega^t))$ denotes the cone generated by $C^{t+1}(\omega^t)$ we get that cone $(C^{t+1}(\omega^t)) = \mathbb{R}^d$. Let $u \neq 0 \in \text{cone}\,(C^{t+1}(\omega^t))$ then $-u \in \text{cone}\,(C^{t+1}(\omega^t))$ and there exist $\lambda_1 > 0$, $\lambda_2 > 0$ and $v_1,\ v_2 \in C^{t+1}(\omega^t)$ such that $u = \lambda_1 v_1$ and $-u = \lambda_2 v_2$. Thus $0 = \frac{\lambda_1}{\lambda_1+\lambda_2}v_1 + \frac{\lambda_2}{\lambda_1+\lambda_2}v_2 \in C^{t+1}(\omega^t)$ by convexity of $C^{t+1}(\omega^t)$.

Now we assume that there exists some $h_0 \in \mathbb{R}^d$, $h_0 \neq 0$ such that $q_{t+1}(h_0 \Delta S_{t+1}(\omega^t,.) \geq 0|\omega^t) = 1$. Note that since $h_0 \in \mathbb{R}^d$ we cannot use (2.5). Introduce the orthogonal projection on $C^{t+1}(\omega^t)$ (recall that $C^{t+1}(\omega^t)$ is a closed convex subset of $\mathbb{R}^d$)

$$p : h \in \mathbb{R}^d \to p(h) \in C^{t+1}(\omega^t).$$

Then p is continuous and we have $(h - p(h))\,(x - p(h)) \leq 0$ for all $x \in C^{t+1}(\omega^t)$. Fix $\omega_{t+1} \in \{\omega_{t+1} \in \Omega_{t+1},\ \Delta S_{t+1}(\omega^t, \omega_{t+1}) \in \widetilde{D}^{t+1}(\omega^t)\} \cap \{\omega_{t+1} \in \Omega_{t+1},\ h_0 \Delta S_{t+1}(\omega^t, \omega_{t+1}) \geq 0\}$ and $\lambda \geq 0$. Let $h = \lambda h_0$ and $x = \Delta S_{t+1}(\omega^t, \omega_{t+1}) \in C^{t+1}(\omega^t)$ in the previous equation, we obtain (recall that $\widetilde{D}^{t+1}(\omega^t) \subset C^{t+1}(\omega^t)$)

$$\begin{aligned} 0 \leq \lambda h_0 \Delta S_{t+1}(\omega^t, \omega_{t+1}) &= (\lambda h_0 - p(\lambda h_0))\, \Delta S_{t+1}(\omega^t, \omega_{t+1}) + p(\lambda h_0) \Delta S_{t+1}(\omega^t, \omega_{t+1}) \\ &\leq (\lambda h_0 - p(\lambda h_0))\, p(\lambda h_0) + p(\lambda h_0) \Delta S_{t+1}(\omega^t, \omega_{t+1}). \end{aligned}$$

As this is true for all $\lambda \geq 0$ we may take the limit when λ goes to zero and use the continuity of p

$$p(0) \Delta S_{t+1}(\omega^t, \omega_{t+1}) \geq |p(0)|^2 \geq 0$$

As $q_{t+1}\left(\left\{\omega_{t+1} \in \Omega_{t+1},\ \Delta S_{t+1}(\omega^t, \omega_{t+1}) \in \widetilde{D}^{t+1}(\omega^t)\right\} |\omega^t\right) = 1$ by definition of $\widetilde{D}^{t+1}(\omega^t)$ and as $q_{t+1}(h_0 \Delta S_{t+1}(\omega^t,.) \geq 0|\omega^t) = 1$ as well we have obtained that

$$q_{t+1}(p(0) \Delta S_{t+1}(\omega^t, \cdot) \geq 0|\omega^t) = 1.$$

The fact that $p(0) \in C^{t+1}(\omega^t)$ together with (2.5) implies that $p(0) = 0$ and $0 \in C^{t+1}(\omega^t)$ follows. □

We introduce now for all $\omega^t \in \Omega^t$ the orthogonal space of $D^{t+1}(\omega^t)$

$$L^{t+1}(\omega^t) := \left(D^{t+1}(\omega^t)\right)^{\perp}. \tag{2.6}$$

and we prove the following lemma that will be used in the proof of Lemma 2.3.6. It corresponds to [99, Lemma 2.5] adapted to our setting.

Lemma 2.3.5 *Let $\omega^t \in \Omega^t$ be fixed. Then for $h \in \mathbb{R}^d$ we have that*

$$q_{t+1}(h \Delta S_{t+1}(\omega^t, \cdot) = 0|\omega^t) = 1 \iff h \in L^{t+1}(\omega^t).$$

Proof. Assume that $h \in L^{t+1}(\omega^t)$. Then $\{\omega \in \Omega_t,\ \Delta S_{t+1}(\omega^t, \omega) \in D^{t+1}(\omega^t)\} \subset \{\omega \in \Omega_t, h \Delta S_{t+1}(\omega^t, \omega) = 0\}$. As by definition of $D^{t+1}(\omega^t)$, $q_{t+1}(\Delta S_{t+1}(\omega^t,.) \in D^{t+1}(\omega^t)|\omega^t) = 1$, we conclude that $q_{t+1}(h \Delta S_{t+1}(\omega^t,.) = 0|\omega^t) = 1$. Conversely, we assume that $h \notin L^{t+1}(\omega^t)$ and we show that $q_{t+1}(h \Delta S_{t+1}(\omega^t,.) = 0|\omega^t) < 1$. We first show that there exists $v \in \widetilde{D}^{t+1}(\omega^t)$ such that $hv \neq 0$. If not, for all $v \in \widetilde{D}^{t+1}(\omega^t)$, $hv = 0$ and for any $w \in D^{t+1}(\omega^t)$ with $w = \sum_{i=1}^m \lambda_i v_i$ where $\lambda_i \in \mathbb{R}$, $\sum_{i=1}^m \lambda_i = 1$ and $v_i \in \widetilde{D}^{t+1}(\omega^t)$, we get that $hw = 0$, a contradiction. Furthermore there exists an open ball centered in v with radius $\varepsilon > 0$, $B(v,\varepsilon)$, such that $hv' \neq 0$ for all $v' \in B(v,\varepsilon)$. Assume that $q_{t+1}(\Delta S_{t+1}(\omega^t,.) \in B(v,\varepsilon)|\omega^t) = 0$ or equivalently that $q_{t+1}(\Delta S_{t+1}(\omega^t,.) \in \mathbb{R}^d \setminus B(v,\varepsilon)|\omega^t) = 1$. By definition of the support, $\widetilde{D}^{t+1}(\omega^t) \subset \mathbb{R}^d \setminus B(v,\varepsilon)$: this contradicts $v \in \widetilde{D}^{t+1}(\omega^t)$. Therefore $q_{t+1}(\Delta S_{t+1}(\omega^t,.) \in B(v,\varepsilon)|\omega^t) > 0$. Let $\omega \in \{\Delta S_{t+1}(\omega^t,.) \in B(v,\varepsilon)\}$, then $h\Delta S_{t+1}(\omega^t,\omega) \neq 0$ *i.e* $q_{t+1}(h\Delta S_{t+1}(\omega^t,.) = 0|\omega^t)) < 1$. □

Lemma 2.3.6 *Assume that the (NA) condition holds true. Then for all* $0 \leq t \leq T-1$*, there exists a full measure set* Ω^t_{NA1} *such that for all* $\omega^t \in \Omega^t_{NA1}$, $0 \in D^{t+1}(\omega^t)$*, i.e* $D^{t+1}(\omega^t)$ *is a vector space of* $\mathbb{R}^d$*. Moreover, for all* $\omega^t \in \Omega^t_{NA1}$ *and all* $h \in \mathbb{R}^d$ *we get that*

$$q_{t+1}(h\Delta S_{t+1}(\omega^t,\cdot) \geq 0|\omega^t) = 1 \Rightarrow q_{t+1}(h\Delta S_{t+1}(\omega^t,\cdot) = 0|\omega^t) = 1. \tag{2.7}$$

In particular, if $\omega^t \in \Omega^t_{NA1}$ *and* $h \in D^{t+1}(\omega^t)$ *we obtain that*

$$q_{t+1}(h\Delta S_{t+1}(\omega^t,\cdot) \geq 0|\omega^t) = 1 \Rightarrow h = 0. \tag{2.8}$$

Proof. Let $0 \leq t \leq T$ be fixed. Recall that $\overline{\mathcal{F}}_t$ is the P_t-completion of $\mathcal{F}_t$ and that $\overline{P}_t$ is the (unique) extension of P_t to $\overline{\mathcal{F}}_t$. We introduce the following random set Π^t

$$\Pi^t := \left\{\omega^t \in \Omega^t,\ \exists h \in D^{t+1}(\omega^t), h \neq 0,\ q_{t+1}(h\Delta S_{t+1}(\omega^t,\cdot) \geq 0|\omega^t) = 1\right\}.$$

Assume for a moment that $\Pi^t \in \overline{\mathcal{F}}_t$ and that $\overline{P}_t(\Pi^t) = 0$ (this will be proven below). Let $\omega^t \in \Omega^t \setminus \Pi^t$. The fact that $0 \in D^{t+1}(\omega^t)$ is a direct consequence of the definition of Π^t and of Lemma 2.3.4. We now prove (2.7). Let $h \in \mathbb{R}^d$ be fixed such that $q_{t+1}(h\Delta S_{t+1}(\omega^t,\cdot) \geq 0|\omega^t) = 1$. We prove that $q_{t+1}(h\Delta S_{t+1}(\omega^t,\cdot) = 0|\omega^t) = 1$. If $h = 0$ this is straightforward. If $h \in D^{t+1}(\omega^t) \setminus \{0\}$, $\omega^t \in \Pi^t$ which is impossible. Now if $h \notin D^{t+1}(\omega^t)$ and $h \neq 0$, let h' be the orthogonal projection of h on $D^{t+1}(\omega^t)$ (recall that since $\omega^t \notin \Pi^t$, $D^{t+1}(\omega^t)$ is a vector subspace). We first show that $q_{t+1}(h'\Delta S_{t+1}(\omega^t,\cdot) \geq 0|\omega^t) = 1$. Indeed, if it were not the case the set $B := \{\omega_{t+1} \in \Omega_{t+1},\ h'\Delta S_{t+1}(\omega^t,\omega_{t+1}) < 0\}$ would verify $q_{t+1}(B|\omega^t) > 0$. Recall that $L^{t+1}(\omega^t)$ is the orthogonal space of $D^{t+1}(\omega^t)$ (see (2.6)) As $(h-h') \in L^{t+1}(\omega^t)$ (recall that $D^{t+1}(\omega^t)$ is a vector subspace), by Lemma 2.3.5 the set $A := \{\omega_{t+1} \in \Omega_{t+1},\ (h-h')\Delta S_{t+1}(\omega^t,\omega_{t+1}) = 0\}$ verify $q_{t+1}(A|\omega^t) = 1$. We would therefore obtain that $q_{t+1}(A \cap B|\omega^t) > 0$ which implies that $q_{t+1}(h\Delta S_{t+1}(\omega^t,.) \geq 0|\omega^t) < 1$, a contradiction. Thus $q_{t+1}(h'\Delta S_{t+1}(\omega^t,\cdot) \geq 0|\omega^t) = 1$. If $h' \neq 0$ as $h' \in D^{t+1}(\omega^t)$, $\omega^t \in \Pi^t$ which is again a contradiction. Thus $h' = 0$ and as $A \cap \{h'\Delta S_{t+1}(\omega^t,\cdot) = 0\} \subset \{h\Delta S_{t+1}(\omega^t,\cdot) = 0\}$, $q_{t+1}(h\Delta S_{t+1}(\omega^t,\cdot) = 0|\omega^t) = 1$.
As $\Omega^t \setminus \Pi^t \in \overline{\mathcal{F}_t}$ there exists $\Omega^t_{NA1} \in \mathcal{F}_t$ and $N^t \in \mathcal{N}_{P_t}$ (the collection of negligible set of (Ω^t, P_t)) such that $\Omega^t \setminus \Pi^t = \Omega^t_{NA1} \cup N^t$ and $P_t(\Omega^t_{NA1}) = \overline{P}_t(\Omega^t \setminus \Pi^t) = 1$. Since $\Omega^t_{NA1} \subset \Omega^t \setminus \Pi^t$, it follows that for all $\omega^t \in \Omega^t_{NA1}$, $0 \in D^{t+1}(\omega^t)$ and for all $h \in \mathbb{R}^d$, (2.7) holds true.

We prove (2.8). Assume now that $\omega^t \in \Omega^t_{NA1}$ and $h \in D^{t+1}(\omega^t)$ are such that $q_{t+1}(h\Delta S_{t+1}(\omega^t,\cdot) \geq 0|\omega^t) = 1$. Using (2.7) and Lemma 2.3.5 we get that $h \in L^{t+1}(\omega^t)$. So $h \in D^{t+1}(\omega^t) \cap L^{t+1}(\omega^t) = \{0\}$ and (2.8) holds true.

It remains to prove that $\Pi^t \in \overline{\mathcal{F}}_t$ and $\overline{P}_t(\Pi^t) = 0$. To do that we introduce the following random set $H^t : \Omega^t \twoheadrightarrow \mathbb{R}^d$

$$H^t(\omega^t) := \left\{h \in D^{t+1}(\omega^t),\ h \neq 0,\ q_{t+1}(h\Delta S_{t+1}(\omega^t,\cdot) \geq 0|\omega^t) = 1\right\}.$$

Then

$$\Pi^t = \left\{\omega^t \in \Omega^t,\ H^t(\omega^t) \neq \emptyset\right\} = \textbf{proj}_{|\Omega^t} Graph(H^t)$$

since $Graph(H^t) = \{(\omega^t, h) \in \Omega^t \times \mathbb{R}^d,\ h \in H^t(\omega^t)\}$.

We prove now that $Graph(H^t) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$. Indeed, we can rewrite that

$$Graph(H^t) = Graph(D^{t+1}) \bigcap \{(\omega^t, h) \in \Omega^t \times \mathbb{R}^d,\ q_{t+1}(h\Delta S_{t+1}(\omega^t,\cdot) \geq 0|\omega^t) = 1\} \bigcap \left(\Omega^t \times \mathbb{R}^d \backslash \{0\}\right).$$

As from Lemma 2.8.9, $\left\{(\omega^t, h) \in \Omega^t \times \mathbb{R}^d,\ q_{t+1}(h\Delta S_{t+1}(\omega^t,\cdot) \geq 0|\omega^t) = 1\right\} \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ and from Lemma 2.3.3, $Graph(D^{t+1}) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$, we obtain that $Graph(H^t) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$. The Projection Theorem (see for example [37, Theorem 3.23]) applies and $\Pi^t = \{H^t \neq \emptyset\} = \textbf{proj}_{|\Omega^t} Graph(H^t) \in \overline{\mathcal{F}}_t$. From the Aumann Theorem (see [119, Corollary 1]) there exists a $\overline{\mathcal{F}}_t$-measurable selector $\overline{h}_{t+1} : \Pi^t \to \mathbb{R}^d$ such that $\overline{h}_{t+1}(\omega^t) \in H^t(\omega^t)$ for every $\omega^t \in \Pi^t$. We now extend $\overline{h}_{t+1}$ on Ω^t by setting $\overline{h}_{t+1}(\omega^t) = 0$ for $\omega^t \in \Omega^t \backslash \Pi^t$. It is clear that $\overline{h}_{t+1}$ remains $\overline{\mathcal{F}}_t$-measurable. Applying Lemma 2.8.10, there exists $h_{t+1} : \Omega^t \to \mathbb{R}^d$ which is $\mathcal{F}_t$-measurable and satisfies $h_{t+1} = \overline{h}_{t+1}$ P_t-almost surely. Then if we set

$$\varphi(\omega^t) = q_{t+1}(h_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t,.) \geq 0|\omega^t),$$
$$\overline{\varphi}(\omega^t) = q_{t+1}(\overline{h}_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t,.) \geq 0|\omega^t),$$

we get from Proposition 2.8.9 that φ is $\mathcal{F}_t$-measurable and from Proposition 2.8.6 *iii)* that $\overline{\varphi}$ is $\overline{\mathcal{F}}_t$-measurable. Furthermore as $\{\omega^t \in \Omega^t,\ \varphi(\omega^t) \neq \overline{\varphi}(\omega^t)\} \subset \{\omega^t \in \Omega^t,\ h_t(\omega^t) \neq \overline{h}_{t+1}(\omega^t)\}$, $\varphi = \overline{\varphi}$ P_t-almost surely. This implies that $\int_{\Omega^t} \overline{\varphi} d\overline{P}_t = \int_{\Omega^t} \varphi dP_t$. Now we define the predictable process $(\phi_t)_{1\leq t\leq T}$ by $\phi_{t+1} = h_{t+1}$ and $\phi_i = 0$ for $i \neq t+1$. Then

$$P(V_T^{0,\phi} \geq 0) = P(h_{t+1}\Delta S_{t+1} \geq 0) = P_{t+1}(h_{t+1}\Delta S_{t+1} \geq 0) = \int_{\Omega^t} \varphi(\omega^t)P_t(d\omega^t) = \int_{\Omega^t} \overline{\varphi}(\omega^t)\overline{P}_t(d\omega^t).$$

Thus

$$\begin{aligned} P(V_T^{0,\phi} \geq 0) &= \int_{\Pi^t} q_{t+1}\left(\overline{h}_t(\omega^t)\Delta S_{t+1}(\omega^t,\cdot) \geq 0|\omega^t\right)\overline{P}_t(d\omega^t) + \int_{\Omega^t\backslash\Pi^t} q_{t+1}\left(0 \geq 0|\omega^t\right)\overline{P}_t(d\omega^t) \\ &= \overline{P}_t(\Pi^t) + \overline{P}_t(\Omega^t \setminus \Pi^t) = 1, \end{aligned}$$

where we have used that if $\omega^t \in \Pi^t$, $\overline{h}_{t+1}(\omega^t) \in H^t(\omega^t)$ and otherwise $\overline{h}_{t+1}(\omega^t) = 0$. With the same arguments we obtain that

$$\begin{aligned}P(V_T^{0,\phi} > 0) &= P_t(h_{t+1}\Delta S_{t+1} > 0)\\ &= \int_{\Pi^t} q_{t+1}\left(\overline{h}_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t,\cdot) > 0|\omega^t\right)\overline{P}_t(d\omega^t) + \int_{\Omega^t\setminus\Pi^t} q_{t+1}\left(0>0|\omega^t\right)\overline{P}_t(d\omega^t)\\ &= \int_{\Pi^t} q_{t+1}\left(\overline{h}_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t,\cdot) > 0|\omega^t\right)\overline{P}_t(d\omega^t).\end{aligned}$$

Let $\omega^t \in \Pi^t$ then $q_{t+1}\left(\overline{h}_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t,\cdot) > 0|\omega^t\right) > 0$. Indeed, if it is not the case then $q_{t+1}\left(\overline{h}_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t,\cdot) \leq 0|\omega^t\right) = 1$. As $\omega^t \in \Pi^t$, $\overline{h}_{t+1}(\omega^t) \in D^{t+1}(\omega^t)$ and $q_{t+1}\left(\overline{h}_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t,\cdot) \geq 0|\omega^t\right) = 1$, Lemma 2.3.5 applies and $\overline{h}_{t+1}(\omega^t) \in L^{t+1}(\omega^t)$. Thus we get that $\overline{h}_{t+1}(\omega^t) \in L^{t+1}(\omega^t) \cap D^{t+1}(\omega^t) = \{0\}$, a contradiction. So if $\overline{P}_t(\Pi^t) > 0$ we obtain that $P(V_T^{0,\phi} > 0) > 0$. This contradicts the (NA) condition and we obtain $\overline{P}_t(\Pi^t) = 0$, the required result. □

Similarly as in [111] and [84], we prove a "quantitative" characterisation of (NA).

Proposition 2.3.7 *Assume that the (NA) condition holds true and let $0 \leq t \leq T$. Then there exists $\Omega^t_{NA} \in \mathcal{F}_t$ with $P_t(\Omega^t_{NA}) = 1$ and $\Omega^t_{NA} \subset \Omega^t_{NA1}$ (see Lemma 2.3.6 for the definition of Ω^t_{NA1}) such that for all $\omega^t \in \Omega^t_{NA}$, there exists $\alpha_t(\omega^t) \in (0,1]$ such that for all $h \in D^{t+1}(\omega^t)$*

$$q_{t+1}\big(h\Delta S_{t+1}(\omega^t,\cdot) \leq -\alpha_t(\omega^t)|h||\omega^t\big) \geq \alpha_t(\omega^t). \tag{2.9}$$

Furthermore $\omega^t \to \alpha_t(\omega^t)$ is $\mathcal{F}_t$-measurable.

Proof. Let $\omega^t \in \Omega^t_{NA1}$ be fixed (Ω^t_{NA1} is defined in Lemma 2.3.6).
Step 1 : Proof of (2.9). Introduce the following set for $n \geq 1$

$$A_n(\omega^t) := \left\{h \in D^{t+1}(\omega^t),\ |h| = 1,\ q_{t+1}\left(h\Delta S_{t+1}(\omega^t,\cdot) \leq -\frac{1}{n}|\omega^t\right) < \frac{1}{n}\right\}. \tag{2.10}$$

Let $\overline{n}_0(\omega^t) := \inf\{n \geq 1, A_n(\omega^t) = \emptyset\}$ with the convention that $\inf\emptyset = +\infty$. Note that if $D^{t+1}(\omega^t) = \{0\}$, then $\overline{n}_0(\omega^t) = 1 < \infty$. We assume now that $D^{t+1}(\omega^t) \neq \{0\}$ and we prove by contradiction that $\overline{n}_0(\omega^t) < \infty$. Assume that $\overline{n}_0(\omega^t) = \infty$ *i.e* for all $n \geq 1$, $A_n(\omega^t) \neq \emptyset$. We thus get $h_n(\omega^t) \in D^{t+1}(\omega^t)$ with $|h_n(\omega^t)| = 1$ and such that

$$q_{t+1}\left(h_n(\omega^t)\Delta S_{t+1}(\omega^t,\cdot) \leq -\frac{1}{n}\,|\,\omega^t\right) < \frac{1}{n}.$$

By passing to a sub-sequence we can assume that $h_n(\omega^t)$ tends to some $h^*(\omega^t) \in D^{t+1}(\omega^t)$ (recall that the set $D^{t+1}(\omega^t)$ is closed by definition) with $|h^*(\omega^t)| = 1$. Introduce

$$B(\omega^t) := \{\omega_{t+1} \in \Omega_{t+1},\ h^*(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}) < 0\}$$
$$B_n(\omega^t) := \{\omega_{t+1} \in \Omega_{t+1},\ h_n(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}) \leq -1/n\}.$$

Then $B(\omega^t) \subset \liminf_n B_n(\omega^t)$. Indeed fix some $\omega_{t+1} \in B(\omega^t)$. Then, there exists some $\varepsilon > 0$ such that $h^*(\omega_{t+1})\Delta S_{t+1}(\omega^t, \omega_{t+1}) < -\varepsilon$. Now there exists some $N \geq 1$ such that for all $n \geq N$, $|h_n(\omega^t) - h^*(\omega^t)| \leq \frac{\varepsilon}{2(1+|\Delta S_{t+1}(\omega^t,\omega_{t+1})|)}$ and $\frac{1}{n} \leq \frac{\varepsilon}{2}$ and it follows that

$$\begin{aligned} h_n(\omega^t)\Delta S_{t+1}(\omega_{t+1}) &= h^*(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}) + (h_n(\omega^t) - h(\omega^t))\Delta S_{t+1}(\omega^t, \omega_{t+1}) \\ &\leq -\varepsilon + |h_n(\omega^t) - h(\omega^t)||\Delta S_{t+1}(\omega^t, \omega_{t+1})| \leq -\frac{\varepsilon}{2} \leq -\frac{1}{n}. \end{aligned}$$

Furthermore as $1_{\liminf_n B_n(\omega^t)} = \liminf_n 1_{B_n(\omega^t)}$, Fatou's Lemma implies that

$$\begin{aligned} q_{t+1}\left(h^*(\omega^t)\Delta S_{t+1}(\omega^t, \cdot) < 0|\omega^t\right) &\leq \int_{\Omega_{t+1}} 1_{\liminf_n B_n(\omega^t)}(\omega_{t+1})q_{t+1}(\omega_{t+1}|\omega^t) \\ &\leq \liminf_n \int_{\Omega_{t+1}} 1_{B_n(\omega^t)}(\omega_{t+1})q_{t+1}(\omega_{t+1}|\omega^t) = 0. \end{aligned}$$

This implies that $q_{t+1}\left(h^*(\omega^t)\Delta S_{t+1}(\omega^t, \cdot) \geq 0|\omega^t\right) = 1$, and thus from (2.8) in Lemma 2.3.6 we get that $h^*(\omega^t) = 0$ which contradicts $|h^*(\omega^t)| = 1$. Thus $\overline{n}_0(\omega^t) < \infty$ and we can set for $\omega^t \in \Omega^t_{NA1}$

$$\overline{\alpha}_t(\omega^t) = \frac{1}{\overline{n}_0(\omega^t)}.$$

It is clear that $\overline{\alpha}_t \in (0, 1]$. Then for all $\omega^t \in \Omega^t_{NA1}$, for all $h \in D^{t+1}(\omega^t)$ with $|h| = 1$, by definition of $A_{\overline{n}_0(\omega^t)}(\omega^t)$ we obtain

$$q_{t+1}\left(h\Delta S_{t+1}(\omega^t, \cdot) \leq -\overline{\alpha}_t(\omega^t)|\omega^t\right) \geq \overline{\alpha}_t(\omega^t). \tag{2.11}$$

Step 2 : measurability issue.
We now construct a function α_t which is $\mathcal{F}_t$-measurable and satisfies (2.9) as well. To do that we use the Aumann Theorem again as in the proof of Lemma 2.3.6 but this time applied to the random set $A_n : \Omega^t \twoheadrightarrow \mathbb{R}^d$ where $A_n(\omega^t)$ is defined in (3.11) if $\omega^t \in \Omega^t_{NA1}$ and $A_n(\omega^t) = \emptyset$ otherwise.

We prove that $graph(A_n) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$. From Lemma 2.8.9, the function $(\omega^t, h) \to q_{t+1}\left(h\Delta S_{t+1}(\omega^t, \cdot) \leq -\frac{1}{n}|\omega^t\right)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. From Lemma 2.3.3, $Graph(D^{t+1}) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ and the result follows from

$$\begin{aligned} Graph(A_n) = Graph(D^{t+1}) \bigcap \left(\Omega^t_{NA1} \times \{h \in \mathbb{R}^d,\ |h| = 1\}\right) \\ \bigcap \left\{(\omega^t, h) \in \Omega^t \times \mathbb{R}^d,\ q_{t+1}\left(h\Delta S_{t+1}(\omega^t, \cdot) \leq -\frac{1}{n}|\omega^t\right) < \frac{1}{n}\right\}. \end{aligned}$$

Using the Projection Theorem (see for example [37, Theorem 3.23]), we get that $\{\omega^t \in \Omega^t,\ A_n(\omega^t) \neq \emptyset\} \in \overline{\mathcal{F}}_t$. We now extend $\overline{n}_0$ to Ω^t by setting $\overline{n}_0(\omega^t) = 1$ if $\omega^t \notin \Omega^t_{NA1}$. Then $\{\overline{n}_0 \geq 1\} = \Omega^t \in \mathcal{F}_t \subset \overline{\mathcal{F}}_t$ and for $k > 1$

$$\{\overline{n}_0 \geq k\} = \Omega^t_{NA1} \cap \bigcap_{1 \leq n \leq k-1} \{A_n \neq \emptyset\} \in \overline{\mathcal{F}}_t,$$

this implies that $\overline{n}_0$ and thus $\overline{\alpha}_t$ is $\overline{\mathcal{F}}_t$-measurable. Using Lemma 2.8.10, we get some $\mathcal{F}_t$-measurable function α_t such that $\alpha_t = \overline{\alpha}_t$ P_t-almost surely, $i.e$ there exists $M^t \in \mathcal{F}_t$ such that $P_t(M^t) = 0$ and $\{\alpha_t \neq \overline{\alpha}_t\} \subset M^t$. We set $\Omega^t_{NA} := \Omega^t_{NA1} \bigcap (\Omega^t \setminus M_t)$. Then $P_t(\Omega^t_{NA}) = 1$ and as α_t is $\mathcal{F}_t$-measurable it remains to check that (2.9) holds true.

For $\omega^t \in \Omega^t_{NA}$, $\alpha_t(\omega^t) = \overline{\alpha}_t(\omega^t)$ (recall that $\omega^t \in \Omega^t \setminus M_t$) and since $\omega^t \in \Omega^t_{NA1}$, (2.11) holds true and consequently (2.9) as well. It is also clear that $\alpha_t(\omega^t) \in (0, 1]$ and the proof is completed. □

Remark 2.3.8 In Definition 2.3.2, Lemmata 2.3.3, 2.3.4, 2.3.6 and Proposition 2.3.7 we have included the case $t = 0$. Note however that since $\Omega^0 = \{\omega_0\}$, the various statements and their respective proofs could be considerably simplified.

Remark 2.3.9 The characterisation of (NA) given by (2.9) works only for $h \in D^{t+1}(\omega^t)$. This is the reason why we will have to project the strategy $\phi_{t+1} \in \Xi_t$ onto $D^{t+1}(\omega^t)$ in our proofs.

2.4 Utility problem and main result

We now describe the investor's risk preferences by a possibly non-concave, random utility function.

Definition 2.4.1 A *random utility* is any function $U :\ \Omega \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ satisfying the following conditions

- for every $x \in \mathbb{R}$, the function $U(\cdot, x) :\ \Omega \to \mathbb{R} \cup \{\pm\infty\}$ is $\mathcal{F}$-measurable,
- for all $\omega \in \Omega$, the function $U(\omega, \cdot) :\ \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is non-decreasing and usc on $\mathbb{R}$,
- $U(\cdot, x) = -\infty$, for all $x < 0$.

We introduce the following notations.

Definition 2.4.2 For all $x \geq 0$, we denote by $\Phi(x)$ the set of all strategies $\phi \in \Phi$ such that $P_T(V_T^{x,\phi}(\cdot) \geq 0) = 1$ and by $\Phi(U,x)$ the set of all strategies $\phi \in \Phi(x)$ such that $EU(\cdot, V_T^{x,\phi})$ exists in a generalised sense, *i.e.* either $EU^+(\cdot, V_T^{x,\phi}(\cdot)) < \infty$ or $EU^-(\cdot, V_T^{x,\phi}(\cdot)) < \infty$.

We prove in the next lemma, that under (NA), if $\phi \in \Phi(x)$ then we have that $P_t(V_t^{x,\phi}(\cdot) \geq 0) = 1$ for all $1 \leq t \leq T$.

Lemma 2.4.3 *Assume that (NA) holds true. Let $\phi \in \Phi(x)$ for some $x \geq 0$, then $V_t^{x,\phi} \geq 0$ P_t-a.s. for all $1 \leq t \leq T$.*

Proof. Assume that there is some t such that $P_t(V_t^{x,\phi} \geq 0) < 1$ or equivalently $P_t(V_t^{x,\phi} < 0) > 0$ and let $n = \sup\{t | P_t(V_t^{x,\phi} < 0) > 0\}$. Then $P_n(V_n^{x,\phi} < 0) > 0$ and for all $s \geq n+1$, $P_s(V_s^{x,\phi} \geq 0) = 1$. Let $\Psi_s(\omega) = 0$ if $s \leq n$ and $\Psi_s(\omega) = 1_A \phi_s(\omega)$ if $s \geq n+1$ with $A = \{V_n^{\Phi} < 0\}$. Then

$$V_s^{0,\Psi} = \sum_{k=1}^{s} \Psi_s \Delta S_s = \sum_{k=n+1}^{s} \Psi_s \Delta S_s = 1_A \left(V_s^{x,\phi} - V_n^{x,\phi} \right).$$

If $s \geq n+1$ $P_s(V_s^{x,\phi} \geq 0) = 1$ and on A, $-V_n^{\Phi} > 0$ thus $P_T(V_T^{0,\Psi} \geq 0) = 1$ and $V_T^{0,\Psi} > 0$ on A. As by the (usual) Fubini Theorem $P_T(A) = P_n(V_n^{x,\phi} < 0) > 0$, we get an arbitrage opportunity. Thus for all $t \leq T$, $P_t(V_t^{x,\phi} \geq 0) = 1$. □

We now formulate the problem which is our main concern in the sequel.

Definition 2.4.4 Let $x \geq 0$. The *non-concave portfolio problem* on a finite horizon T with initial wealth x is

$$u(x) := \sup_{\phi \in \Phi(U,x)} EU(\cdot, V_T^{x,\phi}(\cdot)). \tag{2.12}$$

Remark 2.4.5 Assume that there exists some P-full measure set $\widetilde{\Omega} \in \mathcal{F}$ such that for all $\omega \in \widetilde{\Omega}$, $x \to U(\omega, x)$ is non-decreasing and usc on $[0, +\infty)$, *i.e.* $x \to U(\omega, x)$ is usc on $(0, \infty)$ and for any $(x_n)_{n \geq 1} \subset (0, +\infty)$ converging to 0, $U(\omega, 0) \geq \limsup_n U(\omega, x_n)$. We set $\overline{U} : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$

$$\overline{U}(\omega, x) := U(\omega, x) 1_{\widetilde{\Omega} \times [0,+\infty)}(\omega, x) + (-\infty) 1_{\Omega \times (-\infty, 0)}(\omega, x).$$

Then $\overline{U}$ satisfies Definition 2.4.1, see Lemma 2.8.11 for the second item. Moreover, the value function does not change

$$u(x) = \sup_{\phi \in \Phi(U,x)} E\overline{U}(\cdot, V_T^{x,\phi}(\cdot)),$$

and if there exists some $\phi^* \in \Phi(U,x)$ such that $u(x) = E\overline{U}(\cdot, V_T^{x,\phi^*}(\cdot))$, then ϕ^* is an optimal solution for (2.12).

Remark 2.4.6 Let U be a utility function defined only on $(0,\infty)$ and verifying for every $x \in (0,\infty)$, $U(\cdot,x): \Omega \to \mathbb{R} \cup \{\pm\infty\}$ is $\mathcal{F}$-measurable and for all $\omega \in \Omega$, $U(\omega,\cdot): (0,\infty) \to \mathbb{R} \cup \{\pm\infty\}$ is non-decreasing and usc on $(0,\infty)$. We may extend U on $\mathbb{R}$ by setting, for all $\omega \in \Omega$, $\overline{U}(\omega,0) = \lim_{x \searrow 0} U(\omega,x)$ and for $x<0$, $\overline{U}(\omega,x) = -\infty$. Then, as before, $\overline{U}$ verifies Definition 2.4.1 and the value function has not changed. Note that we could have considered a closed interval $F=[a,\infty)$ of $\mathbb{R}$ instead of $[0,\infty)$, we could have adapted our notion of upper semicontinuity and all the sequel would apply.

We now present conditions on U which allows to assert that if $\phi \in \Phi(x)$ then $EU(\cdot, V_T^{x,\phi}(\cdot))$ is well-defined and that there exists some optimal solution for (2.12).

Assumption 2.4.7 For all $\phi \in \Phi(U,1)$, $EU^+\left(\cdot, V_T^{1,\phi}(\cdot)\right) < \infty$.

Assumption 2.4.8 $\Phi(U,1) = \Phi(1)$.

Remark 2.4.9 Assumptions 2.4.7 and 2.4.8 are connected but play a different role. Assumption 2.4.8 guarantees that $EU\left(\cdot, V_T^{1,\phi}(\cdot)\right)$ is well-defined for all $\Phi \in \Phi(1)$ and allows us to relax [34, Assumption 2.7] on the behavior of U around 0, namely that $EU^-(\cdot,0) < \infty$. Then Assumption 2.4.7 (together with Assumption 2.4.10) is used to show that $u(x) < \infty$ for all $x > 0$. Note that Assumption 2.4.7 is much more easy to verify that the classical assumption that $u(x) < \infty$ (for all or some $x > 0$), which is usually made in the theory of maximisation of the terminal wealth expected utility.

In Proposition 2.6.1, we will show that under Assumptions 2.4.7, 2.4.8 and 2.4.10, $EU^+\left(\cdot, V_T^{x,\phi}(\cdot)\right) < \infty$ for all $x \geq 0$ and $\phi \in \Phi(x)$. Thus $\Phi(U,x) = \Phi(x)$. Note that if there exists some $\Phi \in \Phi(U,x)$ such that $EU^+\left(\cdot, V_T^{x,\phi}(\cdot)\right) = \infty$ and $EU^-\left(\cdot, V_T^{x,\phi}(\cdot)\right) < \infty$ then $u(x) = \infty$ and the problem is ill-posed.

We propose some examples where Assumptions 2.4.7 or 2.4.8 hold true. Example $ii)$ illustrates the distinction between Assumptions 2.4.7 and 2.4.8 and justifies we do not merge both assumptions and postulate that $EU^+\left(\cdot, V_T^{1,\phi}(\cdot)\right) < \infty$ for all $\phi \in \Phi(1)$.

i) If U is bounded from above then both Assumptions are trivially true. We get directly that $\Phi(U,x) = \Phi(x)$ for all $x \geq 0$.

ii) Assume that $EU^-(\cdot,0) < \infty$ holds true. Let $x \geq 0$ and $\phi \in \Phi(x)$ be fixed. Using that U^- is non-decreasing for all $\omega \in \Omega$ we get that

$$EU^-(\cdot, V_T^{x,\phi}(\cdot)) \leq EU^-(\cdot,0) < +\infty,$$

Thus $EU(\cdot, V_T^{x,\phi}(\cdot))$ is well-defined, $\Phi(U,x) = \Phi(x)$ and Assumption 2.4.8 holds true.

iii) Assume that there exists some $\hat{x} \geq 1$ such that $U(\cdot, \hat{x}-1) \geq 0$ P-almost surely and

$$\widehat{u}(\hat{x}) := \sup_{\phi \in \Phi(\hat{x})} EU(\cdot, V_T^{\hat{x},\phi}(\cdot)) < \infty,$$

where we set for $\phi \in \Phi(\hat{x}) \backslash \Phi(U,\hat{x})$, $EU(\cdot, V_T^{\hat{x},\phi}(\cdot)) = -\infty$. Let $\phi \in \Phi(1)$ be fixed. Then using that U is non-decreasing for all $\omega \in \Omega$, we have that P-almost surely

$$U(\cdot, V_T^{1,\phi}(\cdot) + \hat{x} - 1) \geq U(\cdot, \hat{x}-1) \geq 0.$$

Therefore $U(\cdot, V_T^{1,\phi}(\cdot) + \hat{x} - 1) = U^+(\cdot, V_T^{1,\phi}(\cdot) + \hat{x} - 1)$ P-almost surely. Now using that U^+ is non-decreasing for all $\omega \in \Omega$ we get that for all $\phi \in \Phi(1)$

$$EU^+(\cdot, V_T^{1,\phi}(\cdot)) \leq EU^+(\cdot, V_T^{1,\phi}(\cdot) + \hat{x} - 1) = EU(\cdot, V_T^{1,\phi}(\cdot) + \hat{x} - 1) \leq \widehat{u}(\hat{x}) < +\infty$$

and Assumptions 2.4.7 and 2.4.8 are satisfied. Instead of stipulating that $\widehat{u}(\hat{x}) < \infty$ it is enough to assume that $EU(\cdot, V_T^{\hat{x},\phi}(\cdot)) < \infty$ for all $\phi \in \Phi(\hat{x})$.

iv) We will prove in Theorem 2.4.17 that under the (NA) condition and Assumption 2.4.10 below, Assumptions 2.4.7 and 2.4.8 hold true if $EU^+(\cdot, 1) < +\infty$ and if for all $0 \leq t \leq T$ $|\Delta S_t|$, $\frac{1}{\alpha_t} \in \mathcal{W}_t$ (see (2.18) for the definition of $\mathcal{W}_t$).

Assumption 2.4.10 We assume that there exist some constants $\overline{\gamma} \geq 0$, $K > 0$, as well as a random variable C satisfying $C(\omega) \geq 0$ for all $\omega \in \Omega$ and $E(C) < \infty$ such that for all $\omega \in \Omega$, $\lambda \geq 1$ and $x \in \mathbb{R}$, we have

$$U(\omega, \lambda x) \leq K\lambda^{\overline{\gamma}} \left(U\left(\omega, x + \frac{1}{2}\right) + C(\omega) \right). \tag{2.13}$$

Remark 2.4.11 First note that the constant $\frac{1}{2}$ in (2.13) has been chosen arbitrarily to simplify the presentation. This can be done without loss of generality. Indeed, assume there exists some constant $\overline{x} \geq 0$ such that for all $\omega \in \Omega$, $\lambda \geq 1$ and $x \in \mathbb{R}$

$$U(\omega, \lambda x) \leq K\lambda^{\overline{\gamma}} \left(U(\omega, x + \overline{x}) + C(\omega) \right). \tag{2.14}$$

Using the monotonicity of U, we can always assume $\overline{x} > 0$. Set for all $\omega \in \Omega$ and $x \in \mathbb{R}$, $\overline{U}(\omega, x) = U(\omega, 2\overline{x}x)$. Then for all $\omega \in \Omega$, $\lambda \geq 1$ and $x \in \mathbb{R}$, we have that

$$\overline{U}(\omega, \lambda x) = U(\omega, 2\lambda\overline{x}x) \leq K\lambda^{\overline{\gamma}} \left(U(\omega, 2\overline{x}x + \overline{x}) + C(\omega) \right) = K\lambda^{\overline{\gamma}} \left(\overline{U}\left(\omega, x + \frac{1}{2}\right) + C(\omega) \right),$$

and $\overline{U}$ satisfies (2.13). It is clear that if ϕ^* is an optimal solution for the problem $\overline{u}(x) := \sup_{\phi \in \Phi(\overline{U}, \frac{x}{2\overline{x}})} E\overline{U}(\cdot, V_T^{\frac{x}{2\overline{x}},\phi}(\cdot))$ then $2\overline{x}\phi^*$ is an optimal solution for (2.12). Note as well that, since $K > 0$ and $C \geq 0$, it is immediate to see that for all $\omega \in \Omega$, $\lambda \geq 1$ and $x \in \mathbb{R}$

$$U^+(\omega, \lambda x) \leq K\lambda^{\overline{\gamma}} \left(U^+ \left(\omega, x + \frac{1}{2} \right) + C(\omega) \right). \tag{2.15}$$

Remark 2.4.12 We now provide some insight on Assumption 2.4.10. As the inequality (2.13) is used to control the behaviour of $U^+(\cdot, x)$ for large values of x, the usual assumption in the non-concave case (see [34, Assumption 2.7]) is that there exists some $\hat{x} \geq 0$ such that $EU^+(\cdot, \hat{x}) < \infty$ as well as a random variable C_1 satisfying $E(C_1) < \infty$ and $C_1(\omega) \geq 0$ for all ω [3] such that for all $x \geq \hat{x}$, $\lambda \geq 1$ and $\omega \in \Omega$

$$U(\omega, \lambda x) \leq \lambda^{\overline{\gamma}} \left(U(\omega, x) + C_1(\omega) \right). \tag{2.16}$$

We prove now that if (2.16) holds true then (2.14) is verified with $\overline{x} = \hat{x}$, $K = 1$ and $C = C_1$. Indeed, assume that (2.16) is verified. For $x \geq 0$, using the monotonicity of U, we have for all $\omega \in \Omega$ and $\lambda \geq 1$ that

$$U(\omega, \lambda x) \leq U(\omega, \lambda(x + \hat{x})) \leq \lambda^{\overline{\gamma}} \left(U(\omega, x + \hat{x}) + C_1(\omega) \right).$$

And for $x < 0$ this is true as well since $U(\omega, x) = -\infty$.
Therefore (2.14) is a weaker assumption than (2.16). Note as well that if we assume that (2.16) holds true for all $x > 0$, then if $0 < x < 1$ and $\omega \in \Omega$ we have

$$U(\omega, 1) \leq \left(\frac{1}{x} \right)^{\overline{\gamma}} \left(U(\omega, x) + C_1(\omega) \right),$$

and $U(\omega, 0) := \lim_{x \searrow 0} U(\omega, x) \geq -C_1(\omega)$. This excludes for instance the case where U is the logarithm. Furthermore, this also implies that $EU^-(\cdot, 0) \leq EC_1 < \infty$ and we are back to [34, Assumption 2.7]
Alternatively, recalling the way the concave case is handled (see [111, Lemma 2]), we could have assumed that there exists a random variable C_2 satisfying $E(C_2) < \infty$ and $C_2 \geq 0$ such that for all $x \in \mathbb{R}$, $\omega \in \Omega$

$$U^+(\omega, \lambda x) \leq \lambda^{\overline{\gamma}} \left(U^+(\omega, x) + C_2(\omega) \right). \tag{2.17}$$

We have not done so as it is difficult to prove that this inequality is preserved through the dynamic programming procedure when considering non-concave functions unless we assume that $EU^-(\cdot, 0) < \infty$ as in [34].

[3] In the cited paper $C_1 \geq 0$ a.s but this is not an issue, see Remark 2.4.13 below

Remark 2.4.13 If there exists some set $\Omega_{ae} \in \mathcal{F}$ with $P(\Omega_{ae}) = 1$ such that (2.13) holds true only for $\omega \in \Omega_{ae}$, then setting as in Remark 2.4.5, $\overline{U}(\omega, x) := U(\omega, x)1_{\Omega_{ae}\times\mathbb{R}}(\omega, x)$, $\overline{U}$ satisfies (2.13) and the value function in (2.12) does not change. We also assume without loss of generality that $C(\omega) \geq 0$ for all ω in (2.13). Indeed, if $C \geq 0$ P-a.s, we could consider $\widetilde{C} := C\mathbb{I}_{\overline{C}\geq 0}$. Then Assumption 2.4.10 would hold true with $\widetilde{C}$ instead of C.

Remark 2.4.14 In the case where (2.16) holds true, we refer to [33, Remark 2.5] and [34, Remark 2.10] for the interpretation of $\overline{\gamma}$: for $C_1 = 0$, it can be seen as a generalization of the "asymptotic elasticity" of U at $+\infty$ (see [91]). So (2.16) requires that the (generalized) asymptotic elasticity at $+\infty$ is finite. In this case and if U is differentiable there is a nice economic interpretation of the "asymptotic elasticity" as the ratio of "marginal utility": $U'(x)$ and the "average utility": $\frac{U(x)}{x}$, see again [91, Section 6] for further discussions. The case $C_1 > 0$ allows bounded utilities. In [34] it is proved that unlike in the concave case, the fact that U is bounded from above (and therefore satisfies (2.14)) does not implies that the asymptotic elasticity is bounded.
We propose now an example of an unbounded utility function satisfying (2.14) and such that $\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} = +\infty$. This shows (as the counterexample of [34]), that Assumption 2.4.10 is less strong than the usual "asymptotic elasticity". Let $U : \mathbb{R} \to \mathbb{R}$ be defined by

$$U(x) = -\infty 1_{(-\infty,0)}(x) + \sum_{p\geq 0} p1_{[p,p+1-\frac{1}{2^{p+1}})}(x) + f_p(x)1_{[p+1-\frac{1}{2^{p+1}},p+1)}(x)$$

where $f_p(x) = 2^{p+1}x + (p+1)\left(1 - 2^{p+1}\right)$ for $p \in \mathbb{N}$. Then U satisfies Definition 2.4.1 and we have

$$U'(x) = \sum_{p\geq 0} 2^{p+1}1_{[p+1-\frac{1}{2^{p+1}},p+1)}(x).$$

We prove that (2.14) holds true. Note that for all $x \geq 0$ we have $x-1 \leq U(x) \leq x+1$. Let $x \geq 0$ and $\lambda \geq 1$ be fixed. Then we get that

$$U(\lambda x) \leq \lambda x + 1 \leq \lambda \left(U(x+1) + 1\right) + 1 \leq \lambda \left(U(x+1) + 2\right),$$

and (2.14) is true with $K = \overline{x} = 1$ and $C = 2$. Now for $k \geq 0$, let $x_k = k + 1 - \frac{1}{2^{k+2}}$. We have $U(x_k) = f_k(x_k) = k + \frac{1}{2}$ and

$$\frac{x_k U'(x_k)}{U(x_k)} = 2^{k+1} \frac{\left(k+1-\frac{1}{2^{k+2}}\right)}{k+\frac{1}{2}} \to_{k\to\infty} +\infty.$$

Remark 2.4.15 We propose further examples where Assumption 2.4.10 holds true.

i) Assume that U is bounded from above by some integrable random constant $C_1 \geq 0$ and that $EU^-(\cdot, \frac{1}{2}) < \infty$. Then for all $x \geq 0$, $\lambda \geq 1$, $\omega \in \Omega$ we have

$$\begin{aligned} U(\omega, \lambda x) \leq C_1(\omega) &\leq \lambda U\left(\omega, x + \frac{1}{2}\right) + \lambda\left(C_1(\omega) - U\left(\omega, x + \frac{1}{2}\right)\right) \\ &\leq \lambda U\left(\omega, x + \frac{1}{2}\right) + \lambda\left(C_1(\omega) + U^-\left(\omega, \frac{1}{2}\right)\right), \end{aligned}$$

and (2.13) holds true for $x \geq 0$ with $K = 1$, $\overline{\gamma} = 1$ and $C(\cdot) = C_1(\cdot) + U^-(\cdot, \frac{1}{2})$. As $U(\cdot, x) = -\infty$ for $x < 0$, (2.13) is true for all $x \in \mathbb{R}$.

ii) Assume that U satisfies Definition 2.4.1 and that the restriction of U to $[0, \infty)$ is concave and non-decreasing and that $EU^-(\cdot, 1) < \infty$. We use similar arguments as in [112, Lemma 2], see also Proposition 4.5.19 in Chapter 4. Indeed, let $x \geq 2$, $\lambda \geq 1$ be fixed we have

$$\begin{aligned} U(\omega, \lambda x) &\leq U(\omega, x) + \frac{U(\omega, x) - U(\omega, 1)}{x - 1}(\lambda - 1)x \\ &\leq U(\omega, x) + 2(\lambda - 1)\left(U(\omega, x) - U(\omega, 1)\right) \\ &\leq U(\omega, x) + 3(\lambda - \frac{1}{3})\left(U(\omega, x) - U(\omega, 1)\right) \\ &\leq 3\lambda\left(U(\omega, x) + U^-(\omega, 1)\right), \end{aligned}$$

where we have used the concavity of U for the first inequality and the fact that $x \geq 2$ and U is non-decreasing for the other ones. Thus from the proof that (2.16) implies (2.14), we obtain that (2.14) holds true with $K = 3$, $\overline{\gamma} = 1$, $\overline{x} = 2$ and $C(\cdot) = U^-(\cdot, 1)$.

We can now state our main result.

Theorem 2.4.16 *Assume the (NA) condition and that Assumptions 2.4.7, 2.4.8 and 2.4.10 hold true. Let $x \geq 0$. Then, $u(x) < \infty$ and there exists some optimal strategy $\phi^* \in \Phi(U, x)$ such that*

$$u(x) = EU(\cdot, V_T^{x,\phi^*}(\cdot)).$$

Moreover $\phi_t^*(\cdot) \in D^t(\cdot)$ a.s. for all $0 \leq t \leq T$.

We will use dynamic programming in order to prove our main result. We will combine the approach of [111], [112], [33], [34] and [99]. As in [99], we will consider a one period case where the initial filtration is trivial (so that strategies are in $\mathbb{R}^d$) and thus the proofs are much simpler than the ones of [111], [112], [33] and [34]. The price to pay is that in the multi-period case where we use intensively measurable selection arguments (as in [99]) in order to obtain Theorem 2.4.16. In our model, there is only one probability measure, so we don't have to introduce Borel spaces and analytic sets. Thus our modelisation of $(\Omega, \mathcal{F}, \mathfrak{F}, P)$ is more general than the one of [99] restricted to one probability measure. As we are in a non concave setting we use similar ideas to theses of [33] and [34].

Finally, as in [111], [112], [33] and [34], we propose the following result as a simpler but still general setting where Theorem 2.4.16 applies. We introduce for all $0 \leq t \leq T$

$$\mathcal{W}_t := \left\{ X : \Omega^t \to \mathbb{R} \cup \{\pm\infty\},\ \mathcal{F}_t\text{-measurable},\ E|X|^p < \infty \text{ for all } p > 0 \right\} \tag{2.18}$$

Theorem 2.4.17 *Assume the (NA) condition and that Assumption 2.4.10 hold true. Assume furthermore that* $EU^+(\cdot, 1) < +\infty$ *and that for all* $0 \leq t \leq T$ $|\Delta S_t|$, $\frac{1}{\alpha_t} \in \mathcal{W}_t$. *Let* $x \geq 0$. *Then, for all* $\phi \in \Phi(x)$ *and all* $0 \leq t \leq T$, $V_t^{x,\phi} \in \mathcal{W}_t$. *Moreover, there exists some optimal strategy* $\phi^* \in \Phi(U, x)$ *such that*

$$u(x) = EU(\cdot, V_T^{x,\phi^*}(\cdot)) < \infty$$

2.5 One period case

Let $(\overline{\Omega}, \mathcal{H}, Q)$ be a probability space (we denote by E the expectation under Q) and $Y(\cdot)$ a $\mathcal{H}$-measurable $\mathbb{R}^d$-valued random variable. $Y(\cdot)$ could represent the change of value of the price process. Let $D \subset \mathbb{R}^d$ be the smallest affine subspace of $\mathbb{R}^d$ containing the support of the distribution of $Y(\cdot)$. We assume that D contains 0, so that D is in fact a non-empty vector subspace of $\mathbb{R}^d$. The condition corresponding to (NA) in the present setting is

Assumption 2.5.1 There exists some constant $0 < \alpha \leq 1$ such that for all $h \in D$

$$Q(hY(\cdot) \leq -\alpha|h|) \geq \alpha. \tag{2.19}$$

Remark 2.5.2 If $D = \{0\}$ then (2.19) is trivially true.

Remark 2.5.3 below is exactly [33, Remark 8] (see also [99, Lemma 2.6]).

Remark 2.5.3 Let $h \in \mathbb{R}^d$ and let $h' \in \mathbb{R}^d$ be the orthogonal projection of h on D. Then $h - h' \perp D$ hence $\{Y(\cdot) \in D\} \subset \{(h - h')Y(\cdot) = 0\}$. It follows that

$$Q(hY(\cdot) = h'Y(\cdot)) = Q((h - h')Y(\cdot) = 0) \geq Q(Y(\cdot) \in D) = 1$$

by the definition of D. Hence $Q(hY(\cdot) = h'Y(\cdot)) = 1$.

Assumption 2.5.4 We consider a *random utility* $V : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ satisfying the following two conditions

- for every $x \in \mathbb{R}$, the function $V(\cdot, x) : \overline{\Omega} \to \mathbb{R}$ is $\mathcal{H}$-measurable,
- for every $\omega \in \overline{\Omega}$, the function $V(\omega, \cdot) : \mathbb{R} \to \mathbb{R}$ is non-decreasing and usc on $\mathbb{R}$,
- $V(\cdot, x) = -\infty$, for all $x < 0$.

Let $x \geq 0$ be fixed. We define

$$\mathcal{H}_x := \left\{h \in \mathbb{R}^d,\ Q(x + hY(\cdot) \geq 0) = 1\right\}, \tag{2.20}$$
$$D_x := \mathcal{H}_x \cap D. \tag{2.21}$$

It is clear that $\mathcal{H}_x$ and D_x are closed subsets of $\mathbb{R}^d$. We now define the function which is our main concern in the one period case

$$v(x) = (-\infty)1_{(-\infty,0)}(x) + 1_{[0,+\infty)}(x) \sup_{h \in \mathcal{H}_x} EV\left(\cdot, x + hY(\cdot)\right). \tag{2.22}$$

Remark 2.5.5 First note that, from Remark 2.5.3,

$$v(x) = (-\infty)1_{(-\infty,0)}(x) + 1_{[0,+\infty)}(x) \sup_{h \in D_x} EV(\cdot, x + hY(\cdot)). \tag{2.23}$$

Remark 2.5.6 It will be shown in Lemma 2.5.11 that under Assumptions 2.5.1, 2.5.4, 2.5.7 and 2.5.9, for all $h \in \mathcal{H}_x$, $E(V(\cdot, x + hY(\cdot))$ is well-defined and more precisely that $EV^+(\cdot, x + hY(\cdot)) < +\infty$. So, under this set of assumptions, $\Phi(V, x)$, the set of $h \in \mathcal{H}_x$ such that $EV(\cdot, x + hY(\cdot))$ is well-defined, equals $\mathcal{H}_x$.

We present now the assumptions which allow to assert that there exists some optimal solution for (2.22). First we introduce the “asymptotic elasticity" assumption.

Assumption 2.5.7 There exist some constants $\overline{\gamma} \geq 0$, $K > 0$, as well as some $\mathcal{H}$-measurable C with $C(\omega) \geq 0$ for all $\omega \in \overline{\Omega}$ and $E(C) < \infty$, such that for all $\omega \in \overline{\Omega}$, for all $\lambda \geq 1$, $x \in \mathbb{R}$ we have

$$V(\omega, \lambda x) \leq K\lambda^{\overline{\gamma}} \left(V\left(\omega, x + \frac{1}{2}\right) + C(\omega)\right). \tag{2.24}$$

Remark 2.5.8 The same comments as in Remark 2.4.13 apply. Furthermore, note that since $K > 0$ and $C \geq 0$ we also have that for all $\omega \in \overline{\Omega}$, all $\lambda \geq 1$ and $x \in \mathbb{R}$

$$V^+(\omega, \lambda x) \leq K\lambda^{\overline{\gamma}} \left(V^+\left(\omega, x + \frac{1}{2}\right) + C(\omega)\right). \tag{2.25}$$

We introduce now some integrability assumption on V^+.

Assumption 2.5.9 For every $h \in \mathcal{H}_1$,

$$EV^+(\cdot, 1 + hY(\cdot)) < \infty. \tag{2.26}$$

The following lemma corresponds to [112, Lemma 2.1] in the deterministic case.

Lemma 2.5.10 *Assume that Assumption 2.5.1 holds true. Let $x \geq 0$ be fixed. Then $D_x \subset B(0, \frac{x}{\alpha})$ (see (2.21) for the definition of D_x), where $B(0, \frac{x}{\alpha}) = \{h \in \mathbb{R}^d,\ |h| \leq \frac{x}{\alpha}\}$ and D_x is a convex, compact subspace of $\mathbb{R}^d$.*

Note that if $x = 0$, it follows that $D_x = \{0\}$.

Proof. Let $h \in D_x$. Assume that $|h| > \frac{x}{\alpha}$ and let $\omega \in \{hY(\cdot) \leq -\alpha|h|\}$. Then $x + hY(\omega) \leq x - \alpha|h| < 0$ and from Assumption 2.5.1 $Q(x + hY(\cdot) < 0) \geq Q(hY(\cdot) \leq -\alpha|h|) \geq \alpha > 0$, a contradiction. The convexity and the closedness of D_x are clear and the compactness follows from the boundness property. □

This lemma corresponds in the deterministic case to [34, Lemma 4.8] (see also [112, Lemma 2.3] and [99, Lemma 2.8]).

Lemma 2.5.11 *Assume that Assumptions 2.5.1, 2.5.4, 2.5.7 and 2.5.9 hold true. Then there exists a $\mathcal{H}$-measurable $L \geq 0$ satisfying $E(L) < \infty$ and such that for all $x \geq 0$ and $h \in \mathcal{H}_x$*

$$V^+(\cdot, x + hY(\cdot)) \leq \left((2x)^{\overline{\gamma}}K + 1\right) L(\cdot)\ Q - a.s. \tag{2.27}$$

Proof. We start with the proof of (2.27) when $h \in D_x$. Since D is a vectorial subspace of $\mathbb{R}^d$ and $0 \in \mathcal{H}_x$, the affine hull of D_x is also a vector space that we denote by Aff(D_x). If $x \leq 1$ we have by Assumption 2.5.4 that for all $\omega \in \overline{\Omega}$, $h \in D_x$,

$$V^+(\omega, x + hY(\omega)) \leq V^+\left(\omega, 1 + hY(\omega)\right). \tag{2.28}$$

If $x > 1$ using Assumption 2.5.7 (see (2.25) in Remark 2.5.8) we get that for all $\omega \in \overline{\Omega}$, $h \in D_x$

$$\begin{aligned} V^+(\omega, x + hY(\omega)) &= V^+\left(\omega, 2x\left(\frac{1}{2} + \frac{h}{2x}Y(\omega)\right)\right) \\ &\leq (2x)^{\overline{\gamma}}K\left(V^+\left(\omega, 1 + \frac{h}{2x}Y(\omega)\right) + C(\omega)\right). \end{aligned} \tag{2.29}$$

First we treat the case of $Dim(\text{Aff}(D_x)) = 0$, $i.e\ D_x = \{0\}$. For all $\omega \in \overline{\Omega}$, $h \in D_x = \{0\}$, using (2.28) and (2.29), we obtain for all $x \geq 0$ that

$$\begin{aligned} V^+(\omega, x + hY(\omega)) &\leq V^+(\omega, 1) + (2x)^{\overline{\gamma}}K\left(V^+\left(\omega, 1\right) + C(\omega)\right) \\ &\leq ((2x)^{\overline{\gamma}}K + 1)(V^+(\omega, 1) + C(\omega)). \end{aligned} \tag{2.30}$$

We assume now that $Dim(\text{Aff}(D_x)) > 0$. If $x = 0$ then $Y = 0$ Q-a.s. If this is not the case then we should have $D_0 = \{0\}$ a contradiction. Indeed if there exists some $h \in D_0$ with $h \neq 0$, then $Q\left(\frac{h}{|h|}Y(\cdot) < 0\right) > 0$ by Assumption 2.5.1 which contradicts $h \in D_0$. So for $x = 0$, $Y = 0$ Q-a.s and by Assumption 2.5.4 we get that for all $\omega \in \overline{\Omega}$, $h \in D_0$,

$$V^+(\omega, 0 + hY(\omega)) \leq V^+(\omega, 1).$$

From now we assume that $x > 0$. Then as for $g \in \mathbb{R}^d$, $g \in D_x$ if and only if $\frac{g}{x} \in D_1$, we have that $\text{Aff}(D_x) = \text{Aff}(D_1)$. We set $d' := Dim(\text{Aff}(D_1))$. Let $(e_1, \ldots, e_{d'})$ be an orthonormal basis of $\text{Aff}(D_1)$ (which is a sub-vector space of $\mathbb{R}^d$) and let $\varphi : (\lambda_1, \ldots, \lambda_{d'}) \in \mathbb{R}^{d'} \to \Sigma_{i=1}^{d'} \lambda_i e_i \in \text{Aff}(D_1)$. Then φ is an isomorphism (recall that $(e_1, \ldots, e_{d'})$ is a basis of $\text{Aff}(D_1)$). As φ is linear and the spaces considered are of finite dimension, it is also an homeomorphism between $\mathbb{R}^{d'}$ and $\text{Aff}(D_1)$. Since D_1 is compact by Lemma 2.5.10, $\varphi^{-1}(D_1)$ is a compact subspace of $\mathbb{R}^{d'}$. So there exists some $c \geq 0$ such that for all $h = \Sigma_{i=1}^{d'} \lambda_i e_i \in D_1$, $|\lambda_i| \leq c$ for all $i = 1, \ldots, d'$. We complete the family of vector $(e_1, \ldots, e_{d'})$ in order to obtain an orthonormal basis of $\mathbb{R}^d$, denoted by $(e_1, \ldots, e_{d'}, e_{d'+1}, \ldots e_d)$. For all $\omega \in \Omega$, let $(y_i(\omega))_{i=1,\ldots,d}$ be the coordinate of $Y(\omega)$ in this basis.
Now let $h \in D_x$ be fixed. Then $\frac{h}{2x} \in D_{\frac{1}{2}} \subset D_1$ and $\frac{h}{2x} = \Sigma_{i=1}^{d'} \lambda_i e_i$ for some $(\lambda_1, \ldots \lambda_{d'}) \in \mathbb{R}^{d'}$ with $|\lambda_i| \leq c$ for all $i = 1, \ldots, d'$. Note that as $\frac{h}{2x} \in D_1$, $\lambda_i = 0$ for $i \geq d' + 1$. Then as $(e_1, \ldots, e_d)$ is an orthonormal basis of $\mathbb{R}^d$, we obtain for all $\omega \in \overline{\Omega}$

$$1 + \frac{h}{2x} Y(\omega) = 1 + \Sigma_{i=1}^{d'} \lambda_i y_i(\omega) \leq 1 + \Sigma_{i=1}^{d'} |\lambda_i| |y_i(\omega)| \leq 1 + c\Sigma_{i=1}^{d'} |y_i(\omega)|.$$

Thus from Assumption 2.5.4 for all $\omega \in \overline{\Omega}$ we get that

$$V^+\left(\omega, 1 + \frac{h}{2x} Y(\omega)\right) \leq V^+\left(\omega, 1 + c\Sigma_{i=1}^{d'} |y_i(\omega)|\right).$$

We set

$$L(\cdot) := V^+\left(\omega, 1 + c\Sigma_{i=1}^{d'} |y_i(\omega)|\right) 1_{d'>0} + V^+(\cdot, 1) + C(\cdot).$$

As $d' = Dim(\text{Aff}(D_1))$ it is clear that L does not depend on x. It is also clear that L is $\mathcal{H}$-measurable.
Then using (2.28), (2.29) and (2.30) we obtain that for all $\omega \in \overline{\Omega}$

$$V^+(\omega, x + hY(\omega)) \leq ((2x)^{\overline{\gamma}} K + 1) L(\omega).$$

Note that the first term in L is used in the above inequality if $x \neq 0$ and $Dim(\text{Aff}(D_x)) > 0$. The second and the third one are there for both the case of $Dim(\text{Aff}(D_x)) = 0$ and the case of $x = 0$ and $Dim(\text{Aff}(D_x)) > 0$. As by Assumptions 2.5.7 and 2.5.9, $E(V^+(\cdot, 1) + C(\cdot)) < \infty$, it remains to prove that $d' > 0$ implies $E\left(V^+\left(\cdot, 1 + c\Sigma_{i=1}^{d'} |y_i(\cdot)|\right)\right) <$

∞.
Introduce W, the finite set of $\mathbb{R}^d$ whose coordinates on $(e_1, \dots, e_{d'})$ are 1 or -1 and 0 on $(e_{d'+1}, \dots e_d)$. Then $W \subset \text{Aff}(D_1)$ and the vectors of W will be denoted by θ^j for $j \in \{1, \dots, 2^{d'}\}$. Let θ^ω be the vector whose coordinates on $(e_1, \dots, e_{d'})$ are $(sign(y_i(\omega)))_{i=1\dots d'}$ and 0 on $(e_{d'+1}, \dots e_d)$. Then $\theta^\omega \in W$ and we get that

$$V^+\left(\omega, 1 + c\Sigma_{i=1}^{d'}|y_i(\omega)|\right) = V^+(\omega, 1 + c\theta^\omega Y(\omega)) \leq \sum_{j=1}^{2^{d'}} V^+(\omega, 1 + c\theta^j Y(\omega)).$$

So to prove that $EL < \infty$ it is sufficient to prove that if $d' > 0$ for all $1 \leq j \leq 2^{d'}$, $EV^+(\cdot, 1 + c\theta^j Y(\cdot)) < \infty$. Recall that $\theta^j \in \text{Aff}(D_1)$. We introduce now $\text{Ri}(D_1)$ the relative interior of D_1. Recall from [115, Section 6]) that $\text{Ri}(D_1) = \{y \in D_1, \exists \alpha > 0 \, s.t \, \text{Aff}(D_1) \cap B(y, \alpha) \subset D_1\}$. [4] As D_1 is convex and non-empty (recall $d' > 0$), $ri(D_1)$ is also non-empty and convex and we fix some $e^* \in ri(D_1)$. We prove that $\frac{e^*}{2} \in ri(D_1)$. Let $\alpha > 0$ be such that $\text{Aff}(D_1) \cap B(e^*, \alpha) \subset D_1$ and $g \in \text{Aff}(D_1) \cap B(\frac{e^*}{2}, \frac{\alpha}{2})$. Then $2g \in \text{Aff}(D_1) \cap B(e^*, \alpha)$ (recall that $\text{Aff}(D_1)$ is actually a vector space) and thus $2g \in D_1$. As D_1 is convex and $0 \in D_1$, we get that $g \in D_1$ and $\text{Aff}(D_1) \cap B(\frac{e^*}{2}, \frac{\alpha}{2}) \subset D_1$ which proves that $\frac{e^*}{2} \in ri(D_1)$. Now let ε_j be such that $\varepsilon_j(\frac{c}{2}\theta^j - \frac{e^*}{2}) \in B(0, \frac{\alpha}{2})$. It is easy to see that one can chose $\varepsilon_j \in (0, 1)$. Then as $\bar{e}^j := \frac{e^*}{2} + \frac{\varepsilon_j}{2}(c\theta^j - e^*) \in \text{Aff}(D_1) \cap B(\frac{e^*}{2}, \frac{\alpha}{2})$ (recall that $\theta^j \in W \subset \text{Aff}(D_1)$), we deduce that $\bar{e}^j \in D_1$. Using (2.25) we obtain that for Q-almost all ω

$$\begin{aligned}
V^+(\omega, 1 + c\theta^j Y(\omega)) &= V^+(\omega, 1 + e^* Y(\omega) + (c\theta^j - e^*)Y(\omega)) \\
&\leq \left(\frac{2}{\varepsilon_j}\right)^{\overline{\gamma}} K \left[V^+\left(\omega, \frac{\varepsilon_j}{2}(1 + e^* Y(\omega)) + \frac{\varepsilon_j}{2}(c\theta^j - e^*)Y(\omega) + \frac{1}{2}\right) + C(\omega)\right] \\
&\leq \left(\frac{2}{\varepsilon_j}\right)^{\overline{\gamma}} K \left[V^+\left(\omega, \frac{1}{2} + \frac{e^*}{2}Y(\omega) + \frac{\varepsilon_j}{2}(c\theta^j - e^*)Y(\omega) + \frac{1}{2}\right) + C(\omega)\right] \\
&\leq \left(\frac{2}{\varepsilon_j}\right)^{\overline{\gamma}} K \left[V^+(\omega, 1 + \bar{e}^j Y(\omega)) + C(\omega))\right],
\end{aligned}$$

where the second inequality follows from the fact that $1 + e^* Y(\cdot) \geq 0$ Q-a.s (recall that $e^* \in ri(D_1)$) and the monotonicity property of V in Assumption 2.4.1. Note that the above inequalities are true even if $1 + c\theta^j Y(\omega) < 0$ since (2.25) (see remark 2.5.8) and the monotonicity property of V hold true for all $x \in \mathbb{R}$.
From Assumption 2.5.9 we get that $EV^+(\cdot, 1 + \bar{e}^j Y(\cdot)) < \infty$ (recall that $\bar{e}_j \in D_1$) and Assumption 2.5.7 implies $EC < \infty$, therefore $EV^+(\cdot, 1 + c\theta^j Y(\cdot)) < \infty$ and (2.27) is proven for $h \in D_x$. Now let $h \in \mathcal{H}_x$ and h' its orthogonal projection on D, then $hY(\cdot) = h'Y(\cdot)$ Q-a.s (see Remark 2.5.3). It is clear that $h' \in D_x$ thus $V^+(\cdot, x + hY(\cdot)) = V^+(\cdot, x + h'Y(\cdot))$ Q-a.s and (2.27) is true also for $h \in \mathcal{H}_x$. □

[4] Here $B(y, \alpha)$ is the ball of $\mathbb{R}^d$ centered at y and with radius α.

Lemma 2.5.12 *Assume that Assumptions 2.5.1, 2.5.4, 2.5.7 and 2.5.9 hold true. Let $\mathcal{D}$ be the set valued function that assigns to each $x \geq 0$ the set D_x. Then $Graph(\mathcal{D}) := \{(x,h) \in [0,+\infty) \times \mathbb{R}^d,\ h \in D_x\}$ is a closed subset of $\mathbb{R} \times \mathbb{R}^d$. Let $\psi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ be defined by*

$$\psi(x,h) := \begin{cases} EV(\cdot, x + hY(\cdot)), \textit{if } (x,h) \in Graph(\mathcal{D}) \\ -\infty, \textit{otherwise}. \end{cases} \tag{2.31}$$

Then ψ is usc on $\mathbb{R} \times \mathbb{R}^d$ and $\psi < +\infty$ on $Graph(\mathcal{D})$.

Proof. Let $(x_n, h_n)_{n \geq 1} \in Graph(\mathcal{D})$ be a sequence converging to some $(x^*, h^*) \in \mathbb{R} \times \mathbb{R}^d$. We prove first that $(x^*, h^*) \in Graph(\mathcal{D})$, *i.e* that $Graph(\mathcal{D})$ is a closed set. It is clear that $x^* \geq 0$. Set for $n \geq 1$ $E_n := \{\omega \in \overline{\Omega},\ x_n + h_n Y(\omega) \geq 0\}$ and $E^* := \{\omega \in \overline{\Omega},\ x^* + h^* Y(\omega) \geq 0\}$. It is clear that $\limsup_n E_n \subset E^*$ and applying the Fatou Lemma (the *limsup* version) we get

$$Q\left(x^* + h^* Y(\cdot) \geq 0\right) = E1_{E^*}(\cdot) \geq E \limsup_n 1_{E_n}(\cdot) \geq \limsup_n E1_{E_n}(\cdot) = 1,$$

and $h^* \in \mathcal{H}_{x^*}$. Since D is closed by definition we have $h^* \in D_{x^*}$ and $(x^*, h^*) \in Graph(\mathcal{D})$.
We prove now that ψ is usc on $Graph(\mathcal{D})$. The upper semicontinuity on $\mathbb{R} \times \mathbb{R}^d$ will follow immediately from Lemma 2.8.11. By Assumption 2.5.4 $x \in \mathbb{R} \to V(x, \omega)$ is usc on $\mathbb{R}$ for all $\omega \in \overline{\Omega}$ and thus

$$\limsup_n V(\omega, x_n + h_n Y(\omega)) \leq V(\omega, x^* + h^* Y(\omega)).$$

By Lemma 2.5.11 for all $\omega \in \overline{\Omega}$

$$V(\omega, x_n + h_n Y(\cdot)) \leq V^+(\omega, x_n + h_n Y(\cdot)) \leq (|2x_n|^{\overline{\gamma}} K + 1) L(\omega) \leq (|2x^*|^{\overline{\gamma}} K + 2) L(\omega)$$

for n big enough. We can apply Fatou's Lemma (the *limsup* version) and ψ is usc on $Graph(\mathcal{D})$. From Lemma 2.5.11 it is also clear that $\psi < +\infty$ on $Graph(\mathcal{D})$. □
We are now able to state our main result.

Theorem 2.5.13 *Assume that Assumptions 2.5.1, 2.5.4, 2.5.7 and 2.5.9 hold true. Then for all $x \geq 0$, $v(x) < \infty$ and there exists some optimal strategy $\widehat{h} \in D_x$ such that*

$$v(x) = E(V(\cdot, x + \widehat{h} Y(\cdot))).$$

Moreover, $v : \mathbb{R} \to [-\infty, \infty)$ is non-decreasing and usc on $\mathbb{R}$.

Proof. Let $x \geq 0$ be fixed. We show first that $v(x) < \infty$. Indeed, using Lemma 2.5.11,

$$E(V(\cdot, x + hY(\cdot))) \leq E(V^+(\cdot, x + hY(\cdot))) \leq \left((2x)^{\overline{\gamma}} K + 1\right) EL(\cdot),$$

for all $h \in D_x$. Thus, recalling (2.23), $v(x) \leq ((2x)^{\overline{\gamma}}K+1)\, EL(\cdot) < \infty$.
It follows from Lemma 2.5.12 that $h \in \mathbb{R}^d \to \psi(x,h) := E(V(\cdot, x+hY(\cdot)))$ is usc on $\mathbb{R}^d$ and thus on D_x (recall that D_x is closed and see Lemma 2.8.11). Since by (2.23), $v(x) = \sup_{h\in D_x} E(\cdot, V(x+hY(\cdot)))$ and D_x is compact (see Lemma 2.5.10), applying [3, Theorem 2.43] there exists some $\widehat{h} \in D_x$ such that

$$v(x) = E(V(\cdot, x+\widehat{h}Y(\cdot))). \tag{2.32}$$

We show that v is usc on $[0,+\infty)$. As previously, the upper semicontinuity on $\mathbb{R}$ will follow immediately from Lemma 2.8.11. Let $(x_n)_{n\geq 0}$ be a sequence of non-negative numbers converging to some $x^* \in [0,+\infty)$. Let $\widehat{h}_n \in D_{x_n}$ be the associated optimal strategies to x_n in (2.32). Let $(n_k)_{k\geq 1}$ be a subsequence such that $\limsup_n v(x_n) = \lim_k v(x_{n_k})$. By Lemma 2.5.10 $|\widehat{h}_{n_k}| \leq x_{n_k}/\beta \leq (x^*+1)/\beta$ for k big enough. So we can extract a subsequence (that we still denote by $(n_k)_{k\geq 1}$) such that there exists some $\underline{h}^*$ with $\widehat{h}_{n_k} \to \underline{h}^*$. As the sequence $(x_{n_k}, \hat{h}_{n_k})_{k\geq 1} \in Graph(\mathcal{D})$ converges to $(x^*, \underline{h}^*)$ and $Graph(\mathcal{D})$ is closed (see Lemma 2.5.12), we get that $\underline{h}^* \in \mathcal{D}_{x^*}$. Using Lemma 2.5.12

$$\limsup_n v(x_n) = \lim_k v(x_{n_k}) = \lim_k EV(\cdot, x_{n_k} + \widehat{h}_{n_k}Y(\cdot)) \leq EV(\cdot, x^* + \underline{h}^*Y(\cdot)) \leq v(x^*),$$

where the last inequality holds true because $\underline{h}^* \in D_{x^*}$ and therefore v is usc on $[0,+\infty)$. Now as, by Assumption 2.5.4, $V(\omega,\cdot)$ is non-decreasing for all $\omega \in \overline{\Omega}$, v is also non-decreasing on $[0,+\infty)$ and since $v(x) = -\infty$ on $(-\infty, 0)$, v is non-decreasing on $\mathbb{R}$. □

2.6 Multi-period case

We first prove the following proposition.

Proposition 2.6.1 *Let Assumptions 2.4.7, 2.4.8 and 2.4.10 hold true. Then* $EU^+\left(\cdot, V_T^{x,\phi}(\cdot)\right) < \infty$ *for all* $x \geq 0$ *and* $\phi \in \Phi(x)$. *This implies that* $\Phi(U,x) = \Phi(x)$.

Proof. Fix $0 \leq x \leq 1$ and let $\phi \in \Phi(x)$. Then $V_T^{x,\phi} \leq V_T^{1,\phi}$ and $\phi \in \Phi(1) = \Phi(1,U)$ (recall Assumption 2.4.8). For any $\omega \in \Omega$, the function $y \to U(\omega, y)$ is non-decreasing on $\mathbb{R}$, so that $EU^+\left(\cdot, V_T^{x,\phi}(\cdot)\right) \leq EU^+\left(\cdot, V_T^{1,\phi}(\cdot)\right) < \infty$ by Assumption 2.4.7. Now, if $x \geq 1$, let $\phi \in \Phi(x)$ be fixed. From Assumption 2.4.10 we get that for all $\omega \in \Omega$

$$U(\omega, V_T^{x,\phi}(\omega)) = U\left(\omega, 2x\left(\frac{1}{2} + \sum_{t=1}^{T} \frac{\phi_t(\omega^{t-1})}{2x}\Delta S_t(\omega^t)\right)\right) \leq (2x)^{\overline{\gamma}}K\left(U(\omega, V_T^{1,\frac{\phi}{2x}}(\omega)) + C(\omega)\right).$$

By Assumption 2.4.8, $\frac{\phi}{2x} \in \Phi(\frac{1}{2}) \subset \Phi(1) = \Phi(1, U)$. Thus

$$EU^+\left(\cdot, V_T^{x,\phi}(\cdot)\right) \leq (2x)^{\overline{\gamma}} K \left(EU^+\left(\cdot, V_T^{1,\frac{\phi}{2x}}(\cdot)\right) + E(C)\right) < \infty,$$

using Assumption 2.4.7 and the fact that C is integrable (see Assumption 2.4.10). In both cases, we conclude that $\Phi(x) = \Phi(U, x)$. □

We introduce now the dynamic programming procedure. First we set for all $t \in \{0, \dots, T-1\}$, $\omega^t \in \Omega^t$ and $x \geq 0$

$$\mathcal{H}_x^{t+1}(\omega^t) := \left\{h \in \mathbb{R}^d,\ q_{t+1}(x + h\Delta S_{t+1}(\omega^t, \cdot) \geq 0|\omega^t) = 1\right\}, \tag{2.33}$$

$$\mathcal{D}_x^{t+1}(\omega^t) := \mathcal{H}_x^{t+1}(\omega^t) \cap D^{t+1}(\omega^t), \tag{2.34}$$

where D^{t+1} was introduced in Definition 2.3.2. For $x < 0$ we set $\mathcal{H}_x^{t+1}(\omega^t) = \emptyset$.
We define for all $t \in \{0, \dots, T\}$ the following functions U_t from $\Omega^t \times \mathbb{R} \to \mathbb{R}$. Starting with $t = T$, we set for all $x \in \mathbb{R}$, all $\omega^T \in \Omega$

$$U_T(\omega^T, x) := U(\omega^T, x). \tag{2.35}$$

Recall that $U(\omega^T, x) = -\infty$ for all $(\omega^T, x) \in \Omega \times (-\infty, 0)$.
Using for $t \geq 1$ the full-measure set $\widetilde{\Omega}^t \in \mathcal{F}_t$ that will be defined by induction in Propositions 2.6.9 and 2.6.10, we set for all $x \in \mathbb{R}$ and $\omega^t \in \Omega^t$

$$U_t(\omega^t, x) := \begin{cases} \sup_{h \in \mathcal{H}_x^{t+1}(\omega^t)} \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, x + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) q_{t+1}(d\omega_{t+1}|\omega^t), \\ \text{ if } (\omega^t, x) \in \widetilde{\Omega}^t \times [0, +\infty) \\ 0 \text{ if } x \geq 0 \text{ and } \omega^t \notin \widetilde{\Omega}^t \\ -\infty \text{ if } x < 0. \end{cases} \tag{2.36}$$

Finally for $t = 0$

$$U_0(x) := \begin{cases} \sup_{h \in \mathcal{H}_x^1} \int_{\Omega_1} U_1(\omega_1, x + h\Delta S_1(\omega_1)) P_1(d\omega_1) \text{ if } x \geq 0. \\ -\infty \text{ otherwise} \end{cases} \tag{2.37}$$

Remark 2.6.2 We will prove by induction that U_t is well-defined (see (2.39)), $i.e$ the integrals in (2.36) and (2.37) are well-defined in the generalised sense.

Remark 2.6.3 Before going further we provide some explanations on the choice of U_t. The natural definition of U_t should have been

$$\mathcal{U}_t(\omega^t, x) := \begin{cases} \sup_{h \in \mathcal{H}_x^{t+1}(\omega^t)} \int_{\Omega_{t+1}} \mathcal{U}_{t+1}(\omega^t, \omega_{t+1}, x + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) q_{t+1}(d\omega_{t+1}|\omega^t) \\ \text{ if } x \geq 0 \\ -\infty \text{ otherwise.} \end{cases}$$

Introducing the P_t full measure set $\widetilde{\Omega}^t$ in (2.36) is related to measurability issues that will be tackled in Proposition 2.6.11. This is not a surprise as this is related to the use of conditional expectations which are defined only almost everywhere.

Lemma 2.6.4 *Let $0 \leq t \leq T-1$ and H be a fixed $\mathbb{R}$-valued and $\mathcal{F}_t$-measurable random variable. Consider the following random sets*

$$\mathcal{H}_H^{t+1} : \omega^t \in \Omega^t \twoheadrightarrow \mathcal{H}_{H(\omega^t)}^{t+1}(\omega^t) \;\; \textit{and} \;\; \mathcal{D}_H^{t+1} : \omega^t \in \Omega^t \twoheadrightarrow \mathcal{D}_{H(\omega^t)}^{t+1}(\omega^t).$$

Then those random sets are all closed-valued and with graph valued in $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$.

