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Etude mathématique de problèmes inverses non autonomes de types hyperbolique et quantique

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Abstract

My research interests during the preparation of my PhD were focused on the study of inverse coefficients problems for partial differential equations, which is one of the most rapidly growing research areas in Mathematics. There is a wide mathematical literature on this topic, that is mostly concerned with space dependent only unknown coefficients. In this thesis, I rather address the uniqueness and stability issues in the inverse problem of determining time and space dependent unknown coefficients from the knowledge of boundary data. The first part of the thesis is devoted to the study of inverse problems for the wave equation. Namely, we examine the uniqueness and stability issues in the determination of unknown coefficients of the wave equation, from the knowledge of several suitable sets of boundary data. The first inverse problem we address is to determine a first order coefficient appearing in a dissipative wave equation by boundary observations. This is a mathematically challenging problem, as unique determination of time-dependent coefficients in hyperbolic equations is not even guaranteed on the entire time-space domain. Next, we prove logarithmic stable identification of zero-th order unknown coefficients, in the complement of cloacking regions, from Neumann data. We also establish that the non-uniqueness manifested in certain sub-domains of the propagation region can be removed upon imposing suitable initial conditions to the system. Finally, the same type of analysis is carried out for the time dependent zero-th and first order unknown coefficients of the dissipative wave equation. The second part of this thesis deals with the inverse problem of determining the space-dependent magnetic field and the time-space dependent electric potential of the Schrödinger equation. We prove stable identification in the whole domain of these two unknown coefficients, by Neumann data. The derivation of these results boils down to a sufficiently large set of geometric optics solutions to the system under investigation. There is another approach for solving inverse coefficients problems, based on the celebrated Bukhgeim-Klibanov method, which is by means of a Carleman estimate specifically designed for the system under study. We prove with this approach that

the space dependent part of the electric and magnetic potentials of the magnetic Schrödinger equation are simultaneously Lipschitz stably determined, by a finite number of partial boundary measurements of the solution.

Résumé

Cette thèse est consacrée à l'étude de problèmes inverses associés à des équations aux dérivées partielles hyperboliques et de type Schrödinger. Il existe une vaste littérature mathématique sur ce sujet, mais elle concerne surtout l'identification de coefficients dépendant exclusivement de la variable spatiale. Ici, on s'intéresse plutôt à l'étude de l'unicité et de la stabilité dans la détermination de coefficients qui dépendent aussi de la variable temporelle.

La premiere partie de la thèse est consacrée à l'étude de problèmes inverses pour l'équation des ondes. Il s'agit d'examiner les propriétés de stabilité et d'unicité dans l'identification de certains coefficients apparaissant dans l'équation des ondes, à partir de différents types d'observation. Le premier problème traité concerne l'identification d'un coefficient d'ordre un apparaissant dans l'équation des ondes dissipative. Ce problème est difficile mathématiquement parlant, car l'unicité de la détermination de coefficients dépendant du temps n'est pas garantie dans tout le domaine, pour les équations hyperboliques. En fait, nous montrons qu'un coefficient d'ordre zéro peut être déterminé de façon stable dans le complément de la région dite "de cloacking", à partir de mesures de type Neumann de la solution. De plus, nous établissons que la non-unicité manifestée dans certaines parties du domaine de propagation peut être éliminée par la considération de conditions initiales particulières. Cette analyse s'adapte au cas de l'équation des ondes dissipative, pour laquelle nous prouvons que les coefficients dépendant du temps, et d'ordres zéro ou un en espace, peuvent être déterminés de façon stable, par l'observation de données de type Neumann.

La deuxième partie de cette thèse, traite du problème de l'identification du champ magnétique et du potentiel électrique apparaissant dans l'équation du Schrödinger. Nous prouvons que ces coefficients peuvent être déterminés de façon stable dans tout le domaine, à partir de données de type Neumann. La dérivation de ces résultats est basée sur la construction d'un ensemble de solutions de type optique géométrique, adaptées au système étudié. Il existe une méthode alternative pour l'analyse de ce type de problèmes inverses, celle de Bukhgeim-Klibanov, qui utilise une estimation de Carleman spécifique à l'opérateur con-

sidéré. Elle nous a permis de montrer qu'il est possible de récupérer de façon stable et simultanée, la partie spatiale des potentiels électrique et magnétique de l'équation de Schrödinger magnétique, à partir d'un nombre fini de mesures partielles de la solution.

Résumé détaillé des résultats obtenus

0.1 Introduction

Les travaux de recherche présentés dans cette thèse portent sur l'étude de problèmes inverses consistant en l'identification de coefficients apparaissant dans des équations aux dérivées partielles d'évolution. S'il existe déjà une vaste littérature mathématique sur ce sujet, il est à noter qu'elle concerne surtout la détermination de coefficients dépendant exclusivement des variable d'espace.

Dans cette thèse, on s'intéresse plutôt à l'étude de l'unicité et de la stabilité, dans deux types de problèmes inverses aux limites, associés à des coefficients qui dépendent non seulement des variables spatiales, mais aussi de la variable temporelle. Le premier problème inverse étudié est hyperbolique, car il porte sur l'équation des ondes, amortie ou non. Il est décrit dans la section 0.2. Quant au second, qui est associé à l'équation de Schrödinger magnétique, il est traité dans la section 0.3.

0.2 Problèmes inverses hyperboliques

La première partie de la thèse est consacrée à l'étude de problèmes inverses associés à l'équation des ondes dans un domaine borné Ω de $\mathbb{R}^n$, pour $n \geq 2$, de frontière $\Gamma = \partial\Omega$ supposée lisse, c'est-à-dire de classe $\mathcal{C}^\infty$. Etant donné $T > 0$, on pose $Q = \Omega \times (0, T)$, on note $\Sigma = \Gamma \times (0, T)$ la frontière latérale du cylindre

Q, puis l'on considère le système suivant :

$$\begin{cases} \partial_t^2 u - \Delta u + a(x,t)\partial_t u + b(x,t)u = 0, & (x,t) \in Q, \\ u(\,\cdot\,,0) = u_0, \;\; \partial_t u(\,\cdot\,,0) = u_1, & x \in \Omega, \\ u = f, & (x,t) \in \Sigma. \end{cases} \tag{0.1}$$

Ici, x (resp., t) désigne la variable spatiale (resp., temporelle), u_0 et u_1 sont les conditions initiales du système, et f est la condition dite “de Dirichlet", utilisée dans la suite pour perturber le système (0.1). L'objectif principal de cette partie est d'étudier les problèmes d'unicité et de stabilité dans l'identification des coefficients a et b apparaissant dans la première ligne de (0.1), à partir de mesures latérales de la solution u.

Avant de passer à l'étude du problème inverse mentionné ci-dessus, il convient de noter à partir de [**13**]; [**31**]; [**34**] que le système (0.1) admet une unique solution satisfaisant une inégalité d'énergie usuelle, sous des hypothèses de régularité raisonables sur les coefficients a et b, les données initiales u_0 et u_1, et la condition au bord f:

Theorem 0.2.1. *Soient $a \in L^\infty(Q)$ et $b \in L^\infty(Q)$. Alors, pour tous $u_0 \in H^1(\Omega)$, $u_1 \in L^2(\Omega)$ et $f \in H^1(\Sigma)$, satisfaisant la condition de compatibilité*

$$f(\,\cdot\,,0) = u_0 \text{ sur } \Gamma,$$

il existe une unique solution $u \in \mathcal{C}([0,T]; H^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega))$ du système (0.1).

On vérifie ensuite que la dérivée normale $\partial_\nu u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = \nabla u(x,t) \cdot \nu(x)$ de u, où $\nu(x)$ désigne le vecteur unitaire normal sortant, calculé au point $x \in \Gamma$, ∇ désigne l'opérateur gradient par rapport aux variables d'espace $x \in \mathbb{R}^n$, et $\cdot$ le produit scalaire euclidien de $\mathbb{R}^n$, appartient à $L^2(\Sigma)$:

$$\partial_\nu u \in L^2(\Sigma).$$

De plus, on démondre qu'il existe une constante positive $C > 0$, pour laquelle l'inégalité d'énergie

$$\|\partial_\nu u\|_{L^2(\Sigma)} + \|u(\,\cdot\,,t)\|_{H^1(\Omega)} + \|\partial_t u(\,\cdot\,,t)\|_{L^2(\Omega)} \leq C \left(\|f\|_{H^1(\Sigma)} + \|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)} \right),$$

est vraie uniformément par rapport à $t \in [0,T]$.

0.2.1 Un résultat de non unicité

La première contribution "inverse" de cette thèse est en fait un résultat de non unicité locale pour les coefficients a et b, lorsque $(u_0, u_1) = (0, 0)$, par rapport à l'opérateur Dirichlet-Neumann (ou Poincaré-Steklov):

$$\begin{array}{rccc} \Lambda_{a,b}: & H^1(\Sigma) & \longrightarrow & L^2(\Sigma) \\ & f & \longmapsto & \partial_\nu u. \end{array}$$

Autrement dit, $\Lambda_{a,b}$ est l'opérateur qui à toute donnée de Dirichlet $f \in H^1(\Sigma)$, l'espace de Sobolev de premier ordre sur Σ, associe la valeur de la dérivée normale de la solution u du système (0.1) excité par f.

En s'inspirant de l'analyse d'Ikawa dans [**23**] et en utilisant le fait que les conditions initiales u_0 et u_1 sont prises égales à zéro, il est possible de montrer que la solution u de (0.1) est identiquement nulle sur l'ensemble conique

$$\mathscr{C} := \left\{ (x,t) \in Q,\ |x| \le \frac{r}{2} - t, \quad 0 \le t \le \frac{r}{2} \right\}.$$

Ceci permet ensuite de déduire le résultat de non unicité locale suivant.

Theorem 0.2.2. *(Non unicité). Soient $a \in W^{1,\infty}(Q)$ et $b \in L^\infty(Q)$ tels que supp $(a,b) \subset \mathscr{C}$. Alors, on a $\Lambda_{a,b} = \Lambda_{0,0}$.*

Dans ce qui suit, on va supposer que le coefficient d'ordre un est égal à zéro, c'est-à-dire que le "terme d'amortissement" a est identiquement nul dans le cylindre Q, afin d'examiner le problème inverse de la détermination du "potentiel électrique b", à partir de la donnée de $\Lambda_b = \Lambda_{0,b}$. La section 0.2.3 est entièrement consacrée à l'étude de la stabilité de ce problème inverse. Mais tout d'abord, on va commencer par traiter le cas où a n'est pas identiquement nul et $b = 0$ dans la section 0.2.2. Le cas où $(a,b) \neq (0,0)$, lui est traité dans la section 0.2.4. Les résultats de la section 0.2.3 ont été publiés dans [P1], alors que ceux des sections 0.2.2 et 0.2.4 sont décrits dans [P3,P4].

La dérivation des résultats de stabilité des sections 0.2.3 et 0.2.4, et de celui de la section 0.2.2, s'inspirent assez largement de [**10**]; [**11**] Elle repose essentiellement sur la construction d'un ensemble suffisamment riche de solutions particulières de l'équation hyperbolique considérée, appelées "solutions optiques géométriques". Compte tenu de la généralité des coefficients apparaissant dans l'équation hyperbolique étudiée, ces fonctions ne peuvent évidemment pas être décrites explicitement, mais il est néanmoins possible de les caractériser suffisamment précisément, afin de démontrer une famille d'égalités pour le(s) coefficient(s) inconnu(s). Ceci revient *in fine* à exprimer la transformée dite "rayons lumineux" de ces coefficients en fonction des observations disponibles. La deuxième étape clé de l'analyse de ce problème est d'“inverser" cette transformation, en la reliant prélalablement à celle de Fourier en plus d'un résultat technique d'analyse complexe, déjà utilisé dans [48]; [1] dont la démonstration a été complètée dans cette thèse.

0.2.2 Unicité de a en l'absence de perturbation potentielle ($b = 0$)

En supposant que le coefficient b est nul partout (ce qui signifie qu'il n'y a aucune perturbation “électrique" affectant le système), il est possible d'identifier dans un sous ensemble $Q_{r,*} \subset Q$, formé par les lignes faisant un angle $\pi/4$ avec l'axe temporel, et rencontrant les plans $t = 0$ et $t = T$ en dehors de $\overline{Q_r} = \{x \in \mathbb{R}^n, |x| \leq r/2\} \times [0, T]$, le coefficient d'amortissement a, c'est-à-dire le coefficient du terme différentiel du permier ordre en temps apparaissant dans l'équation des ondes de (0.1), à partir de l'opérateur Dirichlet-Neumann $\Lambda_{a,0}$. Et ceci alors que les conditions initiales sont maintenues homogènes dans Ω : $(u_0, u_1) = (0, 0)$.

Theorem 0.2.3. *(Unicité). Soient a_1 et a_2 dans $\mathcal{C}^2(\overline{Q}_r)$, vérifiant*

$$a_1 = a_2 \text{ dans } \overline{Q}_r \setminus Q_{r,*}.$$

Alors, si $T > 2$ Diam (Ω), on a l'implication :

$$(\Lambda_{a_1,0} = \Lambda_{a_2,0}) \Longrightarrow (a_1 = a_2 \text{ dans } Q_{r,*})\,.$$

Le fait que l'idenfication de a ne soit que partielle (puisqu'elle n'a lieu que sur $Q_{r,*}$, qui est un sous-ensemble strict de Q) est évidemment en accord avec le résultat du théorème 0.2.2. D'une certaine façon, le résultat du théorème précdent est l'analogue pour a de celui qu'on va obtenir pour b au théorème 0.2.4, en ce sens que le second coefficient entrant dans l'équation hyperbolique est supposé nul.

0.2.3 Détermination stable de b en l'absence d'amortissement ($a = 0$)

0.2.3.1 Identification partielle

Tout d'abord, lorsque les conditions initiales sont prises égales à zéro, c'est-à-dire lorsque $u_0 = u_1 = 0$, il découle facilement du théorème 0.2.2 que le coefficient inconnu b ne peut être identifié de façon unique sur tout le domaine Q, au moyen de l'opérateur Λ_b. En fait, Ramm et Rakesh ont montré dans [**36**] que b est entièrement déterminé par Λ_b sur un sous-ensemble $Q_{r,*} \subset Q$. Il est donc nécessaire que b soit connu en dehors de $Q_{r,*}$, pour que ce résultat s'applique. De fait, c'est l'hypothèse principale faite dans le théorème suivant, qui améliore le résultat précité de Ramm et Rakesh, en ce sens qu'il établit que Λ_b détermine b de façon stable dans $Q_{r,*}$.

Theorem 0.2.4. *(Stabilité). Soient b_1 et b_2 dans $\mathcal{C}^1(\overline{Q}_r)$, tels que*

$$b_1 = b_2 \text{ dans } \overline{Q}_r \setminus Q_{r,*}, \text{ et } \partial_x b_1 = \partial_x b_2 \text{ sur } \partial Q_r \cap \partial Q_{r,*}. \tag{0.2}$$

Etant donné $M > 0$, on suppose de plus que $\|b_i\|_{W^{1,\infty}(Q)} \leq M$, pour $i = 1, 2$, et on choisit $T > 2\,Diam\,(\Omega)$. Alors, il existe deux constantes $C > 0$ and $\mu_1 \in (0,1)$, qui dépendent uniquement de Ω, M, T et n, telles que

$$\|b_1 - b_2\|_{H^{-1}(Q_{r,*})} \leq C \left(\|\Lambda_{b_1} - \Lambda_{b_2}\|^{\mu_1} + |\log \|\Lambda_{b_1} - \Lambda_{b_2}\||^{-1} \right).$$

On constate à la lumière du théorème 0.2.4, que la non unicité manifestée dans le théorème 0.2.2, limite de fait l'identification stable du potentiel électrique b au cône $\mathscr{C}$. A ce stade, il est donc naturel d'examiner si le domaine d'identifiabilité de b peut être élargi, en augmentant la quantité d'information disponible sur la solution u du système (0.1). L'étude de cette question fait l'objet de la section suivante.

0.2.3.2 Données augmentées

Rappelons que dans cette section, les conditions initiales sont toujours fixées à zéro : $u_0 = u_1 = 0$ dans Ω. L'idée est d'ajoindre à l'information "latérale" donnée par Λ_b, celle fournie par la mesure finale "volumique" de la solution u et de sa dérivée première par rapport au temps. Plus précisément, l'information "augmentée" disponible est définie par l'opérateur linéaire borné

$$\begin{array}{rccl} \mathscr{R}_b : & H^1(\Sigma) & \longrightarrow & L^2(\Sigma) \times H^1(\Omega) \times L^2(\Omega), \\ & f & \longmapsto & (\partial_\nu u,\, u(.,T),\, \partial_t u(.,T)), \end{array}$$

où l'on rappelle que u est la solution du système (0.1) associé à $a = 0$ et $(u_0, u_1) = (0, 0)$. Le fait de bénéficier de l'information supplémentaire $(u(\cdot, T), \partial_t u(\cdot, T))$ mesurée dans Ω, a permis de construire un sous-ensemble $Q_{r,\sharp} \supset Q_{r,*}$, formé par les lignes faisant un angle $\pi/4$ avec l'axe temporel, et rencontrant le plans $t = 0$ en dehors de $\overline{Q_r}$, sur lequel $\mathscr{R}_b$ détermine b de façon stable. Autrement dit, l'énoncé du théorème 0.2.4 reste valable en remplaçant $(Q_{r,*}, \Lambda_b)$ par $(Q_{r,\sharp}, \mathscr{R}_b)$:

Theorem 0.2.5. *(Stabilité avec données augmentées). Avec les notations et sous les conditions du théorème 0.2.4, où (0.2) est remplacée par*

$$b_1 = b_2 \ \textit{dans} \ \overline{Q}_r \setminus Q_{r,\sharp} \ \textit{et} \ \partial_x b_1 = \partial_x b_2 \ \textit{sur} \ \partial Q_r \cap \partial Q_{r,\sharp}, \tag{0.3}$$

il existe deux constantes $C > 0$ and $\mu_1 \in (0, 1)$, telles que :

$$\|b_1 - b_2\|_{H^{-1}(Q_{r,\sharp})} \leq C \left(\|\mathscr{R}_{b_1} - \mathscr{R}_{b_2}\|^{\mu_1} + |\log \|\mathscr{R}_{b_1} - \mathscr{R}_{b_2}\||^{-1} \right).$$

De plus, C et μ ne dépendent que de Ω, M, T et n.

Le théorème 0.2.5 permet donc, en complétant de façon adéquate les données latérales utilisées dans le théorème 0.2.4, d'élargir le domaine de reconstruction stable de b, au sous-domaine strict $Q_{r,\sharp} \subset Q$. En fait, l'analyse développée dans [**36**] montre qu'on ne peut espérer identifier b sur le cylindre Q en entier, en gardant, comme dans le théorème 0.2.5, les conditions initiales fixées à zéro. C'est pourquoi, nous examinons dans la section suivante, le problème inverse de la détermination de b sur tout le domaine Q, à partir de données obtenues en excitant le système (0.1) avec des conditions initiales (u_0, u_1) non-homogènes.

0.2.3.3 Identification complète

Le terme "identification complète" fait ici référence à la détermination du coefficient inconnu b dans le domaine Q en entier. On considère pour cela la collection de toutes les données contenues dans $\mathscr{R}_b$, obtenue en faisant varier les conditions initiales (u_0, u_1) du système (0.1), dans des espaces fonctionnels ad-hoc. Plus précisément, a étant toujours supposée uniformément nulle dans Q, l'ensemble des observations considérées est défini par l'opérateur linéaire borné

$$\begin{array}{rccl} \mathcal{I}_b : & H^1(\Sigma) \times H^1(\Omega) \times L^2(\Omega) & \longrightarrow & L^2(\Sigma) \times H^1(\Omega) \times L^2(\Omega), \\ & (f, u_0, u_1) & \longmapsto & (\partial_\nu u,\ u(., T),\ \partial_t u(., T)), \end{array}$$

où u désigne la solution du système (0.1) associé à la donnée de Dirichlet f et aux données initiales (u_0, u_1). Le potentiel b est alors entièrement déterminé par $\mathcal{I}_b$ dans tout le cylindre Q.

Theorem 0.2.6. *(Reconstruction dans Q). Etant donné $M > 0$, on considère deux potentiels b_1 et b_2 dans $\mathcal{C}^1(\overline{Q})$, vérifiant*

$$\partial_x b_1(x,t) = \partial_x b_2(x,t), \ \ (x,t) \in \Sigma,$$

ainsi que la condition $\|b_i\|_{W^{1,\infty}(Q)} \leq M$, pour $i = 1, 2$. Alors, il existe deux constantes $C > 0$ and $\mu_1 \in (0,1)$, ne dépendant que de Ω, M, T et n, telles que :

$$\|b_1 - b_2\|_{H^{-1}(Q)} \leq C \left(\|\mathcal{I}_{b_1} - \mathcal{I}_{b_2}\|^{\mu_1} + |\log \|\mathcal{I}_{b_1} - \mathcal{I}_{b_2}\||^{-1} \right).$$

Comme on vient de le voir, les sections 0.2.3 et 0.2.2 sont consacrées au problème inverse de la détermination (à partir de divers ensembles de données) de l' **un des deux coefficients** a où b, entrant dans l'équation hyperbolique du système (0.1), puisque le second coefficient est systématiquement pris égal à zéro. La section suivante s'affranchit de cette hypothèse simplificatrice et traite du problème inverse plus général de la détermination de (a, b).

0.2.4 Identification stable de (a, b)

Dans cette partie, on s'intéresse donc au problème d'identification des deux coefficients inconnus a et b, apparaissant dans l'équation hyperbolique du système (0.1). Plus précisément, on ne suppose plus, comme c'était le cas dans les sections 0.2.3 et 0.2.2, que a ou b est uniformément nul dans Q.

Commençons par introduire l'ensemble des coefficients admissibles :

$$\mathcal{A}(M, M') = \{(a,b) \in \mathcal{C}^2(\overline{Q}_r) \times \mathcal{C}^1(\overline{Q}_r); \ \|a\|_{C^2(Q)} \leq M, \ \|b\|_{C^1(Q)} \leq M'\},$$

où M et M' sont deux réels strictement positifs fixés. On procède comme dans la section 0.2.3 en ce sens que l'on examine d'abord le cas où les conditions initiales sont prises égales à zero et l'ensemble de données disponibles est décrit par l' opérateur Dirichlet-Neumann $\Lambda_{a,b}$. Comme il est impossible de recouvrir a et b dans Q dans ce contexte, en vertu du théorème 0.2.2, on augmente ensuite progressivement la taille des données disponibles (en complétant l'information "latérale" apportée par $\Lambda_{a,b}$ par l'observation finale, dans tout Ω, de la solution du système, puis en la “dupliquant" suffisamment en faisant varier les conditions initiales) jusqu'à l'identification totale.

0.2.4.1 Identification partielle de (a, b)

On rappelle que $(u_0, u_1) = (0, 0)$. Le résultat correspondant s'exprime comme suit, les données initiales étant fixées à zéro : $u_0 = u_1 = 0$ dans Ω.

Theorem 0.2.7. *(Stabilité par rapport à l'opérateur DN). Etant donnés $M > 0$ et $M' > 0$, on considère $(a_i, b_i) \in \mathcal{A}(M, M')$, pour $i = 1, 2$, vérifiant $\|a_i\|_{H^p(Q)} \leq M$, pour $p > (n+3)/2$, ainsi que les conditions suivantes :*

$$(a_1, b_1) = (a_2, b_2) \text{ dans } \overline{Q}_r \setminus Q_{r,*} \text{ et } (\partial_x a_1, \partial_x b_1) = (\partial_x a_2, \partial_x b_2) \text{ sur } \partial Q_r \cap \partial Q_{r,*}. \tag{0.4}$$

Alors, si $T > 2\,Diam\,(\Omega)$, il existe $(m, \mu, \tilde{\mu}) \in (0,1)^3$ et $C \in (0, +\infty)$ dépendant uniquement de Ω, M, M', T et n, pour lesquelles, si $\|\Lambda_{a_1,b_1} - \Lambda_{a_2,b_2}\| \leq m$, on a:

$$\|a_1 - a_2\|_{L^\infty(Q_{r,*})} \leq C \left(|\log \|\Lambda_{a_1,b_1} - \Lambda_{a_2,b_2}\||^{-1} \right)^{\mu},$$

et

$$\|b_1 - b_2\|_{H^{-1}(Q_{r,*})} \leq C \left(\log |\log \|\Lambda_{a_1,b_1} - \Lambda_{a_2,b_2}\||^{\tilde{\mu}} \right)^{-1}.$$

De façon similaire à ce qui a été vu dans la section 0.2.3, il est possible d'adjoindre des données supplémentaires à $\Lambda_{a,b}$, pour reconstruire (a, b) dans $Q_{r,\sharp} \supset Q_{r,*}$.

0.2.4.2 Reconstruction stable dans $Q_{r,\sharp}$

Sans surprise, on définit l'opérateur linéaire et continu

$$\begin{array}{rccl} \mathscr{R}_{a,b} : & H^1(\Sigma) & \longrightarrow & L^2(\Sigma) \times H^1(\Omega) \times L^2(\Omega), \\ & f & \longmapsto & (\partial_\nu u,\, u(.,T),\, \partial_t u(.,T)), \end{array}$$

où u est la solution du système (0.1) associée à f et $(u_0, u_1) = (0, 0)$. Il permet d'identifier (a, b) de façon stable dans $Q_{r,\sharp}$, en ce sens que l'énoncé du théorème 0.2.7 reste donc valable en remplaçant $(Q_{r,*}, \Lambda_{a,b})$ par $(Q_{r,\sharp}, \mathscr{R}_{a,b})$.

Theorem 0.2.8. *(Extension de la stabilité à $Q_{r,\sharp}$). Sous les conditions du théorème 0.2.7, à ceci près que la condition* (0.4) *est remplacée par*

$$(a_1, b_1) = (a_2, b_2) \text{ dans } \overline{Q}_r \setminus Q_{r,\sharp} \text{ et } (\partial_x a_1, \partial_x b_1) = (\partial_x a_2, \partial_x b_2) \text{ sur } \partial Q_r \cap \partial Q_{r,\sharp},$$

il existe $(m, \mu, \tilde{\mu}) \in (0,1)^3$ et $C \in (0, +\infty)$ dépendant uniquement de Ω, M, M', T et n, pour lesquelles,si $\|\mathscr{R}_{a_1,b_1} - \mathscr{R}_{a_2,b_2}\| \leq m$, on a:

$$\|a_1 - a_2\|_{L^\infty(Q_{r,\sharp})} \leq C \left(|\log \|\mathscr{R}_{a_1,b_1} - \mathscr{R}_{a_2,b_2}\||^{-1} \right)^{\mu},$$

et

$$\|b_1 - b_2\|_{H^{-1}(Q_{r,\sharp})} \leq C \left(\log |\log \|\mathscr{R}_{a_1,b_1} - \mathscr{R}_{a_2,b_2}\|| ^{\tilde{\mu}} \right)^{-1}.$$

Pour étendre la reconstruction à Q en entier, on procède comme dans la section 0.2.3.3, c'est-à-dire que l'on considère la collection des observations contenues dans l'opérateur $\mathscr{R}_{a,b}$, obtenue pour toutes les données initiales $(u_0, u_1) \in H^1(\Omega) \times L^2(\Omega)$ possibles.

0.2.4.3 Reconstruction totale

L'ensemble des observations est donc défini par l'opérateur

$$\begin{array}{rccc} \mathcal{I}_{a,b} : & H^1(\Sigma) \times H^1(\Omega) \times L^2(\Omega) & \longrightarrow & L^2(\Sigma) \times H^1(\Omega) \times L^2(\Omega), \\ & (f, u_0, u_1) & \longmapsto & (\partial_\nu u,\ u(.,T),\ \partial_t u(.,T)), \end{array}$$

où u est encore une fois la solution du système (0.1). Il permet d'identifier a et b de façon stable dans tout le cylindre Q.

Theorem 0.2.9. *(Détermination dans Q). Etant donnés $M > 0$ et $M' > 0$, on considère $(a_i, b_i) \in \mathcal{C}^2(\overline{Q}) \times \mathcal{C}^1(\overline{Q})$, $i = 1, 2$, vérifiant $\|a_i\|_{H^p(Q)} + \|a_i\|_{\mathcal{C}^2(Q)} \leq M$ pour $p > (n+3)/2$, ainsi que $\|b_i\|_{\mathcal{C}^1(Q)} \leq M'$ et la condition suivante :*

$$(\partial_x a_1, \partial_x b_1) = (\partial_x a_2, \partial_x b_2) \text{ sur } \Sigma.$$

Alors, il existe $(m, \mu, \tilde{\mu}) \in (0,1)^3$ *et* $C \in (0, +\infty)$ *dépendant uniquement de* Ω, M, M', T *et* n, *tels que si* $\|\mathcal{I}_{a_1,b_1} - \mathcal{I}_{a_2,b_2}\| \leq m$, *on a :*

$$\|a_1 - a_2\|_{L^\infty(Q)} \leq C \left(|\log \|\mathcal{I}_{a_1,b_1} - \mathcal{I}_{a_2,b_2}\||^{-1} \right)^{\mu},$$

et

$$\|b_1 - b_2\|_{H^{-1}(Q)} \leq C \left(\log |\log \|\mathcal{I}_{a_1,b_1} - \mathcal{I}_{a_2,b_2}\||^{\tilde{\mu}} \right)^{-1}.$$

On remarque que l'inégalité de stabilité du potentiel b dans les théorèmes 0.2.7, 0.2.8 et 0.2.9, est de type log-log, alors qu'elle est de type log dans les théorèmes 0.2.4, 0.2.5 et 0.2.6.

0.3 Problèmes inverses associés à l'équation de Schrödinger magnétique

La deuxième partie de la thèse est consacrée à l'étude de problèmes inverses aux limites associés à l'équation de Schrödinger magnétique non autonome. On s'intéresse plus particulièrement à la détermination des deux coefficients inconnus apparaissant devant les termes différentiels d'ordres zero et un, par rapport aux variables d'espace, appelés "potentiel électrique" et "potentiel magnétique" dans la terminologie physique.

Deux approches complémentaires de ce problème, mais très différentes du point de vue de l'analyse mathématique, ont été mises en œuvre, qui s'adaptent en fait à la structure des données disponibles. Le premier cas envisagé est celui où l'ensemble des données est constitué par une **infinité d'observations** contenues dans l'opérateur de Dirichlet-Neumann magnétique. L'approche correspondante est ici assez similaire à celle utilisée dans la section 0.2, puisqu'elle procède par construction d'un ensemble adapté de solutions de type "optique géométrique" de l'équation de Schrödinger magnétique. Le second cas est celui où l'ensemble des données ne contient qu'un **nombre fini** ($n+1$ exactement) textbf de mesures latérales partielles de la solution de l'équation de Schrödinger magnétique. Dans ce cas précis, la méthode de résolution s'inspire de celle de Buckheim-Klibanov [**14**] , et s'appuie essentiellment sur une inégalité de Carleman spécifique à l'équation considérée.

Les résultats de cette section sont décrits dans [P2,P5].

0.3.1 Identification à partir de l'opérateur Dirichlet-Neumann magnétique

Soit Ω un ouvert borné et simplement connexe de $\mathbb{R}^n$, avec $n \geq 3$, de frontière $\Gamma = \partial\Omega$, de classe $\mathcal{C}^\infty$. Pour $T > 0$ fixé, on considère le système suivant

$$\left\{ \begin{array}{ll} (i\partial_t + \Delta_A + q(x,t))u = 0, & (x,t) \in Q = \Omega \times (0,T) \\ u(.,0) = u_0, & x \in \Omega, \\ u = f, & (x,t) \in \Sigma = \Gamma \times (0,T), \end{array} \right. \tag{0.5}$$

associé à la condition initiale u_0 et la la donnée de Dirichlet f. Le potentiel électrique $q \in W^{2,\infty}(0,T; W^{1,\infty}(\Omega))$ est à valeurs réelles, et Δ_A désigne l'opérateur de Laplace magnétique associé au potentiel $A = (a_j)_{1\leq j\leq n} \in \mathcal{C}^3(\Omega; \mathbb{R}^n)$,

$$\Delta_A = \sum_{j=1}^{n} (\partial_j + ia_j)^2 = \Delta + 2iA \cdot \nabla + i\,\mathrm{div}(A) - |A|^2.$$

L'objectif principal est donc ici d'étudier le problème de stabilité dans l'identification du potentiel électrique q et de la 2-forme différentielle

$$d\alpha_A = \sum_{i,j=1}^{n} \Big(\frac{\partial a_i}{\partial x_j} - \frac{\partial a_j}{\partial x_i}\Big) dx_j \wedge dx_i,$$

associée au potentiel magnétique A, à partir de la connaissance de l'opérateur Dirichlet-Neumann

$$\Lambda_{A,q} : (u_0, f) \quad\longmapsto\quad \Big(u(.,T), (\partial_\nu + iA\cdot\nu)u\Big).$$

Notons que dans le cas particulier où $n = 3$, la 2-forme $d\alpha_A$ s'identifie au vecteur rot $A \in \mathbb{R}^3$, appelé “champ magnétique" induit par A dans la terminologie physique.

Avant de passer à l'étude du problème inverse il convient de résoudre (en s'inspirant de [5]; [**16**]) le problème direct associé au système (0.5) à l'aide du résultat d'existence et d'unicité suivant.

Theorem 0.3.1. *Soient $q \in W^{1,\infty}(Q)$ et $A \in \mathcal{C}^1(\Omega)$. Alors, pour tout $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ et tout $f \in H^{2,1}(\Sigma) := H^2(0,T; L^2(\Gamma)) \cap L^2(0,T; H^1(\Gamma))$ vérifiant $f(.,0) = \partial_t f(.,0) = 0$ dans Ω, il existe une unique solution $u \in \mathcal{C}(0,T; H^1(\Omega))$ du système (0.5). De plus, $\partial_\nu u \in L^2(\Sigma)$, et il existe une constante $C > 0$, dépendant* *seulement de Ω, T, $\|A\|_{\mathcal{C}^1(\Omega)}$ et $\|q\|_{W^{1,\infty}(Q)}$ telle que l'estimation*

$$\|u(.,t)\|_{H^1(\Omega)} + \|\partial_\nu u\|_{L^2(\Sigma)} \leq C \left(\|u_0\|_{H^2(\Omega)} + \|f\|_{H^{2,1}(\Sigma)} \right)$$

est vraie pour tout $t \in (0,T)$.

En s'inspirant notamment de l'analyse développée dans [**5**]; [**16**]; [**15**] il est ensuite possible de montrer que $\Lambda_{A,q}$ détermine le potentiel q et le champ magnétique $d\alpha_A$, de façon stable.

Theorem 0.3.2. *(Stabilité par rapport à l'opérateur Dirichlet-Neumann magnétique). Etant donné $M > 0$, on considère $q_i \in W^{2,\infty}(0,T; W^{1,\infty}(\Omega))$ et $A_i \in C^3(\Omega)$, $i = 1,2$, vérifiant $\|A_i\|_{H^\beta(\Omega)} \leq M$ pour un certain réel $\beta > (n+3)/2$, $\|q_i\|_{W^{2,\infty}(0,T;W^{1,\infty}(\Omega))} \leq M$, ainsi que la condition*

$$(A_1, q_1) = (A_2, q_2) \text{ sur } \Sigma.$$

Alors, il existe $\varepsilon > 0$ tel que l'inégalité de stabilité suivante

$$\|d\alpha_{A_1} - d\alpha_{A_2}\|_{L^\infty(\Omega)} \leq C \left(\|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\|^{1/2} + |\log \|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\||^{-\mu} \right)^s,$$

a lieu pour des constantes $C > 0$ et $(\mu, s) \in (0,1)^2$, ne dépendant que de Ω, ε, M et T, et ceci à condition que

$$\|A_i\|_{W^{3,\infty}(\Omega)} \leq \varepsilon, \; i = 1,2. \tag{0.6}$$

De plus, dans le cas particulier où div $A_i = 0$ pour tout $i = 1,2$, alors il existe $C > 0$ et $(\mu, m) \in (0,1)^2$, toutes ces constantes ne dépendant que de Ω, ε, M et T, telles que

$$\|q_1 - q_2\|_{H^{-1}(Q)} \leq C\Phi_m(\|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\|),$$

avec

$$\Phi_m(\eta) = \begin{cases} |\log|\log|\log\eta|^{\mu}||^{-1} & \textit{if } \eta < m, \\ \\ \dfrac{1}{m}\eta & \textit{if } \eta \geq m. \end{cases}$$

Notons que la condition (0.6) imposée aux A_i, $i = 1, 2$, dans le théorème 0.3.2, s'introduit en fait naturellement durant la phase de construction des solutions optiques géométriques du système (0.5).

0.3.2 Identification à partir de $n + 1$ observations latérales de la solution

L'objectif de cette section, développée dans [P5], est d'identifier simultanément les coefficients A et q à partir d'un nombre fini de mesures latérales et partielles de la solution du système (0.5). Pour cela, il est en fait nécessaire de supposer que les potentiels électrique et magnétique s'écrivent sous la forme particulière suivante

$$q(x,t) = \beta(t)q(x) \text{ et } A(x,t) = \chi(t)A(x), \tag{0.7}$$

où β et χ sont deux fonctions prises dans $\mathcal{C}^3(0, T; \mathbb{R})$, qui vérifient

$$\chi\left(\frac{T}{2}\right) = \beta\left(\frac{T}{2}\right) = 0,\ \chi'\left(\frac{T}{2}\right) \neq 0,\ \beta'\left(\frac{T}{2}\right) \neq 0. \tag{0.8}$$

Dans ces conditions, le système (0.5) associé à la condition de Dirichlet homogène $f = 0$, et dont la condition “initiale" est imposée (pour des raisons techniques qui ne réduisent pas la généralité des résultats à venir) en $t = T/2$ plutôt qu'en $t = 0$, se met sous la forme suivante :

$$\begin{cases} \Big(-i\partial_t + (i\nabla + \chi(t)A)^2 + \beta(t)q\Big)u = 0 & \text{dans } Q, \\ u\left(\,\cdot\,, \frac{T}{2}\right) = u_0 & \text{dans } \Omega, \\ u = 0 & \text{sur } \Sigma. \end{cases} \tag{0.9}$$

Comme les potentiels électrique et magnétique sont trois fois continûment dérivables par rapport au temps, on obtient un résultat d'existence et d'unicité plus fort que celui du Theorème 0.3.1, pour peu que la condition “initiale" u_0 et le second membre f perturbant le système associé à (0.9), soient suffisamment réguliers.

Theorem 0.3.3. *Soient $q \in L^\infty(\Omega; \mathbb{R})$ et $A \in H^1(\Omega; \mathbb{R}^n)$ tel que div $A = 0$ dans*

Ω. *On rappelle que* β *et* χ *sont fixées dans* $\mathcal{C}^3(0,T;\mathbb{R})$ *et satisfont la condition* (0.8). *Alors, pour tout* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ *tel que*

$$\Delta^k u_0 \in H^2(\Omega) \cap H_0^1(\Omega), \ k = 1, 2,$$

et tout $f \in W^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)$, *il existe une unique solution*

$$u \in \mathcal{C}^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)) \cap \mathcal{C}^3(0,T; L^2(\Omega)),$$

du problème de Cauchy

$$\begin{cases} (-i\partial_t + (i\nabla + \chi A)^2 + \beta q)u = f, & \text{dans } Q = \Omega \times (0,T), \\ u\left(\cdot, \frac{T}{2}\right) = u_0 & \text{dans } \Omega. \end{cases}$$

De plus il existe une constante $C > 0$ *telle que*

$$\|\partial_t^j u(.,t)\|_{H^1(\Omega)} \leq C \sum_{k=0}^{j} \|\Delta^k u_0\|_{H^1(\Omega)}, \quad j = 0, 1, 2, \quad t \in (0,T). \tag{0.10}$$

L'objectif est ici d'améliorer les résultats de stabilité inverse démontrés dans [**3**]; [**18**] en supposant que l'un des deux coefficients a ou q est identiquement nuls. Il s'agit donc d'obtenir un résultat d'identification stable et simultanée de la partie spatiale des potentiels électrique et magnétique, à partir d'un nombre fini d'obeservations de solution du système (0.9).

