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# Fibrés principaux géométriques sur les variétés riemanniennes géométriques

# Geometric principal bundles over geometric Riemannian manifolds

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# Contents

CHAPTER 1

# Introduction

## 1. Geometric structures and locally homogeneous triples. Motivation

The concept of *model geometry* introduced by William Thurston [**Th**] plays a fundamental role in modern Differential Geometry:

DEFINITION 1.1. *An $n$-dimensional model geometry is a pair*

$$(X, \alpha : G \times X \to X)\,,$$

*where $X$ is a simply connected $n$-dimensional manifold, $G$ is a Lie group, and $\alpha$ is an effective, transitive action with compact stabilizers.*

The condition on the compactness of the stabilizers has an important consequence:

REMARK 1.2. Let $(X, \alpha : G \times X \to X)$ be a model geometry. Then $X$ has an $\alpha$-invariant Riemannian metric. Any such a metric is homogeneous, in particular complete.

Many authors implicitly require that $G$ is connected. Under this assumption it follows that all the stabilizers $G_x$, $x \in M$ are also connected [**Fi**, Proposition 1.1.1].

We will not adopt this convention, but (since we focus on geometric structures on orientable manifolds) we will assume that $G$ acts by orientation preserving diffeomorphisms.

We have a natural notion of isomorphism (equivalence) between two geometries (see [**Sc**, p. 474]). A fundamental problem in the theory of geometric structures is the classification of $n$-dimensional geometries, for a given dimension $n$, up to equivalence.

Let $(X, \alpha : G \times X \to X)$, $(X, \alpha' : G' \times X \to X)$ be two model geometries with the same underlying manifold $X$. We say that $(X, \alpha' : G' \times X \to X)$ is *larger* than $(X, \alpha : G \times X \to X)$ (or $(X, \alpha : G \times X \to X)$ is *contained* in $(X, \alpha' : G' \times X \to X)$), if $G$ is a Lie subgroup of $G'$ and $\alpha$ is the restriction of $\alpha'$ to $G \times X$. A geometry $(X, \alpha : G \times X \to X)$ is called *maximal* if, for any larger geometry $(X, \alpha' : G' \times X \to X)$, one has $G = G'$. Any model geometry $(X, \alpha : G \times X \to X)$ is contained in a maximal geometry [**Fi**, Proposition 1.1.2].

Let $(X, \alpha : G \times X \to X)$ be an $n$-dimensional model geometry, and $\mathcal{G}_\alpha$ be the pseudogroup of transformations of $X$ which are restrictions (to open sets) of diffeomorphisms $\alpha_g : X \to X$.

DEFINITION 1.3. *Let $(X, \alpha : G \times X \to X)$ be an $n$-dimensional model geometry, and $M$ be a differentiable $n$-manifold. A geometric $(X, \alpha)$-structure on $M$ is a differentiable $\mathcal{G}_\alpha$-structure on $M$, i.e. a maximal $\mathcal{G}_\alpha$-atlas $\mathcal{A}$ on $M$.*

A differentiable manifold which admits a geometric $(X, \alpha)$-structure will be called $(X, \alpha)$-geometric manifold. Note that

REMARK 1.4. Let $(X, \alpha : G \times X \to X)$ be a model geometry. The following conditions are equivalent:

(1) $G$ has a discrete group $\Gamma$ such that the right coset space $\Gamma\backslash G$ is compact.
(2) $G$ has a discrete group $\Gamma$ such that $X/\Gamma$ is compact.
(3) There exist compact $(X, \alpha)$-geometric manifolds.

Since in this thesis we are interested in the class of *compact* geometric manifolds, the model geometries satisfying these three equivalent conditions will play an important role. Such a model geometry will be called *model geometry with compact quotients.*

Fixing an $\alpha$-invariant Riemannian metric $g$ on $X$ (see Remark 1.2), we see that $(X, \alpha)$-structure $\mathcal{A}$ on $M$ yields a Riemannian metric $g_{\mathcal{A}}$ on $M$ which is *locally homogeneous*, i.e., for any pair $(x, x') \in M \times M$ there exists open neighborhoods $U \ni x$, $U' \ni x'$ and an isometry $\varphi : U \to U'$ with $\varphi(x) = x'$.

Conversely, any *compact* locally homogeneous Riemannian manifold is geometric. More precisely:

REMARK 1.5. Let $(M, g)$ be a compact, connected, orientable, locally homogeneous Riemannian manifold, and $\pi : \tilde{M} \to M$ be its universal cover. There exists a model geometry $(\tilde{M}, \alpha : G \times \tilde{M} \to \tilde{M})$ and a geometric $(\tilde{M}, \alpha)$-structure on $M$.

This is a well-known, classical result in the theory of geometric manifolds. Since the strategy of the proof will play an important role in this thesis, we explain briefly the argument: Putting $\tilde{g} := \pi^* g$, we obtain a complete, simply connected Riemannian manifold $(\tilde{M}, \tilde{g})$, which will also be locally homogeneous. A well known theorem of Ambrose-Singer [**AS**], [**Si**] implies that $(\tilde{M}, \tilde{g})$ is homogeneous, in particular $\tilde{M}$ admits a connected transitive group of isometries. In particular the group $\mathrm{Iso}^+(\tilde{M}, \tilde{g})$ of orientation preserving diffeomorphisms will act transitively on $\tilde{M}$, and the obtained pair $(\tilde{M}, \alpha : \mathrm{Iso}^+(\tilde{M}, \tilde{g}) \times \tilde{M} \to \tilde{M})$ will be a model geometry, and $M$ obviously has a geometric $(\tilde{M}, \alpha)$-structure.

We emphasize the crucial role of Ambrose-Singer's theorem in this proof. It is easy to prove that a *real analytic* locally homogeneous, complete, simply connected Riemannian manifold is homogeneous. This follows from [**KN**, Theorem 6.3]. The statement also holds in the differentiable category, but the proof, due to Ambrose-Singer, is much more difficult (see section 2 in Chapter 5). We will come back later to this important detail, and the role of Ambrose-Singer's techniques in our thesis.

REMARK 1.6. In general the model geometry given by Remark 1.5 is not maximal. Therefore, in general a locally homogeneous metric on a compact manifold $M$ might not be induced by a geometric structure associated with a maximal geometry with underlying manifold $\tilde{M}$. For instance the Berger metrics [**Be**] on $\mathrm{S}^3$ are locally homogenous, but (excepting the standard constant curvature metrics) are not induced by a geometric structure associated with a maximal geometry.

A Berger metric on $\mathrm{S}^3$ can be constructed using a constant curvature metric on $\mathrm{S}^2$, and a homogeneous connection on the Hopf bundle $\mathrm{S}^3 \to \mathrm{S}^2$, which is a principal $\mathrm{S}^1$-bundle on $\mathrm{S}^2$. One of the goals of this thesis is to generalize this construction, and to prove effective results on the classification of locally homogeneous (geometric) compact manifolds obtained using connections on principal bundles over a given geometric base. The main concept introduced and studied in this thesis is

DEFINITION 1.7. *Let $M$ be a differentiable manifold and $K$ a Lie group. A locally homogeneous triple with structure group $K$ on $M$ is a triple $(g, P \xrightarrow{p} M, A)$, where $p : P \to M$ is a principal $K$-bundle on $M$, $g$ is Riemannian metric on $M$, and $A$ is connection on $P$ such that the following locally homogeneity condition is satisfied: for every two points $x$, $x' \in M$ there exists an isometry $\varphi : U \to U'$ between open neighborhoods $U \ni x$, $U' \ni x'$ with $\varphi(x) = x'$, and a $\varphi$-covering bundle isomorphism $\Phi : P_U \to P_{U'}$ such that $\Phi^*(A_{U'}) = A_U$.*

In these formulae, for an open set $U \subset M$, we use the subscript $_U$ to denote the restriction of the indicated objects to $U$. For connections on principal bundles we adopt the conventions of [**KN**], so in Definition 1.7 the symbol $A$ stands for a $K$-invariant horizontal distribution of $P$. We will denote by $V_P \subset T_P$ the vertical distribution of $P$.

Fix now an inner product $\langle\langle \cdot, \cdot \rangle\rangle$ on the Lie algebra $\mathfrak{k}$ of $K$, and let $g_A$ be the Riemannian metric on $P$ characterized by the conditions

(1) The canonical bundle isomorphism $V_P \simeq P \times \mathfrak{k}$ is an orthogonal bundle isomorphism with respect to the inner products defined by $g_A$ and $\langle\langle \cdot, \cdot \rangle\rangle$.
(2) The restriction of $p_*$ to the subbundle $A \subset T_P$ gives an orthogonal bundle isomorphism $A \to p^*(T_M)$,
(3) The direct sum decomposition $T_P = A \oplus V_P$ of the tangent bundle $T_P$ is $g_A$-orthogonal.

The following remark gives the motivation for introducing and studying locally homogeneous triples:

REMARK 1.8. Let $(g, P \xrightarrow{p} M, A)$ be a locally homogeneous triple with structure group $K$, and let $\langle\langle \cdot, \cdot \rangle\rangle$ be an ad-invariant inner product on $\mathfrak{k}$. Then $g_A$ is a locally homogeneous Riemannian metric on $P$.

We are interested in the important case when $K$ is compact. In this case an ad-invariant inner product $\langle\langle \cdot, \cdot \rangle\rangle$ on $\mathfrak{k}$ exists and, according to Remark 1.5, the total space $P$ will be a geometric compact manifold. Therefore a locally homogeneous triple with compact structure group $K$ gives a geometric principal bundle over a geometric base; this justifies the title of our thesis.

The class of geometric manifolds obtained using locally homogeneous triples is larger than one might think: recall that in dimension 3 there exists (up to equivalence) eight maximal model geometries which admit compact quotients: $\mathbf{E}^3$, $\mathbf{S}^3$, $\mathbf{H}^3$, $\mathbf{S}^2 \times \mathbb{R}$, $\mathbf{H}^2 \times \mathbb{R}$, $\widetilde{\mathbf{SL}_2(\mathbb{R})}$, $\mathbf{Nil}$ and $\mathbf{Sol}$. We refer to [**Th**], [**Sc**] for the explicit description of the pair $(X, \alpha)$ corresponding to each symbol above.

Excepting $\mathbf{H}^3$ and $\mathbf{Sol}$, for any model geometry $(X, \alpha)$ in this list there exist compact 3-manifolds with a geometric $(X, \alpha)$-structure which are associated with locally homogeneous triples with structure group $\mathrm{S}^1$. For instance any *non-trivial* $\mathrm{S}^1$-bundle over a Riemann surface of genus $g$ has a geometric $(X, \alpha)$-structure, where

(1) $(X, \alpha) = \mathbf{S}^3$ when $g = 0$,
(2) $(X, \alpha) = \mathbf{Nil}$ when $g = 1$,
(3) $(X, \alpha) = \widetilde{\mathbf{SL}_2(\mathbb{R})}$ when $g \geq 2$.

The trivial $\mathrm{S}^1$-bundles over Riemann surfaces have geometric structures with model geometry $\mathbf{S}^2 \times \mathbb{R}$, $\mathbf{E}^3$, or $\mathbf{H}^2 \times \mathbb{R}$. Moreover any $\mathrm{S}^1$-bundle over a Riemann surface has a geometric metric which is associated with a locally homogeneous triple in the sense of Definition 1.7. These examples shows the abundance of geometric principal bundles in the class of geometric manifolds.

At the end of this introductory section we explain the motivations behind our interest in locally homogeneous triples and their classifcation:

(1) Locally homogeneous triples give a large class of interesting geometric manifolds in any dimension.
(2) The classification of locally homogeneous triples with fixed structure group $K$ on a given base $M$ leads to interesting moduli spaces, so it is related to gauge theory.
(3) The methods used in chapter 2, in which we give an infinitesimal characterization of the homogeneity condition for triples, are very general and can be used for the classification of other classes of locally homogenous structures on manifolds, e.g. for locally homogenous triples consisting of a Riemannian metric, a $\mathrm{Spin}^c$-structure and a spinor.

## 2. Preliminary definitions and notations

Let $(N,h)$ be a Riemannian manifold, and let $G \subset \mathrm{Iso}(N,h)$ be a closed subgroup of the group $\mathrm{Iso}(N,h)$ of isometries of $(N,h)$. Let $K$ be a compact Lie group, and $q : Q \to N$ be a principal $K$-bundle on $N$. The group of $G$-covering bundle isomorphisms of $Q$ is defined by

$$\mathcal{G}_G(Q) := \{(\Phi, \varphi)|\ \varphi \in G,\ \Phi : Q \to Q \text{ is a } \varphi\text{-covering bundle isomorphism } \}.$$

The group $\mathcal{G}_G(Q)$ has a natural topology (induced by the weak $\mathcal{C}^\infty$-topology [**Hi**, section 2.1]), and fits in the short exact sequence

$$\{1\} \to \mathcal{G}(Q) \to \mathcal{G}_G(Q) \xrightarrow{\mathfrak{p}} G \to \{1\},$$

where $\mathcal{G}(Q)$ is the gauge group of $Q$ [**DK**], [**Te**], i.e. the group of $\mathrm{id}$-covering bundle automorphisms of $Q$. Let now $\Gamma \subset G$ be a subgroup of $G$ acting properly discontinuously on $N$. The quotient $M := N/\Gamma$ comes with a natural Riemannian metric, such that the canonical projection $\pi : N \to M$ becomes a locally isometric covering projection. Denote by $g$ this Riemannian metric on $M$ induced by $h$ via $\pi$.

Suppose now the group epimorphism $\mathfrak{p}^{-1}(\Gamma) \to \Gamma$ has a right inverse, i.e. that there exists a group morphism $\mathfrak{j} : \Gamma \to \mathcal{G}_G(Q)$ such that $\mathfrak{p} \circ \mathfrak{j} = \iota_\Gamma$, where $\iota_\Gamma : \Gamma \to G$ is the inclusion monomorphism. If this is the case, we will obtain the commutative diagram

$$\begin{tikzcd}
\{1\} \arrow[r] & \mathcal{G}(Q) \arrow[r] & \mathcal{G}_G(Q) \arrow[r, "\mathfrak{p}"] & G \arrow[r] & \{1\} \\
 & & & \Gamma \arrow[ul, "\mathfrak{j}"] \arrow[u, hook, "\iota_\Gamma"'] &
\end{tikzcd} \quad . \tag{1}$$

The group $\Gamma$ acts (via $\mathfrak{j}$) on $Q$ by bundle isomorphisms, and the quotient $P := Q/\Gamma$ will be a principal $K$-bundle on the quotient manifold $M$.

Let now $B$ be a connection on $Q$ satisfying the following invariance condition:

$$\textit{Any element } \varphi \in G \textit{ has a lift in } \mathcal{G}_G(Q) \textit{ which leaves } B \textit{ invariant.} \tag{$C_G$}$$

In this case we obtain a short exact sequence

$$\{1\} \to \mathcal{G}^B(Q) \to \mathcal{G}^B_G(Q) \xrightarrow{\mathfrak{p}_B} G \to \{1\}\,,$$

where $\mathcal{G}^B(Q)$ ($\mathcal{G}^B_G(Q)$) is the stabilizer of $B$ in the gauge group $\mathcal{G}(Q)$ (respectively in the group $\mathcal{G}_G(Q)$). If, moreover, we can find a lift $\mathfrak{j}:\Gamma\to\mathcal{G}^B_G(Q)$ of the inclusion monomorphism $\iota_\Gamma:\Gamma\to G$, we will obtain the diagram

$$\begin{tikzcd} \{1\} \arrow[r] & \mathcal{G}^B(Q) \arrow[r] & \mathcal{G}^B_G(Q) \arrow[r, "\mathfrak{p}_B"] & G \arrow[r] & \{1\} \\ & & & \Gamma \arrow[ul, "\mathfrak{j}"] \arrow[u, hook, "\iota_\Gamma"'] & \end{tikzcd} , \qquad (2)$$

and the quotient bundle $p:P=Q/\Gamma\to N/\Gamma=M$ will come with an induced connection, which will be denoted by $A$.

Definition 2.1. *The triple $(g, P\xrightarrow{p} M, A)$ obtained in this way will be called the $\Gamma$-quotient of $(h, Q\xrightarrow{q} N, B)$ associated with the lift $\mathfrak{j}:\Gamma\to\mathcal{G}^B_G(Q)$ of $\mathfrak{p}_B$.*

Note that condition ($\mathrm{C}_G$) has an important gauge theoretical interpretation (see [**BiTe**] for details): Let $\mathcal{B}(Q):=\mathcal{A}(Q)/\mathcal{G}(Q)$ be the moduli space of all connections on $Q$, where $\mathcal{A}(Q)$ denotes the space of connections on $Q$. A connection $B'$ on a principal $K$-bundle $Q'\simeq Q$ yields a well defined element $[B']\in\mathcal{B}(Q)$ in the following way: we chose a bundle isomorphism $\psi:Q\to Q'$, and we put $[B]:=\mathcal{G}(Q)\cdot\psi^*(B')$. This gauge class will be independent of $\psi$. On the other hand, since $G$ is connected, we have $\varphi^*(Q)\simeq Q$ for any $\varphi\in G$. Therefore, for any $B\in\mathcal{A}(Q)$ and any $\varphi\in G$, the pull-back connection $\varphi^*(B)\in\mathcal{A}(\varphi^*(Q))$ defines a gauge class $\varphi^*[B]\in\mathcal{B}(Q)$. In other words, we obtain a well-defined action of $G$ on the moduli space $\mathcal{B}(Q)$.

Remark 2.2. A connection $B\in\mathcal{A}(Q)$ satisfies condition $(C_G)$ if and only if the gauge class $[B]\in\mathcal{B}(Q)$ is $G$-invariant.

Remark 2.3. If $G$ acts transitively on $N$, then

- The pair $(Q,B)$ is homogeneous with respect to the Lie group $\mathcal{G}^B_G(Q)$. This Lie group is an extension of $G$ by the compact group $\mathcal{G}^B(Q)$ (which is isomorphic to a closed subgroup of $K$).
- The quotient triple $(g, P\xrightarrow{p} M, A)$ is locally homogeneous.

The first goal of this thesis is to prove that, under certain (very general) conditions, all locally homogeneous triples can be obtained in this way. More precisely, any locally homogeneous triple on $M$ can be obtained as the quotient of a homogeneous triple on $\tilde{M}$. Then we will use this result to obtain explicit classification theorems for locally homogeneous triples.

## 3. The approach. Presentation of results

Our method for the classification of locally homogeneous triples on compact manifolds has two main steps:

(S1) Prove that any *locally homogeneous* triple on $M$ is obtained as a quotient of a *globally homogeneous* triple on the universal cover $\tilde{M}$.

(S2) Use Biswas-Teleman's description [**BiTe**] of the moduli space of homogeneous connections on a manifold endowed with a transitive action.

(S1) The first step is a special application of the following natural, general idea: reduce the classification of a class of locally homogeneous objects on $M$ to the classification of a class of (globally) homogeneous objects on the universal cover $\tilde{M}$.

Suppose for instance that $M$ is compact, and $g$ is a locally homogeneous Riemannian metric on $M$. Then the induced metric $\tilde{g}$ on $\tilde{M}$ will be locally homogeneous and complete, hence homogeneous by the well-known theorem of Ambrose-Singer mentioned above. Therefore, using the correspondence $g \mapsto \tilde{g}$, the classification of locally homogeneous Riemannian metrics $g$ on $M$ reduces to the classification of homogeneous metrics $\tilde{g}$ on $\tilde{M}$ for which the isometry group $\mathrm{Iso}(\tilde{M}, \tilde{g})$ contains the covering transformation group of the universal cover $\tilde{M} \to M$. We will prove a similar result for locally homogeneous triples. Let $\pi : \tilde{M} \to M$ be the universal cover of $M$, and let $\Gamma$ be the corresponding covering transformation group,

THEOREM 3.1. *Let $M$ be a compact manifold, and $K$ be a compact Lie group. Let $\pi : \tilde{M} \to M$ be the universal cover of $M$, $\Gamma$ be the corresponding covering transformation group. Then, for any locally homogeneous triple $(g, P \xrightarrow{p} M, A)$ with structure group $K$ on $M$ there exists*

(1) *A connection $B$ on the pull-back bundle $Q := \pi^*(P)$.*
(2) *A closed subgroup $G \subset \mathrm{Iso}(\tilde{M}, \pi^*(g))$ acting transitively on $\tilde{M}$ which contains $\Gamma$ and leaves invariant the gauge class $[B] \in \mathcal{B}(Q)$.*
(3) *A lift $\mathfrak{j} : \Gamma \to \mathcal{G}^B_G(Q)$ of the inclusion monomorphism $\iota_\Gamma : \Gamma \to G$, where $\mathcal{G}^B_G(Q)$ stands for the group of automorphisms of $(Q, B)$ which lift transformations in $G$.*
(4) *An isomorphism between the $\Gamma$-quotient of $(\pi^*(g), Q, B)$ and the initial triple $(g, P \xrightarrow{p} M, A)$.*

Therefore, any locally homogeneous triple on $M$ is the quotient of a globally homogeneous triple on the universal cover $\tilde{M}$.

(S2) The group $\hat{G} := \mathcal{G}^B_G(Q)$ whose existence is given by Theorem 3.1 is a Lie group which fits in the short exact sequence

$$\{1\} \to \mathcal{G}^B(Q) \to \mathcal{G}^B_G(Q) \xrightarrow{\mathfrak{p}_B} G \to \{1\}, \tag{3}$$

where $\mathcal{G}^B(Q)$ is the stabilizer of $B$ in the gauge group $\mathcal{G}(Q)$ of $Q$. The connection $B$ is $\mathcal{G}^B(Q)$-homogeneous and, in principle, one can use the results of [**BiTe**] for the classification of homogeneous connections.

REMARK 3.2. An important difficulty arise: one has first to classify all possible Lie groups $\hat{G}$, admitting $G$ as a quotient, which intervene in an exact sequence of the form (3). In other words the pull-back pairs $(Q, B)$ we want to classify are not necessarily $G$-homogeneous, but homogeneous with respect to an extension $\hat{G}$ of $G$.

Fixing a point $y_0 \in Q$ the stabilizer $\mathcal{G}^B(Q)$ can be identified with a closed subgroup $L \subset K$, and the conjugacy class of $L$ is independent of $y_0$. This conjugacy class will be called the conjugacy class associated with (the stabilizer of) the connection $B$.

It is important to point out that only a very small class of conjugacy classes of closed subgroups of $K$ are associated with connections in principal $K$-bundles. The reason is the following: the stabiliser of a connection $B$ is the centralizer of its holonomy subgroup (see section 5.1.4). Therefore, if $L \subset K$ is associated with (the stabilizer of) $B$, then it coincides with the centralizer of Lie subgroup of $K$. Such a subgroup of $K$, and also its conjugacy class, will be called *admissible*. For instance

EXAMPLE 3.1. For $K = \mathrm{SU}(2)$ we have only three admissible conjugacy classes of subgroups: the class of the center $Z(\mathrm{SU}(2)) = \{\pm I_2\}$, the conjugacy class of a maximal torus of $\mathrm{SU}(2)$ (which is isomorphic to $\mathrm{S}^1$), and the conjugacy class of $\mathrm{SU}(2)$.

For $K = \mathrm{SO}(3)$ we have five admissible conjugacy classes: the conjugacy class of the trivial subgroup, the conjugacy class of a subgroup isomorphic to $\mathrm{O}(1) = \mu_2$, the conjugacy class of a maximal torus of $\mathrm{SO}(3)$, the conjugacy class of a subgroup isomorphic to $\mathrm{O}(2)$, and the conjugacy class of $\mathrm{SO}(3)$.

For an arbitrary compact Lie group $K$, the minimal admissible conjugacy class is $\{Z(K)\}$, and the maximal admissible conjugacy class is $\{K\}$. If the stabiliser of a connection $A$ is minimal (coincides with $Z(K)$) $A$ is called *irreducible*.

Our method for the classification of locally homogeneous triples on an orientable, compact, connected manifold $M$ consists of the following:

(1) Classify all homogeneous metrics $\tilde{g}$ on $\tilde{M}$ admitting a closed, group of orientation-preserving isometries $G \subset \mathrm{Iso}^+(\tilde{g}, \tilde{M})$ which acts transitively on $\tilde{M}$ and contains $\Gamma$. In many interesting cases these conditions imply $G = \mathrm{Iso}^+(\tilde{g}, \tilde{M})$ (see section 1.1 Ch. 4).

(2) For any pair $(\tilde{g}, G)$ as above classify all Lie group extensions
$$\{1\} \to L \to \hat{G} \to G \to \{1\}\,, \tag{4}$$
where $L$ is an admissible subgroup of $K$, and all lifts $\mathfrak{j} : \Gamma \to \hat{G}$ of the embedding $\iota_\Gamma : \Gamma \to G$.

(3) For any such extension classify, up to isomorphism, all $\hat{G}$-homogeneous pairs $(Q, B)$ on $\tilde{M}$. This classification is obtained using Biswas-Teleman theorem [**BiTe**, Theorem 12]. Select the $\hat{G}$-homogeneous pairs $(Q, B)$ whose stabilizer is $L$, and such that $\hat{G}$ acts effectively on $Q$.

For any Lie group extension of the form (4) we obtain a family of triples $(\mathfrak{j}, Q, B)$, where $\mathfrak{j} : \Gamma \to \hat{G}$ is a lift of $\iota_\Gamma$, and $(Q, B)$ is a $\hat{G}$-homogeneous pair on $\tilde{M}$. Any such triple yields a locally homogeneous triple $(g, P, A)$ obtained as the $\Gamma$-quotient of $(Q, B)$ via $\mathfrak{j}$. The main Theorem 3.1 shows that *all locally homogeneous triples on $M$ can be obtained in this way*. We will illustrate our method for the classification of locally homogeneous triples with structure groups $\mathrm{S}^1$, $\mathrm{PU}(2)$ on hyperbolic Riemann surfaces. We will see that:

(C1) The isomorphism classes of locally homogeneous $S^1$-triples with fixed Chern class $c_1(P) = c$ correspond bijectively to the points of the corresponding moduli space of Yang-Mills connection (which is a torus of dimension $2g$).

(C2) The isomorphism classes of locally homogeneous $\mathrm{PU}(2)$-triples $(g, P, A)$ with $A$ flat, correspond bijectively to the points of the space of conjugacy classes of representations $\pi_1(M) \to \mathrm{PU}(2)$

(C3) The isomorphism classes of locally homogeneous $\mathrm{PU}(2)$-triples $(g, P, A)$ with $\pi^*(A)$ irreducible can be identified with $(0, \infty)$.