Proof. First it is clear that $\mathcal{H}_H^{t+1}$ is closed-valued. As D^{t+1} is closed-valued (see Lemma 2.3.3) it follows that $\mathcal{D}_H^{t+1}$ is closed-valued as well. The fact that $Graph(\mathcal{H}_H^{t+1}) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ follows immediately from

$$Graph(\mathcal{H}_H^{t+1}) = \left\{(\omega^t, h) \in \Omega^t \times \mathbb{R}^d, H(\omega^t) \geq 0,\; q_{t+1}\left(\left\{H(\omega^t) + h\Delta S_{t+1}(\omega^t, .) \geq 0\right\} = 1|\omega^t\right)\right\},$$

and Lemma 2.8.9 (recall that H is $\mathcal{F}_t$-measurable). We know from Lemma 2.3.3 that $Graph(D^{t+1}) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ and it follows that

$$Graph(\mathcal{D}_H^{t+1}) = Graph(D^{t+1}) \cap Graph(\mathcal{H}_H^{t+1}) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d).$$

□

Finally we introduce

$$\begin{aligned} C_T(\omega^T) :=& \quad C(\omega^T), \text{ for } \omega^T \in \Omega^T, \text{ where } C \text{ is defined in Assumption 2.4.10} \\ C_t(\omega^t) :=& \quad \int_{\Omega_{t+1}} C_{t+1}(\omega^t, \omega_{t+1}) q_{t+1}(d\omega_{t+1}|\omega^t) \text{ for } t \in \{0, \dots, T-1\}, \omega^t \in \Omega^t. \end{aligned} \tag{2.38}$$

Lemma 2.6.5 *The functions $\omega^t \in \Omega^t \to C_t(\omega^t)$ are well-defined, non-negative (for all ω^t), $\mathcal{F}_t$-measurable and satisfy $E(C_t) = E(C_T) < \infty$. Furthermore, for all $t \in \{1, \dots, T\}$, there exists $\Omega_C^t \in \mathcal{F}_t$ and with $P_t(\Omega_C^t) = 1$ and such that $C_t(\cdot) < \infty$ on Ω_C^t. For $t = 0$ we have $C_0 < \infty$.*

Proof. We proceed by induction. For $t = T$ by Assumption 2.4.10 $C_T = C$ is $\mathcal{F}_T$-measurable, $C_T \geq 0$ and $E(C_T) < \infty$. Assume now that C_{t+1} is $\mathcal{F}_{t+1}$-measurable, $C_{t+1} \geq 0$ and $E(C_{t+1}) = E(C_T) < \infty$. From Proposition 2.8.6 $i)$ applied to $f = C_{t+1}$ we get that $\omega^t \to C_t(\omega^t) = \int_{\Omega_{t+1}} C_{t+1}(\omega^t, \omega_{t+1}) q_{t+1}(d\omega_{t+1}|\omega^t)$ is $\mathcal{F}_t$-measurable. As $C_{t+1}(\omega^{t+1}) \geq 0$ for all ω^{t+1}, it is clear that $C_t(\omega^t) \geq 0$ for all ω^t. Applying the Fubini Theorem (see Lemma 2.8.1) we get that

$$\begin{aligned} E(C_t) &= \int_{\Omega^t} \int_{\Omega_{t+1}} C_{t+1}(\omega^t, \omega_{t+1}) q_{t+1}(d\omega_{t+1}|\omega^t) P_t(d\omega^t) \\ &= \int_{\Omega^{t+1}} C_{t+1}(\omega^{t+1}) P_{t+1}(d\omega^{t+1}) = E(C_{t+1}) = E(C_T) < \infty \end{aligned}$$

and the induction step is complete. For the second part of the lemma, we apply Lemma 2.8.7 to $f = C_{t+1}$ and we obtain that $\Omega^t_C := \{\omega^t \in \Omega^t,\ C_t(\omega^t) < \infty\} \in \mathcal{F}_t$ and $P_t(\Omega^t_C) = 1$. □

Propositions 2.6.7 to 2.6.11 below solve the dynamic programming procedure and hold true under the following set of conditions. Let $1 \leq t \leq T$ be fixed.

$$U_t\left(\omega^t, \cdot\right) : \mathbb{R} \to \mathbb{R} \text{ is well-defined, non-decreasing and usc on } \mathbb{R} \text{ for all } \omega^t \in \Omega^t, \tag{2.39}$$

$$U_t\left(\cdot, \cdot\right) : \Omega^t \times \mathbb{R} \to \mathbb{R}\{\pm\infty\} \text{ is } \mathcal{F}_t \otimes \mathbb{B}(\mathbb{R})\text{-measurable}, \tag{2.40}$$

$$\int_{\Omega^t} U_t^+(\omega^t, H(\omega^{t-1}) + \xi(\omega^{t-1})\Delta S_t(\omega^t))P_t(d\omega^t) < \infty, \tag{2.41}$$

for all $\xi \in \Xi_{t-1}$ and $H = x + \sum_{s=1}^{t-1} \phi_s \Delta S_s$ where $x \geq 0$, $\phi_1 \in \Xi_0, \ldots, \phi_{t-1} \in \Xi_{t-2}$ and $P_t(H(\cdot) + \xi(\cdot)\Delta S_t(\cdot) \geq 0) = 1$,

$$U_t(\omega^t, \lambda x) \leq \lambda^{\overline{\gamma}} K \left(U_t\left(\omega^t, x + \frac{1}{2}\right) + C_t(\omega^t) \right), \text{ for all } \omega^t \in \Omega^t,\ \lambda \geq 1,\ x \in \mathbb{R}. \tag{2.42}$$

Remark 2.6.6 Note that from (2.39) and (2.40) we have that $-U_t$ is a $\overline{\mathcal{F}_t}$-normal integrand (see Definition 2.8.23 and Remark 2.8.24 in Section 2.8.3). However to prove that this property is preserved in the dynamic programming procedure we need to show separately that (2.39) and (2.40) are true. Furthermore, as our sigma-algebras are not assumed to be complete, obtaining some $\mathcal{F}_t$-normal integrand from $-U_t$ would introduce yet another layer of difficulty. For these reasons we choose to prove (2.39) and (2.40) instead of some normal integrand property. Nevertheless we will use again the properties of normal integrands in the proof of Lemma 2.6.11.

The next proposition is a first step in the construction of $\widetilde{\Omega}^t$.

Proposition 2.6.7 *Let* $0 \leq t \leq T-1$ *be fixed. Assume that (NA) condition holds true and that* (2.39), (2.40), (2.41) *and* (2.42) *hold true at stage* $t+1$. *Then there exists* $\widetilde{\Omega}^t_1 \in \mathcal{F}_t$ *such that* $P_t(\widetilde{\Omega}^t_1) = 1$ *and such that for all* $\omega^t \in \widetilde{\Omega}^t_1$ *the function* $(\omega_{t+1}, x) \to U_{t+1}(\omega^t, \omega_{t+1}, x)$ *satisfies the assumptions of Theorem 2.5.13 with* $\overline{\Omega} = \Omega_{t+1}$, $\mathcal{H} = \mathcal{G}_{t+1}$, $Q(\cdot) = q_{t+1}(\cdot|\omega^t)$, $Y(\cdot) = \Delta S_{t+1}(\omega^t, \cdot)$, $V(\cdot, y) = U_{t+1}(\omega^t, \cdot, y)$ *where* V *is defined on* $\Omega_{t+1} \times \mathbb{R}$.

Remark 2.6.8 Note that Lemmata 2.5.11, 2.5.12 and Theorem 2.5.13 hold true under the same set of assumptions. Therefore we can replace Theorem 2.5.13 by either Lemmata 2.5.11 or 2.5.12 in the above proposition.

Proof. To prove the proposition we will review one by one the assumptions needed to apply Theorem 2.5.13 in the context $\overline{\Omega} = \Omega_{t+1}$, $\mathcal{H} = \mathcal{G}_{t+1}$, $Q(\cdot) = q_{t+1}(\cdot|\omega^t)$, $Y(\cdot) = \Delta S_{t+1}(\omega^t, \cdot)$, $V(\cdot, y) = U_{t+1}(\omega^t, \cdot, y)$ where V is defined on $\Omega_{t+1} \times \mathbb{R}$. In the

sequel we shortly call this the context $t+1$.
From (2.39) at $t+1$ for all $\omega^t \in \Omega^t$ and $\omega_{t+1} \in \Omega_{t+1}$, the function $x \in \mathbb{R} \to U_{t+1}(\omega^t, \omega_{t+1}, x)$ is non-decreasing and usc on $\mathbb{R}$. From (2.40) at $t+1$ for all fixed $\omega^t \in \Omega^t$ and $x \in \mathbb{R}$, the function $\omega_{t+1} \in \Omega_{t+1} \to U_{t+1}(\omega^t, \omega_{t+1}, x)$ is $\mathcal{G}_{t+1}$-measurable and thus Assumption 2.5.4 is satisfied in the context $t+1$ (recall that $U_{t+1}(\omega^t, \omega_{t+1}, x) = -\infty$ for all $x < 0$ by assumption).
We move now to the assumptions that are verified for ω^t chosen in some specific P_t-full measure set. First from Lemma 2.3.6 for all $\omega^t \in \Omega^t_{NA1}$ we have $0 \in D^{t+1}(\omega^t)$ (recall that in Section 2.5 we have assumed that D contains 0). From Proposition 2.3.7, Assumption 2.5.1 holds true for all $\omega^t \in \Omega^t_{NA}$ in the context $t+1$.
We handle now Assumption 2.5.7 on asymptotic elasticity in context $t+1$. Let $\omega^t \in \Omega^t_C$ be fixed where Ω^t_C is defined in Lemma 2.6.5. From (2.42) at $t+1$ we have that for all $\omega_{t+1} \in \Omega_{t+1}$, $\lambda \geq 1$ and $x \in \mathbb{R}$

$$U_{t+1}(\omega^t, \omega_{t+1}, \lambda x) \leq \lambda^{\overline{\gamma}} K \left(U_{t+1}\left(\omega^t, \omega_{t+1}, x + \frac{1}{2}\right) + C_{t+1}(\omega^t, \omega_{t+1}) \right).$$

Now from Lemma 2.6.5 since $\omega^t \in \Omega^t_C$, we get that

$$\int_{\Omega_{t+1}} C_{t+1}(\omega^t, \omega_{t+1}) q_{t+1}(\omega_{t+1}|d\omega^t) = C_t(\omega^t) < \infty$$

and thus Assumption 2.5.7 in context $t+1$ is verified for all $\omega^t \in \Omega^t_C$. We want to show that for ω^t in some P_t full measure set to be determined and for all $h \in \mathcal{H}^{t+1}_1(\omega^t)$ we have that

$$\int_{\Omega_{t+1}} U^+_{t+1}(\omega^t, \omega_{t+1}, 1 + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) q_{t+1}(d\omega_{t+1}|\omega^t) < \infty.$$

We introduce the following random set $I_1 : \Omega^t \twoheadrightarrow \mathbb{R}^d$

$$I_1(\omega^t) := \left\{ h \in \mathcal{H}^{t+1}_1(\omega^t), \int_{\Omega_{t+1}} U^+_{t+1}(\omega^t, \omega_{t+1}, 1 + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) q_{t+1}(d\omega_{t+1}|\omega^t) = \infty \right\}. \tag{2.43}$$

Arguing by contradiction and using measurable selection arguments we will prove that $I_1(\omega^t) = \emptyset$ for P_t-almost all $\omega^t \in \Omega^t$. We show first that $Graph(I_1) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$. It is clear from (2.40) at $t+1$ that $(\omega^t, \omega_{t+1}, h) \to U^+_{t+1}(\omega^t, \omega_{t+1}, 1 + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))$ is $\mathcal{F}_t \otimes \mathcal{G}_{t+1} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. Using Proposition 2.8.6 *ii)* we get that $(\omega^t, h) \to \int_{\Omega_{t+1}} U^+_{t+1}(\omega^t, \omega_{t+1}, 1 + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) q_{t+1}(d\omega_{t+1}|\omega^t)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$-measurable (taking potentially the value $+\infty$). From Lemma 2.6.4, we obtain $Graph(\mathcal{H}^{t+1}_1) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ and $Graph(I_1) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ follows.
Applying the Projection Theorem (see for example [37, Theorem 3.23]) we obtain

that $\{I_1 \neq \emptyset\} \in \overline{\mathcal{F}}_t$ and using the Aumann Theorem (see [119, Corollary 1]) there exists some $\overline{\mathcal{F}}_t$-measurable $\overline{h}_1 : \{I_1 \neq \emptyset\} \to \mathbb{R}^d$ such that for all $\omega^t \in \{I_1 \neq \emptyset\}$, $\overline{h}_1(\omega^t) \in I_1(\omega^t)$. We extend $\overline{h}_1$ on all Ω^t by setting $\overline{h}_1(\omega^t) = 0$ on $\Omega^t \setminus \{I_1 \neq \emptyset\}$. As $\{I_1 \neq \emptyset\} \in \overline{\mathcal{F}}_t$ it is clear that $\overline{h}_1$ remains $\overline{\mathcal{F}}_t$-measurable. Using Lemma 2.8.10 we get some $\mathcal{F}_t$-measurable $h_1 : \Omega^t \to \mathbb{R}^d$ and $\Omega^t_{I_1} \in \mathcal{F}_t$ such that $P_t(\Omega^t_{I_1}) = 1$ and $\Omega^t_{I_1} \subset \{\omega^t \in \Omega^t, h_1(\omega^t) = \overline{h}_1(\omega^t)\}$. Arguing as in the proof of Lemma 2.3.6 and using the Fubini Theorem (see Lemma 2.8.1) we get that

$$\begin{aligned} P_{t+1}\left(1 + h_1(\cdot)\Delta S_{t+1}(\cdot) \geq 0\right) &= \int_{\Omega^t} q_{t+1}(1 + h_1(\omega^t)\Delta S_{t+1}(\omega^t, \cdot) \geq 0|\omega^t)P_t(d\omega^t) \\ &= \int_{\Omega^t} q_{t+1}(1 + \overline{h}_1(\omega^t)\Delta S_{t+1}(\omega^t, \cdot) \geq 0|\omega^t)\overline{P}_t(d\omega^t) \\ &= 1. \end{aligned}$$

Now assume that $\overline{P}_t(\{I_1 \neq \emptyset\}) > 0$. Since $h_1 \in \Xi_t$ and $P_{t+1}(1 + h_1(\cdot)\Delta S_{t+1}(\cdot) \geq 0) = 1$ from (2.41) at $t+1$ applied to $H = 1$

$$\int_{\Omega^{t+1}} U^+_{t+1}(\omega^{t+1}, 1 + h_1(\omega^t)\Delta S_{t+1}(\omega^{t+1}))P_{t+1}(d\omega^{t+1}) < \infty.$$

We argue as in Lemma 2.3.6 again. Let

$$\begin{aligned} \varphi_1(\omega^t) &= \int_{\Omega_{t+1}} U^+_{t+1}(\omega^t, \omega_{t+1}, 1 + h_1(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t), \\ \overline{\varphi}_1(\omega^t) &= \int_{\Omega_{t+1}} U^+_{t+1}(\omega^t, \omega_{t+1}, 1 + \overline{h}_1(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t). \end{aligned}$$

We have already seen that

$$(\omega^t, h) \in \Omega^t \times \mathbb{R}^d \to \int_{\Omega_{t+1}} U^+_{t+1}(\omega^t, \omega_{t+1}, 1 + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t)$$

is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$-measurable (taking potentially value $+\infty$). By composition it is clear that φ_1 is $\mathcal{F}_t$-measurable and that $\overline{\varphi}_1$ is $\overline{\mathcal{F}}_t$-measurable. Furthermore as $\{\omega^t \in \Omega^t,\ \varphi_1(\omega^t) \neq \overline{\varphi}_1(\omega^t)\} \subset \{\omega^t \in \Omega^t,\ h_1(\omega^t) \neq \overline{h}_1(\omega^t)\}$, $\varphi_1 = \overline{\varphi}_1$ P_t-almost surely. This implies that $\int_{\Omega^t} \overline{\varphi}_1 d\overline{P}_t = \int_{\Omega^t} \varphi_1 dP_t$ and using again the Fubini Theorem (see Lemma 2.8.1) we get that

$$\begin{aligned} &\int_{\Omega^{t+1}} U^+_{t+1}(\omega^{t+1}, x + h_1(\omega^t)\Delta S_{t+1}(\omega^{t+1})P_{t+1}(d\omega^{t+1}) \\ &= \int_{\Omega^t}\int_{\Omega_{t+1}} U^+_{t+1}(\omega^t, \omega_{t+1}, 1 + h_1(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t)P_t(d\omega^t) \\ &= \int_{\Omega^t}\int_{\Omega_{t+1}} U^+_{t+1}(\omega^t, \omega_{t+1}, 1 + \overline{h}_1(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t)\overline{P}_t(d\omega^t) \\ &\geq \int_{\{I_1 \neq \emptyset\}} (+\infty)\overline{P}_t(d\omega^t) = +\infty. \end{aligned}$$

Therefore we must have $\overline{P}_t(\{I_1 \neq \emptyset\}) = 0$ $i.e$ $\overline{P}_t(\{I_1 = \emptyset\}) = 1$. Now since $\{I_1 = \emptyset\} \in \overline{\mathcal{F}}_t$ there exists $\Omega^t_{int} \subset \{I_1 = \emptyset\}$ such that $\Omega^t_{int} \in \mathcal{F}_t$ and $P_t(\Omega^t_{int}) = \overline{P}_t(\{I_1 = \emptyset\}) = 1$. For all $\omega^t \in \Omega^t_{int}$, Assumption 2.5.9 in the context $t+1$ is true and we can now define $\widetilde{\Omega}^t_1 \subset \Omega^t$

$$\widetilde{\Omega}^t_1 := \Omega^t_{NA} \cap \Omega^t_{int} \cap \Omega^t_C. \tag{2.44}$$

It is clear that $\widetilde{\Omega}^t_1 \in \mathcal{F}_t$, $P_t(\widetilde{\Omega}^t_1) = 1$ and the proof is complete. □

The next proposition enables us to initialize the induction argument that will be carried on in Proposition 2.6.11.

Proposition 2.6.9 *Assume that the (NA) condition and Assumptions 2.4.7, 2.4.8 and 2.4.10 hold true. Then U_T satisfies* (2.39), (2.40), (2.41) *and* (2.42) *for $t = T$. We set $\widetilde{\Omega}^T = \Omega$.*

Proof. We start with (2.39) for $t = T$. As $U_T = U$ (see (2.35)), using Definition 2.4.1, $x \in \mathbb{R} \to U_T(\omega^T, x)$ is well-defined, non-decreasing and usc on $\mathbb{R}$ and (2.39) for $t = T$ is true. We prove now (2.40) for $t = T$ $i.e$ that $U_T = U$ is $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R})$-measurable. To do that we show that for all $\omega^T \in \Omega^T$, $x \in \mathbb{R} \to U_T(\omega^T, x)$ is right-continuous and for all $x \in \mathbb{R}$, $\omega^T \in \Omega^T \to U_T(x, \omega^T)$ is $\mathcal{F}_T$-measurable (this is just the second point of Definition 2.4.1) so that we can use Lemma 2.8.16 and establish (2.40) for $t = T$. Let $\omega^T \in \Omega^T$ be fixed. From (2.39) at T that we have just proved, $x \in \mathbb{R} \to U_T(\omega^T, x)$ is non-decreasing and usc on $\mathbb{R}$, thus applying Lemma 2.8.12 we get that $x \in \mathbb{R} \to U_T(\omega^T, x)$ is right-continuous on $\mathbb{R}$.
We prove now that (2.41) is true for $t = T$. Let $\xi \in \Xi_{T-1}$ and $H = x + \sum_{t=1}^{T-1} \phi_t \Delta S_t$ where $x \geq 0$, $\phi_1 \in \Xi_0$, $\dots$,$\phi_{T-1} \in \Xi_{T-2}$ and $P_T(H(\cdot) + \xi(\cdot)\Delta S_T(\cdot) \geq 0) = 1$. Let $(\phi^\xi_i)_{1 \leq i \leq T} \in \Phi$ be defined by $\phi^\xi_T = \xi$ and $\phi^\xi_i = \phi_i$ for $1 \leq i \leq T-1$ then $V^{x,\phi^\xi}_T = H + \xi\Delta S_T$ and thus $\phi^\xi \in \Phi(x)$. Using Proposition 2.6.1 we get that $EU^+(\cdot, V^{x,\phi^\xi}_T(\cdot)) = EU^+_T(\cdot, H(\cdot) + \xi(\cdot)\Delta S_T(\cdot)) < \infty$ (recall that $U = U_T$). Therefore (2.41) is verified for $t = T$. Finally, from Assumption 2.4.10, (2.42) for $t = T$ is true. □

The next proposition proves that if (2.39), (2.40), (2.41) and (2.42) hold true at $t+1$ then they are also true at U_t for some well chosen $\widetilde{\Omega}^t$.

Proposition 2.6.10 *Let $0 \leq t \leq T-1$ be fixed. Assume that the (NA) condition holds true and that* (2.39), (2.40), (2.41) *and* (2.42) *are true at $t+1$ (where U_{t+1} is defined from a given $\widetilde{\Omega}^{t+1}$ see* (2.36)*). Then there exists some $\widetilde{\Omega}^t \in \mathcal{F}_t$ with $P_t(\widetilde{\Omega}^t) = 1$ such that* (2.39), (2.40), (2.41) *and* (2.42) *are true for t.*
Moreover for all $H = x + \sum_{s=1}^t \phi_s \Delta S_s$, with $x \geq 0$ and $\phi_1 \in \Xi_0, \dots, \phi_t \in \Xi_{t-1}$, such that $P_t(H \geq 0) = 1$ there exists some $\widetilde{\Omega}^t_H \in \mathcal{F}_t$ such that $P(\widetilde{\Omega}^t_H) = 1$, $\widetilde{\Omega}^t_H \subset \widetilde{\Omega}^t$ and

some $\widehat{h}^H_{t+1} \in \Xi_t$ *such that for all* $\omega^t \in \widetilde{\Omega}^t_H$, $\widehat{h}^H_{t+1}(\omega^t) \in \mathcal{D}^{t+1}_{H(\omega^t)}(\omega^t)$ *and* [5]

$$U_t(\omega^t, H(\omega^t)) = \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, H(\omega^t) + \widehat{h}^H_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t). \tag{2.45}$$

Proof. First we define $\widetilde{\Omega}^t$ and prove that (2.39) and (2.40) are true for U_t. Applying Proposition 2.6.7, we get that for all $\omega^t \in \widetilde{\Omega}^t_1$, the function $(\omega_{t+1}, x) \to U_{t+1}(\omega^t, \omega_{t+1}, x)$ satisfies the assumptions of Lemma 2.5.11 and Theorem 2.5.13 with $\overline{\Omega} = \Omega_{t+1}$, $\mathcal{H} = \mathcal{G}_{t+1}$, $Q = q_{t+1}(\cdot|\omega^t)$, $Y(\cdot) = \Delta S_{t+1}(\omega^t, \cdot)$, $V(\cdot, y) = U_{t+1}(\omega^t, \cdot, y)$ where V is defined on $\Omega_{t+1} \times \mathbb{R}$. In particular, for $\omega^t \in \widetilde{\Omega}^t_1$ and all $h \in \mathcal{H}^{t+1}_x(\omega^t)$, recalling (2.27) we have

$$\int_{\Omega_{t+1}} U^+_{t+1}(\omega^t, \omega_{t+1}, x + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t) < \infty. \tag{2.46}$$

Now, we introduce $\overline{U}_t : \Omega^t \times \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\overline{U}_t(\omega^t, x) := \begin{cases} \sup_{h \in \mathcal{D}^{t+1}_x(\omega^t)} \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, x + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t) \\ \text{ if } (\omega^t, x) \in \widetilde{\Omega}^t_1 \times [0, \infty) \\ 0 \text{ if } x \geq 0 \text{ and } \omega^t \notin \widetilde{\Omega}^t_1 \\ -\infty \text{ if } x < 0. \end{cases} \tag{2.47}$$

From (2.46), $\overline{U}_t$ is well-defined (in the generalised sense). First, we prove that $\overline{U}_t$ is $\overline{\mathcal{F}}_t \otimes \mathbb{R}$-measurable and then we will show that this implies that U_t is $\mathcal{F}_t \otimes \mathbb{R}$-measurable for a well chosen $\widetilde{\Omega}^t$. To show that $\overline{U}_t$ is $\overline{\mathcal{F}}_t \otimes \mathcal{B}(\mathbb{R})$-measurable, we use Lemma 2.8.16 (and Remark 2.8.17) after having proved that it is an extended Carathéodory function (see Definition 2.8.15). Applying Theorem 2.5.13, we get that for all $\omega^t \in \widetilde{\Omega}^t_1$, the function $x \in \mathbb{R} \to \overline{U}_t(\omega^t, x)$ is non-decreasing and usc on $\mathbb{R}$. Actually, this is true for all $\omega^t \in \Omega^t$ since outside $\widetilde{\Omega}^t_1$, $x \in \mathbb{R} \to U_t(\omega^t, x)$ is constant equal to zero on $[0, \infty)$ and to $-\infty$ on $(-\infty, 0)$. Let now $\omega^t \in \Omega^t$ be fixed. As $x \in \mathbb{R} \to \overline{U}_t(\omega^t, x)$ is non-decreasing and usc on $\mathbb{R}$ we can apply Lemma 2.8.12 and we get that $x \in \mathbb{R} \to \overline{U}_t(\omega^t, x)$ is right-continuous on $\mathbb{R}$. For $x \geq 0$ fixed, applying Lemma 2.6.11 with $H = x$ (here $\Omega^t_H = \widetilde{\Omega}^t_1$) we obtain that $\omega^t \in \Omega^t \to \sup_{h \in \mathbb{R}^d} u_x(\omega^t, h)$ is $\overline{\mathcal{F}}_t$-measurable. Finally, from the definitions of $\overline{U}_t$ and u_x, we get that

$$\overline{U}_t(\omega^t, x) = (-\infty)1_{(-\infty,0)} + 1_{[0,\infty)}(x)1_{\widetilde{\Omega}^t_1}(\omega^t) \sup_{h \in \mathbb{R}^d} u_x(\omega^t, h),$$

and this implies that $\omega^t \in \Omega^t \to \overline{U}_t(\omega^t, x)$ is $\overline{\mathcal{F}}_t$-measurable for all $x \in \mathbb{R}$ and thus that $\overline{U}_t$ is an extended Carathéodory function as claimed.

[5]Recall that the integral on the right hand side is defined in the generalised sense.

Finally, we prove the $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R})$-measurability of U_t. To do that we apply Lemma 2.8.13 and we obtain some $\Omega^t_{mes} \in \mathcal{F}_t$ such that $P_t(\Omega^t_{mes}) = 1$ and some $\mathcal{F}_t \otimes \mathbb{R}$-measurable $\widetilde{U}_t : \Omega^t \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ such that for all $x \in \mathbb{R}$, $\{\omega^t \in \Omega^t,\ \overline{U}_t(\omega^t, x) \neq \widetilde{U}_t(\omega^t, x)\} \subset \Omega^t \backslash \Omega^t_{mes}$. We are now in a position to define $\widetilde{\Omega}^t$ and set

$$\widetilde{\Omega}^t := \widetilde{\Omega}^t_1 \cap \Omega^t_{mes}. \tag{2.48}$$

It is clear that $\widetilde{\Omega}^t \in \mathcal{F}_t$ and that $P_t(\widetilde{\Omega}^t) = 1$ Furthermore, recalling (2.36), Remark 2.5.5 (see (2.23)) and the definition of $\overline{U}_t$ we have that for all $x \in \mathbb{R}$, $\omega^t \in \Omega^t$

$$\begin{aligned}
U_t(\omega^t, x) &= (-\infty)1_{(-\infty,0)}(x) + \\
&1_{[0,\infty)}(x)1_{\Omega^t_{mes}}(\omega^t)1_{\widetilde{\Omega}^t_1}(\omega^t) \sup_{h \in \mathcal{H}^{t+1}_x(\omega^t)} \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, x + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t) \\
&= (-\infty)1_{(-\infty,0)}(x) + \\
&1_{[0,\infty)}(x)1_{\Omega^t_{mes}}(\omega^t)1_{\widetilde{\Omega}^t_1}(\omega^t) \sup_{h \in \mathcal{D}^{t+1}_x(\omega^t)} \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, x + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t) \\
&= 1_{\Omega^t_{mes}}(\omega^t)\overline{U}_t(\omega^t, x) + (-\infty)1_{\Omega^t \backslash \Omega^t_{mes}}(\omega^t)1_{(-\infty,0)}(x) \\
&= 1_{\Omega^t_{mes}}(\omega^t)\widetilde{U}_t(\omega^t, x) + (-\infty)1_{\Omega^t \backslash \Omega^t_{mes}}(\omega^t)1_{(-\infty,0)}(x),
\end{aligned}$$

and the $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R})$-measurability of U_t follows immediately, $i.e$ (2.40) is true at t. It is clear as well from the third equality that (2.39) is true for t since we have proven that for all $\omega^t \in \Omega^t$, $x \in \mathbb{R} \to \overline{U}_t(\omega^t, x)$ is well-defined, non-decreasing and usc on $\mathbb{R}$. We turn now to the assumption on asymptotic elasticity $i.e$ (2.42) for t. If $\omega^t \notin \widetilde{\Omega}^t$, then (2.42) is true since $C_t(\omega^t) \geq 0$ for all ω^t. Let $\omega^t \in \widetilde{\Omega}^t$ be fixed. Let $x \geq 0$, $\lambda \geq 1$, $h \in \mathbb{R}^d$ such that $q_{t+1}(\lambda x + h\Delta S_{t+1}(\omega^t, .) \geq 0|\omega^t) = 1$ be fixed. By (2.42) for $t+1$ for all $\omega_{t+1} \in \Omega_{t+1}$, we have that

$$\begin{aligned}
U_{t+1}\left(\omega^t, \omega_{t+1}, \lambda x + h\Delta S_{t+1}(\omega^t, \omega_{t+1})\right) \leq \lambda^{\overline{\gamma}} K U_{t+1}\left(\omega^t, \omega_{t+1}, x + \frac{1}{2} + \frac{h}{\lambda}\Delta S_{t+1}(\omega^t, \omega_{t+1})\right) \\
+ \lambda^{\overline{\gamma}} C_{t+1}(\omega^t, \omega_{t+1}).
\end{aligned}$$

By integrating both sides (recall (2.46)) we get that

$$\begin{aligned}
&\int_{\Omega_{t+1}} U_{t+1}\left(\omega^t, \omega_{t+1}, \lambda x + h\Delta S_{t+1}(\omega^t, \omega_{t+1})\right) q_{t+1}(d\omega_{t+1}|\omega^t) \leq \\
&\lambda^{\overline{\gamma}} K \int_{\Omega_{t+1}} U_{t+1}\left(\omega^t, \omega_{t+1}, x + \frac{1}{2} + \frac{h}{\lambda}\Delta S_{t+1}(\omega^t, \omega_{t+1})\right) q_{t+1}(d\omega_{t+1}|\omega^t) \\
&\qquad + \lambda^{\overline{\gamma}} K \int_{\Omega_{t+1}} C_{t+1}(\omega^t, \omega_{t+1}) q_{t+1}(d\omega_{t+1}|\omega^t).
\end{aligned}$$

Since $C_t(\omega^t) = \int_{\Omega_{t+1}} C_{t+1}(\omega^t, \omega_{t+1}) q_{t+1}(d\omega_{t+1}|\omega^t)$ (see Lemma 2.6.5) and $h \in \mathcal{H}^{t+1}_{\lambda x}(\omega^t)$ implies that $\frac{h}{\lambda} \in \mathcal{H}^{t+1}_{x}(\omega^t) \subset \mathcal{H}^{t+1}_{x+\frac{1}{2}}(\omega^t)$, we obtain by definition of U_t (see (2.36)) that

$$\int_{\Omega_{t+1}} U_{t+1}\left(\omega^t, \omega_{t+1}, \lambda x + h\Delta S_{t+1}(\omega^t, \omega_{t+1})\right) q_{t+1}(d\omega_{t+1}|\omega^t)$$
$$\leq \lambda^{\overline{\gamma}} K U_t\left(\omega^t, x + \frac{1}{2}\right) + \lambda^{\overline{\gamma}} K C_t(\omega^t).$$

Taking the supremum over all $h \in \mathcal{H}^{t+1}_{\lambda x}(\omega^t)$ we conclude that (2.42) is true for t for $x \geq 0$. If $x < 0$, then (2.42) is true by definition of U_t. Note that we might have $\omega^t \in \Omega^t \backslash \Omega^t_C$ and $C_t(\omega^t) = +\infty$ since (2.42) does not require that $C_t(\omega^t) < +\infty$.

We now prove (2.45) for U_t. First, from Proposition 2.6.7 and Theorem 2.5.13 and since $\widetilde{\Omega}^t \subset \widetilde{\Omega}^t_1$, we have for all $\omega^t \in \widetilde{\Omega}^t$ and $x \geq 0$ that there exists some $\xi^* \in \mathcal{D}^{t+1}_x(\omega^t)$ such that

$$U_t(\omega^t, x) = \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, x + \xi^* \Delta S_{t+1}(\omega^t, \omega_{t+1})) q_{t+1}(d\omega_{t+1}|\omega^t), \tag{2.49}$$

where the integral on the right hand side is defined in the generalised sense (recall (2.46) and Lemma 2.5.11). Let $H = x + \sum_{s=1}^{t-1} \phi_s \Delta S_s$, with $x \geq 0$ and $\phi_s \in \Xi_s$ for $s \in \{1, \dots, t-1\}$, be fixed such that $P(H \geq 0) = 1$. Let $\widetilde{\Omega}^t_H := \widetilde{\Omega}^t \cap \{\omega^t \in \Omega^t, H(\omega) \geq 0\}$. Then $\widetilde{\Omega}^t_H \in \mathcal{F}_t$ and $P(\widetilde{\Omega}^t_H) = 1$. We introduce the following random set $\psi : \Omega^t \twoheadrightarrow \mathbb{R}^d$

$$\psi_H(\omega^t) := \left\{ h \in \mathcal{D}^{t+1}_{H(\omega^t)}(\omega^t),\ U_t(\omega^t, H(\omega^t)) = \int_{\Omega^{t+1}} U_{t+1}\left(\omega^t, \omega_{t+1}, H(\omega^t) + h\Delta S_{t+1}(\omega^t, \omega_{t+1})\right) q_{t+1}(d\omega_{t+1}|\omega^t) \right\},$$

for $\omega^t \in \widetilde{\Omega}^t_H$ and $\psi_H(\omega^t) = \emptyset$ otherwise. To prove (2.45) it is enough to find a $\mathcal{F}_t$-measurable selector for ψ_H. From the definitions of ψ_H and u_H (see (2.51)) we obtain that (recall that $\widetilde{\Omega}^t_H \subset \widetilde{\Omega}^t$ and $\widetilde{\Omega}^t_H \subset \Omega^t_H$, see (2.48) and the definition of Ω^t_H in Lemma 2.6.11).

$$Graph(\psi_H) = \left\{ (\omega^t, h) \in \left(\widetilde{\Omega}^t_H \times \mathbb{R}^d \right) \cap Graph(\mathcal{D}^{t+1}_H),\ U_t(\omega^t, H(\omega^t)) = u_H(\omega^t, h) \right\}.$$

From Lemma 2.6.4 we have that $Graph(\mathcal{D}^{t+1}_H) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$. We have already proved that $(\omega^t, y) \to U_t(\omega^t, y)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R})$-measurable and, as H is $\mathcal{F}_t$-measurable, we obtain that $\omega^t \to U_t(\omega^t, H(\omega^t))$ is $\mathcal{F}_t$-measurable. Now applying Lemma 2.6.11 we obtain that u_H is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. The fact that $Graph(\psi_H) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ follows immediately.

So we can apply the Projection Theorem (see for example [37, Theorem 3.23]) and we get that $\{\psi_H \neq \emptyset\} \in \overline{\mathcal{F}}_t$ and using the Aumann Theorem (see [119, Corollary 1]) that there exists some $\overline{\mathcal{F}}_t$-measurable $\overline{h}^H_{t+1} : \{\psi_H \neq \emptyset\} \to \mathbb{R}^d$ such that for all $\omega^t \in \{\psi_H \neq \emptyset\}$, $\overline{h}^H_{t+1}(\omega^t) \in \psi_H(\omega^t)$. Then we extend $\overline{h}^H_{t+1}$ on all Ω^t by setting $\overline{h}^H_{t+1} = 0$ on $\Omega^t \setminus \{\psi_H \neq \emptyset\}$. Now applying Lemma 2.8.10 we get some $\mathcal{F}_t$-measurable $\widehat{h}^H_{t+1}$:

$\Omega^t \to \mathbb{R}^d$ and some $\overline{\Omega}^t_H \in \mathcal{F}_t$ such that $P(\overline{\Omega}^t_H) = 1$ and $\overline{\Omega}^t_H \subset \{\overline{h}^H_{t+1} = \widehat{h}^H_{t+1}\}$. We prove now that the set $\{\psi_H \neq \emptyset\}$ is of full measure. Indeed, let $\omega^t \in \widetilde{\Omega}^t_H$ be fixed. Using (2.49) for $x = H(\omega^t) \geq 0$, there exists $h^*(\omega^t) \in \psi_H(\omega^t)$. Therefore $\widetilde{\Omega}^t_H \subset \{\psi_H \neq \emptyset\}$ and $\overline{P}_t(\{\psi_H \neq \emptyset\}) = 1$. So for all $\omega^t \in \overline{\Omega}^t_H \cap \widetilde{\Omega}^t_H$ we have

$$\begin{aligned} U_t(\omega^t, H(\omega^t)) &= \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, H(\omega^t) + \overline{h}^H_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t) \\ &= \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, H(\omega^t) + \widehat{h}^H_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t). \end{aligned}$$

So setting

$$\widetilde{\Omega}^t_H = \widetilde{\Omega}^t_H \cap \overline{\Omega}^t_H \subset \widetilde{\Omega}^t \tag{2.50}$$

(2.45) is proved for t.

We are now left with the proof of (2.41) for U_t. Let $\xi \in \Xi_{t-1}$ and $H = x + \sum_{s=1}^{t-1} \phi_s \Delta S_s$ where $x \geq 0$ and $\phi_1 \in \Xi_0, \dots, \phi_{t-1} \in \Xi_{t-2}$ and such that $P_t(H(\cdot) + \xi(\cdot)\Delta S_t(\cdot) \geq 0) = 1$. We fix some $\omega^t \in \widetilde{\Omega}^t$. Let $X(\omega^t) = H(\omega^{t-1}) + \xi(\omega^{t-1})\Delta S_t(\omega^t)$ then X is $\mathcal{F}_t$-measurable. We apply (2.45) to $X(\omega^t)$ (and $\mathcal{D}^{t+1}_{X(\omega^t)}(\omega^t)$), and we get some $\omega^t \in \Omega^t \to \widehat{h}_{t+1}(\omega^t)$ which is $\mathcal{F}_t$-measurable and $\widetilde{\Omega}^t_X \in \mathcal{F}_t$ such that $P_t(\widetilde{\Omega}^t_X) = 1$ and such that for all $\omega^t \in \widetilde{\Omega}^t_X$, $q_{t+1}\left(X(\omega^t) + \widehat{h}_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t, \cdot) \geq 0|\omega^t\right) = 1$ and

$$U_t(\omega^t, X(\omega^t)) = \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, X(\omega^t) + \widehat{h}_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t).$$

Using Jensen's Inequality

$$U^+_t(\omega^t, X(\omega^t)) \leq \int_{\Omega_{t+1}} U^+_{t+1}(\omega^t, \omega_{t+1}, X(\omega^t) + \widehat{h}_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t).$$

Thus as $P_t(\widetilde{\Omega}^t_X) = 1$

$$\begin{aligned} \int_{\widetilde{\Omega}^t_X} U^+_t(\omega^t, X(\omega^t))P_t(d\omega^t) &= \int_{\Omega^t} U^+_t(\omega^t, X(\omega^t))P_t(d\omega^t) \\ &\leq \int_{\Omega^{t+1}} U^+_{t+1}(\omega^{t+1}, X(\omega^t) + \widehat{h}_{t+1}(\omega^t)\Delta S_{t+1}(\omega^{t+1}))P_{t+1}(d\omega^{t+1}) < \infty, \end{aligned}$$

because of (2.41) for $t+1$ which applies since $X = x + \sum_{s=1}^{t-1} \phi_s \Delta S_s + \xi \Delta S_t$ where $x \geq 0$, $\phi_1 \in \Xi_1, \dots, \phi_{t-1} \in \Xi_{t-2}$, $\xi \in \Xi_{t-1}$ and $\widehat{h}_{t+1} \in \Xi_t$: (2.41) for t is proved. □

The following lemma was essential to obtain measurability issues in the proof of Lemma 2.6.10.

Lemma 2.6.11 *Fix some* $0 \leq t \leq T-1$ *and* $x \geq 0$. *Let* $H := x + \sum_{s=1}^{t-1} \phi_s \Delta S_s$, *where* $\phi_1 \in \Xi_0, \dots, \phi_{t-1} \in \Xi_{t-2}$ *and* $P_t(H \geq 0) = 1$. *Assume that the (NA) condition holds true and that* (2.39), (2.40), (2.41) *and* (2.42) *are true at* $t+1$. *Let* $u_H : \Omega^t \times \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ *be defined by*

$$u_H(\omega^t, h) := \begin{cases} \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, H(\omega^t) + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t), \\ \quad \textit{if } (\omega^t, h) \in \left(\Omega_H^t \times \mathbb{R}^d\right) \cap Graph(\mathcal{D}_H^{t+1}), \\ -\infty \ \textit{ if } (\omega^t, h) \notin Graph(\mathcal{D}_H^{t+1}), \\ 0 \ \textit{ otherwise}. \end{cases} \tag{2.51}$$

where $\mathcal{D}_H^{t+1}$ *is defined in Lemma 2.6.4 and* $\Omega_H^t := \widetilde{\Omega}_1^t \bigcap \{\omega^t \in \Omega^t, \ H(\omega^t) \geq 0\}$ *(see* (2.44) *for the definition of* $\widetilde{\Omega}_1^t$). *Then* u_H *is well-defined,* $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$*-measurable and for all* $\omega^t \in \Omega^t$, $h \in \mathbb{R}^d \to u_H(\omega^t, h)$ *is usc. Morevover,* $\omega^t \in \Omega^t \to \sup_{h \in \mathbb{R}^d} u_H(\omega^t, h)$ *is* $\overline{\mathcal{F}}_t$*-measurable.*

Remark 2.6.12 In the proof below we will show that for $(\omega^t, h) \in \left(\Omega_H^t \times \mathbb{R}^d\right) \cap Graph(\mathcal{D}_H^{t+1})$ the integral in (2.51) is well-defined. Note that this is not the case for all $(\omega^t, h) \in \Omega^t \times \mathbb{R}^d$. Indeed, let (ω^t, h) be fixed such that $q_{t+1}(H(\omega^t) + h\Delta S_{t+1}(\omega^t, \cdot) < 0|\omega^t) > 0$. Then it is clear that $\int_{\Omega_{t+1}} U_{t+1}^-(\omega^t, \omega_{t+1}, H(\omega^t) + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t) = \infty$ and as without further assumption we cannot prove that $\int_{\Omega_{t+1}} U_{t+1}^+(\omega^t, \omega_{t+1}, H(\omega^t) + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t) < \infty$ (it is easy to find some counterexamples), the integral in (2.51) may fail to be well-defined. We could have circumvented this issue by using the convention $\infty - \infty = -\infty$ but we prefer to refrain from doing so.

Proof. From (2.40) at $t+1$, U_{t+1} is $\mathcal{F}_t \otimes \mathcal{G}_{t+1} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable and since H and ΔS_{t+1} are respectively $\mathcal{F}_t$ and $\mathcal{F}_{t+1}$-measurable, we obtain that $(\omega^t, \omega_{t+1}, h) \in \Omega^t \times \Omega_{t+1} \times \mathbb{R}^d \to U_{t+1}(\omega^t, \omega_{t+1}, H(\omega^t) + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))$ is also $\mathcal{F}_t \otimes \mathcal{G}_{t+1} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. In order to prove that for $(\omega^t, h) \in \left(\Omega_H^t \times \mathbb{R}^d\right) \cap Graph(\mathcal{D}_H^{t+1})$ the integral in (2.51) is well-defined, we introduce $\widetilde{u}_H : \left(\Omega_H^t \times \mathbb{R}^d\right) \cap Graph(\mathcal{D}_H^{t+1}) \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\widetilde{u}_H(\omega^t, h) = \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, H(\omega^t) + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t).$$

First we show that $\widetilde{u}_H$ is well-defined in the generalised sense. Indeed, let $(\omega^t, h) \in \left(\Omega_H^t \times \mathbb{R}^d\right) \cap Graph(\mathcal{D}_H^{t+1})$ be fixed. As ω^t is fixed in Ω_H^t, we can show as in Proposition 2.6.10 that (2.46) holds true (here $H(\omega^t)$ is a fixed number as ω^t is fixed) and thus

$$\int_{\Omega_{t+1}} U_{t+1}^+(\omega^t, \omega_{t+1}, H(\omega^t) + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))q_{t+1}(d\omega_{t+1}|\omega^t) < \infty,$$

So $\widetilde{u}_H$ is well-defined (but may be infinite-valued).
We now prove that u_H is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. We can apply Proposition 2.8.6 $iv)$

to $\mathcal{S} = \left(\Omega^t_H \times \mathbb{R}^d\right) \cap Graph(\mathcal{D}^{t+1}_H)$, with $f(\omega^t, h, \omega_{t+1})$ equal to both $U^{\pm}_{t+1}(\omega^t, \omega_{t+1}, H(\omega^t) + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))$, since $\left(\Omega^t_H \times \mathbb{R}^d\right) \cap Graph(\mathcal{D}^{t+1}_H) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ (see Lemma 2.6.4), and both $(\omega^t, h, \omega_{t+1}) \in \Omega^t \times \mathbb{R}^d \times \Omega_{t+1} \to U^{\pm}_{t+1}(\omega^t, \omega_{t+1}, H(\omega^t) + h\Delta S_{t+1}(\omega^t, \omega_{t+1}))$ are $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G}_{t+1}$-measurable. So we obtain that $\widetilde{u}_H$ is $\left[\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)\right]_{\mathcal{S}}$-measurable, where $\left[\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)\right]_{\mathcal{S}}$ denotes the trace sigma-algebra of $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ on $\mathcal{S}$. Now we extend $\widetilde{u}_H$ to $\Omega^t \times \mathbb{R}^d$ by setting $\widetilde{u}_H(\omega^t, h) = -\infty$ if $(\omega^t, h) \notin Graph(\mathcal{D}^{t+1}_H)$ and $\widetilde{u}_H(\omega^t, h) = 0$ if $(\omega^t, h) \in Graph(\mathcal{D}^{t+1}_H)$ and $\omega^t \notin \Omega^t_H$. Since $\left[\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)\right]_{\mathcal{S}} \subset \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$, $\Omega^t_H \in \mathcal{F}_t$ and $Graph(\mathcal{D}^{t+1}_H) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$, this extension of $\widetilde{u}_H$ is again $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. As it is clear that this extension of $\widetilde{u}_H$ and u_H coincide, the measurability of u_H is proved.

We turn now to the usc property. Let $\omega^t \in \Omega^t_H \subset \widetilde{\Omega}^t_1$ be fixed. We apply Proposition 2.6.7 to U_{t+1} and we get, as $\omega^t \in \widetilde{\Omega}^t_1$, that the function $(\omega_{t+1}, x) \to U_{t+1}(\omega^t, \omega_{t+1}, x)$ satisfies the assumptions of Lemma 2.5.12 (see Remark 2.6.8) with $\overline{\Omega} = \Omega_{t+1}$, $\mathcal{H} = \mathcal{G}_{t+1}$, $Q = q_{t+1}(\cdot|\omega^t)$, $Y(\cdot) = \Delta S_{t+1}(\omega^t, \cdot)$, $V(\cdot, y) = U_{t+1}(\omega^t, \cdot, y)$ where V is defined on $\Omega_{t+1} \times \mathbb{R}$. Therefore the function $\phi_{\omega^t}(\cdot, \cdot)$ defined on $\mathbb{R} \times \mathbb{R}^d$ by

$$\phi_{\omega^t}(x, h) = \begin{cases} \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, x + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) q_{t+1}(d\omega_{t+1}|\omega^t) \\ \text{if } x \geq 0 \text{ and } h \in D^{t+1}_x(\omega^t) \\ -\infty \text{ otherwise.} \end{cases}$$

is usc on $\mathbb{R} \times \mathbb{R}^d$ (see (2.31)). In particular, for $x = H(\omega^t) \geq 0$ fixed, the function $h \in \mathbb{R}^d \to u_H(\omega^t, h) = \phi_{\omega^t}(H(\omega^t), h)$ is usc on $\mathbb{R}^d$. Now for $\omega^t \notin \Omega^t_H$, as u_H is equal to 0 if $h \in \mathcal{D}^{t+1}_{H(\omega^t)}(\omega^t)$ and to $-\infty$ otherwise, Lemma 2.8.11 applies (recall that the random set $\mathcal{D}^{t+1}_H$ is closed-valued) and $h \in \mathbb{R}^d \to u_H(\omega^t, h)$ is usc on all $\mathbb{R}^d$.

Finally, we apply [116, Corollary 14.34] and find that $-u_H$ is a $\overline{\mathcal{F}}_t$-normal integrand [6]. Now from [116, Theorem 14.37], we obtain that $\omega^t \in \Omega^t \to \sup_{h \in \mathbb{R}^d} u_H(\omega^t, h)$ is $\overline{\mathcal{F}}_t$-measurable and this concludes the proof. □

Proof. of Theorem 2.4.16. We proceed in three steps. First, we handle some integrability issues that are essential to the proof. Then, we build by induction a candidate for the optimal strategy and finally we establish its optimality.

Integrability Issues

We fix some $\phi \in \Phi(x) = \Phi(U, x)$ (recall Proposition 2.6.1). Since Proposition 2.6.9 holds true, we can apply Proposition 2.6.10 for $t = T-1$, and by backward induction, we can therefore apply Proposition 2.6.10 for all $t = T-2, \ldots, 0$. In particular, we get that (2.41) holds true for all $0 \leq t \leq T$. So choosing $H = V^{x,\phi}_{t-1}$ and $\xi = \phi_t$ we get

[6] [116, Corollary 14.34] holds true only for complete σ-algebra. That is the reason why $-u_H$ is a $\overline{\mathcal{F}}_t$-normal integrand and not a $\mathcal{F}_t$-normal integrand.

that (recall Lemma 2.4.3, for $\phi \in \Phi(x)$, $P_t(V_t^{x,\phi}(\cdot) \geq 0) = 1$)

$$\int_{\Omega^t} U_t^+ \left(\omega^t, V_t^{x,\phi}(\omega^t)\right) P_t(d\omega^t) < \infty. \tag{2.52}$$

This implies that $\int_{\Omega^t} U_t \left(\omega^t, V_t^{x,\phi}(\omega^t)\right) P_t(d\omega^t)$ is defined in the generalised sense and that we can apply the Fubini Theorem for generalised integral (see Proposition 2.8.4)

$$\int_{\Omega^t} U_t \left(\omega^t, V_t^{x,\phi}(\omega^t)\right) P_t(d\omega^t) = \int_{\Omega^{t-1}} \int_{\Omega_t} U_t \left(\omega^{t-1}, \omega_t, V_t^{x,\phi}(\omega^{t-1}, \omega_t)\right) q_{t-1}(d\omega_t|\omega^{t-1}) P_{t-1}(d\omega^{t-1}). \tag{2.53}$$

Construction of ϕ^*

We fix some $x \geq 0$ and build our candidate for the optimal strategy by induction. We start at $t = 0$ and use (2.45) in Proposition 2.6.10 with $H = x \geq 0$. We set $\phi_1^* := \widehat{h}_1^x$ and we obtain that (recall that $\mathcal{F}_0 = \{\emptyset, \Omega^0\}$)

$$P_1(x + \phi_1^* \Delta S_1(.) \geq 0) = 1 \text{ and } U_0(x) = \int_{\Omega_1} U_1 \left(\omega_1, x + \phi_1^* \Delta S_1(\omega_1\right) P_1(d\omega_1).$$

Recall from (2.52) that the above integral is well-defined in the generalised sense. Assume that until some $t \geq 1$ we have found some $\phi_1^* \in \Xi_0, \dots, \phi_t^* \in \Xi_{t-1}$ and some $\overline{\Omega}^1 \in \mathcal{F}_1, \dots, \overline{\Omega}^{t-1} \in \mathcal{F}_{t-1}$ such that for all $i = 1, \dots, t-1$, $\overline{\Omega}^i \subset \widetilde{\Omega}^i$, $P_i(\overline{\Omega}^i) = 1$, for all $i = 0, \dots, t-1$, $\phi_{i+1}^*(\omega^i) \in D^{i+1}(\omega^i)$ and

$$P_t \left(x + \phi_1^* \Delta S_1(\omega_1) + \dots + \phi_t^*(\omega^{t-1}) \Delta S_t(\omega^{t-1}, \omega_t) \geq 0\right) = 1,$$

and finally, for all $\omega^t \in \overline{\Omega}^t$

$$U_{t-1} \left(\omega^{t-1}, V_{t-1}^{x,\phi^*}(\omega^{t-1})\right) = \int_{\Omega_t} U_t \left(\omega^{t-1}, \omega_t, V_{t-1}^{x,\phi^*}(\omega^{t-1}) + \phi_t^*(\omega^{t-1}) \Delta S_t(\omega^{t-1}, \cdot)\right) q_t(d\omega_t|\omega^{t-1}),$$

where again the integral is well-defined in the generalised sense (see (2.52)). We apply Proposition 2.6.10 with $H(\cdot) = V_t^{x,\phi^*}(\cdot) = V_{t-1}^{x,\phi^*}(\cdot) + \phi_t^*(\cdot)\Delta S_t(\cdot)$ (recall that $P_t(V_t^{x,\phi^*} \geq 0 = 1)$ and there exists $\overline{\Omega}^t := \widetilde{\Omega}^t_{V_t^{x,\phi^*}} \in \mathcal{F}_t$ such that $\overline{\Omega}^t \subset \widetilde{\Omega}^t$, $P_t(\overline{\Omega}^t) = 1$ and some some $\mathcal{F}_t$-measurable $\omega^t \to \phi_{t+1}^*(\omega^t) := \widehat{h}_{t+1}^{V_t^{x,\phi^*}}(\omega^t)$ such that for all $\omega^t \in \overline{\Omega}^t$, $\phi_{t+1}^*(\omega^t) \in D^{t+1}(\omega^t)$

$$\begin{aligned} & q_{t+1}(V_t^{x,\phi^*}(\omega^t) + \phi_{t+1}^*(\omega^t)\Delta S_{t+1}(\omega^t, \cdot) \geq 0|\omega^t) = 1, \\ & U_t \left(\omega^t, V_t^{x,\phi^*}(\omega^t)\right) = \int_{\Omega_{t+1}} U_{t+1} \left(\omega^t, \omega_{t+1}, V_t^{x,\phi^*}(\omega^t) + \phi_{t+1}^*(\omega^t)\Delta S_{t+1}(\omega^t, \cdot)\right) q_{t+1}(d\omega_{t+1}|\omega^t). \end{aligned} \tag{2.54}$$

Now since $P_t(\overline{\Omega}^t) = 1$, we obtain by the Fubini Theorem that

$$P_{t+1}(V_{t+1}^{x,\phi^*} \geq 0) = \int_{\Omega^t} q_{t+1}(V_t^{x,\phi^*}(\omega^t) + \phi_{t+1}^*(\omega^t)\Delta S_{t+1}(\omega^t, \cdot) \geq 0|\omega^t) P_t(d\omega^t) = 1$$

and we can continue the recursion.
Thus, we have found $\phi^* = (\phi_t^*)_{1\leq t\leq T}$ such that for all $t = 0,\dots,T$, $P_t(V_t^{x,\phi^*} \geq 0) = 1$, *i.e* $\phi^* \in \Phi(x)$. We have also found some $\overline{\Omega}^t \in \mathcal{F}_t$, such that $\overline{\Omega}^t \subset \widetilde{\Omega}^t$, $P_t(\overline{\Omega}^t) = 1$ and for all $\omega^t \in \overline{\Omega}^t$, (2.54) holds true for all $t = 0,\dots,T-1$. Moreover, from Proposition 2.6.1, $\phi^* \in \Phi(U,x)$ and we have that $E(U(V_T^{x,\phi^*})) < \infty$.

Optimality of ϕ^*
We prove that ϕ^* is optimal in two steps.
Step 1: Using (2.53) with $\phi = \phi^*$ and the fact that $P_{T-1}(\overline{\Omega^{T-1}}) = 1$, we get that

$$\begin{aligned}E(U(V_T^{x,\phi^*})) &= \int_{\Omega^{T-1}}\int_{\Omega_T} U\left(\omega^{T-1},\omega_T, V_{T-1}^{x,\phi^*}(\omega^{T-1}) + \phi_T^*(\omega^{T-1})\Delta S_T(\omega^{T-1},\omega_T)\right) q_T(d\omega_T|\omega^{T-1})P_{T-1}(d\omega^{T-1})\\ &= \int_{\overline{\Omega}^{T-1}}\int_{\Omega_T} U_T\left(\omega^{T-1},\omega_T, V_{T-1}^{x,\phi^*}(\omega^{T-1}) + \phi_T^*(\omega^{T-1})\Delta S_T(\omega^{T-1},\omega_T)\right) q_T(d\omega_T|\omega^{T-1})P_{T-1}(d\omega^{T-1}).\end{aligned}$$

Using (2.54) for $t = T-1$ and again the fact that $P_{T-1}(\overline{\Omega}^{T-1}) = 1$, we have that

$$E(U(V_T^{x,\phi^*})) = \int_{\Omega^{T-1}} U_{T-1}\left(\omega^{T-1}, V_{T-1}^{x,\phi^*}(\omega^{T-1})\right) P_{T-1}(d\omega^{T-1}).$$

We iterate the process for $T-1$: using the Fubini Theorem (see (2.53)), $P_{T-2}(\overline{\Omega}^{T-2}) = 1$ and (2.54), we obtain that

$$E(U(V_T^{x,\phi^*})) = \int_{\Omega^{T-2}} U_{T-2}\left(\omega^{T-2}, V_{T-2}^{x,\phi^*}(\omega^{T-2})\right) P_{T-2}(d\omega^{T-2}).$$

By backward induction, we therefore obtain that (recall $\Omega^0 := \{\omega_0\}$)

$$E(U(V_T^{x,\phi^*})) = U_0(x).$$

As $\phi^* \in \Phi(U,x)$, we get that $U_0(x) \leq u(x)$. So ϕ^* will be optimal if $U_0(x) \geq u(x)$.
Step 2: We fix again some $\phi \in \Phi(U,x)$ (recall Proposition 2.6.1). We get that $V_t^{x,\phi} \geq 0$ P_t-a.s. for all $t = 1,\dots,T$ (recall Lemma 2.4.3). As $\phi_1 \in \mathcal{H}_x^1$ we obtain that

$$U_0(x) \geq \int_{\Omega_1} U_1(\omega_1, x + \phi_1\Delta S_1(\omega_1))P_1(d\omega_1).$$

As $P_2(V_1^{x,\phi} + \phi_2\Delta S_2 \geq 0) = 1$, there exists some P_1-full measure set $\widehat{\Omega}^1 \in \mathcal{F}_1$ such that for all $\omega_1 \in \widehat{\Omega}^1$, $q_2\left(V_1^{x,\phi}(\omega_1) + \phi_2(\omega_1)\Delta S_2(\omega_1,\cdot)) \geq 0|\omega_1\right) = 1$ *i.e* $q_2\left(\phi_2(\omega_1) \in \mathcal{H}^2_{V_1^{x,\phi}(\omega_1)}(\omega_1)|\omega_1\right) = 1$ (see Lemma 2.8.9). So for $\omega_1 \in \widehat{\Omega}^1$, we have that

$$U_1(\omega_1, V_1^{x,\phi}(\omega_1)) \geq \int_{\Omega_2} U_2\left(\omega_1,\omega_2, V_1^{x,\phi}(\omega_1) + \phi_2(\omega_1)\Delta S_1(\omega_1,\omega_2)\right) q_2(d\omega_2|\omega^1). \qquad (2.55)$$

From (2.52), $\int_{\Omega^2} U_2^+\left(\omega^2, V_2^{x,\phi}(\omega^2)\right) P_2(d\omega^2) < \infty$ and we can apply the Fubini Theorem (see (2.53)) and

$$\begin{aligned}\int_{\Omega^2} U_2\left(\omega^2, V_2^{x,\phi}(\omega^2)\right) P_2(d\omega^2) &= \int_{\Omega^1}\int_{\Omega_2} U_2\left(\omega_1,\omega_2, V_1^{x,\phi}(\omega_1)+\phi_2\Delta S_1(\omega_1,\omega_2)\right) q_2(d\omega_2|\omega_1)P_1(d\omega_1)\\ &= \int_{\widehat{\Omega}^1}\int_{\Omega_2} U_2\left(\omega_1,\omega_2, V_1^{x,\phi}(\omega_1)+\phi_2\Delta S_1(\omega_1,\omega_2)\right) q_2(d\omega_2|\omega_1)P_1(d\omega_1).\end{aligned}$$

Using again (2.52), $\int_{\Omega^1} U_1^+\left(\omega^1, V_1^{x,\phi}(\omega^1)\right) P_1(d\omega^1) < \infty$ and integrating (in the generalised sense) both side of (2.55) we obtain

$$\begin{aligned}\int_{\Omega^1} U_1(\omega_1, V_1^{x,\phi}(\omega_1))P_1(d\omega_1) &= \int_{\widehat{\Omega}^1} U_1(\omega_1, V_1^{x,\phi}(\omega_1))P_1(d\omega_1)\\ &\geq \int_{\widehat{\Omega}^1}\int_{\Omega_2} U_2\left(\omega_1,\omega_2, V_1^{x,\phi}(\omega_1)+\phi_2\Delta S_1(\omega_1,\omega_2)\right) q_2(d\omega_2|\omega_1)P_1(d\omega_1)\\ &= \int_{\Omega^2} U_2\left(\omega^2, V_2^{x,\phi}(\omega^2)\right) P_2(d\omega^2).\end{aligned}$$

Therefore

$$U_0(x) \geq \int_{\Omega^2} U_2\left(\omega^2, V_2^{x,\phi}(\omega^2)\right) P_2(d\omega^2).$$

We can go forward since for P_2-almost all ω^2 we have that $q_3\left(\phi_3(\omega^2) \in \mathcal{H}^3_{V_2^{x,\phi}(\omega^2)}(\omega^2)|\omega^2\right) = 1, \dots,$ for P_{T-1}-almost all ω^{T-1} we have that $q_T\left(\phi_T(\omega^{T-1}) \in \mathcal{H}^T_{V_{T-1}^{x,\phi}(\omega^{T-1})}(\omega^{T-1})|\omega^{T-1}\right) = 1$, we obtain using again (2.52) and the Fubini Theorem (see (2.53)) that

$$U_0(x) \geq \int_{\Omega_1}\int_{\Omega_2}\cdots\int_{\Omega_T} U\left(\omega^T, V_T^{x,\phi}(\omega^T)\right) q_T(d\omega_T|\omega^{T-1})\cdots q_2(d\omega_2|\omega^1)P_1(d\omega_1). \quad (2.56)$$

So we have that $U_0(x) \geq E(U(\cdot, V_T^{x,\phi}(\cdot)))$ for any $\phi \in \Phi(U,x)$ and the proof is complete since $u(x) = E(U(\cdot, V_T^{x,\phi^*}(\cdot))) < \infty$. □

Proof. of Theorem 2.4.17. To prove Theorem 2.4.17, we want to apply Theorem 2.4.16 and thus we need to establish that Assumptions 2.4.7 and 2.4.8 hold true. To do so we will prove (2.59) below. First we show that for all $x \geq 0$, $\phi \in \Phi(x)$ and $0 \leq t \leq T$, we have for P_t-almost all $\omega^t \in \Omega^t$

$$|V_t^{x,\phi}(\omega^t)| \leq x\prod_{s=1}^{t}\left(1+\frac{|\Delta S_s(\omega^s)|}{\alpha_{s-1}(\omega^{s-1})}\right). \quad (2.57)$$

To do so we first fix $x \geq 0$, some $\phi = (\phi_t)_{t=1,\dots T} \in \Phi(x)$ and $1 \leq t \leq T$. For $\omega^{t-1} \in \Omega^{t-1}$ fixed, we denote by $\phi_t^\perp(\omega^{t-1})$ the orthogonal projection of $\phi_t(\omega^{t-1})$ on $D^t(\omega^t)$. Recalling Remark 2.5.3 we have

$$q_t\left(\phi_t^\perp(\omega^{t-1})\Delta S_t(\omega^{t-1},\cdot) = \phi_t(\omega^{t-1})\Delta S_t(\omega^{t-1},\cdot)|\omega^{t-1}\right) = 1,$$

and thus $\phi_t^\perp(\omega^{t-1}) \in \mathcal{D}^t_{V^{x,\phi}_{t-1}(\omega^{t-1})}(\omega^{t-1})$ (see (4.44) for the definition of $\mathcal{D}^t_x$). As the NA condition holds true, Lemma 2.3.6 applies and $0 \in D^t(\omega^{t+1})$. We can then apply Lemma 2.5.10 and we obtain that

$$|\phi_t^\perp(\omega^{t-1})| \leq \frac{|V^{x,\phi}_{t-1}(\omega^{t-1})|}{\alpha_{t-1}(\omega^{t-1})}. \tag{2.58}$$

Furthermore, it is well-know that $\omega^{t-1} \in \Omega^{t-1} \to \phi_t^\perp(\omega^{t-1})$ is $\mathcal{F}_{t-1}$-measurable (see for example [116, Exercice 14.17 p655]) and we obtain, applying the Fubini Theorem (see Lemma 2.8.1), that $P_t\left(\phi_t^\perp \Delta S_t = \phi_t \Delta S_t\right) = 1$ and we denote by Ω^t_{EQ} the P_t-full measure set on which this equality is verified. We need to slightly modify the set Ω^t_{EQ} to use it for different periods. We proceed by induction. We start at $t = 1$ (recall that $\Omega^0 := \{\omega_0\}$) with Ω^1_{EQ}. For $t = 2$ we reset, with an abuse of notation, $\Omega^2_{EQ} = \Omega^2_{EQ} \cap \left(\Omega^1_{EQ} \times \Omega_2\right)$ and we reiterate the process until T. To prove (2.57) we proceed by induction. It is clear at $t = 0$. Fix some $t \geq 0$ and assume that (2.57) holds true at t. Let $\omega^{t+1} \in \Omega^{t+1}_{EQ}$, using (2.57) at t and (2.58) we get that

$$\begin{aligned} |V^{x,\phi}_{t+1}(\omega^{t+1})| &= \left|V^{x,\phi}_t(\omega^t) + \phi_{t+1}(\omega^t)\Delta S_{t+1}(\omega^{t+1})\right| = \left|V^{x,\phi}_t(\omega^t) + \phi^\perp_{t+1}(\omega^t)\Delta S_{t+1}(\omega^{t+1})\right| \\ &\leq \left|V^{x,\phi}_t(\omega^t)\right|\left(1 + \frac{|\Delta S_{t+1}(\omega^{t+1})|}{\alpha_t(\omega^t)}\right) \leq x \prod_{s=1}^{t+1}\left(1 + \frac{|\Delta S_s(\omega^s)|}{\alpha_{s-1}(\omega^{s-1})}\right) \end{aligned}$$

and (2.57) is proven for $t+1$. It follows since for all $0 \leq s \leq t$, $|\Delta S_s| \in \mathcal{W}_s$ and $\frac{1}{\alpha_s} \in \mathcal{W}_s$ that $V^{x,\phi}_t \in \mathcal{W}_t$. We will prove that for all $\Phi \in \Phi(x)$ and ω^T in a full measure set

$$U^+(\omega^T, V^{x,\phi}_T(\omega^T)) \leq 2^{\overline{\gamma}} K \max(x,1)^{\overline{\gamma}} \left(\prod_{s=1}^T \left(1 + \frac{|\Delta S_s(\omega^s)|}{\alpha_{s-1}(\omega^{s-1})}\right)\right)^{\overline{\gamma}} \left(U^+(\omega^T, 1) + C_T(\omega^T)\right). \tag{2.59}$$

Since by assumptions $EU^+(\cdot, 1) < \infty$, $EC_T < \infty$ and since for all $0 \leq t \leq T$, $|\Delta S_t| \in \mathcal{W}_t$ and $\frac{1}{\alpha_t} \in \mathcal{W}_t$, we get that $EU^+(\cdot, V^{x,\phi}_T(\cdot)) < \infty$ for all $\Phi \in \Phi(x)$ and both Assumptions 2.4.7 and 2.4.8 hold true. We prove now (2.59). We fix some $x \geq 0$ and $\phi \in \Phi(x)$. Then from the monotonicity of U^+, (2.57), Assumption 2.4.10, the fact that $\prod_{s=1}^T \left(1 + \frac{|\Delta S_s(\omega^s)|}{\alpha_{s-1}(\omega^{s-1})}\right) \geq 1$, we have for all $\omega^T \in \Omega^T_{EQ} \bigcap \widetilde{\Omega}_T$ that

$$\begin{aligned} U^+\left(\omega^T, V^{x,\phi}_T(\omega^T)\right) &\leq U^+\left(\omega^T, \max(x,1)\prod_{s=1}^T\left(1 + \frac{|\Delta S_s(\omega^s)|}{\alpha_{s-1}(\omega^{s-1})}\right)\right) \\ &\leq K\left(2\max(x,1)\prod_{s=1}^T\left(1 + \frac{|\Delta S_s(\omega^s)|}{\alpha_{s-1}(\omega^{s-1})}\right)\right)^{\overline{\gamma}} \left(U^+(\omega^T, 1) + C_T(\omega^T)\right). \end{aligned}$$

□

2.7 Conclusion

A natural question regarding this chapter is how the results can be generalised for a utility function defined on the whole-real line. Part of the answer was already provided in [33]. However the integrability assumption is not fully satisfying (see [33, Assumption 2.9]) as it is not easy to check. Moreover, from a purely technical point of view (and also from an aesthetic point of view), this makes the proof very painful and cumbersome as the integrability condition has to be verified with an upward induction (unlike in our case) while the others conditions required to apply the dynamic programming are verified by a more natural backward induction. Unfortunately, it seems difficult to replace this assumption with an assumption similar to Assumptions 2.4.7 and 2.4.8 that is not too restrictive. The reason for this is that when the utility function is defined on the whole-real line, the compacity argument (the equivalent of Lemma 2.5.10) requires more work and the bound obtained in the one-period model is a function of some parameters depending on the utility function. Thus it is not clear how to formulate a proper integrability condition at the terminal date T that would be preserved in the dynamic programming and this is left this for further research. Note that this issue will also be present in the multiple-priors case if one want to generalise the result of Chapter 4 for utility functions defined on the whole real line where this is compounded by other difficulties specific to the multiple-priors setting.

2.8 Appendix

In this appendix we report basic facts about generalised integral, measurable selection theorems, random sets and normal integrand. We also provide the proof Lemma 2.2.2 and of some theoretical result not directly used in the chapter.

2.8.1 Generalised integral and Fubini's Theorem

For ease of the reader we provide some well know results on measure theory, stochastic kernels and integrals. The first lemma provides a version of the Fubini Theorem for non-negative functions (see for instance to [23, Theorem 10.7.2]). We then present our definition of generalised integral and provide another version of the Fubini Theorem for generalised integral (see Proposition 2.8.4), which was used throughout the chapter.

Let $(H, \mathcal{H})$ and $(K, \mathcal{K})$ be two measurable spaces, p be a probabilty measure on $(H, \mathcal{H})$ and q a stochastic kernel on $(K, \mathcal{K})$ given $(H, \mathcal{H})$, *i.e* such that for any $h \in H$, $C \in \mathcal{K} \to q(C|h)$ is a probability measure on $(K, \mathcal{K})$ and for any $C \in \mathcal{K}$,

$h \in H \to q(C|h)$ is $\mathcal{H}$-measurable. Furthermore, for any $A \in \mathcal{H} \otimes \mathcal{K}$ and any $h \in H$, the section of A along h is defined by

$$(A)_h := \{k \in K,\ (h,k) \in A\}. \tag{2.60}$$

Lemma 2.8.1 *Let $A \in \mathcal{H} \otimes \mathcal{K}$ be fixed. For any $h \in H$ we have $(A)_h \in \mathcal{K}$ and we define P by*

$$P(A) := \int_H \int_K 1_A(h,k) q(dk|h) p(dh) = \int_H q((A)_h|h) p(dh). \tag{2.61}$$

Then P is a probability measure on $(H \times K, \mathcal{H} \otimes \mathcal{H})$.
Furthermore, if $f : H \times K \to \mathbb{R}_+ \cup \{+\infty\}$ is non-negative and $\mathcal{H} \otimes \mathcal{K}$-measurable then $h \in H \to \int_K f(h,k) q(dk|h)$ is $\mathcal{H}$-measurable with value in $\mathbb{R}_+ \cup \{\infty\}$ and we have

$$\int_{H \times K} f dP := \int_{H \times K} f(h,k) P(dh, dk) = \int_H \int_K f(h,k) q(dk|h) p(dh). \tag{2.62}$$

Proof. Let $h \in H$ be fixed. Let $\mathcal{T} = \{A \in \mathcal{H} \otimes \mathcal{K} \mid (A)_h \in \mathcal{K}\}$. It is easy to see that $\mathcal{T}$ is a sigma algebra on $H \times K$ and is included in $\mathcal{H} \otimes \mathcal{K}$. Let $A = B \times C \in \mathcal{H} \times \mathcal{K}$ then $(A)_h = \emptyset$ if $h \notin B$ and $(A)_h = C$ if $h \in B$. Thus $(A)_h \in \mathcal{K}$ and $\mathcal{H} \times \mathcal{K} \subset \mathcal{T}$. As $\mathcal{T}$ is a sigma-algebra, $\mathcal{H} \otimes \mathcal{K} \subset \mathcal{T}$ and $\mathcal{T} = \mathcal{H} \otimes \mathcal{K}$ follows.
We show now that

$$h \to \int_K 1_A(h,k) q(dk|h) = \int_K 1_{(A)_h}(k) q(dk|h) = q\left((A)_h |h\right)$$

is $\mathcal{H}$-measurable for any $A \in \mathcal{H} \otimes \mathcal{K}$.
Let $\mathcal{E} = \{A \in \mathcal{H} \otimes \mathcal{K} \mid h \in H \to q\left((A)_h|h\right) \text{ is } \mathcal{H}\text{-measurable}\}$. It is easy to see that $\mathcal{E}$ is a sigma algebra on $H \times K$ and is included in $\mathcal{H} \otimes \mathcal{K}$. Let $A = B \times C \in \mathcal{H} \times \mathcal{K}$ then $q\left((A)_h)|h\right)$ equals to 0 if $h \notin B$ and to $q(C|h)$ if $h \in B$. So by definition of $q(\cdot|\cdot)$, $\mathcal{H} \times \mathcal{K} \subset \mathcal{E}$. As $\mathcal{E}$ is a sigma-algebra, $\mathcal{H} \otimes \mathcal{K} \subset \mathcal{E}$ and $\mathcal{E} = \mathcal{H} \otimes \mathcal{K}$ follows. Thus the last integral in (2.61) is well-defined. We verify that P defines a probability measure on $(H \times K, \mathcal{H} \otimes \mathcal{H})$. It is clear that $P(\emptyset) = 0$ and $P(H \times K) = 1$. The sigma-additivity property follows from the monotone convergence theorem.
We prove now that for $f : H \times K \to \mathbb{R}_+ \cup \{+\infty\}$ non-negative and $\mathcal{H} \otimes \mathcal{K}$-measurable, $h \in H \to \int_K f(h,k) q(dk|h)$ is $\mathcal{H}$-measurable and (2.62) holds true. If $f = 1_A$ for $A \in \mathcal{H} \otimes \mathcal{K}$ the claim is proved. By taking linear combinations, it is proved for $\mathcal{H} \otimes \mathcal{K}$-measurable step functions. Then if $f : H \times K \to \mathbb{R} \cup \{+\infty\}$ is non-negative and $\mathcal{H} \otimes \mathcal{K}$-measurable, then there exists some increasing sequence $(f_n)_{n \geq 1}$ such that $f_n : H \times K \to \mathbb{R}$ is a $\mathcal{H} \otimes \mathcal{K}$-measurable step function and $(f_n)_{n \geq 1}$ converge to f. Using the monotone convergence theorem and (2.62) for steps functions, we conclude that (2.62) holds true for f. □

Definition 2.8.2 Let $f : H \times K \to \mathbb{R} \cup \{\pm\infty\}$ be a $\mathcal{H} \otimes \mathcal{K}$-measurable function. If $\int_{H\times K} f^+ dP < \infty$ or $\int_{H\times K} f^- dP < \infty$, we define the generalised integral of f by

$$\int_{H\times K} f dP := \int_{H\times K} f^+ dP - \int_{H\times K} f^- dP.$$

Remark 2.8.3 Note that if both $\int_{H\times K} f^+ dP = \infty$ and $\int_{H\times K} f^- dP = \infty$, the integral above is not defined. We could have introduced some convention to handle this situation, however, as in most of the cases we treat we have $\int_{H\times K} f^+ dP < \infty$, we refrain from doing so.

Proposition 2.8.4 *Let $f : H \times K \to \mathbb{R} \cup \{\pm\infty\}$ be a $\mathcal{H} \otimes \mathcal{K}$-measurable function such that $\int_{H\times K} f^+ dP < \infty$. Then, we have*

$$\int_{H\times K} f dP = \int_H \int_K f(h,k) q(dk|h) p(dh). \tag{2.63}$$

Remark 2.8.5 Note that we can assume instead that $\int_{H\times K} f^- dP < \infty$ and the result holds as well. We will use this in the proof of Lemma 2.2.2 later in the Appendix.

Proof. Using Definition 2.8.2 and applying Lemma 2.8.1 to f^+ and f^- we obtain that

$$\begin{aligned}\int_{H\times K} f dP &= \int_{H\times K} f^+ dP - \int_{H\times K} f^- dP \\ &= \int_H \int_K f^+ q(dk|h) p(dh) + \int_H \int_K f^- q(dk|h) p(dh).\end{aligned}$$

To establish (2.63), assume for a moment that the followng linearity result have been proved: let $g_i : H \times K \to \mathbb{R} \cup \{\pm\infty\}$ be some $\mathcal{H} \otimes \mathcal{K}$-measurable functions such that $\int_{H\times K} g_i^+ dP < \infty$ for $i = 1, 2$. Then

$$\int_H (g_1 + g_2)\, dp = \int_H g_1 dp + \int_H g_2 dp. \tag{2.64}$$

We apply (2.64) with $g_1(h) = \int_K f^+(h,k) q(dh|k)$ and $g_2 = -\int_K f^-(h,k) q(dh|k)$ since by Lemma 2.8.1,

$$\begin{aligned}\int_H g_1^+ dp &= \int_H \left(\int_K f^+(h,k) q(dh|k) \right) p(dh) \\ &= \int_{H\times K} f^+(h,k) q(dh|k) p(dh) = \int_{H\times K} f^+ dP < \infty\end{aligned}$$

and clearly $\int_H g_2^+ dp = 0 < \infty$. So we obtain that

$$\begin{aligned}\int_H \int_K f^+(h,k) q(dk|h) p(dh) - \int_H \int_K f^-(h,k) q(dk|h) p(dh) \\ = \int_H \left(\int_K f^+(h,k) q(dk|h) - \int_K f^-(h,k) q(dk|h) \right) p(dh) \\ = \int_H \int_K f(h,k) q(dk|h) p(dh),\end{aligned}$$

where the second equality comes from the definition of the generalised integral of $f(h,\cdot)$ with respect to $q(\cdot|h)$ and (2.63) is proven.
We prove now (2.64). If $\int_H g_i^- dp < \infty$ for $i = 1, 2$ this is trivial. From $\int_H g_i^+ dp < \infty$ we get that $g_i^+ < \infty$ p-almost surely for $i = 1, 2$, so the sum $g_1 + g_2$ is p-almost surely well-defined, taking its value in $[-\infty, \infty)$. As $(g_1+g_2)^+ \leq g_1^+ + g_2^+$, using the linearity of the integral for non-negative functions we get that

$$\int_H (g_1 + g_2)^+ (h) p(dh) \leq \int_H g_1^+ dp + \int_H g_2^+ dp < \infty.$$

Now from

$$g_1^+ + g_2^+ - g_1^- - g_2^- = g_1 + g_2 = (g_1 + g_2)^+ - (g_1 + g_2)^-,$$

using again the linearity of the integral for non-negative functions we get that

$$\int_H (g_1 + g_2)^+ dp + \int_H g_1^- dp + \int_H g_2^- dp = \int_H (g_1 + g_2)^- dp + \int_H g_1^+ dp + \int_H g_2^+ dp.$$

Checking the different cases, *i.e* $\int_H g_1^- dp = \infty$ and $\int_H g_2^- dp < \infty$ (and the opposite case) as well as $\int_H g_i^- dp = \infty$ for $i = 1, 2$ we get that (2.64) is true. □

2.8.2 Further measure theory issues

We present now specific applications or results that are used throughout the chapter. We start with four extensions of the Fubini results presented previously. As noted in Remark 2.6.12, the introduction of the trace sigma-algebra is the price to pay in order to avoid using the convention $\infty - \infty = -\infty$.

Proposition 2.8.6 *Fix some* $t \in \{1, \dots, T\}$.

i) Let $f : \Omega^t \to \mathbb{R}_+ \cup \{+\infty\}$ *be a non-negative* $\mathcal{F}_t$*-measurable function. Then* $\omega^{t-1} \in \Omega^{t-1} \to \int_{\Omega_t} f(\omega^{t-1}, \omega_t) q_t(d\omega_t|\omega^{t-1})$ *is* $\mathcal{F}_{t-1}$*-measurable with values in* $\mathbb{R}_+ \cup \{+\infty\}$.

ii) Let $f : \Omega^t \times \mathbb{R}^d \to \mathbb{R}_+ \cup \{+\infty\}$ *be a non-negative* $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$*-measurable function. Then* $(\omega^{t-1}, h) \in \Omega^{t-1} \times \mathbb{R}^d \to \int_{\Omega_t} f(\omega^{t-1}, \omega_t, h) q_t(d\omega_t|\omega^{t-1})$ *is* $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)$*-measurable with values in* $\mathbb{R}_+ \cup \{+\infty\}$

iii) Let $f : \Omega^t \to \mathbb{R}_+ \cup \{+\infty\}$ be a non-negative $\overline{\mathcal{F}}_{t-1} \otimes \mathcal{G}_t$-measurable function. Then $\omega^{t-1} \in \Omega^{t-1} \to \int_{\Omega_t} f(\omega^{t-1}, \omega_t) q_t(d\omega_t|\omega^{t-1})$ is $\overline{\mathcal{F}}_{t-1}$-measurable with values in $\mathbb{R}_+ \cup \{+\infty\}$.

iv) Let $\mathcal{S} \in \mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)$. Introduce $\left[\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)\right]_{\mathcal{S}} := \left\{A \cap \mathcal{S},\ A \in \mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)\right\}$ the trace sigma-algebra of $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)$ on $\mathcal{S}$. Let $f : \Omega^{t-1} \times \mathbb{R}^d \times \Omega_t \to \mathbb{R}_+ \cup \{+\infty\}$ be a non-negative $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G}_t$-measurable function. Then $(\omega^{t-1}, h) \in \mathcal{S} \to \int_{\Omega_t} f(\omega^{t-1}, h, \omega_t) q_t(d\omega_t|\omega^{t-1})$ is $\left[\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)\right]_{\mathcal{S}}$-measurable with values in $\mathbb{R}_+ \cup \{+\infty\}$.

Proof. Statement $i)$ is a direct application of Lemma 2.8.1 for $H = \Omega^{t-1}$, $\mathcal{H} = \mathcal{F}_{t-1}$, $K = \Omega_t$, $\mathcal{K} = \mathcal{G}_t$ and $q(\cdot|\cdot) = q_t(\cdot|\cdot)$. To prove statement $ii)$, let $\bar{q}_t$ be defined by

$$\bar{q}_t : (G, \omega^{t-1}, h) \in \mathcal{G}_t \times \Omega^{t-1} \times \mathbb{R}^d \to \bar{q}_t(G|\omega^{t-1}, h) := q_t(G|\omega^{t-1}). \tag{2.65}$$

We first prove that $\bar{q}_t$ is a stochastic kernel on $\mathcal{G}_t$ given $\Omega^{t-1} \times \mathbb{R}^d$ where measurability is with respect to $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)$. Let $(\omega^{t-1}, h) \in \Omega^{t-1} \times \mathbb{R}^d$ be fixed, $B \in \mathcal{G}_t \to \bar{q}_t(B|\omega^{t-1}, h) = q_t(B|\omega^{t-1})$ is a probability measure on $(\Omega_t, \mathcal{G}_t)$ by definition of q_t. Let $B \in \mathcal{G}_t$ be fixed, then $(\omega^{t-1}, h) \in \Omega^{t-1} \times \mathbb{R} \to \bar{q}_t(B|\omega^{t-1}, h) = q_t(B|\omega^{t-1})$ is $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable since for any $B' \in \mathcal{B}(\mathbb{R})$, we have, by definition of q_t,

$$\begin{aligned}&\left\{(\omega^{t-1}, h) \in \Omega^{t-1} \times \mathbb{R}^d, \bar{q}_t(B|\omega^{t-1}, h) \in B'\right\} \\ &\quad = \left\{\omega^{t-1} \in \Omega^{t-1}, q_t(B|\omega^{t-1}) \in B'\right\} \times \mathbb{R}^d \in \mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)\end{aligned}$$

Statement $ii)$ follows by an application of Lemma 2.8.1 for $H = \Omega^{t-1} \times \mathbb{R}^d$, $\mathcal{H} = \mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)$, $K = \Omega_t$, $\mathcal{K} = \mathcal{G}_t$ and $q(\cdot|\cdot) = \bar{q}_t(\cdot|\cdot)$. To prove statement $iii)$ note that since $\mathcal{F}_{t-1} \subset \overline{\mathcal{F}}_{t-1}$ it is clear that q_t is a stochastic kernel on $(\Omega_t, \mathcal{G}_t)$ given $(\Omega^{t-1}, \overline{\mathcal{F}}_{t-1})$ ($i.e$ measurability is with respect to $\overline{\mathcal{F}}_{t-1}$). And statement $iii)$ follows immediately from an application of Lemma 2.8.1 for $H = \Omega^{t-1}$, $\mathcal{H} = \overline{\mathcal{F}}_{t-1}$, $K = \Omega_t$, $\mathcal{K} = \mathcal{G}_t$ and $q(\cdot|\cdot) = q_t(\cdot|\cdot)$. We prove now the last statement. It is well known that $(S, \left[\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)\right]_{\mathcal{S}})$ is a measurable space. Let $\tilde{q}_t$ be defined by

$$\tilde{q}_t : (G, \omega^{t-1}, h) \in \mathcal{G}_t \times S \to \tilde{q}_t(G|\omega^{t-1}, h) := q_t(G|\omega^{t-1}). \tag{2.66}$$

We prove that $\tilde{q}_t$ is a stochastic kernel on $(\Omega_t, \mathcal{G}_t)$ given $\left(\mathcal{S}, \left[\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)\right]_{\mathcal{S}}\right)$. Indeed, let $(\omega^{t-1}, h) \in S$ be fixed, $B \in \mathcal{G}_t \to \tilde{q}_t(B|\omega^{t-1}, h) = q_t(B|\omega^{t-1})$ is a probability measure on $(\Omega_t, \mathcal{G}_t)$, by definition of q_t. Let $B \in \mathcal{G}_t$ be fixed, then $(\omega^{t-1}, h) \in S \to \tilde{q}_t(B|\omega^{t-1}, h) = q_t(B|\omega^{t-1})$ is $\left[\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)\right]_{\mathcal{S}}$-measurable since for any $B' \in \mathcal{B}(\mathbb{R})$, we have, by definition of q_t

$$\begin{aligned}\left\{(\omega^{t-1}, h) \in S,\ \tilde{q}_t(B|\omega^{t-1}, h) \in B'\right\} &= \left(\left\{\omega^{t-1} \in \Omega^{t-1}, q_t(B|\omega^{t-1}) \in B'\right\} \times \mathbb{R}^d\right) \bigcap S \\ &\in \left[\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)\right]_{\mathcal{S}}.\end{aligned}$$

Now let f_S be the restriction of f to $\mathcal{S} \times \Omega_t$. Using similar arguments and the fact that

$$\left[\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G}_t\right]_{\mathcal{S}\times\Omega_t} = \left[\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)\right]_{\mathcal{S}} \otimes \mathcal{G}_t, \tag{2.67}$$

we obtain that f_S is $\left[\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)\right]_{\mathcal{S}} \otimes \mathcal{G}_t$-measurable. Finally, statement $iv)$ follows from another application of Lemma 2.8.1 for $H = S$, $\mathcal{H} = \left[\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}^d)\right]_{\mathcal{S}}$, $K = \Omega_t$, $\mathcal{K} = \mathcal{G}_t$ and $q(\cdot|\cdot) = \tilde{q}_t(\cdot|\cdot)$. □

Lemma 2.8.7 *Let $f : \Omega^{t+1} \to \mathbb{R}_+ \cup \{\infty\}$ be $\mathcal{F}_{t+1}$-measurable, non-negative and such that $\int_{\Omega^{t+1}} f(\omega^{t+1})P_{t+1}(d\omega^{t+1}) < \infty$. Then $\omega^t \in \Omega^t \to \int_{\Omega_{t+1}} f(\omega^t, \omega_{t+1})q_{t+1}(d\omega_{t+1}|\omega^t)$ is $\mathcal{F}_t$-measurable. Furthermore, let*

$$N^t := \left\{\omega^t \in \Omega^t,\ \int_{\Omega_{t+1}} f(\omega^t, \omega_{t+1})q_{t+1}(d\omega_{t+1}|\omega^t) = \infty\right\}.$$

Then $N_t \in \mathcal{F}_t$ and $P_t(N^t) = 0$

Proof. The first assertion of the lemma is a direct application of $i)$ of Proposition 2.8.6. So it is clear that $N^t \in \mathcal{F}_t$. Furthermore, applying the Fubini Theorem (see Lemma 2.8.1) we get that

$$\int_{\Omega^t}\int_{\Omega_{t+1}} f(\omega^t, \omega_{t+1})q_{t+1}(d\omega_{t+1}|\omega^t)P_t(d\omega^t) = \int_{\Omega^{t+1}} f(\omega^{t+1})P_{t+1}(d\omega^{t+1}) < \infty.$$

Assume that $P_t(N^t) > 0$. Then

$$\int_{\Omega^{t+1}} f(\omega^{t+1})P_{t+1}(d\omega^{t+1}) \geq \int_{N^t}\int_{\Omega_{t+1}} f(\omega^t, \omega_{t+1})q_{t+1}(d\omega_{t+1}|\omega^t)P_t(d\omega^t) = \infty.$$

We get a contradiction : $P_t(N^t) = 0$. □

The next lemma, loosely speaking, allows to obtain “nice" sections ($i.e$ set of full measure for a certain probability measure). We use it in the proofs of Theorem 2.4.17 and Lemma 2.8.9.