Theorem 0.3.4. *Soit* $M > 0$ *et soit* $\mathcal{V}$ *un voisinage arbitraire de la frontière* Γ. *On considère* $(A_j, q_j) \in H^1(\Omega; \mathbb{R}^n) \times L^\infty(\Omega; \mathbb{R})$, $j = 1, 2$, *tels que* $\|A_j\|_{H^1(\Omega)} + \|q_j\|_{L^\infty(\Omega)} \leq M$, *div* $A_j = 0$ *dans* Ω, *et*

$$A_1 = A_2 \ \text{et} \ q_1 = q_2 \ \text{dans} \mathcal{V}.$$

Alors, il existe $\Gamma^+ \subset \Gamma$ *ainsi que* $n+1$ *conditions initiales* $u_{0,k}$, $k = 0, ..., n$, *pour* *lesquelles on a l'inéglité de stabilité suivante :*

$$\|A_1 - A_2\|^2_{L^2(\Omega)} + \|q_1 - q_2\|^2_{L^2(\Omega)} \leq C \sum_{k=0}^{n} \|\partial_\nu \partial_t^2 u_{1,k} - \partial_\nu \partial_t^2 u_{2,k}\|^2_{L^2(0,T;L^2(\Gamma^+))}.$$

Ici, la constante $C > 0$ ne depend que de Ω, T, β et χ, et $u_{j,k}$, pour $j = 1, 2$ et $k = 0, \ldots, n$, désigne la solution du système (0.9) où est remplacé par $u_{0,k}$.

Le résultat du théorème précédent est optimal, en ce sens qu'il garantit la détermination (Hölder-stable) de $n + 1$ coefficients inconnus (les n composantes de A et q) à partir de $n+1$ mesures seulement (qui plus est, latérales et partielles) de la solution du système (0.9).

Par ailleurs, il faut noter que la condition de divergence nulle imposée au potentiel magnétique A est inévitable dans le contexte du problème inverse examiné ici, puisqu'il est manifeste que le "champ magnétique" $d\alpha_A$ ne détermine A qu'à un changement de jauge près. En réalité, la condition divergentielle choisie ne fait rien d'autre que fixer la "classe de jauge" des potentiels magnétiques admissibles (en l'occurence, celle dite "de Coulomb"). C'est le prix à payer si l'on veut avoir une chance d'ídentifer directement le potentiel magnétique, plutôt que le "champ magnétique" $d\alpha_A$ qu'il induit, à partir d'observations latérales de la solution.

Ensuite, il convient de préciser, même si cela n'a pas été fait l'énoncé du théorème 0.3.4, que les $n + 1$ conditions initiales $u_{0,k}$, $k = 0, 1, \ldots, n$, mentionnées plus haut, peuvent de fait être décrites de façon totalement explicites.

Enfin, on peut remarquer que si les données employées au théorème 0.3.4 pour identifier A et q, sont nettement plus "économiques" que celles utilisées au théorème 0.3.2 dans le même but, cela implique en autres conditions, que les coefficients considérés aient d'une part la forme préscrite par (0.7), ce qui est très contraignant, et d'autre part une régularité supérieure (d'ordre 3, au minimum) par rapport à la variable temporelle. De plus, le résultat théorème du 0.3.4 ne détermine que la partie spatiale des potentiels électrique et magnétique inconnus.

Contents

Introduction

A lot of physical phenomena can be modeled by partial differential equations describing the evolution of the problem parameters. The coefficients appearing in these equations model important unknown properties of the media under study. The identification of such internal parameters from observed measurements models the so-colled inverse problem.

The study of inverse problems often requieres a good knowlegde of the direct problem which consists in finding solutions to these equations provided the knowledge of the applied sources, the initial and boundary conditions, and the properties of the medium. For the direct problem, the physical properties of the medium are assumed to be known and we rather aim to find the output results.

The notion of inverse problems have been steadily gaining popularity in modern science, since the middle of the 20th century, notably to designate the determination, through input-output or cause-effect experiments, of unknown coefficients in the physics equations. It designates the best possible reconstruction of missing informations and it appears in several fields like medical imaging, geology, sattelites, radar theory... In view of J.B.Keller [**28**] two problems are inverse to each other if the formulation of one puts the other in question. In 1923, Jacques Hadamard [**22**] introduced the notion of well-posed problems. Indeed, It is a problem that satisfies the following conditions:

1. The existence of a solution

2. The uniqueness of the solution

3. The stability of the solution (the continuous dependence on data).

According to Hadamard, the most of inverse problems are ill posed, in the sens that a small error in the measurement can cause a huge error in the identification.

My research interests during the preparation of my PhD were focused on the analysis of inverse problems of determining some coefficients appearing in non-autonomous hyperbolic, and magnetic systems. The main and common objective is to establish uniqueness and stability results for the identification of these coefficients. In a first time, we will focus our interest on the study of inverse problems for a wave equation. Then, we will move to the analysis of inverse problems concerning a magnetic Schrödinger equation. The thesis can then be divided into two principal parts:

Part I: Hyperbolic inverse problems: The first part of the thesis is devoted to the study of inverse problems associated to the wave equation in a bounded domain $Q = \Omega \times (0,T)$, where Ω is a bounded domain of $\mathbb{R}^n$ for $n \geq 2$ with $\mathcal{C}^\infty$ boundary $\Gamma = \partial\Omega$. We denote by $\Sigma = \Gamma \times (0,T)$ and we introduce the following system

$$\begin{cases} \partial_t^2 u - \Delta u + a(x,t)\partial_t u(x,t) + b(x,t)u = 0, & \text{in } Q, \\ u(\,\cdot\,,0) = u_0,\ \partial_t u(\,\cdot\,,0) = u_1, & \text{in } \Omega, \\ u = f, & \text{on } \Sigma, \end{cases} \tag{0.11}$$

where u_0 and u_1 are the initial conditions and f is the Dirichlet data which will be used to probe the system. Our goal in this part is to treat the uniqueness and the stability issues for the determination of the time and space-dependent coefficients a "and/or" b appearing in the equation (0.11), from measurements made on the solution u.

Actually, there is a wide mathematical literature on this topic but it is mostly concerned with space-dependent unknown coefficients. Indeed, In the case where the unknown coefficient is depending only on the spatial variable, Rakesh and Symes [**35**] proved by means of geometric optics solutions, a uniqueness result in recovering a time-independent potential in a wave equation from global Neumann data. The uniqueness by local Neumann data, was considered by Eskin [**21**] and Isakov [**25**] In [**7**] Bellassoued, Choulli and Yamamoto proved a log-type stability estimate, in the case where the Neumann data are observed on any arbitrary subset of the boundary. Isakov and Sun [**27**] proved that the knowledge of local Dirichlet-to-Neumann map yields a stability result of Hölder type in determining a coefficient in a subdomain. As for the stability obtained from global Neumann data, one can see Sun [**44**] Cipolatti and Lopez [**17**] The case of Riemannian manifold was considered by Bellassoued and Dos Santos Ferreira [**8**] Stefanov and Uhlmann [**43**]

All the mentioned papers are concerned only with time-independent coefficients. In the case where the coefficient is also depending on the time variable, there is a uniqueness result proved by Ramm and Rakesh [**36**] in which they showed that a time-dependent coefficient, with a compact support, appearing in a wave equation with zero initial conditions, can be uniquely determined from the knowledge of global Neumann data, but only in a precise subset of the cylindrical domain Q that is made of lines making an angle of $\pi/4$ with the t-axis and meeting the planes $t = 0$ and $t = T$ outside $\overline{Q}$. However, inspired by the work of [**46**] Isakov proved in [**26**] that the time-dependent coefficient can be recovered from the responses of the medium for all possible initial data, over the whole domain Q.

As for uniqueness results, we have also the paper of Stefanov [**41**] in which he proved that a time-dependent potential, tempered in the time variable and appearing in a wave equation can be uniquely recovered from scattering data and the paper of Ramm and Sjöstrand [**37**] in which they proved a uniqueness result for a tempered time-dependent coeffcient on an infinite time-space cylindrical domain $\Omega \times \mathbb{R}_t$.

The stability in the case where the unknown coefficient is also depending on the time variable, was considered by Salazar [**38**] who extended the result of Ramm and Sjöstrand [**37**] to more general coefficients and he established a stability result for compactly supported coefficients provided T is sufficiently large. We also refer to the works of Kian [**29**]; [**30**] who followed techniques used by Bellassoued, Jellali and Yamamoto [**10**]; [**11**] and proved uniqueness and a $\log$-$\log$ type stability estimate from the knowledge of partial Neumann data. As for stability results from global Neumann data we refer to Waters [**49**] who derived, in Riemannian case, conditional Hölder stability estimates for the X-ray transform of the time-dependent potential appearing in the wave equation. As for results of hyperbolic inverse problems dealing with single measurement data, one can see [**2**]; [**4**]; [**12**]; [**19**]; [**24**]; [**42**] and the references therein.

The first inverse problem considered in the thesis is to know whether the knowledge of boundary Neumann measurements can uniquely determine the time-dependent coefficient a appearing in the dissipative wave equation (0.11) with $b = 0$ and $(u_0, u_1) = (0, 0)$. It was actually a challenging problem as unique determination of the coefficient, in the case where the initial conditions are fixed to zero, is not even guarateed on the entire space-time domain Q. Because of this obstruction to uniqueness, we were able to prove a uniqueness theorem for this problem, only in a precise subset of the domain Q. A better description of this obstruction to uniqueness and a rigorous proof of the uniquness result are given in Chapter 2.

The second inverse problem considered in this thesis, is the purpose of Chapter

4 and it consists in the identification of the zero order space and time-dependent coefficient b appearing in (0.11) with $a = 0$, from differents sets of data. We will first derive a log-type stability estimate with respect to the Dirichlet-to-Neumann map for the restriction of b to a suitable subset of the domain Q. This is provided that b is known outside the above subset. Then, by enlarging the set of data, we will prove that b can be stably retrieved in larger subsets of the domain, including the whole domain itself.

This problem was actually treated by Ramm and Rakesh in [**36**] in the case where the initial conditions are fixed to zero and they proved a uniqueness result which is valid in a specific subset of the domain. By studying the problem under consideration, we radher aimed to know whether the time-dependent coefficient b can be stably recovered in some specific subsets of the cylindrical domain Q from boundary measurements made on the solution u of the wave equation. It was also a challenging problem as unique determination of the coefficient, in the homogeneous initial conditions case , is not guarateed on the whole domain Q. Nevertheless, we were able to stably retrieve the coefficient outside the non-uniqueness "**cloaking**" areas.

The results given in Chapitre 2 and 4 were, thereafter, partially generalised and improved by establishing stability estimates for the two space and time-dependent unknown coefficients a and b appearing in the wave equation (0.11) with respect to Neumann boundary measurements. Indeed, inspired by the work of Bellassoued [**5**] we will prove stability estimates in the recovery of the unknown coefficients a and b via different sets of measurements and over different subsets of the domain Q. The reader will find rigerous proofs of the stability results in Chapter 5.

The derivation of the results of Chapter 3, 4 and 5, boils down on building a sufficiently large set of geometric optics solutions for the systems under investigation, and on the relation between the light-ray transform and the Fourier transform. We will give more explicit informations about the construction of the geometric optics solutions and the notion of the light-ray transform in Chapter 1. Chapter 3 will be devoted to prove a technical result that plays a crucial role in proving our main results.

Part II: Inverse problems for magnetic Schrödinger equations: The second part of the thesis is devoted to treat inverse problems that concern magnetic Schrödinger equations. Our primary focus is to study the stability issue in determining the magnetic field induced by the magnetic potential A and the time-dependent electric potential q appearing in the following Schrödinger equation

$$\begin{cases} (i\partial_t + \Delta_A + q(x,t))u = 0, & \text{in } Q, \\ u(.,0) = u_0, & \text{in } \Omega, \\ u = f, & \text{on } \Sigma, \end{cases} \tag{0.12}$$

from the knowledge of Neumann boundary observations. Here u_0 is the initial condition and f is the Dirichlet data which will be used in order to probe the system.

Actually, the problem of recovering coefficients in the magnetic Schrödinger equation was treated by many authors. In [**6**] Bellassoued and Choulli considered the problem of recovering the magnetic potential A from the knowledge of the Dirichlet-to-Neumann map $\Lambda_A(f) = (\partial_\nu + i\nu.A)u$ for $f \in L^2(\Sigma)$, associated to the Schrödinger equation with zero initial data. As it was noted in [**20**] the Dirichlet-to-Neumann map Λ_A is invariant under the gauge transformation of the magnetic potential. Namely, given $\varphi \in \mathcal{C}^1(\overline{\Omega})$ such that $\varphi_{|\Gamma} = 0$, we have

$$e^{-i\varphi}\Delta_A e^{i\varphi} = \Delta_{A+\nabla\varphi}, \quad e^{-i\varphi}\Lambda_A e^{i\varphi} = \Lambda_{A+\nabla\varphi}, \tag{0.13}$$

and $\Lambda_A = \Lambda_{A+\nabla\varphi}$. Therefore, the magnetic potential A can not be uniquely determined by the Dirichlet-to-Neumann map Λ_A. In geometric terms, the magnetic potential A defines the connection given by the one form $\alpha_A = \sum_{j=1}^n a_j dx_j$. The non uniqueness manifested in (0.13) says that the best one can hope to recover from the Dirichlet-to-Neumann map is the 2-form

$$d\alpha_A = \sum_{i,j=1}^{n} \Big(\frac{\partial a_i}{\partial x_j} - \frac{\partial a_j}{\partial x_i} \Big) dx_j \wedge dx_i,$$

called the magnetic field. Bellassoued and Choulli proved in dimension $n \geq 2$ that the knowledge of the Dirichlet-to-Neumann map Λ_A Hölder stably determines the magnetic field $d\alpha_A$.

In the presence of a time-independent electric potential, the inverse problem of determining the magnetic field $d\alpha_A$ and the electric potential q from boundary observations was first considered by Sun [45] in the case $n \geq 3$. He showed that $d\alpha_A$ and q can be uniquely determined when $A \in W^{2,\infty}$, $q \in L^\infty$ and $d\alpha_A$ is small in the L^∞ norm. In [15] Benjoud studied the inverse problem of recovering the magnetic field $d\alpha_A$ and the electric potential q from the knowledge of the Dirichlet-to-Neumann map. Assuming that the potentials are known in a neighborhood of the boundary, she proved a stability estimate with respect to arbitrary partial boundary observations.

In the Riemannian case, Bellassoued [5] proved recently a Hölder-type stability estimate in the recovery of the magnetic field $d\alpha_A$ and the time-independent electric potential q from the knowledge of the Dirichlet-to-Neumann map associated to the Shrödinger equation with zero initial data. In the absence of the magnetic potential A, the problem of recovering the electric potential q on a compact Riemannian manifold was solved by Bellassoued and Dos Santos Ferreira [9]

The problem of determining time-dependent electromagnetic potentials appearing in a Schrödinger equation was treated by Eskin [20] Using a geometric optics construction, he prove the uniqueness for this problem in domains with obstacles. In unbounded domains and in the absence of the magnetic potential, Choulli , Kian and Soccorsi [16] treated the problem of recovering the time-dependent scalar potential q appearing in the Schrödinger equation from boundary observations. Assuming that the domain is a 1-periodic cylindrical waveguide, they proved logarithmic stability for this problem.

In Chapter 7, we will deal with the inverse problem associated to the equation (0.12) and we will prove by means of techniques used in [**5**]; [**15**] and under some hypothesis on the coefficients A and q, a "$\log$-type" stability estimate in the recovery of the magnetic field and a "$\log$-$\log$-$\log$-type" stability inequality in the determination of the time-dependent electric potential from the knowledge of the Dirichlet-to-Neumann map.

The last inverse problem we deal with in this thesis will be the purpose of Chapter 8, wich consists in determining two coefficients appearing in a magnetic Schrödinger equation in a bounded domain $Q = \Omega \times (0, T)$ with lateral boudary Σ, where Ω is a bounded domain of $\mathbb{R}^n$ for $n \geq 1$. So, let us consider the following system

$$\begin{cases} \Big(-i\partial_t + (i\nabla + \chi(t)a(x))^2 + \beta(t)q(x) \Big) u(x,t) = 0, & \text{in } Q, \\ u(x, \frac{T}{2}) = u_0(x) & \text{in } \Omega, \\ u(x,t) = 0, & \text{on } \Sigma, \end{cases}$$

Our gaol is to identify the magnetic potential a and the magnetic field q from a finite number of Neumann measurements taken in a suboundary of the Σ.

To our knowledge, there is a few results on the recovery of coefficients appearing in a Schrödinger equation, from a finite number of boundary measurements. By a method based essentially on an appropriate Carleman estimate, Baudouin and Puel [**3**] showed that the electric potential in the Shrödinger equation can be stably recovered from a single boundary measurement. In [**18**] Cristofol and Soccorsi proved a Lipschitz stability in recovering the magnetic field in the Schrödinger equation from a finite number of observations, measured on a subboundary for different choices of u_0.

In Chapter 8, we improve the two above mentionned results by showing that the electric potential and the magnetic field can be stably and simultaneously recovered from a finite number of boundary observations of the solution. We stress out that the simultaneous identification of the magnetic field and the electric potential in the Schrödinger equation cannot be directly obtained from the results of [**3**] and [**18**] as the electric (resp. magnetic) potential is a two (resp. first) order perturbation of the Laplacian.

Part I

Hyperbolic Inverse problems with time dependent-coefficients

CHAPTER 1

Hyperbolic problems and light-ray transfrom

1.1 Introduction

The understanding of real world phenomena and technology is today in a wide part based on partial differential equations. In the first part of the thesis, we will focus our interest on the analysis of a family of inverse coefficients problems concerning the hyperbolic equation the most classic describing the wave propagation phenomenon.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of $\mathbb{R}^n$,with $n \geq 2$ with smooth boundary Γ. We denote by $Q = \Omega \times (0,T)$ the domain of propagation and by $\Sigma = \Gamma \times (0,T)$ its boundary. Let us consider the following system

$$\begin{cases} \partial_t^2 u - \Delta u + a(x,t)\partial_t u + b(x,t)u = 0 & \text{in } Q, \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x) & \text{in } \Omega, \\ u(x,t) = f(x,t) & \text{on } \Sigma, \end{cases} \tag{1.1}$$

where x and t are respectively the space and the time variables, u_0 and u_1 are the initial conditions and f is the Dirichlet data used to probe the system. Our main goal is to treat the uniqueness and the stability issues in determining the time-dependent coefficients a "*and/or*" b appearing in the wave equation (1.1), from measurements made on the solution u.

The study of inverse problems often requires useful knowledge of the direct problem. So, we shall first treat the well posedness for the initial boundary value problem (1.1). Throughout the first section of this chapter, we focus our attention on a proof of the existence of a unique solution u to the initial boundary value problem (1.1). Moreover, we study its regularity and we prove an energy estimate that u satisfies. The second part of this chapter is devoted to the construction of special geometrical optics solutions to the wave equation (1.1). The construction of such solutions will play a crucial role in the main statements proofs. We end this chapter by introducing the light-ray transform that will also be used in the derivation of the main results.

1.2 Well-posedness

In this section, we aim to prove the existence, uniqueness and smoothness properties of the solution u of the initial boundary value problem (1.1). The materials used in this section are picked up from Lions and Magenes [**34**] Bellassoued and Yamamoto [**13**] and Lasiecka, Lions and Triggiani [**31**]

Theorem 1.2.1. *Let $T>0$, $a\in L^\infty(Q)$ and $b\in L^\infty(Q)$ be given. Assume that*

$$u_0\in H^1(\Omega),\quad u_1\in L^2(\Omega),\quad \textit{and}\quad f\in H^1(\Sigma).$$

Suppose in addition that the following compatibility condition is satisfied that is $f(\,\cdot\,,0)=u_{0|\Gamma}$. Then, there exists a unique solution u to the equation (1.1) such that

$$u\in\mathcal{C}([0,T];H^1(\Omega))\cap\mathcal{C}^1([0,T];L^2(\Omega)).$$

Moreover, $\partial_\nu u\in L^2(\Sigma)$ and there exists a positive constant $C>0$ such that we *have*

$$\|\partial_\nu u\|_{L^2(\Sigma)}+\|u(\,\cdot\,,t)\|_{H^1(\Omega)}+\|\partial_t u(\,\cdot\,,t)\|_{L^2(\Omega)} \leq C\left(\|f\|_{H^1(\Sigma)}+\|u_0\|_{H^1(\Omega)}+\|u_1\|_{L^2(\Omega)}\right). \tag{1.2}$$

Proof. We first split u into two terms $u = v + w$, where v and w solve respectively the following initial boundary value problems

$$\begin{cases} (\partial_t^2-\Delta)v=0 & \text{in } Q,\\ v(\,\cdot\,,0)=u_0, \quad \partial_t v(\,\cdot\,,0)=u_1 & \text{in } \Omega,\\ v=f & \text{on } \Sigma, \end{cases}$$

and

$$\begin{cases} (\partial_t^2-\Delta+a\partial_t+b)w=-(a\partial_t+b)v & \text{in } Q,\\ w(\,\cdot\,,0)=0, \quad \partial_t w(\,\cdot\,,0)=0 & \text{in } \Omega,\\ w=0 & \text{on } \Sigma. \end{cases}$$

Using the fact that $u_0 \in H^1(\Omega)$, $u_1 \in L^2(\Omega)$ and $f \in H^1(\Sigma)$, one can see from **[13]**[Theorem 2.2.5] (see also **[31]**[Theorem 2.1]), that there exists a unique solution $v \in \mathcal{C}(0,T;H^1(\Omega))$ such that $\partial_t v \in \mathcal{C}(0,T;L^2(\Omega))$ and $\partial_\nu v \in L^2(\Sigma)$. Moreover, there exists a positive constant $C>0$ such that

$$\|\partial_\nu v\|_{L^2(\Sigma)}+\|v(\,\cdot\,,t)\|_{H^1(\Omega)}+\|\partial_t v(\,\cdot\,,t)\|_{L^2(\Omega)} \leq C\Big(\|f\|_{H^1(\Sigma)}+\|u_0\|_{H^1(\Omega)}+\|u_1\|_{L^2(\Omega)}\Big). \tag{1.3}$$

Furthermore, since $-(a\partial_t+b)v \in L^2(Q)$, then from the theory developped in **[34]**[chapter 3, Section 8], one can check that there exists a unique solution $w \in \mathcal{C}(0,T;H^1(\Omega)$ such that $\partial_t w \in \mathcal{C}(0,T;L^2(\Omega))$ and satisfying

$$\begin{aligned} \|\partial_t w(\,\cdot\,,t)\|_{L^2(\Omega)}+\|w(\,\cdot\,,t)\|_{H_0^1(\Omega)} &\leq C\|a\partial_t v+bv\|_{L^2(Q)}\\ &\leq C\Big(\|f\|_{H^1(\Sigma)}+\|u_0\|_{H^1(\Omega)}+\|u_1\|_{L^2(\Omega)}\Big) \end{aligned} \tag{1.4}$$

Next, we consider a $\mathcal{C}^2$ vector field N on $\overline{\Omega}$ satisfying

$$N(x) = \nu(x), \quad x \in \Gamma; \qquad |N(x)| \leq 1, \quad x \in \Omega.$$

Multiplying both sides of the second system by $< N, \nabla w >$ and integrate over $Q = \Omega \times (0, T)$, we obtain

$$\begin{aligned} I := -\int_0^T \int_\Omega \quad (a(x,t)\partial_t w + b(x,t)w) < N, \nabla w > \, dx\, dt &= \int_0^T \int_\Omega \partial_t^2 w < N, \nabla w > \, dx\, dt \\ - \quad \int_0^T \int_\Omega \Delta w < N, \nabla w > \, dx\, dt &+ \int_0^T \int_\Omega a(x,t)\, \partial_t w < N, \nabla w > \, dx\, dt \\ + \int_0^T \int_\Omega b(x,t)\, w < N, \nabla w > \, dx\, dt &:= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

From (1.4) and proceeding as in **[13]**[Chapter 2, Section 2.4], one can check that

$$\begin{aligned} \Big| \int_0^T \int_\Gamma |\partial_\nu w|^2 \, d\sigma\, dt \Big| \;\; &\leq \;\; C \Big[|I_2| + \Big(\|f\|_{H^1(\Sigma)} + \|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)} \Big)^2 \Big] \\ &\leq \;\; C \Big[|I| + |I_1| + |I_3| + |I_4| + \Big(\|f\|_{H^1(\Sigma)} + \|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)} \Big)^2 \Big]. \end{aligned}$$

Therefore, using the fact that

$$|I| + |I_1| + |I_3| + |I_4| \leq C \left(\|f\|_{H^1(\Sigma)} + \|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)} \right)^2,$$

we obtain

$$\|\partial_\nu w\|_{L^2(\Sigma)} \leq C \left(\|f\|_{H^1(\Sigma)} + \|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)} \right). \tag{1.5}$$

Collecting (1.3)-(1.5), we get the desired estimate (1.2). $\square$

1.3 Construction of geometric optics solutions

The construction of special solutions to some partial differential equations have played an important role in the resolution of inverse problems. The present section is devoted to the construction of suitable geometrical optics solutions concentrated near lines with any direction $\omega \in \mathbb{S}^{n-1}$, for the dissipative wave equation (1.1), which are key ingredients to the proof of the main results of this thesis.

We shall first state the following lemma which is needed to prove the main result of this section.

Lemma 1.3.1. *(see* **[34]***[Chapter 3, Section 8]) Let T, M_1, $M_2 > 0$, $a \in L^\infty(Q)$ and $b \in L^\infty(Q)$, such that $\|a\|_{L^\infty(Q)} \leq M_1$ and $\|b\|_{L^\infty(Q)} \leq M_2$. Assume that $F \in L^2(0,T;L^2(\Omega))$. Then, there exists a unique solution u to the following equation*

$$\begin{cases} \partial_t^2 u - \Delta u + a(x,t)\partial_t u + b(x,t)u(x,t) = F(x,t) & \text{in } Q, \\ u(x,0) = 0 = \partial_t u(x,0) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Sigma, \end{cases} \tag{1.6}$$

such that

$$u \in \mathcal{C}([0,T];H_0^1(\Omega)) \cap \mathcal{C}^1([0,T];L^2(\Omega)).$$

Moreover, there exists a constant $C > 0$ such that

$$\|\partial_t u(\,\cdot\,,t)\|_{L^2(\Omega)} + \|\nabla u(\,\cdot\,,t)\|_{L^2(\Omega)} \leq C\|F\|_{L^2(0,T;L^2(\Omega))}. \tag{1.7}$$

Armed with the above lemma, we may now construct suitable geometrical optics solutions to the dissipative wave equation (1.1) and to its retrograde problem. For this purpose, we consider $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and notice that for all $\omega \in \mathbb{S}^{n-1}$ the function ϕ given by

$$\phi(x,t) = \varphi(x + t\omega), \tag{1.8}$$

solves the following transport equation

$$(\partial_t - \omega \cdot \nabla)\phi(x,t) = 0. \tag{1.9}$$

We are now in position to prove the following statement

Lemma 1.3.2. *Let $(a,b) \in W^{1,\infty}(Q) \times L^\infty(Q)$, $i = 1,2$. Given $\omega \in \mathbb{S}^{n-1}$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, we consider the function ϕ defined by (1.8). Then, for any $\lambda > 0$, the following equation*

$$\partial_t^2 u - \Delta u + a(x,t)\partial_t u + b(x,t)u = 0 \quad \text{in } Q, \tag{1.10}$$

admits a unique solution

$$u^+ \in \mathcal{C}([0,T]; H^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega)),$$

of the following form

$$u^+(x,t) = \phi(x,t)A^+(x,t)e^{i\lambda(x\cdot\omega+t)} + r_\lambda^+(x,t), \tag{1.11}$$

where $A^+(x,t)$ is given by

$$A^+(x,t) = \exp\Big(-\frac{1}{2}\int_0^t a(x+(t-s)\omega, s)\,ds\Big), \tag{1.12}$$

and $r_\lambda^+(x,t)$ satisfies

$$r_\lambda^+(x,0) = \partial_t r_\lambda^+(x,0) = 0, \ \text{ in } \Omega, \quad r_\lambda^+(x,t) = 0 \ \text{ on } \Sigma. \tag{1.13}$$

Moreover, there exists a positive constant $C > 0$ such that

$$\lambda\|r_\lambda^+(\,\cdot\,,t)\|_{L^2(\Omega)} + \|\partial_t r_\lambda^+(\,\cdot\,,t)\|_{L^2(\Omega)} + \|\nabla r_\lambda^+(\,\cdot\,,t)\|_{L^2(\Omega)} \leq C\|\varphi\|_{H^3(\mathbb{R}^n)}. \tag{1.14}$$

Proof. We put $g(x,t) = -\Big(\partial_t^2 - \Delta + a(x,t)\partial_t + b(x,t)\Big)\Big(\phi(x,t)A^+(x,t)e^{i\lambda(x\cdot\omega+t)}\Big)$.

In light of (1.10) and (1.11), it will be enough to prove the existence of r_λ^+ satisfying

$$\begin{cases} \Big(\partial_t^2 - \Delta + a(x,t)\partial_t + b(x,t)\Big) r_\lambda^+ = g(x,t), \\ r_\lambda^+(x,0) = \partial_t r_\lambda^+(x,0) = 0, \\ r_\lambda^+(x,t) = 0, \end{cases} \tag{1.15}$$

and obeying the estimate (1.14). From (1.9) and using the fact that $A^+(x,t)$ solves the following equation

$$2(\partial_t - 2\omega \cdot \nabla) A^+(x,t) = -a(x,t) A^+(x,t),$$

we obtain the following identity

$$g(x,t) = -e^{i\lambda(x\cdot\omega+t)}\Big(\partial_t^2 - \Delta + a(x,t)\partial_t + b(x,t)\Big)\Big(\phi(x,t)A^+(x,t)\Big) = -e^{i\lambda(x\cdot\omega+t)} g_0(x,t),$$

where $g_0 \in L^2(0,T,L^2(\Omega))$. Thus, in view of Lemma 1.3.1, there exists a unique solution

$$r_\lambda^+ \in \mathcal{C}([0,T];H_0^1(\Omega)) \cap \mathcal{C}^1([0,T];L^2(\Omega)),$$

satisfying (1.15). Let us now define by w the following function

$$w(x,t) = \int_0^t r_\lambda^+(x,s)\,ds. \tag{1.16}$$

We integrate the equation (1.15) over $[0,t]$, for $t \in (0,T)$. Then, in view of (1.16), we have

$$\begin{aligned} \Big(\partial_t^2 - \Delta + a(x,t)\partial_t + b(x,t)\Big) w(x,t) \quad &= \quad \int_0^t g(x,s)\,ds + \int_0^t \Big(b(x,t) - b(x,s)\Big) r_\lambda^+(x,s)\,ds \\ &\quad + \int_0^t \partial_s a(x,s) r_\lambda^+(x,s)\,ds. \end{aligned}$$

Therefore, w is a solution to the following equation

$$\begin{cases} \Big(\partial_t^2 - \Delta + a(x,t)\partial_t + b(x,t)\Big)w(x,t) = F_1(x,t) + F_2(x,t) & \text{in } \ Q, \\ w(x,0) = 0 = \partial_t w(x,0) & \text{in } \ \Omega, \\ w(x,t) = 0 & \text{on } \ \Sigma, \end{cases}$$

where F_1 and F_2 are given by

$$F_1(x,t) = \int_0^t g(x,s)\,ds, \tag{1.17}$$

and

$$F_2(x,t) = \int_0^t \Big(b(x,t) - b(x,s)\Big) r_\lambda^+(x,s)\,ds + \int_0^t \partial_s a(x,s) r_\lambda^+(x,s)\,ds.$$

Let $\tau \in [0,T]$. Applying Lemma 1.3.1 on the interval $[0,\tau]$, we get

$$\|\partial_t w(\,\cdot\,,\tau)\|^2_{L^2(\Omega)} \ \leq \ C\Big(\|F_1\|^2_{L^2(0,T;L^2(\Omega))} + T\,(M_1^2 + 4M_2^2)\int_0^\tau\!\!\int_\Omega\!\int_0^t |r_\lambda^+(x,s)|^2\,ds\,dx\,dt\Big).$$

From (1.16), we get

$$\begin{aligned} \|\partial_t w(\,\cdot\,,\tau)\|^2_{L^2(\Omega)} \ &\leq \ C\Big(\|F_1\|^2_{L^2(0,T;L^2(\Omega))} + \int_0^\tau\!\!\int_0^t \|\partial_s w(\,\cdot\,,s)\|^2_{L^2(\Omega)}\,ds\,dt\Big) \\ &\leq \ C\Big(\|F_1\|^2_{L^2(0,T;L^2(\Omega))} + T\int_0^\tau \|\partial_s w(\,\cdot\,,s)\|^2_{L^2(\Omega)}\,ds\Big). \end{aligned}$$

Therefore, from Gronwall's Lemma, we find out that

$$\|\partial_t w(\,\cdot\,,\tau)\|^2_{L^2(\Omega)} \leq C\|F_1\|^2_{L^2(0,T;L^2(\Omega))}.$$

As a consequence, in light of (1.16), we conclude that $\|r_\lambda^+(\,\cdot\,,t)\|_{L^2(\Omega)} \leq C\|F_1\|_{L^2(0,T;L^2(\Omega))}$. Further, according to (1.17), F_1 can be written as follows

$$F_1(x,t) = \int_0^t g(x,s)\,ds = \frac{1}{i\lambda}\int_0^t g_0(x,s)\partial_s(e^{i\lambda(x\,\cdot\,\omega + s)})\,ds.$$

Integrating by parts with respect to s, we conclude that there exists a positive constant $C > 0$ such that

$$\|r_\lambda^+(\,\cdot\,,t)\|_{L^2(\Omega)} \leq \frac{C}{\lambda}\|\varphi\|_{H^3(\mathbb{R}^n)}.$$

Finally, since $\|g\|_{L^2(0,T;L^2(\Omega))} \leq C\|\varphi\|_{H^3(\mathbb{R}^n)}$, the energy estimate (1.7) associated to the problem (1.15) yields

$$\|\partial_t r_\lambda^+(\,\cdot\,,t)\|_{L^2(\Omega)} + \|\nabla r_\lambda^+(\,\cdot\,,t)\|_{L^2(\Omega)} \leq C\|\varphi\|_{H^3(\mathbb{R}^n)}.$$

This completes the proof of the lemma. $\square$

As a consequence we have the following lemma

Lemma 1.3.3. *Let $(a_i, b_i) \in W^{1,\infty}(Q) \times L^\infty(Q)$, for $i = 1, 2$. Given $\omega \in \mathbb{S}^{n-1}$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, we consider the function ϕ defined by (1.8). Then, the following equation*

$$\partial_t^2 u - \Delta u - a(x,t)\partial_t u + b(x,t)u = 0 \quad \textit{in } Q, \tag{1.18}$$

admits a unique solution

$$u^- \in \mathcal{C}([0,T]; H^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega)),$$

of the following form

$$u^-(x,t) = \varphi(x + t\omega)A^-(x,t)e^{-i\lambda(x\,\cdot\,\omega + t)} + r_\lambda^-(x,t), \tag{1.19}$$

where $A^-(x,t)$ is given by

$$A^-(x,t) = \exp\Big(\frac{1}{2}\int_0^t a(x + (t-s)\omega, s)\,ds\Big), \tag{1.20}$$

and $r_\lambda^-(x,t)$ *satisfies*

$$r_\lambda^-(x,T) = \partial_t r_\lambda^-(x,T) = 0, \;\; \textit{in} \;\; \Omega, \quad r_\lambda^-(x,t) = 0 \;\; \textit{on} \;\; \Sigma. \tag{1.21}$$

Moreover, there exists a constant $C > 0$ *such that*

$$\lambda \| r_\lambda^-(\,\cdot\,,t)\|_{L^2(\Omega)} + \|\partial_t r_\lambda^-(\,\cdot\,,t)\|_{L^2(\Omega)} \leq C\|\varphi\|_{H^3(\mathbb{R}^n)}. \tag{1.22}$$

Proof. We prove this result by proceeding as in the proof of Lemma 1.3.2. Putting

$$\widetilde{g}(x,t) = -\Big(\partial_t^2 - \Delta - a(x,t)\partial_t + b(x,t)\Big)\Big(\phi(x,t)A^-(x,t)e^{-i\lambda(x\,\cdot\,\omega+t)}\Big).$$

Then, it would be enough to see that if $r^- = r_\lambda^-$ is solution to the following system

$$\begin{cases} \Big(\partial_t^2 - \Delta - a(x,t)\partial_t + b(x,t)\Big)r^-(x,t) = \widetilde{g}(x,t) & \text{in } \; Q, \\ r^-(x,T) = 0 = \partial_t r^-(x,T) & \text{in } \; \Omega, \\ r^-(x,t) = 0 & \text{on } \; \Sigma, \end{cases}$$

then, $r_\lambda^+(x,t) = r^-(x,T-t)$ is a solution to (1.15) with $g(x,t) = \widetilde{g}(x,T-t)$ and $a(x,t)$, $b(x,t)$ are replaced by $a(x,T-t)$ and $b(x,T-t)$. □

1.4 The light-ray transform

In this section, we focus our interest on one of the most important key ingredients in the proof of the main results of this thesis: the light-ray transrom. This transform belongs to the class of X-rays transforms. It appears in the study of hyperbolic equations and it allows to recover coefficients from boundary information. The X-ray transform is an integral transform of practical importance which it is introduced by Fritz John in 1938. It is considered as one of the cornerstones of modern integral geometry.

Definition 1.4.1. *(The X-ray transform) Let $\omega \in \mathbb{S}^{n-1}$ and $\omega^\perp$ the hyperplane through the origin orthogonal to ω. We parameterize a line $l(x,\omega)$ in $\mathbb{R}^n$ by specifying its direction $\omega \in \mathbb{S}^{n-1}$ and the point x where the line intersects the hyperplane $\omega^\perp$. The X-ray transform of the function $f \in L^1(\mathbb{R}^n)$ is given by*

$$X(f)(x,\omega) = \int_{\mathbb{R}} f(x - s\omega)\, ds.$$

We see that $X(f)(x,\omega)$ is the integral of f over the line $l(x,\omega)$ parallel to ω which passes through $x \in \omega^\perp$.

The integral transforms most relevant for tomography are the X-ray transform and the Radon transform. In two dimensions these transforms coincide apart from a parametrization: we parameterize $\omega \in \mathbb{S}^1$ by its polar angle φ and define a vector $\omega^\perp$ orthogonal to ω such that

$$\omega = (\cos\varphi, \sin\varphi), \quad \omega^\perp = (-\sin\varphi, \cos\varphi).$$

Then, the points in the subspace $\omega^\perp$ are given by $\omega^\perp = \{s\omega^\perp, \quad s \in \mathbb{R}\}$ and we have the relation

$$X(f)(s\omega^\perp, \omega) = P(f)(s, \omega^\perp),$$

where $P(f)$ is the Radon transform given by

$$P(f)(s,\omega) = \int_{\omega^\perp} f(y - s\omega)\, dy, \quad s \in \mathbb{R}.$$

In this thesis we are radher intereseted in the light-ray transform which is the integral of a function f over lines of direction $\tilde{\omega} = (\omega, 1)$ where $\omega \in \mathbb{S}^{n-1}$. The light-ray transform can be defined as follows:

Definition 1.4.2. *(The light-ray transform) Let $\omega \in \mathbb{S}^{n-1}$. The light-ray transform R maps a function defined in $\mathbb{R}^{n+1}$ into the set of its line integrals and it is* *given by the following formula*

$$R(f)(x,\omega) := \int_{\mathbb{R}} f(x - s\omega, s)\, ds,$$

We see that $R(f)(x,\omega)$ is the integral of f over the lines $\{(x - s\omega, s),\ \ s \in \mathbb{R},\quad x \in \mathbb{R}\}$. There is an immediate relation linking the light-ray transform of a function f to the Fourier transform of f, that will prove to be useful:

Lemma 1.4.1. *Let $\omega \in \mathbb{S}^{n-1}$ and $f \in L^1(\mathbb{R}^{n+1})$. We have for all $\xi \in \mathbb{R}^n$,*

$$\mathcal{F}(R(q)(\,\cdot\,,\omega))(\xi) = \widehat{f}(\xi, \omega \cdot \xi).$$

Proof. By the change of variables $x = y - t\omega$ we have for all $\xi \in \mathbb{R}^n$ and all $\omega \in \mathbb{S}^{n-1}$,

$$\begin{aligned}
\int_{\mathbb{R}^n} R(f)(y,\omega) e^{-iy\cdot\xi}\, dy &= \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}} f(y - t\omega, t)\, dt \Big) e^{-iy\cdot\xi}\, dy \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^n} f(x,t) e^{-ix\cdot\xi} e^{-it(\omega\cdot\xi)}\, dx\, dt \\
&= \widehat{f}(\xi, \omega\cdot\xi).
\end{aligned}$$

□

As we will see in the next chapters, Lemma 1.4.1 will be used in order to adress questions of uniquness, stability and reconstruction...