The $\mathrm{PU}(2)$-triples given by the third statement have not been considered in the literature before (to our knowledge). This leads to interesting locally homogeneous (hence geometric) 5-dimensional manifolds, which are fibre bundles over hyperbolic Riemann surfaces. We will come back to these family of triples in future research. One can ask interesting questions about these triples, for instance: study the Ricci flow on these geometric 5-dimensional manifolds.

The thesis is organized as follows: In Chapter 2 we prove Theorem 3.1 in the general framework of $\mathcal{C}^\infty$ triples. The proof is based on the following characterization of locally homogeneous triples:

THEOREM 3.3. *Let $M$ be a connected manifold, $(g, P \xrightarrow{p} M, A_0)$ be a triple consisting of a Riemannian metric $g$ on $M$, a principal $K$-bundle $P$ on $M$, and a connection $A_0$ on $P$. Denote by $C_0 \in \mathcal{A}(\mathrm{O}(M))$ the Levi-Civita connection on the orthonormal frame bundle $\mathrm{O}(M)$ of $(M, g)$. The following conditions are equivalent:*

*(1) $(g, P \xrightarrow{p} M, A_0)$ is locally homogeneous.*
*(2) $(g, P \xrightarrow{p} M, A_0)$ is infinitesimally homogeneous.*
*(3) There exists a pair $(C, A) \in \mathcal{A}(\mathrm{O}(M)) \times \mathcal{A}(P)$ such that*

$$\nabla^C R^C = 0,\ \nabla^C T^C = 0,\ \nabla^{CA} F^A = 0,\ \nabla^{CA}(A - A_0) = 0\,.$$

In this statement we have denoted by $R^C$ ($T^C$) the curvature (torsion) of the connection $C$ on $\mathrm{O}(M)$, and by $F^A$ the curvature of the connection $A$ on $P$. The concept of infinitesimally homogeneity for triples is inspired by Singer's notion of infinitesimally homogeneous Riemannian metrics [**Si**], [**NT**]. Taking into account the role of Singer's indeas in our thesis, we inserted a section in the Appendix in which we explain briefly the method of proof of this important result.

In Chapter 3 we give a (much simpler) proof of Theorem 3.1 in the real-analytic framework. The main idea in this chapter is to reduce the problem of extending bundle morphisms, which are compatible with a pair of connections, to an extension problem for parallel sections.

In Chapter 4 we make use of our main theorem to classify locally homogeneous triples on hyperbolic Riemann surfaces.

Chapter 5 is an appendix containing definitions and results used in our proofs. The results presented in the appendix are known, of general interest, but not easily available in the literature.

CHAPTER 2

# Locally homogeneous and homogeneous triples. The differentiable case

## 1. Infinitesimally homogeneous sections in associated bundles

Let $M$ be a differentiable $n$-manifold, and $L(M)$ be its frame bundle. Let $\hat{K}$ be a Lie group, $\hat{\mathfrak{k}}$ its Lie algebra, $r : \hat{K} \to \mathrm{GL}(n)$ be a morphism of Lie groups, $\pi : \hat{P} \to M$ be a principal $\hat{K}$-bundle[1] over $M$, and $f : \hat{P} \to L(M)$ be an $r$-morphism of principal bundles.

Let $V$ be a finite dimensional vector space, $\rho : \hat{K} \to \mathrm{GL}(V)$ be a morphism of Lie groups, and $E := \hat{P} \times_\rho V$ be the associated vector bundle. Put

$$W_{ijpq} := (\mathbb{R}^n)^{\otimes i} \otimes (\mathbb{R}^{n*})^{\otimes j} \otimes V^{\otimes p} \otimes V^{*\otimes q}\,,$$

and let

$$R : \hat{K} \to \mathrm{GL}(W_{ijpq})$$

be the linear representation induced by $r$ and $\rho$. The corresponding infinitesimal action $\hat{\mathfrak{k}} \to \mathrm{gl}(W_{ijpq})$ defines a $\hat{K}$-invariant pairing

$$\hat{K} \times W_{ijpq} \to W_{ijpq}\,, \tag{5}$$

which induces a paring of associated vector bundles

$$\mathrm{ad}(\hat{P}) \times \big(T_M^{\otimes i} \otimes (\Lambda^1_M)^{\otimes j} \otimes E^{\otimes p} \otimes E^{\otimes *q}\big) \to T_M^{\otimes i} \otimes (\Lambda^1_M)^{\otimes j} \otimes E^{\otimes p} \otimes E^{\otimes *q} \tag{6}$$

The pairings (5), (6) will be denoted by $(b, \eta) \mapsto b \cdot \eta$ to save on notations. For instance, the pairing

$$\mathrm{ad}(\hat{P}) \times_M \big(\{\Lambda^1_M\}^{\otimes j} \otimes E\big) \xrightarrow{\cdot} \{\Lambda^1_M\}^{\otimes j} \otimes E \tag{7}$$

is given by the formula

$$(b \cdot \eta)(w_1, \dots, w_j) := b \cdot \big(\eta(w_1, \dots, w_j)\big) - \sum_{i=1}^{j} \eta(w_1, \dots, b \cdot w_i, \dots, w_j)\,.$$

The fact that (5) is $\hat{K}$-invariant, has an important consequence:

Remark 1.1. The pairing (6) is parallel with respect to any connection on $\hat{P}$.

We shall also need the pairing

$$\big(\{\Lambda^1_M\}^{\otimes k} \otimes \mathrm{ad}(\hat{P})\big) \times_M \big(\{\Lambda^1_M\}^{\otimes j} \otimes E\big) \xrightarrow{\cdot} \{\Lambda^1_M\}^{\otimes (j+k)} \otimes E$$

given by

$$\big(u \otimes b\big) \cdot \eta := u \otimes (b \cdot \eta)\,.$$

[1] The results of this section will be applied later taking $\hat{K} = \mathrm{O}(n) \times K$, $\hat{P} = \mathrm{O}(M) \times_M P$, where $P$ is a principal $K$-bundle over $M$ as in Chapter 1. This justifies the notations $\hat{K}$, $\hat{P}$ used in this section.

In other words, for $\beta\in\{\Lambda^1_x\}^{\otimes k}\otimes\mathrm{ad}(\hat P_x)$ and $\eta\in\{\Lambda^1_x\}^{\otimes j}\otimes E_x$, one has

$$\begin{aligned}(\beta\cdot\eta)(v_1,\dots,v_k,w_1,\dots,w_j)&=\big\{\beta(v_1,\dots,v_k)\cdot\eta\big\}(w_1,\dots,w_j)=\\ =\beta(v_1,\dots,v_k)\cdot\eta(w_1,\dots,w_j)&-\sum_{i=1}^{j}\eta(w_1,\dots,\beta(v_1,\dots,v_k)\cdot w_i,\dots,w_j)\,.\end{aligned}\tag{8}$$

For a tensor monomial $\beta=(\omega_1\otimes\dots\otimes\omega_k)\otimes b$ with $\omega_i\in\Lambda^1_x$ and $b\in\mathrm{ad}(\hat P)_x$, one has

$$\big((\omega_1\otimes\dots\otimes\omega_k)\otimes b\big)\cdot\eta=(\omega_1\otimes\dots\otimes\omega_k)\otimes(b\cdot\eta)\,.$$

For a connection $B\in\mathcal{A}(\hat P)$ we define the following associated linear connections:

- $\nabla^B_M$ is the linear connection on $T_M$ associated with $f_*(B)$. This connection corresponds (via the the isomorphism $\hat P\times_r\mathbb{R}^n\simeq T_M$ induced by $f$) to the linear connection associated with $B$ on the associated vector bundle $\hat P\times_r\mathbb{R}^n$.
- $\nabla^B_E$ is the linear connection on $E=\hat P\times_\rho V$ associated with $B$.
- $\nabla^B_{\mathrm{ad}}$ is the linear connection on $\mathrm{ad}(\hat P)=\hat P\times_{\mathrm{ad}}\hat{\mathfrak{k}}$ associated with $B$.
- $\nabla^B_{ijpq}$ is the linear connection on $T_M^{\otimes i}\otimes(\Lambda^1_M)^{\otimes j}\otimes E^{\otimes p}\otimes E^{\otimes *q}$ associated with $B$. In other words
$$\nabla^B_{ijpq}=(\nabla^B_M)^{\otimes i}\otimes(\nabla^B_M)^{*\otimes j}\otimes(\nabla^B_E)^{\otimes p}\otimes(\nabla^B_E)^{*\otimes q}\,.$$

Taking into account Remark 1.1 it follows

REMARK 1.2. For any $x\in M$ and tangent vector $\xi\in T_xM$ the following Leibniz rule holds:

$$\nabla^B_{ijpq,\xi}(b\cdot\eta)=(\nabla^B_{\mathrm{ad},\xi}b)\cdot\eta+b\cdot\nabla^B_{ijpq,\xi}\eta\,.$$

In particular one has

$$\nabla^B_{M,\xi}(b\cdot X)=(\nabla^B_{\mathrm{ad},\xi}b)\cdot X+b\cdot(\nabla^B_{M,\xi}X)\,,$$
$$\nabla^B_{E,\xi}(b\cdot y)=(\nabla^B_{\mathrm{ad}}b)\cdot y+b\cdot(\nabla^B_E y)\,,$$

for any pairs $(b,X)\in\Gamma(\mathrm{ad}(\hat P))\times\mathcal{X}(M)$, $(b,y)\in\Gamma(\mathrm{ad}(\hat P))\times\Gamma(E)$.

Using tensor product of connections we obtain connections $(\nabla^B_M)^{\otimes j}\otimes\nabla^B_E$, $(\nabla^B_M)^{\otimes j}\otimes\nabla^B_{\mathrm{ad}}$ on the vector bundles $\{\Lambda^1_M\}^{\otimes j}\otimes E$, $\{\Lambda^1_M\}^{\otimes j}\otimes\mathrm{ad}(\hat P)$.

Using the pairing (8) we obtain the following variation formula for the connection $(\nabla^B_M)^{\otimes j}\otimes\nabla^B_E$ with respect to $B$:

REMARK 1.3. Let $B\in\mathcal{A}(\hat P)$ and $\beta\in A^1(\mathrm{ad}(\hat P))$. Put $B':=B+\beta$. For any $\eta\in\Gamma\big(\{\Lambda^1_M\}^{\otimes j}\otimes E\big)$ one has

$$\big((\nabla^{B'}_M)^{\otimes j}\otimes\nabla^{B'}_E\big)\eta=\big((\nabla^B_M)^{\otimes j}\otimes\nabla^B_E\big)\eta+\beta\cdot\eta\,.\tag{9}$$

In other words, for any $x\in M$ and $\xi\in T_xM$ one has

$$\big((\nabla^{B'}_M)^{\otimes j}\otimes\nabla^{B'}_E\big)_\xi\eta=\big((\nabla^B_M)^{\otimes j}\otimes\nabla^B_E\big)_\xi\eta+\beta(\xi)\cdot\eta\,.$$

LEMMA 1.4. *(Leibniz formula) With the notations introduced above the following holds: For any $\beta\in\Gamma\big(\{\Lambda^1_M\}^{\otimes k}\otimes\mathrm{ad}(\hat P)\big)$, $\eta\in\Gamma\big(\{\Lambda^1_M\}^{\otimes j}\otimes E\big)$, and any tangent vector $\xi\in T_xM$ one has*

$$\big((\nabla^B_M)^{\otimes(k+j)}\otimes\nabla^B_E\big)_\xi\big(\beta\cdot\eta\big)=\Big(\big((\nabla^B_M)^{\otimes k}\otimes\nabla^B_{\mathrm{ad}}\big)_\xi\beta\Big)\cdot\eta+\beta\cdot\big((\nabla^B_M)^{\otimes j}\otimes\nabla^B_E\big)_\xi\eta\,.\tag{10}$$

PROOF. We give the proof in the case $k=1$, which will be used later. We may suppose that $\beta$ is tensor monomial, so it has the form $\beta=\omega\otimes b$, where $\omega\in\Gamma(\Lambda^1_M)$, and $b\in\Gamma(\mathrm{ad}(\hat{P}))$. Using Remark 1.2 we obtain

$$\begin{aligned}\big((\nabla^B_M)^{\otimes(1+j)}\otimes\nabla^B_E\big)_\xi\big(\beta\cdot\eta\big)&=\big((\nabla^B_M)^{\otimes(1+j)}\otimes\nabla^B_E\big)_\xi\big((\omega\otimes b)\cdot\eta\big)\\&=\big((\nabla^B_M)\otimes((\nabla^B_M)^{\otimes j}\otimes\nabla^B_E)\big)_\xi\big(\omega\otimes(b\cdot\eta)\big)\\&=\nabla^B_{M,\xi}\omega\otimes(b\cdot\eta)+\omega\otimes\big((\nabla^B_{\mathrm{ad},\xi}b)\cdot\eta+b\cdot((\nabla^B_M)^{\otimes j}\otimes\nabla^B_E)_\xi\eta\big)\\&=\big(\nabla^B_{M,\xi}\omega\otimes b+\omega\otimes\nabla^B_{\mathrm{ad},\xi}b\big)\cdot\eta+(\omega\otimes b)\cdot((\nabla^B_M)^{\otimes j}\otimes\nabla^B_E)_\xi\eta\big)\\&=\big(\big(\nabla^B_M\otimes\nabla^B_{\mathrm{ad}}\big)_\xi\beta\big)\cdot\eta+\beta\cdot((\nabla^B_M)^{\otimes j}\otimes\nabla^B_E)_\xi\eta\big)\,.\end{aligned}$$

■

Let $B_0$ be a fixed connection on $\hat{P}$, and $\sigma\in\Gamma(E)$. Put

$$\sigma^{(i)}_{B_0}:=((\nabla^{B_0}_M)^{\otimes(i-1)}\otimes\nabla^{B_0}_E)\otimes\cdots\otimes(\nabla^{B_0}_M\otimes\nabla^{B_0}_E)(\nabla^{B_0}_E)\sigma\in\Gamma((\Lambda^1_M)^{\otimes i}\otimes E)\,.$$

For any $k\in\mathbb{N}$ and $x\in M$ we put

$$\mathfrak{h}^\sigma_x(k):=\{b\in\mathrm{ad}(\hat{P}_x)|\ b\cdot\sigma^{(i)}_{B_0}=0\text{ for }0\le i\le k\}\,,$$

and note that $\mathfrak{h}^\sigma_x(k)$ is a Lie subalgebra of $\mathrm{ad}(\hat{P}_x)$. One has $\mathfrak{h}^\sigma_x(k+1)\subset\mathfrak{h}^\sigma_x(k)$ for any $k$. Put

$$k^\sigma_x:=\min\{k\in\mathbb{N}|\ \mathfrak{h}^\sigma_x(k+1)=\mathfrak{h}^\sigma_x(k)\}\,.$$

Any $\hat{K}$-equivariant isomorphism $\theta:\hat{P}_x\to\hat{P}_{x'}$ defines linear isomorphisms

$$\theta_V:E_x\to E_{x'}\,,\ \theta_{\hat{\mathfrak{k}}}:\mathrm{ad}(\hat{P}_x)\to\mathrm{ad}(\hat{P}_{x'})\,.$$

and (via the bundle morphism $f$) it also defines a linear isomorphism

$$\theta_M:T_x\to T_{x'}\,.$$

Denote by $\theta^k$ the induced isomorphism $\{\Lambda^1_x\}^{\otimes k}\otimes E_x\to\{\Lambda^1_{x'}\}^{\otimes k}\otimes E_{x'}$. We can now define

DEFINITION 1.5. *Let $B_0$ be a connection on $\hat{P}$. A section $\sigma\in\Gamma(E)$ is called infinitesimally homogeneous with respect to $B_0$ if for any pair $(x,x')\in M\times M$ there exists a $\hat{K}$-equivariant isomorphism $\theta:\hat{P}_x\to\hat{P}_{x'}$ such that*

$$\theta^i\big(\{\sigma^{(i)}_{B_0}\}_x\big)=\{\sigma^{(i)}_{B_0}\}_{x'}\ \textit{for}\ 0\le i\le k^\sigma_x+1\,.\tag{11}$$

Let $\theta:\hat{P}_x\to\hat{P}_{x'}$ be a $\hat{K}$-equivariant isomorphism such that (11) holds. Then $\theta_{\hat{\mathfrak{k}}}$ applies isomorphically $\mathfrak{h}^\sigma_x(k)$ on $\mathfrak{h}^\sigma_{x'}(k)$ for $0\le k\le k^\sigma_x+1$. This implies

REMARK 1.6. Let $\sigma\in\Gamma(E)$ be an infinitesimally homogeneous section with respect to $B_0$. Then $k^\sigma_x$ is independent of $x$.

We will denote by $k^\sigma$ the obtained constant.

PROPOSITION 1.7. *Suppose that $\sigma$ is infinitesimally homogeneous with respect to $B_0$. Let $\mathfrak{B}\in\mathcal{A}(\hat{P})$ be a connection such that*

$$((\nabla^{\mathfrak{B}}_M)^{\otimes k}\otimes\nabla^{\mathfrak{B}}_E)\sigma^{(k)}_{B_0}=0\quad\textit{for}\ 0\le k\le k^\sigma+1\,.\tag{12}$$

*Then*

(1) *The union $\mathfrak{h}^\sigma:=\bigcup_{x\in M}\mathfrak{h}^\sigma_x(k^\sigma+1)$ is an $\nabla^{\mathfrak{B}}_{\mathrm{ad}}$-parallel subbundle of $\mathrm{ad}(\hat{P})$.*
(2) *One has $(\nabla^{\mathfrak{B}}_M\otimes\nabla^{\mathfrak{B}}_{\mathrm{ad}})(\mathfrak{B}-B_0)\in\Gamma(\Lambda^1_M\otimes\Lambda^1_M\otimes\mathfrak{h}^\sigma)\,.$*

PROOF. (1) Let $\nu:[0,1]\to M$ be a smooth path in $M$. It suffices to note that, by Remark 1.2, the parallel transport with respect to the connection $\nabla^{\mathfrak{B}}_{\mathrm{ad}}$ maps isomorphically $\mathfrak{h}^{\sigma}_{\nu(0)}$ onto $\mathfrak{h}^{\sigma}_{\nu(1)}$.

(2) Put $\beta:=\mathfrak{B}-B_0\in A^1(\mathrm{ad}(\hat P))$. Using Remark 1.3 we obtain, for $0\leq k\leq k^\sigma+1$

$$0=((\nabla^{\mathfrak{B}}_{M})^{\otimes k}\otimes\nabla^{\mathfrak{B}}_{E})\sigma^{(k)}_{B_0}=((\nabla^{B_0}_{M})^{\otimes k}\otimes\nabla^{B_0}_{E})\sigma^{(k)}_{B_0}+\beta\cdot\sigma^{(k)}_{B_0}=\sigma^{(k+1)}_{B_0}+\beta\cdot\sigma^{(k)}_{B_0}\,. \quad (13)$$

Let $x\in M$, $\xi\in T_xM$. Taking $0\leq k\leq k^\sigma$, applying $((\nabla^{\mathfrak{B}}_{M})^{\otimes k}\otimes\nabla^{\mathfrak{B}}_{E})_\xi$ on both terms of (13), noting that for these values of $k$ the first term on the right will still vanish, and using the Leibniz rule (Lemma 1.4), one obtains

$$\big((\nabla^{\mathfrak{B}}_{M}\otimes\nabla^{\mathfrak{B}}_{\mathrm{ad}})_\xi\beta\big)\cdot\sigma^{(k)}_{B_0}=0\ \text{ for } 0\leq k\leq k^\sigma\,.$$

Taking into account formula (8) it follows that, for any $v\in T_xM$ one has

$$\big(\big((\nabla^{\mathfrak{B}}_{M}\otimes\nabla^{\mathfrak{B}}_{\mathrm{ad}})_\xi\beta\big)(v)\big)\cdot\sigma^{(k)}=0\,.$$

Therefore for any $(\xi,v)\in T_xM\times T_xM$ one has

$$\big((\nabla^{\mathfrak{B}}_{M}\otimes\nabla^{\mathfrak{B}}_{\mathrm{ad}})\beta\big)(\xi,v)\in\mathfrak{h}^\sigma_x(k^\sigma_x)=\mathfrak{h}^\sigma_x(k^\sigma_x+1)\,,$$

which shows that

$$(\nabla^{\mathfrak{B}}_{M}\otimes\nabla^{\mathfrak{B}}_{\mathrm{ad}})\beta\in\Gamma(\Lambda^1_M\otimes\Lambda^1_M\otimes\mathfrak{h}^\sigma(k^\sigma+1))=\Gamma(\Lambda^1_M\otimes\Lambda^1_M\otimes\mathfrak{h}^\sigma)\,.$$

■

The following result shows that, assuming that $\hat K$ is compact, any connection $\mathfrak{B}$ satisfying (12) can be modified, by adding a section in $\Gamma(\Lambda^1_M\otimes\mathfrak{h}^\sigma)$, such that the modified connection $B$ still satisfies (12), and also satisfies $(\nabla^B_M\otimes\nabla^B_{\mathrm{ad}})(B-B_0)=0$, which is a much stronger property than Proposition 1.7 (2).

PROPOSITION 1.8. *Suppose that $\sigma$ is infinitesimally homogeneous with respect to $B_0$. Let $\mathfrak{B}\in\mathcal{A}(\hat P)$ be a connection such that*

$$((\nabla^{\mathfrak{B}}_{M})^{\otimes k}\otimes\nabla^{\mathfrak{B}}_{E})\sigma^{(k)}_{B_0}=0\ \text{ for } 0\leq k\leq k^\sigma+1\,.$$

*Suppose that $\hat K$ is compact. Then there exists a section $\beta\in\Gamma(\Lambda^1_M\otimes\mathfrak{h}^\sigma)$ such that the connection $B:=\mathfrak{B}+\beta$ has the properties:*

*(1)*

$$((\nabla^{B}_{M})^{\otimes k}\otimes\nabla^{B}_{E})(\sigma^{(k)}_{B_0})=0\ \text{for } 0\leq k\leq k^\sigma+1\,. \quad (14)$$

*(2)*

$$(\nabla^{B}_{M}\otimes\nabla^{B}_{\mathrm{ad}})(B-B_0)=0\,. \quad (15)$$

PROOF. Note first that, in fact, for any $\beta\in\Gamma(\Lambda^1_M\otimes\mathfrak{h}^\sigma)$ the connection $B:=\mathfrak{B}+\beta$ has the property (14). The problem is to find $\beta$ such that the second conclusion holds.

Since $\hat K$ is compact, we can endow its Lie algebra with an ad-invariant inner product. Therefore $\mathrm{ad}(\hat P)$ becomes an Euclidian vector bundle, and any connection on $\hat P$ induces an Euclidian connection on $\mathrm{ad}(\hat P)$. We obtain orthogonal decompositions

$$\mathrm{ad}(\hat P)=\mathfrak{h}^\sigma\oplus(\mathfrak{h}^\sigma)^\perp, \quad (16)$$

$$\Lambda^1\otimes\mathrm{ad}(\hat P)=(\Lambda^1_M\otimes\mathfrak{h}^\sigma)\oplus(\Lambda^1_M\otimes(\mathfrak{h}^\sigma)^\perp)\,. \quad (17)$$

Put $\mathfrak{b} := \mathfrak{B} - B_0$, and let

$$\mathfrak{b} = \mathfrak{b}' + \mathfrak{b}''$$

the decomposition of $\mathfrak{b}$ with respect to the splitting (17). We define

$$B := \mathfrak{B} - \mathfrak{b}' = B_0 + \mathfrak{b}'' .$$

Since $\mathfrak{b}'$ is a section of $\mathfrak{h}^\sigma$, we see that the new connection $B := \mathfrak{B} - \mathfrak{b}'$ still satisfies (14). On the other hand, by Proposition 1.7, the decomposition (16) is parallel with respect to any connection $B$ on $\hat{P}$ which satisfies (14). Similarly, for any such connection the decomposition (17) will be $(\nabla^B_M \otimes \nabla^B_{\mathrm{ad}})$-parallel, in particular $\Lambda^1_M \otimes (\mathfrak{h}^\sigma)^\perp$ is a $(\nabla^B_M \otimes \nabla^B_{\mathrm{ad}})$-parallel subbundle of $\Lambda^1_M \otimes \mathrm{ad}(\hat{P})$. Since $\mathfrak{b}''$ is a section of $\Lambda^1_M \otimes (\mathfrak{h}^\sigma)^\perp$ we obtain

$$(\nabla^B_M \otimes \nabla^B_{\mathrm{ad}})_\xi \mathfrak{b}'' \in \Gamma(\Lambda^1_M \otimes \Lambda^1_M \otimes (\mathfrak{h}^\sigma)^\perp) \ \ \forall \xi \in T_M .$$

On the other hand, taking $\beta = \mathfrak{b}''$ in Proposition 1.7, we obtain

$$(\nabla^B_M \otimes \nabla^B_{\mathrm{ad}})_\xi \mathfrak{b}'' \in \Gamma(\Lambda^1_M \otimes \Lambda^1_M \otimes \mathfrak{h}^\sigma) \ \ \forall \xi \in T_M .$$

Therefore $(\nabla^B_M \otimes \nabla^B_{\mathrm{ad}})_\xi \mathfrak{b}'' = 0$. But $\mathfrak{b}'' = B - B_0$.

■

We prove now that a connection $\mathfrak{B}$ satisfying the hypothesis of Proposition 1.8 exists always. Therefore, under the additional assumption that $\hat{K}$ is compact, we will obtain the existence of a connection $B$ satisfying the conclusion of Proposition 1.8.

PROPOSITION 1.9. *Suppose that $\sigma$ is infinitesimally homogeneous with respect to $B_0$. There exists a connection $\mathfrak{B} \in \mathcal{A}(\hat{P})$ such that*

$$((\nabla^{\mathfrak{B}}_M)^{\otimes k} \otimes \nabla^{\mathfrak{B}}_E)\sigma^{(k)}_{B_0} = 0 \ \ \textit{for } 0 \le k \le k^\sigma + 1 .$$

PROOF. We use the infinitesimal homogeneity condition to obtain a Lie subgroup $\mathfrak{K} \subset \hat{K}$ and a principal $\mathfrak{K}$-bundle $\mathfrak{P} \subset \hat{P}$ (a reduction of the structure group of $\hat{P}$ from $\hat{K}$ to $\mathfrak{K}$) such that the sections $\sigma^{(k)}_{B_0}$ $(0 \le k \le k^\sigma + 1)$ are all defined by *constant* $\mathfrak{K}$-equivariant maps on $\mathfrak{P}$. It will follow that all theses sections are parallel with respect to any connection on $\mathfrak{P}$, so the claim will follow by choosing $\mathfrak{B}$ to be a connection on $\hat{P}$ associated with any connection on $\mathfrak{P}$.