Lemma 2.8.8 *Fix some $t \in \{1, \dots, T\}$. Let $\widetilde{\Omega}^t \in \mathcal{F}_t$ such that $P_t(\widetilde{\Omega}^t) = 1$ and $\widetilde{\Omega}^{t-1} \in \mathcal{F}_{t-1}$ such that $P_{t-1}(\widetilde{\Omega}^{t-1}) = 1$ and set*

$$\overline{\Omega}^{t-1} := \left\{\omega^{t-1} \in \widetilde{\Omega}^{t-1},\ q_t\left(\left(\widetilde{\Omega}^t\right)_{\omega^{t-1}}|\omega^{t-1}\right) = 1\right\}$$

see Lemma 2.8.1 for the definition of $\left(\widetilde{\Omega}^t\right)_{\omega^{t-1}}$. Then $\overline{\Omega}^{t-1} \in \mathcal{F}_{t-1}$ and $P_t(\overline{\Omega}^{t-1}) = 1$.

Proof. From Lemma 2.8.1 we know $\omega^{t-1} \to q_t\left(\left(\widetilde{\Omega}^t\right)_{\omega^{t-1}} |\omega^{t-1}\right)$ is $\mathcal{F}_{t-1}$-measurable and the fact that $\overline{\Omega}^{t-1} \in \mathcal{F}_{t-1}$ follows immediately.
Furthermore, using the Fubini Theorem (see Lemma 2.8.1) we have that

$$\begin{aligned}
1 = P_t(\widetilde{\Omega}^t) &= \int_{\Omega^{t-1}} \int_{\Omega_t} 1_{\widetilde{\Omega}^t}(\omega^{t-1}, \omega_t) q_t(d\omega_t|\omega^{t-1}) P_{t-1}(d\omega^{t-1}) \\
&= \int_{\Omega^{t-1}} \int_{\Omega_t} 1_{\left(\widetilde{\Omega}^t\right)_{\omega^{t-1}}}(\omega_t) q_t(d\omega_t|\omega^{t-1}) P_{t-1}(d\omega^{t-1}) \\
&= \int_{\widetilde{\Omega}^{t-1}} \int_{\Omega_t} 1_{\left(\widetilde{\Omega}^t\right)_{\omega^{t-1}}}(\omega_t) q_t(d\omega_t|\omega^{t-1}) P_{t-1}(d\omega^{t-1}) \\
&= \int_{\widetilde{\Omega}^{t-1}} q_t\left(\left(\widetilde{\Omega}^t\right)_{\omega^{t-1}} |\omega^{t-1}\right) P_{t-1}(d\omega^{t-1}) \\
&= \int_{\overline{\Omega}^{t-1}} 1 \times P_{t-1}(d\omega^{t-1}) + \int_{\widetilde{\Omega}^{t-1}\setminus\overline{\Omega}^{t-1}} q_t\left(\left(\widetilde{\Omega}^t\right)_{\omega^{t-1}} |\omega^{t-1}\right) P_{t-1}(d\omega^{t-1}),
\end{aligned}$$

where we have used for the third line the fact that $P(\widetilde{\Omega}^{t-1}) = 1$.
But if $P(\widetilde{\Omega}^{t-1}\setminus\overline{\Omega}^{t-1}) > 0$ then we have that by definition of $\overline{\Omega}^{t-1}$ that

$$\int_{\widetilde{\Omega}^{t-1}\setminus\overline{\Omega}^{t-1}} q_t\left(\left(\widetilde{\Omega}^t\right)_{\omega^{t-1}} |\omega^{t-1}\right) P_{t-1}(d\omega^{t-1}) < P_{t-1}(\widetilde{\Omega}^{t-1}\setminus\overline{\Omega}^{t-1}),$$

and thus

$$1 < P_{t-1}(\overline{\Omega}^{t-1}) + P_{t-1}(\widetilde{\Omega}^{t-1}\setminus\overline{\Omega}^{t-1}) = 1,$$

which is absurd and thus $P_{t-1}(\widetilde{\Omega}^{t-1}\setminus\overline{\Omega}^{t-1}) = 0$. We conclude using again that $P_{t-1}(\widetilde{\Omega}^{t-1}) = 1$. □

The following lemma is used throughout the chapter. In particular, the last statement is used in the proof of the main theorem

Lemma 2.8.9 *Let $0 \leq t \leq T-1$, $B \in \mathcal{B}(\mathbb{R})$, $H : \Omega^t \to \mathbb{R}$ and $h_t : \Omega^t \to \mathbb{R}^d$ be $\mathcal{F}_t$-measurable be fixed. Then the functions*

$$(\omega^t, h) \in \Omega^t \times \mathbb{R}^d \to q_{t+1}(H(\omega^t) + h\Delta S_{t+1}(\omega^t, \cdot) \in B|\omega^t), \tag{2.68}$$
$$\omega^t \in \Omega^t \to q_{t+1}(H(\omega^t) + h_t(\omega^t)\Delta S_{t+1}(\omega^t, \cdot) \in B|\omega^t), \tag{2.69}$$

are respectively $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$-measurable and $\mathcal{F}_t$-measurable. Furthermore, assume that $P_{t+1}\left(H(\cdot) + h_t(\cdot)\Delta S_{t+1}(\cdot) \in B\right) = 1$, then there exists some P_t-full measure set $\overline{\Omega}^t$ such that for all $\omega^t \in \overline{\Omega}^t$, $q_{t+1}(H(\omega^t) + h_t(\omega^t)\Delta S_{t+1}(\omega^t, \cdot) \in B|\omega^t) = 1$.

Proof. As $h \in \mathbb{R}^d \to h\Delta S_{t+1}(\omega^t, \omega_{t+1})$ is continuous for all $(\omega^t, \omega_{t+1}) \in \Omega^t \times \Omega_{t+1}$ and $(\omega^t, \omega_{t+1}) \in \Omega^t \times \Omega_{t+1} \to h\Delta S_{t+1}(\omega^t, \omega_{t+1})$ is $\mathcal{F}_{t+1} = \mathcal{F}_t \otimes \mathcal{G}_{t+1}$-measurable for all $h \in \mathbb{R}^d$

(recall that S_t and S_{t+1} are respectively $\mathcal{F}_t$ and $\mathcal{F}_{t+1}$ measurable by assumption), $(\omega^t, \omega_{t+1}, h) \in \Omega^t \times \Omega_{t+1} \times \mathbb{R}^d \to h\Delta S_{t+1}(\omega^t, \omega_{t+1})$ is $\mathcal{F}_t \otimes \mathcal{G}_{t+1} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable as a Carathéodory function (see [3, Definition 4.50, Lemma 4.51] for instance). As H is $\mathcal{F}_t$-measurable we obtain that $\psi : (\omega^t, \omega_{t+1}, h) \in \Omega^t \times \Omega_{t+1} \times \mathbb{R}^d \to H(\omega^t) + h\Delta S_{t+1}(\omega^t, \omega_{t+1})$ is also $\mathcal{F}_t \otimes \mathcal{G}_{t+1} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. Therefore, for any $B \in \mathcal{B}(\mathbb{R})$, $f_B : (\omega^t, \omega_{t+1}, h) \in \Omega^t \times \Omega_{t+1} \times \mathbb{R}^d \to 1_{\psi(\cdot,\cdot,\cdot)\in B}(\omega^t, \omega_{t+1}, h)$ is $\mathcal{F}_t \otimes \mathcal{G}_{t+1} \otimes \mathcal{B}(\mathbb{R}^d)$. We conclude using statement $i)$ of Proposition 2.8.6 applied to f_B and (2.68) is proved. We prove (2.69) using similar arguments. Since h_t is $\mathcal{F}_t$-measurable, it is clear that $\psi_{h_t} : (\omega^t, \omega_{t+1}) \in \Omega^t \times \Omega_{t+1} \to H(\omega^t) + h_t(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1})$ is $\mathcal{F}_t \otimes \mathcal{G}_{t+1}$-measurable. Therefore, for any $B \in \mathcal{B}(\mathbb{R})$, $f_{B,h_t} : (\omega^t, \omega_{t+1}) \in \Omega^t \times \Omega_{t+1} \to 1_{\psi_{h_t}(\cdot,\cdot)\in B}(\omega^t, \omega_{t+1})$ is $\mathcal{F}_t \otimes \mathcal{G}_{t+1}$-measurable. We conclude applying $i)$ of Proposition 2.8.6 to f_{B,h_t}.
For the last statement, we set

$$\widetilde{\Omega}^{t+1} := \left\{\omega^{t+1} = (\omega^t, \omega_{t+1}) \in \Omega^t \times \Omega_{t+1},\ H(\omega^t) + h_t(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}) \in B\right\}.$$

It is clear that $\widetilde{\Omega}^{t+1} \in \mathcal{F}_{t+1}$ and that $P_{t+1}(\widetilde{\Omega}^{t+1}) = 1$. We can then apply Lemma 2.8.8 and we obtain some P_t-full measure set $\overline{\Omega}^t$ such that for all $\omega^t \in \overline{\Omega}^t$, $q_{t+1}(H(\omega^t) + h_t(\omega^t)\Delta S_{t+1}(\omega^t, \cdot) \in B|\omega^t) = 1$. □

Lemma 2.8.10 is often used in conjunction with the Aumann Theorem (see [119, Corollary 1]) to obtain a $\mathcal{F}_t$-measurable selector.

Lemma 2.8.10 *Let $f : \Omega^t \to \mathbb{R}$ be $\overline{\mathcal{F}}_t$-measurable. Then there exists $g : \Omega^t \to \mathbb{R}$ that is $\mathcal{F}_t$-measurable and such that $f = g$ P_t-almost surely, $i.e$ there exists $\Omega^t_{fg} \in \mathcal{F}_t$ with $P_t\left(\Omega^t_{fg}\right) = 1$ and $\Omega^t_{fg} \subset \{f = g\}$.*

Proof. Let $f = 1_B$ with $B \in \overline{\mathcal{F}}_t$ then $B = A \cup N$, with $A \in \mathcal{F}_t$ and $N \in \mathcal{N}_{P_t}$ (see (2.2). Let $g = 1_A$. Then g is $\mathcal{F}_t$-measurable. Clearly, $\{f \neq g\} = N \in \mathcal{N}_{P_t}$, thus $f = g$ P_t a.s. By taking linear combinations, the lemma is proven for step functions using the same argument for each indicator function. Then it is always possible to approximate some $\overline{\mathcal{F}}_t$-measurable function f by a sequence of step function $(f_n)_{n\geq 1}$. From the preceding step for all $n \geq 1$, we get some $\mathcal{F}_t$-measurable step functions g_n such that $f_n = g_n$ P_t-almost surely. Let $g = \limsup g_n$, g is $\mathcal{F}_t$-measurable and we conclude since $\{f \neq g\} \subset \cup_{n\geq 1}\{f_n \neq g_n\}$ which is again in $\mathcal{N}_{P_t}$. □
Next we provide some simple but useful results on usc functions.

Lemma 2.8.11 *Let C be a closed subset of $\mathbb{R}^m$ for some $m \geq 1$. Let $g : \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$ be such that $g = -\infty$ on $\mathbb{R}^m \backslash C$. Then g is usc on $\mathbb{R}^m$ if and only if g is usc on C.*

Proof. We prove that if g is usc on C then it is usc on $\mathbb{R}^m$ as the reverse implication is trivial. Let $\alpha \in \mathbb{R}$ be fixed. We prove that $S_\alpha := \{x \in \mathbb{R}^m,\ g(x) \geq \alpha\}$ is closed in

$\mathbb{R}^m$. Let $(x_n)_{n\geq 1} \subset S_\alpha$ converge to $x \in \mathbb{R}^m$. Then $x_n \in C$ for all $n \geq 1$ and as C is a closed set, $x \in C$. As g is usc on C, ($i.e$ the set $\{x \in C,\ g(x) \geq \alpha\}$ is closed for the induced topology of $\mathbb{R}^m$ on C) we get that $g(x) \geq \alpha$, $i.e$ $x \in S_\alpha$ and g is usc on $\mathbb{R}^m$. □

Lemma 2.8.12 *Let $S \subset \mathbb{R}$ be a closed subset of $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ be such that f is usc and non-decreasing on S. Then f is right-continuous on S.*

Proof. Let $(x_n)_{n\geq 1} \subset S$ be a sequence converging to some x^* from above. Then $x^* \in S$ since S is closed. As $x \in S \to f(x)$ is non-decreasing, for all $n \geq 1$ we have that $f(x_n) \geq f(x^*)$ and thus $\liminf_n f(x_n) \geq f(x^*)$. Now as f is usc on S, we get that $\limsup_n f(x_n) \leq f(x^*)$. The right-continuity of f on S follows immediately. □

We now establish a useful extension of Lemma 2.8.10.

Lemma 2.8.13 *Let $f : \Omega^t \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ be an $\overline{\mathcal{F}}_t \otimes \mathcal{B}(\mathbb{R})$-measurable function such that for all $\omega^t \in \Omega^t$, $x \in \mathbb{R} \to f(\omega^t, x)$ is usc and non-decreasing. Then, there exists some $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R})$-measurable function g from $\Omega^t \times \mathbb{R}$ to $\mathbb{R} \cup \{\pm\infty\}$ and some $\Omega^t_{mes} \in \mathcal{F}_t$ such that $P_t(\Omega^t_{mes}) = 1$ and $f(\omega^t, x) = g(\omega^t, x)$ for all $(\omega^t, x) \in \Omega^t_{mes} \times \mathbb{R}$.*

Remark 2.8.14 In particular, for all $\omega^t \in \Omega^t_{mes}$, $x \in \mathbb{R} \to g(\omega^t, x)$ is usc and non-decreasing.

Proof. Let $n \geq 1$ and $k \in \mathbb{Z}$ be fixed. We apply Lemma 2.8.10 to $f(\cdot) = f(\cdot, \frac{k}{2^n})$ that is $\overline{\mathcal{F}}_t$-measurable by assumption and we get some $\mathcal{F}_t$-measurable $g_{n,k} : \Omega^t \to \mathbb{R} \cup \{\pm\infty\}$ and some $\Omega^t_{n,k} \in \mathcal{F}_t$ such that $P_t(\Omega^t_{n,k}) = 1$ and $\Omega^t_{n,k} \subset \left\{\omega^t \in \Omega^t,\ f(\omega^t, \frac{k}{2^n}) = g_{n,k}(\omega^t)\right\}$. We set

$$\Omega^t_{mes} := \bigcap_{n\geq 1, k\in\mathbb{Z}} \Omega^t_{n,k}. \tag{2.70}$$

It is clear that $\Omega^t_{mes} \in \mathcal{F}_t$ and that $P_t(\Omega^t_{mes}) = 1$.

Now, we define for all $n \geq 1$, $g_n : \Omega^t \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ by

$$g_n(\omega^t, x) := \sum_{k\in\mathbb{Z}} 1_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right]}(x) g_{n,k}(\omega^t).$$

It is clear that g_n is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R})$-measurable for all $n \geq 1$. Finally, we define $g : \Omega^t \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ by

$$g(\omega^t, x) := \lim_n g_n(\omega^t, x). \tag{2.71}$$

Then g is again $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R})$-measurable and it remains to prove that $f(\omega^t, x) = g(\omega^t, x)$ for all $(\omega^t, x) \in \Omega^t_{mes} \times \mathbb{R}$. Let $(\omega^t, x) \in \Omega^t_{mes} \times \mathbb{R}$ be fixed. For all $n \geq 1$, there exists $k_n \in \mathbb{Z}$ such that $\frac{k_n - 1}{2^n} < x \leq \frac{k_n}{2^n}$ and such that $g_n(\omega^t, x) = g_{n,k_n}(\omega^t) = f(\omega^t, \frac{k_n}{2^n})$. Applying Lemma 2.8.12 to $f(\cdot) = f(\omega^t, \cdot)$ (and $S = \mathbb{R}$), we get that $x \in \mathbb{R} \to f(\omega^t, x)$ is right-continuous on $\mathbb{R}$. As $\left(\frac{k_n}{2^n}\right)_{n\geq 1}$ converges to x from above, it follows that $g(\omega^t, x) = \lim_n f(\omega^t, \frac{k_n}{2^n}) = f(\omega^t, x)$ and this concludes the proof. □

Finally, we introduce the following definition.

Definition 2.8.15 Let S be a closed interval of $\mathbb{R}$. A function $f : \Omega^t \times S \to \mathbb{R}$ is an extended Carathéodory function if

i) for all $\omega^t \in \Omega^t$, $x \in S \to f(\omega^t, x)$ is right-continuous,

ii) for all $x \in S$, $\omega^t \in \Omega^t \to f(\omega^t, x)$ is $\mathcal{F}_t$-measurable.

And we prove the following lemma that is an extension of a well-know result on Carathéodory functions (see [3, Definition 4.50, Lemma 4.51]).

Lemma 2.8.16 *Let $S \subset \mathbb{R}$ be a closed interval of $\mathbb{R}$ and $f : \Omega^t \times S \to \mathbb{R}$ be an extended Carathéodory function. Then f is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R})$-measurable.*

Proof. We define for all $n \geq 1$, $f_n : \Omega^t \times \mathbb{R} \to \mathbb{R}$ by

$$f_n(\omega^t, x) := \sum_{k \in \mathbb{Z}} 1_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right]}(x) 1_S\left(\frac{k}{2^n}\right) f\left(\omega^t, \frac{k}{2^n}\right).$$

It is clear that f_n is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R})$-measurable. From the right continuity of f, we can show as in the proof of Lemma 2.8.13 that $f(\omega^t, x) = \lim_n f_n(\omega^t, x)$ for all $(\omega^t, x) \in \Omega^t \times S$ and the proof is complete (recall that $\Omega \times S \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R})$ as S is a closed subset of $\mathbb{R}$). □

Remark 2.8.17 Note that we have the same result if we replace $\mathcal{F}_t$ with $\overline{\mathcal{F}_t}$.

2.8.3 Random sets, normal integrands and related results

In this section, we first recall the definition of random set, of the measurability of a random set as well as the definition of normal integrands. These notions were used in Chapter 2 and will be used again in different setting throughout the dissertation. We propose also in Lemmata 2.8.21, 2.8.22 and 2.8.25 some theoretical results that are not directly used in the chapter but still worth mentioning. Let $(\Omega, \mathcal{F})$ be a measurable space and $(T, \mathcal{T})$ be a topological space.

Definition 2.8.18 A random set R is a set valued function that assigns to each $\omega \in \Omega$ a subset $R(\omega)$ of T. We write $R : \Omega \twoheadrightarrow T$.

Definition 2.8.19 Let $R : \Omega \twoheadrightarrow T$ be a random set. We say that R is $\mathcal{F}$-measurable if for any open set O in T the set $\{\omega \in \Omega,\ R(\omega) \cap O \neq \emptyset\} \in \mathcal{F}$.

It is possible to introduce alternative definitions of the measurability of a random set. This is related to the fact that for a set $A \subset T$, $R^{-1}(A)$ can be defined in different ways. We refer to [116, Chapter 14] and [3, Chapter 17-18] for more details. For instance [116, Theorem 14.3] proposes various equivalent definitions of the measurability of a random set when $T = \mathbb{R}^d$ and R is closed valued ($i.e$ for all ω, $R(\omega)$ is a closed subset of $\mathbb{R}^d$). The open sets of T in Definition 2.8.19 can be replaced with the closed sets or the Borel sets of T for example.

An other different definition of the measurability of a random set that we will use throughout the disseration involves $\text{Graph}(R) := \{(\omega^t, t) \in \Omega \times T, t \in R(\omega)\}$. If $T = \mathbb{R}^n$ and $\mathcal{B}(\mathcal{T}) = \mathcal{B}(\mathbb{R}^n)$ is the usual Borel sigma-algebra on $\mathbb{R}^n$ and if R is closed valued then

$$\mathcal{F}\text{-measurable} \Rightarrow \ \text{Graph}(R) \in \mathcal{F} \otimes \mathcal{B}(\mathcal{T}) \tag{2.72}$$

If furthermore $(\Omega, \mathcal{F})$ is complete for some measure μ then the reverse implication holds also true see [116, Theorem 14.8], even if R is not closed valued.

We now recall the following well-know measurability related result, that was used in the proof of Lemma 2.3.3 and will be used again below.

Proposition 2.8.20 *Castaing representation*
Let $(\Omega, \mathcal{F})$ a measurable space. Let $R : \Omega \twoheadrightarrow \mathbb{R}^d$ be a closed valued random set. Then R is $\mathcal{F}$-measurable if and only if there exists a countable family $(f_n)_{n\geq 1}$ of $\mathcal{F}$-measurable functions $\Omega \to \mathbb{R}^d$ (called Castaing representation) such that for all $\omega \in \Omega$, $R(\omega) = \overline{\{f_n(\omega),\ n \geq 1\}}$ (where the closure is taken in $\mathbb{R}^d$ with respect to the usual topology)

Proof. See [116, Theorem 14.5] or [3, Corollary 18.14]. □
Using Proposition 2.8.20, we prove following lemma which is the pendant of Lemma 2.8.10 for random sets. We fix some probability P on $(\Omega, \mathcal{F})$ and denote by $\overline{\mathcal{F}}$ the P completion of $\mathcal{F}$.

Lemma 2.8.21 *Fix some $m \geq 1$ and let $S : \Omega \twoheadrightarrow \mathbb{R}^m$ be a $\overline{\mathcal{F}}$-measurable and closed valued random set. Then, there exists some $\mathcal{F}$-measurable and closed valued random set $\widetilde{S} : \Omega \twoheadrightarrow \mathbb{R}^m$ and some $\widetilde{\Omega} \in \mathcal{F}$ such that $P(\widetilde{\Omega}) = 1$ and $S(\omega) = \widetilde{S}(\omega)$ for all $\omega \in \widetilde{\Omega}$.*

Proof. As S is $\overline{\mathcal{F}}$-measurable and closed valued, we apply Proposition 2.8.20 and get a countable family $(\widehat{f_p} : \Omega \to \mathbb{R}^m)_{p\geq 1}$ of $\overline{\mathcal{F}}$-measurable functions such that for all $\omega \in \Omega$

$$S(\omega) = \overline{\left\{\widehat{f_p}(\omega),\ p \geq 1\right\}},$$

where the closure is taken in $\mathbb{R}^m$. Fix some $p \geq 1$. We apply Lemma 2.8.10 to $\widehat{f_p}$ and obtain some $\mathcal{F}$-measurable $f_p : \Omega \to \mathbb{R}^m$, some $\Omega_p \in \mathcal{F}_t$ such that $P(\Omega_p) = 1$ and

$f_p(\omega) = \widehat{f}_p(\omega)$ for all $\omega \in \Omega_p$. Introduce

$$\widetilde{\Omega} := \bigcap_{p \geq 1} \Omega_p.$$

It is clear that $\widetilde{\Omega} \in \mathcal{F}$ and $P(\widetilde{\Omega}) = 1$. We modify f_p by setting with a slight abuse of notation $f_p(\omega) = 1_{\widetilde{\Omega}}(\omega) f_p(\omega)$ for all $\omega \in \Omega$ and finally, we define $\widetilde{S} : \Omega \twoheadrightarrow \mathbb{R}^m$ by

$$\widetilde{S}(\omega) := \overline{\{f_p(\omega),\ p \geq 1\}},$$

where the closure is taken again in $\mathbb{R}^m$. Note that for $\omega \notin \widetilde{\Omega}$ we have $\widetilde{S}(\omega) = \{0\}$. As the countable family $(f_p : \Omega \to \mathbb{R}^m)_{p \geq 1}$ is $\mathcal{F}$-measurable, applying again Proposition 2.8.20, we obtain that $\widetilde{S}$ is $\mathcal{F}$-measurable. Finally, as for all $\omega \in \widetilde{\Omega}$, $f_p(\omega) = \widehat{f}_p(\omega)$ for all $p \geq 1$, we have that $S(\omega) = \widetilde{S}(\omega)$ and the proof is complete. □

We propose an application of the above lemma. Recall that when the sigma-algebra $\mathcal{F}$ is not-complete, we only have that (loosely speaking) "measurability implies that the graph is measurable" (see (2.72)). In the following lemma we study the reverse implication.

Lemma 2.8.22 *Fix some $m \geq 1$ and let some closed valued random set $S : \Omega \twoheadrightarrow \mathbb{R}^m$ be such that Graph$(S) \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^n)$. Then, there exists some $\mathcal{F}$-measurable and closed valued random set $\widetilde{S} : \Omega \twoheadrightarrow \mathbb{R}^m$ and $\widetilde{\Omega} \in \mathcal{F}$ such that $P(\widetilde{\Omega}) = 1$ and $S(\omega) = \widetilde{S}(\omega)$ for all $\omega \in \widetilde{\Omega}$.*

Proof. First, we establish that S is $\overline{\mathcal{F}}$-measurable. Let $\mathcal{O}$ be an open set of $\mathbb{R}^m$. Then $\Omega \times \mathcal{O} \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^m)$ and as S is $\mathcal{F}$-graph measurable we have $G := Graph(S) \bigcap (\Omega \times \mathcal{O}) \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^m)$. Then

$$\begin{aligned}
\{\omega \in \Omega,\ S(\omega) \cap \mathcal{O} \neq \emptyset\} &= \{\omega \in \Omega,\ \exists\, h \in \mathbb{R}^m,\ h \in S(\omega) \cap \mathcal{O}\} \\
&= \{\omega \in \Omega,\ \exists\, h \in \mathbb{R}^m,\ (\omega, h) \in G\} \\
&= proj_{\Omega} G,
\end{aligned}$$

and applying the Projection Theorem (see for example Theorem 3.23 in [37]) we obtain that $\{\omega \in \Omega,\ S(\omega) \cap \mathcal{O} \neq \emptyset\} \in \overline{\mathcal{F}}$: S is $\overline{\mathcal{F}}$-measurable.

Now, we can apply Lemma 2.8.21 and obtain a $\mathcal{F}$-measurable and closed valued random set $\widetilde{S} : \Omega \twoheadrightarrow \mathbb{R}^m$, $\widetilde{\Omega} \in \mathcal{F}$ such that $P(\widetilde{\Omega}) = 1$ and $S(\omega) = \widetilde{S}(\omega)$, for all $\omega \in \widetilde{\Omega}$. This achieves the proof. □

We finish this section with the definition of a normal integrand (see also [116, Definition 14.27] or [97, Section 3, Chapter 5]) which is a fundamental underlying concept also this chapter and in Chapter 4. The relevance of the normal integrand concept in this type of optimisation problems was illustrated for instance in [106] or [107].

Definition 2.8.23 A function $N : \Omega \times \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ is a $\mathcal{F}$-normal integrand if the following random set $S_N : \Omega \twoheadrightarrow \mathbb{R} \times \mathbb{R}^d$ defined by

$$S_N(\omega) := \left\{(l,h) \in \mathbb{R} \times \mathbb{R}^d,\ N(\omega,h) \leq l\right\},$$

is closed valued and $\mathcal{F}$-measurable.

Remark 2.8.24 [116, Corollary 14.34] provides interesting insight: a normal integrand is jointly-measurable and lower semicontinus in h and the reverse holds true if $\mathcal{F}$ is complete. As our sigma-algebra are not complete we will not be able to use this result and the essential ingredient will be [116, Theorem 14.37] that guarantees that $\omega \in \Omega \to \inf_{h\in\mathbb{R}^n} N(\omega^t,h)$ is $\mathcal{F}$-measurable and is used in the proof of of Lemma 2.6.11.

We have just seen that a normal integrand is always jointly-measurable and lower semicontinous in h and that the reverse is true only if the sigma-algebra is complete. In the following lemma, we study the reverse implication when the sigma-algebra is not complete.

Lemma 2.8.25 *Let $N : \Omega \times \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$, for some $m \geq 1$ fixed, be a $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^m)$-measurable random set such that for all $\omega \in \Omega$, $h \in \mathbb{R}^m \to N(\omega,h)$ is lower semicontinous. Then there exists $\widetilde{\Omega} \in \mathcal{F}$ such that $P(\widetilde{\Omega}) = 1$ and such that $N1_{\widetilde{\Omega}\times\mathbb{R}^m}$ is a $\mathcal{F}$-normal integrand.*

Proof. We introduce $S_N : \Omega \twoheadrightarrow \mathbb{R} \times \mathbb{R}^m$ (see Definition 2.8.23)

$$S_N(\omega) := \{(l,h) \in \mathbb{R} \times \mathbb{R}^m,\ N(\omega,h) \leq l\}.$$

As for all $\omega \in \Omega$, $h \in \mathbb{R}^m \to N(\omega,h)$ is lower semicontinous, S_N is closed valued. Set

$$G := \bigcap_{n\in\mathbb{N},\ n\geq 1} \bigcup_{q\in\mathbb{Q}} \left\{(\omega^t,l,h) \in \Omega^t \times \mathbb{R} \times \mathbb{R}^m,\ q \leq l \leq q + \frac{1}{n}, N(\omega,h) \leq q + \frac{1}{n}\right\}$$

As N is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^m)$-measurable we have for some $q \in \mathbb{Q}$, $n \geq$ fixed that

$$\left\{(\omega,h) \in \Omega \times \mathbb{R}^m,\ N(\omega,h) \leq q + \frac{1}{n}\right\} \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^m),$$

therefore $\left\{(\omega^t,l,h) \in \Omega^t \times \mathbb{R} \times \mathbb{R}^m,\ q \leq l \leq q + \frac{1}{n}, N(\omega,h) \leq q + \frac{1}{n}\right\} \in \mathcal{F}\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}^m)$ and $G \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^m)$ follows immediately. The fact that S_N is $\mathcal{F}$-graph measurable follows from $G = \text{Graph}(S_N)$. Indeed, Let $(\omega^t,l,h) \in \text{Graph}(S_N)$. It is clear that for all $n \geq 1$, there exists some rational q_n such that $q_n \leq l \leq q_n + \frac{1}{n}$ and such that $N(\omega,h) \leq l \leq q_n + \frac{1}{n}$ and $(\omega^t,l,h) \in G$. Now, let $(\omega^t,l,h) \in G$. There exists a sequence $(q_n)_{n\geq1}$ of rational numbers converging to l, such that $q_n \leq l \leq q_n + \frac{1}{n}$ and

$N(\omega,h) \leq q_n + \frac{1}{n}$ for all n. Thus $N(\omega,h) \leq l$ and $(\omega^t, x, h) \in \text{Graph}(S_N)$ follows. Therefore, we can apply Lemma 2.8.22 to S_N (with $m = m+1$) and we obtain some $\mathcal{F}$-measurable and closed valued random set $\widetilde{S} : \Omega \twoheadrightarrow \mathbb{R}^m$ and $\widetilde{\Omega} \in \mathcal{F}$ such that $P(\widetilde{\Omega}) = 1$ and $S_N(\omega) = \widetilde{S}(\omega)$ for all $\omega \in \widetilde{\Omega}$. For $\omega \notin \widetilde{\Omega}$ we set $\widetilde{S}(\omega) = \mathbb{R}^+ \times \mathbb{R}^m$. It is easy to see that $\widetilde{S}_N$ is still closed valued and $\mathcal{F}$-measurability. We prove now that $N1_{\widetilde{\Omega}\times\mathbb{R}^m}$ is a $\mathcal{F}$-normal integrand. For ease of notation, we set $\widetilde{N} := N1_{\widetilde{\Omega}\times\mathbb{R}^m}$. Then

$$\begin{aligned} S_{\widetilde{N}}(\omega) &:= \left\{(l,h) \in \mathbb{R}\times\mathbb{R}^m,\ \widetilde{N}(\omega,h) \leq l\right\} \\ &= \begin{cases} \{(l,h) \in \mathbb{R}\times\mathbb{R}^m,\ N(\omega,h) \leq l\} = S_N(\omega) = \widetilde{S}(\omega), & \text{if } \omega \in \widetilde{\Omega} \\ \{(l,h) \in \mathbb{R}\times\mathbb{R}^m,\ 0 \leq l\} = \mathbb{R}^+ \times \mathbb{R}^m = \widetilde{S}(\omega), & \text{otherwise} \end{cases} \\ &= \widetilde{S}(\omega). \end{aligned}$$

Recalling Definition 2.8.23, we have proven that $\widetilde{N}$ is a $\mathcal{F}$-normal integrand. □

2.8.4 Proof of technical results

Finally, we provide the missing result of the chapter: the proof of Lemma 2.2.2

Proof of Lemma 2.2.2. We refer to [33, Section 6.1] for the definition and various properties of generalized conditional expectations. In particular since $E(h^+) = \int_{\Omega^t} h^+ dP_t < \infty$, $E(h|\mathcal{F}_s)$ is well-defined (in the generalised sense) for all $0 \leq s \leq t$ (see [33, Lemma 6.2]). Similarly, from Proposition 2.8.4 we have that $\varphi : \Omega^s \to \mathbb{R} \cup \{\pm\infty\}$ is well-defined (in the generalised sense) and $\mathcal{F}_s$-measurable.
As $\varphi(X_1,\dots,X_s)$ is $\mathcal{F}_s$-measurable, it remains to prove that $E(gh) = E(g\varphi(X_1,\dots,X_s))$ for all $g : \Omega^s \to \mathbb{R}_+$ non-negative, $\mathcal{F}_s$-measurable and such that $E(gh)$ is well-defined in the generalised sense, $i.e$ such that $E\,(gh)^+ < \infty$ or $E\,(gh)^- < \infty$. Recalling the notations of the beginning of Section 2.2 and using the Fubini Theorem for the third and fourth equality (see Proposition 2.8.4 and Remark 2.8.5), we get that

$$\begin{aligned} E(gh) &= E(g(X_1,\dots,X_s)h(X_1,\dots,X_t)) = \int_{\Omega^T} g(\omega_1,\dots,\omega_s)h(\omega_1,\dots,\omega_t)P(d\omega^T) \\ &= \int_{\Omega^t} g(\omega_1,\dots,\omega_s)h(\omega_1,\dots,\omega_t)q_t(\omega_t|\omega^{t-1})\dots q_{s+1}(\omega_{s+1}|\omega^s)P_s(d\omega^s) \\ &= \int_{\Omega^s} g(\omega_1,\dots,\omega_s)\left(\int_{\Omega_{s+1}\times\ldots\times\Omega_t} h(\omega_1,\dots,\omega_s,\omega_{s+1},\dots,\omega_t)q_t(\omega_t|\omega^{t-1})\dots q_{s+1}(\omega_{s+1}|\omega^s)\right)P_s(d\omega^s) \\ &= \int_{\Omega^s} g(\omega_1,\dots,\omega_s)\varphi(\omega_1,\dots,\omega_s)P_s(d\omega^s) \\ &= E(g(X_1,\dots,X_s)\varphi(X_1,\dots,X_t)), \end{aligned}$$

which concludes the proof. □

CHAPTER 3

No-arbitrage with multiple-priors

This chapter presents some results on the no-arbitrage condition with multiple-priors (*i.e.* in the presence of uncertainty) obtained in [20] (submitted for publication) and in [21] which is in preparation.

3.1 Introduction

In this chapter, we study some characterisation of no-arbitrage with multiple-priors that can be seen as a prelude to Chapter 4. Indeed, one of the main reason why we focus on the question of no-arbitrage in the presence of uncertainty is its central role to solve the maximisation of worst-case expected utility problem by primal approach (as already evident in Chapter 2 in a mono-prior setting) and as we will see in Chapter 4. Recall from Chapter 1 that by uncertainty we refers to Knightian uncertainty a concept that distinguishes between the "unknown unknown", or uncertainty, and the "known unknown", or risk.
First, we revisit the definxition introduced in [25] (the $NA(\mathcal{Q}^T)$ condition or quasi-sure no-arbitrage, see Assumption 3.4.1) which is a natural extension of the classical arbitrage. This is justified by the extension in [25, Fundamental Pricing Theorem] of the classical and well-know FTAP (where a set of martingale measures equivalent, in a certain sense, to the initial family is introduced) together with subsequent results on worst-case utility maximisation (see [99], [7], [98] and Chapter 4). We recall however that Assumption 3.4.1 is by far not the only solution: model independent arbitrage as in [1] or a scenarii based approach as in [26], [27] and [28] are alternative choices (see Chapter 1 for a more detailed discussion on the subject).
In this chapter, we are first interested in providing alternative but equivalent formulation of the $NA(\mathcal{Q}^T)$ condition: namely a quantitative (Definition 3.4.4) and a geometric (Definition 3.4.6) characterisation. Both the quantitative and geometric interpretations are not new and can be seen as generalisations of [112, Proposition 3.3] and [84, Theorem 3] (see also [62, Theorem 1.48 p30]) in the presence of uncertainty and in the framework of [25]. As already evident in Chapter 2 and as we will see in Chapter 4, the quantitative characterisation is particularly relevant in the context of expected utility maximisation for utility function defined on the half real line. Indeed, in the primal approach, the proof of the existence of a maximiser relies on some compacity argument provided by the no-arbitrage condition. In

the case of unbounded utility function defined on the whole real line, the quantitative characterisation (together with some condition on asymptotic elasticity) is also essential: the idea was introduced in [112] and will be illustrated in Section 3.5 when we use it in a simple one-period model with multiple-priors and prove Theorem 3.5.13. The geometric interpretation is somehow a different formulation of the quantitative interpretation that is actually very intuitive. It is related to the concept of the conditional support of the traded asset price increments which plays an important role when looking at arbitrage question (see Section 3.3) and will be useful to obtain some measurability result in Proposition 3.4.9.

We also focus on some of the measurability issues underpinning the framework introduced in [25]. A key element of this framework is the equivalence between the global and local version of the no-arbitrage: namely [25, Theorem 5.5] that uses measurable selection and Castaing's representation (see Proposition 2.8.20 in Chapter 2). We will rely on this theorem to prove the equivalence of the different assumptions in Theorem 3.4.7. Then, in Proposition 3.4.9, we obtain as well some key measurability properties. This property (together with some integrability condition) is important to control the behaviour of the optimal strategy in the case maximisation of expected utility function defined on the half real-line: this was the case in mono-prior situation (see Theorem 2.4.17 in Chapter 2) and we will use the result in Chapter 5. Finally, we introduce another stronger no-arbitrage definition that will be used in Chapter 4. In Theorem 3.6.4 we establish the equivalence between the local and global version of the strong no-arbitrage condition and prove that the full-measure set on which the local strong no-arbitrage holds is coanalytic (see Section 3.2.2) while in the case of the no-arbitrage of [25] the full-measure set is only universally-measurable. We illustrate how the fact that we obtain a coanalytic set can be useful in Propositions 3.6.6 and 3.6.7. These questions are related to issues arising when handling non-dominated set of probability measures in the dynamic programming and will be at the heart of the difficulties of Chapter 4. In the same spirit, [7, Theorem 3.3, Remark 3.6] proposes a stronger version of Assumption 3.4.1: the local no-arbitrage condition is assumed to hold for all ω^t which is an other way to simplify measurability issues (see also Remark 3.6.9). Finally, in Proposition 3.6.11 and 3.6.12 we propose a quantitative characterisation of the strong no-arbitrage condition and in Proposition, we present an extension of the classical FTAP in a setting that does not require technical measurability assumptions.

As in [25] and [99] our proof relies heavily on measure theory tools, namely on analytic sets that will be used again in Chapter 4.

The chapter is structured as follows: section 3.2 recall some important definitions and some key properties of analytic sets before our framework presentation. Section 3.3 introduces the multiple-priors conditional support and establish important measurabiliy properties. Section 3.4 presents the various no-arbitrage conditions and proves their equivalence in Theorem 3.4.7. As a simple application in Section 3.5, Theorem 3.4.7 is used in a one period model to solve the problem of maximising worst-case expected utility for an unbounded utility function defined on $\mathbb{R}$. In section 3.6 we revisit the strong no-arbitrage condition $sNA(\mathcal{Q}^T)$ and illustrate some of its properties. Finally, section 3.7 collects some technical results and proofs.

3.2 Definitions and set-up

To model uncertainty and describe the multi-period financial market where investors can trade actively at each period we work under the framework introduced in [25] that we describe together with our notations below.
The same framework and notation will also be used in Chapters 4 and 5 where some of these elements will be recalled for the convenience of the reader.

3.2.1 Polar sets and universal sigma-algebra

For any Polish space X ($i.e$ complete and separable metric space), we denote by $\mathcal{B}(X)$ its Borel sigma-algebra and by $\mathfrak{P}(X)$ the set of all probability measures on $(X, \mathcal{B}(X))$. We recall that $\mathfrak{P}(X)$ endowed with the weak topology is a Polish space (see [13, Propositions 7.20 p127, 7.23 p131]). If P in $\mathfrak{P}(X)$, $\mathcal{B}_P(X)$ will be the completion of $\mathcal{B}(X)$ with respect to P and the universal sigma-algebra is defined by

$$\mathcal{B}_c(X) := \bigcap_{P \in \mathfrak{P}(X)} \mathcal{B}_P(X).$$

It is clear that $\mathcal{B}(X) \subset \mathcal{B}_c(X)$. In the rest of the chapter we will use the same notation for P in $\mathfrak{P}(X)$ and for its (unique) extension on $\mathcal{B}_c(X)$. A function $f : X \to Y$ (where Y is an other Polish space) is universally-measurable or $\mathcal{B}_c(X)$-measurable (resp. Borel-measurable or $\mathcal{B}(X)$-measurable) if for all $B \in \mathcal{B}(Y)$, $f^{-1}(B) \in \mathcal{B}_c(X)$ (resp. $f^{-1}(B) \in \mathcal{B}(X)$).
For a given $\mathcal{P} \subset \mathfrak{P}(X)$, a set $N \subset X$ is called a $\mathcal{P}$-polar if for all $P \in \mathcal{P}$, there exists some $A_P \in \mathcal{B}_c(X)$ such that $P(A_P) = 0$ and $N \subset A_P$. We say that a property holds true $\mathcal{P}$-quasi-surely (q.s.), if it is true outside a $\mathcal{P}$-polar set. Finally we say that a set is of $\mathcal{P}$-full measure if its complement is a $\mathcal{P}$-polar set.

3.2.2 Analytic sets

An analytic set of X is the continuous image of a Polish space, see [3, Theorem 12.24 p447]. We denote by $\mathcal{A}(X)$ the set of analytic sets of X and recall some key properties that will be often used in the rest of the chapter. The projection of an analytic set is an analytic set see [13, Proposition 7.39 p165]) and a countable union or intersection of analytic sets is an analytic set (see [13, Corollary 7.35.2 p160]). However the complement of an analytic set does not need to be an analytic set. We denote by $\mathcal{CA}(X) := \{A \in X,\ X \backslash A \in \mathcal{A}(X)\}$ the set of all coanalytic sets of X. We have that (see [13, Proposition 7.36 p161, Corollary 7.42.1 p169])

$$\mathcal{B}(X) \subset \mathcal{A}(X) \cap \mathcal{CA}(X) \text{ and } \mathcal{A}(X) \cup \mathcal{CA}(X) \subset \mathcal{B}_c(X). \tag{3.1}$$

Now, for $D \in \mathcal{A}(X)$, a function $f : D \to \mathbb{R} \cup \{\pm\infty\}$ is lower-semianalytic (resp. upper-semianalytic) on X if $\{x \in X\ f(x) < c\} \in \mathcal{A}(X)$ (resp. $\{x \in X\ f(x) > c\} \in \mathcal{A}(X)$) for all $c \in \mathbb{R}$. We denote by $\mathcal{LSA}(X)$ (resp. $\mathcal{USA}(X)$) the set of all lower-semianalytic (resp. upper-semianalytic) functions on X. From (3.1) it is clear that if $f \in \mathcal{LSA}(X) \cup \mathcal{USA}(X)$ then f is $\mathcal{B}_c(X)$-measurable. Finally, a function $f : X \to Y$ (where Y is another Polish space) is analytically-measurable if for all $B \in \mathcal{B}(Y)$, $f^{-1}(B)$ belongs to the sigma-algebra generated by $\mathcal{A}(X)$. From (3.1), it is clear that if f is analytically-measurable, then f is universally-measurable.

3.2.3 The measurable spaces

We fix a time horizon $T \in \mathbb{N}$ and introduce a sequence $(\Omega_t)_{1 \le t \le T}$ of Polish spaces. We denote by

$$\Omega^t := \Omega_1 \times \cdots \times \Omega_t,$$

with the convention that Ω^0 is reduced to a singleton. An element of Ω^t will be denoted by $\omega^t = (\omega_1, \dots, \omega_t) = (\omega^{t-1}, \omega_t)$ for $(\omega_1, \dots, \omega_t) \in \Omega_1 \times \cdots \times \Omega_t$ and $(\omega^{t-1}, \omega_t) \in \Omega^{t-1} \times \Omega_t$ (to avoid heavy notation we drop the dependancy in ω_0). It is well know that $\mathcal{B}(\Omega^t) = \mathcal{B}(\Omega^{t-1}) \otimes \mathcal{B}(\Omega_t)$, see [3, Theorem 4.44 p149]. However we have only that $\mathcal{B}_c(\Omega^{t-1}) \otimes \mathcal{B}_c(\Omega_t) \subset \mathcal{B}_c(\Omega^t)$.

3.2.4 Stochastic kernels and definition of $\mathcal{Q}^T$

For all $0 \le t \le T-1$, we denote by $\mathcal{SK}_{t+1}$ the set of universally-measurable stochastic kernel on Ω_{t+1} given Ω^t (see [13, Definition 7.12 p134, Lemma 7.28 p174]). Fix some $1 \le t \le T$, $P_{t-1} \in \mathfrak{P}(\Omega^{t-1})$ and $p_t \in \mathcal{SK}_t$. Using Fubini's Theorem, see [13, Proposition 7.45 p175], we define a probability on Ω^t by setting for all $A \in \mathcal{B}(\Omega^t)$

$$P_{t-1} \otimes p_t(A) := \int_{\Omega^{t-1}} \int_{\Omega_t} 1_A(\omega^{t-1}, \omega_t) p_t(d\omega_t, \omega^{t-1}) P_{t-1}(d\omega^{t-1}). \tag{3.2}$$

For all $0 \le t \le T-1$, we consider a random set $\mathcal{Q}_{t+1} : \Omega^t \twoheadrightarrow \mathfrak{P}(\Omega_{t+1})$: $\mathcal{Q}_{t+1}(\omega^t)$ can be seen as the set of possible priors for the t-th period given the state ω^t until time t.

Assumption 3.2.1 For all $0 \le t \le T-1$, $\mathcal{Q}_{t+1}$ is a non-empty and convex valued random set such that

$$\text{Graph}(\mathcal{Q}_{t+1}) = \left\{(\omega^t, P) \in \Omega^t \times \mathfrak{P}(\Omega_{t+1}),\ P \in \mathcal{Q}_{t+1}(\omega^t)\right\} \in \mathcal{A}\left(\Omega^t \times \mathfrak{P}(\Omega_{t+1})\right).$$

From the Jankov-von Neumann Theorem, see [13, Proposition 7.49 p182], there exists some analytically-measurable and thus $\mathcal{B}_c(\Omega^t)$-measurable $q_{t+1} : \Omega^t \to \mathfrak{P}(\Omega_{t+1})$ such that for all $\omega^t \in \Omega^t$, $q_{t+1}(\cdot, \omega^t) \in \mathcal{Q}_{t+1}(\omega^t)$ (recall that for all $\omega^t \in \Omega^t$, $\mathcal{Q}_{t+1}(\omega^t) \neq \emptyset$). In other words q_{t+1} is a universally-measurable selector of $\mathcal{Q}_{t+1}$. Note as well that $q_{t+1} \in \mathcal{S}K_{t+1}$. For all $1 \leq t \leq T$ we define $\mathcal{Q}^t \subset \mathfrak{P}(\Omega^t)$ by

$$\mathcal{Q}^t := \{Q_1 \otimes q_2 \otimes \cdots \otimes q_t,\ Q_1 \in \mathcal{Q}_1,\ q_{s+1} \in \mathcal{S}K_{s+1}, q_{s+1}(\cdot, \omega^s) \in \mathcal{Q}_{s+1}(\omega^s)\ Q_s\text{-a.s. } \forall\, 1 \leq s \leq t-1\ \}, \tag{3.3}$$

where if $Q_t = Q_1 \otimes q_2 \otimes \cdots \otimes q_t \in \mathcal{Q}^t$ we denote by $Q_s := Q_1 \otimes q_2 \otimes \cdots \otimes q_s$ for any $2 \leq s \leq t$. It is clear that $Q_s \in \mathcal{Q}^s$. We will often use in the chapter the following construction: let $Q = Q_0 \otimes q_1 \cdots \otimes q_t \in \mathcal{Q}^t$ be fixed and let some $q^*_{t+1} \in \mathcal{S}K_{t+1}$ be such that there exists $\widetilde{\Omega}^t \in \mathcal{B}_c(\Omega^t)$ with $Q(\widetilde{\Omega}^t) = 1$ and $q^*_{t+1}(\cdot, \omega^t) \in \mathcal{Q}_{t+1}(\omega^t)$ for all $\omega^t \in \widetilde{\Omega}^t$. We define $Q^* \in \mathfrak{P}(\Omega^{t+1})$ by

$$Q^* = Q_0 \otimes q_1 \cdots \otimes q_t \otimes q^*_{t+1} = Q \otimes q^*_{t+1}.$$

Then, it is clear that $Q^* \in \mathcal{Q}^{t+1}$.

3.2.5 The traded assets and strategies

Let $S := \{S_t,\ 0 \leq t \leq T\}$ be a $(\mathcal{B}_c(\Omega^t))_{0 \leq t \leq T}$-adapted d-dimensional process where for $0 \leq t \leq T$, $S_t = (S^i_t)_{1 \leq i \leq d}$ represents the price of d risky securities in the financial market in consideration. We make the following assumptions already stated in [99].

Assumption 3.2.2 We have that S is $(\mathcal{B}(\Omega^t))_{0 \leq t \leq T}$-adapted.

Remark 3.2.3 If we do not assume Assumption 3.2.2, we cannot obtain some crucial measurability properties in Lemma 3.3.2

There exists also a riskless asset for which we assume a price constant equal to 1, for sake of simplicity. Without this assumption, all the developments below could be carried out using discounted prices. The notation $\Delta S_t := S_t - S_{t-1}$ will often be used. If $x, y \in \mathbb{R}^d$ then the concatenation xy stands for their scalar product. The symbol $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$ (or on $\mathbb{R}$).
Trading strategies are represented by d-dimensional processes $\phi := \{\phi_t, 1 \leq t \leq T\}$ where for all $1 \leq t \leq T$, $\phi_t = (\phi^i_t)_{1 \leq i \leq d}$ represents the investor's holdings in each of the d assets at time t. We assume that ϕ_t is $\mathcal{B}_c(\Omega^{t-1})$-measurable for all $1 \leq t \leq T$. The family of all such trading strategies is denoted by Φ. We assume that trading is self-financing. As the riskless asset's price is constant 1, the value at time t of a portfolio ϕ starting from initial capital $x \in \mathbb{R}$ is given by

$$V^{x,\phi}_t = x + \sum_{s=1}^{t} \phi_s \Delta S_s.$$

3.3 The multiple-priors conditional support of ΔS_{t+1} and related results

We introduce the following definitions which is the pendant in the multiple-priors setting of Definition 2.3.2 in Chapter 2.

Definition 3.3.1 For all $0 \leq t \leq T-1$, the random sets $\widetilde{D}^{t+1}$, $\widehat{D}^{t+1}$ and $D^{t+1} : \Omega^t \twoheadrightarrow \mathbb{R}^d$ are defined by

$$\widetilde{D}^{t+1}(\omega^t) := \bigcap \left\{ A \subset \mathbb{R}^d, \text{ closed}, \ P_{t+1}\left(\Delta S_{t+1}(\omega^t,.) \in A\right) = 1, \ \forall P_{t+1} \in \mathcal{Q}_{t+1}(\omega^t) \right\},$$
$$\widehat{D}^{t+1}(\omega^t) := \overline{\text{Conv}\widetilde{D}^{t+1}(\omega^t)} \text{ and } D^{t+1}(\omega^t) := \text{Aff}\left(\widetilde{D}^{t+1}(\omega^t)\right),$$

where the closure in taken in $\mathbb{R}^d$ for the usual topology.

The reasons for introducing D^{t+1} are twofold. First, from a theoretical standpoint, we want to relate the properties of the support of ΔS_{t+1} to the no-arbitrage definition. Secondly, and this will be important in Chapter 4, for a strategy $\phi \in \Phi$ such that $\phi^{t+1}(\omega^t) \in D^{t+1}(\omega^t)$ one have nice properties (see (3.5) in Definition 3.4.4). If $D^{t+1}(\omega^t) = \mathbb{R}^d$ then, intuitively, there are no redundant assets for all model specification. Otherwise, for any $\mathcal{B}_c(\Omega^t)$-measurable strategy ϕ_{t+1}, one may always replace ϕ_{t+1} by its orthogonal projection $\phi_{t+1}^{\perp}(\omega^t, \cdot)$ on $D^{t+1}(\omega^t)$ without changing the portfolio value (see Remark 4.5.5 in Chapter 4 below and [99, Lemma 2.6]). .
The following lemma, similar to [25, Lemma 4.3] or Lemma 2.3.3 in Chapter 2 establishes some important measurability properties of D^{t+1}.

Lemma 3.3.2 *Let Assumption 3.2.1 and 3.2.2 hold true and $0 \leq t \leq T-1$ be fixed. Then $\widetilde{D}^{t+1}$, $\widehat{D}^{t+1}$ and D^{t+1} are non-empty, closed valued and for all open set $O \subset \mathbb{R}^d$*

$$\left\{\omega^t \in \Omega^t, \ O \cap \widetilde{D}^{t+1}(\omega^t) \neq \emptyset\right\} \in \mathcal{A}(\Omega^t). \tag{3.4}$$

Furthermore $\widetilde{D}^{t+1}$, $\widehat{D}^{t+1}$ and D^{t+1} are $\mathcal{B}_c(\Omega^t)$-measurable and we have that Graph$(D^{t+1}) \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$.

Remark 3.3.3 It is possible to show that $\{\omega^t \in \Omega^t, \ O \cap D^{t+1}(\omega^t) \neq \emptyset\} \in \mathcal{A}(\Omega^t)$ and that $\text{Graph}(D^{t+1}) \in \mathcal{A}(\Omega^t) \times \mathcal{B}(\mathbb{R}^d) \subset \mathcal{A}(\Omega^t \times \mathbb{R}^d)$. As this will not be used we have not included the proof which is technically not difficult but rather long and tedius.

Proof. We use similar arguments as in the proof of [25, Lemma 4.3]. The proof relies also on Lemma 4.8.5 stated in the appendix of Chapter 4. It is clear from its definition that for all $\omega^t \in \Omega^t$, $\widetilde{D}^{t+1}(\omega^t)$ is a non-empty and closed subset of $\mathbb{R}^d$. Let O be an open set in $\mathbb{R}^d$. We set

$$\mu_O : (\omega^t, P) \in \Omega^t \times \mathfrak{P}(\Omega_{t+1}) \to \mu_O(\omega^t, P) := P\left(\Delta S_{t+1}(\omega^t,.) \in O\right) = F_O(\omega^t, P, 1, 0)$$

(see (4.92) for the definition of F_O) and we prove that (3.4) holds true. Using Lemma 4.8.5, we get that μ_O is $\mathcal{B}(\Omega^t)\otimes\mathcal{B}(\mathfrak{P}(\Omega_{t+1}))$-measurable. By definition of $\widetilde{D}^{t+1}(\omega^t)$ we get that

$$\begin{aligned}\{\omega^t \in \Omega^t,\, \widetilde{D}^{t+1}(\omega^t)\cap O \neq \emptyset\} &= \{\omega^t \in \Omega^t, \exists P \in \mathcal{Q}_{t+1}(\omega^t),\ \mu_O(\omega^t,P)>0\}\\ &= Proj_{\Omega^t}\left(\left\{(\omega^t,P)\in\Omega^t\times\mathfrak{P}(\Omega_{t+1}),\ \mu_O(\omega^t,P)>0\right\}\cap \text{Graph}(\mathcal{Q}_{t+1})\right).\end{aligned}$$

We have that

$$\left\{(\omega^t,P)\in\Omega^t\times\mathfrak{P}(\Omega_{t+1}),\ \mu_O(\omega^t,P)>0\right\}\in\mathcal{B}(\Omega^t)\otimes\mathcal{B}(\mathfrak{P}(\Omega_{t+1}))\subset\mathcal{A}(\Omega^t\times\mathfrak{P}(\Omega_{t+1})),$$

see (3.1) and as Graph$(\mathcal{Q}_{t+1})\in\mathcal{A}(\Omega^t\times\mathfrak{P}(\Omega_{t+1}))$ (see Assumption 3.2.1, [13, Proposition 7.39] we obtain that $\{\omega^t \in \Omega^t,\ \widetilde{D}^{t+1}(\omega^t)\cap O\neq\emptyset\}\in\mathcal{A}(\Omega^t)\subset\mathcal{B}_c(\Omega^t)$ and therefore (3.4) is true. Recalling the definition of measurability for random set (see Definition 2.8.19 in Chapter 2) and (3.1), $\widetilde{D}^{t+1}$ is $\mathcal{B}_c(\Omega^t)$-measurable . We apply now [116, Exercise 14.12] and obtain that D^{t+1} is $\mathcal{B}_c(\Omega^t)$-measurable [1]. Similarly using [116, Exercise 14.12, Proposition 14.2], we get that $\widehat{D}^{t+1}$ is $\mathcal{B}_c(\Omega^t)$-measurable. The fact that Graph$(D^{t+1})\in\mathcal{B}_c(\Omega^t)\otimes\mathcal{B}(\mathbb{R}^d)$ follows from [116, Theorem 14.8]. □

3.4 Quantitative no-arbitrage, geometric no-arbitrage and $NA(\mathcal{Q}^T)$

3.4.1 Different notions of no-arbitrage and main results

First, we recall the no-arbitrage condition $NA(\mathcal{Q}^T)$ (also referred to as quasi-sure no-arbitrage) as introduced in [25].

Definition 3.4.1 The $NA(\mathcal{Q}^T)$ condition holds true if for all $\phi\in\Phi$, $V_T^{0,\phi}\geq 0$ $\mathcal{Q}^T$-q.s. $\Rightarrow$ $V_T^{0,\phi}=0$ $\mathcal{Q}^T$-q.s.

Defintion 3.4.1 is a natural and intuitive extension of the classical mono-prior arbitrage condition. This argument is strengthened by the FTAP generalisation proved in [25]: under appropriate measurability conditions the $NA(\mathcal{Q}^T)$ is equivalent to the following: for all $Q\in\mathcal{Q}^T$, there exists some $P\in\mathcal{R}^T$ such that $Q\ll P$ where

$$\mathcal{R}^T := \{P\in\mathfrak{P}(\Omega^T),\ \exists Q^{'}\in\mathcal{Q}^T, P\ll Q^{'} \text{ and } R \text{ is a martingale measure}\}.$$

The classical notion of equivalent martingale measure is replaced by the fact that for all prior $Q\in\mathcal{Q}^T$, there exists a martingale measure P such that Q is absolutely

[1] Recall that in the proof of Lemma 2.3.3 in Chapter 2 we have given the details of the arguments of [116, Exercise 14.12]

continuous with respect to P and one can find an other prior $Q' \in \mathcal{Q}^T$ such that P is absolutely continuous with respect to Q'.
The following local version of the no-arbitrage will be very useful.

Definition 3.4.2 For $\omega^t \in \Omega^t$ fixed, we say that the $NA(\mathcal{Q}_{t+1}(\omega^t))$ condition holds true if for all $h \in \mathbb{R}^d$, $h\Delta S_{t+1}(\omega^t, \cdot) \geq 0$ $\mathcal{Q}_{t+1}(\omega^t)$-q.s. implies that $h\Delta S_{t+1}(\omega^t, \cdot) = 0$ $\mathcal{Q}_{t+1}(\omega^t)$-q.s. We define also

$$\Omega^t_{NA} := \{\omega^t \in \Omega^t,\ NA(\mathcal{Q}_{t+1}(\omega^t)) \text{ holds true}\}.$$

We recall the first part of [25, Theorem 4.5] which establishes the link between the global version $NA(\mathcal{Q}^T)$ and the local version of the no-arbitrage that we will use below.

Theorem 3.4.3 *Assume that Assumptions 3.2.1 and 3.2.2 hold true. Then the following statements are equivalent*
1. $NA(\mathcal{Q}^T)$ *hold true.*
2. For all $0 \leq t \leq T-1$, $\Omega^t_{NA} \in \mathcal{B}_c(\Omega^t)$ *is a* $\mathcal{Q}^t$*-full measure set.*

We propose now our alternative formulations of no-arbitrage definition. First, in the spirit of [111] and [84] (see also Chapter 2), we introduce the following quantitative no-arbitrage condition.

Definition 3.4.4 The quantitative no-arbitrage condition holds true if for all $0 \leq t \leq T-1$, there exists some $\mathcal{Q}^t$-full measure set $\Omega^t_{qNA} \in \mathcal{B}_c(\Omega^t)$ such that for all $\omega^t \in \Omega^t_{qNA}$, there exists $\alpha_t(\omega^t) > 0$ such that for all $h \in D^{t+1}(\omega^t)$, $h \neq 0$ there exists $P_h \in \mathcal{Q}_{t+1}(\omega^t)$ satisfying

$$P_h\left(\frac{h}{|h|}\Delta S_{t+1}(\omega^t, \cdot) < -\alpha_t(\omega^t)\right) > \alpha_t(\omega^t). \tag{3.5}$$

In the case where there is only one risky asset and one period, the interpretation of (3.5) is straightforward. It simply means that there exists a prior (i.e. some probability P^+) for which the price of the risky asset increases enough and an other one (P^-) for which the price decreases, $i.e$ $P^{\pm}\left(\pm\Delta S(\cdot) < -\alpha\right) > \alpha$ where $\alpha > 0$. The number α serves as a measure of the gain/loss and of their size. Note that for an agent buying or selling some quantity of the risky asset, there is always a prior in which she is exposed to a potential loss.

Remark 3.4.5 Note that (3.5) in Definition 3.4.4 can be equivalently written using large or strict inequalities. Indeed, if (3.5) is verified: fix some $\omega^t \in \Omega^t_{qNA}$, $h \in D^{t+1}(\omega^t)$, $h \neq 0$. Then it is immediate to see that

$$P_h\left(\frac{h}{|h|}\Delta S_{t+1}(\omega^t, \cdot) \leq -\alpha_t(\omega^t)\right) \geq P_h\left(\frac{h}{|h|}\Delta S_{t+1}(\omega^t, \cdot) < -\alpha_t(\omega^t)\right) \geq \alpha_t(\omega^t).$$

Now, if (3.5) is verified but with large inequalities, then again for some $\omega^t \in \Omega^t_{qNA}$, $h \in D^{t+1}(\omega^t)$, $h \neq 0$ fixed, we have that $\left\{\frac{h}{|h|}\Delta S_{t+1}(\omega^t,\cdot) \leq -\alpha_t(\omega^t)\right\} \subset \left\{\frac{h}{|h|}\Delta S_{t+1}(\omega^t,\cdot) < -\frac{\alpha_t(\omega^t)}{2}\right\}$. Thus

$$P_h\left(\left\{\frac{h}{|h|}\Delta S_{t+1}(\omega^t,\cdot) < -\frac{\alpha_t(\omega^t)}{2}\right\}\right) > \frac{\alpha_t(\omega^t)}{2}.$$

Before introducing the geometric no-arbitrage condition, we recall that for a convex set $C \subset \mathbb{R}^d$, the relative interior of C (see [115, Section 6]) is $\mathrm{Ri}(C) = \{y \in C,\ \exists \varepsilon > 0,\ \mathrm{Aff}(C) \cap B(y,\varepsilon) \subset C\}$ (the relative interior was already introduced in the proof of Lemma 2.5.11 in Chapter 2).

Definition 3.4.6 The geometric no-arbitrage condition holds true if for all $0 \leq t \leq T-1$, there exists some $\mathcal{Q}^t$-full measure set $\Omega^t_{gNA} \in \mathcal{B}_c(\Omega^t)$ such that for all $\omega^t \in \Omega^t_{gNA}$, $0 \in \mathrm{Ri}\left(\widehat{D}^{t+1}(\omega^t)\right)$. In this case there exists $\varepsilon_t(\omega^t) > 0$ such that

$$B(0,\varepsilon_t(\omega^t)) \cap D^{t+1}(\omega^t) \subset \widehat{D}^{t+1}(\omega^t) \tag{3.6}$$

Theorem 3.4.7 *Assume that Assumptions 3.2.1 and 3.2.2 hold true. Then the quantitative no-arbitrage (see Definition 3.4.4), the geometric no-arbitrage (see Definition 3.4.6) and the $NA(\mathcal{Q}^T)$ conditions (see Definition 3.4.1) are equivalent and $\Omega^t_{NA} = \Omega^t_{qNA} = \Omega^t_{gNA}$ for all $0 \leq t \leq T$. Moreover, for all $\omega^t \in \Omega^t_{NA}$ one can choose $\alpha_t(\omega^t) = \varepsilon(\omega^t)$ and* (3.5) *and* (3.6) *holds true (where Ω^t_{NA} was introduced in Definition 3.4.2).*

Remark 3.4.8 It is clear that in this case for all $\omega^t \in \Omega^t_{NA}$, $D^{t+1}(\omega^t)$ is a vector space.

The proof of Theorem 3.4.7 is presented in Section 3.4.2 just below. In a first step, we look at a one-period model with determinist initial data where the problem is easier to formulate since strategies are vectors in $\mathbb{R}^d$. We prove the equivalence of the different no-arbitrage condition in this setting in Proposition 3.4.14. The second step extend the result to the multi-period: this is an immediate application of Theorem 3.4.3 that we have recalled above. Note that the proof of Theorem 3.4.3 requires heavy technical considerations related to measurability issues and relies on the framework introduced in Section 3.2 and in particular Assumptions 3.2.1 and 3.2.2.
To conclude this section, we propose the following proposition that establishes some tricky measurability properties. It will be used in Chapter 5.

Proposition 3.4.9 *Assume that Assumptions 3.2.1 and 3.2.2 hold true. Under one of the no-arbitrage condition (see Definitions 3.4.1, 3.4.4 and 3.4.6) one can choose α_t in* (3.5) *such that $\omega^t \in \Omega^t \to \alpha_t(\omega^t)$ is $\mathcal{B}_c(\Omega^t)$-measurable.*

3.4.2 Proof of Theorem 3.4.7

We start with a one-period model. Let $(\overline{\Omega}, \mathcal{G})$ be a measured space, $\mathfrak{P}(\overline{\Omega})$ the set of all probabilities on $\overline{\Omega}$ defined on $\mathcal{G}$ and $\mathcal{Q}$ a non-empty convex subset of $\mathfrak{P}(\overline{\Omega})$. For $P \in \mathcal{Q}$ fixed, we denote by E_P the expectation under P. Let $Y(\cdot) := (Y_i(\cdot),\ i = 1 \cdots d)$ a $\mathcal{G}$-measurable $\mathbb{R}^d$-valued random variable. We introduce the following sets which are the pendant in the one-period case of Definition 3.3.1. The support of the distribution of $Y(\cdot)$ under P for all $P \in \mathcal{Q}$ is the following set

$$\widetilde{D} := \bigcap \left\{ A \subset \mathbb{R}^d,\ \text{closed},\ P\left(Y(\cdot) \in A\right) = 1,\ \forall P \in \mathcal{Q} \right\}$$

We also introduce the closed (for the usual topology in $\mathbb{R}^d$) convex hull and the affine hull of $\widetilde{D}$

$$\widehat{D} := \overline{\text{Conv}(\widetilde{D})},\ D := \text{Aff}\left(\widetilde{D}\right). \tag{3.7}$$

It is clear that $\widetilde{D} \neq \emptyset$ and that $\widetilde{D} \subset \widehat{D} \subset D$. We then introduce the following definitions of no-arbitrage in this setting: the first one is the one-period pendant of $NA(\mathcal{Q}^{t+1}(\omega^t))$ for ω^t fixed while the two others are the pendant of Definitions 3.4.4 and 3.4.6.

Definition 3.4.10 The one-period no-arbitrage condition holds true if for all $h \in \mathbb{R}^d$, $hY \geq 0$ $\mathcal{Q}$-q.s implies that $hY = 0$ $\mathcal{Q}$-q.s.

Definition 3.4.11 The one-period quantitative no-arbitrage condition holds true if there exists some constant $0 < \alpha \leq 1$ such that for all $h \in D$, $h \neq 0$ there exists $P_h \in \mathcal{Q}$ satisfying

$$P_h(hY(\cdot) < -\alpha|h|) > \alpha. \tag{3.8}$$

Definition 3.4.12 The one-period geometric no-arbitrage condition holds true if $0 \in \text{Ri}(\widehat{D})$.

If Assumption 3.4.12 holds true, $0 \in \widehat{D} \subset D$ and D is a vector subspace. Moreover, there exists some ε such that

$$(B(0,\varepsilon) \cap D) \subset \widehat{D}, \tag{3.9}$$

where $B(0,\varepsilon)$ is the open ball in $\mathbb{R}^d$ centred in 0 with radius ε.
In Proposition 3.4.14 we establish that these three assumptions are actually equivalent. We first establish the following lemma that will be used in Proposition 3.4.14 and is based on well-know separation results for convex sets in $\mathbb{R}^d$.

Lemma 3.4.13 *If $0 \notin Ri\left(\widehat{D}\right)$, there exists some $h^* \in D$, $h^* \neq 0$ such that $h^*Y \geq 0$ $\mathcal{Q}$-q.s.*

Proof. First note that since $\widehat{D} \neq \emptyset$, we get that $\text{Ri}\left(\widehat{D}\right) \neq \emptyset$ (see [115, Theorem 6.2 p45]). Now to prove the lemma, note that it is enough to find some $h^* \in D$, $h^* \neq 0$ such that $h^* y \geq 0$ for all $y \in \widehat{D}$. Indeed if $h^* y \geq 0$ for all $y \in \widehat{D}$, as $\widetilde{D} \subset \widehat{D}$, $P\left(\{Y(\cdot) \in \widetilde{D}\}\right) = 1$ for all $P \in \mathcal{Q}$ and $\{Y(\cdot) \in \widetilde{D}\} \subset \{h^* Y(\cdot) \geq 0\}$ we get that $h^* Y(\cdot) \geq 0$ $\mathcal{Q}$-q.s.
We now build h^*. As $0 \notin \text{Ri}\left(\widehat{D}\right)$, we apply [115, Theorem 11.1, 11.3 p97] and obtain some $h_1 \in \mathbb{R}^d$, $h_1 \neq 0$, such that $h_1 y \geq 0$ for all $y \in \widehat{D}$ and some $y_0 \in \widehat{D}$ such that $h_1 y_0 > 0$ (and it is clear that $y_0 \neq 0$). For sake of completeness we give the details of the arguments. From [115, Theorem 11.3], there is a hyperplan separating properly the two convex sets $\widehat{D}$ and $\{0\}$. Now from [115, Theorem 11.1] we get some $h_1 \in \mathbb{R}^d$ such that $\inf_{y \in \widehat{D}}(h_1 y) \geq \sup_{y \in \{0\}}(h_1 y) = 0.$ and $\sup_{y \in \widehat{D}}(h_1 y) > 0$ and thus there exists some $y_0 \in \widehat{D}$ such that $h_1 y_0 > 0$.
We distinguishes two cases. If $h_1 \in D$, then we set $h^* = h_1 \neq 0$ and we are done. Assume now that $h_1 \notin D$. We introduce the orthogonal projection on D $p : h \in \mathbb{R}^d \to p(h) \in D$. We have that

$$(h - p(h))\,(y - p(h)) \leq 0$$

for all $h \in \mathbb{R}^d$, $y \in D$. Thus, for all $\lambda \geq 0$, $y \in D$ we get that $\lambda h_1 y \leq (\lambda h_1 - p(\lambda h_1))p(\lambda h_1) + y p(\lambda h_1)$. Taking the limit when λ goes to zero and recalling that p is continuous, we find that $p(0) y \geq |p(0)|^2$ for all $y \in D$ and thus for all $y \in \widehat{D} \subset D$. If $p(0) \neq 0$, then we set $h^* = p(0) \in D$, $h^* \neq 0$, $h^* y \geq 0$ for all $y \in \widehat{D}$: we are done. If $p(0) = 0$, then $0 \in D$ and D is actually a vector space. Thus $p(h_1) y = h_1 y$ for all $y \in D$ and in particular, for all $y \in \widehat{D} \subset D$ we have that $p(h_1) y = h_1 y \geq 0$. For $y = y_0 \neq 0$ we also get that $p(h_1) y_0 = h_1 y_0 > 0$. Thus $p(h_1) \neq 0$, $p(h_1) \in D$, $p(h_1) y \geq 0$ for all $y \in \widehat{D}$. We set $h^* = p(h_1)$ and we are done. □

Proposition 3.4.14 *The three different notions of no-arbitrage (see Definition 3.4.10, 3.4.11 and 3.4.12) are equivalent. Furthermore if one of this condition holds true, one can choose* $\alpha = \varepsilon$ *in* (3.8) *and* (3.9)*.*

Proof. We will prove first that Definition 3.4.10 implies that both Definitions 3.4.12 and 3.4.11 hold true. In a second step we prove the reverse implication for both Definitions 3.4.12 and 3.4.11.
First note that from Definition 3.4.10 we have that for all $h \in D$

$$hY(\cdot) \geq 0 \ \mathcal{Q}\text{-q.s.} \Rightarrow h = 0. \tag{3.10}$$

Indeed, assume that there exists some $h \in D$, $h \neq 0$ such that $hY(\cdot) \geq 0$ $\mathcal{Q}$-q.s. From Definition 3.4.10, we get that $hY(\cdot) = 0$ $\mathcal{Q}$-q.s. and thus $h \in \widetilde{L} := \{h \in \mathbb{R}^d, hy = 0 \text{ for all } y \in \widetilde{D}\} = \left(\widetilde{D}\right)^{\perp}$ the orthogonal space of $\widetilde{D}$. Indeed, if $h \notin \widetilde{L}$, there

exists some $y_1 \in \widetilde{D}$ such that either $hy_1 < 0$ (resp. $hy_1 > 0$) and some $\delta > 0$ such that $hy < 0$ (resp. $hy > 0$) for all $y \in B(y_1, \delta)$. By the minimal property of $\widetilde{D}$, there exists some $P \in \mathcal{Q}$ such that $P(Y \in B(y_1, \delta)) > 0$ but this contradicts the fact that $hY = 0$ $\mathcal{Q}$-q.s. (see also the proof [99, Lemma 2.6]). We claim that $\widetilde{L} = L := (D)^{\perp}$ the orthogonal space of D. Indeed as $\widetilde{D} \subset D$, if $h \in L$ it is clear that $h \in \widetilde{L}$. But if $h \in \widetilde{L}$, then by affine combination it is also clear that $h \in L$. Thus $h \in D \cap L \subset \{0\}$, a contradiction.
We prove now that Definition 3.4.10 implies that Definition 3.4.12 holds true. Indeed if $0 \notin \text{Ri}\left(\widehat{D}\right)$, from Lemma 3.4.13, there exists some $h^* \in D$, $h^* \neq 0$ such that $h^*Y \geq 0$ $\mathcal{Q}$-q.s. which contradicts (3.10).
Then, we show that Definition 3.4.10 implies that Definition 3.4.11 holds true. We introduce for $n \geq 1$

$$A_n := \left\{h \in D,\ |h| = 1,\ P\left(hY(\cdot) < -\frac{1}{n}\right) < \frac{1}{n},\ \forall P \in \mathcal{Q}\right\} \tag{3.11}$$

and we define $n_0 := \inf\{n \geq 1, A_n = \emptyset\}$ with the convention that $\inf \emptyset = +\infty$. From the previous step, Definition 3.4.12 holds also true and thus we have that $0 \in \text{Ri}\left(\widehat{D}\right) \subset D$: *i.e* D is a vector space. If $D = \{0\}$, then $n_0 = 1 < \infty$. We assume now that $D \neq \{0\}$ and we prove by contradiction that $n_0 < \infty$. Assume that $n_0 = \infty$. For all $n \geq 1$, we get some $h_n \in D$ with $|h_n| = 1$ and such that for all $P \in \mathcal{Q}$ $P\left(h_nY(\cdot) \leq -\frac{1}{n}\right) \geq P\left(h_nY(\cdot) < -\frac{1}{n}\right) \geq -\frac{1}{n}$. By passing to a sub-sequence we can assume that h_n tends to some $h^* \in D$ (D is closed by definition) with $|h^*| = 1$. Fix some $P \in \mathcal{Q}$ and introduce $B := \{\omega \in \overline{\Omega},\ h^*Y(\omega) < 0\}$ and $B_n := \{\omega \in \overline{\Omega},\ h_nY(\omega) \leq -1/n\}$. Then $B \subset \liminf_n B_n$. Indeed fix some $\omega^t \in B$. Then, there exists some $\varepsilon > 0$ such that $h^*Y(\omega) < -\varepsilon$. Now there exists some $N \geq 1$ such that for all $n \geq N$ $|h_n - h| \leq \frac{\varepsilon}{2(1+|Y(\omega))}$ and such that $\frac{1}{n} \leq \frac{\varepsilon}{2}$ and it follows that

$$h_nY(\omega) = hY(\omega) + (h_n - h)Y(\omega) \leq -\varepsilon + |h_n - h||Y(\omega)| \leq \frac{\varepsilon}{2} \leq -\frac{1}{n}.$$

Furthermore as $1_{\liminf_n B_n} = \liminf_n 1_{B_n}$, Fatou's Lemma implies that

$$P\left(h^*Y(\cdot) < 0\right) \leq \int_{\overline{\Omega}} 1_{\liminf_n B_n}(\omega)P(d\omega) \leq \liminf_n \int_{\overline{\Omega}} 1_{B_n}(\omega)P(d\omega) = 0.$$

This implies that $P\left(h^*Y(\cdot) \geq 0\right) = 1$. As this is true for all $P \in \mathcal{Q}$, we get from (3.10) that $h^* = 0$ which contradicts $|h^*| = 1$. Thus $n_0 < \infty$ and we can set for $\alpha = \frac{1}{n_0}$. It is clear that $\alpha \in (0, 1]$ and by definition of A_{n_0}, (3.8) holds true.
We prove now that Definition 3.4.11 implies Definition 3.4.12. Indeed, if this is not the case, using Lemma 3.4.13, we find some $h^* \in D$, $h^* \neq 0$ such that $h^*Y(\cdot) \geq 0$ for $\mathcal{Q}$-q.s.: a contradiction with (3.8).
Finally, we prove that Definition 3.4.12 implies Definition 3.4.10. We fix some $h \in \mathbb{R}^d$ such that $hY \geq 0$ $\mathcal{Q}$-q.s. We claim that this implies that $hy \geq 0$ for all $y \in \widetilde{D}$.