CHAPTER 2

The uniqueness issue for hyperbolic inverse problems

The contents of this chapter are collected in a paper accepted at *ARIMA*.

2.1 Introduction

In this chapter, we study of the uniqueness issue for inverse coefficients problems for non-autonomous hyperbolic equations. Our primary focus is to deepen the concept of local non-uniqueness in the recovery of time-dependent coefficients in a wave equation and give a better description of the non-uniqueness cloaking area.

We emphasise on a basic property of a second order hyperbolic equation and show how the value of the solution of a second order hyperbolic equation can be effected by its value in some finite domain of the data, which is actually due to a fundamental concept, concerning only solutions of hyperbolic equations, called the domain of dependence.

In the second part of this chapter, we deal with an inverse problem for a dissipative wave equation and we prove a uniqueness result, outside the cloacking region, for the determination of an absorbing coefficient of first order, appearing in this equation.

Throughout the rest, the domain of propagation is modeled by the cylinder $Q = \Omega \times (0,T)$, where Ω is assumed to be a bounded domain of $\mathbb{R}^n$, $n \geq 2$ with a smooth boundary $\Gamma = \partial\Omega$. We assume without loss of generality that Ω is of origin 0. We denote by $\Sigma = \Gamma \times (0,T)$ its lateral boundary.

2.2 Local non-uniqueness in determining time-dependent coefficients

We consider the following second order initial boundary value problem

$$\begin{cases} \partial_t^2 u - \Delta u + a(x,t)\partial_t u + b(x,t)u = 0 & \text{in } Q, \\ u(\,\cdot\,,0) = 0,\ \partial_t u(\,\cdot\,,0) = 0 & \text{in } \Omega, \\ u = f & \text{on } \Sigma, \end{cases} \tag{2.1}$$

with $f \in H^1(\Sigma)$, $b \in \mathcal{C}^1(Q)$ and $a \in \mathcal{C}^2(Q)$. Our goal is to show that there is no hope to recover the time-dependent unknown coefficients a and b over the whole domain Q since the initial conditions are zero.

Before stating the main result of this section, let us first start by introducing some notations. We denote by $\mathscr{C}$ the following conic set

$$\mathscr{C} = \Big\{ (x,t) \in Q, \quad |x| \leq \frac{\text{Diam}\,(\Omega)}{2} - t, \quad 0 \leq t \leq \frac{\text{Diam}\,(\Omega)}{2} \Big\}.$$

Moreover, for $t' \in \left(0, \dfrac{\text{Diam}(\Omega)}{2}\right)$, we define the set $\mathscr{C}_\alpha$ as follows

$$\mathscr{C}_\alpha = \bigcup_{0 \le \alpha \le t'} \mathscr{D}(\alpha) = \bigcup_{0 \le \alpha \le t'} \mathscr{C} \cap \{t = \alpha\}.$$

Finally, we denote by $S = \partial \mathscr{C} \cap (\Omega \times (0, t'))$ and $\partial \mathscr{C}_\alpha = S \cup \mathscr{D}(t') \cup \mathscr{D}(0)$.

Our main interest lies in showing a non-uniqueness result in the cloaking area $\mathscr{C}$, for the identification of the time-dependent coefficients a and b arising in the equation (2.1), from measurements given by the so called Dirichlet-to-Neumann map defined as follows:

Definition 2.2.1. *Let $a \in W^{1,\infty}(Q)$ and $b \in L^\infty(Q)$. The hyperbolic Dirichlet-to-Neumann map is defined as follows*

$$\begin{array}{rcl} \Lambda_{a,b} : H^1(\Sigma) & \longrightarrow & L^2(\Sigma) \\ f & \longmapsto & \partial_\nu u, \end{array}$$

with u is the solution of (2.1). We recall that here ν denotes the unit outward normal to Γ a at x and ∂_ν stands for $\nabla u \cdot \nu$.

In view of Theorem 1.2.1, one can see that $\Lambda_{a,b}$ is continuous from $H^1(\Sigma)$ to $L^2(\Sigma)$. We denote by $\|\Lambda_{a,b}\|$ its norm in $\mathcal{L}(H^1(\Sigma), L^2(\Sigma))$.

2.2.1 Preliminaries

In this section, we prove a fundamental result borrowed from [**23**] We need first to state the following classical inequality which can be found in [**13**]

Lemma 2.2.1. *(Gronwall's inequality) Let an interval I be $[a, +\infty)$ or $[a, c]$ or $[a, c)$ with $a < c$. Let α, β and u be real valued functions defined in the interval I. Suppose that β and u are continuous and that the negative part of α is integrable on any closed and bounded subinterval of I. Then, we have*

1. If the function β is non-negative and u satisfies the integral inequality

$$u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)\, ds, \quad \textit{for any } t \in I,$$

then u satisfies

$$u(t) \leq \alpha(t) + \int_a^t \alpha(s)\beta(s) \, \exp\Big(\int_s^t \beta(r)\, dr\Big)\, ds, \quad \textit{for any } t \in I.$$

2. If, in addition the function α is non-decreasing, then,

$$u(t) \leq \alpha(t) \, \exp\Big(\int_a^t \beta(s)\, ds\Big).$$

Proposition 2.2.1. *Let us denote by u the solution of the dissipative wave equation (2.1). Then, $u(x,t) = 0$ on the set $\mathscr{C}$.*

Proof. We denote by $P = \partial_t^2 - \Delta + a(x,t)\partial_t + b(x,t)$. A simple calculation gives us

$$\begin{aligned}
\int_{\mathscr{C}_\alpha} 2Pu(x,t)\,\partial_t u(x,t)\, dx\, dt &= \int_{\mathscr{C}_\alpha} 2\partial_t^2 u(x,t)\partial_t u(x,t)\, dx\, dt - \int_{\mathscr{C}_\alpha} 2\Delta u(x,t)\,\partial_t u(x,t)\, dx\, dt \\
&\quad + \int_{\mathscr{C}_\alpha} 2a(x,t)|\partial_t u(x,t)|^2\, dx\, dt + \int_{\mathscr{C}_\alpha} 2b(x,t)u\partial_t u\, dx\, dt \\
&= \int_{\mathscr{C}_\alpha} \partial_t(|\partial_t|^2 + |\nabla u|^2)\, dx\, dt + \int_{\mathscr{C}_\alpha} \sum_{j=1}^n \partial_j(\partial_t u\, \partial_j u)\, dx\, dt \\
&\quad + \int_{\mathscr{C}_\alpha} 2a(x,t)|\partial_t u(x,t)|^2\, dx\, dt + \int_{\mathscr{C}_\alpha} 2b(x,t)u\partial_t u\, dx\, dt.
\end{aligned}$$

Then, using the above identity, we see that

$$\begin{aligned}
&\int_{\mathscr{C}_\alpha} 2Pu(x,t)\partial_t u(x,t)\, dx\, dt = \int_{\mathscr{C}_\alpha} \partial_t e(x,t)\, dx\, dt + \int_{\mathscr{C}_\alpha} \sum_{j=1}^n \partial_j X_j(x,t)\, dx\, dt \\
&+ \int_{\mathscr{C}_\alpha} 2a(x,t)|\partial_t u(x,t)|^2\, dx\, dt, + \int_{\mathscr{C}_\alpha} 2b(x,t)u(x,t)\partial_t u(x,t)\, dx\, dt,
\end{aligned}$$

where $e(x,t) = |\partial_t u(x,t)|^2 + |\nabla u(x,t)|^2$ and $X_j(x,t) = -2\partial_t u(x,t)\partial_j u(x,t)$. Next, by applying the divergence theorem, one gets

$$\int_{\mathscr{C}_\alpha} 2Pu(x,t)\partial_t u(x,t)\,dx\,dt = \int_S \Big(e(x,t)\eta + \sum_{j=1}^{n} X_j(x,t)\mu_j\Big)\,d\sigma + \int_{\mathscr{D}(t')} e(x,t')\,dx$$
$$- \int_{\mathscr{D}(0)} e(x,0)\,dx + \int_{\mathscr{C}_\alpha} 2a(x,t)|\partial_t u(x,t)|^2\,dx\,dt + \int_{\mathscr{C}_\alpha} 2b(x,t)u\partial_t u\,dx\,dt, \qquad (2.2)$$

where $d\sigma$ denotes the surface element of S and the vector $(\eta, \mu_1, \mu_2, ..., \mu_n) \in \mathbb{R}^{n+1}$ is the outward unit normal vector at $(x,t) \in S$ such that

$$\eta = (\sum_{j=1}^{n} \mu_j^2)^{1/2}. \qquad (2.3)$$

On the other hand, from Cauchy-Schwartz inequality and (2.3), we can see that

$$\int_S (e(x,t)\eta + \sum_{j=1}^{n} X_j(x,t)\mu_j) d\sigma$$
$$\geq \int_S (|\partial_t u(x,t)|^2 + |\nabla u(x,t)|^2)\,\eta - 2|\partial_t u(x,t)||\nabla u(x,t)|\,\eta\,d\sigma \geq 0. \qquad (2.4)$$

Then, since $e(x,0) = 0$ we get from (2.2) and (2.4) this estimation

$$\int_{\mathscr{D}(t')} e(x,t')\,dx \;\leq\; \int_{\mathscr{C}_\alpha} 2Pu\,\partial_t u(x,t)\,dx\,dt - \int_{\mathscr{C}_\alpha} 2a(x,t)|\partial_t u(x,t)|^2\,dx\,dt$$
$$- \int_{\mathscr{C}_\alpha} 2b(x,t)(x,t)u(x,t)\partial_t(x,t)\,dx\,dt.$$

Now, using the fact that $Pu(x,t) = 0$ for any $(x,t) \in \mathscr{C}_\alpha$, we get

$$\int_{\mathscr{D}(t')} e(x,t')\,dx \leq C \int_0^{t'} \int_{\mathscr{D}(t)} \Big(e(x,t) + |u(x,t)|^2\Big)\,dx\,dt. \qquad (2.5)$$

Now bearing in mind that

$$|u(x,t')|^2 = |u(x,0)|^2 + \int_0^{t'} \partial_t(|u(x,t)|^2)\,dt \leq \int_0^{t'} e(x,t)\,dx. \qquad (2.6)$$

Thus, from (2.5) and (2.6) we deduce that

$$\int_{\mathscr{D}(t')} \left(e(x,t') + |u(x,t')|^2 \right) dx \leq \int_0^{t'} \int_{\mathscr{D}(t)} \left(e(x,t) + |u(x,t)|^2 \right) dx\, dt.$$

In view of Lemma 2.2.1, we end up deducing that $u(x,t) = 0$ for any $x \in \mathscr{D}(t')$ and $t' \in (0, \mathrm{Diam}(\Omega)/2)$. This completes the proof of the lemma. □

2.2.2 The non uniqueness result

From proposition 2.2.1, one can see that the unique determination of the coefficients a and b appearing in the wave equation (2.1) from the knowledge of the Dirichlet-to-Neumann map is not guaranteed in the conic set $\mathscr{C}$, which can be stated as follows.

Theorem 2.2.1. *Let $a \in W^{1,\infty}(Q)$ and $b \in L^\infty(Q)$. such that Supp $(a) \subset \mathscr{C}$ and Supp $(b) \subset \mathscr{C}$. Then, we have $\Lambda_{a,b} = \Lambda_{0,0}$.*

Proof. Let $f \in H^1(\Sigma)$ and u satisfy

$$\begin{cases} \partial_t^2 u - \Delta u + a(x,t)\partial_t u + b(x,t)u = 0 & \text{in } \ Q, \\ u(x,0) = 0, \ \partial_t u(x,0) = 0 & \text{in } \ \Omega, \\ u = f & \text{on } \ \Sigma. \end{cases}$$

From Proposition 2.2.1, we have $u = 0$ in the conic set $\mathscr{C}_r$. Then, using the fact that Supp$a \subset \mathscr{C}$ and Supp $(b) \subset \mathscr{C}$, we deduce that u solves also the following hyperbolic boundary-value problem

$$\begin{cases} \partial_t^2 v - \Delta v = 0 & \text{in } \ Q, \\ v(x,0) = 0 = \partial_t v(x,0) & \text{in } \ \Omega, \\ v = f & \text{on } \ \Sigma. \end{cases}$$

Then, we conclude that $\Lambda_{a,b}(f) = \Lambda_{0,0}(f)$ for all $f \in H^1(\Sigma)$. □

2.3 Uniqueness in determining a first order time-dependent coefficient

In this section, we aim to treat the inverse problem of determining a time-dependent coefficient of first order appearing in a dissipative wave equation. More precisely, our objective is to adress the uniqness issue for the inverse problem of determining the absorbing coefficient a appearing in the following equation

$$\begin{cases} \partial_t^2 u - \Delta u + a(x,t)\partial_t u = 0 & \text{in } Q, \\ u(\,\cdot\,,0) = 0,\ \ \partial_t u(\,\cdot\,,0) = 0 & \text{in } \Omega, \\ u = f & \text{on } \Sigma, \end{cases} \tag{2.7}$$

where $f \in H^1(\Sigma)$, and the coefficient $a \in \mathcal{C}^2(\overline{Q})$ is assumed to be real valued. In light of Theorem 1.2.1, it is well known that if the compatibility condition is satisfied, that is $f(\,\cdot\,,0) = 0$, then, there exists a unique solution u to the equation (2.7) that belongs to the following space

$$u \in \mathcal{C}([0,T], H^1(\Omega)) \cap \mathcal{C}^1([0,T], L^2(\Omega)).$$

Moreover, there exists a constant $C > 0$ such that we have

$$\|\partial_\nu u\|_{L^2(\Sigma)} \leq \|f\|_{H^1(\Sigma)}, \quad f \in H^1(\Sigma), \tag{2.8}$$

where ν denotes the unit outward normal to Γ at x and $\partial_\nu u$ stands for $\nabla u \cdot \nu$. In the present section, we prove that the knowledge of the Dirichlet-to-Neumann map $\Lambda_a := \Lambda_{a,0}$ can uniquely determine the time-dependent absorbing coefficient a, but only in precise subset of the domain Q and provided that it is known outside of this region.

2.3.1 Backward and forward light cones

This section is devoted to set some notations that we will use in what follows. Let $r > 0$ be such that $T > 2\,r$ and $\Omega \subseteq B(0, r/2) = \Big\{x \in \mathbb{R}^n,\ |x| < r/2\Big\}$. We set $Q_r = B(0, r/2) \times (0, T)$. We consider the annular region around the domain Ω,

$$\mathscr{A}_r = \left\{x \in \mathbb{R}^n,\ \frac{r}{2} < |x| < T - \frac{r}{2}\right\},$$

and the forward and backward light cones:

$$\mathscr{C}_r^+ = \left\{(x,t) \in Q_r,\ |x| < t - \frac{r}{2},\, t > \frac{r}{2}\right\},$$

$$\mathscr{C}_r^- = \left\{(x,t) \in Q_r,\ |x| < T - \frac{r}{2} - t,\, T - \frac{r}{2} > t\right\},$$

$$\mathscr{C}_r = \left\{(x,t) \in Q_r,\ |x| \leq \frac{r}{2} - t,\ 0 \leq t \leq \frac{r}{2}\right\}.$$

Moreover, we denote

$$Q_r^* = \mathscr{C}_r^+ \cap \mathscr{C}_r^- \text{ and } Q_{r,*} = Q \cap Q_r^*, \quad \text{and} \quad Q_{r,\sharp} = Q \cap \mathscr{C}_r^+.$$

We remark that the open subset $Q_{r,*}$ is made of lines making an angle of $45°$ with the t-axis and meeting the planes $t = 0$ and $t = T$ outside $\overline{Q}_r$. We notice that $Q_{r,*} \subset Q$ and that in the particular case where $\Omega = B(0, r/2)$, we have $Q_{r,*} = Q_r^*$ (see Figure 2.1).

Finally, we introduce the admissible set of the absorbing coefficients a. Given $a_0 \in \mathcal{C}^2(\overline{Q}_r)$ and $M > 0$ we define

$$\mathcal{A}(a_0, M) = \Big\{a \in \mathcal{C}^2(\overline{Q}_r), a = a_0 \quad \text{in} \quad \overline{Q}_r \setminus Q_{r,*}, \quad \|a\|_{\mathcal{C}^2(Q)} \leq M\Big\}.$$

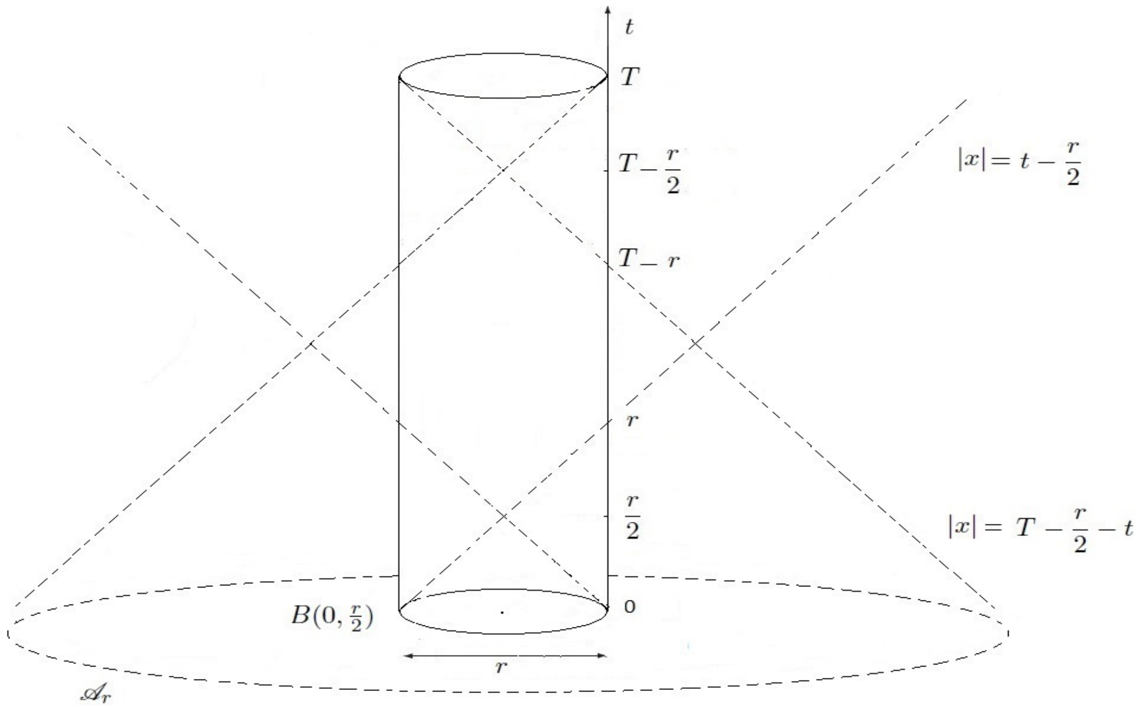

Figure 2.1: Particular case: $\Omega = B(0, r/2)$

Having said that, we move now to give a preliminary identity:

2.3.2 A preliminary identity

The main purpose of this section is to establish a preliminary identity for the absorbing coefficient a appearing in the wave equation (2.7), by means of the geometric optics solutions constructed in Chapter 1. Let $\omega \in \mathbb{S}^{n-1}$, $a_1,\ a_2 \in \mathcal{A}(a_0, M)$. We set

$$\widetilde{a}(x,t) = (\widetilde{a}^{-}\widetilde{a}^{+})(x,t) = \exp\Big(-\frac{1}{2}\int_0^t a(x+(t-s)\omega, s)\,ds\Big),$$

where $\widetilde{a}_1^{-}$ and $\widetilde{a}_2^{+}$ are given by

$$\widetilde{a}_1^{-}(x,t) = \exp\Big(\frac{1}{2}\int_0^t a_1(x+(t-s)\omega, s)\,ds\Big), \quad \widetilde{a}_2^{+}(x,t) = \exp\Big(-\frac{1}{2}\int_0^t a_2(x+(t-s)\omega, s)\,ds\Big).$$

Moreover, we define a in $\mathbb{R}^{n+1}$ by $a = a_2 - a_1$ in $\overline{Q}_r$ and $a = 0$ on $\mathbb{R}^{n+1} \setminus \overline{Q}_r$.

Lemma 2.3.1. *Let $\varphi \in \mathcal{C}_0^\infty(\mathscr{A}_r)$ and $a_1, a_2 \in \mathcal{A}(a_0, M)$. Assume that $\Lambda_{a_2} = \Lambda_{a_1}$, then, the following identity holds*

$$\int_Q a(x,t)\varphi^2(x+t\omega)\widetilde{a}(x,t)\,dx\,dt = 0. \tag{2.9}$$

Proof. In light of Lemma 1.3.2, there exists a geometrical optics solution u^+ to the equation

$$\begin{cases} \partial_t^2 u^+ - \Delta u^+ + a_2(x,t)\partial_t u^+ = 0 & \text{in } \ Q, \\ u^+(x,0) = \partial_t u^+(x,0) = 0 & \text{in } \ \Omega, \end{cases}$$

in the following form

$$u^+(x,t) = \varphi(x+t\omega)\widetilde{a}_2^+(x,t)e^{i\lambda(x\cdot\omega+t)} + r_\lambda^+(x,t), \tag{2.10}$$

corresponding to the coefficients a_2, where $r_\lambda^+(x,t)$ satisfies (1.13), (1.14). We denote by f_λ the function

$$f_\lambda(x,t) = u^+(x,t)_{|\Sigma} = \varphi(x+t\omega)\widetilde{a}_2^+(x,t)e^{i\lambda(x\cdot\omega+t)}.$$

We denote by u_1 the solution of

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 + a_1(x,t)\partial_t u_1 = 0 & \text{in } \ Q, \\ u_1(x,0) = \partial_t u_1(x,0) = 0 & \text{in } \ \Omega, \\ u_1(x,t) = f_\lambda(x,t) & \text{on } \ \Sigma. \end{cases}$$

Putting $u = u_1 - u^+$. Then, u is a solution to the following system

$$\begin{cases} \partial_t^2 u - \Delta u + a_1(x,t)\partial_t u = a(x,t)\partial_t u^+ & \text{in } \ Q, \\ u(x,0) = \partial_t u(x,0) = 0 & \text{in } \ \Omega, \\ u(x,t) = 0, & \text{on } \ \Sigma, \end{cases} \tag{2.11}$$

where $a = a_2 - a_1$. On the other hand Lemma 1.3.3 guarantees the existence of a geometrical optic solution u^- to the adjoint problem of (2.7)

$$\begin{cases} \partial_t^2 u^- - \Delta u^- - a_1(x,t)\partial_t u^- - \partial_t a_1(x,t) u^- = 0 & \text{in } Q, \\ u^-(x,T) = \partial_t u^-(x,T) = 0 & \text{in } \Omega, \end{cases}$$

corresponding to the coefficients a_1 and $-\partial_t a_1$, in the form

$$u^-(x,t) = \varphi(x+t\omega)e^{-i\lambda(x\cdot\omega+t)}\widetilde{a}_1{}^-(x,t) + r_\lambda^-(x,t), \tag{2.12}$$

where $r_\lambda^-(x,t)$ satisfies (1.21), (1.22). Multiplying the first equation of (2.11) by u^-, integrating by parts and using Green's formula, we obtain

$$\int_Q a(x,t)\partial_t u^+ \, u^- \, dx\, dt = \int_\Sigma (\Lambda_{a_2} - \Lambda_{a_1})(f_\lambda)\, u^- \, d\sigma\, dt. \tag{2.13}$$

On the other hand, by replacing u^+ and u^- by their expressions, we get

$$\begin{aligned} \int_Q a(x,t)\partial_t u^+ \, u^- \, dx\, dt &= \int_Q a(x,t)\partial_t \varphi(x+t\omega)e^{i\lambda(x\cdot\omega+t)}\widetilde{a}_2^+ r_\lambda^- \, dx\, dt \\ + \int_Q a(x,t)\varphi(x+t\omega)e^{i\lambda(x\cdot\omega+t)}\partial_t \widetilde{a}_2^+ r_\lambda^- \, dx\, dt &+ \int_Q a(x,t)\partial_t\varphi(x+t\omega)\varphi(x+t\omega)(\widetilde{a}_2^+\widetilde{a}_1^-) dx\, dt \\ + \int_Q a(x,t)\varphi^2(x+t\omega)\partial_t\widetilde{a}_2^+\widetilde{a}_1^- \, dx\, dt &+ i\lambda \int_0^T\int_\Omega a(x,t)\varphi(x+t\omega)e^{i\lambda(x\cdot\omega+t)}\widetilde{a}_2^+ r_\lambda^- \, dx\, dt \\ + \int_Q a(x,t)\varphi(x+t\omega)e^{-i\lambda(x\cdot\omega+t)}\widetilde{a}_1^-\partial_t r_\lambda^+ dx\, dt &+ i\lambda\int_Q a(x,t)\varphi^2(x+t\omega)(\widetilde{a}_2^+\widetilde{a}_1^-)\, dx\, dt \\ + \int_Q a(x,t)\partial_t r_\lambda^+ r_\lambda^- \, dx\, dt &= i\lambda\int_Q a(x,t)\varphi^2(x+t\omega)\widetilde{a}\, dx\, dt + \mathcal{I}(\lambda), \end{aligned}$$

where $\widetilde{a} = \widetilde{a}_2^+\widetilde{a}_1^-$. Then, in light of (2.13), we have

$$i\lambda\int_Q a(x,t)\varphi^2(x+t\omega)\widetilde{a}(x,t)\, dx\, dt = \int_\Sigma(\Lambda_{a_2} - \Lambda_{a_1})(f_\lambda)\, u^- \, d\sigma\, dt - \mathcal{I}(\lambda). \tag{2.14}$$

Note that for λ sufficiently large, we have

$$|\mathcal{I}_\lambda| \leq C\|\varphi\|^2_{H^3(\mathbb{R}^n)}. \tag{2.15}$$

Hence, using the fact that $\Lambda_{a_2} = \Lambda_{a_1}$, we deduce from (2.14) and (2.15) and by taking $\lambda \to +\infty$ the desired result. $\square$

2.3.3 The uniqueness result

We move now to give and prove the main statement of this section which lies in the unique determination of the coefficient a appearing in the wave equation (2.7) from the Dirichlet-to-Neumann map Λ_a. We need first to define the following conic set

$$E = \{(\xi, \tau) \in \mathbb{R}^n \setminus \{0_{\mathbb{R}^n}\} \times \mathbb{R},\ |\tau| \leq |\xi|\} \tag{2.16}$$

Theorem 2.3.1. *Let $T > 2Diam(\Omega)$ and $a_i \in \mathcal{A}(a_0, M)$, $i = 1, 2$. Then, we have*

$$\Lambda_{a_2} = \Lambda_{a_1} \quad \textit{implies} \quad a_2 = a_1, \quad \textit{on } Q_{r,*}.$$

Proof. The proof is based on the result we have already obtained in the previous section. In light of (2.9), we have as λ goes to $+\infty$, the following identity

$$\int_Q a(x,t)\varphi^2(x+t\omega)\exp\Big(-\frac{1}{2}\int_0^t a(x+(t-s)\omega, s)\,ds\Big)\,dx\,dt = 0. \tag{2.17}$$

Then, using the fact $a(x,t) = 0$ outside $Q_{r,*}$ and making this change of variables $y = x + t\omega$, one gets

$$\int_0^T \int_{\mathbb{R}^n} a(y - t\omega, t)\varphi^2(y)\exp\Big(-\frac{1}{2}\int_0^t a(y - s\omega, s)\,ds\Big)\,dy\,dt = 0.$$

Bearing in mind that

$$\int_0^T \int_{\mathbb{R}^n} a(y - t\omega, t)\ \varphi^2(y)\exp\Big(-\frac{1}{2}\int_0^t a(y - s\omega, s)\,ds\Big)\,dy\,dt$$
$$= -2\int_0^T \int_{\mathbb{R}^n} \varphi^2(y)\frac{d}{dt}\Big[\exp\Big(-\frac{1}{2}\int_0^t a(y - s\omega, s)\,ds\Big)\Big]\,dy\,dt$$

$$= -2 \int_{\mathbb{R}^n} \varphi^2(y) \Big[\exp\Big(-\frac{1}{2} \int_0^T a(y - s\omega, s)\, ds \Big) - 1 \Big]\, dy.$$

we conclude that

$$\int_{\mathbb{R}^n} \varphi^2(y) \Big[\exp\Big(-\frac{1}{2} \int_0^T a(y - s\omega, s)\, ds \Big) - 1 \Big]\, dy = 0. \tag{2.18}$$

Now, we consider a positive function $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ supported in the unit ball $B(0,1)$ such that $\|\psi\|_{L^2(\mathbb{R}^n)} = 1$. We define

$$\varphi_h(x) = h^{-n/2} \psi\Big(\frac{x - y}{h} \Big), \tag{2.19}$$

where $y \in \mathscr{A}_r$. Then, for $h > 0$ sufficiently small one can see that Supp $\varphi_h \subset \mathcal{C}_0^\infty(\mathscr{A}_r)$ and satisfies

$$\operatorname{Supp} \varphi_h \cap \Omega = \emptyset, \quad \text{and} \quad \operatorname{Supp} \varphi_h \pm T\omega \cap \Omega = \emptyset.$$

Then, as h goes to 0 we deduce from (2.18) with $\varphi = \varphi_h$ that

$$\exp\Big(-\frac{1}{2} \int_0^T a(y - s\omega, s)\, ds \Big) - 1 = 0.$$

Since $a = a_2 - a_1 = 0$ outside $Q_{r,*}$, we conclude that

$$\int_{\mathbb{R}} a(y - t\omega, t)\, dt = 0, \quad \text{a.e} \;\; y \in \mathscr{A}_r, \;\; \omega \in \mathbb{S}^{n-1}. \tag{2.20}$$

On the other hand, if $|y| \leq \dfrac{r}{2}$, we notice that

$$a(y - t\omega, t) = 0, \;\; \forall\, t \in \mathbb{R}. \tag{2.21}$$

Indeed, we have

$$|y - t\omega| \geq |t| - |y| \geq t - \frac{r}{2}, \tag{2.22}$$

hence, $(y-t\omega,t)\notin\mathscr{C}_r^+$ if $t>r/2$, from (2.22). As $(y-t\omega,t)\notin\mathscr{C}_r^+$ if $t\leq r/2$, then we have $(y-t\omega,t)\notin\mathscr{C}_r^+\supset Q_{r,*}$ for $t\in\mathbb{R}$. This and the fact that $a=a_2-a_1=0$ outside $Q_{r,*}$, yield (2.21), and consequently,

$$\int_{\mathbb{R}} a(y-t\omega,t)\,dt=0,\quad |y|\leq\frac{r}{2}.$$

By a similar way, we prove for $|y|\geq T-r/2$, that $(y-t\omega,t)\notin\mathscr{C}_r^-\supset Q_{r,*}$, for $t\in\mathbb{R}$. Then we obtain

$$\int_{\mathbb{R}} a(y-t\omega,t)\,dt=0,\quad \text{a.e. } y\notin\mathscr{A}_r,\ \ \omega\in\mathbb{S}^{n-1}. \tag{2.23}$$

Thus, by (2.20) and (2.23) we find

$$R(a)(y,\omega)=0,\quad \text{a.e }\ y\in\mathbb{R}^n,\ \ \omega\in\mathbb{S}^{n-1}.$$

We now turn our attention to the fourier transform of a. Let $\xi\in\mathbb{R}^n$. In light of Lemma 1.4.1, one can see that $\widehat{a}(\xi,\omega\cdot\xi)=0$. Let us consider $\xi'\in\mathbb{S}^{n-1}$ such that $\xi\cdot\xi'=0$. Setting

$$\omega=\frac{\tau}{|\xi|^2}\cdot\xi+\sqrt{1-\frac{\tau^2}{|\xi|^2}}\cdot\xi'\in\mathbb{S}^{n-1},$$

then $(\xi,\tau)=(\xi,\omega\cdot\xi)\in E$. We then deduce that $\widehat{a}(\xi,\tau)=0$ in the set E. By an argument of analyticity, we extend this result to $\mathbb{R}^{n+1}$. Hence, by the injectivity of the Fourier transform we get the diserd result. This completes the proof of Theorem 2.3.1.

$\square$

Chapter 3

Quantitative estimate of the unique continuation for analytic functions

The main interest of this Chapter lies in establishing an important technical result for analytic functions. Taking inspiration from estimates given in [**1**][Theorem 3] (see also [**48**] and Chapter 3 in [**32**]), we will show an observability inequality that plays a crucial role in proving the main stability results given in the next chapters.

3.1 Preliminaries

In order to express the main goal of this scetion, we start first by recalling Hadamard's three cercle theorem that will be used in what follows.

Theorem 3.1.1. *Let r, r_1, r_2 satisfying $0 < r_1 \leq r \leq r_2$. Let F be a holomrphic function of a complex variable in the ball $B(0, r_2)$. Then, we have*

$$\|F\|_{L^\infty(B_r)} \leq \|F\|^{\theta}_{L^\infty(B_{r_1})} \|F\|^{1-\theta}_{L^\infty(B_{r_2})},$$

where $\theta = \frac{\log \frac{r_2}{r}}{\log \frac{r_2}{r_1}}$.

we shall now prove the following Lemma

Lemma 3.1.1. *Let J be an open interval in $[-\frac{1}{5},\frac{1}{5}]$, and g be an holomorphic function in the unit disc $D(0,1)\subset\mathbb{C}$ satisfying*

$$|g(z)|\leq 1,\quad |z|<1. \tag{3.1}$$

Then, there exist $\gamma\in(0,1)$ and $N>0$ such that the following estimate holds

$$\|g\|_{L^\infty(B(0,\frac{1}{2}))}\leq N\|g\|^\gamma_{L^\infty(J)},$$

where N and γ are depending only on $|J|$.

Proof. We should first notice that for all $n\geq 1$, there exist $(n+1)$ points such that

$$-\frac{1}{5}\leq x_0<...<x_n\leq\frac{1}{5},$$

with $x_i\in\overline{J}$, $i=0,..,n$, and satisfying the following estimation

$$x_i-x_{i-1}\geq\frac{|J|}{n+1},\quad \text{for } i=1,...,n. \tag{3.2}$$

Let $z\in\mathbb{C}$. We denote by

$$P_n(z)=\sum_{i=0}^n g(x_i)\prod_{j\neq i}(z-x_j)\prod_{j\neq i}(x_i-x_j)^{-1}.$$

In order to prove this lemma, we need first to find an upper bound for $|P_n(z)|$. To do that we first notice that for $l'>l$ we have $x_{l'}-x_l=\sum\limits_{i=l+1}^{l'}(x_i-x_{i-1})$. Hence, (3.2) entails that

$$\begin{cases}(x_j-x_i)\geq(j-i)\,\dfrac{|J|}{n+1} & j>i,\\ (x_i-x_j)\geq(i-j)\,\dfrac{|J|}{n+1} & j<i.\end{cases}$$

As a consequence we have the following estimation

$$\prod_{j\neq i}|x_i - x_j| \geq \prod_{j=0}^{i-1}(i-j)\frac{|J|}{(n+1)}\prod_{j=i+1}^{n}(j-i)\frac{|J|}{(n+1)} \geq i!\frac{|J|^i}{(n+1)^i}(n-i)!\frac{|J|^{n-i}}{(n+1)^{n-i}} \quad (3.3)$$

On the other hand, it is easy to see that for $|z| \leq \frac{1}{2}$ and $x_j \in \overline{J}$, $j = 0, .., n$, we have

$$\prod_{j\neq i}|z - x_j| \leq \prod_{j\neq i}(|z| + |x_j|) \leq 1,$$

Putting this together with (3.3), we end up getting this result

$$|P_n(z)| \leq \sum_{i=0}^{n} C_n^i \frac{(n+1)^n}{n!|J|^n}\|g\|_{L^\infty(J)} \leq e\left(\frac{6}{|J|}\right)^n \|g\|_{L^\infty(J)}. \quad (3.4)$$

The next step of the proof is to control $|g(z) - P_n(z)|$. For this purpose, let us introduce the following function: for all $\xi \in \mathbb{C}$, such that $|\xi| = 1$, we denote by

$$G(\xi) = g(\xi)(\xi - z)^{-1}\prod_{j=0}^{n}(\xi - x_j)^{-1}.$$

Applying the residue Theorem, one obtains the following identity

$$\frac{1}{2i\pi}\int_{|\xi|=1} G(\xi)\,d\xi = \left(\mathrm{Res}(G, z) + \sum_{k=0}^{n}\mathrm{Res}(G, x_k)\right) = \Big(g(z) - P_n(z)\Big)\prod_{j=0}^{n}(z - x_j)^{-1}.$$

From this and the hypothesis (3.1), it follows that for $|z| \leq \frac{1}{2}$, and $x_i \in \overline{J}$, we have

$$|g(z) - P_n(z)| \leq 2\left(\frac{1}{2} + \frac{1}{5}\right)^{n+1}\left(1 - \frac{1}{5}\right)^{-(n+1)} \leq 2\left(\frac{7}{8}\right)^n. \quad (3.5)$$

Combining (3.4) with (3.5), one gets

$$||g||_{L^\infty(B(0,1/2))} \leq 2\left(\frac{7}{8}\right)^n + e\left(\frac{6}{|J|}\right)^n \|g\|_{L^\infty(J)}, \quad n \geq 1.$$

To complete the proof of the lemma, we need to minimize the right hand side of the last estimate with respect to n. To this end, let us define the following function

$$\psi(x) = 2e^{-x\log(8/7)} + e\,\|g\|_{L^\infty(J)}\, e^{x\ \log(6/|J|)}, \quad x \in \mathbb{R}.$$

A simple calculation show that the function ψ reaches a minimum at this point

$$x_0 = \left[\log\left(\frac{48}{7|J|}\right)\right]^{-1} \log\left[\frac{\log(8/7)}{e\,\|g\|_{L^\infty(J)}\log(6/|J|)}\right].$$

Then, we end up getting the desired result. $\square$

We move now to establish the second result by the use of Hadamard's three-circle theorem and Lemma 3.1.1.