For $x \in M$ we will identify the fibre $L(M)_x$ of the frame bundle $L(M)$ with the space of linear isomorphisms $\mathbb{R}^n \to T_xM$. Therefore, with the notations introduced at the beginning of this section, a point $y \in \hat{P}$ defines a linear isomorphism

$$f(y) : \mathbb{R}^n \to T_xM .$$

Using the $k$-order covariant derivative $\sigma^{(k)}_{B_0}$ of $\sigma$ we obtain a $\hat{K}$-equivariant map

$$\varphi_k : \hat{P} \to L^k(\mathbb{R}^n, V)$$

defined by the formula

$$\sigma^{(k)}_{B_0}(f(y)(\xi_1), \dots, f(y)(\xi_k)) = [y, \varphi_k(y)(\xi_1, \dots, \xi_k)] .$$

In other words, $\varphi_k$ is the equivariant map $\hat{P} \to L^k(\mathbb{R}^n, V)$ associated with $\sigma^{(k)}_{B_0}$ regarded as a section in the associated bundle

$$(\Lambda^1_M)^{\otimes k} \otimes E = \hat{P} \times_{\hat{K}} L^k(\mathbb{R}^n, V) .$$

Put $W := \bigoplus_{k=0}^{k^\sigma+1} L^k(\mathbb{R}^n, V)$ and define a $\hat{K}$-equivariant map $\Phi : \hat{P} \to W$ by

$$\Phi(y) := (\varphi_k(y))_{0 \le k \le k^\sigma+1}\,. \tag{18}$$

Since the section $\sigma \in \Gamma(E)$ is infinitesimally homogeneous, it follows that $\Phi(\hat{P})$ is a single $\hat{K}$-orbit of $W$. Indeed, let $x_0 \in M$, $y_0 \in \hat{P}_{x_0}$. For a point $y \in \hat{P}$, let $x = \pi(y)$ and $\theta : \hat{P}_{x_0} \to \hat{P}_x$ be a $\hat{K}$-equivariant isomorphism such that

$$\theta^k\big(\{\sigma^{(k)}_{B_0}\}_{x_0}\big) = \{\sigma^{(k)}_{B_0}\}_x \text{ for } 0 \le k \le k^\sigma + 1\,. \tag{19}$$

(see Definition 1.5). This implies the equality

$$[\theta(y_0), \varphi_k(y_0)] = [y, \varphi_k(y)]$$

in $\hat{P}_x \times_{\hat{K}} L^k(\mathbb{R}^n, V)$. Choosing $a \in \hat{K}$ such that $\theta(y_0) = ya$, we obtain

$$\varphi_k(y) = a\varphi_k(y_0) \text{ for } 0 \le k \le k^\sigma + 1\,,$$

which shows that $y \in \hat{K}\Phi(y_0)$. Therefore $\Phi(\hat{P}) \subset \hat{K}\Phi(y_0)$. Using the $\hat{K}$-equivariance property of $\Phi$ we get $\Phi(\hat{P}) = \hat{K}\Phi(y_0)$, as claimed. Put

$$\mathfrak{K} = \hat{K}_{\Phi(y_0)}\,,\ \mathfrak{P} := \Phi^{-1}(\Phi(y_0))\,.$$

Using a well-known theorem in the theory of fibre bundles (see Lemma 1.10 below), it follows that $\mathfrak{P}$ is a $\mathfrak{K}$-reduction of $\hat{P}$. Since $\Phi$ is obviously constant on $\mathfrak{P}$, it follows that the restrictions $\varphi_{k\,|\mathfrak{P}}$ are all constant on $\mathfrak{P}$, so the corresponding sections will be parallel with respect to any connection on $\mathfrak{P}$. Therefore $\sigma^{(k)}_{B_0}$ will be parallel with respect to any connection $\mathfrak{B}$ on $\hat{P}$ which is associated with a connection on $\mathfrak{P}$.

■

LEMMA 1.10. *Let $\pi : \hat{P} \to M$ be a principal $\hat{K}$-bundle over a manifold $M$, and let $\mathfrak{K}$ be a closed subgroup of $\hat{K}$. There is a bijection between the set of $\mathfrak{K}$-reductions of $\hat{P}$ and the set of pairs*

$$(\varphi, u) \in \Gamma(M, \hat{P} \times_{\hat{K}} (\hat{K}/\mathfrak{K})) \times (\hat{K}/\mathfrak{K})\,.$$

*The $\mathfrak{K}$-reduction associated with a pair $(\varphi, u)$ is the pre-image $\Phi^{-1}(u)$, where*

$$\Phi : \hat{P} \to {}^{\hat{K}}\!/_{\mathfrak{K}}$$

*is the equivariant map associated with $\varphi$.*

PROOF. See the proof of [**KN**, Proposition 5.6]. ■

Combining Proposition 1.8 and Proposition 1.9 we obtain the main theorem of this section

THEOREM 1.11. *Suppose that $\sigma$ is infinitesimally homogeneous with respect to $B_0$ and the structure group $\hat{K}$ is compact. Then there exists a connection $B \in \mathcal{A}(\hat{P})$ with the properties:*

*(1)*

$$((\nabla^B_M)^{\otimes k} \otimes \nabla^B_E)(\sigma^{(k)}_{B_0}) = 0 \textit{ for } 0 \le k \le k^\sigma + 1\,. \tag{20}$$

*(2)*

$$(\nabla^B_M \otimes \nabla^B_{\mathrm{ad}})(B - B_0) = 0\,. \tag{21}$$

## 2. Infinitesimally homogeneous triples

Let $(M, g)$ be an $n$-dimensional Riemannian manifold, $K$ be a compact Lie group, $P$ be a principal $K$-bundle on $M$, and $A_0$ be a connection on $P$. Put

$$\hat{K} := \mathrm{O}(n) \times K\ ,\ \hat{P} := \mathrm{O}(M) \times_M P\ ,\ V := (\mathbb{R}^{n*})^{\otimes 4} \oplus \big((\mathbb{R}^{n*})^{\otimes 2} \otimes \mathfrak{k}\big)\,.$$

Note that $\hat{K}$ comes with obvious representations $r : \hat{K} \to \mathrm{GL}(n)$, $\rho : \hat{K} \to \mathrm{GL}(V)$, and $\hat{P}$ comes with an obvious bundle morphism $f : \hat{P} \to L(M)$ of type $r$ given by the projection $\mathrm{O}(M) \times_M P \to \mathrm{O}(M) \subset L(M)$.

The connection on the principal bundle $\mathrm{O}(M)$ which corresponds to the Levi-Civita connection will be denoted by $C_0$. The Riemannian curvature $R$ of $g$ will be regarded as a section of $(\Lambda^1_M)^{\otimes 4}$, and the curvature $F^{A_0}$ of $A_0$ will be regarded as a section of the vector bundle $\Lambda^2_M \otimes \mathrm{ad}(P) \subset (\Lambda^1_M)^{\otimes 2} \otimes \mathrm{ad}(P)$. The pair $(R, F^{A_0})$ can be regards as a section in the associated vector bundle $E := \hat{P} \times_{\hat{K}} V$. Note also that $\hat{P}$ comes with a connection $B_0$ defined by the pair $(C_0, A_0)$.

Definition 2.1. *The triple $(g, P \xrightarrow{p} M, A_0)$ will be called infinitesimally homogeneous if the pair $(R, F^{A_0})$ (regarded as a section in the associated vector bundle $E = \hat{P} \times_{\hat{K}} V$) is infinitesimally homogeneous with respect to $B_0$ in the sense of Definition 1.5 .*

This condition can be reformulated explicitly as follows:

Let $\nabla^{C_0}$ denotes the Levi-Civita connection of $g$ on $M$ and, let $\nabla^{A_0}$ denote the associted connection on adjoint bundle $\mathrm{ad}(P)$. We will denote by $\nabla^{C_0 A_0}$ the tensor product connection $\nabla^{C_0} \otimes \nabla^{A_0}$ on $\Lambda^2_M \otimes \mathrm{ad}(P)$. More generally, we obtain a tensor product connection $(\nabla^{C_0 A_0})^i = (\nabla^{C_0})^{\otimes(i-1)} \otimes \nabla^{C_0 A_0}$ on $(\Lambda^1_M)^{\otimes(i-1)} \otimes \Lambda^2_M \otimes \mathrm{ad}(P)$. For $x \in M$ let $\mathrm{so}(T_x)$ be the Lie algebra of skew-symmetric endomorphism of the Euclidian space $(T_x, g_x)$. For any $k \in \mathbb{N}$ and $x \in M$ we define

$$\begin{aligned}\mathfrak{h}^{g,A_0}_x(k) := \{(u, v) \in \mathrm{so}(T_x) \oplus \mathrm{ad}(P_x)|\ & u \cdot \big((\nabla^{C_0})^i R\big)_x = 0, \\ & (u, v) \cdot \big((\nabla^{C_0 A_0})^i F^{A_0}\big)_x = 0 \text{ for } 0 \le i \le k\}\,,\end{aligned}$$

and note that $\mathfrak{h}^{g,A_0}_x(k)$ is a Lie subalgebra of $\mathrm{so}(T_x) \oplus \mathrm{ad}(P_x)$. For any $k$ one has

$$\mathfrak{h}^{g,A_0}_x(k+1) \subset \mathfrak{h}^{g,A_0}_x(k)\,.$$

Put

$$k^{g,A_0}_x := \min\{k \in \mathbb{N}|\ \mathfrak{h}^{g,A_0}_x(k+1) = \mathfrak{h}^{g,A_0}_x(k)\}\,.$$

With these definitions we see that

Remark 2.2. The triple $(g, P \xrightarrow{p} M, A_0)$ is infinitesimally homogeneous if and only if for any $(x, x') \in M \times M$, there exists a pair $(f, \phi)$ where $f : T_x M \to T_{x'} M$ is a linear isometry, and $\phi : P_x \to P_{x'}$ is $K$-equivariant isomorphism, such that for any $0 \le k \le k^{g,A_0}_x + 1$ one has:

(1) $f\Big(\big((\nabla^{C_0})^k R\big)_x\Big) = \big((\nabla^{C_0})^k R\big)_{x'}$.
(2) $(f, \phi)\Big(\big((\nabla^{C_0 A_0})^k F^{A_0}\big)_x\Big) = \big((\nabla^{C_0 A_0})^k F^{A_0}\big)_{x'}$.

The second condition can be reformulates explicitly as follows: $(\nabla^{C_0 A_0})^k F^{A_0}$ is a section of the vector bundle $(\Lambda^1_M)^{\otimes k} \otimes \Lambda^2_M \otimes \mathrm{ad}(P)$. The fiber $\mathrm{ad}(P)_x$ at $x$ of the adjoint bundle $\mathrm{ad}(P)$ is given by

$$\mathrm{ad}(P)_x = P_x \times_K \mathfrak{k} = {}^{P_x \times \mathfrak{k}}\!/_{K}\,,$$

so a $K$-equivariant isomorphism $\phi : P_x \to P_{x'}$ induces a linear isomorphism

$$\mathrm{ad}(\phi) : \mathrm{ad}(P)_x \to \mathrm{ad}(P)_{x'}$$

by the formula $[y,\alpha] \mapsto [\phi(y),\alpha]$. The second condition in Remark 2.2 can be written as

$$\begin{aligned}\big((\nabla^{C_0 A_0})^k F^{A_0}\big)_{x'}(f(v_1),\dots,f(v_k),f(w_1),f(w_2)) =\\ = \mathrm{ad}(\phi)\big((\nabla^{C_0 A_0})^k F^{A_0})_x(v_1,\dots,v_k,w_1,w_2)\big)\end{aligned}$$

in $\mathrm{ad}(P)_{x'}$ for any tangent vectors $v_i \in T_x M$, $w_1$, $w_2 \in T_x M$.

The isomorphism pair $(f,\phi)$ defines a Lie algebra isomorphism

$$\mathrm{so}(T_x) \oplus \mathrm{ad}(P_x) \to \mathrm{so}(T_{x'}) \oplus \mathrm{ad}(P_{x'})\,,$$

which isomorphically maps $\mathfrak{h}^{g,A_0}_x(k)$ onto $\mathfrak{h}^{g,A_0}_{x'}(k)$ for $0 \le k \le k^{g,A_0}_x + 1$. This implies

Remark 2.3. Let $(g, P \xrightarrow{p} M, A_0)$ be an infinitesimally homogeneous triple. Then $k^{g,A_0}_x$ is independent of $x$. We will denote by $k^{g,A_0}$ the obtained constant.

Applying Theorem 1.11 to our situation we obtain:

Theorem 2.4. *Let $(g, P \xrightarrow{p} M, A_0)$ be an infinitesimally homogeneous triple on $M$. Then, there exist a pair of connection $B = (C,A) \in \mathcal{A}(\mathrm{O}(M)) \times \mathcal{A}(P) = \mathcal{A}(\hat{P})$ with the following properties:*

*(1)*

$$\nabla^C R = 0\,,\ \nabla^{CA} F^{A_0} = 0\,,\ \nabla^C(C - C_0) = 0\,,\ \nabla^{CA}(A - A_0) = 0\,.$$

*(2)*

$$\nabla^C T^C = 0\,,\ \nabla^C R^C = 0\,,\ \nabla^{CA} F^A = 0\,,$$

In these formulae $R^C$, $T^C$ stand for the curvature, respectively the torsion tensor of $\nabla^C$.

Proof. Let $\sigma$ be the section of $E := \hat{P} \times_{\hat{K}} V$ defined by the pair $(R, F^{A_0})$. Theorem 1.11 gives a connection $B = (C,A) \in \mathcal{A}(\mathrm{O}(M)) \times \mathcal{A}(P) = \mathcal{A}(\hat{P})$ such that

$$((\nabla^B_M)^{\otimes k} \otimes \nabla^B_E)(\sigma^{(k)}_{B_0}) = 0 \text{ for } 0 \le k \le k^{g,A_0} + 1\,,\ (\nabla^B_M \otimes \nabla^B_{\mathrm{ad}})(B - B_0) = 0\,.$$

In particular, for $k = 0$ we obtain $(\nabla^B_E)(R, F^{A_0}) = 0$, i.e.

$$\nabla^C R = 0,\ \nabla^{CA} F^{A_0} = 0\,.$$

The difference $B - B_0$ can be identified with the pair $(C - C_0, A - A_0)$. Therefore the above formula gives

$$\nabla^C(C - C_0) = 0\,,\ \nabla^{CA}(A - A_0) = 0\,,$$

which proves (1). For (2) note first that $T^C$ is the image of $C - C_0$ under the bundle isomorphism $\Lambda^1\mathrm{so}(T_M) \to L^2_{\mathrm{alt}}(T_M, T_M)$ given by $S \mapsto T_S$, where

$$T_S(X,Y) := S(X)(Y) - S(Y)(X)\,.$$

This isomorphism is induced by an $\mathrm{O}(n)$-equivariant isomorphism

$$\mathbb{R}^{n*} \otimes \mathrm{so}(n) \to L^2_{\mathrm{alt}}(\mathbb{R}^n, \mathbb{R}^n).$$

So it is parallel with respect to any metric connection on $M$. Therefore, the condition $\nabla^C(C - C_0) = 0$ is equivalent to $\nabla^C T^C = 0$. Using $\nabla^C T^C = 0$, $\nabla^C R = 0$, it follows $\nabla^C R^C = 0$ [**TV**, Page. 14].

Now, in order to prove that $\nabla^{CA} F^A = 0$, put $\alpha := A - A_0$. Therefore we have,

$$\nabla^{CA} F^{A_0} = 0 \; , \; \nabla^{CA}\alpha = 0 \, .$$

Using [**Te**, Proposition 3.2.8] we have

$$F^{A_0} = F^{A-\alpha} = F^A - d^A\alpha + \frac{1}{2}[\alpha \wedge \alpha] \, . \tag{22}$$

Let $X$, $Y$, $Z \in \mathcal{X}(M)$ be arbitrary vector fields on $M$. We have

$$(d^A\alpha)(X,Y) = \nabla^A_X(\alpha(Y)) - \nabla^A_Y(\alpha(X)) - \alpha([X,Y]) \, . \tag{23}$$

Using the fact that $\alpha \in A^1(\mathrm{ad}(P))$ is $\nabla^{CA}$-parallel, we obtain

$$0 = (\nabla^{CA}\alpha)(X,Y) = (\nabla^{CA}_X\alpha)(Y) = \nabla^A_X(\alpha(Y)) - \alpha(\nabla^C_X Y) \, . \tag{24}$$

Using (23) and (24) to obtain,

$$(d^A\alpha)(X,Y) = \alpha(\nabla^C_X Y) - \alpha(\nabla^C_Y X) - \alpha([X,Y]) = \alpha(T^C(X,Y)) \, . \tag{25}$$

On one hand

$$0 = (\nabla^{CA}_X\alpha)(T^C(Y,Z)) = \nabla^A_X(\alpha(T^C(Y,Z)) - \alpha(\nabla^C_X T^C(Y,Z)) \, , \tag{26}$$

and on the other hand (24) and (25) imply that

$$\begin{aligned}
(\nabla^{CA} d^A\alpha)(X,Y,Z) &= \nabla^A_X((d^A\alpha)(Y,Z)) - (d^A\alpha)(\nabla^C_X Y, Z) - (d^A\alpha)(Y, \nabla^C_X Z) \\
&= \nabla^A_X(\alpha(T^C(Y,Z))) - \alpha(T^C(\nabla^C_X Y, Z)) - \alpha(T^C(Y, \nabla^C_X Z)) \\
&= \alpha(\nabla^C_X T^C(Y,Z)) - \alpha(T^C(\nabla^C_X Y, Z)) - \alpha(T^C(Y, \nabla^C_X Z)) \\
&= \alpha((\nabla^C_X T^C)(Y,Z)) = 0.
\end{aligned}$$

Finally, if we put $\eta = \frac{1}{2}[\alpha \wedge \alpha]$, then for any $Y, Z \in \mathcal{X}(M)$

$$2\eta(Y,Z) = [\alpha \wedge \alpha](Y,Z) = [\alpha(Y), \alpha(Z)] - [\alpha(Z), \alpha(Y)] = 2[\alpha(Y), \alpha(Z)] \, . \tag{27}$$

Therefore,

$$\nabla^A_X(\eta(Y,Z)) = \nabla^A_X([\alpha(Y), \alpha(Z)]) = [\nabla^A_X(\alpha(Y)), \alpha(Z)] + [\alpha(Y), \nabla^A_X(\alpha(Z))] \, . \tag{28}$$

Using (22), (24) and (28) we obtain

$$\begin{aligned}
(\nabla^{CA}_X\eta)(Y,Z) =& \nabla^A_X(\eta(Y,Z)) - \eta(\nabla^C_X Y, Z) - \eta(Y, \nabla^C_X Z) \\
=& [\nabla^A_X(\alpha(Y)), \alpha(Z)] + [\alpha(Y), \nabla^A_X(\alpha(Z))] - [\alpha(\nabla^C_X Y), \alpha(Z)] \\
& -[\alpha(Y), \alpha(\nabla^C_X Z)] = [(\nabla^{CA}_X\alpha)(Y), \alpha(Z)] + [\alpha(Y), (\nabla^{CA}_X\alpha)(Z)] = 0 \, .
\end{aligned}$$

Therefore $\nabla^{CA} F^A = 0$ as claimed. ∎

Using Singer's theorem (see Chapter 5.2.9) we obtain

Remark 2.5. Let $(g, P \xrightarrow{p} M, A_0)$ be a infinitesimally homogeneous triple. Then $(M,g)$ is locally homogeneous.

A pair $(C, A) \in \mathcal{A}(\mathrm{O}(M)) \times \mathcal{A}(P)$ defines in a canonical way a linear connection $\bar\nabla = \bar\nabla^{CA}$ on the tangent bundle $T_P$. This construction will play an important role in our arguments, so we explain this construction in detail:

Recall that, in general, we identify a connection on a principal bundle with the corresponding horizontal distribution. Using the bundle isomorphism

$$J : p^* T_M \to A$$

(defined by the inverse of $p_{*|A}$), we obtain a linear connection

$$\nabla^{h,CA} := J(p^* \nabla^C)$$

on the horizontal subbundle $A \subset T_P$. Its curvature $F^{\nabla^{h,CA}}$ is given by

$$F^{\nabla^{h,CA}} = j(p^*(R^C))\,,$$

where $j : \mathrm{gl}(p^*(T_M)) = p^*(\mathrm{gl}(T_M)) \to \mathrm{gl}(A)$ is the bundle isomorphism induced by $J$.

On the other hand we can define a linear connection $\nabla^{v,A}$ on the vertical bundle $V_P$ via the canonical bundle isomorphism $V_P \simeq P \times \mathfrak{k}$. More precisely, if $a^\#$, $b^\#$ denote the fundamental fields corresponding to $a, b \in \mathfrak{k}$ and if $\tilde X$ denotes the $A$-horizontal lift of $X \in \mathcal{X}(M)$, then we define

$$\nabla^{v,A}_{a^\#} b^\# = [a,b]^\#,\ \nabla^{v,A}_{\tilde X} a^\# = 0\,.$$

Using the direct sum decomposition $T_P = A \oplus V_P$, the linear connection $\nabla^{h,CA}$ on the horizontal subbundle $A$, and the linear connection $\nabla^{v,A}$ on vertical subbundle $V_P$, we obtain a linear connection $\bar\nabla = \bar\nabla^{CA} := \nabla^{h,CA} \oplus \nabla^{v,A}$ on $P$. The connection $\bar\nabla$ has the following properties:

Proposition 2.6. *Let $\bar\nabla = \bar\nabla^{CA}$ be the linear connection on the tangent bundle $T_P$ defined as above. Let $a^\#$, $b^\#$ be the fundamental fields corresponding to $a, b \in \mathfrak{k}$, and let $\tilde X$, $\tilde Y$ denote the $A$-horizontal lifts of $X, Y \in \mathcal{X}(M)$. Let $Z \in \mathcal{X}^v(P)$ be a vertical vector field on $P$. Then one has*

*(1)* $\bar\nabla_{\tilde X} \tilde Y = (\nabla^C_X Y)^\sim$.
*(2)* $\bar\nabla_{\tilde X} a^\# = \bar\nabla_{a^\#} \tilde X = 0$.
*(3)* $\bar\nabla_{a^\#} b^\# = [a,b]^\#$.
*(4)* $\bar\nabla_{\tilde X} Z = [\tilde X, Z]$.
*(5)* $\bar\nabla_{a^\#} Z = [a^\#, Z]$.

Proof. The first three properties follow from the definition of $\bar\nabla$. We prove (4). Since $Z \in \mathcal{X}^v(P)$ is a vertical vector field, there exists a family of fundamental fields $b_1^\#, \dots, b_r^\#$ and a family of differentiable functions $f_1, \dots, f_r$ on $P$ such that $Z = \sum_{i=1}^r f_i b_i^\#$. Therefore, we have

$$\bar\nabla_{\tilde X} Z = \bar\nabla_{\tilde X} \sum_{i=1}^r f_i b_i^\# = \sum_{i=1}^r (\tilde X(f_i) b_i^\# + f_i \bar\nabla_{\tilde X} b_i^\#) = \sum_{i=1}^r \tilde X(f_i) b_i^\#.$$

Also we have

$$[\tilde X, Z] = [\tilde X, \sum_{i=1}^r f_i b_i^\#] = \sum_{i=1}^r (\tilde X(f_i) b_i^\# + f_i [\tilde X, b_i^\#]) = \sum_{i=1}^r \tilde X(f_i) b_i^\#\,,$$

which proves (4).

To prove (5) let $Z=\sum_{i=1}^r f_i b_i^{\#}$ be as above. We have

$$\begin{aligned}\bar\nabla_{a^\#} Z = \bar\nabla_{a^\#}\sum_{i=1}^r f_i b_i^{\#} &= \sum_{i=1}^r (a^{\#}(f_i) b_i^{\#} + f_i \bar\nabla_{a^\#} b_i^{\#})\\ &= \sum_{i=1}^r \big(a^{\#}(f_i) b_i^{\#} + f_i [a^{\#}, b_i^{\#}]\big) = \sum_{i=1}^r [a^{\#}, f_i b_i^{\#}] = [a^{\#}, Z]\,.\end{aligned}$$

■

The space $A^0(\mathrm{ad}(P))$ has a natural Lie algebra structure, and can be identified with the Lie algebra of the gauge group $\mathcal{G}(P)=\Gamma(M, P\times_{\mathrm{Ad}} K)$ of $P$. For an element $\nu\in A^0(\mathrm{ad}(P))$ we define a vertical vector field $\xi^\nu\in\mathcal{X}^v(P)$ by

$$\xi^\nu_y = \frac{d}{dt}\Big|_{t=0}\big\{y\,\exp(t\nu(y))\big\}.$$

The next lemma gives two important properties of the vertical vector field $\xi^\nu$.

Lemma 2.7. *Let $p: P\to M$ be a principal $K$-bundle and $A$ a connection on $P$. Let $a\in\mathfrak{k}$ and let $\tilde X$ be the $A$-horizontal lift of vector field $X\in\mathcal{X}(M)$. For any section $\nu\in A^0(\mathrm{ad}(P))$ one has*

$$[a^{\#}, \xi^\nu]=0\,,\ \ [\tilde X, \xi^\nu] = \xi^{\nabla^A_X \nu}.$$

Proof. To prove the first formula let $(\varphi_t)_{t\in\mathbb{R}}$ be the one parameter group of diffeomorphisms associated with the fundamental field $a^{\#}$. Therefore one has $\varphi_t(y):=y\,\exp(ta)$. Let $(f_s)_{s\in\mathbb{R}}$ be the one parameter group of diffeomorphisms associated with the field $\xi^\nu$. By [**KN**, Proposition 1.11] one has $[a^{\#},\xi^\nu]=0$ if and only if $\varphi_t\circ f_s = f_s\circ\varphi_t$ for every $s$ and $t$. But for each $y\in P$ one has

$$f_s(\varphi_t(y)) = f_s(y\,\exp(ta)) = f_s(y)\,\exp(ta) = \varphi_t(f_s(y))\,.$$

because $f_s$ is a bundle isomorphism, so it commutes with right translations.