Indeed, if this is not the case there exists some $y_0 \in \widetilde{D}$ and some $\delta > 0$ such that $hy_0 < 0$ for all $y \in B(y_0, \delta)$. By the minimal property of $\widetilde{D}$, there exists some $P \in \mathcal{Q}$ such that $P(Y \in B(y_0, \delta)) > 0$ but this contradicts the fact that $hY \geq 0$ $\mathcal{Q}$-q.s. By convex combination and closure, we also have that

$$hy \geq 0 \text{ for all } y \in \widehat{D}. \tag{3.12}$$

Now let again $p(h)$ be the orthogonal projection of h on D (recall that here D is a vector space since $0 \in \text{Ri}(\widehat{D}) \subset D$) and thus we have that $p(h)y = hy \geq 0$ for all $y \in \widehat{D} \subset D$. As $P(\{Y \in \widetilde{D}\}) = 1$ for all $P \in \mathcal{Q}$ and $\widetilde{D} \subset D$, we also have that $hY = p(h)Y$ $\mathcal{Q}^T$-q.s. If $p(h) = 0$ we are done. We assume that $p(h) \neq 0$. As $0 \in \text{Ri}(\widehat{D})$, there exists some $\varepsilon > 0$ such that $B(0, \varepsilon) \cap D \subset \widehat{D}$. Therefore (recall that we assume that $p(h) \neq 0$) we have that $-\varepsilon \frac{p(h)}{|p(h)|} \in \widehat{D}$ and $-\varepsilon \frac{p(h)}{|p(h)|} p(h) < 0$. This contradicts (3.12). Thus we must have $p(h) = 0$ and it follows that $hY = p(h)Y = 0$ $\mathcal{Q}$-q.s.
We prove the last statement of the proposition. If any of the definition holds true we know that there exists some $0 < \varepsilon < 1$ such that $(B(0, \varepsilon) \cap D) \subset \widehat{D}$ and some $0 < \alpha < 1$ such that (3.8) holds true. It is easy to verify that (3.8) and (3.9) both hold true for $\min(\alpha, \varepsilon)$. □

Finally we conclude with the proof of Theorem 3.4.7.
Proof. of Theorem 3.4.7. This is a straightforward application of Theorem 3.4.3 together with Proposition 3.4.14. Indeed from Theorem 3.4.3, we get that Definition 3.4.1 is equivalent to the fact that $\Omega^t_{NA} = \{\omega^t \in \Omega^t,\ NA(\mathcal{Q}_{t+1}(\omega^t)) \text{ holds true}\}$ is a $\mathcal{Q}^t$-full measure set and belongs to $\mathcal{B}_c(\Omega^t)$ and from Proposition 3.4.14 for all $\omega^t \in \Omega^t_{NA}$, we have that $NA(\mathcal{Q}_{t+1}(\omega^t))$ is equivalent to (3.5) and (3.6). It is also clear from Proposition 3.4.14 that for all $\omega^t \in \Omega^t_{NA}$ one can choose $\alpha_t(\omega^t) = \varepsilon_t(\omega^t)$ such that (3.5) and (3.6) holds true. □

3.4.3 Proof of Proposition 3.4.9

Proof. of Proposition 3.4.9
To prove that we fin a version of α_t that is $\mathcal{B}_c(\Omega^T)$ measurable, we use Theorem 3.4.7 and the fact that for all $\omega^t \in \Omega^t_{NA}$ one can choose $\alpha_t(\omega^t) = \varepsilon(\omega^t)$ such that (3.5) and (3.6) holds true. Thus we use the geometric no-arbitrage condition (see Definition 3.4.6) to build a measurable version of α_t. We fix some $0 \leq t \leq T-1$ and set for all $\omega^t \in \Omega^t_{NA}$

$$\Gamma^{t+1}(\omega^t) := \left\{\alpha \in \mathbb{Q},\ \alpha > 0,\ B(0,\alpha) \cap D^{t+1}(\omega^t) \subset \widetilde{D}^{t+1}(\omega^t)\right\}$$

and we set $\Gamma^{t+1}(\omega^t) = \emptyset$ outside of Ω^t_{NA}. We assume that Graph $\Gamma^{t+1} \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$ (this will be proven below). From the Aumann Theorem (see [119, Corollary 1])

there exists a $\mathcal{B}_c(\Omega^t)$-measurable selector $\alpha_t : \{\Gamma^{t+1} \neq \emptyset\} \to \mathbb{R}$ such that $\alpha_t(\omega^t) \in \Gamma^{t+1}(\omega^t)$ for every $\omega^t \in \{\Gamma^{t+1} \neq \emptyset\}$. Now, using Theorem 3.4.7, we get that $\Omega^t_{NA} \subset \{\Gamma(\omega^t) \neq \emptyset\}$: indeed for $\omega^t \in \Omega^t_{NA}$, Definition 3.4.6 holds true, hence there exists some $\alpha \in \mathbb{R}$, $\alpha > 0$ such that $B(0,\alpha) \cap D^{t+1}(\omega^t) \subset \widetilde{D}^{t+1}(\omega^t)$ and we can always choose $\alpha_q \in \mathbb{Q}$ with $0 < \alpha_q \leq \alpha$ such that $B(0,\alpha_q) \cap D^{t+1}(\omega^t) \subset B(0,\alpha) \cap D^{t+1}(\omega^t) \subset \widetilde{D}^{t+1}(\omega^t)$ (note that as $\Gamma^{t+1}(\omega^t) = \emptyset$ outside of Ω^t_{NA} we have actually that $\Omega^t_{NA} = \{\Gamma(\omega^t) \neq \emptyset\}$). We set $\alpha_t = 0$ outside of Ω^t_{NA} and thus we have found a $\mathcal{B}_c(\Omega^t)$-measurable version of α_t. Hence, if Graph $\Gamma^{t+1} \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$ the theorem is proved.
We prove now that Graph $\Gamma^{t+1} \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$. For all $\alpha > 0$, $\alpha \in \mathbb{Q}$, we set

$$A_\alpha := \left\{\omega^t \in \Omega^t_{NA},\ B(0,\alpha) \cap D^{t+1}(\omega^t) \subset \widetilde{D}^{t+1}(\omega^t)\right\}.$$

It is clear that

$$\text{Graph } \Gamma = \bigcup_{\alpha\in\mathbb{Q},\ \alpha>0} A_\alpha \times \{\alpha\}.$$

Thus to prove that Graph $\Gamma^{t+1} \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$ it is enough to prove that $A_\alpha \in \mathcal{B}_c(\Omega^t)$. To do so, we need a bit of preparatory work. We first define for all $\omega^t \in \Omega^t$

$$\delta^{t+1}(\omega^t) := 1_{\Omega^t_{NA}}(\omega^t)\textbf{Dim}(D^{t+1}(\omega^t)) \tag{3.13}$$

(recall that if $\omega^t \in \Omega^t_{NA}$, $D^{t+1}(\omega^t)$ is a vector space). From Lemma 3.4.15, we obtain that δ^{t+1} is $\mathcal{B}_c(\Omega^t)$-measurable and that there exists a family of $\mathcal{B}_c(\Omega^t)$-measurable functions $(e_j)_{1\leq j\leq d}$ (with $e_j : \omega^t \in \Omega^t \to \mathbb{R}^d$ for all $1 \leq j \leq d$) such that for all $0 \leq i \leq d$, $\omega^t \in \Omega^t_{NA} \cap \{\delta^{t+1} = i\}$, $(e_j(\omega^t))_{1\leq j\leq i}$ is an orthonormal basis of $D^{t+1}(\omega^t)$ and $e_{i+1}(\omega^t) = \cdots = e_d(\omega^t) = 0$ if $i < d$. We can now introduce the following random sets

$$D^{t+1}_{\mathbb{Q}}(\omega^t) := \left\{\sum_{i=1}^{\delta^{t+1}(\omega^t)} \lambda_i e_i(\omega^t),\ \lambda_i \in \mathbb{Q},\ i \in \{1,\cdots,\delta^{t+1}(\omega^t)\}\right\} \tag{3.14}$$

for all $\omega^t \in \Omega^t_{NA}$ and $D^{t+1}_{\mathbb{Q}}(\omega^t) = \emptyset$ for $\omega^t \notin \Omega^t_{NA}$. It is clear that for all $\omega^t \in \Omega^t_{NA}$ $\overline{D^{t+1}_{\mathbb{Q}}(\omega^t)} = D^{t+1}(\omega^t)$ [2]. Indeed it is clear that $\overline{D^{t+1}_{\mathbb{Q}}(\omega^t)} \subset D^{t+1}(\omega^t)$. We prove the reverse inclusion and fix some $h \in D^{t+1}(\omega^t)$. From Lemma 3.4.15, $\delta^{t+1}(\omega^t) = i$ for some $0 \leq i \leq d$ and $(e_j(\omega^t))_{1\leq j\leq i}$ is an orthonormal basis of $D^{t+1}(\omega^t)$. Thus there exists $(\lambda_j)_{1\leq j\leq i}$ such that $h = \sum_{j=1}^i \lambda_j e_j(\omega^t)$. Now for each $1 \leq j \leq i$, there exists a sequence $(\lambda_{j,q})_{q\geq 1}$ in $\mathbb{Q}$ such that $\lim_q \lambda_{j,q} = \lambda_j$. Set for all $q \geq 1$, $h_q = \sum_{j=1}^i \lambda_{j,q} e_j(\omega^t) \in D^{t+1}_{\mathbb{Q}}(\omega^t)$ and it is clear that $\lim_q h_q = h$.
Now, we claim that for $\alpha > 0$, $\alpha \in \mathbb{Q}$

$$\omega^t \in A_\alpha \iff \forall h \in D^{t+1}_{\mathbb{Q}}(\omega^t), |h| \leq \alpha,\ h \in \widetilde{D}^{t+1}(\omega^t). \tag{3.15}$$

[2]where the closure is taken in $\mathbb{R}^d$ with respect to the euclidien topology.

As for all $\omega^t \in \Omega^t_{NA}$, $\widetilde{D}^{t+1}_{\mathbb{Q}}(\omega^t) \subset \widetilde{D}^{t+1}(\omega^t)$ the first implication is trivial. Assume now that for some $\omega^t \in \Omega^t_{NA}$, for all $h \in D^{t+1}_{\mathbb{Q}}(\omega^t)$, $|h| \leq \alpha$ then $h \in \widetilde{D}^{t+1}(\omega^t)$. We prove that $\omega^t \in A_\alpha$. Let $h \in D^{t+1}(\omega^t)$ such that $|h| \leq \alpha$. If $h \in D^{t+1}_{\mathbb{Q}}(\omega^t)$ there is nothing to prove. Otherwise, we use again Lemma 3.4.15: $\delta^{t+1}(\omega^t) = i$ for some $0 \leq i \leq d$ and $(e_j(\omega^t))_{1\leq j\leq i}$ is an orthonormal basis of $D^{t+1}(\omega^t)$. Thus there exists $(\lambda_j)_{1\leq j\leq i}$ such that $h = \sum_{j=1}^i \lambda_j e_j(\omega^t)$ and $|h|^2 = \sum_{j=1}^i \lambda_j^2 \leq \alpha^2$. Now for each $1 \leq j \leq i$, there exists a sequence $(\lambda_{j,q})_{q\geq 1}$ in $\mathbb{Q}$ such that $\lim_q \lambda_{j,q} = \lambda_j$ and $\lambda_{j,q} \leq \lambda_j$ for all $q \geq 1$. Set for all $q \geq 1$, $h_q = \sum_{j=1}^i \lambda_{j,q} e_j(\omega^t) \in D^{t+1}_{\mathbb{Q}}(\omega^t)$, it is clear that $\lim_q h_q = h$ and that $|h_q| = \sum_{j=1}^d \lambda_{j,q}^2 \leq \alpha$ for all $q \geq 1$. By assumption for all $q \geq 1$, $h_q \in \widetilde{D}^{t+1}(\omega^t)$ and since $\widetilde{D}^{t+1}(\omega^t)$ is a closed subset of $\mathbb{R}^d$, $h \in \widetilde{D}^{t+1}(\omega^t)$. As this is true for all $h \in D^{t+1}(\omega^t)$ such that $|h| \leq \alpha$, $\omega^t \in A_\alpha$ follows.
From (3.15) we get that

$$A_\alpha := \bigcap_{(\lambda_1,\cdots\lambda_d)\in\mathbb{Q}^d, \sum_{i=1}^d \lambda_i^2 \leq \alpha^2} \left\{ \omega^t,\ \left\{ \sum_i^d \lambda_i e_i(\omega^t) \right\} \cap \widetilde{D}^{t+1}(\omega^t) \neq \emptyset \right\}.$$

Indeed, $\omega^t \in A_\alpha$ if and only if for all $h \in D^{t+1}_{\mathbb{Q}}(\omega^t)$ such that $|h| \leq \alpha$, $h \in \widetilde{D}^{t+1}(\omega^t)$ which is equivalent to the fact that for $(\lambda_1, \cdots \lambda_q) \in \mathbb{Q}^d$ such that $\sum_{i=1}^d \lambda_i^2 \leq \alpha$, $h := \sum_{i=1}^d \lambda_i e_i(\omega^t) \in \widetilde{D}^{t+1}(\omega^t)$.
We prove now for that $(\lambda_1, \cdots, \lambda_q) \in \mathbb{Q}^d$ fixed, $\left\{\omega^t,\ \left\{\sum_i^d \lambda_i e_i(\omega^t)\right\} \cap \widetilde{D}^{t+1}(\omega^t) \neq \emptyset\right\} \in \mathcal{B}_c(\Omega^t)$ and the fact that $A_\alpha \in \mathcal{B}_c(\Omega^t)$ will follow immediately. We fix some $(\lambda_1, \cdots, \lambda_q) \in \mathbb{Q}^d$. From Lemma 3.4.15, $e : \omega^t \to \sum_{i=1}^d \lambda_i e_i(\omega^t)$ is $\mathcal{B}_c(\Omega^t)$-measurable and using [3, Theorem 4.45], we obtain that $\text{Graph}\, e \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$. From Lemma 3.3.2 we have that $\text{Graph}\, \widetilde{D}^{t+1} \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$. Thus $\text{Graph}\, e \bigcap \text{Graph}\, \widetilde{D}^{t+1} \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$ and from the Projection Theorem (see [37, Theorem 3.23 p75]), $\text{Proj}_{\Omega^t}\left(\text{Graph} e \bigcap \text{Graph} \widetilde{D}^{t+1}\right) \in \mathcal{B}_c(\Omega^t)$. Now it is easy to see that

$$\omega^t \in \text{Proj}_{\Omega^t}\left(\text{Graph}\, e \bigcap \text{Graph}\, \widetilde{D}^{t+1}\right) \iff \left\{ \sum_i^d \lambda_i e_i(\omega^t) \right\} \cap \widetilde{D}^{t+1}(\omega^t) \neq \emptyset,$$

Indeed $\omega^t \in \text{Proj}_{\Omega^t}\left(\text{Graph}\, e \bigcap \text{Graph}\, \widetilde{D}^{t+1}\right)$ if and only if there exists some $h \in \mathbb{R}^d$, such that $h = \sum_i^d \lambda_i e_i(\omega^t)$ and $h \in \widetilde{D}^{t+1}(\omega^t)$, *i.e.* if and only if $\left\{\sum_i^d \lambda_i e_i(\omega^t)\right\} \cap \widetilde{D}^{t+1}(\omega^t) \neq \emptyset$. This concludes the proof. □

The following lemma was used in the previous proof. Its purpose is to build a measurable orthonormal basis for the random set D^{t+1}.

Lemma 3.4.15 *Assume that Assumptions 3.2.1, 3.2.2 hold true. Then for all* $0 \leq t \leq T-1$, δ^{t+1} *(see* (3.13)*) is* $\mathcal{B}_c(\Omega^t)$*-measurable and there exists a family of* $\mathcal{B}_c(\Omega^t)$*-measurable functions* $(e_j)_{1\leq j\leq d}$ *(with* $e_j : \omega^t \in \Omega^t \to \mathbb{R}^d$ *for all* $1 \leq j \leq d$*) such that*

for all $0 \leq i \leq d$, $\omega^t \in \Omega^t_{NA} \cap \{\delta^{t+1} = i\}$, $(e_j(\omega^t))_{1 \leq j \leq i}$ ***is an orthonormal basis of*** $D^{t+1}(\omega^t)$ ***and*** $e_{i+1}(\omega^t) = \cdots = e_d(\omega^t) = 0$ ***if*** $i < d$.

Proof. From Theorem 3.4.7 (see Remark 3.4.8), we get that $0 \in D^{t+1}(\omega^t)$ and $D^{t+1}(\omega^t)$ is a vector space for $\omega^t \in \Omega^t_{NA}$.
We construct the family $(e_j)_{1 \leq j \leq d}$ by induction on the dimension δ^{t+1}. We start with $i = 1$ and set $\widetilde{D}^{t+1}_1(\omega^t) := D^{t+1}(\omega^t) \backslash \{0\}$ for all $\omega^t \in \Omega^t$. From Lemma 3.3.2, we get that $\text{Graph}(\widetilde{D}^{t+1}_1) = \text{Graph}(D^{t+1}) \backslash (\Omega^t \times \{0\}) \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$. So we can apply the Projection Theorem (see [37, Theorem 3.23 p75]) and the Auman Theorem (see [119, Corollary 1]) and we get that $\{\widetilde{D}^{t+1}_1 \neq \emptyset\} \in \mathcal{B}_c(\Omega^t)$ and that there exists some $\mathcal{B}_c(\Omega^t)$-measurable $\widetilde{e}_1 : \{\widetilde{D}^{t+1}_1 \neq \emptyset\} \to \mathbb{R}^d$ such that $\widetilde{e}_1(\omega^t) \in \widetilde{D}^{t+1}_1(\omega^t)$ for all $\omega^t \in \{\widetilde{D}^{t+1}_1 \neq \emptyset\}$. We set for all $\omega^t \in \Omega^t$, $e_1(\omega^t) := 1_{\{\widetilde{D}^{t+1}_1 \neq \emptyset\}}(\omega^t) \frac{\widetilde{e}_1(\omega^t)}{|\widetilde{e}_1(\omega^t)|}$. It is clear that e_1 is $\mathcal{B}_c(\Omega^t)$-measurable and that $e_1(\omega^t) \in D^{t+1}(\omega^t)$ for all $\omega^t \in \Omega^t_{NA}$ (recall that $0 \in D^{t+1}(\omega^t)$). Moreover it is easy to verify that $\{\omega^t \in \Omega^t,\ \delta^{t+1}(\omega^t) = 0\} = \{\omega^t \in \Omega^t_{NA},\ e_1(\omega^t) = 0\} \cup \Omega^t \backslash \Omega^t_{NA} \in \mathcal{B}_c(\Omega^t)$. Finally, for all $\omega^t \in \Omega^t_{NA}$ such that $\delta^{t+1}(\omega^t) = 1$, $e_1(\omega^t)$ is an orthonormal basis of $D^{t+1}(\omega^t)$.
Assume now that for some $i \geq 1$ we have build a family of $\mathcal{B}_c(\Omega^t)$-measurable functions $(e_j)_{j=1,\dots,i}$ such that

1) $e_j(\omega^t) \in D^{t+1}(\omega^t)$ and $e_p(\omega^t) e_j(\omega^t) = 0$ for all $\omega^t \in \Omega^t_{NA}$, $1 \leq j, p \leq i$ with $p \neq j$,
2) if $\delta^{t+1}(\omega^t) = l$ for some $1 \leq l \leq i$ and $\omega^t \in \Omega^t_{NA}$, then $(e_1(\omega^t), \cdots, e_l(\omega^t))$ is an orthonormal basis of $D^{t+1}(\omega^t)$,
3) $\{\omega^t \in \Omega^t,\ \delta^{t+1}(\omega^t) = l-1\} = \{\omega^t \in \Omega^t_{NA},\ e_p(\omega^t) \neq 0,\ p \in \{1, \cdots, l-1\},\ e_k(\omega^t) = 0,\ k \in \{l, \cdots, i\}\} \in \mathcal{B}_c(\Omega^t)$ for all $2 \leq l \leq i$ and $\{\omega^t \in \Omega^t_{NA},\ \delta^{t+1}(\omega^t) = 0\} = \{\omega^t \in \Omega^t_{NA},\ e_k(\omega^t) = 0,\ k \in \{1, \cdots, i\}\} \in \mathcal{B}_c(\Omega^t)$.

We set
$$\widetilde{D}^{t+1}_{i+1}(\omega^t) = \left\{ h \in \mathbb{R}^d, h \in \widetilde{D}^{t+1}_1(\omega^t), h e_j(\omega^t) = 0,\ j \in \{1, \dots, i\} \right\}.$$
It is clear that $\widetilde{D}^{t+1}_{i+1}(\omega^t) \subset \widetilde{D}^{t+1}_i(\omega^t)$ for all $\omega^t \in \Omega^t$. Since e_j is $\mathcal{B}_c(\Omega^t)$-measurable for all $j = 1, \cdots, i$, we get that

$$\text{Graph}(\widetilde{D}^{t+1}_{i+1}) = \left(\bigcap_{j=1}^{i} \{(\omega^t, h) \in \Omega^t \times \mathbb{R}^d, h e_j(\omega^t) = 0\} \right) \bigcap \text{Graph}(\widetilde{D}^{t+1}_1) \in \mathcal{B}_c(\Omega^t) \times \mathcal{B}(\mathbb{R}^d).$$

As before we can apply the Projection Theorem and Auman Theorem and we obtain that $\{\widetilde{D}^{t+1}_{i+1} \neq \emptyset\} \in \mathcal{B}_c(\Omega^t)$ and some $\mathcal{B}_c(\Omega^t)$-measurable $\widetilde{e}_{i+1} : \{\widetilde{D}^{t+1}_{i+1} \neq \emptyset\} \to \mathbb{R}^d$ such that $\widetilde{e}_{i+1}(\omega^t) \in \widetilde{D}^{t+1}_{i+1}(\omega^t)$ for all $\omega^t \in \{\widetilde{D}^{t+1}_{i+1} \neq \emptyset\}$. We set $e_{i+1}(\omega^t) := 1_{\{\widetilde{D}^{t+1}_{i+1} \neq \emptyset\}}(\omega^t) \frac{\widetilde{e}_{i+1}(\omega^t)}{|\widetilde{e}_{i+1}(\omega^t)|}$ for all $\omega^t \in \Omega^t$. It is clear that e_{i+1} is $\mathcal{B}_c(\Omega^t)$-measurable and that $e_{i+1}(\omega^t) \in D^{t+1}(\omega^t)$ for all $\omega^t \in \Omega^t_{NA}$. Furthermore it is easy to verify that for all $\omega^t \in \Omega^t_{NA}$, $e_p(\omega^t)e_{i+1}(\omega^t) = 0$ for $1 \leq p \leq i$ and that $e_1(\omega^t), \cdots, e_{i+1}(\omega^t)$ are linearly independent. Thus, for all $\omega^t \in \Omega^t_{NA}$ such that $\delta^{t+1}(\omega^t) = i+1$, $(e_1(\omega^t), \cdots, e_{i+1}(\omega^t))$ is an orthonormal basis of $D^{t+1}(\omega^t)$. Items 1) and 2) are proved. We show now that for all $2 \leq l \leq i+1$

$$\{\omega^t \in \Omega^t,\ \delta^{t+1}(\omega^t) = l-1\} = \{\omega^t \in \Omega^t_{NA},\ e_p(\omega^t) \neq 0,\ p \in \{1, \cdots, l-1\}$$
$$e_k(\omega^t) = 0,\ k \in \{l, \cdots, i+1\}\}.$$

Fix some $2 \leq l \leq i+1$ and let $\omega^t \in \Omega^t$ be such that $\delta^{t+1}(\omega^t) = l-1$. As $l \geq 2$, this implies that $\omega^t \in \Omega^t_{NA}$ and by induction hypothesis we know that $(e_1(\omega^t), \cdots, e_{l-1}(\omega^t))$ is an orthonormal basis of $D^{t+1}(\omega^t)$, thus $e_j(\omega^t) \neq 0$ for $j \in \{1, \cdots, l-1\}$. So, $\widetilde{D}^{t+1}_l(\omega^t) = \emptyset$ and $\widetilde{D}^{t+1}_k(\omega^t) = \emptyset$ for $k \in \{l+1, \cdots, i+1\}$ which implies that $e_k(\omega^t) = 0$ for $k = l, \cdots, i+1$ and the first inclusion is proved. Now let $\omega^t \in \Omega^t_{NA}$ be such that $e_j(\omega^t) \neq 0$ for $j \in \{1, \cdots, l-1\}$ and $e_k(\omega^t) = 0$ for $k \in \{l, \cdots, i+1\}$. By induction hypothesis again, the family $(e_j(\omega^t))_{1 \leq j \leq l-1}$ is linearly independent thus $\delta^{t+1}(\omega^t) \geq l-1$. As $e_l(\omega^t) = 0$, we have that $\widetilde{D}^{t+1}_l = \emptyset$ (recall that $\widetilde{e}_l(\omega^t) \neq 0$) and therefore $\delta^{t+1}(\omega^t) = l-1$ and the second inclusion is proved. The second part of 3) can be proved using the same arguments. Now the measurability of δ^{t+1} follows directly from 3) and the $\mathcal{B}_c(\Omega^t)$-measurability of the $(e_j)_{1 \leq j \leq d}$. □

3.5 The quantitative no-arbitrage condition for maximising worst-case expected utility defined on $\mathbb{R}$

In this section, we illustrate how the quantitative no-arbitrage comes into play when maximising worst-case expected utility for unbounded utility functions defined on $\mathbb{R}$. As we want to focus on a relatively simple application of Assumption 3.4.11, we keep the presentation in a one-period model framework without trying to get the sharpest result or to formulate assumptions adapted to the multi-period case and the application of dynamic programming. As already mentioned (see also Chapter 4), some difficult measurability and integrability issues arise in the multi-period: they are left for further research.
We use the same one-period framework as in Section 3.4.2.

First, to simplify the issue of redundant assets in this section we make the following assumption (recall the definition of D in (3.7)).

Assumption 3.5.1 We have that $D = \mathbb{R}^d$.

Assumption 3.5.2 We consider a *random utility* $V : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfying the following conditions

- for every $x \in \mathbb{R}$, the function $V(\cdot, x) : \overline{\Omega} \to \mathbb{R}$ is $\mathcal{G}$-measurable,
- for every $\omega \in \overline{\Omega}$, the function $V(\omega, \cdot) : \mathbb{R} \to \mathbb{R}$ is non-decreasing and concave on $\mathbb{R}$,

Remark 3.5.3 Note that for all $\omega \in \overline{\Omega}$, as $V(\omega, \cdot)$ is concave and finite, it is continuous on $\mathbb{R}$. The result could be for instance extended to $\mathbb{R} \cup \{\pm\infty\}$-valued upper semi-continuous concave functions.

Assumption 3.5.4 The function V satisfies the so-called Asymptotic Elasticity Conditions at $+\infty$ and $-\infty$, *i.e.* there exist some constant $0 < \gamma < 1$ and a non negative $\mathcal{G}$-measurable random variable C verifying $\sup_{P \in \mathcal{Q}} E_P(C) < \infty$ such that for all $\omega \in \overline{\Omega}$, all $x \in \mathbb{R}$ and $\lambda \geq 1$:

$$V(\omega, \lambda x) \leq \lambda V(\omega, x) + C(\omega)\lambda^{\gamma}, \tag{3.16}$$
$$V(\omega, \lambda x) \leq \lambda^{\gamma} V(\omega, x) + C(\omega)\lambda^{\gamma}. \tag{3.17}$$

Remark 3.5.5 The above condition is inspired from [111, (10)], see also the slightly more general [33, Assumption 2.3, Proposition 4.2] for non-concave functions. We refer also to [34] for some economic interpretation on the Asymptotic Elasticity for random utility functions. We give some further insight on this assumption. As V is not bounded from above, (3.16) is used to control the behaviour of $V^-(\cdot, x)$ while (3.17) is used for $V^+(\cdot, x)$. Indeed from (3.16), recalling that C is non-negative, we get that

$$V^-(\omega, \lambda x) + C(\omega)\lambda^{\gamma} \geq \max(-V(\omega, \lambda x) + C(\omega)\lambda^{\gamma}, 0) \geq \max(-\lambda V(\omega, x), 0) = \lambda V^-(\omega, x)$$

and from (3.17), we have that

$$V^+(\omega, \lambda x) \leq \lambda^{\gamma} V^+(\omega, x) + C(\omega)\lambda^{\gamma}.$$

Roughly speaking the fact that $\gamma < 1$ means that the value of $V(\omega, x)$ decreases more quickly for large negative value of x than it increases for large positive value of x. In other words if a trading strategy yields very large outcome (both positive and negative), the utility of the negative outcome will dominate the utility of the positive outcome and the strategy is unlikely to be optimal. The fact that any trading strategy yielding very large positive outcome will also yields very large negative outcome is provided by the no-arbitrage condition. This will be proved precisely in Lemma 3.5.12.

We also need some integrability assumptions. We introduce

$$W := \{-1, 1\}^d \tag{3.18}$$

Assumption 3.5.6 The following conditions are satisfied

$$\sup_{P \in \mathcal{Q}} E_P V^+ (., 1 + \theta Y(.)) < \infty, \ \forall \, \theta \in W \tag{3.19}$$

$$\sup_{P \in \mathcal{Q}} E_P V^-(., x) < \infty, \ \forall x \in \mathbb{R} \tag{3.20}$$

Remark 3.5.7 If (3.19) is not true it is possible to find a counterexample as in [99, Example 2.3] where $v(x) < \infty$ but the supremum is not attained in (3.22).

Assumption 3.5.8 There exists some $n_0 \geq 1$ such that

$$P\left(V(., n_0) < -2\frac{\sup_{Q \in \mathcal{P}} E_Q(C)}{\alpha} - 1\right) \geq 1 - \frac{\alpha}{2}, \ \forall \, P \in \mathcal{Q}. \tag{3.21}$$

where α was introduced in Assumption 3.4.11.

Note that if (3.21) is true we have as well $P(V(., n_0) < -1) \geq 1 - \frac{\alpha}{2}$ for all $P \in \mathcal{Q}$.

Remark 3.5.9 In a multi-period framework, to prove that Assumption 3.5.8 is preserved in the dynamic programming, one can borrow ideas used in [33] where a similar situation arise. However note that we will need some measurability property for the α introduced in Assumptions 3.4.11. This illustrate the importance of Proposition 3.4.9.

Remark 3.5.10 Remark that if V is a non-random utility function then this assumption is trivially satisfied. Indeed we know that any concave function on $\mathbb{R}$ goes to $-\infty$ as x goes to $-\infty$.

Our main concern in the one period case is the following optimisation problem

$$v(x) := \sup_{h \in \mathbb{R}^d} \inf_{P \in \mathcal{Q}} E_P V\left(\cdot, x + hY(\cdot)\right). \tag{3.22}$$

Remark 3.5.11 We will prove in Lemma 3.5.12, that if Assumption 3.5.6 holds true, then the integral in (3.22) are well-defined, *i.e.* $E_P V^+ \left(\cdot, x + hY(\cdot)\right) < \infty$, but potentially equals to $-\infty$.

Lemma 3.5.12 *Assume that Assumptions 3.4.11, 3.5.1 3.5.2, 3.5.4, 3.5.6 and 3.5.8 hold true. Let* $\psi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ *be defined by*

$$\psi(x,h) := \inf_{P \in \mathcal{Q}} E_P V(\cdot, x + hY(\cdot)) \tag{3.23}$$

First, $E_P V(\cdot, x+hY(\cdot))$ *is well defined (potentially equals to* $-\infty$*) for all* $P \in \mathcal{Q}$ *and all* $(x,h) \in \mathbb{R} \times \mathbb{R}^d$*. Then,* ψ *is usc and concave on* $\mathbb{R} \times \mathbb{R}^d$*,* $\psi < +\infty$ *on* $\mathbb{R} \times \mathbb{R}^d$ *and* $\psi(x,0) > -\infty$ *for all* $x \in \mathbb{R}$*. Finally, there exists some constant* $B \geq 0$ *such that for all* $x \in \mathbb{R}$ *there exists* $K(x, n_0, \alpha, \gamma) := \max\left(1, x^+, \frac{x^+ + n_0}{\alpha}, (\frac{x^+ + n_0}{\alpha})^{\frac{2}{1-\gamma}}\right)$ *such that for all* $h \in \mathbb{R}^d$

$$|h| > K(x, n_0, \alpha, \gamma) \Rightarrow \psi(x,h) \leq B|h|^\gamma - \frac{\alpha}{2}|h|^{\frac{1+\gamma}{2}}. \tag{3.24}$$

Proof. For all $P \in \mathcal{Q}$, we introduce, $\psi_P : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\psi_P(x,h) := E_P V(\cdot, x + hY(\cdot))$$

First, for some fixed $P \in \mathcal{Q}$, we establish that for all $(x,h) \in \mathbb{R} \times \mathbb{R}^d$, $E_P V(x + hY(\cdot))$ is well-defined potentially equals to $-\infty$. We fix some $(x,h) \in \mathbb{R} \times \mathbb{R}^d$. For some $\omega \in \overline{\Omega}$, let $\theta^\omega := (sign(Y_i(\omega)))_{i=1\ldots d} \in \mathbb{R}^d$. We have that

$$|Y(\omega)| = \left(\sum_{i=1}^d Y_i(\omega)^2\right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^d Y_i(\omega)^2 + \sum_{i \neq j} |Y_i(\omega)| Y_j(\omega)|\right)^{\frac{1}{2}} = \sum_{i=1}^d |Y_i(\omega)| = \theta^\omega Y(\omega).$$

As $\theta^\omega \in W$ (see (3.18)), using the monotonicity of V and (3.16) (see Remark 3.5.5), we obtain that

$$\begin{aligned} V^+(\omega, x + hY(\omega)) &\leq V^+\left(\omega, x^+ + |h||Y(\omega)|\right) \\ &\leq K_{x,h} V^+\left(\omega, 1 + \theta^\omega Y(\omega)|\right) + K_{x,h}^\gamma C(\omega) \\ &\leq \sum_{\theta \in W} K_{x,h} V^+\left(\omega, 1 + \theta Y(\omega)\right) + K_{x,h}^\gamma C(\omega) \end{aligned} \tag{3.25}$$

where $K_{x,h} = x^+ + |h|$. Now, $E_P V^+(x + hY(\cdot)) < \infty$ follows from Assumption 3.5.6 (see (3.19)) and $E_P(C) < \infty$ (see Assumption 3.5.4).
Now, it is clear from the concavity of V that ψ_P is concave, thus $\psi = \inf_{P \in \mathcal{Q}} \psi_P$ is also concave as the infimum of concave functions. We now prove that ϕ_P is usc. The fact that ψ is usc will follow from the same argument. Note that as $\mathrm{Dom}(\psi_P)$ might not be equal to $\mathbb{R} \times \mathbb{R}^d$, we cannot directly conclude that ϕ_P is continuous using the concavity of ϕ_P. We fix again some $(x,h) \in \mathbb{R} \times \mathbb{R}^d$ and let $(x_n, h_n)_{n \in \mathbb{N}}$ be a sequence converging to (x,h). Let $\varepsilon > 0$ and $N \geq 1$ be such that $\max(|x_n - x|, |h_n - h|) \leq \varepsilon$ for

all $n \geq N$. Then $x_n + h_n Y(\omega) \leq x^+ + \varepsilon + (\varepsilon + |h|)|Y|(\omega)$ for all $\omega \in \overline{\Omega}$. From (3.25) we obtain for all $n \geq N$

$$V(\omega, x_n + h_n Y(\omega)) \leq V^+\left(\omega, x^+ + \varepsilon + (\varepsilon + |h|)|Y(\omega)|\right) \leq \sum_{\theta \in W} K_{x^+ + 2\varepsilon, h} V^+\left(\omega, 1 + \theta Y(\omega)\right) + K^{\gamma}_{x^+ + 2\varepsilon, h} C(\omega).$$

Thus, using (3.19) (see Assumption 3.5.6), the (lim sup) Fatou Lemma applies and as V is continuous (see Remark 3.5.3) we get that

$$\limsup_n \psi_P(x_n, h_n) = \limsup_n E_P\left(V(., x_n + h_n Y)\right) \leq E_P\left(V(., x + hY)\right) = \psi_P(x, h),$$

and ψ_P is usc as claimed. Again from Assumption 3.5.6 (see (3.20)) we also have that $\psi(x, 0) > -\infty$.

We prove now (3.24). Let again $(x, h) \in \mathbb{R} \times \mathbb{R}^d$ be fixed and assume that $|h| \geq max(1, x^+)$. From the monotonicity of V and (3.17) (see Assumption 3.5.4 and Remark 3.5.5) we have for all $\omega \in \overline{\Omega}$ that

$$V^+(\omega, x + hY(\omega)) = V^+\left(\omega, |h|\left(\frac{x}{|h|} + \frac{h}{|h|}Y(\omega)\right)\right) \leq |h|^{\gamma}\left(V^+\left(\omega, \frac{x^+}{|h|} + \frac{h}{|h|}Y(\omega)\right) + C(\omega)\right).$$

As $|h| \geq \max(1, x^+)$, using as previously the monotonicity of V, again $\theta^{\omega} \in W$ (see (3.18)) , we get for all $\omega \in \overline{\Omega}$ that

$$V^+\left(\omega, \frac{x^+}{|h|} + \frac{h}{|h|}Y(\omega)\right) \leq V^+\left(\omega, 1 + |Y(\omega)|\right) \leq V^+\left(\omega, 1 + \theta^{\omega} Y(\omega)\right) \leq \sum_{\theta \in W} V^+\left(\omega, 1 + \theta Y(\omega)\right).$$

Setting $L(\omega) := \sum_{\theta \in W} V^+\left(\omega, 1 + \theta Y(\omega)\right)$, for all $\omega \in \overline{\Omega}$, we get that for all P in $\mathcal{Q}$

$$\mathbb{E}_P\left(V^+\left(., x + hY(.)\right)\right) \leq |h|^{\gamma}\left(\sup_{P \in \mathcal{P}} E_P L(.) + \sup_{P \in \mathcal{P}} E_P C(.)\right) < \infty \tag{3.26}$$

using again (3.19) (see Assumption 3.5.6) and the fact that $\sup_{P \in \mathcal{P}} E_P C(\cdot) < \infty$ (see Assumption 3.5.4).

To study $V^-(\omega, x + hY(\omega))$ we use the quantitative version of the no-arbitrage (Assumption 3.4.11) together with (3.16) (see Assumption 3.5.4) and Assumption 3.5.8. Recall that x is fixed and that $|h| \geq max(1, x^+)$. We first consider the following set

$$B_{n_0, h} := \left\{\omega \in \overline{\Omega},\ \frac{h}{|h|}Y(\omega) \leq -\alpha,\ V(\omega, -n_0) < -2\frac{\sup_{P \in \mathcal{P}} E_P(C)}{\alpha} - 1\right\}.$$

It is clear that $B_{n_0,h}$ is $\mathcal{G}$-measurable. Using Assumption 3.4.11 (see (3.8)), there exists $P_h \in \mathcal{Q}$ such that $P\left(\frac{h}{|h|}Y(.) \leq -\alpha\right) \geq \alpha$ and together with (3.21) in Assumption 3.5.8 we get that

$$\begin{aligned} P_h(B_{n_0,h}) &\geq P_h\left(\frac{h}{|h|}Y(.) \leq -\alpha\right) + P_h\left(V(., -n_0) < -2\frac{\sup_{P\in\mathcal{P}} E_P(C)}{\alpha} - 1\right) - 1 \\ &\geq \frac{\alpha}{2}. \end{aligned} \tag{3.27}$$

In the rest of the proof we will use that if $\omega \in B_{n_0,h}$ then $V(\omega, -n_0) < -1$. Assume now that $x^+ - \alpha|h| \leq -n_0$ then $B_{n_0,h} \subset \{\omega \in \overline{\Omega},\ V(\omega, x + hY(\omega)) < 0\}$ since V is monotone. Thus, for all $\omega \in \overline{\Omega}$ we have that

$$\begin{aligned} V^-\left(\omega, x + hY(\omega)\right) &= -1_{\{V(.,x+hY(.))<0\}}(\omega)V\left(\omega, x + hY(\omega)\right) \\ &\geq -1_{B_{n_0,h}}(\omega)V\left(\omega, x + hY(\omega)\right). \end{aligned} \tag{3.28}$$

And using condition (3.16) in Assumption 3.5.4 together with the monotonicity of V we get that

$$\begin{aligned} & 1_{B_{n_0,h}}(\omega)V\left(\omega, x + hY(\omega)\right) \\ &= 1_{B_{n_0,h}}(\omega)V\left(\omega, |h|^{\frac{1+\gamma}{2}}\left(\frac{x}{|h|^{\frac{1+\gamma}{2}}} + |h|^{\frac{1-\gamma}{2}}\frac{h}{|h|}Y(\omega)\right)\right) \\ &\leq 1_{B_{n_0,h}}(\omega)\left[|h|^{\frac{1+\gamma}{2}}V\left(\omega, \frac{x^+}{|h|^{\frac{1+\gamma}{2}}} + |h|^{\frac{1-\gamma}{2}}\frac{h}{|h|}Y(\omega)\right) + C(\omega)|h|^{\frac{\gamma(1+\gamma)}{2}}\right]. \end{aligned}$$

Now if we impose as well that $\frac{x^+}{|h|^{\frac{1+\gamma}{2}}} - \alpha|h|^{\frac{1-\gamma}{2}} < -n_0$, recalling that $\gamma < 1$ we obtain that

$$\begin{aligned} 1_{B_{n_0,h}}(\omega)V\left(\omega, x + hY(\omega)\right) &\leq 1_{B_{n_0,h}}(\omega)|h|^{\frac{1+\gamma}{2}}V(\omega, -n_0) + C(\omega)|h|^{\frac{\gamma(1+\gamma)}{2}} \\ &\leq -1_{B_{n_0,h}}(\omega)|h|^{\frac{1+\gamma}{2}} + C(\omega)|h|^{\gamma} \end{aligned}$$

and together with (3.28) we obtain that

$$V^-\left(\omega, x + hY(\omega)\right) \geq 1_{B_{n_0,h}}(\omega)|h|^{\frac{1+\gamma}{2}} - C(\omega)|h|^{\gamma} \tag{3.29}$$

To conclude if h satisfies that

$$x^+ - \alpha|h| \leq -n_0,\ \frac{x^+}{|h|^{\frac{1+\gamma}{2}}} - \alpha|h|^{\frac{(1-\gamma)}{2}} < -n_0,\ |h| \geq \max(1, x^+),$$

using (3.27) and (3.29) we obtain that

$$E_{P_h}\left(V^-(,.x + hY(.))\right) \geq \frac{\alpha}{2}|h|^{\frac{1+\gamma}{2}} - \sup_{P\in\mathcal{P}} E_P C(.)|h|^{\gamma}. \tag{3.30}$$

This is the case if $|h| \geq K(x, n_0, \alpha, \gamma) := \max(1, x^+, \frac{x^+ + n_0}{\alpha}, (\frac{x^+ + n_0}{\alpha})^{\frac{2}{1-\gamma}})$. Indeed if $|h| \geq 1$, then $\frac{x^+}{|h|^{\frac{1+\gamma}{2}}} - \alpha|h|^{\frac{(1-\gamma)}{2}} \leq x^+ - \alpha|h|^{\frac{(1-\gamma)}{2}}$, thus if $|h| \geq (\frac{x^+ + n_0}{\alpha})^{\frac{2}{1-\gamma}}$, it is clear that $\frac{x^+}{|h|^{\frac{1+\gamma}{2}}} - \alpha|h|^{\frac{(1-\gamma)}{2}} \leq -n_0$. The other conditions are immediate.
Note that $K(x, n_0, \alpha, \gamma)$ does not depend of P_h since α, n_0 and γ don't and thus $K(x, n_0, \alpha, \gamma)$ does not depend of h. Finally, combining (3.26) and (3.30) for a given x and for all $h \in \mathbb{R}^d$ such that $|h| > K(x, n_0, \alpha, \gamma)$ there exists P_h in $\mathcal{Q}$ such that

$$\psi(x, h) \leq \psi_{P_h}(x, h) = E_{P_h}\left(V(., x + h.Y(.))\right) \leq B|h|^{\gamma} - \frac{\alpha}{2}|h|^{\frac{1+\gamma}{2}}.$$

where $0 \leq B := \sup_{P \in \mathcal{P}} E_P(L) + 2 \sup_{P \in \mathcal{P}} E_P C < \infty$ and (3.24) is proven. □
We can now state our main result.

Theorem 3.5.13 *Assume that Assumptions 3.4.11, 3.5.1, 3.5.2, 3.5.4, 3.5.6 and 3.5.8 hold true. Then v is finite and there exists some optimal strategy $\widehat{h} \in \mathbb{R}^d$ such that*

$$v(x) = \inf_{P \in \mathcal{Q}} E_P(V(\cdot, x + \widehat{h}Y(\cdot))). \tag{3.31}$$

Moreover v is concave and non-decreasing.

Remark 3.5.14 As in Remark 3.5.3, v is continuous.

Proof. Let $x \in \mathbb{R}$ be fixed. Applying Lemma 3.5.12, we obtain that $v(x) \geq \psi(x, 0) > -\infty$, that ψ is concave, usc on $\mathbb{R} \times \mathbb{R}^d$ and from (3.24) for all $|h| > K(x, n_0, \alpha, \gamma)$ we have $\psi(x, h) \leq B|h|^{\gamma} - \frac{\alpha}{2}|h|^{\frac{1+\gamma}{2}}$. Since $B \geq 0$, $\alpha > 0$ and $0 < \gamma < 1$ (see Assumption 3.5.4) we obtain that $\lim_{\lambda \to \infty} \psi(x, \lambda h) = -\infty$. Therefore we can apply [62, Lemma 3.5 p113] and we obtain $\widehat{h}$ such that $\sup_{h \in \mathbb{R}^d} \psi(x, h) = \psi(x, \widehat{h}) < +\infty$.
As V is non-decreasing, it is clear that v is also non-decreasing. The proof of the concavity of v on $\mathbb{R}$ relies on a midpoint concavity argument and on Ostrowski Theorem, see [55, p12]. It is very similar to [112, Proposition 2] and [99, Lemma 3.5] and thus omitted (see also Lemma 4.5.17 in Chapter 4). □

3.6 The strong no-arbitrage condition: $sNA(\mathcal{Q}^T)$

In this section we introduce the strong no-arbitrage condition $sNA(\mathcal{Q}^T)$ which will be used in Chapter 4 in order to apply a theorem of multiple-priors expected utility maximisation for unbounded function defined on $(0, \infty)$ (namely Theorem 4.4.14) in a large range of setting as it is more suited to the integrability assumption required in this case.
A strategy $\phi \in \Phi$ will be a so called p-arbitrage if there exists some $P \in \mathcal{Q}^T$ such that $V_T^{0,\phi} \geq 0$ P-a.s. and $P(V_T^{0,\phi} > 0) > 0$.

3.6.1 Local characterisation and applications

Definition 3.6.1 We say that the $sNA(\mathcal{Q}^T)$ condition holds true if for all $\phi \in \Phi$ and $P \in \mathcal{Q}^T$ $V_T^{0,\phi} \geq 0$ P-a.s. implies that $V_T^{0,\phi} = 0$ P-a.s.

In other words, the $sNA(\mathcal{Q}^T)$ condition holds true if for all $P \in \mathcal{Q}^T$, the "classical" no-arbitrage condition in model P, $NA(P)$ holds true. Note as well that if $\mathcal{Q}^T = \{P\}$ then $sNA(\mathcal{Q}^T) = NA(\mathcal{Q}^T) = NA(P)$. It is clear that the $sNA(\mathcal{Q}^T)$ condition is stronger than the $NA(\mathcal{Q}^T)$ condition. Indeed if the $NA(\mathcal{Q}^T)$ condition fails, there exists some $\phi \in \Phi$ and $P \in \mathcal{Q}^T$ such that $V_T^{0,\phi} \geq 0$ $\mathcal{Q}^T$-q.s. and $P(V_T^{0,\phi} > 0) > 0$. Then, it is clear that ϕ is as well a p-arbitrage, hence the $sNA(\mathcal{Q}^T)$ condition also fails.
First we study the local characterisation of the $sNA(\mathcal{Q}^T)$ and introduce the following definition

Definition 3.6.2 For $\omega^t \in \Omega^t$ fixed, we say that the $sNA(\mathcal{Q}_{t+1}(\omega^t))$ condition holds true if for all $h \in \mathbb{R}^d$ and $P \in \mathcal{Q}_{t+1}(\omega^t)$, $h\Delta S_{t+1}(\omega^t, \cdot) \geq 0$ P-a.s $\Rightarrow h\Delta S_{t+1}(\omega^t, \cdot) = 0$ P-a.s and we define

$$\Omega^t_{sNA} := \{\omega^t \in \Omega^t,\ sNA(\mathcal{Q}_{t+1}(\omega^t)) \text{ holds true}\}.$$

Remark 3.6.3 It is clear that if for some $\omega^t \in \Omega^t$ fixed, the $sNA(\mathcal{Q}_{t+1}(\omega^t))$ condition holds true, then the $NA(\mathcal{Q}_{t+1}(\omega^t))$ condition (see Definition 3.4.2) holds true also. In particular, recalling Remark 3.4.8, in this case $0 \in D^{t+1}(\omega^t)$.

We prove the following proposition which is the pendant of local characterisation in Theorem 3.4.3.

Theorem 3.6.4 *Assume that Assumptions 3.2.1 and 3.2.2 hold true. Then the following statements are equivalent*
1. $sNA(\mathcal{Q}^T)$ *hold true.*
2. For all $0 \leq t \leq T-1$, $\Omega^t_{sNA} \in \mathcal{CA}(\Omega^t)$ *is a* $\mathcal{Q}^t$*-full measure set.*

Proof. We fixe some $1 \leq t \leq T-1$ and introduce the following random set $N_t : \Omega^t \twoheadrightarrow \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1})$ defined by

$$N_t(\omega^t) := \{(h, P) \in \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1}),\ P \in \mathcal{Q}_{t+1}(\omega^t),$$
$$P(h\Delta S_{t+1}(\omega^t, \cdot) \geq 0) = 1,\ P(h\Delta S_{t+1}(\omega^t, \cdot) \neq 0) > 0\}.$$

Then $\{N_t = \emptyset\} = \{\omega^t \in \Omega^t,\ sNA(\mathcal{Q}_{t+1}(\omega^t)) \text{ holds true}\}$. We prove first that $\text{Graph}(N_t) \in \mathcal{A}(\Omega^t \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1}))$. From Assumption 3.2.1, we have that $\{(\omega^t, h, P) \in \Omega^t \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1}), P \in \mathcal{Q}_{t+1}(\omega^t)\} \in \mathcal{A}(\Omega^t \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1}))$. Now using Lemma 4.8.5 in Chapter 4, it is clear that $\{(\omega^t, h, P) \in \Omega^t \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1}),\ P(h\Delta S_{t+1}(\omega^t, \cdot) \geq 0) = 1,\ P(h\Delta S_{t+1}(\omega^t, \cdot) \neq 0) > 0\} \in \mathcal{B}(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}\left(\mathfrak{P}(\Omega_{t+1})\right)$. The fact that

Graph$(N_t) \in \mathcal{A}(\Omega^t \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1}))$ follows from [13, Proposition 7.36]. We can thus apply the Jankov-von Neumann Projection Theorem (see [13, Proposition 7.49 p182]) and we obtain that $\{N_t \neq \emptyset\} \in \mathcal{A}(\Omega^t)$. Thus $\Omega^t_{sNA} \in \mathcal{CA}(\Omega^t)$. We also obtain some analytically-measurable and therefore $\mathcal{B}_c(\Omega^t)$-measurable function $\omega^t \in \{N_t \neq \emptyset\} \to (h^a(\omega^t), p^a(\cdot,\omega^t)) \in \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1})$ such that for all $\omega^t \in \{N_t \neq \emptyset\}$, $(h^a(\omega^t), p^a(\cdot,\omega^t)) \in N_t(\omega^t)$. We extend h^a and p^a on all Ω^t by setting for all $\omega^t \in \Omega^t \setminus \{N_t \neq \emptyset\}$, $h^a(\omega^t) = 0$ and $p^a(\cdot,\omega^t) = \widehat{p}(\cdot,\omega^t)$ where $\widehat{p}(\cdot,\omega^t)$ is a given $\mathcal{B}_c(\Omega^t)$-measurable selector of Graph$(\mathcal{Q}_{t+1})$. As $\{N_t \neq \emptyset\} \in \mathcal{B}_c(\Omega^t)$, it is clear that h^a and p^a remain $\mathcal{B}_c(\Omega^t)$-measurable.
We prove now by contradiction that the $sNA(\mathcal{Q}^T)$ implies that Ω^T_{sNA} is a $\mathcal{Q}^t$-full measure set. We assume that there exists some $1 \leq t \leq T-1$ and some $P^*_t \in \mathcal{Q}^t$ such that $P^*_t(\{N_t \neq \emptyset\}) > 0$. Using the selector (h^a, p^a) defined previously, we set $P^* = P^*_t \otimes p^a \otimes p_{t+2} \otimes \cdots \otimes p_T$ where p_s is a given $\mathcal{B}_c(\Omega^{s-1})$-measurable selector of Graph$(\mathcal{Q}_s)$ for $s = t+2, \cdots, T$. We also set $\phi_{t+1} = h^a$ and $\phi_s = 0$ for $s \neq t+1$. It is clear that $\phi = (\phi_s)_{1 \leq s \leq T} \in \Phi$. Recalling the definition of N_t, of $(h^a, p^a)$and that $P^*_t(\{N_t \neq \emptyset\}) > 0$ we obtain that

$$\begin{aligned} P^*(V_T^{0,\phi} \geq 0) &= P^*_{t+1}(h^a(\cdot)\Delta S_{t+1} \geq 0) \\ &= \int_{\Omega^t \setminus \{N_t \neq \emptyset\}} 1 P^*_t(d\omega^t) + \int_{\{N_t \neq \emptyset\}} p^a(h^a(\omega^t)\Delta S_{t+1}(\omega^t,\cdot) \geq 0, \omega^t) P^*_t(d\omega^t) = 1 \end{aligned}$$

while at the same time

$$P^*(V_T^{0,\phi} > 0) = P^*_{t+1}(h^a(\cdot)\Delta S_{t+1} > 0) = \int_{\{N \neq \emptyset\}} p^a(h^a(\omega^t)\Delta S_{t+1}(\omega^t,\cdot) > 0, \omega^t) P^*_t(d\omega^t) > 0$$

which means that the $sNA(\mathcal{Q}^T)$ is violated.
The proof of the reverse implication is based on Lemmata 3.7.1 and 3.7.2 presented in the appendix. We assume that $\Omega^T_{sNA} \in \mathcal{CA}(\Omega^t)$ is a $\mathcal{Q}^t$-full measure set. Let $\phi \in \Phi$ and $P \in \mathcal{Q}^T$ be such that $P(V_T^{0,\phi} \geq 0) = 1$. From Lemma 3.7.1 below, we get that $V_t^{0,\phi} \geq 0$ P_t-a.s. for all $1 \leq t \leq T$. Now, we prove by upward induction that $V_T^{0,\phi} = 0$ P-a.s. For $t = 1$, we have that $V_1^{0,\phi} = \phi_1 \Delta S_1 \geq 0$ P_1-a.s. and from Lemma 3.7.2 (see (3.37)), we obtain that $\phi_1 \Delta S_1 = 0$ P_1-a.s. Then, $V_2^{0,\phi} = \phi_2 \Delta S_2 \geq 0$ P_2-a.s. and again from Lemma 3.7.2 (see (3.37)) we get that $\phi_2 \Delta S_2 = 0$ P_2-a.s. Finally, by induction we obtain that $V_T^{0,\phi} = 0$ P-a.s. as claimed. □

Remark 3.6.5 Note that the proof of the direct implication is very similar to the proof of Theorem 3.4.3. (see [25, Theorem 4.5]) but the measurable-selection argument is somehow simplified by the fact that $\{N_t \neq \emptyset\} \in \mathcal{A}(\Omega^t)$ which avoids the use of Castaing's representation.

From a technical point of view, the fact that $\Omega^t_{sNA} \in \mathcal{CA}(\Omega^t)$ is sometimes useful. We provide two illustrations.

Proposition 3.6.6 *Fix some* $1 \leq t \leq T-1$ *and let* $F : \Omega^t \to \mathbb{R} \cup \{\pm\infty\} \in \mathcal{LSA}(\Omega^t)$, $A \in \mathcal{A}(\Omega^t)$ *and* $C \in \mathcal{CA}(\Omega^t)$ *be fixed. Then* $F_A := +\infty 1_{\Omega^t \backslash A} + 1_A F \in \mathcal{LSA}(\Omega^t)$ *and* $F_C := -\infty 1_{\Omega^t \backslash C} + 1_C F \in \mathcal{LSA}(\Omega^t)$

Proof. The proof is straightforward. Lets $c \in \mathbb{R}$, then $\{F_A \leq c\} = \{F \leq c\} \cap A \in \mathcal{A}(\Omega^t)$ and similarly $\{F_C \leq c\} = (\Omega^t \backslash C) \cup (\{F \leq c\} \cap C) = (\Omega^t \backslash C^t) \cup \{F \leq c\} \in \mathcal{A}(\Omega^t)$. □

It is easy to obtain the equivalent result if $F \in \mathcal{USA}(\Omega^t)$. Note that for some $B \in \mathcal{B}_c(\Omega^t)$ then whatever value we put outside of B is it not possible to preserve the lower (or upper) semi-analyticity of F. This kind of problematic will arise for instance in Chapter 4 where the full-measure sets induced by the $NA(\mathcal{Q}^T)$ condition are only in $\mathcal{B}_c(\Omega^t)$.
The next lemma could be used in the dynamic programming resolution of maximizing worst-case expected utility under the $sNA(\mathcal{Q}^T)$ condition. Roughly speaking assume that you have found at time t a $\mathcal{Q}^t$-full-measure C^t such that a given property $H(\omega^t)$ is true if you take $\omega^t \in C^t$. Now, you want to find a $\mathcal{Q}^{t-1}$-full-measure set C^{t-1} at time $t-1$ such that if you take $\omega^{t-1} \in C^{t-1}$ the property $H(\omega^{t-1}, \cdot)$ is valid on a $\mathcal{Q}_t(\omega^{t-1})$-full measure set and this is the case in particular if the section of C^t along ω^{t-1} is a $\mathcal{Q}_t(\omega^{t-1})$-full measure set.
In the next lemma, we denote by $(C^t)_{\omega^{t-1}} := \{\omega_t \in \Omega_t, (\omega^{t-1}, \omega_t) \in C^t\}$ the section of C^t along ω^{t-1}.

Lemma 3.6.7 *Fix some* $1 \leq t \leq T-1$ *and let Assumption 3.2.1 holds true. Let* $C^t \in \mathcal{CA}(\Omega^t)$ *such that* $P(C^t) = 1$ *for all* $P \in \mathcal{Q}^t$ *and set*

$$C^{t-1} := \left\{\omega^{t-1} \in \Omega^{t-1},\ P\left((C^t)_{\omega^{t-1}}\right) = 1,\ \forall\, P \in \mathcal{Q}_t(\omega^{t-1})\right\}. \tag{3.32}$$

Then $C^{t-1} \in \mathcal{CA}(\Omega^{t-1})$ *and* $P(C^{t-1}) = 1$ *for all* $P \in \mathcal{Q}^{t-1}$.

Proof. As $C^t \in \mathcal{CA}(\Omega^t)$, $F_C : \omega^t \in \Omega^t \to 1_{C^t}(\omega^t) \in \mathcal{LSA}(\Omega^t)$. Using [13, Proposition 7.48 p180], we get that $(\omega^{t-1}, P) \in \Omega^{t-1} \times \mathfrak{P}(\Omega_t) \to \int_{\Omega_t} 1_{C^t}(\omega^{t-1}, \omega_t) P(d\omega_t) \in \mathcal{LSA}(\Omega^{t-1} \times \mathfrak{P}(\Omega_t))$. Since $\mathrm{Graph}(\mathcal{Q}_t) \in \mathcal{A}(\Omega^{t-1} \times \mathfrak{P}(\Omega_t))$ (see Assumption 3.2.1), applying [13, Proposition 7.47 p179] we get that

$$\begin{aligned} F^*_{C^t}(\omega^{t-1}) &:= \inf_{P \in \mathcal{Q}_t(\omega^{t-1})} \int_{\Omega_t} 1_{C^t}(\omega^{t-1}, \omega_t) P(d\omega_t) \\ &= \inf_{P \in \mathcal{Q}_t(\omega^{t-1})} P\left((C^t)_{\omega^{t-1}}, \omega^{t-1}\right) \in \mathcal{LSA}(\Omega^{t-1}). \end{aligned}$$

Then using the definition of C^{t-1} in (3.32)

$$C^{t-1} = \left\{\omega^{t-1} \in \Omega^{t-1},\ F^*_{C^t}(\omega^{t-1}) \geq 1\right\} \in \mathcal{CA}(\Omega^{t-1}).$$

We prove now that $P(C^{t-1}) = 1$ for all $P \in \mathcal{Q}^{t-1}$. Assume that this is not the case. Then, there exists some $\widetilde{P} \in \mathcal{Q}^{t-1}$, such that $\widetilde{P}(C^{t-1}) < 1$. For $n \geq 1$, Set $B_n^{t-1} := \{\omega^{t-1} \in \Omega^{t-1},\ F^*_{C^t}(\omega^{t-1}) \leq 1 - \frac{1}{n}\}$. Then $\Omega^{t-1} \backslash C^{t-1} = \cup_{n\geq 1} B_n^{t-1}$ Hence, there exist also some $n \geq 1$ such that $\widetilde{P}(B_n^{t-1}) > 0$. Now, from [13, Proposition 7.50 p184], there exists some analytically-measurable (and therefore $\mathcal{B}_c(\Omega^{t-1})$-measurable) $p_n : \omega^{t-1} \in \Omega^{t-1} \to \mathfrak{P}(\Omega_t)$, such that for all $\omega^{t-1} \in C^{t-1}$, $p_n(\cdot, \omega^{t-1}) \in \mathcal{Q}_t(\omega^{t-1})$ and

$$p_n\left(\left(C^t\right)_{\omega^{t-1}}, \omega^{t-1}\right) \leq F^*_{C^t}(\omega^{t-1}) + \frac{1}{2n}$$

(since $F^*_{C^t}(\cdot) \geq 0$). Setting $\widetilde{P}_n := \widetilde{P} \otimes p_n \in \mathcal{Q}^t$ (see (3.3)) we get that

$$\begin{aligned}
\widetilde{P}_n(C^t) &= \int_{\Omega^{t-1}} p_n\left(\left(C^t\right)_{\omega^{t-1}}, \omega^{t-1}\right) \widetilde{P}(d\omega^{t-1}) \\
&= \int_{B_n^{t-1}} p_n\left(\left(C^t\right)_{\omega^{t-1}}, \omega^{t-1}\right) \widetilde{P}(d\omega^{t-1}) + \int_{\Omega^{t-1}\backslash B_n^{t-1}} p_n\left(\left(C^t\right)_{\omega^{t-1}}, \omega^{t-1}\right) \widetilde{P}(d\omega^{t-1}) \\
&\leq \widetilde{P}(B_n^{t-1})(1 - \frac{1}{2n}) + 1 - \widetilde{P}(B_n^{t-1}) < 1,
\end{aligned}$$

a contradiction. Therefore $P(C^{t-1}) = 1$ for all $P \in \mathcal{Q}^{t-1}$. □

Remark 3.6.8 Note that the previous argument do no apply if $A^t \in \mathcal{A}(\Omega^t)$. Indeed using the same notation as in the proof of Lemma 3.6.7, one cannot prove that $F^*_{A^t} \in \mathcal{LSA}(\Omega^{t-1})$. However, using again [13, Propositions 7.47, 7.48 p179, p180], we get that

$$\begin{aligned}
F^+_{A^t}(\omega^{t-1}) &:= \sup_{P \in \mathcal{Q}_t(\omega^{t-1})} \int_{\Omega_t} 1_{A^t}(\omega^{t-1}, \omega_t) P(d\omega_t) \\
&= \sup_{P \in \mathcal{Q}_t(\omega^{t-1})} P\left(\left(A^t\right)_{\omega^{t-1}}, \omega^{t-1}\right) \in \mathcal{USA}(\Omega^{t-1}).
\end{aligned}$$

And it follows that

$$\begin{aligned}
A_2^{t-1} &:= \left\{\omega^{t-1} \in \Omega^{t-1},\ \exists\, P \in \mathcal{Q}_t(\omega^{t-1}), P\left(\left(A^t\right)_{\omega^{t-1}}\right) \geq 1\right\} \\
&= \left\{\omega^{t-1} \in \Omega^{t-1},\ F^+_{A^t}(\omega^{t-1}) \geq 1\right\} \in \mathcal{A}(\Omega^{t-1}).
\end{aligned}$$

But this set is not the one we are interested in.

Finally, if $A^t \in \mathcal{B}_c(\Omega^t)$ the argument simply does not apply: indeed one can show that $(\omega^{t-1}, P) \to \int_{\Omega_t} 1_{A^t}(\omega^{t-1}, \omega_t) P(d\omega_t)$ is $\mathcal{B}_c(\Omega^{t-1} \times \mathfrak{P}(\Omega_t))$-measurable (see [13, Proposition 7.46 p177]) but as the projection of an universally-measurable set is not universally-measurable in general one cannot obtain any measurability of the infimum or supremum over P. This problem will arise again in Chapter 4.

Remark 3.6.9 As already mentioned in the introduction in [7, Theorem 3.3] an other (stronger) alternative no-arbitrage is introduced where the local-no arbitrage $NA(\mathcal{Q}^{t+1}(\omega^t))$ is assumed to be true for all $\omega^t \in \Omega^t$. This is an other way to solve the measurability issues. Note that if $NA(\mathcal{Q}^T)$ hold true and that we modify S_t outside of the full-measure set Ω^t_{NA} (setting it to 0 for instance), we obtain that the local-no arbitrage is valid for all ω^t but we loose the Borel-measurability of S_t (Assumption 3.2.2) which is essential to obtain some other measurability results such as Lemma 3.3.2.

3.6.2 Quantitative characterisation of the $sNA(\mathcal{Q}^T)$

We propose now the following quantitative characterisation of the $sNA(\mathcal{Q}^T)$ that will be used in Chapter 4. First, as in Definition 2.3.2 in Chapter 2 and also Definition 3.3.1, we introduce for all $P = Q_1 \otimes q_2 \otimes \cdots \otimes q_T \in \mathcal{Q}^T$ (see (3.3)) and for all $1 \leq t \leq T-1$

$$\widetilde{D}^{t+1}_P(\omega^t) := \bigcap \left\{ A \subset \mathbb{R}^d, \text{ closed}, \; q_{t+1}\left(\Delta S_{t+1}(\omega^t, .) \in A, \omega^t\right) = 1 \right\} \tag{3.33}$$

$$D^{t+1}_P(\omega^t) := \text{Aff}\left(\widetilde{D}^{t+1}_P(\omega^t)\right). \tag{3.34}$$

The case $t = 0$ is obtained by replacing $q_{t+1}(\cdot, \omega^t)$ by $Q_1(\cdot)$. We have the following result which is the pendant of Lemma 2.3.3 in Chapter 2 in our setting.

Lemma 3.6.10 *Let Assumption 3.2.1 and 3.2.2 hold true and* $0 \leq t \leq T-1$ *be fixed. Then* D^{t+1}_P *is a non-empty, closed valued and* $\mathcal{B}_c(\Omega^t)$*-measurable random set and Graph*$(D^{t+1}_P) \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$.

Proof. The proof is a simple adaptation of Lemma 2.3.3 in Chapter 2 (as $\mathcal{B}_c(\Omega^s)$ is not a product sigma-algebra) and is provided for sake of completeness. It is clear that for all $\omega^t \in \Omega^t$, $\widetilde{D}^{t+1}_P(\omega^t)$ is a non-empty and closed subset of $\mathbb{R}^d$. We now show that $\widetilde{D}^{t+1}_P$ is $\mathcal{B}_c(\Omega^t)$-measurable. Since $P \in \mathcal{Q}^T$ and Assumption 3.2.1 holds true, $q_{s+1} \in \mathcal{S}K_{s+1}$ for all $1 \leq s \leq T-1$ (see (3.3)), in other words the q_{s+1} are $\mathcal{B}_c(\Omega^s)$-measurable stochastic kernels on Ω_{s+1} given Ω^s. Let O be a fixed open set in $\mathbb{R}^d$. Recalling Assumption 3.2.2, $\omega^{t+1} \to 1_{\{\Delta S_{t+1}(\cdot) \in O\}}(\omega^{t+1})$ is $\mathcal{B}(\Omega^{t+1})$-measurable (and thus also $\mathcal{B}_c(\Omega^t)$-measurable) and as $q_{t+1} \in \mathcal{S}K_{t+1}$, using [13, Proposition 7.46 p177], we get that $\omega^t \in \Omega^t \to q_{t+1}\left(\Delta S_{t+1}(\omega^t, .) \in O | \omega^t\right)$ is $\mathcal{B}_c(\Omega^t)$-measurable. By definition of $\widetilde{D}^{t+1}_P(\omega^t)$ we get that

$$\{\omega^t \in \Omega^t, \; \widetilde{D}^{t+1}_P(\omega^t) \cap O \neq \emptyset\} = \{\omega^t \in \Omega^t, \; q_{t+1}\left(\Delta S_{t+1}(\omega^t, .) \in O | \omega^t\right) > 0\} \in \mathcal{B}_c(\Omega^t).$$

It is immediate to verify that D^{t+1}_P is a non-empty, closed-valued and applying [116, Exercise 14.12] we obtain that D^{t+1}_P is $\mathcal{B}_c(\Omega^t)$-measurable. Finally, from [116, Theorem 14.8] we get that $Graph(D^{t+1}_P) \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$. □

Proposition 3.6.11 *Assume that the $sNA(\mathcal{Q}^T)$ condition and Assumptions 3.2.1 and 3.2.2 hold true and let $0 \leq t \leq T-1$. Fix some $P = Q_1 \otimes q_2 \otimes \cdots \otimes q_T \in \mathcal{Q}^T$. Then there exists $\Omega_P^t \in \mathcal{B}(\Omega^t)$ with $P_t(\Omega_P^t) = 1$ such that for all $\omega^t \in \Omega_P^t$, $D_P^{t+1}(\omega^t)$ is a vector subspace and there exists $\alpha_t^P(\omega^t) \in (0,1]$ such that for all $h \in D_P^{t+1}(\omega^t)$, $h \neq 0$*

$$q_{t+1}\big(h\Delta S_{t+1}(\omega^t, \cdot) \leq -\alpha_t^P(\omega^t)|h|, \omega^t\big) \geq \alpha_t^P(\omega^t). \tag{3.35}$$

Furthermore $\omega^t \to \alpha_t^P(\omega^t)$ is $\mathcal{B}(\Omega^t)$-measurable.

Proof. We want to apply Propositions 2.3.6 and 2.3.7 of Chapter 2. Since $P \in \mathcal{Q}^T$ and Assumption 3.2.1 holds true, $q_{s+1} \in \mathcal{S}K_{s+1}$ for all $1 \leq s \leq T-1$ (see (3.3)), in other words the q_{s+1} are $\mathcal{B}_c(\Omega^s)$-measurable stochastic kernels on Ω_{s+1} given Ω^s. However, as $\mathcal{B}_c(\Omega^s)$ is not a product sigma-algebra, we need to be a bit cautious. For all $1 \leq s \leq T-1$, we apply first [13, Lemma 7.28 (c) p174] and we obtain some $\mathcal{B}(\Omega^s)$-measurable stochastic kernels $\overline{q}_{s+1}$ (see [13, Definition 7.12 p134]), and some $\widehat{\Omega}^s \subset \mathcal{B}_c(\Omega^s)$ such that $P_s(\widehat{\Omega}^s) = 1$ and $\overline{q}_{s+1}(\cdot, \omega^s) = q_{s+1}(\cdot, \omega^s)$ for all $\omega^s \in \widehat{\Omega}^s$. We set $\overline{P} = Q_1 \otimes \overline{q}_2 \otimes \cdots \otimes \overline{q}_T$. Then, it is easy to prove by induction that $P_s(A_s) = \overline{P}_s(A_s)$ for all $A_s \in \mathcal{B}(\Omega^s)$, $1 \leq s \leq T$. Indeed, for $s = 1$ there is nothing to prove. Assume that until some $1 \leq s < T$, $P_s(A_s) = \overline{P}_s(A_s)$ for all $A_s \in \mathcal{B}(\Omega^s)$. Fix some $A_{s+1} \in \mathcal{B}(\Omega^{s+1})$. Using (3.2) we find that

$$\begin{aligned}
P_{s+1}(A_{s+1}) &= \int_{\Omega^s} \int_{\Omega_{s+1}} 1_{A_{s+1}}(\omega^s, \omega_{s+1}) q_{s+1}(d\omega_{s+1}, \omega^s) P_s(d\omega^s) \\
&= \int_{\overline{\Omega}^s} \int_{\Omega_{s+1}} 1_{A_{s+1}}(\omega^s, \omega_{s+1}) \overline{q}_{s+1}(d\omega_{s+1}, \omega^s) \overline{P}_s(d\omega^s) = \overline{P}_{s+1}(A_{s+1}),
\end{aligned}$$

where we have used the induction hypothesis for the second equality. Since this is true for all $A_{s+1} \in \mathcal{B}(\Omega^{s+1})$, the induction is complete.
So, as $NA(P)$ holds true, we get that $NA(\overline{P})$ holds also true. Since $\widehat{\Omega}^t \in \mathcal{B}_c(\Omega^t)$, there exists some $\overline{\Omega}^t \in \mathcal{B}(\Omega^t)$ as well as a $\overline{P}_t$-null set N^t such that $\widehat{\Omega}^t = \overline{\Omega}^t \cup N^t$. We have also that for all $\omega^t \in \overline{\Omega}^t$, $D_P^{t+1}(\omega^t) = D_{\overline{P}}^{t+1}(\omega^t)$. We can now apply Propositions 2.3.6 and 2.3.7 of Chapter 2 for $\overline{P}$ and $\mathcal{F}_t = \mathcal{B}(\Omega_1) \otimes \cdots \otimes \mathcal{B}(\Omega_t) = \mathcal{B}(\Omega^t)$: $D_{\overline{P}}^{t+1}(\omega^t)$ is a vector subspace and there exists some $\Omega_{\overline{P}}^t \in \mathcal{B}(\Omega^t)$ with $\overline{P}_t(\Omega_{\overline{P}}^t) = 1$ and some $\mathcal{B}(\Omega^t)$-measurable $\omega^t \to \alpha_t^{\overline{P}}(\omega^t)$ such that (3.35) holds true for all $\omega^t \in \Omega_{\overline{P}}^t$ and $h \in D_{\overline{P}}^{t+1}(\omega^t)$. We finally set $\Omega_P^t = \Omega_{\overline{P}}^t \cap \overline{\Omega}^t$, $\alpha_t^P = \alpha_t^{\overline{P}}$. It is easy to verify that $P_t(\Omega_P^t) = 1$, that $D_P^{t+1}(\omega^t)$ is a vector subspace for $\omega^t \in \Omega_P^t$, that (3.35) holds true and that α_t^P is $\mathcal{B}(\Omega^t)$-measurable. □

3.6.3 First Fundamental Theorem for the $sNA(\mathcal{Q}^T)$

The next proposition can be seen as the pendant of [25, First Fundamental Theorem] for the $sNA(\mathcal{Q}^T)$ condition. We propose it in a slightly more general setting and in particular we do not use the technical assumptions related to measurability issues such as Assumptions 3.2.1 and 3.2.2. Note that it is a straightforward extension of the FTAP in a mono-prior setting.

Proposition 3.6.12 *We consider a general measurable space* $(\Omega, \mathcal{F}_T)$ *and a filtration* $\mathcal{F} := (\mathcal{F}_t)_{1\leq t\leq T}$. *We denote by* $\mathfrak{P}$ *the set of all probability measures defined on* $(\Omega, \mathcal{F}_T)$. *Let* $S := \{S_t,\ 0 \leq t \leq T\}$ *be a* $(\mathcal{F}_t)_{1\leq t\leq T}$*-adapted* d*-dimensional process representing the price process of the* d *traded assets while the trading strategies are represented by* d*-dimensional processes* $\phi := \{\phi_t, 1 \leq t \leq T\}$ *where for all* $1 \leq t \leq T$, ϕ_t *is* $\mathcal{F}_{t-1}$*-measurable (the set of all such trading strategies is still denoted by* Φ*). Let also* $\mathcal{Q}$ *be a subset of* $\mathfrak{P}$. *For any* $P \in \mathcal{Q}$ *we define*

$$\mathcal{R}(P) := \{R \in \mathfrak{P},\ R \sim P \text{ such that } S \text{ is a martingale-measure under } R\}.$$

Then the following statements are equivalent

i) For all $P \in \mathcal{Q}$, $\phi \in \Phi$, $V_T^{0,\phi} \geq 0$ P*-a.s. implies that* $V_T^{0,\phi} = 0$ P*-a.s. (*$i.e$ *Assumption 3.6.1 holds true in the setting of this proposition).*

ii) For all $P \in \mathcal{Q}$, $\mathcal{R}(P) \neq \emptyset$.

Proof. This is a simple application of [62, Theorem 5.17] for all $P \in \mathcal{Q}$. □

So, under the $sNA(\mathcal{Q}^T)$ condition, the set of martingale measures obtained is

$$\bigcup_{P\in\mathcal{Q}} \mathcal{R}(P) = \{R \in \mathfrak{P},\ \exists\, P \in \mathcal{Q},\ R \sim P \text{ and such that } S \text{ is a martingale-measure under } R\}\}$$

and is a subset of the the one obtained in [25, First Fundamental Theorem] which can be written as

$$\mathcal{R}^T := \{R \in \mathfrak{P},\ \exists\, P_1, P_2 \in \mathcal{Q},\ P_1 \ll R \ll P_2\ S \text{ is a martingale-measure under } R\}.$$

Note also that in a continuous time framework, [14] the no-arbitrage condition introduced (namely the $NA_1(\mathcal{P})$) holds true if and only if for every $P \in \mathcal{Q}$, there exists a local martingale measure Q such that Q and P are equivalent prior to ξ (where ξ is a random time, corresponding to the time where the process jump into a cemetery state that is invisible from all model $P \in \mathcal{P}$).

Remark 3.6.13 In the spirit of [47], we could also introduce the notion of weak-arbitrage (or model-dependant arbitrage). There is a weak-arbitrage if for all prior (or model) $P \in \mathcal{Q}$, we can find a strategy ϕ_P that is an arbitrage for this specific prior (or model). In other words, we say that the $NwA(\mathcal{Q}^T)$ condition holds true if there exists some $P^* \in \mathcal{Q}^T$ such that for all $\phi \in \Phi$ $V_T^{0,\phi} \geq 0$ P^*-a.s. implies that $V_T^{0,\phi} = 0$ P^*-a.s.. It is clear that the $NwA(\mathcal{Q}^T)$ condition is weaker than the $sNA(\mathcal{Q}^T)$ condition. However the local characterisation is fundamentally different as we do not obtain a $\mathcal{Q}^T$-full measure set.

3.7 Appendix

The following two lemmata were used in the proof of Theorem 3.6.4. Recall that for $P = Q_1 \otimes q_2 \otimes \cdots \otimes q_T \in \mathcal{Q}^T$ we denote by $P_t := Q_1 \otimes q_2 \otimes \cdots \otimes q_t$ for any $1 \leq t \leq T$.

Lemma 3.7.1 *Assume that Assumptions 3.2.1, 3.2.2 and that for all* $0 \leq t \leq T-1$, Ω^t_{sNA} *(see Definition 3.6.2) is a* $\mathcal{Q}^t$*-full measure set. Let* $x \geq 0$, $P \in \mathcal{Q}^T$ *be fixed and let* $\phi \in \Phi$ *such that* $V_T^{x,\phi} \geq 0$ P*-a.s. Then for all* $1 \leq t \leq T$, $V_t^{x,\phi} \geq 0$ P*-a.s.*

Proof. We prove the lemma by backward induction. This is obviously true for $t = T$. We fix some $t \leq T-1$ and assume that $V_{t+1}^{x,\phi} \geq 0$ P_{t+1}-a.s. We set $A_t := \{\omega^t \in \Omega^t,\ V_t^{x,\phi}(\omega^t) < 0\} \in \mathcal{B}_c(\Omega^t)$ and prove that $P_t(A_t) = 0$. To do so, we proceed by contradiction and assume that $P_t(A_t) > 0$. We set for all $\omega^t \in \Omega^t$, $\phi^A_{t+1}(\omega^t) := 1_{A_t}(\omega^t)\phi_{t+1}(\omega^t)$. It is clear that ϕ^A_{t+1} is $\mathcal{B}_c(\Omega^t)$-measurable. We prove that $\phi^A_{t+1}\Delta S_{t+1} \geq 0$ P_{t+1}-a.s. Let $A_{t+1} := \{\omega^{t+1} \in \Omega^{t+1},\ V_{t+1}^{x,\phi}(\omega^{t+1}) \geq 0\} \in \mathcal{B}_c(\Omega^{t+1})$. By induction hypothesis A_{t+1} is a P_{t+1}-full measure set. We fix some $\omega^{t+1} = (\omega^t, \omega_{t+1}) \in A_{t+1}$. If $\omega^t \notin A_t$, then $\phi^A_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}) = 0$ and if $\omega^t \in A_t$, we have that $\phi^A_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}) = \phi_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}) = V_{t+1}^{x,\phi}(\omega^t, \omega_{t+1}) - V_t^{x,\phi}(\omega^t) > 0$. Thus we can use Lemma 3.7.2 (see (3.37)) and we get that

$$\phi^A_{t+1}\Delta S_{t+1} = 0 \ P_{t+1}\text{-a.s.} \tag{3.36}$$

As $A_{t+1} \in \mathcal{B}_c(\Omega^{t+1})$ is a P_{t+1}-full measure set, using Lemma 3.7.3, we get that

$$\widetilde{\Omega}^t := \left\{\omega^t \in \Omega^t,\ q_{t+1}\left(\left(A_{t+1}\right)_{\omega^t}, \omega^t\right) = 1\right\} \in \mathcal{B}_c(\Omega^t)^3$$

is a P_t-full measure set. Thus, we have that

$$P_{t+1}(\phi^A_{t+1}\Delta S_{t+1} > 0) \geq \int_{A_t \cap \widetilde{\Omega}^t} q_{t+1}(\phi^A_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t, \cdot) > 0, \omega^t) P_t(d\omega^t) > 0$$

where we have used the fact that $P_t(A_t \cap \widetilde{\Omega}^t) = P_t(A_t) > 0$ and that for all $\omega^t \in A_t \cap \widetilde{\Omega}^t$, $q_{t+1}(\phi^A_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t, \cdot) > 0, \omega^t) > 0$. Indeed $q_{t+1}\left(\left(A_{t+1}\right)_{\omega^t}, \omega^t\right) = 1$ and for all $\omega_{t+1} \in \left(A_{t+1}\right)_{\omega^t}$, $\phi^A_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}) \geq -V_t^{x,\phi}(\omega^t) > 0$ since $\omega^t \in A_t$ and $\omega^{t+1} \in A_{t+1}$. But this contradicts (3.36). Thus we must have that $P_t(A_t) = 0$, $i.e$ $V_t^{x,\phi} \geq 0$ P_t-q.s. □

Lemma 3.7.2 *Assume that Assumptions 3.2.1, 3.2.2 hold true and that for all* $0 \leq t \leq T-1$, Ω^t_{sNA} *(see Definition 3.6.2) is a* $\mathcal{Q}^t$*-full measure set. Then for all* $1 \leq t \leq T$, *all* $P \in \mathcal{Q}^T$ *and any* $\mathbb{R}^d$*-valued* $\mathcal{B}_c(\Omega^{t-1})$*-measurable* ϕ_t,

$$\phi_t \Delta S_t \geq 0 \ P_t\text{-a.s.} \ \Rightarrow \phi_t \Delta S_t = 0 \ P_t\text{-a.s.} \tag{3.37}$$

[3] Recall that $\left(A_{t+1}\right)_{\omega^t} := \{\omega_{t+1} \in \Omega_{t+1}, (\omega^t, \omega_{t+1}) \in A_{t+1}\}$ is the section of A_{t+1} along ω^t.

Proof. We fix some $P \in \mathcal{Q}^T$, some $1 \leq t \leq T$ and some $\mathcal{B}_c(\Omega^{t-1})$-measurable ϕ_t such that $\phi_t \Delta S_t \geq 0$ P_t-a.s. For $\omega^{t-1} \in \Omega^{t-1}$ fixed, we denote by $\phi_t^{\perp}(\omega^{t-1})$ the orthogonal projection of $\phi_t(\omega^{t-1})$ on $D_P^t(\omega^{t-1})$ (a convex and closed set of $\mathbb{R}^d$, see (3.34)). As Assumptions 3.2.1 and 3.2.2 hold true, from Lemma 3.6.10 we get that the random set D_P^t is $\mathcal{B}_c(\Omega^{t-1})$-measurable and Graph$(D_P^t) \in \mathcal{B}_c(\Omega^{t-1}) \otimes \mathcal{B}(\mathbb{R}^d)$. Applying for example [116, Exercice 14.17 p655], we obtain that $\omega^{t-1} \in \Omega^{t-1} \to \phi_t^{\perp}(\omega^{t-1})$ is $\mathcal{B}_c(\Omega^{t-1})$-measurable. We reset $\Omega_{sNA}^{t-1} := \Omega_{sNA}^{t-1} \cap \Omega_P^t$ where Ω_P^t was introduced in Proposition 3.6.11. In particular it follows that for all $\omega^{t-1} \in \Omega_{sNA}^{t-1}$, $D_P^t(\omega^{t-1})$ is a vector space, thus we have that

$$\phi_t(\omega^{t-1})\Delta S_t(\omega^{t-1}, \cdot) = \phi_t^{\perp}(\omega^{t-1})\Delta S_t(\omega^{t-1}, \cdot) \; q_t(\omega^{t-1})\text{-a.s.} \tag{3.38}$$

Indeed, for $\omega^{t-1} \in \Omega_{sNA}^{t-1}$, we have that $\{\Delta S_t(\omega^{t-1}, \cdot) \in D_P^t(\omega^{t-1})\} \subset \{\phi_t(\omega^{t-1})\Delta S_t(\omega^{t-1}, \cdot) = \phi_t^{\perp}(\omega^{t-1})\Delta S_t(\omega^{t-1}, \cdot)\}$ and by definition of $D_P^t(\omega^{t-1})$, it follows that $q_t(\{\phi_t(\omega^{t-1})\Delta S_t(\omega^{t-1}, \cdot) = \phi_t^{\perp}(\omega^{t-1})\Delta S_t(\omega^{t-1}, \cdot)\} = 1$. We set now

$$E_{t-1} := \{\omega^{t-1} \in \Omega_{sNA}^{t-1}, \; \phi_t^{\perp}(\omega^{t-1}) \neq 0\} \in \mathcal{B}_c(\Omega^{t-1}).$$

Note that from Proposition 3.6.11, $q_t\left(\phi_t^{\perp}(\omega^{t-1})\Delta S_t(\omega^{t-1}, \cdot) \neq 0, \omega^{t-1}\right) > 0$ for $\omega^{t-1} \in E_{t-1}$.

Assume for a moment that we have proved that $P_{t-1}(E_{t-1}) = 0$. Then using Fubini's Theorem (see [13, Proposition 7.45 p175]), we obtain that

$$\begin{aligned} P_t(\phi_t \Delta S_t = 0) &= \int_{\Omega^{t-1}} q_t\left(\phi_t(\omega^{t-1})\Delta S_t(\omega^{t-1}, \cdot) = 0, \omega^{t-1}\right) P_{t-1}(d\omega^{t-1}) \\ &= \int_{\Omega^{t-1} \setminus E_{t-1}} q_t\left(\phi_t^{\perp}(\omega^{t-1})\Delta S_t(\omega^{t-1}, \cdot) = 0, \omega^{t-1}\right) P_{t-1}(d\omega^{t-1}) = 1 \end{aligned} \tag{3.39}$$

where we have used that $q_t(\cdot, \omega^{t-1}) \in \mathcal{Q}_t(\omega^{t-1})$, $P_{t-1}(E_{t-1}) = 0$, (3.38) and $q_t\left(\phi_t^{\perp}(\omega^{T-1})\Delta S_t(\omega^{T-1}, \cdot) = 0, \omega^{t-1}\right) = 1$ for all $\omega^{t-1} \in \Omega^{t-1} \setminus E_{t-1}$. Thus, we get that $\phi_t \Delta S_t = 0$ P_t-a.s. and (3.37) is proved.

We establish now that $P_{t-1}(E_{t-1}) = 0$. If $P_{t-1}(E_{t-1}) > 0$ using the same argument as in (3.39), we obtain that

$$\begin{aligned} P_t\left(\phi_t(\cdot)\Delta S_t(\cdot) \neq 0\right) &= P_t\left(\phi_t^{\perp}(\cdot)\Delta S_t(\cdot) \neq 0\right) \\ &\geq \int_{E_{t-1}} q_t\left(\phi_t^{\perp}(\omega^{t-1})\Delta S_t(\omega^{t-1}, \cdot) \neq 0, \omega^{t-1}\right) P_{t-1}(d\omega^{t-1}) > 0 \end{aligned}$$

where we have used the fact that for $\omega^{t-1} \in E_{t-1}$ $q_t\left(\phi_t^{\perp}(\omega^{t-1})\Delta S_t(\omega^{t-1}, \cdot) \neq 0\right) > 0$: thus $sNA(\mathcal{Q}^t(\omega^{t-1}))$ is violated. Thus $P_{t-1}(E_{t-1}) = 0$ as claimed. □

The following lemma was used in the proof of Lemma 3.7.1. It is a slight modification of Lemma 2.8.8 in Chapter 2 to deal with the fact that $\mathcal{B}_c(\Omega^{t+1})$ is not a product sigma-algebra.