Lemma 3.1.2. *Let φ be an analytic function in $[-1,1]$, and I an open interval in $[-1,1]$. We assume that there exist positive constants M and ρ such that*

$$|\varphi^{(k)}(s)| \leq \frac{Mk!}{(2\rho)^k}, \qquad k \geq 0,\ \ s \in [-1,1]. \tag{3.6}$$

Then, there exist $N = N(\rho, |I|)$ and $\gamma = \gamma(\rho, |I|)$ such that we have

$$|\varphi(s)| \leq N\|\varphi\|^{\gamma}_{L^\infty(I)} M^{1-\gamma}, \quad \textit{for any}\ \ s \in [-1,1]. \tag{3.7}$$

Proof. In light of (3.6), we have for all $s \in [-1,1]$,

$$\left|\sum_{k\geq 0}\varphi^{(k)}(s)\frac{1}{k!}(z-s)^k\right| \ \leq\ \sum_{k\geq 0} M(2\rho)^{-k}|z-s|^k.$$

This entails that for all $s \in [-1,1]$ and for all $z \in B(s,\rho)$, we have the following estimation

$$\left|\sum_{k\geq 0}\varphi^{(k)}(s)\frac{1}{k!}(z-s)^k\right| \leq M\sum_{k\geq 0}(2\rho)^{-k}\rho^k \leq 2M, \tag{3.8}$$

which implies that φ can be extended to an holomorphic function in $D_\rho = \cup B(s,\rho)$

for $-1 \leq s \leq 1$. We need first to construct a specific open interval in $[-\frac{1}{5}, \frac{1}{5}]$ to apply Lemma 3.1.1. To this end, we notice that

$$[-1,1] \subset \bigcup_{1\leq j\leq n_0} I_j = \bigcup_{1\leq j\leq n_0} \Big[s_j - \frac{\rho}{5}, s_j + \frac{\rho}{5} \Big[, \tag{3.9}$$

where we have putted $s_j = -1 + (2j-1)\rho/5$, $5/\rho \leq n_0 \leq 5/\rho + 1/2$ and assumed that $I_j \cap I_{j'} = \emptyset$, for all j, $j' = 1, ...n_0$, $j \neq j'$. Therefore, the open interval I can be written as the meeting of $(I_j \cap I)$, for $1 \leq j \leq n_0$ where

$$(I_j \cap I) \bigcap_{j\neq j'} (I_{j'} \cap I) = \emptyset, \text{ for } j,\ j' = 1, ..., n_0.$$

Now, we fix $j_0 \in \{1, ..., n_0\}$ such that $|I_{j_0} \cap I| = \max_{1\leq j\leq n_0} |I_j \cap I|$. We define $J_{s_{j_0},\rho} = \frac{1}{\rho}(I_{j_0} \cap I - s_{j_0})$. In light of (3.9) , we deduce that $J_{s_{j_0},\rho}$ is an open interval of $[-\frac{1}{5}, \frac{1}{5}]$. Next, we consider the function g defined on $D(0,1)$ as follows

$$g(z) = \frac{\varphi(s_{j_0} + \rho z)}{2M}.$$

The estimate (3.8) entails that $|g(z)| \leq 1$ for $|z| \leq 1$. Bearing in mind that the function g is holomorphic in the unit disc, we deduce from Lemma 3.1.1 the existence of two constants $N = N(|I|)$ and $\gamma = \gamma(|I|)$ such that the following estimate holds

$$\|g\|_{L^\infty(B(0,1/2))} \leq N\|g\|^{\gamma}_{L^\infty(J_{s_{j_0},\rho})} \leq N(2M)^{-\gamma}\|\varphi\|_{L^\infty(I_{j_0}\cap I)}.$$

This combined with the fact that $\|g\|_{L^\infty(B(0,1/2))} = (2M)^{-1}\|\varphi\|_{L^\infty(B(s_{j_0},\rho/2))}$ yield the following result

$$\|\varphi\|_{L^\infty(B(s_{j_0},\rho/2))} \leq N\|\varphi\|^{\gamma}_{L^\infty(I)} M^{1-\gamma}. \tag{3.10}$$

Now, we aim to extend this result to the interval $[-1,1]$. To this end, let $r > 0$, satisfying

$$\frac{\rho}{2} \leq r \leq 2r \leq \rho. \tag{3.11}$$

Let $(a_j)_{j\geq 1} = (s_j)_{j\geq 1}$ be a sequence such that $[-1,1] \subset \bigcup_{1\leq j\leq n_0} B(a_j, 2r)$ and satisfying

$$\begin{cases} B(a_{j+1}, r) \subset B(a_j, 2r) \quad \text{for} \quad j \in \{j_0, ..., n_0\} \\ \\ B(a_{j-1}, r) \subset B(a_j, 2r) \quad \text{for } j \in \{1, ..., j_0\}. \end{cases} \tag{3.12}$$

In view of Hamdamard's three-circle theorem, using (3.10) and (3.11) we get

$$\|\varphi\|_{L^\infty(B(a_{j_0},2r))} \leq \|\varphi\|^{\theta}_{L^\infty(B(a_{j_0},\frac{\rho}{2}))} \|\varphi\|^{1-\theta}_{L^\infty(B(a_{j_0},\rho))} \leq N\|\varphi\|^{\gamma}_{L^\infty(I)} M^{1-\gamma}, \tag{3.13}$$

where $\theta = \frac{\log \rho/2r}{\log 2}$. Then, using the fact that $B(a_{j+1}, r) \subset B(a_j, 2r)$ for $j \in \{j_0, ..., n_0\}$, we deduce

$$\|\varphi\|_{L^\infty(B(a_{j_0+1},r))} \leq \|\varphi\|_{L^\infty(B(a_{j_0},2r))} \leq N\|\varphi\|^{\gamma}_{L^\infty(I)} M^{1-\gamma}.$$

From this and Hadamard's three-circle theorem, we obtain

$$\|\varphi\|_{L^\infty(B(a_{j_0+1},2r))} \leq \|\varphi\|^{\theta'}_{L^\infty(B(a_{j_0+1},r))} \|\varphi\|^{1-\theta'}_{L^\infty(B(a_{j_0+1},\rho)} \leq N\|\varphi\|^{\gamma}_{L^\infty(I)} M^{1-\gamma},$$

where $\theta' = \frac{\log \rho/2r}{\log \rho/r}$. So, from (3.12) and a repeated application of Hadamard's three circle theorem, we get

$$\|\varphi\|_{L^\infty(B(a_j,2r))} \leq N\|\varphi\|^{\gamma}_{L^\infty(I)} M^{1-\gamma}, \quad j \in \{j_0 + 2, ...n_0\}.$$

By a similar way, we prove that

$$\|\varphi\|_{L^\infty(B(a_j,2r))} \leq N\|\varphi\|^{\gamma}_{L^\infty(I)} M^{1-\gamma}, \quad j \in \{1, ...j_0\}.$$

As a consequence, we obtain

$$\|\varphi\|_{L^\infty([-1,1]))} \leq \sum_{j=1}^{n_0} \|\varphi\|_{L^\infty(B(a_j,2r))} \leq N\|\varphi\|^\gamma_{L^\infty(I)} M^{1-\gamma}.$$

This completes the proof of the Lemma. $\square$

3.2 The quantitative estimate

We now move to prove the main interest of this chapter which claims conditional stability for the analytic continuation. For $\rho > 0$ and $\kappa \in (\mathbb{N} \cup \{0\})^{n+1}$, we put

$$|\kappa| = k_1 + ... + k_{n+1}, \qquad B(0,\rho) = \{x \in \mathbb{R}^n, |x| < \rho\}.$$

Theorem 3.2.1. *Let $\mathcal{O}$ be a non empty open set of the unit ball $B(0,1) \subset \mathbb{R}^d$, $d \geq 2$, and let F be an analytic function in $B(0,2)$, that satisfy*

$$\|\partial^\kappa F\|_{L^\infty(B(0,2))} \leq \frac{M|\kappa|!}{(2\rho)^{|\kappa|}}, \quad \forall \kappa \in (\mathbb{N} \cup \{0\})^d,$$

for some $M > 0$, $\rho > 0$ and $N = N(\rho)$. Then, we have

$$\|F\|_{L^\infty(B(0,1))} \leq NM^{1-\gamma}\|F\|^\gamma_{L^\infty(\mathcal{O})},$$

where $\gamma \in (0,1)$ depends on d, ρ and $|\mathcal{O}|$.

Proof. Notice first that there exists a sequence of open intervals $(I_j)_j$ such that

$$E = I_1 \times ... \times I_j \times ... \times I_d \subset \mathcal{O} \subset B(0,1).$$

Let $x = (x_1, x_2, ..., x_d)$ be fixed in B(0,1). We consider the analytic function φ_j

defined as follows

$$\varphi_j(s) = F(x_1, ..., x_{j-1}, s, x_{j+1}, ..., x_d), \quad s \in [-1, 1]. \tag{3.14}$$

Assume that there exist positive constants M and ρ such that

$$|\varphi_j(s)^{(k)}| \le \frac{Mk!}{(2\rho)^k}, \quad s \in [-1, 1].$$

Then, in view of lemma 3.1.2, we conclude the existence of of $N = N(\rho, |I_j|)$ and $\gamma = \gamma(\rho, |I|)$ such that we have

$$|\varphi_j(s)| \le N \|\varphi_j\|_{L^\infty(I_j)}^{\gamma_j} M^{1-\gamma_j}, \quad s \in [-1, 1],$$

This and (3.14) yield

$$|F(x)| \le N_j \sup_{x_j \in I_j} |F(x)|^{\gamma_j} M^{1-\gamma_j}. \tag{3.15}$$

Therefore, by iterating (3.15), we get

$$|F(x)| \le N_1 N_2^{\gamma_1} ... N_d^{\gamma_1 \cdots \gamma_{d-1}} \sup_{x \in E} |F(x)|^{\gamma_1 \cdots \gamma_d} M^{1-\gamma_1 \cdots \gamma_d}.$$

This completes the proof of the Theorem. $\square$

Chapter 4

Stable determination of a zero-th order time-dependent coefficient

The contents of this chapter are available in a paper published at *Inverse problems 31, 2015 (21pp)*.

4.1 Introduction

In this chapter, we deal with an inverse problem that concerns the wave propagation phenomenon described by the following hyperbolic partial differential equation with a constant speed which is taken equal to one

$$\begin{cases} \partial_t^2 u - \Delta u + b(x,t)u = 0 & \text{in } Q := \Omega \times (0,T), \\ u(\,\cdot\,,0) = u_0,\ \partial_t u(\,\cdot\,,0) = u_1 & \text{in } \Omega, \\ u = f & \text{on } \Sigma := \Gamma \times (0,T), \end{cases} \tag{4.1}$$

where u_0 and u_1 are the initial conditions and f is the Dirichlet data which is used to probe the system. This equation is disturbed by an electric potential b which is a function of both varibales: the spatial variable x that is assumed to live in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary Γ and the time variable $t \in (0,T)$. Here, the time-dependent coefficient of order zero b models some of the physical properties of the medium where the wave propagates and we are interseted in the recovery of this coefficient from various sets of data. From Chapter 1, it is well known that if $b \in \mathcal{C}^1(\overline{Q})$, the initial conditions $(u_0,u_1) \in H^1(\Omega)\times L^2(\Omega)$ and the Dirichlet data $f \in H^1(\Sigma)$ and satisfies the compatibility condition, then (4.1) is well posed.

The inverse problem we deal with in this chapter, is to know wether the time-dependent potential b can be stably recovered in some specific subsets of the cylindrical domain Q, from measurements made on the solution u of the wave equation (4.1). We will consider three different sets of data and we will treat the stability issue for the inverse problem under consideration in three different cases. In the first and the second case, we will assume that the initial conditions (u_0,u_1) are fixed at zero and we will prove that in these two cases it is possible to recover the time-dependent coefficient b but only in some specific subsets of the cylindrical domain. And if we want to recover the unkown coefficient over the whole domain, we need to further vary the initial conditions.

In order to give the main statements of this chapter, we need first to introduce the set of the admissible coefficients b: Given $b_0 \in \mathcal{C}^1(\overline{Q_r})$ and $M>0$, we define

$$\mathcal{A}^*(b_0,M)=\left\{b\in \mathcal{C}^1(\overline{Q}_r),\ b=b_0 \text{ in } \overline{Q}_r\setminus Q_{r,*},\ \|b\|_{L^\infty(Q)}\leq M\right\},$$

and

$$\mathcal{A}^\sharp(b_0,M)=\left\{b\in \mathcal{C}^1(\overline{Q}_r),\ b=b_0 \text{ in } \overline{Q}_r\setminus Q_{r,\sharp},\ \|b\|_{L^\infty(Q)}\leq M\right\}.$$

where the sets Q_r, $Q_{r,*}$ and $Q_{r,\sharp}$ are defined in Section 2.3.1.

4.2 Determination of the electric potential from boundary measurements

We start by considering the first case, in which we treat the inverse problem of determining the electric potential b appearing in the wave equation (4.1) with $(u_0, u_1) = (0, 0)$, from measurements given by the Dirichlet-to-Neumann map $\Lambda_b := \Lambda_{0,b}$ defined as in Defintion 2.2.1 with $a = 0$. Obviously, the determination of the coefficient is not guaranteed on the entire time-space domain Q since the initial conditions are zero (see Chapter 2, Section 2.2).

Actually, this problem was treated by Ramm and Rakesh [**36**] and they proved a uniqueness result which is valid only in the region $Q_{r,*}$ and provided that the time-dependent coefficient is known outside of this region. Since we know that it is hopeless to recover the potential b everywhere, we will then focus on the region arised by Ramm and Rakesh and we will prove that we can even have a stability result in determining b from the Dirichlet-to-Neumann map Λ_b. We start by considering the geometric optics solutions constructed in Chapter 1 of the following form

$$u^{\pm}(x,t) = \phi(x,t)e^{\pm i\lambda(x\cdot\omega+t)} + r^{\pm}_{\lambda}(x,t),$$

where ϕ is given by (1.8) and satisfies supp $\varphi \subset \mathscr{A}_r$, in such a way we have

$$\text{supp}\,\varphi \cap \Omega = \emptyset \;\; \text{and} \;\; (\text{supp}\,\varphi \pm T\omega) \cap \Omega = \emptyset, \;\; \forall\,\omega \in \mathbb{S}^{n-1}.$$

Here $\mathscr{A}_r$ is the annulus set defined in Section 2.3.1. In the rest of this section, for b_1, $b_2 \in \mathcal{A}^*(b_0, M)$, we define b in $\mathbb{R}^{1+n}$ by $b = b_2 - b_1$ in $\overline{Q}_r$ and $b = 0$ on $\mathbb{R}^{n+1} \setminus \overline{Q}_r$.

4.2.1 An estimate for the light ray transform

This section is devoted to establish an estimate linking the light-ray transform of the time-dependent coefficient b to the Dirichlet-to-Neumann map Λ_b. We start by showing the following preliminary estimate

Lemma 4.2.1. *There exists $C>0$, such that for any $\omega\in\mathbb{S}^{n-1}$ and $\varphi\in\mathcal{C}_0^\infty(\mathscr{A}_r)$, the following estimate*

$$\left|\int_0^T\int_{\mathbb{R}^n} b(x-t\omega,t)\,\varphi^2(x)\,dx\,dt\right|\leq C\,\left(\lambda^3\|\Lambda_{b_2}-\Lambda_{b_1}\|+\frac{1}{\lambda}\right)\,\|\varphi\|^2_{H^3(\mathbb{R}^n)},\tag{4.2}$$

holds true for any sufficiently large $\lambda>0$.

Proof. In view of Lemma 1.3.2 and using the fact that supp $\varphi\cap\Omega=\emptyset$, there exists a geometrical optics solution u^+ to the equation

$$\begin{cases}(\partial_t^2-\Delta+b_2(t,x))\,u(t,x)=0 & \text{in }\ Q,\\ u(.,0)=\partial_t u(.,0)=0 & \text{in }\ \Omega,\end{cases}$$

of the form

$$u^+(x,t)=\phi(x,t)e^{i\lambda(x\,\cdot\,\omega+t)}+r^+_\lambda(x,t),\tag{4.3}$$

where r^+_λ satisfies

$$\partial_t r^+_{\lambda|t=0}=r^+_{\lambda|t=0}=0,\quad r^+_{\lambda|\Sigma}=0,$$

and

$$\|r^+_\lambda\|_{L^2(Q)}\leq\frac{C}{\lambda}\,\|\varphi\|_{H^3(\mathbb{R}^n)}.\tag{4.4}$$

Let us denote by f_λ the function

$$f_\lambda(t,x)\equiv u^+(x,t)=\phi(x,t)e^{i\lambda(x\,\cdot\,\omega+t)},\quad (x,t)\in\Sigma,$$

and denote by u^-, the solution of

$$\begin{cases} (\partial_t^2 - \Delta + b_1(x,t))\, u^-(x,t) = 0 & \text{in } \ Q, \\ u^-(.,0) = \partial_t u^-(x,0) = 0 & \text{in } \ \Omega, \\ u^-(x,t) = f_\lambda(x,t) & \text{on } \ \Sigma. \end{cases}$$

Putting $u(x,t) = u^-(x,t) - u^+(x,t)$, we get that

$$\begin{cases} (\partial_t^2 - \Delta + b_1(x,t))\, u(x,t) = b(t,x) u^+(x,t) & \text{in } \ Q, \\ u(.,0) = \partial_t u(.,0) = 0 & \text{in } \ \Omega, \\ u(x,t) = 0 & \text{on } \ \Sigma. \end{cases}$$

Applying Lemma 1.3.3, for λ large enough and using the fact that supp $\varphi \pm T\omega \cap \Omega = \emptyset$, we may find a geometrical optic solution u^- to the backward wave equation

$$\begin{cases} (\partial_t^2 - \Delta + b_1(t,x))\, u^-(t,x) = 0 & \text{in } \ Q, \\ u^-(.,T) = \partial_t u^-(.,T) = 0 & \text{in } \ \Omega, \end{cases}$$

of the form

$$u^-(x,t) = \phi(x,t) e^{-i\lambda(x\cdot\omega + t)} + r_\lambda^-(x,t), \tag{4.5}$$

where r_λ^- satisfies

$$\partial_t r^-_{\lambda|t=T} = r^-_{\lambda|t=T} = 0, \quad r^-_{\lambda|\Sigma} = 0,$$

and

$$\|r_\lambda^-\|_{L^2(Q)} \leq \frac{C}{\lambda} \|\varphi\|_{H^3(\mathbb{R}^n)}. \tag{4.6}$$

Consequently, by integrating by parts and using Green's formula, we obtain

$$\begin{aligned} \int_Q b(x,t) u^+(x,t) u^-(x,t)\, dx\, dt &= \int_Q \left(\partial_t^2 - \Delta + b_1(x,t)\right) u(x,t) u^-(x,t)\, dx\, dt \\ &= \int_\Sigma (\Lambda_{b_2} - \Lambda_{b_1}) f_\lambda(x,t) u^-(x,t)\, d\sigma\, dt, \end{aligned} \tag{4.7}$$

So, (4.3), (4.5) and (4.7) yield

$$\begin{aligned}
& \int_Q b(t,x)\phi^2(x,t)\,dx\,dt + \int_Q b(t,x) r_\lambda^-(x,t) r_\lambda^+(x,t)\,dx\,dt \\
& + \int_Q b(t,x)\phi(x,t)\left(r_\lambda^+(x,t)e^{-i\lambda(x\cdot\omega+t)} + r_\lambda^-(x,t)e^{i\lambda(x\cdot\omega+t)}\right)\,dx\,dt \\
= & \int_\Sigma (\Lambda_{b_2}-\Lambda_{b_1}) f_\lambda(x,t)\ u^-(x,t)\ d\sigma\ dt. \qquad (4.8)
\end{aligned}$$

From (4.8), (4.4) and (4.6) it follows that

$$\left|\int_Q b(x,t)\,\phi^2(x,t)\,dx\,dt\right| \le \int_\Sigma |(\Lambda_{q_2}-\Lambda_{q_1}) f_\lambda(x,t)\,v_\lambda(x,t)|\,d\sigma\,dt + \frac{C}{\lambda}\,\|\varphi\|^2_{H^3(\mathbb{R}^n)},$$

where the constant $C>0$ does not depend on λ. Hence from the Cauchy-Schwarz inequality and the identity $f_\lambda(x,t)=u_{2,\lambda}(x,t)$ on Σ, we obtain

$$\left|\int_Q q(x,t)\phi^2(x,t)\,dx\,dt\right| \le \|\Lambda_{b_2}-\Lambda_{b_1}\|\ \|u_\lambda^+\|_{H^1(\Sigma)}\,\|u_\lambda^-\|_{L^2(\Sigma)} + \frac{C}{\lambda}\,\|\varphi\|^2_{H^3(\mathbb{R}^n)}. \qquad (4.9)$$

Further, as $r_\lambda^\pm = 0$, on Σ, we deduce from (4.9) that

$$\left|\int_Q b(x,t)\,a^2(x,t)\,dx\,dt\right| \le C\left(\|\Lambda_{b_2}-\Lambda_{b_1}\|\ \|u_\lambda^+ - r_\lambda^+\|_{H^2(Q)}\,\|u^- - r_\lambda^-\|_{H^1(Q)} + \frac{1}{\lambda}\,\|\varphi\|^2_{H^3(\mathbb{R}^n)}\right).$$

Bearing in mind that

$$\|u_\lambda^- - r_\lambda^-\|_{H^1(Q)} \le C\lambda\,\|\varphi\|_{H^3(\mathbb{R}^n)},$$

$$\|u^+ - r_\lambda^+\|_{H^2(Q)} \le C\lambda^2\,\|\varphi\|_{H^3(\mathbb{R}^n)},$$

we end up getting that

$$\left|\int_Q b(x,t)\,\phi^2(x,t)\,dx\,dt\right| \le C\left(\lambda^3\|\Lambda_{b_2}-\Lambda_{b_1}\| + \frac{1}{\lambda}\right)\,\|\varphi\|^2_{H^3(\mathbb{R}^n)}.$$

Therefore using the fact that $b(x,t)=0$ outside Q_*, we get

$$\left|\int_0^T\int_{\mathbb{R}^n} b(x-t\omega,t)\,\varphi^2(x)\,dx\,dt\right| \leq C\left(\lambda^3\|\Lambda_{b_2}-\Lambda_{b_1}\|+\frac{1}{\lambda}\right)\|\varphi\|^2_{H^3(\mathbb{R}^n)}.$$

This completes the proof of the lemma. $\square$

Using the above lemma, we can estimate the light-ray transform of b as follows:

Lemma 4.2.2. *There exist $C>0$, $\beta>0$, $\delta>0$, and $\lambda_0>0$ such that for all $\omega\in\mathbb{S}^{n-1}$, we have*

$$|R(b)(y,\omega)|\leq C\left(\lambda^\beta\|\Lambda_{b_2}-\Lambda_{b_1}\|+\frac{1}{\lambda^\delta}\right),\quad a.e.\ y\in\mathbb{R}^n,$$

for any $\lambda\geq\lambda_0$.

Proof. Let $\phi\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ be a positive function which is supported in the unit ball $B(0,1)$ and such that $\|\phi\|_{L^2(\mathbb{R}^n)}=1$. Define

$$\varphi_\varepsilon(x)=\varepsilon^{-n/2}\phi\left(\frac{x-y}{\varepsilon}\right), \tag{4.10}$$

where $y\in\mathscr{A}_r$. Then for $\varepsilon>0$ sufficiently small we can verify that

$$\operatorname{supp}\varphi_\varepsilon\cap\Omega=\emptyset,\ \text{ and }\ \operatorname{supp}\varphi_\varepsilon\pm T\omega\cap\Omega=\emptyset.$$

Moreover we have

$$\begin{aligned}&\left|\int_0^T b(y-t\omega,t)\,dt\right| = \left|\int_0^T\int_{\mathbb{R}^n} b(y-t\omega,t)\,\varphi_\varepsilon^2(x)\,dx\,dt\right|\\ \leq\ &\left|\int_0^T\int_{\mathbb{R}^n} b(x-t\omega,t)\,\varphi_\varepsilon^2(x)\,dx\,dt\right| + \left|\int_0^T\int_{\mathbb{R}^n}\left(b(y-t\omega,t)-b(x-t\omega,t)\right)\,\varphi_\varepsilon^2(x)\,dx\,dt\right|.\end{aligned}$$

Since $b\in\mathcal{C}^1([0,T]\times\mathbb{R}^n)$, we have $|b(y-t\omega,t)-b(x-t\omega,t)|\leq C\,|x-y|$. So upon applying Lemma 4.2.1 with $\varphi = \varphi_\varepsilon$, we obtain

$$\left|\int_0^T b(y - t\omega, t)\, dt\right| \leq C\left(\lambda^3 \|\Lambda_{b_2} - \Lambda_{b_1}\| + \frac{1}{\lambda}\right)\|\varphi_\varepsilon\|^2_{H^3(\mathbb{R}^n)} + C\int_{\mathbb{R}^n} |x - y|\, \varphi_\varepsilon^2(x)\, dx. \tag{4.11}$$

On the other hand, we have

$$\|\varphi_\varepsilon\|_{H^3(\mathbb{R}^n)} \leq C\,\varepsilon^{-3}, \quad \int_{\mathbb{R}^n} |x - y|\, \varphi_\varepsilon^2(x)\, dx \leq C\varepsilon.$$

So we infer, from (4.11) that

$$\left|\int_0^T b(y - t\omega, t)\, dt\right| \leq C\left(\lambda^3 \|\Lambda_{b_2} - \Lambda_{b_1}\| + \frac{1}{\lambda}\right)\varepsilon^{-6} + C\varepsilon.$$

Taking $\varepsilon = \varepsilon^{-6}/\lambda$, we find two constants $\delta > 0$ and $\beta > 0$ such that

$$\left|\int_0^T b(y - t\omega, t)\, dt\right| \leq\ C\ \left(\lambda^\beta \|\Lambda_{b_2} - \Lambda_{b_1}\| + \frac{1}{\lambda^\delta}\right).$$

Since $b = 0$ outside Q_*, this entails that

$$\left|\int_{\mathbb{R}} b(t, y - t\omega)\, dt\right| \leq\ C\ \left(\lambda^\beta \|\Lambda_{b_2} - \Lambda_{b_1}\| + \frac{1}{\lambda^\delta}\right), \quad \text{a.e.}\, y \in \mathscr{A}_r,\ \omega \in \mathbb{S}^{n-1}. \tag{4.12}$$

On the other hand, if $|y| \leq \dfrac{r}{2}$, we notice that

$$b(t, y - t\omega) = 0,\ \ \forall\, t \in \mathbb{R}. \tag{4.13}$$

Indeed, we have

$$|y - t\omega| \geq |t| - |y| \geq t - \frac{r}{2}, \tag{4.14}$$

hence, $(y - t\omega, t) \notin \mathscr{C}_r^+$ if $t > \dfrac{r}{2}$, from (4.14). As $(y - t\omega, t) \notin \mathscr{C}_r^+$ if $t \leq \dfrac{r}{2}$, then we have

$$(y - t\omega, t) \notin \mathscr{C}_r^+ \supset Q_*, \quad t \in \mathbb{R}.$$

This and the fact that $b = b_2 - b_1 = 0$ outside Q_*, yield (4.13), and consequently,

$$\int_{\mathbb{R}} b(t, y - t\omega)\, dt = 0, \quad |y| \leq \frac{r}{2}.$$

By a similar way, we prove for $|y| \geq T - \frac{r}{2}$, that

$$(y - t\omega, t) \notin \mathscr{C}_r^- \supset Q_*, \quad t \in \mathbb{R},$$

and then obtain

$$\int_{\mathbb{R}} b(y - t\omega, t)\, dt = 0, \quad \text{a.e. } y \notin \mathscr{A}_r, \;\; \omega \in \mathbb{S}^{n-1}. \tag{4.15}$$

Thus we get,

$$|R(b)(y,\omega)| = \left| \int_{\mathbb{R}} b(y - t\omega, t)\, dt \right| \leq C \left(\lambda^\beta \|\Lambda_{b_2} - \Lambda_{b_1}\| + \frac{1}{\lambda^\delta} \right), \;\; \text{a.e. } y \in \mathbb{R}^n, \;\; \omega \in \mathbb{S}^{n-1},$$

by (4.12) and (4.15). This completes the proof of the lemma. □

This allows us to deduce an estimate for the Fourier transform of the coefficient b which is the goal of the next section.

4.2.2 An estimate for the Fourier transform

In this section, we aim to control the Fourier transform of b with respect to the Dirichlet-to-Neumann map for all $(\xi, \tau) \in E$, where E is the conic set given by (2.16).

Lemma 4.2.3. *There exist $C > 0$, $\beta > 0$, $\delta > 0$ and $\lambda_0 > 0$, such that the following estimate*

$$|\widehat{b}(\xi, \tau)| \leq C \left(\lambda^\beta \|\Lambda_{b_2} - \Lambda_{b_1}\| + \frac{1}{\lambda^\delta} \right),$$

holds for any $(\xi,\tau)\in E$ *and* $\lambda\geq\lambda_0$.

Proof. Let $(\tau,\xi)\in E$ and $\zeta\in\mathbb{S}^{n-1}$ be such that $\xi.\zeta=0$. By defining

$$\omega=\frac{\tau}{|\xi|^2}\cdot\xi+\sqrt{1-\frac{\tau^2}{|\xi|^2}}\cdot\zeta,$$

we have $\omega\in\mathbb{S}^{n-1}$ and $\omega\cdot\xi=\tau$. From Lemma 1.4.1 one can see that

$$\int_{\mathbb{R}^n}R(b)(y,\omega)\,e^{-iy\cdot\xi}\,dy\;=\;\widehat{b}(\xi,\omega\cdot\xi)=\widehat{b}(\xi,\tau)$$

We set $(\xi,\tau)=(\xi,\omega\cdot\xi,)\in E$. Since $\operatorname{supp}b(t,.)\subset\Omega\subset B(0,\frac{r}{2})$, then we have

$$\int_{\mathbb{R}^n\cap B(0,\frac{r}{2}+T)}R(b)(y,\omega)\,e^{-iy\cdot\xi}\,dy=\widehat{b}(\xi,\tau).$$

Applying Lemma 4.2.2, we obtain the desired result. □

4.2.3 Stability estimate for the electric potential

We are now in position to state and prove the first main result of this chapter.

Theorem 4.2.1. *Let* $T>2\,Diam\,(\Omega)$ *and* $b_1,\,b_2\in\mathcal{A}^*(b_0,M)$ *satisfy the following identity*

$$\partial_x b_1(t,x)=\partial_x b_2(t,x),\quad (t,x)\in\ \partial Q_r\cap\partial Q_{r,*}.\tag{4.16}$$

Then, there exist two constants $C>0$ *and* $\mu_1\in(0,1)$, *such that we have*

$$\|b_1-b_2\|_{H^{-1}(Q_{r,*})}\leq C\,\left(\|\Lambda_{b_1}-\Lambda_{b_2}\|^{\mu_1}+|\log\|\Lambda_{b_1}-\Lambda_{b_2}\||^{-1}\right),$$

where C *depends only on* Ω, M, T, *and* n. *If we assume in addition that* $b_1,\,b_2\in H^{s+1}(Q)$, $s>\dfrac{n}{2}$, *verify* $\|b_i\|_{H^{s+1}(Q)}\leq M$, *for* $i=1,2$, *and some constant* $M>0$.

Then there exist two constant $C'>0$ and $\mu_2\in(0,1)$ such that we have

$$\|b_1-b_2\|_{L^\infty(Q_{r,*})}\leq C'\left(\|\Lambda_{b_1}-\Lambda_{b_2}\|+|\log\|\Lambda_{b_1}-\Lambda_{b_2}\||^{-1}\right)^{\mu_2}, \tag{4.17}$$

We will prove the above theorem by the use of the analytic argument Theorem 3.2.1.

Proof. For fixed $\alpha>0$, let us set $F_\alpha(\xi,\tau)=\widehat{b}(\alpha(\xi,\tau)),\quad (\xi,\tau)\in\mathbb{R}^{n+1}$. It is easily seen that F_α is analytic and we have for $\kappa\in(\mathbb{N}\cup\{0\})^{n+1}$

$$\begin{aligned}|\partial^\kappa F_\alpha(\xi,\tau)|=\left|\partial^\kappa\widehat{b}(\alpha(\xi,\tau))\right| &= \left|\partial^\kappa\int_{\mathbb{R}^{n+1}}b(x,t)\,e^{-i\alpha(x,t)\cdot(\xi,\tau)}\,dx\,dt\right|\\ &=\left|\int_{\mathbb{R}^{n+1}}b(x,t)(-i)^{|\kappa|}\alpha^{|\kappa|}(x,t)^\kappa e^{-i\alpha(x,t)\cdot(\xi,\tau)}\,dx\,dt\right|\end{aligned} \tag{4.18}$$

Therefore, from (4.18) one gets

$$|\partial^\kappa F_\alpha(\xi,\tau)|\leq\int_{\mathbb{R}^{n+1}}|b(t,x)|\alpha^{|\kappa|}(|x|^2+t^2)^{\frac{|\kappa|}{2}}\,dx\,dt\leq\|b\|_{L^1(Q_{r,*})}\;\alpha^{|\gamma|}\;(2T^2)^{\frac{|\kappa|}{2}}\leq C\,\frac{e^\alpha|\kappa|!}{(T^{-1})^{|\kappa|}}.$$

Applying Theorem 3.2.1 to the set $\mathcal{O}=\mathring{E}\cap B(0,1)$ with $M=Ce^\alpha$, $2\rho=T^{-1}$, and where

$$\mathring{E}=\{(\xi,\tau)\in\mathbb{R}\times(\mathbb{R}^n\setminus\{0_{\mathbb{R}^n}\})\,,\;\;|\tau|<|\xi|\},$$

we may find a constant $\gamma\in(0,1)$ such that we have

$$|F_\alpha(\xi,\tau)|=|\widehat{b}(\alpha(\xi,\tau))|\leq Ce^{\alpha(1-\gamma)}\|F_\alpha\|^\gamma_{L^\infty(\mathcal{O})},\qquad(\xi,\tau)\in B(0,1).$$

Hence, since $\alpha\,\mathring{E}=\{\alpha(\xi,\tau),\,(\xi,\tau)\in\mathring{E}\}=\mathring{E}$, we get for $(\xi,\tau)\in B(0,\alpha)$ that

$$\begin{aligned}|\widehat{b}(\xi,\tau)|=|F_\alpha(\alpha^{-1}(\xi,\tau)| &\leq Ce^{\alpha(1-\gamma)}\,\|F_\alpha\|^\gamma_{L^\infty(\mathcal{O})}\\ &\leq Ce^{\alpha(1-\gamma)}\|\,\widehat{b}\,\|^\gamma_{L^\infty(B(0,\alpha)\cap\mathring{E})}\\ &\leq Ce^{\alpha(1-\gamma)}\|\,\widehat{b}\,\|^\gamma_{L^\infty(\mathring{E})}.\end{aligned} \tag{4.19}$$

On the other hand we have

$$\begin{aligned}\|b\|^{2/\gamma}_{H^{-1}(\mathbb{R}^{n+1})} =& \left(\int_{|(\xi,\tau)|<\alpha}(1+|(\xi,\tau)|^2)^{-1}|\widehat{b}(\xi,\tau)|^2\,d\tau d\xi + \int_{|(\xi,\tau)|\geq\alpha}(1+|(\xi,\tau)|^2)^{-1}|\widehat{b}(\xi,\tau)|^2\,d\tau d\xi\right)^{1/\gamma}\\ \leq& \;C\left(\alpha^{n+1}\;\|\widehat{b}\|^2_{L^\infty(B(0,\alpha))} + \alpha^{-2}\;\|b\|^2_{L^2(\mathbb{R}^{n+1})}\right)^{1/\gamma}.\end{aligned}$$

Thus it follows from (4.19) and Lemma 4.2.3, that

$$\begin{aligned}\|b\|^{2/\gamma}_{H^{-1}(\mathbb{R}^{n+1})} &\leq C\left(\alpha^{n+1}\,e^{2\alpha(1-\gamma)}\,(\lambda^\beta\|\Lambda_{b_2}-\Lambda_{b_1}\| + \frac{1}{\lambda^\delta})^{2\gamma} + \alpha^{-2}\right)^{1/\gamma}\\ &\leq C\left(\alpha^{\frac{n+1}{\gamma}}\,e^{\frac{2\alpha(1-\gamma)}{\gamma}}\lambda^{2\beta}\|\Lambda_{b_2}-\Lambda_{b_1}\|^2 + \alpha^{\frac{n+1}{\gamma}}\,e^{\frac{2\alpha(1-\gamma)}{\gamma}}\,\lambda^{-2\delta} + \alpha^{-2/\gamma}\right).\end{aligned}$$

Let $\alpha_0 > 0$ be sufficiently large and take $\alpha > \alpha_0$. Set $\lambda = \alpha^{\frac{n+3}{2\gamma\delta}}\,e^{\frac{\alpha(1-\gamma)}{\gamma\delta}}$. Since $\alpha > \alpha_0$, we can assume that $\lambda > \lambda_0$, so we have

$$\alpha^{\frac{n+1}{\gamma}}\,e^{\frac{2\alpha(1-\gamma)}{\gamma}}\,\lambda^{-2\delta} = \alpha^{-2/\gamma}.$$

Therefore it holds true that

$$\begin{aligned}\|b\|^{2/\gamma}_{H^{-1}(\mathbb{R}^{n+1})} &\leq C\left(\alpha^{\frac{\delta(n+1)+\beta(n+3)}{\delta\gamma}}\;e^{\frac{2\alpha(\delta+\beta)(1-\gamma)}{\delta\gamma}}\|\Lambda_{b_2}-\Lambda_{b_1}\|^2 + \alpha^{-2/\gamma}\right)\\ &\leq C\left(e^{N\alpha}\|\Lambda_{b_2}-\Lambda_{b_1}\|^2 + \alpha^{-2/\gamma}\right),\end{aligned}$$

where N depends on δ, β, n, and γ. In order to minimize the right hand-side of the above inequality with respect to α, we set

$$\alpha = \frac{1}{N}|\,\log\|\Lambda_{b_2}-\Lambda_{b_1}\|\,|,$$

where we assume that $0 < \|\Lambda_{b_2}-\Lambda_{b_1}\| < c$. We obtain that

$$\begin{aligned}\|b\|_{H^{-1}(Q_{r,*})} \leq \|b\|_{H^{-1}(\mathbb{R}^{n+1})} &\leq C\left(\|\Lambda_{b_2}-\Lambda_{b_1}\| + |\,\log\|\Lambda_{b_2}-\Lambda_{b_1}\|\,|^{-2/\gamma}\right)^{\gamma/2}\\ &\leq C\left(\|\Lambda_{b_2}-\Lambda_{b_1}\|^{\gamma/2} + |\log\|\Lambda_{b_2}-\Lambda_{b_1}\||^{-1}\right)\qquad(4.20)\end{aligned}$$

Now if $\|\Lambda_{b_2}-\Lambda_{b_1}\|\geq c$, we have

$$\|b\|_{H^{-1}(Q_{r,*})}\leq C\|b\|_{L^\infty(Q_{r,*})}\leq \frac{2CMc^{\gamma/2}}{c^{\gamma/2}}\leq \frac{2CM}{c^{\gamma/2}}\|\Lambda_{b_2}-\Lambda_{b_1}\|^{\gamma/2},$$

hence (4.20) holds. The estimate (4.17), is now an easy consequence of the Sobolev embedding theorem and the following interpolation inequality. If $\delta'>0$ is such that $s=n/2+2\delta'$, then, we have

$$\begin{aligned}\|b\|_{L^\infty(Q_{r,*})} &\leq C\|q\|_{H^s(Q_{r,*})}\\ &\leq C\,\|b\|^{1-\beta}_{H^{-1}(Q_{r,*})}\,\|b\|^{\beta}_{H^{s+1}(Q_{r,*})}\\ &\leq C\,\|b\|^{1-\beta}_{H^{-1}(Q_{r,*})},\end{aligned}$$

for some $\beta\in(0,1)$. This completes the proof of Theorem 4.2.1. □

4.3 Determination of the electric potential from boundary and final data

In this section, we aim to extend the stability estimate given by Theorem 4.2.1 to a larger region $Q_{r,\sharp}\supset Q_{r,*}$. We will consider geometric optics solutions similar to the one used in the previous section, except in this time, we assume that the function φ obeys $\operatorname{supp}\varphi\cap\Omega=\emptyset$. In particular, we do not assume that supp $\varphi\pm T\omega\cap\Omega=\emptyset$ anymore. Our observations are given by the following operator

Definition 4.3.1. *We define the boundary operator $\mathscr{R}_b$ as follows*

$$\begin{aligned}\mathscr{R}_b: H^1(\Sigma) &\longrightarrow L^2(\Sigma)\times H^1(\Omega)\times L^2(\Omega),\\ f &\longmapsto (\partial_\nu u,\, u(T,.),\, \partial_t u(T,.)),\end{aligned}$$

with u is a solution of the equation (4.1) with $(u_0,u_1)=(0,0)$.

From Theorem 1.2.1, one can see that $\mathscr{R}_b$ is continuous from $H^1(\Sigma)$ to $L^2(\Sigma)\times$

$H^1(\Omega)\times L^2(\Omega)$. We denote by $\|\mathscr{R}_b\|$ its norm in $\mathcal{L}\Big(H^1(\Sigma),\ L^2(\Sigma)\times H^1(\Omega)\times L^2(\Omega)\Big)$.
We start by showing the following lemma

Lemma 4.3.1. *Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be such that $\mathrm{supp}\,\varphi \cap \Omega = \emptyset$. Then, there exists $C > 0$, such that for any $\omega \in \mathbb{S}^{n-1}$ the following estimate*

$$\left| \int_0^T \int_{\mathbb{R}^n} b(x - t\omega, t)\, \varphi^2(x)\, dx\, dt \right| \leq C\, \left(\lambda^3 \|\mathscr{R}_{b_2} - \mathscr{R}_{b_1}\| + \frac{1}{\lambda} \right) \|\varphi\|^2_{H^3(\mathbb{R}^n)} \tag{4.21}$$

holds true for any sufficiently large $\lambda > 0$.