To prove the second formula, let $t\mapsto x_t = x(t)$ be an integral curve of $X$ defined for $t\in(-\varepsilon,\varepsilon)$ for some $\varepsilon>0$ with the initial condition $x(0)=x_0$. The $A$-horizontal lift $t\mapsto y_t$ of the path $t\mapsto x_t$ through a point $y_0\in P_{x_0}$ is the integral curve of $\tilde X$ with initial condition $y(0)=y_0$. A smooth section $\nu\in A^0(\mathrm{ad}(P))$ can be regarded as a $K$-equivariant map $\nu: P\to\mathfrak{k}$. Put $a_t:=\nu(y_t)\in\mathfrak{k}$ and note that, for each $t\in(-\varepsilon,\varepsilon)$, the vertical vector field $\xi^\nu_{y_t}$ is defined by

$$\xi^\nu_{y_t} = (a_t)^{\#}_{y_t} = \frac{d}{ds}\Big|_{s=0}\big\{y_t\,\exp(sa_t)\big\}\,.$$

Let $(\beta_t)_{t\in(-\varepsilon,\varepsilon)}$ be the one parameter group of local diffeomorphism associated with $\tilde X$. We have

$$\begin{aligned}[\tilde X,\xi^\nu]_{y_0} &= \lim_{t\to 0}\frac{1}{t}\{\xi^\nu_{y_0} - (\beta_t)_*(\xi^\nu_{y_{-t}})\}\\ &= \lim_{t\to 0}\frac{1}{t}\{(a_0)^{\#}_{y_0} - (\beta_t)_*(a_{-t})^{\#}_{y_{-t}}\}\end{aligned}$$

Since, $(\beta_t)_*(a_{-t})^{\#}_{y_{-t}} = (a_{-t})^{\#}_{y_0}$ we have

$$\begin{aligned}[\tilde X,\xi^\nu]_{y_0} &= \lim_{t\to 0}\frac{1}{t}\{(a_0)^{\#}_{y_0} - (a_{-t})^{\#}_{y_0}\} = \{\lim_{t\to 0}\frac{1}{t}(a_0 - a_{-t})\}^{\#}_{y_0} = \{(\dot a_t)_{t=0}\}^{\#}_{y_0}\\ &= \{\frac{d}{dt}\nu(y_t)_{t=0}\}^{\#}_{y_0} = \{d\nu(\tilde X_{y_0})\}^{\#}_{y_0} = \{\nabla^A_{X_{y_0}}\nu\}^{\#}_{y_0} = \xi^{\nabla^A_{X_{y_0}}\nu}.\end{aligned}$$

■

PROPOSITION 2.8. *Let $\nu \in A^0(\mathrm{ad}(P))$ be a section of the adjoint bundle and denote by $\xi^\nu$ the corresponding vertical field. Let $\tilde{X}$ be the $A$-horizontal lift of the vector field $X \in \mathcal{X}(M)$ and let $a^\#$ denote the fundamental vector field corresponding to $a \in \mathfrak{k}$. Then we have*

$$\bar{\nabla}_{\tilde{X}}\xi^\nu = \xi^{\nabla^A_X \nu}, \quad \bar{\nabla}_{a^\#}\xi^\nu = 0\,.$$

PROOF. The result follows from Proposition 2.6 and Lemma 2.7.

■

The following theorem states that if $\nabla^C R^C = 0$, $\nabla^C T^C = 0$, $\nabla^{CA} F^A = 0$, then the associated connection $\bar{\nabla} = \bar{\nabla}^{CA}$ satisfies the hypothesis of the Singer Theorem (see Theorem 2.9 in Chapter 5):

THEOREM 2.9. *Let $(C, A) \in \mathcal{A}(\mathrm{O}(M)) \times \mathcal{A}(P)$ such that $\nabla^C R^C = 0$, $\nabla^C T^C = 0$, $\nabla^{CA} F^A = 0$. Then, the associated connection $\bar{\nabla} = \bar{\nabla}^{CA}$ satisfies the following conditions:*

$$\bar{\nabla} R^{\bar{\nabla}} = \bar{\nabla} T^{\bar{\nabla}} = 0.$$

PROOF. Since the connection $\nabla^{\nu,A}$ on $V_P \simeq P \times \mathfrak{k}$ is flat, we have

$$R^{\bar{\nabla}} = j(p^*(R^C)) \oplus F^{\nabla^{\nu,A}} = j(p^*(R^C))\,.$$

Since $\mathrm{gl}(A)$ is a $\bar{\nabla}$-parallel subbundle of $\mathrm{gl}(T_P)$, and the curvature tensor $R^C$ is $\nabla^C$-parallel one has

$$\bar{\nabla}(j(p^*(R^C))) = \nabla^{h,CA}(j(p^*(R^C))) = j(p^*(\nabla^C))\big(j(p^*(R^C))\big) =$$
$$= j\big(p^*(\nabla^C)(p^*(R^C))\big) = j\big(p^*(\nabla^C R^C)\big) = 0\,.$$

In this formulae we used the same symbols for the connections induced by $\bar{\nabla}$, $\nabla^C$ on the bundles of endomorphisms. The claimed formula $\bar{\nabla} R^{\bar{\nabla}} = 0$ is proved. For the second, denote the torsion tensor of linear connection $\bar{\nabla}$ by $\bar{T}$ to save on notation. Let $a^\#$, $b^\#$ be the fundamental fields corresponding to $a, b \in \mathfrak{k}$, and let $\tilde{X}$, $\tilde{Y}$ denote the $A$-horizontal lifts of $X$, $Y \in \mathcal{X}(M)$. Using the properties of $\bar{\nabla}$ given by Proposition 2.6, and formula [**GH**, P. 257] for the vertical component of the Lie bracket of two horizontal lifts, we obtain the following formulae:

(i) $\bar{T}(\tilde{X}, \tilde{Y}) = \bar{\nabla}_{\tilde{X}}\tilde{Y} - \bar{\nabla}_{\tilde{Y}}\tilde{X} - [\tilde{X}, \tilde{Y}] =$
$= (\nabla^C_X Y - \nabla^C_Y X - [X, Y])^\sim + \xi^{F^A(X,Y)} = T^C(X, Y)^\sim + \xi^{F^A(X,Y)}$.

(ii) $\bar{T}(a^\#, b^\#) = \bar{\nabla}_{a^\#} b^\# - \bar{\nabla}_{b^\#} a^\# - [a^\#, b^\#] = [a^\#, b^\#]\,.$

(iii) $\bar{T}(\tilde{X}, a^\#) = \bar{\nabla}_{\tilde{X}} a^\# - \bar{\nabla}_{a^\#}\tilde{X} - [\tilde{X}, a^\#] = 0\,.$

It suffices to prove $(\bar{\nabla}\bar{T})(U, V, W) = 0$ for vector fields $U$, $V$, $W \in \mathcal{X}(P)$ in the special cases when each one of these three vector fields is either a horizontal lift, or a fundamental field. We have

$$(\bar{\nabla}_U \bar{T})(V, W) = \bar{\nabla}_U \bar{T}(V, W) - \bar{T}(\bar{\nabla}_U V, W) - \bar{T}(V, \bar{\nabla}_U W)\,. \tag{29}$$

(1) Suppose $U = \tilde{X}$, $V = \tilde{Y}$, $W = \tilde{Z}$. Then

$$\begin{aligned}\bar{\nabla}_U \bar{T}(V, W) &= \bar{\nabla}_{\tilde{X}}\big((T^C(Y, Z)^\sim) + \xi^{F^A(Y,Z)}\big)\\ &= \big(\nabla^C_X T^C(Y, Z)\big)^\sim + \xi^{\nabla^A_X F^A(Y,Z)}.\end{aligned} \tag{30}$$

In the same way one obtains

$$\bar{T}(\bar{\nabla}_U V, W) = (T^C(\nabla^C_X Y, Z))^\sim + \xi^{F^A(\nabla^C_X Y, Z)}\,, \tag{31}$$

$$\bar{T}(V, \bar{\nabla}_U W) = (T^C(Y, \nabla^C_X Z))^\sim + \xi^{F^A(Y, \nabla^C_X Z)}\,. \tag{32}$$

Using the equations (29), (30), (31), (32) and Proposition 2.8 we obtain

$$\begin{aligned}(\bar{\nabla}_U \bar{T})(V, W) &= \bar{\nabla}_U \bar{T}(V, W) - \bar{T}(\bar{\nabla}_U V, W) - \bar{T}(V, \bar{\nabla}_U W)\\ &= (\nabla^C_X T^C(Y, Z))^\sim - (T^C(\nabla^C_X Y, Z))^\sim - (T^C(Y, \nabla^C_X Z))^\sim\\ &\quad + \xi^{\nabla^A_X F^A(Y,Z)} - \xi^{F^A(\nabla^C_X Y, Z)} - \xi^{F^A(Y, \nabla^C_X Z)}\\ &= \big((\nabla^C T^C)(X, Y, Z)\big)^\sim + \xi^{(\nabla^{CA}_X F^A)(Y,Z)}\,.\end{aligned}$$

Since $\nabla^C T^C = 0$ and $\nabla^{CA} F^A = 0$, the right hand side of the above equation vanishes.

(2) Suppose $U = a^\sharp$, $V = b^\sharp$, $W = c^\sharp$, where $a$, $b$, $c \in \mathfrak{k}$. By the Jacobi identity

$$\begin{aligned}(\bar{\nabla}_{a^\sharp} \bar{T})(b^\sharp, c^\sharp) &= \bar{\nabla}_{a^\sharp} \bar{T}(b^\sharp, c^\sharp) - \bar{T}(\bar{\nabla}_{a^\sharp} b^\sharp, c^\sharp) - \bar{T}(b^\sharp, \bar{\nabla}_{a^\sharp} c^\sharp)\\ &= \big([a, [b, c]] + [b, [c, a]] + [c, [a, b]]\big)^\# = 0\end{aligned}$$

(3) Suppose $U = a^\sharp$ and $V = \tilde{Y}$, $W = \tilde{Z}$ are the $A$-horizontal lifts of vector fields $Y, Z \in \mathcal{X}(M)$, then using Proposition 2.8 we obtain

$$\bar{\nabla}_{a^\#}((T^C(Y, Z))^\sim + \xi^{F^A(Y,Z)}) - \bar{T}(\bar{\nabla}_{a^\#} \tilde{Y}, \tilde{Z}) - \bar{T}(\tilde{Y}, \bar{\nabla}_{a^\#} \tilde{Z}) = 0.$$

(4) In any other cases the claim follows from the definition of $\bar{\nabla}$.

■

Recall that the space of connections $\mathcal{A}(P)$ is an affine space with model space $A^1(\mathrm{ad}(P))$ **[DK]**, **[Te]**. For a connection $A_0 \in \mathcal{A}(P)$, and a 1-form $\alpha \in A^1(\mathrm{ad}(P))$ the connection form $\omega_A$ of $A := A_0 + \alpha$ is given by $\omega_A = \omega_{A_0} + \alpha$, where the second term on the right as been identified with the associated tensorial 1-form of type $\mathrm{ad}$ (see [**KN**, Example 5.2 p.76]) on $P$. In other words, for a tangent vector $v \in T_y P$ the element $\alpha_y(v) \in \mathfrak{k}$ is defined by the equality $\alpha(p_*(v)) = [y, \alpha_y(v)] \in \mathrm{ad}(P_{p(y)})$. In other words, regarding $\alpha(p_*(v))$ as a $K$-equivariant map $P_{p(y)} \to \mathfrak{k}$, one has $\alpha_y(v) = \alpha(p_*(v))(y)$.

LEMMA 2.10. *Let $A = A_0 + \alpha$. Let $Z \in \mathcal{X}(M)$, $\tilde{Z}^{A_0}$ be its $A_0$-horizontal lift, and $\tilde{Z}$ be its $A$-horizontal lift (which coincides with the $A$-horizontal projection of $\tilde{Z}^{A_0}$). Then*

$$\tilde{Z} = \tilde{Z}^{A_0} - \xi^{\alpha(Z)}\,.$$

PROOF. For any field $X \in \mathcal{X}(P)$ the $A$-horizontal projection of $X$ is given by $X^A = X - \xi^{\omega_A(X)}$. If now $X = \tilde{Z}^{A_0}$, we have $\omega_{A_0}(X) = 0$, so for any $y \in P$ we have $\omega_{A,y}(X) = \alpha_y(X) = \alpha(Z)(y)$. ■

LEMMA 2.11. *Let $(C, A) \in \mathcal{A}(\mathrm{O}(M)) \times \mathcal{A}(P)$ such that*

$$\nabla^C R^C = 0,\ \nabla^C T^C = 0,\ \nabla^{CA} F^A = 0,\ \nabla^{CA}(A - A_0) = 0\,.$$

*Then the distributions $A$, $A_0$ are $\bar{\nabla}$-parallel.*

PROOF. To prove that $A$ is $\bar{\nabla}$-parallel, let $\tilde{X}$ be the $A$-horizontal lift of vector field $X \in \mathcal{X}(M)$. We have to show that for any $Y \in \mathcal{X}(P)$ the vector field $\bar{\nabla}_Y \tilde{X}$ is $A$-horizontal. If $Y := \tilde{Z}$ is $A$-horizontal lift of a vector fields $Z$ on $M$ then $\bar{\nabla}_Y X$ is $A$-horizontal lift of $\nabla^C_Z X$. If $Y \in \Gamma(V_P)$ is a vertical fields, then $\bar{\nabla}_Y \tilde{X} = 0$ and therefore $\bar{\nabla}_Y \tilde{X} \in \Gamma(A)$.

For the second property, we will denote by $\tilde{Z}^{A_0}$ the $A_0$-horizontal lift of a vector field $Z \in \mathcal{X}(M)$. Since the vector fields of the form $\tilde{Z}^{A_0}$ generate $\Gamma(A_0)$ as a $\mathcal{C}^\infty(P, \mathbb{R})$ module, it suffices to prove that, for any $Y \in \mathcal{X}(P)$, $Z \in \mathcal{X}(M)$ one has $\bar{\nabla}_Y(\tilde{Z}^{A_0}) \in \Gamma(A_0)$. By Lemma 2.10 we have

$$\tilde{Z} = \tilde{Z}^{A_0} - \xi^{\alpha(Z)}.$$

If $Y = \tilde{U}$ is the $A$-horizontal lift of a vector field $U \in \mathcal{X}(M)$, then, using the parallelism assumption $\nabla^{CA}\alpha = 0$, the definition of $\bar{\nabla}$, and Proposition 2.8, we obtain:

$$\begin{aligned}\bar{\nabla}_Y \tilde{Z}^{A_0} &= \bar{\nabla}_Y(\tilde{Z} + \xi^{\alpha(Z)}) = \widetilde{(\nabla^C_U Z)} + \bar{\nabla}_{\tilde{U}} \xi^{\alpha(Z)} \\ &= \widetilde{(\nabla^C_U Z)} + \xi^{\nabla^A_U(\alpha(Z))} = \widetilde{(\nabla^C_U Z)} + \xi^{\alpha(\nabla^C_U Z)}.\end{aligned}$$

But, by Lemma 2.10, the right hand side of the above equation is the $A_0$-horizontal lift of $\nabla^C_U Z$, which proves the claim in this case. If now $Y = a^\#$ is a fundamental field, then

$$\begin{aligned}\bar{\nabla}_Y \tilde{Z}^{A_0} &= \bar{\nabla}_{a^\#}(\tilde{Z} + \xi^{\alpha(Z)}) \\ &= \bar{\nabla}_{a^\#} \tilde{Z} + \bar{\nabla}_{a^\#} \xi^{\alpha(Z)}.\end{aligned}$$

Using the Proposition 2.6 we obtain $\bar{\nabla}_{a^\#} \tilde{Z} = 0$. Also, since $\alpha(Z) \in A^0(\mathrm{ad}(P))$ one can use the Proposition 2.8 to see that $\bar{\nabla}_{a^\#} \xi^{\alpha(Z)} = 0$. Therefore in all cases $\bar{\nabla}_Y \tilde{Z}^{A_0}$ is $A_0$-horizontal. ∎

In conclusion, using Theorem 2.4, Theorem 2.9, Lemma 2.11 and Proposition 2.6 we obtain

THEOREM 2.12. *Suppose that the triple $(g, P \xrightarrow{p} M, A_0)$ is infinitesimally homogeneous. There exists a pair $(C, A) \in \mathcal{A}(\mathrm{O}(M)) \times \mathcal{A}(P)$ such that*

$$\nabla^C R^C = 0,\ \nabla^C T^C = 0,\ \nabla^{CA} F^A = 0,\ \nabla^{CA}(A - A_0) = 0\,.$$

*The linear connection $\bar{\nabla} = \bar{\nabla}^{CA}$ on $T_P$ associated with this pair has the properties:*

*(1) $\bar{\nabla} R^{\bar{\nabla}} = \bar{\nabla} T^{\bar{\nabla}} = 0$,*
*(2) The vector fields $a^\#$ are $\bar{\nabla}$ parallel along the $A$-horizontal curves.*
*(3) The distributions $A$, $A_0$ are $\bar{\nabla}$-parallel.*

## 3. The main theorems

We begin by recalling the results proved in [**KN**, Ch.VI Section 7] on the existence and extension properties of local affine isomorphisms with respect to a linear connection satisfying the conditions $\nabla R^\nabla = 0$, $\nabla T^\nabla = 0$. Compared to [**KN**], our presentation uses a new formalism: the space of germs of $\nabla$-affine isomorphisms.

Let $M$ be a differentiable $n$-manifold, and let $\nabla$ be a linear connection on $M$ satisfying the conditions $\nabla R^\nabla = 0$, $\nabla T^\nabla = 0$. Let $\mathfrak{G}^\nabla$ be the space of germs of $\nabla$-affine isomorphisms defined between open sets of $M$, and let $\mathfrak{s} : \mathfrak{G}^\nabla \to M$,

$\mathfrak{t} : \mathfrak{G}^\nabla \to M$ be the source, respectively the target map on this space. $\mathfrak{G}^\nabla$ has a canonical structure of a differentiable manifold and, with respect to this structure, $\mathfrak{s}$ and $\mathfrak{t}$ are local diffeomorphisms.

A germ $\varphi \in \mathfrak{G}^\nabla$ defines an isomorphism $\varphi_* : T_{\mathfrak{s}(\varphi)}M \to T_{\mathfrak{t}(\varphi)}M$ with the property

$$\varphi_*(R^\nabla_{\mathfrak{s}(\varphi)}) = R^\nabla_{\mathfrak{t}(\varphi)},\ \varphi_*(T^\nabla_{\mathfrak{s}(\varphi)}) = T^\nabla_{\mathfrak{t}(\varphi)}\,.$$

Conversely, [**KN**, Theorem 7.4] can be reformulated as follows

REMARK 3.1. Let $(u, v) \in M \times M$. For any linear isomorphism $f : T_uM \to T_vM$ satisfying

$$f(R^\nabla_u) = R^\nabla_v,\ f(T^\nabla_u) = T^\nabla_v \tag{33}$$

there exists a unique germ $\varphi_f \in (\mathfrak{s}, \mathfrak{t})^{-1}(u, v)$ (of a $\nabla$-affine isomorphism) such that $(\varphi_f)_* = f$.

Let now $\sigma : M\times M \to M$, $\tau : M\times M \to M$ be the projections on the two factors, and let $\mathrm{Iso}(\sigma^*(T_M), \tau^*(T_M)) \subset \mathrm{Hom}(\sigma^*(T_M), \tau^*(T_M))$ be the (locally trivial) fibre bundle of isomorphisms between the two pull-backs of $T_M$. Conditions (33) define a closed, locally trivial subbundle $S^\nabla \subset \mathrm{Iso}(\sigma^*(T_M), \tau^*(T_M))$. Remark 3.1 shows that

REMARK 3.2. The natural map $\delta : \mathfrak{G}^\nabla \to S^\nabla$ given by $\varphi \mapsto \varphi_*$ is bijective.

This map is an injective immersion, and it is bijective, but it is *not* a diffeomorphism. $\mathfrak{G}^\nabla$ can be identified with the union of leaves of a foliation of $S^\nabla$ with $n$-dimensional leaves. The topology of $\mathfrak{G}^\nabla$ is finer than the topology of $S^\nabla$. The leaves of this foliation are the integrable submanifolds of the involutive distribution $\mathcal{D}^\nabla \subset T_{S^\nabla}$ defined in the following way: Let $f \in S^\nabla$. Put $u := \sigma(f)$, $v := \tau(f)$, $\varphi := \delta^{-1}(f)$. For a tangent vector $\xi \in T_uM$, let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a smooth curve such that $\gamma(0) = u$, $\dot\gamma(0) = \xi$. Using parallel transport with respect to $\nabla$ along the curves $\gamma$, $\varphi \circ \gamma$ we obtain, for any sufficiently small $t \in (-\varepsilon, \varepsilon)$, isomorphisms $a_t : T_uM \to T_{\gamma(t)}M$, $b_t : T_vM \to T_{\varphi(\gamma(t))}M$. Define

$$\lambda_f(\xi) := \frac{d}{dt}\Big|_{t=0}(b_t \circ f \circ a_t^{-1}) \in T_{(u,v)}(S^\nabla)\,.$$

The distribution $\mathcal{D}^\nabla$ is defined by

$$\mathcal{D}^\nabla_f := \{\lambda_f(\xi) |\ \xi \in T_{\sigma(f)}M\}\,.$$

The curve $t \mapsto b_t \circ f \circ a_t^{-1}$ will be an integral curve of this distribution. Note that we may take $\gamma$ to be the $\nabla$-geodesic with initial condition $(u, \xi)$ , and then $\varphi\circ\gamma$ will be the $\nabla$-geodesic with initial condition $(v, f(\xi))$. Using this remark, we obtain

REMARK 3.3. Let $\varphi \in \mathfrak{G}^\nabla$, and $\xi \in T_{\mathfrak{s}(\varphi)}$. Suppose that $\nabla$-geodesics $\gamma$, $\eta$ with initial conditions $(\mathfrak{s}(\varphi), \xi)$, $(\mathfrak{t}(\varphi), \varphi_*(\xi))$ respectively can be both extended on the interval $(\alpha, \beta) \ni 0$. Then $\gamma$ has a smooth lift in $\mathfrak{G}^\nabla$ with initial condition $\varphi$ via the source map $\mathfrak{s} : \mathfrak{G}^\nabla \to M$.

Using this remark one can prove:

PROPOSITION 3.4. *Let $\nabla$ be a connection on a connected manifold $M$ such that $\nabla R^\nabla = 0$, $\nabla T^\nabla = 0$. Suppose that $\nabla$ is complete. Then the source map $\mathfrak{s} : \mathfrak{G}^\nabla \to M$ is a covering map. In particular, when $\nabla$ is complete and $M$ is simply connected, any element $\varphi \in \mathfrak{G}^\nabla$ extends to a unique global $\nabla$-affine isomorphism*

*$M \to M$. In particular, for any pair $(x, x') \in M \times M$ there exists a unique global $\nabla$-affine isomorphism mapping $x$ to $x'$.*

PROOF. It suffices to note, that for a $\nabla$-convex open set $U \subset M$ the following holds: any germ $\varphi \in \mathfrak{S}^\nabla$ with $\mathfrak{s}(\varphi) \in U$ has an extension on $U$. The point is that, since $\nabla$ is complete, for any geodesic $\gamma : (\alpha, \beta) \to U$ passing through $\mathfrak{s}(\varphi)$, the composition $\varphi \circ \gamma$ can be extended on the whole $(\alpha, \beta)$. Therefore all $\nabla$- geodesics in $U$ passing through $\mathfrak{s}(\varphi)$ admit lifts with initial condition $\varphi$. Using these lifts it follows that the connected components of $\mathfrak{s}^{-1}(U)$ are identified with $U$ via $\mathfrak{s}$. ■

An important special case (which intervenes in the proof of Singer's theorem) concerns a connection $\nabla$ satisfying the conditions $\nabla R^\nabla = 0$, $\nabla T^\nabla = 0$ which is a metric connection, i.e. there exists a Riemannian metric $g$ on $M$ such that $\nabla g = 0$. In this case one defines submanifolds

$$\mathfrak{S}^\nabla_g := \{\varphi \in \mathfrak{S}^\nabla |\ \varphi_* \text{ is an isometry}\},\ S^\nabla_g := \{f \in S^\nabla |\ f \text{ is an isometry}\}$$

of $\mathfrak{S}^\nabla$, $S^\nabla$ respectively. $\mathfrak{S}^\nabla_g$ is open in $\mathfrak{S}^\nabla$. In other words $S^\nabla_g$ is a union of integral submanifolds (of maximal dimension) of the involutive distribution $\mathcal{D}^\nabla$. On the other hand, an important result in Riemannian geometry states [**TV**, Proposition 1.5]:

PROPOSITION 3.5. *Let $(M, g)$ be a complete Riemannian manifold. Then any metric connection $\nabla$ on $M$ is complete.*

Using these facts Proposition 3.4 gives

COROLLARY 3.6. *Let $(M, g)$ be a connected Riemannian manifold endowed with a metric connection $\nabla$ such that $\nabla R^\nabla = 0$, $\nabla T^\nabla = 0$. Suppose that $(M, g)$ is complete. Then the source map $\mathfrak{s} : \mathfrak{S}^\nabla_g \to M$ is a covering map. In particular, when $(M, g)$ is complete and simply connected, any element $\varphi \in \mathfrak{S}^\nabla_g$ extends to a unique global $\nabla$-affine isometry $M \to M$. In particular, for any pair $(x, x') \in M \times M$ there exists a unique global $\nabla$-affine isometry mapping $x$ to $x'$.*

Now we come back to the connection $\bar\nabla = \bar\nabla^{CA}$ on $T_P$ associated with a pair $(C, A) \in \mathcal{A}(\mathrm{O}(M)) \times \mathcal{A}(P)$ satisfying the conditions

$$\nabla^C R^C = 0,\ \nabla^C T^C = 0,\ \nabla^{CA} F^A = 0,\ \nabla^{CA}(A - A_0) = 0\,.$$

We know that $\bar\nabla R^{\bar\nabla} = 0$, $\bar\nabla T^{\bar\nabla} = 0$, so all constructions and results above apply to $\bar\nabla$. Using the additional structure we have on $P$ (the $K$-action, the two connections $A$, $A_0$, and the metric $g$ on $P/K$) we will define an open submanifold $\mathfrak{S}^{\bar\nabla}_{g,K}$ of $\mathfrak{S}^{\bar\nabla}$ consisting of germs of affine transformations which are compatible with this structure:

Since $\bar\nabla$ is $K$-invariant, the manifolds $\mathfrak{S}^{\bar\nabla}$, $S^{\bar\nabla}$ come with natural right $K$-actions given by $(\varphi, k) \mapsto R_k \circ \varphi \circ R_k^{-1}$, $(f, k) \mapsto (R_k)_* \circ f \circ (R_k^{-1})_*$, and the maps $\mathfrak{s}$, $\mathfrak{t} : \mathfrak{S}^{\bar\nabla} \to P$, $\sigma$, $\tau : S^{\bar\nabla} \to P$ are $K$-equivariant.