Lemma 3.7.3 *Fix some* $0 \leq t \leq T-1$. *For all* $A \in \mathcal{B}_c(\Omega^{t+1})$ *and* $\omega^t \in \Omega^t$ *we denote by* $(A)_{\omega^t} := \{\omega_{t+1} \in \Omega_{t+1},\ (\omega^t, \omega_{t+1}) \in A\}$ *the section of* A *along* ω^t. *Let* $P_t \in \mathfrak{P}(\Omega^t)$ *and* $q_{t+1} \in \mathcal{S}K_{t+1}$. *We set* $P_{t+1} = P_t \otimes q_{t+1} \in \mathfrak{P}(\Omega^{t+1})$. *Let* $\widetilde{\Omega}^{t+1} \in \mathcal{B}_c(\Omega^{t+1})$ *such that* $P_{t+1}(\widetilde{\Omega}^{t+1}) = 1$. *Then*

$$\widetilde{\Omega}^t := \left\{\omega^t \in \Omega^t,\ q_{t+1}\left(\left(\widetilde{\Omega}^{t+1}\right)_{\omega^t}, \omega^t\right) = 1\right\} \in \mathcal{B}_c(\Omega^t)$$

and is a P_t*-full measure set*

Proof. We prove first that $\omega^t \in \Omega^t \to q_{t+1}\left(\left(\widetilde{\Omega}^{t+1}\right)_{\omega^t}, \omega^t\right)$ is $\mathcal{B}_c(\Omega^t)$-measurable. The fact that $\widetilde{\Omega}^t \in \mathcal{B}_c(\Omega^t)$ will follow immediately. As $(\omega^t, \omega_{t+1}) \to 1_{\widetilde{\Omega}^{t+1}}(\omega^t, \omega_{t+1})$ is $\mathcal{B}_c(\Omega^{t+1})$ measurable and q_{t+1} is a $\mathcal{B}_c(\Omega^t)$-measurable stochastic kernel, we can apply [13, Proposition 7.46 p177] and we obtain that $\omega^t \in \Omega^t \to \int_{\Omega_{t+1}} 1_{\widetilde{\Omega}^{t+1}}(\omega^t, \omega_{t+1}) q_{t+1}(d\omega_{t+1}, \omega^t) = q_{t+1}\left(\left(\widetilde{\Omega}^{t+1}\right)_{\omega^t}, \omega^t\right)$ is $\mathcal{B}_c(\Omega^t)$-measurable. Assume that $P_t(\widetilde{\Omega}^t) < 1$ we obtain that

$$\begin{aligned}
1 = P_{t+1}(\widetilde{\Omega}^{t+1}) &= \int_{\Omega^t} q_{t+1}\left(\left(\widetilde{\Omega}^{t+1}\right)_{\omega^t}, \omega^t\right) P_t(d\omega^t) \\
&= P_t(\widetilde{\Omega}^t) + \int_{\Omega^t \setminus \widetilde{\Omega}^t} q_{t+1}\left(\left(\widetilde{\Omega}^{t+1}\right)_{\omega^t}, \omega^t\right) P_t(d\omega^t) \\
&< P_t(\widetilde{\Omega}^t) + P_t(\Omega^t \setminus \widetilde{\Omega}^t) = 1
\end{aligned}$$

where we have used that $q_{t+1}\left(\left(\widetilde{\Omega}^{t+1}\right)_{\omega^t}, \omega^t\right) < 1$ on $\Omega^t \setminus \widetilde{\Omega}^t$. This contradiction shows that $P_t(\widetilde{\Omega}^t) = 1$. □

CHAPTER 4

Multiple-priors optimal investment in discrete time for unbounded utility function

This chapter is an extended version of the results on multiple-priors utility maximisation obtained in [20] that has been submitted for publication.

This chapter investigates the problem of maximizing multiple-priors expected terminal utility in a discrete-time financial market model with a finite horizon under non-dominated model uncertainty. We use a dynamic programming framework together with measurable selection arguments to prove that under mild integrability conditions, an optimal portfolio exists for an unbounded utility function defined on the half-real line.

4.1 Introduction

We consider investors trading in a multi-period and discrete-time financial market. We study the problem of terminal wealth expected utility maximisation under Knightian uncertainty. It was first introduced by F. Knight ([90]) and refers to the "unknown unknown", or uncertainty, as opposed to the "known unknown", or risk. This concept is very appropriate in the context of financial mathematics as it describes accurately market behaviors which are becoming more and more surprising. The belief of investors are modeled with a set of probability measures rather than a single one. This can be related to model mispecification issues or model risk and has triggered a renewed and strong interest by practitioners and academics alike.
The axiomatic theory of the classical expected utility was initiated by [126]. They provided conditions on investor preferences under which the expected utility of a contingent claim X can be expressed as $E_P U(X)$ where P is a given probability measure and U is a so-called utility function. The problem of maximising the von Neumann and Morgenstern expected utility has been extensively studied, we refer to [111] and [112] for the discrete-time case and to [91] and [121] for the continuous-time one. In the presence of Knightian uncertainty, [69] provided a pioneering con-

tribution by extending the axiomatic of von Neumann and Morgenstern. In this case, under suitable conditions on the investor preferences, the utility functional is of the form $\inf_{P\in\mathcal{Q}^T} E_P U(X)$ where $\mathcal{Q}^T$ is the set of all possible probability measures representing the agent beliefs. Most of the literature on the so-called multiple-priors or robust expected utility maximisation assumes that $\mathcal{Q}^T$ is dominated by a reference measure. We refer to [63] for an extensive survey.
However assuming the existence of a dominating reference measure does not always provide the required degree of generality from an economic and practical perspective. Indeed, uncertain volatility models (see [5], [53], [93]) are concrete examples where this hypothesis fails. On the other hand, assuming a non-dominated set of probability measures significantly raises the mathematical difficulty of the problem as some of the usual tools of probability theory do not apply. In the multiple-priors non-dominated case, [52] obtained the existence of an optimal strategy, a worst case measure as well as some "minmax" results under some compacity assumption on the set of probability measures and with a bounded (from above and below) utility function. This result is obtained in the continuous-time case. In the discrete-time case, [99] (where further references to multiple-priors non-dominated problematic can be found) obtained the first existence result without any compacity assumption on the set of probability measures but for a bounded (from above) utility function. We also mention two articles subsequent to our contribution. The first one (see [7]) provides a dual representation in the case of an exponential utility function with a random endowment and the second one (see [98]) studies a market with frictions in the spirit of [106] for a bounded from above utility function.
To the best of our knowledge, this chapter provides the first general result for unbounded utility functions assuming a non-dominated set of probability measures (and without compacity assumption). This includes for example, the useful case of Constant Relative Risk Aversion utility functions ($i.e$ logarithm or power functions). In Theorem 4.4.14, we give sufficient conditions for the existence of an optimizer to our "maxmin" problem (see Definition 4.4.8). We work under the framework of [25] and [99]. The market is governed by a non-dominated set of probability measures $\mathcal{Q}^T$ that determines which events are relevant or not. Assumption 4.2.1, which is related to measurability issues, is the only assumption made on $\mathcal{Q}^T$ and is the cornerstone of the proof. We introduce two integrability assumptions. The first one (Assumption 4.4.2) is related to measurability and continuity issues. The second one (Assumption 4.4.12) replaces the boundedness assumption of [99] and allows us to use auxiliary functions which play the role of properly integrable bounds for the value functions at each step. The no-arbitrage condition is essential as well, we use the $NA(\mathcal{Q}^T)$ introduced in [25] (see also Chapter 3) and use its equivalent "quantitative" characterisation (see Definition 3.4.4 and Theorem 3.4.7 in Chapter 3). Finally, we will use also the "strong" no-arbitrage condition $sNA(\mathcal{Q}^T)$ introduced in Definition 3.6.1 in Chapter 3 and prove in Theorem 4.4.15 that under

the $sNA(\mathcal{Q}^T)$ condition, Theorem 4.4.14 applies to a large range of settings.
As in [25] and [99] our proof relies heavily on measure theory tools, namely on analytic sets. Those sets display the nice property of being stable by projection or countable unions and intersections. However they fail to be stable by complementation, hence the sigma-algebra generated by analytic sets contains sets that are not analytic which leads to significant measurability issues. Such difficulties arise for instance in Lemma 4.6.5, where we are still able to prove some tricky measurability properties, as well as in Proposition 4.6.12 which is pivotal in solving the dynamic programming. Note as well, that we have identified (and corrected) a small issue in [25, Lemma 4.12] which is also used in [99] to prove some important measurability properties. Indeed it is not enough in order to have joint-measurability of a function $\theta(\omega, x)$ to assume that $\theta(\cdot, x)$ is measurable and $\theta(\omega, \cdot)$ is lower-semicontinuous, one has to assume for example that $\theta(\omega, \cdot)$ is convex (see Lemma 4.8.3 as well as the counterexample 4.8.2).
To solve our optimisation problem we follow a similar approach as [99]. We first consider a one-period case with strategy in $\mathbb{R}^d$. To "glue" together the solutions found in the one-period case we use dynamic programming as in [111], [112], [33], [34], [99] (and also Chapter 2) together with measurable selection arguments (Auman and Jankov-von Neumann Theorems).
The chapter is structured as follows. In Section 4.2, we recall some important properties of analytic sets, present our framework and state our main result. In section 4.3 we recall some useful results from Chapter 3 and in particular the quantitative characterisation of the quasi-sure no-arbitrage condition. Section 4.4 presents the main theorem on terminal wealth worst-case expected utility maximisation; section 4.5 establishes the existence of an optimal strategy for the one period case and the main theorem is proved in section 4.6. Finally, section 4.8 collects some technical results and propose a correction to a measurability issue identified in [25, Lemma 4.12].

4.2 Definitions and set-up

This section is similar to Section 3.2 in Chapitre 3. Section 4.2.4 was added in order to define generalised integrals that are used throughout this chapter and the next one. Assumption 4.2.4 is also specific to the optimisation problem.

4.2.1 Polar sets and universal sigma-algebra

For any Polish space X ($i.e$ complete and separable metric space), we denote by $\mathcal{B}(X)$ its Borel sigma-algebra and by $\mathfrak{P}(X)$ the set of all probability measures on $(X, \mathcal{B}(X))$. We recall that $\mathfrak{P}(X)$ endowed with the weak topology is a Polish space (see [13, Propositions 7.20 p127, 7.23 p131]). If P in $\mathfrak{P}(X)$, $\mathcal{B}_P(X)$ will be the completion of $\mathcal{B}(X)$ with respect to P and the universal sigma-algebra is defined by

$$\mathcal{B}_c(X) := \bigcap_{P \in \mathfrak{P}(X)} \mathcal{B}_P(X).$$

It is clear that $\mathcal{B}(X) \subset \mathcal{B}_c(X)$. In the rest of the chapter we will use the same notation for P in $\mathfrak{P}(X)$ and for its (unique) extension on $\mathcal{B}_c(X)$. A function $f : X \to Y$ (where Y is an other Polish space) is universally-measurable or $\mathcal{B}_c(X)$-measurable (resp. Borel-measurable or $\mathcal{B}(X)$-measurable) if for all $B \in \mathcal{B}(Y)$, $f^{-1}(B) \in \mathcal{B}_c(X)$ (resp. $f^{-1}(B) \in \mathcal{B}(X)$).
For a given $\mathcal{P} \subset \mathfrak{P}(X)$, a set $N \subset X$ is called a $\mathcal{P}$-polar if for all $P \in \mathcal{P}$, there exists some $A_P \in \mathcal{B}_c(X)$ such that $P(A_P) = 0$ and $N \subset A_P$. We say that a property holds true $\mathcal{P}$-quasi-surely (q.s.), if it is true outside a $\mathcal{P}$-polar set. Finally we say that a set is of $\mathcal{P}$-full measure if its complement is a $\mathcal{P}$-polar set.

4.2.2 Analytic sets

An analytic set of X is the continuous image of a Polish space, see [3, Theorem 12.24 p447]. We denote by $\mathcal{A}(X)$ the set of analytic sets of X and recall some key properties that will be often used in the rest of the chapter. The projection of an analytic set is an analytic set see [13, Proposition 7.39 p165]) and a countable union or intersection of analytic sets is an analytic set (see [13, Corollary 7.35.2 p160]). However the complement of an analytic set does not need to be an analytic set. We denote by $CA(X) := \{A \in X,\ X \backslash A \in \mathcal{A}(X)\}$ the set of all coanalytic sets of X. We have that (see [13, Proposition 7.36 p161, Corollary 7.42.1 p169])

$$\mathcal{B}(X) \subset \mathcal{A}(X) \cap \mathcal{CA}(X) \text{ and } \mathcal{A}(X) \cup \mathcal{CA}(X) \subset \mathcal{B}_c(X). \tag{4.1}$$

Now, for $D \in \mathcal{A}(X)$, a function $f : D \to \mathbb{R} \cup \{\pm\infty\}$ is lower-semianalytic (resp. upper-semianalytic) on X if $\{x \in X\ f(x) < c\} \in \mathcal{A}(X)$ (resp. $\{x \in X\ f(x) > c\} \in \mathcal{A}(X)$) for all $c \in \mathbb{R}$. We denote by $\mathcal{LSA}(X)$ (resp. $\mathcal{USA}(X)$) the set of all lower-semianalytic (resp. upper-semianalytic) functions on X. From (4.1) it is clear that if $f \in \mathcal{LSA}(X) \cup \mathcal{USA}(X)$ then f is $\mathcal{B}_c(X)$-measurable. Finally, a function $f : X \to Y$ (where Y is another Polish space) is analytically-measurable if for all $B \in \mathcal{B}(Y)$, $f^{-1}(B)$ belongs to the sigma-algebra generated by $\mathcal{A}(X)$. From (4.1), it is clear that if f is analytically-measurable, then f is universally-measurable.

4.2.3 The measurable spaces

We fix a time horizon $T \in \mathbb{N}$ and introduce a sequence $(\Omega_t)_{1 \leq t \leq T}$ of Polish spaces. We denote by

$$\Omega^t := \Omega_1 \times \cdots \times \Omega_t,$$

with the convention that Ω^0 is reduced to a singleton. An element of Ω^t will be denoted by $\omega^t = (\omega_1, \dots, \omega_t) = (\omega^{t-1}, \omega_t)$ for $(\omega_1, \dots, \omega_t) \in \Omega_1 \times \dots \times \Omega_t$ and $(\omega^{t-1}, \omega_t) \in \Omega^{t-1} \times \Omega_t$ (to avoid heavy notation we drop the dependancy in ω_0). It is well know that $\mathcal{B}(\Omega^t) = \mathcal{B}(\Omega^{t-1}) \otimes \mathcal{B}(\Omega_t)$, see [3, Theorem 4.44 p149]. However we have only that $\mathcal{B}_c(\Omega^{t-1}) \otimes \mathcal{B}_c(\Omega_t) \subset \mathcal{B}_c(\Omega^t)$.

4.2.4 Generalised integral

From now on the positive (resp. negative) part of some number or random variable Y is denoted by Y^+ (resp. Y^-). We will also write $f^{\pm}(Y)$ for $(f(Y))^{\pm}$ for any random variable Y and (possibly random) function f.
We fix some $1 \leq t \leq T$ and $P_t \in \mathfrak{P}(\Omega^t)$. For some $\mathcal{B}_c(\Omega^t)$-measurable function $g_t : \Omega^t \to [0, \infty]$, applying [13, Lemma 7.27 p173], there exists some $\mathcal{B}(\Omega^t)$-measurable function $\overline{g}_t : \Omega^t \to [0, \infty]$ such that $\overline{g}_t = g_t$ P_t-almost surely and we set

$$\int_{\Omega^t} g_t(\omega^t) P_t(d\omega^t) := \int_{\Omega^t} \overline{g}_t(\omega^t) P_t(d\omega^t). \tag{4.2}$$

It is easy to verify that (4.2) does not depend on the choice of $\overline{g}_t$.
In the rest of the chapter we will use generalised integrals. For some $\mathcal{B}_c(\Omega^t)$-measurable function $f_t : \Omega^t \to \mathbb{R} \cup \{\pm\infty\}$, we define

$$\int_{\Omega^t} f_t(\omega^t) P_t(d\omega^t) := \int_{\Omega^t} f_t^+(\omega^t) P_t(d\omega^t) - \int_{\Omega^t} f_t^-(\omega^t) P_t(d\omega^t), \tag{4.3}$$

using (4.2) for f_t^+ and f_t^-. Note that if both $\int_{\Omega^t} f_t^+(\omega^t) P_t(d\omega^t) = \infty$ and $\int_{\Omega^t} f_t^-(\omega^t) P_t(d\omega^t) = \infty$ the integral above is not defined. In this case we set $\int_{\Omega^t} f_t(\omega^t) P_t(d\omega^t) = +\infty$: see also Remark 4.4.9 for further discussion on this. Note that unlike in Chapter 2 we could not avoid the use of this convention.

4.2.5 Stochastic kernels and definition of $\mathcal{Q}^T$

For all $0 \leq t \leq T-1$, we denote by $\mathcal{S}K_{t+1}$ the set of universally-measurable stochastic kernel on Ω_{t+1} given Ω^t (see [13, Definition 7.12 p134, Lemma 7.28 p174]). Fix some $1 \leq t \leq T$, $P_{t-1} \in \mathfrak{P}(\Omega^{t-1})$ and $p_t \in \mathcal{S}K_t$. Using Fubini's Theorem, see [13, Proposition 7.45 p175], we define a probability on Ω^t by setting for all $A \in \mathcal{B}(\Omega^t)$

$$P_{t-1} \otimes p_t(A) := \int_{\Omega^{t-1}} \int_{\Omega_t} 1_A(\omega^{t-1}, \omega_t) p_t(d\omega_t, \omega^{t-1}) P_{t-1}(d\omega^{t-1}). \tag{4.4}$$

For all $0 \leq t \leq T-1$, we consider a random set $\mathcal{Q}_{t+1} : \Omega^t \twoheadrightarrow \mathfrak{P}(\Omega_{t+1})$: $\mathcal{Q}_{t+1}(\omega^t)$ can be seen as the set of possible priors for the t-th period given the state ω^t until time t.

Assumption 4.2.1 For all $0 \leq t \leq T-1$, $\mathcal{Q}_{t+1}$ is a non-empty and convex valued random set such that

$$\text{Graph}(\mathcal{Q}_{t+1}) = \left\{(\omega^t, P) \in \Omega^t \times \mathfrak{P}(\Omega_{t+1}),\ P \in \mathcal{Q}_{t+1}(\omega^t)\right\} \in \mathcal{A}\left(\Omega^t \times \mathfrak{P}(\Omega_{t+1})\right).$$

From the Jankov-von Neumann Theorem, see [13, Proposition 7.49 p182], there exists some analytically-measurable and thus $\mathcal{B}_c(\Omega^t)$-measurable $q_{t+1} : \Omega^t \to \mathfrak{P}(\Omega_{t+1})$ such that for all $\omega^t \in \Omega^t$, $q_{t+1}(\cdot, \omega^t) \in \mathcal{Q}_{t+1}(\omega^t)$ (recall that for all $\omega^t \in \Omega^t$, $\mathcal{Q}_{t+1}(\omega^t) \neq \emptyset$). In other words q_{t+1} is a universally-measurable selector of $\mathcal{Q}_{t+1}$. Note as well that $q_{t+1} \in \mathcal{S}K_{t+1}$. For all $1 \leq t \leq T$ we define $\mathcal{Q}^t \subset \mathfrak{P}(\Omega^t)$ by

$$\mathcal{Q}^t := \{Q_1 \otimes q_2 \otimes \cdots \otimes q_t,\ Q_1 \in \mathcal{Q}_1,\ q_{s+1} \in \mathcal{S}K_{s+1}, q_{s+1}(\cdot, \omega^s) \in \mathcal{Q}_{s+1}(\omega^s)\ Q_s\text{-a.s. } \forall\, 1 \leq s \leq t-1\}, \tag{4.5}$$

where if $Q_t = Q_1 \otimes q_2 \otimes \cdots \otimes q_t \in \mathcal{Q}^t$ we denote by $Q_s := Q_1 \otimes q_2 \otimes \cdots \otimes q_s$ for any $2 \leq s \leq t$. It is clear that $Q_s \in \mathcal{Q}^s$. We will often use in the chapter the following construction: let $Q = Q_0 \otimes q_1 \cdots \otimes q_t \in \mathcal{Q}^t$ be fixed and let some $q^*_{t+1} \in \mathcal{S}K_{t+1}$ be such that there exists $\widetilde{\Omega}^t \in \mathcal{B}_c(\Omega^t)$ with $Q(\widetilde{\Omega}^t) = 1$ and $q^*_{t+1}(\cdot, \omega^t) \in \mathcal{Q}_{t+1}(\omega^t)$ for all $\omega^t \in \widetilde{\Omega}^t$. We define $Q^* \in \mathfrak{P}(\Omega^{t+1})$ by

$$Q^* = Q_0 \otimes q_1 \cdots \otimes q_t \otimes q^*_{t+1} = Q \otimes q^*_{t+1}.$$

Then, it is clear that $Q^* \in \mathcal{Q}^{t+1}$.

4.2.6 The traded assets and strategies

Let $S := \{S_t,\ 0 \leq t \leq T\}$ be a $(\mathcal{B}_c(\Omega^t))_{0 \leq t \leq T}$-adapted d-dimensional process where for $0 \leq t \leq T$, $S_t = (S^i_t)_{1 \leq i \leq d}$ represents the price of d risky securities in the financial market in consideration. We make the following assumptions already stated in [99].

Assumption 4.2.2 We have that S is $(\mathcal{B}(\Omega^t))_{0 \leq t \leq T}$-adapted.

Remark 4.2.3 As already stated in Remark 3.2.3 in Chapter 3, if we do not assume Assumption 4.2.2, we cannot obtain some crucial measurability properties in Lemma 4.3.2. In this chapter Assumption 4.2.2 will also be needed to obtain (4.57) and (4.58) and to use [13, Lemma 7.30 (3) p178]). Note that we do no need this assumption in the one period case.

The next assumption was not present in Chapter 3 as it is specific to the optimisation problem (it appears also in [99]).

Assumption 4.2.4 There exists some $0 \leq s < \infty$ such that $-s \leq S^i_t(\omega^t) < +\infty$ for all $1 \leq i \leq d$, $\omega^t \in \Omega^t$ and $0 \leq t \leq T$.

Remark 4.2.5 If S is $(\mathcal{B}(\Omega^t))_{0\leq t\leq T}$-adapted and for $t = 1, \cdots, T$ there exists some $\mathcal{Q}^t$-full measure set $\Omega^t_S \in \mathcal{B}(\Omega^t)$ such that $-s \leq S^i_t(\omega^t) < +\infty$ for all $1 \leq i \leq d$ and $\omega^t \in \Omega^t_S$, we set $\overline{S}_t = 1_{\Omega^t_S} S_t$ and $\overline{S} := \left\{\overline{S}_t,\ 0 \leq t \leq T\right\}$ satisfies Assumptions 4.2.2 and 4.2.4.

There exists also a riskless asset for which we assume a price constant equal to 1, for sake of simplicity. Without this assumption, all the developments below could be carried out using discounted prices. The notation $\Delta S_t := S_t - S_{t-1}$ will often be used. If $x, y \in \mathbb{R}^d$ then the concatenation xy stands for their scalar product. The symbol $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$ (or on $\mathbb{R}$).
Trading strategies are represented by d-dimensional processes $\phi := \{\phi_t, 1 \leq t \leq T\}$ where for all $1 \leq t \leq T$, $\phi_t = (\phi^i_t)_{1\leq i\leq d}$ represents the investor's holdings in each of the d assets at time t. We assume that ϕ_t is $\mathcal{B}_c(\Omega^{t-1})$-measurable for all $1 \leq t \leq T$. The family of all such trading strategies is denoted by Φ. We assume that trading is self-financing. As the riskless asset's price is constant 1, the value at time t of a portfolio ϕ starting from initial capital $x \in \mathbb{R}$ is given by

$$V^{x,\phi}_t = x + \sum_{s=1}^{t} \phi_s \Delta S_s.$$

4.3 Conditional support and no-arbitrage: useful results

For ease of reading, this section collects some important definitions and useful results from Chapter 3.

4.3.1 Conditional support

First, we recall the following definitions and propositions from Section 3.3 in Chapter 3 concerning the multiple-priors conditional support of the price increments and its affine hull (denoted by Aff).

Definition 4.3.1 For all $0 \leq t \leq T-1$, we define the random sets $\widetilde{D}^{t+1} : \Omega^t \twoheadrightarrow \mathbb{R}^d$ and $D^{t+1} : \Omega^t \twoheadrightarrow \mathbb{R}^d$ by

$$\widetilde{D}^{t+1}(\omega^t) := \bigcap \left\{ A \subset \mathbb{R}^d,\ \text{closed},\ P_{t+1}\left(\Delta S_{t+1}(\omega^t, .) \in A\right) = 1,\ \forall\, P_{t+1} \in \mathcal{Q}_{t+1}(\omega^t) \right\},$$
$$D^{t+1}(\omega^t) := \text{Aff}\left(\widetilde{D}^{t+1}(\omega^t)\right),$$

The following measurability results were proved in Lemma 3.3.2 in Chapter 3.

Lemma 4.3.2 *Let Assumption 4.2.1 and 4.2.2 hold true and* $0 \leq t \leq T-1$ *be fixed. Then* $\widetilde{D}^{t+1}$ *and* D^{t+1} *are non-empty, closed valued and* $\mathcal{B}_c(\Omega^t)$*-measurable (recall Definition 2.8.19 in Chapter 2) random sets. We have also that Graph*$(D^{t+1}) \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$.

4.3.2 No-arbitrage conditions

We recall now the definition of the $NA(\mathcal{Q}^T)$ condition (also referred to as quasi-sure no-arbitrage) introduced in [25] (see also Definition 3.4.1 in Chapter 3).

Definition 4.3.3 The $NA(\mathcal{Q}^T)$ condition holds true if for $\phi \in \Phi$, $V_T^{0,\phi} \geq 0$ $\mathcal{Q}^T$-q.s. $\Rightarrow$ $V_T^{0,\phi} = 0$ $\mathcal{Q}^T$-q.s.

The following one-period version (see Definition 3.4.2 in Chapter 3) will often be used. For $\omega^t \in \Omega^t$ fixed we will say that $NA(\mathcal{Q}_{t+1}(\omega^t))$ condition holds true if for all $h \in \mathbb{R}^d$

$$h\Delta S_{t+1}(\omega^t, \cdot) \geq 0 \ \mathcal{Q}_{t+1}(\omega^t)\text{-q.s.} \Rightarrow h\Delta S_{t+1}(\omega^t, \cdot) = 0 \ \mathcal{Q}_{t+1}(\omega^t)\text{-q.s.} \quad (4.6)$$

And we denote

$$\Omega^t_{NA} := \{\omega^t \in \Omega^t, \ NA(\mathcal{Q}_{t+1}(\omega^t)) \text{ holds true}\}.$$

We state now the following "quantitative" characterisation of $NA(\mathcal{Q}^T)$ condition. This is a direct application of Theorem 3.4.7 in Chapter 3 where it was proved that there is in fact an equivalence between the $NA(\mathcal{Q}^T)$ condition and (4.7) (but the reverse implication is not required in this chapter).

Proposition 4.3.4 *Assume that the* $NA(\mathcal{Q}^T)$ *condition and Assumptions 4.2.1, 4.2.2 hold true. Then for all* $0 \leq t \leq T-1$, $\Omega^t_{NA} \in \mathcal{B}_c(\Omega^t)$ *is a* $\mathcal{Q}^t$*-full measure set and for all* $\omega^t \in \Omega^t_{NA}$, $NA(\mathcal{Q}_{t+1}(\omega^t))$ *holds true and* $D^{t+1}(\omega^t)$ *is a vector space. For all* $\omega^t \in \Omega^t_{NA}$ *there exists* $\alpha_t(\omega^t) > 0$ *such that for all* $h \in D^{t+1}(\omega^t)$ *there exists* $P_h \in \mathcal{Q}_{t+1}(\omega^t)$ *satisfying*

$$P_h\left(\frac{h}{|h|}\Delta S_{t+1}(\omega^t, .) < -\alpha_t(\omega^t)\right) > \alpha_t(\omega^t). \quad (4.7)$$

Proof. The proof is a direct application of Theorem 3.4.7 in Chapter 3. □

Finally, we recall the alternative notion of strong no-arbitrage already introduced in Definition 3.6.1 in Chapter 3. This will be use in Theorem 4.4.15.

Definition 4.3.5 We say that the $sNA(\mathcal{Q}^T)$ condition holds true if for all $\phi \in \Phi$ and $P \in \mathcal{Q}^T$

$$V_T^{0,\phi} \geq 0 \ P\text{-a.s.} \Rightarrow V_T^{0,\phi} = 0 \ P\text{-a.s.} \quad (4.8)$$

In Chapter 3 (see (3.33) and (3.34)), for all $P = Q_1 \otimes q_2 \otimes \cdots \otimes q_T \in \mathcal{Q}^T$ and for all $1 \leq t \leq T-1$ the conditional support of ΔS_{t+1} with respect to P and its corresponding affine hull were defined as follow

$$\widetilde{D}_P^{t+1}(\omega^t) := \bigcap \left\{ A \subset \mathbb{R}^d, \text{ closed}, \ q_{t+1}\left(\Delta S_{t+1}(\omega^t, .) \in A, \omega^t\right) = 1 \right\}$$
$$D_P^{t+1}(\omega^t) := \text{Aff}\left(\widetilde{D}^{t+1}(\omega^t)\right),$$

(where the case $t = 0$ is obtained by replacing $q_{t+1}(\cdot, \omega^t)$ by $Q_1(\cdot)$) and the following quantitative characterisation of the $sNA(\mathcal{Q}^T)$ was obtained in Proposition 3.6.11 in Chapter 3.

Proposition 4.3.6 *Assume that the* $sNA(\mathcal{Q}^T)$ *condition and Assumptions 4.2.1 and 4.2.2 hold true and let* $0 \leq t \leq T-1$. *Fix some* $P = Q_1 \otimes q_2 \otimes \cdots \otimes q_T \in \mathcal{Q}^T$. *Then there exists* $\Omega_P^t \in \mathcal{B}(\Omega^t)$ *with* $P_t(\Omega_P^t) = 1$ *such that for all* $\omega^t \in \Omega_P^t$, $D_P^{t+1}(\omega^t)$ *is a vector subspace and there exists* $\alpha_t^P(\omega^t) \in (0, 1]$ *such that for all* $h \in D_P^{t+1}(\omega^t)$

$$q_{t+1}\big(h \Delta S_{t+1}(\omega^t, \cdot) \leq -\alpha_t^P(\omega^t)|h|, \omega^t\big) \geq \alpha_t^P(\omega^t). \tag{4.9}$$

Furthermore $\omega^t \to \alpha_t^P(\omega^t)$ *is* $\mathcal{B}(\Omega^t)$*-measurable.*

4.4 Utility problem and main result

We now describe the investor's risk preferences by a concave, random utility function.

Definition 4.4.1 A *random utility* is any function $U : \Omega^T \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ satisfying the following conditions

- for every $x \in \mathbb{R}$, the function $U(\cdot, x) : \Omega^T \to \mathbb{R} \cup \{\pm\infty\}$ is $\mathcal{B}(\Omega^T)$-measurable,
- for all $\omega^T \in \Omega^T$, the function $U\left(\omega^T, \cdot\right) : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is non-decreasing, usc and concave on $\mathbb{R}$,
- $U(\cdot, x) = -\infty$, for all $x < 0$.

Fix some $\omega^T \in \Omega^T$ and let $\text{Dom}\, U(\omega^T, \cdot) := \{x \in \mathbb{R}, U(\omega^T, x) > -\infty\}$ be the domain of $U(\omega^T, \cdot)$. It is clear that $\text{Dom}\, U(\omega^T, \cdot) \subset [0, \infty)$. We need some assumptions ensuring that $\text{Ri}(\text{Dom}\, U(\omega^T, \cdot))$, the relative interior of the domain of $U(\omega^T, \cdot)$, is in fact equal to $(0, \infty)$ $\mathcal{Q}^T$-q.s.

Assumption 4.4.2 For all $r \in \mathbb{Q}$, $r > 0$ $\sup_{P \in \mathcal{Q}^T} E_P U^-(\cdot, r) < +\infty$.

Remark 4.4.3 In the mono-prior case, this assumption does not appear (see Chapter 2). The reason for introducing Assumption 4.4.2 is related to the dynamic programming part. First and crucially, Assumption 4.4.2 ensures that the functions U_t defined in (4.47) and (4.48) are versions of the value functions which have “good" measurability properties. We will come back to this in Remark 4.5.11. Moreover we will prove (see Proposition 4.6.7) that Assumption 4.4.2 is preserved through the dynamic programming procedure. Note that in the case of non-random utility function, Assumption 4.4.2 is equivalent to Ri(Dom U) $= (0, \infty)$. If U does not satisfy this assumption, then one can apply the arguments of Remark 4.4.10: in other words, Assumption 4.4.2 is superfluous in the case of a non-random utility function.

Example 4.4.4 We propose the following example where Assumption 4.4.2 holds true. Assume that there exists some $x_0 > 0$ such that $\sup_{P \in \mathcal{Q}^T} E_P U^-(\cdot, x_0) < \infty$. Assume also that there exists some functions $f_1, f_2 : (0,1] \to (0, \infty)$ as well as some non-negative $\mathcal{B}_c(\Omega^T)$-measurable random variable D verifying $\sup_{P \in \mathcal{Q}^T} E_P D(\cdot) < \infty$ such that for all $\omega^T \in \Omega^T$, $x \geq 0$, $0 < \lambda \leq 1$

$$U(\omega^T, \lambda x) \geq f_1(\lambda) U(\omega^T, x) - f_2(\lambda) D(\omega^T). \tag{4.10}$$

This condition is a kind of elasticity assumption around zero. It is satisfied for example by the logarithm function. Fix some $r \in \mathbb{Q}$, $r > 0$. If $r \geq x_0$, it is clear from Definition 4.4.1 that $\sup_{P \in \mathcal{Q}^T} E_P U^-(\cdot, r) < \infty$. If $r < x_0$, we have for all $\omega^T \in \Omega^T$, $U(\omega^T, r) \geq f_1(\frac{r}{x_0}) U(\omega^T, x_0) - f_2(\frac{r}{x_0}) D(\omega^T)$ and $\sup_{P \in \mathcal{Q}^T} E_P U^-(\cdot, r) < \infty$ follows immediatly.

Proposition 4.4.5 *Assume that Assumption 4.4.2 holds true. Set* $\Omega^T_{Dom} := \{\omega^T \in \Omega^T,\ U(\omega^T, r) > -\infty,\ \forall r \in \mathbb{Q},\ r > 0\}$. *Then* $P(\Omega^T_{Dom}) = 1$ *for all* $P \in \mathcal{Q}^T$. *Furthermore, for all* $\omega^T \in \Omega^T_{Dom}$

- *Ri*(*Dom* $U(\omega^T, \cdot)) = (0, \infty)$,
- $U(\omega^T, \cdot)$ *is continuous on* $(0, \infty)$ *and right-continuous in* 0.

Proof. From Definition 4.4.1 it is clear that $\Omega^T_{Dom} \in \mathcal{B}(\Omega^T)$. From Assumption 4.4.2, for all $P \in \mathcal{Q}^T$ there exists some $\Omega^T_P \subset \Omega^T_{Dom}$ such that $P(\Omega^T_P) = 1$. Hence $P(\Omega^T_{Dom}) = 1$ for all $P \in \mathcal{Q}^T$. Now fix $\omega^T \in \Omega^T_{Dom}$. As $U(\omega^T, \cdot)$ is non-decreasing we get that $(0, \infty) \subset$ Dom $U(\omega^T, \cdot) \subset [0, \infty)$ and thus Ri(Dom $U(\omega^T, \cdot)) = (0, \infty)$. As $U(\omega^T, \cdot)$ is concave, it is continuous on $(0, \infty)$, see [116, Theorem 2.35 p59]. Finally as $U(\omega^T, \cdot)$ is usc and non-decreasing it is right-continuous in 0 (see for example Lemma 2.8.12 of Chapter 2). $\square$

We introduce the following notations.

Definition 4.4.6 Fix some $x \geq 0$. For $P \in \mathfrak{P}(\Omega^T)$ fixed, we denote by $\Phi(x, P)$ the set of all strategies $\phi \in \Phi$ such that $V_T^{x,\phi}(\cdot) \geq 0$ P-a.s. and by $\Phi(x, U, P)$ the set of all strategies $\phi \in \Phi(x, P)$ such that either $E_P U^+(\cdot, V_T^{x,\phi}(\cdot)) < \infty$ or $E_P U^-(\cdot, V_T^{x,\phi}(\cdot)) < \infty$. We denote by $\Phi(x, \mathcal{Q}^T)$ the set of all strategies $\phi \in \Phi$ such that $V_T^{x,\phi}(\cdot) \geq 0$ $\mathcal{Q}^T$-q.s. and by $\Phi(x, U, \mathcal{Q}^T)$ the set of all strategies $\phi \in \Phi(x, \mathcal{Q}^T)$ such that either $E_P U^+(\cdot, V_T^{x,\phi}(\cdot)) < \infty$ or $E_P U^-(\cdot, V_T^{x,\phi}(\cdot)) < \infty$ for all $P \in \mathcal{Q}^T$. In other words

$$\Phi(x, \mathcal{Q}^T) = \bigcap_{P \in \mathcal{Q}^T} \Phi(x, P) \text{ and } \Phi(x, U, \mathcal{Q}^T) = \bigcap_{P \in \mathcal{Q}^T} \Phi(x, U, P). \tag{4.11}$$

The following lemma shows that under $NA(\mathcal{Q}^T)$, if $\phi \in \Phi(x, \mathcal{Q}^T)$ then $P_t(V_t^{x,\phi}(\cdot) \geq 0) = 1$ for all $P \in \mathcal{Q}^t$ and $1 \leq t \leq T$. Note that in [99, Definition of $\mathcal{H}_x$, top of p10], this intertemporal budget imposed.

Lemma 4.4.7 *Assume that the $NA(\mathcal{Q}^T)$ condition holds true. Let $x \geq 0$ and $\phi \in \Phi$ such that $V_T^{x,\phi} \geq 0$ $\mathcal{Q}^T$-q.s., then $V_t^{x,\phi} \geq 0$ $\mathcal{Q}^t$-q.s.*

Proof. We assume that this is not the case. Let $n = \sup\{t,\ \exists P_t \in \mathcal{Q}^t,\ P_t(V_t^{x,\phi} < 0) > 0\}$. Thus $n < T$ and there exists some $\widehat{P}_n \in \mathcal{Q}^n$ such that $\widehat{P}_n(V_n^{x,\phi} < 0) > 0$ and for all $s \geq n+1$, $P \in \mathcal{Q}^s$, $P(V_s^{x,\phi} \geq 0) = 1$. Let $\Psi_s(\omega^{s-1}) = 0$ if $1 \leq s \leq n$ and $\Psi_s(\omega^{s-1}) = 1_A(\omega^n)\phi_s(\omega^{s-1})$ if $s \geq n+1$ with $A = \{\omega^n \in \Omega^n, V_n^{\Phi}(\omega^n) < 0\} \in \mathcal{B}_c(\Omega^n)$. It is clear that $\Psi \in \Phi$. Then

$$V_s^{0,\Psi} = \sum_{k=1}^{s} \Psi_s \Delta S_s = \sum_{k=n+1}^{s} \Psi_s \Delta S_s = 1_A \left(V_s^{x,\phi} - V_n^{x,\phi} \right)$$

If $s \geq n+1$ $P(V_s^{x,\phi} \geq 0) = 1$ for all $P \in \mathcal{Q}^s$ and on A, $-V_n^{\Phi} > 0$ thus $P_T(V_T^{0,\Psi} \geq 0) = 1$ and $V_T^{0,\Psi} > 0$ on A. Let $\widehat{P}_T := \widehat{P}_n \otimes p_{n+1} \cdots \otimes p_T \in \mathcal{Q}^T$ where for $s = n+1, \cdot, T$, $p_s(\cdot, \cdot)$ is a given universally-measurable selector of $\mathcal{Q}^s$ (see (4.5)). It is clear that $\widehat{P}_T(A) = \widehat{P}_n(V_n^{x,\phi} < 0) > 0$, hence we get an arbitrage opportunity. Thus for all $t \leq T$, $V_t^{x,\phi} \geq 0$ $\mathcal{Q}^t$-q.s. □

We now state our main concern.

Definition 4.4.8 Let $x \geq 0$, the *multiple-priors portfolio problem* on a finite horizon T with initial wealth x is

$$u(x) := \sup_{\phi \in \Phi(x, U, \mathcal{Q}^T)} \inf_{P \in \mathcal{Q}^T} E_P U(\cdot, V_T^{x,\phi}(\cdot)). \tag{4.12}$$

Remark 4.4.9 As already mentioned, we use the convention $+\infty - \infty = +\infty$ throughout the chapter. This choice is rather unnatural when studying maximisation problem. It is justified by the fact that we will use [13, Proposition 7.48 p180] (which relies on [13, Lemma 7.30 (4) p177]) for lower-semianalytic function.

Remark 4.4.10 We propose some alternative solution when we do not have that $\text{Ri}(\text{Dom}\, U(\omega^t,\cdot)) = (0,\infty)$ for $\mathcal{Q}^T$-q.s. all $\omega^T \in \Omega^T$ (see Proposition 4.4.5). We introduce

$$m(\omega^T) := \inf\{x \in \mathbb{R},\ U(\omega^T,x) > -\infty\} \geq 0.$$

As $U(\omega^T,\cdot)$ is a non-decreasing function, $(m(\omega^T),+\infty) \subset \text{Dom}\, U(\omega^T,\cdot) \subset [m(\omega^T),\infty)$ and therefore $\text{Ri}(\text{Dom}\, U(\omega^T,\cdot)) = (m(\omega^T),\infty)$. First m is $\mathcal{B}(\Omega^T)$-measurable. Indeed, for any $c \in \mathbb{R}$ be fixed we have that

$$\{\omega^T \in \Omega^T, m(\omega^T) > c\} = \bigcup_{n\in\mathbb{N}\setminus\{0\}} \left\{\omega^T \in \Omega^T,\ U\left(\omega^T, c+\frac{1}{n}\right) = -\infty\right\} \in \mathcal{B}(\Omega^t),$$

recalling Definition 4.4.1. Now we set for all $\omega^T \in \Omega^T$, $x \in \mathbb{R}$, $\overline{U}(\omega^T,x) = U(\omega^T, x + m(\omega^T))$. It is clear that $(0,\infty) \subset \text{Dom}\, \overline{U}(\omega^T,\cdot) \subset [0,\infty)$ and that $\text{Ri}(\text{Dom}\, \overline{U}(\omega^T,\cdot)) = (0,\infty)$. We show that $\overline{U}$ satisfies Definition 4.4.1. For $\omega^T \in \Omega^T$ fixed, $U(\omega^T,\cdot)$ is usc and non-decreasing and thus right-continuous (see Lemma 2.8.12 of Chapter 2). For $x \in \mathbb{R}$ fixed, $U(\cdot,x)$ is $\mathcal{B}(\Omega^t)$-measurable and we deduce that U is $\mathcal{B}(\Omega^t)\otimes\mathcal{B}(\mathbb{R})$-measurable applying Lemma 2.8.16 of Chapter 2. So for all $x \in \mathbb{R}$ fixed, $\overline{U}(\cdot,x)$ is $\mathcal{B}(\Omega^T)$-measurable (recall that m is $\mathcal{B}(\Omega^T)$-measurable). The fact that for $\omega^T \in \Omega^T$, $\overline{U}(\omega^T,\cdot)$ is non-decreasing, concave and usc is clear. Consider now

$$\overline{u}(x) := \sup_{\phi\in\Phi(x,\overline{U},\mathcal{Q}^T)} \inf_{P\in\mathcal{Q}^T} E_P \overline{U}(\cdot, V_T^{x,\phi}(\cdot)) \tag{4.13}$$

Unfortunately, without further assumption, we cannot deduce from a solution $\overline{\phi}^*$ of $\overline{u}(x)$ some solution for $u(x)$. However, assume that there is some $x_m \geq 0$ and $\phi_m \in \Phi(x_m,U,\mathcal{Q}^T)$ such that $m = V_T^{x_m,\phi_m}$. Then as $\phi \in \Phi(x,\overline{U},\mathcal{Q}^T)$ if and only if $\phi+\phi_m \in \Phi(x+x_m,U,\mathcal{Q}^T)$ we have that

$$\overline{u}(x) = \sup_{\phi,\ \phi+\phi_m\in\Phi(x+x_m,U,\mathcal{Q}^T)} \inf_{P\in\mathcal{Q}^T} E_P U(\cdot, V_T^{x_m+x,\phi+\phi_m}(\cdot)) = u(x+x_m).$$

Thus if $\overline{\phi}^*$ is a solution of (4.13) with an initial wealth x, then $\overline{\phi}^* + \phi_m$ will be a solution for (4.12) starting from $x+x_m$. A simple example of a replicable m is obtained for a non-random utility function U with $\text{Ri}(\text{Dom}\, U) = (m,\infty)$, as m is constant in this case.

Remark 4.4.11 Note that if we study a utility function U defined only on $(0,\infty)$ and such that there exists some $\mathcal{Q}^T$-full measure set $\widetilde{\Omega}^T \in \mathcal{B}(\Omega^T)$ such that for all $\omega^T \in \widetilde{\Omega}^T$, $x \to U(\omega^T,x)$ is non-decreasing, usc and concave on $(0,+\infty)$. We extend U by (right) continuity in 0 by setting $U(\cdot,0) = \lim_{\substack{x\to 0\\ x>0}} U(\cdot,0)$ and we set $\overline{U} : \Omega^T \times \mathbb{R} \to \mathbb{R}\cup\{\pm\infty\}$

$$\overline{U}(\omega^T,x) := U(\omega^T,x)1_{\widetilde{\Omega}^T\times[0,+\infty)}(\omega^T,x) + (-\infty)1_{\Omega^T\times(-\infty,0)}(\omega_T,x).$$

Then $\overline{U}$ satisfies Definition 4.4.1. Moreover, the value function does not change and if there exists some $\phi^* \in \Phi(x, U, \mathcal{Q}^T)$ such that $u(x) = \inf_{P \in \mathcal{Q}^T} E_P \overline{U}(\cdot, V_T^{x,\phi^*}(\cdot))$, then ϕ^* is an optimal solution for (4.12).

We now present condition on U which allow to assert that if $\phi \in \Phi(x, \mathcal{Q}^T)$ then $E_P U(\cdot, V_T^{x,\phi}(\cdot))$ is well defined for all $P \in \mathcal{Q}^T$. It also allows us to work with auxiliary functions which play the role of properly integrable bounds for the value functions at each step (see (4.50), (4.58), (4.59) and (4.60))

Assumption 4.4.12 We have that $\sup_{P \in \mathcal{Q}^T} \sup_{\phi \in \Phi(1,P)} E_P U^+(\cdot, V_T^{1,\phi}(\cdot)) < \infty$.

Remark 4.4.13 From Assumption 4.4.12 we get that $\Phi(1, P) = \Phi(1, U, P)$ for all $P \in \mathcal{Q}^T$ and therefore $\Phi(1, \mathcal{Q}^T) = \Phi(1, U, \mathcal{Q}^T)$. In Proposition 4.6.2, we will show that under Assumption 4.4.12, for all $x \geq 0$, $\sup_{P \in \mathcal{Q}^T} \sup_{\phi \in \Phi(x,P)} E_P U^+(\cdot, V_T^{x,\phi}(\cdot)) < \infty$. Thus $\Phi(x, P) = \Phi(x, U, P)$ for all $P \in \mathcal{Q}^T$ and $x \geq 0$ and also $\Phi(x, \mathcal{Q}^T) = \Phi(x, U, \mathcal{Q}^T)$ for all $x \geq 0$.

We can now state our main result.

Theorem 4.4.14 *Assume that the $NA(\mathcal{Q}^T)$ condition and Assumptions 4.2.1, 4.2.2, 4.2.4, 4.4.2 and 4.4.12 hold true. Let $x \geq 0$. Then, there exists some optimal strategy $\phi^* \in \Phi(x, U, \mathcal{Q}^T)$ such that*

$$u(x) = \inf_{P \in \mathcal{Q}^T} E_P U(\cdot, V_T^{x,\phi^*}(\cdot)) < \infty.$$

Moreover $\phi_t^*(\cdot) \in D^t(\cdot)$ $\mathcal{Q}^t$-q.s. for all $0 \leq t \leq T$.

Assumption 4.4.12 is not easy to verify: we propose an application of Theorem 4.4.14 in the following fairly general set-up where Assumption 4.4.12 is automatically satisfied. We introduce for all $1 \leq t \leq T$, $r > 0$,

$$\mathcal{W}_t^r := \left\{ X : \Omega^t \to \mathbb{R} \cup \{\pm\infty\},\ \mathcal{B}(\Omega^t)\text{-measurable},\ \sup_{P \in \mathcal{Q}^t} E_P|X|^r < \infty \right\} \quad (4.14)$$

$$\mathcal{W}_t := \bigcap_{r>0} \mathcal{W}_t^r. \quad (4.15)$$

In [51, Proposition 14] it is proved that $\mathcal{W}_t^r$ is a Banach space (up to the usual quotient identifying two random variables that are $\mathcal{Q}^t$-q.s. equal) for the norm $||X|| := (\sup_{P \in \mathcal{Q}^t} E_P|X|^r)^{\frac{1}{r}}$. Hence, the space $\mathcal{W}_t$ is the "natural" extension of the one introduced in the mono-prior case (see [33] or see Chapter 2). We introduce as well

$$\widehat{\mathcal{W}_t} := \left\{ X : \Omega^t \to \mathbb{R} \cup \{\pm\infty\},\ \mathcal{B}_c(\Omega^t)\text{-measurable},\ \sup_{P \in \mathcal{Q}^t} E_P|X|^r < \infty \text{ for all } r > 0 \right\}.$$

Theorem 4.4.15 *Assume that the $sNA(\mathcal{Q}^T)$ condition and Assumptions 4.2.1, 4.2.2, 4.2.4 and 4.4.2 hold true. Assume furthermore that $U^+(\cdot, 1), U^-(\cdot, \frac{1}{4}) \in \mathcal{W}_T$ and that for all $1 \leq t \leq T$, $P \in \mathcal{Q}^t$, $\Delta S_t, \frac{1}{\alpha_t^P} \in \mathcal{W}_t$ (recall Proposition 4.3.6 for the definition of α_t^P). Let $x \geq 0$. Then, for all $P \in \mathcal{Q}^T$, $\phi \in \Phi(x, P)$ and $0 \leq t \leq T$, $V_t^{x,\phi} \in \widehat{\mathcal{W}_t}$. Moreover, there exists some optimal strategy $\phi^* \in \Phi(x, U, \mathcal{Q}^T)$ such that*

$$u(x) = \inf_{P \in \mathcal{Q}^T} E_P U(\cdot, V_T^{x,\phi^*}(\cdot)) < \infty.$$

4.5 One period case

Let $(\overline{\Omega}, \mathcal{G})$ be a measured space, $\mathfrak{P}(\overline{\Omega})$ the set of all probability measures on $\overline{\Omega}$ defined on $\mathcal{G}$ and $\mathcal{Q}$ a non-empty convex subset of $\mathfrak{P}(\overline{\Omega})$. For $P \in \mathcal{Q}$ fixed, we denote by E_P the expectation under P. Let $Y(\cdot) := (Y_1(\cdot), \cdots Y_d(\cdot))$ be a $\mathcal{G}$-measurable $\mathbb{R}^d$-valued random variable. The random variable $Y(\cdot)$ could represent the change of value of the price process.

Assumption 4.5.1 We assume that there exists some constant $0 < b < \infty$ such that $Y_i(\omega) \geq -b$ for all $\omega \in \overline{\Omega}$ and $i = 1, \cdots, d$.

Finally, let $D \subset \mathbb{R}^d$ be the smallest affine subspace of $\mathbb{R}^d$ containing the support of the distribution of $Y(\cdot)$ under P for all $P \in \mathcal{Q}$, $i.e$

$$D = \text{Aff}\left(\bigcap \left\{A \subset \mathbb{R}^d, \text{ closed}, \; P\left(Y(\cdot) \in A\right) = 1, \; \forall P \in \mathcal{Q}\right\}\right).$$

Assumption 4.5.2 We assume that D contains 0, so that D is in fact a non-empty vector subspace of $\mathbb{R}^d$.

The pendant of the $NA(\mathcal{Q}^T)$ condition in the one-period model is given by

Assumption 4.5.3 There exists some constant $0 < \alpha \leq 1$ such that for all $h \in D$ there exists $P_h \in \mathcal{Q}$ satisfying

$$P_h(hY(\cdot) < -\alpha|h|) > \alpha. \tag{4.16}$$

Remark 4.5.4 If $D = \{0\}$ then (4.16) is trivially true.

Remark 4.5.5 Let $h \in \mathbb{R}^d$ and $h' \in \mathbb{R}^d$ be the orthogonal projection of h on D. Then $h - h' \perp D$ hence

$$\{Y(\cdot) \in D\} \subset \{(h - h')Y(\cdot) = 0\} = \{hY(\cdot) = h'Y(\cdot)\}.$$

By definition of D we have $P(Y(\cdot) \in D) = 1$ for all $P \in \mathcal{Q}$ and therefore $hY = h'Y$ $\mathcal{Q}$-q.s.

For $x \geq 0$ and $a \geq 0$ we define

$$\mathcal{H}_x^a := \left\{h \in \mathbb{R}^d,\ x + hY \geq a\ \mathcal{Q}\text{-q.s.}\right\} \tag{4.17}$$

$$\mathcal{H}_x := \mathcal{H}_x^0 = \left\{h \in \mathbb{R}^d,\ x + hY \geq 0\ \mathcal{Q}\text{-q.s.}\right\} \tag{4.18}$$

$$D_x := \mathcal{H}_x \cap D. \tag{4.19}$$

It is clear that for all $a \geq 0$, $x \geq 0$, $\mathcal{H}_x^a$ and D_x are closed subset of $\mathbb{R}^d$.

Remark 4.5.6 Note that we have for $x \geq 0$, $a \geq 0$

$$\mathcal{H}_x^a = \left\{h \in \mathbb{R}^d,\ P(x + hY(\cdot) \geq a) = 1,\ \forall P \in \mathcal{Q}\right\} = \left\{h \in \mathbb{R}^d, \inf_{P \in \mathcal{Q}} P(x + hY(\cdot) \geq a) = 1\right\}.$$

Lemma 4.5.7 *Assume that Assumption 4.5.3 holds true. Then for all* $x \geq 0$*,* $D_x \subset B(0, \frac{x}{\alpha})$ *where* $B(0, \frac{x}{\alpha}) = \{h \in \mathbb{R}^d,\ |h| \leq \frac{x}{\alpha}\}$ *and* D_x *is a convex and compact subspace of* $\mathbb{R}^d$ *.*

Proof. For $x \geq 0$, the convexity and the closedness of D_x are clear. Let $h \in D_x$ be fixed. Assume that $|h| > \frac{x}{\alpha}$ and let $\omega \in \{hY(\cdot) < -\alpha|h|\}$. Then $x + hY(\omega) < x - \alpha|h| < 0$ and from Assumption 4.5.3, there exists $P_h \in \mathcal{Q}$ such that $P_h(x + hY(\cdot) < 0) \geq P_h(hY(\cdot) \leq -\alpha|h|) > \alpha > 0$, a contradiction. The compactness of D_x follows from the boundness property. □

Assumption 4.5.8 We consider a *random utility* $V : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ satisfying the following conditions

- for every $x \in \mathbb{R}$, the function $V(\cdot, x) : \overline{\Omega} \to \mathbb{R}$ is $\mathcal{G}$-measurable,
- for every $\omega \in \overline{\Omega}$, the function $V(\omega, \cdot) : \mathbb{R} \to \mathbb{R}$ is non-decreasing, concave and usc on $\mathbb{R}$,
- $V(\cdot, x) = -\infty$, for all $x < 0$.

As in the previous section we introduce the following assumption, which ensures that Ri(Dom $V(\omega, \cdot)) = (0, \infty)$ for $\mathcal{Q}$-q.s. all $\omega \in \overline{\Omega}$.

Assumption 4.5.9 For all $r \in \mathbb{Q}$, $r > 0$, $\sup_{P \in \mathcal{Q}} E_P V^-(\cdot, r) < \infty$.

Remark 4.5.10 Set

$$\Omega_{Dom} := \{\omega \in \overline{\Omega},\ V(\omega, r) > -\infty,\ \forall r \in \mathbb{Q},\ r > 0\}. \tag{4.20}$$

Under Assumptions 4.5.8 and 4.5.9, applying Proposition 4.4.5 we get that $P(\Omega_{Dom}) = 1$ for all $P \in \mathcal{Q}$. Furthermore, for all $\omega \in \Omega_{Dom}$, Ri($Dom\, V(\omega, \cdot)) = (0, \infty)$ and $V(\omega, \cdot)$ is continuous on $(0, \infty)$ and right-continuous in 0.

Remark 4.5.11 Assumption 4.5.9 is the one-period pendant of Assumption 4.4.2. This assumption is essential to prove in Theorem 4.5.23 that (4.39) holds true as it allows to prove that $\mathbb{Q}^d$ is dense in $\mathrm{Ri}\left(\{h \in \mathcal{H}_x,\ \inf_{P \in \mathcal{Q}} EV(\cdot, x + hY(\cdot)) > -\infty\}\right)$. However, the one-period optimisation problem in (4.21) could be solved without this assumption.

Our main concern in the one period case is the following optimisation problem

$$v(x) := \begin{cases} \sup_{h \in \mathcal{H}_x} \inf_{P \in \mathcal{Q}} E_P V\left(\cdot, x + hY(\cdot)\right), & \text{if } x \geq 0 \\ -\infty, \text{ otherwise.} \end{cases} \tag{4.21}$$

Remark 4.5.12 Recall (see (4.3)) that we set $E_P V\left(\cdot, x + hY(\cdot)\right) = +\infty$ in (4.21) if the integral is not well-defined for some $P \in \mathcal{Q}$. It will be shown in Lemma 4.5.20 that under Assumptions 4.5.2, 4.5.3, 4.5.8, 4.5.9 and 4.5.14, $E_P(V(\cdot, x + hY(\cdot))$ is well-defined and more precisely that $E_P V^+(\cdot, x + hY(\cdot)) < +\infty$ for all $h \in \mathcal{H}_x$ and $P \in \mathcal{Q}$.

Remark 4.5.13 Note as well that from Remark 4.5.5, for $x \geq 0$

$$v(x) = \sup_{h \in D_x} \inf_{P \in \mathcal{Q}} E_P V(\cdot, x + hY(\cdot)). \tag{4.22}$$

We present now some integrability assumptions on V^+ which allow to assert that there exists some optimal solution for (4.21).

Assumption 4.5.14 For every $P \in \mathcal{Q}$, $h \in \mathcal{H}_1$, $E_P V^+(\cdot, 1 + hY(\cdot)) < \infty$.

Remark 4.5.15 If Assumption 4.5.14 is not true, [99, Example 2.3] shows that one can find a counterexample where $v(x) < \infty$ but the supremum is not attained in (4.21). So one cannot use the “natural" extension of the mono-prior approach, which should be that there exists some $P \in \mathcal{Q}$ such that $E_P V^+(\cdot, 1 + hY(\cdot)) < \infty$ for all $h \in \mathcal{H}_1$ (see 2.5.9 in Chapter 2).

We define now

$$v^{\mathbb{Q}}(x) := \begin{cases} \sup_{h \in \mathcal{H}_x \cap \mathbb{Q}^d} \inf_{P \in \mathcal{Q}} E_P V\left(\cdot, x + hY(\cdot)\right), & \text{if } x \geq 0 \\ -\infty, \text{ otherwise.} \end{cases} \tag{4.23}$$

Finally, we introduce the closure of $v^{\mathbb{Q}}$ denoted by $\mathrm{Cl}(v^{\mathbb{Q}})$ which is the smallest usc function $w : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ such that $w \geq v^{\mathbb{Q}}$. We know that (see for example [116, 1(7) p14])

$$\mathrm{Cl}(v^{\mathbb{Q}})(x) = \limsup_{y \to x} v^{\mathbb{Q}}(y). \tag{4.24}$$

We will show in Theorem 4.5.23 that under Assumptions 4.5.1, 4.5.2, 4.5.3, 4.5.8, 4.5.9 and 4.5.14

$$v(x) = v^{\mathbb{Q}}(x) = \mathrm{Cl}(v^{\mathbb{Q}})(x).$$

Remark 4.5.16 The reasons for introducing $v^{\mathbb{Q}}$ and $\mathrm{Cl}(v^{\mathbb{Q}})$ are related to the multi-period setting and the issues arising from the definition of U_t in (4.47) and (4.48): It allows in the multiperiod case to work with a countable supremum (for measurability issues) and an usc function.

We now provide two lemmata which are stated under Assumption 4.5.8 only. They will be used in the multi-period part to prove that the value function is usc, concave (see (4.55) and (4.56)) and dominated (see (4.59)) for all ω^t. This avoid difficult measurability issues when proving (4.57) and (4.58) coming from full-measure sets which are not Borel and on which Assumptions 4.5.2, 3.4.10, 4.5.9 and 4.5.14 hold true. This can be seen for example in the beginning of the proof of Proposition 4.6.12 where we need to apply Lemma 4.5.17 using only Assumption 4.5.8. They are somehow technical lemmata used to solved measurability issues while not directly related to the optimisation problem.

Lemma 4.5.17 *Assume that Assumption 4.5.8 holds true. Then $v^{\mathbb{Q}}$ and Cl($v^{\mathbb{Q}}$) are concave and non-decreasing on $\mathbb{R}$ and*

$$\mathit{Cl}(v^{\mathbb{Q}})(x) = \lim_{\substack{\delta \to 0 \\ \delta > 0}} v^{\mathbb{Q}}(x+\delta). \tag{4.25}$$

Proof. Assume for a moment that $v^{\mathbb{Q}}$ is concave and non-decreasing on $\mathbb{R}$. Using [116, Proposition 2.32 p57], we obtain that $\mathrm{Cl}(v^{\mathbb{Q}})$ is concave on $\mathbb{R}$. Then, recalling (4.24), for all $x \in \mathbb{R}$ we have that

$$\mathrm{Cl}(v^{\mathbb{Q}})(x) = \lim_{\delta \to 0} \sup_{|y-x|<\delta} v^{\mathbb{Q}}(y) = \lim_{\substack{\delta \to 0 \\ \delta > 0}} v^{\mathbb{Q}}(x+\delta). \tag{4.26}$$

Hence (4.25) holds true and the fact that $\mathrm{Cl}(v^{\mathbb{Q}})$ is non-decreasing follows immediately. We prove now that $v^{\mathbb{Q}}$ is concave and non-decreasing on $\mathbb{R}$. As V is non-decreasing (see Assumption 4.5.8), $v^{\mathbb{Q}}$ is clearly non-decreasing. To prove that $v^{\mathbb{Q}}$ is concave, we use similar arguments as in the proof of [99, Lemma 3.5] and [112, Proposition 2] and we first establish that $v^{\mathbb{Q}}$ is midpoint concave. Let $x_1, x_2 \in \mathrm{Dom}\, v^{\mathbb{Q}}$ and $\varepsilon > 0$ be fixed. Assume that $v^{\mathbb{Q}}(x_i) < \infty$ for $i = 1, 2$. Recalling (4.23), for $i = 1, 2$ there exists $h_i \in \mathcal{H}_{x_i} \cap \mathbb{Q}^d$ such that

$$v^{\mathbb{Q}}(x_i) - \varepsilon \leq \inf_{P \in \mathcal{Q}} E_P V(\cdot, x_i + h_i Y(\cdot)). \tag{4.27}$$

Therefore using the concavity of V (see Assumption 4.5.8) we get

$$\begin{aligned}\frac{v^{\mathbb{Q}}(x_1)+v^{\mathbb{Q}}(x_2)}{2} &\leq \varepsilon + \frac{\inf_{P\in\mathcal{Q}} E_P V(\cdot, x_1+h_1Y(\cdot)) + \inf_{P\in\mathcal{Q}} E_P V(\cdot, x_2+h_2Y(\cdot))}{2}\\ &\leq \varepsilon + \inf_{P\in\mathcal{Q}} \frac{E_P V(\cdot, x_1+h_1Y(\cdot)) + E_P V(\cdot, x_2+h_2Y(\cdot))}{2}\\ &\leq \varepsilon + \inf_{P\in\mathcal{Q}} E_P V\left(\cdot, \frac{x_1+x_2}{2} + \frac{h_1+h_2}{2}Y(\cdot)\right)\\ &\leq \varepsilon + \sup_{h\in\mathcal{H}_{\frac{x_1+x_2}{2}}\cap\mathbb{Q}^d} \inf_{P\in\mathcal{Q}} E_P V\left(\cdot, \frac{x_1+x_2}{2} + hY(\cdot)\right)\\ &= \varepsilon + v^{\mathbb{Q}}\left(\frac{x_1+x_2}{2}\right).\end{aligned}$$

As this is true for all $\varepsilon > 0$ we obtain that $\frac{v^{\mathbb{Q}}(x_1)+v^{\mathbb{Q}}(x_2)}{2} \leq v^{\mathbb{Q}}\left(\frac{x_1+x_2}{2}\right)$. Fix again some $\varepsilon > 0$. If now $v^{\mathbb{Q}}(x_1) = +\infty$ and $v^{\mathbb{Q}}(x_2) < \infty$, then there exists some h_1 such that $\frac{1}{\varepsilon} \leq \inf_{P\in\mathcal{Q}} E_P V(\cdot, x_1+h_1Y(\cdot))$. And combining this with (4.27) for $i=2$, we get as previously that $\frac{1}{2\varepsilon} + \frac{v^{\mathbb{Q}}(x_2)}{2} \leq \frac{\varepsilon}{2} + v^{\mathbb{Q}}\left(\frac{x_1+x_2}{2}\right)$. As this is true for all $\varepsilon > 0$ we get that $v^{\mathbb{Q}}\left(\frac{x_1+x_2}{2}\right) = \infty$ and $\frac{v^{\mathbb{Q}}(x_1)+v^{\mathbb{Q}}(x_2)}{2} \leq v^{\mathbb{Q}}\left(\frac{x_1+x_2}{2}\right)$ holds true again. The case $v^{\mathbb{Q}}(x_1) = +\infty$ and $v^{\mathbb{Q}}(x_2) = +\infty$ is solved similarly.
We introduce $M := \sup\{x \geq 0,\ v^{\mathbb{Q}}(x) < \infty\}$. Assume that $0 \leq M < \infty$. Then we have $v^{\mathbb{Q}} = +\infty$ on (M,∞). If $v^{\mathbb{Q}} = -\infty$ on $(-\infty, M)$ then $v^{\mathbb{Q}}$ is concave on $\mathbb{R}$. If there exists some $0 \leq x < M$ such that $v^{\mathbb{Q}}(x)$ is finite, we choose $\varepsilon > 0$ such that $x+\varepsilon < M$ (thus $\frac{M+\varepsilon+x}{2} < M$). From the mid-point concavity we have that

$$v^{\mathbb{Q}}\left(\frac{M+\varepsilon+x}{2}\right) \geq \frac{v^{\mathbb{Q}}(x)+v^{\mathbb{Q}}(M+\varepsilon)}{2} = \infty.$$

But this implies $\frac{M+\varepsilon+x}{2} \geq M$: a contradiction. Therefore we must have $M = +\infty$. Let now $a < b$ with $a \in \operatorname{Dom} v^{\mathbb{Q}}$. As $v^{\mathbb{Q}}$ is non-decreasing, $b \in \operatorname{Dom} v^{\mathbb{Q}}$ and $-\infty < v^{\mathbb{Q}}(a) \leq v^{\mathbb{Q}}(x) \leq v^{\mathbb{Q}}(b) < \infty$ for all $x \in [a,b]$. So applying Ostrowski Theorem, see [55, p12], $v^{\mathbb{Q}}$ is concave on $[a,b]$ and therefore on all $\operatorname{Dom} v^{\mathbb{Q}}$. Applying [115, Theorem 4.2 p25], $v^{\mathbb{Q}}$ is concave on $\mathbb{R}$. □

Let $x \geq 0$ and $P \in \mathcal{Q}$ be fixed. We introduce $H_x(P) := \left\{h \in \mathbb{R}^d,\ x+hY \geq 0\ P\text{-a.s.}\right\}$. Note that $\mathcal{H}_x = \bigcap_{P\in\mathcal{Q}} H_x(P)$ (see (4.18)).

Lemma 4.5.18 *Assume that Assumption 4.5.8 holds true. Let $I : \overline{\Omega}\times\mathbb{R} \to [0,\infty]$ be a function such that for all $x \in \mathbb{R}$ and $h \in \mathbb{R}^d$, $I(\cdot, x+hY(\cdot))$ is $\mathcal{G}$-measurable, $I(\omega,\cdot)$ is non-decreasing and non-negative for all $\omega \in \overline{\Omega}$ and $V \leq I$. Set*

$$i(x) := 1_{[0,\infty)}(x) \sup_{h\in\mathbb{R}^d} \sup_{P\in\mathcal{Q}} 1_{H_x(P)}(h) E_P I(\cdot, x+hY(\cdot)).$$

Then i is non-decreasing, non-negative on $\mathbb{R}$ and Cl$(v^{\mathbb{Q}})(x) \leq i(x+1)$ for all $x \in \mathbb{R}$.

Proof. Since $\omega \in \overline{\Omega} \to I(\omega, x + hY(\omega))$ is $\mathcal{G}$-measurable for all $x \in \mathbb{R}$ and $I(\omega, x) \geq 0$ for $(\omega, x) \in \overline{\Omega} \times \mathbb{R}$, the integral in the definition of i is well-defined (potentially equals to $+\infty$). It is clear that i is non-decreasing and non-negative on $\mathbb{R}$.
As $V(\omega, x) \leq I(\omega, x)$ for all $(\omega, x) \in \overline{\Omega} \times \mathbb{R}$ and $\mathcal{H}_x \subset H_x(P)$ if $P \in \mathcal{Q}$, it is clear that $v^{\mathbb{Q}}(x) \leq i(x)$ for $x \geq 0$. And since $v^{\mathbb{Q}}(x) = -\infty < i(x) = 0$ for $x < 0$, $v^{\mathbb{Q}} \leq i$ on $\mathbb{R}$ (note that $v \leq i$ on $\mathbb{R}$ for the same reasons). Applying Lemma 4.5.17 (see (4.25)), we have that $\mathrm{Cl}(v^{\mathbb{Q}})(x) \leq v^{\mathbb{Q}}(x+1) \leq i(x+1)$ for all $x \in \mathbb{R}$. Note that if Assumptions 4.5.1, 4.5.2, 4.5.3, 4.5.9 and 4.5.14 hold true as well, then from Theorem 4.5.23 we have directly that $\mathrm{Cl}(v^{\mathbb{Q}})(x) = v^{\mathbb{Q}}(x) \leq i(x) \leq i(x+1)$ for all $x \in \mathbb{R}$. □

Proposition 4.5.19 *Assume that Assumptions 4.5.8 and 4.5.9 hold true. Then there exists some $\mathcal{G}$-measurable C with $\sup_{P \in \mathcal{Q}} E_P(C) < \infty$ and $C(\omega) \geq 0$ for all $\omega \in \overline{\Omega}$, such that for all $\omega \in \Omega_{Dom}$ (see (4.20)), $\lambda \geq 1$, $x \in \mathbb{R}$ we have*

$$V(\omega, \lambda x) \leq 2\lambda \left(V\left(\omega, x + \frac{1}{2}\right) + C(\omega) \right), \tag{4.28}$$

Proof. We use similar arguments as [112, Lemma 2]. We fix $\omega \in \Omega_{Dom}$, $x \geq \frac{1}{2}$ and $\lambda \geq 1$. As $\omega \in \Omega_{Dom}$, we have that $\mathrm{Ri}(\mathrm{Dom}\, V(\omega, \cdot)) = (0, \infty)$ (recall Remark 4.5.10). We assume first that there exists some $x_0 \in \mathrm{Dom}\, V(\omega, \cdot)$ such that $V(\omega, x_0) < \infty$. Since $V(\omega, \cdot)$ is usc and concave, using similar arguments as in [115, Corollary 7.2.1 p53], we get that $V(\omega, \cdot) < \infty$ on $\mathbb{R}$. Using the concavity property we get that (recall that $x \geq \frac{1}{2}$)

$$V(\omega, \lambda x) - V(\omega, x) \leq \frac{V(\omega, x) - V\left(\omega, \frac{1}{4}\right)}{x - \frac{1}{4}} (\lambda - 1) x \leq 2(\lambda - 1) \left(V(\omega, x) - V\left(\omega, \frac{1}{4}\right) \right).$$

If follows that

$$\begin{aligned} V(\omega, \lambda x) &\leq V(\omega, x) + 2\left(\lambda - \frac{1}{2}\right)\left(V(\omega, x) - V\left(\omega, \frac{1}{4}\right)\right) \\ &\leq 2\lambda \left(V(\omega, x) + V^{-}\left(\omega, \frac{1}{4}\right) \right) \\ &\leq 2\lambda \left(V\left(\omega, x + \frac{1}{2}\right) + V^{-}\left(\omega, \frac{1}{4}\right) \right) \end{aligned} \tag{4.29}$$

where we have used that $V(\omega, \cdot)$ is non-decreasing. Fix now $0 \leq x \leq \frac{1}{2}$ and $\lambda \geq 1$. Using again the fact that $V(\omega, \cdot)$ is non-decreasing and (4.29)

$$V(\omega, \lambda x) \leq V\left(\omega, \lambda\left(x + \frac{1}{2}\right)\right) \leq 2\lambda \left(V\left(\omega, x + \frac{1}{2}\right) + V^{-}\left(\omega, \frac{1}{4}\right) \right).$$

We set $C(\omega) = V^{-}\left(\omega, \frac{1}{4}\right)$. As $C(\omega) \geq 0$ for all $\omega \in \overline{\Omega}$ and $\sup_{P \in \mathcal{Q}} E_P C < \infty$ (see Assumption 4.5.9), (4.28) holds true for all $x \geq 0$ if there exists some $x_0 \in \mathrm{Dom}\, V(\omega, \cdot)$

such that $V(\omega, x_0) < \infty$. Now, if this is not the case, we know that $V(\omega, x) = \infty$ for all $x \in \operatorname{Dom} V(\omega, \cdot)$. Then $C(\omega) = V^-\left(\omega, \frac{1}{4}\right) = 0$ and (4.28) also holds true for all $x \geq 0$. Finally, as in all cases $V(\omega, x) = -\infty$ for $x < 0$, (4.28) holds true for all $x \in \mathbb{R}$ and the proof is complete. □

The following lemma is the pendant of Lemma 2.5.11 in Chapter 2.

Lemma 4.5.20 *Assume that Assumptions 4.5.2, 4.5.3, 4.5.8, 4.5.9 and 4.5.14 hold true. Then there exists a $\mathcal{G}$-measurable $L \geq 0$ satisfying for all $P \in \mathcal{Q}$, $E_P(L) < \infty$ and such that for all $x \geq 0$ and $h \in \mathcal{H}_x$*

$$V^+(\cdot, x + hY(\cdot)) \leq (4x + 1)\, L(\cdot) \ \mathcal{Q}\text{-q.s.} \tag{4.30}$$

Proof. The proof use similar arguments as in Lemma 2.8 in [99] and is almost a copy word for word of to Lemma 2.5.11 in Chapter 2. We start with the proof of (4.30) when $h \in D_x$. Since D is a vectorial subspace of $\mathbb{R}^d$ (see Assumption 4.5.2) and $0 \in \mathcal{H}_x$, the affine hull of D_x is also a vector space that we denote by Aff(D_x). If $x \leq 1$ we have by Assumption 4.5.8 that for all $\omega \in \overline{\Omega}$, $h \in D_x$,

$$V^+(\omega, x + hY(\omega)) \leq V^+\left(\omega, 1 + hY(\omega)\right). \tag{4.31}$$

If $x > 1$ from Proposition 4.5.19, we get that for all $\omega \in \Omega_{Dom}$, $h \in D_x$

$$\begin{aligned} V^+(\omega, x + hY(\omega)) &= V^+\left(2x\left(\frac{1}{2} + \frac{h}{2x}Y(\omega)\right)\right) \\ &\leq 4x\left(V^+\left(\omega, 1 + \frac{h}{2x}Y(\omega)\right) + C(\omega)\right). \end{aligned} \tag{4.32}$$

First we treat the case of $Dim(\text{Aff}(D_x)) = 0$, *i.e* $D_x = \{0\}$. For all $\omega \in \Omega_{Dom}$, $h \in D_x = \{0\}$, using (4.31) and (4.32), we obtain for all $x \geq 0$ that

$$\begin{aligned} V^+(\omega, x + hY(\omega)) &\leq V^+(\omega, 1) + 4x\left(V^+(\omega, 1) + C(\omega)\right) \\ &\leq (4x + 1)(V^+(\omega, 1) + C(\omega)). \end{aligned} \tag{4.33}$$

We assume now that $Dim(\text{Aff}(D_x)) > 0$. If $x = 0$ then $Y = 0$ $\mathcal{Q}$-q.s. If this is not the case then we should have $D_0 = \{0\}$ a contradiction. Indeed if not, there exists some $h \in D_0$ with $h \neq 0$ and by Assumption 4.5.3 there exist $P_h \in \mathcal{Q}$ such that $P_h\left(\frac{h}{|h|}Y(\cdot) < 0\right) > 0$ which contradicts $h \in D_0$. So for $x = 0$, $Y = 0$ $\mathcal{Q}$-q.s. and by Assumption 4.5.8 we get that for all $\omega \in \overline{\Omega}$, $h \in D_0$,

$$V^+(\omega, 0 + hY(\omega)) \leq V^+(\omega, 1).$$

From now we assume that $x > 0$. Then as for $g \in \mathbb{R}^d$, $g \in D_x$ if and only if $\frac{g}{x} \in D_1$, we have that Aff$(D_x) = $ Aff(D_1). We set $d' := Dim(\text{Aff}(D_1))$. Let $(e_1, \ldots, e_{d'})$

be an orthonormal basis of Aff(D_1) (which is a sub-vector space of $\mathbb{R}^d$) and $\varphi : (\lambda_1, \dots, \lambda_{d'}) \in \mathbb{R}^{d'} \to \Sigma_{i=1}^{d'} \lambda_i e_i \in$ Aff(D_1). Then φ is an isomorphism (recall that $(e_1, \dots, e_{d'})$ is a basis of Aff(D_1)). As φ is linear and the spaces considered are of finite dimension, it is also an homeomorphism between $\mathbb{R}^{d'}$ and Aff(D_1). Since D_1 is compact by Lemma 4.5.7, $\varphi^{-1}(D_1)$ is a compact subspace of $\mathbb{R}^{d'}$. So there exists some $c \geq 0$ such that for all $h = \Sigma_{i=1}^{d'} \lambda_i e_i \in D_1$, $|\lambda_i| \leq c$ for all $i = 1, \dots, d'$. We complete the family of vector $(e_1, \dots, e_{d'})$ in order to obtain an orthonormal basis of $\mathbb{R}^d$, denoted by $(e_1, \dots, e_{d'}, e_{d'+1}, \dots e_d)$. For all $\omega \in \overline{\Omega}$, let $(y_i(\omega))_{i=1,\dots,d}$ be the coordinate of $Y(\omega)$ in this basis.

Now let $h \in D_x$ be fixed. Then $\frac{h}{2x} \in D_{\frac{1}{2}} \subset D_1$ and $\frac{h}{2x} = \Sigma_{i=1}^{d'} \lambda_i e_i$ for some $(\lambda_1, \dots \lambda_{d'}) \in \mathbb{R}^{d'}$ with $|\lambda_i| \leq c$ for all $i = 1, \dots, d'$. Note that as $\frac{h}{2x} \in D_1$, $\lambda_i = 0$ for $i \geq d' + 1$. Then as $(e_1, \dots, e_d)$ is an orthonormal basis of $\mathbb{R}^d$, we obtain for all $\omega \in \overline{\Omega}$

$$1 + \frac{h}{2x} Y(\omega) = 1 + \Sigma_{i=1}^{d'} \lambda_i y_i(\omega) \leq 1 + \Sigma_{i=1}^{d'} |\lambda_i||y_i(\omega)| \leq 1 + c\Sigma_{i=1}^{d'} |y_i(\omega)|.$$

Thus from Assumption 4.5.8 for all $\omega \in \overline{\Omega}$ we get that

$$V^+ \left(\omega, 1 + \frac{h}{2x} Y(\omega) \right) \leq V^+ \left(\omega, 1 + c\Sigma_{i=1}^{d'} |y_i(\omega)| \right).$$

We set for all $\omega \in \overline{\Omega}$

$$L(\omega) := V^+ \left(\omega, 1 + c\Sigma_{i=1}^{d'} |y_i(\omega)| \right) 1_{d'>0} + V^+(\omega, 1) + C(\omega).$$

As $d' = Dim(\text{Aff}(D_1))$ it is clear that L does not dependent on x. It is also clear that L is $\mathcal{G}$-measurable.

Then using (4.31), (4.32) and (4.33) we obtain that for all $\omega \in \Omega_{Dom}$

$$V^+(\omega, x + hY(\omega)) \leq (4x + 1)L(\omega).$$

Note that the first term in L is used in the above inequality if $x \neq 0$ and $Dim(\text{Aff}(D_x)) > 0$. The second and the third one are there for both the case of $Dim(\text{Aff}(D_x)) = 0$ and the case of $x = 0$ and $Dim(\text{Aff}(D_x)) > 0$. Now by Assumptions 4.5.14 and Proposition 4.5.19, we get that $E_P(V^+(\cdot, 1) + C(\cdot)) < \infty$ for all $P \in \mathcal{Q}$, so it remains to prove if $d' > 0$ that for all $P \in \mathcal{Q}$

$$E_P \left(V^+ \left(\cdot, 1 + c\Sigma_{i=1}^{d'} |y_i(\cdot)| \right) \right) < \infty.$$

Introduce W the finite set of $\mathbb{R}^d$ whose coordinates on $(e_1, \dots, e_{d'})$ are 1 or -1 and 0 on $(e_{d'+1}, \dots e_d)$. Then $W \subset$ Aff(D_1) and the vectors of W will be denoted by θ^j for $j \in \{1, \dots, 2^{d'}\}$. Let θ^ω be the vector whose coordinates on $(e_1, \dots, e_{d'})$ are $(sign(y_i(\omega)))_{i=1\dots d'}$ and 0 on $(e_{d'+1}, \dots e_d)$. Then $\theta^\omega \in W$ and we get that

$$V^+ \left(\omega, 1 + c\Sigma_{i=1}^{d'} |y_i(\omega)| \right) = V^+(\omega, 1 + c\theta^\omega Y(\omega)) \leq \sum_{j=1}^{2^{d'}} V^+(\omega, 1 + c\theta^j Y(\omega)).$$

So to prove that for all $P \in \mathcal{Q}$, $E_P L < \infty$ it is sufficient to prove that if $d' > 0$, $E_P V^+(\cdot, 1 + c\theta^j Y(\cdot)) < \infty$ for all $1 \leq j \leq 2^{d'}$ and $P \in \mathcal{Q}$. Recall that $\theta^j \in \text{Aff}(D_1)$. As D_1 is convex and non-empty (recall $d' > 0$), $\text{Ri}(D_1)$ is also non-empty and convex and we fix some $e^* \in \text{Ri}(D_1)$. It is easy to prove that $\frac{e^*}{2} \in \text{Ri}(D_1)$. (see the proof of Lemma 2.5.11 in Chapter 2). Now let $\beta > 0$ be such that $\text{Aff}(D_1) \cap B(e^*, \beta) \subset D_1$ and ε_j be such that $\varepsilon_j(\frac{c}{2}\theta^j - \frac{e^*}{2}) \in B(0, \frac{\beta}{2})$ where one can chose $\varepsilon_j \in (0,1)$. Then as $\bar{e}^j := \frac{e^*}{2} + \frac{\varepsilon_j}{2}(c\theta^j - e^*) \in \text{Aff}(D_1) \cap B(\frac{e^*}{2}, \frac{\beta}{2})$ (recall that $\theta^j \in W \subset \text{Aff}(D_1)$), we deduce that $\bar{e}^j \in D_1$. Using (4.28) we obtain that for $\mathcal{Q}$-almost all ω

$$
\begin{aligned}
V^+(\omega, 1 + c\theta^j Y(\omega)) &= V^+(\omega, 1 + e^* Y(\omega) + (c\theta^j - e^*)Y(\omega)) \\
&\leq \left(\frac{4}{\varepsilon_j}\right)\left[V^+\left(\omega, \frac{\varepsilon_j}{2}(1 + e^* Y(\omega)) + \frac{\varepsilon_j}{2}(c\theta^j - e^*)Y(\omega) + \frac{1}{2}\right) + C(\omega)\right] \\
&\leq \left(\frac{4}{\varepsilon_j}\right)\left[V^+(\omega, 1 + \bar{e}^j Y(\omega)) + C(\omega)\right],
\end{aligned}
$$

where the second inequality follows from the fact that $1 + e^* Y(\cdot) \geq 0$ $\mathcal{Q}$-q.s. (recall that $e^* \in \text{Ri}(D_1)$) and the monotonicity property of V in Assumption 4.5.8. Note that the above inequalities are true even if $1 + c\theta^j Y(\omega) < 0$ since (4.28) and the monotonicity property of V holds true for all $x \in \mathbb{R}$.
For all $P \in \mathcal{Q}$, Assumption 4.5.14 implies that $E_P V^+(\cdot, 1 + \bar{e}^j Y(\cdot)) < \infty$ (recall that $\bar{e}_j \in D_1$) and Proposition 4.5.19 implies that $E_P C < \infty$, therefore $E_P V^+(\cdot, 1 + c\theta^j Y(\cdot)) < \infty$ and (4.30) is proven for $h \in D_x$.
Now let $h \in \mathcal{H}_x$ and h' its orthogonal projection on D, then $hY(\cdot) = h'Y(\cdot)$ $\mathcal{Q}$-q.s. (see Remark 4.5.5). It is clear that $h' \in D_x$ thus $V^+(\cdot, x + hY(\cdot)) = V^+(\cdot, x + h'Y(\cdot))$ $\mathcal{Q}$-q.s. and (4.30) is true also for $h \in \mathcal{H}_x$. □

Lemma 4.5.21 *Assume that Assumptions 4.5.2, 4.5.3, 4.5.8, 4.5.9 and 4.5.14 hold true. Let $\mathcal{H}$ be the set valued function that assigns to each $x \geq 0$ the set $\mathcal{H}_x$. Then Graph$(\mathcal{H}) := \{(x,h) \in [0, +\infty) \times \mathbb{R}^d,\ h \in \mathcal{H}_x\}$ is a closed and convex subset of $\mathbb{R} \times \mathbb{R}^d$. Let $\psi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ be defined by*

$$
\psi(x,h) := \begin{cases} \inf_{P \in \mathcal{Q}} E_P V(\cdot, x + hY(\cdot)) \text{ if } (x,h) \in \text{Graph}(\mathcal{H}), \\ -\infty \text{ otherwise}. \end{cases} \tag{4.34}
$$

Then ψ is usc and concave on $\mathbb{R} \times \mathbb{R}^d$, $\psi < +\infty$ on Graph$(\mathcal{H})$ and $\psi(x,0) > -\infty$ for all $x > 0$.

