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Etude sur le contrôle/régulation automatique des systèmes non-linéaires hyperboliques

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Résumé

Dans cette étude on s'intéresse à la dynamique d'une classe de systèmes non-linéaires décrits par des équations aux dérivées partielles (EDP) du type hyperbolique. L'objectif de l'étude est de construire des lois de contrôle par feedback dynamique de la sortie afin de stabiliser le système autour d'un point d'équilibre d'une part, et, d'autre part, de réguler la sortie vers le point de consigne.

Nous considérons la classe des systèmes de dimension infinie gouvernés par des
EDP quasi-linéaires à deux variables indépendantes (une variable temporelle et une variable spatiale). Le modèle mathématique des systèmes prend la forme suivante en termes de EDP :

$$\frac{\partial \Phi(x,t)}{\partial t} + A\big(\Phi(x,t)\big)\frac{\partial \Phi(x,t)}{\partial x} = 0, \qquad x \in \Omega, t > 0$$

où Ω désigne un intervalle connexe borné de R, l'état $\Phi(x,t) \in \mathrm{R}^{\mathrm{n}}$, et la matrice $A\big(\Phi(x,t)\big) \in \mathrm{R}^{n.n}$.
Nous supposons que la matrice $A\big(\Phi(x,t)\big)$ est diagonalisable ou plus simplement diagonale ayant toutes les valeurs propres réelles et non nulles. Pour le bien-posé du système dynamique non seulement l'état initial mais aussi certaines conditions frontières doivent être prescrites en cohérence avec les EDP. Nous supposons que l'observation et le contrôle sont ponctuels. Autrement dit l'action du contrôle intervient dans le système via les conditions frontières et l'observation est effectuée aux points de la frontière. La trajectoire du système évolue en fonction du temps dans un espace fonctionnel défini sur l'intervalle Ω.

Notre étude est motivée par l'observation que de nombreux processus physiques sont modélisés par ce type d'équations EDP. Nous citons, par exemple, des processus tels que flux trafique en transport, flux de gaz dans un réseau de pipeline, échangeurs thermiques en génie des procédés, équations de télégraphe dans des lignes de transmission, canaux d'irrigation en génie civil etc.

Nous commençons l'étude par une EDP non-linéaire scalaire. Dans ce cas-là nous proposons un correcteur intégral stabilisant qui assure la régulation de la sortie avec l'erreur statique nulle. Nous prouvons la stabilisation locale du système non-linéaire par le correcteur intégral en construisant une fonctionnelle de Lyapunov appropriée.

La conception des correcteurs proportionnels et intégraux (PI) que nous proposons est étendue dans un cadre de systèmes de deux EDP. Nous prouvons la stabilisation du système en boucle fermée à l'aide d'une nouvelle fonctionnelle de Lyapunov.

La synthèse des correcteurs PI stabilisants se poursuit dans un cadre de réseaux formés d'un nombre fini de systèmes à deux EDP : réseau étoilé et réseau série en cascade. Les contrôles et les observations se trouvent localisés aux différents nœuds de connexion. Pour ces configurations nous présentons un ensemble de correcteurs PI stabilisants qui assurent la régulation vers le point de consigne. Les correcteurs PI que nous concevons sont validés par des simulations numériques à partir des modèles non-linéaires EDP.

La contribution de la thèse, par rapport à la littérature existante, consiste en l'élaboration de nouvelles fonctionnelles de Lyapunov pour une classe de systèmes stabilisés par correcteur PI. En effet une grande quantité de résultats ont été obtenus sur la stabilisation des systèmes hyperboliques par feedback statique de la sortie. Toutefois il existe encore peu de résultats sur la stabilisation de ces systèmes par feedback dynamique de la sortie. L'étude de la thèse est consacrée sur l'élaboration des fonctionnelles de Lyapunov permettant d'obtenir des correcteurs PI stabilisants. L'approche de Lyapunov direct que nous avons proposée a pour l'avantage de permettre d'étudier la robustesse des lois de feedback de la sortie PI vis-à-vis de la non-linéarité.

Une autre contribution de la thèse consiste en la construction des programmes de Malab permettant d'effectuer des simulations numériques pour la validation des correcteurs conçus. Pour la résolution numérique des EDP hyperboliques nous avons discrétisé nos systèmes par le schéma numérique de Preissmann. Nous avons chaque fois un système d'équations algébriques non-linéaires à résoudre de façon récurrente. L'apport des simulations numériques nous permet de mieux comprendre la méthodologie applicative de la théorie du contrôle en dimension infinie.

Mots-Clés

Systèmes EDP hyperboliques – systèmes non-linéaires – contrôle au bord – feedback dynamique - stabilité – régulation de mesure – PI contrôleur – fonctionnelle de Lyapunov – réseau en étoile – réseau en cascade – simulation numérique – schéma Preissmann.

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Notations

$\mathbb{R}$	The set of real numbers
$\mathbb{R}^{m\times n}$	The set of real matrices of dimension $m \times n$
C^m	Class of functions whose derivatives of order of to m are continuous
$L^2(a,b)$	Class of functions that are square integrable on $[a,b]$
$H^m(a,b)$	Space of absolutely continuous functions whose derivatives of order of to m are in $L^2(a,b)$

Chapter 1

General introduction

1.1 Hyperbolic systems and boundary control

1.1.1 PDE hyperbolic systems of conservation laws

In nature, a large number of phenomena or physical processes are modelized by systems of conservation laws. For example they describe the state evolution of open channels in hydraulic engineering such as the Saint-Venant equations in [12, 19, 56, 68, 77], traffic flow (see the Burger equation or Lighthill-Whitham-Richards model in [7, 33, 49]), gas flow in pipeline networks (see the Euler equations in [8, 9, 27, 34]), heat exchanger processes in [29, 74, 79], and the telegrapher equation of current and voltage along electrical transmission lines in [36], to cite but a few. A very nice literature on the modelling applications by systems of conservation laws can be seen in [4, Chapter 1].

Systems of conservation laws are often considered in three main forms as follows:

- Derivative form:
$$\frac{\partial}{\partial t}\phi(x,t) + \frac{\partial}{\partial x} f(\phi(x,t)) \ = 0$$
where space variable $x \in \mathbb{W}$, time variable $t \in [0, \infty)$, the conserved state $\phi(x,t) \in \mathbb{R}^n$, and the flux function $f : \mathbb{R}^n \to \mathbb{R}^n$.

- Hyperbolic form: in the case that function f from the derivative form above is differentiable, by noting $A(\phi)$ the $n \times n$ Jacobien matrix of function f at point ϕ, we get the following hyperbolic form
$$\frac{\partial}{\partial t}\phi(x,t) + A(\phi) \ \frac{\partial}{\partial x}\phi(x,t) = 0$$

- Integral form: by integrating the derivative form over a given space interval $[x_1, x_2]$ one obtains
$$\frac{\partial}{\partial t}\int_{x_1}^{x_2}\phi(x,t)dx = f\left(\phi(x_1,t)\right) - f\left(\phi(x_2,t)\right)$$
This equation implies that the quantity ϕ is conserved because the total amount of ϕ in any interval $[x_1, x_2]$ can change only due to the flow f at boundary points (see in [14]).

In the thesis, we consider the systems of conservation laws in the hyperbolic form, i.e the dynamic systems governed by 1D homogeneous first-order hyperbolic partial differential equations (PDE) as follows:

$$\begin{aligned}&\frac{\partial}{\partial t}\phi(x,t) + A(\phi)\ \frac{\partial}{\partial x}\phi(x,t) = 0\\ &\phi(0,x) = \phi^0(x)\ ,\end{aligned} \tag{1.1}$$

where $x \in [0, L]$ and $t \in [0, \infty)$ are respectively space and time variables; $\phi : [0, L] \times [0, \infty] \to \mathbb{R}^n$ is the vector state, for each $(x, t) \mapsto \phi(x, t)$ is a mapping which is in $\mathbb{X}$ the state space of the associated Cauchy problem; and matrix $A : \mathbb{X} \to \mathbb{R}^{n\times n}$ such that $A(\phi)$ has n real eigenvalues $\lambda_i(\phi)$ for $i = \overline{1, n}$. In the case that these eigenvalues are all distinct, then system (1.1) is strictly hyperbolic. We distinguish two cases: (i) if $A(\phi)$ does not depend on ϕ, system (1.1) is called linear hyperbolic; (ii) if $A(\phi)$ depends on ϕ, we call (1.1) nonlinear system or quasi linear system.

We consider some examples of PDE hyperbolic equations of conservation laws in details:

- The first one is the Saint-Venant equations (see [63]) derived from the conservation of mass and conservation of linear momentum, which describe an open channel of neglected friction slope,
$$\frac{\partial}{\partial t}\begin{pmatrix}H\\Q\end{pmatrix} + \begin{pmatrix}0 & \frac{1}{B}\\ \frac{Q^2}{BH} + gB & \frac{2Q}{BH}\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}H\\Q\end{pmatrix} = 0\ ,$$
where $H(x, t)$ and $Q(x, t)$ represent the water level and the flow discharge respectively; B and g denote the base width and the Newton gravitation constant. This model will be studied as the direct application with simulations in Chapter 3 for a single system and in Chapter 4 for networks.

- The second one is the Lighthill-Witham-Richards equation (see [49]) which modelizes traffic flow on long road
$$\frac{\partial}{\partial t}\rho + f(\rho)\ \frac{\partial}{\partial x}\rho = 0\ ,$$

with conserved quantity $\rho(x,t)$ represents the vehicle density. This scalar system will be studied more in Chapter 2.

It should be remarked that in many cases, system (1.1) can be diagonalized by an appropriate change of coordinates (usually using Riemann invariants) to obtain a characteristic form. It means that we can find a change of coordinates $\psi(\phi) = \Big(\psi_1(\phi), \cdots, \psi_n(\phi)\Big)$ such that

$$\frac{\partial}{\partial t}\psi(x,t) + D(\psi)\frac{\partial}{\partial x}\psi(x,t) = 0$$

where $D(\psi) = diag\Big(\lambda_1(\psi), \lambda_2(\psi), \cdots, \lambda_n(\psi)\Big)$.
Note also that this change of coordinates always exits if system (1.1) is linear and all eigenvalues of matrix A are non-zero, or if system (1.1) is quasi-linear with distinct eigenvalues of A and $n = 2$, see in [4]. This characteristic form will be used in Chapter 3 and Chapter 4 for linear hyperbolic systems.

1.1.2 Boundary control problem

One of the popular topic in studying PDE hyperbolic systems is concerning boundary conditions. Firstly, the boundary conditions play an important role to guarantee the existence and unique solution for the PDE hyperbolic system (1.1) in the state space $\mathbb{X}$. It implies that in our case, special conditions on the boundary interval $x \in [0, L]$ need to be added. In addition, these boundary conditions are also useful to control PDE hyperbolic systems, for example to obtain the stability of un equilibrium for controlled closed-loop systems, or the output regulation with the presence of disturbances.

To be more specific, we consider system (1.1) with the following boundary condition

$$\mathbb{F}\Big(\phi(0,t), \phi(L,t), U(t)\Big) = 0 \tag{1.2}$$

where $\mathbb{F} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$ is C^1 function which is obtained from physical constraints on the system (see [4, 44]). The function $U(t) \in \mathbb{R}^q$ denotes the feedback control action on the boundary. We distinguish here two kinds of control action U:

- Static state feedback control law: if the control action is a function of state on the boundary, i.e $U(t) = U\Big(\phi(0,t), \phi(L,t)\Big)$ (see for instantce [20, 22, 37, 38, 46]). The kind of control is not robust to disturbances (for example constant disturbances).
- Dynamic state feedback control law: if some other dynamic terms are added in the control action, i.e $U(t) = g\Big(\phi(0,t), \phi(L,t), z(t)\Big)$ where $\dot{z}(t) = h\Big(\phi(0,t), \phi(L,t)\Big)$, with h and g are functions in $\mathbb{R}^q$ (see for example [3, 55, 60, 68, 77]). With this

control type, the closed-loop system is more complicated because it becomes a couple system of PDE and ordinary differential equation (ODE). However, and as this is already the case for finite dimensional controled system, adding this new freedom degree in the control law allows to extend the class of system for which output feedback stabilization may be achieved. Moreover, it may increase the robustness of the closed loop system. This robustness aspect is the aspect in which we are interested in the following.

To resume, the general objective of boundary control problem is to find boundary conditions in the form (1.2) such that:

- **the well-posed problem :** System (1.1)- (1.2) has a unique solution in a normed vector state space $\mathbb{X}$ with the norm denoted by $|\cdot|_{\mathbb{X}}$.
- **The stability problem :** The equilibrium point of system (1.1)- (1.2) is stable in one of three kinds as follows:
  - Globally asymptotically stable in $\mathbb{X}$ if there exists a class $\mathcal{KL}$ function β such that for all initial condition φ in $\mathbb{X}$ for which there exists a (weak) solution denoted $\phi(x,t)$
$$|\phi(x,t)|_{\mathbb{X}} \leqslant \beta(|\varphi|_{\mathbb{X}},t) \ , \ \forall t \geqslant 0.$$
  - Locally asymptotically stable in $\mathbb{X}$ if there exists a positive real number d such that former property is true only for $|\varphi|_{\mathbb{X}} \leqslant d$
  - Locally and globally Exponentially stable if there exists k and λ such that the function $\beta(s,t) = k\exp(-\lambda t)s$.

On top of these two classical problems in boundary control problem, we can consider another one which is the problem of robustness with respect to external disturbances. This aspect will be addressed in the thesis and precisely defined in the following Section.

Let us now mention some important results in the literature concerning the boundary control problem.
Almost all results of boundary control for general quasi-linear systems are given by static control law. In this case, supposing that the eigenvalues $\lambda_i > 0$ for $i = \overline{1,k}$ and $\lambda_i < 0$ for $i = \overline{k+1,n}$, boundary condition (1.2) with static state feedback control law is written under the following form (see in [22])

$$\begin{pmatrix} \phi_+(0,t) \\ \phi_-(L,t) \end{pmatrix} = G \begin{pmatrix} \phi_+(L,t) \\ \phi_-(0,t) \end{pmatrix} \tag{1.3}$$

where the $G : \mathbb{R}^n \to \mathbb{R}^n$ satisfies that $G(0) = 0$, $\phi_+ \in \mathbb{R}^k$ and $\phi_-(L,t) \in \mathbb{R}^{n-k}$ are two vector elements of ϕ. Many interesting results in the literature are based on finding sufficient conditions for G in (1.3) such that the equilibrium point $\phi \equiv 0$ is exponentially stable in terms of some norms.

The first result in this topic is given by Li in [46], and de Halleux et al. in [35] by the method of characteristic. A sufficient condition for the exponential stability in norm $C^1(0,L)$ is introduced as follows: *The equilibrium point $\phi \equiv 0$ of quasi-linear system (1.1) and (1.3) is exponentially stable in the state space $C^1(0,L)$ if the following sufficient condition is satisfied:*

$$\rho_\infty\Big(G'(0)\Big) < 1 \tag{1.4}$$

where

$$\rho_\infty : \mathbb{R}^{n\times n} \to \mathbb{R}\ ,\ \rho_\infty(P) := Inf\left\{R_\infty(\triangle P \triangle^{-1}),\ \triangle \in \mathbb{D}_{n,+}\right\}$$

with $R_p(K) := Max_{||\sigma||_p=1}\{\ ||K\sigma||_p\}$, and $\mathbb{D}_{n,+}$ is space of all diagonal matrix with positive diagonal terms; here $||\sigma||_p := \Big(\sum\limits_{j=1}^{n} \sigma_i^p\Big)^{\frac{1}{p}}$ and $1 \leqslant p \leqslant \infty$.

Regarding the sufficient condition (1.4), in [22], the authors give another proof by using Lyapunov direct method to prove the exponential stability of $\phi \equiv 0$ in also C^1 norm.

Another remarkable result is developed by Coron et al. in [20]. In this work, another sufficient boundary condition is proposed to have the stability of the equilibrium solution $\phi \equiv 0$ in the norm $H^2(0,L)$: *The equilibrium solution $\phi \equiv 0$ of quasi-linear system (1.1) and (1.3) is exponentially stable in the state space $H^2(0,L)$ if we have the following sufficient condition:*

$$\rho_2\Big(G'(0)\Big) < 1 \tag{1.5}$$

where

$$\rho_2 : \mathbb{R}^{n\times n} \to \mathbb{R}\ ,\ \rho_2(P) := Inf\left\{R_2(\triangle P \triangle^{-1}),\ \triangle \in \mathbb{D}_{n,+}\right\}$$

It is proved in [20] that the condition in (1.5) is weaker than the one in (1.4), i.e $\rho_2\Big(G'(0)\Big) \leqslant \rho_\infty\Big(G'(0)\Big)\ \forall n \geqslant 1$.

In the particular case that system (1.1) is linear where A is a nonzero distinct diagonal matrix and boundary condition (1.3) becomes

$$\begin{pmatrix} \phi_+(0,t) \\ \phi_-(L,t) \end{pmatrix} = K \begin{pmatrix} \phi_+(L,t) \\ \phi_-(0,t) \end{pmatrix}$$

with $K \in \mathbb{R}^{n\times n}$, it should be mentioned the result of Silkowski (see in [64]) and also of Hale and Verduyn Lunel in [38]), in which they gave a necessary and sufficient boundary condition for the exponential stability of the zero equilibrium point $\phi \equiv 0$ in the norm $L^2(0,L)$: *The equilibrium point $\phi \equiv 0$ of linear system is exponentially stable in the L^2*

norm if and only if

$$\rho_3(K) := max\Big(\mathbb{S}\left(diag(e^{j\theta_1}, \cdots, e^{j\theta_n})K\right); \theta_i \in \mathbb{R}\ \forall i = \overline{1,n}\Big) < 1\ , \tag{1.6}$$

where $j \in \mathbb{C}$ *and* $j^2 = -1$*, and* $\mathbb{S}(N)$ *is the spectral radius of matrix* $N \in \mathbb{C}^{n\times n}$*.*
Note that in general $\rho_3(K) \leqslant \rho_2(K)$ (see in [22]).

Concerning boundary control problem of hyperbolic systems with dynamic control laws, in the case of quasi-linear systems it is still an open question up to now to find appropriate conditions for exponential stability of the equilibrium points. There exist only a few results for some particular systems such as:

- nonlinear scalar systems (i.e $n = 1$) in [65, Chapter 5] by using Proportional-Integral (PI) control with damping terms, and in [57] by applying a boundary integrator with predictive control.
- 2×2 quasi-linear systems (i.e $n = 2$) in [60] with a dynamic boundary control called finite-time stabilizer.

Most of the stabilization results with dynamic control laws in the literature are concerned with linear hyperbolic systems inspired from the studies of general infinite-dimensional linear systems with implementation of integral control action (see [58] and [62]). Most results have been developed in recent publications for linearized hyperbolic systems: the predictive control combined with integral action for a class of 2×2 linear hyperbolic systems and their cascaded network in [55, 56]; multivariable PI control design for some classes of 2×2 systems by using Lyapunov techniques in [24, 68], and integral controller design by using Laplace transformation frequency method in [3, 21], and integral control laws by using operator and semi-group approach in [71, 77]. The backstepping method has been exploited in [26, 43] to elaborate output regulators in the same context. These approaches will be considered in details in Section 1.3.

1.2 Output regulation with PI controllers and motivation

1.2.1 Output regulation problem with PI control design

In the above Section, we have described the boundary control problem with main objective as stabilization of closed-loop systems with boundary control feedback. In this Section, we consider another problem which is the output regulation, i.e., the output

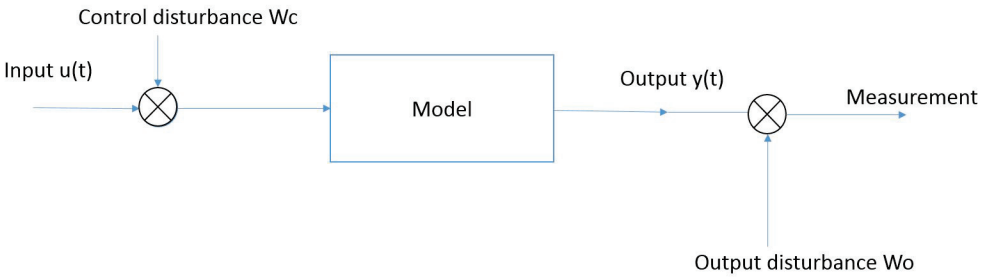

FIGURE 1.1: Example of disturbances in practical model

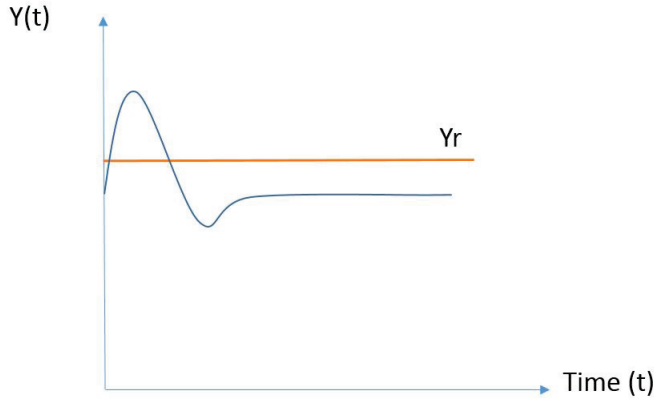

FIGURE 1.2: Error between output $y(t)$ and reference y_r

measurement needs to track a reference defined a priori despite of the presence of constant disturbances. Therefore, the control input must be designed such that the controlled disturbed system is robust in the senses that the output measurement is regulated to the desired reference.

At first, we explain how the constant disturbances lead us to considered output regulation problem. As a matter of fact, the disturbances are inevitable in practical model, that may come from the error of modelling, linearisation, sensors,... as in Figure 1.1. These disturbances result in errors between the measured output noted by $y(t) \in \mathbb{R}^q$ and the references $y_r \in \mathbb{R}^q$: $e(t) = y(t) - y_r$, see in Figure 1.2. Our task is to find input control law $u(t) \in \mathbb{R}^q$ such that $\lim_{t\to\infty} y(t) = y_r$ (i.e $\lim_{t\to\infty} ||e(t)||_{\mathbb{R}^q} = 0$).

In industrial applications, the PI controller is considered as one of the most effective approaches to remove steady state errors and oscillations of the controlled systems, see in [1]. This motivates us to introduce the integral action control in the form of PI controllers to resolve the output regulation problem. A PI controller admits the general form as follows:

$$u(t) = K_P\Big(y(t) - y_r\Big) + K_I \int_0^t \Big(y(\tau) - y_r\Big)d\tau + w_c \ ,$$

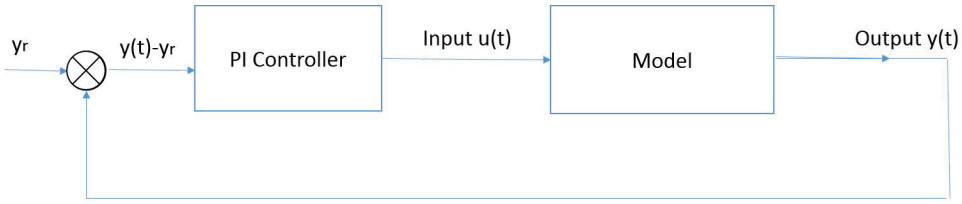

FIGURE 1.3: Schema of controlled systems with PI controllers

where K_P and K_I are respectively proportional matrix gain and integral matrix gain to be designed, w_c is constant control disturbance.
The ideas of designing PI control for PDE hyperbolic systems have been introduced in the literature by the works of Pohjolainen, Xu and Jerbi [58, 74] for some classes of linear hyperbolic systems. Later on, many results on the same topic were developed for nonlinear hyperbolic systems [65, 67] and for linear systems [3, 24, 68, 77] with different techniques that will be analysed in Section 1.3.

1.2.2 Motivation

The motivation here is employing PI control as a dynamic feedback control law to resolve simultaneously two tasks: boundary feedback stabilization problem and output regulation problem, i.e., (i) the asymptotic stabilization of closed-loop system controlled by PI controllers, (ii) the output regulation to the desired references.
The application of PI controllers in controlled models is illustrated in Figure 1.3. More precisely we consider the PDE hyperbolic system (1.1) with the boundary condition (1.2), and the output measurement taken on the boundary, i.e.,

$$y(t) := l\Big(\phi(0,t), \phi(L,t), w_o\Big)$$

where $w_o \in \mathbb{R}^q$ is the constant output disturbance, function $l: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^q$. We write the control law by

$$u(t) = K_P\Big(l\left(\phi(0,t), \phi(L,t), w_o\right) - y_r\Big) + K_I z(t) + w_c\ ,\ \dot{z}(t) = l\Big(\phi(0,t), \phi(L,t), w_o\Big) - y_r$$

The main objective here is to find the conditions for boundary function l and matrix gains K_P, K_I such that the equilibrium point of the coupled PDE/ODE closed-loop system is asymptotically stable, and $y(t) \to y_r$ as $t \to \infty$.
In the next Section we will give a general introduction about some popular methods to prove the stability of the the coupled PDE/ODE closed-loop systems.

1.3 Stability analysis methods in the literature

1.3.1 Lyapunov direct method

The Lyapunov direct approach called also the second method of Lyapunov in [52] is widely used to analyze the stability of hyperbolic PDE systems in both linear cases and nonlinear ones. The advantage of this method is: (i) The stability proof is less complicate because of using more direct computations compared to other approaches; and (ii) it gives a direct extension from linear systems to nonlinear systems (note that in PDE systems, the stability of linearized models does not imply the one of nonlinear systems with even a very small variation, see in [22]).
The pioneer result of using Lyapunov direct method for 1D hyperbolic PDE systems goes back to [19] where entropies are used as Lyapunov functionals. By this Lyapunov functional type, their time derivatives are only semi-definite negative. Therefore the precompactness of trajectories is required to apply the LaSalle invariant principle to get the asymptotic stability. However, this condition is difficult to obtain in nonlinear PDEs (see [22]). Following this, the strict Lyapunov functionals [18, 20, 22, 24, 61, 67] are proposed to avoid the drawback of the former Lyapunov functionals so that the time derivative along the trajectories is strictly definite negative. To be detailed, let denote extended state $\varphi(.,t)$ of extended closed-loop systems in state space $\mathbb{X}$, which is equipped with a norm $||.||_{\mathbb{X}}$. One must construct a strict Lyapunov functional $V(\varphi(t)) \in C^1(\mathbb{X},\mathbb{R})$ such that with the boundary conditions the followings hold:

- There exists a constant $M > 0$ such that

$$\frac{1}{M}||\varphi||_{\mathbb{X}} \leqslant V(\varphi) \leqslant M||\varphi||_{\mathbb{X}}$$

- There exists some constant $\alpha > 0$ such that the time derivative of $V(\varphi(t))$ along the trajectories of the system satisfies

$$\frac{dV(\varphi(t))}{dt} \leqslant -\alpha V(\varphi(t))$$

A large number of works using the strict Lyapunov functional technique can be found in the literature. Some Lyapunov functionals of nonlinear hyperbolic systems with static state feedback control laws, in which the Lyapunov inequalities are only derived with $||\varphi||_{\mathbb{X}}$ small enough have been considered in [18, 20, 22] equivalent to H^2-norm or C^1-norm. Other Lyapunov functionals (equivalent to H^2-norm) have been proposed in [65, 67] for scalar nonlinear hyperbolic systems with integral control action. In the linear case, Lyapunov functionals (equivalent to L^2 norm) for a general class of linear

symmetric hyperbolic systems with static boundary control have been constructed [78], for the 2×2 linear hyperbolic systems with boundary PI control [24, 43, 68], and for some networks of linear hyperbolic systems in [6, 69].
The Lyapunov direct method is the principal tool utilized along this thesis.

1.3.2 Frequency method with Laplace transform

Another method to be mentioned here is the frequency analysis method with Laplace transform. This method is only used for liner hyperbolic systems, and has the advantage that we can establish necessary and sufficient conditions for the stability analysis. The principal idea is that, by employing Laplace transform, one gets a characteristic equation (i.e, denominator of the transfer function). In the case of 1D linear hyperbolic system, the controlled system is exponentially stable if and only if all roots of the characteristic equation (called poles of the system) have negative real parts (see [21, 48]). Some works with this approach can be found in [4, Chapter 2] for hyperbolic systems of two conservation laws with static boundary control, for the control of 2×2 linear hyperbolic systems with a PI control action in [3], and for linear scalar hyperbolic systems with integral control action in [21, 67].
As a matter of fact, the closed-loop systems in these works can be considered as the delay-differential equations of neutral type, and we have the characteristic equation in the form of a polynomial which contains some exponential terms. To guarantee all roots of a characteristic equation to be in the left half-plane, one can apply the direct computation as in [67, Appendix], or some results presented in [38, Appendix] and [11, Chapter 13], or a variant of the Walton-Marshall procedure as in [3, 73].
The frequency method will be used in Section 2.3.1.1 Chapter 2 to study exponential stability of the linear scalar hyperbolic system with integral controller.

1.3.3 Operator and semi-group method

Concerning the stability analysis of linear hyperbolic systems, operator and semi-group approach is also a well-known approach. Indeed, this approach is developed for general linear infinite-dimensional systems, and therefore applied for linear hyperbolic systems to determine stabilities. In general, the closed-loop systems is written by the extended system $\dot{\varphi} = \mathcal{A}\varphi$ where φ is extended state containing all state and dynamic terms of control action, and the linear operator $\mathcal{A}$ is the generator of a C_0 semi-group on the extended state space of the controlled system. Then, with the boundary control action and characteristics of the original system, one may prove that $\mathcal{A}$ generates an exponentially stable C_0 semigroup on the extended state space.

The approach has been made use of to construct robust proportional integral (PI) controllers in [58] for a class of infinite holomorphic semigroup systems, and then for strongly continuous semigroup systems in [59, 74]. The method has been applied to a particular model of 2×2 hyperbolic systems governed by the Saint-Venant equations with constant control in [76], and with PI control in [77].

1.3.4 Backstepping method

The last method mentioned to analyze the stability of hyperbolic systems with boundary control is so-called backstepping technique. A nice introduction about this approach can be found in [40] and [4, Chapter 7]. The key idea of the method is that, one needs to find a coordinate transformation along with an adaptive mapping for control action to transform the controlled system to a target system for which the stability condition is well-known a priori. From the stability criteria of the target system, by inverse transformation, one can find the solution of boundary control for the original system. Many results in the literature have been developed based on this technique, such as in [72] for the static boundary control of a class of 2×2 linear hyperbolic systems, or in [26, 42, 43] for integral control actions of some class of 2×2 linear hyperbolic systems.

1.4 Contribution of the thesis

In this thesis, we consider the boundary control and output regulation problems with dynamic control laws for some classes of PDE hyperbolic systems as follows:

$$\begin{aligned}
&\frac{\partial}{\partial t}\phi(x,t) + A(\phi)\ \frac{\partial}{\partial x}\phi(x,t) = 0\\
&\phi(0,x) = \phi^0(x)\ ,\\
&\mathbb{F}\Big(\phi(0,t), \phi(L,t), U(t)\Big) = 0\\
&y(t) = l\Big(\phi(0,t), \phi(L,t), w_o\Big)
\end{aligned}$$

Firstly, in Chapter 2, we consider a nonlinear scalar hyperbolic system where $A(.) > 0$, $\phi \in H^2(0,L)$, $\mathbb{F}\Big(\phi(0,t), \phi(L,t), U(t)\Big) = \phi(0,t) - U(t)$, and the disturbed measured output $y(t) = \phi(L,t) + w_o$ with unknown constant disturbance w_o. The aim of the chapter is to design an integral controller to guarantee asymptotic stability of the equilibrium of the nonlinear closed-loop system, and the output regulation to a given set-point. The main result of the chapter is to propose a systematic design of boundary integral output feedback controllers with small integral gain. The main result is firstly validated on the

the linearized model by using frequency domain method with Laplace transformation. Then, we construct a new Lyapunov functional to prove the exponential stabilization of the nonlinear closed-loop system with the designed integral controller. The regulation effect with zero static error offered by the integral action is automatically guaranteed once the stabilization is obtained. Furthermore the Preissmann scheme is used to discretize the nonlinear PDE model and numerical simulations are carried out to validate the designed integral controllers for the nonlinear closed-loop system.