We define a submanifold $S^{\bar\nabla}_{K,g} \subset S^{\bar\nabla}$ by

$$\begin{aligned} S^{\bar\nabla}_{g,K} :=& \{f \in S^{\bar\nabla} |\ f(A_{\sigma(f)}) = A_{\tau(f)},\ f(A_{0,\sigma(f)}) = A_{0,\tau(f)}, \\ & f(a^\#_{\sigma(f)}) = a^\#_{\tau(f)}\ \forall a \in \mathfrak{k}, f \text{ induces an isometry } T_{p(\sigma(f))}M \to T_{p(\tau(f))}M\}\,. \end{aligned} \tag{34}$$

LEMMA 3.7. *$S^{\bar\nabla}_{g,K}$ is $K$-invariant, and is a union of integral submanifolds of the distribution $\mathcal{D}^{\bar\nabla}$. In particular $\mathfrak{S}^{\bar\nabla}_{g,K} := \delta^{-1}(S^{\bar\nabla}_{g,K})$ is a $K$-invariant open submanifold of $\mathfrak{S}^{\bar\nabla}$.*

PROOF. We have to prove that for any $f \in S^{\bar\nabla}_{g,K}$ one has $\mathcal{D}_f \subset T_f S^{\bar\nabla}_{g,K}$, i.e. that for any $\xi \in T_{\sigma(f)}P$ we have $\lambda_f(\xi) \in T_f S^{\bar\nabla}_{g,K}$ . It suffices to prove that for any $a \in \mathfrak{k}$ and any $\zeta \in T_{p(\sigma(f))}M$ one has (denoting by $\tilde\zeta_{\sigma(f)}$ the $A$-horizontal lift of $\zeta$ at $\sigma(f)$):

$$\lambda_f(a^{\#}_{\sigma(f)}) \in T_f S^{\bar\nabla}_{g,K}\,,\ \lambda_f(\tilde\zeta_{\sigma(f)}) \in T_f S^{\bar\nabla}_{g,K}\,.$$

The first formula is obtained using the curve $t \mapsto \sigma(f)\exp(ta)$, and the second is obtained using the curve $t \mapsto \tilde\eta_{\sigma(f)}(t)$, where $\eta : (-\varepsilon, \varepsilon) \to M$ is a $\nabla^C$-geodesic such that $\eta(0) = p(\sigma(f))$, $\dot\eta(0) = \zeta$. One uses the fact that the vector fields $a^{\#}$ are $\bar\nabla$-parallel along $A$-horizontal curves, and that the distributions $A$, $A_0$ is $\bar\nabla$-parallel. ■

LEMMA 3.8. *The restrictions,*

$$(p\circ\sigma, p\circ\tau)\big|_{S^{\bar\nabla}_{g,K}} : S^{\bar\nabla}_{g,K} \to M\times M\,,\ (p\circ\mathfrak{s}, p\circ\mathfrak{t})\big|_{\mathfrak{S}^{\bar\nabla}_{g,K}} : \mathfrak{S}^{\bar\nabla}_{g,K} \to M\times M$$

*are surjective.*

PROOF. Let $x_0$, $x_1 \in M$, and let $\eta : [0,1] \to M$ be a smooth path in $M$ such that $\eta(0) = x_0$, $\eta(1) = x_1$. Choose a point $y_0 \in P_{x_0}$, let $\tilde\eta$ be the $A$-horizontal lift of $\eta$ with the initial condition $\tilde\eta(0) = y_0$, and let $y_1 := \tilde\eta(1)$. Using $\bar\nabla$-parallel transport along $\tilde\eta$, we obtain an element $f \in S^{\bar\nabla}$ with $\sigma(f) = y_0$, $\tau(f) = y_1$. Using Theorem 2.12, we see that $f \in S^{\bar\nabla}_{g,K}$ ■

THEOREM 3.9. *Let $M$ be a connected manifold, $(g, P \xrightarrow{p} M, A_0)$ be a triple consisting of a Riemannian metric $g$ on $M$, a principal $K$-bundle $P$ on $M$, and a connection $A_0$ on $P$. Denote by $C_0 \in \mathcal{A}(\mathrm{O}(M))$ the Levi-Civita connection on the orthonormal frame bundle $\mathrm{O}(M)$ of $(M, g)$. The following conditions are equivalent:*

*(1) $(g, P \xrightarrow{p} M, A_0)$ is locally homogeneous.*
*(2) $(g, P \xrightarrow{p} M, A_0)$ is infinitesimally homogeneous.*
*(3) There exists a pair $(C, A) \in \mathcal{A}(\mathrm{O}(M)) \times \mathcal{A}(P)$ such that*

$$\nabla^C R^C = 0,\ \nabla^C T^C = 0,\ \nabla^{CA} F^A = 0,\ \nabla^{CA}(A - A_0) = 0\,.$$

PROOF. The implication (1)⇒(2) is obvious, and the implication (2)⇒(3) stated by Theorem 2.4. For the implication (3)⇒(1), let $\bar\nabla$ be the connection associated with the pair $(C, A)$. Let $(x_0, x_1) \in M \times M$. By Lemma 3.8 there exists $\varphi \in S^{\bar\nabla}_{g,K}$ such that $y_0 := \mathfrak{s}(\varphi) \in P_{x_0}$, $y_1 := \mathfrak{t}(\varphi) \in P_{x_1}$. The orbit $\varphi K$ is a submanifold of $\mathfrak{S}^{\bar\nabla}_{g,K}$ which is mapped diffeomorphically onto $y_0K$ via $\mathfrak{s}$, and onto $y_1K$ via $\mathfrak{t}$. Using [**God**, Théorème 3.3.1] it follows that there exists an open neighbourhood $\mathfrak{U} \subset \mathfrak{S}^{\bar\nabla}_{g,K}$ of $\varphi K$ in $\mathfrak{S}^{\bar\nabla}_{g,K}$ which is mapped injectively onto an open neighbourhood $\mathcal{U}$ of $y_0K$ via $\mathfrak{s}$, and is mapped injectively onto an open neighbourhood $\mathcal{V}$ of $y_1K$ via $\mathfrak{t}$. Since $K$ is compact, we may suppose that $\mathfrak{U}$ (hence also $\mathcal{U}$ and $\mathcal{V}$) is $K$-invariant. $\mathfrak{U}$ defines a $K$-equivariant, $\bar\nabla$-affine isomorphism $\mathcal{U} \to \mathcal{V}$ which maps $A_{0\mathcal{U}}$ onto $A_{0\mathcal{V}}$ and induced an isometry $p(\mathcal{U}) := U \to V := p(\mathcal{V})$. This shows that $(g, P \xrightarrow{p} M, A_0)$ is locally homogeneous. ■

For the case when $(M, g)$ is complete, we have

THEOREM 3.10. *Let $(g, P \xrightarrow{p} M, A_0)$ be a locally homogeneous triple with $M$ connected. If $(M, g)$ is complete, then the map $\mathfrak{S}^{\bar\nabla}_{g,K}/K \to M$ induced by $\mathfrak{s}$ is a covering map. If $(M, g)$ is complete and $M$ is simply connected, then any germ $\varphi \in \mathfrak{S}^{\bar\nabla}_{g,K}$ can be extended to a unique bundle isomorphism $\Phi : P \to P$ which covers an isometry $M \to M$, and has the property $\Phi_*(A_0) = A_0$. In particular, for any $(x, x') \in M \times M$ there exists such a bundle isomorphism with $\Phi(P_x) = P_{x'}$.*

PROOF. We use the same method as in the proof of Proposition 3.4, Corollary 3.6. The fact that $\mathfrak{S}^{\bar\nabla}_{g,K}/K \to M$ is a covering map is obtained using parallel transport with respect to $\bar\nabla$ along $A$-horizontal lifts of $\nabla^C$-geodesics in $M$. ■

We can prove now our main theorem stated in the introduction

THEOREM 3.11. *Let $M$ be a compact manifold, and $K$ be a compact Lie group. Let $\pi : \tilde M \to M$ be the universal cover of $M$, $\Gamma$ be the corresponding covering transformation group. Then, for any locally homogeneous triple $(g, P \xrightarrow{p} M, A)$ with structure group $K$ on $M$ there exists*

*(1) A connection $B$ on the pull-back bundle $Q := \pi^*(P)$.*
*(2) A closed subgroup $G \subset \mathrm{Iso}(\tilde M, \pi^*(g))$ acting transitively on $\tilde M$ which contains $\Gamma$ and leaves invariant the gauge class $[B] \in \mathcal{B}(Q)$.*
*(3) A lift $\mathfrak{j} : \Gamma \to \mathcal{G}^B_G(Q)$ of the inclusion monomorphism $\iota_\Gamma : \Gamma \to G$, where $\mathcal{G}^B_G(Q)$ stands for the group of automorphisms of $(Q, B)$ which lift transformations in $G$.*
*(4) An isomorphism between the $\Gamma$-quotient of $(\pi^*(g), Q, B)$ and the initial triple $(g, P \xrightarrow{p} M, A)$.*

PROOF. Let $G \subset \mathrm{Iso}(\tilde M, \tilde g)$ be the subgroup defined by

$$G := \big\{\psi \in \mathrm{Iso}(\tilde M, \tilde g)|\ \exists\, \Psi : Q \to Q\ \psi\text{-covering bundle isom.}, \Psi^*(B) = B\big\}.$$

Using the fact that $K$ is compact, it follows by Lemma 3.12 below, that $G$ is a closed subgroup of the Lie group $\mathrm{Iso}(\tilde M, \tilde g)$. Note that Lemma 3.12 applies because the action of the Lie group $\mathrm{Iso}(\tilde M, \tilde g)$ on $\tilde M$ is smooth.

Applying Theorem 3.10 to $(\pi^*(g), Q, B)$, it follows that $G$ acts transitively on $\tilde M$, and leaves invariant the gauge class $[B]$. Moreover, the definition of $G$ shows that it contains $\Gamma$. The lift $\mathfrak{j}$ is obtained as follows: for $\varphi \in \Gamma$ we define $\mathfrak{j}(\varphi) : Q \to Q$ by

$$\mathfrak{j}(\varphi)(\tilde x, y) := (\varphi(\tilde x), y) \forall (\tilde x, y) \in Q := \tilde M \times_\pi P\,.$$

Note that the map $\Gamma \ni \varphi \mapsto \mathfrak{j}(\varphi) \in \mathcal{G}^B_G(Q)$ is group morphism (as required). ■

LEMMA 3.12. *Let $M$ be a differentiable manifold, $K$ be a compact Lie group, $p : P \to M$, $p' : P' \to M$ be principal $K$-bundles on $M$, and $A \in \mathcal{A}(P)$, $A' \in \mathcal{A}(P')$ be connections on $P$, $P'$ respectively. Let $\alpha : L \times M \to M$ be a a smooth action of a Lie group $L$ on $M$. For $l \in L$ denote by $\varphi_l : M \to M$ the corresponding diffeomorphism. The subspace*

$$L_{AA'} := \{l \in L|\ \exists \Phi \in \mathrm{Hom}_{\varphi_l}(P, P')\ \textit{such that}\ \Phi^*(A') = A\}$$

*is closed in $L$. In the special case $P = P'$, $A = A'$, the obtained subset $L_{AA}$ is a Lie subgroup of $L$.*

PROOF. Let $(l_n)_{n\in\mathbb{N}}$ be sequence in $L_{AA'}$ converging to an element $l_\infty \in L$. We will prove that $l_\infty \in L_{AA'}$.

Let $V \subset T_eL$ a sufficiently small convex neighborhood of 0 in the tangent space $\mathfrak{l} = T_eL$ of $L$ at its unit element $e$, such that the exponential map $\exp : T_eL \to L$ induces a diffeomorphism $E : V \to U$. The map $\eta : [0,1] \times U \to U$ defined by

$$\eta(t,l) = E(tE^{-1}(l)))$$

is a smooth homotopy joining the constant map $e$ on $U$ to $\mathrm{id}_U$. Moreover one has

$$\lim_{l\to e} \eta(t,l) = e \tag{35}$$

uniformly on $[0,1]$. For any $l \in U$ and $x \in M$ we obtain a smooth path $\gamma^l_x : [0,1] \to M$ given by

$$\gamma^l_x(t) = \varphi_{\eta(t,l)}(x)$$

which joins $x$ to $\varphi_l(x)$. For $l \in U$ denote by $\gamma^l : [0,1] \times M \to M$ the map $(t,x) \mapsto \gamma^l_x(t)$. Denoting by $\mathrm{p}_M : [0,1] \times M \to M$ the projection on the $M$-factor, and using (35) we get

$$\lim_{l\to e} \gamma^l = \mathrm{p}_M \tag{36}$$

in the weak topology $\mathcal{C}^\infty_w([0,1]\times M, M)$.

Fix any connection $B'$ on $P'$. For any $l \in U$ we get a $\varphi_l$-covering automorphism $\Psi_l \in \mathrm{Hom}_{\varphi_l}(P', P')$ defined by

$$\Psi_l(y') = \{\widetilde{\gamma^l_{p'(y')}}\}^{B'}_{y'}$$

where $\{\widetilde{\gamma^l_{p'(y')}}\}^{B'}_{y'}$ is the $B'$-horizontal lift of $\gamma^l_{p'(y')}$ with initial $\{\widetilde{\gamma^l_{p'(y')}}\}^{B'}_{y'}(0) = y'$. Using (36) we get

$$\lim_{l\to e} \Psi_l = \mathrm{id}_{P'} \tag{37}$$

in the weak topology $\mathcal{C}^\infty_w(P', P')$. Put

$$\lambda_n := l_\infty l_n^{-1}.$$

Since $\lim_{n\to\infty} l_n = l_\infty$ we may suppose that $\lambda_n \in U$ for any $n$. Let $\Phi_n \in \mathrm{Hom}_{\varphi_{l_n}}(P, P')$ such that $\Phi_n^*(A') = A$, and note that

$$\Sigma_n := \Psi_{\lambda_n} \circ \Phi_n$$

is a $\varphi_{l_\infty}$-covering bundle morphism with the property

$$(\Sigma_n)_*(A) = (\Psi_{\lambda_n})_*(A').$$

Using (37) and $\lim_{n\to\infty} \lambda_n = e$, we obtain

$$\lim_{n\to\infty} (\Sigma_n)_*(A) = A'$$

in the Fréchet affine space $\mathcal{A}(P')$. The claim follows now from Lemma 3.13 below. ∎

LEMMA 3.13. *Let $K$ be a compact Lie group, $p : P \to M$, $p' : P' \to M'$ be principal $K$-bundles on $M$ and $M'$, and $A \in \mathcal{A}(P)$, $A' \in \mathcal{A}(P')$ be connections on $P$, $P'$ respectively. Let $\varphi : M \to M'$ be a smooth map and $(\Phi_n)_n$ be a sequence in $\mathrm{Hom}_\varphi(P, P')$ such that $\lim_{n\to\infty} \Phi_n^*(A') = A$. Then there exists a subsequence $(\Phi_{n_k})_k$ of $(\Phi_n)_n$ which converges (in the weak $\mathcal{C}^\infty$-topology) to a morphism $\Phi_\infty \in \mathrm{Hom}_\varphi(P, P')$ satisfying $\Phi_\infty^*(A') = A$.*

Proof. It suffices to prove the claim in the special case when $M' = M$, $\varphi = \mathrm{id}_M$. Indeed, using Remark 1.4 (see section 1.2) we obtain morphisms $\Phi_n^0 \in \mathrm{Hom}_{\mathrm{id}_M}(P, \varphi^*(P'))$ such that

$$\lim_{n\to\infty} (\Phi_n^0)^*(\varphi^*(A')) = A$$

Since the bijection given by Remark 1.4 is a homeomorphism with respect to the weak $\mathcal{C}^\infty$-topology it suffices to prove the claim for the sequence of $\mathrm{id}_M$-covering morphisms $(\Phi_n^0)_n$.

From now we suppose that $M' = M$, $\varphi = \mathrm{id}_M$, $\Phi_n \in \mathrm{Hom}_{\mathrm{id}_M}(P, P')$. We will use ideas form gauge theory [**DK**, section 2.3.7]. Note that

$$F_n := \Phi_1^{-1} \circ \Phi_n \in \mathrm{Hom}_{\mathrm{id}_M}(P, P) = \mathcal{G}(P) = \Gamma(P \times_\iota K),$$

where $\mathcal{G}(P)$ is the gauge group of $P$ and $\iota : K \to \mathrm{Aut}(K)$ is the interior morphism defined by $\iota(k)(u) = kuk^{-1}$. Therefore we have a sequence $(F_n)_n$ of gauge transformations of $P$ with the property

$$\lim_{n\to\infty} (F_n)_*(A) = \Phi_1^*(A') =: A_1. \tag{38}$$

Fix an embedding $\rho : K \to \mathrm{O}(N)$, and let $E := P \times_\rho \mathbb{R}^N$ be the Euclidian associated bundle. Any gauge transformation $F \in \mathcal{G}(P)$ can be identified with a section in the endomorphism bundle $\mathrm{End}(E)$, and via this identification one has

$$F_*(A) = A - (\nabla^A F)F^{-1},$$

where $\nabla^A$ is the linear connection induced by $A$ on $\mathrm{End}(E)$. Therefore, putting $B_n := (F_n)_*(A)$, we get

$$\nabla^A F_n = (A - B_n)F_n \tag{39}$$

where, by (38),

$$\lim_{n\to\infty} B_n = A_1 \tag{40}$$

in the Fréchet $\mathcal{C}^\infty$-topology of the affine space $\mathcal{A}(P)$. Since $K$ is compact, $F_n$ is uniformly bounded on $M$. On the other hand by (40) the sequence $(A - B_n)_n$ is bounded in the Fréchet space $\Gamma(\Lambda^1_M \otimes \mathrm{ad}(P)) \subset \Gamma(\Lambda^1_M \otimes \mathrm{End}(E))$. Using (39) and a standard bootstrapping procedure we see that all partial derivatives of $F_n$ (with respect to local coordinates in $M$ and trivializations of $P$) are uniformly bounded on any compact subset of $M$. Using a well-known combination of the Arzela-Ascoli theorem and the diagonal argument we obtain a subsequence $(F_{n_k})_k$ of $(F_n)$ which converges in $\Gamma(\mathrm{End}(E))$ (with respect to its Fréchet $\mathcal{C}^\infty$-topology) to a section $F_\infty \in \Gamma(\mathrm{End}(E))$. Since $\rho(K)$ is closed in $\mathrm{gl}(N, \mathbb{R})$, it follows that $\mathcal{G}(P)$ is closed in $\Gamma(\mathrm{End}(E))$, hence $F_\infty \in \mathcal{G}(P)$. Replacing $n$ by $n_k$ and taking $k \to \infty$ in (39) we get $\nabla^A F_\infty = (A - A_1)F_\infty$, i.e. $A_1 = (F_\infty)_*(A)$. It suffices to put $\Phi_{n_k} = \Phi_1 \circ F_{n_k}$, $\Phi_\infty := \Phi_1 \circ F_\infty$. ■

Remark 3.14. Theorem 3.11 yields a closed subgroup $G \subset \mathrm{Iso}(\tilde{M}, \tilde{g})$ which acts transitively on $\tilde{M}$. In general this group will not be connected. The connected component $G_0 \subset G$ of $\mathrm{id} \in G$ still acts transitively on $\tilde{M}$ (see Lemma 3.16 below) but, in general, it will not contain $\Gamma$. On the other hand, if $M$ is orientable it follows that $G \cap \mathrm{Iso}^+(\tilde{M}, \tilde{g})$ still satisfies the conclusion of the theorem. Therefore, under this assumption Theorem 3.11 holds with $G \subset \mathrm{Iso}^+(\tilde{M}, \tilde{g})$.

REMARK 3.15. With the notations of Remark 3.14 put $\Gamma_0 := \Gamma \cap G_0$. It is easy to see that $\Gamma_0 \backslash G_0$ can be identified with the connected component of $\Gamma \in \Gamma\backslash G$, therefore the compactness of $\Gamma\backslash G$ implies the compactness of $\Gamma_0\backslash G_0$. This shows that $G_0$ is a connected, subgroup of $\mathrm{Iso}^+(\tilde{M}, \tilde{g})$ which acts transitively on $\tilde{M}$ and is unimodular. In many interesting cases these properties implies $G_0 = \mathrm{Iso}^+(\tilde{M}, \tilde{g})$.

LEMMA 3.16. *Let $(M, g)$ be a complete, connected Riemannian manifold, and $G \subset \mathrm{Iso}(M, g)$ be a closed subgroup of the group of isometries $\mathrm{Iso}(M, g)$. If $G$ acts transitively on $M$, then the connected component of identity $G_0 \subset G$ also acts transitively on $M$.*

PROOF. For every class $\xi \in G/G_0$, choose a representative $\psi_\xi \in \xi$. In other words one has

$$\xi = G_0\psi_\xi \ \forall \xi \in G/G_0,$$

hence $G = \bigcup_{\xi \in G/G_0} G_0\psi_\xi$. Fix $x_0 \in M$. Since $G$ acts transitively on $M$, we have $M = Gx_0$. Therefore

$$M = \bigcup_{\xi \in G/G_0} G_0\psi_\xi(x_0). \tag{41}$$

But $G_0$ is a closed subgroup of $\mathrm{Iso}(M, g)$, hence its action on $M$ is proper [**Ra**, Theorem 4]. Therefore the $G_0$-orbits are embedded closed submanifolds of $M$. If (by reductio ad absurdum) the $G_0$-action on $M$ were not transitive, all these orbits would be submanifolds of dimension strictly smaller than $\dim(M)$. But the quotient set $G/G_0$ is at most countable, hence formula (41) would lead to a contradiction. ■

Theorem 3.11 shows that:

COROLLARY 3.17. *Let $M$ be a connected compact manifold, and $K$ be a compact Lie group. Let $\pi : \tilde{M} \to M$ be the universal cover of $M$, $\Gamma$ be the corresponding covering transformation group. Then any locally homogeneous triple $(g, P \xrightarrow{p} M, A)$ with structure group $K$ on $M$ can be identified with a $\Gamma$-quotient of a homogeneous triple $(\tilde{g} = \pi^*(g), Q := \pi^*(P) \xrightarrow{q} \tilde{M}, B)$ on the universal cover $\tilde{M}$.*

CHAPTER 3

# Locally homogeneous and homogeneous triples. The real-analytic case

The main result of this chapter is Theorem 3.11 of Chapter 2 in the real analytic category. The proof is based on an extension theorem (Theorem 2.2) for real analytic locally defined bundle morphisms which are compatible with a real analytic connection. The main idea in the proof of this extension theorem is to reduce the extension problem for bundle morphisms (which are compatible with a pair of connections) to an extension problem for sections in an associated bundle (which are parallel with respect to an associated connection). The proofs make use of [**KN**, Lemma 2, p. 253] for analytic maps between manifolds endowed with analytic distributions, so the analyticity condition plays a crucial role. The results in this chapter appeared in [**Ba**].

## 1. Extension of local parallel sections

**1.1. The space of parallel sections.** Let $p : P \to M$ be a real analytic principal $K$-bundle over a real analytic manifold $M$. Let $\lambda : K \times F \to F$ be an analytic action of $K$ on an analytic manifold $F$, and let $E := P \times_\lambda F$ be the associated bundle with fibre $F$. Let $\mathcal{E} \xrightarrow{\mu} M$ be the projection map of the étale space of the sheaf of (locally defined) analytic sections of the bundle $p_E : E \to M$. In other words, a point of $\mathcal{E}$ is a germ $[s]_x$ where $x \in X$, and $s \in \Gamma^{\mathrm{an}}(U, E)$ is an analytic section defined on an open neighbourhood $U$ of $x$ in $M$. Two pairs

$$(s : U \to E, x),\ (s' : U' \to E, x')$$

define the same germ (hence one has $[s]_x = [s']_x$) if $x = x'$ and there exists $U_0 \subset U \cap U'$ such that $s_{|U_0} = s'|_{U_0}$. The projection map $\mu$ is given by $[s]_x \mapsto x$. The space $\mathcal{E}$ has a natural structure of a real analytic manifold, which is Hausdorff, but does not have countable basis. The topology of $\mathcal{E}$ can be easily described as follows: Any section $s \in \Gamma^{\mathrm{an}}(U, E)$ defines a section $\tilde{s} : U \to \mathcal{E}$ of $\mathcal{E} \xrightarrow{\mu} M$ given by

$$\tilde{s}(x') = [s : U \to E, x'].$$

The sets of the form $\tilde{s}(U)$ (where $U$ is open in $M$ and $s \in \Gamma^{\mathrm{an}}(U, E)$) give a basis for the topology of $\mathcal{E}$. Note also that, for any open set $U \subset M$, the restriction $\mu_{|\tilde{s}(U)} : \tilde{s}(U) \to U$ is a real analytic diffeomorphism.

Let now $A$ be a real analytic connection on $P$, and let $\Gamma^A \subset T_E$ be the associated connection on the associated bundle $E$. Denote by $\mathcal{E}^A \subset \mathcal{E}$ the open submanifold of $\mathcal{E}$ whose points are germs of $\Gamma^A$-parallel sections, and by $\mu^A : \mathcal{E}^A \to M$ the restriction of $\mu$ to $\mathcal{E}^A$. With these notations we state

THEOREM 1.1. *Suppose that $M$ is connected. If $\mathcal{E}^A$ is non-empty, then the map $\mu^A : \mathcal{E}^A \to M$ is a covering map.*

PROOF. We will prove that $\mu^A$ satisfies condition (1) in Lemma 1.2 below. Let $x_0 \in M$, and let $h : U_0 \to B(0,r) \subset \mathbb{R}^n$ be local chart of $M$ around $x_0$ such that $h(x_0) = 0$. We will show that $\mu^A$ maps diffeomorphically any connected component of $(\mu^A)^{-1}(U_0)$ onto $U_0$. Using Lemma 1.3 below, we obtain, for every point $\sigma = [s]_{x_0} \in (\mu^A)^{-1}(x_0)$ a $\Gamma^A$-parallel section $s_\sigma : U_0 \to A$ defining the germ $\sigma$. We claim that the connected components of $(\mu^A)^{-1}(U_0)$ are precisely the sets $\tilde{s}_\sigma(U_0)$, which are obviously mapped diffeomorphically onto $U_0$ via $\mu^A$. We will first prove that

*Claim:* The family $(\tilde{s}_\sigma(U_0))_{\sigma \in (\mu^A)^{-1}(x_0)}$ is a partition of $(\mu^A)^{-1}(U_0)$.