Proof. For all $P \in \mathcal{Q}$, we introduce, $\psi_P : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$
\psi_P(x,h) := \begin{cases} E_P V(\cdot, x + hY(\cdot)) \text{ if } (x,h) \in \text{Graph}(\mathcal{H}), \\ -\infty \text{ otherwise}. \end{cases} \tag{4.35}
$$

As in Lemma 2.5.12 in Chapter 2, Graph($\mathcal{H}$) is a closed convex subset of $\mathbb{R} \times \mathbb{R}^d$, ψ_P is usc on $\mathbb{R} \times \mathbb{R}^d$ and $\psi_P < \infty$ on Graph($\mathcal{H}$) for all $P \in \mathcal{Q}$. Furthermore the concavity of ψ_P follows immedialty from the one of V.
The function $\psi = \inf_{P \in \mathcal{Q}} \psi_P$ is usc (resp. concave) as the infimum of usc (resp. concave) functions. As $\psi_P < \infty$ on Graph($\mathcal{H}$) for all $P \in \mathcal{Q}$, it is clear that $\psi < +\infty$ on Graph($\mathcal{H}$). Finally let $x > 0$ be fixed and let $r \in \mathbb{Q}$, $r < x$. From Assumptions 4.5.8 and 4.5.9 we have $-\infty < \psi(r, 0) \leq \psi(x, 0)$ and this concludes the proof. □
The next results is used in the proof of Theorem 4.5.23 (and also in the multi-period part in the proof of Proposition 4.6.6).

Lemma 4.5.22 *Assume that Assumption 4.5.1 holds true. For all $x > 0$, we have Aff($\mathcal{H}_x$) $= \mathbb{R}^d$, Ri($\mathcal{H}_x$) is an open set in $\mathbb{R}^d$ and $\mathbb{Q}^d$ is dense in Ri($\mathcal{H}_x$) [1] . Moreover*

$$Ri(\mathcal{H}_x) \subset \bigcup_{r \in \mathbb{Q},\, r>0} \mathcal{H}_x^r \subset \mathcal{H}_x \tag{4.36}$$

and therefore $\overline{\bigcup_{r \in \mathbb{Q},\, r>0} \mathcal{H}_x^r} = \mathcal{H}_x$, where the closure is taken in $\mathbb{R}^d$. If furthermore, we assume that there exists some $0 \leq c < \infty$ such that $Y_i(\omega) \leq c$ for all $i = 1, \cdots, d$, $\omega \in \overline{\Omega}$ (recalling Assumption 4.5.1, $|Y|$ is bounded) then

$$Ri(\mathcal{H}_x) = \bigcup_{r \in \mathbb{Q},\, r>0} \mathcal{H}_x^r. \tag{4.37}$$

Proof. Fix some $x > 0$. Let $\varepsilon > 0$ be such that $x - \varepsilon > 0$. Using Assumption 4.5.1, for $h \in \mathbb{R}^d$ such that $0 \leq h_i < \frac{x-\varepsilon}{db}$ for all $i = 1, \cdots, d$, we get that for all $\omega \in \overline{\Omega}$, $x + hY(\omega) \geq x - b\sum_{i=1}^d h_i \geq \varepsilon$. Hence $h \in \mathcal{H}_x^\varepsilon \subset \mathcal{H}_x$. Thus the set $R := \{h \in \mathbb{R}^d,\ 0 \leq h_i \leq \frac{x-\varepsilon}{db}\} \subset \mathcal{H}_x$. As Aff($R$) $= \mathbb{R}^d$, we obtain that Aff($\mathcal{H}_x$) $= \mathbb{R}^d$ (recall that $0 \in \mathcal{H}_x$). Therefore Ri($\mathcal{H}_x$) is the interior of $\mathcal{H}_x$ in $\mathbb{R}^d$ and thus an open set in $\mathbb{R}^d$ and the fact that $\mathbb{Q}^d$ is dense in Ri($\mathcal{H}_x$) follows immediatly.
We prove now (4.36). The second inclusion is trivial. Fix now some $h \in$ Ri($\mathcal{H}_x$). As $0 \in \mathcal{H}_x$, there exists some $\varepsilon > 0$ such that $(1+\varepsilon)h \in \mathcal{H}_x$, see [115, Theorem 6.4 p47], *i.e* $x + (1+\varepsilon)hY(\cdot) \geq 0$ $\mathcal{Q}$-q.s. It follows that $x + hY(\cdot) \geq \frac{\varepsilon}{1+\varepsilon}x > 0$ $\mathcal{Q}$-q.s., hence for $0 < r \leq \frac{\varepsilon}{1+\varepsilon}x$, $r \in \mathbb{Q}$, we have that $h \in \mathcal{H}_x^r$ and (4.36) is proved. The last equation follows from the fact that $\overline{\text{Ri}(\mathcal{H}_x)} = \mathcal{H}_x$ (recall that Ri($\mathcal{H}_x$) the interior of $\mathcal{H}_x$ in $\mathbb{R}^d$).
We prove now (4.37) under the assumption that $|Y|$ is bounded by some constant $K > 0$. We fix some $h \in \bigcup_{r \in \mathbb{Q},\, r>0} \mathcal{H}_x^r$ and we establish that h belongs to the interior of $\mathcal{H}_x$ and therefore to Ri($\mathcal{H}_x$). Let $r \in \mathbb{Q}$, $r > 0$ be such that $h \in \mathcal{H}_x^r$ and set $\varepsilon := \frac{r}{2K}$. Then for any $g \in B(0, \varepsilon)$, we have for $\mathcal{Q}$-almost all $\omega \in \overline{\Omega}$ that $x + (h+g)Y(\omega) \geq r + gY(\omega) \geq r - |g||Y(\omega)| \geq \frac{r}{2}$, hence $h + g \in \mathcal{H}_x$ and $B(h, \varepsilon) \subset \mathcal{H}_x$.

[1] For a Polish space X, we say that a set $D \subset X$ is dense in $B \subset X$ if for all $\varepsilon > 0$, $b \in B$, there exists $d \in D \cap B$ such that $d(b, d) < \varepsilon$ where d is a metric on X consistant with its topology.

□

We are now able to state our main result.

Theorem 4.5.23 *Assume that Assumptions 4.5.1, 4.5.2, 4.5.3, 4.5.8, 4.5.9 and 4.5.14 hold true. Then for all $x \geq 0$, $v(x) < \infty$ and there exists some optimal strategy $\widehat{h} \in D_x$ such that*

$$v(x) = \inf_{P \in \mathcal{Q}} E_P(V(\cdot, x + \widehat{h}Y(\cdot))). \tag{4.38}$$

Moreover v is usc, concave and non-decreasing, Dom $v = (0, \infty)$. For all $x \in \mathbb{R}$

$$v(x) = v^{\mathbb{Q}}(x) = \mathit{Cl}(v^{\mathbb{Q}})(x). \tag{4.39}$$

Proof. Let $x \geq 0$ be fixed. Fix some $P \in \mathcal{Q}$. Using Lemma 4.5.20 we have that $E_P V(\cdot, x + hY(\cdot)) \leq E_P V^+(\cdot, x + hY(\cdot)) \leq (4x + 1)\, E_P L(\cdot) < \infty$, for all $h \in \mathcal{H}_x$. Thus $v(x) < \infty$. Now if $x > 0$, $v(x) \geq \psi(x, 0) > -\infty$ (see Lemma 4.5.21). Using Lemma 4.5.17, v is concave and non-decreasing. Thus v is continuous on $(0, \infty)$.
From Lemma 4.5.21, $h \to \psi(x, h)$ is usc on $\mathbb{R}^d$ and thus on D_x (recall that D_x is closed and use Lemma 2.8.11 in Chapter 2. Since D_x is compact (see Lemma 4.5.7), recalling (4.22) and applying [3, Theorem 2.43 p44], we find that there exists some $\widehat{h} \in D_x$ such that (3.31) holds true.
We prove now that v is usc in 0 (the proof works as well for all $x^* \geq 0$). Let $(x_n)_{n \geq 0}$ be a sequence of non-negative numbers converging to 0. Let $\widehat{h}_n \in D_{x_n}$ be the optimal strategies associated to x_n in (3.31). Let $(n_k)_{k \geq 1}$ be a subsequence such that $\limsup_n v(x_n) = \lim_k v(x_{n_k})$. Using Lemma 4.5.7, $|\widehat{h}_{n_k}| \leq x_{n_k}/\alpha \leq 1/\alpha$ for k big enough. So we can extract a subsequence (that we still denote by $(n_k)_{k \geq 1}$) such that there exists some $\underline{h}^*$ with $\widehat{h}_{n_k} \to \underline{h}^*$. As $(x_{n_k}, \hat{h}_{n_k})_{k \geq 1} \in \text{Graph}(\mathcal{H})$ which is a closed subset of $\mathbb{R} \times \mathbb{R}^d$ (see Lemma 4.5.21), $\underline{h}^* \in \mathcal{H}_0$. Thus using that ψ is usc, we get that

$$\begin{aligned} \limsup_n v(x_n) &= \lim_k \inf_{P \in \mathcal{Q}} E_P V(\cdot, x_{n_k} + \widehat{h}_{n_k} Y(\cdot)) = \lim_k \psi(x_{n_k}, h_{n_k}) \\ &\leq \psi(0, \underline{h}^*) = \inf_{P \in \mathcal{Q}} E_P V(\cdot, \underline{h}^* Y(\cdot)) \leq v(0). \end{aligned}$$

For $x < 0$ all the equalities in (4.39) are trivial. We prove the first equality in (4.39) for $x \geq 0$ fixed. We start with the case $x = 0$. If $Y = 0$ $\mathcal{Q}$-q.s. then the first equality is trivial. If $Y \neq 0$ $\mathcal{Q}$-q.s., then it is clear that $D_0 = \{0\}$ (recall Assumption 4.5.2) and the first equality in (4.39) is true again. We assume now that $x > 0$. From Lemma 4.5.21, $\psi_x : h \to \psi(x, h)$ is concave, $0 \in \text{Dom}\, \psi_x$. Thus $\text{Ri}(\text{Dom}\, \psi_x) \neq \emptyset$ (see [115, Theorem 6.2 p45]) and we can apply Lemma 4.8.1. Assume for a moment that we have proved that $\mathbb{Q}^d$ is dense in $\text{Ri}(\text{Dom}\, \psi_x)$. As ψ_x is continuous on $\text{Ri}(\text{Dom}\, \psi_x)$

(recall that ψ_x is concave), we obtain that

$$\begin{aligned}v(x) = \sup_{h\in\mathcal{H}_x} \psi_x(h) = \sup_{h\in \mathrm{Dom}\,\psi_x} \psi_x(h) &= \sup_{h\in \mathrm{Ri}(\mathrm{Dom}\,\psi_x)} \psi_x(h)\\ &= \sup_{h\in \mathrm{Ri}(\mathrm{Dom}\,\psi_x)\cap\mathbb{Q}^d} \psi_x(h)\\ &\leq \sup_{h\in\mathcal{H}_x\cap\mathbb{Q}^d} \psi_x(h) = v^{\mathbb{Q}}(x),\end{aligned}$$

since $\mathrm{Ri}(\mathrm{Dom}\,\psi_x) \subset \mathcal{H}_x$ and the first equality in (4.39) is proved. It remains to prove that $\mathbb{Q}^d$ is dense in $\mathrm{Ri}(\mathrm{Dom}\,\psi_x)$. Fix some $h \in \mathrm{Ri}(\mathcal{H}_x)$. From Lemma 4.5.22, there is some $r \in \mathbb{Q}$, $r > 0$ such that $h \in \mathcal{H}_x^r$. Using Lemma 4.5.21 we obtain that $\psi_x(h) \geq \psi(r,0) > -\infty$ thus $h \in \mathrm{Dom}\,\psi_x$ and $\mathrm{Ri}(\mathcal{H}_x) \subset \mathrm{Dom}\,\psi_x$. Recalling that $0 \in \mathrm{Dom}\,\psi_x$ and that $\mathrm{Ri}(\mathcal{H}_x)$ is an open set in $\mathbb{R}^d$ (see Lemma 4.5.22) we obtain that $\mathrm{Aff}(\mathrm{Dom}\,\psi_x) = \mathbb{R}^d$. Then $\mathrm{Ri}(\mathrm{Dom}\,\psi_x)$ is an open set in $\mathbb{R}^d$ and the fact that $\mathbb{Q}^d$ is dense in $\mathrm{Ri}(\mathrm{Dom}\,\psi_x)$ follows easily.
The second equality in (4.39) follows immediately : $v^{\mathbb{Q}}(x) = v(x)$ for all $x \geq 0$ and v is usc on $[0,\infty)$ thus $\mathrm{Cl}(v^{\mathbb{Q}})(x) = v^{\mathbb{Q}}(x)$ for all $x \geq 0$. □

4.6 Multiperiod case

Proposition 4.6.1 *Assume that Assumption 4.4.2 holds true. Then there exists some $\Omega^T_{Dom} \in \mathcal{B}(\Omega^T)$ such that $P(\Omega^T_{Dom}) = 1$ for all $P \in \mathcal{Q}^T$ and a $\mathcal{B}(\Omega^T)$-measurable random variable C_T, satisfying $C_T(\omega^T) \geq 0$ for all $\omega^T \in \Omega^T$, $\sup_{P\in\mathcal{Q}^T} E_P(C_T) < \infty$ and such that for all $\omega^T \in \Omega^T_{Dom}$, $\lambda \geq 1$ and $x \in \mathbb{R}$, we have*

$$\begin{aligned}U(\omega^T,\lambda x) &\leq 2\lambda \left(U\left(\omega^T, x+\frac{1}{2}\right) + C_T(\omega^T)\right)\\ U^+(\omega^T,\lambda x) &\leq 2\lambda \left(U^+\left(\omega^T, x+\frac{1}{2}\right) + C_T(\omega^T)\right).\end{aligned}$$

Proof. This is just Proposition 4.5.19 for $V = U$ and $\mathcal{G} = \mathcal{B}(\Omega^T)$. Here we set (see (4.20))

$$\Omega^T_{Dom} := \{\omega^T \in \Omega^T,\ U(\omega^T,r) > -\infty,\ \forall r \in \mathbb{Q},\ r > 0\} \tag{4.40}$$

and $C_T(\omega^T) = U^-\left(\omega^T, \frac{1}{4}\right)$. From Assumption 4.4.2, it is clear that $\sup_{P\in\mathcal{Q}^T} E_P(C_T) < \infty$. The second inequality follows immediatly since C_T is non-negative. □

Proposition 4.6.2 *Let Assumptions 4.4.2 and 4.4.12 hold true and fix some $x \geq 0$. Then*

$$M_x := \sup_{P \in \mathcal{Q}^T} \sup_{\phi \in \phi(x,P)} E_P U^+(\cdot, V_T^{x,\phi}(\cdot)) < \infty. \tag{4.41}$$

Moreover, $\Phi(x, U, P) = \Phi(x, P)$ *for all* $P \in \mathcal{Q}^T$ *and thus* $\Phi(x, U, \mathcal{Q}^T) = \Phi(x, \mathcal{Q}^T)$.

Proof. Fix some $P \in \mathcal{Q}^T$. From Assumption 4.4.12 we know that $\Phi(1, P) = \Phi(1, U, P)$. Let $x \geq 0$ and $\phi \in \Phi(x, P)$ be fixed. If $x \leq 1$ then $V_T^{x,\phi} \leq V_T^{1,\phi}$ and therefore using Assumption 4.4.12, $M_x \leq M_1 < \infty$ and $\Phi(x, P) = \Phi(x, U, P)$. If $x \geq 1$, from Proposition 4.6.1 we get that for all $\omega^T \in \Omega^T_{Dom}$

$$\begin{aligned} U^+(\omega^T, V_T^{x,\phi}(\omega^T)) &= U^+\left(\omega^T, 2x\left(\frac{1}{2} + \sum_{t=1}^T \frac{\phi_t(\omega^{t-1})}{2x} \Delta S_t(\omega^t)\right)\right) \\ &\leq 4x \left(U^+(\omega^T, V_T^{1,\frac{\phi}{2x}}(\omega)) + C_T(\omega^T)\right). \end{aligned}$$

As since $\frac{\phi}{2x} \in \Phi(\frac{1}{2}, P) \subset \Phi(1, P) = \Phi(1, U, P)$, we find using Proposition 4.6.1 again that $M_x \leq 4x\left(M_1 + \sup_{P \in \mathcal{Q}^T} E_P C_T\right) < \infty$. Thus $\Phi(x, P) = \Phi(x, U, P)$ and the last assertion follows from (4.11). □

We introduce now the dynamic programming procedure. First we set for all $t \in \{0, \ldots, T-1\}$, $\omega^t \in \Omega^t$, $P \in \mathfrak{P}(\Omega_{t+1})$ and $x \geq 0$

$$H_x^{t+1}(\omega^t, P) := \left\{h \in \mathbb{R}^d,\ x + h\Delta S_{t+1}(\omega^t, \cdot) \geq 0\ P\text{-a.s.}\right\}, \tag{4.42}$$

$$\mathcal{H}_x^{t+1}(\omega^t) := \left\{h \in \mathbb{R}^d,\ x + h\Delta S_{t+1}(\omega^t, \cdot) \geq 0\ \mathcal{Q}_{t+1}(\omega^t)\text{-q.s.}\right\}, \tag{4.43}$$

$$\mathcal{D}_x^{t+1}(\omega^t) := \mathcal{H}_x^{t+1}(\omega^t) \cap D^{t+1}(\omega^t), \tag{4.44}$$

where D^{t+1} was introduced in Definition 4.3.1. For all $t \in \{0, \ldots, T-1\}$, $\omega^t \in \Omega^t$, $P \in \mathfrak{P}(\Omega_{t+1})$ and $x < 0$, we set $H_x^{t+1}(\omega^t, P) = \mathcal{H}_x^{t+1}(\omega^t) = \emptyset$. We introduce the following notation. Let $F : \Omega^t \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ and fix $\omega^t \in \Omega^t$. Then, $x \in \mathbb{R} \to F_{\omega^t}(x) := F(\omega^t, x)$ is a real-valued function and we denote its closure by $\mathrm{Cl}\,(F_{\omega^t})$. Now $\mathrm{Cl}(F) : \Omega^t \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is defined for all $\omega^t \in \Omega^t$, $x \in \mathbb{R}$ by

$$\mathrm{Cl}(F)(\omega^t, x) := \mathrm{Cl}\,(F_{\omega^t})\,(x). \tag{4.45}$$

We define for all $t \in \{0, \ldots, T\}$ the following functions U_t from $\Omega^t \times \mathbb{R} \to \mathbb{R}$. Starting with $t = T$, we set for all $x \in \mathbb{R}$, $\omega^T \in \Omega^T$

$$U_T(\omega^T, x) := U(\omega^T, x). \tag{4.46}$$

Recall that $U(\omega^T, x) = -\infty$ for all $(\omega^T, x) \in \Omega^T \times (-\infty, 0)$. By Definition 4.4.1, it is clear that $U_T(\omega^T, x) = \mathrm{Cl}(U)(\omega^T, x)$. For $0 \leq t \leq T-1$, we set for all $x \in \mathbb{R}$ and $\omega^t \in \Omega^t$

$$\mathcal{U}_t(\omega^t, x) := \begin{cases} \sup_{h \in \mathcal{H}_x^{t+1}(\omega^t) \cap \mathbb{Q}^d} \inf_{P \in \mathcal{Q}_{t+1}(\omega^t)} \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, x + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) P(d\omega_{t+1}), \\ \text{if } x \geq 0 \text{ and } -\infty \text{ if } x < 0 \end{cases} \tag{4.47}$$

$$U_t(\omega^t, x) := \mathrm{Cl}(\mathcal{U}_t)(\omega^t, x). \tag{4.48}$$

As already mentioned for $t = 0$ we drop the dependancy in ω_0 and note $\mathcal{U}_0(x) = \mathcal{U}_0(\omega^0, x)$ and $U_0(x) = U_0(\omega^0, x)$.

Remark 4.6.3 Recall that in (4.47) if for some $\omega^t \in \Omega^t$ and $P \in \mathcal{Q}_{t+1}(\omega^t)$, $\int_{\Omega_{t+1}} U_{t+1}^+(\omega^t, \omega_{t+1}, x + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) P(d\omega_{t+1}) = +\infty$ and $\int_{\Omega_{t+1}} U_{t+1}^-(\omega^t, \omega_{t+1}, x + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) P(d\omega_{t+1}) = \infty$, we set $\int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, x + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) P(d\omega_{t+1}) = \infty$.

Remark 4.6.4 The natural definition of U_t should have been (for $x \geq 0$)

$$\mathfrak{U}_t(\omega^t, x) = \sup_{h \in \mathcal{H}_x^{t+1}(\omega^t)} \inf_{P \in \mathcal{Q}_{t+1}(\omega^t)} \int_{\Omega_{t+1}} \mathfrak{U}_{t+1}(\omega^t, \omega_{t+1}, x + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) P(d\omega_{t+1}).$$

However proving directly some measurability properties for $\mathfrak{U}_t$ is problematic. Hence we define U_t thanks to (4.47) and (4.48).

We also introduce the function $I_t : \Omega^t \times \mathbb{R} \to [0, \infty]$ which will be used for integrability issues. Starting with $t = T$, we set for all $x \in \mathbb{R}$, $\omega^T \in \Omega^T$,

$$I_T(\omega^T, x) := U^+(\omega^T, x). \tag{4.49}$$

And for $0 \leq t \leq T-1$, $\omega^t \in \Omega^t$,

$$I_t(\omega^t, x) := 1_{[0,\infty)}(x) \sup_{h \in \mathbb{R}^d, P \in \mathcal{Q}_{t+1}(\omega^t)} 1_{H_x^{t+1}(\omega^t, P)}(h) \int_{\Omega_{t+1}} I_{t+1}(\omega^t, \omega_{t+1}, x + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) P(d\omega_{t+1}). \tag{4.50}$$

Lemma 4.6.5 *Assume that Assumptions 4.2.1 and 4.2.2 hold true. Let $0 \leq t \leq T-1$ be fixed, G be a fixed non-negative real-valued, $\mathcal{B}_c(\Omega^t)$-measurable random variable and consider the following random sets:*

$$\begin{aligned} H^{t+1} &: (\omega^t, x, P) \in \Omega^t \times \mathbb{R} \times \mathfrak{P}(\Omega_{t+1}) \twoheadrightarrow H_x^{t+1}(\omega^t, P), \\ \mathcal{H}^{t+1} &: (\omega^t, x) \in \Omega^t \times \mathbb{R} \twoheadrightarrow \mathcal{H}_x^{t+1}(\omega^t), \\ \mathcal{H}_G^{t+1} &: \omega^t \in \Omega^t \twoheadrightarrow \mathcal{H}_{G(\omega^t)}^{t+1}(\omega^t), \\ \mathcal{D}_G^{t+1} &: \omega^t \in \Omega^t \twoheadrightarrow \mathcal{D}_{G(\omega^t)}^{t+1}(\omega^t). \end{aligned}$$

Then those random sets are all closed valued. Furthermore we have that Graph$(H^{t+1}) \in \mathcal{B}(\Omega^t) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathfrak{P}(\Omega_{t+1})) \otimes \mathcal{B}(\mathbb{R}^d)$, *Graph*$(\mathcal{H}^{t+1}) \in \mathcal{CA}(\Omega^t \times \mathbb{R} \times \mathbb{R}^d)$. *Graph*$(\mathcal{H}^{t+1}) \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$, *Graph*$(\mathcal{H}_G^{t+1}) \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$ *and Graph*$(\mathcal{D}_G^{t+1}) \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$.

Remark 4.6.6 From Lemma 4.6.5 we get that $(\omega^t, P, h, x) \in \Omega^t \times \mathfrak{P}(\Omega_{t+1}) \times \mathbb{R}^d \times \mathbb{R} \to 1_{H_x^{t+1}(\omega^t,P)}(h)$ is $\mathcal{B}(\Omega^t) \otimes \mathcal{B}(\mathfrak{P}(\Omega_{t+1})) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R})$-measurable.

Proof. It is clear that H^{t+1}, $\mathcal{H}^{t+1}$, $\mathcal{H}_G$ and $\mathcal{D}_G$ are closed valued. Now, using Lemma 4.8.5 (see (4.92)) we get that

$$\begin{aligned}\text{Graph}(H^{t+1}) = &\left\{(\omega^t, x, P, h) \in \Omega^t \times \mathbb{R} \times \mathfrak{P}(\Omega_{t+1}) \times \mathbb{R}^d,\ P(x + h\Delta S_{t+1}(\omega^t, \cdot) \geq 0) = 1\right\} \\ &\in \mathcal{B}(\Omega^t) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathfrak{P}(\Omega_{t+1})) \otimes \mathcal{B}(\mathbb{R}^d).\end{aligned}$$

Using Lemma 4.8.5 (see (4.93)), we get that

$$\begin{aligned}\text{Graph}(\mathcal{H}^{t+1}) = &\left\{(\omega^t, x, h) \in \Omega^t \times \mathbb{R} \times \mathbb{R}^d, \inf_{P \in \mathcal{Q}_{t+1}(\omega^t)} P\left(x + h\Delta S_{t+1}(\omega^t, \cdot) \geq 0\right) = 1\right\} \\ &\in \mathcal{CA}(\Omega^t \times \mathbb{R} \times \mathbb{R}^d).\end{aligned}$$

We prove now the second part of the lemma. Fix some $x \in \mathbb{R}$. For any integer $k \geq 1$, $r \in \mathbb{Q}$, $r > 0$ we introduce the following $\mathbb{R}^d$-valued random variable and random sets

$$\begin{aligned}\Delta S_{k,t+1}(\omega^{t+1}) &:= \Delta S_{t+1}(\omega^{t+1}) 1_{\{|\Delta S_{t+1}(\cdot)| \leq k\}}(\omega^{t+1}), \\ \mathcal{H}_{k,x}^{r,t+1}(\omega^t) &:= \left\{h \in \mathbb{R}^d,\ x + \Delta S_{k,t+1}(\omega^t, \cdot) \geq r\ \mathcal{Q}_{t+1}(\omega^t)\text{-q.s.}\right\},\end{aligned}$$

for all $\omega^{t+1} \in \Omega^t$, $\omega^t \in \Omega^t$. In the sequel, we will write $\mathcal{H}_{k,x}^{t+1}(\omega^t)$ instead of $\mathcal{H}_{k,x}^{0,t+1}(\omega^t)$. We first prove that Graph $(\mathcal{H}_x^{t+1}) \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$. We fix some $k \geq 1$. As $\Delta S_{k,t+1}$ is bounded, we can apply Lemma 4.5.22 and we get that for all $\omega^t \in \Omega^t$, $\text{Ri}(\mathcal{H}_{k,x}^{t+1}(\omega^t))$ is the interior of $\mathcal{H}_{k,x}^{t+1}(\omega^t)$ and that

$$\text{Ri}(\mathcal{H}_{k,x}^{t+1})(\omega^t) = \bigcup_{r \in \mathbb{Q},\, r > 0} \mathcal{H}_{k,x}^{r,t+1}(\omega^t). \tag{4.51}$$

Now, using Lemma 4.8.5 (see (4.93)) and Lemma 4.8.4 (it is clear that $\Delta S_{k,t+1}$ is $\mathcal{B}(\Omega^{t+1})$-measurable) we obtain that for all $r \in \mathbb{Q}$, $r > 0$,

$$\begin{aligned}\text{Graph}\left(\mathcal{H}_{k,x}^{r,t+1}\right) = &\left\{(\omega^t, h) \in \Omega^t \times \mathbb{R}^d, \inf_{P \in \mathcal{Q}_{t+1}(\omega^t)} P\left(x + h\Delta S_{k,t+1}(\omega^t, \cdot) \geq r\right) = 1\right\} \\ &\in \mathcal{CA}(\Omega^t \times \mathbb{R}^d).\end{aligned}$$

So from (4.51) and [13, Corollary 7.35.2 p160], we obtain that Graph $\left(\text{Ri}(\mathcal{H}_{k,x}^{t+1})\right) \in \mathcal{CA}(\Omega^t \times \mathbb{R}^d)$. We can now apply Lemma 4.8.6 *ii)* to $\text{Ri}(\mathcal{H}_{k,x}^{t+1})$ and we obtain that

Graph $\left(\mathrm{Ri}(\mathcal{H}_{k,x}^{t+1})\right) \in \mathcal{B}_c(\Omega^t)\otimes\mathcal{B}(\mathbb{R}^d)$. Using that $\overline{\mathrm{Ri}(\mathcal{H}_{k,x}^{t+1})(\omega^t)} = \mathcal{H}_{k,x}^{t+1}(\omega^t)$ for all $\omega^t \in \Omega^t$ and applying Lemma 4.8.6 $i)$ we get that Graph $\left(\mathcal{H}_{k,x}^{t+1}\right) \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$. Then for all $\omega^t \in \Omega^t$ we have that $\mathcal{H}_x^{t+1}(\omega^t) = \bigcap_{k\in\mathbb{N},\, k\geq 1} \mathcal{H}_{k,x}^{t+1}(\omega^t)$, since for any $h \in \mathbb{R}^d$

$$\begin{aligned}&\left\{\omega_{t+1} \in \Omega_{t+1},\ x + h\Delta S_{t+1}(\omega^t, \omega_{t+1}) \geq 0)\right\}\\ &= \bigcap_{k\geq 1} \left\{\omega_{t+1} \in \Omega_{t+1},\ x + h\Delta S_{k,t+1}(\omega^t, \omega_{t+1}) \geq 0)\right\}.\end{aligned}$$

So $Graph(\mathcal{H}_x^{t+1}) \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$ follows immediately. We set

$$B := \bigcap_{n\in\mathbb{N},\, n\geq 1} \bigcup_{q\in\mathbb{Q},\, q\geq 0} \left\{(\omega^t, x, h) \in \Omega^t \times \mathbb{R} \times \mathbb{R}^d,\ q \leq x \leq q + \frac{1}{n}, h \in \text{Graph}\left(\mathcal{H}_{q+\frac{1}{n}}^{t+1}\right)\right\}$$

For some integer $n \geq 1$ and $q \in \mathbb{Q},\ q \geq 0$ fixed, since $Graph(\mathcal{H}_q^{t+1}) \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$, we get that

$$\left\{(\omega^t, x, h) \in \Omega^t \times \mathbb{R} \times \mathbb{R}^d,\ q \leq x \leq q + \frac{1}{n}, h \in Graph(\mathcal{H}_q^{t+1})\right\} \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$$

and thus $B \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$. We prove now that Graph$(\mathcal{H}^{t+1}) = B$ and thus Graph$(\mathcal{H}^{t+1}) \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$. Let $(\omega^t, x, h) \in$ Graph$(\mathcal{H}^{t+1})$. It is clear that $x \geq 0$, hence for all $n \geq 1$, there exists some non-negative rational q_n such that $q_n \leq x \leq q_n + \frac{1}{n}$. It follows that $q_n + \frac{1}{n} + h\Delta S_{t+1}(\omega^t, \cdot) \geq x + h\Delta S_{t+1}(\omega^t, \cdot) \geq 0$ $\mathcal{Q}^{t+1}(\omega^t)$-q.s for all $n \geq 1$ and $(\omega^t, x, h) \in B$. Now, let $(\omega^t, x, h) \in B$. There exists a sequence $(q_n)_{n\geq 1}$ of rational numbers converging to x, such that $q_n \leq x \leq q_n + \frac{1}{n}$ and $h \in$ Graph $\left(\mathcal{H}_{q_n+\frac{1}{n}}^{t+1}\right)$. Set $C_n := \{q_n + \frac{1}{n} + h\Delta S_{t+1}(\omega^t, \cdot) \geq 0\}$ and $C := \{x + h\Delta S_{t+1}(\omega^t, \cdot) \geq 0\}$. It is clear that $\cap_{n\geq 1} C_n \subset C$, therefore for all $P \in \mathcal{Q}_{t+1}(\omega^t)$ we have that $P(C) \geq P\left(\cap_{n\geq 1} C_n\right) = 1$ and $(\omega^t, x, h) \in$ Graph$(\mathcal{H}^{t+1})$ follows.

We prove now that Graph$(\mathcal{H}_G^{t+1}) \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$. Using similar arguments as before and the fact that G is $\mathcal{B}_c(\Omega^t)$-measurable we obtain that

$$\begin{aligned}&\text{Graph}(\mathcal{H}_G^{t+1})\\ &= \bigcap_{n\in\mathbb{N},\, n\geq 1} \bigcup_{q\in\mathbb{Q},\, q\geq 0} \left\{(\omega^t, h) \in \Omega^t \times \mathbb{R} \times \mathbb{R}^d,\ q \leq G(\omega^t) \leq q + \frac{1}{n}, h \in \text{Graph}(\mathcal{H}_{q+\frac{1}{n}}^{t+1})\right\}\\ &\in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d).\end{aligned}$$

Finally using Lemma 4.3.2 and since Graph$(\mathcal{D}_G^{t+1}) =$ Graph$(\mathcal{H}_G^{t+1}) \cap$ Graph$(\mathcal{D})$, we obtain that Graph$(\mathcal{D}_G^{t+1}) \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$ and this concludes the proof □

We introduce for all $r \in \mathbb{Q}$, $r > 0$

$$J_T^r(\omega^T) := U^-(\omega^T, r),\ \text{for } \omega^T \in \Omega^T, \tag{4.52}$$

$$J_t^r(\omega^t) := \sup_{P\in\mathcal{Q}_{t+1}(\omega^t)} \int_{\Omega_{t+1}} J_{t+1}^r(\omega^t, \omega_{t+1}) P(d\omega_{t+1}) \text{ for } t \in \{0, \ldots, T-1\}, \omega^t \in \Omega^t. \tag{4.53}$$

As usual we will denote $J_0^r = J_0^t(\omega^0)$.

Proposition 4.6.7 *Assume that Assumptions 4.2.1 and 4.4.2 hold true. Then for any $t \in \{0,\dots,T\}$, $r \in \mathbb{Q}$, $r > 0$, the function $\omega^t \in \Omega^t \to J_t^r(\omega^t)$ is well defined, non-negative, belongs to $\mathcal{USA}(\Omega^t)$ and verifies $\sup_{P\in\mathcal{Q}^t} E_P J_t^r < \infty$. Furthermore, there exists $\widehat{\Omega}^t \in \mathcal{CA}(\Omega^t) \subset \mathcal{B}_c(\Omega^t)$, with $P(\widehat{\Omega}^t) = 1$ for all $P \in \mathcal{Q}^t$ and satisfaying $J_t^r(\omega^t) < \infty$ for all $\omega^t \in \widehat{\Omega}^t$.*

Proof. We proceed by induction on t. Fix some $r \in \mathbb{Q}$, $r > 0$. For $t = T$, $J_T^r(\omega^T) = U^-(\omega^T, r)$ is $\mathcal{B}(\Omega^T)$-measurable by Definition 4.4.1 and $J_T^r \in \mathcal{USA}(\Omega^T)$ (see (4.1)). It is clear that $J_T^r(\omega^T) \geq 0$ for all $\omega^T \in \Omega^T$ and we have that $\sup_{P\in\mathcal{Q}^T} E_P(J_T^r) < \infty$ by Assumption 4.4.2. We set now $\widehat{\Omega}^T := \Omega^T_{Dom}$ (see (4.40)). Using Proposition 4.6.1, we have that $\widehat{\Omega}^T \in \mathcal{B}(\Omega^T) \subset \mathcal{CA}(\Omega^T) \subset \mathcal{B}_c(\Omega^T)$ (see (4.1)) and $P\left(\widehat{\Omega}^T\right) = 1$ for all $P \in \mathcal{Q}^T$. Moreover it is clear that $J_T^r < \infty$ on $\widehat{\Omega}^T$ (see (4.40)). Assume now that for some $t \leq T-1$, $J_{t+1}^r \in \mathcal{USA}(\Omega^{t+1})$, $J_{t+1}^r(\omega^{t+1}) \geq 0$ for all $\omega^{t+1} \in \Omega^{t+1}$ and that $\sup_{P\in\mathcal{Q}^{t+1}} E_P(J_{t+1}^r) < \infty$.
We apply [13, Proposition 7.48 p180] [2] with $X = \Omega^t \times \mathfrak{P}(\Omega_{t+1})$, $Y = \Omega_{t+1}$, $f(\omega^t, P, \omega_{t+1}) = J_{t+1}^r(\omega^t, \omega_{t+1})$ and $q(d\omega_{t+1}|\omega^t, P) = P(d\omega_{t+1})$. First as $J_{t+1}^r \in \mathcal{USA}(\Omega^{t+1})$, it is clear that $f \in \mathcal{USA}(\Omega^t \times \mathfrak{P}(\Omega_{t+1}) \times \Omega_{t+1})$, see [13, Proposition 7.38 p165]. Furthermore $(\omega^t, P) \in \Omega^t \times \mathfrak{P}(\Omega_{t+1}) \to P(d\omega_{t+1}) \in \mathfrak{P}(\Omega_{t+1})$ is a $\mathcal{B}(\Omega^t) \otimes \mathcal{B}(\mathfrak{P}(\Omega_{t+1}))$-measurable stochastic kernel (see [13, Definition 7.12 p134], it is even continuous). So [13, Proposition 7.48 p180] applies and we get that

$$j_t^r : (\omega^t, P) \in \Omega^t \times \mathfrak{P}(\Omega_{t+1}) \to \int_{\Omega_{t+1}} J_{t+1}^r(\omega^t, \omega_{t+1}) P(d\omega_{t+1}) \in \mathcal{USA}(\Omega^t \times \mathfrak{P}(\Omega_{t+1})).$$

As Assumption 4.2.1 holds true ($\text{Proj}_{\Omega^t}(\text{Graph}(\mathcal{Q}_{t+1})) = \Omega^t$ since $\mathcal{Q}_{t+1}$ is a non-empty random set), we can apply [13, Proposition 7.47 p179] and we get that

$$\omega^t \in \Omega^t \to \sup_{P\in\mathcal{Q}_{t+1}(\omega^t)} j_t^r(\omega^t, P) = J_t^r(\omega^t) \in \mathcal{USA}(\Omega^t).$$

As $J_{t+1}^r(\omega^{t+1}) \geq 0$ for all $\omega^{t+1} \in \Omega^{t+1}$, it is clear that $J_t^r(\omega^t) \geq 0$ for all $\omega^t \in \Omega^t$.
We prove now that $\sup_{P\in\mathcal{Q}^t} E_P J_t^r < \infty$ and that there exists $\Omega_r^t \in \mathcal{CA}(\Omega^t)$ with $P(\Omega_r^t) = 1$ for all $P \in \mathcal{Q}^t$ and such that $J_t^r < \infty$ on Ω_r^t. We set $\Omega_r^t := \{\omega^t \in \Omega^t,\ J_t^r(\omega^t) < \infty\}$. Using [13, Corollary 7.35.2 p160], we get that $\Omega_r^t = \bigcup_{n\geq 1}\{\omega^t \in \Omega^t,\ J_t^r(\omega^t) \leq n\} \in \mathcal{CA}(\Omega^t) \subset \mathcal{B}_c(\Omega^t)$.
Fix some $\varepsilon > 0$. From [13, Proposition 7.50 p184] (recall Assumption 4.2.1), there exists some analytically-measurable $p_\varepsilon : \omega^t \in \Omega^t \to \mathfrak{P}(\Omega_{t+1})$, such that $p_\varepsilon(\cdot, \omega^t) \in$

[2] As we will often use similar arguments in the rest of the chapter, we provide some details at this stage.

$\mathcal{Q}_{t+1}(\omega^t)$ for all $\omega^t \in \Omega^t$ and

$$j_t^r(\omega^t, p_\varepsilon(\cdot, \omega^t)) = \int_{\Omega_{t+1}} J_{t+1}^r(\omega^t, \omega_{t+1}) p_\varepsilon(d\omega_{t+1}, \omega^t) \geq \begin{cases} J_t^r(\omega^t) - \varepsilon, \text{ if } J_t^r(\omega^t) < \infty \\ \frac{1}{\varepsilon}, \text{ otherwise.} \end{cases} \tag{4.54}$$

We prove now that $P(\Omega_r^t) = 1$ for all $P \in \mathcal{Q}^t$. Assume this is not the case and that there exists some $P^* \in \mathcal{Q}^t$ such that $P^*(\Omega_r^t) < 1$. Set $P_\varepsilon^* := P^* \otimes p_\varepsilon$. As p_ε is analytically-measurable, $p_\varepsilon \in \mathcal{S}K_{t+1}$. Moreover, $p_\varepsilon(\cdot, \omega^t) \in \mathcal{Q}_{t+1}(\omega^t)$ for all $\omega^t \in \Omega^t$ and we get that $P_\varepsilon^* \in \mathcal{Q}^{t+1}$ (see (4.5)). Then we have that

$$\begin{aligned} \sup_{P \in \mathcal{Q}^{t+1}} E_P J_{t+1}^r \geq E_{P_\varepsilon^*} J_{t+1}^r &= \int_{\Omega^t} \int_{\Omega_{t+1}} J_{t+1}^r(\omega^t, \omega_{t+1}) p_\varepsilon(d\omega_{t+1}, \omega^t) P^*(d\omega^t) \\ &\geq \frac{1}{\varepsilon} \int_{\Omega^t} 1_{\{J_t^r(\cdot)=\infty\}}(\omega^t) P^*(d\omega^t) + \int_{\Omega^t} 1_{\{J_t^r(\cdot)<\infty\}}(\omega^t) \left(J_t^r(\omega^t) - \varepsilon\right) P^*(d\omega^t) \\ &\geq \frac{1}{\varepsilon}(1 - P^*(\Omega_r^t)) - \varepsilon P^*(\Omega_r^t). \end{aligned}$$

As the previous inequality holds true for all $\varepsilon > 0$, we obtain $\sup_{P \in \mathcal{Q}^{t+1}} E_P(J_{t+1}^r) = +\infty$ letting ε go to 0, a contradiction. Thus $P(\Omega_r^t) = 1$ for all $P \in \mathcal{Q}^t$.
Now, for all $P \in \mathcal{Q}^t$, we set $P_\varepsilon = P \otimes p_\varepsilon \in \mathcal{Q}^{t+1}$ (see (4.5)). Then, using (4.54) we get that

$$E_P J_t^r - \varepsilon = E_P 1_{\Omega_r^t} J_t^t - \varepsilon \leq E_{P_\varepsilon} J_{t+1}^r \leq \sup_{P \in \mathcal{Q}^{t+1}} E_P(J_{t+1}^r).$$

Again, as this is true for all $\varepsilon > 0$ and all $P \in \mathcal{Q}^t$ we obtain that $\sup_{P \in \mathcal{Q}^t} E_P(J_t^r) \leq \sup_{P \in \mathcal{Q}^{t+1}} E_P(J_{t+1}^r) < \infty$. Finally we set $\widehat{\Omega}^t = \bigcap_{r \in \mathbb{Q},\, r>0} \Omega_r^t$. It is clear that $\widehat{\Omega}^t \in \mathcal{CA}(\Omega^t) \subset \mathcal{B}_c(\Omega^t)$, that $P(\widehat{\Omega}^t) = 1$ for all $P \in \mathcal{Q}^t$ and that $J_t^r < \infty$ on $\widehat{\Omega}^t$ for all $r \in \mathbb{Q}$, $r > 0$. □

Let $1 \leq t \leq T$ be fixed. We introduce the following notation: for any $\mathcal{B}_c(\Omega^{t-1})$-measurable random variable G and any $P \in \mathcal{Q}^t$, $\phi_t(G, P)$ is the set of all $\mathcal{B}_c(\Omega^{t-1})$-measurable random variable ξ (one-step strategy), such that $G(\cdot) + \xi \Delta S_t(\cdot) \geq 0$ P-a.s. Propositions 4.6.8 to 4.6.12 solve the dynamic programming procedure and hold true under the following set of conditions.

$$\forall\, \omega^t \in \Omega^t,\ U_t\left(\omega^t, \cdot\right) : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\} \text{ is non-decreasing, usc and concave on } \mathbb{R}, \tag{4.55}$$

$$\forall\, \omega^t \in \Omega^t,\ I_t\left(\omega^t, \cdot\right) : \mathbb{R} \to \mathbb{R} \cup \{+\infty\} \text{ is non-decreasing and non-negative on } \mathbb{R}, \tag{4.56}$$

$$U_t \in \mathcal{LSA}(\Omega^t \times \mathbb{R}), \tag{4.57}$$

$$I_t \in \mathcal{USA}(\Omega^t \times \mathbb{R}), \tag{4.58}$$

$$U_t\left(\omega^t, x\right) \leq I_t(\omega^t, x+1) \text{ for all } (\omega^t, x) \in \Omega^t \times \mathbb{R}, \tag{4.59}$$

$$\sup_{P \in \mathcal{Q}^t} \sup_{\xi \in \phi_t(G,P)} \int_{\Omega^t} I_t(\omega^t, G(\omega^{t-1}) + \xi(\omega^{t-1})\Delta S_t(\omega^t)) P(d\omega^t) < \infty, \tag{4.60}$$

for any $G := x + \sum_{s=1}^{t-1} \phi_s \Delta S_s$, where $x \geq 0$, $(\phi_s)_{1 \leq s \leq t-1}$ is $\left(\mathcal{B}_c(\Omega^{s-1})\right)_{1 \leq s \leq t-1}$-adapted,

$$U_t(\omega^t, r) \geq -J_t^r(\omega^t) \text{ for all } \omega^t \in \Omega^t, \text{ all } r \in \mathbb{Q},\ r > 0. \tag{4.61}$$

Proposition 4.6.8 *Let* $0 \leq t \leq T-1$ *be fixed. Assume that the (NA) condition and Assumptions 4.2.1, 4.2.2, 4.2.4 hold true and that* (4.55)*,* (4.56)*,* (4.57)*,* (4.58)*,* (4.59)*,* (4.60) *and* (4.61) *hold true at stage* $t+1$*. Then there exists* $\widetilde{\Omega}^t \in \mathcal{B}_c(\Omega^t)$*, such that* $P(\widetilde{\Omega}^t) = 1$ *for all* $P \in \mathcal{Q}^t$ *and such that for all* $\omega^t \in \widetilde{\Omega}^t$ *the function* $(\omega_{t+1}, x) \to U_{t+1}(\omega^t, \omega_{t+1}, x)$ *satisfies the assumptions of Theorem 4.5.23 with* $\overline{\Omega} = \Omega_{t+1}$*,* $\mathcal{G} = \mathcal{B}_c(\Omega^{t+1})$*,* $\mathcal{Q} = \mathcal{Q}_{t+1}(\omega^t)$*,* $Y(\cdot) = \Delta S_{t+1}(\omega^t, \cdot)$*,* $V(\cdot,\cdot) = U_{t+1}(\omega^t, \cdot, \cdot)$ *where* V *is defined on* $\Omega_{t+1} \times \mathbb{R}$*.*

Remark 4.6.9 Note that under the assumptions of Proposition 4.6.8 (see (4.39), (4.47) and (4.48)), for all $\omega^t \in \widetilde{\Omega}^t$ and $x \geq 0$ we have that

$$U_t(\omega^t, x) = \mathcal{U}_t(\omega^t, x) = \sup_{h \in \mathcal{H}_x^{t+1}(\omega^t)} \inf_{P \in \mathcal{Q}_{t+1}(\omega^t)} \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, x + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) P(d\omega_{t+1}). \tag{4.62}$$

Remark 4.6.10 Note that Lemmata 4.5.20, 4.5.21 hold true under a weaker set of assumptions than Theorem 4.5.23. Therefore we can replace Theorem 4.5.23 by either Lemmata 4.5.20 or 4.5.21 in the above proposition.

Proof. To prove the proposition we will review one by one the assumptions needed to apply Theorem 4.5.23 in the context $\overline{\Omega} = \Omega_{t+1}$, $\mathcal{G} = \mathcal{B}_c(\Omega_{t+1})$, $\mathcal{Q} = \mathcal{Q}_{t+1}(\omega^t)$, $Y(\cdot) = \Delta S_{t+1}(\omega^t, \cdot)$, $V(\cdot,\cdot) = U_{t+1}(\omega^t, \cdot, \cdot)$ where V is defined on $\Omega_{t+1} \times \mathbb{R}$. In the sequel we shortly call this the context $t+1$.
First from Assumption 4.2.4 for $\omega^t \in \Omega^t$ fixed we have $Y(\cdot) = \Delta S_{t+1}(\omega^t, \cdot) \geq -s - S_t(\omega^t)$. So setting $b := \max(1 + s + S_t^i(\omega^t), i \in \{1, \ldots, d\})$ we have that $Y_i(\cdot) = \Delta S_{t+1}^i(\omega^t, \cdot) \geq -b$ and $0 < b < \infty$: Assumption 4.5.1 holds true.
From (4.55) at $t+1$ for all $\omega^t \in \Omega^t$ and $\omega_{t+1} \in \Omega_{t+1}$, $U_{t+1}(\omega^t, \omega_{t+1}, \cdot)$ is non-decreasing, concave and usc on $\mathbb{R}$. From (4.57) at $t+1$ and (4.1), U_{t+1} is $\mathcal{B}_c(\Omega^{t+1} \times \mathbb{R})$-measurable.

Fix some $x \in \mathbb{R}$ and $\omega^t \in \Omega^t$, then $\omega_{t+1} \in \Omega_{t+1} \to U_{t+1}(\omega^t, \omega_{t+1}, x)$ is $\mathcal{B}_c(\Omega_{t+1})$-measurable, see [13, Lemma 7.29 p174]. Thus Assumption 4.5.8 is satisfied in the context $t+1$ (recall that $U_{t+1}(\omega^t, \omega_{t+1}, x) = -\infty$ for all $x < 0$).

We now prove the assumptions that are verified for ω^t in some well chosen $\mathcal{Q}^t$-full measure set. First from Proposition 4.3.4, for all $\omega^t \in \Omega^t_{NA}$, Assumptions 4.5.2 and 4.5.3 hold true in the context $t+1$.

We handle now Assumption 4.5.9. Fix $\omega^t \in \widehat{\Omega}^t$ and some $r \in \mathbb{Q}$, $r > 0$ ($\widehat{\Omega}^t$ has been defined in Proposition 4.6.7). Since $J^r_{t+1}(\omega^{t+1}) \geq 0$ for all $\omega^{t+1} \in \Omega^{t+1}$ (see Proposition 4.6.7), using (4.61) at $t+1$ we get for all $\omega_{t+1} \in \Omega_{t+1}$ that $U^-_{t+1}(\omega^t, \omega_{t+1}, r) \leq J^r_{t+1}(\omega^t, \omega_{t+1})$. Thus since $\omega^t \in \widehat{\Omega}^t$ (see Proposition 4.6.7)

$$\begin{aligned}\sup_{P \in \mathcal{Q}_{t+1}(\omega^t)} \int_{\Omega_{t+1}} U^-_{t+1}(\omega^t, \omega_{t+1}, r) P(d\omega^t) &\leq \sup_{P \in \mathcal{Q}_{t+1}(\omega^t)} \int_{\Omega_{t+1}} J^r_{t+1}(\omega^t, \omega_{t+1}) P(d\omega^t) \\ &= J^r_t(\omega^t) < \infty,\end{aligned}$$

and Assumption 4.5.9 in context $t+1$ is verified for all $\omega^t \in \widehat{\Omega}^t$.

We finish with Assumption 4.5.14 in context $t+1$ whose proof is more involved. We want to show that for ω^t in some $\mathcal{Q}^t$-full measure set to be determined and for all $h \in \mathcal{H}^{t+1}_1(\omega^t)$ and $P \in \mathcal{Q}_{t+1}(\omega^t)$ we have that

$$\int_{\Omega_{t+1}} U^+_{t+1}(\omega^t, \omega_{t+1}, 1 + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) P(d\omega_{t+1}) < \infty. \tag{4.63}$$

We introduce

$$i_t : (\omega^t, h, P) \in \Omega^t \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1}) \to \int_{\Omega_{t+1}} I_{t+1}(\omega^t, \omega_{t+1}, 2 + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) P(d\omega_{t+1})$$

$$I_1(\omega^t) := \left\{(h, P) \in \mathbb{R}^d \times \mathcal{Q}_{t+1}(\omega^t),\ P\left(1 + h\Delta S_{t+1}(\omega^t, \cdot) \geq 0\right) = 1,\ i_t(\omega^t, h, P) = \infty\right\}. \tag{4.64}$$

Arguing by contradiction and using measurable selection arguments we will prove that $I_1(\omega^t) = \emptyset$ for $\mathcal{Q}^t$-almost all $\omega^t \in \Omega^t$. Then from (4.56) and (4.59) at $t+1$ we have that

$$\left\{(h, P) \in \mathcal{H}^{t+1}_1(\omega^t) \times \mathcal{Q}_{t+1}(\omega^t),\ \int_{\Omega_{t+1}} U^+_{t+1}(\omega^t, \omega_{t+1}, 1 + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) P(d\omega_{t+1}) = \infty\right\} \tag{4.65}$$

$$\subset I_1(\omega^t), \text{ for all } \omega^t \in \Omega^t.$$

So (4.63) holds true if $\omega^t \in \{I_1 = \emptyset\}$.

We first prove that Graph$(I_1) \in \mathcal{A}(\Omega^t \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1}))$. From (4.58) at $t+1$ and [13, Lemma 7.30 (3) p178], $(\omega^t, h, \omega_{t+1}) \to I_{t+1}(\omega^t, \omega_{t+1}, 2 + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) \in \mathcal{USA}(\Omega^t \times$

$\mathbb{R}^d \times \Omega_{t+1}$) (recall that ΔS_{t+1} is $\mathcal{B}(\Omega^{t+1})$-measurable, see Assumption 4.2.2). Then using [13, Proposition 7.48 p180] (which can be used with similar arguments as in the proof of Proposition 4.6.7), we get that $i_t \in \mathcal{USA}(\Omega^t \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1}))$. It follows that, see [13, Corollary 7.35.2 p160],

$$i_t^{-1}(\{\infty\}) = \bigcap_{n\geq 1} \left\{(\omega^t, h, P) \in \Omega^t \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1}),\ i_t(\omega^t, h, P) > n\right\} \in \mathcal{A}(\Omega^t \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1})).$$

Now using Assumption 4.2.1, [13, Proposition 7.38 p165] together with (4.92) in Lemma 4.8.5 and (4.1) we get that

$$\{(\omega^t, h, P) \in \Omega^t \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1}),\ P \in \mathcal{Q}_{t+1}(\omega^t),\ P\left(1 + h\Delta S_{t+1}(\omega^t, \cdot) \geq 0\right) = 1\} \in \mathcal{A}(\Omega^t \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1}))$$

and $\text{Graph}(I_1) \in \mathcal{A}(\Omega^t \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1}))$ follows immediately.
Applying [13, Proposition 7.39 p165] and the Jankov-von Neumann Projection Theorem, see [13, Proposition 7.49 p182], we obtain that

$$Proj_{\Omega^t}\left(\text{Graph}(I_1)\right) = \{I_1 \neq \emptyset\} \in \mathcal{A}(\Omega^t)$$

and that there exists some analytically-measurable and therefore $\mathcal{B}_c(\Omega^t)$-measurable function $\omega^t \in \{I_1 \neq \emptyset\} \to (h^*(\omega^t), p^*(\cdot, \omega^t)) \in \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1})$ such that for all $\omega^t \in \{I_1 \neq \emptyset\}$, $(h^*(\omega^t), p^*(\cdot, \omega^t)) \in I_1(\omega^t)$. We extend h^* and p^* on all Ω^t by setting for all $\omega^t \in \Omega^t \setminus \{I_1 \neq \emptyset\}$, $h^*(\omega^t) = 0$ and $p^*(\cdot, \omega^t) = \widehat{p}(\cdot, \omega^t)$ where $\widehat{p}(\cdot, \omega^t)$ is a given $\mathcal{B}_c(\Omega^t)$-measurable selector of $\text{Graph}(\mathcal{Q}_{t+1})$. As $\{I_1 \neq \emptyset\} \in \mathcal{B}_c(\Omega^t)$ (see (4.1)) it is clear that h^* and p^* remain $\mathcal{B}_c(\Omega^t)$-measurable.
We prove now that $P(\{I_1 \neq \emptyset\}) = 0$ for all $P \in \mathcal{Q}^t$. We proceed by contradiction and assume that there exists some $\widetilde{P} \in \mathcal{Q}^t$ such that $\widetilde{P}(\{I_1 \neq \emptyset\}) > 0$. We set $\widetilde{P}^* = \widetilde{P} \otimes p^*$. Since $p^* \in SK_{t+1}$ and $p^*(\cdot, \omega^t) \in \mathcal{Q}_{t+1}(\omega^t)$ for all $\omega^t \in \Omega^t$, it is clear that $\widetilde{P}^* \in \mathcal{Q}^{t+1}$ (see (4.5)) and that $\widetilde{P}^*\left(2 + h^*(\cdot)\Delta S_{t+1}(\cdot) \geq 0\right) = 1$. For all $\omega^t \in \{I_1 \neq \emptyset\}$, we have that

$$\int_{\Omega_{t+1}} I_{t+1}(\omega^t, \omega_{t+1}, 2 + h^*(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}))p^*(d\omega_{t+1}, \omega^t) = i_t(\omega^t, h^*(\omega^t), p^*(\cdot, \omega^t)) = \infty.$$

Finally

$$\int_{\Omega^{t+1}} I_{t+1}\left(\omega^{t+1}, 2 + h^*(\omega^t)\Delta S_{t+1}(\omega^{t+1})\right)\widetilde{P}^*(d\omega^{t+1}) \geq \int_{\{I_1 \neq \emptyset\}} (+\infty)\widetilde{P}(d\omega^t) = +\infty$$

a contradiction with (4.60) at $t+1$. Therefore we must have $P(\{I_1 \neq \emptyset\}) = 0$ for all $P \in \mathcal{Q}^t$ as claimed. Thus, recalling (4.65), for $\omega^t \in \{I_1 = \emptyset\}$, Assumption 4.5.14 in the context $t+1$ is true. We can now define $\widetilde{\Omega}^t := \{I_1 = \emptyset\} \cap \widehat{\Omega}^t \cap \Omega^t_{NA} \subset \widehat{\Omega}^t$. It is clear, recalling Propositions 4.3.4 and 4.6.7, that $\widetilde{\Omega}^t \in \mathcal{B}_c(\Omega^t)$ and that $P(\widetilde{\Omega}^t) = 1$ for all $P \in \mathcal{Q}^t$ and the proof is complete. □

The next proposition enables us to initialize the induction proof that will be carried on in the proof of the main theorem.

Proposition 4.6.11 *Assume that the (NA) condition, Assumptions 4.4.2 and 4.4.12 hold true. Then* (4.55), (4.56), (4.57), (4.58), (4.59), (4.60) *and* (4.61) *hold true for* $t = T$.

Proof. Using Definition 4.4.1, as $U_T = U$ (see (4.46)) and $I_T = U^+$ (see (4.49)), (4.55), (4.56), (4.59) and (4.61) (recall (4.52)) for $t = T$ are true. As (4.55) is true at T, for all $\omega^T \in \Omega^T$, $U(\omega^T, \cdot)$ is non-decreasing and usc, hence is right-continuous (see Lemma 2.8.12 in Chapter 2). From Definition 4.4.1 again, $U(\cdot, x)$ is $\mathcal{B}(\Omega^T)$-measurable for all $x \in \mathbb{R}$ and Lemma 2.8.16 in Chapter 2implies that U is $\mathcal{B}(\Omega^T) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable and thus (4.57) and (4.58) hold true for $t = T$ (recall (4.1)). It remains to prove that (4.60) is true for $t = T$. Let $G := x + \sum_{t=1}^{T-1} \phi_t \Delta S_t$ where $x \geq 0$, $(\phi_s)_{1 \leq s \leq T-1}$ is $(\mathcal{B}_c(\Omega^{s-1}))_{1 \leq s \leq T-1}$-adapted. Fix some $P \in \mathcal{Q}^T$ and $\xi \in \phi_T(G, P)$. Let $(\phi_i^\xi)_{1 \leq i \leq T} \in \Phi$ be defined by $\phi_T^\xi = \xi$ and $\phi_s^\xi = \phi_s$ for $1 \leq s \leq T-1$ then $V_T^{x,\phi^\xi} = G + \xi \Delta S_T$ and $\phi^\xi \in \Phi(x, P)$. We have $\int_{\Omega^T} I_T\left(\omega^T, G(\omega^{T-1}) + \xi(\omega^{T-1})\Delta S_T(\omega^T)\right) P(d\omega^T) = E_P U^+(\cdot, V_T^{x,\phi^\xi}(\cdot))$ and (4.60) follows from Proposition 4.6.2. □

The next proposition proves the induction step.

Proposition 4.6.12 *Let* $0 \leq t \leq T-1$ *be fixed. Assume that the (NA) condition holds true as well as Assumptions 4.2.1, 4.2.2, 4.2.4 and* (4.55), (4.56), (4.57), (4.58), (4.59), (4.60) *and* (4.61) *at* $t+1$. *Then* (4.55), (4.56), (4.57), (4.58), (4.59), (4.60) *and* (4.61) *are true for* t.
Moreover for all $X = x + \sum_{s=1}^{t} \phi_s \Delta S_s$, *where* $x \geq 0$, $(\phi_s)_{1 \leq s \leq t}$ *is* $(\mathcal{B}_c(\Omega^{s-1}))_{1 \leq s \leq t}$-*adapted, such that* $P(X \geq 0) = 1$ *for all* $P \in \mathcal{Q}^t$, *there exists some* $\Omega_X^t \in \mathcal{B}_c(\Omega^t)$ *such that* $P(\Omega_X^t) = 1$ *for all* $P \in \mathcal{Q}^t$, $\Omega_X^t \subset \widetilde{\Omega}^t$ *(see Proposition 4.6.8 for the definition of* $\widetilde{\Omega}^t$*) and some* $\mathcal{B}_c(\Omega^t)$-*measurable random variable* $\widehat{h}_{t+1}^X$ *such that for all* $\omega^t \in \Omega_X^t$, $\widehat{h}_{t+1}^X(\omega^t) \in \mathcal{D}_{X(\omega^t)}^{t+1}(\omega^t)$ *and*

$$U_t(\omega^t, X(\omega^t)) = \inf_{P \in \mathcal{Q}_{t+1}(\omega^t)} \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, X(\omega^t) + \widehat{h}_{t+1}^X(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}))P(d\omega_{t+1}). \tag{4.66}$$

Proof. First we prove that (4.55) is true at t. We fix some $\omega^t \in \Omega^t$. From (4.55) at $t+1$, the function $U_{t+1}(\omega^t, \omega_{t+1}, \cdot)$ is usc, concave and non-decreasing on $\mathbb{R}$ for all $\omega_{t+1} \in \Omega_{t+1}$. From (4.47) and (4.48), $U_{t+1}(\omega^t, \omega_{t+1}, x) = -\infty$ for all $x < 0$ and $\omega_{t+1} \in \Omega_{t+1}$. Then using (4.57) at $t+1$, Lemma 4.8.4 and (4.1), we find that $U_{t+1}(\omega^t, \cdot, x)$ is $\mathcal{B}_c(\Omega_{t+1})$-measurable for all $x \in \mathbb{R}$. Hence, Assumption 4.5.8 of Lemma 4.5.17 holds true for $\overline{\Omega} = \Omega_{t+1}$, $\mathcal{G} = \mathcal{B}_c(\Omega_{t+1})$, $\mathcal{Q} = \mathcal{Q}_{t+1}(\omega^t)$, $Y(\cdot) = \Delta S_{t+1}(\omega^t, \cdot)$, $V(\cdot, \cdot) =$

$U_{t+1}(\omega^t,\cdot,\cdot)$ and we obtain that $x \in \mathbb{R} \to U_t(\omega^t,x) = \mathrm{Cl}(\mathcal{U}_t)(\omega^t,x)$ (see (4.47) and (4.48)) is usc, concave and non-decreasing. As this is true for all $\omega^t \in \Omega^t$, (4.55) at t is proved. Note that we also obtain that $x \in \mathbb{R} \to \mathcal{U}_t(\omega^t,x)$ is non-decreasing for all $\omega^t \in \Omega^t$. Now we prove that (4.57) is true for U_t. Since integrals might not always be well defined we need to be a bit cautious. We introduce first $u_t : \Omega^t \times \mathbb{R}^d \times [0,\infty) \times \mathfrak{P}(\Omega_{t+1}) \to \mathbb{R} \cup \{\pm\infty\}$

$$u_t(\omega^t,h,x,P) = \int_{\Omega_{t+1}} U_{t+1}(\omega^t,\omega_{t+1},x+h\Delta S_{t+1}(\omega^t,\omega_{t+1}))P(d\omega_{t+1}). \tag{4.67}$$

Recall that we set $u_t(\omega^t,h,x,P) = +\infty$ if $\int_{\Omega_{t+1}} U^+_{t+1}(\omega^t,\omega_{t+1},x+h\Delta S_{t+1}(\omega^t,\omega_{t+1}))P(d\omega_{t+1}) = \infty$ and $\int_{\Omega_{t+1}} U^-_{t+1}(\omega^t,\omega_{t+1},x+h\Delta S_{t+1}(\omega^t,\omega_{t+1}))P(d\omega_{t+1}) = \infty$. We prove that $u_t \in \mathcal{LSA}(\Omega^t \times \mathbb{R}^d \times \mathbb{R} \times \mathfrak{P}(\Omega_{t+1}))$. As ΔS_{t+1} is $\mathcal{B}(\Omega^{t+1})$-measurable (see Assumption 4.2.2) and $U_{t+1} \in \mathcal{LSA}(\Omega^t \times \Omega_{t+1} \times \mathbb{R}^d)$ (see (4.57) at $t+1$), applying [13, Lemma 7.30 (3) p177], we obtain that

$$\begin{aligned}(\omega^t,\omega_{t+1},h,x) \in \Omega^t \times \Omega_{t+1} \times \mathbb{R}^d \times \mathbb{R} \to U_{t+1}(\omega^t,\omega_{t+1},x+h\Delta S_{t+1}(\omega^t,\omega_{t+1}))\\ \in \mathcal{LSA}(\Omega^t \times \Omega_{t+1} \times \mathbb{R}^d \times \mathbb{R}).\end{aligned}$$

Applying [13, Proposition 7.48 p180], we get that $u_t \in \mathcal{LSA}(\Omega^t \times \mathbb{R}^d \times \mathbb{R} \times \mathfrak{P}(\Omega_{t+1}))$[3] [4]. We define $\widehat{u}_t : (\omega^t,h,x,P) \in \Omega^t \times \mathbb{R}^d \times [0,+\infty) \times \mathfrak{P}(\Omega_{t+1}) \to \mathbb{R} \cup \{\pm\infty\}$ by

$$\widehat{u}_t(\omega^t,h,x,P) = 1_{\mathcal{H}^{t+1}_x(\omega^t)}(h)u_t(\omega^t,h,x,P) + (-\infty)1_{\mathbb{R}^d \setminus \mathcal{H}^{t+1}_x(\omega^t)}(h). \tag{4.68}$$

We prove that $\widehat{u}_t \in \mathcal{LSA}(\Omega^t \times \mathbb{R}^d \times \mathbb{R} \times \mathfrak{P}(\Omega_{t+1}))$. Fix some $c \in \mathbb{R}$ and set

$$\begin{aligned}\widehat{C} &:= \{(\omega^t,h,x,P) \in \Omega^t \times \mathbb{R}^d \times \mathbb{R} \times \mathfrak{P}(\Omega_{t+1}),\ \widehat{u}_t(\omega^t,h,x,P) < c\},\\ C &:= \left\{(\omega^t,h,x,P) \in \Omega^t \times \mathbb{R}^d \times \mathbb{R} \times \mathfrak{P}(\Omega_{t+1}),\ u_t(\omega^t,h,x,P) < c\right\}.\\ A &:= \left\{(\omega^t,h,x) \in \Omega^t \times \mathbb{R}^d \times \mathbb{R},\ h \in \mathcal{H}^{t+1}_x(\omega^t)\right\} \times \mathfrak{P}(\Omega_{t+1}),\\ A^c &:= \left\{(\omega^t,h,x) \in \Omega^t \times \mathbb{R}^d \times \mathbb{R},\ h \notin \mathcal{H}^{t+1}_x(\omega^t)\right\} \times \mathfrak{P}(\Omega_{t+1}).\end{aligned}$$

Then we have that $\widehat{C} = (C \cap A) \cup A^c = C \cup A^c$. As $u_t \in \mathcal{LSA}(\Omega^t \times \mathbb{R}^d \times \mathbb{R} \times \mathfrak{P}(\Omega_{t+1}))$, we get that $C \in \mathcal{A}(\Omega^t \times \mathbb{R}^d \times \mathbb{R} \times \mathfrak{P}(\Omega_{t+1}))$. Using Lemma 4.6.5 we get that

$$\begin{aligned}\left\{(\omega^t,h,x) \in \Omega^t \times \mathbb{R}^d \times \mathbb{R},\ h \notin \mathcal{H}^{t+1}_x(\omega^t)\right\} = \{(\omega^t,h,x) \in \Omega^t \times \mathbb{R}^d \times \mathbb{R},\ (\omega^t,x,h) \notin \mathrm{Graph}(\mathcal{H}^{t+1})\}\\ \in \mathcal{A}(\Omega^t \times \mathbb{R}^d \times \mathbb{R}),\end{aligned}$$

and therefore $A^c \in \mathcal{A}(\Omega^t \times \mathbb{R}^d \times \mathbb{R} \times \mathfrak{P}(\Omega_{t+1}))$ (see [13, Proposition 7.38 p165]). It follows that $\widehat{C} \in \mathcal{A}(\Omega^t \times \mathbb{R}^d \times \mathbb{R} \times \mathfrak{P}(\Omega_{t+1}))$ and $\widehat{u}_t \in \mathcal{LSA}(\Omega^t \times \mathbb{R}^d \times \mathbb{R} \times \mathfrak{P}(\Omega_{t+1}))$

[3] Note that [13, Proposition 7.48 p180] relies on [13, Lemma 7.30 (4) p177] applied for lower-semianalytic functions where the convention $+\infty - \infty = +\infty$ needs to be used.

[4] Note as well that it is clear that $\Omega^t \times \mathbb{R}^d \times [0,\infty) \times \mathfrak{P}(\Omega_{t+1}) \in \mathcal{A}(\Omega^t \times \mathbb{R}^d \times \mathbb{R} \times \mathfrak{P}(\Omega_{t+1}))$.

as claimed. Since $\text{Graph}(\mathcal{Q}_{t+1}) \in \mathcal{A}(\Omega^t \times \mathfrak{P}(\Omega_{t+1}))$ (see Assumption 4.2.1), we can apply [13, Proposition 7.47 p179] and we get that

$$\widetilde{u}_t : (\omega^t, h, x) \in \Omega^t \times \mathbb{R}^d \times [0, \infty) \to \inf_{P \in \mathcal{Q}_{t+1}(\omega^t)} \widehat{u}_t(\omega^t, h, x, P) \in \mathcal{LSA}(\Omega^t \times \mathbb{R}^d \times \mathbb{R}). \tag{4.69}$$

Then from [13, Lemma 7.30 (2) p178], we get that

$$\widetilde{\mathcal{U}}_t : (\omega^t, x) \in \Omega^t \times [0, \infty) \to \sup_{h \in \mathbb{Q}^d} \widetilde{u}_t(\omega^t, h, x) \in \mathcal{LSA}(\Omega^t \times \mathbb{R}). \tag{4.70}$$

Let $x \geq 0$ and $\omega^t \in \Omega^t$, recalling (4.47), (4.68), (4.69) and (4.70) we have that

$$\begin{aligned}
\widetilde{\mathcal{U}}_t(\omega^t, x) &= \sup_{h \in \mathbb{Q}^d} \inf_{P \in \mathcal{Q}_{t+1}(\omega^t)} \Big(1_{\mathcal{H}_x^{t+1}(\omega^t)}(h) u_t(\omega^t, h, x, P) + (-\infty) 1_{\mathbb{R}^d \setminus \mathcal{H}_x^{t+1}(\omega^t)}(h) \Big) \\
&= \sup_{h \in \mathbb{Q}^d} \left(1_{\mathcal{H}_x^{t+1}(\omega^t)}(h) \inf_{P \in \mathcal{Q}_{t+1}(\omega^t)} u_t(\omega^t, h, x, P) + (-\infty) 1_{\mathbb{R}^d \setminus \mathcal{H}_x^{t+1}(\omega^t)}(h) \right) \\
&= \sup \big(\mathcal{U}_t(\omega^t, x), -\infty \big) = \mathcal{U}_t(\omega^t, x).
\end{aligned}$$

As $\mathcal{U}_t = -\infty$ if $x < 0$, it follows that $\mathcal{U}_t \in \mathcal{LSA}(\Omega^t \times \mathbb{R})$. It remains to prove that $U_t \in \mathcal{LSA}(\Omega^t \times \mathbb{R})$. We have already seen that $\omega^t \in \Omega^t$, $x \in \mathbb{R} \to \mathcal{U}_t(\omega^t, x)$ is non-decreasing, thus, for all $\omega^t \in \Omega^t$ and $x \in \mathbb{R}$ we get that (recall (4.48))

$$U_t(\omega^t, x) = \text{Cl}(\mathcal{U}_t(\omega^t, x)) = \limsup_{y \to x} \mathcal{U}_t(\omega^t, x) = \lim_{n \to \infty} \mathcal{U}_t(\omega^t, x + \frac{1}{n}).$$

As $(\omega^t, x) \in \Omega^t \times \mathbb{R} \to \mathcal{U}_t(\omega^t, x + \frac{1}{n}) \in \mathcal{LSA}(\Omega^t \times \mathbb{R})$, [13, Lemma 7.30 (3) p178] implies that $U_t \in \mathcal{LSA}(\Omega^t \times \mathbb{R})$.

We prove now that U_t is $\mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R})$ as we will need it below. Since $U_t \in \mathcal{LSA}(\Omega^t \times \mathbb{R})$, U_t is $\mathcal{B}_c(\Omega^t \times \mathbb{R})$-measurable. Applying [13, Lemma 7.29 p177] for $x \in \mathbb{R}$ fixed, we get that $U_t(\cdot, x)$ is $\mathcal{B}_c(\Omega^t)$-measurable. Now for $\omega^t \in \Omega^t$ fixed, we have just proved (see (4.55) for t) that $U_t(\omega^t, \cdot)$ is usc and non-decreasing. Thus, from Lemma 2.8.12 in Chapter 2, we get that U_t is $\mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R})$-measurable.

We prove now that (4.58) holds true for t. We introduce $\hat{i}_t : \Omega^t \times \mathbb{R}^d \times [0, \infty) \times \mathfrak{P}(\Omega_{t+1}) \to \mathbb{R} \cup \{+\infty\}$ (recall (4.42) for the definition of $H_x^{t+1}(\omega^t, P)$)

$$\hat{i}_t(\omega^t, h, x, P) = 1_{H_x^{t+1}(\omega^t, P)}(h) \int_{\Omega_{t+1}} I_{t+1}(\omega^t, \omega_{t+1}, x + 1 + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) P(d\omega_{t+1}). \tag{4.71}$$

Note that, using (4.56) at $t+1$, for $\omega^t \in \Omega^t$, $x \geq 0$, $h \in \mathbb{R}^d$, $P \in \mathfrak{P}(\Omega_{t+1})$ fixed $I_{t+1}(\omega^t, \omega_{t+1}, x + 1 + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) \geq 0$ for all $\omega_{t+1} \in \Omega_{t+1}$ and the integral in (4.71) is well defined (potentially infinite valued). From (4.58) at $t+1$, $I_{t+1} \in \mathcal{USA}(\Omega^{t+1} \times \mathbb{R})$ and since $(\omega^{t+1}, h, x, P) \in \Omega^{t+1} \times \mathbb{R}^d \times \mathbb{R} \times \mathfrak{P}(\Omega_{t+1}) \to x + 1 + h\Delta S_{t+1}(\omega^{t+1})$ is $\mathcal{B}(\Omega^{t+1}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathfrak{P}(\Omega_{t+1}))$- measurable (recall Assumption 4.2.2), using

[13, Lemma 7.30 (3) p178] we find that $(\omega^{t+1}, h, x, P) \in \Omega^{t+1} \times \mathbb{R}^d \times \mathbb{R} \times \mathfrak{P}(\Omega_{t+1}) \to I_{t+1}(\omega^t, \omega_{t+1}, x+1+h\Delta S_{t+1}(\omega^t, \omega_{t+1})) \in \mathcal{USA}(\Omega^{t+1} \times \mathbb{R}^d \times \mathbb{R} \times \mathfrak{P}(\Omega_{t+1})$. We apply now [13, Proposition 7.48 p179] [5] and we obtain that

$$(\omega^t, h, x, P) \to \int_{\Omega_{t+1}} I_{t+1}(\omega^t, \omega_{t+1}, x+1+h\Delta S_{t+1}(\omega^t, \omega_{t+1}))P(d\omega_{t+1}) \in \mathcal{USA}(\Omega^t \times \mathbb{R}^d \times \mathbb{R} \times \mathfrak{P}(\Omega_{t+1})).$$

Finally, [13, Lemma 7.30 (3) p177] together with Lemma 4.6.5 (see Remark 4.6.6) imply that $\hat{i}_t \in \mathcal{USA}(\Omega^t \times \mathbb{R}^d \times \mathbb{R} \times \mathfrak{P}(\Omega_{t+1}))$. As $\{(\omega^t, h, x, P) \in \Omega^t \times \mathbb{R}^d \times \mathbb{R} \times \mathfrak{P}(\Omega_{t+1}),\ P \in \mathcal{Q}_{t+1}(\omega^t)\} \in \mathcal{A}(\Omega^t \times \mathbb{R}^d \times \mathbb{R} \times \mathfrak{P}(\Omega_{t+1}))$, we get from [13, Proposition 7.47 p179, Lemma 7.30 (3) p178] (recalling (4.50) and (4.71)), that

$$I_t(\omega^t, x) = 1_{[0,\infty)}(x) \sup_{h \in \mathbb{R}^d} \sup_{P \in \mathcal{Q}_{t+1}(\omega^t)} \hat{i}_t(\omega^t, h, x, P) \in \mathcal{USA}(\Omega^t \times \mathbb{R}) \tag{4.72}$$

and (4.58) for t is proved. For later purpose, we set $\bar{i}_t : \Omega^t \times \mathbb{R}^d \times [0,\infty) \times \mathfrak{P}(\Omega_{t+1}) \to \mathbb{R} \cup \{\pm\infty\}$

$$\bar{i}_t(\omega^t, h, x, P) := \hat{i}_t(\omega^t, h, x, P) + (-\infty)1_{\mathbb{R}^d \setminus H_x^{t+1}(\omega^t, P)}(h). \tag{4.73}$$

Using again Lemma 4.6.5, it is easy to see that $\bar{i}_t \in \mathcal{USA}(\Omega^t \times \mathbb{R}^d \times \mathbb{R} \times \mathfrak{P}(\Omega_{t+1}))$ and using as before [13, Proposition 7.47 p179]

$$\overline{I}_t(\omega^t, x) = 1_{[0,\infty)}(x) \sup_{h \in \mathbb{R}^d} \sup_{P \in \mathcal{Q}_{t+1}(\omega^t)} \bar{i}_t(\omega^t, h, x, P) \in \mathcal{USA}(\Omega^t \times \mathbb{R}).$$

Furthemore as $\hat{i}_t \geq 0$ we have that for all $(\omega^t, x) \in \Omega^t \times \mathbb{R}$

$$\overline{I}_t(\omega^t, x) = 1_{[0,\infty)}(x) \sup_{h \in \mathbb{R}^d} \sup_{P \in \mathcal{Q}_{t+1}(\omega^t)} \hat{i}_t(\omega^t, h, x, P) = I_t(\omega^t, x). \tag{4.74}$$

We prove (4.56) and (4.59) at t. Fix some $\omega^t \in \Omega^t$. We want to apply Lemma 4.5.18 to $V(\omega_{t+1}, x) = U_{t+1}(\omega^t, \omega_{t+1}, x)$, $I(\omega_{t+1}, x) = I_{t+1}(\omega^t, \omega_{t+1}, x+1)$ and $\mathcal{G} = \mathcal{B}_c(\Omega_{t+1})$. We have already proved (see the proof of (4.55) at t) that Assumption 4.5.8 holds true for V. From (4.56) and (4.59) at $t+1$, for all $(\omega^t, \omega_{t+1}) \in \Omega^{t+1}$, the function $x \in \mathbb{R} \to I_{t+1}(\omega^t, \omega_{t+1}, x+1)$ is non-decreasing and non-negative on $\mathbb{R}$ and $U_{t+1}(\omega^t, \omega_{t+1}, x) \leq I_{t+1}(\omega^t, \omega_{t+1}, x+1)$ for all $x \in \mathbb{R}$. Now for $x \in \mathbb{R}$, $h \in \mathbb{R}^d$ fixed, the function $\omega_{t+1} \in \Omega_{t+1} \to x+1+h\Delta S_{t+1}(\omega^t, \omega_{t+1})$ is $\mathcal{B}(\Omega_{t+1})$-measurable (see Assumption 4.2.2) and from (4.58) at $t+1$, we have that $(\omega_{t+1}, y) \in \Omega_{t+1} \times \mathbb{R} \to I_{t+1}(\omega^t, \omega_{t+1}, y) \in \mathcal{USA}(\Omega_{t+1} \times \mathbb{R})$ (see Lemma 4.8.4). Using [13, Lemma 7.30 p177] and (4.1), we obtain that

[5] As already mentioned, [13, Proposition 7.48 p180] relies on [13, Lemma 7.30 (4) p177] applied for upper-semianalytic functions where the convention $-\infty + \infty = -\infty$ needs to be used. But here, as we deal with a non-negative function we do not need to use the convention.