Secondly, in Chapter 3, we are concerned with a class of linear 2×2 hyperbolic systems with multivariable inputs and multivariable outputs on the boundary, where $\phi = (\phi_1, \phi_2) \in (L^2(0,L))^2$, $A(\phi) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}$ with two real positive constants λ_1 and λ_2, $\mathbb{F}\Big(\phi(0,t), \phi(L,t), U(t)\Big) = \begin{pmatrix} \phi_1(0,t) - R_0\phi_2(0,t) - u_1(t) \\ \phi_2(L,t) - R_L\phi_1(L,t) - u_2(t) \end{pmatrix}$ with two constants R_0 and R_L, and output measurement $y(t) = \begin{pmatrix} a\phi_1(0,t) + b\phi_2(0,t) \\ c\phi_1(L,t) + d\phi_2(L,t) \end{pmatrix} + y_r$ with constants a, b, c, d such that $abcd < 0$. Our contribution is to design explicit multivariable PI controllers for a general class of 2×2 hyperbolic systems, and to propose a proof of the stability and regulation of the closed-loop system by using a novel Lyapunov functional. Then we apply the designed PI controller for a fluid flow model and carry out numerical simulations to validate the theoretical results.

In addition, in Chapter 4, two kinds of network are studied: star-shaped form and cascaded one of n linear 2×2 hyperbolic systems. The boundary conditions and output measurements depend on the network type. The contribution of the chapter is: (i) modelling each network in details with corresponding boundary conditions and output measurements; (ii) proposing systematic PI controllers for each network; (iii) constructing new strict Lyapunov functionals to prove the exponential stability of closed-loop networks with the output regulation guaranteed; (iv) applying PI control designs for corresponding practical models and implementing numerical simulations to study the performances of designed PI controllers.

Finally, in Appendix, we develop a numerical Preissmann scheme that can be used for general PDE hyperbolic systems of conservation laws. Based on an Preissmann scheme, we discretize the PDE models to get an implicit discrete equation. Then we use Newton Raphson method to find numerical solution of the discrete equation. This approach is employed in details to carry out all the simulations in Chapters 2-4.

1.5 Publications

During the thesis, the following publications have been realized:

1. N.-T Trinh, V. Andrieu, and C.Z. Xu. Design of integral controllers for nonlinear systems governed by scalar hyperbolic partial differential equations. *IEEE Transactions on Automatic Control*, vol. 62, 2017.

2. N.-T. Trinh, V. Andrieu, and C.Z. Xu. Stability and output regulation for a cascaded network of 2x2 hyperbolic systems.
   Provisionally accepted to Automatica, 2017.

3. N.-T. Trinh, V. Andrieu, and C.Z. Xu. Boundary PI controllers for a star-shaped network of 2 x 2 systems governed by hyperbolic partial differential equations. In *Proceeding of 20th IFAC World Congress*, Toulouse, France, 2017.

4. N.-T. Trinh, V. Andrieu, and C.Z. Xu. Multivariable PI controller design for 2 x 2 systems governed by hyperbolic partial differential equations with Lyapunov techniques. In *Proceeding of 55th IEEE Conference on Decision and Control (CDC)*, pp. 5654-5659, Las Vegas, USA, 2016.

5. N.-T. Trinh, V. Andrieu, and C.Z. Xu. PI regulation control of a fluid flow model governed by hyperbolic partial differential equations. In *International Conference on Systems Engineering*, Coventry, England, 2015.

Chapter 2

Scalar hyperbolic systems with boundary integral controllers

2.1 Introduction

The chapter deals with the control and regulation by integral controllers for the nonlinear systems governed by scalar quasi-linear hyperbolic partial differential equations. Both the control input and the measured output are located on the boundary. This type of systems appear in many industrial applications and in study of traffic flow. For instance, scalar quasi-linear hyperbolic equations include Burgers equations [17] which are employed in modeling turbulent fluid motion. Another example is given by the equation employed by Lighthill-Whitham in [49] to describe traffic flow on long crowded roads. Finally scalar conservation laws can also be regarded as a particular simpler case of quasi-linear hyperbolic systems under some regularity assumptions (see for instance [4, 13]).

In this work, we consider a 1D scalar conservation law with strictly increasing flux. We are interested in the classical solutions of the system around an equilibrium state. The objective is to propose a boundary integral output feedback controller to asymptotically stabilize the nonlinear closed-loop system and regulate the disturbed output measurement to the desired reference. The local stability and the regulation effect for the nonlinear closed-loop system are proven in the H^2 topology by using Lyapunov techniques. To be more detailed, the closed-loop stabilization of the linearized model with the designed integral controller is proved first by using the method of spectral analysis and then by the Lyapunov direct method. Based on the elaborated Lyapunov function inspired from [18], we prove local exponential stability of the nonlinear closed-loop system with the same controller. The output regulation to the set-point with zero static error by the

integral controller is shown upon the nonlinear system. Then, numerical simulations by the Preissmann scheme are carried out to validate the robustness performance of the designed controller in nonlinear closed-loop PDE system to face to unknown constant disturbances.

The chapter is organized as follows. Section 2.2 is devoted to the statement of the regulation problem and the announcement of the main result. Section 2.3 presents the proof of the main result. More precisely, in Section 2.3.1 asymptotic stabilization of the linearized system is considered by proposing a Lyapunov functional; In Section 2.3.2 the stabilization problem is solved for the nonlinear system by extending the proposed Lyapunov functional. Numerical simulations to validating the theoretical results are implemented in Section 2.4 by applying the Preissmann numerical scheme. Finally Section 2.5 is devoted to our conclusions.

2.2 Problem statement and existing results

2.2.1 Statement of the problem

In this chapter, we consider a 1D quasi-linear hyperbolic system of the form :

$$\frac{\partial \psi}{\partial t}(x,t) + \widetilde{F}(\psi(x,t))\frac{\partial \psi}{\partial x}(x,t) = 0,\ x \in (0,L),\ t \in \mathbb{R}_+, \tag{2.1}$$

where L is a positive constant, $\psi : (0,L) \times \mathbb{R}_+ \to \mathbb{R}$ is the state in $C([0,\infty), H^2(0,L))$, and $\widetilde{F} : \mathbb{R} \to \mathbb{R}$ is a C^2 function such that $\widetilde{F}(\sigma) > 0\ \forall\ \sigma \in \mathbb{R}$. The initial condition is given by $\psi(\cdot,0) = \psi_0 \in H^2(0,L)$. Notice that $H^2(0,L)$ is the usual Sobolev space defined by

$$H^2(0,L) = \{f \in L^2(0,L) \mid f', f'' \in L^2(0,L)\}$$

where $L^2(0,L)$ denotes the usual Hilbert space of square summable functions on the open set $(0,L)$. The Sobolev space $H^2(0,L)$ is normed by

$$\|f\|_{H^2}^2 = \int_0^L (|f(x)|^2 + |f'(x)|^2 + |f''(x)|^2)dx$$

$$\forall\ f \in H^2(0,L).$$

We consider the control u on the boundary $x = 0$, i.e.,

$$\psi(0,t) = u(t)\ ,\ t \in \mathbb{R}_+\ .$$

The output we wish to regulate is also located on the boundary and eventually corrupted by an additive unknown disturbance, i.e.

$$y(t) = \psi(L,t) + w_o \ , \ t \in \mathbb{R}_+ \ ,$$

where $w_o \in \mathbb{R}$ is an unknown constant. Our control objective is to design a dynamic output feedback control law in order to achieve asymptotic stabilization of the closed loop system and to ensure that *the output* $y(t)$ converges to a desired set-point $y_r \in \mathbb{R}$, as $t \to \infty$.

In our study, the control action $u(t)$ has the structure of an integral controller. We assume that an unknown constant disturbance may corrupt the control. Hence we write the control law as follows

$$u(t) = -k_I \zeta(t) + w_c \ , \dot{\zeta}(t) = y(t) - y_r$$

where $w_c \in \mathbb{R}$ is an unknown constant and k_I is a positive constant called tuning parameter.

To summarize, the closed-loop system with disturbances is governed by the following PDE :

$$\begin{cases} \dfrac{\partial \psi}{\partial t}(x,t) &= -\widetilde{F}(\psi(x,t))\dfrac{\partial \psi}{\partial x}(x,t) \\ \dot{\zeta}(t) &= \psi(L,t) - y_r + w_o \\ \psi(0,t) &= -k_I \zeta(t) + w_c \\ \psi(x,0) &= \psi_0(x), \quad \zeta(0) = \zeta_0. \end{cases} \tag{2.2}$$

We are studying a nonlinear infinite-dimensional system controlled by an integral controller faced with unknown constant disturbances on the control and the output. The purpose of the chapter is to find sufficient conditions on the control parameter $k_I > 0$ such that the three objectives are realized : (a) the closed-loop system (2.2) is well posed; (b) asymptotic stability of the closed-loop system is guaranteed; and (c) the regulation property holds

$$\lim_{t \to \infty} |y(t) - y_r| = 0. \tag{2.3}$$

As only the classical solutions are considered, in the following we restrict ourselves to study the solutions from the initial data (ψ_0, ζ_0) in $H^2(0,L) \times \mathbb{R}$ which satisfy the C^0 and C^1 compatibility conditions :

$$\begin{cases} \psi_0(0) &= -k_I \zeta_0 + w_c \\ \widetilde{F}(\psi_0(0))\psi_0'(0) &= k_I(\psi_0(L) - y_r + w_o). \end{cases} \tag{2.4}$$

2.2.2 Existing results

There are many results in literature concerning this topic. The first work to be mentioned is in [21], in which a PI controller with damping for linear scalar PDE hyperbolic systems is addressed with Lyapunov direct method and frequency method. By frequency analysis, the authors also prove that PID controllers in general with the derivative terms can not be applied because of the instability of closed-loop system. Moreover in [65, Chapter 5], the same PI controller with damping is extended for nonlinear scalar PDE hyperbolic systems. However in both works, they loose the regulation effect because of the damping required in the PI controller.

It is also interesting to note that asymptotic stabilization of entropy solutions to scalar conservation laws has been recently studied by Perrollaz in [54] and by Balandin *et al.* in [10]. In particular they have considered the stabilization problem of weak entropy solutions by boundary control and internal control around a constant equilibrium state for a scalar 1D conservation law with strictly convex flux. In [54] an internal state feedback control law has been designed to asymptotically stabilize the entropy solutions around a constant equilibrium in the topologies L^1 and L^∞. A stabilizing nonlocal boundary control law (depending on time and the whole initial data) has been proposed in [10] to get asymptotic stability of the constant equilibrium in the L^2 topology.

2.3 Main result and its proof

To be simple the initial data with the compatibility condition satisfied up to the required order are called compatible initial data throughout the chapter. From now on, the state space X for (2.2) is the Hilbert space $X = H^2(0, L) \times \mathbb{R}$ equipped with the norm $\|(f, z)\|_X^2 = \|f\|_{H^2}^2 + z^2$. Note that due to the constants w_0, w_c and y_r, $(\psi, \zeta) = (0, 0)$ is not a steady state of the closed-loop system. In fact the equilibrium denoted $(\psi_\infty, \zeta_\infty)$ is defined as follows

$$\psi_\infty = y_r - w_o, \quad \zeta_\infty = k_I^{-1}(w_o + w_c - y_r). \tag{2.5}$$

Let $B_X((f, z), \delta)$ denote the open ball in X centered at (f, z) with radius $\delta > 0$, i.e.,

$$B_X((f, z), \delta) = \{(\psi, \zeta) \in X \mid \|(\psi, \zeta) - (f, z)\|_X < \delta\}.$$

Then the main result of the chapter is stated as follows.

Theorem 2.1. *There exist positive real constants k_I^* and δ such that, for each $k_I \in (0, k_I^*)$, and for every $(y_r, w_o, w_c) \in \mathbb{R}^3$ and every compatible $(\psi_0, \zeta_0) \in B_X((\psi_\infty, \zeta_\infty), \delta)$, the following assertions hold true :*

1. *The closed-loop system (2.2) has a unique solution $(\psi, \zeta) \in C([0, \infty), X)$;*
2. *The solution of the closed-loop system (2.2) converges exponentially to the equilibrium state $(\psi_\infty, \zeta_\infty)$ in the state space X as $t \to \infty$, and the disturbed output is regulated to the desired set-point y_r, i.e.,*

$$\lim_{t\to\infty} |y(t) - y_r| = 0 \ .$$

3. *There exist real constants $M > 0$ and $\omega > 0$ such that*

$$\|(\psi(\cdot, t) - \psi_\infty, \zeta(t) - \zeta_\infty)\|_X \leqslant Me^{-\omega t}\|(\psi_0 - \psi_\infty, \zeta_0 - \zeta_\infty)\|_X \ \ \forall t \geqslant 0.$$

Remark : The equilibrium state is some constant state determined by y_r, w_o and w_c. Though it is unknown *a priori*, the state of the closed-loop system is bounded because of the asymptotic stability property of the equilibrium. Moreover the output is always regulated to the set-point independently of the unknown disturbances. It is the virtue of the integral controller that allows to suppress the static error and hence achieves output regulation. A more general situation is explained in the chapter [77]. ◀

Remark : The solution considered in Theorem 2.1 is a classical solution in the sense of Li and Yu [47]. However the topology used here is the topology induced by the Hilbert H^2 norm instead of the C^1 norm. Moreover, we have only local exponential stability of the equilibrium of the closed-loop system. For initial compatible data outside some neighborhood of the equilibrium, the classical solution to the Cauchy problem (2.2) may not be extended on the whole positive time axis. ◀

The proof of Theorem 2.1 is given in the next section. Our proof is based on the construction of an appropriate Lyapunov functional. The direct Lyapunov approach allows us to consider the tuning parameter k_I relatively bigger than the classical method (see [23], [58] and [74]). The proposed upper bound on k_I is computed directly from the given system. This may be the advantage of our approach with respect to that of the literature [58, 74, 77].

To prove Theorem 2.1 we consider the following transformation :

$$\phi(x,t) = \psi(x,t) - \psi_\infty \ , \ \xi(t) = \zeta(t) - \zeta_\infty \tag{2.6}$$

where ψ_∞ and ζ_∞ are defined in (2.5). Then we obtain a perturbation free nonlinear closed-loop system as follows :

$$\begin{aligned}
\phi_t(x,t) &= -F(\phi(x,t))\phi_x(x,t) && (2.7)\\
\dot{\xi}(t) &= \phi(L,t) && (2.8)\\
\phi(0,t) &= -k_I\xi(t) && (2.9)\\
\phi(x,0) &= \phi_0(x) = \psi_0(x) - \psi_\infty && (2.10)\\
\xi(0) &= \xi_0 = \zeta_0 - \zeta_\infty && (2.11)
\end{aligned}$$

where $\phi_t(x,t)$ denotes the time partial derivative of $\phi(x,t)$, and we have defined $F(\phi) = \widetilde{F}(\phi + \psi_\infty)$. In the new coordinates, the output is written as

$$y(t) = \phi(L,t) + y_r.$$

Hence the output regulation to y_r is achieved if

$$\lim_{t\to\infty} |\phi(L,t)| = 0.$$

To guarantee the output regulation of the disturbed nonlinear system (2.2), we design the integral controller so as to ensure local asymptotic stabilization to the origin of the equivalent system (2.7)-(2.11).

In the following, the integral stabilization problem of the equivalent system is considered first for the linearized case in Section 2.3.1 and then for the nonlinear case in Section 2.3.2. Finally the complete proof of Theorem 2.1 is presented in Section 2.3.3.

2.3.1 Linear hyperbolic system

The purpose of this Section is to study stability property of the origin for the nonlinear hyperbolic system with an integral controller on the boundary as described in (2.7)-(2.11). To begin with, we consider the particular case where the system is linear, i.e., F does not depend on ϕ. This is the case if for instance the considered system is obtained by the tangent linearization of the nonlinear system around the equilibrium state. In

this subsection, we consider the following linear system :

$$\begin{aligned} \phi_t &= -r\phi_x \ , \ r > 0 && (2.12)\\ \dot{\xi} &= \phi(L,t), \ \ \phi(0,t) = -k_I\xi(t) && (2.13)\\ \phi(x,0) &= \phi_0(x), \ \ \xi(0) = \xi_0. && (2.14) \end{aligned}$$

To the system (2.12)-(2.14) is associated the state space Z which is the Hilbert space $Z = L^2(0,L) \times \mathbb{R}$ equipped with the scalar product

$$\langle (\phi_a, \xi_a), (\phi_b, \xi_b) \rangle_Z = \int_0^L \phi_a(x)\phi_b(x)dx + \xi_a\xi_b \ , \forall \ ((\phi_a, \xi_a), (\phi_b, \xi_b)) \in Z^2.$$

and we denote by $\| \cdot \|_Z$ its associated norm.

2.3.1.1 Frequency analysis method

The first stability result is obtained by employing the Laplace transform approach.

Proposition 2.2. *The closed-loop linear system (2.12)-(2.14) is exponentially stable in Z w.r.t. L^2 norm if and only if $k_I \in \left(0, \dfrac{r\pi}{2L}\right)$.*

The proof of this result can also be found in [11, p.444, Chapter 13] or in [38, Appendix Theorem A.5]. Our another proof is given in the following

Proof : A necessary and sufficient condition for exponential stability of the system (2.12)-(2.14) is that all the poles of the transfer function have negative real part (see [79], [11, Chapter 13], or [38, Appendix Theorem 3.5]). To formulate the transfer function, we set $v(t)$ as the new control input with $y(t)$ as the output :

$$\phi(0,t) = -k_I\xi(t) + v(t), \ y(t) = \phi(L,t). \tag{2.15}$$

By taking the Laplace transform in (2.12)-(2.14), we obtain:

$$s\hat{\phi} + r\hat{\phi}_x = 0 \tag{2.16}$$

$$s\hat{\xi} = \hat{\phi}(L,s), \tag{2.17}$$

$$\hat{\phi}(0,s) = -k_I\hat{\xi}(s) + \hat{v}(s) \ , \ \ \hat{y}(s) = \hat{\phi}(L,s) \tag{2.18}$$

From (2.16) we have the solution $\hat{\phi}(x,s) = \hat{\phi}(0,s)e^{-sr^{-1}x}$. Combining it with (2.17) and (2.18), we obtain:

$$\begin{aligned}\hat{y}(s) &= \hat{\phi}(L,s) = \hat{\phi}(0,s)e^{-sLr^{-1}} \\ &= e^{-sLr^{-1}}(-k_I\hat{\xi}(s) + \hat{v}(s)) = e^{-sLr^{-1}}\left(-k_I\frac{\hat{y}(s)}{s} + \hat{v}(s)\right)\end{aligned}$$

Hence,

$$\left(1 + \frac{k_I}{s}e^{-sLr^{-1}}\right)\hat{y}(s) = e^{-sLr^{-1}}\hat{v}(s)$$

Therefore we get the transfer function as follows:

$$G(s) = \frac{\hat{y}(s)}{\hat{v}(s)} = \frac{s}{k_I + se^{sLr^{-1}}}$$

The poles of transfer function are solutions of the following equation :

$$k_I + se^{sLr^{-1}} = 0 \tag{2.19}$$

We set

$$\mu = sLr^{-1} \text{ and } \alpha = k_I Lr^{-1}. \tag{2.20}$$

Note that $\alpha > 0$. The characteristic equation now becomes

$$\alpha + \mu e^{\mu} = 0 \tag{2.21}$$

The proposition is proved if we show that the equation (2.21) has all the solutions μ in the left-half complex plane $\Re e(\mu) < 0$ if and only if $\alpha \in (0, \frac{\pi}{2})$.
Let set $\mu = \sigma + i\eta$, where $\sigma, \eta \in \mathbb{R}$. Then (2.21) is rewritten as follows :

$$(\sigma + i\eta)e^{\sigma+i\eta} + \alpha = 0$$

By separating the real part and the imaginary part, we obtain:

$$-e^{\sigma}(\sigma cos(\eta) - \eta sin(\eta)) = \alpha, \tag{2.22}$$

$$\eta cos(\eta) + \sigma sin(\eta) = 0. \tag{2.23}$$

We consider the following two cases.

- If $sin(\eta) = 0$, by (2.23), $\eta\ cos(\eta) = 0$ implies $\eta = 0$. From (2.22), we have $\alpha = -\sigma e^{\sigma}$. The last equation has no solution $\sigma \geqslant 0$ whatever is $\alpha > 0$. Hence each solution σ is negative if and only if $\alpha \in (0, \pi/2)$.

- If $sin(\eta) \neq 0$, from (2.23),

$$\sigma = -\frac{\eta cos(\eta)}{sin(\eta)} \tag{2.24}$$

Thus $\alpha = H(\eta)$ where

$$H(\eta) = \frac{\eta}{sin(\eta)} \exp\left(\frac{-\eta cos(\eta)}{sin(\eta)}\right)$$

Because $H(\eta)$ is a pair function, we only need to consider the case where $\eta > 0$. Thus $\alpha > 0$ if and only if $sin(\eta) > 0$. As $\eta > 0$ and $sin(\eta) > 0$, we set $\eta = \gamma + 2k\pi$, where $\gamma \in (0, \pi)$ and $k \in \mathbb{N}$.
Now considering the function $H(\eta)$, we have :

$$\frac{\partial H(\eta)}{\partial \eta} = e^{\frac{-\eta cos(\eta)}{sin(\eta)}} \frac{sin^2(\eta) - \eta sin(2\eta) + \eta^2}{sin^3(\eta)}$$

One can easily check that $sin^2(\eta) - \eta sin(2\eta) + \eta^2 \geqslant 0$ for all $\eta > 0$. Therefore, $\frac{\partial H(\eta)}{\partial \eta} \geqslant 0$. Hence, on each interval $(2k\pi, 2k\pi+\pi)$, the function $H(\eta)$ is continuous and monotonic increasing. Moreover we have

$$\lim_{\eta \to (2k+1)\pi^-} H(\eta) = +\infty, \quad \lim_{\eta \to 2k\pi + \frac{\pi}{2}} H(\eta) = 2k\pi + \frac{\pi}{2}.$$

In addition, $\lim_{\eta \to 2k\pi^+} H(\eta) = 0 \ \forall \ k \in \mathbb{N}$ and $k > 0$, and $\lim_{\eta \to 0} H(\eta) = e^{-1}$. By (2.24) and $\eta = \gamma + 2k\pi$,we have $\sigma < 0$ if and only if $\gamma \in (0, \frac{\pi}{2})$. Obviously $\sigma \geqslant 0$ if $\gamma \in [\frac{\pi}{2}, \pi)$. Therefore $\sigma < 0$ if and only if the equation $H(\eta) - \alpha = 0$ has all its solutions in $\cup_{k=0}^{\infty}(2k\pi, 2k\pi + \frac{\pi}{2})$. Since α needs to be in one interval including 0, we have $\alpha \in (0, \frac{\pi}{2})$.

From the two cases, the proposition is proved. ■

Remark : By frequency-domain analysis it is possible to establish some necessary and sufficient conditions on the parameter k_I for asymptotic stability of the equilibrium to the linear closed-loop system (2.12)-(2.14). However the approach is no longer applicable when dealing with a general nonlinear system. This is the reason why we introduce a Lyapunov functional for the linear system which allows us to tackle the nonlinear hyperbolic system in the following section. ◄

2.3.1.2 Lyapunov direct method

In this Section, the following Lyapunov functional candidate $V : Z \to \mathbb{R}$ is considered:

$$V(\phi,\xi) = \int_0^L \left[\phi^2(x)e^{-\mu x} + q_1\xi\phi(x)e^{-\frac{\mu x}{2}}\right] dx + q_2\xi^2 \tag{2.25}$$

where $\mu > 0$ and $q_i > 0 \ \forall \ i = 1,2$. Consider the function $\Pi : [0,2] \to \mathbb{R}$ such that $\Pi(z) = \sqrt{z(2-z)}e^{-z/2}$. We have $\Pi(2-\sqrt{2}) \simeq 0.3395$ that is the maximum value of $\Pi(z)$ in $[0,2]$.

Given $T > 0$ and a function $\phi : (0,L)\times(0,T) \to \mathbb{R}$, we use the notation $\phi(t) := \phi(\cdot,t)$ when there is no ambiguity. Assume that the initial condition is smooth enough so that the solution of (2.12)-(2.14) is continuously differentiable with respect to time t and space x. Then, by differentiating $V(\phi(t),\xi(t))$ with time along the solution and by using integration by parts we get

$$\begin{aligned}\dot{V}(\phi(t),\xi(t)) &= -re^{-\mu L}\phi^2(L,t) - k_I\, r(q_1-k_I)\xi^2(t) - \mu r\int_0^L e^{-\mu x}\phi^2(x,t)dx \\ &+ \left(2q_2 - q_1 re^{-\frac{\mu L}{2}}\right)\xi(t)\phi(L,t) - \frac{\mu q_1 r}{2}\xi(t)\int_0^L e^{-\frac{\mu x}{2}}\phi(x,t)dx \\ &+ q_1\phi(L,t)\int_0^L e^{-\frac{\mu x}{2}}\phi(x,t)dx.\end{aligned} \tag{2.26}$$

Lemma 2.3. *Let $k_I^* = \left(\frac{r}{2L}\right)\Pi(2-\sqrt{2})$. Take $k_I \in (0,k_I^*)$ and $\mu \in (0,(2-\sqrt{2})/L]$ such that $\left(\frac{r}{2L}\right)\Pi(\mu L) > k_I$. Let $q_1 = 2k_I$ and let $q_2 = rk_Ie^{-\mu L/2}$. Then there exist positive constants $M \geqslant 1$ and $\alpha > 0$ such that*

$$M^{-1}||(\phi,\xi)||_Z^2 \leqslant V(\phi,\xi) \leqslant M||(\phi,\xi)||_Z^2 \ \ \forall\ (\phi,\xi) \in Z, \tag{2.27}$$

and for every smooth compatible $(\phi_0,\xi_0) \in Z$

$$\dot{V}(\phi(t),\xi(t)) \leqslant -\alpha V(\phi(t),\xi(t)) - \left(\frac{re^{-\mu L}}{2}\right)\phi^2(L,t). \tag{2.28}$$

Proof : Rewrite $V(\phi,\xi)$ as follows

$$V(\phi,\xi) = \int_0^L \begin{bmatrix} \phi(x)e^{-\mu x/2} \\ \frac{\xi}{\sqrt{L}} \end{bmatrix}^\top P \begin{bmatrix} \phi(x)e^{-\mu x/2} \\ \frac{\xi}{\sqrt{L}} \end{bmatrix} dx$$

where

$$P = \begin{bmatrix} 1 & \frac{\sqrt{L}q_1}{2} \\ \frac{\sqrt{L}q_1}{2} & q_2 \end{bmatrix}.$$

We claim that the matrix P is positive definite. Indeed, we have

$$\det(P) = Lk_I \left\{ \frac{r\Pi(\mu L)}{2L} - k_I + \frac{r}{2L} e^{-\mu L/2} \left(2 - \sqrt{\mu L(2-\mu L)} \right) \right\}.$$

Since $\left(\frac{r}{2L}\right)\Pi(\mu L) > k_I$ and $\mu L < 2$, it is easy to see that $\det(P) \geqslant \frac{rk_I}{2} e^{-\frac{\mu L}{2}}$. Hence there is some real constant $M \geqslant 1$ such that the inequality (2.27) holds.

By substituting the given q_1 and q_2 into (2.26) we have the following

$$\begin{aligned} \dot{V}(\phi(t),\xi(t)) &= -re^{-\mu L}\phi^2(L,t) - \mu r \int_0^L e^{-\mu x}\phi^2(x,t)dx - k_I^2 r\xi^2(t) \\ &- \mu r k_I \xi(t) \int_0^L e^{-\frac{\mu x}{2}}\phi(x,t)dx + 2k_I\phi(L,t)\int_0^L e^{-\frac{\mu x}{2}}\phi(x,t)dx. \end{aligned} \tag{2.29}$$

By using the Young and Cauchy-Schwarz inequalities we get

$$2k_I\phi(L,t)\int_0^L e^{-\frac{\mu x}{2}}\phi(x,t)dx \leqslant \left(\frac{re^{-\mu L}}{2}\right)\phi^2(L,t) + \left(\frac{2Lk_I^2 e^{\mu L}}{r}\right)\int_0^L e^{-\mu x}\phi^2(x,t)dx \tag{2.30}$$

and

$$\left|\mu r k_I \xi(t)\int_0^L e^{-\frac{\mu x}{2}}\phi(x,t)dx\right| \leqslant \left(\frac{rk_I^2}{2}\right)\xi^2(t) + \left(\frac{r\mu^2 L}{2}\right)\int_0^L e^{-\mu x}\phi^2(x,t)dx. \tag{2.31}$$

Substituting (2.30) and (2.31) into (2.29) leads us to the following inequality

$$\dot{V}(\phi(t),\xi(t)) \leqslant -\left(\frac{re^{-\mu L}}{2}\right)\phi^2(L,t) - \left(\frac{k_I^2 r}{2}\right)\xi^2(t) - \left(\frac{r}{2L}\Pi(\mu L) + k_I\right)\left(\frac{r}{2L}\Pi(\mu L) - k_I\right) J_{\phi,\xi} \tag{2.32}$$

where

$$J_{\phi,\xi} = \left(\frac{2L}{r}\right)e^{\mu L}\int_0^L e^{-\mu x}\phi^2(x,t)dx.$$

By the choice of μ, we have $\frac{r}{2L}\Pi(\mu L) - k_I > 0$. It follows from (2.32) that there exists a positive real number M_1 such that

$$\dot{V}(\phi(t),\xi(t)) \leqslant -M_1\left(\xi^2(t) + \int_0^L e^{-\mu x}\phi^2(x,t)dx\right) - \left(\frac{re^{-\mu L}}{2}\right)\phi^2(L,t). \tag{2.33}$$

The required inequality (2.28) is true by (2.33) and (2.27). ■

Remark : It can be noticed that the set of parameter k_I which makes the Lyapunov functional decreasing along solutions is smaller than the set of parameter obtained from Proposition 2.2. Hence, in the linear context our Lyapunov approach is conservative.

However the Lyapunov functional allows us to deal with nonlinear systems as it will be shown in the next Section. ◀

Remark : In our Lyapunov functional $V(\phi,\xi)$ in 2.25, the new added couple term between state ϕ and dynamic feedback ξ has a key role. To explain its role, it should be mentioned the following result whose proof can be found in [65, Theorem 4.7, Chapter 4] or [21, Theorem 4.1]:

Theorem 2.4. *Assume that $k_I \neq 0$. Then there exist no $\mu > 0$ and $q_2 > 0$ such that the following function (derived from function V without the coupled term, i.e $q_1 = 0$)*

$$U(\phi,\xi) = \int_0^L \phi^2(x,t)e^{-\mu x}dx + q_2\xi^2(t)$$

is a strict Lyapunov function for the closed-loop system (2.12)-(2.14).