Indeed, we obviously have $\tilde{s}_\sigma(U_0) \subset (\mu^A)^{-1}(U_0)$ for any $\sigma \in (\mu^A)^{-1}(x_0)$, hence

$$\bigcup_{\sigma \in (\mu^A)^{-1}(x_0)} \tilde{s}_\sigma(U_0) \subset (\mu^A)^{-1}(U_0).$$

The opposite inclusion is obtained as follows: let $\nu \in (\mu^A)^{-1}(U_0)$, therefore there exists $x \in U_0$ and a $\Gamma^A$-parallel section $s$ defined around $x$ such that $\nu = [s]_x$. Note now that the pair $(x, U_0)$ also satisfies the assumption of Lemma 1.3, because there obviously exists a chart $h_x : U_0 \to B(x,r)$ such that $h_x(x) = 0$. Therefore the germ $\nu$ extends to a $\Gamma^A$-parallel section $s_\nu : U_0 \to E$. Note that $[s_\nu]_{x_0} \in (\mu^A)^{-1}(x_0)$, and that the sections $s_\nu$, $s_{[s_\nu]_{x_0}}$ coincide, because they are both parallel and their germ at $x_0$ coincide. This proves that $\nu \in \tilde{s}_{[s_\nu]_{x_0}}(U_0)$, which proves the inclusion

$$(\mu^A)^{-1}(U_0) \subset \bigcup_{\sigma \in (\mu^A)^{-1}(x_0)} \tilde{s}_\sigma(U_0).$$

In order to complete the proof of the claim, it remains to note that the sets $\tilde{s}_\sigma(U_0)$ are pairwise disjoint. This follows using unique continuation for parallel sections.

Note now that any set $\tilde{s}_\sigma(U_0)$ is open in $(\mu^A)^{-1}(U_0)$. Using the claim we see that any such set is also closed in $(\mu^A)^{-1}(U_0)$ (as complement of an open set). Since any such set is obviously connected, the theorem follows by Lemma 1.2.

■

LEMMA 1.2. *Let $M$, $N$ be differentiable manifolds, and $f : N \to M$ be a locally diffeomorphic map satisfying the following properties:*

*(1) any point $x_0 \in M$ has an open neighborhood $U_0$ such that, for any connected component $\tilde{U}_0$ of $f^{-1}(U_0)$, the map $\tilde{U}_0 \to U_0$ induced by $f$ is a diffeomorphism,*

*(2) $N$ is non-empty and $M$ is connected.*

*Then $f$ is a covering projection.*

Usually in the definition of a covering projection, condition (1) and the surjectivity of $f$ are required. In our case, since $N$ is non-empty and $M$ is connected, the first condition implies the surjectivity of $f$.

LEMMA 1.3. *Let $x_0 \in M$, let $[s]_{x_0} \in (\mu^A)^{-1}(x_0)$, where $s : U \to E$ is $\Gamma^A$-parallel, and let $h : U_0 \to B(0,r) \subset \mathbb{R}^n$ be an analytic local chart of $M$ around $x_0$ such that $h(x_0) = 0$. Then $e := [s]_{x_0}$ extends to a $\Gamma^A$-parallel section $s_0 \in \Gamma^{\mathrm{an}}(U_0, E)$, i.e. one has $e = [s_0]_{x_0}$, for a $\Gamma^A$-parallel section $s_0 \in \Gamma^{\mathrm{an}}(U_0, E)$.*

PROOF. Let $s : U \to E$ be a $\Gamma^A$-parallel section defining the germ $e$, where $U$ is an open neighborhood of $x_0$ in $M$. Let $\varepsilon \in (0, r)$ be sufficiently small such that $U_\varepsilon := h^{-1}(B(0,\varepsilon)) \subset U$. We will construct a real analytic section $s_0 \in \Gamma^{\mathrm{an}}(U_0, E)$ whose restriction to $U_\varepsilon$ coincides with $s_{|U_\varepsilon}$. Using [**KN**, Lemma 2, p. 253] it follows that $s_0$ is $\Gamma^A$-parallel, which completes the proof.

The construction of $s_0$ uses parallel transport along "radial" curves in $U_0$. More precisely, for any $u \in U_0$ define the path $\gamma_u : [0,1] \to U_0$ by

$$\gamma_u(t) := h^{-1}(th(u)),$$

and note that $\gamma_u$ is a path in $U_0$ joining $x_0$ to $u$. Put $e_0 := s(x_0) \in E_{x_0}$. For any $u \in U_0$ let $\tilde\gamma_u : [0,1] \to E$ be the $\Gamma^A$-horizontal lift of $\gamma_u : [0,1] \to M$ with initial condition $\tilde\gamma_u(0) = e_0$ (see section 1.1). Therefore, $\tilde\gamma_u(0) = e_0$, and for every $t \in [0,1]$, one has

$$\frac{d}{dt}\tilde\gamma_u(t) \in \Gamma^A_{\tilde\gamma_u(t)},\ p_E \circ \tilde\gamma_u(t) = \gamma_u(t).$$

Using the analycity of the map $(t,u) \mapsto \tilde\gamma_u(t)$, and a standard theorem on the analycity of solutions of ordinary differential equations with respect to parameters, we see that the map $u \mapsto s_0(u) := \tilde\gamma_u(1)$ is analytic. On the other hand one has

$$(p_E \circ s_0)(u) = p_E(\tilde\gamma_u(1)) = \gamma_u(1) = u.$$

It remains to prove that $s_{0|U_\varepsilon} = s_{|U_\varepsilon}$. For this, it suffices to note that for any $u \in U_\varepsilon$ the paths $s_0 \circ \gamma_u$, $s \circ \gamma_u$ are both $\Gamma^A$-horizontal lifts of $\gamma_u$ with the same initial condition $e_0$.

■

**1.2. The extension theorem for local parallel sections.** In this section we prove an extension theorem for locally defined parallel sections in analytic associated bundles. This result can be obtained using analytic continuation along paths. This method is used for instance in [**KN**] for the problem of extending affine mappings and isometric immersions. We will use a different method, which is base on Theorem 1.1 proved in the previous section.

THEOREM 1.4. *Let $p : P \to M$ be a real analytic principal $K$-bundle over a real analytic manifold $M$, $\lambda : K \times F \to F$ a real analytic action of $K$ on a real analytic manifold $F$, and $E := P \times_\lambda F$ the associated bundle with fibre $F$. Let $A$ be an analytic connection on $P$ and $\Gamma^A \subset T_E$ the associated connection on $E$.*

*If $M$ is simply connected, then any $\Gamma^A$-parallel section $s : U \to E$ defined on a connected, non-empty open set $U \subset M$ admits a unique $\Gamma^A$-parallel extension on $M$.*

PROOF. The $\Gamma^A$-parallel section $s$ defines a section $\tilde s \in \Gamma(U, \mathcal{E}^A)$. Let $\mathcal{C} \subset \mathcal{E}^A$ be connected component of $\mathcal{E}^A$ which contains the image $\tilde s(U)$ of $\tilde s$. By Theorem 1.1 we know that $\mu^A : \mathcal{E}^A \to M$ is a covering projection, hence the restriction $\mu^A|_{\mathcal{C}} : \mathcal{C} \to M$ of $\mu^A$ to $\mathcal{C}$ will also have this property.

But $M$ is simply connected, hence this restriction will be an analytic diffeomorphism. The inverse map $\left(\mu^A|_{\mathcal{C}}\right)^{-1} : M \to \mathcal{C} \subset \mathcal{E}^A$ will define a parallel extension of $s$. ■

## 2. Extension of bundle isomorphisms

We will prove first an extension theorem for locally defined, id-covering bundle isomorphisms intertwining two global connections:

THEOREM 2.1. *Let $p : P \to M$, $p' : P' \to M$ be real analytic principal $K$-bundles over a real analytic manifold $M$, and $A$, $A'$ be analytic connections on $P$, $P'$ respectively. Let $U \subset M$ be a nonempty, connected open set, and $\Phi : P_U \to P'_U$ be an $\mathrm{id}_U$-covering analytic isomorphism such that $\Phi^*(A'_U) = A_U$.*

*If $M$ is simply connected, then $\Phi$ has a unique* id*-covering analytic extension $\tilde{\Phi} : P \to P'$ such that $\Phi^*(A') = A$.*

PROOF. For a point $x \in M$ let $I(P, P')_x$ be the set of all isomorphisms $\psi : P_x \to P'_x$ of right $K$-spaces. Fixing a pair $(y, y') \in P_x \times P'_x$, we obtain a diffeomorphism $I(P, P')_x \simeq K$. The union

$$I(P, P') := \bigcup_{x \in M} I(P, P')_x$$

has a natural manifold structure, and the obvious projection $I(P, P') \to M$ is a locally trivial fibre bundle over $M$ with fiber $K$. More precisely one can identify $I(P, P')$ (as a bundle over $M$) with the associated bundle

$$(P \times_M P') \times_\tau K,$$

where $P \times_M P'$ is regarded as a principal $(K \times K)$-bundle over $M$, and

$$\tau : (K \times K) \times K \to K$$

is the action of $K \times K$ on $K$ defined by

$$\tau((k_1, k_2), k) := k_2 k k_1^{-1}.$$

The pair of connections $(A, A')$ defines a connection $A \times A'$ on the product principal bundle $P \times_M P'$. For a pair $(y, y') \in P \times_M P'$ the horizontal space $(A \times A')_{(y,y')}$ is just

$$\{(v, v') \in A_y \times A'_{y'} |\ p_*(v) = p'_*(v')\}.$$

The data of an $\mathrm{id}_U$-covering morphism $\Phi : P_U \to P'_U$ is equivalent to the data of a section $s^\Phi \in \Gamma(U, I(P, P'))$ (see Proposition 1.6 in the Appendix) . Moreover, by the same proposition, one can prove that $\Phi^*(A') = A$ if and only if the section $s^\Phi$ is $\Gamma^{A \times A'}$-parallel. With this remark the theorem follows from Theorem 1.4. ■

We can treat now the general case of a locally defined bundle morphism covering a globally defined map between the base manifolds:

THEOREM 2.2. *Let $p : P \to M$, $p' : P' \to M'$ be real analytic principal $K$-bundles, and $A$, $A'$ be analytic connections on $P$, $P'$ respectively. Let $\varphi : M \to M'$ be a real analytic map, $U \subset M$ be a non-empty, connected open set, and $\Phi : P_U \to P'$ be a $\varphi$-covering analytic morphism such that $\Phi^*(A') = A_U$.*

*If $M$ is simply connected, then $\Phi$ has a unique $\varphi$-covering analytic extension $\tilde{\Phi} : P \to P'$ for which $\Phi^*(A') = A$.*

PROOF. By Remark 1.4 in section 1.2, the data of a $\varphi$-covering analytic bundle morphism $\phi : P_U \to P'$ for which $\Phi^*(A') = A_U$ is equivalent to the data of an $\mathrm{id}_U$-covering analytic bundle isomorphism $\Phi_0 : P_U \to \varphi^*(P')_U$ for which $\Phi_0^*((\varphi^* A')_U) =$

$A_U$. Here we denoted by $\varphi^*(P')$ the bundle on $M$ obtained as the pull-back of $P'$ via $\varphi : M \to M'$. In other words one has

$$\varphi^*(P') := M \times_{M'} P'$$

regarded as a principal $K$-bundle on $M$ via the projection on the first factor. Similarly $\varphi^*(A')$ stands for the pull-back connection of $A'$ via $\varphi$. The connection form of this pull-back connection is the pull-back of the connection form of $A'$ via the second projection $M \times_{(\varphi,p')} P' \to P'$.

By Theorem 2.1 it follows that there exists an $\mathrm{id}_M$-covering analytic bundle isomorphism $\tilde{\Phi}_0 : P \to \varphi^*(P')$, which extends $\Phi_0$ such that

$$(\tilde{\Phi}_0)^*(\varphi^*(A')) = A.$$

But the data of such an isomorphism is equivalent to the data of a $\varphi$-covering analytic bundle morphism $\Phi : P \to P'$ such that $\Phi^*(A') = A$. ■

## 3. Analytic locally homogeneous triples on compact manifolds

Using our results we can now give a new proof (valid in the real-analytic framework) of our main theorem (Theorem 3.11). More precisely we prove:

THEOREM 3.1. *Let $M$ be a compact real analytic manifold, and $(g, P \xrightarrow{p} M, A)$ be a real analytic locally homogeneous triple on $M$, where $P \xrightarrow{p} M$ is a $K$-principal bundle with $K$ compact. Let $\pi : \tilde{M} \to M$ be the universal cover of $M$, $\Gamma$ be the corresponding covering transformation group, $\tilde{g} := \pi^*(g)$, $q : Q := \pi^*(P) \to \tilde{M}$, and $B := \pi^*(A)$. Then there exists*

1. *A closed subgroup $G \subset \mathrm{Iso}(\tilde{M}, \tilde{g})$ acting transitively on $\tilde{M}$ which leaves invariant the gauge class $[B] \in \mathcal{B}(Q)$ and contains $\Gamma$,*
2. *A lift $\mathrm{j} : \Gamma \to \mathcal{G}^B_G(Q)$ of $\iota_\Gamma : \Gamma \to G$.*

PROOF. Let $G \subset \mathrm{Iso}(\tilde{M}, \tilde{g})$ be the subgroup defined by

$$G := \left\{\psi \in \mathrm{Iso}(\tilde{M}, \tilde{g}) |\ \exists\, \Psi : Q \to Q\ \psi\text{-covering bundle isom.},\ \Psi^*(B) = B\right\}.$$

Using the fact that $K$ is compact, it follows by Lemma 3.12 of Chapter 2 , that $G$ is a closed subgroup of the Lie group $\mathrm{Iso}(\tilde{M}, \tilde{g})$. Note that Lemma 3.12 applies because the action of the Lie group $\mathrm{Iso}(\tilde{M}, \tilde{g})$ on $\tilde{M}$ is smooth. Using Theorem 2.2 one can prove that $G$ acts transitively on $\tilde{M}$ as follows:

We have to show that for any two points $\tilde{x}$, $\tilde{x}' \in \tilde{M}$ there exists $\psi \in G$ such that $\psi(\tilde{x}) = \tilde{x}'$. Put $x := \pi(\tilde{x})$, $x' := \pi(\tilde{x}')$. Since $(g, P \xrightarrow{p} M, A)$ is locally homogeneous, there exists open neighborhoods $U$, $U'$ of $x$ and $x'$, an isometry $\varphi : U \to U'$ with $\varphi(x) = x'$, and a $\varphi$-covering bundle isomorphism $\Phi : P_U \to P_{U'}$ such that $\Phi^*(A_{U'}) = A_U$. We can assume of course that $U$ and $U'$ are simply connected. Let $\tilde{U}$ ($\tilde{U}'$) be the connected component of $\pi^{-1}(U)$ (respectively $\pi^{-1}(U')$) which contains $\tilde{x}$ (respectively $\tilde{x}'$). The map $\pi$ induces diffeomorphisms $\tilde{U} \simeq U$, $\tilde{U}' \simeq U'$, and $\pi$-covering bundle isomorphisms $Q_{\tilde{U}} \simeq P_U$, $Q_{\tilde{U}'} \simeq P_{U'}$. Using these identifications we obtain:

(1) an isometry $\tilde{\varphi} : \tilde{U} \to \tilde{U}'$ with $\tilde{\varphi}(\tilde{x}) = \tilde{x}'$,
(2) a $\tilde{\varphi}$-covering bundle isomorphism $\tilde{\Phi} : Q_{\tilde{U}} \to Q_{\tilde{U}'}$ with the property

$$\tilde{\Phi}^*(B_{\tilde{U}'}) = B_{\tilde{U}}.$$

Since $\tilde{M}$ is simply connected and complete, we obtain a global isometry

$$\psi : \tilde{M} \to \tilde{M}$$

extending $\tilde{\varphi}$ (see [**Si**], [**KN**, Theorem 6.3]). Applying Theorem 2.2 we obtain a $\psi$-covering bundle isomorphism $\Psi : Q \to Q$ such that $\Psi^*(B) = B$. Therefore one has $\psi \in G$. Since $\psi(\tilde{x}) = \tilde{x}'$ transitivity is proved.

Using the same arguments as in the proof of Theorem 3.11, we complete the proof. ∎

CHAPTER 4

# Examples and applications. Moduli spaces of locally homogeneous triples on Riemann surfaces

Many of the results presented here will appear in [**BaTe**].

## 1. Homogeneous connections. The moduli space of Biswas-Teleman

We begin with the following

DEFINITION 1.1. [**Ya**, p. 30] *A pair $(G,H)$ consisting of a Lie group $G$ and a closed subgroup $H\subset G$ is called reductive if $\mathfrak{h}$ has a $\mathrm{ad}_H$-invariant complement in $\mathfrak{g}$.*

Note that any pair $(G,H)$ with $H$ compact is reductive. Let $(G,H)$ be a reductive pair. An $\mathrm{ad}_H$-invariant complement $\mathfrak{s}$ of $\mathfrak{h}$ in $\mathfrak{g}$ defines a left invariant connection on the principal $H$-bundle $G\to G/H$.

Let $(G,K)$ be a pair of Lie groups, and $H$ be a closed subgroup of $G$. Put

$$\mathcal{A}(G,H,K):=\{(\chi,\mu)\in\mathrm{Hom}(H,K)\times\mathrm{Hom}(\mathfrak{g}/\mathfrak{h},\mathfrak{k}):\mu\circ\mathrm{ad}_h=\mathrm{ad}_{\chi(h)}\circ\mu,\forall h\in H\}$$

Following [**BiTe**] we define

$$\mathcal{M}(G,H,K):={}^{\mathcal{A}(G,H,K)}\big/_{K}\,,$$

where $K$ acts on $\mathrm{Hom}(H,K)\times\mathrm{Hom}(\mathfrak{g}/\mathfrak{h},\mathfrak{k})$ by

$$k\cdot(\chi,\mu)=(\iota_k\circ\chi,\mathrm{ad}_k\circ\mu)\,,$$

leaving the real algebraic variety $\mathcal{A}(G,H,K)\subset\mathrm{Hom}(H,K)\times\mathrm{Hom}(\mathfrak{g}/\mathfrak{h},\mathfrak{k})$ invariant. $\mathcal{M}(G,H,K)$ will be called the moduli space associated with $(G,H,K)$.

Let now $N$ be connected differentiable $n$-manifold, $G$ a connected Lie group and $\alpha:G\times N\to N$ be transitive left action of $G$ on $N$. Fix $x_0\in N$, and let $H\subset G$ be the stabiliser of $x_0$ in $G$. The map $g\mapsto gx_0$ induces a diffeomorphism $G/H\to N$, and $G$ can be regarded as a principal $H$-bundle on $N$.

We will denote by $\alpha_g:N\to N$ the diffeomorphism associated with $g\in G$. Let $K$ be a Lie group. We are interested in the classification (up to isomorphism in the obvious sense) of triples $(P,A,\beta)$, where $P$ is principal $K$-bundle on $N$, $A$ a connection on $P$, and $\beta:G\times P\to P$ a $G$-action on $P$ by $K$-bundle isomorphisms, which covers $\alpha$ and is compatible with $A$. In other words $\beta$ is a $G$ action on $P$ such that

(1) For any $g\in G$, the associated diffeomorphism $\beta_g$ is an $\alpha_g$-covering bundle isomorphism.
(2) For any $g\in G$, $\beta_g$ leaves the connection $A$ invariant, i.e. $\beta_g^*A=A$.

DEFINITION 1.2. [**BiTe**] *A triple $(P,A,\beta)$ satisfying the above conditions will be called a $G$-invariant (or $G$-homogeneous) $K$-connection on $N$.*

The group $G$ acts on the set of all gauge classes of $K$-connections (on principal $K$-bundles) on $N$ **[BiTe]**. If $(P, A, \beta)$ is $G$-homogeneous $K$-connection, then the gauge class $[A]$ is $G$-invariant, but, in general, the $G$-invariance of $[A]$ does not imply the existence of an action $\beta$ for which $(P, A, \beta)$ is $G$-homogeneous. The classification of $G$-homogeneous $K$-connections is substantially more difficult than the classification of $G$-invariant gauge classes.

We will denote by $\Phi_{\alpha,K}$ the set of isomorphism classes of $G$-homogeneous $K$-connections on $N$.

THEOREM 1.3. *Suppose that $\alpha : G \times N \to N$ is a transitive action of $G$ on $N$. Fix $x_0 \in N$, and let $H \subset G$ be its stabiliser. Suppose that the pair $(G, H)$ is reductive. Then there exists a natural identification $\mathcal{M}(G, H, K) \to \Phi_{\alpha,K}$.*

The identification given by Theorem 1.3 is explicit. A pair $(\chi, \mu) \in \mathcal{A}(G, H, K)$ defines a principal $K$-bundle $P_\chi$ endowed with an $\alpha$-covering $G$-action by bundle isomorphisms, and an invariant connection $A_{\chi,\mu}$ on this bundle. $P_\chi$ is just the associated bundle $G \times_\chi K$. This bundle comes with a distinguished point $y_0 := [e_G, e_K]$. Identifying the tangent space $T_{x_0}N$ with $\mathfrak{s}$ one has an explicit formula for the curvature form $F_{A_{\chi,\mu}}$ at the point $y_0 \in P_\chi$:

$$F_{\chi,\mu}(u, v) = -\chi_*([u, v]^{\mathfrak{h}}) + [\mu(u), \mu(v)] - \mu([u, v]) \,. \tag{42}$$

where $u,\ v \in T_{y_0}(P_\chi)$. Since $A_{\chi,\mu}$ is $G$-invariant, this formula determines the curvature of $A_{\chi,\mu}$ at any point.

**1.1. Examples: $\mathrm{PSL}(2, \mathbb{R})$-homogeneous connections on the hyperbolic plane.** Let $\mathbb{H}$ be the hyperbolic plane. We recall that the group of orientation preserving isometries of $\mathbb{H}$ can be identified with $G := \mathrm{PSL}(2, \mathbb{R})$. Choosing a base point $x_0 \in \mathbb{H}$ in a appropriate way, the stabiliser $H = G_{x_0}$ will coincide with the image of $\mathrm{SO}(2)$ in $\mathrm{PSL}(2, \mathbb{R})$ (which is the quotient $\mathrm{SO}(2, \mathbb{R})$ by $\{\pm I_2\}$). Therefore

$$H = \left\{ h_t := \left[\begin{pmatrix} \cos(t/2) & -\sin(t/2) \\ \sin(t/2) & \cos(t/2) \end{pmatrix}\right] \middle|\ t \in [0, 2\pi] \right\} \subset \mathrm{PSL}(2, \mathbb{R}) \,. \tag{43}$$

Identifying $\mathfrak{g}$ with $\mathrm{sl}(2, \mathbb{R})$, we have $\mathfrak{h} = \mathbb{R}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and an $\mathrm{ad}_H$-invariant complement of $\mathfrak{h}$ in $\mathfrak{g}$ is

$$\mathfrak{s} := \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle .$$

Put

$$a := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ b := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ c := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

We have

$$[a, b] = 2c,\ [a, c] = -2b,\ [b, c] = -2a \,.$$

The action $\mathrm{ad}_H$ on $\mathfrak{s}$ is given by

$$\mathrm{ad}_{h_t}(ub + vc) = (b, c) \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}\ \forall \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2 \,,$$

so $H$ acts on $\mathfrak{s}$ by rotations. Let $K$ be Lie group. Identifying $H$ with $\mathrm{S}^1$ via the map $h_t \mapsto e^{it}$, we obtain an identification

$$\mathcal{A}(\mathrm{PSL}(2, \mathbb{R}), H, K) = \{(\chi, \mu) \in \mathrm{Hom}(\mathrm{S}^1, K) \times \mathrm{Hom}_{\mathbb{R}}(\mathfrak{s}, \mathfrak{k}) |\ \mu \in \mathrm{Hom}^{\chi}_{\mathrm{S}^1}(\mathfrak{s}, \mathfrak{k})\} \,, \tag{44}$$

where

$$\mathrm{Hom}^{\chi}_{\mathrm{S}^1}(\mathfrak{s},\mathfrak{k}) := \{\mu \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{s},\mathfrak{k}) |\ \mu\circ R_\zeta = \mathrm{ad}_{\chi(\zeta)}\mu\ \forall \zeta\in \mathrm{S}^1\}\,.$$

In this formula we used the notation

$$R_{e^{it}}(ub+vc) = \mathrm{ad}_{h_t}(ub+vc) = (b,c)\begin{pmatrix}\cos(t) & -\sin(t)\\ \sin(t) & \cos(t)\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}\,.$$

We identify $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{s},\mathfrak{k})$ with the complexification $\mathfrak{k}^{\mathbb{C}}$ of $\mathfrak{k}$ using the isomorphism $I$ : $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{s},\mathfrak{k}) \to \mathfrak{k}^{\mathbb{C}}$ given by $I(\mu)=\mu(c)+i\mu(b)$. Using the identity $I(\mu\circ R_\zeta)=\zeta I(\mu)$ and putting

$$\mathfrak{k}^{\mathbb{C}}_{\chi} := \{Z\in \mathfrak{k}^{\mathbb{C}} |\ \mathrm{ad}_{\chi(\zeta)}(Z)=\zeta Z,\ \forall \zeta\in\mathrm{S}^1\}$$

(the maximal complex linear subspace of $\mathfrak{k}^{\mathbb{C}}$ on which $\mathrm{S}^1$ acts with weight 1 via $\mathrm{ad}_\chi$), we obtain a further identification

$$\mathcal{A}(\mathrm{PSL}(2,\mathbb{R}),H,K) = \{(\chi,Z)\in \mathrm{Hom}(\mathrm{S}^1,K)\times \mathfrak{k}^{\mathbb{C}} |\ Z\in \mathfrak{k}^{\mathbb{C}}_{\chi}\}\,. \tag{45}$$

For $k\in\mathbb{Z}$ let $\chi_k:\mathrm{S}^1\to\mathrm{S}^1$ be the group morphism defined by $\chi_k(\zeta)=\zeta^k$, and let

$$\tau:\mathrm{S}^1\to\mathrm{SU}(2),\ \theta:\mathrm{S}^1\to\mathrm{PU}(2)=\mathrm{SU}(2)/\{\pm I_2\}$$

be the standard monomorphisms $\mathrm{S}^1\to\mathrm{SU}(2)$, $\mathrm{S}^1\to\mathrm{PU}(2)$ given respectively by:

$$\tau(\zeta) := \begin{pmatrix}\zeta & 0\\ 0 & \bar\zeta\end{pmatrix},\ \theta(e^{it}) = [\tau(e^{it/2})]\,.$$

The image of $\tau$ ($\theta$) is a maximal torus of $\mathrm{SU}(2)$ (respectively of $\mathrm{PU}(2)$)), and any monomorphism $\mathrm{S}^1\to\mathrm{SU}(2)$ ($\mathrm{S}^1\to\mathrm{PU}(2)$) is equivalent with $\tau$ ($\theta$) modulo an interior automorphism of $\mathrm{SU}(2)$ (respectively $\mathrm{PU}(2)$). Moreover, any morphism $\chi:\mathrm{S}^1\to\mathrm{SU}(2)$ ($\chi:\mathrm{S}^1\to\mathrm{PU}(2)$) is equivalent with a morphism of the form $\tau_k := \tau\circ\chi_k$ (respectively $\theta_k:=\theta\circ\chi_k$) with $k\in\mathbb{N}$ (modulo an interior automorphism). Using these remarks we obtain

$$\mathcal{M}(\mathrm{PSL}(2,\mathbb{R}),H,\mathrm{S}^1) = \{[\chi_k,0] |\ k\in\mathbb{Z}\}\,. \tag{46}$$

It is important to describe explicitly the homogeneous $\mathrm{PSL}(2,\mathbb{R})$-connection which corresponds to the class $[\chi_k,0]$. For $k=1$ the bundle $P_{\chi_1}$ can be identified with the $\mathrm{S}^1$-bundle $\mathrm{PSL}(2,\mathbb{R})\to\mathrm{PSL}(2,\mathbb{R})/H=\mathbb{H}$, which is the principal bundle of unit length tangent vectors of $\mathbb{H}$ or, equivalently, the principal bundle associated with the oriented Euclidian 2-bundle $T_{\mathbb{H}}$. The connection $A_{\chi_1,0}$ coincides with Levi-Civita connection $C_{\mathbb{H}}$ of this principal bundle [**BaTe**], which is obviously $\mathrm{PSL}(2,\mathbb{R})$-homogeneous. The connection $A_{\chi_k,0}$ can be identified with the tensor power $C_{\mathbb{H}}^{\otimes k}$ of this $\mathrm{S}^1$-connection.