Proof. In view of Lemma 1.3.2, and using the fact that supp $\varphi \cap \Omega = \emptyset$, there exists a geometrical optics solutions u^+ to the equation

$$\begin{cases} (\partial_t^2 - \Delta + b_2(x,t))\, u(x,t) = 0 & \text{in } \ Q, \\ u(\,\cdot\,, 0) = \partial_t u(\,\cdot\,, 0) = 0 & \text{in } \ \Omega, \end{cases}$$

in the following form

$$u^+(x,t) = \phi(x,t) e^{i\lambda(x\,\cdot\,\omega + t)} + r^+_\lambda(x,t), \tag{4.22}$$

where r^+_λ satisfies

$$\partial_t r^+_{\lambda|t=0} = r^+_{\lambda|t=0} = 0, \quad r^+_{\lambda|\Sigma} = 0,$$

and

$$\|r^+_\lambda\|_{L^2(Q)} \leq \frac{C}{\lambda}\, \|\varphi\|_{H^3(\mathbb{R}^n)}. \tag{4.23}$$

Let us define by f_λ the function

$$f_\lambda(x,t) \equiv u^+(x,t) = \phi(x,t) e^{i\lambda(x\,\cdot\,\omega + t)}, \quad (x,t) \in \Sigma,$$

and denote by u_1, the solution of

$$\begin{cases} \left(\partial_t^2 - \Delta + b_1(x,t)\right) u_1(x,t) = 0 & \text{in } \; Q, \\ u_1(x,0) = \partial_t u_{1,\lambda}(x,0) = 0 & \text{in } \; \Omega, \\ u_1(x,t) = f_\lambda(x,t) & \text{on } \; \Sigma. \end{cases}$$

Putting $u(t,x) = u_1(x,t) - u^+(x,t)$, we get that

$$\begin{cases} \left(\partial_t^2 - \Delta + b_1(x,t)\right) u(x,t) = b(x,t)u^+(x,t) & \text{in } \; Q \\ u(x,0) = \partial_t u(x,0) = 0 & \text{in } \; \Omega \\ u(x,t) = 0 & \text{on } \; \Sigma. \end{cases}$$

Applying Lemma 1.3.3, we find for λ large enough, a geometrical optic solution u^- to the backward wave equation

$$\left(\partial_t^2 - \Delta + b_1(x,t)\right) u^-(x,t) = 0, \; \text{ in } Q,$$

of the form

$$u^-(x,t) = \phi(x,t)e^{-i\lambda(x\cdot\omega+t)} + r_\lambda^-(x,t), \tag{4.24}$$

where r_λ^- satisfies

$$\partial_t r^-_{\lambda|t=T} = r^-_{\lambda|t=T} = 0, \quad r^-_{\lambda|\Sigma} = 0,$$

and

$$\|r_\lambda^-\|_{L^2(Q)} \leq \frac{C}{\lambda} \|\varphi\|_{H^3(\mathbb{R}^n)}. \tag{4.25}$$

Consequently, by integrating by parts and using Green's formula we obtain

$$\begin{aligned} \int_Q b(x,t)u_{2,\lambda}(x,t)u^-(x,t)\,dx\,dt \;\; &= \;\; \int_\Sigma (\mathscr{R}^1_{b_2} - \mathscr{R}^1_{b_1})(f_\lambda)u^-(x,t)\,d\sigma\,dt \\ &+ \int_\Omega \left(\mathscr{R}^2_{b_2} - \mathscr{R}^2_{b_1}\right)(f_\lambda)\,\partial_t u_\lambda^-(x,T)\,dx \\ &- \int_\Omega \left(\mathscr{R}^3_{b_2} - \mathscr{R}^3_{b_1}\right)(f_\lambda)\,u^-(x,T)\,dx. \end{aligned} \tag{4.26}$$

By replacing u^+ (resp., u^-) by the right hand side of (4.22) (resp., (4.24)) in the left hand side of (4.26), we get from (4.23), (4.25) and the Cauchy-Schwarz inequality that

$$\begin{aligned}\left|\int_Q b(x,t)\,\phi^2(x,t)\,dx\,dt\right| &\leq \|(\mathscr{R}^1_{b_2}-\mathscr{R}^1_{b_1})(f_\lambda)\|_{L^2(\Sigma)}\|u^-\|_{L^2(\Sigma)}+\frac{C}{\lambda}\|\varphi\|^2_{H^3(\mathbb{R}^n)}\\ &+\|(\mathscr{R}^2_{b_2}-\mathscr{R}^2_{b_1})(f_\lambda)\|_{L^2(\Omega)}\|\partial_t u^-(T,.)\|_{L^2(\Omega)}\\ &+\|(\mathscr{R}^3_{b_2}-\mathscr{R}^3_{b_1})(f_\lambda)\|_{L^2(\Omega)}\|u^-(T,.)\|_{L^2(\Omega)},\end{aligned}$$

which entails

$$\left|\int_Q b(x,t)\,\phi^2(x,t)\,dx\,dt\right| \leq \Big(\|(\mathscr{R}^1_{b_2}-\mathscr{R}^1_{b_1})(f_\lambda)\|^2_{L^2(\Sigma)}+\|(\mathscr{R}^2_{b_2}-\mathscr{R}^2_{b_1})(f_\lambda)\|^2_{H^1(\Omega)}$$
$$+\|(\mathscr{R}^3_{b_2}-\mathscr{R}^3_{b_1})(f_\lambda)\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}\Big(\|u^-\|^2_{L^2(\Sigma)}+\|u^-(T,.)\|^2_{L^2(\Omega)}+\|\partial_t u^-(T,.)\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}+\frac{C}{\lambda}\|\varphi\|^2_{H^3(\mathbb{R}^n)}.$$

Next, setting

$$\phi_\lambda=(u^-|_\Sigma,\,u^-(T,.),\,\partial_t u^-(T,.)),$$

we may rewrite the above inequality as

$$\left|\int_Q b(x,t)\phi^2(t,x)\,dx\,dt\right| \leq \|(\mathscr{R}_{b_2}-\mathscr{R}_{b_1})(f_\lambda)\|_{L^2(\Sigma)\times H^1(\Omega)\times L^2(\Omega)}\|\phi_\lambda\|_{L^2(\Sigma)\times L^2(\Omega)\times L^2(\Omega)}+\frac{C}{\lambda}\|\varphi\|^2_{H^3(\mathbb{R}^n)}.$$

Since $f_\lambda(t,x)=u^+(x,t)$ on Σ, this yields that

$$|\int_Q b(x,t)\phi^2(x,t)\,dx\,dt| \leq \|\mathscr{R}_{b_2}-\mathscr{R}_{b_1}\|\;\|u^+\|_{H^1(\Sigma)}\;\|\phi_\lambda\|_{L^2(\Sigma)\times L^2(\Omega)\times L^2(\Omega)}+\frac{C}{\lambda}\;\|\varphi\|^2_{H^3(\mathbb{R}^n)}.$$

Further, as $r^\pm_\lambda=0$ on Σ we obtain

$$\left|\int_Q b(x,t)\,\phi^2(x,t)\,dx\,dt\right| \leq C\Big(\|\mathscr{R}_{b_2}-\mathscr{R}_{b_1}\|\|u^+-r^+_\lambda\|_{H^2(Q)}\|\phi_{1,\lambda}\|_{H^1(Q)\times L^2(\Omega)\times L^2(\Omega)}+\frac{1}{\lambda}\|\varphi\|^2_{H^3(\mathbb{R}^n)}\Big),$$

where

$$\phi_{1,\lambda} = \left(u^- - r_\lambda^-,\, u^-(T,.),\, \partial_t u^-(T,.)\right).$$

Since $r_\lambda^-(T,.) = \partial_t r_\lambda^-(T,.) = 0$ in Ω, we have

$$\begin{aligned} \|\phi_{1,\lambda}\|_{H^1(Q)\times L^2(\Omega)\times L^2(\Omega)} &\leq \|u^- - r_\lambda^-\|_{H^1(Q)} + \|u^-_{|t=T}\|_{L^2(\Omega)} + \|\partial_t u^-_{|t=T}\|_{L^2(\Omega)} \\ &\leq C\lambda \,\|\varphi\|_{H^3(\mathbb{R}^n)}. \end{aligned}$$

Moreover we have

$$\|u^+ - r_\lambda^+\|_{H^2(Q)} \leq C\lambda^2 \,\|\varphi\|_{H^3(\mathbb{R}^n)},$$

Then, since b vanishes outside $Q_\sharp$, then,

$$\left|\int_0^T \int_{\mathbb{R}^n} b(x-t\omega,t)\,\varphi^2(x)\,dx\,dt\right| \leq C\left(\lambda^3\|\mathscr{R}_{b_2}-\mathscr{R}_{b_1}\| + \frac{1}{\lambda}\right)\|\varphi\|^2_{H^3(\mathbb{R}^n)}.$$

This completes the proof of the Lemma. $\square$

As a consequence, we can control the light-ray transform of b as follows:

Lemma 4.3.2. *There exists four constants $C>0$, $\beta>0$, $\delta>0$ and $\lambda_0>0$, such that the estimate*

$$|R(b)(y,\omega)| \leq C\left(\lambda^\beta\|\mathscr{R}_{b_2}-\mathscr{R}_{b_1}\| + \frac{1}{\lambda^\delta}\right), \quad a.e.\ y\in\mathbb{R}^n.$$

holds for all $\omega\in\mathbb{S}^{n-1}$ and for any $\lambda \geq \lambda_0$.

Proof. In order to prove this lemma, it will be enough to consider the sequence $(\varphi_\varepsilon)_\varepsilon$ defined by (4.10) for a fixed $y\in\mathbb{R}^n\setminus\Omega$, in a such way supp $\varphi_\varepsilon\cap\Omega=\emptyset$ for sufficiently small $\varepsilon>0$. From this, Lemma 4.3.1 and the fact that $b=b_2-b_1$ vanishes outside $Q_\sharp$, we obtain upon arguing as in the derivation of Lemma 4.2.2 the desired result. $\square$

Having said that, we are now in position to state our second result.

Theorem 4.3.1. *Let $T > 2\,Diam\,(\Omega)$ and $b_1,\, b_2 \in \mathcal{A}^\sharp(b_0, M)$ satisfy the following identity*

$$\partial_x b_1(x,t) = \partial_x b_2(x,t), \quad (t,x) \in \partial Q_r \cap \partial Q_{r,\sharp}. \tag{4.27}$$

Then, there exist two constants $C > 0$ and $\mu_1 \in (0,1)$, such that we have

$$\|b_1 - b_2\|_{H^{-1}(Q_{r,\sharp})} \leq C \left(\|\mathscr{R}_{b_1} - \mathscr{R}_{b_2}\|^{\mu_1} + |\ \log \|\mathscr{R}_{b_1} - \mathscr{R}_{b_2}\||^{-1} \right),$$

where C depends only on Ω, M, T, and n. If we assume in addition that $b_1,\, b_2 \in H^{s+1}(Q)$, $s > \dfrac{n}{2}$, verify $\|b_i\|_{H^{s+1}(Q)} \leq M$, for $i = 1,2$, and some constant $M > 0$. Then there exist two constants $C' > 0$ and $\mu_2 \in (0,1)$ such that we have

$$\|b_1 - b_2\|_{L^\infty(Q_{r,\sharp})} \leq C' \left(\|\mathscr{R}_{b_1} - \mathscr{R}_{b_2}\| + |\log \|\mathscr{R}_{b_1} - \mathscr{R}_{b_2}\||^{-1} \right)^{\mu_2}.$$

Proof. Armed with Lemma 4.3.2, we prove Theorem 4.3.1 by repeating the same steps followed in the proof of Theorem 4.2.1. □

4.4 Determination of the electric potential from boundary and final data by varying the initial conditions

In this section we deal with the same problem as in the previous sections, except that the set of data is made of the responses of the medium for all possible initial states given by the following operator:

Definition 4.4.1. *We define the boundary operator $\mathcal{I}_b$ as follows*

$$\begin{aligned} \mathcal{I}_b : H^1(\Sigma) \times H^1(\Omega) \times L^2(\Omega) &\longrightarrow L^2(\Sigma) \times H^1(\Omega) \times L^2(\Omega), \\ (f, u_0, u_1) &\longmapsto (\partial_\nu u,\, u(T,.),\, \partial_t u(T,.)), \end{aligned}$$

with u is a solution of the equation (4.1).

From Theorem 1.2.1, one can see that the linear boudary operator $\mathcal{I}_b$ is continuous from $H^1(\Sigma) \times H^1(\Omega) \times (L^2(\Omega)$ to $L^2(\Sigma) \times H^1(\Omega) \times L^2(\Omega)$. From the observations given by the boundary operator $\mathcal{I}_b$, we were able to stably recover the electric potential b everywhere.

Theorem 4.4.1. *Let $T > 2\,Diam\,(\Omega)$ and $b_1,\, b_2 \in \mathcal{C}^1(\overline{Q})$ such that $\|b_i\|_{W^{1,\infty}(Q)} \leq M$ and*

$$\partial_x b_1(x,t) = \partial_x b_2(x,t), \quad (x,t) \in \Sigma. \tag{4.28}$$

Then, there exist two constants $C > 0$ and $\mu_1 \in (0,1)$, such that we have

$$\|b_1 - b_2\|_{H^{-1}(Q)} \leq C\left(\|\mathcal{I}_{b_1} - \mathcal{I}_{b_2}\|^{\mu_1} + |\ \log \|\mathcal{I}_{b_1} - \mathcal{I}_{b_2}\||^{-1}\right),$$

where C depends only on Ω, M, T, and n. If we assume in addition that $b_1,\, b_2 \in H^{s+1}(Q)$, $s > \dfrac{n}{2}$, verify $\|b_i\|_{H^{s+1}(Q)} \leq M$, for $i = 1,2$, and some constant $M > 0$. Then there exist two constant $C' > 0$ and $\mu_2 \in (0,1)$ such that we have

$$\|b_1 - b_2\|_{L^\infty(Q)} \leq C'\left(\|\mathcal{I}_{b_1} - \mathscr{I}_{b_2}\| + |\log \|\mathcal{I}_{b_1} - \mathcal{I}_{b_2}\||^{-1}\right)^{\mu_2}.$$

Proof. We use the same tools as in the derivation of Theorem 4.2.1 and Theorem 4.3.1, that is geometric optics solutions and light -ray transform. For $b_1,\, b_2 \in \mathcal{C}^1(\overline{Q})$, we define b in $\mathbb{R}^{n+1}$ by $b = b_2 - b_1$ in Q and $b = 0$ on $\mathbb{R}^{n+1} \setminus Q$. Notice that from (4.28) we have $b \in \mathcal{C}^1(\mathbb{R}^n \times (0,T))$.
We consider a function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and we proceed as in the proof of Lemma 4.3.1. Then we find out that

$$\left|\int_0^T \int_{\mathbb{R}^n} b(x - t\omega, t)\, \varphi^2(x)\, dx\, dt\right| \leq C\left(\lambda^3 \|\mathcal{I}_{b_2} - \mathcal{I}_{b_1}\| + \frac{1}{\lambda}\right) \|\varphi\|^2_{H^3(\mathbb{R}^n)}, \quad \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n).$$

Next, in order to derive an estimation of the light-ray transform $R(b)(y,\omega)$ of b, we fix $y \in \mathbb{R}^n$, we use the fact that $b \in \mathcal{C}^1(\mathbb{R}^n \times (0,T))$ and proceed as in the proof of Lemma 4.2.2 for the above sequence $(\varphi_\varepsilon)_\varepsilon$. The result of Theorem 4.4.1 follows by repeating the arguments of the two previous sections. $\square$

CHAPTER 5

Stable determination of two time-dependent coefficients

The statement of this chapter are collected in a paper that will appear in *Mathematical Analysis and Applications*

5.1 Introduction

In this chapter, we aim to generalize and improve the previous results by studying the inverse problem of determining two time-dependent unknown coefficients of order zero and one appearing in a wave equation. We consider the following system

$$\begin{cases} \partial_t^2 u - \Delta u + a(x,t)\partial_t u + b(x,t)u = 0 & \text{in } Q := \Omega \times (0,T), \\ u(\,\cdot\,,0) = u_0,\ \partial_t u(\,\cdot\,,0) = u_1 & \text{in } \Omega, \\ u = f & \text{on } \Sigma := \Gamma \times (0,T), \end{cases} \tag{5.1}$$

where $f \in H^1(\Sigma)$, $u_0 \in H^1(\Omega)$, $u_1 \in L^2(\Omega)$ and the coefficients $a \in \mathcal{C}^2(Q)$ and $b \in \mathcal{C}^1(Q)$ are assumed to be real valued. In light of Theorem 1.2.1, it is well known that if $f(\cdot,0) = u_{0|\Gamma}$, there exists a unique solution u to the equation (5.1) satisfying

$$u \in \mathcal{C}([0,T], H^1(\Omega)) \cap \mathcal{C}^1([0,T], L^2(\Omega)).$$

In the present chapter, we address the stability issues in the study of an inverse problem for the equation (5.1), in the presence of an absorbing coefficient a and an electric potential b that depend on both space and time variables. Inspired by the work of Bellassoued [5] and following the same strategy as in the previous chapter, we prove stability estimates in the recovery of the unknown coefficients a and b via different types of measurements and over different subsets of the domain Q. In the first and the second case, we will assume that the initial conditions (u_0, u_1) are fixed at zero and we will prove that in these two cases it is possible to recover the time-dependent coefficients a and b but only in some specific subsets of the cylindrical domain. And if we want to recover the unkown coefficients over the whole domain, we need to further vary the initial conditions.

In order to give the main statements of this chapter, we need first to introduce the set of the admissible coefficients a and b: Given $M_1,\ M_2 > 0$, we consider the set of admissible coefficients a and b:

$$\mathcal{A}(M_1, M_2) = \{(a,b) \in \mathcal{C}^2(\overline{Q}_r) \times \mathcal{C}^1(\overline{Q}_r);\ \|a\|_{C^2(Q)} \leq M_1,\ \|b\|_{C^1(Q)} \leq M_2\}.$$

Here the sets Q_r, $Q_{r,*}$ and $Q_{r,\sharp}$ are defined in Section 2.3.1.

5.2 Determination of the coefficients from boundary measurements

In the first case, we will assume that the initial conditions (u_0, u_1) are zero and our set of data will be given by boundary measurements *enclosed* by the

Dirichlet-to-Neumann map $\Lambda_{a,b}$ given by (2.2.1). Note that from Chapter 1 we have $\Lambda_{a,b}$ is continuous from $\mathcal{H}_0^1(\Sigma)$ to $L^2(\Sigma)$. We denote by $\|\Lambda_{a,b}\|$ its norm in $\mathcal{L}(\mathcal{H}_0^1(\Sigma), L^2(\Sigma))$.

From Chapter 2, Section 2.2, one can see that it is hopeless to determine a and b everywhere since the initial conditions are zero. So, we will focus on the region arised by Ramm and Rakesh [**36**] and prove stability estimates for the absorbing coefficient a and the potential b appearing in the initial boundary value problem (5.1), by the use of the geometrical optics solutions constructed in Chapter 1 and the light-ray transform. Throughout the rest of this section, we assume that Supp $\varphi \subset \mathscr{A}_r$, in such a way we have

$$\text{Supp}\varphi\cap\Omega=\emptyset \quad \text{and} \quad (\text{Supp}\varphi\pm T\omega)\cap\Omega=\emptyset,\ \ \forall\,\omega\in\mathbb{S}^{n-1},$$

where $\mathscr{A}_r$ is defined in Section 2.3.1.

5.2.1 Determination of the absorbing coefficient

Our goal here is to show that the time dependent coefficient a depends stably on the Dirichlet-to-Neumann map $\Lambda_{a,b}$. Let $\omega\in\mathbb{S}^{n-1}$, and $(a_i,b_i)\in\mathcal{A}(M_1,M_2)$ such that $(a_1,b_1)=(a_2,b_2)$ in $\overline{Q}_r\setminus Q_{r,*}$. we define a in $\mathbb{R}^{n+1}$ by $a=a_2-a_1$ in $\overline{Q}_r$ and $a=0$ on $\mathbb{R}^{n+1}\setminus\overline{Q}_r$ and we set

$$b=b_2-b_1,\ \text{and}\ A(x,t)=(A^-A^+)(x,t)=\exp\Big(-\frac{1}{2}\int_0^t a(x+(t-s)\omega,s)\,ds\Big).$$

Here, we recall the definition of A^- and A^+

$$A^-(x,t)=\exp\Big(\frac{1}{2}\int_0^t a_1(x+(t-s)\omega,s)\,ds\Big),\quad A^+(x,t)=\exp\Big(-\frac{1}{2}\int_0^t a_2(x+(t-s)\omega,s)\,ds\Big).$$

In the rest of this section, We start by collecting a preliminary estimate which relates the difference of the absorbing coefficients to the Dirichlet-to-Neumann map.

Lemma 5.2.1. *Let* $(a_i, b_i) \in \mathcal{A}(M_1, M_2)$, $i = 1, 2$. *There exists* $C > 0$ *such that for any* $\omega \in \mathbb{S}^{n-1}$ *and* $\varphi \in \mathcal{C}_0^\infty(\mathscr{A}_r)$, *the following estimate holds true*

$$\Big| \int_{\mathbb{R}^n} \varphi^2(y) \Big[\exp \Big(-\frac{1}{2} \int_0^T a(y - s\omega, s)\, ds \Big) - 1 \Big]\, dy \Big| \leq C \Big(\lambda^2 \| \Lambda_{a_2,b_2} - \Lambda_{a_1,b_1} \| + \frac{1}{\lambda} \Big) \| \varphi \|^2_{H^3(\mathbb{R}^n)},$$

for any sufficiently large $\lambda > 0$. *Here* C *depends only on* Ω, T, M_1 *and* M_2.

Proof. In view of Lemma 1.3.2, and using the fact that $\mathrm{Supp}\,\varphi \cap \Omega = \emptyset$, there exists a geometrical optics solution u^+ to the equation

$$\begin{cases} \partial_t^2 u^+ - \Delta u^+ + a_2(x,t)\partial_t u^+ + b_2(x,t) u^+ = 0 & \text{in } Q, \\ u^+(x,0) = \partial_t u^+(x,0) = 0 & \text{in } \Omega, \end{cases}$$

in the following form

$$u^+(x,t) = \varphi(x + t\omega) A^+(x,t) e^{i\lambda(x\cdot\omega + t)} + r_\lambda^+(x,t), \tag{5.2}$$

corresponding to the coefficients a_2 and b_2, where $r_\lambda^+(x,t)$ satisfies (1.13), (1.14). Next, let us denote by f_λ the function

$$f_\lambda(x,t) = u^+(x,t)_{|\Sigma} = \varphi(x + t\omega) A^+(x,t) e^{i\lambda(x\cdot\omega + t)}.$$

We denote by u_1 the solution of

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 + a_1(x,t)\partial_t u_1 + b_1(x,t) u_1 = 0 & \text{in } Q, \\ u_1(x,0) = \partial_t u_1(x,0) = 0 & \text{in } \Omega, \\ u_1 = f_\lambda & \text{on } \Sigma. \end{cases}$$

Putting $u = u_1 - u^+$. Then, u is a solution to the following system

$$\begin{cases} \partial_t^2 u - \Delta u + a_1(x,t)\partial_t u + b_1(x,t)u = a(x,t)\partial_t u^+ + b(x,t)u^+ & \text{in } \ Q, \\ u(x,0) = \partial_t u(x,0) = 0 & \text{in } \ \Omega, \\ u(x,t) = 0 & \text{on } \ \Sigma. \end{cases} \tag{5.3}$$

where $a = a_2 - a_1$ and $b = b_2 - b_1$. On the other hand Lemma 1.3.3 and the fact that $(\text{Supp } \varphi \pm T\omega) \cap \Omega = \emptyset$, guarantee the existence of a geometrical optic solution u^- to the backward problem of (5.1)

$$\begin{cases} \partial_t^2 u^- - \Delta u^- - a_1(x,t)\partial_t u^- + (b_1(x,t) - \partial_t a_1(x,t))u^- = 0 & \text{in } \ Q, \\ u^-(x,T) = 0 = \partial_t u^-(x,T) & \text{in } \ \Omega, \end{cases}$$

corresponding to the coefficients a_1 and $(-\partial_t a_1 + b_1)$, in the form

$$u^-(x,t) = \varphi(x + t\omega) e^{-i\lambda(x \cdot \omega + t)} A^-(x,t) + r_\lambda^-(x,t), \tag{5.4}$$

where $r_\lambda^-(x,t)$ satisfies (1.21), (1.22). Multiplying the first equation of (5.3) by u^-, integrating by parts and using Green's formula, we obtain

$$\int_0^T\!\!\int_\Omega a(x,t)\partial_t u^+ \, u^- \, dx \, dt + \int_0^T\!\!\int_\Omega b(x,t)u^+ \, u^- \, dx \, dt = \int_0^T\!\!\int_\Gamma (\Lambda_{a_2,b_2} - \Lambda_{a_1,b_1})(f_\lambda) \, u^- \, d\sigma \, dt. \tag{5.5}$$

On the other hand, by replacing u^+ and u^- by their expressions, we have

$$\begin{aligned} \int_0^T\!\!\int_\Omega a(x,t)\partial_t u^+ \, u^- \, dx \, dt &= \int_0^T\!\!\int_\Omega a(x,t)\partial_t \varphi(x+t\omega) e^{i\lambda(x \cdot \omega + t)} A^+ r_\lambda^- \, dx \, dt \\ &+ \int_0^T\!\!\int_\Omega a(x,t)\varphi(x+t\omega) e^{i\lambda(x \cdot \omega + t)} \partial_t A^+ r_\lambda^- \, dx \, dt + \int_0^T\!\!\int_\Omega a(x,t)\partial_t\varphi(x+t\omega)\varphi(x+t\omega)(A^+A^-) dx \, dt \\ &+ \int_0^T\!\!\int_\Omega a(x,t)\varphi(x+t\omega) e^{-i\lambda(x.\omega+t)} A^- \partial_t r_\lambda^+ dx \, dt + i\lambda \int_0^T\!\!\int_\Omega a(x,t)\varphi^2(x+t\omega)(A^+A^-) \, dx \, dt \\ &+ \int_0^T\!\!\int_\Omega a(x,t)\varphi^2(x+t\omega)\partial_t A^+ A^- \, dx \, dt + \ i\lambda \int_0^T\!\!\int_\Omega a(x,t)\varphi(x+t\omega) e^{i\lambda(x \cdot \omega + t)} A^+ r_\lambda^- \, dx \, dt \\ &+ \int_0^T\!\!\int_\Omega a(x,t)\partial_t r_\lambda^+ r_\lambda^- \, dx \, dt = i\lambda \int_0^T\!\!\int_\Omega a(x,t)\varphi^2(x+t\omega) A \, dx \, dt + \mathcal{I}_\lambda, \end{aligned}$$

where $A = A^+A^-$. In light of (5.5), we have

$$i\lambda \int_0^T \int_\Omega a(x,t)\varphi^2(x+t\omega)A(x,t)\,dx\,dt = \int_0^T \int_\Gamma (\Lambda_{a_2,b_2} - \Lambda_{a_1,b_1})(f_\lambda)\,u^-\,d\sigma\,dt - \int_0^T \int_\Omega b(x,t)u^+u^-\,dx\,dt - \mathcal{I}_\lambda. \quad (5.6)$$

Note that for λ sufficiently large, we have

$$\|u^+u^-\|_{L^1(Q)} \leq C\|\varphi\|^2_{H^3(\mathbb{R}^n)}, \quad \text{and} \quad |\mathcal{I}_\lambda| \leq C\|\varphi\|^2_{H^3(\mathbb{R}^n)}. \quad (5.7)$$

On the other hand, since on Σ, we have $u^+ = f_\lambda$ and $r_\lambda^- = r_\lambda^+ = 0$, then, we get the following estimate

$$\begin{aligned} \Big|\int_0^T\int_\Gamma (\Lambda_{a_2,b_2} - \Lambda_{a_1,b_1})(f_\lambda)\,u^-\,d\sigma\,dt\Big| &\leq \|\Lambda_{a_2,b_2} - \Lambda_{a_1,b_1}\|\,\|f_\lambda\|_{H^1(\Sigma)}\|u^-\|_{L^2(\Sigma)} \\ &\leq \|\Lambda_{a_2,b_2} - \Lambda_{a_1,b_1}\|\,\|u^+ - r_\lambda^+\|_{H^2(Q)}\,\|u^- - r_\lambda^-\|_{H^1(Q)} \\ &\leq C\lambda^3\|\Lambda_{a_2,b_2} - \Lambda_{a_1,b_1}\|\,\|\varphi\|^2_{H^3(\mathbb{R}^n)}. \end{aligned} \quad (5.8)$$

Consequently, by (5.6), (5.7) and (5.8), we obtain

$$\Big|\int_0^T\int_\Omega a(x,t)\varphi^2(x+t\omega)A(x,t)\,dx\,dt\Big| \leq C\Big(\lambda^2\|\Lambda_{a_2,b_2} - \Lambda_{a_1,b_1}\| + \frac{1}{\lambda}\Big)\|\varphi\|^2_{H^3(\mathbb{R}^n)},$$

where $A(x,t) = \exp\Big(-\frac{1}{2}\int_0^t a(x+(t-s)\omega, s)\,ds\Big)$. Then, using the fact $a(x,t) = 0$ outside $Q_{r,*}$ and making this change of variables $y = x + t\omega$, one gets the following estimation

$$\Big|\int_0^T\int_{\mathbb{R}^n} a(y-t\omega,t)\varphi^2(y)\exp\Big(-\frac{1}{2}\int_0^t a(y-s\omega,s)\,ds\Big)\,dy\,dt\Big| \leq C\Big(\lambda^2\|\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1}\|+\frac{1}{\lambda}\Big)\|\varphi\|^2_{H^3(\mathbb{R}^n)}.$$

Bearing in mind that

$$\int_0^T\!\!\int_{\mathbb{R}^n} a(y-t\omega,t)\ \varphi^2(y)\ \exp\Big(-\frac{1}{2}\int_0^t a(y-s\omega,s)\,ds\Big)\,dy\,dt$$
$$= -2\int_0^T\!\!\int_{\mathbb{R}^n} \varphi^2(y)\frac{d}{dt}\Big[\exp\Big(-\frac{1}{2}\int_0^t a(y-s\omega,s)\,ds\Big)\Big]\,dy\,dt$$
$$= -2\int_{\mathbb{R}^n} \varphi^2(y)\Big[\exp\Big(-\frac{1}{2}\int_0^T a(y-s\omega,s)\,ds\Big)-1\Big]\,dy,$$

we conclude the desired estimate given by

$$\Big|\int_{\mathbb{R}^n} \varphi^2(y)\Big[\exp\Big(-\frac{1}{2}\int_0^T a(y-s\omega,s)\,ds\Big)-1\Big]\,dy\Big| \leq C\Big(\lambda^2\|\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1}\|+\frac{1}{\lambda}\Big)\|\varphi\|^2_{H^3(\mathbb{R}^n)}.$$

This completes the proof of the lemma. □

Our next goal is to obtain an estimate that links the light-ray transform of the absorbing coefficient $a=a_2-a_1$ to the measurement $\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1}$ on a precise set. Using the above lemma, we can control the light-ray transform of a as follows:

Lemma 5.2.2. *Let $(a_i,b_i)\in\mathcal{A}(M_1,M_2)$, $i=1,2$. There exist $C>0$, $\delta>0$, $\beta>0$ and $\lambda_0>0$ such that for all $\omega\in\mathbb{S}^{n-1}$, we have*

$$|\mathcal{R}(a)(y,\omega)|\leq C\Big(\lambda^\delta\|\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1}\|+\frac{1}{\lambda^\beta}\Big),\quad a.e.\, y\in\mathbb{R}^n,$$

for any $\lambda\geq\lambda_0$. Here C depends only on Ω, T, M_1 and M_2.

Proof. Let $\psi\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ be a positive function which is supported in the unit ball $B(0,1)$ and such that $\|\psi\|_{L^2(\mathbb{R}^n)}=1$. Define

$$\varphi_h(x)=h^{-n/2}\psi\Big(\frac{x-y}{h}\Big),\tag{5.9}$$

where $y\in\mathscr{A}_r$. Then, for $h>0$ sufficiently small we can verify that

$$\mathrm{Supp}\,\varphi_h\cap\Omega=\emptyset,\quad \text{and}\ \ \mathrm{Supp}\,\varphi_h\pm T\omega\cap\Omega=\emptyset.$$

Moreover, we have

$$\begin{aligned}\Big|\exp\Big[-\frac{1}{2}\int_0^T a(y-s\omega,s)\,ds\Big]-1\Big| &= \Big|\int_{\mathbb{R}^n}\varphi_h^2(x)\Big[\exp\Big(-\frac{1}{2}\int_0^T a(y-s\omega,s)\,ds\Big)-1\Big]\,dx\Big| \\ \leq \quad & \Big|\int_{\mathbb{R}^n}\varphi_h^2(x)\Big[\exp\Big(-\frac{1}{2}\int_0^T a(y-s\omega,s)\,ds\Big)-\exp\Big(-\frac{1}{2}\int_0^T a(x-s\omega,s)\,ds\Big)\Big]\,dx\Big| \\ & +\Big|\int_{\mathbb{R}^n}\varphi_h^2(x)\Big[\exp\Big(-\frac{1}{2}\int_0^T a(x-s\omega,s)\,ds\Big)-1\Big]dx\Big|. \qquad (5.10)\end{aligned}$$

Therefore, since we have

$$\begin{aligned}&\Big|\exp\Big(-\frac{1}{2}\int_0^T a(y-s\omega,s)ds\Big)\exp\Big(-\frac{1}{2}\int_0^T a(x-s\omega,s)ds\Big)\Big| \\ &\qquad \leq C\Big|\int_0^T a(y-s\omega,s)-a(x-s\omega,s)ds\Big|,\end{aligned}$$

and using the fact that $\Big|\int_0^T\Big(a(y-s\omega,s)-a(x-s\omega,s)\Big)ds\Big|\leq C\,|y-x|$, we deduce upon applying Lemma 5.2.1 with $\varphi=\varphi_h$ the following estimation

$$\Big|\exp\Big(-\frac{1}{2}\int_0^T a(y-s\omega,s)\,ds\Big)-1\Big|\leq C\int_{\mathbb{R}^n}\varphi_h^2(x)\,|y-x|\,dx+C\Big(\lambda^2\|\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1}\|+\frac{1}{\lambda}\Big)\|\varphi_h\|^2_{H^3(\mathbb{R}^n)}.$$

On the other hand, we have

$$\|\varphi_h\|_{H^3(\mathbb{R}^n)}\leq Ch^{-3} \quad \text{and} \quad \int_{\mathbb{R}^n}\varphi_h^2(x)|y-x|\,dx\leq Ch.$$

So that we end up getting the following inequality

$$\Big|\exp\Big(-\frac{1}{2}\int_0^T a(y-s\omega,s)\,ds\Big)-1\Big|\leq C\,h+C\Big(\lambda^2\|\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1}\|+\frac{1}{\lambda}\Big)h^{-6}.$$

Selecting h small such that $h=1/\lambda h^6$, that is $h=\lambda^{-1/7}$, we find two constants $\delta>0$ and $\beta>0$ such that

$$\Big|\exp\Big(-\frac{1}{2}\int_0^T a(y-s\omega,s)\,ds\Big)-1\Big|\leq C\Big[\lambda^\delta\|\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1}\|+\frac{1}{\lambda^\beta}\Big].$$

Now, using the fact that $|X| \le e^M |e^X - 1|$ for any $|X| \le M$, we deduce that

$$\Big| - \frac{1}{2} \int_0^T a(y - s\omega, s)\, ds \Big| \le e^{M_1 T} \Big| \exp\Big(- \frac{1}{2} \int_0^T a(y - s\omega, s)\, ds \Big) - 1 \Big|.$$

Hence, we conclude that for all $y \in \mathscr{A}_r$ and $\omega \in \mathbb{S}^{n-1}$ we have

$$\Big| \int_0^T a(y - s\omega, s)\, ds \Big| \le C \Big(\lambda^\delta \| \Lambda_{a_2,b_2} - \Lambda_{a_1,b_1} \| + \frac{1}{\lambda^\beta} \Big).$$

Since $a = a_2 - a_1 = 0$ outside $Q_{r,*}$, this entails that for all $y \in \mathscr{A}_r$, and $\omega \in \mathbb{S}^{n-1}$, we have

$$\Big| \int_{\mathbb{R}} a(y - t\omega, t)\, dt \Big| \le C \left(\lambda^\delta \| \Lambda_{a_2,b_2} - \Lambda_{a_1,b_1} \| + \frac{1}{\lambda^\beta} \right). \tag{5.11}$$

Moreover, if $y \in B(0, r/2)$, we have $|y - t\omega| \ge |t| - |y| \ge |t| - \dfrac{r}{2}$. Hence, one can see that $(y - t\omega, t) \notin \mathscr{C}_r^+$ if $t > r/2$. On the other hand, we have $(y - t\omega, t) \notin \mathscr{C}_r^+$ if $t \le \dfrac{r}{2}$. Thus, we conclude that $(y - t\omega, t) \notin \mathscr{C}_r^+ \supset Q_{r,*}$ for $t \in \mathbb{R}$. This and the fact that $a = a_2 - a_1 = 0$ outside $Q_{r,*}$, entails that for all $y \in B(0, r/2)$ and $\omega \in \mathbb{S}^{n-1}$, we have

$$a(y - t\omega, t) = 0, \ \forall\, t \in \mathbb{R}.$$

By a similar way, we prove for $|y| \ge T - r/2$, that $(y - t\omega, t) \notin \mathscr{C}_r^- \supset Q_{r,*}$ for $t \in \mathbb{R}$ and then $a(y - t\omega, t) = 0$. Hence, we conclude that

$$\int_{\mathbb{R}} a(y - t\omega, t)\, dt = 0, \quad \text{a.e. } y \notin \mathscr{A}_r, \ \ \omega \in \mathbb{S}^{n-1}. \tag{5.12}$$

Thus, by (5.11) and (5.12) we finish the proof of the lemma by getting

$$|\mathcal{R}(a)(y, \omega)| = \Big| \int_{\mathbb{R}} a(t, y - t\omega)\, dt \Big| \le C \left(\lambda^\delta \| \Lambda_{a_2,b_2} - \Lambda_{a_1,b_1} \| + \frac{1}{\lambda^\beta} \right), \ \ \text{a.e. } y \in \mathbb{R}^n, \ \ \omega \in \mathbb{S}^{n-1}.$$

The proof of Lemma 5.2.2 is complete. $\square$

Our goal now is to obtain an estimate linking the Fourier transform with respect to (x, t) of the absorbing coefficient $a = a_2 - a_1$ to the measurement

$\Lambda_{a_2,b_2} - \Lambda_{a_1,b_1}$ in the conic set E given by (2.16). Namely, we aim for proving that the Fourier transform of a is bounded as follows:

Lemma 5.2.3. *Let $(a_i, b_i) \in \mathcal{A}(M_1, M_2)$, $i = 1, 2$. There exist $C > 0$, $\delta > 0$, $\beta > 0$ and $\lambda_0 > 0$, such that the following estimate*

$$|\widehat{a}(\xi, \tau)| \leq C \left(\lambda^{\delta} \|\Lambda_{a_2,b_2} - \Lambda_{a_1,b_1}\| + \frac{1}{\lambda^{\beta}} \right),$$

holds for any $(\xi, \tau) \in E$ and $\lambda \geq \lambda_0$.

Proof. Let $(\xi, \tau) \in E$ and $\zeta \in \mathbb{S}^{n-1}$ be such that $\xi \cdot \zeta = 0$. Setting

$$\omega = \frac{\tau}{|\xi|^2} \cdot \xi + \sqrt{1 - \frac{\tau^2}{|\xi|^2}} \cdot \zeta.$$

Then, one can see that $\omega \in \mathbb{S}^{n-1}$ and $\omega \cdot \xi = \tau$. On the other from Lemma 1.4.1, one can see that

$$\int_{\mathbb{R}^n} \mathcal{R}(a)(y, \omega)\, e^{-iy \cdot \xi}\, dy = \widehat{a}(\xi, \omega \cdot \xi) = \widehat{a}(\xi, \tau),$$

where we have set $(\xi, \tau) = (\xi, \omega \cdot \xi) \in E$. Bearing in mind that for any $t \in \mathbb{R}$, $\mathrm{Supp}\, a(\,\cdot\,, t) \subset \Omega \subset B(0, r/2)$, we deduce that

$$\int_{\mathbb{R}^n \cap B(0, \frac{r}{2} + T)} \mathcal{R}(a)(\omega, y)\, e^{-iy \cdot \xi}\, dy = \widehat{a}(\xi, \tau).$$

Then, in view of Lemma 5.2.2, we finish the proof of this lemma. $\square$

Let us state the main statement of this section.