Therefore, by adding the new coupled term $\int_0^L q_1\xi(t)\phi(x,t)e^{-\frac{\mu x}{2}}dx$ allows us to construct strict Lypunov functionals for the closed-loop system (2.12)-(2.14). ◀

2.3.2 Nonlinear system

In this section, we consider the problem for the nonlinear system (2.7)-(2.11) with $F(0) = r > 0$. By the designed integral controller the nonlinear closed-loop system (2.7)-(2.11) is written as follows

$$\begin{aligned} &\phi_t + F(\phi)\phi_x = 0 \\ &\dot{\xi} = \phi(L,t) \\ &\phi(0,t) = -k_I\xi \\ &\phi(x,0) = \phi_0(x), \quad \xi(0) = \xi_0. \end{aligned} \tag{2.34}$$

Let us set :

$$s(x,t) = \phi_x(x,t)\ ,\ p(x,t) = \phi_{xx}(x,t).$$

By successive derivatives and compatibility conditions we find that the dynamics of $s(x,t)$ and $p(x,t)$ are governed by the following PDE, respectively,

$$\begin{aligned} s_t + F(\phi)s_x &= -F'(\phi)\,s^2 \\ F(\phi(0,t))s(0,t) &= k_I\phi(L,t) \\ s(x,0) &= \phi_0'(x) \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} p_t + F(\phi)p_x &= -3F'(\phi)\, s\, p - F''(\phi)\, s^3 \\ F^2(\phi(0,t))p(0,t) &= k_I F(\phi(L,t))s(L,t) - 2k_I F'(\phi(0,t))\phi(L,t)s(0,t) \\ p(x,0) &= \phi_0''(x). \end{aligned} \tag{2.36}$$

Now we use the idea presented in [18] to extend the Lyapunov functional from the linear system (in L^2 norm) to the nonlinear system (in H^2 norm). Therefore local asymptotic stability of the equilibrium state and the set-point output regulation will be proved for the nonlinear closed-loop system (2.34). To do that, we consider the Lyapunov functional candidate $S : X \to \mathbb{R}_+$ such that

$$S(\phi,\xi) = V(\phi,\xi) + q_3 V_1(\phi_x) + q_4 V_1(\phi_{xx}) \tag{2.37}$$

where $V(\phi,\xi)$ is defined in (2.25) with q_1 and q_2 given in Lemma 2.3 and

$$V_1(\phi_x) = \int_0^L e^{-\mu x}\phi_x^2(x)dx \tag{2.38}$$

with the real positive constants q_3 and q_4 to be determined later.

For the moment we assume that all the required regularity is satisfied and carry out formal computations.

Lemma 2.5. *The time derivative of $V(\phi(t),\xi(t))$ along each regular trajectory of the nonlinear system (2.34) is written as follows*

$$\begin{aligned} \dot V(\phi(t),\xi(t)) = &-re^{-\mu L}\phi^2(L,t) - k_I^2 r\xi^2(t) - \mu r\int_0^L e^{-\mu x}\phi^2(x,t)dx \\ &- \mu r k_I \xi(t)\int_0^L e^{-\frac{\mu x}{2}}\phi(x,t)dx + 2k_I\phi(L,t)\int_0^L e^{-\frac{\mu x}{2}}\phi(x,t)dx \\ &- \phi^3(L,t)F_1(\phi(L,t))e^{-\mu L} + \phi^3(0,t)F_1(\phi(0,t)) \\ &+ \int_0^L e^{-\mu x}\left[F'(0) + \phi(x,t)F_2(\phi(x,t))\right]\phi_x(x,t)\phi^2(x,t)dx \\ &- \mu\int_0^L e^{-\mu x}F_1(\phi(x,t))\phi^3(x,t)dx \\ &- 2k_I\xi(t)\int_0^L e^{-\mu x/2}F_1(\phi(x,t))\phi(x,t)\phi_x(x,t)dx \end{aligned} \tag{2.39}$$

where

$$\begin{aligned} F(z) &= F(0) + F_1(z)z \\ F'(z) &= F'(0) + F_2(z)z \end{aligned} \tag{2.40}$$

with $F_1(z)=\int_0^1 F'(\lambda z)d\lambda$ *and* $F_2(z)=\int_0^1 F''(\lambda z)d\lambda$.

Proof : By differentiating $V(\phi(t),\xi(t))$ along each regular trajectory of (2.34) the following identity holds true

$$\begin{aligned}\dot V(\phi(t),\xi(t)) = &-\int_0^L 2e^{-\mu x}\phi(x,t)F(\phi(x,t))\phi_x(x,t)dx\\ &-q_1\int_0^L e^{-\mu x/2}F(\phi(x,t))\phi_x(x,t)\xi(t)dx\\ &+q_1\int_0^L e^{-\mu x/2}\phi_x(x,t)dx\phi(L,t)+2q_2\xi(t)\phi(L,t).\end{aligned}$$

By integration by parts and by using the boundary condition (2.34) and the parameters q_1 and q_2 given in Lemma 2.3 as well as the relations (2.40) we prove the required identity (2.39). ■

Similarly we may prove the following lemmas.

Lemma 2.6. *With the same notations as in Lemma 2.5, the time derivative of* $V_1(\phi_x(t))$ *along every regular trajectory of the nonlinear system (2.34)-(2.35) is written as follows*

$$\begin{aligned}\dot V_1(\phi_x(t)) = &-re^{-\mu L}s^2(L,t) \;+\; r^{-1}k_I^2\phi^2(L,t)\\ &-r\mu\int_0^L e^{-\mu x}s^2(x,t)dx-k_I^2F_3(\phi(0,t))\phi(0,t)\phi^2(L,t)\\ &-e^{-\mu L}F_1(\phi(L,t))\phi(L,t)s^2(L,t)\\ &-\int_0^L\left[(F(\phi(x,t)))_x+\mu F_1(\phi(x,t))\phi(x,t)\right]e^{-\mu x}s^2(x,t)dx\end{aligned}\tag{2.41}$$

where $F_3(z)=\displaystyle\int_0^1\frac{F'(\lambda z)}{F^2(\lambda z)}d\lambda$.

Lemma 2.7. *With the same notations as in Lemma 2.5, the time derivative of* $V_1(\phi_{xx}(t))$ *along each regular trajectory of the nonlinear system (2.34)-(2.36) is written as follows*

$$\begin{aligned}\dot V_1(\phi_{xx}(t)) = &-e^{-\mu L}F(\phi(L,t))p^2(L,t)+\frac{k_I^2F^2(\phi(L,t))}{F^3(\phi(0,t))}s^2(L,t)-\mu\int_0^L e^{-\mu x}F(\phi(x,t))p^2(x,t)dx\\ &+\frac{4k_I^2(F'(\phi(0,t))^2}{F^3(\phi(0,t))}\phi^2(L,t)s^2(0,t)\;-\;\frac{4k_I^3F(\phi(L,t))F'(\phi(0,t))}{F^4(\phi(0,t))}s(L,t)\phi^2(L,t)\\ &-\;5\int_0^L e^{-\mu x}F'(\phi(x,t))s(x,t)p^2(x,t)dx\;-2\int_0^L e^{-\mu x}F''(\phi(x,t))s^3(x,t)p(x,t)dx.\end{aligned}\tag{2.42}$$

Let $T > 0$. For each function $(\phi, \xi) \in C([0,T]; C^1[0,L] \times \mathbb{R})$ we define

$$\|(\phi, \phi_x, \xi)\|_{T,\infty} = \sup_{t\in[0,T]} |\xi(t)| + \sup_{\substack{x \in [0,L] \\ t \in [0,T]}} |\phi(x,t)| + \sup_{\substack{x \in [0,L] \\ t \in [0,T]}} |\phi_x(x,t)|.$$

By combining results of Lemmas 2.3-2.7 the following theorem is obtained.

Theorem 2.8. *Let the parameters k_I, μ, q_1 and q_2 be determined as in Lemma 2.3. Then there are positive real constants q_3, q_4, δ and β such that, for each function $(\phi, \xi) \in C([0,T]; C^3[0,L] \times \mathbb{R}) \cap C^1([0,T]; C^2[0,L] \times \mathbb{R})$ satisfying the PDE (2.34)-(2.36) and the condition $\|(\phi, \phi_x, \xi)\|_{T,\infty} < \delta$, the following differential inequality holds true*

$$\dot{S}(\phi(t), \xi(t)) \leqslant -\beta S(\phi(t), \xi(t)) \quad \forall\, t \in [0,T]. \tag{2.43}$$

Moreover there exists a positive constant $K \geqslant 1$ such that

$$K^{-1}\|(\phi,\xi)\|_X^2 \leqslant S(\phi,\xi) \leqslant K\|(\phi,\xi)\|_X^2 \quad \forall\, (\phi,\xi) \in X. \tag{2.44}$$

Proof : Without loss of generality we assume that $\delta \leqslant 1$. For the sake of simplicity we write $C_T = \|(\phi, \phi_x, \xi)\|_{T,\infty}$. By Lemma 2.5, Lemma 2.3 and the Cauchy-Schwarz inequality there exists a positive constant $K_1 > 0$ such that

$$\begin{aligned}\dot{V}(\phi(t), \xi(t)) \leqslant -\alpha V(\phi(t), \xi(t)) - (r/2)e^{-\mu L}\phi^2(L,t) \\ + K_1 C_T \left[\int_0^L e^{-\mu x}\phi^2(x,t)dx + \xi^2(t) + \phi^2(L,t)\right].\end{aligned} \tag{2.45}$$

Similarly, by Lemma 2.6 there exists a positive constant $K_2 > 0$ such that

$$\begin{aligned}\dot{V}_1(\phi_x(t)) \leqslant -(re^{-\mu L} - K_2 C_T)s^2(L,t) + r^{-1}k_I^2\phi^2(L,t) \\ - r\mu\int_0^L e^{-\mu x}s^2(x,t)dx + K_2 C_T\left(\phi^2(L,t) + \int_0^L e^{-\mu x}s^2(x,t)dx\right).\end{aligned} \tag{2.46}$$

Similarly, by Lemma 2.7 there exists a positive constant $K_3 > 0$ such that

$$\begin{aligned}\dot{V}_1(\phi_{xx}(t)) \leqslant -(re^{-\mu L} - K_3 C_T)p^2(L,t) + (r^{-2}k_I^2 + K_3 C_T)s^2(L,t) + K_3 C_T\phi^2(L,t) \\ + K_3 C_T\int_0^L e^{-\mu x}s^2(x,t)dx - (r\mu - K_3 C_T)\int_0^L e^{-\mu x}p^2(x,t)dx.\end{aligned} \tag{2.47}$$

As C_T can be made as small as we like with δ, adding the inequalities (2.45)-(2.47) and taking δ, q_3 and q_4 sufficiently small lead us directly to the following differential relation

$$\dot{S}(\phi(t),\xi(t)) \leqslant -\frac{\alpha}{2}V(\phi(t),\xi(t)) - \frac{q_3 r\mu}{2}\int_0^L e^{-\mu x}s^2(x,t)dx - \frac{q_4 r\mu}{2}\int_0^L e^{-\mu x}p^2(x,t)dx. \tag{2.48}$$

Therefore the theorem is proved by using (2.27), (2.37) and (2.48). ■

2.3.3 Proof of Theorem 2.1

With Theorem 2.8, we are now ready to prove the main result of the chapter.

Proof of Theorem 2.1 : We first prove the local existence of a unique solution to the closed-loop system (2.7)-(2.11) for each compatible initial state (ϕ_0,ξ_0) in $H^2(0,L)\times\mathbb{R}$. The closed-loop control system (2.7)-(2.11) is governed by the following PDE coupled with an ODE through the boundary as follows:

$$\begin{cases} \phi_t = -F(\phi)\phi_x, \;\; \dot{\xi} = \phi(L,t) \\ \phi(0,t) = -k_I\xi, \;\; (\phi(x,0),\xi(0)) = (\phi_0(x),\xi_0). \end{cases} \tag{2.49}$$

Recall that $X = H^2(0,L)\times\mathbb{R}$ is equipped with the norm $\|(f,z)\|_X^2 = \|f\|_{H^2}^2 + z^2$. Assume that the initial condition (ϕ_0,ξ_0) is in $B_X(0,\delta)$, $\delta>0$ and satisfies the C^0 and C^1 compatibility conditions as in (2.34) and (2.35).

By using the Theorem 1.2 and the Propositions 1.3-1.5 in [66, pp.362-365], or [39, Theorem II] we deduce the existence of a unique solution to (2.49) for some $\delta>0$ and $T>0$:

$$(\phi,\xi)\in C([0,T];H^2(0,L)\times\mathbb{R})\cap C^1([0,T];H^1(0,L)\times\mathbb{R}).$$

The reader is referred to [18] and [4, Appendix B] for a rigorous proof to the initial boundary case.

Now we prove local exponential stability of the null state to (2.49). Notice that each compatible $w_0=(\phi_0,\xi_0)\in X$ admits a sequence of $w_{0,n}=(\phi_{0,n},\xi_{0,n})\in H^4(0,L)\times\mathbb{R}$ satisfying the C^k compatibility condition, $k=0,1,2,3$, such that $\lim_{n\to\infty}\|w_{0,n}-w_0\|_X=0$ (cf. [15, p.130]). Hence it is sufficient for us to prove the exponential stability for $w_0\in H^4(0,L)\times\mathbb{R}$. As the solution depends continuously on the initial condition (see [39, Theorem III]), then the exponential decay of solution from compatible $w_0\in X$ is proved by taking the limit.

Indeed, take a compatible $w_0 \in (H^4(0,L)\times\mathbb{R})\cap B_X(0,\delta)$. As stated above the system (2.49) has a unique solution $w(t)$ in $H^4(0,L)\times\mathbb{R}$ (cf. [66]) such that

$$w \in C([0,T];H^4(0,L)\times\mathbb{R})\cap C^1([0,T];H^3(0,L)\times\mathbb{R})$$

where $w(t)=(\phi(t),\xi(t))$. By the continuous embedding (cf. [15, p.167]) $H^n(0,L)\hookrightarrow C^{n-1}[0,L]\ \forall\, n\geqslant 1$ integer, we have the solution

$$w \in C([0,T];C^3[0,L]\times\mathbb{R})\cap C^1([0,T];C^2[0,L]\times\mathbb{R}).$$

Let $\|w_0\|_X<\delta_1$ for some $\delta_1>0$. We choose $\delta_1>0$ sufficiently small such that $\|w_0\|_X<K\delta_1$ implies $\|(\phi,\phi_x,\xi)\|_{T,\infty}<\delta$ with smaller T if necessary (cf. [53, Theorem 2.2 , p.46]). Notice that K and δ are defined in our Theorem 2.8. Then direct application of Theorem 2.8 allows us to get $\|w(T)\|_X<K\delta_1$. Since the system is autonomous, the same argument can be used on the time interval $[T,2T]$. By successive iterations we obtain the differential inequality (2.43) satisfied for all $t\geqslant 0$. By (2.43)-(2.44) we find positive constants M and ω such that

$$\|w(t)\|_X \leqslant Me^{-\omega t}\|w_0\|_X \ \ \forall\, w_0\in B_X(0,\delta_1).$$

The regulation effect is automatically guaranteed, since $w\in C([0,\infty);H^2(0,L)\times\mathbb{R})$. Hence the proof of Theorem 2.1 is complete. □

2.4 Numerical simulations

2.4.1 Discretization with Preissmann scheme and numerical solutions

To make numerical simulations, we firstly use the Preissmann scheme (see in Appendix) to discretize the nonlinear system (2.1) looped by the integral controller designed from Theorem 2.1.

Let recall some principal parameters used in Preissmann scheme: N is the number of discretized space intervals; Δt and Δx are time discretization step and space discretization step, respectively; and $\theta\in[0.5,1]$ is the weight parameter. In this Section, the nonlinear system (2.1) is considered with the following flux function $\widetilde{F}(\psi)=\psi^2+3$.

The following nonlinear equation is obtained with the Preissmann scheme

$$G\Big(X(k+1)\Big)=0 \tag{2.50}$$

where $F : \mathbb{R}^N \to \mathbb{R}^N$ defined by $G = \begin{pmatrix} g_1 \\ g_2 \\ ... \\ g_{N-1} \\ g_N \end{pmatrix}$, and $X(k+1) = \begin{pmatrix} \psi_1^{k+1} \\ \psi_2^{k+1} \\ ... \\ \psi_N^{k+1} \end{pmatrix}$.

with the $N-1$ sub-functions g_i (i $= \overline{1, N-1}$) received from the discretization of $N-1$ PDE equations in (2.1)

$$g_i = \psi_{i+1}^{k+1} + \psi_i^{k+1} - \psi_{i+1}^k - \psi_i^k + \frac{\Delta t}{\Delta x} M_i \Big(\theta(\psi_{i+1}^{k+1} - \psi_i^{k+1}) + (1-\theta)(\psi_{i+1}^k - \psi_i^k) \Big) , \tag{2.51}$$

where

$$M_i = \Big[6 + \theta \Big((\psi_{i+1}^{k+1})^2 + (\psi_i^{k+1})^2 \Big) + (1-\theta) \Big((\psi_{i+1}^k)^2 + (\psi_i^k)^2 \Big) \Big] ,$$

and a boundary conditions

$$g_N = \psi_1^{k+1} - u.$$

Note that the values at instant k of ψ_i are supposed to be computed previously and considered known values.

The next step is to resolve the implicit nonlinear equation (3.25) by Newton-Raphson method, see the detail of this method in Appendix. To apply this iterative approach, it is obligatory to compute the Jacobien matrix of function G as following:

$$J = \begin{pmatrix} \frac{\partial g_1}{\partial \psi_1^{k+1}} & \frac{\partial g_1}{\partial \psi_2^{k+1}} & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & \frac{\partial g_2}{\partial \psi_2^{k+1}} & \frac{\partial g_2}{\partial \psi_3^{k+1}} & 0 & ... & 0 & 0 & 0 \\ ... & & & & & & & \\ 0 & 0 & 0 & 0 & ... & 0 & \frac{\partial g_{N-1}}{\partial \psi_{N-1}^{k+1}} & \frac{\partial g_{N-1}}{\partial \psi_N^{k+1}} \\ \frac{\partial g_N}{\partial \psi_1^{k+1}} & 0 & 0 & 0 & ... & 0 & 0 & 0 \end{pmatrix}$$

where $\frac{\partial g_N}{\partial \psi_1^{k+1}} = 1$; and $\frac{\partial g_i}{\partial \psi_i^{k+1}}$, $\frac{\partial g_i}{\partial \psi_i^{k+1}}$ are computed from the formula (2.51) ($\forall\ i = \overline{1, N-1}$).

Finally, we receive a numerical solution at instant $k+1$ of ψ_1, ψ_n,$\cdots$, ψ_n. With a sufficient time horizon, we get a numerical solution for the nonlinear system (2.1).

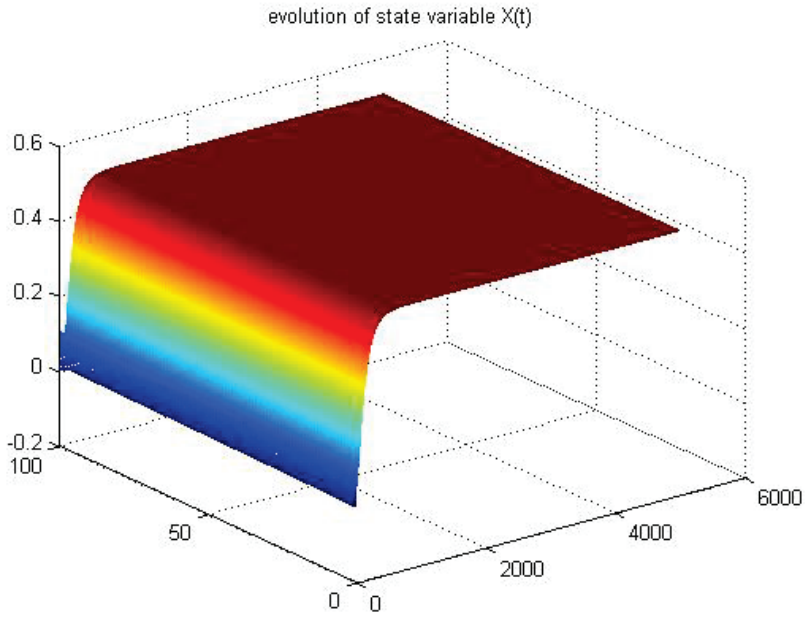

FIGURE 2.1: Evolution of $\psi(x,t)$

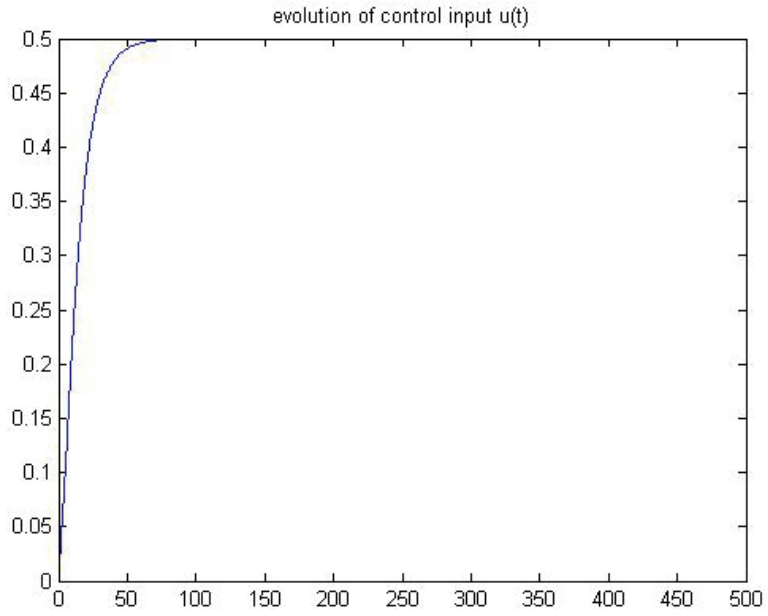

FIGURE 2.2: Evolution of input $u(t)$

2.4.2 Simulations and results

The simulations in the chapter are done with following parameters: $L = 50m$, $N = 100$, $\theta = 0.55$, and $\dfrac{\Delta t}{\Delta x} = 0.5$. The gain of integral controller and the reference set-point are applied: $k_I = 0.05$, and $y_r = 0.5$. The constant perturbations on the output and on the control are given by $w_o = 0.1$ and $w_c = 0.05$, respectively.

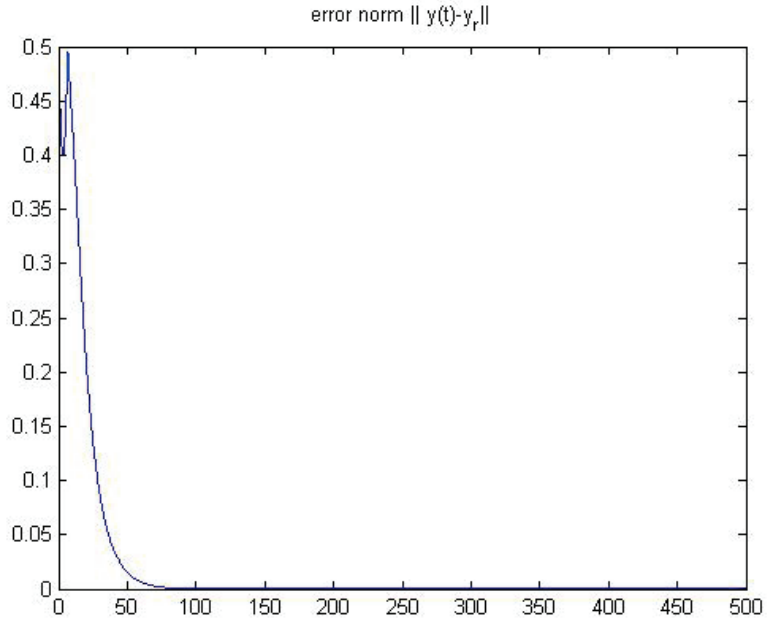

FIGURE 2.3: Evolution of error $|y(t) - y_r|$

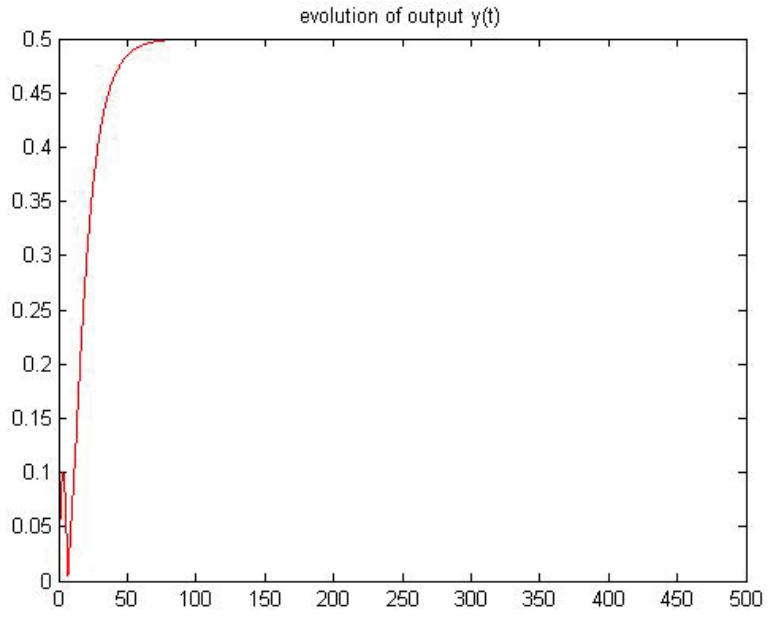

FIGURE 2.4: Evolution of output $y(t)$

Figure 2.1 shows asymptotic stability of the nonlinear closed-loop system and illustrates the evolution of the state $\psi(x,t)$. Moreover the regulation of the output $y(t)$ to the desired set-point y_r is illustrated by Figure 2.3 and Figure 2.4. Finally Figure 2.2 shows the evolution of the control input $u(t)$ perturbed by w_c. As clearly indicated by the simulations, by virtue of the integral action the output converges to the set-point as $t \to \infty$ independently of the constant perturbations.

2.5 Conclusions

We have considered the design of stabilizing integral controllers for the nonlinear systems described by scalar hyperbolic PDE. First we have proposed an interval of integral gain for stabilization and then proved exponential stability of the linearized system controlled by the designed controller. Moreover, for the linearized system we have been able to establish a necessary and sufficient condition for the integral gain to get exponential stability of the controlled system in the L^2 norm. Then we have proved local exponential stability of the nonlinear controlled system by the same integral controller in the H^2 norm. Both of two main proofs have used Lyapunov techniques with the Lyapunov functions in the quadratic form. The regulation of the output to the set-point is automatically guaranteed from the local exponential stability of the closed-loop system in H^2 norm. Numerical simulations for the nonlinear closed-loop system have been carried out to validate the performance of the controlled system. In the future, the work is to extend the design of stabilizing integral controllers for networks of scalar systems governed by nonlinear hyperbolic PDE.

The results of this chapter can also be found in the publication [67].

Chapter 3

Multiple input - multiple output 2×2 systems with PI control

3.1 Introduction

This chapter is concerned with the boundary control and output regulation of a class of linear multiple input - multiple output (MIMO) systems described by two hyperbolic partial differential equations with multivariable PI controllers. Both the input control and the output measurement are situated on the boundary. First, the system is transformed into the characteristic form of Riemann invariants, and the PI controller design is proposed for the Riemann invariant system. Then, a Lyapunov functional is constructed to prove stabilization and regulation of the closed-loop system. Furthermore, we apply the designed PI controller for a nonlinear Saint-Venant model and carry out numerical simulations to evaluate the performances of the designed PI controller.

We consider the following linear system governed by two hyperbolic partial differential equations:

$$\begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} z_1(x,t) \\ z_2(x,t) \end{pmatrix} + G\,\dfrac{\partial}{\partial x}\begin{pmatrix} z_1(x,t) \\ z_2(x,t) \end{pmatrix} = 0, \\ h_1(z_1(0,t), z_2(0,t)) = U_0(t), \\ h_2(z_1(L,t), z_2(L,t)) = U_L(t), \\ y(t) = (z_1(0,t), z_2(L,t)), \end{cases} \tag{3.1}$$

with the initial condition:

$$z_1(x,0) = z_1^0(x),\ z_2(x,0) = z_2^0(x),$$

where $x \in [0, L]$ and $t \in \mathbb{R}_+$ are respectively space and time variable. The two state variables are $z_i : [0, L] \times \mathbb{R}_+ \to \mathbb{R}$ for $i = 1, 2$. The matrix $G \in \mathbb{R}^{2\times 2}$ has two real eigenvalues with opposite signs. Two control inputs on the boundary are $U_0(t)$ and $U_L(t)$, with $h_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $h_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The output measurement $y(t) \in \mathbb{R}^2$ is determined by two state space variables, the first one evaluated at the boundary $x = 0$ and the other one at $x = L$.

System (3.1) is a MIMO system and represents for dynamics of many linearized models in industrial engineering. For instance, it can be used as the linearized model of the open channel in the field of hydraulics (see in [12], [19], [76] and [71]), or the model of counter flow heat exchanger process in [74], [75].

Our control problem is to find two dynamic feedback control inputs on the boundary $U_0(t)$ and $U_L(t)$ which depend only on the two available measured outputs $y(t) = (z_1(0, t), z_2(L, t))$ such that the closed-loop system is stable and the output $y(t)$ is regulated to the desired set-point $y_r = (y_{r1}, y_{r2})$.

The studied problem and existing results are described in the Section 3.2. In Section 4.4.3, we present the main result of the chapter and its proof based on Lyapunov techniques. Section 4.5.2.3 is devoted to numerical simulations in which we apply the PI controller design for a nonlinear fluid flow model. The conclusions are addressed in Section 3.5.

3.2 Problem statement and review of existing results

3.2.1 Statement of the problem

Following the usual approach when dealing with hyperbolic systems, we first perform a change of coordinates (for example Riemann coordinates) around the desired set-point y_r, and we rewrite the system in *characteristic form*

$$\frac{\partial}{\partial t}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} + \begin{pmatrix}\lambda_1 & 0\\ 0 & -\lambda_2\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = 0, \tag{3.2}$$

where $\phi_i : [0, L] \times \mathbb{R}_+ \to \mathbb{R}$, and $\lambda_i > 0$ for $i = 1, 2$. In addition, combining with the first order approximation, the boundary conditions and the measured outputs now have the following forms:

$$\begin{cases} \phi_1(0, t) &= R_0\phi_2(0, t) + u_1(t),\\ \phi_2(L, t) &= R_L\phi_1(L, t) + u_2(t), \end{cases} \tag{3.3}$$

$$y(t) - y_r = \begin{pmatrix} a\phi_1(0,t) + b\phi_2(0,t) \\ c\phi_1(L,t) + d\phi_2(L,t) \end{pmatrix}, \tag{3.4}$$

where R_0, R_L, a, b, c and d are real constants obtained from the linearization around the set-point. The new control input $u(t) = (u_1(t), u_2(t)) \in \mathbb{R}^2$ is obtained by a linear transformation from (U_0, U_L) (see in Section 4.5.2.3 as an example). In the chapter, we consider the case in which $abcd < 0$. Notice that the system (3.2), (3.3), (3.4) is the general form of the linearized systems used in [76], [71], [3], [24] and [77] (when the viscous friction slope is null).