REMARK 1.4. Let $\Gamma\subset\mathrm{PSL}(2)$ be a discrete subgroup acting properly discontinuously on $\mathbb{H}$ with compact quotient, and let $(M,g_M)$ be the hyperbolic Riemann surface $M:=\mathbb{H}/\Gamma$. The $\Gamma$-quotient of $(P_{\chi_1},C_{\mathbb{H}})$ can be identified with $(P_M,C_M)$ where $P_M$ is $\mathrm{SO}(2)$-frame bundle of $M$ (regarded as a principal $\mathrm{S}^1$-bundle), and $C_M$ is the Levi-Civita connection of the hyperbolic Riemann surface $M$. Similarly the $\Gamma$-quotient of $(P_{\chi_k},A_{\chi_k,0})$ can be identified with the tensor power $(P_M^{\otimes k},C_M^{\otimes k})$. The Chern class of $P_M^{\otimes k}$ is $k(2-2g)$.

In this way (using quotients of $\mathrm{PSL}(2,\mathbb{R})$-homogeneous $\mathrm{S}^1$-connections on $\mathbb{H}$) we obtain only very special locally homogeneous $\mathrm{S}^1$-triples on $M$: only the tensor powers of the *canonical* locally homogeneous triple $(g_M,P_M,C_M)$ given by the Levi-Civita connection on $M$. In the next section we will see that any $\mathrm{S}^1$-triple

$(g_M, P, A)$ with $A$ Yang-Mills is locally homogeneous, and we will show how all these Yang-Mills triples can be obtained as quotients of homogeneous connections on $\mathbb{H}$. We will have to replace the group $\mathrm{PSL}(2,\mathbb{R})$ by an $\mathrm{S}^1$-extension of it (see Remark 3.2 in Section 3 Chapter 1).

In a similar way we obtain:

$$\mathcal{M}(\mathrm{PSL}(2,\mathbb{R}), H, \mathrm{SU}(2)) = \{[\tau_k, 0] |\ k \in \mathbb{N}\}\,.$$

This shows that any $\mathrm{PSL}(2,\mathbb{R})$-homogeneous $\mathrm{SU}(2)$-connection on $\mathrm{SU}(2)$ has a $\mathrm{PSL}(2,\mathbb{R})$-homogeneous $\mathrm{S}^1$-reduction, so it coincides with the $\mathrm{SU}(2)$-extension of a $\mathrm{PSL}(2,\mathbb{R})$-homogeneous $\mathrm{S}^1$-connection.

The case $K = \mathrm{PU}(2)$ is more interesting: for $k \in \mathbb{N}\setminus\{1\}$ one has $\mathrm{Hom}^{\theta_k}_{\mathrm{S}^1}(\mathbb{C}, \mathfrak{k}) = \{0\}$, but

$$\mathrm{Hom}^{\theta_1}_{\mathrm{S}^1}(\mathfrak{s}, \mathfrak{k}) = \mathbb{C}\mu_0\,, \tag{47}$$

where $\mu_0 : \mathfrak{s} \to \mathrm{su}(2)$ is given by

$$\mu_0(ub + vc) := \begin{pmatrix} 0 & u + iv \\ -(u - iv) & 0 \end{pmatrix}\,.$$

In (47) we have used the complex structure on $\mathrm{Hom}(\mathfrak{s}, \mathrm{su}(2))$ given by the isomorphism $I : \mathrm{Hom}(\mathfrak{s}, \mathrm{su}(2)) \to \mathrm{su}(2)^{\mathbb{C}}$ introduced above. The centraliser of $\theta_1$ is $\mathrm{im}(\theta_1) \simeq \mathrm{S}^1$, and it acts with weight 1 on the complex line $\mathrm{Hom}^{\theta_1}_{\mathrm{S}^1}(\mathfrak{s}, \mathfrak{k})$. This shows that

$$\mathcal{M}(\mathrm{PSL}(2,\mathbb{R}), H, \mathrm{PU}(2)) = \{[\theta_k, 0] |\ k \in \mathbb{N}\} \cup \mathcal{M}_1\,,$$

where

$$\mathcal{M}_1 = \{[\theta_1, r\mu_0] |\ r \in (0,\infty)\} \simeq (0,\infty)\,.$$

The curvature of the connection $A_{\theta_1, r\mu_0}$ can be computed using formula (42), and the result is

$$F_{\theta_1, r\mu_0}(b, c) = (1 + r^2)\begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}. \tag{48}$$

## 2. Locally homogeneous $\mathrm{S}^1$ and $\mathrm{PU}(2)$-triples on hyperbolic Riemann surfaces

We start with a simple consequence of Theorem 3.9 of Chapter 2.

PROPOSITION 2.1. *Let $(M, g)$ be a connected, oriented, compact Riemann surface endowed with a Riemannian metric with constant curvature, let $P$ a principal $\mathrm{S}^1$-bundle on $M$ and $A$ a connection on $M$. Then $(g, P, A)$ is locally homogeneous if and only if $A$ is Yang-Mills.*

PROOF. Recall that, by definition, an $\mathrm{S}^1$-connection $A$ is Yang-Mills if and only if its curvature $F_A \in iA^2(M)$ is harmonic [**DK**], [**Te**]. Writing $F_A = i f_A \mathrm{vol}_g$, with $f_A \in \mathcal{C}^\infty(M, \mathbb{R})$, we see that $A$ is Yang-Mills if and only of $f_A$ is constant. Therefore, if $(g, P, A)$ is locally homogeneous we see (using the fact that $\mathrm{vol}_g$ is invariant under isometries up to sign) that $|f_A|$ is constant, so $f_A$ is constant, so $A$ is Yang-Mills. Conversely if $f_A$ is constant, then $F_A$ is $\nabla^{C_0} \otimes \nabla^A$-parallel, so $(g, P, A)$ is infinitesimally homogeneous, so locally homogeneous by Theorem 3.9 of Chapter 2. ■

This shows that, for a fixed integer $k \in \mathbb{Z}$, the set of isomorphism classes of locally homogeneous triples $(g, P, A)$ with $c_1(P) = k$ can be identified with the moduli space of Yang-Mills connections on a Hermitian line bundle of Chern class $k$, which is a torus of dimension $2g$ (see [**Te**]).

The Yang-Mills $\mathrm{S}^1$-connections on hyperbolic Riemann surfaces can be obtained as quotients of homogeneous $\mathrm{S}^1$-connections on $\mathbb{H}$ as follows (see [**BaTe**]):

Let $q : \widetilde{\mathrm{PSL}(2,\mathbb{R})} \to \mathrm{PSL}(2,\mathbb{R})$ be the universal cover of $\mathrm{PSL}(2,\mathbb{R})$, and let $\gamma_0$ be the generator of $\ker(q) = \pi_1(\mathrm{PSL}(2,\mathbb{R}))$ given by $[0, 2\pi] \ni t \mapsto h_t$ (see (43)). For any $a \in \mathbb{R}$ define the group $\hat{G}_a$ by

$$\hat{G}_a := \widetilde{\mathrm{PSL}(2,\mathbb{R})} \times \mathrm{S}^1 \Big/ \{(\gamma_0^k, e^{-2\pi a k i}) |\ k \in \mathbb{Z}\} .$$

We obtain a short exact sequence

$$1 \to \mathrm{S}^1 \xrightarrow{j_a} \hat{G}_a \xrightarrow{q_a} \mathrm{PSL}(2,\mathbb{R}) \to 1 ,$$

where $j_a(z) = [e, z]$, $q_a([\gamma, \zeta]) := q(\gamma)$. The pre-image $\tilde{H} := q^{-1}(H) \subset \widetilde{\mathrm{PSL}(2,\mathbb{R})}$ can be identified with the universal cover of $H$, and is isomorphic to $\mathbb{R}$. We write this group as $\tilde{H} = \{\tilde{h}_t |\ t \in \mathbb{R}\}$, where $t \mapsto \tilde{h}_t$ is a group isomorphism, and $\tilde{h}_t$ is a lift of $h_t$. Therefore $\gamma_0 = \tilde{h}_{2\pi}$. The pre-image $\hat{H}_a := q_a^{-1}(H)$ can be identified with the quotient

$$\tilde{H} \times \mathrm{S}^1 \Big/ \{(\gamma_0^k, e^{-2\pi a k i}) |\ k \in \mathbb{Z}\} .$$

This group is abelian, and fits in the short exact sequence

$$1 \to \mathrm{S}^1 \xrightarrow{i_a} \hat{H}_a \xrightarrow{p_a} H \to 1 ,$$

where $i_a$, $p_a$ are defined as $j_a$, $q_a$. This short exact sequence has a left splitting $\chi_a : \hat{H}_a \to \mathrm{S}^1$ given by

$$\chi_a([\tilde{h}_t, \zeta]) := e^{ita}\zeta ,$$

so $\hat{H}_a \simeq \mathrm{S}^1 \times \mathrm{S}^1$. We endow $\mathbb{H}$ with the action $\alpha_a : \hat{G}_a \times \mathbb{H} \to \mathbb{H}$ induced by the standard $\mathrm{PSL}(2,\mathbb{R})$-action via $q_a$. The stabiliser of $x_0$ is the group $q_a^{-1}(H) = \hat{H}_a$. According to the general Theorem 1.3, the pair $(\chi_a, 0)$ defines a $\hat{G}_a$-homogeneous connection $(P_{\chi_a}, A_{\chi_a,0}, \beta_a)$. We refer to [**BaTe**] for the following result, which describes all Yang-Mills $\mathrm{S}^1$-connections on hyperbolic Riemann surfaces as quotients of homogeneous connections on $\mathbb{H}$.

PROPOSITION 2.2. *Let $\Gamma \subset \mathrm{PSL}(2)$ be a discrete subgroup acting properly discontinuously on $\mathbb{H}$ with compact quotient, and let $(M, g_M)$ be the hyperbolic Riemann surface $M := \mathbb{H}/\Gamma$. Let $P$ be a principal $\mathrm{S}^1$-bundle on $M$, and $A$ be a Yang-Mills connection on $P$. Put $a := \frac{c_1(P)}{2-2g}$. There exists a lift $\mathfrak{j} : \Gamma \to \hat{G}_a$ of the embedding monomorphism $\Gamma \hookrightarrow \mathrm{PSL}(2,\mathbb{R})$, and an isomorphism*

$$(P, A) \simeq (P_{\chi_a}, A_{\chi_a,0}) \big/ \Gamma ,$$

*where $\Gamma$ acts on $P_{\chi_a}$ via $\mathfrak{j}$.*

Note that, by Proposition 2.1 and Remark 3.15, this result describes all locally homogeneous $\mathrm{S}^1$-triples on hyperbolic Riemann surfaces as quotients of of homogeneous connections on $\mathbb{H}$ on hyperbolic Riemann surfaces.

In a similar way we obtain the classification of all locally homogeneous $\mathrm{PU}(2)$-triples [**BaTe**]:

PROPOSITION 2.3. *Let $\Gamma \subset \mathrm{PSL}(2)$ be a discrete subgroup acting properly discontinuously on $\mathbb{H}$ with compact quotient, and let $(M, g_M)$ be the hyperbolic Riemann surface $M := \mathbb{H}/\Gamma$. Let $P$ be a principal $\mathrm{PU}(2)$-bundle on $M$, and $A$ be a locally homogeneous connection on $P$. Let $B$ be the pull-back connection on the pull-back bundle $Q$ on $\mathbb{H}$. Then*

*(1) If $B$ is irreducible (i.e. it has trivial stabiliser), then there exists $r > 0$ and an isomorphism between $(P, A)$ and the $\Gamma$-quotient of $(P_{\theta_1}, A_{\theta_1, r\mu_0})$.*
*(2) If the stabiliser of $B$ is $\mathrm{S}^1$, then $(g_M, P, A)$ can be identified with the $\mathrm{PU}(2)$-extension of a locally homogeneous $\mathrm{S}^1$-triple.*
*(3) If the stabiliser of $B$ is $\mathrm{PU}(2)$, then $A$ is flat, so the pair $(P, A)$ is defined by a representation $\pi_1(M) = \Gamma \to \mathrm{PU}(2)$.*

CHAPTER 5

# Appendix

## 1. Connections in associated bundles. Bundle isomorphisms

**1.1. Parallel transport in associated bundles.** Let $M$ be a differentiable $n$-manifold, $K$ be a Lie group, and $p : P \to M$ be a principal $K$-bundle on $M$, endowed with a connection $A$. Let $\alpha : [t_0, t_1] \to M$ be a smooth curve in $M$. By a well known, fundamental theorem in Differential Geometry it follows that any point $u_0 \in P_{\alpha(t_0)} := p^{-1}(\alpha(t_0))$ there is a unique horizontal lift $\alpha_{u_0} : [t_0, t_1] \to P$ such that $\alpha_{u_0}(t_0) = u_0$ [**KN**, Proposition 3.1].

We also have an existence and unicity horizontal lift theorem for associated bundles. More precisely, let $p : P \to M$ be a principal $K$-bundle endowed with a connection $A$, $\lambda : K \times F \to F$ be a smooth action of $K$ on a differentiable manifold $F$, and let $E := P \times_\lambda F$ be the associated bundle with fibre $F$. The connection $A$ is a horizontal rank $n$-distribution on $P$, which defines in the obvious way a rank $n$-distribution on $P \times F$, whose projection on $E = (P \times F)/K$ is a well defined horizontal rank $n$-distribution on $E$. This distribution will be called the connection on $E$ induced by $A$, and will be denoted by $\Gamma^A$. With this definitions we have:

PROPOSITION 1.1. *Let $p : P \to M$ be a principal $K$-bundle endowed with a connection $A$, $\lambda : K \times F \to F$ be a smooth action of $K$ on a manifold $F$, and let $E := P \times_\lambda F$ be the associated bundle with fibre $F$. Let $\alpha : [t_0, t_1] \to M$ be a smooth curve in $M$. For any point $e_0 \in E_{\alpha(t_0)}$ there is a unique $\Gamma^A$-horizontal lift $\alpha_{e_0} : [t_0, t_1] \to E$ such that $\alpha_{e_0}(t_0) = e_0$.*

A proof of this results is sketched in [**KN**, p. 88]. In this section we give a detailed proof for completeness.

PROOF. The Lie group $K$ acts freely on $P \times F$ from the right by

$$(u, y, k) \mapsto (uk, k^{-1}y).$$

By definition the associated bundle $E := P \times_\lambda F$ is the quotient manifold $(P \times F)/K$, and the projection $q : P \times F \to E$ is principal $K$-bundle, which comes with a natural connection $B$ given by

$$B_{(u,y)} := A_u \times T_y F \ \forall (u, y) \in P \times F.$$

The bundle projection $p : P \to M$ induces a locally trivial submersion $p_E : E \to M$, such that the following diagram is commutative:

$$\begin{tikzcd} P \times F \arrow[r, "q"] \arrow[d, "p_1"'] & E \arrow[d, "p_E"] \\ P \arrow[r, "p"] & M \end{tikzcd} .$$

In other words the quotient map $q : P \times F \to E$ is fiber preserving map over $M$. For a pair $(u,y) \in P \times F$ put $[u,y] := q(u,y)$. With this notation the map $p_E : E \to M$ is given by $p_E([u,y]) := p(u)$. By the definition of $\Gamma^A$, for a point $e := [u,y] \in E$ we have

$$\Gamma^A_e := q_{*(u,y)}(A_u \times \{0_y\})$$

where $\{0_y\}$ is the zero subspace of $T_yF$. Put $x_0 := \alpha(t_0)$, and let $e_0 = [u_0, y_0]$ be a point in the fiber $E_{x_0}$. It follows $u_0 \in p^{-1}(x_0)$.

By the classical horizontal lift theorem [**KN**, Proposition 3.1], there exists a $A$-horizontal lift $\alpha_{u_0} : [t_0,t_1] \to P$. Therefore for every $t \in [t_0,t_1]$ we have $\alpha'_{u_0}(t) \in A_{\alpha_{u_0}(t)}$. It suffices to note that the curve $\alpha_{e_0} : [t_0,t_1] \to E$ defined by

$$\alpha_{e_0}(t) := [\alpha_{u_0}(t), y_0]$$

is a $\Gamma^A$-horizontal lift of $\alpha$ satisfying the initial condition $\alpha_{e_0}(t_0) = e_0$.

To prove unicity, let $\beta : [t_0,t_1] \to E$ be a $\Gamma^A$-horizontal lift of $\alpha$ with $\beta(t_0) = e_0$, and let

$$\beta_{(u_0,y_0)} = (\beta^P_{(u_0,y_0)}, \beta^F_{(u_0,y_0)}) : [t_0,t_1] \to P \times F$$

be a $B$-horizontal lift of $\beta$ satisfying the initial condition $\beta_{(u_0,y_0)}(t_0) = (u_0,y_0)$. Since $\beta_{(u_0,y_0)}$ is a $B$-horizontal curve, it follows that

$$\beta'_{(u_0,y_0)}(t) \in B_{\beta_{(u_0,y_0)}(t)} \ \forall t \in [t_0,t_1], \tag{49}$$

and, since $\beta_{(u_0,y_0)}(t_0) = (u_0,y_0)$ is a lift of a $\Gamma^A$-horizontal curve, it follows that

$$\beta'_{(u_0,y_0)}(t) \in \left\{(q_*)^{-1}(\Gamma^A)\right\}_{\beta_{(u_0,y_0)}(t)} \ \forall t \in [t_0,t_1]. \tag{50}$$

But it is easy to show that the intersection $B \cap (q_*)^{-1}(\Gamma_A)$ of the distributions $B$, $(q_*)^{-1}(\Gamma^A) \subset T_{P\times F}$ coincides with the distribution $\tilde{A} \subset T_{P\times F}$ given by

$$\tilde{A}_{(u,y)} := A_u \times \{0_y\}.$$

Therefore, by (49), (50) it follows that $\beta^P_{(u_0,y_0)}$ is $A$-horizontal, and $\beta^F_{(u_0,y_0)}$ is constant. Using the unicity part of [**KN**, Proposition 3.1], we obtain $\beta^P_{(u_0,y_0)} = \alpha_{u_0}$, hence $\beta_{(u_0,y_0)}(t) = (\alpha_{u_0}(t), y_0)$, hence $\beta_{(u_0,y_0)} = \alpha_{e_0}$.

■

REMARK 1.2. A similar theorem holds for general fiber bundles endowed with a connection, but one has to assume that the fiber is compact [**Mo**].

**1.2. Pull back bundles and pull back connections.** Let $M$ and $M'$ be smooth manifolds, $K$ a Lie group, and $\varphi : M \to M'$ be a smooth map. Let $p' : P' \to M'$ be a principal $K$-bundle on $M'$. Recall that the pull-back bundle $\varphi^*(P')$ is defined by

$$\varphi^*(P') := M \times_{M'} P' = (\varphi \times p')^{-1}(\Delta_{M'}),$$

where $\varphi \times p' : M \times P' \to M' \times M'$ denotes the product map, and $\Delta_{M'} \subset M' \times M'$ is the diagonal submanifold. One can check that $\varphi \times p'$ is transversal to $\Delta_{M'}$, hence $M \times_{M'} P'$ is a submanifold of $M \times P'$, whose tangent space at a point $(x, y')$ is

$$T_{(x,y')}(M \times_{M'} P') = \{(u,w) \in TM \times TP' \mid \varphi_*(u) = p'_*(w)\}.$$

Note that set theoretically one has

$$\varphi^*(P') = \bigsqcup_{x \in M} P'_{\varphi(x)}.$$

The projections $p_1 : \varphi^*(P') \to M'$, $p_2 : \varphi^*(P') \to P'$ fit in the commutative diagram

$$\begin{tikzcd} \varphi^*(P') \arrow[r, "p_2"] \arrow[d, "p_1"'] & P' \arrow[d, "p'"'] \\ M \arrow[r, "\varphi"] & M' \end{tikzcd}$$

in which $p_1$ is a $K$-principal bundle on $M$ with respect to the right $K$-action induced from $P'$.

Let $A'$ be a connection on $P'$ and $\omega^{A'} \in A^1(P, \mathfrak{k})$ its connection form. By definition the pull-back connection $\varphi^*(A')$ is the connection on $\varphi^*(P')$ defined by the connection form $p_2^*(\omega^{A'})$. With this definition we have for any pair $(x, y') \in \varphi^*(P')$:

$$\begin{aligned} \varphi^*(A')_{(x,y')} &= \{(u,w) \in T_xM \times T_{y'}(P) |\ \varphi_{*x}(u) = p'_{*y'}(w),\ w \in A'_{y'}\} \\ &= \{(u,w) \in T_xM \times A'_{y'} |\ \varphi_{*x}(u) = p'_{*y'}(w)\}. \end{aligned}$$

DEFINITION 1.3. *Let $M$, $M'$ be smooth manifolds, $K$ a Lie group, and $\varphi : M \to M'$ be a smooth map. Let $P \xrightarrow{p} M$, $P' \xrightarrow{p'} M'$ be principal $K$-bundles on $M$, $M'$ respectively. We denote by* $\mathrm{Hom}_\varphi(P, P')$ *the set of $\varphi$-covering bundle morphisms $P \to P'$ which are compatible with the group morphism* $\mathrm{id}_K$ [**KN**, section I.5].

For a morphism $\Phi \in \mathrm{Hom}_\varphi(P, P')$ and a connection $A'$ on $P'$ one defines the pull-back connection $\Phi^*(A')$ using the connection form $\Phi^*(\omega^{A'})$ [**KN**, Proposition 6.2].

REMARK 1.4. With the notations above the following holds:

(1) The map $\mathrm{Hom}_\varphi(P, P') \to \mathrm{Hom}_{\mathrm{id}_M}(P, \varphi^*(P'))$ given by $\Phi \mapsto \Phi_0$, where
$$\Phi_0(y) := (p(y), \Phi(y))$$
is bijective.

(2) For any connection $A'$ on $P'$ and bundle morphism $\Phi \in \mathrm{Hom}_\varphi(P, P')$ one has
$$\Phi^*(A') = \Phi_0^*(\varphi^*(A')).$$

**1.3. Bundle isomorphisms compatible with a pair of connections.** Let $P$, $P'$ be $K$-principal bundles over a manifold $M$, and let $(A, A') \in \mathcal{A}(P) \times \mathcal{A}(P')$ be a pair of connections. The goal of this section is the proof of Proposition 1.6, which shows that the data of an $\mathrm{id}$-covering bundle morphism $\Phi : P \to P'$ with the property $\Phi^*(A') = A$ is equivalent to the data of an $(A, A')$-parallel section in a bundle $I(P, P')$ associated with the fibre product $P \times_M P'$ and the action $\tau : (K \times K) \times K \to K$ given by $((k_1, k_2), k) \mapsto k_2 k k_1^{-1}$ (see section 2). The first part of the following proposition is well-known. The second part can be checked easily.

PROPOSITION 1.5. *Let $p : P \to M$ be a principal $K$-bundle, $\alpha : K \times F \to F$ be a smooth left action of $K$ on a manifold $F$, and $E := P \times_K F$ be the associated fiber bundle with fiber $F$. Denote by $\mathcal{C}^K(P, F)$ the space of smooth $K$-equivariant maps $P \to F$:*

$$\mathcal{C}^K(P, F) := \{\sigma : P \to F |\ \sigma(yk) = k^{-1}\sigma(y)\ \forall y \in P,\ \forall k \in K\}.$$

*Then*

(1) *The map* $\mathcal{F}:\mathcal{C}^K(P,F)\to\Gamma(M,E)$ *given by*

$$\sigma\mapsto s_\sigma,\ \ s_\sigma(x):=[y,\sigma(y)],\ \ \textit{where } y\in P_x,$$

*is bijective.*

(2) *Let* $\sigma\in\mathcal{C}^K(P,F)$, $A$ *be a connection on* $P$, *and* $\Gamma^A$ *be the induced connection on* $E$. *The following conditions are equivalent:*

(i) *The section* $s_\sigma$ *is* $\Gamma^A$*-parallel.*

(ii) *The restriction of* $\sigma_{*y}$ *to the horizontal distribution of* $A$ *vanishes.*

Proof. (1) This result is well known. We mention only that the inverse of $\mathcal{F}$ is the map $s\mapsto\sigma^s$ where, for a section $s\in\Gamma(E)$, the equivariant map $\sigma^s$ is defined by the identity

$$s(p(y))=[y,\sigma^s(y)],\forall y\in P. \tag{51}$$

(2) Let $q:P\times F\to E$ be the quotient map. By (51) we know that

$$s_\sigma(x)=[y,\sigma(y)]=q(y,\sigma(y)). \tag{52}$$

Let $v\in T_xM$, $y\in p^{-1}(x)$, and let $w$ be a lift of $v$ in $T_yP$. Using (52) we obtain easily:

$$s_{*x}(v)=q_{*(y,\sigma(y))}(w,\sigma_{*y}(w)). \tag{53}$$

Recall that, by the definition of $\Gamma^A$, we have $q_{*(y,\sigma(y))}(A_y\times\{0\})=\Gamma^A_{s(x)}$. Therefore, if $\sigma_{*\,|A}=0$, then, choosing $w$ to be the horizontal lift of $v$ at $y$, we get $s_{*x}(v)\in\Gamma^A_{s(x)}$. Conversely, supposing that for every $v\in T_xM$ we have $s_{*x}(v)\in\Gamma^A_{s(x)}$, we show that $\sigma_{*\,|A}=0$. Let $y\in P$, $w\in A_y$, and $v=p_*(w)$. Using again (52) we see that

$$s_*(v)=q_*(w,\sigma_*(w))\in q_*(A_y\times\{0\}).$$

Therefore, there exists $u\in A_y$ such that

$$(w-u,\sigma_*(w))\in\ker(q_{*(y,\sigma(y))}).$$

The projection on the first factor induces an isomorphism $\ker(q_{*(y,\sigma(y))})\to V_y$, where $V_y\subset T_yP$ denotes the vertical tangent space at $y$. Taking into account that $w-u\in A_y$, we get $w-u=0$, and $\sigma_*(w)=0$. ■

Proposition 1.6. *Let* $p:P\to M$, $p':P'\to M$ *be* $K$*-principal bundles over* $M$, *and let* $I(P,P')$ *be the associated bundle* $(P\times_M P')\times_\tau K$. *There exist a natural bijection* $S:\mathrm{Hom}_{\mathrm{id}}(P,P')\to\Gamma(M,I(P,P'))$ *between the space of* id*-covering* $K$*-bundle isomorphisms and the space of sections* $\Gamma(M,I(P,P'))$ *with the following property: For any pair of connections* $(A,A')\in\mathcal{A}(P)\times\mathcal{A}(P')$ *the following conditions are equivalent:*

(i) $\Phi^*(A')=A$.