Theorem 5.2.1. *Let $T > 2\,\mathrm{Diam}(\Omega)$. There exist $C > 0$ and m, $\mu_1 \in (0, 1)$ such that if $\|\Lambda_{a_1,b_1} - \Lambda_{a_2,b_2}\| \leq m$, we have*

$$\|a_2 - a_1\|_{L^\infty(Q_{r,*})} \leq C |\log \|\Lambda_{a_1,b_1} - \Lambda_{a_2,b_2}\||^{-\mu_1},$$

for any $(a_i, b_i) \in \mathcal{A}(M_1, M_2)$ *such that* $\|a_i\|_{H^p(Q)} \leq M_1$, *for some* $p > n/2 + 3/2$, $(a_1, b_1) = (a_2, b_2)$ *in* $\overline{Q}_r \setminus Q_{r,*}$ *and* $(\partial_x a_1, \partial_x b_1) = (\partial_x a_2, \partial_x b_2)$ *on* $\partial Q_r \cap \partial Q_{r,*}$. *Here* C *depends only on* Ω, M_1, M_2, T *and* n.

We move now to prove our result by the use of the result we have already obtained and the analytic argument given by Theorem 3.2.1, which is inspired by [**1**] and adapted for our case.

Proof. For a fixed $\alpha > 0$, we set $F_\alpha(\tau, \xi) = \widehat{a}(\alpha(\xi, \tau))$, for all $(\xi, \tau) \in \mathbb{R}^{n+1}$. It is easy to see that F_α is analytic and we have for $\kappa \in (\mathbb{N} \cup \{0\})^{n+1}$

$$\begin{aligned} |\partial^\kappa F_\alpha(\xi, \tau)| = |\partial^\kappa \widehat{a}(\alpha(\xi, \tau))| &= \left| \partial^\kappa \int_{\mathbb{R}^{n+1}} a(x,t)\, e^{-i\alpha(t,x)\,\cdot\,(\xi,\tau)}\, dx\, dt \right| \\ &= \left| \int_{\mathbb{R}^{n+1}} a(x,t)(-i)^{|\kappa|} \alpha^{|\kappa|} (x,t)^\kappa e^{-i\alpha(x,t)\,\cdot\,(\xi,\tau)}\, dx\, dt \right|. \end{aligned}$$

This entails that

$$|\partial^\kappa F_\alpha(\xi, \tau)| \leq \int_{\mathbb{R}^{n+1}} |a(x,t)| \alpha^{|\kappa|} (|x|^2 + t^2)^{\frac{|\kappa|}{2}}\, dx\, dt \leq \|a\|_{L^1(Q_{r,*})}\, \alpha^{|\kappa|}\, (2T^2)^{\frac{|\kappa|}{2}} \leq C\, \frac{|\kappa|!}{(T^{-1})^{|\kappa|}}\, e^\alpha.$$

The, upon applying Theorem 3.2.1 with $M = Ce^\alpha$, $2\rho = T^{-1}$, and $\mathcal{O} = \mathring{E} \cap B(0,1)$, where

$$\mathring{E} = \{(\xi, \tau) \in \mathbb{R} \times (\mathbb{R}^n \setminus \{0_{\mathbb{R}^n}\}),\ \ |\tau| < |\xi|\},$$

one may find a constant $\mu \in (0,1)$ such that we have for all $(\xi, \tau) \in B(0,1)$, the following estimation

$$|F_\alpha(\xi, \tau)| = |\widehat{a}(\alpha(\xi, \tau))| \leq C e^{\alpha(1-\gamma)} \|F_\alpha\|^\gamma_{L^\infty(\mathcal{O})}.$$

Now the idea is to find an estimate for the Fourier transform of a in a suitable ball. Using the fact that $\alpha\, \mathring{E} = \{\alpha(\xi, \tau),\, (\xi, \tau) \in \mathring{E}\} = \mathring{E}$, we obtain for all $(\xi, \tau) \in B(0, \alpha)$

$$|\widehat{a}(\xi, \tau)| = |F_\alpha(\alpha^{-1}(\xi, \tau))| \ \leq\ C e^{\alpha(1-\gamma)}\, \|F_\alpha\|^\gamma_{L^\infty(\mathcal{O})}$$

$$\begin{aligned} &\leq Ce^{\alpha(1-\gamma)} \| \, \widehat{a} \, \|^{\mu}_{L^\infty(B(0,\alpha)\cap \mathring{E})} \\ &\leq Ce^{\alpha(1-\gamma)} \| \, \widehat{a} \, \|^{\gamma}_{L^\infty(\mathring{E})}. \end{aligned} \tag{5.13}$$

The next step in the proof is to deduce an estimate that links the unknown coefficient a to the measurement $\Lambda_{a_2,b_2} - \Lambda_{a_1,b_1}$. To obtain such estimate, we need first to decompose the $H^{-1}(\mathbb{R}^{n+1})$ norm of a into the following way

$$\begin{aligned} \|a\|^{2/\gamma}_{H^{-1}(\mathbb{R}^{n+1})} &= \Big(\int_{|(\tau,\xi)|<\alpha} (1+|(\tau,\xi)|^2)^{-1} |\widehat{a}(\xi,\tau)|^2 \, d\xi d\tau + \int_{|(\xi,\tau)|\geq\alpha} (1+|(\tau,\xi)|^2)^{-1} |\widehat{a}(\xi,\tau)|^2 \, d\xi d\tau \Big)^{1/\gamma} \\ &\leq C\Big(\alpha^{n+1} \, \|\widehat{a}\|^2_{L^\infty(B(0,\alpha))} + \alpha^{-2} \, \|a\|^2_{L^2(\mathbb{R}^{n+1})} \Big)^{1/\gamma}. \end{aligned}$$

Hence, in light of (5.13) and Lemma 5.2.3, we get

$$\begin{aligned} \|a\|^{2/\gamma}_{H^{-1}(\mathbb{R}^{n+1})} &\leq C\Big(\alpha^{n+1} \, e^{2\alpha(1-\gamma)} \, (\lambda^\delta \|\Lambda_{a_2,b_2} - \Lambda_{a_1,b_1}\| + \frac{1}{\lambda^\beta})^{2\gamma} + \alpha^{-2} \Big)^{1/\gamma} \\ &\leq C\Big(\alpha^{\frac{n+1}{\gamma}} \, e^{\frac{2\alpha(1-\gamma)}{\gamma}} \lambda^{2\beta} \|\Lambda_{a_2,b_2} - \Lambda_{a_1,b_1}\|^2 + \alpha^{\frac{n+1}{\gamma}} \, e^{\frac{2\alpha(1-\gamma)}{\gamma}} \, \lambda^{-2\beta} + \alpha^{-2/\gamma} \Big). \end{aligned}$$

Let $\alpha_0 > 0$ be sufficiently large and assume that $\alpha > \alpha_0$. Setting

$$\lambda = \alpha^{\frac{n+3}{2\mu\gamma}} \, e^{\frac{\alpha(1-\mu)}{\mu\gamma}},$$

and using the fact that $\alpha > \alpha_0$, one can see that $\lambda > \lambda_0$ and $\alpha^{\frac{n+1}{\gamma}} \, e^{\frac{2\alpha(1-\gamma)}{\gamma}} \, \lambda^{-2\beta} = \alpha^{-2/\gamma}$. This entails that

$$\begin{aligned} \|a\|^{2/\gamma}_{H^{-1}(\mathbb{R}^{n+1})} &\leq C\Big(\alpha^{\frac{\beta(n+1)+\delta(n+3)}{\beta\gamma}} \, e^{\frac{2\alpha(\beta+\delta)(1-\gamma)}{\beta\gamma}} \|\Lambda_{a_2,b_2} - \Lambda_{a_1,b_1}\|^2 + \alpha^{-2/\gamma} \Big) \\ &\leq C\Big(e^{N\alpha} \|\Lambda_{a_2,b_2} - \Lambda_{a_1,b_1}\|^2 + \alpha^{-2/\gamma} \Big), \end{aligned}$$

where N depends on δ, β, n, and γ. The next step is to minimize the right hand-side of the above inequality with respect to α. We need to take α sufficiently large. So, there exists a constant $m > 0$ such that if $0 < \|\Lambda_{a_2,b_2} - \Lambda_{a_1,b_1}\| < m$, and

$$\alpha = \frac{1}{N} | \, \log \|\Lambda_{a_2,b_2} - \Lambda_{a_1,b_1}\| \, |,$$

then, we have the following estimation

$$\begin{aligned}\|a\|_{H^{-1}(Q_{r,*})} \leq \|a\|_{H^{-1}(\mathbb{R}^{n+1})} &\leq C\Big(\|\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1}\|+|\log\|\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1}\||^{-2/\gamma}\Big)^{\gamma/2} \\ &\leq C\Big(\|\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1}\|^{\gamma/2}+|\log\|\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1}\||^{-1}\Big). \quad (5.14)\end{aligned}$$

Let us now consider $\theta > 1$ such that $p := s-1 = \frac{n+1}{2}+2\theta$. Use Sobolev's embedding theorem we find by interpolating

$$\begin{aligned}\|a\|_{L^\infty(Q_{r,*})} &\leq C\|a\|_{H^{\frac{n}{2}+\theta}(Q_{r,*})} \\ &\leq C\,\|a\|^{1-\eta}_{H^{-1}(Q_{r,*})}\,\|a\|^{\eta}_{H^{s-1}(Q_{r,*})} \\ &\leq C\,\|a\|^{1-\eta}_{H^{-1}(Q_{r,*})},\end{aligned}$$

for some $\eta\in(0,1)$. This completes the proof of Theorem 5.2.1. □

This will be a key ingredient in proving the result of the next section.

5.2.2 Determination of the electric potential

By means of the geometrical optics solutions constructed in Chapter 1, we will show using the stability estimate we have already obtained for the absorbing coefficient a, that the time dependent potential b depends stably on the Dirichlet-to-Neumann map $\Lambda_{a,b}$. As before, given $\omega \in \mathbb{S}^{n-1}$, $(a_i, b_i) \in \mathcal{A}(M_1, M_2)$ such that $(a_1, b_1) = (a_2, b_2)$ in $\overline{Q}_r \setminus Q_{r,*}$, we set

$$a = a_2-a_1, \quad b = b_2-b_1 \quad \text{and} \quad A(x,t) = (A^-A^+)(x,t) = \exp\Big(-\frac{1}{2}\int_0^t a(x+(t-s)\omega, s)\,ds\Big),$$

where A^- and A^+ are given by

$$A^-(x,t) = \exp\Big(\frac{1}{2}\int_0^t a_1(x+(t-s)\omega, s)\,ds\Big), \quad A^+(x,t) = \exp\Big(-\frac{1}{2}\int_0^t a_2(x+(t-s)\omega, s)\,ds\Big).$$

In the rest of this section, we define b in $\mathbb{R}^{n+1}$ by $b = b_2 - b_1$ in $\overline{Q}_r$ and $b = 0$ on $\mathbb{R}^{n+1} \setminus \overline{Q}_r$. We start by giving a preliminary estimate that will be used to prove the main statement of this section.

Lemma 5.2.4. *Let $(a_i, b_i) \in \mathcal{A}(M_1, M_2)$, $i = 1, 2$. There exists $C > 0$ such that for any $\omega \in \mathbb{S}^{n-1}$ and $\varphi \in \mathcal{C}_0^\infty(\mathscr{A}_r)$, the following estimate holds*

$$\left|\int_0^T\int_{\mathbb{R}^n} b(y-t\omega,t)\varphi^2(y)\,dy\,dt\right| \leq C\Big(\lambda^3\|\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1}\| + \lambda\|a\|_{L^\infty(Q_{r,*})} + \frac{1}{\lambda}\Big)\|\varphi\|^2_{H^3(\mathbb{R}^n)},$$

for any $\lambda > 0$ sufficiently large. Here C depends only on Ω, M_1, M_2 and T.

Proof. We start with the identity (5.5), except this time we will isolate the electric potential

$$\int_0^T\int_\Omega b(x,t)u^+u^-\,dx\,dt = \int_0^T\int_\Gamma(\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1})(f_\lambda)u^-\,d\sigma\,dt - \int_0^T\int_\Omega a(x,t)\partial_t u^+u^-\,dx\,dt.$$

By replacing u^+ and u^- by their expressions we get

$$\begin{aligned}
&\int_0^T\int_\Omega b(x,t)\varphi^2(x+t\omega)A(x,t)\,dx\,dt = \int_0^T\int_\Gamma(\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1})(f_\lambda)u^-\,d\sigma\,dt \\
&-\int_0^T\int_\Omega b(x,t)\varphi(x+t\omega)A^-(x,t)e^{-i\lambda(x\cdot\omega+t)}r_\lambda^+(x,t)\,dx\,dt - \int_0^T\int_\Omega b(x,t)r_\lambda^+(x,t)r_\lambda^-(x,t)\,dx\,dt \\
&-\int_0^T\int_\Omega a(x,t)\partial_t u^+u^-\,dx\,dt - \int_0^T\int_\Omega b(x,t)\varphi(x+t\omega)A^+(x,t)e^{i\lambda(x\cdot\omega+t)}r_\lambda^-(x,t)\,dx\,dt \\
&\qquad = \int_0^T\int_\Gamma(\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1})(f_\lambda)u^-\,d\sigma\,dt + I'_\lambda.
\end{aligned} \tag{5.15}$$

Then, in view of (5.15), we have

$$\begin{aligned}
\int_0^T\int_\Omega b(x,t)\varphi^2(x+t\omega)dxdt &= \int_0^T\int_\Omega b(x,t)\varphi^2(x+t\omega)(1-A)dxdt \\
&\quad + \int_0^T\int_\Gamma(\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1})(f_\lambda)u^-d\sigma dt + I'_\lambda.
\end{aligned}$$

From (1.14), (1.22) and using the fact that $a = a_2 - a_1 = 0$ outside $Q_{r,*}$, we find

$$|I'_\lambda| \leq C\Big(\lambda\,\|a\|_{L^\infty(Q_{r,*})} + \frac{1}{\lambda}\Big)\|\varphi\|^2_{H^3(\mathbb{R}^n)}. \tag{5.16}$$

By the trace theorem, we get

$$\begin{aligned}\left|\int_0^T\int_\Gamma(\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1})(f_\lambda)u^-\,d\sigma\,dt\right| &\leq \|\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1}\|\|f_\lambda\|_{H^1(\Sigma)}\|u^-\|_{L^2(\Sigma)}\\ &\leq C\lambda^3\|\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1}\|\,\|\varphi\|^2_{H^3(\mathbb{R}^n)}. \quad (5.17)\end{aligned}$$

On the other hand, we have

$$\left|\int_0^T\int_\Omega b(x,t)\varphi^2(x+t\omega)(1-A)\,dx\,dt\right| \leq C\|a\|_{L^\infty(Q_{r,*})}\,\|\varphi\|^2_{H^3(\mathbb{R}^n)}. \qquad (5.18)$$

Then, in light of (5.16)-(5.18), taking to account that $b=b_2-b_1=0$ outside $Q_{r,*}$ and using the change of variables $y=x+t\omega$ we get

$$\left|\int_0^T\int_{\mathbb{R}^n} b(y-t\omega,t)\varphi^2(y)\,dy\,dt\right| \leq C\Big(\lambda^3\|\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1}\|+\lambda\|a\|_{L^\infty(Q_{r,*})}+\frac{1}{\lambda}\Big)\|\varphi\|^2_{H^3(\mathbb{R}^n)}.$$

This completes the proof of the Lemma. □

Now the idea is to deduce an estimate for the light ray transform of the time-dependent unknown coefficient b in order to control thereafter its Fourier transform.

Lemma 5.2.5. *Let $(a_i,b_i)\in\mathcal{A}(M_1,M_2)$, $i=1,2$. There exists $C>0$, $\delta>0$, $\beta>0$ and $\lambda_0>0$ such that for all $\omega\in\mathbb{S}^{n-1}$, the following estimate holds*

$$\Big|\mathcal{R}(b)(y,\omega)\Big| \leq C\Big(\lambda^\delta\|\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1}\|+\lambda^\delta\|a\|_{L^\infty(Q_{r,*})}+\frac{1}{\lambda^\beta}\Big), \qquad a.\,e\,y\in\mathbb{R}^n,$$

for any $\lambda>\lambda_0$. Here C depends only on Ω, T, M_1 and M_2.

Proof. We proceed as in the proof of Lemma 5.2.2. We consider the sequence $(\varphi_h)_h$ defined by (5.9) with $y\in\mathscr{A}_r$. Since we have

$$\begin{aligned}\left|\int_0^T b(y-t\omega,t)\;\;dt\right| &= \left|\int_0^T\int_{\mathbb{R}^n} b(y-t\omega,t)\,\varphi_h^2(x)\,dx\,dt\right|\\ &\leq \left|\int_0^T\int_{\mathbb{R}^n} b(x-t\omega,t)\varphi_h^2(x)dx\,dt\right| + \left|\int_0^T\int_{\mathbb{R}^n}\Big(b(y-t\omega,t)-b(x-t\omega,t)\Big)\varphi_h^2(x)dx\,dt\right|.\end{aligned}$$

Then, by applying Lemma 5.2.4 with $\varphi = \varphi_h$, and since $|b(y-t\omega,t)-b(x-t\omega,t)| \le C|y-x|$, we obtain

$$\Big|\int_0^T b(y-t\omega,t)dt\Big| \le C\Big(\lambda^3\|\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1}\| + \lambda\|a\|_{L^\infty(Q_{r,*})} + \frac{1}{\lambda}\Big)\|\varphi_h\|^2_{H^3(\mathbb{R}^n)} + C\int_{\mathbb{R}^n}|x-y|\varphi_h^2(x)dx.$$

On the other hand, since $\|\varphi_h\|_{H^3(\mathbb{R}^n)} \le Ch^{-3}$ and $\displaystyle\int_{\mathbb{R}^n}|x-y|\varphi_h^2(x)\,dx \le C\,h$, we conclude that

$$\Big|\int_0^T b(y-t\omega,t)\,dt\Big| \le C\Big(\lambda^3\|\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1}\| + \lambda\|a\|_{L^\infty(Q_{r,*})} + \frac{1}{\lambda}\Big)h^{-6} + C\,h.$$

Selecting h small such that $h = h^{-6}/\lambda$. Then, we find two constants $\delta > 0$ and $\beta > 0$ such that

$$\Big|\int_0^T b(y-t\omega,t)\Big| \le C\Big(\lambda^\delta\|\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1}\| + \lambda^\delta\|a\|_{L^\infty(Q_{r,*})} + \frac{1}{\lambda^\beta}\Big).$$

Using the fact that $b = b_2 - b_1 = 0$ outside $Q_{r,*}$, we then conclude that for all $y \in \mathscr{A}_r$ and $\omega \in \mathbb{S}^{n-1}$,

$$\Big|\int_{\mathbb{R}} b(y-t\omega,t)\,dt\Big| \le C\Big(\lambda^\delta\|\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1}\| + \lambda^\delta\|a\|_{L^\infty(Q_{r,*})} + \frac{1}{\lambda^\beta}\Big).$$

Next, by arguing as in the derivation of Lemma 5.2.2, we end up upper bounding the light-ray transform of b, for all $y \in \mathbb{R}^n$. □

By proceeding by a similar way as in the previous section, we can control the Fourier transform of b as follows

Lemma 5.2.6. *Let $(a_i, b_i) \in \mathcal{A}(M_1, M_2)$, $i = 1, 2$. There exists $C > 0$, $\delta > 0$, $\beta > 0$ and $\lambda_0 > 0$, such that the following estimate*

$$|\widehat{b}(\xi,\tau)| \le C\Big(\lambda^\delta\|\Lambda_{a_2,b_2}-\Lambda_{a_1,b_1}\| + \lambda^\delta\|a\|_{L^\infty(Q_{r,*})} + \frac{1}{\lambda^\beta}\Big), \quad a.\ e, (\xi,\tau) \in E,$$

for any $\lambda > \lambda_0$. Here C depends only on Ω, T, M_1 and M_2.

Theorem 5.2.2. *Let $T > 2\,Diam(\Omega)$. There exist $C > 0$ and $m,\, \mu \in (0,1)$ such that if $\|\Lambda_{a_1,b_1} - \Lambda_{a_2,b_2}\| \leq m$, we have*

$$\|b_2 - b_1\|_{H^{-1}(Q_{r,*})} \leq C\Big(\log|\log\|\Lambda_{a_2,b_2} - \Lambda_{a_1,b_1}\||^{\mu}\Big)^{-1},$$

for any $(a_i, b_i) \in \mathcal{A}(M_1, M_2)$ such that $\|a_i\|_{H^p(Q)} \leq M_1$, for some $p > n/2 + 3/2$, $(a_1, b_1) = (a_2, b_2)$ in $\overline{Q}_r \setminus Q_{r,*}$ and $(\partial_x a_1, \partial_x b_1) = (\partial_x a_2, \partial_x b_2)$ on $\partial Q_r \cap \partial Q_{r,*}$. Here C depends only on Ω, M_1, M_2, T and n.

This mentioned result is the main statement of this section and it shows that the time-dependent potential b can also be stably determined, from the knowledge of the boundary measurements $\Lambda_{a,b}$ in the same subset $Q_{r,*} \subset Q$, provided it is known outside this region.

Proof. Using Lemma 5.2.6 as well as the analytic continuation argument Theorem 3.2.1, we upper bound the Fourier transform of b in a suitable ball $B(0,\alpha)$ as follows

$$|\widehat{b}(\xi,\tau)| \leq C e^{\alpha(1-\gamma)} \Big(\lambda^{\delta}\|\Lambda_{a_2,b_2} - \Lambda_{a_1,b_1}\| + \lambda^{\delta}\|a\|_{L^\infty(Q_{r,*})} + \frac{1}{\lambda^{\beta}}\Big)^{\gamma}, \tag{5.19}$$

for some $\gamma \in (0,1)$ and where $\alpha > 0$ is assumed to be sufficiently large. Then, in order to deduce an estimate linking the unknown coefficient b to the measurement $\Lambda_{a_2,b_2} - \Lambda_{a_1,b_1}$, we control the $H^{-1}(\mathbb{R}^{n+1})$ norm of b as follows

$$\|b\|^{\frac{2}{\gamma}}_{H^{-1}(\mathbb{R}^{n+1})} \leq C\Big[\alpha^{n+1}\|\widehat{b}\|^2_{L^\infty(B(0,\alpha))} + \alpha^{-2}\|b\|^2_{L^2(\mathbb{R}^{n+1})}\Big]^{\frac{1}{\gamma}}.$$

So, by the use of (5.19), we obtain the following inequality

$$\|b\|^{\frac{2}{\gamma}}_{H^{-1}(\mathbb{R}^{n+1})} \leq C\Big[\alpha^{\frac{n+1}{\gamma}} e^{\frac{2\alpha(1-\gamma)}{\gamma}} \Big(\lambda^{2\delta}\epsilon^2 + \lambda^{2\delta}\|a\|^2_{L^\infty(Q_{r,*})} + \lambda^{-2\beta}\Big) + \alpha^{\frac{-2}{\gamma}}\Big], \tag{5.20}$$

where we have set $\epsilon = \|\Lambda_{a_2,b_2} - \Lambda_{a_1,b_1}\|$. In light of Theorem 5.2.1, one gets

$$\|b\|_{H^{-1}(\mathbb{R}^{n+1})}^{\frac{2}{\gamma}} \leq C\Big[\alpha^{\frac{n+1}{\gamma}} e^{\frac{2\alpha(1-\gamma)}{\gamma}} \left(\lambda^{2\delta}\epsilon^2 + \lambda^{2\delta}|\log\, \epsilon|^{-2\mu_1} + \lambda^{-2\beta}\right) + \alpha^{-\frac{2}{\gamma}}\Big],$$

for some γ, $\mu_1 \in (0,1)$ and δ, $\beta > 0$. Let $\alpha_0 > 0$ be sufficiently large and we take $\alpha > \alpha_0$. Setting

$$\lambda = \alpha^{\frac{n+3}{2\gamma\beta}} e^{\frac{\alpha(1-\gamma)}{\gamma\beta}}.$$

By $\alpha > \alpha_0$, we can assume that $\lambda > \lambda_0$. Therefore, the estimate (5.20) yields

$$\|b\|_{H^{-1}(\mathbb{R}^{n+1})}^{\frac{2}{\gamma}} \leq C\Big[e^{N\alpha} \left(\epsilon^2 + |\log \epsilon|^{-2\mu_1}\right) + \alpha^{-\frac{2}{\gamma}}\Big],$$

for some s, $\mu_1 \in (0,1)$, and where N is depending on n, γ, δ and β. Thus, since ϵ is small, we have

$$\|b\|_{H^{-1}(\mathbb{R}^{n+1})}^{\frac{2}{\gamma}} \leq C\Big(e^{N\alpha}|\log \epsilon|^{-2\mu_1} + \alpha^{-\frac{2}{\gamma}}\Big). \tag{5.21}$$

In order to minimize the right hand side of the above inequality with respect to α, we need to take α sufficiently large. So, we select α as follows

$$\alpha = \frac{1}{N} \log |\log \epsilon|^{\mu_1}.$$

Then, the estimate (5.21) yields

$$\|b\|_{H^{-1}(Q_{r,*})} \leq \|b\|_{H^{-1}(\mathbb{R}^{n+1})} \leq C\Big(\log |\log \|\Lambda_{a_2,b_2} - \Lambda_{a_1,b_1}\||^{\mu_1}\Big)^{-1}.$$

This completes the proof of Theorem 5.2.2. $\square$

5.3 Determination of the coefficients from boundary and final data

In order to extend the above results to a larger region $Q_{r,\sharp} \supset Q_{r,*}$, we require more information about the solution u of the wave equation 5.1 with $(u_0, u_1) = (0, 0)$. So, in this case we will add the final data of the solution u. This leads to defining the following boundary operator (response operator):

Definition 5.3.1. *We define the boundary operator $\mathscr{R}_{a,b}$ as follows*

$$\begin{array}{rcl} \mathscr{R}_{a,b} : H^1(\Sigma) & \longrightarrow & \mathcal{K} := L^2(\Sigma) \times H^1(\Omega) \times L^2(\Omega) \\ f & \longmapsto & (\partial_\nu u, u(\,\cdot\,, T), \partial_t u(\,\cdot\,, T)), \end{array}$$

with u is a solution to the equation (5.1) with $(u_0, u_1) = (0, 0)$.

We conclude from Theorem 1.2.1, that $\mathscr{R}_{a,b}$ is a continuous operator form $H^1(\Sigma)$ to $\mathcal{K}$. We denote by $\|\mathscr{R}_{a,b}\|$ its norm in $\mathcal{L}(H^1(\Sigma), \mathcal{K})$. In this case, we shall consider the geometric optics solutions constructed in Chapter 1, associated with a function φ obeying supp $\varphi \cap \Omega = \emptyset$. Note that this time, we have more flexibility on the support of the function φ and we don't need to assume that supp $\varphi \pm T\omega \cap \Omega = \emptyset$ anymore. Throughout the rest of this section, we denote by

$$\mathscr{R}^1_{a,b}(f) = \partial_\nu u, \quad \mathscr{R}^2_{a,b}(f) = u(\,\cdot\,, T), \quad \mathscr{R}^3_{a,b}(f) = \partial_t u(\,\cdot\,, T).$$

5.3.1 Determination of the absorbing coefficient

In this section, we will prove that the absorbing coefficient a can be stably recovered in a larger region if we further know the final data of the solution u of the dissipative wave equation (5.1) with $(u_0, u_1) = (0, 0)$. In the rest of this section, we define $a = a_2 - a_1$ in $\overline{Q}_r$ and $a = 0$ on $\mathbb{R}^{n+1} \setminus \overline{Q}_r$. We shall first prove the following statement

Theorem 5.3.1. *Let $T > 2\,Diam(\Omega)$. There exist $C > 0$ and m, $\mu_1 \in (0,1)$ such that if $\|\Lambda_{a_1,b_1} - \Lambda_{a_2,b_2}\| \leq m$, we have*

$$\|a_2 - a_1\|_{L^\infty(Q_{r,\sharp})} \leq |\log \|\mathscr{R}_{a_1,b_1} - \mathscr{R}_{a_2,b_2}\||^{-\mu_1},$$

for any $(a_i, b_i) \in \mathcal{A}(M_1, M_2)$ such that $\|a_i\|_{H^p(Q)} \leq M_1$, for some $p > n/2 + 3/2$, $(a_1, b_1) = (a_2, b_2)$ in $\overline{Q}_r \setminus Q_{r,\sharp}$ and $(\partial_x a_1, \partial_x b_1) = (\partial_x a_2, \partial_x b_2)$ on $\partial Q_r \cap \partial Q_{r,\sharp}$. Here C depends only on Ω, M_1, M_2, T and n.

Proof. In view of Lemma 1.3.2 and using the fact that supp $\varphi \cap \Omega = \emptyset$, there exists a geometrical optic solution u^+ to the wave equation

$$\begin{cases} \Big(\partial_t^2 - \Delta + a_2(x,t)\partial_t + b_2(x,t)\Big)u^+ = 0 & \text{in } Q, \\ u^+(x,0) = \partial_t u^+(x,0) = 0 & \text{in } \Omega, \end{cases}$$

in the following form

$$u^+(x,t) = \varphi(x + t\omega)A^+(x,t)e^{i\lambda(x\cdot\omega+t)} + r_\lambda^+(x,t), \tag{5.22}$$

corresponding to the coefficients a_2 and b_2, where $r_\lambda^+(x,t)$ satisfies (1.13) and (1.14). We denote

$$f_\lambda(x,t) = u^+(x,t)_{|\Sigma} = \varphi(x + t\omega)A^+(x,t)e^{i\lambda(x\cdot\omega+t)}.$$

Let u_1 be the solution of

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 + a_1(x,t)\partial_t u_1 + b_1(x,t)u_1 = 0 & \text{in } Q, \\ u_1(x,0) = \partial_t u_1(x,0) = 0 & \text{in } \Omega, \\ u_1 = f_\lambda & \text{on } \Sigma. \end{cases}$$

Putting $u = u_1 - u^+$. Then, u is a solution to the following system

$$\begin{cases} \partial_t^2 u - \Delta u + a_1(x,t)\partial_t u + b_1(x,t)u = a(x,t)\partial_t u^+ + b(x,t)u^+ & \text{in } Q, \\ u(x,0) = \partial_t u(x,0) = 0 & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Sigma, \end{cases} \tag{5.23}$$

where $a = a_2 - a_1$ and $b = b_2 - b_1$. On the other hand, Lemma 1.3.3 guarantees the existence of a geometrical optic solution u^- to the adjoint problem

$$\partial_t^2 u^- - \Delta u^- - a_1(x,t)\partial_t u^- + (b_1(x,t) - \partial_t a_1(x,t))u^- = 0 \quad \text{in } Q,$$

corresponding to the coefficients a_1 and $(-\partial_t a_1 + b_1)$, in the form

$$u^-(x,t) = \varphi(x + t\omega)e^{-i\lambda(x\cdot\omega+t)}A^-(x,t) + r_\lambda^-(x,t), \tag{5.24}$$

where $r_\lambda^-(x,t)$ satisfies (1.21) and (1.22). Multiplying the first equation of (5.23) by u^-, integrating by parts and using Green's formula, we get

$$\begin{aligned} \int_0^T\!\!\int_\Omega a(x,t)\partial_t u^+ u^- dxdt &= -\int_\Omega (\mathscr{R}^2_{a_2,b_2} - \mathscr{R}^2_{a_1,b_1})(f_\lambda)\Big[a_1(x,T)u^-(x,T) - \partial_t u^-(x,T)\Big]\,dx \\ &+ \int_0^T\!\!\int_\Gamma (\mathscr{R}^1_{a_2,b_2} - \mathscr{R}^1_{a_1,b_1})(f_\lambda)u^-(x,t)d\sigma dt - \int_\Omega (\mathscr{R}^3_{a_2,b_2} - \mathscr{R}^3_{a_1,b_1})(f_\lambda)u^-(x,T)\,dx \\ &- \int_0^T\!\!\int_\Omega b(x,t)u^+(x,t)u^-(x,t)\,dx\,dt. \end{aligned} \tag{5.25}$$

By replacing u^+ and u^- by their expressions, using (5.7) and the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} \Big|\int_0^T\!\!\int_\Omega a(x,t)\varphi^2(x+t\omega)A(x,t)\,dx\,dt\Big| \le \frac{C}{\lambda}\Big[\Big(\|u^-\|^2_{L^2(\Sigma)} + \|u^-(\,\cdot\,,T)\|^2_{L^2(\Omega)} + \|\partial_t u^-(\,\cdot\,,T)\|^2_{L^2(\Omega)}\Big)^{\frac12} \\ \Big(\|(\mathscr{R}^1_{a_2,b_2} - \mathscr{R}^1_{a_1,b_1})(f_\lambda)\|^2_{L^2(\Sigma)} + \|(\mathscr{R}^2_{a_2,b_2} - \mathscr{R}^2_{a_1,b_1})(f_\lambda)\|^2_{H^1(\Omega)} \\ + \|(\mathscr{R}^3_{a_2,b_2} - \mathscr{R}^3_{a_1,b_1})(f_\lambda)\|^2_{L^2(\Omega)}\Big)^{\frac12} + \|\varphi\|^2_{H^3(\mathbb{R}^n)}\Big]. \end{aligned}$$

Then, by setting $\phi_\lambda = \Big(u^-_{|\Sigma},\, u^-(\,\cdot\,,T),\, \partial_t u^-(\,\cdot\,,T)\Big)$, one can see that

$$\Big|\int_0^T\!\!\int_\Omega a(x,t)\varphi^2(x+t\omega)A(x,t)\,dx\,dt\Big| \leq \frac{C}{\lambda}\Big(\|\mathscr{R}_{a_2,b_2}-\mathscr{R}_{a_1,b_1}\|\|f_\lambda\|_{H^1(\Sigma)}\|\phi_\lambda\|_{\mathcal{K}}+\|\varphi\|^2_{H^3(\mathbb{R}^n)}\Big).$$

Therefore, by the trace theorem we get

$$\Big|\int_0^T\!\!\int_\Omega a(x,t)\varphi^2(x+t\omega)A(x,t)\,dx\,dt\Big| \leq C\Big(\lambda^2\|\mathscr{R}_{a_2,b_2}-\mathscr{R}_{a_1,b_1}\|+\frac{1}{\lambda}\Big)\|\varphi\|^2_{H^3(\mathbb{R}^n)}.$$

Finally, we use the fact that $a=a_2-a_1=0$ outside $Q_{r,\sharp}$ and we get by arguing as in the proof of Lemma 5.2.1 the following estimate

$$\Big|\int_{\mathbb{R}^n}\varphi^2(y)\Big[\exp\Big(-\frac{1}{2}\int_0^T a(y-s\omega,s)\,ds\Big)-1\Big]\,dy\Big| \leq C\Big(\lambda^2\|\mathscr{R}_{a_2,b_2}-\mathscr{R}_{a_1,b_1}\|+\frac{1}{\lambda}\Big)\|\varphi\|^2_{H^3(\mathbb{R}^n)}.$$

Next, by considering the sequence φ_h defined by (5.9) with $y\notin\Omega$, taking to account that $a=a_2-a_1=0$ outside $Q_{r,\sharp}$ and arguing as in the proof of Theorem 5.2.1, we complete the proof of Theorem 5.3.1. □

5.3.2 Determination of the electric potential

We aim to show by the use of Theorem 5.3.1, that the potential b can be stably recovered in the region $Q_{r,\sharp}$, with respect to the operator $\mathscr{R}_{a,b}$. In the rest of this section, we define b in $\mathbb{R}^{n+1}$ by $b=b_2-b_1$ in $\overline{Q}_r$ and $b=0$ on $\mathbb{R}^{n+1}\setminus\overline{Q}_r$. The main result of this section can be stated as follows

Theorem 5.3.2. *Let $T>2\,Diam(\Omega)$. There exist $C>0$ and $m,\,\mu\in(0,1)$ such that if $\|\Lambda_{a_1,b_1}-\Lambda_{a_2,b_2}\|\leq m$, we have*

$$\|b_2-b_1\|_{H^{-1}(Q_{r,\sharp})}\leq C\Big(\log|\log\|\mathscr{R}_{a_2,b_2}-\mathscr{R}_{a_1,b_1}\||^{\mu}\Big)^{-1},$$

for any $(a_i,b_i)\in\mathcal{A}(M_1,M_2)$ such that $\|a_i\|_{H^p(Q)}\leq M_1$, for some $p>n/2+3/2$, $(a_1,b_1)=(a_2,b_2)$ in $\overline{Q}_r\setminus Q_{r,\sharp}$ and $(\partial_x a_1,\partial_x b_1)=(\partial_x a_2,\partial_x b_2)$ on $\partial Q_r\cap\partial Q_{r,\sharp}$. Here

C depends only on Ω, M_1, M_2, T and n.

Proof. We start with the identity (5.25), except this time we isolate the potential b, we get

$$\int_0^T\int_\Omega b(x,t)u^+u^-dxdt = \int_0^T\int_\Gamma(\mathscr{R}^1_{a_2,b_2}-\mathscr{R}^1_{a_1,b_1})(f_\lambda)u^-(x,t)d\sigma dt - \int_\Omega(\mathscr{R}^3_{a_2,b_2}-\mathscr{R}^3_{a_1,b_1})(f_\lambda)u^-(x,T)\,dx$$
$$-\int_\Omega(\mathscr{R}^2_{a_2,b_2}-\mathscr{R}^2_{a_1,b_1})(f_\lambda)\Big[a_1(x,T)u^-(x,T)-\partial_t u^-(x,T)\Big]\,dx$$
$$-\int_0^T\int_\Omega a(x,t)\partial_t u^+(x,t)u^-(x,t)\,dx\,dt.$$

So, by replacing u^+ and u^- by their expressions, taking to account (5.16), (5.18) and the fact that $a = a_2 - a_1 = 0$ outside $Q_{r,\sharp}$, and making the change of variables $y = x + t\omega$, we obtain

$$\Big|\int_0^T\int_{\mathbb{R}^n} b(y-t\omega,t)\varphi^2(y)\,dy\,dt\Big| \leq C\Big(\lambda^3\|\mathscr{R}_{a_2,b_2}-\mathscr{R}_{a_1,b_1}\|+\lambda\|a\|_{L^\infty(Q_{r,\sharp})}+\frac{1}{\lambda}\Big)\|\varphi\|^2_{H^3(\mathbb{R}^n)}.$$

Then, in order to complete the proof of Theorem 5.3.2, it will be enough to consider the sequence (φ_h) defined by (5.9), with, $y \notin \Omega$, use the fact $b = b_2 - b_1 = 0$ outside $Q_{r,\sharp}$ and proceed as in the proof of Theorem 5.2.2.

□

5.4 Determination of the coefficients from boundary and final data by varying the initial conditions

In the first and the second case, we can see that there is no hope to recover the unknown coefficients a and b over the whole domain, since the initial data (u_0, u_1) are zero. However, we shall prove that this is no longer the case by considering all possible initial data. For $(a_i, b_i) \in \mathcal{C}^2(\overline{Q}) \times \mathcal{C}^1(\overline{Q})$, $i = 1, 2$, we define $(a,b) = (a_2 - a_1, b_2 - b_1)$ in Q and $(a,b) = (0,0)$ on $\mathbb{R}^{n+1} \setminus Q$.