In this work, the control action $u(t) = (u_1(t), u_2(t))$ designed by the PI controller on the boundary has the following form:

$$u(t) = K_p(y(t) - y_r) + K_I \int_0^t (y(\tau) - y_r)d\tau, \tag{3.5}$$

where $K_p \in \mathbb{R}^{2\times 2}$ and $K_I \in \mathbb{R}^{2\times 2}$ are matrix gains which are the parameters of PI controller to be designed. They have to be selected to guarantee the asymptotic stability and regulation of the closed-loop system (3.2)-(3.5).

3.2.2 Existing results

In the simple case where $u_1(t) = u_2(t) = 0$, the system is said to be in open-loop. It is shown in [76] by operator and semi-group method that the origin of open-loop system is exponentially stable in C^1 norm if and only if the following boundary condition is satisfied $|R_0 R_L| < 1$.

In other work [18], they consider $u_1(t)$ and $u_2(t)$ as static feedback control laws, and after the linearization of boundary control in (3.3), they obtain the following boundary condition $\phi_1(0,t) = R_0^*\phi_2(0,t)$ and $\phi_2(L,t) = R_L^*\phi_1(L,t)$ where R_0^* and R_L^* are two real constants. With a strict Lypunov functional, they prove that the origin of nonlinear closed-loop network is exponentially stable in H^2 norm if $|R_0^* R_L^*| < 1$.

Furthermore, linear hyperbolic systems with boundary PI control only from one side (in other word, single input single output hyperbolic system) are considered in [3, 43]. In [3], by frequency method with Laplace transform, the authors give a necessary and sufficient condition of PI controllers at boundary $x = 0$ to obtain exponential stability of the closed-loop system. In [43], by using Lyapunov direct method, a scalar PI controller is constructed at boundary $x = L$ from scalar output measurement.

It should be mentioned that a multivariable PI control design has been proposed for linear 2×2 hyperbolic systems by using operator and semi-group techniques [77]. Moreover, with Lyapunov techniques, Dos Santos et al. [24] have addressed a multivariable PI controller design for a special class of 2×2 linear hyperbolic system.

3.3 Main result and its proof with Lyapunov technique

3.3.1 Main result

To describe the closed-loop system, let us introduce new state variable $\xi(t) = (\xi_1(t), \xi_2(t))$ where $\dot{\xi}(t) = y(t) - y_r$. The closed-loop system (3.2) - (3.5) can be rewritten as follows:

$$\begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}-\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\dfrac{\partial}{\partial x}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \\ \dfrac{\partial}{\partial t}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}a\phi_1(0,t) + b\phi_2(0,t)\\ c\phi_1(L,t) + d\phi_2(L,t)\end{pmatrix}, \\ \begin{pmatrix}\phi_1(0,t)\\ \phi_2(L,t)\end{pmatrix} = \begin{pmatrix}R_0\phi_2(0,t)\\ R_L\phi_1(L,t)\end{pmatrix} + K_p(y(t) - y_r) + K_I\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}, \\ y(t) - y_r = (a\phi_1(0,t) + b\phi_2(0,t), c\phi_1(L,t) + d\phi_2(L,t)). \end{cases} \tag{3.6}$$

Let $\mathbb{X} = (L^2(0,L))^2 \times \mathbb{R}^2$ be the state space of the closed-loop system (3.6) equipped with the following norm:

$$||v||_{\mathbb{X}}^2 = ||v_1||_{L^2(0,L)}^2 + ||v_2||_{L^2(0,L)}^2 + |v_3|^2 + |v_4|^2$$

where $v = (v_1, v_2, v_3, v_4) \in \mathbb{X}$.

We consider now the closed-loop system (3.6), under the initial condition $(\phi_1^0(x), \phi_2^0(x), \xi_1^0, \xi_2^0)$. For each initial condition $(\phi_1^0(x), \phi_2^0(x), \xi_1^0, \xi_2^0) \in \mathbb{X}$ satisfying the C^0 and C^1 compatibility conditions, the closed-loop system (3.6) has a unique solution[1] $(\phi_1, \phi_2, \xi_1, \xi_2) \in \mathbb{X}$ (see [20] and [77] for a proof with dynamic boundary conditions). In the chapter, we look for asymptotic stability of the closed-loop system in $\mathbb{X}$ with the output regulation guaranteed.

The system (3.6) is called exponentially stable in $\mathbb{X}$ if there exits positive constants $M > 0$ and $\alpha > 0$ such that for each initial condition $(\phi_1^0(x), \phi_2^0(x), \xi_1^0, \xi_2^0) \in \mathbb{X}$, the solution of (3.6) satisfies the following inequalities:

$$||(\phi_1(\cdot,t), \phi_2(\cdot,t), \xi_1(t), \xi_2(t))||_{\mathbb{X}}^2 \leqslant Me^{-\alpha t}||(\phi_1^0, \phi_2^0, \xi_1^0, \xi_2^0)||_{\mathbb{X}}^2 \ , \ \forall t \in \mathbb{R}_+.$$

Remark : It is interesting to note that in our work, the PI controller ensures stabilization of the origin in $\mathbb{X}$ without requiring a priori the classical constraint $|R_0R_L| < 1$.

◀

[1] Without the compatibility condition, it is possible to show following [20] and [77] that weak solutions exist.

The main result of the chapter is stated as follows:

Theorem 3.1. *Let* a, b, c *and* d *be real constants such that their product* $abcd < 0$. *Then there exists* $\mu^* > 0$ *such that, for each* $\mu \in (0, \mu^*)$, *and each PI controller with proportional gain* K_p *and integral gain* K_I *as follows:*

$$\begin{aligned} K_p &= \begin{pmatrix} \frac{-R_0}{b} & 0 \\ 0 & \frac{-R_L}{c} \end{pmatrix}, \\ K_I &= -\mu \begin{pmatrix} (\frac{aR_0}{b} + 1)a & 0 \\ 0 & (\frac{dR_L}{c} + 1)d \end{pmatrix}, \end{aligned} \tag{3.7}$$

we have:

- *the closed-loop system (3.6) with the PI controller design in (3.7) is exponentially stable toward the origin in* $\mathbb{X}$.
- *for each initial condition* $(\phi_1^0(x), \phi_2^0(x), \xi_1^0, \xi_2^0) \in (H^1(0, L))^2 \times \mathbb{R}^2$ *satisfying the* C^0 *and* C^1 *compatibility conditions, the output* $y(t)$ *is regulated to the desired set-point* y_r, *i.e,*

$$\lim_{t \to \infty} ||y(t) - y_r||^2_{\mathbb{R}^2} = 0.$$

Remark : Since the weak solution depends continuously on the initial data, the exponential decay of the weak solution can be proved by a Lyapunov functional approach together with the classical density argument. ◀

3.3.2 Lyapunov functional for the closed-loop system

By applying the PI controller from (3.7), the closed-loop system (3.6) is governed by the following partial differential equations :

$$\begin{cases} \frac{\partial}{\partial t} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} &= \begin{pmatrix} -\lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \\ \frac{\partial}{\partial t} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} &= \begin{pmatrix} a\phi_1(0,t) + b\phi_2(0,t) \\ c\phi_1(L,t) + d\phi_2(L,t) \end{pmatrix}, \\ \phi_1(0,t) &= -k_1\xi_1, \\ \phi_2(L,t) &= -k_2\xi_2, \\ y(t) - y_r &= (a\phi_1(0,t) + b\phi_2(0,t), c\phi_1(L,t) + d\phi_2(L,t)), \end{cases} \tag{3.8}$$

where

$$k_1 = a\mu, \quad k_2 = d\mu, \tag{3.9}$$

with $\mu > 0$; $\phi_1, \phi_2 \in L^2(0, L)$ and $\xi_1, \xi_2 \in \mathbb{R}$.

The candidate Lyapunov function $V : \mathbb{X} \to \mathbb{R}$ is defined as follows:

$$V(\phi_1, \phi_2, \xi_1, \xi_2) = \int_0^L \begin{pmatrix} \phi_1 e^{-\frac{\mu x}{2}} \\ \phi_2 e^{\frac{\mu x}{2}} \\ \xi_1 \\ \xi_2 \end{pmatrix}^T P \begin{pmatrix} \phi_1 e^{-\frac{\mu x}{2}} \\ \phi_2 e^{\frac{\mu x}{2}} \\ \xi_1 \\ \xi_2 \end{pmatrix} dx \ , \tag{3.10}$$

where

$$P = \begin{pmatrix} 1 & 0 & 0 & q_4 \\ 0 & q_1 & q_5 & 0 \\ 0 & q_5 & q_2 & 0 \\ q_4 & 0 & 0 & q_3 \end{pmatrix}, \tag{3.11}$$

and $q_i \in \mathbb{R}$ with $i = 1, \cdots, 5$ are parameters that have to be designed.

In the following, by properly selecting the parameters μ and the (q_i)'s, it is shown that the functional V is positive definite and that its time derivative along the solution of the system (3.8) is negative. The proof of Theorem 3.1 is established thereafter.

In order to show that the functional V is positive definite, note that the following lemma may be easily obtained.

Lemma 3.2. *The matrix P is symmetric positive definite if and only if the following three conditions are satisfied :*

$$q_1 > 0, \ q_1 q_2 > q_5^2, \ q_3 > q_4^2. \tag{3.12}$$

Proof : Applying the Sylvester criterion, the matrix P defined in (3.11) is symmetric positive definite if the following three determinants are positive:

$$P_1 = det \begin{pmatrix} 1 & 0 \\ 0 & q_1 \end{pmatrix}, \ P_2 = det \begin{pmatrix} 1 & 0 & 0 \\ 0 & q_1 & q_5 \\ 0 & q_5 & q_2 \end{pmatrix},$$

$$P_3 = det(P) \ .$$

Direct computations give us

$$P_1 = q_1, \ P_2 = q_1 q_2 - q_5^2, \ P_3 = (q_3 - q_4^2) P_2 \ .$$

This leads to the proof of Lemma 3.2. ■

In order to simplify the reading, all along the chapter, the notation

$$V(t) := V(\phi_1(.,t), \phi_2(.,t), \xi_1(t), \xi_2(t))$$

is employed. The following result can be obtained.

Lemma 3.3. *Assume* $abcd < 0$. *Let* k_1, k_2 *and* q_i, *for* $i = 1, \ldots, 5$ *be defined as follows:*

$$\begin{aligned} k_1 &= a\mu \ , \ k_2 = d\mu, \\ q_1 &= -\frac{bd\lambda_1}{ac\lambda_2} e^{-\mu L}, \ q_2 = \frac{\lambda_1 \mu}{L} \ , \ q_3 = -\frac{bd\lambda_1}{acL}\mu \ , \\ q_4 &= -\frac{bd}{a}\mu e^{\frac{\mu L}{2}}, \ q_5 = \frac{\lambda_1 b}{\lambda_2}\mu. \end{aligned} \tag{3.13}$$

Then there exists $\mu^* > 0$ *such that for every* $\mu \in (0, \mu^*)$, *the following holds:*

1. *there exists* $K > 0$ *such that* $\forall\ (\phi_1, \phi_2, \xi_1, \xi_2)$ *in* $\mathbb{X}$*:*

$$\frac{1}{K} V(\phi_1, \phi_2, \xi_1, \xi_2) \leqslant ||(\phi_1, \phi_2, \xi_1, \xi_2)||^2_{\mathbb{X}} \leqslant K V(\phi_1, \phi_2, \xi_1, \xi_2) \ . \tag{3.14}$$

2. *there exists* $\alpha > 0$ *such that along the solution of (3.8), for all* t *at which the solution is well defined*

$$\dot{V}(t) \leqslant -\alpha V(t) - M\Big(\phi_1^2(L,t) + \phi_2^2(0,t)\Big) \ . \tag{3.15}$$

Proof : To begin with, we prove the first assertion. Since $abcd < 0$, it yields that $\rho = -\dfrac{bd}{ac} > 0$. Employing (3.13), one can find that:

$$q_1 = \rho \frac{\lambda_1}{\lambda_2} e^{-\mu L} \ ,$$

$$q_1 q_2 - q_5^2 = \mu(\rho \frac{\lambda_1^2}{L\lambda_2} e^{-\mu L} - \mu \frac{\lambda_1^2 b^2}{\lambda_2^2}) \ ,$$

$$q_3 - q_4^2 = \mu(\frac{\rho\lambda_1}{L} - \mu\rho^2 c^2 e^{\mu L}) \ .$$

It is clearly that when μ is small enough, $q_1 > 0$, $q_1 q_2 - q_5^2 > 0$ and $q_3 - q_4^2 > 0$. Therefore, using Lemma 3.2, it yields that the matrix P is symmetric positive definite.

Hence, there exits $\sigma_1, \sigma_2 > 0$ such that

$$\sigma_1 \int_0^L \begin{pmatrix} \phi_1 e^{-\frac{\mu x}{2}} \\ \phi_2 e^{\frac{\mu x}{2}} \\ \xi_1 \\ \xi_2 \end{pmatrix}^T \begin{pmatrix} \phi_1 e^{-\frac{\mu x}{2}} \\ \phi_2 e^{\frac{\mu x}{2}} \\ \xi_1 \\ \xi_2 \end{pmatrix} dx \leqslant V(\phi_1, \phi_2, \xi_1, \xi_2)$$
$$\leqslant \sigma_2 \int_0^L \begin{pmatrix} \phi_1 e^{-\frac{\mu x}{2}} \\ \phi_2 e^{\frac{\mu x}{2}} \\ \xi_1 \\ \xi_2 \end{pmatrix}^T \begin{pmatrix} \phi_1 e^{-\frac{\mu x}{2}} \\ \phi_2 e^{\frac{\mu x}{2}} \\ \xi_1 \\ \xi_2 \end{pmatrix} dx.$$

As a result, there exists $K > 0$ such that (3.14) holds.

Now we prove the second part. Substituting q_i and k_1, k_2 from (3.13), the time derivative of V along the solution of the system (3.8) has the following form:

$$\dot{V} = -\int_0^L \begin{pmatrix} \phi_1(x,t) e^{-\frac{\mu x}{2}} \\ \phi_2(x,t) e^{\frac{\mu x}{2}} \\ \xi_1(t) \\ \xi_2(t) \\ \phi_1(L,t) \\ \phi_2(0,t) \end{pmatrix}^T Q \begin{pmatrix} \phi_1(x,t) e^{-\frac{\mu x}{2}} \\ \phi_2(x,t) e^{\frac{\mu x}{2}} \\ \xi_1(t) \\ \xi_2(t) \\ \phi_1(L,t) \\ \phi_2(0,t) \end{pmatrix} dx$$

where

$$Q = \frac{1}{L} \begin{pmatrix} \mu\lambda_1 L & 0 & 0 & T_{\phi_1,\xi_2} & T_{\phi_1,\phi_1 L} & 0 \\ 0 & \mu q_1 \lambda_2 L & T_{\phi_2,\xi_1} & 0 & 0 & T_{\phi_2,\phi_2 0} \\ 0 & T_{\phi_2,\xi_1} & T_{\xi_1} & T_{\xi_1,\xi_2} & 0 & T_{\phi_2 0,\xi_1} \\ T_{\phi_1,\xi_2} & 0 & T_{\xi_1,\xi_2} & T_{\xi_2} & T_{\phi_1 L,\xi_2} & 0 \\ T_{\phi_1,\phi_1 L} & 0 & T_{\phi_1 L,\xi_1} & T_{\phi_1 L,\xi_2} & \lambda_1 e^{-\mu L} & 0 \\ 0 & T_{\phi_2,\phi_2 0} & T_{\phi_2 0,\xi_1} & T_{\phi_2 0,\xi_2} & 0 & \lambda_2 q_1 \end{pmatrix}$$

with

$$\begin{aligned} T_{\xi_1} &= 2aq_2 L k_1 - k_1^2 \lambda_1, \; T_{\xi_2} = 2dq_3 L k_2 - k_2^2 \lambda_2 q_1 e^{\mu L}, \\ T_{\phi_1,\xi_2} &= dk_2 q_4 L + \frac{q_4 \lambda_1}{2} \mu L, \; T_{\phi_1,\phi_1 L} = -cq_4 L, \\ T_{\phi_2,\xi_1} &= ak_1 q_5 L + \frac{q_5 \lambda_2}{2} \mu L, \quad T_{\phi_2,\phi_2 0} = -bq_5 L, \\ T_{\xi_1,\xi_2} &= \lambda_2 q_5 k_2 e^{\frac{\mu L}{2}} + \lambda_1 k_1 q_4, \; T_{\phi_2 0,\xi_1} = -q_2 bL + \lambda_2 q_5, \\ T_{\phi_1 L,\xi_2} &= -q_3 cL + \lambda_1 q_4 e^{\frac{-\mu L}{2}}. \end{aligned}$$

Therefore, $\dot{V}$ can be rewritten as follows

$$\dot{V} = -\int_0^L \begin{pmatrix} \phi_1(x,t)e^{-\frac{\mu x}{2}} \\ \xi_2(t) \\ \phi_1(L,t) \end{pmatrix}^T Q_1 \begin{pmatrix} \phi_1(x,t)e^{-\frac{\mu x}{2}} \\ \xi_2(t) \\ \phi_1(L,t) \end{pmatrix} dx$$
$$-\int_0^L \begin{pmatrix} \phi_2(x,t)e^{\frac{\mu x}{2}} \\ \xi_1(t) \\ \phi_2(0,t) \end{pmatrix}^T Q_2 \begin{pmatrix} \phi_2(x,t)e^{\frac{\mu x}{2}} \\ \xi_1(t) \\ \phi_2(0,t) \end{pmatrix} dx,$$

where

$$Q_1 = \frac{1}{L}\begin{pmatrix} \mu\lambda_1 L & dk_2q_4L + \frac{q_4\lambda_1}{2}\mu L & -cq_4L \\ dk_2q_4L + \frac{q_4\lambda_1}{2}\mu L & \rho\lambda_1 e^{\mu L}d^2\mu^2 & 0 \\ -cq_4L & 0 & \lambda_1 e^{-\mu L} \end{pmatrix},$$

$$Q_2 = \frac{1}{L}\begin{pmatrix} \mu q_1\lambda_2 L & ak_1q_5L + \frac{q_5\lambda_2}{2}\mu L & -bq_5L \\ ak_1q_5L + \frac{q_5\lambda_2}{2}\mu L & \lambda_1 a^2\mu^2 & 0 \\ -bq_5L & 0 & \lambda_2 q_1 \end{pmatrix}.$$

In the following it is proven that by picking μ small enough, the two matrices Q_1 and Q_2 are symmetric positive definite.

Regarding matrix Q_1, from the Sylvester criterion, Q_1 is symmetric positive definite if and only if

$$D_1 = det\begin{pmatrix} \mu\lambda_1 L & dk_2q_4L + \frac{q_4\lambda_1}{2}\mu L \\ dk_2q_4L + \frac{q_4\lambda_1}{2}\mu L & \rho\lambda_1 e^{\mu L}d^2\mu^2 \end{pmatrix} > 0,$$
$$D_2 = det(Q_1) > 0.$$

From (3.13) we have that

$$D_1 = \mu^3 L d^2 e^{\mu L}\left(\rho\lambda_1^2 - \mu\frac{b^2L}{4a^2}(2d^2+\lambda_1)^2\right),$$

and

$$D_2 = \lambda_1 e^{-\mu L}D_1 - c^2q_4^2L^2\lambda_1\rho e^{\mu L}d^2\mu^2,$$
$$= \mu^3 L d^2\lambda_1\left(\rho\lambda_1^2 - \mu\frac{b^2L}{4a^2}(2d^2+\lambda_1)^2 - \mu L e^{2\mu L}\frac{b^2d^2c^2}{a^2}\right).$$

Taking μ small enough, it yields that the two terms D_1, D_2 are both positive. Consequently, the matrix Q_1 is symmetric positive definite.

Similarly, it can be shown that the matrix Q_2 is symmetric positive definite when μ is small enough.

Therefore, if μ is small enough, there exists a positive real number $M > 0$ such that for all t at which the solution is well defined we have:

$$\dot{V}(t) \leqslant -M\Big(||(\phi_1, \phi_2, \xi_1, \xi_2)||^2_{\mathbb{X}} + \phi_1^2(L,t) + \phi_2^2(0,t)\Big).$$

With (3.14) the former inequality implies that we can find $\alpha > 0$ such that (3.15) holds. This completes the proof of Lemma 3.3. ■

Remark : Our result is an extension of the one in [18] in which a static feedback is considered and a Lyapunov function is constructed. In our work, since we have added some new dynamics $\xi_1(t)$ and $\xi_2(t)$ the Lyapunov function has to be modified to take into account these new states. This is also the case in [24] in which a similar topic is studied. Note however that the Lyapunov function that we have introduced in (3.10) is different than the one of [24] since it contains coupling terms between the state variables (ϕ_1, ϕ_2) and the feedback variables (ξ_1, ξ_2). This allows to consider a larger class of system since the unique condition on the physical parameters that has to be considered is that $abcd < 0$. This is an improvement compared to the result in [24] in which the constraints on the parameter are : $a > 0$, $b > 0$, $c > 0$, $d = -c$, $\dfrac{b}{a} < 1$, $a + b = 1$.

Note that another PI control for the case of $a = b = c = 0$ and $d = 1$ is solved with Lyapunov functional in [43]. ◀

3.3.3 Proof of the main result

In this section, by using the results of Lemma 3.3, we state the proof of Theorem 1.

3.3.3.1 Stability analysis

Firstly, we prove that the closed-loop system (3.8) is exponentially stable toward the origin in $\mathbb{X}$. From Lemma 3.3, there exists $\alpha > 0$ such that:

$$V(t) \leqslant V(0)e^{-\alpha t} .$$

Employing (3.14), it implies that there exists $S > 0$ such that for all initial conditions $(\phi_1^0(x), \phi_2^0(x), \xi_1^0, \xi_2^0) \in \mathbb{X}$ and satisfying the C^0 and C^1 compatibility conditions, the

solution are defined for all positive time and satisfies:

$$||(\phi_1, \phi_2, \xi_1, \xi_2)||^2_{\mathbb{X}} \leqslant Se^{-\alpha t}||(\phi_1^0, \phi_2^0, \xi_1^0, \xi_2^0)||^2_{\mathbb{X}}. \tag{3.16}$$

As a result, the origin of the closed-loop system (3.6) with PI controller design in (3.7) is exponentially stable in $\mathbb{X}$.

3.3.3.2 Proof of output regulation

It remains to show that the regulation is also obtained. First of all, if the initial condition $(\phi_1^0(x), \phi_2^0(x), \xi_1^0, \xi_2^0)$ is in $(H^1(0,L))^2 \times \mathbb{R}^2$ and satisfies C^0 and C^1 compatibility conditions, then the solution of (3.8) $(\phi_1(\cdot,t), \phi_2(\cdot,t), \xi_1(t), \xi_2(t))$ is in $(H^1(0,L))^2 \times \mathbb{R}^2$ for all t.

Now, by adding some high-order terms to the Lyapunov functional in (3.10), we prove the following lemma

Lemma 3.4. *If the initial condition* $(\phi_1^0(x), \phi_2^0(x), \xi_1^0, \xi_2^0)$ *is in* $(H^1(0,L))^2 \times \mathbb{R}^2$ *and satisfies* C^0 *and* C^1 *compatibility conditions, then the following holds true:*

$$\lim_{t\to\infty} \left(||\phi_1(x,t)||_{H^1(0,L)} + ||\phi_2(x,t)||_{H^1(0,L)} \right) = 0 \ . \tag{3.17}$$

Proof : Let us denote $\phi_{1x} = \partial_x \phi_1$ and $\phi_{2x} = \partial_x \phi_2$. The dynamics of $\phi_{1x}(x,t)$ and $\phi_{2x}(x,t)$ are given by

$$\begin{cases} \partial_t \begin{pmatrix} \phi_{1x} \\ \phi_{2x} \end{pmatrix} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} \partial_x \begin{pmatrix} \phi_{1x} \\ \phi_{2x} \end{pmatrix} = 0 \ , \\ \phi_{1x}(0,t) = \dfrac{k_1}{\lambda_1} \left(b\phi_2(0,t) - ak_1\xi_1(t) \right) \\ \phi_{2x}(L,t) = -\dfrac{k_2}{\lambda_2}(c\phi_1(L,t) - d\xi_2(t)) \end{cases} \tag{3.18}$$

with the initial conditions

$$(\phi_{1x}(x,0), \phi_{2x}(x,0)) \in (L^2(0,L))^2.$$

Now, considering the functional of ϕ_{1x} and ϕ_{2x}

$$V_1(\phi_{1x}, \phi_{2x}) = q \int_0^L \left(\phi_{1x}^2 e^{-\mu x} + \phi_{2x}^2 e^{\mu x} \right) dx$$

where q is a positive number.

Analysis the time derivative of V_1 along the solution of (3.18), one gets that

$$\dot{V}_1(t) = -\mu q \int_0^L \left(\lambda_1 \phi_{1x}^2 e^{-\mu x} + \lambda_2 \phi_{2x}^2 e^{\mu x}\right) dx - \phi_{1x}^2(L,t) e^{-\mu L} q\lambda_1 - \phi_{2x}^2(0,t)\lambda_2 q + \frac{k_1^2 q}{\lambda_1}\left(b\phi_2(0,t) - ak_1\xi_1(t)\right)^2 + \frac{k_2^2 q}{\lambda_2}\left(c\phi_1(L,t) - d\xi_2(t)\right)^2 . \quad (3.19)$$

With (3.19), one can find $N > 0$, $K > 0$ such that

$$\dot{V}_1(t) \leqslant -qN\Big(||\phi_{1x}||_{L^2(0,L)} + ||(\phi_{2x}||_{L^2(0,L)} + \phi_{1x}^2(L,t) + \phi_{2x}^2(0,t)\Big) + qK\Big(\phi_1^2(L,t) + \phi_2^2(0,t) + \xi_1^2(t) + \xi_2^2(t)\Big) \quad (3.20)$$

With (3.15), there exists $q > 0$ and $\alpha > 0$ such that

$$\dot{V}(t) + \dot{V}_1(t) \leqslant -\alpha V - qN\Big(||\phi_{1x}||_{L^2(0,L)} + ||(\phi_{2x}||_{L^2(0,L)}\Big)$$

It implies that there exists $\beta > 0$ such that

$$\partial_t(V + V_1) \leqslant -\beta(V + V_1)$$

With the similar analysis in the proof of stability, it is easily to prove that the origin of (3.18) is also exponentially stable in $L^2(0,L)$. This leads to the proof of (3.17). ■

Finally, we have the following lemma which is a special case of the Sobolev embedding theorem.

Lemma 3.5. *Let $f : [0,L] \to \mathbb{R}$ be a smooth function in $H^1(0,L)$, then there exists a positive real number C such that*

$$||f||_{L^\infty(0,L)} \leqslant C||f||_{H^1(0,L)} .$$

Employing Lemma 3.4 and Lemma 3.5 for $\phi_1(\cdot,t)$ and $\phi_2(\cdot,t)$, one can find that for $\forall x \in [0,L]$

$$\lim_{t\to\infty} \phi_1(x,t) = 0, \lim_{t\to\infty} \phi_2(x,t) = 0 ,$$

As a result, $\lim_{t\to\infty} ||y(t) - y_r||_{\mathbb{R}^2}^2 = 0$.
This completes the proof of Theorem 3.1.

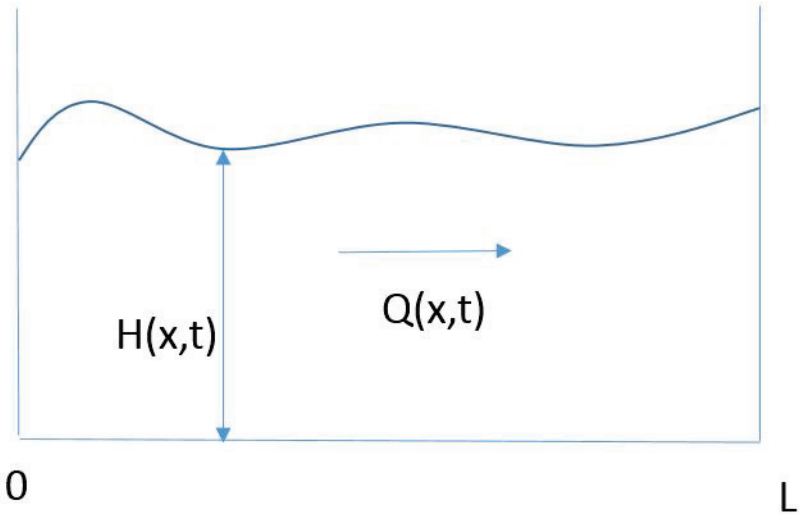

FIGURE 3.1: Saint-Venant model

3.4 Application and numerical simulations

3.4.1 Application to the Saint Venant nonlinear model

In this Section, we apply our PI control design to the Saint Venant model of a reach canal for shallow water (see in Figure 3.1) studied in [24, 56, 77] with the friction slope neglected:

$$\begin{cases} \dfrac{\partial H(x,t)}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{Q(x,t)}{B}\right) = 0, \\ \dfrac{\partial Q(x,t)}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{Q^2(x,t)}{BH(x,t)}\right) + gBH(x,t)\dfrac{\partial H(x,t)}{\partial x} = 0 \\ y(t) = (H(0,t), Q(L,t)) \end{cases} \tag{3.21}$$

equipped with two boundary conditions:

$$Q(0,t) = U_0(t)\ ,\ H(L,t) = U_L(t)$$

where $H(x,t)$ and $Q(x,t)$ represent the water level and the flow discharge respectively; B and g denote the base width and the Newton gravitation constant; $y(t)$ is the measured output on the boundary; U_0 and U_L are control actions for the system.

Let define $y_r = (H_r, Q_r)$ be the desired set-point satisfying the sub-critical condition $gB^2H_r^3 - Q_r^2 > 0$, and denoting $h = H - H_r$ and $q = Q - Q_r$, the linearized model of

(3.21) around the desired set-point is given as follows:

$$\begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} h \\ q \end{pmatrix} + \begin{pmatrix} 0 & \dfrac{1}{B} \\ \dfrac{-Q_r^2}{BH_r^2} + gBH_r & \dfrac{2Q_r}{BH_r} \end{pmatrix} \dfrac{\partial}{\partial x}\begin{pmatrix} h \\ q \end{pmatrix} = 0 \\ U_0(t) = Q_r + q(0,t) \\ U_L(t) = H_r + h(L,t) \\ y(t) = (h(0,t), q(L,t)) + y_r, \end{cases} \tag{3.22}$$

In addition, the characteristic form in (3.2) is obtained by using the following change of coordinates:

$$h = \phi_1 + \phi_2,\ q = (B\sqrt{gH_r} + \frac{Q_r}{H_r})\phi_1 - (B\sqrt{gH_r} - \frac{Q_r}{H_r})\phi_2$$

The two boundary conditions and the output $y(t)$ are rewritten in the following form:

$$\phi_1(0,t) = R_0\phi_2(0,t) + u_1(t)$$

$$\phi_2(L,t) = R_L\phi_1(L,t) + u_2(t)$$

$$y(t) - y_r = \begin{pmatrix} a\phi_1(0,t) + b\phi_2(0,t) \\ c\phi_1(L,t) + d\phi_2(L,t) \end{pmatrix}$$

where

$$R_0 = \frac{BH_r\sqrt{gH_r} - Q_r}{BH_r\sqrt{gH_r} + Q_r},\quad R_L = -1,$$
$$a = b = 1,\quad c = B\sqrt{gH_r} + \frac{Q_r}{H_r},\ d = -B\sqrt{gH_r} + \frac{Q_r}{H_r},$$

and the new control $u(t) = (u_1(t), u_2(t))$ defined by

$$u_1(t) = \frac{U_0(t) - Q_r}{B\sqrt{gH_r} + \dfrac{Q_r}{H_r}},\quad u_2(t) = U_L(t) - H_r \tag{3.23}$$

Finally, applying Theorem 3.1, we design the PI controller as follows:

$$u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = K_p(y(t) - y_r) + K_I\int_0^t (y(\tau) - y_r)d\tau, \tag{3.24}$$

where the matrices K_p and K_I are given by

$$K_p = \begin{pmatrix} \dfrac{Q_r - BH_r\sqrt{gH_r}}{Q_r + BH_r\sqrt{gH_r}} & 0 \\ 0 & \dfrac{H_r}{BH_r\sqrt{gH_r} + Q_r} \end{pmatrix},$$

$$K_I = \mu \begin{pmatrix} \dfrac{-2BH_r\sqrt{gH_r}}{Q_r + BH_r\sqrt{gH_r}} & 0 \\ 0 & \dfrac{2Q_r(Q_r - BH_r\sqrt{gH_r})}{H_rQ_r + BH_r^2\sqrt{gH_r}} \end{pmatrix}.$$

When the tuning parameter μ is chosen small enough, the PI controller stabilizes the linearized model (4.56) and regulates the output to the desired set-point.