(ii) $S(\Phi)$ *is* $\Gamma^{A\times A'}$*-parallel.*

Proof. By Proposition 1.5 the space of sections $\Gamma(M,I(P,P'))$ is naturally identified with the space $\mathcal{C}^{K\times K}(P\times_M P',K)$ of $(K\times K)$-equivariant maps $P\times_M P'\to K$. Using this identification we will define a natural bijection

$$S:\mathrm{Hom}_{\mathrm{id}}(P,P')\to\mathcal{C}^{K\times K}(P\times_M P',K).$$

Let $\Phi\in\mathrm{Hom}_{\mathrm{id}}(P,P')$, and $(y,y')\in P\times_M P'$. Since $\Phi(y)$ and $y'$ are in the same fiber, there exists a unique element $\sigma^\Phi(y,y')\in K$ such that $\Phi(y)=y'.\sigma^\Phi(y,y')$. It is easy to see that the map $P\times_M P'\ni(y,y')\mapsto\sigma^\Phi(y,y')$ is $(K\times K)$-equivariant, hence it gives an element $\sigma^\Phi \in \mathcal{C}^{K\times K}(P\times_M P', K)$. Therefore, by definition, we get the identity

$$\Phi(y) = y'.\sigma^\Phi(y,y'). \tag{54}$$

Conversely, for an element $\sigma \in \mathcal{C}^{K\times K}(P\times_M P', K)$, it is easy to check that the right hand side of (54) depends only on $y$ and defines an id-covering bundle morphism $P \to P'$. Our bijection $S$ is $\Phi \mapsto \sigma^\Phi$.

We prove now that, for any pair $(A, A') \in \mathcal{A}(P)\times\mathcal{A}(P')$, the conditions (i), (ii) are equivalent.

(2) Let $\Phi \in \mathrm{Hom}_{\mathrm{id}}(P, P')$. Let $\lambda : P'\times K \to P'$ be the right action of $K$ on $P'$. For a pair $(y', k) \in P'\times K$ denote by $\lambda_{y'} : K \to P'$, $\lambda_k : P' \to P'$ the corresponding maps obtained from $\lambda$ by fixing an argument. Using (54) we obtain for a pair $(w, w') \in T(P\times_M P')$

$$\Phi_{*y}(w) = (\lambda_{y'})_*\sigma^\Phi_*(w,w') + (\lambda_{\sigma^\Phi(y,y')})_*(w'). \tag{55}$$

Let $\mathcal{H} \subset T_{P\times_M P'}$ be the $(A, A')$-horizontal distribution. Recall that one has $\mathcal{H}_{(y,y')} = A_y \times_{T_xM} A'_y$. Suppose that $\sigma^\Phi_*|_{\mathcal{H}} = 0$, let $w \in A_y$ and let $w'$ be the $A'$-horizontal lift of $p_*(w)$. Using (55) we obtain

$$\Phi_{*y}(w) = (\lambda_{\sigma^\Phi(y,y')})_*(w') \in A'_{y'.\sigma^\Phi(y,y')} = A'_{\Phi(y)}$$

Therefore $\Phi_{*y}(A_y) = A'_{y'}$. Since this holds for any $(y, y') \in P\times_M P'$ we get $\Phi^*(A') = A$, as claimed. Conversely, suppose $\Phi^*(A') = A$. Then for any $(w, w') \in \mathcal{H}_{(y,y')}$ we have

$$(\lambda_{y'})_*\sigma^\Phi_*(w,w') = -(\lambda_{\sigma^\Phi(y,y')})_*(w') + \Phi_{*y}(w) \in A'_{\Phi(y)},$$

where the left hand side is vertical, and the right hand side is $A'$-horizontal. This implies $(\lambda_{y'})_*\sigma^\Phi_*(w,w') = 0$, hence $\sigma^\Phi_*(w,w') = 0$. ■

**1.4. The stabilizer of a connection.** Let $\pi : P \to M$ be a principal $K$-bundle over $M$. A gauge transformation of $P$ is a $\mathrm{id}_M$-covering bundle isomorphism $f : P \to P$ such that for all $(y,k) \in P\times K$, $f(yk) = f(y)k$. The gauge transformations of $P$ form a group, called the gauge group of $P$, and is denoted by $\mathcal{G}(P)$.

Let $\mathrm{Ad} : K\times K \to K$ be the action of $K$ on itself defined by $(k,k') \mapsto k^{-1}k'k$. The space of all Ad-equivariant maps $\varphi : P \to K$ is defined by

$$\mathcal{C}^K(P,K) = \{\varphi : P \to K \mid \varphi(yk) = k^{-1}\varphi(y)k \ \ \forall y \in P, \forall k \in K\}\,.$$

The space $\mathcal{C}^K(P,K)$ has a group structure inherited from $K$ and can be identified with $\mathcal{G}(P)$ in a natural way. For any $\varphi \in \mathcal{C}^K(P,K)$ the corresponding gauge transformation is defined by $f^\varphi(y) = y\varphi(y)$. Conversely, if $f \in \mathcal{G}(P)$, define $\varphi^f : P \to K$ by the relation $f(y) = y\varphi^f(y)$. Also there exists a natural identification between the space $\mathcal{C}^K(P,K)$ and the space of smooth sections of the associated bundle $P\times_{\mathrm{Ad}} K \to M$.

The space of connections on $P$, denoted by $\mathcal{A}(P)$, is an affine space modeled on the vector space $A^1(\mathrm{ad}P) \simeq A^1_{\mathrm{ad}}(P,\mathfrak{k})$ [**DK**], [**Te**]. The gauge group $\mathcal{G}(P)$ acts on $\mathcal{A}(P)$ by $(f,A) \mapsto f_*A$. The next proposition is stated in [**DK**] (see [**DK**, Lemma 4.2.8]) but the proof is left to the reader as an “exercise". We give a proof below:

Proposition 1.7. *Suppose that $M$ is connected. Let $A \in \mathcal{A}(P)$ be a connection on $P$, $\mathcal{G}^A(P)$ denote the stabilizer subgroup of $A$ in gauge group $\mathcal{G}(P)$ and $\mathrm{Hol}^A_{y_0}$ be the holonomy group of $P$ with respect to connection $A$ at a fixed point $y_0 \in P$. The evaluation morphism $e_{y_0} : \mathcal{G}(P) \to K$ given by $f \mapsto \varphi^f(y_0)$ maps $\mathcal{G}^A(P)$ isomorphically onto the centralizer $Z_K(\mathrm{Hol}^A_{y_0})$ of $\mathrm{Hol}^A_{y_0}$ in $K$.*

Proof. Put $x_0 := \pi(y_0)$. Let $P^A_{y_0}$ denote the holonomy bundle of $A$ through $y_0 \in P_{x_0}$ [**KN**, p. 85]. This bundle is the submanifold of $P$ consisting of all points $y \in P$ which can be joined to $y_0$ by a smooth $A$-horizontal curve. Let $\gamma : [0,1] \to P$ be a $A$-horizontal curve with initial point $\gamma(0) = y_0$. Let $f \in \mathcal{G}$. We claim that $f \in \mathcal{G}^A$ if and only if $\varphi^f$ is constant on $P^A_{y_0}$. Indeed, if $f \in \mathcal{G}^A$ then the curves $f \circ \gamma$ and $R_{\varphi^f(y_0)} \circ \gamma$ are both $A$-horizontal with the same initial point $f(y_0)$. Thus for any $t \in [0,1]$ we have

$$\gamma(t)\varphi^f(\gamma(t)) = f(\gamma(t)) = R_{\varphi^f(y_0)}(\gamma(t)) = \gamma(t)\varphi^f(y_0). \tag{56}$$

Therefore $\varphi^f(\gamma(t)) = \varphi^f(y_0)$, so $\varphi^f$ is constant on $P^A_{y_0}$.

Conversely, if $\varphi^f$ is constant on $P^A_{y_0}$, it will be constant on any $A$-horizontal curve $\gamma$ with $\gamma(0) = y_0$. Thus for any $t \in [0,1]$ we have $\varphi(\gamma(t)) = \varphi(y_0)$ and thus (56) is satisfied. Therefore $f \circ \gamma$ is $A$-horizontal for any $A$-horizontal curve $\gamma$ with $\gamma(0) = y_0$. Since $A$ is invariant, it follows that $f \circ \gamma$ is $A$-horizontal for any $A$-horizontal curve $\gamma$ with $\gamma(0) \in P_{x_0}$. Since $M$ is connected, for any $y \in P$ and any $v \in A_y$ there exists an $A$-horizontal curve $\gamma : [0,1] \to P$ such that $\gamma(0) \in P_{x_0}$ and, $\gamma(1) = y$, $\dot\gamma(1) = v$. Therefore the claim is proved.

We have to prove

(1) the restriction $e_{y_0}|_{\mathcal{G}^A(P)}$ is injective,
(2) $e_{y_0}(\mathcal{G}^A(P)) = Z_K(\mathrm{Hol}^A_{y_0})$.

For the first claim, $f \in \mathcal{G}^A(P)$ such that $\varphi^f(y_0) = e_K$. Then $\varphi^f \equiv e_K$ on $P^A_{y_0}$. Since $\varphi^f$ is $K$-equivariant, it follows that $\varphi^f \equiv e_K$ on the whole $P$. For the second claim, let $z \in Z_K(H)$. Define $\varphi_z : P^A_{y_0} \to K$ by $\varphi(y) = z$. This map is $\mathrm{Hol}^A_{y_0}$-equivariant. Thus $\varphi_z : P^A_{y_0} \to K$ has a unique $K$-equivariant extension $\varphi : P \to K$. Let $f \in \mathcal{G}(P)$ such that $\varphi^f = \varphi$. The restriction of $\varphi^f$ on $P^A_{y_0}$ is constant, so $f \in \mathcal{G}^A(P)$.

■

## 2. Infinitesimally homogeneous Riemannian metrics. The theorem of Singer

Let $(M, g)$ be a Riemannian manifold endowed with a locally homogeneous metric. This means that, for any pair $(x, x') \in M \times M$ there exists open neighborhoods $U \ni x$, $U' \ni x'$ and an isometry $f : U \to U'$ such that $f(x) = x'$.

A natural question is: under which conditions on $(M, g)$ can one prove that any local isometry $f : U \to U'$ as above, with $U$, $U'$ connected, can be uniquely extended to a global isometry $\tilde f : M \to M$. The answer is given by the following theorem of Ambrose-Singer [**AS**], [**Si**]:

Theorem 2.1. *Any locally homogeneous, connected, simply connected, complete Riemannian manifold is globally homogeneous.*

If $(M, g)$ is a real analytic Riemannian manifold this result follows from a well-know extension theorem for local isometric immersions [**KN**, Theorem 6.3]. The result holds in full generality in the category of differentiable Riemannian manifolds, so real-analyticity is not necessary. The proof in the differentiable category is much more difficult and requires new ideas. Since these techniques play an important role in Chapter 2, we explain here, in an original way, the strategy of the proof.

An important ingredient in Ambrose Singer's proof is the following classical result [**KN**, Theorem 7.4, Theorem 7.8, Corollary 7.9].

THEOREM 2.2. *Let $M$ and $M'$ be a differentiable manifolds endowed with linear connections $\nabla$, $\nabla'$. Assume: $\nabla T^{\nabla} = 0$, $\nabla R^{\nabla} = 0$, $\nabla' T^{\nabla'} = 0$, $\nabla' R^{\nabla'} = 0$. Let $F : T_{x_0}M \to T_{x'_0}M'$ be a linear isomorphism such that $F(T^{\nabla}_{x_0}) = T^{\nabla'}_{x'_0}$ and $F(R^{\nabla}_{x_0}) = R^{\nabla'}_{x'_0}$. Then there exists an affine isomorphism $f : U \to U'$ of an open neighborhood $U$ of $x_0$ onto an open neighborhood $U'$ of $x'_0$ such that $f(x_0) = x'_0$ and that $f_{*x_0} = F$.*

*If, moreover, $M$ and $M'$ are simply connected and $(M, \nabla)$, $(M', \nabla')$ are complete, then there exists a global affine isomorphism $f : M \to M'$ such that $f(x_0) = x'_0$ and that $f_{*x_0} = F$.*

In this theorem we used the notations

$$T^{\nabla} \in A^0(L^2_{\mathrm{alt}}(T_M, T_M)) \ , \ R^{\nabla} \in A^2(\mathrm{gl}(T_M))$$

for the torsion, respectively the curvature tensor of $\nabla$. Note that this theorem holds for arbitrary linear connections, not only for metric connections. In the special case $(M, \nabla) = (M', \nabla')$ one obtains [**KN**, Corollary 7.9]:

COROLLARY 2.3. *Let $M$ be a connected differentiable manifold endowed with a linear connection. Let $\nabla$, $T$, $R$ be the covariant derivative, torsion, respectively the curvature of this connection. Suppose that $\nabla T = 0$, $\nabla R = 0$. For any pair $(x_0, x'_0) \in M \times M$ there exists open neighborhoods $U \ni x_0$, $U' \ni x'_0$ and an affine isomorphism $f : U \to U'$ such that $f(x_0) = x'_0$.*

*If, moreover, $M$ is simply connected and $(M, \nabla)$ is complete, the group of affine automorphisms of $(M, \nabla)$ acts transitively on $M$.*

This important result follows from Theorem 2.2 using a linear isomorphism $F : T_{x_0}M \to T_{x'_0}M$ obtained by parallel transport along a curve joining $x_0$ to $x'_0$.

This result is applied in the special case when $M$ is endowed with a Riemannian metric $g$, and $\nabla$ is a metric connection (but *not* necessarily the Levi-Civita connection of $g$). The conditions concerning the completeness of the connection $\nabla$ is equivalent to the completeness of $g$ [**TV**, Proposition 1.5]:

PROPOSITION 2.4. *Let $(M, g)$ be complete Riemannian manifold. Then each metric connection $\nabla$ on $M$ is complete.*

Combining these results we obtain the following important homogeneity criterion for Riemannian manifolds:

PROPOSITION 2.5. *Let $(M, g)$ be a connected, simply connected, complete Riemannian manifold endowed with a linear metric connection $\nabla$ such that $\nabla T^{\nabla} = 0$, $\nabla R^{\nabla} = 0$, then $(M, g)$ is homogeneous.*

As explained above the homogeneity of $M$ follows from the classical results proved in [**KN**]. However the method of Ambrose is more effective: it gives an explicit construction, in terms of a connection $\nabla$ satisfying the hypothesis of 2.5, of Lie group $G$ acting transitively and effectively on $(M,g)$ by isometries:

Let $\mathrm{O}(M)$ be the bundle of orthonormal frames of $(M,g)$, and fix $u\in \mathrm{O}(M)$. Let $P^\nabla_u$ denote the holonomy bundle of $\nabla$ through $u$. $P^\nabla_u$ is the submanifold of $O(M)$ consisting of all points $u'\in \mathrm{O}(M)$ which can be joined to $u$ by a smooth $\nabla$-horizontal curve. Ambrose and Singer showed that $G=P^\nabla_u$ has a natural Lie group structure, and it acts on $M$ as an effective, transitive group of isometries ([**AS**], [**TV**, Theorem 1.18]).

The main difficulty in the proof of the Ambrose-Singer theorem is the construction of a metric connection $\nabla$ satisfying the hypothesis of the homogeneity criterion 2.5. For this construction one needs a condition on $g$ which is *apparently* weaker than (but a posteriori equivalent to) local homogeneity. In order to introduce this condition we need some preparations:

For a Riemannian manifold $(M,g)$, a point $x_0\in M$ and a non-negative integer $k\in\mathbb{N}$ let $\mathfrak{g}(x_0;k)$ be the Lie subalgebra of $\mathrm{so}(T_{x_0}M)$ defined by

$$\mathfrak{g}(x_0;k) := \{F\in \mathrm{so}(T_{x_0}M)|\ F\cdot D^iR_{x_0}=0 \text{ for } 0\le i\le k\}\,.$$

Since $\mathfrak{g}(x_0;k+1)\subset\mathfrak{g}(x_0;k)$ for any $k\in\mathbb{N}$, there exists a first non-negative integer $k^g_{x_0}$ such that, $\mathfrak{g}(x_0;k^g_{x_0})=\mathfrak{g}(x_0;k^g_{x_0}+1)$. With this notation we can define

DEFINITION 2.6. *Let $(M,g)$ be a Riemannian manifold, and let $D$, $R$ denote the covariant derivative associated with the Levi-Civita connection of $g$, and the Riemannian curvature tensor. $(M,g)$ is called infinitesimally homogeneous if for any pair $(x_0,x_0')\in M\times M$, there exists a linear isometry $F:T_{x_0}M\to T_{x_0'}M$ such that*

$$F^*(D^iR_{x_0'})=D^iR_{x_0}\ \textit{for } 0\le i\le k^g_{x_0}+1. \tag{57}$$

The linear isometry $F:T_{x_0}M\to T_{x_0'}M$ defines a Lie algebra isomorphism $\mathrm{so}(T_{x_0}M)\to\mathrm{so}(T_{x_0'}M)$ which applies isomorphically $\mathfrak{g}(x_0;k)$ onto $\mathfrak{g}(x_0';k)$ for each $0\le k\le k^g_{x_0}+1$. This implies

REMARK 2.7. If $(M,g)$ is infinitesimally homogeneous, then $k^g_{x_0}$ is independent of $x_0$. We will denoted by $k^g$ the obtained constant.

REMARK 2.8. If $(M,g)$ is locally homogeneous, then it is also infinitesimally homogeneous.

The difficult part of the proof of the Ambrose-Singer Theorem is the following theorem of Singer [**Si**]:

THEOREM 2.9. *Let $(M,g)$ be an infinitesimally homogeneous Riemannian manifold. Then there exists a Rimannian connection $\nabla$ on $M$ such that $\nabla T^\nabla=0$, $\nabla R^\nabla=0$.*

PROOF. This construction of $\nabla$ is obtained in two steps:

S1. Construct a Riemannian linear connection $\mathfrak{D}$ with respect to which the tensors $D^iR$ are parallel for $0\le i\le k^g+1$.
This connection will be obtained using a reduction of the structure group of $\mathrm{O}(M)$

from $\mathrm{O}(n)$ (where $n := \dim(M)$) to a carefully chosen Lie subgroup $H \subset \mathrm{O}(n)$. For $x \in M$ we identify the fiber $\mathrm{O}(M)_x$ with the manifold of linear isometries $u : \mathbb{R}^n \to T_x$. A point $u \in \mathrm{O}(M)_x$ defines an isomorphism

$$\tau_u : L^4(T_x, \mathbb{R}) \to L^4(\mathbb{R}^n, \mathbb{R}) .$$

Here we used the notation $L^k(V, W)$ for the space of $k$-multilinear maps

$$V \times \cdots \times V \to W .$$

The covariant derivative $D^i R$ is a section in the bundle $(\Lambda^1_M)^{\otimes i} \otimes L^4(T_M, \mathbb{R})$. Therefore, for any $i \in \mathbb{N}$ we obtain a map

$$\phi_i : \mathrm{O}(M) \to L^i(\mathbb{R}^n, L^4(\mathbb{R}^n, \mathbb{R})) \simeq L^{4+i}(\mathbb{R}^n, \mathbb{R})$$

given by

$$\phi_i(u)(\xi_1, \ldots, \xi_i) := \tau_u\big((D^i R)(u(\xi_1), \ldots, u(\xi_i))\big) .$$

Now define the $\mathrm{O}(n)$-equivariant map

$$\Phi : \mathrm{O}(M) \to \bigoplus_{i=0}^{k_g+1} L^i(\mathbb{R}^n, L^4(\mathbb{R}^n, \mathbb{R}))$$

by

$$\Phi(u) := (\phi_i(u))_{0 \leq i \leq k^g+1} .$$

The main observation in this first step of the proof is: since $(M, g)$ is infinitesimally homogeneous with associated constant $k^g$, the image $\Phi(\mathrm{O}(M))$ is an orbit with respect to the natural $\mathrm{O}(n)$ action on the vector space

$$\bigoplus_{i=0}^{k_g+1} L^i(\mathbb{R}^n, L^4(\mathbb{R}^n, \mathbb{R})).$$

Fixing $u_0 \in \mathrm{O}(M)$, denoting by $H$ the stabiliser of $\Phi(u_0)$ in $\mathrm{O}(n)$, identifying $\Phi(\mathrm{O}(M))$ with the homogeneous space $\mathrm{O}(n)/H$, we obtain (see [**KN**, Proposition 5.6]) a $H$-reduction

$$\mathfrak{Q} := \Phi^{-1}(\Phi(u_0)) \subset \mathrm{O}(M)$$

of $\mathrm{O}(M)$. With this definition we see that the restriction $\Phi|_{\mathfrak{Q}}$ is constant. This shows that, identifying $(\Lambda^1_M)^{\otimes i} \otimes L^4(T_M, \mathbb{R})$ with a vector bundle associated with $\mathfrak{Q}$, the sections $D^i R$ ($0 \leq i \leq k^g + 1$) are $\mathfrak{B}$-parallel, for any connection $\mathfrak{B}$ on $\mathfrak{Q}$. It suffices to define $\mathfrak{D}$ to be the linear connection on $T_M$ associated with any connection $\mathfrak{B}$ on $\mathfrak{Q}$, and the first step is complete.

S2. Finding $\beta \in A^1(\mathrm{so}(T_M))$ such that $\nabla := \mathfrak{D} + \beta$ satisfies the conclusion of Theorem 2.9.

In this step we use essentially the definition of $k^g$. For any $k \in \mathbb{N}$ and $x \in M$ put

$$\mathfrak{g}_x(k) := \{b \in \mathrm{so}(T_x M) |\ b \cdot (D^i R)_x = 0 \text{ for } 0 \leq i \leq k\} ,$$

where $b \mapsto b\,\cdot$ stands for the infinitesimal action of the Lie algebra $\mathrm{so}(T_x M)$ on the tensor algebra of $T_x M$. Since the sections $D^i R$ are $\mathfrak{D}$-parallel for $0 \leq i \leq k^g + 1$, it follows that

$$\mathfrak{g}(k) := \bigcup_{x \in M} \mathfrak{g}_x(k)$$

is a $\mathfrak{D}$-parallel subbundle of $\mathrm{so}(T_M)$ for any $0 \leq k \leq k^g+1$. Moreover, the definition of $k^g$ gives the equality $\mathfrak{g}(k^g) = \mathfrak{g}(k^g+1)$. Put $\mathfrak{g} := \mathfrak{g}(k^g+1)$ to save on notations, and let $\mathfrak{g}^\perp$ be the orthogonal complement of $\mathfrak{g}$ with respect to the standard inner product on $\mathrm{so}(T_M)$. Let $\mathfrak{b} := \mathfrak{D} - D \in A^1(\mathrm{so}(T_M))$, and let

$$\mathfrak{b} = \mathfrak{b}' + \mathfrak{b}''$$

be the decomposition of $\mathfrak{b}$ with respect to the orthogonal decomposition

$$\Lambda^1 \otimes \mathrm{so}(T_M) = (\Lambda^1 \otimes \mathfrak{g}) \oplus (\Lambda^1 \otimes \mathfrak{g}^\perp) \,. \tag{58}$$

Put $\nabla = \mathfrak{D} - \mathfrak{b}' = D + \mathfrak{b}''$. Since $\mathfrak{D}(D^k R) = \mathfrak{b}' \cdot (D^k R) = 0$ for $0 \leq k \leq k^g + 1$ it follows that one also has

$$\nabla(D^k R) = 0 \text{ for } 0 \leq k \leq k^g + 1 \,, \tag{59}$$

in particular the orthogonal decomposition

$$\mathrm{so}(T_M) = \mathfrak{g} \oplus \mathfrak{g}^\perp \tag{60}$$

is both $\mathfrak{D}$ and $\nabla$-parallel.

We use now the formula

$$0 = \nabla(D^k R) = (D + \mathfrak{b}'')(D^k R) = D^{k+1}R + \mathfrak{b}'' \cdot (D^k R) \text{ for } 0 \leq k \leq k^g \,.$$

Applying $\nabla_\xi$ on both sides (for a tangent vector $\xi \in T_x M$), one proves that $\nabla \mathfrak{b}''$ is a section of

$$\Lambda^1_M \otimes \Lambda^1_M \otimes \mathfrak{g}(k^g) = \Lambda^1_M \otimes \Lambda^1_M \otimes \mathfrak{g}(k^g+1) = \Lambda^1_M \otimes \Lambda^1_M \otimes \mathfrak{g}$$

(see section 2.1 for details and a generalization of this argument). But, since the decomposition (60) is $\nabla$-parallel, and $\mathfrak{b}'' \in \Gamma(\Lambda^1 \otimes \mathfrak{g}^\perp)$ it follows that $\nabla \mathfrak{b}''$ is also a section of $\Lambda^1_M \otimes \Lambda^1_M \otimes \mathfrak{g}^\perp$. Therefore $\nabla \mathfrak{b}'' = 0$. This gives $\nabla T^\nabla = 0$. Taking $k = 0$ in (59) we also have $\nabla R = 0$. Expressing $R^\nabla$ in terms of $R$ and $\mathfrak{b}''$, and using again $\nabla \mathfrak{b}'' = 0$, we obtain $\nabla R^\nabla = 0$. Therefore, taking $\beta := -\mathfrak{b}'$, the connection $\nabla := \mathfrak{D} + \beta$ satisfies the conclusion of Theorem 2.9. ■

Using now Corollary 2.3 and the homogeneity criterion 2.5 we obtain [**Si**, Main Theorem, p. 692]:

COROLLARY 2.10. *A Riemannian manifold $(M, g)$ is infinitesimally homogeneous if and only if it is locally homogeneous. Any connected, simply connected, complete infinitesimally homogeneous Riemannian manifold is homogeneous.*

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