By proceeding as in the derivation of Theorem 5.2.1 and Theorem 5.3.1, we prove a $\log$-type stability estimate in the determination of the absorbing coefficient a over the whole domain Q, from the knowledge of the measurement $\mathcal{I}_{a,b}$ defined as follows

Definition 5.4.1. *We define the boudary operator $\mathcal{I}_{a,b}$ as follows*

$$\begin{array}{rcccl} \mathcal{I}_{a,b}: & \mathcal{F} & \longrightarrow & \mathcal{K} \\ & (f,u_0,u_1) & \longmapsto & (\partial_\nu u, u(\,\cdot\,,T), \partial_t u(\,\cdot\,,T)), \end{array}$$

where $\mathcal{F} = H^1(\Sigma)\times H^1(\Omega)\times L^2(\Omega)$ and $\mathcal{K} = L^2(\Sigma)\times H^1(\Omega)\times L^2(\Omega)$. From Theorem 1.2.1, we deduce that $\mathcal{I}_{a,b}$ is continuous from $\mathcal{F}$ into $\mathcal{K}$, we denote by $\|\mathcal{I}_{a,b}\|$ its norm in $\mathcal{L}(\mathcal{F},\mathcal{K})$.

The main statements of this section are:

Theorem 5.4.1. *There exist $C>0$ and m, $\mu_1\in(0,1)$ such that if $\|\Lambda_{a_1,b_1}-\Lambda_{a_2,b_2}\|\leq m$, the following estimate holds*

$$\|a_2-a_1\|_{L^\infty(Q)}\leq C|\log\|\mathcal{I}_{a_1,b_1}-\mathcal{I}_{a_2,b_2}\||^{-\mu_1},$$

for any $(a_i,b_i)\in\mathcal{C}^2(\overline{Q})\times\mathcal{C}^1(\overline{Q})$, such that $\|a_i\|_{\mathcal{C}^2(Q)}+\|a_i\|_{H^p(Q)}\leq M_1$ for some $p>n/2+3/2$, $\|b_i\|_{\mathcal{C}^1(Q)}\leq M_2$ and $(\partial_x a_1,\partial_x b_1)=(\partial_x a_2,\partial_x b_2)$ on Σ. Here C depends only on Ω,M_1,M_2,T and n.

To prove such estimate, it will be enough to proceed as in the proof of Theorem 5.2.1 and 5.3.1, except this time, we have more flexibility on the support of the function φ_h defined by (5.9). Namely, we don't need to impose any condition on its support anymore (we fix $y\in\mathbb{R}^n$).

The same thing for the determination of the time-dependent potential b. we argue as in the proof of Theorem 5.2.2 and 5.3.2, to prove a $\log$-$\log$-type stability estimate in recovering the time dependent coefficient b with respect to the operator $\mathcal{I}_{a,b}$, over the whole domain Q.

Theorem 5.4.2. *There exist $C > 0$ and m, $\mu \in (0,1)$ such that if $\|\Lambda_{a_1,b_1} - \Lambda_{a_2,b_2}\| \leq m$, the following estimate holds*

$$\|b_2 - b_1\|_{H^{-1}(Q)} \leq C\Big(\log|\log\|\mathcal{I}_{a_2,b_2} - \mathcal{I}_{a_1,b_1}\||^{\mu}\Big)^{-1},$$

for any $(a_i, b_i) \in \mathcal{C}^2(\overline{Q}) \times \mathcal{C}^1(\overline{Q})$, such that $\|a_i\|_{\mathcal{C}^2(Q)} + \|a_i\|_{H^p(Q)} \leq M_1$ for some $p > n/2 + 3/2$, $\|b_i\|_{\mathcal{C}^1(Q)} \leq M_2$ and $(\partial_x a_1, \partial_x b_1) = (\partial_x a_2, \partial_x b_2)$ on Σ. Here C depends only on Ω,M_1,M_2,T and n.

Part II

Inverse problem for magnetic Schrödinger equations

CHAPTER 6

Determination of coefficients in magnetic Schrödinger equations

6.1 Introduction and main results

My PhD work was not only focused on hyperbolic equations, as I also adressed the same type of analysis for magnetic Schrödinger systems. This section is devoted to introduce the inverse problems we deal with in the second part of the thesis and give the main obtained results.

The first work in this part turns around studying an inverse problem concerning the following magnetic Schrödinger equation posed on a bounded cylindrical domain $Q = \Omega \times (0,T)$ of lateral boundary $\Sigma = \partial\Omega \times (0,T)$

$$\begin{cases} (i\partial_t + \Delta_A + q(x,t))u = 0 & \text{in } Q, \\ u(.,0) = u_0 & \text{in } \Omega, \\ u = f & \text{on } \Sigma, \end{cases}$$

where Ω is a subdomain of $\mathbb{R}^n$, $n \geq 3$ which is assumed to be bounded and simply connected, u_0 is the initial condition and f is the Dirichlet data used to probe the system. Here Δ_A is the Laplace operator associated to the real valued magnetic potential A which is defined as follows

$$\Delta_A = \sum_{j=1}^{n} (\partial_j + i a_j)^2 = \Delta + 2iA \cdot \nabla + i\,\text{div}(A) - |A|^2. \tag{6.1}$$

The inverse problem is to determine the real valued time-dependent electric potential q and the magnetic field $d\alpha_A$ induced by the magnetic potential A and given by the following formula

$$d\alpha_A = \sum_{i,j=1}^{n} \Big(\frac{\partial a_i}{\partial x_j} - \frac{\partial a_j}{\partial x_i} \Big) dx_j \wedge dx_i. \tag{6.2}$$

More precisely, our objective is to treat the stability issue in determining the time-dependent coefficient q and the magnetic field $d\alpha_A$ from the knowledge of the Dirichlet-to-Neumann map which is defined as follows

$$\begin{array}{rcl} \Lambda_{A,q} : H^2(\Omega) \times H^{2,1}(\Sigma) & \longrightarrow & H^1(\Omega) \times L^2(\Sigma) \\ (u_0, f) & \longmapsto & \Big(u(., T), (\partial_\nu + iA \cdot \nu) u \Big), \end{array}$$

were $\nu(x)$ denotes the unit outward normal to Γ at x, and $\partial_\nu u$ stands for $\nabla u \cdot \nu$. Here $H^{2,1}(\Sigma)$ is a Sobolev space we shall define in the next chapter.

From a physical view point, the inverse problem consists in determining the magnetic field $d\alpha_A$ induced by the magnetic potential A, and the electric potential q of an inhomogeneous medium by probing it with disturbances generated on the boundary. Here we assume that the medium is quiet initially and f denotes the disturbance used to probe the medium. Our data are the response $(\partial_\nu + iA.\nu)u$ performed on the whole boundary Σ, and the measurement $u(., T)$, for different choices of f and for all possible initial data u_0.

So, by means of techniques used in [**5**]; [**15**] we prove a "log-type" stability estimate in the recovery of the magnetic field and a "log-log-log-type" stability inequality in the determination of the time-dependent electric potential which belong to the following sets: for $\varepsilon > 0$, $M > 0$, we set

$$\mathcal{A}_\varepsilon = \{A \in C^3(\Omega), \ \ \|A\|_{W^{3,\infty}(\Omega)} \le \varepsilon, \quad A_1 = A_2 \text{ in}\,\Gamma\}, \tag{6.3}$$

$$\mathcal{Q}_M = \{q \in \mathcal{X} = W^{2,\infty}(0,T;W^{1,\infty}(\Omega)), \ \ \|q\|_{\mathcal{X}} \le M, \quad q_1 = q_2 \ \text{ in}\,\Gamma\}. \tag{6.4}$$

The main results can be stated as follows:

Result 1: *Let $\alpha > \frac{n}{2}+1$. Let $q_i \in \mathcal{Q}_M$, $A_i \in \mathcal{A}_\varepsilon$, such that $\|A_i\|_{H^\alpha(\Omega)} \le M$, for $i = 1, 2$. Then, there exist three constants $C > 0$ and $\mu, s \in (0,1)$, such that we have*

$$\|d\alpha_{A_1} - d\alpha_{A_2}\|_{L^\infty(\Omega)} \le C\left(\|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\|^{1/2} + |\log \|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\||^{-\mu}\right)^s.$$

Here C depends only on Ω, ε, M and T.

As a consequence, we can retrieve a stability estimate for the electric potential by assuming that the magnetic potential A is divergence free:

Result 2: *Let $q_i \in \mathcal{Q}_M$, $A_i \in \mathcal{A}_\varepsilon$, for $i = 1, 2$. Assume that div $A_i = 0$. Then there exist three constants $C > 0$, and $m, \mu \in (0,1)$, such that we have*

$$\|q_1 - q_2\|_{H^{-1}(Q)} \le C\Phi_m(\eta),$$

where

$$\Phi_m(\eta) = \begin{cases} |\log\,|\log|\log \eta|^\mu|\,|^{-1} & \text{if } \ \eta < m, \\ \\ \dfrac{1}{m}\eta & \text{if } \ \|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\| \ge m. \end{cases}$$

Here $\eta = \|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\|$,and C depends on Ω, M, ε and T.

The proofs of the mentionned results are the purpose of Chapter 7. The derivation of such results is essentially based on building a sufficiently large set of geometric optics solutions for the system under investigation.

There is another approach for solving inverse coefficients problems, based on the celebrated Bukhgeim. Klibanov method, which is by means of a Carleman estimate. In this case, only a finite number of boundary measurements are requiered:

In Chapter 8, we study the inverse problem of determining simultaneously two coefficients appearing in a magnetic Schrödinger equation posed in a bounded domain $Q = \Omega \times (0,T)$ with smooth boundary $\Sigma = \Gamma \times (0,T)$, where Ω is a subdomain of $\mathbb{R}^n$ with $n \geq 1$. More precisely, our goal here is to stably determine the coefficients a and q arising in this equation

$$\begin{cases} \Big(-i\partial_t + (i\nabla + \chi(t)a(x))^2 + \beta(t)q(x)\Big)u(x,t) = 0 & \text{in } Q, \\ u(x,\frac{T}{2}) = u_0(x) & \text{in } \Omega, \\ u(x,t) = 0, & \text{on } \Sigma, \end{cases}$$

from the knowledge of a finite number of lateral observations taken on an open subset of the boundary $\Gamma^+ \times (0,T) \subset \Sigma$ that satisfies an appropriate geometrical condition we shall precise later. Here the functions $\beta, \chi \in \mathcal{C}^3(0,T;\mathbb{R})$ are assumed to be known functions satisfying

$$\chi(\frac{T}{2}) = \beta(\frac{T}{2}) = 0,\ \chi'(\frac{T}{2}) \neq 0,\ \beta'(\frac{T}{2}) \neq 0.$$

By means of a Carleman estimate, it was possible to establish a stability estimate of Lipschitz type for the determination of $n+1$ unknown functions by exactly $n+1$ observations. Let us denote by $\mathcal{V}$, an arbitrary neighborhood of the boundary Γ, and by $A := H^1(\Omega)^n \cap \{a \in L^\infty(\Omega,\mathbb{R}^n), \quad \nabla\cdot a = 0\}$. For $M > 0$, and $(a_0,q_0) \in A \times L^\infty(\Omega)$, we define the admissible set of the unknown coefficients a and q:

$$\mathcal{S}_M(a_0,q_0) := \{(a,q) \in A \times L^\infty(\Omega), \text{such that } a = a_0, \ \text{ and } q = q_0 \text{ in } \mathcal{V}\}. \quad (6.5)$$

The main result of Chapter 8 can be stated as follows:

Result 3: *Let $M>0$ and (a_j,q_j), $j=1,2$ be in $\mathcal{S}_M(a_0,q_0)$, where (a_0,q_0) are the same as above. Then, there exists $n+1$ initial conditions $u_{0,k}$, $k=0,...,n$, such that we have*

$$\|a_1-a_2\|_{L^2(\Omega)}+\|q_1-q_2\|_{L^2(\Omega)}\leq C\Big(\sum_{k=0}^{n}\|\partial_\nu\partial_t^2 u_{1,k}-\partial_\nu\partial_t^2 u_{2,k}\|^2_{L^2(0,T;L^2(\Gamma^+))}\Big).$$

Here $C>0$ is a constant depending only on Ω, T, χ and β and $u_{j,k}$, $j=1,\,2$, is the solution of the magnetic Schrödinger equation where $u_{0,k}$ is substituted for u_0.

In chapter 8, we will developp the proof of this result.

6.2 The basic tools

This section is devoted to give the main tools needed to prove the above mentionned results.

6.2.1 Geometric optics solutions

The first and the second result mentionned above ensue mainly from the construction of special geometric optics solutions for the system under investigation. For this purpose, we consider a vector $\omega=\omega_{\Re}+i\,\omega_{\Im}$, such that $\omega_{\Re},\,\omega_{\Im}\in\mathbb{S}^{n-1}$ and $\omega_{\Re}\,.\,\omega_{\Im}=0$. For $\sigma>1$, we define the complex variable ρ as follows

$$\rho=\sigma\omega+y,$$

where $y\in B(0,1)$ is fixed and independent of σ. We shall see that the differential operator $N_\omega=\omega\cdot\nabla$ is invertible and we have

$$N_\omega^{-1}(g)(x)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{-ix\,\cdot\,\xi}\left(\frac{\hat{g}(\xi)}{\omega\cdot\xi}\right)d\xi=\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{1}{y_1+iy_2}g(x-y_1\omega_{\Re}-y_2\omega_{\Im})\,dy_1\,dy_2.$$

Notice that the differential operator $\overline{\partial}$ corresponds to N_ω with $\omega = (0, 1)$.

Inspired by techniques used in elliptic problems, and by assuming that the magnetic potential A is of small norm, it was possible to construct solutions to the magnetic Schrödinger equation

$$(i\partial_t + \Delta_A + q(x,t))u(x,t) = 0, \quad \text{in } Q,$$

for σ sufficiently large, of the form

$$u(x,t) = e^{-i\Big((\rho\cdot\rho)t + x\cdot\rho\Big)}\Big(e^{i\phi(x)} + w(x,t)\Big),$$

in such away that

$$\omega\cdot\nabla\phi(x) = -\omega\cdot A(x), \quad x\in\mathbb{R}^n.$$

Here $\phi = N_\omega^{-1}(-\omega.A)$, and the finction w denotes the error term that satisfies the following estimation

$$\sigma\|w\|_{H^2(0,T;H^1(\Omega))} + \|w\|_{L^2(0,T;H^2(\Omega))} \leq C.$$

We give more details about the construction of geometric optics solutions in Chapter 8.

6.2.2 A Carleman estimate

In this section, we recall the global Carleman inequality for vanishing solution on the boundary Σ, which can be found in [**3**][Section3](see also [**18**]). This estimate is the main tool needed for the derivation of Result 3.

Given the Schrödinger operator

$$L := i\partial_t + \Delta, \tag{6.6}$$

we define a function $\psi \in \mathcal{C}^4(\Omega, \mathbb{R}_+)$, satisfying the following conditions:

(i) $|\nabla\psi(x)| \geq \beta > 0, \ \forall x \in \Omega$.

(ii) $\nabla\psi \cdot \nu < 0$ for all $x \in \Gamma \setminus \Gamma^+$.

(iii) $\exists \Lambda_1 > 0$, $\exists \varepsilon > 0$ such that for all $\xi \in \mathbb{C}^n$, and for all $\lambda > \Lambda_1$, we have

$$\lambda|\nabla\psi \cdot \xi|^2 + D^2\psi(\xi, \overline{\xi}) \geq \varepsilon|\xi|^2,$$

where $D^2\psi = \Big(\dfrac{\partial^2\psi}{\partial x_i \partial x_j}\Big)_{1\leq i,j\leq n}$ and $D^2\psi(\xi, \overline{\xi})$ denotes the $\mathbb{C}^n$-scalar product of $D^2\widetilde{\psi}\xi$ with ξ.

Notice that there are actual functions ψ verifying the above assumptions, such as $x \mapsto |x - x_0|^2$, for an arbitrary $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ and a subboundary $\Gamma^+ \supset \{x \in \Gamma,\ (x - x_0) \cdot \nu \geq 0\}$. Furthermore, for $\lambda > 0$ the following weight functions:

$$\theta(x,t) = \frac{e^{\lambda\psi(x)}}{t(T-t)}, \quad \text{and} \quad \eta(x,t) = \frac{\alpha - e^{\lambda\psi(x)}}{t(T-t)}, \tag{6.7}$$

where $\alpha > \|e^{\lambda\psi}\|_{L^\infty(\Omega)}$. Finaly, we introduce the two operators P_1 and P_2 acting in $\mathcal{C}_0^\infty(Q)'$, as follows:

$$P_1 := i\partial_t + \Delta + s^2|\nabla\eta|^2, \quad \text{and} \quad P_2 := is\,\partial_t\eta + 2s\nabla\eta \cdot \nabla + s(\Delta\eta), \tag{6.8}$$

in such a way that $P_1 + P_2 = e^{-s\eta}Le^{s\eta}$.

Proposition 6.2.1. *(see* [**3**]*) Assume that ψ and Γ^+ satisfy the above conditions. Let η and θ be as in (6.7), and let P_j, $j = 1, 2$ be defined by (6.8). Then, there are two constants $s_0 > 0$ and $C > 0$, depending only on T, Ω and Γ^+, such that the estimate*

$$\begin{aligned} &s\|e^{-s\eta}\nabla u\|^2_{L^2(Q)} + s^3\|e^{-s\eta}u\|^2_{L^2(Q)} + \sum_{j=1,2}\|P_je^{-s\eta}u\|^2_{L^2(Q)} \\ &\leq C\Big(s\|e^{-s\eta}\theta^{1/2}(\partial_\nu\psi)^{1/2}\partial_\nu u\|^2_{L^2(\Sigma^+)} + \|e^{-s\eta}Lu\|^2_{L^2(Q)}\Big), \end{aligned}$$

holds for all $s \geq s_0$, *and for any function* $u \in L^2(0,T;H_0^1(\Omega))$ *such that* $Lu \in L^2(Q)$ *and* $\partial_\nu u \in L^2(0,T;L^2(\Gamma^+))$. *Here* Σ_T^+ *stands for* $\Gamma^+ \times (0,T)$.

CHAPTER 7

Determination of coefficients in a magnetic Schrödinger equation from Dirichlet-to-Neumann map

The contents of this chapter are collected in a paper submitted at *JMP*.

7.1 Introduction

In this chapter, we deal with the inverse problem of determining the magnetic field and the time-dependent electric potential in the magnetic Schrödinger equation from the knowledge of boundary observations. Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded and simply connected domain with $\mathcal{C}^\infty$ boundary Γ. Given $T > 0$, we denote by $Q = \Omega \times (0,T)$ and $\Sigma = \Gamma \times (0,T)$. We consider the following initial boundary problem for the Schrödinger equation

$$\begin{cases} (i\partial_t + \Delta_A + q(x,t))u = 0 & \text{in } Q, \\ u(.,0) = u_0 & \text{in } \Omega, \\ u = f & \text{on } \Sigma, \end{cases} \tag{7.1}$$

where Δ_A is given by (6.1), the real valued bounded function $q \in W^{2,\infty}(0,T;W^{1,\infty}(\Omega))$ is the electric potential and $A \in \mathcal{C}^3(\overline{\Omega})$ is the magnetic potential. We define the Dirichlet-to-Neumann map associated to the magnetic Schrödinger equation (7.1) as follows

$$\begin{aligned} \Lambda_{A,q} : H^2(\Omega) \times H^{2,1}(\Sigma) &\longrightarrow \qquad H^1(\Omega) \times L^2(\Sigma) \\ (u_0, f) \qquad &\longmapsto \Big(u(.,T), (\partial_\nu + iA\cdot\nu)u\Big), \end{aligned}$$

where $\nu(x)$ denotes the unit outward normal to Γ at x, and $\partial_\nu u$ stands for $\nabla u \cdot \nu$. Here $H^{2,1}(\Sigma)$ is a Sobolev space we shall make precise below. In the present chapter, we address the stability issue for the inverse problem of recovering the magnetic field $d\alpha_A$ defined by (6.2) and the time-dependent potential q appearing in the dynamical Schrödinger equation (7.1), from the knowledge of the operator $\Lambda_{A,q}$.

7.2 Well posedness

The study of an inverse problem often requires a good knowledge of the direct problem. So let us first establish the existence, uniqueness and continuous dependence with respect to the data, of the solution u of the Schrödinger equation (7.1) with non-homogeneous Dirichlet-boundary condition $f \in H_0^{2,1}(\Sigma)$ and an initial data $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$. To this end, we introduce the following Sobolev space

$$H^{2,1}(\Sigma) = H^2(0,T;L^2(\Gamma)) \cap L^2(0,T;H^1(\Gamma)),$$

equipped with the norm

$$\|f\|_{H^{2,1}(\Sigma)} = \|f\|_{H^2(0,T;L^2(\Gamma))} + \|f\|_{L^2(0,T;H^1(\Gamma))},$$

and we set

$$H_0^{2,1}(\Sigma) = \{f \in H^{2,1}(\Sigma),\ f(.,0) = \partial_t f(.,0) = 0\}.$$

Then we have the following theorem.

Theorem 7.2.1. *Let $T > 0$ and let $q \in W^{1,\infty}(Q)$, $A \in \mathcal{C}^1(\Omega)$ and $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$. Suppose that $f \in H_0^{2,1}(\Sigma)$. Then, there exists a unique solution $u \in \mathcal{C}(0,T;H^1(\Omega))$ of the Shrödinger equation (7.1). Furthermore, we have $\partial_\nu u \in L^2(\Sigma)$ and there exists a constant $C > 0$ such that*

$$\|u(.,t)\|_{H^1(\Omega)} + \|\partial_\nu u\|_{L^2(\Sigma)} \leq C\left(\|u_0\|_{H^2(\Omega)} + \|f\|_{H^{2,1}(\Sigma)}\right).$$

One can consequently note that the Dirichlet-to-Neumann map $\Lambda_{A,q}$ is bounded from $H^2(\Omega) \times H^{2,1}(\Sigma)$ to $H^1(\Omega) \times L^2(\Sigma)$.

Proof. We decompose the solution u of the Schrödinger equation (7.1) as $u = u_1 + u_2$, with u_1 and u_2 are respectively solutions to

$$\begin{cases} (i\partial_t + \Delta_A)u_1 = 0 & \text{in } Q \\ u_1(x,0) = 0 & \text{in } \Omega \\ u_1(x,t) = f & \text{on } \Sigma \end{cases}, \qquad \begin{cases} (i\partial_t + \Delta_A + q)u_2 = -qu_1 & \text{in } Q \\ u_2(x,0) = u_0 & \text{in } \Omega \\ u_2 = 0 & \text{on } \Sigma. \end{cases}$$

Using the fact that $f \in H_0^{2,1}(\Sigma)$, we can see from **[5]**[Theorem 1.1] that

$$u_1 \in \mathcal{C}^1(0,T;H^1(\Omega)), \tag{7.2}$$

and

$$\|u_1\|_{\mathcal{C}^1(0,T;H^1(\Omega))} \leq C\|f\|_{H^{2,1}(\Sigma)}. \tag{7.3}$$

Moreover, we have $\partial_\nu u_1 \in L^2(\Sigma)$, and we get a constant $C > 0$ such that

$$\|\partial_\nu u_1\|_{L^2(\Sigma)} \leq C\|f\|_{H^{2,1}(\Sigma)}. \tag{7.4}$$

On the other hand, from **[16]**[Lemma 2.1] , we conclude the existence of a unique solution

$$u_2 \in \mathcal{C}^1(0,T;L^2(\Omega)) \cap \mathcal{C}(0,T;H^2(\Omega) \cap H_0^1(\Omega)), \tag{7.5}$$

that satisfies

$$\begin{aligned} \|u_2(.,t)\|_{H_0^1(\Omega)} &\leq C\left(\|qu_1\|_{W^{1,1}(0,T;L^2(\Omega)}+\|u_0\|_{H_0^1\cap H^2}\right). \\ &\leq C\left(\|u_0\|_{H_0^1\cap H^2}+\|f\|_{H^{2,1}(\Sigma)}\right). \end{aligned} \tag{7.6}$$

Next, we consider a $\mathcal{C}^2$ vector field N satisfying

$$N(x) = \nu(x), \quad x \in \Gamma, \quad |N(x)| \leq 1, \;\; x \in \Omega.$$

Multiplying the second Schrödinger equation by $N.\nabla\overline{u}_2$ and integrating over $Q = \Omega \times (0,T)$ we get

$$\begin{aligned} -\int_0^T\int_\Omega q\,u_1\,N.\nabla\overline{u}_2\,dx\,dt &= i\int_0^T\int_\Omega \partial_t u_2\,N.\nabla\overline{u}_2\,dx\,dt + \int_0^T\int_\Omega \Delta u_2\,N.\nabla\overline{u}_2\,dx\,dt \\ &\quad + \int_0^T\int_\Omega (2iA.\nabla + i\mathrm{div}\,A - |A|^2 + q)u_2\,N.\nabla\overline{u}_2\,dx\,dt = I_1 + I_2 + I_3. \end{aligned}$$

By integrating with respect to t in the first term I_1, we get

$$\begin{aligned} I_1 &= i\int_\Omega\Big[u_2(x,T)\,N.\nabla\overline{u}_2(x,T) - u_2(x,0)\,N.\nabla\overline{u}_2(x,0)\Big]\,dx \\ &\quad -i\int_0^T\int_\Omega N.\nabla(u_2\,\partial_t\overline{u}_2)\,dx\,dt + i\int_0^T\int_\Omega \partial_t\overline{u}_2\,N.\nabla u_2\,dx\,dt. \end{aligned}$$

Therefore, bearing in mind that $i\partial_t\overline{u}_2 = -q\,\overline{u}_1 - \overline{\Delta_A u_2} - q\overline{u}_2$, we get

$$\begin{aligned}
2\Re\, I_1 \;&=\; i\int_\Omega \Big[u_2(x,T)\, N.\nabla\overline{u}_2(x,T) - u_0\, N.\nabla\overline{u}_0\Big]\, dx - \int_0^T\int_\Omega \mathrm{div}\, N\, q\, u_2\overline{u}_1\, dx\, dt \\
&\quad - \int_0^T\int_\Omega \mathrm{div}\, N\, q\, |u_2|^2\, dx\, dt + \int_0^T\int_\Omega \nabla_A(\mathrm{div}\, N\, u_2).\nabla_A\overline{u}_2\, dx\, dt \\
&\quad - i\int_0^T\int_\Gamma u_2\partial_t\overline{u}_2\, d\sigma\, dt - \int_0^T\int_\Sigma \partial_\nu\overline{u}_2(u_2\,\mathrm{div}\, N)\, d\sigma\, dt.
\end{aligned}$$

As the last term vanishes since $u_2 = 0$ on Σ, we deduce from (7.6) that

$$|\Re\, I_1| \le C\left(\|f\|^2_{H^{2,1}(\Sigma)} + \|u_0\|^2_{H^1_0\cap H^2}\right).$$

On the other hand, by Green's Formula, we have

$$\begin{aligned}
I_2 \;&=\; -\int_0^T\int_\Omega \nabla u_2\nabla(N.\nabla\overline{u}_2)\, dx\, dt + \int_0^T\int_\Gamma \partial_\nu u_2(N.\nabla\overline{u}_2)\, d\sigma\, dt \\
&=\; -\int_0^T\int_\Omega \nabla u_2.\nabla(N.\nabla\overline{u}_2)\, dx\, dt + \int_0^T\int_\Gamma |\partial_\nu u_2|^2\, d\sigma\, dt.
\end{aligned}$$

So, we get

$$\begin{aligned}
I_2 \;&=\; \int_0^T\int_\Gamma |\partial_\nu u_2|^2\, d\sigma\, dt - \frac{1}{2}\int_0^T\int_\Omega \mathrm{div}(|\nabla u_2|^2 N)\, dx\, dt \\
&\quad + \frac{1}{2}\int_0^T\int_\Omega |\nabla u_2|^2\,\mathrm{div}\, N\, dx\, dt - \int_0^T\int_\Omega DN(\nabla u_2, \nabla\overline{u_2})\, dx\, dt.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
I_2 \;&=\; \int_0^T\int_\Gamma |\partial_\nu u_2|^2 - \frac{1}{2}\int_0^T\int_\Gamma |\nabla u_2|^2\, N.\nu\, d\sigma\, dt \\
&\quad + \frac{1}{2}\int_0^T\int_\Omega |\nabla u_2|^2\,\mathrm{div}\, N\, dx\, dt - \int_0^T\int_\Omega DN(\nabla u_2, \nabla u_2)\, dx\, dt.
\end{aligned}$$

Next, using the fact that

$$|\nabla u_2|^2 = |\partial_\nu u_2|^2 + |\nabla_\tau u_2|^2 = |\partial_\nu u_2|^2, \qquad x\in\Gamma,$$

where ∇_τ is the tangential gradient on Γ, we obtain

$$\begin{aligned}\Re I_2 &= \frac{1}{2}\int_0^T\int_\Omega |\partial_\nu u_2|^2 d\sigma\, dt + \frac{1}{2}\int_0^T |\nabla u_2|^2 \,\mathrm{div}\, N\, dx\, dt \\ &\quad - \int_0^T\int_\Omega DN(\nabla u_2, \nabla \overline{u}_2)\, dx\, dt.\end{aligned}$$

Moreover, by (7.6), it is easy to see that

$$|\Re I_3| \leq C\left(\|f\|^2_{H^{2,1}(\Sigma)} + \|u_0\|^2_{H_0^1\cap H^2}\right),$$

so that, we deduce from the above statements that

$$\|\partial_\nu u_2\|_{L^2(\Sigma)} \leq C\left(\|f\|_{H^{2,1}(\Sigma)} + \|u_0\|_{H_0^1\cap H^2}\right).$$

From the above reasoning, we conclude that $u = u_1 + u_2 \in \mathcal{C}(0,T; H^1(\Omega))$, $\partial_\nu u \in L^2(\Sigma)$ and we have

$$\|u(.,t)\|_{H^1(\Omega)} + \|\partial_\nu u\|_{L^2(\Sigma)} \leq C\left(\|f\|_{H^{2,1}(\Sigma)} + \|u_0\|_{H_0^1\cap H^2}\right).$$

$\square$

7.3 Geometric optics solutions

The present section is devoted to the construction of suitable geometrical optics solutions to the magnetic Schrödinger equation (7.1), inspired by techniques used for elliptic problems.

7.3.1 Preliminaries

In this section, we collect several technical results that are needed to construct geometric optics solutions to the equation (7.1). We first introduce the following notations. Let $P(D)$ be a differential operator with $D = -i(\partial_t, \partial_x)$. We denote by

$$\widetilde{P}(\xi,\tau) = \Big(\sum_{k\in\mathbb{N}}\sum_{\alpha\in\mathbb{N}^n} |\partial_\tau^k \partial_\xi^\alpha P(\xi,\tau)|^2\Big)^{\frac{1}{2}}, \qquad \xi \in \mathbb{R}^n,\ \tau \in \mathbb{R}.$$

For $1 \le p \le \infty$, we define the space

$$B_{p,\widetilde{P}} = \{f \in S'(\mathbb{R}^{n+1}),\ \ \widetilde{P}\mathcal{F}(f) \in L^p(\mathbb{R}^{n+1})\},$$

equipped with the following norm

$$\|f\|_{B_{p,\widetilde{P}}} = \|\widetilde{P}\mathcal{F}(f)\|_{L^p(\mathbb{R}^{n+1})}.$$

We finally denote by

$$B^{loc}_{p,\widetilde{P}} = \{f \in S'(\mathbb{R}^{n+1}),\ \varphi f \in B_{p,\widetilde{P}},\ \forall\, \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^{n+1})\}.$$

We start by recalling some known results of Hörmander:

Lemma 7.3.1. *Let $u \in B_{\infty,\widetilde{P}}$ and $v \in \mathcal{C}_0^\infty(\mathbb{R}^{n+1})$. Then, we have $uv \in B_{\infty,\widetilde{P}}$, and*

$$\|uv\|_{\infty,\widetilde{P}} \le C\|u\|_{\infty,\widetilde{P}},$$

where the positive constant C depends only on v, n and the degree of P.

Lemma 7.3.2. *Any differential operator $P(D)$ admits a fundamental solution $F \in B^{loc}_{\infty,\widetilde{P}}$ satisfying $\frac{F}{\cosh|(x,t)|} \in B_{\infty,\widetilde{P}}$. Moreover, it verifies*

$$\|\frac{F}{\cosh|(x,t)|}\|_{\infty,\widetilde{P}} \le C,$$

where C is a positive constant that depends only on n and the degree of P.

Our first goal in this section is to prove the following theorem:

Theorem 7.3.1. *Let $P \neq 0$ be a differential operator. Then for all $k \in \mathbb{N}$, there exists a linear operator*

$$E : L^2(0,T; H^k(\Omega)) \to L^2(0,T; H^k(\Omega)),$$

such that:

1. *$P(D)Ef = f$, for any $f \in L^2(0,T; H^k(\Omega))$.*

2. *For any linear differential operator with constant coefficient $Q(D)$ such that $\dfrac{|Q(\xi,\tau)|}{\widetilde{P}(\xi,\tau)}$ is bounded, we have $Q(D)E \in B(L^2(0,T; H^k(\Omega)))$ and*

$$\|Q(D)Ef\|_{L^2(0,T;H^k(\Omega))} \leq C \sup_{\mathbb{R}^{n+1}} \frac{|Q(\xi,\tau)|}{\widetilde{P}(\xi,\tau)} \|f\|_{L^2(0,T;H^k(\Omega))},$$

where C depends only on the degree of P, Ω and T.

Proof. Let $f \in L^2(0,T; H^k(\Omega))$. There exists an extension operator

$$\begin{array}{rcl} S : L^2(0,T; H^k(\Omega)) & \longrightarrow & L^2(0,T; H^k(\mathbb{R}^n)) \\ f & \longmapsto & \widetilde{f}, \end{array}$$

such that for all $t \in (0,T)$, we have $\widetilde{f}(.,t)_{|\Omega} = f(.,t)$. Next, we introduce

$$\widetilde{f}_0 = \begin{cases} \widetilde{f}, & t \in (0,T), \quad x \in \mathbb{R}^n \\ \\ 0, & t \notin (0,T), \quad x \in \mathbb{R}^n. \end{cases}$$

So, we have $\widetilde{f}_{0|Q} = f$. Let $R > 0$ and V be a neighborhood of $\overline{Q}$. We consider $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^{n+1})$ such that $\psi_{|V} = 1$ and satisfying supp $\psi \subset B(0,R) \subset \mathbb{R}^{n+1}$. Let F be a fundamental solution of P. We consider the following operator

$$\begin{array}{rcl} E : L^2(0,T; H^k(\Omega)) & \longrightarrow & L^2(0,T; H^k(\Omega)) \\ f & \longmapsto & E(f) = (F * \psi \widetilde{f}_0)_{|Q} \end{array}$$

Since $P(D)(F * \psi\widetilde{f}_0) = \psi\widetilde{f}_0$, then we clearly have

$$P(D)Ef = (\psi\widetilde{f}_0)_{|Q} = f.$$

We turn now to proving the second point. For this purpose, we consider $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^{n+1})$ such that $\varphi = 1$ on a neighborhood of the closure of $\{x - y,\ \ x,\, y \in Q\}$. We can easily verify that

$$(F * \psi\widetilde{f}_0)_{|Q} = (\varphi F * \psi\widetilde{f}_0)_{|Q}.$$

The last identity entails that for all $\alpha \in \mathbb{N}^n$, such that $|\alpha| \leq k$, we have

$$\begin{aligned}
\|\partial^\alpha Q(D)Ef\|_{L^2(Q)} &= \|Q(D)\partial^\alpha(F * \psi\widetilde{f}_0)\|_{L^2(Q)} \\
&= \|Q(D)\varphi F * \partial^\alpha(\psi\widetilde{f}_0)\|_{L^2(Q)} \\
&\leq \|Q(D)\varphi F * \partial^\alpha(\psi\widetilde{f}_0)\|_{L^2(\mathbb{R}^{n+1})} \\
&\leq \|\mathcal{F}\Big(Q(D)\varphi F * \partial^\alpha(\psi\widetilde{f}_0)\Big)\|_{L^2(\mathbb{R}^{n+1})} \\
&\leq \|Q(\xi,\tau)\mathcal{F}(\varphi F)\mathcal{F}(\partial^\alpha(\psi\tilde{f}_0))\|_{L^2(\mathbb{R}^{n+1})} \\
&\leq \|Q(\xi,\tau)\mathcal{F}(\varphi F)\|_{L^\infty(\mathbb{R}^{n+1})}\|\partial^\alpha(\psi\tilde{f}_0)\|_{L^2(\mathbb{R}^{n+1})} \\
&\leq \|Q(\xi,\tau)\mathcal{F}(\varphi F)\|_{L^\infty(\mathbb{R}^{n+1})}\|\partial^\alpha f\|_{L^2(Q)}.
\end{aligned} \tag{7.7}$$

Using the fact that

$$Q(\xi,\tau)\mathcal{F}(\varphi F) = \frac{Q(\xi,\tau)}{\widetilde{P}(\xi,\tau)}\widetilde{P}(\xi,\tau)\mathcal{F}\Big(\varphi \cosh|(x,t)|\frac{F}{\cosh|(x,t)|}\Big),$$

we deduce from Lemma 7.3.1 and Lemma 7.3.2 that

$$\|Q(\xi,\tau)\mathcal{F}(\varphi F)\|_{L^\infty(\mathbb{R}^{n+1})} \leq C \sup_{(\xi,\tau)\in\mathbb{R}^{n+1}} \frac{|Q(\xi,\tau)|}{\widetilde{P}(\xi,\tau)}. \tag{7.8}$$

Then from (7.7) and (7.8), we get

$$\|\partial^\alpha Q(D)Ef\|_{L^2(Q)} \leq C \sup_{(\xi,\tau)\in\mathbb{R}^{n+1}} \frac{|Q(\xi,\tau)|}{\widetilde{P}(\xi,\tau)} \|f\|_{L^2(0,T;H^k(\Omega))}, \quad \forall\, \alpha \in \mathbb{N}^n,\ |\alpha| \leq k. \tag{7.9}$$

Thus, we find that

$$\|Q(D)Ef\|_{L^2(0,T;H^k(\Omega))} \leq C \sup_{\mathbb{R}^{n+1}} \frac{|Q(\xi,\tau)|}{\widetilde{P}(\xi,\tau)} \|f\|_{L^2(0,T;H^k(\Omega))},$$

which completes the proof of the lemma. $\square$

Let $\omega = \omega_{\Re} + i\,\omega_{\Im}$ be a vector such that $\omega_{\Re},\, \omega_{\Im} \in \mathbb{S}^{n-1}$ and $\omega_{\Re} \,.\, \omega_{\Im} = 0$. We state the following known lemma from [**40**]

Lemma 7.3.3. *Let $r > 0$, $k > 0$ and let $g \in W^{k,\infty}(\mathbb{R}^n)$ be such that Supp $g \subseteq B(0,r) = \{x \in \mathbb{R}^n,\ |x| \leq r\}$. Then the function $\phi = N_\omega^{-1}(g) \in W^{k,\infty}(\mathbb{R}^n)$ solves $N_\omega(\phi) = \omega \cdot \nabla\phi = g$, and satisfies the following estimate*

$$\|\phi\|_{W^{k,\infty}(\mathbb{R}^n)} \leq C\, \|g\|_{W^{k,\infty}(\mathbb{R}^n)},$$

where C is a positive constant depending only on r.

7.3.2 Construction of geometric optics solutions

For $\sigma > 1$, we define the complex variable ρ as follows

$$\rho = \sigma\omega + y, \tag{7.10}$$

where $y \in B(0,1)$ is fixed and independent of σ. In what follows, $P(D)$ denotes a differential operator with constant coefficients:

$$P(D) = \sum_{|\alpha|\leq m} a_\alpha\, D^\alpha, \qquad D = -i(\partial_t, \partial_x).$$

We associate to the operator $P(D)$ its symbol $p(\xi,\tau)$ defined by

$$p(\xi,\tau)=\sum_{|\alpha|\leq m} a_\alpha(\xi,\tau)^\alpha, \qquad (\xi,\tau)\in\mathbb{R}^{n+1}.$$

Moreover, we introduce the operators

$$\Delta_\rho=\Delta-2i\rho\cdot\nabla \quad \text{and} \quad \nabla_\rho=\nabla-i\rho.$$

We turn now to building particular solutions to the magnetic Shrödinger equation. We proceed with a succession of lemmas. As a consequence to Theorem 7.3.1, we have

Corollary 7.3.1. *Let $P\neq 0$ be an operator. There exists a linear operator $E\in\mathcal{L}(L^2(0,T;H^1(\Omega)))$, such that:*

$$P(D)Ef=f, \quad \textit{for any } f\in L^2(0,T;H^1(\Omega)).$$

Moreover, for any linear operator S with constant coefficients such that $\dfrac{|S(\xi,\tau)|}{\tilde{p}(\xi,\tau)}$ is bounded in $\mathbb{R}^{n+1}$, we have the following estimate

$$\|S(D)Ef\|_{L^2(0,T;H^1(\Omega))}\leq C\sup_{\mathbb{R}^{n+1}}\frac{|S(\xi,\tau)|}{\tilde{p}(\xi,\tau)}\|f\|_{L^2(0,T;H^1(\Omega))}. \tag{7.11}$$

Here C depends only on the degree of P, Ω and T.