3.4.2 Numerical simulations and results

3.4.2.1 Discretization by Preissmann scheme and numerical solution

In this Section, applying Preissmann scheme (see in Appendix), we give the discretized nonlinear equations of the model (3.21) by

$$F\Big(X(k+1)\Big) = 0 \tag{3.25}$$

where $F : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ defined by $F = \begin{pmatrix} f_1 \\ g_1 \\ \dots \\ f_{N-1} \\ g_{N-1} \\ f_N \\ g_N \end{pmatrix}$, and $X(k+1) = \begin{pmatrix} Z_1^{k+1} \\ Q_1^{k+1} \\ \dots \\ Z_N^{k+1} \\ Q_N^{k+1} \end{pmatrix}$.

with the $2N-2$ sub-functions f_i and g_i ($\mathrm{i} = \overline{1, N-1}$) received from the discretization of $2N-2$ PDE equations in (3.21)

$$\begin{aligned} f_i &= Z_{i+1}^{k+1} + Z_i^{k+1} - Z_{i+1}^k - Z_i^k + 2\frac{\Delta t}{\Delta x}\Big(\theta(Q_{i+1}^{k+1} - Q_i^{k+1}) + (1-\theta)(Q_{i+1}^k - Q_i^k\Big), \\ g_i &= Q_{i+1}^{k+1} + Q_i^{k+1} - Q_{i+1}^k - Q_i^k + 2\frac{\Delta t}{\Delta x}\left[\theta M_i + (1-\theta)H_i\right]. \end{aligned} \tag{3.26}$$

and two ODE boundary conditions

$$f_N = Z_N^{k+1} - U_L,$$

$$g_N = Q_1^{k+1} - U_0,$$

where

$$
\begin{aligned}
M_i &= (\frac{Q^2}{BZ})_{i+1}^{k+1} - (\frac{Q^2}{BZ})_i^{k+1} + gB(\theta\frac{Z_{i+1}^{k+1} + Z_i^{k+1}}{2} + (1-\theta)\frac{Z_{i+1}^k + Z_i^k}{2})(Z_{i+1}^{k+1} - Z_i^{k+1}),\\
H_i &= (\frac{Q^2}{BZ})_{i+1}^{k} - (\frac{Q^2}{BZ})_i^{k} + gB(\theta\frac{Z_{i+1}^{k+1} + Z_i^{k+1}}{2} + (1-\theta)\frac{Z_{i+1}^k + Z_i^k}{2})(Z_{i+1}^{k} - Z_i^{k}).
\end{aligned}
$$

Note also that the value at instant k of Z and Q are supposed to be computed previously and considered known values.

To continue, the implicit nonlinear equation (3.25) is resolved by Newton Raphson method, see the details in Appendix. To employ this iterative method, we need to compute the Jacobien matrix of function F as follows:

$$
J = \begin{pmatrix}
\frac{\partial f_1}{\partial Z_1^{k+1}} & \frac{\partial f_1}{\partial Q_1^{k+1}} & \frac{\partial f_1}{\partial Z_2^{k+1}} & \frac{\partial f_1}{\partial Q_2^{k+1}} & \dots & 0 & 0 & 0 & 0 \\
\frac{\partial g_1}{\partial Z_1^{k+1}} & \frac{\partial g_1}{\partial Q_1^{k+1}} & \frac{\partial g_1}{\partial Z_2^{k+1}} & \frac{\partial g_1}{\partial Q_2^{k+1}} & \dots & 0 & 0 & 0 & 0 \\
\dots & & & & & & & & \\
0 & 0 & 0 & 0 & \dots & \frac{\partial g_{N-1}}{\partial Z_{N-1}^{k+1}} & \frac{\partial g_{N-1}^{k+1}}{\partial Q_{N-1}^{k+1}} & \frac{\partial g_{N-1}}{\partial Z_N^{k+1}} & \frac{\partial g_{N-1}}{\partial Q_N} \\
0 & 0 & 0 & 0 & \dots & \frac{\partial f_{N-1}}{\partial Z_{N-1}^{k+1}} & \frac{\partial f_{N-1}}{\partial Q_{N-1}^{k+1}} & \frac{\partial f_{N-1}}{\partial Z_N^{k+1}} & \frac{\partial f_{N-1}}{\partial Q_N^{k+1}} \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & \frac{\partial f_N}{\partial Z_N^{k+1}} & \frac{\partial f_N}{\partial Q_N^{k+1}} \\
\frac{\partial g_N}{\partial Z_1^{k+1}} & \frac{\partial g_N}{\partial Q_1^{k+1}} & 0 & 0 & \dots & 0 & 0 & 0 & 0
\end{pmatrix}
$$

where

$$
\begin{aligned}
\frac{\partial f_i}{\partial Z_i^{k+1}} &= 1\ ,\ \frac{\partial f_i}{\partial Z_{i+1}^{k+1}} = 1\ ,\ \frac{\partial f_i}{\partial Q_i^{k+1}} = -2\frac{\Delta t}{\Delta x}\theta\ ,\ \frac{\partial f_i}{\partial H_{i+1}^{k+1}} = 2\frac{\Delta t}{\Delta x}\theta\ ,\\
\frac{\partial g_i}{\partial Z_i^{k+1}} &= 2\frac{\Delta t}{\Delta x}(\theta\frac{\partial M_i}{\partial Z_i^{k+1}} + (1-\theta)\frac{\partial H_i}{\partial Z_i^{k+1}})\ ,\ \frac{\partial g_i}{\partial Z_{i+1}^{k+1}} = 2\frac{\Delta t}{\Delta x}(\theta\frac{\partial M_i}{\partial Z_{i+1}^{k+1}} + (1-\theta)\frac{\partial H_i}{\partial Z_{i+1}^{k+1}}),\\
\frac{\partial g_i}{\partial Q_i^{k+1}} &= 1 + 2\frac{\Delta t}{\Delta x}(\theta\frac{\partial M_i}{\partial Q_i^{k+1}} + (1-\theta)\frac{\partial H_i}{\partial Q_i^{k+1}})\ ,\ \frac{\partial g_i}{\partial Q_{i+1}^{k+1}} = 1 + 2\frac{\Delta t}{\Delta x}(\theta\frac{\partial M_i}{\partial Q_{i+1}^{k+1}} + (1-\theta)\frac{\partial H_i}{\partial Q_{i+1}^{k+1}})\ ,\\
\frac{\partial f_N}{\partial Z_N^{k+1}} &= 1\ ,\ \frac{\partial f_N}{\partial Q_N^{k+1}} = 0\\
\frac{\partial g_N}{\partial Z_1^{k+1}} &= 0\ ,\ \frac{\partial g_N}{\partial Q_1^{k+1}} = 1\ .
\end{aligned}
$$

With

$$\begin{aligned}
\frac{\partial M_i}{\partial Z_i^{k+1}} &= \frac{(Q_i^{k+1})^2}{B(Z_i^{k+1})^2} - gB\Big(\theta Z_i^{k+1} + (1-\theta)\frac{Z_{i+1}^k + Z_i^k}{2}\Big) \ , \ \frac{\partial M_i}{\partial Q_i^{k+1}} = -\frac{2Q_i^{k+1}}{BZ_i^{k+1}} \ , \\
\frac{\partial M_i}{\partial Z_{i+1}^{k+1}} &= -\frac{(Q_{i+1}^{k+1})^2}{B(Z_{i+1}^{k+1})^2} + gB(\theta Z_{i+1}^{k+1} + (1-\theta)\frac{Z_{i+1}^k + Z_i^k}{2}) \ , \ \frac{\partial M_i}{\partial Q_{i+1}^{k+1}} = \frac{2Q_{i+1}^{k+1}}{BZ_{i+1}^{k+1}} \ , \\
\frac{\partial H_i}{\partial Z_i^{k+1}} &= gB\frac{\theta}{2}(Z_{i+1}^{k+1} - Z_i^{k+1}) \ , \ \frac{\partial H_i}{\partial Q_i^{k+1}} = 0 \ , \\
\frac{\partial H_i}{\partial Z_{i+1}^{k+1}} &= gB\frac{\theta}{2}(Z_{i+1}^{k+1} - Z_i^{k+1}) \ , \ \frac{\partial H_i}{\partial Q_{i+1}^{k+1}} = 0 \ .
\end{aligned}$$

3.4.2.2 Simulation results

In this section, we perform some numerical simulations for the model (3.21). The data used are the following consider the following: $L = 50m$, $H_r = 10m$, $Q_r = 5m^3/s$, $B = 1m$. Note that the subcritical condition is satisfied. The parameters for Preismann schema to make the simulations are weighting parameter $\theta = 0.6$, the ratio between space and time discretization steps $\Delta t/\Delta x = 0.5$, and the number of the discretized space point $N = 100$.

The PI controller designed in (3.24) with tunning with $\mu = 0.001$. Figure 3.2 and figure 3.3 show the evolutions of two states: water level $H(x,t)$ and water discharge Q(x,t) respectively. With the simulations, one can conclude about the stability obtained for the closed-loop system with PI controller.

Figure 3.5 and Figure 3.6 present the performance of the PI controller against the unknown constant disturbances. We consider the system with the constant output disturbance w_o and the control disturbance w_c, that means $y(t) = (H(0,t), Q(L,t)) + w_o$ and $(U_0, U_L) = (H(L,t), Q(0,t)) + w_c$, where $w_o = (0.1,\ 0.05)$ and $w_c = (0.02,\ 0.05)$. These disturbances lead to the error between the output $y(t)$ and the desired set-point $y_r = (H_r, Q_r)$ in the open-loop system at right simulations. However, by using PI controller in closed-loop system, this error is totally cancelled as seeing in simulations on the left.

In addition, the evolutions of two boundary control inputs $U_0(t) = Q(0,t)$ and $U_L(t) = H(L,t)$ are depicted in figure 3.4. Another remark is that, the bigger the integral gain μ is, the more the regulation is rapid, but the more the profile is oscillatory; and a big value of μ (for instance $\mu = 0.005$) results to the instability of the closed-loop system.

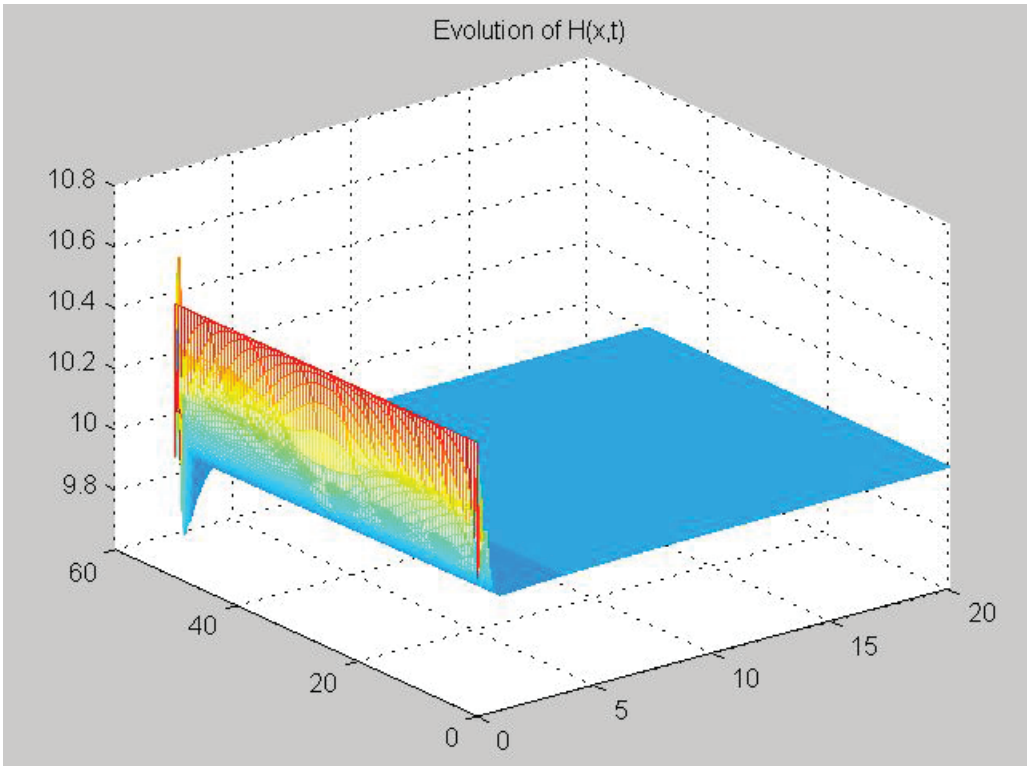

FIGURE 3.2: Evolution of H(x,t)

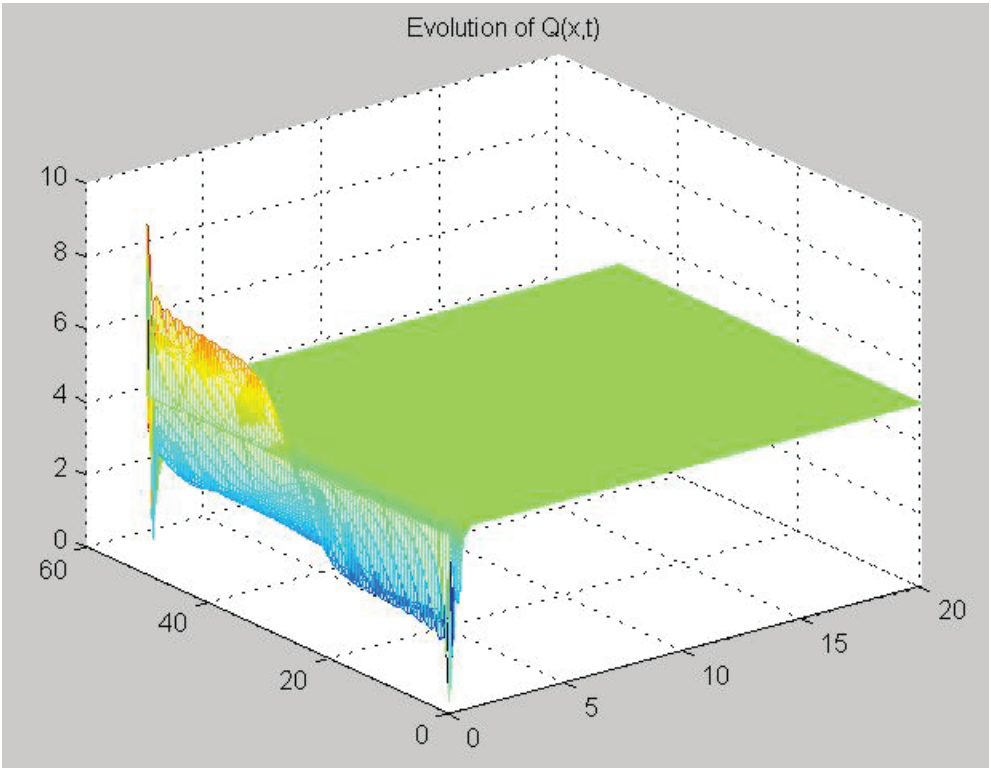

FIGURE 3.3: Evolution of Q(x,t)

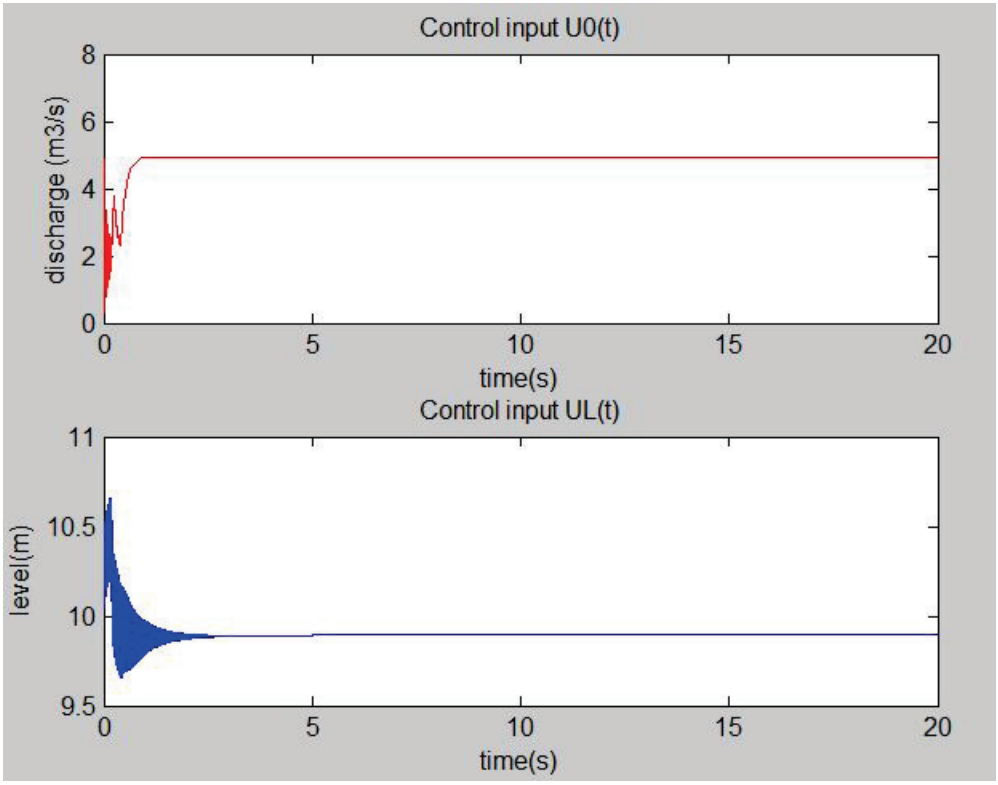

FIGURE 3.4: Evolution of U(t)

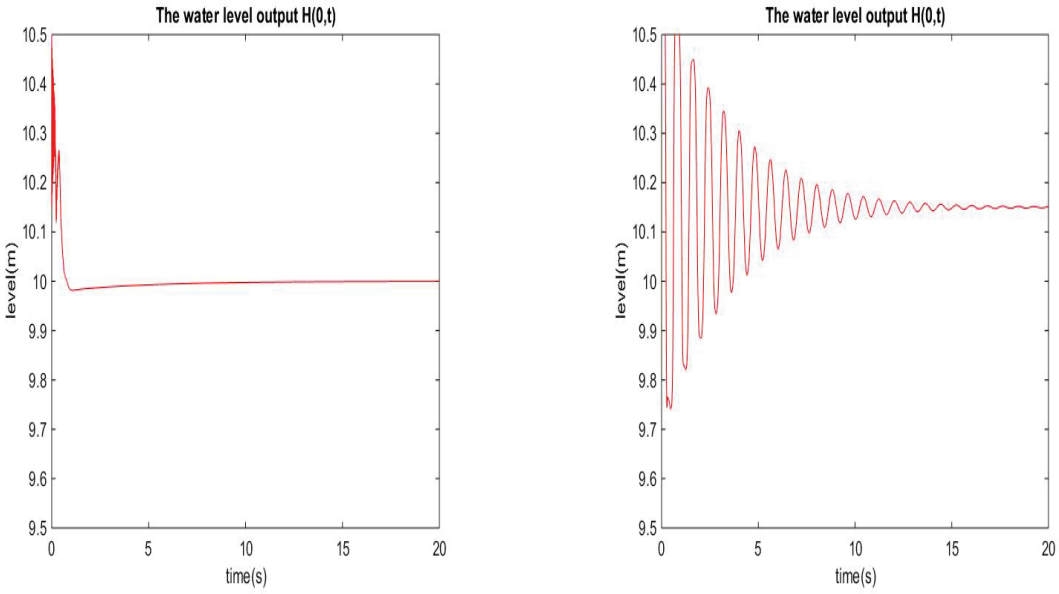

FIGURE 3.5: Output $H(0,t)$ with output disturbance w_o and control disturbance w_c of closed-loop system (left) and open-loop system (right).

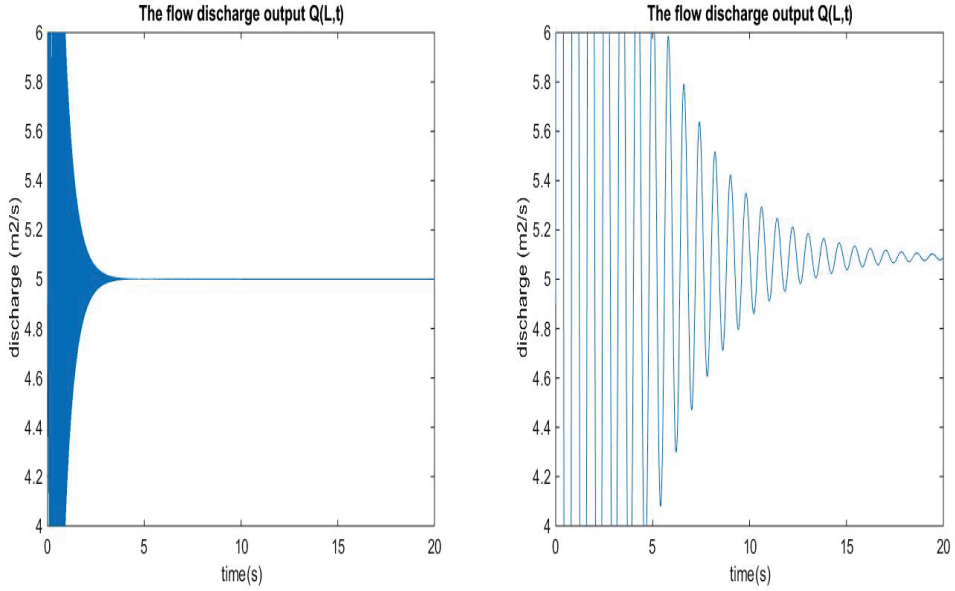

FIGURE 3.6: Output $Q(L,t)$ with output disturbance w_o and control disturbance w_c of closed-loop system (left) and open-loop system (right).

3.5 Conclusions

In this chapter, we have considered a class of linear 2×2 multiple input multiple output systems governed by hyperbolic partial differential equations with the problem of designing dynamic feedback under the form of multivariable PI controller on the boundary. The Lyapunov functional in the general quadratic form of state variables and feedback variables is used to prove the stability of the closed-loop system in L^2 norm and the regulation of the output to the desired set-point. The idea of constructing the Lyapunov function can be used for the higher order systems, not only 2×2 systems. Furthermore, an explicit PI controller given in the Theorem 3.1 is easily applied for many linearized model of hyperbolic systems, such as the Saint-Venant linerized model in the Section 4.5.2.3. In the future, the work is to extend for the nonlinear models with the nonlinear boundary conditions; and the optimization in the performance of the closed-loop system by controller will be considered as well.

The problem studied in this chapter was almost published in [68].

Chapter 4

Networks of hyperbolic systems: Stability and Output regulation

4.1 Introduction

In this chapter, we consider some networks of n systems described by the following two hyperbolic partial differential equations (PDE)

$$\begin{cases} \partial_t \psi_{i1}(x,t) + \lambda_{i1}\ \partial_x \psi_{i1}(x,t) & = 0 \\ \partial_t \psi_{i2}(x,t) - \lambda_{i2}\ \partial_x \psi_{i2}(x,t) & = 0 \end{cases} , \ x \in [0,L],\ t \in [0,\infty),\ i = \overline{1,n} \tag{4.1}$$

under the initial conditions:

$$\psi_{i1}(x,0) = \psi_{i1}^0(x)\ ,\ \psi_{i2}(x,0) = \psi_{i2}^0(x).$$

where $\psi_{i1},\ \psi_{i2} :\ [0,L]\times[0,\infty] \to \mathbb{R}$ are two states and λ_{i1}, λ_{i2} are two positive constants. Here ∂_t, ∂_x denote the time and space partial derivative respectively.

Each system in (4.1) is a 2×2 linear hyperbolic PDE model. This type of dynamics may be used to describe various physical phenomena. Indeed, a large number of models can be transformed into the form (4.1) by some change of coordinates as shown for example in[3, 4, 25, 56, 60].

Our goal in the chapter is to study two kinds of network: star-shaped network and cascaded one; and to design the proportional- integral (PI) dynamic control law in order to guarantee the stability of closed-loop networks and regulate output measurements to desired set-points.

The chapter is organized as follows. Section 4.2 is to review existing results in the literatures and present our contribution in general. In Section 4.3 and Section 4.4, we give the main results and their proofs based on Lyapunov direct method for the star-shaped network and cascaded one, respectively. Then we explain in details in Section 4.5.2.3 how to apply our general PI control designs for practical networks of n fluid flow models. Moreover, numerical simulations are carried out to validate the theoretical results. Finally, conclusions are addressed in Section 4.6.

4.2 Existing results and motivation

The problem of controlling hyperbolic networks with both inputs and outputs on the junctions becomes an important subject with a lot of studies in the literature. Most of results have established the boundary conditions at each junction to stabilize all sub-systems around the equilibrium state. Let us review significant results in the literature in this domain:

- In [35], a simple network of water flow in open channels with three reaches in cascade is studied. The static control law is given by using the Li Ta-Tsien theorem (Theorem 1.3 in [46, Chapter 5] to prove the stability of the closed-loop system in C^1 norm.

  In addtion, by employing the same theorem of Li Ta-Tsien, a static stabilization feedback control law is done for fan-shaped networks of gas flow pipe in [34].

- Based on Lyapunov technique, in [5],[6] they have demonstrated the stability in L^2 norm of the open-water channel cascaded network described by Saint-Venant equations with the static boundary control design.

- The finite time stabilization method is considered in [60] for tree-shaped networks of 2×2 hyperbolic systems of conservation laws.

- The PI boundary control was applied for the networks of density-flow systems governed by linear hyperbolic PDE equations with the frequent approach in [3].

- In [55], the receding horizon optimal control approach which is analysed for a network of 2×2 hyperbolic systems in cascade.

In the chapter, control inputs $U(t)$ and measured outputs $Y(t)$ are on the boundary for the star-shaped network, and on junctions for cascaded network. Our purpose is to design dynamic feedback control laws $U(t)$ with structure of PI controllers such that the networks of closed-loop systems are exponentially stable, and output measurements

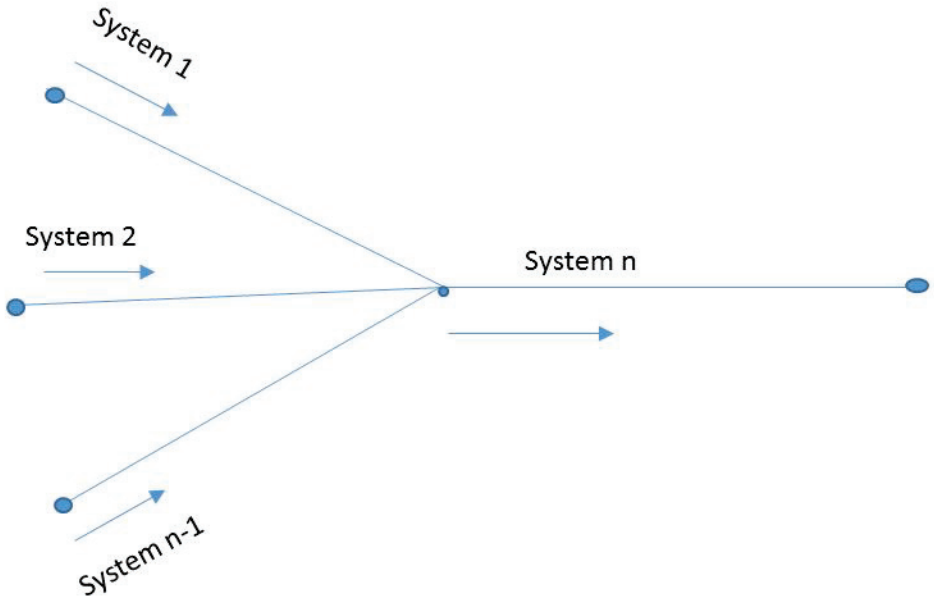

FIGURE 4.1: Star-shaped networks of n channels

$Y(t)$ are regulated to the desired references Y_r. To be more detail, we must design PI controllers with proportional gain matrix K_P, and integral oneK_I, i.e

$$U(t) = K_P\left(Y(t) - Y_r\right) + K_I \int_0^t (Y(s) - Y_r)ds \tag{4.2}$$

in order to obtain the exponential stability of the network of closed-loop systems and $\lim_{t\to\infty} Y(t) = Y_r$. The stability of closed-loop networks is considered by Lyapunov direct method, and the output regulation is proven based on the stability analysis combined with Sobolev embedding theorem.

In the following, we give the results and their proofs in details, for the star-shaped network in Section 4.3, and for cascaded network in Section 4.4.

4.3 Star-shaped network

4.3.1 Problem statement

In this section, we consider a star-shaped network of n PDE systems in (4.1). The connection between systems is depicted in Fig. 4.1, with $n-1$ "inlet" systems (1th to $(n-1)$th) and an "outlet" system (nth channel). To describe the star-shaped network, we suppose the following boundary conditions:

- For $n-1$ inlet systems: $(j = \overline{1, n-1})$

$$\begin{aligned} \psi_{j1}(0,t) &= R_{j1}\psi_{j2}(0,t) + u_j(t), \\ \psi_{j2}(L,t) &= R_{j2}\psi_{j1}(L,t), \end{aligned} \tag{4.3}$$

- For the outlet system:

$$\begin{aligned} \psi_{n1}(0,t) &= R_{n1}\psi_{n2}(0,t) + \sum_{j=1}^{n-1} \alpha_j \psi_{j1}(L,t), \\ \psi_{n2}(L,t) &= R_{n2}\psi_{n1}(L,t) \end{aligned} \tag{4.4}$$

where α_j, R_{j1}, R_{j2}, R_{n1} and R_{n2} are real constants depending on physical models; $u_j(t)$ is the dynamic feedback control law located at $x = 0$, which we want to design by PI controller for jth channel.

Moreover, $n-1$ measured outputs are considered for each inlet system:

$$y_j(t) - y_{jr} = a_j\psi_{j1}(0,t) + b_i\psi_{j2}(0,t), \tag{4.5}$$

where a_j, b_j are real constants, and y_{jr} is desired reference for channel j.

To summarize, the dynamic feedback PI control laws $u_j(t)$ are as follows

$$u_j(t) = K_{jP}(y_j(t) - y_{jr}) + K_{jI}\int_0^L (y_j(s) - y_{jr})ds \tag{4.6}$$

such that the network of closed-loop systems is stabilized toward the origin and the outputs $y_j(t)$ of inlet channels are regulated to the reference y_{jr} $(\forall j = \overline{1, n-1})$.