$\omega_{t+1} \in \Omega_{t+1} \to I_{t+1}(\omega^t, \omega_{t+1}, x+1+h\Delta S_{t+1}(\omega^t,\omega_t))$ is $\mathcal{B}_c(\Omega_{t+1})$-measurable. Therefore we can apply Lemma 4.5.18 and we get that $x \in \mathbb{R} \to I_t(\omega^t, x)$ (recall (4.50)) is non-decreasing and non-negative on $\mathbb{R}$ and that $U_t(\omega^t, x) \leq I_t(\omega^t, x+1)$ for all $x \in \mathbb{R}$. As this is true for all $\omega^t \in \Omega^t$, (4.56) and (4.59) are true at t.
We prove now (4.61) at t. Fix some $r \in \mathbb{Q}$, $r > 0$. We have from the definition of U_t (see (4.47) and (4.48))

$$U_t\left(\omega^t, r\right) \geq \mathcal{U}_t\left(\omega^t, r\right) \geq \inf_{P \in \mathcal{Q}_{t+1}(\omega^t)} \int_{\Omega_{t+1}} U_{t+1}\left(\omega^t, \omega_{t+1}, r\right) P(d\omega_{t+1}). \tag{4.75}$$

Using (4.61) at $t+1$ and the definition of J_t^r (see (4.53)) we have that

$$\begin{aligned}\inf_{P \in \mathcal{Q}_{t+1}(\omega^t)} \int_{\Omega_{t+1}} U_{t+1}\left(\omega^t, \omega_{t+1}, r\right) P(d\omega_{t+1}) &\geq \inf_{P \in \mathcal{Q}_{t+1}(\omega^t)} \int_{\Omega_{t+1}} -J_{t+1}^r(\omega^t, \omega_{t+1}) P(d\omega_{t+1}) \\ &= -J_t^r(\omega^t).\end{aligned}$$

And therefore using (4.75) we obtain (4.61) at t.
We prove now (4.60) at t. Let $x \geq 0$ and $(\phi_s)_{1 \leq s \leq t-1}$ $(\mathcal{B}_c(\Omega^{s-1}))_{1 \leq s \leq t-1}$-adapted random variables be fixed. We set $\overline{G} := x + \sum_{s=1}^{t-1} \phi_s \Delta S_s$. Furthermore, we fix some $P \in \mathcal{Q}^t$, $\xi \in \phi_t(\overline{G}, P)$ and set $G(\cdot) := \overline{G}(\cdot) + \xi(\cdot)\Delta S_t(\cdot)$.
We fix now some $\varepsilon > 0$ and we apply [13, Proposition 7.50 p184] to $\bar{i}_t$ (see (4.73)) in order to obtain a selector $S^\varepsilon : (\omega^t, x) \in \Omega^t \times \mathbb{R} \to (h^\varepsilon(\omega^t, x), p^\varepsilon(\cdot, \omega^t, x)) \in \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1})$ that is analytically-measurable (and therefore $\mathcal{B}_c(\Omega^t \times \mathbb{R})$-measurable) such that $p^\varepsilon(\cdot, \omega^t, x) \in \mathcal{Q}_{t+1}(\omega^t)$ for all $\omega^t \in \Omega^t$, $x \geq 0$ and (recall (4.74))

$$\bar{i}_t(\omega^t, h^\varepsilon(\omega^t, x), x, p^\varepsilon(\cdot, \omega^t, x)) \geq \begin{cases} \frac{1}{\varepsilon}, \text{ if } I_t(\omega^t, x) = \infty \\ I_t(\omega^t, x) - \varepsilon, \text{ otherwise.} \end{cases} \tag{4.76}$$

Let $h_G^\varepsilon(\omega^t) := h^\varepsilon(\omega^t, 1_{\{G \geq 0\}}(\omega^t) G(\omega^t))$ and $p_G^\varepsilon(\cdot, \omega^t) := p^\varepsilon(\cdot, \omega^t, 1_{\{G \geq 0\}}(\omega^t) G(\omega^t))$. Using [13, Proposition 7.44 p172], both h_G^ε and p_G^ε are $\mathcal{B}_c(\Omega^t)$-measurable. For some $\omega^t \in \Omega^t$, $y \geq 0$ fixed, if $h^\varepsilon(\omega^t, y) \notin H_y^{t+1}(\omega^t, p^\varepsilon(\cdot, \omega^t, y))$, using (4.73), we have $\bar{i}_t(\omega^t, h^\varepsilon(\omega^t, y), y, p^\varepsilon(\cdot, \omega^t, y)) = -\infty < \min\left(\frac{1}{\varepsilon}, I_t(\omega^t, y) - \varepsilon\right)$. This contradicts (4.76) (indeed from (4.56) at t, $I_t \geq 0$) and therefore we must have that $h^\varepsilon(\omega^t, y) \in H_y^{t+1}(\omega^t, p^\varepsilon(\cdot, \omega^t, y))$ and also that $h_G^\varepsilon(\omega^t) \in H_{G(\omega^t)}^{t+1}(\omega^t, p_G^\varepsilon(\cdot, \omega^t))$ for $\omega^t \in \{G \geq 0\}$. We set $P_G^\varepsilon := P \otimes p_G^\varepsilon \in \mathcal{Q}^{t+1}$ (recall that $p_G^\varepsilon(\cdot, \omega^t) \in \mathcal{Q}_{t+1}(\omega^t)$ for all $\omega^t \in \Omega^t$ and see (4.5)), we get that

$$\begin{aligned}P_G^\varepsilon(G(\cdot) + h_G^\varepsilon(\cdot)\Delta S_{t+1}(\cdot) \geq 0) &= \int_{\{G \geq 0\}} \int_{\Omega_{t+1}} p_G^\varepsilon(G(\omega^t) + h_G^\varepsilon(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1}) \geq 0, \omega^t) P(d\omega^t) \\ &= 1,\end{aligned}$$

since $\{G \geq 0\}$ is a P-full measure set and thus we have that $h_G^\varepsilon \in \phi_{t+1}(G, P_G^\varepsilon)$. We have as well for $\omega^t \in \{G \geq 0\}$ that (see (4.71) and (4.73))

$$\bar{i}_t(\omega^t, h_G^\varepsilon(\omega^t), G(\omega^t), p_G^\varepsilon(\cdot, \omega^t)) = \int_{\Omega_{t+1}} I_{t+1}(\omega^t, \omega_{t+1}, G(\omega^t) + 1 + h_G^\varepsilon(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1})) p_G^\varepsilon(d\omega_{t+1}, \omega^t)$$

and using again that $\{G \geq 0\}$ is a P-full measure set, we obtain that

$$\int_{\Omega^t} \bar{i}_t(\omega^t, h_G^\varepsilon(\omega^t), p_G^\varepsilon(\omega^t), G(\omega^t))P(d\omega^t) = \int_{\Omega^{t+1}} I_{t+1}(\omega^{t+1}, G(\omega^t) + 1 + h_G^\varepsilon(\omega^t)\Delta S_{t+1}(\omega^{t+1}))P_G^\varepsilon(d\omega^{t+1}) \leq A_x < \infty$$

where we have used (4.60) at $t+1$ ($\phi_{t+1}(G,P) \subset \phi_{t+1}(G+1,P)$) and we have set

$$A_x := \sup_{P \in \mathcal{Q}^{t+1}} \sup_{\xi \in \phi_{t+1}(G+1,P)} \int_{\Omega^{t+1}} I_{t+1}\left(\omega^{t+1}, G(\omega^t) + 1 + \xi(\omega^t)\Delta S_{t+1}(\omega^{t+1})\right) P(d\omega^{t+1}).$$

Combining with (4.76) we find that

$$\begin{aligned}&\frac{1}{\varepsilon}\int_{\{I_t(\cdot,G(\cdot))=\infty\}} P(d\omega^t) + \int_{\{I_t(\cdot,G(\cdot))<\infty\}} \left(I_t(\omega^t, G(\omega^t)) - \varepsilon\right) P(d\omega^t) \\ &\leq \int_{\Omega^t} \bar{i}_t(\omega^t, h_G^\varepsilon(\omega^t), G(\omega^t), p_G^\varepsilon(\cdot,\omega^t))P(d\omega^t) \leq A_x < \infty. \end{aligned} \tag{4.77}$$

As this is true for all $\varepsilon > 0$, $P(\{I_t(\cdot, G(\cdot)) = \infty\}) = 0$ follows and thus using again (4.77) we find that $\int_{\Omega^t} I_t(\omega^t, \overline{G}(\omega^{t-1}) + \xi\Delta S_t(\omega^t))P(d\omega^t) \leq A_x$. Finally as this is true for all $P \in \mathcal{Q}^t$ and $\xi \in \phi_t(\overline{G}, P)$, we obtain that $\sup_{P\in\mathcal{Q}^t} \sup_{\xi\in\phi_t(\overline{G},P)} \int_{\Omega^t} I_t(\omega^t, \overline{G}(\omega^{t-1}) + \xi(\omega^{t-1})\Delta S_t(\omega^t))P(d\omega^t) \leq A_x < \infty$ and (4.60) is true for t.
We are left with the proof of (4.66) for U_t. Let $X = x + \sum_{s=1}^{t-1} \phi_s \Delta S_{s+1}$, with $x \geq 0$ and $(\phi_s)_{1\leq s\leq t-1}$ some $(\mathcal{B}_c(\Omega^{s-1}))_{1\leq s\leq t-1}$-adapted random variables, be fixed such that $X \geq 0$ $\mathcal{Q}^t$-q.s. Let $\Omega_X^t := \widetilde{\Omega}^t \cap \{\omega^t \in \Omega^t, X(\omega^t) \geq 0\}$. Then $\Omega_X^t \in \mathcal{B}_c(\Omega^t)$ and $P(\Omega_X^t) = 1$ for all $P \in \mathcal{Q}^t$. We introduce the following random set $\psi_X : \Omega^t \twoheadrightarrow \mathbb{R}^d$

$$\begin{aligned}\psi_X(\omega^t) :=& \{h \in \mathcal{D}_{X(\omega^t)}^{t+1}(\omega^t),\ U_t(\omega^t, X(\omega^t)) = \\ & \inf_{P\in\mathcal{Q}_{t+1}(\omega^t)} \int_{\Omega^{t+1}} U_{t+1}\left(\omega^t, \omega_{t+1}, X(\omega^t) + h\Delta S_{t+1}(\omega^t, \omega_{t+1})\right) P(d\omega_{t+1})\},\end{aligned} \tag{4.78}$$

for $\omega^t \in \Omega_X^t$ and $\psi_X(\omega^t) = \emptyset$ otherwise (see (4.44) for the definition of $\mathcal{D}_{X(\omega^t)}^{t+1}(\omega^t)$). First we prove that $\Omega_X^t \subset \{\psi_X \neq \emptyset\}$ and consequently $P(\{\psi_X \neq \emptyset\}) = 1$ for all $P \in \mathcal{Q}^t$. Indeed, from Proposition 4.6.8 and Theorem 4.5.23 (see (4.38), (4.39), (4.47) and (4.48)), we have for all $\omega^t \in \widetilde{\Omega}^t$ and $x \geq 0$ that there exists some $\xi^* \in \mathcal{D}_x^{t+1}(\omega^t)$ such that

$$U_t(\omega^t, x) = \inf_{P\in\mathcal{Q}_{t+1}(\omega^t)} \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, x + \xi^*\Delta S_{t+1}(\omega^t, \omega_{t+1}))P(d\omega_{t+1}), \tag{4.79}$$

Thus for $\omega^t \in \Omega_X^t \subset \widetilde{\Omega}^t$ fixed, using (4.79) for $x = X(\omega^t) \geq 0$, there exists $\xi^* \in \psi_X(\omega^t)$. To prove (4.66), we want to find some $\mathcal{B}_c(\Omega^t)$-measurable selector for ψ_X. Let $u_X : \Omega^t \times \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$

$$u_X(\omega^t, h) = 1_{\Omega_X^t}(\omega^t)\widetilde{u}_t(\omega^t, h, X(\omega^t)), \tag{4.80}$$

(see (4.69) for the definition of $\widetilde{u}_t$). We first establish that u_X is $\mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. To do that we prove that $-u_X$ is a $\mathcal{B}_c(\Omega^t)$-normal integrand, see Definition 2.8.23 in Chapter 2. Since $\widetilde{u}_t \in \mathcal{LSA}(\Omega^t \times \mathbb{R}^d \times \mathbb{R})$ (and therefore is $\mathcal{B}_c(\Omega^t \times \mathbb{R}^d \times \mathbb{R})$-measurable), X is $\mathcal{B}_c(\Omega^t)$-measurable and $\Omega^t_X \in \mathcal{B}_c(\Omega^t)$, we can use [13, Proposition 7.44 p172] and u_X is $\mathcal{B}_c(\Omega^t \times \mathbb{R}^d)$-measurable. So for $h \in \mathbb{R}^d$ fixed, $\omega^t \in \Omega^t \to u_X(\omega^t, h)$ is $\mathcal{B}_c(\Omega^t)$-measurable, see [13, Lemma 7.29 p177]). Now we fix $\omega^t \in \Omega^t$. If $\omega^t \notin \Omega^t_X$, it is clear that $h \in \mathbb{R}^d \to u_X(\omega^t, h)$ is usc and concave. Now if $\omega^t \in \Omega^t_X \subset \widetilde{\Omega}^t$, we know from Proposition 4.6.8 and Remark 4.6.10, that we can apply Lemma 4.5.21 and the function $\phi_{\omega^t}(\cdot, \cdot)$ defined on $\mathbb{R} \times \mathbb{R}^d$ by

$$\phi_{\omega^t}(x,h) = \begin{cases} \inf_{P \in \mathcal{Q}_{t+1}(\omega^t)} \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, x + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) P(d\omega_{t+1}) \\ \text{if } x \geq 0 \text{ and } h \in \mathcal{H}^{t+1}_x(\omega^t) \\ -\infty \text{ otherwise,} \end{cases} \tag{4.81}$$

is usc and concave. In particular for $\omega^t \in \Omega^t_X$ and $x = X(\omega^t)$ we get that $h \in \mathbb{R}^d \to \phi_{\omega^t}(X(\omega^t), h) = u_X(\omega^t, h)$ is usc and concave (see (4.68), (4.69) and (4.80)). We apply now [116, Proposition 14.39 p666, Corollary 14.34 p664] and obtain that $-u_X$ is a $\mathcal{B}_c(\Omega^t)$-normal integrand and u_X is $\mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable.
Now, from the definitions of ψ_X and u_X (see (4.78) and (4.80)) we obtain that for $\omega^t \in \Omega^t_X$

$$\psi_X(\omega^t) = \left\{ h \in \mathcal{D}^{t+1}_{X(\omega^t)}(\omega^t),\ U_t(\omega^t, X(\omega^t)) = u_X(\omega^t, h) \right\}.$$

From Lemma 4.6.5, we have that Graph $\left(\mathcal{D}^{t+1}_X\right) \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$. Since we have already proved that U_t is $\mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R})$-measurable and X is $\mathcal{B}_c(\Omega^t)$-measurable, we obtain that $\omega^t \to U_t(\omega^t, X(\omega^t))$ is $\mathcal{B}_c(\Omega^t)$-measurable, see [13, Proposition 7.44 p172]. It follows that Graph$(\psi_X) \in \mathcal{B}_c(\Omega^t) \otimes \mathcal{B}(\mathbb{R}^d)$ and we can apply the Projection Theorem (see [37, Theorem 3.23 p75]) and we get that $\{\psi_X \neq \emptyset\} \in \mathcal{B}_c(\Omega^t)$. Using the Auman Theorem (see [119, Corollary 1]) there exists some $\mathcal{B}_c(\Omega^t)$-measurable $\widehat{h}^X_{t+1} : \{\psi_X \neq \emptyset\} \to \mathbb{R}^d$ such that for all $\omega^t \in \{\psi_X \neq \emptyset\}$, $\widehat{h}^X_{t+1}(\omega^t) \in \psi_X(\omega^t)$. Then we extend $\widehat{h}^X_{t+1}$ on all Ω^t by setting $\widehat{h}^X_{t+1} = 0$ on $\Omega^t \setminus \{\psi_X \neq \emptyset\}$. This concludes the proof of (4.66) since $\Omega^t_X \subset \{\psi_X \neq \emptyset\}$ and thus $\widehat{h}^X_{t+1}$ is a $\mathcal{B}_c(\Omega^t)$-measurable selector for ψ_X on Ω^t_X. □

Proof. of Theorem 4.4.14. We proceed in three steps. First, we handle some integrability issues that are essential to the proof. Then, we build by induction a candidate for the optimal strategy and finally we establish its optimality. The proof is very similar to the one of [99].

Integrability Issues

First from Proposition 4.6.2 and (4.12), $u(x) \leq M_x < \infty$. We fix some $x \geq 0$ and $\phi \in \Phi(x, \mathcal{Q}^T) = \Phi(x, U, \mathcal{Q}^T)$ (recall Assumptions 4.4.2 and 4.4.12 and Proposition 4.6.2). From Proposition 4.6.11, we can apply by backward induction Proposition

4.6.12 for $t = T-1, T-2, \dots, 0$. In particular, we get that (4.60) holds true for all $0 \leq t \leq T$. So choosing $G = V_{t-1}^{x+1,\phi}$ and $\xi = \phi_t$ (recall from Lemma 4.4.7 that $\phi \in \Phi(x, \mathcal{Q}^T)$ implies that $P_t(V_t^{x,\phi}(\cdot) \geq 0) = 1$ for all $P \in \mathcal{Q}^t$), we get using (4.59) and (4.60) that for all $P \in \mathcal{Q}^t$,

$$\int_{\Omega^t} U_t^+ \left(\omega^t, V_t^{x,\phi}(\omega^t)\right) P(d\omega^t) < \infty. \tag{4.82}$$

So for all $P = P_{t-1} \otimes p \in \mathcal{Q}^t$ (see (4.5)) we can use [13, Proposition 7.45 p175] and we get that

$$\int_{\Omega^t} U_t \left(\omega^t, V_t^{x,\phi}(\omega^t)\right) P(d\omega^t) = \int_{\Omega^{t-1}} \int_{\Omega_t} U_t \left(\omega^{t-1}, \omega_t, V_t^{x,\phi}(\omega^{t-1}, \omega_t)\right) p(d\omega_t, \omega^{t-1}) P_{t-1}(d\omega^{t-1}). \tag{4.83}$$

Recall as well (see (4.62)) that for all $1 \leq t \leq T$, $x \in \mathbb{R}$, $\omega^t \in \widetilde{\Omega}^t$, $U_t(\omega^t, x) = \mathcal{U}_t(\omega^t, x)$ and also $U_0(x) = \mathcal{U}_0(x)$.

Construction of ϕ^*

We fix some $x \geq 0$ and build by induction our candidate for the optimal strategy. We start at $t = 0$ and use (4.66) in Proposition 4.6.12 with $X = x \geq 0$. We set $\phi_1^* := \widehat{h}_1^x \in \mathcal{D}_x^1$ and we obtain that $P_1(x + \phi_1^* \Delta S_1(.) \geq 0) = 1$ for all $P \in \mathcal{Q}^1$ and

$$U_0(x) = \inf_{P \in \mathcal{Q}^1} \int_{\Omega} U_1 \left(\omega_1, x + \phi_1^* \Delta S_1(\omega_1)\right) P(d\omega_1).$$

Assume that until some $t \geq 1$ we have found some $(\phi_s^*)_{1 \leq s \leq t}$ $(\mathcal{B}_c(\Omega^{s-1}))_{1 \leq s \leq t}$-adapted random variables and some $\overline{\Omega}^1 \in \mathcal{B}_c(\Omega^1), \dots, \overline{\Omega}^{t-1} \in \mathcal{B}_c(\Omega^{t-1})$ such that $P_s(\overline{\Omega}^s) = 1$, $\phi_{s+1}^*(\omega^s) \in D^{s+1}(\omega^s)$ for all $\omega^s \in \overline{\Omega}^s$, $s = 1, \dots, t-1$, for all $P \in \mathcal{Q}^t$
$P\left(x + \phi_1^* \Delta S_1(\cdot) + \dots + \phi_t^*(\cdot) \Delta S_t(\cdot) \geq 0\right) = 1$ and for all $\omega^t \in \overline{\Omega}^{t-1}$

$$\begin{aligned} & U_{t-1}\left(\omega^{t-1}, V_{t-1}^{x,\phi^*}(\omega^{t-1})\right) \\ & = \inf_{P \in \mathcal{Q}_t(\omega^{t-1})} \int_{\Omega_t} U_t \left(\omega^{t-1}, \omega_t, V_{t-1}^{x,\phi^*}(\omega^{t-1}) + \phi_t^*(\omega^{t-1}) \Delta S_t(\omega^{t-1}, \omega_{t+1})\right) P(d\omega_t). \end{aligned}$$

We apply Proposition 4.6.12 with $X = V_t^{x,\phi^*}(\cdot) = V_{t-1}^{x,\phi^*}(\cdot) + \phi_t^*(\cdot)\Delta S_t(\cdot)$ (recall that $P\left(V_t^{x,\phi^*} \geq 0\right) = 1$ for all $P \in \mathcal{Q}^t$) and there exists $\overline{\Omega}^t := \Omega^t_{V_t^{x,\phi^*}} \in \mathcal{B}_c(\Omega^t)$ such that $P(\overline{\Omega}^t) = 1$ for all $P \in \mathcal{Q}^t$ and some $\mathcal{B}_c(\Omega^t)$-measurable $\omega^t \to \phi_{t+1}^*(\omega^t) := \widehat{h}_{t+1}^{V_t^{x,\phi^*}}(\omega^t)$ such that $\phi_{t+1}^*(\omega^t) \in \mathcal{D}^{t+1}_{V_t^{x,\phi^*}(\omega^t)}(\omega^t)$ for all $\omega^t \in \overline{\Omega}^t$ and

$$U_t\left(\omega^t, V_t^{x,\phi^*}(\omega^t)\right) = \inf_{P \in \mathcal{Q}_{t+1}(\omega^t)} \int_{\Omega_{t+1}} U_{t+1}\left(\omega^t, \omega_{t+1}, V_t^{x,\phi^*}(\omega^t) + \phi_{t+1}^*(\omega^t)\Delta S_{t+1}(\omega^t, \omega_{t+1})\right) P(d\omega_{t+1}). \tag{4.84}$$

Let $P^{t+1} = P \otimes p \in \mathcal{Q}^{t+1}$ where $P \in \mathcal{Q}^t$ and $p \in \mathcal{S}K_{t+1}$ with $p(\cdot, \omega^t) \in \mathcal{Q}_{t+1}(\omega^t)$ for all $\omega^t \in \overline{\Omega}^t$ (see (4.5)). From [13, Proposition 7.45 p175] we get that

$$P_{t+1}(V_{t+1}^{x,\phi^*} \geq 0) = \int_{\Omega^t} p(V_t^{x,\phi^*}(\omega^t) + \phi^*_{t+1}(\omega^t)\Delta S_{t+1}(\omega^t, \cdot) \geq 0, \omega^t) P(d\omega^t) = 1,$$

where we have used that $\phi^*_{t+1}(\omega^t) \in \mathcal{H}^{t+1}_{V_t^{x,\phi^*}(\omega^t)}(\omega^t)$ for all $\omega^t \in \overline{\Omega}^t$ and $P(\overline{\Omega}^t) = 1$ and we can continue the recursion.
Thus, we have found $\phi^* = (\phi^*_t)_{1 \leq t \leq T}$ such that for all $t = 0, \ldots, T$, all $P \in \mathcal{Q}^T$, $P_t(V_t^{x,\phi^*} \geq 0) = 1$, *i.e* $\phi^* \in \Phi(x, \mathcal{Q}^T)$. We have also found some $\overline{\Omega}^t \in \mathcal{B}_c(\Omega^t)$, with $P_t(\overline{\Omega}^t) = 1$ for all $P \in \mathcal{Q}^T$ and such that (4.84) holds true for all $\omega^t \in \overline{\Omega}^t$, all $t = 0, \ldots, T-1$. Moreover from Proposition 4.6.2, $\phi^* \in \Phi(x, U, \mathcal{Q}^T)$ and $E_P U_T(\cdot, V_T^{x,\phi^*}(\cdot)) < \infty$ for all $P \in \mathcal{Q}^T$.
Optimality of ϕ^*
We prove in two steps that ϕ^* is an optimal strategy.
Step 1: Fix some $P = P_{T-1} \otimes p_T \in \mathcal{Q}^T$. Using (4.83), $P_{T-1}(\overline{\Omega}^{T-1}) = 1$ and (4.84) for $t = T-1$ we get that

$$\begin{aligned}
&E_P U(\cdot, V_T^{x,\phi^*}(\cdot)) \\
&= \int_{\overline{\Omega}^{T-1}} \int_{\Omega_T} U_T\left(\omega^{T-1}, \omega_T, V_{T-1}^{x,\phi^*}(\omega^{T-1}) + \phi^*_T(\omega^{T-1})\Delta S_T(\omega^{T-1}, \omega_T)\right) p_T(d\omega_T, \omega^{T-1}) P_{T-1}(d\omega^{T-1}) \\
&\geq \int_{\Omega^{T-1}} U_{T-1}\left(\omega^{T-1}, V_{T-1}^{x,\phi^*}(\omega^{T-1})\right) P_{T-1}(d\omega^{T-1}).
\end{aligned}$$

We iterate the process by backward induction and obtain that (recall that $\Omega^0 := \{\omega_0\}$)

$$U_0(x) \leq E_P U(\cdot, V_T^{x,\phi^*}(\cdot)). \tag{4.85}$$

As the preceding equality holds true for all $P \in \mathcal{Q}^T$ and as $\phi^* \in \Phi(x, U, \mathcal{Q}^T)$, we get that $U_0(x) \leq u(x)$ (see (4.12)). So ϕ^* will be optimal if $U_0(x) \geq u(x)$.
Step 2: Now we fix some $\phi \in \Phi(x, U, \mathcal{Q}^T)$ and also some $0 \leq t \leq T-1$. For all $\omega^t \in \widetilde{\Omega}^t \cap \{\omega^t \in \Omega^t,\ V_t^{x,\phi}(\omega^t) \geq 0\}$, we get that

$$\begin{aligned}
&\inf_{P \in \mathcal{Q}_{t+1}(\omega^t)} \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, V_{t+1}^{x,\phi}(\omega^t, \omega_{t+1})) P(d\omega_{t+1}) \\
&\leq \sup_{h \in \mathcal{H}^{t+1}_{V_t^{x,\phi}(\omega^t)}(\omega^t)} \inf_{P \in \mathcal{Q}_{t+1}(\omega^t)} \int_{\Omega_{t+1}} U_{t+1}(\omega^t, \omega_{t+1}, V_t^{x,\phi}(\omega^t) + h\Delta S_{t+1}(\omega^t, \omega_{t+1})) P(d\omega_{t+1}) \\
&= U_t(\omega^t, V_t^{x,\phi}(\omega^t)),
\end{aligned} \tag{4.86}$$

where we have used Remark 4.6.9. Now fix some $\varepsilon > 0$. As $(\omega^t, y, h, P, \omega_{t+1}) \in \Omega^t \times \mathbb{R} \times \mathbb{R}^d \times \mathfrak{P}(\Omega_{t+1}) \times \Omega_{t+1} \to y + h\Delta S_{t+1}(\omega^t, \omega_{t+1})$ is $\mathcal{B}(\Omega^t) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathfrak{P}(\Omega_{t+1})) \otimes$

$\mathcal{B}(\Omega_{t+1})$-measurable (recall Assumption 4.2.2) and U_{t+1} is $\mathcal{L}SA(\Omega^{t+1}\times\mathbb{R})$ (see (4.57)), we can use [13, Lemma 7.30 (3) p178, Proposition 7.48 p180] in order to obtain that

$$(\omega^t,y,h,P)\to\int_{\Omega_{t+1}}U_{t+1}(\omega^t,\omega_{t+1},y+h\Delta S_{t+1}(\omega^t,\omega_{t+1}))P(d\omega_{t+1})\in\mathcal{L}SA(\Omega^t\times\mathbb{R}\times\mathbb{R}^d\times\mathfrak{P}(\Omega_{t+1})).$$

We then apply [13, Proposition 7.50 p184] and there exists some $\widetilde{p}_{t+1}^{\varepsilon}:(\omega^t,y,h)\in\Omega^t\times\mathbb{R}\times\mathbb{R}^d\to\mathfrak{P}(\Omega_{t+1})$ that is $\mathcal{B}_c(\Omega^t\times\mathbb{R}\times\mathbb{R}^d)$-measurable and such that $\widetilde{p}_{t+1}^{\varepsilon}(\cdot,\omega^t,y,h)\in\mathcal{Q}_{t+1}(\omega^t)$ for all $(\omega^t,y,h)\in\Omega^t\times\mathbb{R}\times\mathbb{R}^d$ and

$$\int_{\Omega_{t+1}}U_{t+1}(\omega^t,\omega_{t+1},y+h\Delta S_{t+1}(\omega^t,\omega_{t+1}))\widetilde{p}_t^{\varepsilon}(d\omega_{t+1},\omega^t,y,h)$$
$$\leq\begin{cases}\inf_{P\in\mathcal{Q}_{t+1}(\omega^t)}\int_{\Omega_{t+1}}U_{t+1}(\omega^t,\omega_{t+1},y+h\Delta S_{t+1}(\omega^t,\omega_{t+1})))P(d\omega_{t+1})+\varepsilon,\\ \text{if }\inf_{P\in\mathcal{Q}_{t+1}(\omega^t)}\int_{\Omega_{t+1}}U_{t+1}(\omega^t,\omega_{t+1},y+h\Delta S_{t+1}(\omega^t,\omega_{t+1})))P(d\omega_{t+1})>-\infty\\ -\frac{1}{\varepsilon},\ \text{otherwise}.\end{cases}\tag{4.87}$$

We define $p_{t+1}^{\varepsilon}:\Omega^t\to\mathfrak{P}(\Omega_{t+1})$ by $p_{t+1}^{\varepsilon}(\cdot,\omega^t)=\widetilde{p}_{t+1}^{\varepsilon}\left(\cdot,\omega^t,V_t^{x,\phi}(\omega^t),\phi_{t+1}(\omega^t)\right)$ and [13, Proposition 7.44 p172] implies that p_{t+1}^{ε} is $\mathcal{B}_c(\Omega^t)$-measurable. Now, choosing $y=V_t^{x,\phi}(\omega^t)$, $h=\phi_{t+1}(\omega^t)$ in (4.87) and using (4.86), we find that for all $\omega^t\in\widetilde{\Omega}^t\cap\{\omega^t\in\Omega^t,\ V_t^{x,\phi}(\omega^t)\geq 0\}$

$$\int_{\Omega_{t+1}}U_{t+1}(\omega^t,\omega_{t+1},V_{t+1}^{x,\phi}(\omega^t,\omega_{t+1}))p_{t+1}^{\varepsilon}(d\omega_{t+1},\omega^t)-\varepsilon\leq\max\left(U_t(\omega^t,V_t^{x,\phi}(\omega^t)),-\frac{1}{\varepsilon}-\varepsilon\right).\tag{4.88}$$

Fix some $Q\in\mathcal{Q}^t$ and set $P^{\varepsilon}:=Q\otimes p_{t+1}^{\varepsilon}\in\mathcal{Q}^{t+1}$ (see (4.5)). Using (4.83) and since $\widetilde{\Omega}^t\cap\{\omega^t\in\Omega^t,\ V_t^{x,\phi}(\omega^t)\geq 0\}$ is a $\mathcal{Q}^t$ full measure set (recall that $\phi\in\Phi(x,\mathcal{Q}^T)$ and Lemma 4.4.7) , we get from (4.88) that

$$\inf_{P\in\mathcal{Q}^{t+1}}E_PU_{t+1}(\cdot,V_{t+1}^{x,\phi}(\cdot))-\varepsilon\leq E_{P^{\varepsilon}}U_{t+1}(\cdot,V_{t+1}^{x,\phi}(\cdot))-\varepsilon\leq E_Q\max\left(U_t(\cdot,V_t^{x,\phi}(\cdot)),-\frac{1}{\varepsilon}-\varepsilon\right).$$

Since for all $0<\varepsilon<1$, $\max\left(U_t(\cdot,V_t^{x,\phi}(\cdot)),-\frac{1}{\varepsilon}-\varepsilon\right)\leq -1+U_t^{+}(\cdot,V_t^{x,\phi}(\cdot))$, recalling (4.82), letting ε go to zero and applying Fatou's Lemma, we obtain that

$$\inf_{P\in\mathcal{Q}^{t+1}}E_PU_{t+1}(\cdot,V_{t+1}^{x,\phi}(\cdot))\leq E_QU_t(\cdot,V_t^{x,\phi}(\cdot)).$$

As this holds true for all $Q\in\mathcal{Q}^t$ we get that

$$\inf_{P\in\mathcal{Q}^{t+1}}E_PU_{t+1}(\cdot,V_{t+1}^{x,\phi}(\cdot))\leq\inf_{Q\in\mathcal{Q}^t}E_QU_t(\cdot,V_t^{x,\phi}(\cdot)).$$

So recursively we obtain that (recall (4.47), (4.48) and Remark 4.6.9)

$$\inf_{P\in\mathcal{Q}^T}E_PU_T(\cdot,V_T^{x,\phi}(\cdot))\leq\inf_{Q\in\mathcal{Q}^1}E_QU_1(\cdot,V_1^{x,\phi}(\cdot))\leq U_0(x).$$

As this is true for all $\phi \in \Phi(x, U, \mathcal{Q}^T)$ we have that $u(x) \leq U_0(x) < \infty$ and the proof is complete. □

Proof. of Theorem 4.4.15. Since the $sNA(\mathcal{Q}^T)$ condition holds true, we know that the $NA(\mathcal{Q}^T)$ condition holds true as well. Hence, to apply Theorem 4.4.14 it remains to prove that Assumption 4.4.12 holds true. We fix some $P \in \mathcal{Q}^T$ $x \geq 0$ and some $\phi \in \phi(x, P)$. Since the $NA(P)$ condition holds true, using similar arguments as in the proof of Theorem 2.4.17 in Chapter 4, we find that for $\overline{P}_t$-almost all $\omega^t \in \Omega^t$ (see the proof of Proposition 4.3.6 for the definition of $\overline{P}_t$)

$$|V_t^{x,\phi}(\omega^t)| \leq \prod_{s=1}^{t} \left(x + \frac{|\Delta S_s(\omega^s)|}{\alpha_{s-1}^P(\omega^{s-1})} \right). \tag{4.89}$$

As $\Delta S_s,\ \frac{1}{\alpha_s^P} \in \mathcal{W}_s$ for all $s \geq 1$, we obtain that $V_t^{x,\phi} \in \widehat{\mathcal{W}_t}$ (recall that the trading strategies are universally-measurable). Now we use (5.38) for $x = 1$, $t = T$, the monotonicity of U^+, the fact that $2\prod_{s=1}^T \left(1 + \frac{|\Delta S_s(\omega^s)|}{\alpha_{s-1}^P(\omega^{s-1})}\right) \geq 1$ and Proposition 4.6.1 and we obtain that for $\overline{P}_t$-almost all $\omega^t \in \Omega^t$

$$U^+(\omega^T, V_T^{1,\phi}(\omega^T)) \leq 4 \left(\prod_{s=1}^{T} \left(1 + \frac{|\Delta S_s(\omega^s)|}{\alpha_{s-1}^P(\omega^{s-1})} \right) \right) \left(U^+(\omega^T, 1) + C_T(\omega^T) \right). \tag{4.90}$$

We set $N := 4 \sup_{P \in \mathcal{Q}^T} E_P \left(\prod_{s=1}^T \left(1 + \frac{|\Delta S_s(\omega^s)|}{\alpha_{s-1}^P(\omega^{s-1})} \right) \right) \left(U^+(\omega^T, 1) + C_T(\omega^T) \right).$
Since $U^+(\cdot, 1),\ U^-(\cdot, \frac{1}{4}) \in \mathcal{W}_T$ and $\Delta S_s,\ \frac{1}{\alpha_s^P} \in \mathcal{W}_s$ for all $s \geq 1$, we obtain that $N < \infty$ (recall the definition of C_T in Proposition 4.6.1). Using (4.90) we find that $E_P U^+(\cdot, V_T^{1,\phi}(\cdot)) \leq N < \infty$ and as this is true for all $P \in \mathcal{Q}^T$ and $\phi \in \Phi(1, P)$, Assumption 4.4.12 holds true. □

4.7 Conclusion

As a conclusion we propose a list of potential improvements to the current results. First we could introduce a random endowment *i.e.* some $G : \Omega^T \to [0, \infty)$ and study $\sup_{\phi \in \Phi_G(x,U,\mathcal{Q}^T)} \inf_{P \in \mathcal{Q}^T} E_P U(\cdot, V_T^{x,\phi}(\cdot) - G(\cdot))$ where $\phi_G(x, U, \mathcal{Q}^T) = \{\phi \in \phi(x, U, \mathcal{Q}^T),\ V_T^{x,\phi}(\cdot) - G(\cdot) \geq 0\ \mathcal{Q}^T\text{-q.s}\}$. This is a useful generalisation, especially in the context of Chapter 5 where we could obtain some optimal strategy to realise the utility indifference price. In the spirit of [112] assuming that G is usa and bounded and using [25, SuperHedging Theorem, Lemma 4.10], the current proof can be adapted introducing $G_t(\omega^t) = \sup_{P \in \mathcal{Q}^{t+1}(\omega^t)} E_P G_{t+1}(\omega^t, \cdot)$ and replacing $\mathcal{H}^{t+1}(\omega^t)$ with $\mathcal{H}^{t+1}(\omega^t) := \left\{ h \in \mathbb{R}^d,\ x + h\Delta S_{t+1}(\omega^t, \cdot) \geq G_t(\omega^t)\ \mathcal{Q}_{t+1}(\omega^t)\text{-q.s.} \right\}$. The precise and careful adaptation of the proof is left for further research.

An other significant improvement would be to remove the concavity assumption in Definition 4.4.1. Indeed, this assumption is not required in the mono-prior case (see Chapter 2) and as already discussed is not always verified in practice. Unfortunately in the multiple-priors framework, even if it is not essential to the optimisation problem in itself, it is still crucial to obtain measurability property. This occurs for instance in (4.39) where the concavity is essential so that our version of U_t ((4.47), (4.48)) is lsa. And this is also the case in Example 4.8.2 and Lemma 4.8.3 which are used to prove that u_X (see (4.80)) is a normal-integrand. We have explored alternative assumptions but none of them was conclusive and the question remains open.
Lastly, it is not clear what the final word could be in term of integrability condition (see (4.4.12)). We were initially aiming for the following sharper (since $\Phi(1, \mathcal{Q}^T) \subset \Phi(1, P)$ for all $P \in \mathcal{Q}^T$) and nicer condition to Assumption 4.4.12

$$\sup_{P \in \mathcal{Q}^T} \sup_{\phi \in \Phi(1, \mathcal{Q}^T)} E_P U^+(\cdot, V_T^{1,\phi}(\cdot)) < \infty. \tag{4.91}$$

Unfortunately this raises measurability issues when establishing that I_t is usa (whose definition has to be slightly adapted) that we could not solve.
For utility function defined on $\mathbb{R}$, [98] obtained some results for utility function bounded from above. Note however that in the unbounded case the question remains open: as pointed out in the conclusion of Chapter 2 already in the mono-prior case it seems difficult to replace the usual assumption: $u(x) < \infty$ with an assumption on $E_P U(V_T^{x,\phi})$ that would not be too restrictive. In the multiple-priors case this is compounded by measurability issues. Note that even $u(x) < \infty$ cannot work for similar reasons as in Remark 4.5.15. In the same spirit as what we have done, an integrability condition that would look like $I_0(x) < \infty$ (see (4.49) and (4.50)) might work but is clearly not very satisfying as very difficult to establish in practice. We have provided some details in Chapter 3 on how the one-period model can be tackled but the study of the multi-period model is left for further research.
As already eluded to before, introducing on top of the uncertainty some distortions on the probability would also be an other very interesting generalisation.
As a a last remark, note that while in Chapter 2 in a mono-prior approach all the measurability questions could be worked out relatively smoothly, this is not always the case in the multiple-priors framework. The main reason is that universally measurable sets are not stable by projection: if X, Y are Polish spaces and $A \in \mathcal{B}_c(X \times Y)$, we don't know if $Proj_X(A) \in \mathcal{B}_c(X)$ and this is the reason why the class of analytic sets is introduced. The price to pay is that analytic sets are not stable by complementation. Is there a way to avoid the use of analytic sets: if yes this is far from trivial. Note however that if we have $O \subset X \times Y$ open in $X \times Y$ (for the product topology) then $Proj_{X \times Y}(O)$ is open in X. So an alternative to analytics sets is to use open sets and thus introduce some continuity assumption. This would

greatly simplify all measurability arguments.

4.8 Appendix

4.8.1 Technical results

The following lemma is a well-known result on concave functions which proof is given since we did not find some reference.

Lemma 4.8.1 *Let* $f : \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ *be a concave function such that* $Ri(\mathrm{Dom}\, f) \neq \emptyset$. *Then*

$$\sup_{h \in \mathrm{Dom}\, f} f(h) = \sup_{h \in \mathrm{Ri}(\mathrm{Dom}\, f)} f(h).$$

Proof. Let $C := \sup_{h \in \mathrm{Ri}(\mathrm{Dom}\, f)} f(h)$ and $h_1 \in \mathrm{Dom}\, f \backslash \mathrm{Ri}(\mathrm{Dom}\, f)$ be fixed. We have to prove that $f(h_1) \leq C$. If $C = \infty$ there is nothing to show. So we assume that $C < +\infty$. Let $h_0 \in \mathrm{Ri}(\mathrm{Dom}\, f)$ and introduce $\phi : t \in \mathbb{R} \to f(th_1 + (1-t)h_0)$ if $t \in [0,1]$ and $-\infty$ otherwise. From [115, Theorem 6.1 p45], $th_1 + (1-t)h_0 \in \mathrm{Ri}(\mathrm{Dom}\, f)$ if $t \in [0,1)$ and thus $[0,1) \subset \{t \in [0,1], \phi(t) \leq C\}$. Clearly, ϕ is concave on $\mathbb{R}$. Furthermore from the convexity of Dom f, we have that $\mathrm{Dom}\, \phi = [0,1]$. So, using [62, Proposition A.4 p400], we get that ϕ is lsc on $[0,1]$ and $\{t \in [0,1],\ \phi(t) \leq C\}$ is a closed set in $\mathbb{R}$. It follows that $1 \in \{t \in [0,1], \phi(t) \leq C\}$, *i.e* $\phi(1) = f(h_1) \leq C$ and the proof is complete. □

4.8.2 Measure theory issues

In this section, we first provide some counterexamples to [25, Lemma 4.12] and propose an alternative to this lemma. Our counterexample 4.8.2 is based on a result from [66] originally due [123]. An other counterexample can be found [116, Proposition 14.28 p661].

Example 4.8.2 We denote by $\mathcal{L}(\mathbb{R}^2)$ the Lebesgue sigma-algebra on $\mathbb{R}^2$. Recall that $\mathcal{B}(\mathbb{R}^2) \subset \mathcal{L}(\mathbb{R}^2)$. Let $A \notin \mathcal{L}(\mathbb{R}^2)$ be such that every line has at most two common points with A (see [66, Example 22 p142] for the proof of the existence of A) and define $F : \mathbb{R}^2 \to \mathbb{R}$ by $F(x,y) := 1_A(x,y)$. We fix some $x \in \mathbb{R}$ and prove that the function $F(x,\cdot)$ is usc. Let $A_x^1 := \{y \in \mathbb{R},\ (x,y) \in A\}$. Since every line has at most two common points with A, A_x^1 contains at most two points: thus it is a closed subset of $\mathbb{R}$. It follows that $\{y \in \mathbb{R},\ F(x,y) \geq c\}$ is a closed subset of $\mathbb{R}$ for all $c \in \mathbb{R}$ and $F(x,\cdot)$ is usc as claimed.
Similarly the function $F(\cdot,y)$ is usc and thus $\mathcal{B}(\mathbb{R})$-measurable for all $y \in \mathbb{R}$ fixed. But since $A \notin \mathcal{L}(\mathbb{R}^2)$, F is not $\mathcal{L}(\mathbb{R}^2)$-measurable and therefore not $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$-measurable.

We propose now the following correction to [25, Lemma 4.12]. Note that Lemma 4.8.3 can be applied in the proof of [99, Lemma 3.7] since the considered function is concave (as well as in the proof of [25, Lemma 4.10] where the considered function is convex).

Lemma 4.8.3 *Let $(A, \mathcal{A})$ be a measurable space and let $\theta : \mathbb{R}^d \times A \to \mathbb{R} \cup \{\pm\infty\}$ be a function such that $\omega \to \theta(y, \omega)$ is $\mathcal{A}$-measurable for all $y \in \mathbb{R}^d$ and $y \to \theta(y, \omega)$ is lsc and convex for all $\omega \in A$. Then θ is $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{A}$-measurable.*

Proof. It is a direct application of [116, Proposition 14.39 p666, Corollary 14.34 p664]. □

We give the following three useful lemmata related to measure theory issues that are used throughout the chapter.

Lemma 4.8.4 *Let X, Y be two Polish spaces and $F : X \times Y \to \mathbb{R} \cup \{\pm\infty\} \in \mathcal{USA}(X \times Y)$ (resp. $\mathcal{LSA}(X \times Y)$). Then, for $x \in X$ fixed, the function $F_x : y \in Y \to F(x, y) \in \mathbb{R} \cup \{\pm\infty\}$ belongs to $\mathcal{USA}(Y)$ (resp. $\mathcal{LSA}(Y)$).*

Proof. Fix some $c \in \mathbb{R}$. By assumption we have that $C := \{(x, y) \in X \times Y,\ F(x, y) > c\} \in \mathcal{A}(X \times Y)$. Fix now some $x \in X$. Since the function $I_x : y \in Y \to (x, y) \in X \times Y$ is $\mathcal{B}(Y)$-measurable (it is even continuous), applying [13, Proposition 7.40 p165], we get that $\{y \in Y,\ F_x(y) > c\} = \{y \in Y,\ (x, y) \in C\} = I_x^{-1}(C) \in \mathcal{A}(Y)$. □

Lemma 4.8.5 *Assume that Assumptions 4.2.1 and 4.2.2 hold true. Let $0 \leq t \leq T-1$, $B \in \mathcal{B}(\mathbb{R})$. Introduce the following functions*

$$F_B : (\omega^t, P, h, x) \in \Omega^t \times \mathfrak{P}(\Omega_{t+1}) \times \mathbb{R}^d \times \mathbb{R} \to P\left(x + h\Delta S_{t+1}(\omega^t, \cdot) \in B\right), \tag{4.92}$$

$$H_B : (\omega^t, h, x) \in \Omega^t \times \mathbb{R}^d \times \mathbb{R} \to \inf_{P \in \mathcal{Q}_{t+1}(\omega^t)} P(x + h\Delta S_{t+1}(\omega^t, \cdot) \in B), \tag{4.93}$$

$$K_B : (\omega^t, h) \in \Omega^t \times \mathbb{R}^d \to \sup_{P \in \mathcal{Q}_{t+1}(\omega^t)} P(x + h\Delta S_{t+1}(\omega^t, \cdot) \in B). \tag{4.94}$$

Then F_B is $\mathcal{B}(\Omega^t) \otimes \mathcal{B}(\mathfrak{P}(\Omega_{t+1})) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R})$-measurable, $H_B \in \mathcal{LSA}(\Omega^t \times \mathbb{R}^d \times \mathbb{R})$ and $K_B \in \mathcal{USA}(\Omega^t \times \mathbb{R}^d)$.

Proof. Applying [13, Proposition 7.29 p144], with $Y = \Omega_{t+1}$, $X = \Omega^t \times \mathfrak{P}(\Omega_{t+1}) \times \mathbb{R}^d \times \mathbb{R}$, $f(\omega_{t+1}, \omega^t, P, x, h) = 1_{x + h\Delta S_{t+1}(\omega^t, \omega_{t+1}) \in B}(\omega_{t+1})$ (recall Assumption 4.2.2) and the Borel-measurable stochastic kernel $q(d\omega_{t+1}|\omega^t, P, x, h) = P(d\omega_{t+1})$, we get that F_B is $\mathcal{B}(\Omega^t) \otimes \mathcal{B}(\mathfrak{P}(\Omega_{t+1})) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R})$-measurable.
To prove that $H_B \in \mathcal{LSA}(\Omega^t \times \mathbb{R}^d \times \mathbb{R})$, we apply [13, Proposition 7.47 p179] to F_B with $X = \Omega^t \times \mathbb{R}^d \times \mathbb{R}$, $Y = \mathfrak{P}(\Omega_{t+1})$ and

$$D = \left\{(\omega^t, h, x, P) \in \Omega^t \times \mathbb{R}^d \times \mathbb{R} \times \mathfrak{P}(\Omega_{t+1}),\ (\omega^t, P) \in \text{Graph}(\mathcal{Q}_{t+1})\right\} \\ \in \mathcal{A}(\Omega^t \times \mathbb{R}^d \times \mathbb{R} \times \mathfrak{P}(\Omega_{t+1}))$$

(recall Assumption 4.2.1 and [13, Proposition 7.38 p165]). Using Lemma 4.8.4 and the fact that $\sup_{P\in\mathcal{Q}_{t+1}(\omega^t)} P(x + h\Delta S_{t+1}(\omega^t,\cdot) \in B) = 1 - \inf_{P\in\mathcal{Q}_{t+1}(\omega^t)} P(x + h\Delta S_{t+1}(\omega^t,\cdot) \in B^c)$ we obtain (4.94). □

Lemma 4.8.6 *Let X be a Polish space and Λ be an $\mathbb{R}^d$-valued random.*

i) Assume that Graph$(\Lambda) \in \mathcal{B}_c(X) \otimes \mathcal{B}(\mathbb{R}^d)$. Then Graph$(\overline{\Lambda}) \in \mathcal{B}_c(X) \otimes \mathcal{B}(\mathbb{R}^d)$ where $\overline{\Lambda}$ is defined by $\overline{\Lambda}(x) = \overline{\Lambda(x)}$ for all $x \in X$ (where the closure is taken in $\mathbb{R}^d$).

ii) Assume now that Λ is open valued and that Graph$(\Lambda) \in \mathcal{CA}(X \times \mathbb{R}^d)$. Then Graph$(\Lambda) \in \mathcal{B}_c(X) \otimes \mathcal{B}(\mathbb{R}^d)$.

Proof. We prove $i)$. Applying [116, Theorem 14.8 p648] (recall that $\mathcal{B}_c(X)$ is complete) we obtain that Λ is $\mathcal{B}_c(X)$-measurable (see Definition 2.8.19 in Chapter 2) and using then [3, Theorem 18.6 p596] we get that Graph$(\overline{\Lambda}) \in \mathcal{B}_c(X) \otimes \mathcal{B}(\mathbb{R}^d)$.
We prove now $ii)$. Fix some open set $O \subset \mathbb{R}^d$ and let $\Lambda^c(x) = \mathbb{R}^d \backslash \Lambda(x)$. Then applying [13, Proposition 7.39 p165] we get that

$$\{x \in X,\ \Lambda^c(x) \cap O \neq \emptyset\} = Proj_X\left((X \times O) \cap \text{Graph}(\Lambda^c)\right) \in \mathcal{A}(X) \subset \mathcal{B}_c(X),$$

since $X \times O \in \mathcal{B}(X) \otimes \mathcal{B}(\mathbb{R}^d) \subset \mathcal{A}(X \times \mathbb{R}^d)$ (see (4.1)) and Graph$(\Lambda^c) = \left(X \times \mathbb{R}^d\right) \backslash$Graph$(\Lambda) \in \mathcal{A}(X \times \mathbb{R}^d)$. Thus Λ^c is $\mathcal{B}_c(X)$-measurable and as Λ^c is closed valued, we can apply [116, Theorem 14.8 p648] and we get that Graph$(\Lambda^c) \in \mathcal{B}_c(X) \otimes \mathcal{B}(\mathbb{R}^d)$. The fact that Graph$(\Lambda) \in \mathcal{B}_c(X) \otimes \mathcal{B}(\mathbb{R}^d)$ follows immediatly. □

CHAPTER 5

Asymptotic of utility indifference prices to the superreplication price in a multiple-priors framework

The content of this chapter is very close to [19] which will be submitted soon for publication.

This chapter formulates an utility indifference pricing model for investors trading in a discrete time financial market under non-dominated model uncertainty. The investors preferences are described by strictly increasing concave random functions defined on the positive axis. We prove that under suitable conditions the multiple-priors utility indifference prices of a contingent claim converge to its multiple-priors superreplication price when the investors absolute risk-aversion tends to infinity. We also revisit the notion of certainty equivalent for random utility functions and establish its relation with the absolute risk aversion.

5.1 Introduction

In this chapter, we are interested in different notions of prices for a contingent claim and their relation in the context of uncertainty. Risk and uncertainty are at the heart of economic life and modeling the way an agent will react to them is a central thematic of the economic research (see for instance [68]). By uncertainty we refer to Knightian uncertainty and we distinguish between the know unknown (risk) and unknown unknown (uncertainty) as introduced by F. Knight ([90]). In different words the agent cannot be certain about the choice of a given prior to model the outcome of a situation. Issues related to uncertainty arise in various concrete situations in social science and economics such as policy making but also in many aspects of modern finance such as model risk when pricing and risk managing complex derivatives products or capital requirement quantification when looking at the banks regulation and others financial entities. As illustrated for instance in the Ellsberg Paradox (see [58]), when facing uncertainty an agent displays uncertainty aversion: she tends to prefer a situation where the unknown unknown is reduced. This is the pendant of the risk aversion when the agent faces only risk. And it is well know that if one want to represent the preferences of the agent in this context, the axiomatic of the von Neumann and Morgenstern expected utility criterium (see [126]) are not verified even if the Savage's extension (see [120]), where subjective probability measures depending on each agent are introduced, is considered. In this chapter, we follows the pioneering approach introduced by [69] where under suitable axiomatic on the investor preferences, the utility functional is of the form of a worst case expected utility: $\inf_{P\in\mathcal{Q}} E_P U(\cdot, X)$, where $\mathcal{Q}$ is the set of all possible probability measures representing the agent beliefs. Note that this approach can also be used for robustness considerations where a set of models resulting from small perturbations of the initial reference model is taken. This is related for instance to the work of [73] where a term corresponding to the relative entropy given a certain reference probability measure is added to the utility functional. The framework of [69] was extended by [94] who introduced a penalty term to the utility functional. Finally, [39] represent the preferences by a more general functional $\inf_{P\in\mathcal{Q}} G(E_P U(X), P)$ where G is a so-called uncertainty index reflecting the decision maker's attitudes toward uncertainty.

From an economical and practical point of view an important and welcomed feature is to consider a set of probability measures $\mathcal{Q}$ which is a non-dominated. This means that no probability measure determines the set of events that can happen or not. The relevance of this idea is illustrated by the concrete example of volatility uncertainty, see [5], [93] and [59]. However this raises significantly the mathematical difficulty as some of the classical tools of probability theory such as conditional expectation or essential supremum are ill-suited (since they are defined with respect to a given probability measure). These type of issues have contributed to the development of innovative mathematical tools such as quasi-sure stochastic analysis, non-linear expectations, G-Brownian motions. On these topics, we refer amongst others to [104], [124] or [43].

The No Arbitrage (NA) notion is central to many problems in quantitative finance. It asserts that starting from a zero wealth it is not possible to reach a positive one (non negative almost surely and strictly positive with strictly positive probability). The characterisation of this condition or of the No Free Lunch condition is called the Fundamental Theorem of Asset Pricing (FTAP in short) and makes the link between those notions and the existence of equivalent risk-neutral probability measures (also called martingale measures or pricing measures) which are equivalent probability measures that turn the (discounted) asset price process into a martingale. This was initially formalised in [74], [75] and [92] while [46] obtain the FTAP in a general discrete time setting under the NA condition. The literature on the subject is huge and we refer to [49] for a general overview. The martingale measures allow for pricing issues and an other fundamental result, the Superhedging Theorem (see for instance [57], [44]) relates those pricing measures to the set of no-arbitrage prices for a given contingent claim. The so-called superreplication price is the minimal amount needed for an agent selling a claim in order to superreplicate it by trading in the market. To the best of our knowledge it was first introduced in [11] in the context of transaction costs. This is a hedging price with no risk but unfortunately it is not always of practical use as it is often to onerous: for example the superreplication price of a call option may be equal to the underlying initial price in an incomplete market (see [45]).

All these concepts have seen a renewed interest in the context of uncertainty, see amongst others [114], [1], [10], [54], [25], [38], [6], [28] and [40]. In this chapter, we have chosen to work under the discrete time framework introduced in [25]. We outline briefly in Section 5.2.1 some of the interesting features of this framework, in particular with respect to time-consistency.

Utility indifference price or reservation price was first introduced in [82] in the context of transaction costs as the minimal amount of money to be paid to an agent selling a contingent claim G so that added to her initial capital, her utility when selling G and hedging it by trading dynamically in the market is greater or equal to the one she would get without selling this product (see Definition 5.2.13). This notion of price is linked to certainty equivalent and allows to take into account the preferences of the agent. Unfortunately it is very difficult to compute outside the constant absolute risk aversion case (i.e. for exponential utility functions).

Note also that the notions of superreplication or utility indifference pricing can be related to the concept of risk measures introduced by [4] as illustrated in Propositions 5.2.18 and 5.2.19. For more details on risk measures in the context of multiple-priors we refer for instance to [61], [17] and more recently [8].

Intuitively speaking, the utility indifference price allows for some preference base risk-seeking behavior while the superreplication price corresponds to a totally risk averse agent (see Theorem 5.4.8). In the chapter we investigate the effect of increasing risk aversion on utility based prices: when absolute risk aversion tends to infinity, the reservation price should tend to the superreplication price. Our main contribution is presented in Theorem 5.4.8. We consider a sequence of investors whose preferences are represented by a sequence of random utility functions (see Definition 5.2.11) trading in a discrete-time financial market in the presence of uncertainty. We establish that under suitable conditions, the multiple-priors utility indifference prices of a given contingent claim (for the seller) converge to its multiple-priors superreplication price. For non random utility functions those conditions are implied by the convergence to infinity of the absolute risk-aversion (see Definition 5.3.1) of the agents.

In the mono-prior case for constant absolute risk averse agents, the convergence of reservation prices to the superreplication price was shown by [118] for Brownian models and in [48] in a general semimartingale setting. A nonexponential case was treated in [24], but with severe restrictions on the utility functions. The case of general utilities was considered in [30] in discrete-time market models and in [32] for continuous time ones.

To the best of your knowledge Theorem 5.4.8 is the first general asymptotic result in the multiple-priors framework. Note that simultaneously [7] obtains some convergence result for constant absolute risk averse agents. In Theorem 5.4.10, we apply Theorem 5.4.8 and prove that the convergence result occurs for a large class of non-random utility functions while in Proposition 5.4.11 we obtain the pendant asymptotic result for the multiple-priors subreplication and utility indifference buyer prices. We also show the convergence of the associated risk measures (see Proposition 5.4.9). To solve our problem, we use some arguments of [30] that are adapted to our multiple-priors framework together with results of [25] (namely Theorems 2.2 and 2.3). We also use some elements of quasi-sure stochastic analysis as developed in [53] and [51].

We revisit as well in a static context, the notion of certainty equivalent introduced in [108]. We extend it for random utility functions and in the presence of multiple-priors and we establish that the absolute risk aversion allows to rank the multiple-priors certainty equivalent despite the presence of uncertainty aversion (see Proposition 5.3.4). This part is also related to [67] where an alternative notion of indifference buyer (and seller) prices are introduced for non-random utility functions in a static setting under the representation of [39].

The chapter is structured as follows: Section 5.2 presents our framework, some definitions and results needed in the rest of the chapter. Section 5.3 revisits the link between certainty equivalent and absolute risk aversion in our set-up. Section 5.4 presents the main theorem on the convergence of the utility indifference prices to the superreplication price. Finally, section 5.5 contains the remaining proofs and technical results.

5.2 The model

This section presents our multiple-priors framework and the definitions of the sub- and superreplication prices and of the utility buyer and seller indifference prices.

5.2.1 Uncertainty modelisation

We model uncertainty as in [25] and [99]. We use the same framework and notations as in in Chapter 3 and 4.
For any Polish space X ($i.e$ complete and separable metric space), we denote by $\mathcal{B}(X)$ its Borel sigma-algebra and by $\mathfrak{P}(X)$ the set of all probability measures on $(X, \mathcal{B}(X))$. For some $P \in \mathfrak{P}(X)$ fixed, we denote by $\mathcal{B}_P(X)$ the completion of

$\mathcal{B}(X)$ with respect to P and we introduce the universal sigma-algebra defined by $\mathcal{B}_c(X) := \bigcap_{P\in\mathfrak{P}(X)} \mathcal{B}_P(X)$. It is clear that $\mathcal{B}(X) \subset \mathcal{B}_c(X)$. In the rest of the chapter, we will use the same notation for $P \in \mathfrak{P}(X)$ and for its (unique) extension on $\mathcal{B}_c(X)$. For a given $\mathcal{Q} \subset \mathfrak{P}(X)$, a set $N \subset X$ is called a $\mathcal{Q}$-polar if for all $P \in \mathcal{Q}$, there exists some $A_P \in \mathcal{B}_c(X)$ such that $P(A_P) = 0$ and $N \subset A_P$. We say that a property holds true $\mathcal{Q}$-quasi-surely (q.s.), if it is true outside a $\mathcal{Q}$-polar set. Finally we say that a set is of $\mathcal{Q}$-full measure if its complement is a $\mathcal{Q}$-polar set. A function $f : X \to Y$ (where Y is an other Polish space) is universally-measurable or $\mathcal{B}_c(X)$-measurable (resp. Borel-measurable or $\mathcal{B}(X)$-measurable) if for all $B \in \mathcal{B}(Y)$, $f^{-1}(B) \in \mathcal{B}_c(X)$ (resp. $f^{-1}(B) \in \mathcal{B}(X)$). Similarly we will speak of universally-adapted or universally-predictable (resp. Borel-adapted or Borel-predictable) processes.
We fix a time horizon $T \in \mathbb{N}$ and introduce a sequence $(\Omega_t)_{1\leq t\leq T}$ of Polish spaces. We denote by $\Omega^t := \Omega_1 \times \cdots \times \Omega_t$, with the convention that Ω^0 is reduced to a singleton. An element of Ω^t will be denoted by $\omega^t = (\omega_1, \ldots, \omega_t) = (\omega^{t-1}, \omega_t)$ for $(\omega_1, \ldots, \omega_t) \in \Omega_1 \times \cdots \times \Omega_t$ and $(\omega^{t-1}, \omega_t) \in \Omega^{t-1} \times \Omega_t$ (to avoid heavy notation we drop the dependency in ω_0). For all $0 \leq t \leq T-1$, we denote by $\mathcal{S}K_{t+1}$ the set of universally-measurable stochastic kernel on Ω_{t+1} given Ω^t (see [13, Definition 7.12 p134, Lemma 7.28 p174]). Fix some $1 \leq t \leq T$, $P_{t-1} \in \mathfrak{P}(\Omega^{t-1})$ and $p_t \in \mathcal{S}K_t$. Using Fubini's Theorem, see [13, Proposition 7.45 p175], we define a probability measure on $\mathcal{B}_c(\Omega^t)$ as follows

$$P_{t-1} \otimes p_t(A) := \int_{\Omega^{t-1}} \int_{\Omega_t} 1_A(\omega^{t-1}, \omega_t) p_t(d\omega_t, \omega^{t-1}) P_{t-1}(d\omega^{t-1}), \tag{5.1}$$

where $A \in \mathcal{B}_c(\Omega^t)$. To model the uncertainty we consider a family of random sets $\mathcal{Q}_{t+1} : \Omega^t \twoheadrightarrow \mathfrak{P}(\Omega_{t+1})$, for all $0 \leq t \leq T-1$. The set $\mathcal{Q}_{t+1}(\omega^t)$ can be seen as the set of all possible models for the $t+1$-th period given the state ω^t until time t. From the random sets $(\mathcal{Q}_{t+1})_{0\leq t\leq T-1}$ we build the sets of probability measures $(\mathcal{Q}^t)_{1\leq t\leq T}$ where $\mathcal{Q}^t$ governs the market until time t and determines which events are relevant or not in $\mathcal{B}_c(\Omega^t)$. To do that, as in [25], [99], we have to make the following assumption which is now classical in the recent litterature on multiple-priors model.

Assumption 5.2.1 For all $0 \leq t \leq T-1$, $\mathcal{Q}_{t+1}$ is a non-empty and convex valued random set such that $\mathrm{Graph}(\mathcal{Q}_{t+1}) = \{(\omega^t, P) \in \Omega^t \times \mathfrak{P}(\Omega_{t+1}),\ P \in \mathcal{Q}_{t+1}(\omega^t)\}$ is an analytic set.

Recall that an analytic set is the continuous image of some Polish space, see [3, Theorem 12.24 p447]), see also [13, Chapter 7] for more details on analytic sets. Assumption 5.2.1 allows to apply the Jankov-von Neumann Theorem (see [13, Proposition 7.49 p182]) and gets some universally-measurable selector q_{t+1} of $\mathcal{Q}_{t+1}$. Then, for each time $1 \leq t \leq T$, the set $\mathcal{Q}^t \subset \mathfrak{P}(\Omega^t)$ is completely determined by the random sets of one-step models $\mathcal{Q}_{s+1}$ (for $s = 0, \cdots, t-1$) in the following way

$$\mathcal{Q}^t := \{Q_1 \otimes q_2 \otimes \cdots \otimes q_t,\ Q_1 \in \mathcal{Q}_1,\ q_{s+1} \in \mathcal{S}K_{s+1},\\ q_{s+1}(\cdot, \omega^s) \in \mathcal{Q}_{s+1}(\omega^s)\ Q_s\text{-a.s. } s \in \{1, \ldots, t-1\}\}, \tag{5.2}$$

where we denote $Q_s := Q_1 \otimes q_2 \otimes \cdots \otimes q_s$ for any $2 \leq s \leq t$.
The technical Assumption 5.2.1 plays a key role to obtain measurability properties required to prove the FTAP, the Superreplication Theorem and also to apply a dynamic programming procedure in multiple-priors utility maximisation problem (see for instance [99], [7] or [98] and also Chapter 4). More precisely, if $X_{t+1} : \Omega^{t+1} \to \mathbb{R}$ is lower-semianalytic (see [13, Definition 7.21]), then $X_t : \Omega^t \to \mathbb{R}$ defined by $X_t(\omega^t) = \inf_{P \in \mathcal{Q}_{t+1}(\omega^t)} \int_{\Omega_{t+1}} X_{t+1}(\omega^t, \omega_{t+1}) P(d\omega_{t+1})$ remains lower-semianalytic. More generally, this framework allows to construct families of dynamic sublinear expectations (see [25, Lemma 4.10] and also [8]). For similar issues in the continuous-time setting we also refer also amongst others to [102], [101] and [59]. Apart from Assumption 5.2.1, we make no specific assumptions on the set of priors: $\mathcal{Q}^T$ is neither assumed to be dominated by a given reference probability measure nor to be weakly compact. For example, in the continuous-time case, dominated set of priors can arise when there is uncertainty on the drift of the underlying process while non-dominated set of priors may arise if there is uncertainty on the volatility of this process (see for example [59]). In the case of volatility uncertainty, the corresponding set is however weakly compact (see for instance [5], [93], [52, Proposition 3]) and also [59].

We focus briefly on the time-consistency issue: how are the agent decisions or risk evaluations at different times interrelated once the information has been updated. Roughly speaking, time-consistency means that a decision taken tomorrow will satisfies today's objective. Recall that this issue appears already in mono-prior setting, in the study of dynamic risk measures for instance, and is linked with the law of iterated conditional expectations and the dynamic programming principle. We refer to the surveys [2] and [16] for detailed overviews.
Now when introducing multiple-priors one has to be even more careful with time-consistency. In [113, Appendix D] a simple example illustrates what can happen if one is not cautious on the structure of the initial set of priors: one cannot hope to find an optimal solution using the dynamic programming principle when trying for instance to maximise a worst case expected utility problem. To deal with this, one has to assume that the set of prior is stable under pasting which roughly means that different priors can be mixed together (see [113, Assumption 4]). Given (5.2), it is clear that our set of priors are stable under pasting. Indeed, if $Q^1, Q^2 \in \mathcal{Q}^T$ with $Q^1 = Q_1^1 \otimes q_2^1 \otimes \cdots \otimes q_T^1$, $Q^2 = Q_1^2 \otimes q_2^2 \otimes \cdots \otimes q_T^2$, then $R := Q_1^1 \otimes q_2^1 \cdots \otimes q_{t-1}^1 \otimes q_t^2 \otimes \cdots \otimes q_T^2 \in \mathcal{Q}^T$ for all $2 \leq t \leq T-1$. In other words,

the set $\mathcal{Q}^T$ is large enough (unlike in the example considered in [113, Appendix D]). In [60, Definition 3.1] the equivalent notion of rectangularity is introduced (see also [60, Sections 3, 4] for more details and a graphical interpretation). An important feature of rectangularity is that it implies that the set of priors is uniquely determined by the set of one-step-ahead priors: this is an other way to verify that our approach satisfies this property (see (5.2)).

For $1 \leq t \leq T$ fixed, we introduce the following spaces

$$\mathcal{W}_t^0 := \left\{ X : \Omega^t \to \mathbb{R} \cup \{\pm\infty\},\ \mathcal{B}_c(\Omega^t)\text{-measurable} \right\},$$
$$\mathcal{W}_t^\infty := \mathcal{W}_t^0 \cap \left\{ X,\ \exists M\ \geq 0,\ |X| \leq M\ \mathcal{Q}^t\text{-q.s.} \right\}.$$

Finally, we will add a superscript $^+$ when considering non-negative elements (it will be also used for denoting positive parts).

5.2.2 The traded assets and the trading strategies

Let $S := \{S_t,\ 0 \leq t \leq T\}$ be a universally-adapted d-dimensional process where for $0 \leq t \leq T$, $S_t = (S_t^i)_{1 \leq i \leq d}$ represents the price of d risky securities in the financial market in consideration. To solve measurability issues, we make the following assumption already present in [25] and [99].

Assumption 5.2.2 The price process S is Borel-adapted.

Trading strategies are represented by universally-adapted d-dimensional processes $\phi := \{\phi_t, 1 \leq t \leq T\}$ where for all $1 \leq t \leq T$, $\phi_t = (\phi_t^i)_{1 \leq i \leq d}$ represents the investor's holdings in each of the d assets at time t. The set of trading strategies is denoted by Φ.
We assume that trading is self-financing and that the riskless asset's price is constant equal to 1. The value at time t of a portfolio ϕ starting from initial capital $x \in \mathbb{R}$ is given by $V_t^{x,\phi} = x + \sum_{s=1}^t \phi_s \Delta S_s$.

5.2.3 Multiple-priors no-arbitrage condition

As already eluded to in the introduction, the issue of no-arbitrage in the context of uncertainty has seen a renewed interest. In this chaptuer we follow the definition introduced by [25] that we recall below. We outline briefly some of the interesting feature of this definition. First it looks like a natural and intuitive extension of the classical mono-prior arbitrage condition. This argument is strengthened by the FTAP generalisation proved by [25]. Under appropriated measurability conditions the $NA(\mathcal{Q}^T)$ is equivalent to the following: for all $Q \in \mathcal{Q}^T$, there exists some $P \in \mathcal{R}^T$ such that $Q \ll P$ where

$$\mathcal{R}^T := \{P \in \mathfrak{P}(\Omega^T),\ \exists Q' \in \mathcal{Q}^T, P \ll Q' \text{ and } P \text{ is a martingale measure}\}. \tag{5.3}$$

The classical notion of equivalent martingale measure is replaced by the fact that for all priors $Q \in \mathcal{Q}^T$, there exists a martingale measure P such that Q is absolutely continuous with respect to P and one can find an other prior $Q' \in \mathcal{Q}^T$ such that P is absolutely continuous with respect to Q'. The extension in the same multiple-priors setting of the Superhedging Theorem and subsequent results on worst-case expected utility maximisation (see [99], [7], [98] and Chapter 4) is an other convincing element.

Assumption 5.2.3 The $NA(\mathcal{Q}^T)$ condition holds true if for $\phi \in \Phi$, $V_T^{0,\phi} \geq 0$ $\mathcal{Q}^T$-q.s. implies that $V_T^{0,\phi} = 0$ $\mathcal{Q}^T$-q.s.

For the convenience of the reader we recall the following definition and proposition from Chapter 3 (see Sections 3.3 and 3.4) concerning the multiple-priors conditional support of the price increments or more precisely of its affine hull (denoted by Aff).

Definition 5.2.4 For all $0 \leq t \leq T-1$ we define the random set $D^{t+1} : \Omega^t \twoheadrightarrow \mathbb{R}^d$ by

$$D^{t+1}(\omega^t) := \text{Aff}\left(\bigcap \left\{A \subset \mathbb{R}^d,\ \text{closed},\ P_{t+1}\left(\Delta S_{t+1}(\omega^t, .) \in A\right) = 1,\ \forall\, P_{t+1} \in \mathcal{Q}_{t+1}(\omega^t)\right\}\right). \tag{5.4}$$

Proposition 5.2.5 *Assume that the $NA(\mathcal{Q}^T)$ condition and Assumptions 5.2.1, 5.2.2 hold true. Then for all $0 \leq t \leq T-1$, there exists some $\mathcal{Q}^t$-full measure set $\Omega^t_{NA} \in \mathcal{B}_c(\Omega^t)$ such that for all $\omega^t \in \Omega^t_{NA}$, $D^{t+1}(\omega^t)$ is a vector space. For all $\omega^t \in \Omega^t_{NA}$ there exists $\alpha_t(\omega^t) > 0$ such that for all $h \in D^{t+1}(\omega^t)$ there exists $P_h \in \mathcal{Q}_{t+1}(\omega^t)$ satisfying*

$$P_h\left(\frac{h}{|h|}\Delta S_{t+1}(\omega^t, .) < -\alpha_t(\omega^t)\right) > \alpha_t(\omega^t). \tag{5.5}$$

Note that in Theorem 3.4.7 Chapter 3, the equivalence between Assumption 5.2.3 and condition 5.5 is established.
For all $x \geq 0$, we introduce the set of terminal wealth including the possibility of throwing away money defined by

$$\mathcal{C}_x^T := \{V_T^{x,\phi},\ \phi \in \Phi\} - \mathcal{W}_T^{0,+}. \tag{5.6}$$

In the sequel we will write $X \in \mathcal{C}_x^T$ if there exists some $\phi \in \Phi$ and $Z \in \mathcal{W}_T^{0,+}$ such that $X = V_T^{x,\phi} - Z$ $\mathcal{Q}^T$ q.s. Under the assumptions of Lemma 5.2.6 (which will be crucial in Section 5.4), the set $\mathcal{C}_x^T$ has a classical closure property (in the $\mathcal{Q}^T$ quasi-sure sense, see [25, Theorem 2.2]).

Lemma 5.2.6 *Assume that Assumptions 5.2.2 and 5.2.3 hold true. Fix some* $z \geq 0$ *and let* $B \in \mathcal{W}_T^0$ *such that* $B \notin \mathcal{C}_z^T$*. Then there exists some* $\varepsilon > 0$ *such that*

$$\inf_{\phi \in \Phi} \sup_{P \in \mathcal{Q}^T} P(V_T^{z,\phi} < B - \varepsilon) > \varepsilon. \tag{5.7}$$

Proof. Assume that (5.7) does not hold true. Then, for all $n \geq 1$, there exist some $\phi_n \in \Phi$ such that $P(V_n < B - \frac{1}{n}) \leq \frac{1}{n}$ for all $P \in \mathcal{Q}^T$, where $V_n := V_T^{z,\phi_n}$. Set $K_n := \left(V_n - (B - \frac{1}{n})\right) 1_{\{V_n \geq B - \frac{1}{n}\}} \in \mathcal{W}_T^{0,+}$, then $V_n - K_n \in \mathcal{C}_z^T$. Moreover $P(|V_n - K_n - B| > \frac{1}{n}) = P(V_n < B - \frac{1}{n}) \leq \frac{1}{n}$ for all $P \in \mathcal{Q}^T$. Thus $\lim_{n\to\infty} \sup_{P \in \mathcal{Q}^T} P(|V_n - K_n - B| > \frac{1}{n}) = 0$ for any $\frac{1}{n} > 0$ and using Proposition 5.5.1, there exists a subsequence $(n_k)_{k \geq 1}$ such that $(V_{n_k} - K_{n_k})_{k \geq 1}$ converges to B $\mathcal{Q}^T$-q.s. (i.e. on a $\mathcal{Q}^T$-full measure set). Applying [25, Theorem 2.2], we get that $B \in \mathcal{C}_z^T$, a contradiction. □

Remark 5.2.7 Note that to apply [25, Theorem 2.2] we do not need Assumptions 5.2.1 to hold true. Similarly it applies under a weaker assumption than Assumption 5.2.2.

5.2.4 Multiple-priors superreplication and subreplication prices

The multiple-priors superreplication price is the minimal initial amount that an agent will ask for delivering some contingent claim $G \in \mathcal{W}_T^0$ so that she is fully hedged at T when trading in the market. The multiple-priors subreplication price is the maximal amount an agent will accept to pay in order to receive some contingent claim while being fully hedge at T by trading in the market. Note that the superreplication price is a seller price while the subreplication price is a buyer price. We also introduce the set of strategies which dominate G $\mathcal{Q}^T$-q.s. starting from a given wealth $x \in \mathbb{R}$

$$\mathcal{A}(G, x) := \left\{ \phi \in \Phi,\ V_T^{x,\phi} \geq G\ \mathcal{Q}^T \text{q.s.} \right\}. \tag{5.8}$$

Definition 5.2.8 Let $G \in \mathcal{W}_T^0$. The multiple-priors superreplication price of G is defined by

$$\pi(G) := \inf \{ z \in \mathbb{R},\ \mathcal{A}(G, z) \neq \emptyset \}, \tag{5.9}$$

and $\pi(G) = +\infty$ if $\mathcal{A}(G, z) = \emptyset$ for all $z \in \mathbb{R}$. The multiple-priors subreplication price of G is defined by

$$\pi^{sub}(G) := \sup \{ z \in \mathbb{R},\ \mathcal{A}(-G, -z) \neq \emptyset \}, \tag{5.10}$$

and $\pi^{sub}(G) = -\infty$ if $\mathcal{A}(-G, -z) = \emptyset$ for all $z \in \mathbb{R}$.

We recall now for the convenience of the reader [25, Theorem 2.3] slightly adapted to our setup (the same comment as in Remark 5.2.7 applies)

Theorem 5.2.9 *Assume that Assumptions 5.2.2 and 5.2.3 hold true and let* $G \in \mathcal{W}_T^0$ *be fixed. Then* $\pi(G) > -\infty$ *and* $\mathcal{A}(G, \pi(G)) \neq \emptyset$*, i.e. there exists some* $\phi_G \in \Phi$ *such that* $V_T^{\pi(G),\phi_G} \geq G$ $\mathcal{Q}^T$*-q.s.*

If G is replicable, *i.e.* if there exists some x_G and $\phi_G \in \Phi$ such that $G = V_T^{x_G,\phi_G}$ $\mathcal{Q}^T$-q.s., then $\pi(G) = x_G = \pi\left(V_T^{x_G,\phi_G}\right)$. Note that under some measurability assumption on G, the Superreplication Theorem is still true: $\pi(G) = \sup_{P \in \mathcal{R}^T} E_P G$ (see [25, Superhedging Theorem] and (5.3) for the definition of $\mathcal{R}^T$).

Note that if $G \in \mathcal{W}_T^\infty$, it is clear that $\pi(G) \leq ||G||_\infty$. This is the case if G represents the payoff of a put option or a digital but not for a call option. This illustrates that the case $G \in \mathcal{W}_T^\infty$ can be sometimes too restrictive especially in a multiple-priors setting and explains why in the rest of the chapter, we will try to avoid results limited to $\mathcal{W}_T^\infty$. The price to pay is often related to integrability issues. The next lemma resumes some basic results on superreplication prices.

Lemma 5.2.10 *Let* $G \in \mathcal{W}_T^0$ *then* $\pi^{sub}(G) = -\pi(-G)$*. Moreover* $\pi(G) = +\infty$ *if and only if* $\mathcal{A}(G, z) = \emptyset$ *for all* $z \in \mathbb{R}$*. If Assumption 5.2.3 holds true then* $\pi(0) = 0$ *and if* $G \in \mathcal{W}_T^{0,+}$*, then* $\pi(G) = 0$ *implies that* $G = 0$ $\mathcal{Q}^T$*-q.s.*

Proof. The two asssertions are clear. Indeed by definition if $\mathcal{A}(G, z) = \emptyset$ for all $z \in \mathbb{R}$, then $\pi(G) = +\infty$. Assume that there exits some $z \in \mathbb{R}$ such that $\mathcal{A}(G, z) \neq \emptyset$, then $\pi(G) \leq z < +\infty$.
By definition $\pi(0) \leq 0$. Assume that $\pi(0) < 0$ and let $\varepsilon > 0$ such that $\pi(0) < -\varepsilon$. Then there exists some $\phi \in \Phi$ such that $V_T^{0,\phi} \geq \varepsilon > 0$ $\mathcal{Q}^T$-q.s. a contradiction. For the last assertion, assume that there exists some $P \in \mathcal{Q}^T$ such that $P(G(\cdot) > 0) > 0$. Using Theorem 5.2.9 there exists $\phi_0 \in \Phi$ such that $V_T^{\pi(G),\phi_0} \geq G$ $\mathcal{Q}^T$-q.s. Thus $P(V_T^{0,\phi_0} > 0) > 0$ which contradics $NA(\mathcal{Q}^T)$. □

We now turn to some pricing rules which takes into account the preferences of the agents.

5.2.5 Utility functions and utility indifference prices

In this chapter we focus on utility function defined on the half-real line whose definition follows.

Definition 5.2.11 A random utility function $U : \Omega^T \times (0, \infty) \to \mathbb{R} \cup \{-\infty\}$ satisfies the following conditions

i) for every $x > 0$, $U(\cdot, x) : \Omega^T \to \mathbb{R}$ is universally-measurable,

ii) for all $\omega^T \in \Omega^T$, $U\left(\omega^T, \cdot\right) : (0, \infty) \to \mathbb{R}$ is concave, strictly increasing and twice continuously differentiable on $(0, \infty)$.

We extend U by (right) continuity in 0 and set $U(\cdot, x) = -\infty$ if $x < 0$.

Example 5.2.12 We give some concrete examples of random utility functions. The first one arises if the agent analyzes her gain or loss with respect to a (random) reference point B rather than with respect to zero has suggested for instance by [88]. Formally, let $\overline{U}$ be a non-random function satisfying Definition 5.2.11 and $B \in \mathcal{W}_T^{\infty,+}$. Set for all $\omega^T \in \Omega^T$, $x \geq 0$, $U(\omega^T, x) = \overline{U}(x + ||B||_\infty - B(\omega^T))$ and $U(\omega^T, x) = -\infty$ for $x < 0$. Then it is clear that U satisfies also the condition of Definition 5.2.11.
The second example allows to consider random absolute risk aversion (see Definition 5.3.1 for the precise statement of this concept). The idea is to use classical utility functions but with random coefficients. For example, we can consider $U(\omega^T, x) = x^{\beta_1(\omega^T)}$ or $U(\omega^T, x) = -e^{-\beta_2(\omega^T)x}$ for $x \geq 0$ (and $U(\cdot, x) = -\infty$ for $x < 0$) where $\beta_1, \beta_2 \in \mathcal{W}_T^0$ and $0 < \beta_1(\cdot) < 1$, $\beta_2(\cdot) > 0$ $\mathcal{Q}^T$-q.s. We can imagine various situations for β_2 (which can be easily adapted for β_1): β_2 can be uniformly distributed on $[\beta_{min}^P, \beta_{max}^P]$ for all $P \in \mathcal{Q}^T$ (with $\beta_{max}^P \geq \beta_{min}^P > 0$) or alternatively it could follow a Poisson law of parameter $\lambda_P > 0$ for all $P \in \mathcal{Q}^T$.

We now turn to pricing issues and first define some particular sets of strategies for a claim $G \in \mathcal{W}_T^0$ and some $x \in \mathbb{R}$ (recall (5.8))

$$\begin{aligned} \Phi(U, G, x) &:= \left\{\phi \in \Phi,\ E_P U^+(\cdot, V_T^{x,\phi}(\cdot) - G(\cdot)) < +\infty, \forall P \in \mathcal{Q}^T\right\} \\ \mathcal{A}(U, G, x) &:= \Phi(U, G, x) \cap \mathcal{A}(G, x). \end{aligned}$$

Note that for $x \geq \pi(G)$, $\mathcal{A}(U, G, x)$ might be empty. Indeed, from Theorem 5.2.9 there exists some $\phi \in \mathcal{A}(G, x)$, but ϕ might not belong to $\Phi(U, G, x)$. In Lemma 5.4.14 we will prove that under suitable conditions, $\mathcal{A}(G, x) = \mathcal{A}(U, G, x)$ for all $x \geq 0$. This is the reason why in $\Phi(U, G, x)$ we do not consider strategies such that $E_P U^-(\cdot, V_T^{x,\phi}(\cdot) - G(\cdot)) < \infty$.

We now introduce the quantity $u(G, x)$ which represents the maximum worst-case expected utility starting from initial capital x and delivering G at the terminal date

$$u(G, x) := \sup_{\phi \in \mathcal{A}(U,G,x)} \inf_{P \in \mathcal{Q}^T} E_P U\left(\cdot, V_T^{x,\phi}(\cdot) - G(\cdot)\right), \tag{5.11}$$

where we set $u(G, x) = -\infty$ if $\mathcal{A}(U, G, x)$ is empty.
We are now in position to define the (seller) multiple-priors utility indifference price or reservation price, which generalizes in the presence of uncertainty, the

concept introduced by [82]. It represents the minimal amount of money to be paid to an agent selling a contingent claim G so that added to her initial capital, her multiple-priors utility when selling G and hedging it by trading dynamically in the market is greater or equal than the one she would get without selling this product. Similarly the (buyer) multiple-priors utility indifference price represents the maximum amount of money an agent is ready to pay in order to buy G so that substracted to her initial capital, her multiple-priors expected utility when buying G and hedging it by trading dynamically in the market is greater or equal than the one she would get without buying this product.

Definition 5.2.13 Let $G \in \mathcal{W}_T^0$ be a contingent claim. The (seller) multiple-priors utility indifference price is given by

$$p(G,x) := \inf \{z \in \mathbb{R},\ u(G, x+z) \geq u(0,x)\}, \tag{5.12}$$

where we set $p(G,x) = +\infty$ if $u(G, x+z) < u(0,x)$, for all $z \in \mathbb{R}$. The (buyer) multiple-priors utility indifference price is given by

$$p^B(G,x) := \sup \{z \in \mathbb{R},\ u(-G, x-z) \geq u(0,x)\}, \tag{5.13}$$

where we set $p^B(G,x) = -\infty$ if $u(-G, x-z) < u(0,x)$, for all $z \in \mathbb{R}$.

It is easy to see that $p^B(G,x) = -p(-G,x)$. We will see in Lemma 5.4.14, that under suitable integrability conditions, $p(G,x) \leq \pi(G)$ for all $G \in \mathcal{W}_T^{0,+}$. Whatever the preference of the agent is, she will always evaluate a reservation price which is lower than the superreplication price. The superreplication price is, in the sense that we will precise below, the price corresponding to an infinite absolute risk averse agent. The following proposition presents some other properties.

Proposition 5.2.14 *We fixe some $x \geq 0$ and assume that Assumptions 5.2.2 and 5.2.3 hold true and that $u(0,x) > -\infty$.*
1. *For any $G \in \mathcal{W}_T^0$, $p(G,x) \geq \pi(G) - x > -\infty$, $p^B(G,x) \leq \pi^{sub}(G) + x < \infty$. In particular $-x \leq p(0,x) \leq 0$.*
2. *If $\mathcal{A}(U,G,x) = \mathcal{A}(G,x)$ then $p(G,x) \geq p(0,x)$ for any $G \in \mathcal{W}_T^{0,+}$.*
3. *If $u(0,x-\delta) < u(0,x)$ for all $\delta > 0$, then $p(0,x) = p^B(0,x) = 0$.*

Proof. 1. For any $G \in \mathcal{W}_T^0$ since Assumptions 5.2.2 and 5.2.3 hold true, Theorem 5.2.9 yields to $\pi(G) > -\infty$. Let $z \in \mathbb{R}$ be such that $x + z < \pi(G)$. By definition of $\pi(G)$, $\mathcal{A}(G, x+z) = \emptyset$ and thus $u(G, x+z) = -\infty$ (see (5.11)). This implies that $u(G, x+z) < u(0,x)$ and recalling (5.12), we get that $p(G,x) > z$. Letting z go to $\pi(G) - x$, we obtain that $p(G,x) \geq \pi(G) - x > -\infty$. Applying the preceding inequality to $-G$ and recalling (5.10) and (5.13), $p^B(G,x) \leq \pi^{sub}(G) + x < +\infty$. By definition $p(0,x) \leq 0$. Since from Lemma 5.2.10 $\pi(0) = 0$, $-x \leq p(0,x)$ is immediate.