Remark : It should be mentioned here that the stability of the origin of closed-loop network implies the output regulation. This is because if we can prove that $\lim_{t\to\infty} \psi_{i1}(.,t) = 0$ and $\lim_{t\to\infty} \psi_{i2}(.,t) = 0$ for all $x \in [0, L]$, by employing the output form in (4.29), one deduces that $\lim_{t\to\infty} y_j(t) = y_{jr}$ for all $j = \overline{1, n-1}$. ◀

4.3.2 Main result

Denoting the new state variables $\xi_j(t)$ where $\partial_t \xi_j = y_j(t) - y_{jr}$, the network of closed-loop systems (4.1), (4.3), (4.4) and (4.6) is governed by :

$$\begin{cases} \partial_t \psi_{i1}(x,t) &= -\lambda_{i1}\partial_x \psi_{i1}, \\ \partial_t \psi_{i2}(x,t) &= \lambda_{i2}\partial_x \psi_{i2}, \\ \partial_t \xi_j &= a_j \psi_{j1}(0,t) + b_j \psi_{j2}(0,t) \\ \psi_{i2}(L,t) &= R_{i2}\psi_{i1}(L,t), \\ \psi_{j1}(0,t) &= R_{j1}\psi_{j2}(0,t) + K_{jP}(y_j(t) - y_{jr}) + K_{jI}\xi_j(t), \\ \psi_{n1}(0,t) &= R_{n1}\psi_{n2}(0,t) + \sum_{j=1}^{n-1} \alpha_j \psi_{j1}(L,t). \\ y_j(t) - y_{jr} &= a_j \psi_{j1}(0,t) + b_j \psi_{j2}(0,t). \end{cases} \tag{4.7}$$

The closed-loop system (4.7) is completed by the following initial conditions

$$\left(\psi_{11}^0(x), \psi_{12}^0(x), \cdots, \psi_{n1}^0(x), \psi_{n2}^0(x), \xi_1^0, \cdots, \xi_{n-1}^0\right) \in (L^2(0,L))^{2n} \times \mathbb{R}^{n-1} \tag{4.8}$$

Let $\mathbb{E} = (L^2(0,L))^{2n} \times \mathbb{R}^{n-1}$ be the state space of the closed-loop system (4.7) equipped with the following norm :

$$||Y||_{\mathbb{E}}^2 = \sum_{i=1}^{2n} ||Y_i(.,t)||_{L^2(0,L)}^2 + \sum_{j=2n+1}^{3n-1} |Y_j(t)|^2 \ ,$$

where $Y = (Y_1, Y_2, \cdots, Y_{3n-1}) \in \mathbb{E}$.

The main result of this Section is given in the following theorem

Theorem 4.1. *There exists $\mu^* > 0$ such that for each $\mu \in (0, \mu^*)$, each $R_{n2} \in \left(-\frac{\lambda_{n1}}{\lambda_{n2}}, \frac{\lambda_{n1}}{\lambda_{n2}}\right)$ and for each PI controller with the following proportional gain K_{jP} and the integral gain K_{jI}*

$$K_{jP} = \frac{-R_{j1}}{b_j} \ , \ K_{jI} = -\mu \ \frac{(a_j + b_j R_{j2} e^{\mu L})(b_j + a_j R_{j1})}{b_j} \tag{4.9}$$

then the two following properties hold true :

- *The network of closed-loop systems (4.7) with the initial condition in (4.8) is exponentially stable toward the origin in $\mathbb{E}$.*

- *If the initial conditions of each inlet systems* $(\psi_{j1}(x,0),\psi_{j2}(x,0)) \in (H^1(0,L))^2$, *then the measured outputs* $y_j(t)$ *are regulated to the desired set-points* y_{jr}, *i.e*

$$\lim_{t\to\infty} |y_j(t) - y_{jr}| = 0$$

4.3.3 Proof of Theorem 4.1

To prove Theorem 4.1, we construct the following Lyapunov candidate functional for the closed-loop network (4.7):

$$\mathbb{S} = \sum_{j=1}^{n-1} V_j + V_n \tag{4.10}$$

where

$$V_j = \int_0^L F_j^T P_j F dx$$

with $F_j = \begin{pmatrix} \psi_{j1}e^{-\frac{\mu x}{2}} \\ \psi_{j2}e^{\frac{\mu x}{2}} \\ \xi_j \end{pmatrix}$, $P_j = \begin{pmatrix} 1 & 0 & q_{j3} \\ 0 & q_{j1} & q_{j4} \\ q_{j3} & q_{j4} & q_{j2} \end{pmatrix}$, and

$$V_n = q_n \int_0^L \left(q\psi_{n1}^2 e^{-\mu x} + \psi_{n2}^2 e^{\mu x}\right) dx$$

Note that each sub-functional V_j is independent on others. Therefore we consider at first each sub-functional V_j and design the parameter q_{j1}, q_{j2}, q_{j3}, q_{j4} such that each V_j is a Lyapunov functional. Then, by taking appropriate values of q_n and q, we prove that $\mathbb{S}$ is also a Lyapunov functional.

4.3.3.1 Lyapunov candidate functional for each inlet system

To simplify the writing, in this Section we omit the symbol j, and all notations used imply for jth system. At first, applying the PI controller design in (4.9), each closed-loop of each inlet system can be described as follows

$$\begin{cases} \partial_t \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} &= \begin{pmatrix} -\lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \partial_x \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \\ \partial_t \xi &= a\psi_1(0,t) + b\psi_2(0,t) \\ \psi_1(0,t) &= -k_I \xi(t) \\ \psi_2(L,t) &= R_2 \psi_1(L,t) \\ y(t) - y_r &= a\psi_1(0,t) + b\psi_2(0,t). \end{cases} \tag{4.11}$$

where

$$k_I = \mu(a + bR_2 e^{\mu L}) \tag{4.12}$$

Let denote the set $\mathbb{X} = (L^2(0,L))^2 \times \mathbb{R}$ associated with the following norm

$$||v||_{\mathbb{X}}^2 = ||v_1||_{L^2(0,L)}^2 + ||v_2||_{L^2(0,L)}^2 + |v_3|^2$$

where $v = (v_1, v_2, v_3) \in \mathbb{X}$. The following lemma gives the design of each Lyapunov sub-functional.

Lemma 4.2. *Let k_I be defined in (4.12) and $q_1,\ q_2,\ q_3,\ q_4$ be defined by*

$$\begin{aligned} q_1 &= \frac{\lambda_1 e^{-2\mu L}}{2\lambda_2 R_2^2}\ ,\ q_2 = \mu\lambda_1, \\ q_3 &= \mu b R_2 e^{\mu L}\ ,\ q_4 = \mu\frac{b\lambda_1}{\lambda_2}. \end{aligned} \tag{4.13}$$

Then there exists $\gamma > 0$ and $\mu^* > 0$ such that for every $\mu \in (0, \mu^*)$, we have :

1. *There exists $M > 0$ such that $\forall\ (\psi_1, \psi_2, \xi)$ in $\mathbb{X}$:*

$$\frac{1}{M}V(\psi_1, \psi_2, \xi) \leqslant ||(\psi_1, \psi_2, \xi)||_{\mathbb{X}}^2 \leqslant MV(\psi_1, \psi_2, \xi). \tag{4.14}$$

2. *There exists $\beta > 0$ such that along the solution of each closed-loop inlet system, for all t at which the solution is well defined*

$$\frac{dV}{dt} \leqslant -\beta V - \psi_1^2(L,t)\frac{\lambda_1 e^{-\mu L}}{2}. \tag{4.15}$$

Proof : Proof of the first part

To begin with, we prove that matrix P is definite positive. Applying the Sylvester criterion, matrix P is positive definite if and only if :

$$q_1 > 0,\ q_2 > q_3^2 + \frac{q_4^2}{q_1}.$$

Employing (4.13), one can find that :

$$q_1 > 0,\ q_2 > 0\ ,$$

$$q_2 - q_3^2 - \frac{q_4^2}{q_1} = \mu\left(\lambda_1 - \mu b^2 R_2^2 e^{2\mu L} - \mu b^2 R_2^2 e^{2\mu L}\frac{2\lambda_1}{\lambda_2}\right).$$

It is clearly that if μ is small enough, $q_2 - q_3^2 - \frac{q_4^2}{q_1} > 0$. It therefore yields that the matrix P is symmetric positive definite. Hence, there exits $\sigma_1, \sigma_2 > 0$ such that

$$\sigma_1 \int_0^L \begin{pmatrix} \psi_1 e^{-\frac{\mu x}{2}} \\ \psi_2 e^{\frac{\mu x}{2}} \\ \xi \end{pmatrix}^T \begin{pmatrix} \psi_1 e^{-\frac{\mu x}{2}} \\ \psi_2 e^{\frac{\mu x}{2}} \\ \xi \end{pmatrix} dx \leqslant V(\psi_1, \psi_2, \xi) \leqslant \sigma_2 \int_0^L \begin{pmatrix} \psi_1 e^{-\frac{\mu x}{2}} \\ \psi_2 e^{\frac{\mu x}{2}} \\ \xi \end{pmatrix}^T \begin{pmatrix} \psi_1 e^{-\frac{\mu x}{2}} \\ \psi_2 e^{\frac{\mu x}{2}} \\ \xi \end{pmatrix} dx.$$

As a result, there exists $M > 0$ such that (4.14) holds.

Proof of the second part

The time derivative of V along the solution of each inlet closed-loop system has the following form:

$$\dot{V} = -\int_0^L \mathbf{\Lambda}^T Q \mathbf{\Lambda} - \quad L(t) \ ,$$

where $\mathbf{\Lambda} = \begin{pmatrix} \psi_1 e^{-\frac{\mu x}{2}} \\ \psi_2 e^{\frac{\mu x}{2}} \\ \xi \\ \psi_2(0,t) \end{pmatrix}$, and

$$L(t) = \psi_1^2(L,t)\frac{\lambda_1 e^{-\mu L}}{2} \ + \psi_2^2(0,t)\frac{\lambda_1 e^{-2\mu L}}{4R_2^2} + \ \xi^2(t)\frac{\lambda_1 k_i^2}{2} \ - \frac{\gamma k_i^2}{\lambda_1}\left(b\psi_2(0,t) - ak_i\xi(t)\right)^2,$$

$$Q = \frac{1}{L}\begin{pmatrix} \mu\lambda_1 L & 0 & T_{\psi_1,\xi} & T_{\psi_1,\psi_{20}} \\ 0 & \mu q_1 \lambda_2 L & T_{\psi_2,\xi} & T_{\psi_2,\psi_{20}} \\ T_{\psi_1,\xi} & T_{\psi_2,\xi} & T_\xi & 0 \\ T_{\psi_1,\psi_{20}} & T_{\psi_2,\psi_{20}} & 0 & \frac{\lambda_2 q_1}{2} \end{pmatrix}$$

with

$$T_\xi = \mu^2\frac{\lambda_1 c^2}{2} \ , \ T_{\psi_1,\psi_20} = -\mu L b^2 R_2 e^{\mu L},$$
$$T_{\psi_2,\psi_20} = -\mu L\frac{b^2\lambda_1}{\lambda_2} \ , T_{\psi_1,\xi} = \mu^2 L b R_2 e^{\mu L} \ \frac{2ac+\lambda_1}{2} \ ,$$
$$T_{\psi_2,\xi} = \mu^2 L b\lambda_1 \ \frac{2ac+\lambda_2}{2\lambda_2}, c = a + bR_2 e^{\mu L}.$$

At first, one can easily see that if taking γ small enough, we have $L(t) \geqslant \psi_1^2(L,t)\frac{\lambda_1 e^{-\mu L}}{2}$. In the following, we prove that by picking μ small enough, matrix Q is symmetric positive

definite. From the Sylvester criterion, Q is symmetric positive definite if and only if

$$D_1 = det\begin{pmatrix} \mu\lambda_1 L & 0 & T_{\psi_1,\xi} \\ 0 & \mu q_1\lambda_2 L & T_{\psi_2,\xi} \\ T_{\psi_1,\xi} & T_{\psi_2,\xi} & T_\xi \end{pmatrix} > 0,$$

$$D_2 = det\begin{pmatrix} \mu\lambda_1 L & 0 & T_{\psi_1,\xi} & T_{\psi_1,\psi_{20}} \\ 0 & \mu q_1\lambda_2 L & T_{\psi_2,\xi} & T_{\psi_2,\psi_{20}} \\ T_{\psi_1,\xi} & T_{\psi_2,\xi} & T_\xi & 0 \\ T_{\psi_1,\psi_{20}} & T_{\psi_2,\psi_{20}} & 0 & \frac{\lambda_2 q_1}{2} \end{pmatrix} > 0.$$

By the direct computing, we have that

$$D_1 = \mu^4 f_1(\mu)\ ,\ D_2 = \mu^4 f_2(\mu)$$

with $\lim_{\mu\to 0} f_1(\mu) = \frac{1}{2}L^2\lambda_1^2\lambda_2 c^2 q_1 > 0$ and $\lim_{\mu\to 0} f_2(\mu) = \frac{1}{4}L^2\lambda_1^2\lambda_2^2 c^2 q_1^2 > 0$.
Taking μ small enough, it yields that the two terms D_1, D_2 are both positive. Consequently, the matrix Q is symmetric positive definite.
Therefore, with the adaptive choice of γ and μ, there exists a positive real number $K > 0$ such that for all t at which the solution is well defined we have:

$$\dot{V}(t) \leqslant -K||(\psi_1,\psi_2,\xi)||^2_{\mathbb{X}} \ -\ \psi_1^2(L,t)\frac{\lambda_1 e^{-\mu L}}{2}.$$

With (3.14) the former inequality implies that we can find $\beta > 0$ such that (4.15) holds. This completes the proof of Lemma 4.2. ■

Remark : Lemma 4.2 shows a Lypunov functional constructed for a single system in the case of single input-single output. A more complex situation of multi inputs-multi outputs is considered in chapter 3. In [3], by frequency method, another result of PI control for single system is presented. Another work in the same context with Lyapunov direct method is given in [4]. However the Lyapunov functional constructed in this work has not coupled term (the term q_3, q_4 in our Lyapunov functional), therefore the result obtained is restrained for only the case of parameter $|R_2| < 1$. ◄

4.3.3.2 Proof of the theorem

Proof of the stability

Employing the Lemma 4.2, we can design the parameters $q_{j1}, \cdots, q_{j4}$ for every $j = \overline{1, n-1}$ such that

- There exists $M_j > 0$ such that

$$\frac{1}{M_j} V_j \leqslant ||(\psi_{j1}, \psi_{j2}, \xi_j)||^2_{\mathbb{X}} \leqslant M_j V_j. \tag{4.16}$$

- There exists $\beta_j > 0$ such that

$$\dot{V}_j(t) \leqslant -\beta_j V_j(t) - \psi^2_{j1}(L,t)\frac{\lambda_{j1} e^{-\mu L}}{2}. \tag{4.17}$$

Now analysis the time derivative of V_n along the solution of the closed-loop network,

$$\begin{aligned}\dot{V}_n(t) = -\mu q_n \int_0^L \left(\lambda_{n1} q \psi^2_{n1} e^{-\mu x} + \lambda_{n2} \psi^2_{n2} e^{\mu x}\right) dx \\ - \psi^2_{n1}(L,t) e^{-\mu L} q_n (\lambda_{n1} q - \lambda_{n2} R^2_{n2}) - \psi^2_{n2}(0,t)\lambda_{2n} q_n \\ + \lambda_{n1} q_n q \left(R_{n1}\psi_{n2}(0,t) + \sum_{j=1}^{n-1} \alpha_j \psi_{j1}(L,t)\right)^2 .\end{aligned} \tag{4.18}$$

Since $|R_{n2}| < \dfrac{\lambda_{n1}}{\lambda_{n2}}$, one can choose q such that $\dfrac{R^2_{n2}\lambda_{n2}}{\lambda_{n1}} < q < \dfrac{\lambda_{n1}}{\lambda_{n2}}$, and take $q_n > 0$ small enough, we have

$$\begin{aligned}-\sum_{j=1}^{n-1} \psi^2_{j1}(L,t)\frac{\lambda_{j1} e^{-\mu L}}{2} - \psi^2_{n2}(0,t)\lambda_{n2} q_n + \lambda_{n1} q_n q \left(R_{n1}\psi_{n2}(0,t) + \sum_{j=1}^{n-1} \alpha_j \psi_{j1}(L,t)\right)^2 \leqslant 0 \,, \\ - \psi^2_{n1}(L,t) e^{-\mu L}(\lambda_{n1} q - \lambda_{n2} R^2_{n2}) q_n \leqslant 0 \,.\end{aligned} \tag{4.19}$$

From (4.17), (4.18) and (4.51), there exits $\delta > 0$ such that

$$\sum_{j=1}^{n-1} \dot{V}_j + \dot{V}_n \leqslant -\delta \left(\sum_{j=1}^{n-1} V_j + V_n \right)$$

This implies that

$$\dot{\mathbb{S}}(t) \leqslant -\delta \mathbb{S}(t) \tag{4.20}$$

Moreover, one can find $A > 0$ such that

$$\frac{1}{A}\mathbb{S} \leqslant ||\left(\psi_{11}, \psi_{12}, \cdots, \psi_{n1}, \psi_{n2}, \xi_1, \cdots, \xi_{n-1}\right)||_{\mathbb{E}} \leqslant A\ \mathbb{S} \tag{4.21}$$

From (4.20) and (4.21), it leads to the exponential stability toward the origin of network closed-loop system (4.7) in $\mathbb{E}$.

Proof of the output regulation

We use the same technique with the output regulation proof in Section 3.3.3.2 from Chap 3. We prove that with the initial condition $(\psi_{j1}(x,0), \psi_{j2}(x,0)) \in (H^1(0,L))^2$, then $\forall x \in [0,L]$

$$\lim_{t\to\infty} ||\psi_{j1}(x,t)||_{H^1(0,L)} = 0\ ,\ \lim_{t\to\infty} ||\psi_{j2}(x,t)||_{H^1(0,L)} = 0. \tag{4.22}$$

Let us put $\psi_{j1x} = \partial_x\psi_{j1}$ and $\psi_{j2x} = \partial_x\psi_{j2}$. The dynamics of $\psi_{j1x}(x,t)$ and $\psi_{j2x}(x,t)$ are given by

$$\begin{aligned}
&\partial_t \begin{pmatrix}\psi_{j1x}\\ \psi_{j2x}\end{pmatrix} + \begin{pmatrix}\lambda_{j1} & 0\\ 0 & -\lambda_{j2}\end{pmatrix}\partial_x \begin{pmatrix}\psi_{j1x}\\ \psi_{j2x}\end{pmatrix} = 0\ ,\\
&(\psi_{j1x}(x,0), \psi_{j2x}(x,0)) \in (L^2(0,L))^2\ ,\\
&\psi_{j1x}(0,t) = \frac{k_I}{\lambda_{j1}}(a_j\psi_{j1}(0,t) + b\psi_{j2}(0,t))\\
&\psi_{j2x}(L,t) = -\frac{R_{j2}\lambda_{j1}}{\lambda_{j2}}\psi_{j1x}(L,t)
\end{aligned} \tag{4.23}$$

Now, considering the following functional of ψ_{j1x} and ψ_{j2x}

$$V_{j1}(\psi_{j1x}, \psi_{j2x}) = \gamma\int_0^L \left(q_5\psi_{j1x}^2 e^{-\mu x} + \psi_{j2x}^2 e^{\mu x}\right) dx$$

where γ and q_5 are positive numbers.

Analysis the time derivative of V_{j1} along the solution of closed-loop system j, we have

$$\begin{aligned}
\frac{dV_{j1}}{dt} = -\mu\gamma\int_0^L \left(q_5\lambda_{j1}\psi_{j1x}^2 e^{-\mu x} + \lambda_{j2}\psi_{j2x}^2 e^{\mu x}\right) dx - \psi_{j1x}^2(L,t)\gamma\Big(e^{-\mu L}q_5\lambda_{j1} - \frac{R_{j2}^2\lambda_{j1}^2}{\lambda_{j2}^2}\Big)\\
- \psi_{j2x}^2(0,t)\gamma\lambda_{j2} + \frac{k_I^2\gamma q_5}{\lambda_{j1}}\left(-k_I a_j\xi_j(t) + b_j\psi_{j2}(0,t)\right)^2.
\end{aligned} \tag{4.24}$$

From (4.24), if choosing $q_5 > \dfrac{R_{j2}^2\lambda_{j1}}{\lambda_{j2}^2}e^{\mu L}$ one can find $N_j > 0$, $K_j > 0$ such that

$$\frac{dV_{j1}}{dt} \leqslant -\gamma N_j\Big(||\psi_{j1x}||_{L^2(0,L)} + ||(\psi_{j2x}||_{L^2(0,L)} + \psi_{j1x}^2(L,t) + \psi_{j2x}^2(0,t)\Big) + \gamma K_j\Big(\psi_{j2}^2(0,t) + \xi_j^2(t)\Big) \quad (4.25)$$

With (4.17), there exists $\zeta > 0$ such that

$$\frac{d(V_j + V_{j1})}{dt} \leqslant -\zeta(V + V_{j1})$$

With the similar analysis in the proof of stability, it is easily to prove that the origin of each closed-loop inlet system is also exponentially stable in $H^1(0,L)$. Employing the Sobolev embedding theorem in Lemma 3.5, this leads to the proof of (4.22).

As a result, $\lim\limits_{t\to\infty}|y_j(t) - y_{jr}| = 0$. $\forall i = \overline{1, n-1}$.

This completes the proof of Theorem 4.1.

4.4 Cascaded network

4.4.1 Problem statement

In this Section we are concerned with a class of cascaded networks of n PDE hyperbolic systems (4.1) (see Figure 4.2). In addition, we suppose that the boundary conditions define a cascaded network. More precisely, the following boundary conditions are defined at the junctions

- For the first junction,
$$\psi_{11}(0,t) = R_{11}\psi_{12}(0,t) \quad (4.26)$$
- For n-1 intermediate junctions,
$$\begin{aligned}\psi_{i2}(L,t) &= R_{i2}\psi_{i1}(L,t) + u_i(t)\\ \psi_{(i+1)1}(0,t) &= R_{(i+1)1}\psi_{(i+1)2}(0,t) + \alpha_i\psi_{i1}(L,t) + \delta_i\psi_{i2}(L,t)\end{aligned}, \ i = \overline{1, n-1} \quad (4.27)$$
- For the last junction,
$$\psi_{n2}(L,t) = R_{n2}\psi_{n1}(L,t) + u_n(t) \quad (4.28)$$



FIGURE 4.2: Cascaded network of n systems

where u_i are control inputs located at $x = L$ of each junction; R_{i1}, R_{i2}, α_i and δ_i are real constants, with $i = \overline{1,n}$.

The measured outputs that need to be regulated are located at the junctions and disturbed by some unknown output perturbations, i.e

$$y_i(t) = a_i\psi_{i1}(L,t) + b_i\psi_{i2}(L,t) + w_{io} \tag{4.29}$$

where a_i, b_i are unknown constants dependant on chosen outputs and $w_{io} \in \mathbb{R}$ are unknown constant perturbations.

As mentioned in Section 4.2 above, we design n PI controllers located at the junctions with real gain parameters K_{iP}, K_{iI}, and corrupted by some unknown constant control disturbances w_{ic}, i.e

$$u_i(t) = K_{iP}(y_i(t) - y_{ir}) + K_{iI}\int_0^t (y_i(s) - y_{ir})ds + w_{ic} \tag{4.30}$$

such that the network of closed-loop systems (4.1)-(4.30) is exponentially stable and $\lim_{t\to\infty} y_i(t) = y_{ir}$.

4.4.2 Main result

Denoting the new state variables $z_i(t)$ where $\partial_t z_i = y_i(t) - y_{ir}$, the network of closed-loop systems (4.1)-(4.30) becomes a PDE-ODE system and is governed by ($i = \overline{1,n}$ and

$j = \overline{2,n}$):

$$\begin{cases} \partial_t \psi_{i1}(x,t) &= -\lambda_{i1}\partial_x \psi_{i1}, \\ \partial_t \psi_{i2}(x,t) &= \lambda_{i2}\partial_x \psi_{i2}, \\ \partial_t z_i &= y_i(t) - y_{ir}, \\ \psi_{11}(0,t) &= R_{11}\psi_{12}(0,t), \\ \psi_{i2}(L,t) &= R_{i2}\psi_{i1}(L,t) + K_{iP}(y_i(t) - y_{ir}) + K_{iI}z_i(t) + w_{ic}, \\ \psi_{j1}(0,t) &= R_{j1}\psi_{j2}(0,t) + \alpha_j \psi_{(j-1)1}(L,t) + \delta_j \psi_{(j-1)2}(L,t), \\ y_i(t) &= a_i\psi_{i1}(L,t) + b_i\psi_{i2}(L,t) + w_{io}. \end{cases} \tag{4.31}$$

The closed-loop system (4.31) is completed by the following initial conditions

$$\left(\psi_{11}^0(x), \psi_{12}^0(x), z_1^0, \cdots, \psi_{n1}^0(x), \psi_{n2}^0(x), z_n^0\right) \in \left((L^2(0,L))^2 \times \mathbb{R}\right)^n, \quad \forall x \in [0,L]$$

which satisfy the C^0 and C^1 compatibility conditions in (4.31).

Let $\mathbb{E} = \left((L^2(0,L))^2 \times \mathbb{R}\right)^n$ be the state space of the closed-loop system (4.31) equipped with the following norm:

$$||Y||_{\mathbb{E}}^2 = \sum_{i=1}^{n} \left(||Y_{3i-2}||_{L^2(0,L)}^2 + ||Y_{3i-1}||_{L^2(0,L)}^2 + Y_{3i}^2 \right)$$

where $Y = (Y_1, Y_2, \cdots, Y_{3n}) \in \mathbb{E}$.

For each initial condition $\left(\psi_{11}^0(x), \psi_{12}^0(x), z_1^0, \cdots, \psi_{n1}^0(x), \psi_{n2}^0(x), z_n^0\right)$ in $\mathbb{E}$ satisfying C^0 and C^1 compatibility conditions, then there exits a unique smooth solution of (4.31) in $\mathbb{E}$ for all t (see in [20]). We study therefore the exponential stability of the closed-loop system (4.31) in $\mathbb{E}$ and the output regulation to the desired references.

Let us define iteratively equilibrium points $\psi_{i1\infty}$, $\psi_{i2\infty}$ and $z_{i\infty}$ as follows

$$\psi_{i2\infty} = \frac{y_{ir} - w_{io} + a_i D_i}{b_i + a_i R_{i1}}\ ,\ \psi_{i1\infty} = R_{i1}\psi_{i2\infty} - D_i\ ,\ z_{i\infty} = \frac{\psi_{i2\infty} - R_{i2}\psi_{i1\infty} - w_{ic}}{K_{iI}}. \tag{4.32}$$

where

$$D_1 = 0\ ,\ D_j = \alpha_j \psi_{(j-1)1\infty} + \delta_j \psi_{(j-1)2\infty}\ ,\forall\ j = \overline{2,n}.$$

We suppose the following hypothesises for the output measures:

H1: $a_i \neq 0\ \forall i = \overline{1,n}$.

H2: $a_i + b_i R_{i2} \neq 0\ \forall i = \overline{1,n}$.

H3: $b_i + a_i R_{i1} \neq 0 \ \forall i = \overline{1,n}$.

Then the main result of our paper is given in the following theorem.

Theorem 4.3. *Assume that the three hypotheses H1, H2 and H3 are satisfied. Then, there exists $\mu^* > 0$ such that for each initial condition in $\mathbb{E}$ satisfying the C^0 and C^1 compatibility conditions, each $\mu \in (0, \mu^*)$ and each PI controller with the following proportional gain K_{iP} and the integral gain K_{iI}:*

$$K_{iP} = \frac{-R_{i2}}{a_i} \ , \ K_{iI} = -\mu \ \frac{(b_i + a_i R_{i1} e^{\mu L})(a_i + b_i R_{i2})}{a_i} \quad , \ \forall i = \overline{1,n} \tag{4.33}$$

the following two properties hold true:

- *The equilibrium state $(\psi_{11\infty}, \psi_{12\infty}, z_{1\infty}, \cdots, \psi_{n1\infty}, \psi_{n2\infty}, z_{n\infty})$ defined in (4.32) of network of closed-loop systems (4.31) is exponentially stable in the state space $\mathbb{E}$.*
- *For smooth initial condition*

$$\left(\psi_{11}^0(x), \psi_{12}^0(x), z_1^0, \cdots, \psi_{n1}^0(x), \psi_{n2}^0(x), z_n^0\right) \in \left((H^1(0,L))^2 \times \mathbb{R}\right)^n$$

satisfying the C^0 and C^1 compatibility conditions, the n measured outputs $y_i(t)$ are regulated to the desired set-points y_{ir}, i.e

$$\lim_{t\to\infty} |y_i(t) - y_{ir}| = 0 \ , \forall i = \overline{1,n}.$$

Remark : We need to impose Hypothesis H1 for our PI controller to exist. On the contrary, if H2 is not satisfied, then our PI controller can still be implemented. However the integral term K_i disappears and consequently the PI control law becomes only proportional control law. In that case, even so stability of the closed loop system may still be obtained, the integral effect of our PI controllers that leads to the output regulation is lost. Up to now, removing these conditions is an open question.

In addition, hypothesis H3 is a necessary condition for the existence of an equilibrium point of the closed-loop system (4.31) for all values of disturbances w_{io} and w_{ic}. Without this assumption, the procedure introduced in (4.32) is no longer valid. Hence, H3 is a necessary condition for the output regulation by integral action. ◀

4.4.3 Proof of the main result

To prove Theorem 4.3, the following coordinate transformation is considered

$$\phi_{i1}(x,t) = \psi_{i1}(x,t) - \psi_{i1\infty} \ , \ \phi_{i2}(x,t) = \psi_{i2}(x,t) - \psi_{i2\infty} \ , \ \xi_i(t) = z_i(t) - z_{i\infty} \tag{4.34}$$

where $\psi_{i1\infty}$, $\psi_{i2\infty}$ and $z_{i\infty}$ are defined in (4.32). With the new coordinates defined in (4.34) and applying the PI controller design (4.33) in Theorem 4.3, then we obtains the following network of closed-loop systems without perturbations

$$\begin{cases} \partial_t \phi_{i1}(x,t) & = -\lambda_{i1}\partial_x \phi_{i1}, \\ \partial_t \phi_{i2}(x,t) & = \lambda_{i2}\partial_x \phi_{i2}, \\ \partial_t \xi_i & = a_i\phi_{i1}(L,t) + b_i\phi_{i2}(L,t) \\ \phi_{i2}(L,t) & = -k_i\xi_i(t), \\ \phi_{i1}(0,t) & = R_{i1}\phi_{i2}(0,t) + \alpha_i\phi_{(i-1)1}(L,t) + \beta_i\xi_{i-1}(t). \\ y_i(t) - y_{ir} & = a_i\phi_{i1}(L,t) + b_i\phi_{i2}(L,t). \end{cases} \tag{4.35}$$

completed by the initial conditions in $\mathbb{E}$:

$$\phi_{i1}(x,0) = \phi_{i1}^0(x) \ , \ \phi_{i2}(x,0) = \phi_{i2}^0(x) \ , \ \xi_i(0) = \xi_i^0 \ ,$$

where

$$k_i = \mu(b_i + a_i R_{i1} e^{\mu L}) \tag{4.36}$$

and $\alpha_1 = 0$, $\beta_1 = 0$, $\beta_j = -k_{j-1}\delta_j$ for $j = \overline{2,n}$.

In the new coordinates, $y_i(t) - y_{ir} = a_i\phi_{i1}(L,t) + b_i\phi_{i2}(L,t)$, the output regulation is obtained if

$$\lim_{t\to\infty} |a_i\phi_{i1}(L,t) + b_i\phi_{i2}(L,t)| = 0 \ , \forall i = \overline{1,n}.$$

Hence, to prove the asymptotic stability and the output regulation for the disturbed network systems in (4.31) with the PI control design in (4.30), we must naturally prove the stability of equivalent system (4.35) to the origin.

In the following, the Lyapunov candidate function is given in Section 4.4.3.1 and then the proof of the Theorem 4.3 by using direct Lyapunov method is presented in Section 4.4.3.2.

4.4.3.1 Lyapunov candidate functional

In the paper, we construct the following Lyapunov candidate functional:

$$\mathbb{V}(\phi_{11}, \phi_{12}, \xi_1, \cdots, \phi_{n1}, \phi_{n2}, \xi_n) = \sum_{i=1}^{n} p_i V_i \tag{4.37}$$

where V_i is defined by

$$V_i(\phi_{i1}, \phi_{i2}, \xi_i) = \int_0^L F_i^T P_i F_i \, dx$$

with $F_i = \begin{pmatrix} \phi_{i1} e^{-\frac{\mu x}{2}} \\ \phi_{i2} e^{\frac{\mu x}{2}} \\ \xi_i \end{pmatrix}$ and $P_i = \begin{pmatrix} 1 & 0 & q_{i3} \\ 0 & q_{i1} & q_{i4} \\ q_{i3} & q_{i4} & q_{i2} \end{pmatrix}$.

Here p_i and $q_{i1}, \cdots, q_{i4}$ are positive real number that be designed later on.

To begin with, consider the set $\mathbb{X} = (L^2(0, L))^2 \times \mathbb{R}$. The following lemma for the constructing of each sub-function V_i can be obtained.

Lemma 4.4. *Let k_i be defined in (4.36) and $q_{i1}, q_{i2}, q_{i3}, q_{i4}$ be defined as follows:*

$$\begin{aligned} q_{i1} &> \frac{3\lambda_{i1} R_{i1}^2}{\lambda_{i2}} \ , \ q_{i2} = \mu e^{\mu L} \lambda_{i2} q_{i1}, \\ q_{i3} &= \mu e^{\frac{3\mu L}{2}} \ \frac{a_i \lambda_{i2} q_{i1}}{\lambda_{i1}} \ , \ q_{i4} = \mu e^{\frac{3\mu L}{2}} \ a_i R_{i1} q_{i1}. \end{aligned} \tag{4.38}$$

Then there exists $\mu^* > 0$ such that for every $\mu \in (0, \mu^*)$, we have

1. *There exists $M_i > 0$ such that $\forall \ (\phi_{i1}, \phi_{i2}, \xi_i)$ in $\mathbb{X}$:*

$$\frac{1}{M_i} V_i(\phi_{i1}, \phi_{i2}, \xi_i) \leqslant ||\phi_{i1}(., t)||^2_{L^2(0,L)} + ||\phi_{i2}(., t)||^2_{L^2(0,L)} + \xi_i^2(t) \leqslant M_i V_i(\phi_{i1}, \phi_{i2}, \xi_i) \tag{4.39}$$

2. *There exists $\gamma_i > 0$ such that along the solution of (4.35), for all t such that the solution is well defined*

$$\dot{V}_i(t) \leqslant -\gamma_i V_i(t) - \frac{1}{4} \xi_i^2(t) \ k_i^2 \lambda_{i2} q_{i1} e^{\mu L} - \phi_{i1}^2(L, t) \frac{\lambda_{i1} e^{-\mu L}}{2} + \phi_{(i-1)1}^2(L, t) A_i + \xi_{i-1}^2(t) B_i \ , \tag{4.40}$$

where $A_i = \lambda_{i1} \alpha_i^2 \left(3 + \frac{4\lambda_{i1}^2 q_{i3}^2 e^{-\mu L}}{k_i^2 \lambda_{i2} q_{i1}} \right) \ , \ B_i = \lambda_{i1} \beta_i^2 \left(3 + \frac{4\lambda_{i1}^2 q_{i3}^2 e^{-\mu L}}{k_i^2 \lambda_{i2} q_{i1}} \right) .$

Proof : Proof of the first property 4.39

To begin with, we prove that the matrix P_i is positive definite. Making use of the Sylvester criterion, matrix P_i is positive definite if and only if:

$$q_{i1} > 0,\ q_{i2} > q_{i3}^2 + \frac{q_{i4}^2}{q_{i1}}.$$

Employing (4.38), one can find that:

$$q_{i1} > 0\ ,\ q_{i2} - q_{i3}^2 - \frac{q_{i4}^2}{q_{i1}} = \mu e^{\mu L} q_{i1} \left(\lambda_{i2} - \mu e^{2\mu L} \frac{a_i^2 \lambda_{i2}^2 q_{i1}}{\lambda_{i1}^2} - \mu e^{2\mu L} a_i^2 R_{i1}^2 \right).$$

It is clearly that if μ is small enough, $q_{i2} - q_{i3}^2 - \dfrac{q_{i4}^2}{q_{i1}} > 0$. It therefore yields that the matrix P_i is symmetric positive definite. Hence, there exits $\sigma_{i1}, \sigma_{i2} > 0$ such that

$$\sigma_{i1} \int_0^L \begin{pmatrix} \phi_{i1} e^{-\frac{\mu x}{2}} \\ \phi_{i2} e^{\frac{\mu x}{2}} \\ \xi_i \end{pmatrix}^T \begin{pmatrix} \phi_{i1} e^{-\frac{\mu x}{2}} \\ \phi_{i2} e^{\frac{\mu x}{2}} \\ \xi_i \end{pmatrix} dx \leqslant V_i(\phi_{i1}, \phi_{i2}, \xi_i) \leqslant \sigma_{i2} \int_0^L \begin{pmatrix} \phi_{i1} e^{-\frac{\mu x}{2}} \\ \phi_{i2} e^{\frac{\mu x}{2}} \\ \xi_i \end{pmatrix}^T \begin{pmatrix} \phi_{i1} e^{-\frac{\mu x}{2}} \\ \phi_{i2} e^{\frac{\mu x}{2}} \\ \xi_i \end{pmatrix} dx.$$

As a result, there exists $M_i > 0$ such that (4.39) holds.

Proof of the second property 4.40

The time derivative of V along the solution of the system (4.35) has the following form:

$$\dot{V}_i(t) = -\int_0^L \begin{pmatrix} \phi_{i1}(x,t) e^{-\frac{\mu x}{2}} \\ \phi_{i2}(x,t) e^{\frac{\mu x}{2}} \\ \xi_i(t) \\ \phi_{i1}(L,t) \end{pmatrix}^T Q_i \begin{pmatrix} \phi_{i1}(x,t) e^{-\frac{\mu x}{2}} \\ \phi_{i2}(x,t) e^{\frac{\mu x}{2}} \\ \xi_i(t) \\ \phi_{i1}(L,t) \end{pmatrix} dx \ - \ F(t) \tag{4.41}$$

where

$$\begin{aligned} F(t) = {} & \phi_{i1}^2(L,t) \frac{\lambda_{i1} e^{-\mu L}}{2} + \phi_{i2}^2(0,t) \lambda_{i2} q_{i1} \ + \frac{3}{4} \xi_i^2(t) k_i^2 \lambda_{i2} q_{i1} e^{\mu L} \\ & - \lambda_{i1} \left(R_{i1} \phi_{i2}(0,t) + \alpha_i \phi_{(i-1)1}(L,t) + \beta_i \xi_{i-1}(t) \right)^2 - 2 \xi_i(t) q_{i3} \lambda_{i1} \left(\alpha_i \phi_{(i-1)1}(L,t) + \beta_i \xi_{i-1}(t) \right), \end{aligned} \tag{4.42}$$

and,

$$Q_i = \begin{pmatrix} \mu \lambda_{i1} & 0 & T_{\phi_{i1},\xi_i} & T_{\phi_{i1},\phi_{i1L}} \\ 0 & \mu q_{i1} \lambda_{i2} & T_{\phi_{i2},\xi_i} & T_{\phi_{i2},\phi_{i1L}} \\ T_{\phi_{i1},\xi_i} & T_{\phi_{i2},\xi_i} & T_{\xi_i} & 0 \\ T_{\phi_{i1},\phi_{i1L}} & T_{\phi_{i2},\phi_{i1L}} & 0 & \dfrac{\lambda_{i1} e^{-\mu L}}{2L} \end{pmatrix}$$

with

$$T_{\xi_i} = \mu^2 \ \frac{c^2\lambda_{i2}q_{i1}e^{\mu L}}{4L} \ , \ T_{\phi_{i1},\phi_{i1L}} = -\mu e^{\frac{3\mu L}{2}} \ \frac{a_i^2\lambda_{i2}q_{i1}}{\lambda_{i1}} \ , \ T_{\phi_{i2},\phi_{i1L}} = -\mu e^{\frac{3\mu L}{2}} \ a_i^2 R_{i1}q_{i1} \ ,$$

$$T_{\phi_{i1},\xi_i} = \mu^2 e^{\frac{3\mu L}{2}} \ \frac{a_i\lambda_{i2}q_{i1}(2b_ic+\lambda_{i1})}{2\lambda_{i1}} \ , \ T_{\phi_{i2},\xi_i} = \mu^2 e^{\frac{3\mu L}{2}} \ a_i R_{i1}q_{i1} \ \frac{2b_ic+\lambda_{i2}}{2\lambda_{i2}},$$

and $c = b_i + a_iR_{i1}e^{\mu L}$.

At first, we consider the boundary terms $F(t)$ in (4.42). Applying the Cauchy-Schwarz inequality, one can find that

$$\big(R_{i1}\phi_{i2}(0,t) + \alpha_i\phi_{(i-1)1}(L,t) + \beta_i\xi_{i-1}(t)\big)^2 \leqslant 3\left(R_{i1}^2\phi_{i2}^2(0,t) + \alpha_i^2\phi_{(i-1)1}^2(L,t) + \beta_i^2\xi_{i-1}^2(t)\right) \tag{4.43}$$

$$\begin{aligned} 2\xi_i(t)q_{i3}\lambda_{i1}\left(\alpha_i\phi_{(i-1)1}(L,t) + \beta_i\xi_{i-1}(t)\right) \leqslant \frac{1}{2}\xi_i^2(t) \ k_i^2\lambda_{i2}q_{i1}e^{\mu L} \\ + \ \frac{4\lambda_{i1}^2q_{i3}^2e^{-\mu L}}{k_i^2\lambda_{i2}q_{i1}}\left(\alpha_i^2\phi_{(i-1)1}^2(L,t) + \beta_i^2\xi_{i-1}^2(t)\right) \end{aligned} \tag{4.44}$$

From (4.42), (4.43), (4.44) and with the choice of q_{i1} in (4.38) it can be deduced that

$$F(t) \geqslant \phi_{i1}^2(L,t)\frac{\lambda_{i1}e^{-\mu L}}{2} + \frac{1}{4}\xi_i^2(t) \ k_i^2\lambda_{i2}q_{i1}e^{\mu L} - \phi_{(i-1)1}^2(L,t)A_i - \xi_{i-1}^2(t)B_i \ , \tag{4.45}$$

where

$$A_i = \lambda_{i1}\alpha_i^2\left(3 + \frac{4\lambda_{i1}^2q_{i3}^2e^{-\mu L}}{k_i^2\lambda_{i2}q_{i1}}\right) \ , \ B_i = \lambda_{i1}\beta_i^2\left(3 + \frac{4\lambda_{i1}^2q_{i3}^2e^{-\mu L}}{k_i^2\lambda_{i2}q_{i1}}\right)$$

In the following, we prove that by picking μ small enough, matrix Q is symmetric positive definite. From the Sylvester criterion, Q is symmetric positive definite if and only if

$$D_1 = det\begin{pmatrix} \mu\lambda_{i1} & 0 & T_{\phi_{i1},\xi_i} \\ 0 & \mu q_{i1}\lambda_{i2} & T_{\phi_{i2},\xi_i} \\ T_{\phi_{i1},\xi_i} & T_{\phi_{i2},\xi_i} & T_{\xi_i} \end{pmatrix} > 0,$$

$$D_2 = det(Q_i) > 0.$$

By the direct computing, we have that

$$D_1 = \mu^4 f_1(\mu) \ , \ D_2 = \mu^4 f_2(\mu) \ ,$$

with $\lim\limits_{\mu\to 0} f_1(\mu) = \frac{1}{L}\lambda_{i1}\lambda_{i2}^2q_{i1}^2c^2e^{\mu L} > 0$ and $\lim\limits_{\mu\to 0} f_2(\mu) = \frac{1}{2L}\lambda_{i1}^2\lambda_{i2}^2q_{i1}^2c^2 > 0$.

Taking μ small enough, it yields that the two terms D_1, D_2 are both positive. Consequently, the matrix Q_i is symmetric positive definite.

Therefore, with the adaptive choice of μ, there exists a positive real number $K_i > 0$ such that for all t at which the solution is well defined we have:

$$\dot{V}_i(t) \leqslant -K_i||(\phi_{i1}, \phi_{i2}, \xi_i)||^2_{\mathbb{X}} - \frac{1}{4}\xi_i^2(t)\, k_i^2\lambda_{i2}q_{i1}e^{\mu L} - \phi_{i1}^2(L,t)\frac{\lambda_{i1}e^{-\mu L}}{2} + \phi^2_{(i-1)1}(L,t)A_i + \xi^2_{i-1}(t)B_i.$$

With (4.39) the former inequality implies that we can find $\gamma_i > 0$ such that (4.40) holds. This completes the proof of Lemma 4.4. ■

Now, by employing Lemma (4.4), the following lemma for the design of Lypunov candidate function $\mathbb{V}$ is given,

Lemma 4.5. *Let k_i and $q_{i1}, q_{i2}, \cdots, q_{i4}$ be defined in Lemma 4.4 and p_i be defined as follows*

$$p_1 > 0\ ,\ p_{i+1} = \epsilon p_i\ . \tag{4.46}$$

Then there exists $\epsilon > 0$ and $\mu^* > 0$ such that for every $\mu \in (0, \mu^*)$, we have :

1. *There exists $M > 0$ such that $\forall\ (\phi_{11}, \phi_{12}, \xi_1, \cdots, \phi_{n1}, \phi_{n2}, \xi_n)$ in $\mathbb{E}$:*

$$\begin{aligned}\frac{1}{M}\mathbb{V}(\phi_{11}, \phi_{12}, \xi_1, \cdots, \phi_{n1}, \phi_{n2}, \xi_n) &\leqslant ||(\phi_{11}, \phi_{12}, \xi_1, \cdots, \phi_{n1}, \phi_{n2}, \xi_n)||^2_{\mathbb{E}} \\ &\leqslant M\ \mathbb{V}(\phi_{11}, \phi_{12}, \xi_1, \cdots, \phi_{n1}, \phi_{n2}, \xi_n)\ .\end{aligned} \tag{4.47}$$

2. *There exists $\gamma > 0$ such that along the solution of (4.31), for all t at which the solution is well defined*

$$\dot{\mathbb{V}}(t) \leqslant -\gamma\mathbb{V}(t)\ - \delta\sum_{i=1}^{n}\left(\xi_i^2(t) + \phi_{i1}^2(L,t)\right), \tag{4.48}$$

we have used the notation $\mathbb{V}(t) = \mathbb{V}(\phi_{11}(\cdot,t), \phi_{12}(\cdot,t), \xi_1(t), \cdots, \phi_{n1}(\cdot,t), \phi_{n2}(\cdot,t), \xi_n(t))$.

Proof : To begin with, applying the property (4.39) of Lemma 4.4, with μ small enough, one easily finds that

$$\sum_{i=1}^{n}\frac{p_i}{M_i}V_i \leqslant ||(\phi_{11}, \phi_{12}, \xi_1, \cdots, \phi_{n1}, \phi_{n2}, \xi_n)||^2_{\mathbb{E}} \leqslant \sum_{i=1}^{n} p_i M_i V_i. \tag{4.49}$$

This leads to the existence of $M > 0$ such that (4.47) holds.

To prove (4.48), we analyze the time derivative of $\mathbb{V}$ along the solution of the network of closed-loop systems in (4.35),

$$\dot{\mathbb{V}}(t) = \sum_{i=1}^{n} p_i \dot{V}_i(t)$$

To simplify the writing in the following, we denote p_{n+1}, A_{n+1} and B_{n+1} such that $p_{n+1} = A_{n+1} = B_{n+1} = 0$. Now, by employing the property (4.40) of Lemma 4.4, one finds that

$$\begin{aligned}\dot{\mathbb{V}}(t) \leqslant -\sum_{i=1}^{n} p_i \gamma_i V_i(t) - \sum_{i=1}^{n} \xi_i^2(t) \left(p_i \ \frac{k_i^2 \lambda_{i2} q_{i1} e^{\mu L}}{4} - p_{i+1} B_{i+1} \right) \\ - \sum_{i=1}^{n} \phi_{i1}^2(L,t) \left(p_i \ \frac{\lambda_{i1} e^{-\mu L}}{2} - p_{i+1} A_{i+1} \right). \end{aligned} \tag{4.50}$$

Since $p_{i+1} = \epsilon p_i$, by taking ϵ small enough, it is clearly to see that

$$\begin{aligned} p_i \ \frac{k_i^2 \lambda_{i2} q_{i1} e^{\mu L}}{4} - p_{i+1} B_{i+1} > 0 \\ p_i \ \frac{\lambda_{i1} e^{-\mu L}}{2} - p_{i+1} A_{i+1} > 0 \end{aligned} \tag{4.51}$$

From (4.50) and (4.51), there exits $\gamma > 0$ and $\delta > 0$ such that

$$\dot{\mathbb{V}}(t) \leqslant -\gamma \sum_{i=1}^{n} p_i V_i(t) \ - \delta \sum_{i=1}^{n} \left(\xi_i^2(t) + \phi_{i1}^2(L,t) \right)$$

This inequality implies that (4.48) holds. ■

Remark : In our Lyapunov functional (4.37), compared to the one in [6], dynamic feedbacks are added with n new states $\xi_i(t)$. Some Lyapunov functional for dynamic feedback states have been studied for single hyperbolic PDE systems in literature [21, 25], but it should be pointed out that our Lyapunov function is different since it contains coupling terms of the states ϕ_i and feedback states ξ_i. These coupling terms allows to avoid the damping term in PI controller as in [21]. It allows also to consider a larger classe of hyperbolic PDE systems than the one considered in [25].

Note that in [68], a Lyapunov functional with the dynamic feedback terms and the coupled terms is considered only for the single PDE hyperbolic system, but cannot be extended for a cascaded network. ◀

4.4.3.2 Proof of the theorem 4.3

In this section, by using the result of Lemma 4.5, we present the proof of Theorem 4.3.

First of all, we prove the exponential stability of the closed-loop system (4.31) to the equilibrium state $(\psi_{11\infty}, \psi_{12\infty}, z_{1\infty}, \cdots, \psi_{n1\infty}, \psi_{n2\infty}, z_{n\infty})$. From Lemma 4.5, there exists $\gamma > 0$ such that

$$\mathbb{V}(t) \leqslant \mathbb{V}(0)e^{-\gamma t} \,. \tag{4.52}$$

With (4.47), it implies that there exists $S > 0$ such that for all initial conditions

$$\left(\phi_{11}^0(x), \phi_{12}^0(x), \xi_1^0, \cdots, \phi_{n1}^0(x), \phi_{n2}^0(x), \xi_n^0\right) \in \mathbb{E}, \quad \forall x \in [0, L]$$

and satisfying C^0, C^1 compatibility conditions, the solution of (4.35) is defined for all positive time and satisfies that for all $t \geqslant 0$

$$\begin{aligned} &||(\phi_{11}(\cdot,t), \phi_{12}(\cdot,t), \xi_1(t), \cdots, \phi_{n1}(\cdot,t), \phi_{n2}(\cdot,t), \xi_n(t))||_{\mathbb{E}}^2 \\ &\qquad \leqslant Se^{-\gamma t}||(\phi_{11}^0, \phi_{12}^0, \xi_1^0, \cdots, \phi_{n1}^0, \phi_{n2}^0, \xi_n^0)||_{\mathbb{E}}^2. \end{aligned} \tag{4.53}$$

Inequality (4.53) implies that the origin of the closed-loop system (4.35) is exponentially stable in $\mathbb{E}$. This leads to the proof of the first property in Theorem 4.3.

Secondly, we prove the output regulation property of Theorem 4.3. Let denote $v_{i1} = -\lambda_{i1}\partial_x\phi_{i1}$, $v_{i2} = \lambda_{i2}\partial_x\phi_{i2}$, and $s_i(t)$ such that $\partial_t s_i = a_i v_{i1}(L,t) + b_i v_{i2}(L,t)$. It can be easily found that the dynamics of v_{i1}, v_{i2} and s_i are the same as the ones of ϕ_{i1}, ϕ_{i2} and ξ_i in (4.35):

$$\begin{cases} \partial_t v_{i1}(x,t) &= -\lambda_{i1}\partial_x v_{i1}, \\ \partial_t v_{i2}(x,t) &= \lambda_{i2}\partial_x v_{i2}, \\ \partial_t s_i &= a_i v_{i1}(L,t) + b_i v_{i2}(L,t) \\ v_{i2}(L,t) &= -k_i s_i(t), \\ v_{i1}(0,t) &= R_{i1}v_{i2}(0,t) + \alpha_i v_{(i-1)1}(L,t) + \beta_i s_{i-1}(t). \end{cases} \tag{4.54}$$

From the hypothesis in theorem, we deduce that (4.54) has also smooth initial condition $(v_{i1}^0(x), v_{i2}^0(x), s_i^0)$ in $\mathbb{E}$ $(\forall i = \overline{1,n})$ and satisfying the C^0 and C^1 compatibility conditions. Employing similar analysis, one finds that the origin of (4.54) is also exponentially stable in $\mathbb{E}$. This allows us to prove that:

$$\lim_{t\to\infty} ||\phi_{i1}(\cdot,t)||_{H^1(0,L)} = 0 \,, \quad \lim_{t\to\infty} ||\phi_{i2}(\cdot,t)||_{H^1(0,L)} = 0.$$

By the Sobolev embedding theorem, one gives that

$$\lim_{t\to\infty} \phi_{i1}(x,t) = 0, \lim_{t\to\infty} \phi_{i2}(x,t) = 0 \ \forall x \in [0,L].$$

Therefore, the output regulation is obtained, i.e $\lim_{t\to\infty} |y_i(t) - y_{ir}| = 0$, $\forall i = \overline{1,n}$.

This completes the proof of the Theorem 4.3.

4.5 Numerical simulations

In this section, we consider networks with the structure of n horizontal channels which are described by Saint Venant equations with the neglected friction slope, studied for example in [25, 56, 68] for a single channel and [55, 60] for networks of n channels. Without loss of generality, the lengths of channels are assumed identical and equal L. The dynamics of each channel is governed by

$$\begin{cases} B_i\partial_t H_i(x,t) + \partial_x Q_i(x,t) & = 0 \\ \partial_t Q_i(x,t) + \partial_x \left(\dfrac{Q_i^2(x,t)}{B_i H_i(x,t)} + \dfrac{1}{2} g B_i H_i^2(x,t) \right) & = 0 \end{cases} , \ x \in [0,L], \ t \in [0,\infty), \ i = \overline{1,n} \tag{4.55}$$

where $H_i(x,t)$ denote water level, $Q_i(x,t)$ water discharge at the position x and time t respectively, B_i the channel width and g the gravitational constant for the channel i.

Now considering the linearization of the network (4.55) around a steady state (H_i^*, Q_i^*) with $i = \overline{1,n}$ (the steady state must satisfy subcritical condition $gB_i^2(H_i^*)^3 - (Q_i^*)^2 > 0$, and is more discussed in the following for each kind of network). By the notation $h_i = H_i - H_i^*$ and $q_i = Q_i - Q_i^*$, the linearized network is described as follows

$$\frac{\partial}{\partial t}\begin{pmatrix} h_i \\ q_i \end{pmatrix} + \begin{pmatrix} 0 & \dfrac{1}{B_i} \\ \dfrac{-(Q_i^*)^2}{B_i(H_i^*)^2} + gB_iH_i^* & \dfrac{2Q_i^*}{B_iH_i^*} \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} h_i \\ q_i \end{pmatrix} = 0 \tag{4.56}$$

Then by using the following change of coordinates

$$h_i = \psi_{i1} + \psi_{i2}, \ q_i = (B_i\sqrt{gH_i^*} + \frac{Q_i^*}{H_i^*})\psi_{i1} - (B_i\sqrt{gH_i^*} - \frac{Q_i^*}{H_i^*})\psi_{i2} \tag{4.57}$$

One obtains the following form that is similar to the one in (4.1):

$$\begin{cases} \partial_t\psi_{i1}(x,t) + \lambda_{i1}\ \partial_x\psi_{i1}(x,t) & = 0 \\ \partial_t\psi_{i2}(x,t) - \lambda_{i2}\ \partial_x\psi_{i2}(x,t) & = 0 \end{cases} \tag{4.58}$$

where $\lambda_{i1} = \sqrt{gH_i^*} + \dfrac{Q_i^*}{B_i H_i^*} > 0$, $\lambda_{i2} = \sqrt{gH_i^*} - \dfrac{Q_i^*}{B_i H_i^*} > 0$.

In the following, we present in details two kinds of network: star-shaped model and cascaded one.

4.5.1 Simulations for star-shaped network

4.5.1.1 Modeling of star network

In this section, we aim to realise simulations for a star-shaped network of n channels ($n \geqslant 3$) whose dynamics are considered in (4.55). The connection between channels can be seen in Fig. 4.1, with $n-1$ inlet channels ($j = \overline{1, n-1}$) and an outlet channel (nth channel).

The following notations are used for the next : $i = \overline{1, n}$ and $j = \overline{1, n-1}$.
The $n-1$ online measured outputs at each time t are

$$y_j(t) = H_j(0, t). \tag{4.59}$$

We assume that the network is controlled by $2n-1$ input controls, in which each channel is controlled by the discharge at $x = L$, i.e

$$Q_i(L, t) = Q_{ic}(t), \tag{4.60}$$

and $n-1$ inlet channels is actively controlled at $x = 0$ by

$$Q_j(0, t) = U_j(t), \tag{4.61}$$

where inputs U_j are at $x = 0$, and $Q_{jc}(t)$ are inputs at $x = L$.
Note however that at the junction, the constraint of flow-rate conservation is

$$Q_n(0, t) = \sum_{j=1}^{n-1} Q_j(L, t). \tag{4.62}$$

The equation (4.62) implies that, the boundary $x = 0$ of the outlet channel (nth channel) cannot be controlled, but its dynamic can be deduced from the conservation of the flow.

4.5.1.2 PI controllers for the model

At first, we consider the equilibrium state of the network model. Each $2n$ constant values H_i^*, Q_i^* satisfying the subcritical conditions $gB_i^2(H_i^*)^3 - (Q_i^*)^2 > 0$ and the conservation

law condition $Q_n^* = \sum_{j=1}^{n-1} Q_j^*$ can be chosen as an equilibrium state with the appropriate controls U_j and Q_{ic}.

The objective is to stabilize the linearised model of the network (4.56) and (4.59)-(4.62) to the set-points H_i^*, V_i^* and regulate the output measurements of each inlet channel to the desired references $y_{jr} = H_j^*$.
With the change of coordinates in (4.57), we return to the familiar form in (4.58). Inspired by results in Theorem 4.1, we then rewrite all outputs and boundary conditions in the new coordinates as follow:
$n-1$ measured outputs

$$y_j(t) - H_j^* = \phi_{j1}(0,t) + \phi_{j2}(0,t) \tag{4.63}$$

and $2n-1$ boundary conditions

$$\begin{aligned} \phi_{j1}(0,t) &= R_{j1}\phi_{j2}(0,t) + u_j(t), \\ \phi_{i2}(L,t) &= R_{i2}\phi_{i1}(L,t), \end{aligned} \tag{4.64}$$

where $R_{j1} = \dfrac{\lambda_{j2}}{\lambda_{j1}}$, R_{i2} are arbitrary constants depending on Q_{ic}, and $u_j(t)$ are new dynamic feedback control law deduced from the control inputs $U_j(t)$ by

$$u_j(t) = \frac{U_j(t) - Q_j^*}{B_j \lambda_{j1}} \tag{4.65}$$

The last condition comes from the conservation constraint at the junction (4.62)

$$\phi_{n1}(0,t) = R_{n1}\phi_{n2}(0,t) + \sum_{j=1}^{n-1} \alpha_j \phi_{j1}(L,t) \ , \tag{4.66}$$

where

$$R_{n1} = \frac{\lambda_{n2}}{\lambda_{n1}} \ , \ \alpha_j = \frac{B_j(\lambda_{j1} + \lambda_{j2})}{B_n \lambda_{jn}}.$$

Remark : Here, to implement control inputs U_j, we only need $n-1$ online measurements y_j which are water levels $H_j(0,t)$ of inlet channels. It is practical because measure flow discharge in reality is difficult. ◀

Now employing the Theorem 4.1, $n-1$ dynamic feedback PI controllers $u_j(t)$ are designed as follows

$$u_j(t) = K_{jP}(y_j(t) - H_j^*) + K_{jI} \int_0^L (y_j(s) - H_j^*) ds \tag{4.67}$$

where

$$K_{jP} = -R_{j1} \ , \ K_{jI} = -\mu(1 + R_{j2}e^{\mu L})(1 + R_{j1})$$

with μ is tuning parameter chosen small enough.

4.5.1.3 Numerical simulations

In this Section, we give numerical simulations for a star-shaped network of three channels with the following data:

- Length of each channel is $L = 100\ m$, with the same width $B = 1\ m$.
- Level set-points $H_1^* = 10\ m$, $H_2^* = 8\ m$, $H_3^* = 12\ m$, and discharge set-points $Q_1^* = 5\ m^3/s$, $Q_2^* = 4\ m^3/s$, $Q_3^* = 9\ m^3/s+$.
- Output disturbances for two inlet channels $w_{1o} = 0.02$, $w_{2o} = 0.05$.

Clearly, subcritical conditions and conservation condition are all satisfied with above data. Tuning parameter μ is chosen by 0.001. The simulations are based on Preissmann schema with the weighting coefficient $\theta = 0.6$ and space discretization $\Delta x = 1\ m$ (see in Appendix and Chapter 3).

Figure 4.3- Figure 4.8 verify the stability of linearized closed-loop network and show the evolution states of the water level and water discharge for each channel in closed-loop. The evolutions of input with PI controllers are seen in Figure 4.9.

In Figure 4.10, we see that two output measurements $y_j(t) = H_j(0,t)$ are regulated to the desired references H_j^* in spite of output disturbances w_{jo}.

4.5.2 Simulation for cascaded network

4.5.2.1 Modelling of network

In this section, we make simulations for a cascaded network of n horizontal channels in (4.55), see in Figure 4.11.

The network is controlled by n underflow gates at the end of each channel, i.e

$$Q_i^2(L,t) = \alpha_i \left(H_i(L,t) - U_i(t)\right) \ , \ i = \overline{1,n} \tag{4.68}$$

where α_i are positive constant coefficient of gates and $U_i(t)$ represent the openings of the gates considering as control inputs.