2. The fact that $p(G,x) \geq p(0,x)$ for $G \in \mathcal{W}_T^{0,+}$ follows from the monotonicity property that will be proven in Proposition 5.2.19 below.
3. We assume now that $u(0,x-\delta) < u(0,x)$ for all $\delta > 0$. Then (5.12) implies that $p(0,x) \geq 0$ and $p(0,x) = p^B(0,x) = 0$ follows immediately. □

Remark 5.2.15 Assume that $\mathcal{A}(U,0,y) = \mathcal{A}(0,y)$ for all $y \in \mathbb{R}$, that $u(0,x) > -\infty$ and that

$$l(x) := \inf_{\phi \in \mathcal{A}(U,0,x)} \inf_{P \in \mathcal{Q}^T} E_P U'(\cdot, V_T^{x,\phi}(\cdot)) > 0$$

then $u(0,x-\delta) < u(0,x)$ for all $\delta > 0$. Indeed fix some $\delta > 0$, then

$$U(\cdot, V_T^{x-\delta,\phi}(\cdot)) + \delta U'(\cdot, V_T^{x,\phi}(\cdot)) \leq U(\cdot, V_T^{x,\phi}(\cdot)).$$

For all $\phi \in \mathcal{A}(U,0,x-\delta) = \mathcal{A}(0,x-\delta) \subset \mathcal{A}(0,x) = \mathcal{A}(U,0,x)$ we obtain that

$$\begin{aligned}\inf_{P \in \mathcal{Q}^T} E_P U(\cdot, V_T^{x-\delta,\phi}(\cdot)) + \delta l(x) &\leq \inf_{P \in \mathcal{Q}^T} E_P U(\cdot, V_T^{x-\delta,\phi}(\cdot)) + \delta \inf_{P \in \mathcal{Q}^T} E_P U'(\cdot, V_T^{x,\phi}(\cdot)) \\ &\leq \inf_{P \in \mathcal{Q}^T} E_P U(\cdot, V_T^{x,\phi}(\cdot)).\end{aligned}$$

If $\mathcal{A}(U,0,x-\delta) = \emptyset$, then $u(0,x-\delta) = -\infty$ and there is nothing to prove. Otherwise, for all $n \geq 1$, there exists some $\phi_n \in \mathcal{A}(U,0,x-\delta)$ such that

$$\begin{aligned}u(0,x-\delta) &\leq \inf_{P \in \mathcal{Q}^T} E_P U(\cdot, V_T^{x-\delta,\phi_n}(\cdot)) + \frac{1}{n} \\ &\leq \inf_{P \in \mathcal{Q}^T} E_P U(\cdot, V_T^{x,\phi_n}(\cdot)) + \frac{1}{n} - \delta l(x) \leq u(0,x) + \frac{1}{n} - \delta l(x).\end{aligned}$$

and taking $n > \frac{1}{\delta l(x)}$, we obtain that $u(0,x-\delta) < u(0,x)$.

Remark 5.2.16 We discuss briefly the issue of finding a trading strategy that can realize the (seller) multiple-priors utility indifference price. To do so, the two sides of the inequation in (5.12) need to be evaluated. To evaluate $u(0,x)$, in other word the maximise the multiple-priors expected utility without trading G, Theorem 4.4.14 in Chapter 4 provides some conditions under which $u(0,x) < \infty$ and an optimal strategy exists. However proving the existence of an optimal strategy to $u(G,x+z)$ remains an open question: see the Conclusion section in Chapter 4 for some discussion on this problem.

5.2.6 Risk measures

We make the link with monetary risk measures introduced in [4], see also [36]. Recall that a risk measure allows to quantify by some number $\rho(X)$ a financial position described by some $X \in \mathcal{X}$ where $\mathcal{X} \subset \mathcal{W}_T^0$ is a linear space of random variables (containing the constant random variables) and $X(\omega^T)$ represents the discounted net worth of the position at the end of the trading period if the scenario $\omega^T \in \Omega^T$ is realized. More precisely,

Definition 5.2.17 A monetary risk measure is a mapping $\rho : G \in \mathcal{X} \to \rho(G) \in \mathbb{R} \cup \{\pm\infty\}$ that verifies
1. for $G, H \in \mathcal{X}$, if $G \geq H$ $\mathcal{Q}^T$-q.s., then $\rho(G) \leq \rho(H)$ (monotonicity),
2. if $m \in \mathbb{R}$, then $\rho(G+m) = \rho(G) - m$ (cash invariance).
The measure ρ is said to be a normalized if $\rho(0) = 0$ and convex if
3. for all $0 \leq \lambda \leq 1$, $G, H \in \mathcal{X}$, $\rho(\lambda G + (1-\lambda)H) \leq \lambda\rho(G) + (1-\lambda)\rho(H)$ (convexity).

We refer to [62, Section 4] (where similar definitions are introduced) for a detailed interpretation of these properties.
We now discuss the relations with the multiple-priors sub- and superreplication prices as well as with the buying and selling prices. From the cash invariance property, a risk-measure can also be seen as a capital requirement: $\rho(G)$ is the amount of cash to be held in addition to the financial instrument G for the aggregate position to be acceptable (from the point of view of a risk-manager, regulator,...). With this in mind, the acceptance set of ρ is often defined by $\{G \in \mathcal{X}, \rho(G) \leq 0\}$. In our context, to measure the risk of a position one can set for some $x \geq 0$

$$\rho_x : G \in \mathcal{X} \to p(-G, x), \tag{5.14}$$

see for example [36, Definition 1.2]. We also consider the following measure

$$\rho : G \in \mathcal{X} \to \pi(-G). \tag{5.15}$$

Assume for a moment that ρ_x and ρ verify the cash invariance property of Definition 5.2.17. We consider an agent with initial capital x who is willing to buy and hedge (by trading dynamically in the market) an option whose non-negative payoff is represented by some $G \in \mathcal{X}^+$ for a price p_b. Then we have that $\rho_x(G - p_b) = \rho_x(G) + p_b = p(-G, x) + p_b = -p^B(G, x) + p_b$ (see (5.12) and (5.13)) and the position is acceptable for the measure ρ_x as long as she can buy the contingent claim at or below her buyer multiple-priors utility indifference price $p^B(G, x)$. From the point of view of the measure ρ, the position is acceptable as soon as $\rho(G - p_b) = \pi(-G) + p_b = -\pi^{sub}(G) + p_b$ (see Lemma 5.2.10) and the position is acceptable as long as she can buy the contingent claim at or below her multiple-priors subreplication price $\pi^{sub}(G)$.

Alternatively, if the agent is now considering selling the option at price p_s and hedging it, the short position in the contingent claim is represented by $-G \in \mathcal{X}^-$ and in this case the risk of her position is measured by $\rho_x(-G + p_s) = \rho_x(-G) - p_s = p(G, x) - p_s$: this position will be acceptable if she can sell the option at or above her seller multiple-priors utility indifference price. From the point of view of the measure ρ the position will be acceptable if she can sell the option at or above her multiple-priors superreplication price $\pi(G)$.

Note finally that if ρ_x is a normalized convex monetary measure of risk on $\mathcal{W}_T^0$, we have that $0 = \rho_x(0) \leq \frac{1}{2}(\rho_x(G) + \rho_x(-G))$. Recalling Definition 5.2.13, this implies that $p^{sub}(G,x) \leq p(G,x)$. Similarly one gets $\pi^{sub}(G) \leq \pi(G)$ (see Lemma 5.2.10).

The two following propositions establish under which conditions ρ and ρ_x (see (5.14) and (5.15)) are normalized convex monetary measures of risk.

Proposition 5.2.18 *If Assumptions 5.2.2 and 5.2.3 holds true then ρ is a normalized convex monetary measure of risk on $\mathcal{W}_T^0$.*

Proof. We prove 1. of Definition 5.2.17. Fix some $G, H \in \mathcal{W}_T^0$ such that $G \geq H$ $\mathcal{Q}^T$-q.s. From Theorem 5.2.9 there exists $\phi \in \Phi$ such that $V_T^{\pi(-H),\phi} \geq -H \geq -G$ $\mathcal{Q}^T$-q.s. and $\pi(-H) \geq \pi(-G)$. The cash invariance and the convexity are also straightforward and the normalization condition follows from Lemma 5.2.10. □

Proposition 5.2.19 *Let $x \geq 0$ be fixed. Assume that $\mathcal{A}(U,G,x) = \mathcal{A}(G,x)$ for all $G \in \mathcal{W}_T^0$.*
1. *The mapping ρ_x is a monetary measure of risk on $\mathcal{W}_T^0$.*
2. *If Assumptions 5.2.2 and 5.2.3 holds true and $u(0,x) > -\infty$ then ρ_x is a convex monetary measure of risk on $\{G \in \mathcal{W}_T^0,\ u(-G,z) < \infty, \forall z \in \mathbb{R}\}$.*
3. *If furthermore we assume that $u(0,x) > -\infty$, $u(0,x-\delta) < u(0,x)$ for all $\delta > 0$, then ρ_x is normalized.*

Remark 5.2.20 To prove the cash invariance property it is not necessary to assume that $\mathcal{A}(U,G,x) = \mathcal{A}(G,x)$ for all $G \in \mathcal{W}_T^0$. We will give in Proposition 5.4.13 and Lemma 5.4.14 some conditions under which $\mathcal{A}(U,G,x) = \mathcal{A}(G,x)$ and $u(G,z) < \infty$. Those conditions are needed in order to prove our asymptotic result (see Theorem 5.4.8). Note that if we assume that U is bounded from above the two preceding conditions are obviously satisfied.

Proof. We prove 1. of Definition 5.2.17. Fix some $G, H \in \mathcal{W}_T^0$ such that $G \geq H$ $\mathcal{Q}^T$-q.s. As $\mathcal{A}(U,-H,x) = \mathcal{A}(-H,x) \subset \mathcal{A}(-G,x) = \mathcal{A}(U,-G,x)$, it is easy to check that for all $z \in \mathbb{R}$, $u(-H,x+z) \leq u(-G,x+z)$. If $\rho_x(H) = +\infty$, there is nothing to prove, while if $\rho_x(H) = -\infty$, it is clear that $\rho_x(G) = -\infty$ also. Assume now that $\rho_x(H)$ is finite and fix some $\varepsilon > 0$. Then $u(0,x) \leq u(-H, x + p(-H,x) + \varepsilon) \leq u(-G, x + p(-H,x) + \varepsilon)$ and we get that $p(-G,x) \leq p(-H,x) + \varepsilon$. As this is true for all ε, $\rho_x(G) \leq \rho_x(H)$ follows immediately.
We prove 2. of Definition 5.2.17. We fix some $m \in \mathbb{R}$. It is clear (without the assumption that $\mathcal{A}(U,G,x) = \mathcal{A}(G,x)$ for all $I \in \mathcal{W}_T^0$) that $\mathcal{A}(U, -(G+m), x) = \mathcal{A}(U, -G, x+m)$ and it follows that $u(-(G+m),x) = u(-G,x+m)$. One can easily

see that $\rho_x(G+m),x) = +\infty$ (resp. $= -\infty$) if and only if $\rho_x(G,x) = +\infty$ (resp. $= -\infty$). So we may assume that $\rho_x(G+m,x)$ is finite and fix some $\varepsilon > 0$. Then

$$u(0,x) \leq u(-(G+m), x+p(-(G+m),x)+\varepsilon) = u(-G, x+p(-(G+m),x)+m+\varepsilon).$$

Thus $p(-G,x) \leq p(-(G+m),x)+m+\varepsilon$ follows. We have also that

$$u(0,x) \leq u(-G, x+p(-G,x)+\varepsilon) = u(-(G+m), x-m+p(-G,x)+\varepsilon)$$

and $p(-(G+m),x) \leq p(-G,x)-m+\varepsilon$. letting $\varepsilon > 0$ go to 0, we get that $p(-G,x)-m = p(-(G+m),x)$ as claimed.
We prove now the second part of the proposition. From Proposition 5.2.14, one get that $p(-G,x) > -\infty$ and $p(-H,x) > -\infty$ for some fixed $G,H \in \{X \in \mathcal{W}_T^0,\ u(-X,z) < \infty, \forall z \in \mathbb{R}\}$. We want to show that for all $0 < \lambda < 1$

$$\lambda p(-G,x) + (1-\lambda)p(-H,x) \geq p(-(\lambda G + (1-\lambda)H), x). \tag{5.16}$$

First if either $p(-G,x) = +\infty$ or $p(-H,x) = +\infty$, (5.16) is immediate (recall that $p(-G,x) > -\infty$ and $p(-H,x) > -\infty$). So assume that $p(-G,x)$ and $p(-H,x)$ are both finite and fix some $\varepsilon > 0$. Then

$$u(0,x) \leq u(-G, x+\varepsilon+p(-G,x)) \text{ and } u(0,x) \leq u(-H, x+\varepsilon+p(-H,x)).$$

Since both right-hand side are finite we get that

$$u(0,x) \leq \lambda u(-G, x+\varepsilon+p(-G,x)) + (1-\lambda)u(-H, x+\varepsilon+p(-H,x)).$$

Set $z_G = x+\varepsilon+p(-G,x)$ and $z_H = x+\varepsilon+p(-H,x)$. It remains to prove that

$$\lambda u(-G,z_G) + (1-\lambda)u(-H,z_H) \leq u(-(\lambda G + (1-\lambda)H), \lambda z_G + (1-\lambda)z_H). \tag{5.17}$$

If $u(-G,z_G) = -\infty$ or $u(-H,z_H) = -\infty$ there is nothing to prove (recall that $u(-G,z_G) < +\infty$ and $u(-H,z_H) < +\infty$). We assume now that $u(-G,z_G)$ and $u(-H,z_H)$ are both finite. Recalling (5.11), there exists some $\phi_G \in \mathcal{A}(U,-G,z_G)$, $\phi_H \in \mathcal{A}(U,-H,z_H)$ such that

$$\begin{aligned}
u(-G,z_G) - \varepsilon &\leq \inf_{P \in \mathcal{Q}^T} E_P U\left(\cdot, V_T^{z_G,\phi_G}(\cdot) + G(\cdot)\right), \\
u(-H,z_H) - \varepsilon &\leq \inf_{P \in \mathcal{Q}^T} E_P U\left(\cdot, V_T^{z_H,\phi_H}(\cdot) + H(\cdot)\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\lambda u(-G,z_G) + (1-\lambda)u(-H,z_H) - \varepsilon \\
&\leq \lambda \inf_{P \in \mathcal{Q}^T} E_P U\left(\cdot, V_T^{z_G,\phi_G}(\cdot) + G(\cdot)\right) + (1-\lambda) \inf_{P \in \mathcal{Q}^T} E_P U\left(\cdot, V_T^{z_H,\phi_H}(\cdot) + H(\cdot)\right) \\
&\leq \inf_{P \in \mathcal{Q}^T} E_P U\left(\cdot, V_T^{\lambda z_G + (1-\lambda)z_H, \lambda\phi_G + (1-\lambda)\phi_H}(\cdot) + (\lambda G(\cdot) + (1-\lambda)H(\cdot))\right) \\
&\leq u(-(\lambda G + (1-\lambda)H), \lambda z_G + (1-\lambda)z_H),
\end{aligned}$$

where we have used the concavity of U and the fact that if $\phi_G \in \mathcal{A}(U, -G, z_G) = \mathcal{A}(-G, z_G)$ and $\phi_H \in \mathcal{A}(U, -H, z_H) = \mathcal{A}(-H, z_H)$, then $\lambda\phi_G + (1-\lambda)\phi_H \in \mathcal{A}(-(\lambda G + (1-\lambda)H), \lambda z_H + (1-\lambda)z_G) = \mathcal{A}(U, -(\lambda G + (1-\lambda)H), \lambda z_G + (1-\lambda)z_H)$ by assumption. As the previous inequality is true for all ε, (5.17) is proven. So

$$u(0, x) \leq u(-(\lambda G + (1-\lambda)H), x + \varepsilon + \lambda p(-G, x) + (1-\lambda)p(-H, x)).$$

It follows that $\varepsilon + \lambda p(-G, x) + (1-\lambda)p(-H, x) \geq p(-(\lambda G + (1-\lambda)H), x)$ and as this is true for all $\varepsilon > 0$, the convexity of ρ_x is proven.
The third part of the proposition follows from Proposition 5.2.14 under the additional assumption. □

5.3 Absolute risk aversion and certainty equivalent

We present now a formal definition of the notion of absolute risk aversion for a general random utility function.

Definition 5.3.1 For any function U satisfying Definition 5.2.11, the absolute risk aversion is defined for all $(\omega^T, x) \in \Omega^T \times (0, +\infty)$ by

$$r(\omega^T, x) := -\frac{U''(\omega^T, x)}{U'(\omega^T, x)}.$$

In the mono-prior case, $i.e$ when $\mathcal{Q}^T = \{P\}$, the absolute risk aversion is related to the notion of certainty equivalent. If the preferences of an agent are represented by a non-random utility function U and given an asset whose payoff at maturity is G, the certainty equivalent $e(G, P)$ is the amount of cash that will make her indifferent (in the sense of the expected utility evaluation) between receiving the cash and the asset G

$$E_P U(e(G, P)) = U(e(G, P)) = E_P U(G(\cdot)).$$

The risk premium $\rho(G, P) := E_P G(\cdot) - e(G, P)$ is the amount that the agent is ready to loose in order to be indifferent (in the sense of the expected utility evaluation) between the sure quantity $E_P G(\cdot) - \rho(G, P)$ and the random variable G since

$$E_P U(E_P G(\cdot) - \rho(G, P)) = U(e(G, P)) = E_P U(G(\cdot)).$$

We will see in Proposition 5.3.2 that under suitable assumptions $\rho(G, P) \geq 0$. The risk premium is thus a measure of the risk-aversion of the agent: the higher the

risk premium, the more risk-adverse the agent is as she will accept a smaller amount of cash rather than G.
The following proposition recalls the definition of the certainty equivalent in a mono-prior framework but for random utility functions and proposes an extension to the multiple-priors framework.

Proposition 5.3.2 *Let $G \in \mathcal{W}_T^{0,+}$ such that $G(\cdot) < +\infty$ $\mathcal{Q}^T$-q.s.*
1. *Assume that U is an utility function verifying Definition 5.2.11, such that $\sup_{P\in\mathcal{Q}^T} E_P U^-(\cdot, y) < +\infty$ for all $y > 0$, $E_P U^+(\cdot, 1) < +\infty$ and $E_P|U(\cdot, G(\cdot))| < +\infty$ for all $P \in \mathcal{Q}^T$.*
1.a. *For all $P \in \mathcal{Q}^T$, there exists a unique constant $e(G, P) \in [0, +\infty)$ such that*

$$E_P U(\cdot, e(G, P)) = E_P U(\cdot, G(\cdot)). \tag{5.18}$$

1.b *If furthermore $G \in \mathcal{W}_T^{\infty,+}$, $\sup_{P\in\mathcal{Q}^T} E_P U^-(\cdot, G(\cdot)) < \infty$ and $\inf_{P\in\mathcal{Q}^T} E_P U'(\cdot, z) > 0$ for all $z > 0$, then there exists also an unique $e(G) \in [0, ||G||_\infty]$ such that*

$$\inf_{P\in\mathcal{Q}^T} E_P U(\cdot, e(G)) = \inf_{P\in\mathcal{Q}^T} E_P U(\cdot, G(\cdot)) \tag{5.19}$$

and in this case, we have that $e(G) \geq \inf_{P\in\mathcal{Q}^T} e(G, P)$. We call $e(G)$ the multiple-priors certainty equivalent of G.
2. *Assume now that U is a non-random utility function verifying Definition 5.2.11 such that Dom $U = \{x \in \mathbb{R}, U(x) > -\infty\} = (0, \infty)$, $E_P U^+(G(\cdot)) < \infty$ for all $P \in \mathcal{Q}^T$ and $\sup_{P\in\mathcal{Q}^T} E_P U^-(G(\cdot)) < \infty$. Then, there exists some unique $e(G, P)$ and $e(G)$ in $[0, \infty)$ such that*

$$U(e(G, P)) = E_P U(G(\cdot)), \ \forall P \in \mathcal{Q}^T \tag{5.20}$$

$$U(e(G)) = \inf_{P\in\mathcal{Q}^T} E_P U(G(\cdot)). \tag{5.21}$$

Moreover, $e(G, P) \leq E_P G(\cdot)$ for all $P \in \mathcal{Q}^T$ and

$$e(G) = \inf_{P\in\mathcal{Q}^T} e(G, P) \leq \inf_{P\in\mathcal{Q}^T} E_P G(\cdot).$$

Furthermore the multiple-priors risk premium defined by $\rho(G) := \sup_{P\in\mathcal{Q}^T} \rho(G, P)$ satisfies

$$0 \leq \rho(G) \leq \sup_{P\in\mathcal{Q}^T} E_P G(\cdot) - e(G).$$

Remark that (5.20) is true assuming only that $E_P U^-(G(\cdot)) < \infty$ for all $P \in \mathcal{Q}^T$.

Proof. 1. We set for all $y \geq 0$, $P \in \mathcal{Q}^T$ $\psi_P(y) = E_P U(\cdot, y)$ and $\psi(y) = \inf_{P\in\mathcal{Q}^T} \psi_P(y)$. In the rest of the proof, the properties concerning ψ_P will be stated for all $P \in \mathcal{Q}^T$. It is clear that ψ_P is concave, strictly increasing and that ψ is concave and non-decreasing. As for all $y > 0$, $\sup_{P\in\mathcal{Q}^T} E_P U^-(\cdot, y) < +\infty$, we have that Ri (Dom ψ) =

Ri (Dom ψ_P) $= (0, +\infty)$ (where Dom $\psi = \{y, \psi(y) > -\infty\}$ and Ri(Dom ψ) is its relative interior). So ψ and ψ_P are continuous on $(0, \infty)$. Using the monotonicity of U, for all $0 \leq y \leq 1$ $U(\cdot, y) \leq U^+(\cdot, 1)$ and as $E_P U^+(\cdot, 1) < +\infty$, the monotone convergence theorem applies and we get that $\lim_{y \searrow 0} \psi_P(y) = \psi_P(0)$. Thus the function ψ_P is right-continuous in 0 and it follows easily that ψ is also right-continuous in 0. Indeed let again $(x_n)_{n \geq 1}$ be a sequence of positive real number converging to 0, then for all $n \geq 1$ $\psi(x_n) \geq \psi(0)$ and therefore $\liminf_n \psi(x_n) \geq \psi(0)$. Now, for all $P \in \mathcal{Q}^T$, $n \geq 1$ we also have that $\psi(x_n) \leq \psi_P(x_n)$, thus $\limsup_n \psi(x_n) \leq \limsup_n \psi_P(x_n) = \psi_P(0)$ and $\limsup_n \psi(x_n) \leq \inf_{P \in \mathcal{Q}^T} \psi_P(0) = \psi(0)$ follows.
Now let $F(\cdot) := \lim_{y \to +\infty} U(\cdot, y) \in (-\infty, \infty]$. Since $E_P U^-(\cdot, 1) < +\infty$ by assumption, the monotone convergence theorem applied and we get that

$$\lim_{y \nearrow +\infty} \psi_P(y) = E_P F(\cdot) \in (-\infty, +\infty]. \tag{5.22}$$

Moreover as $P(G(\cdot) < \infty) = 1$ and U is strictly increasing

$$F(\cdot) - U(\cdot, G(\cdot)) > 0 \ P\text{-a.s.} \tag{5.23}$$

$1.a$. Set for all $y \geq 0$, $\overline{\psi}_P(y) = \psi_P(y) - E_P U(\cdot, G(\cdot))$ which is well-defined since $E_P |U(\cdot, G(\cdot))| < +\infty$ for all $P \in \mathcal{Q}^T$. It is clear that $\overline{\psi}_P$ is continuous on $(0, +\infty)$ and right-continuous in 0. Thus

$$\overline{\psi}_P(0) = E_P U(\cdot, 0) - E_P U(\cdot, G(\cdot)) \leq 0, \tag{5.24}$$

since U is non-decreasing and $G \in \mathcal{W}_T^{0,+}$. As $E_P U(\cdot, G(\cdot)) \leq E_P |U(\cdot, G(\cdot))| < \infty$ by assumption, if $\lim_{y \nearrow +\infty} \psi_P(y) = +\infty$ then for y large enough $\overline{\psi}_P(y) > 0$. Now if $\lim_{y \nearrow +\infty} \psi_P(y) < +\infty$, then (5.22) and (5.23) imply that $E_P U(\cdot, G(\cdot)) < E_P F(\cdot) = \lim_{y \nearrow +\infty} \psi_P(y)$ and $\overline{\psi}_P(y) > 0$ again for some y large enough. In both cases, the intermediate value theorem gives a unique $e(G, P) \in [0, +\infty)$ such that $\overline{\psi}_P(e(G, P)) = 0$ and (5.18) is proved.

$1.b$. Set $\overline{\psi}(y) = \psi(y) - \inf_{P \in \mathcal{Q}^T} E_P U(\cdot, G(\cdot))$ which is well-defined since we have that $\sup_{P \in \mathcal{Q}^T} E_P U^-(\cdot, G(\cdot)) < \infty$ and $\inf_{P \in \mathcal{Q}^T} E_P U^+(\cdot, G(\cdot)) < \infty$. The function $\overline{\psi}$ is continuous on $(0, +\infty)$ and right-continuous in 0 and using (5.24), we get that

$$\inf_{P \in \mathcal{Q}^T} E_P U(\cdot, G(\cdot)) \geq \inf_{P \in \mathcal{Q}^T} E_P U(\cdot, 0) = \psi(0),$$

$\overline{\psi}(0) \leq 0$ follows. Since $\inf_{P \in \mathcal{Q}^T} E_P U'(\cdot, z) > 0$ for all $z > 0$, ψ is strictly increasing on $(0, +\infty)$. Indeed let $0 < x < y$, then $U(\cdot, x) + (y - x) U'(\cdot, y) \leq U(\cdot, y)$ and this implies that

$$\inf_{P \in \mathcal{Q}^T} E_P U(\cdot, x) + (y - x) \inf_{P \in \mathcal{Q}^T} E_P U'(\cdot, y) \leq \inf_{P \in \mathcal{Q}^T} E_P U(\cdot, y). \tag{5.25}$$

As $G \in \mathcal{W}_T^{\infty,+}$, using the monotonicity of U we obtain for any $\varepsilon > 0$

$$\inf_{P \in \mathcal{Q}^T} E_P U(\cdot, G(\cdot)) \leq \inf_{P \in \mathcal{Q}^T} E_P U(\cdot, ||G||_\infty) = \psi(||G||_\infty) < \psi(||G||_\infty + \varepsilon)$$

and thus $\overline{\psi}(||G||_\infty + \varepsilon) > 0$. We apply again the intermediate value theorem and there exist a unique $e(G) \in [0, ||G||_\infty]$ such that $\overline{\psi}(e(G)) = 0$ and (5.19) is proved. Now for any $Q \in \mathcal{Q}^T$, (5.18) implies that

$$E_Q U(\cdot, \inf_{P \in \mathcal{Q}^T} e(G, P)) \leq E_Q U(\cdot, e(G, Q)) = E_Q U(\cdot, G(\cdot)). \quad (5.26)$$

Therefore using (5.19), $\psi(\inf_{P \in \mathcal{Q}^T} e(G, P)) \leq \psi(e(G))$ and $e(G) \geq \inf_{P \in \mathcal{Q}^T} e(G, P)$ follows since ψ is strictly increasing.

2. From Definition 5.2.11 and $\text{Dom}(U) = (0, \infty)$, U is continuous on $(0, \infty)$ and right-continuous in 0. Fix some $P \in \mathcal{Q}^T$. As $E_P U^-(G(\cdot)) < +\infty$, $G \in \mathcal{W}_T^{0,+}$ and U is non-decreasing, $E_P U(G(\cdot)) - U(0) \geq 0$ (recall that U is non random). Now as before P-a.s $U(G(\omega^T)) < \lim_{y \to +\infty} U(y)$ and since $E_P U^+(G(\cdot)) < +\infty$, one can always conclude that $E_P U(G(\cdot)) - U(y) < 0$ for y large enough and the intermediate value theorem implies that (5.20) holds true.

Now since $E_P U(G(\cdot)) \geq U(0)$, $\inf_{P \in \mathcal{Q}^T} E_P U(G(\cdot)) - U(0) \geq 0$. Moreover since $\inf_{P \in \mathcal{Q}^T} E_P U(G(\cdot)) - U(y) \leq E_P U(G(\cdot)) - U(y) < 0$, the intermediate value theorem implies (5.21). As before (see (5.26)) one can prove that $\inf_{P \in \mathcal{Q}^T} e(G, P) \leq e(G)$. For some $P \in \mathcal{Q}^T$ fixed, using Jensen's inequality we have that

$$U(e(G)) \leq E_P U(G(\cdot)) = U(e(G, P)) \leq U\left(E_P G(\cdot)\right).$$

Thus, by strict monotonicity of U, $e(G) \leq e(G, P) \leq E_P G(\cdot)$ and since this is true for all $P \in \mathcal{Q}^T$, we find that

$$e(G) \leq \inf_{P \in \mathcal{Q}^T} e(G, P) \leq \inf_{P \in \mathcal{Q}^T} E_P G(\cdot).$$

□

Note that it is easy to find an example where $\overline{\psi}$ is constant and thus $e(G)$ is not unique if $\inf_{P \in \mathcal{Q}^T} E_P U'(\cdot, z) = 0$

Example 5.3.3 We illustrate with a simple example why in Proposition 5.3.2, when dealing with random utility function we need to impose that $\inf_{P \in \mathcal{Q}^T} E_P U'(\cdot, z) > 0$ for all $z > 0$ in order to get a unique $e(G)$ such that (5.21) holds true (otherwise ψ might be constant). We consider a one-period model and set $\Omega = (0, \infty)$ and $\mathcal{Q} = \text{Conv}\{P_n, n \geq 1\}$ where $P_n = \delta_n$ for all $n \geq 1$. We choose the random utility function $U(\omega, x) = \frac{ln(1+x)}{\omega}$ defined for all $(\omega, x) \in \Omega \times [0, \infty)$. For all $z > 0$ we have that $\inf_{P \in \mathcal{Q}} E_P U'(\omega, z) = \inf_{n \geq 1} \frac{1}{n(1+z)} = 0$. Now, using the notation introduced in the proof of Proposition 5.3.2, we have for some $n \geq 1$ and all $x > 0$

$\psi_{P_n}(x) = E_{P_n}U(\cdot,x) = \frac{ln(1+x)}{n}$ while $\psi(x) = \inf_{P\in\mathcal{Q}} E_P U(\cdot,x) = \inf_{n\geq 1}\frac{ln(1+x)}{n} = 0$. Hence if we consider the (bounded) random variable $G(\omega) = e^{-\omega}$, then $\overline{\psi}_{P_n}(x) = \frac{ln(1+x)}{n} - E_{P_n}U(\cdot, G(\cdot)) = \frac{ln(1+x)}{n} - \frac{ln(1+e^{-n})}{n}$ and we get obviously that $e(G,P_n) = e^{-n}$. However $\overline{\psi}(x) = \psi(x) - \inf_n E_{P_n}U(\cdot,G(\cdot)) = 0$ for all $x \geq 0$.

Finally, we consider two investors A and B with respective non-random utility functions U_A and U_B satisfying Definition 5.2.11. Recall that in the mono-prior case with $\mathcal{Q}^T = \{P\}$ investor A has greater absolute risk-aversion than investor B (*i.e.* $r_A(x) \geq r_B(x)$ for all $x > 0$) if and only if investor A is globally more risk averse than investor B, in the sense that the certainty equivalent of every contingent claim is smaller for A than for B (*i.e* $e_A(G,P) \leq e_B(G,P)$ for any $G \in \mathcal{W}_T^{0,+}$), see [108]. We propose the following generalization of this result in the multiple-priors framework.

Proposition 5.3.4 *Let U_A, U_B be non-random utility functions with domain equal to $(0,\infty)$ verifying Definition 5.2.11. Let*

$$\mathcal{W}_T^{+}(U) := \mathcal{W}_T^{0,+}\cap\left\{G,\, G(\cdot) < +\infty = \ \mathcal{Q}^T - \textbf{q.s}\, E_P U^+(G(\cdot)) < \infty, \forall P \in \mathcal{Q}^T,\, \sup_{P\in\mathcal{Q}^T} E_P U^-(G(\cdot)) < \infty\right\}.$$

1. *If for all $x > 0$, $r_A(x) \geq r_B(x)$, then $e_A(G) \leq e_B(G)$ for all $G \in \mathcal{W}_T^+(U)$.*
2. *If for all $G \in \mathcal{W}_T^+(U)$, $e_A(G) < e_B(G)$ then $r_A(x) \geq r_B(x)$ for all $x > 0$.*

We prove in Proposition 5.3.4 that the absolute risk aversion allows to rank the multiple-priors certainty equivalent despite the presence of uncertainty (and thus uncertainty aversion). The reason for this is related to the specific multiple-priors representation we have chosen. For more details we refer to [67, Theorem 5, Example 2].

Proof. The proof is a straightforward adaptation of [62, Proposition 2.47]. 1. We first show that if for all $x > 0$, $r_A(x) \geq r_B(x)$, then $e_A(G,P) \leq e_B(G,P)$ for all $P \in \mathcal{Q}^T$ and $G \in \mathcal{W}_T^+(U)$. This will imply that $e_A(G) \leq e_B(G)$ using the second item of Proposition 5.3.2. We fix some $G \in \mathcal{W}_T^+(U)$ and $P \in \mathcal{Q}^T$. Let $D := U_B((0,\infty)) \subset (-\infty,\infty)$ and define $F : D \to \mathbb{R}$ by $F(y) = U_A\left(U_B^{-1}(y)\right)$. Then on D

$$F'(\cdot) = \frac{U_A'(U_B^{-1}(\cdot))}{U_B'(U_B^{-1}(\cdot))} \text{ and } F''(\cdot) = \frac{U_A'(U_B^{-1}(\cdot))}{\left(U_B'(U_B^{-1}(\cdot))\right)^2}\left(r_B(U_B^{-1}(\cdot)) - r_A(U_B^{-1}(\cdot))\right). \quad (5.27)$$

As $U_B^{-1}(\cdot) > 0$ on D, F is increasing and concave on D and $U_A(x) = F(U_B(x))$ for all $x > 0$. Now let $d := U_B(0) \in [-\infty,\infty)$ be the lower bound of D. We distinguish between two cases. First if $d > -\infty$, we extend F by continuity in d, setting $F(d) = U_A\left(U_B^{-1}(d)\right) = U_A(0) \in [-\infty,\infty)$. It is clear that $F(d) \leq F(y)$ for all $y \in [d,+\infty)$, that

F is concave on $[d, +\infty)$ and that $U_A(x) = F(U_B(x))$ holds also true for all $x \geq 0$. Now, using the fact that U_A and U_B are non-random, (5.20) and Jensen's inequality, we get that

$$U_A(e_A(G,P)) = E_P U_A(G(\cdot)) = E_P F(U_B(G(\cdot))) \leq F(E_P(U_B(G(\cdot)))) \\ = F(U_B(e_B(G,P))) = U_A(e_B(G,P)) \tag{5.28}$$

and since U_A is strictly increasing, we obtain that $e_A(G,P) \leq e_B(G,P)$ as claimed. Now we treat the case where $d = -\infty$. First $P(G > 0) = 1$. Indeed if $P(G = 0) > 0$, $E_P U_B^-(G(\cdot)) = E_P U_B^-(G(\cdot)) 1_{\{G>0\}}(\cdot) + U_B^-(0) P(G = 0) = +\infty$, a case that we have excluded. Thus $P(G > 0) = 1$ and the previous argument applies, we also obtain $e_A(G,P) \leq e_B(G,P)$.

2. Assume that there exists some $x_0 > 0$ such that $r_A(x_0) < r_B(x_0)$. By continuity, there exists $\alpha > 0$, such that $r_A(x) < r_B(x)$ on $(x_0 - \alpha, x_0 + \alpha)$. We can choose α such that $x_0 - \alpha > 0$. Let $I := (U_B(x_0 - \alpha), U_B(x_0 + \alpha)) \subset D$, then F is strictly convex on I (see (5.27)). Fix $\widetilde{G} \in \mathcal{W}_T^+(U)$ and set $G := x_0 - \alpha + 2\alpha \frac{\widetilde{G}}{\widetilde{G}+1} \in \mathcal{W}_T^+(U)$. It is clear that $G(\cdot) \in (x_0 - \alpha, x_0 + \alpha)$. As in (5.28), using Jensen inequality, the fact that F is (strictly) convex on I we get that for any $P \in \mathcal{Q}^T$

$$U_A(e_A(G,P)) = E_P F(U_B(G(\cdot))) \geq F(E_P(U_B(G(\cdot))) = F(U_B(e_B(G,P))) = U_A(e_B(G,P)). \tag{5.29}$$

This implies that $e_A(G,P) \geq e_B(G,P)$ for all $P \in \mathcal{Q}^T$, thus $e_A(G) \geq e_B(G)$: a contradiction. Note that if P is such that one can find some $\widetilde{G}$ which is not constant then the first inequality in (5.29) is strict and one gets that $e_A(G,P) > e_B(G,P)$. □

5.4 Convergence of utility indifference prices

Intuitively speaking an agent which is totally risk averse will use the superreplication price : whatever the possible outcome (where possible outcome are defined by a set of probability measures), she doesn't want to incur any loss (see (5.9)). We are going to prove that under suitable conditions, the multiple-priors utility indifference prices of a given contingent claim (for the seller) converge to its multiple-priors superreplication price. For non random utility functions those conditions are implied by the convergence of the absolute risk-aversion (see Definition 5.3.1) of the agents to infinity.

First we give some intuition of this result and show that for a utility function that has a sort of infinite absolute risk aversion, the utility indifference price is equal to the superreplication price for some contingent claim $G \in \mathcal{W}_T^{0,+}$. Fix some $x \geq \pi(G)$ and introduce the following utility function $U_\infty : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ defined by $U_\infty(y) = -\infty 1_{(-\infty,x)}(y)$. Note that the absolute risk aversion of U_∞ is not defined.

However $U_n(y) = -e^{-n(y-x)}$ for $y \geq 0$ and $U_n(y) = -\infty$ for $y < 0$ satisfies Definition 5.2.11 and for $y \geq 0$ fixed with $y \neq x$, $\lim_{n\to+\infty} U_n(y) = U_\infty(y)$. Then the absolute risk aversion of the utility function U_n satisfies $\lim_{n\to+\infty} r_n(y) = +\infty$ for all $y \geq 0$. Hence, one may say that U_∞ has an infinite absolute risk aversion. We now show that the superreplication price of $G \in \mathcal{W}_T^{0,+}$ is equal it utility indifference price evaluated with the function U_∞. Since for all $\phi \in \Phi$, $y \in \mathbb{R}$, $U_\infty^+(V_T^{y,\phi}(\cdot) - G(\cdot)) = 0$, we have that $\Phi(U_\infty, G, y) = \Phi(U_\infty, 0, y) = \Phi$ and $\mathcal{A}(U_\infty, y) = \mathcal{A}(U_\infty, G, y)$. Moreover $\mathcal{A}(U_\infty, G, y)$ is not empty for all $y \geq \pi(G)$ (see Theorem 5.2.9). First, it is easy to see that $u_\infty(0, x) = 0$. Now we fix some $0 \leq z < \pi(G)$ and $\phi \in \mathcal{A}(U_\infty, G, x+z)$. There exists some $P \in \mathcal{Q}^T$ such that $P(V_T^{z,\phi}(\cdot) - G(\cdot) < 0) > 0$ or equivalently $P(V_T^{z+x,\phi}(\cdot) - G(\cdot) < x) > 0$ which implies that $E_P U_\infty(\cdot, V_T^{x+z,\phi}(\cdot) - G(\cdot)) = -\infty$. Hence for all $\phi \in \mathcal{A}(U_\infty, G, x+z)$, $\inf_{P\in\mathcal{Q}^T} E_P U_\infty(\cdot, V_T^{x+z,\phi}(\cdot) - G(\cdot)) = -\infty$ and it follows that $u_\infty(G, x+z) = -\infty < u_\infty(0, x)$. From the definition of $p(G, x)$ we get that $p(G, x) \geq z$ and letting z go to $\pi(G)$, $p(G, x) \geq \pi(G)$. Combining this with Lemma 5.4.14 below we have that $p(G, x) = \pi(G)$.

Now we state precisely our convergence result. We consider a sequence of utility functions (see Definition 5.2.11) $U_n : \Omega^T \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, $n \geq 1$ and some contingent claim $G \in \mathcal{W}_T^{0,+}$. We denote for all $n \geq 1$, $x \geq 0$, $\omega^T \in \Omega^T$

$$u_n(G, x) := \sup_{\phi\in\mathcal{A}(U_n,G,x)} \inf_{P\in\mathcal{Q}^T} E_P U_n\left(\cdot, V_T^{x,\phi}(\cdot) - G(\cdot)\right) \tag{5.30}$$

$$p_n(G, x) := \inf\{z \in \mathbb{R},\ u_n(G, x+z) \geq u_n(0, x)\} \tag{5.31}$$

$$r_n(\omega^T, x) := -\frac{U_n''(\omega^T, x)}{U_n'(\omega^T, x)}. \tag{5.32}$$

We review the assumptions needed in Theorem 5.4.8 in order to have the convergence result.

Assumption 5.4.1 We have that $\Delta S_t, \frac{1}{\alpha_t} \in \mathcal{W}_t^r$ for all $1 \leq t \leq T$ and $0 < r < \infty$ where $\mathcal{W}_t^r := \{X : \Omega^t \to \mathbb{R} \cup \{\pm\infty\},\ \mathcal{B}_c(\Omega^t)\text{-measurable},\ \sup_{P\in\mathcal{Q}^t} E_P|X|^r < \infty\}$.

Note that as in [51, Propositions 14, 15] one can prove that for all $r \in [1, \infty]$, $\mathcal{W}_t^r$ is a Banach space (up to the usual quotient identifying two random variables that are $\mathcal{Q}^t$-q.s. equal) for the norm $||X||_{r,t} := (\sup_{P\in\mathcal{Q}^t} E_P|X|^r)^{\frac{1}{r}}$ if $r < \infty$ and $||X||_{\infty,t} := \inf\{M \geq 0, X(\cdot) \leq M\ \mathcal{Q}^t\text{q.s.}\}$. We will omit the index t when $t = T$.
In the light of Proposition 5.2.5, the condition $\frac{1}{\alpha_t} \in \mathcal{W}_t^r$ is a kind of strong form of no-arbitrage. Note that if α_t is not constant, then even in the mono-prior case utility maximisation problem may be ill posed (see Example 3.3 in [31]), so an integrability assumption on $\frac{1}{\alpha_t}$ looks reasonable. Assumption 5.4.1 could be weakened to the existence of the $\mathcal{W}_T^N$-th moment for N large enough but this would lead to

complicated book-keeping with no essential gain in generality, which we prefer to avoid.

The asymptotic result for general random utility functions will be stated for some fixed $x_0 > 0$. However in the case of non-random utility functions we can avoid Assumption 5.4.2 below and obtain the convergence result for all $x > 0$. We can also use the natural assumption that $\lim_{n \to +\infty} r_n(x) = +\infty$ instead of Assumption 5.4.4, see Theorem 5.4.10. The first assumption states that U_n is sufficiently measurable and regular in x_0.

Assumption 5.4.2 We have that $\sup_n ||U_n^{\pm}(\cdot, x_0)||_1 < \infty$ and for some $q > 1$ that $\sup_n ||U_n'(\cdot, x_0)||_q < \infty$.

Remark 5.4.3 If we assume that $\sup_n U_n^{\pm}(\cdot, x_0) \in \mathcal{W}_T^1$ and that there exists some $q > 1$ such that $\sup_n ||U_n'(\cdot, x_0)||_q < \infty$, then Assumption 5.4.2 is verified. Indeed let's prove for instance that $\sup_n ||E_P U_n^+(\cdot, x_0)||_1 < +\infty$. For all $P \in \mathcal{Q}^T$, $n \geq 1$ we have that $E_P U_n^+(\cdot, x_0) \leq E_P \sup_n U_n^+(\cdot, x_0)$ and it follows that $\sup_n \sup_{P \in \mathcal{Q}^T} E_P U_n^+(\cdot, x_0) \leq || \sup_n U_n^+(\cdot, x_0)||_1$.

We postulate now the assumption which will play the role of the convergence of the absolute risk aversion to infinity for random utility functions.

Assumption 5.4.4 For all $\varepsilon > 0$ such that $x_0 > \varepsilon$ and all $C \geq 0$

$$\lim_{n \to \infty} \inf_{P \in \mathcal{Q}^T} P\left(\left\{ \int_{x_0 - \frac{\varepsilon}{2}}^{x_0} U_n''(\cdot, v) dv \leq -\frac{C}{\varepsilon} \right\} \right) = 1. \tag{5.33}$$

We first provide a lemma which gives natural sets of assumptions under which Assumption 5.4.4 is satisfied. It is stated under the assumption that U_n is strictly increasing in x_0 uniformly in n.

Lemma 5.4.5 *Let $\varepsilon > 0$ such that $x_0 > \varepsilon$ and $C \geq 0$. Assume that there exists a random variable λ such that $\lambda > 0$ $\mathcal{Q}^T$-q.s, $U_n'(\omega^T, x_0) \geq \lambda(\omega^T)$ for all $\omega^T \in \Omega^T$, $n \geq 1$ and that either* 1. *or* 2. *below are satisfied. Then* (5.33) *in Assumption 5.4.4 is satisfied for these ε, x_0 and C.*

1. *For all n and $\omega^T \in \Omega^T$, $U_n''(\omega^T, \cdot)$ is non decreasing, and*

$$\lim_{n \to \infty} \inf_{P \in \mathcal{Q}^T} P\left(\left\{ \lambda(\cdot) r_n(\cdot, x_0) \geq \frac{2C}{\varepsilon^2} \right\} \right) = 1. \tag{5.34}$$

2. *We have that*

$$\lim_{n \to \infty} \inf_{P \in \mathcal{Q}^T} P\left(\lambda(\cdot) \int_{x_0 - \frac{\varepsilon}{2}}^{x_0} r_n(\cdot, v) dv \geq \frac{C}{\varepsilon} \right) = 1. \tag{5.35}$$

Note that in 2. if we only assume for all $x \in (0, x_0]$, $\omega^T \in \Omega^T$, that $\lim_{n \to +\infty} r_n(\omega^T, x) = +\infty$, applying Fatou's Lemma for all $\omega^T \in \Omega^T$, there exists N_{ω^T} such that for all $k \geq N_{\omega^T}$, $\lambda(\omega^T) \int_{x_0 - \frac{\varepsilon}{2}}^{x_0} r_n(\omega^T, v) dv \geq \frac{C}{\varepsilon}$, this means that

$$\Omega^T = \cup_n \cap_{k>n} \left\{ \lambda(\cdot) \int_{x_0 - \frac{\varepsilon}{2}}^{x_0} r_k(\cdot, v) dv > \frac{C}{\varepsilon} \right\}$$

and using [51, Theorem 1] this implies that

$$\lim_{n \to \infty} \sup_{P \in \mathcal{Q}^T} P \left(\lambda(\cdot) \int_{x_0 - \frac{\varepsilon}{2}}^{x_0} r_n(\cdot, v) dv \geq \frac{C}{\varepsilon} \right) = 1.$$

Before proving Lemma 5.4.5 we discuss the conditions 1. and 2.

Remark 5.4.6 In this remark, we assume that there exists some constant $l > 0$ such that $\lambda(\cdot) \geq l$, which means that U_n is strictly increasing in x_0 uniformly in n and in ω^T. We start with 1. Power utility functions or exponential utility functions (with random coefficients) are examples where $U_n''(\omega^T, \cdot)$ is non decreasing for all n and $\omega^T \in \Omega^T$. Then (5.34) means that $r_n(\cdot, x_0) \to \infty$ with respect to $\inf_{P \in \mathcal{Q}^T} P$ or equivalently that $\lim_{n \to \infty} \sup_{P \in \mathcal{Q}^T} P(\{r_n(\cdot, x_0) \leq M\}) = 0$ for all $M > 0$.
We now discuss 2. Assume that there exists some deterministic functions $(r_n)_{n \geq 1}$ satisfying $\lim_n r_n(x) = +\infty$ and $r_n(\omega^T, x) \geq r_n(x)$ for all $0 < x \leq x_0$. Then 2. holds true. Indeed Fatou's lemma implies that $\lim_{n \to \infty} \int_{x_0 - \frac{\varepsilon}{2}}^{x_0} r_n(v) dv = +\infty$ and we conclude since $\{\lambda \int_{x_0 - \frac{\varepsilon}{2}}^{x_0} r_n(v) dv \geq \frac{C}{\varepsilon}\} \subset \{\lambda(\cdot) \int_{x_0 - \frac{\varepsilon}{2}}^{x_0} r_n(\cdot, v) dv \geq \frac{C}{\varepsilon}\}$.

Proof. of Lemma 5.4.5 We start with 1. Since for all n and $\omega^T \in \Omega^T$, $U_n''(\omega^T, \cdot)$ is non decreasing and $U_n'(\omega^T, x_0) \geq \lambda(\omega^T)$, we get that

$$\int_{x_0 - \frac{\varepsilon}{2}}^{x_0} U_n''(\cdot, v) dv \leq \frac{\varepsilon}{2} U_n''(\cdot, x_0) = -\frac{\varepsilon}{2} U_n'(\cdot, x_0) r_n(\cdot, x_0) \leq -\frac{\varepsilon}{2} \lambda(\cdot) r_n(\cdot, x_0).$$

Thus (5.34) implies (5.33).
For 2. since for all n and $\omega^T \in \Omega^T$, $U_n'(\omega^T, x_0) \geq \lambda(\omega^T)$, we get that

$$\int_{x_0 - \frac{\varepsilon}{2}}^{x_0} U_n''(\cdot, v) dv = -\int_{x_0 - \frac{\varepsilon}{2}}^{x_0} U_n'(\cdot, v) r_n(\cdot, v) dv \leq -\lambda(\cdot) \int_{x_0 - \frac{\varepsilon}{2}}^{x_0} r_n(\cdot, v) dv.$$

Thus (5.35) implies (5.33). □

Note that it is easy to see that Definition 5.2.11 $ii)$ and the fact that U_n is strictly increasing in x_0 uniformly in n can be postulate only on a $\mathcal{Q}^T$-full measure set.

Example 5.4.7 We give a concrete example of some random utility functions satisfying Definition 5.2.11 and Assumptions 5.4.2 and 5.4.4. For all $n \geq 1$, let R_n be a random variable uniformly distributed in $[b_n, B_n]$ for all $P \in \mathcal{Q}^T$ with $b_n > 0$, $\lim_{n\to\infty} b_n = \infty$ and $B_n^3 - b_n^3 < A$ for some $A > 0$. Note that as $b_n > 0$ and $\lim_{n\to\infty} b_n = \infty$, there exists some $b > 0$ such that $b_n \geq b$ for all $n \geq 1$. Set now $U_n(\omega^T, x) = -e^{-R_n(\omega^T)(x-1)}$ for $x \geq 0$ and $U_n(\omega^T, x) = -\infty$ for $x < 0$. We choose $x_0 = 1$. As $U_n'(\cdot, 1) = R_n(\cdot) \geq b > 0$ $\mathcal{Q}^T$ q.s., $U_n(\cdot, 1)$ is uniformly decrasing in n and ω^T. Now $||U_n'(\cdot,1)||_2^2 = \frac{B_n^3 - b_n^3}{3}$, thus $\sup_n ||U_n'(\cdot,1)||_2 < \infty$. Then as $U_n(\cdot,1) = -1$, it is clear that $\sup_n U_n^{\pm}(\cdot,1) \in \mathcal{W}_T^1$ and the fact that Assumption 5.4.2 holds true follows from Remark 5.4.3. Finally for all $n \geq 1$, $x > 0$ and $\omega^T \in \Omega^T$ we have $r_n(\omega^T, x) = R_n(\omega^T) \geq b_n$ and $\lim_n b_n = +\infty$, so using Lemma 5.4.5 and Remark 5.4.6, Assumption 5.4.4 is verified.

Theorem 5.4.8 *Let $G \in \mathcal{W}_T^{0,+}$ and $G \neq 0$ $\mathcal{Q}^T$-q.s. Assume that Assumptions 5.2.1, 5.2.2, 5.2.3 and 5.4.1, holds true as well as Assumptions 5.4.2 and 5.4.4 for some $x_0 > 0$. For all $n \geq 1$, $p_n(G, x_0)$ is well defined and $\lim_{n\to+\infty} p_n(G, x_0) = \pi(G)$.*

If $G = 0$ $\mathcal{Q}^T$-q.s. then $\pi(G) = 0$ (see Lemma 5.2.10) but in order to have that $p_n(G, x_0) = 0$, one have to make further assumptions (see Proposition 5.2.14). Applying Theorem 5.4.8 and Propositions 5.2.18 and 5.2.19 and we obtain immediately the following proposition on the convergence of risk measures.

Proposition 5.4.9 *Assume that Assumptions 5.2.1, 5.2.2, 5.2.3 and 5.4.1 holds true as well as Assumptions 5.4.2 and 5.4.4 for some $x_0 > 0$. For all $n \geq 1$, $\lim_{n\to+\infty} \rho_{x_0}^n(G) = \rho(G)$ for all $G \in \mathcal{W}_T^{0,+}$ such that $G \neq 0$ $\mathcal{Q}^T$-q.s., where $\rho_{x_0}^n$ and ρ are the monetary risk measures defined in* (5.14) *and* (5.15).

Theorem 5.4.10 *Let $G \in \mathcal{W}_T^{0,+}$ $G \neq 0$ $\mathcal{Q}^T$-q.s. Assume that Assumptions 5.2.1, 5.2.2, 5.2.3, 5.4.1 hold true. Assume furthermore that U_n is a non-random utility function for all $n \geq 1$ and that $\lim_{n\to\infty} r_n(x) = +\infty$ for all $x > 0$. Then, $\lim_{n\to+\infty} p_n(G, x) = \pi(G)$ for all $x > 0$.*

Proof of Theorem 5.4.10. We fix some $x > 0$. As in [30], if we replace U_n by $\hat{U}_n := \alpha_n U_n + \beta_n$ for some $\alpha_n > 0$ and $\beta_n \in \mathbb{R}$, $\hat{U}_n$ is still a non-random, concave, strictly increasing and twice continuously differentiable function. The absolute risk aversion and the utility indifference price for U_n and $\hat{U}_n$ (see (5.31)) are the same, thus $\lim_{n\to\infty} r_n(x) = +\infty$ for all $x > 0$ for $\hat{U}_n$. Now, if we choose, $\alpha_n = \frac{1}{U_n'(x)}$ and $\beta_n = -\frac{U_n(x)}{U_n'(x)}$ we have that $\hat{U}_n(x) = 0$ and $\hat{U}_n'(x) = 1$ for all $n \geq 1$. Thus $\hat{U}_n$ satisfies Assumptions 5.4.2 for $x_0 = x$. Using Lemma 5.4.5 and Remark 5.4.6, Assumption 5.4.4 holds true for $x_0 = x$ since $\lim_{n\to\infty} r_n(x) = +\infty$ for all $x > 0$ and Theorem 5.4.8 applies to $\hat{U}_n$. □

Recalling the definition of the subreplication price (see (5.10)) and the buyer multiple-priors utility indifference price (see (5.13)), the following proposition is a simple consequence of Theorem 5.4.8.

Proposition 5.4.11 *Let $G \in \mathcal{W}_T^{\infty,+}$ such that $G \neq 0$ $\mathcal{Q}^T$-q.s.Assume that Assumptions 5.2.1, 5.2.2, 5.2.3 and 5.4.1 holds true as well as Assumptions 5.4.2 and 5.4.4 for some $x_0 > 0$. Then, for all $n \geq 1$, $p_n^B(G, x_0)$ is well defined and $\lim_{n\to+\infty} p_n^B(G, x_0) = \pi^{sub}(G)$.*

Proof of Proposition 5.4.11. We set $\widehat{G} = -G + ||G||_\infty \in \mathcal{W}_T^{0,+}$. From the cash invariance property in Proposition 5.2.18, we get that $\pi(\widehat{G}) = \pi(-G) + ||G||_\infty = -\pi^{sub}(G) + ||G||_\infty$, see Lemma 5.2.10. Thus, applying Theorem 5.4.8 to $\widehat{G}$, we get that $\lim_{n\to\infty} p_n(\widehat{G}, x_0) = \pi(\widehat{G})$. Now for $n \geq 1$, using the cash invariance property in Proposition 5.2.19 and recalling (5.13) we obtain that $p_n(\widehat{G}, x_0) = p_n(-G + ||G||_\infty, x_0) = p_n(-G, x_0) + ||G||_\infty = -p_n^B(G, x_0) + ||G||_\infty$ and the result follows. □

To prove Theorem 5.4.8 we borrow some ideas of [30] adapted to the multiple-priors case.

Lemma 5.4.12 *Suppose that for all $n \geq 1$, U_n verifies Assumptions 5.4.2 and 5.4.4. Then for all $\varepsilon > 0$ such that $x_0 \geq \varepsilon$ and $M \geq 0$,*

$$\lim_{n\to+\infty} \inf_{P\in\mathcal{Q}^T} P\left(U_n(\cdot, x_0 - \varepsilon) \leq -M\right) = 1. \tag{5.36}$$

Proof. Fix some $\varepsilon > 0$ such that $x_0 > \varepsilon$ and $M \geq 0$. For all $\omega^T \in \Omega^T$ $U_n(\omega^T, x_0 - \varepsilon) = U_n(\omega^T, x_0) - \int_{x_0-\varepsilon}^{x_0} U_n'(\omega^T, u)du$. Using that $U_n'(\omega^T, \cdot)$ is non negative and non decreasing (see Definition 5.2.11), we obtain that

$$\begin{aligned} U_n(\omega^T, x_0 - \varepsilon) + \frac{\varepsilon}{2} U_n'\left(\omega^T, x_0 - \frac{\varepsilon}{2}\right) &\leq U_n(\omega^T, x_0 - \varepsilon) + \int_{x_0-\varepsilon}^{x_0-\frac{\varepsilon}{2}} U_n'(\omega^T, v)dv \\ &\leq U_n(\omega^T, x_0 - \varepsilon) + \int_{x_0-\varepsilon}^{x_0} U_n'(\omega^T, v)dv = U_n(\omega^T, x_0). \end{aligned}$$

Thus

$$U_n(\omega^T, x_0 - \varepsilon) \leq U_n(\omega^T, x_0) - \frac{\varepsilon}{2} U_n'\left(\omega^T, x_0 - \frac{\varepsilon}{2}\right).$$

Now

$$U_n'\left(\omega^T, x_0 - \frac{\varepsilon}{2}\right) = U_n'(\omega^T, x_0) - \int_{x_0-\frac{\varepsilon}{2}}^{x_0} U_n''(\omega^T, v)dv \geq -\int_{x_0-\frac{\varepsilon}{2}}^{x_0} U_n''(\omega^T, v)dv,$$

and all together

$$U_n(\omega^T, x_0 - \varepsilon) \leq |U_n(\omega^T, x_0)| + \frac{\varepsilon}{2} \int_{x_0 - \frac{\varepsilon}{2}}^{x_0} U_n''(\omega^T, v)dv. \tag{5.37}$$

We fix some $\eta > 0$ and show now that there exists some $N_\eta > 0$ such that

$$\inf_{P \in \mathcal{Q}^T} P\left(|U_n(\cdot, x_0)| \leq N_\eta\right) > 1 - \frac{\eta}{2}$$

for all n. Indeed using [51, Lemma 13] and Assumption 5.4.2, we get that

$$\sup_{P \in \mathcal{Q}^T} P\left(|U_n(\cdot, x_0)| > k\right) \leq \frac{1}{k} \sup_{P \in \mathcal{Q}^T} E_P\left(|U_n(\cdot, x_0)|\right) \leq \frac{1}{k} \sup_n \|U_n(\cdot, x_0)\|_1.$$

Thus there exists $N_\eta > 0$ such that $\sup_{P \in \mathcal{Q}^T} P\left(|U_n(\cdot, x_0)| > N_\eta\right) < \frac{\eta}{2}$ for all n. From (5.33) with $C = 2(N_\eta + M)$ together with (5.37), there exists $N := N(q, M, \varepsilon, \eta)$ such that for all $n \geq N$,

$$\begin{aligned}
&\inf_{P \in \mathcal{Q}^T} P(U_n(\cdot, x_0 - \varepsilon) \leq -M) \\
&\geq \inf_{P \in \mathcal{Q}^T} P\left(\{|U_n(\cdot, x_0)| \leq N_\eta\} \cap \left\{\int_{x_0 - \frac{\varepsilon}{2}}^{x_0} U_n''(\omega^T, v)dv \leq -\frac{2(N_\eta + M)}{\varepsilon}\right\}\right) \\
&\geq \inf_{P \in \mathcal{Q}^T} P\left(\{|U_n(\cdot, x_0)| \leq N_\eta\}\right) + \inf_{P \in \mathcal{Q}^T} P\left(\left\{\int_{x_0 - \frac{\varepsilon}{2}}^{x_0} U_n''(\omega^T, v)dv \leq -\frac{2(N_\eta + M)}{\varepsilon}\right\}\right) - 1 \\
&> 1 - \eta.
\end{aligned}$$

Thus, (5.36) is proved for all $x_0 > \varepsilon$. Since U_n is (strictly) increasing (5.36) is also true for $x_0 = \varepsilon$ and this concludes the proof. □

The following proposition shows that whatever the strategy is, the wealth is uniformly bounded.

Proposition 5.4.13 *Fix some $x \geq 0$. Assume that Assumptions 5.2.1, 5.2.2, 5.2.3, 5.4.1 and 5.4.2 hold true. Then for all $\phi \in \mathcal{A}(0, x)$ and for all $0 \leq t \leq T$, we have for $\mathcal{Q}^t$-q.s. all $\omega^t \in \Omega^t$ that*

$$|V_t^{x,\phi}(\omega^t)| \leq x \prod_{s=1}^{t} \left(1 + \frac{|\Delta S_s(\omega^s)|}{\alpha_{s-1}(\omega^{s-1})}\right) := x M_t(\omega^t), \tag{5.38}$$

where $M_1 = 1$. Furthermore for all $1 \leq t \leq T$, we have that $M_t \geq 1$, that $M_t, V_t^{x,\phi} \in \mathcal{W}_t^r$ for all $r \in [0, \infty)$ and that for all $P \in \mathcal{Q}^T$ and $n \geq 1$

$$E_P U_n^+(\cdot, V_T^{x,\phi}(\cdot)) \leq K_x, \tag{5.39}$$

where $K_x := \sup_n ||U_n^+(\cdot, x_0)||_1 + x||M_T(\cdot)||_p \sup_n ||U_n'(\cdot, x_0)||_q < \infty$ and where q is defined in Assumption 5.4.2 and p is such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We use similar arguments as in the proof of Theorem 2.4.17 in Chapter 2. We fix $x \geq 0$, $\phi = (\phi_t)_{1 \leq t \leq T} \in \Phi$ such that $\phi \in \mathcal{A}(0,x)$. For all $1 \leq t \leq T$ and $\omega^{t-1} \in \Omega^{t-1}_{NA}$, we denote by $\phi^{\perp}_t(\omega^{t-1})$ the orthogonal projection of $\phi_t(\omega^{t-1})$ on the vector space $D^t(\omega^{t-1})$ (recall Proposition 5.2.5). We have for all $\omega^{t-1} \in \Omega^{t-1}_{NA}$, that

$$\phi_t(\omega^{t-1})\Delta S_t(\omega^{t-1},\cdot) = \phi^{\perp}_t(\omega^{t-1})\Delta S_t(\omega^{t-1},\cdot) \ \mathcal{Q}_t(\omega^{t-1})\text{-q.s.} \tag{5.40}$$

As $V^{x,\phi}_T \geq 0$ $\mathcal{Q}^T$-q.s. and as Assumptions 5.2.1, 5.2.2 and 5.2.3 hold true, Lemma 4.4.7 in Chapter 4 applies together with [99, Lemma 3.4] and we obtain that the set $\mathcal{H}^{t-1} := \{\omega^{t-1} \in \Omega^{t-1},\ V^{x,\phi}_{t-1}(\omega^{t-1}) + \phi_t(\omega^{t-1})\Delta S_t(\omega^{t-1},\cdot) \geq 0\ \mathcal{Q}_t(\omega^{t-1})\text{-q.s.}\}$ is a $\mathcal{Q}^{t-1}$-full measure set. We fix now some $1 \leq t \leq T$, $\omega^{t-1} \in \mathcal{H}^{t-1} \cap \Omega^{t-1}_{NA}$ and we prove that

$$|\phi^{\perp}_t(\omega^{t-1})| \leq \frac{|V^{x,\phi}_{t-1}(\omega^{t-1})|}{\alpha_{t-1}(\omega^{t-1})}. \tag{5.41}$$

If $\phi^{\perp}_t(\omega^{t-1}) = 0$ there is nothing to prove. So we can assume that $\phi^{\perp}_t(\omega^{t-1}) \neq 0$. First, using (5.40), since $\omega^{t-1} \in \mathcal{H}^{t-1} \cap \Omega^{t-1}_{NA}$, we get that

$$V^{x,\phi}_{t-1}(\omega^{t-1}) + \phi^{\perp}_t(\omega^{t-1})\Delta S_t(\omega^{t-1},\cdot) \geq 0 \ \mathcal{Q}_t(\omega^{t-1})\text{-q.s.} \tag{5.42}$$

Now, we proceed by contradiction and assume that (5.41) does not hold true. We set $B := \{\phi^{\perp}_t(\omega^{t-1})\Delta S_t(\omega^{t-1},\cdot) < -\alpha_{t-1}(\omega^{t-1})|\phi^{\perp}_t(\omega^{t-1})|\}$. From Proposition 5.2.5, there exists some $P_\phi \in \mathcal{Q}_t(\omega^{t-1})$ such that $P_\phi(B) > \alpha_{t-1}(\omega^{t-1}) > 0$. But, for all $\omega_t \in B$ we have that

$$V^{x,\phi}_{t-1}(\omega^{t-1}) + \phi^{\perp}_t(\omega^{t-1})\Delta S_t(\omega^{t-1},\omega_t) < |V^{x,\phi}_{t-1}(\omega^{t-1})| - \alpha_{t-1}(\omega^{t-1})|\phi^{\perp}_t(\omega^{t-1})| < 0,$$

a contradiction with (5.42) and therefore (5.41) holds true.
We now establish (5.38) by induction. For $t = 0$ this is trivial. Assume now that for some $t \geq 1$, there exists some $\mathcal{Q}^{t-1}$-full measure set $\widetilde{\Omega}^{t-1} \in \mathcal{B}_c(\Omega^{t-1})$ on which (5.38) is true at stage $t-1$. We denote by

$$\Omega^t_{EQ} := \{(\omega^{t-1},\omega_t) \in \Omega^{t-1} \times \Omega_t,\ \phi^{\perp}_t(\omega^{t-1})\Delta S_t(\omega^{t-1},\omega_t) = \phi_t(\omega^{t-1})\Delta S_t(\omega^{t-1},\omega_t)\}.$$

It is clear that $\Omega^t_{EQ} \in \mathcal{B}_c(\Omega^t)$. For some $P = P_{t-1} \otimes p_t \in \mathcal{Q}^t$, recalling (5.40) and applying Fubini's Theorem (see [13, Proposition 7.45 p175]), we have that

$$\begin{aligned}
P(\Omega^t_{EQ}) &= \int_{\Omega^{t-1}} \int_{\Omega_t} 1_{\Omega^t_{EQ}}(\omega^{t-1},\omega_t) p_t(d\omega_t,\omega^{t-1}) P_{t-1}(d\omega^{t-1}) \\
&= \int_{\Omega^{t-1}_{NA}} \int_{\Omega_t} p_t\left(\phi^{\perp}_t(\omega^{t-1})\Delta S_t(\omega^{t-1},\cdot) = \phi_t(\omega^{t-1})\Delta S_t(\omega^{t-1},\cdot),\omega^{t-1}\right) P_{t-1}(d\omega^{t-1}) \\
&= 1.
\end{aligned}$$

Set $\widehat{\Omega}^{t-1} := \widetilde{\Omega}^{t-1} \cap \mathcal{H}^{t-1} \cap \Omega^{t-1}_{NA}$ and $\widetilde{\Omega}^t = \Omega^t_{EQ} \cap \left(\widehat{\Omega}^{t-1} \times \Omega_t\right)$. It is clear that $\widetilde{\Omega}^t \in \mathcal{B}_c(\Omega^t)$ and is a $\mathcal{Q}^t$-full measure set. We have that for all $\omega^t = (\omega^{t-1}, \omega_t) \in \widetilde{\Omega}^t$

$$\begin{aligned}|V^{x,\phi}_t(\omega^{t-1}, \omega_t)| &= |V^{x,\phi}_{t-1}(\omega^{t-1}) + \phi^{\perp}_t(\omega^{t-1})\Delta S_t(\omega^{t-1}, \omega_t)| \\ &\le |V^{x,\phi}_{t-1}(\omega^{t-1})| \left(1 + \frac{|\Delta S_t(\omega^{t-1}, \omega_t)|}{\alpha_{t-1}(\omega^{t-1})}\right) \\ &\le x M_{t-1}(\omega^{t-1}) \left(1 + \frac{|\Delta S_t(\omega^{t-1}, \omega_t)|}{\alpha_{t-1}(\omega^{t-1})}\right),\end{aligned}$$

where we have used the fact that $\omega^t \in \Omega^t_{EQ}$ for the first equality, $\omega^{t-1} \in \mathcal{H}^{t-1} \cap \Omega^{t-1}_{NA}$ and (5.41) for the second inequality and $\omega^{t-1} \in \widehat{\Omega}^{t-1}$ for the last one: (5.38) is proved. For all $0 \le r < \infty$ and $1 \le s \le T$, as ΔS_s, $\frac{1}{\alpha_s} \in \mathcal{W}^r_s$ (see Assumption 5.4.1), so both M_t and $V^{x,\phi}_t$ belong to $\mathcal{W}^r_t$ for all $1 \le t \le T$.
Fix now some $n \ge 1$. Using the monotonicity, concavity and differentiability of $U_n(\omega^T, \cdot)$, we get for all $\omega^T \in \Omega^T$ that

$$U_n(\omega^T, x) \le U_n(\omega^T, \max(x, x_0)) \le U_n(\omega^T, x_0) + \max(x - x_0, 0) U'_n(\omega^T, x_0).$$

Thus

$$U^+_n(\omega^T, x) \le U^+_n(\omega^T, x_0) + |x| U'_n(\omega^T, x_0). \tag{5.43}$$

And it follows that for all $P \in \mathcal{Q}^T$

$$\begin{aligned}&E_P U^+_n(\omega^T, V^{x,\phi}_T(\omega^T)) \\ &\le \sup_{P \in \mathcal{Q}^T} E_P U^+_n(\cdot, x_0) + \sup_{P \in \mathcal{Q}^T} E_P \left(\left|V^{x,\phi}_T(\cdot)\right| U'_n(\cdot, x_0)\right) \\ &\le \sup_{P \in \mathcal{Q}^T} E_P U^+_n(\cdot, x_0) + x \left(\sup_{P \in \mathcal{Q}^T} E_P \left(M_T(\cdot)\right)^p\right)^{\frac{1}{p}} \left(\sup_{P \in \mathcal{Q}^T} E_P \left(U'_n(\cdot, x_0)\right)^q\right)^{\frac{1}{q}} \\ &\le \sup_n ||U^+_n(\cdot, x_0)||_1 + x ||M_T(\cdot)||_p \sup_n ||U'_n(\cdot, x_0)||_q = K_x < \infty,\end{aligned}$$

where we have used (5.43), (5.38), $M_T \in \mathcal{W}^r_T$ for all $r \ge 1$, Assumption 5.4.2, [51, Proposition 16] (p verifies $\frac{1}{p} + \frac{1}{q} = 1$) and finally, again Assumption 5.4.2 for the last inequality. As this is true for all $P \in \mathcal{Q}^T$ and as K_x does not depend on P and n, (5.39) is proved. □

Lemma 5.4.14 *Assume that Assumptions 5.2.2 and 5.2.3 hold true. Fix some $G \in \mathcal{W}^0_T$, $x \ge 0$ and some random utility function U verifying Definition 5.2.11.*
1. *Assume that $\mathcal{A}(U, G, \pi(G) + x) = \mathcal{A}(G, \pi(G) + x)$. Then $p(G, x) \le \pi(G)$.*
2. *Assume that $\mathcal{A}(U, -G, \pi(-G) + x) = \mathcal{A}(-G, \pi(-G) + x)$. Then $\pi^{sub}(G) \le p^B(G, x)$.*
3. *Assume that Assumptions 5.4.1 and 5.4.2 hold true and that $G \in \mathcal{W}^{0,+}_T$. Then for all $n \ge 1$, $\mathcal{A}(U_n, G, x) = \mathcal{A}(G, x)$ and $u_n(G, x) < \infty$.*

Proof. 1. We apply Theorem 5.2.9 and obtain some $\phi_G \in \mathcal{A}(G, \pi(G))$. As U is non-decreasing we have that

$$\begin{aligned} u(0,x) &= \sup_{\phi \in \mathcal{A}(U,0,x)} \inf_{P \in \mathcal{Q}^T} E_P U(\cdot, V_T^{x,\phi}(\cdot)) \\ &\leq \sup_{\phi \in \mathcal{A}(U,0,x)} \inf_{P \in \mathcal{Q}^T} E_P U\left(\cdot, V_T^{x+\pi(G),\phi+\phi_G}(\cdot) - G(\cdot)\right) \\ &\leq \sup_{\phi \in \mathcal{A}(U,G,x+\pi(G))} \inf_{P \in \mathcal{Q}^T} E_P U\left(\cdot, V_T^{x+\pi(G),\phi}(\cdot) - G(\cdot)\right) \\ &= u(G, x+\pi(G)), \end{aligned}$$

where the second inequality follows from the fact that when $\phi \in \mathcal{A}(U, 0, x) \subset \mathcal{A}(0, x)$, $\phi + \phi_G \in \mathcal{A}(G, \pi(G)+x) = \mathcal{A}(U, G, \pi(G)+x)$ by assumption. So $p(G, x) \leq \pi(G)$ follows from (5.12).
2. Now if $\mathcal{A}(U, -G, \pi(-G)+x) = \mathcal{A}(-G, \pi(-G)+x)$, we obtain that $p(-G, x) \leq \pi(-G)$ and recalling (5.10) and (5.13), we get that $\pi^{sub}(G) \leq p^B(G, x)$.
3. If $\mathcal{A}(G, x) = \emptyset$ then $\mathcal{A}(U_n, G, x) = \emptyset$ and $u_n(G, x) = -\infty < \infty$ for all $n \geq 1$. We assume now that $\mathcal{A}(G, x) \neq \emptyset$. For all $n \geq 1$, using the monotonicity of U_n, the fact that $G \geq 0$ $\mathcal{Q}^T$-q.s., Proposition 5.4.13 (see (5.39)), we get that for any $\phi \in \mathcal{A}(G, x)$ and $P \in \mathcal{Q}^T$

$$E_P U_n^+(\cdot, V_T^{x,\phi}(\cdot) - G(\cdot)) \leq E_P U_n^+(\cdot, V_T^{x,\phi}(\cdot)) \leq K_x < \infty \tag{5.44}$$

Hence the integrals in (5.11) are well defined and we get that $\mathcal{A}(U_n, G, x) = \mathcal{A}(G, x)$. The fact that $u_n(G, x) < \infty$ for all $n \geq 1$ follows immediately (5.44). □

Proof. of Theorem 5.4.8 Since $G \in \mathcal{W}_T^{0,+}$ is such that $G \neq 0$ $\mathcal{Q}^T$-q.s., $\pi(G) > 0$ (see Lemma 5.2.10 and the monotonicity property in Proposition 5.2.18).

We treat first the case where $\pi(G) = +\infty$. We have seen that for all $x \in \mathbb{R}$, $n \geq 1$, $\emptyset = \mathcal{A}(G, x) = \mathcal{A}(U_n, G, x)$ and recalling (5.11), $u_n(G, x_0 + z) = -\infty$ for all $z \in \mathbb{R}$. From Assumption 5.4.2, $\sup_n ||U_n^+(\cdot, x_0)||_1 < \infty$, so that $0 \in \mathcal{A}(U_n, 0, x_0)$ (recall that $x_0 > 0$) for all $n \geq 1$. This implies that for all $n \geq 1$

$$u_n(0, x_0) \geq \inf_{P \in \mathcal{Q}^T} E_P U_n(\cdot, x_0) \geq -\sup_n ||U_n^-(\cdot, x_0)||_1 > -\infty, \tag{5.45}$$

using Assumption 5.4.2 again. Recalling (5.12) we get that $p_n(G, x_0) = +\infty$ for all $n \geq 1$.

We assume now that $\pi(G) < \infty$. Using Lemma 5.4.14 we have that $p_n(G, x_0) \leq \pi(G) < \infty$ for all $n \geq 1$. Thus, to prove that $\lim_{n \to \infty} p_n(G, x_0) = \pi(G)$ it is enough to show that $\liminf_n p_n(G, x_0) \geq \pi(G)$. Assume that this is not the case. Hence we can find a subsequence $(n_k)_{k \geq 1}$ and some $\eta > 0$ such that $p_{n_k}(G, x_0) \leq \pi(G) - \eta$ for all $k \geq 1$. Since $x_0 > 0$ and $\pi(G) > 0$, we may and will assume that $\eta < \pi(G)$ and $\eta < x_0$. By definition of $p_{n_k}(G, x_0)$ we have that

$$u_{n_k}(G, x_0 + \pi(G) - \eta) \geq u_{n_k}(0, x_0).$$

Assume that $\lim_{k \to +\infty} u_{n_k}(G, x_0 + \pi(G) - \eta) = -\infty$ is proved. Then we have that $\liminf_{k \to +\infty} u_{n_k}(0, x_0) = -\infty$. But using Assumption 5.4.2 (see (5.45)) $\liminf_{k \to +\infty} u_{n_k}(0, x_0) > -\infty$, a contradiction.
It remains to prove that $\lim_{k \to +\infty} u_{n_k}(G, y) = -\infty$ with $y = x_0 + \pi(G) - \eta < x_0 + \pi(G)$. For ease of notation, we will prove that $\lim_{n \to +\infty} u_n(G, y) = -\infty$. First we show that $x_0 + G \notin \mathcal{C}_y^T$ (see (5.6) for the definition of $\mathcal{C}_y^T$). Indeed if this is not the case, there exists some $X \in \mathcal{W}_T^{0,+}$ and $\phi \in \Phi$ such that $x_0 + G = V_T^{y,\phi} - X$ $\mathcal{Q}^T$-q.s., hence $G \leq V_T^{y-x_0,\phi}$ $\mathcal{Q}^T$-q.s. Therefore we must have $y - x_0 \geq \pi(G)$: a contradiction. Applying Lemma 5.2.6, we get some $\varepsilon > 0$ such that $\inf_{\phi \in \Phi} \sup_{P \in \mathcal{Q}^T} P(A_\phi) > \varepsilon$, where $A_\phi := \{V_T^{y,\phi}(\cdot) < x_0 + G(\cdot) - \varepsilon\}$. Note that we can always assume that $x_0 \geq \varepsilon$. Hence for all $\phi \in \Phi$, there exists some $P_{\varepsilon,\phi} \in \mathcal{Q}^T$ such that $P_{\varepsilon,\phi}(A_\phi) > \varepsilon$. From Lemma 5.4.14 and Theorem 5.2.9, we get that $\mathcal{A}(U_n, G, y) = \mathcal{A}(G, y) \neq \emptyset$ since $y \geq \pi(G)$. We choose some $\phi \in \mathcal{A}(G, y)$. Using the monotonicity of U_n and recalling (5.39) (as $G(\cdot) \geq 0$ $\mathcal{Q}^T$-q.s., $\phi \in \mathcal{A}(0, y)$) we get that

$$E_{P_{\varepsilon,\phi}} 1_{\Omega^T \setminus A_\phi} U_n(\cdot, V_T^{y,\phi}(\cdot) - G(\cdot)) \leq E_{P_{\varepsilon,\phi}} U_n^+(\cdot, V_T^{y,\phi}(\cdot)) \leq K_y \leq K_{x_0 + \pi(G)}. \tag{5.46}$$

Fix some $J > 0$ and set $C_J := \frac{2}{\varepsilon}\left(J + K_{x_0} + K_{x_0 + \pi(G)}\right)$ and $B_{J,n} := \{U_n(\cdot, x_0 - \varepsilon) \leq -C_J\}$. We apply Lemma 5.4.12 and obtain that there exists some $N_J \geq 1$ (which does not depend on ϕ) such that for all $n \geq N_J$,

$$P_{\varepsilon,\phi}(B_{J,n}) \geq \inf_{P \in \mathcal{Q}^T} P(B_{J,n}) > 1 - \frac{\varepsilon}{2}.$$

It follows that for all $n \geq N_J$, $P_{\varepsilon,\phi}(B_{J,n} \cap A_\phi) > \frac{\varepsilon}{2}$ and we get that

$$\begin{aligned}
E_{P_{\varepsilon,\phi}} 1_{A_\phi} U_n\left(\cdot, V_T^{y,\phi}(\cdot) - G(\cdot)\right) &\leq E_{P_{\varepsilon,\phi}} 1_{A_\phi} U_n(\cdot, x_0 - \varepsilon) \\
&= E_{P_{\varepsilon,\phi}} 1_{A_\phi \cap B_{J,n}} U_n(\cdot, x_0 - \varepsilon) + E_{P_{\varepsilon,\phi}} 1_{A_\phi \setminus B_{J,n}} U_n(\cdot, x_0 - \varepsilon) \\
&\leq -E_{P_{\varepsilon,\phi}} 1_{A_\phi \cap B_{J,n}} C_J + E_{P_{\varepsilon,\phi}} U_n^+(\cdot, x_0) \\
&\leq \frac{-\varepsilon C_J}{2} + K_{x_0} = -J - K_{x_0 + \pi(G)},
\end{aligned}$$

using (5.39) and the definition of C_J. Combining the previous equation with (5.46), we obtain for all $n \geq N_J$ that

$$\inf_{P \in \mathcal{Q}^T} E_P U_n\left(\cdot, V_T^{y,\phi}(\cdot) - G(\cdot)\right) \leq E_{P_{\varepsilon,\phi}} U_n\left(\cdot, V_T^{y,\phi}(\cdot) - G(\cdot)\right) \leq -J.$$

As N_J doesn't depend on ϕ, recalling the definition of u_n (see (5.30)), we obtain for all that $n \geq N_J$, $u_n(y, G) \leq -J$. Since this is true for all $J \geq 0$, $\lim_{n \to \infty} u_n(G, y) = -\infty$ and the proof is complete. □

5.5 Appendix

Proposition 5.5.1 *Let $(X_n)_{n\geq 1}$ and X be $\mathbb{R}^d$-valued and $\mathcal{B}_c(\Omega^T)$-measurable random variables. If $\lim_{n\to\infty} \sup_{P\in\mathcal{Q}^T} P(|X_n - X| > \frac{1}{n}) = 0$, then there exists a sub-sequence $(X_{n_k})_{k\geq 1}$ that converges to X $\mathcal{Q}^T$-q.s. (i.e. on a $\mathcal{Q}^T$-full measure set).*

Proof. Consider the sub-sequence $(X_{n_k})_{k\geq 1}$ such that $\sup_{P\in\mathcal{Q}^T} P(A_k) \leq \frac{1}{2^k}$ where $A_k := \{|X_{n_k}(\cdot) - X(\cdot)| \geq \frac{1}{k}\}$. As $\sum_{k\geq 1} \sup_{P\in\mathcal{Q}^T} P(A_k) < \infty$, using Borel-Cantelli's Lemma for capacity (see [51, Lemma 5]), we obtain that
$\sup_{P\in\mathcal{Q}^T} P(\limsup_k A_k) = 0$. Hence $\Omega^T \backslash \limsup_k A_k$ is a $\mathcal{Q}^T$-full measure set on which $\lim_k X_{n_k}(\cdot) = X(\cdot)$ holds true. □

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Application du contrôle stochastique en théorie de la décision avec croyances multiples et non dominées à temps discret

Cette dissertation traite des thématiques suivantes: incertitude, fonctions d'utilité et non-arbitrage.

Dans le premier chapitre, il n'y a pas d'incertitude sur les croyances. Nous établissons l'existence d'un portefeuille optimal pour un investisseur qui opère dans un marché financier multi-période à temps discret et maximise son espérance terminale d'utilité. Les fonctions d'utilité sont aléatoires, non concaves, non continues et définies sur l'axe réel positif. La preuve repose sur de la programmation dynamique et des outils de théorie de la mesure.

Dans les chapitres suivants nous introduisons le concept d'incertitude Knightienne et adoptons le modèle de marché financier multi-période à temps discret avec croyances multiples et non dominées introduit par [25]. Dans le second chapitre, nous étudions la notion de non-arbitrage quasi-sûre introduite dans [25] et en proposons deux formulations équivalentes: une version quantitative et une version géométrique. Nous proposons aussi une condition forte de non-arbitrage permettant de simplifier des difficultés techniques.

Nous utilisons ces résultats dans le troisième chapitre pour résoudre le problème de la maximisation d'espérance d'utilité, sous la plus défavorable des croyances, pour des fonctions d'utilité concaves, définies sur l'axe positif réel et non-bornées. La preuve utilise de la programmation dynamique et des techniques de sélection mesurable.

Finalement, dans le dernier chapitre, nous développons un modèle d'évaluation par méthode d'indifférence d'utilité et démontrons que sous de bonnes conditions, le prix d'indifférence d'un actif contingent converge vers son prix de sur-réplication.

Incertitude Knightienne, arbitrage, maximisation d'utilité, prix d'indifférence d'utilité, croyances multiples non dominées , programmation dynamique, théorie de la mesure, sélection mesurable, ensemble analytique

Stochastic control applied in the theory of decision in a discrete time non-dominated multiple-priors framework

This dissertation evolves around the following thematics: uncertainty, utility functions and no-arbitrage.

In the first chapter, there is no uncertainty and we establish the existence of an optimal portfolio for an investor trading in a multi-period and discrete-time financial market and maximising its terminal wealth expected utility. We consider general non-concave and non-smooth random utility function defined on the half real-line. The proof is based on dynamic programming and measure theory tools.

In the next chapters, we introduce the concept of Knightian uncertainty and adopt the non-dominated multi-priors framework introduced in [25] in discrete time. We first study in the second chapter the notion of quasi-sure no-arbitrage introduced in [25] and propose two equivalent definitions: a quantitative and geometric characterisation. We also introduce a stronger no-arbitrage condition that simplifies some of the measurability difficulties.

In the third chapter, we build on these results to study the maximisation of non-dominated multiple-priors worst-case expected utility for investors trading in a multi-period and discrete-time financial for general concave utility functions defined on the half-real line and unbounded from above. The proof uses again a dynamic programming framework together with measurable selection tools.

Finally the last chapter formulates a utility indifference pricing model for investor trading in a multi-period and discrete-time financial market. We prove that under suitable conditions the utility indifference prices of a contingent claim converge to its superreplication price.

Knightian uncertainty, arbitrage, utility maximisation, utility indifference prices, multiple-priors, dynamic programing, measure theory, measurable selection, analytic set

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