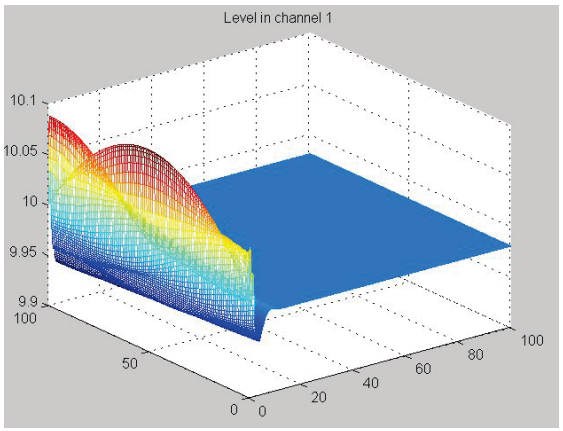

FIGURE 4.3: $H_1(x,t)$ Star-shaped network

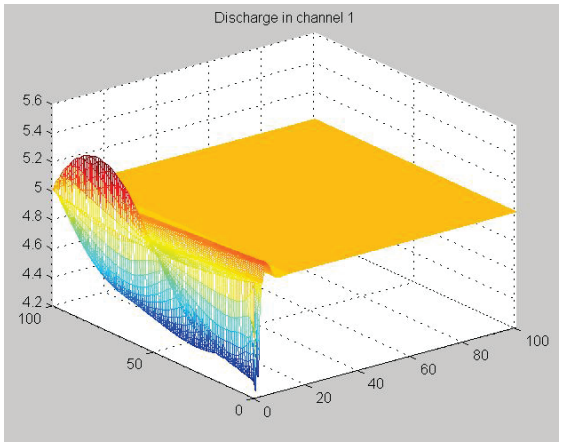

FIGURE 4.4: $Q_1(x,t)$ Star-shaped network

Moreover, n channels are interconnected by $n-1$ following discharge conservation constrains

$$Q_i(L,t) = Q_{i+1}(0,t) \ , \ i = \overline{1,n-1} \tag{4.69}$$

The last boundary condition comes from the control of the inflow discharge by an appropriate constant value Q_0, i.e

$$Q_1(0,t) = Q_0. \tag{4.70}$$

The n online disturbed outputs $y_i(t)$ are water levels at $x = L$, i.e

$$y_i(t) = H_i(L,t) + w_{io} \tag{4.71}$$

where w_{io} is unknown output disturbance.

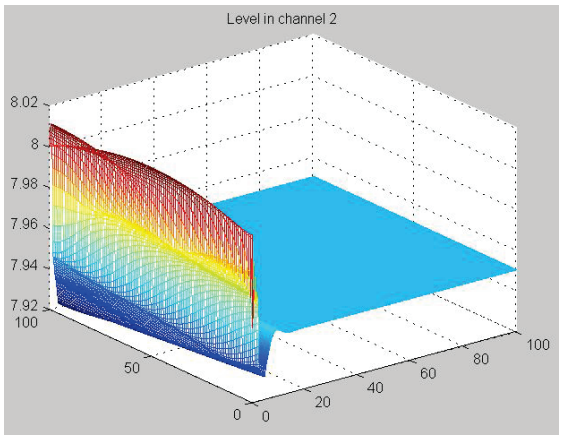

FIGURE 4.5: $H_2(x,t)$ Star-shaped network

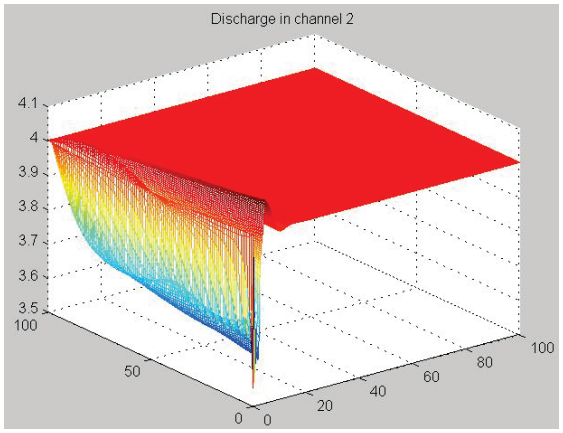

FIGURE 4.6: $Q_2(x,t)$ Star-shaped network

4.5.2.2 PI control for linerized model

Let us firsly consider n set-points H_i^* for each channels satisfying the subcritical conditions $gB_i^2(H_i^*)^3 - Q_0^2 > 0 \; \forall i = \overline{1,n}$. One can easily see that (H_i^*, Q_0) is also steady state for each channel of the network (4.55), (4.68), (4.70), and (4.71).

Then, we consider the linearized model analysed in (4.56), and rewrite all boundary conditions in (4.68)-(4.71) in the new coordinates

$$\begin{aligned} \psi_{i2}(L,t) &= R_{i2}\psi_{i1}(L,t) \; + u_i(t) \; , \\ \psi_{i1}(0,t) &= R_{i1}\psi_{i2}(0,t) + \alpha_i\psi_{(i-1)1}(L,t) + \delta_i\psi_{(i-1)2}(L,t) \; , \\ y_i(t) &= \psi_{i1}(L,t) + \psi_{i2}(L,t) + H_i^* + w_{io} \; , \end{aligned} \tag{4.72}$$

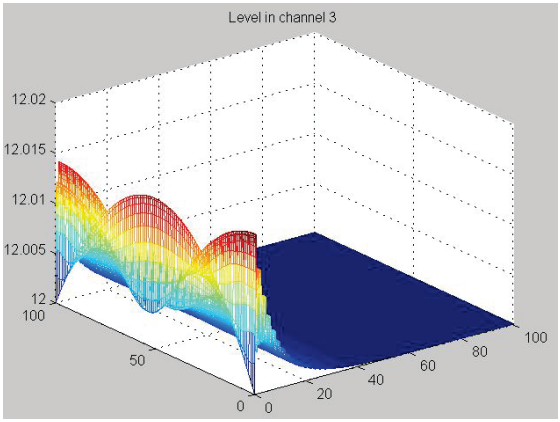

FIGURE 4.7: $H_3(x,t)$ Star-shaped network

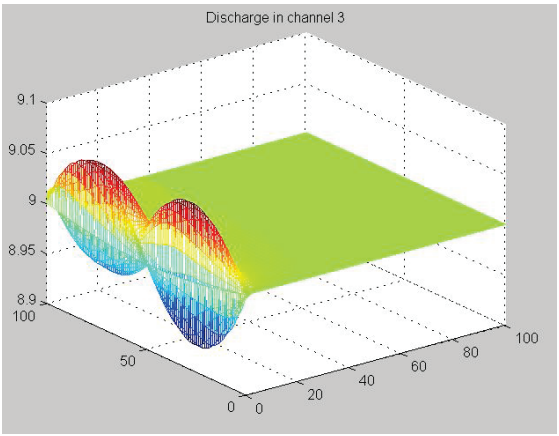

FIGURE 4.8: $Q_3(x,t)$ Star-shaped network

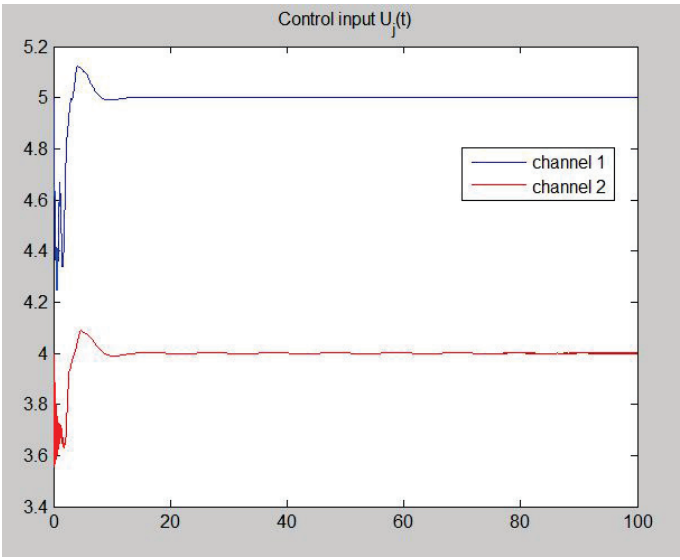

FIGURE 4.9: Control inputs $Q_j(t)$ Star-shaped network

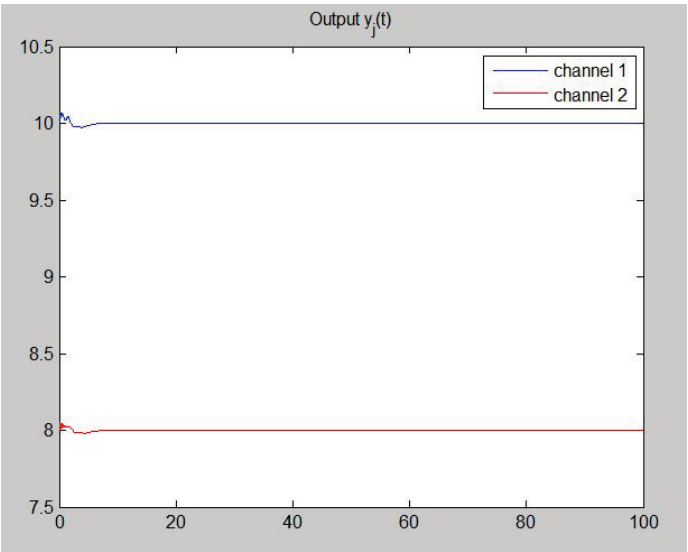

FIGURE 4.10: Output measurements $y_i(t)$ in Star-shaped network

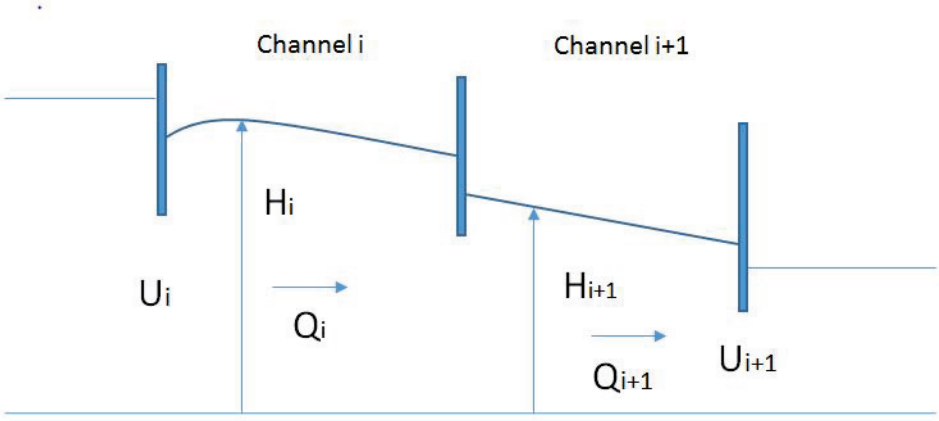

FIGURE 4.11: Cascaded network of n fluid flow channels

where

$$R_{i1} = \frac{\lambda_{i2}}{\lambda_{i1}} \ , R_{i2} = \frac{2Q_0(B_i\sqrt{gH_i^*} + \dfrac{Q_0}{H_i^*}) - \alpha_i}{2Q_0(B_i\sqrt{gH_i^*} - \dfrac{Q_0}{H_i^*}) + \alpha_i} \ , \forall i = \overline{1,n}$$

$$\alpha_1 = \delta_1 = 0 \ , \ \alpha_k = \frac{\lambda_{(k-1)1}}{\lambda_{k1}} \ , \ \delta_k = -\frac{\lambda_{(k-1)2}}{\lambda_{k1}} \ , \forall k = \overline{2,n}$$

and n new control inputs

$$u_i(t) = (U_i(t) - \overline{U_i}) \ \frac{2Q_0(B_i\sqrt{gH_i^*} - \dfrac{Q_0}{H_i^*}) + \alpha_i}{\alpha_i} \tag{4.73}$$

with $\overline{U_i} = H_i^* - \dfrac{Q_0^2}{\alpha_i}$ is the constant equilibrium opening of the ith gate.

Inspired by the result in Theorem 4.3, we design n feedback control laws $u_i(t)$ at junctions by the form of PI controllers with unknown control disturbance w_{ic} as follows

$$u_i(t) = K_{iP}(y_i(t) - H_i^*) + K_{iI}\int_0^t (y_i(s) - H_i^*)ds + w_{ic}$$

where K_{iP} and K_{iI} are computed by

$$\begin{aligned} K_{iP} &= \frac{-2Q_0(B_i\sqrt{gH_i^*} + \dfrac{Q_0}{H_i^*}) + \alpha_i}{2Q_0(B_i\sqrt{gH_i^*} - \dfrac{Q_0}{H_i^*}) + \alpha_i} \\ K_{iI} &= -\mu\ (1 + e^{\mu L}\frac{\sqrt{gH_i^*} + \dfrac{Q_0}{B_iH_i^*}}{\sqrt{gH_i^*} - \dfrac{Q_0}{B_iH_i^*}})\frac{4Q_0(B_i\sqrt{gH_i^*}}{2Q_0(B_i\sqrt{gH_i^*} - \dfrac{Q_0}{H_i^*}) + \alpha_i} \end{aligned} \quad , \ \forall i = \overline{1,n} \tag{4.74}$$

Here, the tuning parameter μ is chosen small enough. Then n PI controllers in (4.74) stabilize the linearized model (4.56) of channel network, and n measured outputs are regulated to the desired references.

Remark : To control this network of n cascaded models, we only need n control inputs represented by n underflow gates at junctions. Moreover, to implement controllers, only water levels at the end of each channel are required for the output measurements. This is an advantage because in practical engineering, water level is much simpler to measure than water discharge. ◀

4.5.2.3 Numerical simulations

In this Section, in order to validate theoretical results we make numerical simulations for a cascade network of three channels with the following data:

- Length of each channel is $L = 100\ m$, with the same width $B = 4\ m$.
- Level set-points $H_1^* = 10\ m$, $H_2^* = 8\ m$, $H_3^* = 6.5\ m$, and constant discharge $Q_0 = 7\ m^3/s$.
- Output disturbances $w_{1o} = 0.1$, $w_{2o} = 0.2$, $w_{2o} = 0.15$; and control disturbances $w_{1c} = 0.02$, $w_{2c} = 0.03$, $w_{2c} = 0.01$.

Obviously, subcritical conditions are satisfied with above data. Tuning parameter μ is chosen by 0.001. The simulations are based on Preissmann schema with the weighting

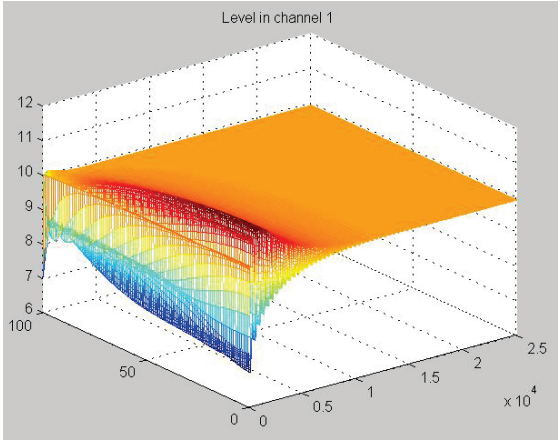

FIGURE 4.12: $H_1(x,t)$

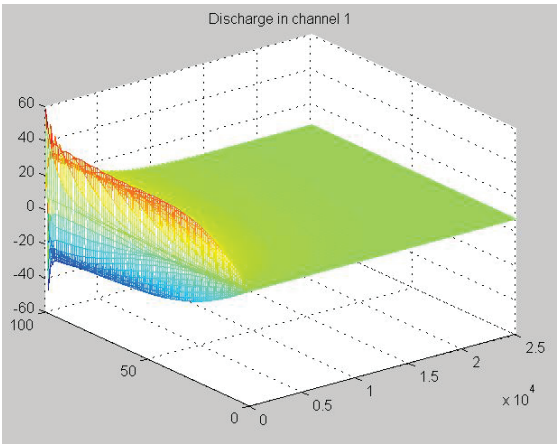

FIGURE 4.13: $Q_1(x,t)$

coefficient $\theta = 0.6$ and space discretization $\Delta x = 1\ m$ (see in Appendix and Chapter 3).

Figure 4.12- Figure 4.17 verify the stability of linearized closed-loop network and show the evolutions states of the water level and water discharge for each channel controlled by three PI controllers at junctions.
In Figure 4.18, we see that three output measurements $y_i(t) = H_i(L,t)$ are regulated to the desired references H_i^* in spite of output disturbances w_{io} and control disturbances w_{ic} $(i = \overline{1,3})$.
Finally, note that with the big value of tuning parameter μ (bigger than 0.002) in simulations, the controlled network becomes unstable.

4.6 Conclusions

In this chapter, we have study two kinds of PDE hyperbolic network: star-shaped one and cascaded one, accompanied by boundary control and output regulation with PI

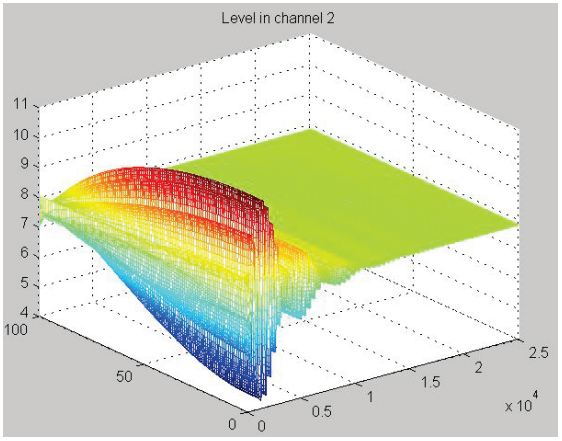

FIGURE 4.14: $H_2(x,t)$

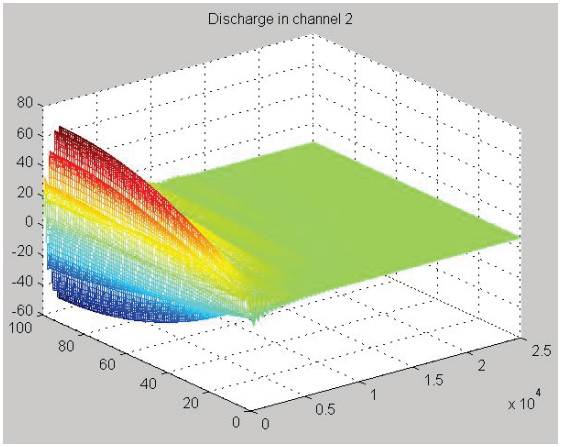

FIGURE 4.15: $Q_2(x,t)$

control. The PI controllers are designed at free extremities of each inlet channel for star-shaped network; and at junctions of each subsystem for cascaded network. The exponential stability for the closed-loop network and the output regulation are proven by using Lyapunov direct method. The PI control design can be applied for many practical networks with the same steps as presented in Section 4.5.2.3. Although the stability analysis in the chapter is considered almost for linear system networks in L^2 norm, it is possible to extend our Lyapunov functional for the nonlinear case with the idea in [18]. In the future, the objective is to study the PI control design for other network types of PDE hyperbolic systems, such as tree-shaped network; and consider the performance of closed-loop systems controlled by PI controllers.

The results of this chapter are published in [69] and [70]

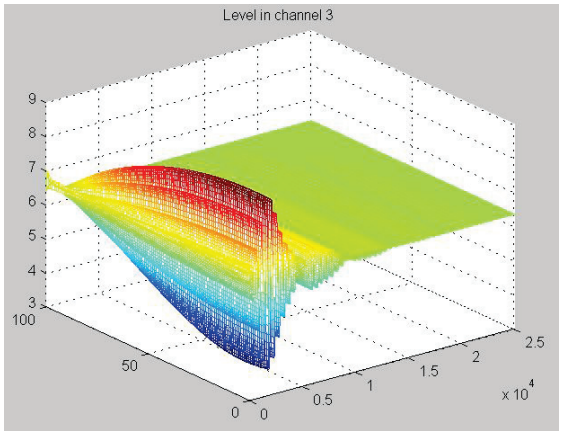

FIGURE 4.16: $H_3(x,t)$

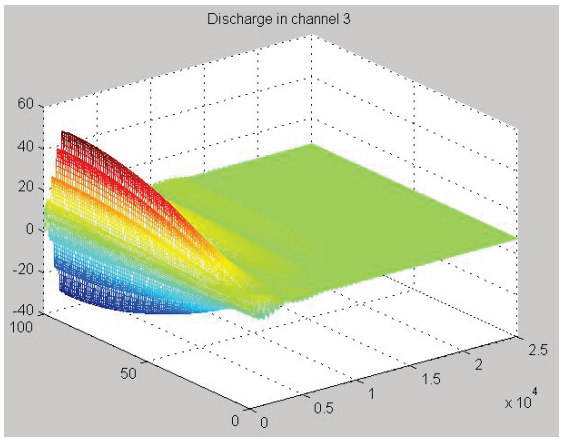

FIGURE 4.17: $Q_3(x,t)$

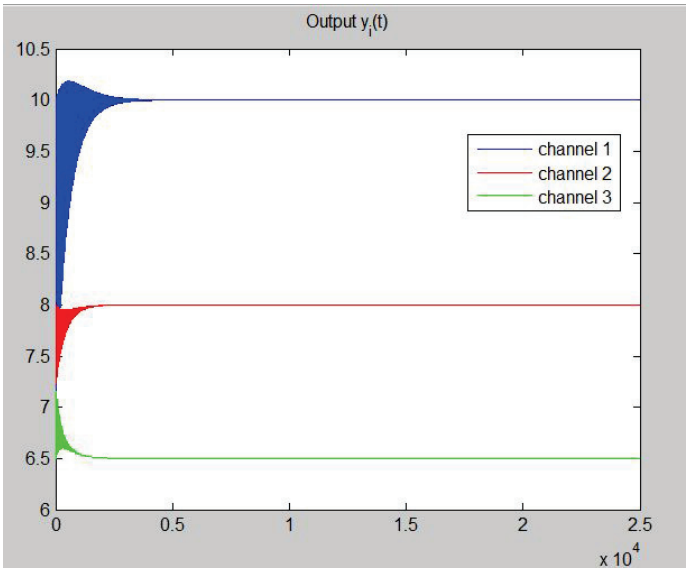

FIGURE 4.18: Output measurements $y_i(t)$

Chapter 5

Conclusions and Perspectives

5.1 Summary

In the thesis, we have studied the control and output regulation of infinite dimensional systems governed by hyperbolic PDEs of conservation laws. By the dynamic control laws in the form of PI controllers, two main objectives are considered and resolved: boundary control problem, and output regulation issue.

In Chapter 2, we have addressed the problems of boundary control and output regulation for a class of nonliear scalar hyperbolic systems with boundary integral controllers. The exponential stability and regulation effect of nonlinear closed-loop systems coupled by PDE/ODE are proven by Lyapunov direct method in the norm H^2. Moreover, we have tried to proposed an interval of integral gain for controllers to guarantee the stabilization property. Some numerical simulations are implemented to validate the theoretical control design.

In Chapter 3, we have considered a class of 2×2 linear hyperbolic systems and the multivariable PI control design on the boundary. Based on strict Lyapunov functionals constructed, we prove the exponential stabilization of the closed-loop system controlled by PI controllers in the norm L^2. Then the output regulation is obtained by analysis of extended stability in the norm H^1. Finally, numerical simulations of a Saint-Venant fluid flow model are given as a direct application of our control design.

In Chapter 4, we are concerned with the star-shaped network and the cascaded one of multi-systems described by 2×2 linear hyperbolic PDEs. The issues of stabilization and output regulation are studied for both two kinds of network with systematic PI control designs by using Lyapunov direct method. Then elaborate applications of our

control design for Saint-Venant channel models are presented with numerical simulations to evaluate the performance of the control methodology.

5.2 Perspectives

There are still many directions to extend and deepen the results obtained in this thesis concerning the control and output regulation of hyperbolic systems.

The first perspective to be mentioned is how to extend the control design in the thesis to the case of inhomogeneous hyperbolic systems, i.e the such systems

$$\frac{\partial}{\partial t}\phi(x,t) + A(\phi)\ \frac{\partial}{\partial x}\phi(x,t) = B(\phi)\ ,$$

This is an interesting question because the equilibrium point in this case is no longer constant value but dependent on x, and stability analysis for nonlinear systems is more challenging, in particularly with Lyapunov direct method.

The second perspective worthy studying in the future is how to find a strict Lyapunov function for general hyperbolic systems with dynamic control laws. In fact, in the case of static control law, Coron et al. have proposed a general Lyapunov functional in their works [20, 22]. And in the thesis, this issue is considered for some special classes of hyperbolic systems as nonlinear scalar systems, MIMO linear 2×2 systems, some networks of $2n$ states with PI control, based on strict Lyapunov functions that contain the coupled terms between states and feedback variables. However, constructing a such Lyapunov functional for general closed-loop systems coupled by PDE/ODE is until an open question.

Another direction that we can deepen from the work in the thesis is to improve the PI control design for hyperbolic systems. The first improvement that could be considered is concerning the optimal value of integral gain of PI controllers. For example one would like to find an exact interval of integral gain μ to guarantee the stability of closed-loop systems, or an optimal value in the sense of minimizing some cost function defined a priori. The second improvement may be how to extend the PI control design in the thesis for the case of inconstant disturbances.

The last perspective is that how to adapt our control design method in the thesis to apply for other types of infinite dimensional systems such as parabolic systems, elliptic systems. This issue is interesting because it allows to enlarge the domain of applications in the practical models.

Appendix A

Numerical simulations of PDE hyperbolic systems with Preisssmann schema

A.1 Introduction

In this Section, a methodology of numerical simulations for models described by hyperbolic partial differential equations (PDE) is presented. The main steps of the simulations are summarized as follows:

- Studying an appropriate scheme for discretizing hyperbolic PDE models. In the thesis, we choose the Preissmann scheme which has been considered in the literature, see in [31], [2], [51] and [71].
- Based on the approximate formulas of Preissmann schema, we compute the discretized expressions of functions and theirs derivatives.
- Formulating the problem under the form of a nonlinear algebraic equation $F(X) = 0$, where X is the unknown state variable.
- Applying the Newton-Raphson iterative method to find a numerical solution X of above equation.

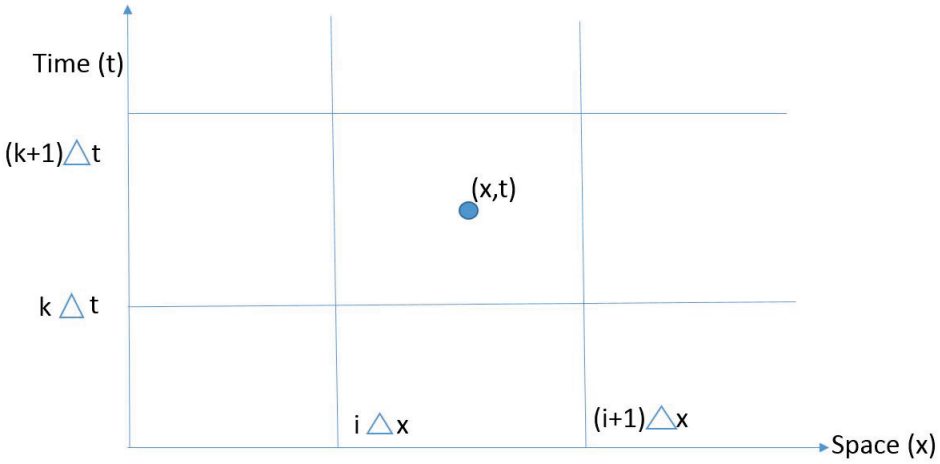

FIGURE A.1: Preissmann discretization scheme

A.2 Preissmann schema

Preissmann scheme is an implicit finite difference method. Given a continuous function $f(x,t)$ and a real parameter $\theta \in (0,1)$, let denote i for space index and k for time index, and write $f_i^k = f(i\Delta x, k\Delta t)$ where Δx and Δt are respectively space and time dicretization steps, see in Figure A.1. The following formulas are employed to discretize function f and their derivatives:

$$\frac{\partial f}{\partial t}(x,t) = \frac{f_{i+1}^{k+1} + f_i^{k+1} - f_{i+1}^k - f_i^k}{2\Delta t} \tag{A.1}$$

$$\frac{\partial f}{\partial x}(x,t) = \theta \frac{f_{i+1}^{k+1} - f_i^{k+1}}{\Delta x} + (1-\theta)\frac{f_{i+1}^k - f_i^k}{\Delta x} \tag{A.2}$$

$$f(x,t) = \theta \frac{f_{i+1}^{k+1} + f_i^{k+1}}{2} + (1-\theta)\frac{f_{i+1}^k + f_i^k}{2} \tag{A.3}$$

A.3 Discretizing hyperbolic PDE models

We consider the following 1D hyperbolic PDE system:

$$\frac{\partial Y(x,t)}{\partial t} + \frac{\partial f(Y(x,t))}{\partial x} = 0. \tag{A.4}$$

with boundary condition

$$h(Y(0,t), Y(L,t), u(t)) = 0 \tag{A.5}$$

where $x \in [0, L]$ denotes the space variable, $t \in \mathbb{R}_+$ the time variable, $Y : [0, L] \times \mathbb{R}_+ \to \mathbb{R}^n$ state, $u(t) \in \mathbb{R}^m$ input control, and nonlinear functions $f : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$

Note that N denotes the number of spatial elements in $[0, L]$. We discretize the PDE equation (A.4) and boundary equation (A.5) with Preissmann scheme (A.1)-(A.3), one gets that

$$\frac{Y_{i+1}^{k+1} + Y_i^{k+1} - Y_{i+1}^k - Y_i^k}{2\Delta t} + \frac{1}{\Delta x}\left\{\theta\left(f(Y_{i+1}^{k+1}) - f(Y_i^{k+1})\right) + (1-\theta)\left(f(Y_{i+1}^k - f(Y_i^k)\right)\right\} = 0 \quad \forall i = \overline{1, N-1}\ , \quad \text{(A.6)}$$

$$h(Y_1^{k+1}, Y_N^{k+1}, u^{k+1}) = 0 \quad \text{(A.7)}$$

where control input u^{k+1} is taken online.

With equations (A.6) and (A.7), we obtain $n \times N$ nonlinear equations with $n \times N$ unknown variables, which are written in the form as follows

$$F(X) = 0 \quad \text{(A.8)}$$

where $X = (Y_i^{k+1}) \in \mathbb{R}^{n \times N}$ $\forall i = \overline{1, N}$ includes all unknown variables at instant $k+1$, and an nonlinear function $F : \mathbb{R}^{n \times N} \to \mathbb{R}^{n \times N}$.

In the following, we apply the Newton-Raphson method to compute a numerical solution X for the implicit nonlinear equation (A.8).

A.4 Newton-Raphson method

The Newton-Raphson method is a very popular iterative approach used to resolve an implicit nonlinear equation by giving an approximate solution with chosen tolerance.

To construct a solution for nonlinear equation (A.8), we follow the following steps

- Take an initial value $X_0 \in \mathbb{R}^{n \times N}$ for X and a small tolerance ε.
- Make successive iterations by

$$X_{p+1} = X_p - J^{-1}(X_p)F(X_p)$$

where $p = 0, 1, 2, ...$ denotes the iteration index and $J(X_p)$ is the Jacobian matrix of function F computed at X_p, i.e ,

$$J(X_p) = \frac{\partial F}{\partial X}(X_p).$$

- For each iteration, compute the error vector $X_{p+1}(k) - X_p(k)$ and verify that whether. The algorithm stops when

$$||X_{p+1}(k) - X_p(k)|| < \varepsilon$$

Then, we set the approximate value for X:

$$X = X_{p+1}$$

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