Lemma 7.3.4. *There exists a bounded operator $E_\rho: L^2(0,T;H^1(\Omega))\longrightarrow L^2(0,T;H^2(\Omega))$ such that*

$$P_\rho(D)E_\rho f=(i\partial_t+\Delta_\rho)E_\rho f=f \quad \textit{for any} \quad f\in L^2(0,T;H^1(\Omega)).$$

Moreover, there exists a constant $C(\Omega,T)>0$ such that

$$\|E_\rho f\|_{L^2(0,T;H^k(\Omega))}\leq\frac{C}{\sigma^{2-k}}\|f\|_{L^2(0,T;H^1(\Omega))}, \quad k=1,\,2. \tag{7.12}$$

Proof. From Corollary 7.3.1, we deduce the existence of a linear operator $E_\rho \in \mathcal{L}\Big(L^2(0,T;H^1(\Omega))\Big)$ such that $P_\rho(D)E_\rho f = f$. Moreover, since $|\widetilde{p_\rho}(\xi,\tau)| > \sigma$, we get from (7.11)

$$\|E_\rho f\|_{L^2(0,T;H^1(\Omega))} \leq \frac{C}{\sigma}\|f\|_{L^2(0,T;H^1(\Omega))}. \tag{7.13}$$

Similarly, since $\dfrac{|\xi|}{\widetilde{p_\rho}(\xi,\tau)}$ is bounded on $\mathbb{R}^{n+1}$, we get

$$\|\nabla E_\rho f\|_{L^2(0,T;H^1(\Omega))} \leq C\|f\|_{L^2(0,T;H^1(\Omega))}.$$

From this and (7.13) we see that E_ρ is bounded from $L^2(0,T;H^1(\Omega))$ into $L^2(0,T;H^2(\Omega))$. $\square$

Let us now deduce the coming statementfrom the above lemma.

Lemma 7.3.5. *There exists $\varepsilon > 0$ such that for all $A \in W^{1,\infty}(\Omega)$ obeying $\|A\|_{W^{1,\infty}(\Omega)} \leq \varepsilon$, we may build a bounded operator $F_\rho : L^2(0,T;H^1(\Omega)) \longrightarrow L^2(0,T;H^2(\Omega))$ such that:*

$$\Big(i\partial_t + \Delta_\rho + 2iA\cdot\nabla\Big)F_\rho f = f, \quad \textit{for any} \quad f \in L^2(0,T;H^1(\Omega)). \tag{7.14}$$

Moreover, there exists a constant $C(\Omega,T) > 0$ such that

$$\|F_\rho f\|_{L^2(0,T;H^k(\Omega))} \leq \frac{C}{\sigma^{2-k}}\|f\|_{L^2(0,T;H^1(\Omega))}, \quad k = 1,\ 2. \tag{7.15}$$

Proof. Let $f \in L^2(0,T;H^1(\Omega))$. We start by introducing the following operator

$$\begin{aligned} S_\rho : L^2(0,T;H^2(\Omega)) &\longrightarrow \quad L^2(0,T;H^2(\Omega)) \\ g &\longmapsto \quad E_\rho(-2iA\cdot\nabla g + f). \end{aligned}$$

Since $\|A\|_{W^{1,\infty}(\Omega)} \leq \varepsilon$, we deduce from (7.12) with $k = 2$ that

$$\|S_\rho(h) - S_\rho(g)\|_{L^2(0,T;H^2(\Omega))} \leq C\varepsilon\|h - g\|_{L^2(0,T;H^2(\Omega))}, \tag{7.16}$$

for any $h,\, g \in L^2(0,T;H^2(\Omega))$. Thus, S_ρ is a contraction from $L^2(0,T;H^2(\Omega))$ into $L^2(0,T;H^2(\Omega))$ for ε small enough. Then, S_ρ admits a unique fixed point $g \in L^2(0,T;H^2(\Omega))$. Put $F_\rho f = g$. It is clear that $F_\rho f$ is a solution to (7.14). Then, taking into account the identity $S_\rho F_\rho f = E_\rho(-2iA\cdot\nabla F_\rho f + f)$ and the estimate (7.16), we get

$$\begin{aligned}\|F_\rho f\|_{L^2(0,T;H^2(\Omega))} &= \|S_\rho F_\rho f - S_\rho(0)\|_{L^2(0,T;H^2(\Omega))} + \|S_\rho(0)\|_{L^2(0,T;H^2(\Omega))}\\ &\leq C\varepsilon\|F_\rho f\|_{L^2(0,T;H^2(\Omega))} + \|E_\rho f\|_{L^2(0,T;H^2(\Omega))}.\end{aligned}$$

From this and (7.12) with $k=2$, we end up getting for ε small enough

$$\|F_\rho f\|_{L^2(0,T;H^2(\Omega))} \leq C\|f\|_{L^2(0,T;H^1(\Omega))}. \tag{7.17}$$

This being said, it remains to show (7.15) for $k=1$. To see this, we notice from (7.12) with $k=1$ that

$$\begin{aligned}\|F_\rho f\|_{L^2(0,T;H^1(\Omega))} &\leq \|E_\rho(-2iA\cdot\nabla F_\rho f + f)\|_{L^2(0,T;H^1(\Omega))}\\ &\leq \frac{C}{\sigma}\Big(\varepsilon\|F_\rho f\|_{L^2(0,T;H^2(\Omega))} + \|f\|_{L^2(0,T;H^1(\Omega))}\Big).\end{aligned}$$

Then the estimate (7.15) for $k=1$ follows readily from this and (7.17).

□

Lemma 7.3.6. *There exists $\varepsilon > 0$ such that for all $A \in W^{1,\infty}(\Omega)$ obeying $\|A\|_{W^{1,\infty}(\Omega)} \leq \varepsilon$, we may build a bounded operator $G_\rho : L^2(0,T;H^1(\Omega)) \longrightarrow L^2(0,T;H^2(\Omega))$ such that:*

$$\Big(i\partial_t + \Delta_\rho + 2iA\cdot\nabla_\rho\Big)G_\rho f = f \quad \textit{for any} \quad f \in L^2(0,T;H^1(\Omega)). \tag{7.18}$$

Moreover, there exists a constant $C(\Omega,T) > 0$ such that

$$\|G_\rho f\|_{L^2(0,T;H^k(\Omega))} \leq \frac{C}{\sigma^{2-k}}\|f\|_{L^2(0,T;H^1(\Omega))}, \quad k = 1,\, 2. \tag{7.19}$$

Proof. Let $f \in L^2(0,T;H^1(\Omega))$. We introduce the following operator

$$\begin{array}{rcl} R_\rho : L^2(0,T;H^1(\Omega)) & \longrightarrow & L^2(0,T;H^1(\Omega)) \\ g & \longmapsto & F_\rho(-2\rho \cdot Ag + f) \end{array}$$

From (7.10), we see that $|\rho| < 3\sigma$. Thus, arguing as in the proof of Lemma 7.3.5, we prove the existence of a unique solution $G_\rho f = g$ to the equation (7.18). Moreover there exists a positive constants $C > 0$ such that we have

$$\|u\|_{L^2(0,T,H^1(\Omega))} \leq \frac{C}{\sigma}\|f\|_{L^2(0,T;H^1(\Omega))}. \tag{7.20}$$

Further, combining the definition of R_ρ with (7.14) we deduce (7.19) for $k = 2$. $\square$

Armed with lemma 7.3.6, we are now in position to establish the main result of this section, which can be stated as follows

Lemma 7.3.7. *Let $M > 0$, $\varepsilon > 0$, $\omega \in \mathbb{S}^{n-1}$ and $A \in \mathcal{A}_\varepsilon$ satisfy $\|A\|_{W^{1,\infty}(\Omega)} \leq \varepsilon$. Put $\phi = N_\omega^{-1}(-\omega.A)$. Then, for all $\sigma \geq \sigma_0 > 0$ the magnetic Schrödinger equation*

$$(i\partial_t + \Delta_A + q(x,t))u(x,t) = 0, \quad \text{in } Q \tag{7.21}$$

admits a solution $u \in H^2(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$, of the form

$$u(x,t) = e^{-i\left((\rho\cdot\rho)t + x\cdot\rho\right)}\left(e^{i\phi(x)} + w(x,t)\right), \tag{7.22}$$

in such a way that

$$\omega \cdot \nabla\phi(x) = -\omega \cdot A(x), \quad x \in \mathbb{R}^n. \tag{7.23}$$

Moreover, $w \in H^2(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ satisfies

$$\sigma\|w\|_{H^2(0,T;H^1(\Omega))} + \|w\|_{L^2(0,T;H^2(\Omega))} \leq C, \tag{7.24}$$

where the constants C and σ_0 depend only on Ω, T and M.

Here we extended A by zero outside Ω.

Proof. To prove our lemma, it is enough to show that $w \in H^2(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ satisfies the estimate (7.24). Substituting (7.22) into the equation (7.21), one gets

$$\Big(i\partial_t + \Delta_\rho + 2iA(x)\cdot\nabla_\rho + h(x,t)\ \Big)w(x,t) = -e^{i\phi(x)}\Big(i\Delta\phi(x) - |\nabla\phi(x)|^2 + 2\sigma\omega\cdot\nabla\phi(x)$$
$$+2\sigma\omega\cdot A(x) + 2y\cdot\nabla\phi(x) + 2A(x)\cdot y - 2A(x)\cdot\nabla\phi(x) + h(x,t)\Big),$$

where $h(x,t) = i\mathrm{div}A(x) - |A(x)|^2 + q(x,t)$. Equating coefficients of power of $|\sigma|$ to zero, we get $\omega\cdot\nabla\phi(x) = -\omega\cdot A(x)$ for all $x \in \mathbb{R}^n$. Then w solves the following equation

$$(i\partial_t + \Delta_\rho + 2iA(x)\cdot\nabla_\rho + h(x,t))\,w(x,t) = L(x,t), \tag{7.25}$$

where

$$L(x,t) = -e^{i\phi(x)}\Big(i\Delta\phi(x) - |\nabla\phi(x)|^2 + 2y\cdot\nabla\phi(x) + 2A(x)\cdot y - 2A\cdot\nabla\phi(x) + h(x,t)\Big). \tag{7.26}$$

In light of (7.25), we introduce the following map

$$\begin{array}{rcl} U_\rho : L^2(0,T;H^1(\Omega)) & \longrightarrow & L^2(0,T;H^1(\Omega)), \\ w & \longmapsto & G_\rho(-w\,h + L). \end{array}$$

Applying (7.19) with $k = 1$ and $f = h\,(w - \tilde{w})$, we get for all $w, \tilde{w} \in L^2(0,T;H^1(\Omega))$ that

$$\begin{array}{rcl} \|U_\rho(w) - U_\rho(\tilde{w})\|_{L^2(0,T;H^1(\Omega))} & = & \|G_\rho(h\,(w-\tilde{w}))\|_{L^2(0,T;H^1(\Omega))} \\ & \leq & \dfrac{C}{\sigma}\|h\|_{\mathcal{X}}\|w-\tilde{w}\|_{L^2(0,T;H^1(\Omega))}. \end{array}$$

Taking σ_0 sufficiently large so that $\sigma_0 > 2C\|h\|_{\mathcal{X}}$, then, for each $\sigma > \sigma_0$, U_ρ admits a unique fixed point $w \in L^2(0,T;H^1(\Omega))$ such that $U_\rho(w) = w$. Again, applying

(7.19) with $k = 1$ and $f = -hw + L$, one gets

$$\begin{aligned}\|w\|_{L^2(0,T;H^1(\Omega))} &= \|G_\rho(-hw + L)\|_{L^2(0,T;H^1(\Omega))}\\ &\leq \frac{1}{2}\|w\|_{L^2(0,T;H^1(\Omega)} + \frac{C}{\sigma}\|L\|_{L^2(0,T;H^1(\Omega))}.\end{aligned}$$

Therefore, in view of Lemma 7.3.3 and (7.26), we get

$$\|w\|_{L^2(0,T;H^1(\Omega))} \leq \frac{C}{\sigma}. \tag{7.27}$$

Next, differentiating the equation (7.25) twice with respect to t, taking into account that $\|h\|_{\mathcal{X}}$ is uniformly bounded with respect to σ, and proceeding as before, we show that

$$\|\partial_t^k w\|_{L^2(0,T;H^1(\Omega))} \leq \frac{C}{\sigma}, \quad k = 1, 2. \tag{7.28}$$

Finally, from (7.27) and Lemma 7.3.3, we obtain

$$\begin{aligned}\|w\|_{L^2(0,T;H^2(\Omega))} &\leq C\| - wh + L\|_{L^2(0,T;H^1(\Omega))}\\ &\leq C\Big(\frac{C}{\sigma}\|h\|_{\mathcal{X}} + C\Big) \leq C, \end{aligned}\tag{7.29}$$

by applying (7.19) with $k = 2$ and $f = -wh + L$. Thus, we get the desired result by combining (7.27)-(7.29).

□

7.4 Stable determination of the magnetic field

This section is devoted to establish a stability estimate for the determination of the magnetic field from the Dirichlet-to-Neumann map $\Lambda_{A,q}$. We prove the main statement of this section, by means of the geometrical optics solutions constructed in the previous section of the form

$$u_j(x,t) = e^{-i\left((\rho_j\cdot\rho_j)t + x\cdot\rho_j\right)}\left(e^{i\phi_j(x)} + w_j(x,t)\right), \quad j=1,2, \tag{7.30}$$

associated A_j and q_j. Here we choose $\rho_j = \sigma\omega_j$ and we recall that the correction term w_j satisfies (7.24) and that $\phi_j(x) = N^{-1}_{\omega_j^*}(-\omega_j^*.A_j)$ solves the transport equation

$$\omega_j^*.\nabla\phi_j(x) = -\omega_j^*.A(x), \quad x\in\mathbb{R}^n.$$

Let us specify the choice of ρ_j: we consider $\xi\in\mathbb{R}^n$ and $\omega = \omega_{\Re} + i\omega_{\Im}$ with $\omega_{\Re},\, \omega_{\Im}\in\mathbb{S}^{n-1}$ and $\omega_{\Re}.\omega_{\Im} = \xi.\omega_{\Re} = \xi.\omega_{\Im} = 0$. for each $\sigma > |\xi|/2$, we denote

$$\rho_1 = \sigma\left(i\omega_{\Im} + \left(-\frac{\xi}{2\sigma} + \sqrt{1-\frac{|\xi|^2}{4\sigma^2}}\omega_{\Re}\right)\right) = \sigma\omega_1^*, \tag{7.31}$$

$$\rho_2 = \sigma\left(-i\omega_{\Im} + \left(\frac{\xi}{2\sigma} + \sqrt{1-\frac{|\xi|^2}{4\sigma^2}}\omega_{\Re}\right)\right) = \sigma\omega_2^*. \tag{7.32}$$

Notice that $\rho_j.\rho_j = 0$. In this section, we aim for recovering the magnetic field $d\alpha_A$ from the boundary operator

$$\begin{array}{rcl}\Lambda_{A,q} : L^2(\Omega)\times H^{2,1}(\Sigma) & \longrightarrow & H^1(\Omega)\times L^2(\Sigma)\\ g=(u_0,f) & \longmapsto & \left(u(.,T),(\partial_\nu + iA\cdot\nu)u\right).\end{array}$$

We denote by

$$\Lambda^1_{A,q} = u(.,T), \quad \Lambda^2_{A,q} = (\partial_\nu + iA\cdot\nu)u.$$

Let us recall the following technical result from [**39**] that will be used later

Lemma 7.4.1. *Let $A\in C_c(\mathbb{R}^n)$, $\xi\in\mathbb{R}^n$, and $\omega=\omega_{\Re}+i\omega_{\Im}$ with $\omega_{\Re},\,\omega_{\Im}\in\mathbb{S}^{n-1}$ and $\omega_{\Re}\cdot\omega_{\Im} = \omega_{\Re}\cdot\xi = \omega_{\Im}\cdot\xi = 0$. Then we have the following identity*

$$\int_{\mathbb{R}^n}\omega\cdot A(x)e^{iN_\omega^{-1}(-\omega\cdot A)(x)}e^{i\xi\cdot x}\,dx = \int_{\mathbb{R}^n}\omega\cdot A(x)e^{i\xi\cdot x}dx.$$

We start by establishing an orthogonality identity for the magnetic potential $A = A_1 - A_2$.

7.4.1 A basic identity for the magnetic potential

In this section, we derive an identity relating the magnetic potential A to the solutions u_j.

Lemma 7.4.2. *Let $\varepsilon > 0$, $A_j \in \mathcal{A}_\varepsilon$ and u_j be the solutions given by (7.30) $j = 1, 2$. Then, for all $\xi \in \mathbb{R}^n$ and $\sigma > max(\sigma_0, |\xi|/2)$, we have*

$$\int_Q iA(x) \cdot \left(\overline{u_1}\nabla u_2 - u_2\nabla\overline{u_1}\right) dx\, dt = \int_Q A(x) \cdot (\rho_2 + \overline{\rho_1})e^{-ix\cdot\xi}e^{i(\phi_2 - \overline{\phi_1})(x)} + I(\xi, \sigma),$$

where the remaining term $I(\xi, \sigma)$ is uniformly bounded with respect to σ and ξ.

Proof. In light of (7.30), we have by direct computation

$$\begin{aligned}
\overline{u_1}\nabla u_2 - u_2\nabla\overline{u_1} \;\; = \;\; & e^{-ix\cdot(\rho_2 - \overline{\rho_1})}\Big[- i\rho_2 e^{i(\phi_2 - \overline{\phi_1})} - i\overline{\rho_1}e^{i(\phi_2 - \overline{\phi_1})} \\
& + i\nabla\phi_2 e^{i(\phi_2 - \overline{\phi_1})} + i\nabla\overline{\phi_1}e^{i(\phi_2 - \overline{\phi_1})} - i\rho_2 w_2 e^{-i\overline{\phi_1}} - i\overline{\rho_1 w_1}e^{i\phi_2} \\
& + \nabla w_2 e^{-i\overline{\phi_1}} - \nabla\overline{w_1}e^{i\phi_2} - i\rho_2\overline{w_1}e^{i\phi_2} - i\overline{\rho_1}w_2 e^{-i\overline{\phi_1}} + i\nabla\phi_2\overline{w_1}e^{i\phi_2} \\
& + iw_2\nabla\overline{\phi_1}e^{-i\overline{\phi_1}} - i\rho_2 w_2\overline{w_1} - i\overline{\rho_1 w_1}w_2 + \nabla w_2\overline{w_1} - \nabla\overline{w_1}w_2\Big].
\end{aligned}$$

Therefore, as we have $\rho_2 - \overline{\rho_1} = \xi$, this yields that

$$\int_Q iA(x) \cdot (\overline{u_1}\nabla u_2 - u_2\nabla\overline{u_1})\, dx\, dt \;\; = \;\; \int_Q A(x) \cdot (\rho_2 + \overline{\rho_1})e^{-ix.\xi}e^{i(\phi_2 - \overline{\phi_1})}\, dx\, dt + I(\xi, \sigma),$$

where $I(\xi, \sigma) = \displaystyle\int_Q iA(x) \cdot \Big(\psi_1(x, t) + \psi_2(x, t)\Big)\, dx\, dt$, and ψ_1, ψ_2 stand for

$$\psi_1 = -i(\rho_2 + \overline{\rho_1})\left(w_2 e^{-i\overline{\phi_1}} + \overline{w_1}e^{i\phi_2} + w_2\overline{w_1}\right),$$

$$\begin{aligned}
\psi_2 \;\; = \;\; & e^{i\phi_2}\Big(i\nabla\phi_2\overline{w_1} - \nabla\overline{w_1}\Big) + e^{-i\overline{\phi_1}}\Big(\nabla w_2 + i\nabla\overline{\phi_1}w_2\Big) \\
& + \nabla w_2\overline{w_1} - \nabla\overline{w_1}w_2 + i\Big(\nabla\phi_2 + \nabla\overline{\phi_1}\Big)e^{i(\phi_2 - \overline{\phi_1})}.
\end{aligned}$$

In view of bounding $|I(\xi,\sigma)|$ uniformly with respect to ξ and σ, we use the fact that A is extended by zero outside Ω and use Lemma 7.3.3 to get

$$\|\phi_j\|_{L^\infty(\Omega)} \le C\|A_j\|_{L^\infty(\mathbb{R}^n)} \le C\varepsilon, \quad j=1,\,2.$$

Recalling (7.23) and (7.24) and applying Lemma 7.3.3, we get

$$\|\psi_j\|_{L^1(Q)} \le C\left(C+\frac{1}{\sigma}\right) \le C, \quad j=1,\,2, \tag{7.33}$$

which yields the desired result. □

With the help of the above lemma we may now derive the following orthogonality identity for the magnetic potential.

Lemma 7.4.3. *Let $\xi \in \mathbb{R}^n$ and $\sigma > max(\sigma_0, |\xi|/2)$. Then, we have the following identity*

$$\int_Q A(x)\cdot(\rho_2+\overline{\rho_1})e^{-ix\cdot\xi}e^{i(\phi_2-\overline{\phi_1})}\,dx\,dt = 2\sigma T\int_\Omega \overline{\omega}\cdot A(x)e^{-ix\cdot\xi}\,dx + J(\xi,\sigma),$$

with $|J(\xi,\sigma)| \le C|\xi|$, where C is independent of σ and ξ.

Proof. In view of (7.31) and (7.32), we have

$$\int_Q A(x)\cdot(\rho_2+\overline{\rho_1})e^{-ix\cdot\xi}e^{i(\phi_2-\overline{\phi_1})}\,dx\,dt = 2\sigma\int_Q \overline{\omega}\cdot A(x)e^{-ix\cdot\xi}e^{i(\phi_2-\overline{\phi_1})}\,dx\,dt$$
$$-2\sigma\left(1-\sqrt{1-|\xi|^2/4\sigma^2}\right)\int_Q \omega_{\Re}\cdot A(x)e^{-ix\cdot\xi}e^{i(\phi_2-\overline{\phi_1})}\,dx\,dt, \tag{7.34}$$

where we recall that

$$\overline{\phi_1} = N^{-1}_{\omega_1^*}(-\overline{\omega_1^*}\cdot A_1), \qquad \phi_2 = N^{-1}_{\omega_2^*}(-\omega_2^*\cdot A_2).$$

Set $\overline{\Psi_1} = N^{-1}_{\overline{\omega}}(-\overline{\omega}\cdot A_1)$ and $\Psi_2 = N^{-1}_{\overline{\omega}}(-\overline{\omega}\cdot A_2)$ in such away that we have

$$\Psi_2-\overline{\Psi_1} = N^{-1}_{\overline{\omega}}(-(-\overline{\omega}\cdot A)) = -N^{-1}_{\overline{\omega}}(-\overline{\omega}\cdot A).$$

Then, we infer from (7.34) that

$$\int_Q A(x) \cdot (\rho_2 + \overline{\rho_1}) e^{-ix\cdot\xi} e^{i(\phi_2 - \overline{\phi_1})} dxdt \quad = \quad J_1(\xi,\sigma) + J_2(\xi,\sigma) + J_3(\xi,\sigma),$$

where we have set

$$J_1(\xi,\sigma) = 2\sigma \int_Q \varpi \cdot A(x) e^{-ix\cdot\xi} e^{i(\Psi_2 - \overline{\Psi_1})}\, dx\, dt,$$

$$J_2(\xi,\sigma) = -2\sigma \int_Q \varpi \cdot A(x) e^{-ix\cdot\xi} \left(e^{i(\Psi_2 - \overline{\Psi_1})} - e^{i(\phi_2 - \overline{\phi_1})} \right) dx\, dt,$$

and

$$J_3(\xi,\sigma) = -2\sigma \left(1 - \sqrt{1 - |\xi|^2/4\sigma^2} \right) \int_Q \omega_{\Re} \cdot A(x) e^{-ix\cdot\xi} e^{i(\phi_2 - \overline{\phi_1})}\, dx\, dt.$$

Using Lemma 7.4.1, one can see that

$$\begin{aligned} J_1(\xi,\sigma) \quad &= \quad 2\sigma T \int_\Omega \varpi \cdot A(x) e^{iN_{\varpi}^{-1}(-\varpi\cdot(-A))} e^{-ix\cdot\xi}\, dx \\ &= \quad 2\sigma T \int_\Omega \varpi \cdot A(x) e^{-ix\cdot\xi}\, dx\, dt. \end{aligned}$$

Now it remains to upper bound the absolute value of $J := J_2 + J_3$. We start by inserting $e^{i(\Psi_2 - \overline{\phi_1})}$ into $J_2(\xi,\sigma)$, getting

$$J_2(\xi,\sigma) \quad = \quad -2\sigma T \int_\Omega \varpi \cdot A(x) e^{-ix\cdot\xi} \left(e^{i\Psi_2} \left(e^{-i\overline{\Psi_1}} - e^{-i\overline{\phi_1}} \right) + e^{-i\overline{\phi_1}} \left(e^{i\Psi_2} - e^{i\phi_2} \right) \right) dx.$$

Further, as $N_\omega^{-1}(-\omega\cdot A)$ depends continuously on ω, according to Lemma 2.4 in **[47]** we get for all $|\xi| \leq 2\sigma$

$$|J_2(\xi,\sigma)| \leq C_T \sigma \left(|\varpi - \overline{\omega_1^*}| + |\varpi - \omega_2^*| \right).$$

Hence, as $1 - \sqrt{1 - |\xi|^2/4\sigma^2} \leq |\xi|^2/4\sigma^2$ for all $|\xi| \leq 2\sigma$, we deduce from (7.31), (7.32) and the above inequality that

$$|J_2(\xi,\sigma)| \leq C_T \left(\sigma \frac{|\xi|^2}{4\sigma^2} + |\xi| \right) \leq C_T\,|\xi|.$$

Arguing in the same way, we find that $|J_3(\xi,\sigma)| \leq C_T|\xi|$, for some positive constant C_T which is independent of ξ and σ. $\square$

7.4.2 An estimate for the Fourier transform of the magnetic field

We aim to relate the Fourier transform of the magnetic field $d\alpha_{A_1} - d\alpha_{A_2}$ to the measurement $\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2}$. To this end, we introduce the following notation: we put

$$a_k(x) = (A_1 - A_2)(x)\cdot e_k = A(x)\cdot e_k,$$

where $(e_k)_k$ is the canonical basis of $\mathbb{R}^n$, and

$$\sigma_{j,k}(x) = \frac{\partial a_k}{\partial x_j}(x) - \frac{\partial a_j}{\partial x_k}(x), \quad j,\, k = 1, ..., n. \tag{7.35}$$

We recall that the Green formula for the magnetic Laplacian

$$\int_\Omega (\Delta_A u\overline{v} - u\overline{\Delta_A v})\,dx = -\int_\Gamma \left((\partial_\nu + i\nu.A)u\overline{v} - u\overline{(\partial_\nu + iA.\nu)v} \right) d\sigma_x, \tag{7.36}$$

holds for any $u,\, v \in H^1(\Omega)$ such that $\Delta u,\, \Delta v \in L^2(\Omega)$. Here $d\sigma_x$ is the Euclidean surface measure on Γ. We estimate the Fourier transform of $\sigma_{j,k}$ as follows.

Lemma 7.4.4. *Let $\xi \in \mathbb{R}^n$ and $\sigma > \max(\sigma_0, |\xi|/2)$, where σ_0 is as in Lemma 7.3.7. Then we have*

$$<\xi>^{-1} |\widehat{\sigma}_{j,k}(\xi)| \leq C \left(e^{C\sigma} \|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\| + \frac{1}{\sigma} + \frac{|\xi|}{|\sigma|} \right),$$

where C is independent of ξ and σ.

Proof. First, for $\sigma > \sigma_0$, Lemma 7.3.7 guarantees the existence of a geometrical optic solution u_2, of the form

$$u_2(x,t) = e^{-ix.\rho_2}(e^{i\phi_2(x)} + w_2(x,t))$$

to the magnetic Schrödinger equation

$$\begin{cases} (i\partial_t + \Delta_{A_2} + q_2(x,t))u_2(x,t) = 0 & \text{in } Q, \\ u_2(x,0) = u_0 & \text{in } \Omega, \end{cases} \tag{7.37}$$

where ρ_2 is given by (7.32). Let us denote by $f_\sigma := u_{2|\Sigma}$. We consider a solution v to the following non homogeneous boundary value problem

$$\begin{cases} (i\partial_t + \Delta_{A_1} + q_1(x,t))v = 0 & \text{in } Q, \\ v(.,0) = u_2(.,0) = u_0 & \text{in } \Omega, \\ v = u_2 = f_\sigma & \text{on } \Sigma. \end{cases} \tag{7.38}$$

Then, $u = v - u_2$ is a solution to the following homogenous boundary value problem for the magnetic Schrödinger equation

$$\begin{cases} (i\partial_t + \Delta_{A_1} + q_1(x,t))u = 2iA\cdot\nabla u_2 + h(x,t)u_2 & \text{in } Q, \\ u(x,0) = 0 & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Sigma, \end{cases}$$

where

$$A = A_1 - A_2, \quad q = q_1 - q_2 \quad \text{and} \quad h = i\,\mathrm{div}A - (|A_1|^2 - |A_2|^2) + q.$$

On the other hand, with reference to Lemma 7.3.7 we consider a solution u_1 to the magnetic Shrödinger equation (7.21), associated with the potentials A_1 and q_1, of the form

$$u_1(x,t) = e^{-ix.\rho_1}(e^{i\phi_1(x)} + w_1(x,t)),$$

where ρ_1 is given by (7.31). Integrating by parts in the following integral, and using the Green Formula (7.36), we get

$$\begin{aligned}\int_Q (i\partial_t + \Delta_{A_1} + q_1)u\overline{u_1}dxdt =& \int_Q 2iA\cdot\nabla u_2\overline{u_1}dxdt + \int_Q\Big(i\text{div}A - (|A_1|^2 - |A_2|^2) + q\Big)u_2\overline{u_1}dxdt \\ =& \; i\int_\Omega u(.,T)\overline{u_1}(.,T)\,dx - \int_\Sigma(\partial_\nu + iA_1.\nu)u\overline{u_1}\,d\sigma_x\,dt. \qquad (7.39)\end{aligned}$$

This entails that

$$\begin{aligned}\int_Q 2iA\cdot\nabla u_2\overline{u_1}dx\,dt \;=& -i\int_\Omega(\Lambda^1_{A_2,q_2} - \Lambda^1_{A_1,q_1})(g)\overline{u_1}(.,T)\,dx + \int_\Sigma(\Lambda^2_{A_2,q_2} - \Lambda^2_{A_1,q_1})(g)\overline{u_1}\,d\sigma_x\,dt \\ & - \int_Q\Big(i\text{div}A - (|A_1|^2 - |A_2|^2) + q\Big)u_2\overline{u_1}dxdt,\end{aligned}$$

where $g = (u_{2|t=0}, u_{2|\Sigma})$. Upon applying the Stokes formula and using the fact that $A_{|\Gamma} = 0$, we get

$$\begin{aligned}\int_Q iA\cdot\Big(\overline{u_1}\nabla u_2 - u_2\nabla\overline{u_1}\Big)dxdt =& -i\int_\Omega\Big(\Lambda^1_{A_2,q_2} - \Lambda^1_{A_1,q_1}\Big)(g)\,\overline{u_1}(.,T)\,dx + \int_\Sigma\Big(\Lambda^2_{A_2,q_2} - \Lambda^2_{A_1,q_1}\Big)(g)\,\overline{u_1}d\sigma_x dt \\ & + \int_Q\Big(|A_1|^2 - |A_2|^2 + q\Big)u_2\overline{u_1}dxdt. \qquad (7.40)\end{aligned}$$

This, Lemma 7.4.2 and Lemma 7.4.3, yield

$$\Big|\int_\Omega \overline{\omega}.A(x)e^{-ix.\xi}\,dx\Big| \leq \frac{C_T}{\sigma}\Big(\|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\|\,\|g\|_{H^2(\Omega)\times H^{2,1}(\Sigma)}\|\phi\|_{L^2(\Sigma)\times L^2(\Omega)} + C + |\xi|\Big),$$

where $\phi = (\overline{u_1}_{|\Sigma}, \overline{u_1}_{t=T})$. Here we used the fact that $\|u_2\overline{u_1}\|_{L^1(Q)} \leq C_T$, for σ sufficiently large. Hence, bearing in mind that

$$\|g\|_{H^2(\Omega)\times H^{2,1}(\Sigma)} \leq Ce^{C\sigma}, \quad \text{and} \quad \|\phi\|_{L^2(\Sigma)\times L^2(\Omega)} \leq Ce^{C\sigma},$$

we get for $\sigma > |\xi|/2$,

$$\left| \int_\Omega \overline{\omega} \cdot A(x) e^{-ix \cdot \xi} \, dx \right| \leq C \left(e^{C\sigma} \|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\| + \frac{1}{\sigma} + \frac{|\xi|}{\sigma} \right). \tag{7.41}$$

Arguing as in the derivation of (7.41), we prove by replacing $\overline{\omega}$ by $-\omega$, that

$$\left| \int_\Omega -\omega \cdot A(x) e^{-ix \cdot \xi} \, dx \right| \leq C \left(e^{C\sigma} \|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\| + \frac{1}{\sigma} + \frac{|\xi|}{\sigma} \right). \tag{7.42}$$

Thus, choosing $\omega_{\Im} = \frac{\xi_j e_k - \xi_k e_j}{|\xi_j e_k - \xi_k e_j|}$, multiplying (7.41) and (7.42) by $|\xi_j e_k - \xi_k e_j|$, and adding the obtained inequalities together, we find that

$$\left| \int_\Omega e^{-ix \cdot \xi} \left(\xi_j \tilde{a}_k(x) - \xi_k \tilde{a}_j(x) \right) \, dx \right| \leq C \, |\xi_j e_k - e_j \xi_k| \left(e^{C\sigma} \|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\| + \frac{1}{\sigma} + \frac{|\xi|}{\sigma} \right).$$

From this and (7.35) we deduce that

$$|\widehat{\sigma}_{j,k}(\xi)| \leq C < \xi > \left(e^{C\sigma} \|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\| + \frac{1}{\sigma} + \frac{|\xi|}{\sigma} \right), \quad j,\, k \in \mathbb{N}.$$

This ends the proof. $\square$

7.4.3 Stability estimate for the magnetic field

We are now in position to show the first result of this chapter which can be stated as follows

Theorem 7.4.1. *Let $\alpha > \frac{n}{2} + 1$. Let $q_i \in \mathcal{Q}_M$, $A_i \in \mathcal{A}_\varepsilon$, such that $\|A_i\|_{H^\alpha(\Omega)} \leq M$, for $i = 1, 2$. Then, there exist three constants $C > 0$ and $\mu, s \in (0,1)$, such that we have*

$$\|d\alpha_{A_1} - d\alpha_{A_2}\|_{L^\infty(\Omega)} \leq C \left(\|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\|^{1/2} + |\log \|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\||^{-\mu} \right)^s.$$

Here C depends only on Ω, ε, M and T and $\mathcal{A}_\varepsilon$(resp.,$\mathcal{Q}_M$) is given by (6.3) (resp., *6.4).*

Proof. In order to prove this theorem, we first need to bound the $H^{-1}(\mathbb{R}^n)$ norm of $d\alpha_{A_1} - d\alpha_{A_2}$. In light of the above reasoning , this can be achieved by taking $\sigma > R > 0$ and decomposing the $H^{-1}(\mathbb{R}^n)$ norm of $\sigma_{j,k}$ as

$$\|\sigma_{j,k}\|^2_{H^{-1}(\mathbb{R}^n)} = \int_{|\xi|\leq R} |\widehat{\sigma}_{j,k}(\xi)|^2 < \xi >^{-2} \, d\xi + \int_{|\xi|>R} |\widehat{\sigma}_{j,k}(\xi)|^2 < \xi >^{-2} \, d\xi.$$

Then, we have

$$\|\sigma_{j,k}\|^2_{H^{-1}(\mathbb{R}^n)} \leq C\Big[R^n \| < \xi >^{-1} \widehat{\sigma}_{j,k}\|^2_{L^\infty(B(0,R))} + \frac{1}{R^2}\|\sigma_{j,k}\|^2_{L^2(\mathbb{R}^n)}\Big],$$

which entails that

$$\|\sigma_{j,k}\|^2_{H^{-1}(\mathbb{R}^n)} \leq C\Big[R^n \left(e^{C\sigma}\|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\|^2 + \frac{1}{\sigma^2} + \frac{R^2}{\sigma^2}\right) + \frac{1}{R^2}\Big],$$

by Lemma 7.4.4. The next step is to choose $R > 0$ in such away $\frac{R^{n+2}}{\sigma^2} = \frac{1}{R^2}$. In this case we get for $\sigma > \max(\sigma_0, |\xi|/2)$, that

$$\begin{aligned}
\|\sigma_{j,k}\|^2_{H^{-1}(\mathbb{R}^n)} &\leq C\left(\sigma^{\frac{2n}{n+4}} e^{C\sigma}\|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\|^2 + \sigma^{\frac{-4}{n+4}}\right) \\
&\leq C\left(e^{C_0\sigma}\|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\|^2 + \frac{1}{\sigma^\mu}\right), \qquad (7.43)
\end{aligned}$$

where $\mu \in (0,1)$. Thus, assuming that $\|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\| \leq c = e^{-C_0 \max(\sigma_0,|\xi|/2)}$, and taking $\sigma = \frac{1}{C_0}|\log\|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\||$ in (7.43), we get that

$$\|\sigma_{j,k}\|_{H^{-1}(\mathbb{R}^n)} \leq C\left(\|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\|^{1/2} + |\log\|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\||^{-\mu'}\right),$$

for some positive $\mu' \in (0,1)$. Since the above estimate remains true when $\|\Lambda_{A_2,q_2} - \lambda_{A_1,q_1}\| \geq c$, as we have

$$\|\sigma_{j,k}\|_{H^{-1}(\mathbb{R}^n)} \leq \frac{2M}{c^{1/2}}c^{1/2} \leq \frac{2M}{c^{1/2}}\|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\|^{1/2},$$

we have obtained that

$$\|d\alpha_{A_1} - d\alpha_{A_2}\|_{H^{-1}(\Omega)} \leq C\left(\|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\|^{1/2} + |\log \|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\||^{-\mu'}\right).$$

In order to complete the proof of the theorem, we consider $\delta > 0$ such that $\alpha := s - 1 = \frac{n}{2} + 2\delta$, use Sobolev's embedding theorem and we find

$$\begin{aligned}
\|d\alpha_{A_1} - d\alpha_{A_2}\|_{L^\infty(\Omega)} &\leq C\|d\alpha_{A_1} - d\alpha_{A_2}\|_{H^{\frac{n}{2}+\delta}(\Omega)} \\
&\leq C\|d\alpha_{A_1} - d\alpha_{A_2}\|^{1-\beta}_{H^{-1}(\Omega)}\|d\alpha_{A_1} - d\alpha_{A_2}\|^{\beta}_{H^{s-1}(\Omega)} \\
&\leq C\left(\|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\|^{1/2} + |\log \|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\||^{-\mu}\right)^{1-\beta},
\end{aligned}$$

by interpolating with $\beta \in (0,1)$. This completes the proof of Theorem 7.4.1. $\square$

This theorem is a key ingredient in the proof of the result of the next section.

7.5 Determination of the electric potential

Using the geometric optics solutions constructed in Section 7.3, we will prove with the aid of the stability estimate obtained for the magnetic field, that the time-dependent electric potential depends stably on the Dirichlet-to-Neuamnn map $\Lambda_{A,q}$.

7.5.1 Preliminaries

Lemma 7.5.1. *Let $\Omega \subset \mathbb{R}^n$ be a simply connected domain, and let $A \in \mathcal{C}^2(\Omega, \mathbb{R}^n)$ be such that $A_{|\Gamma} = 0$. Then, for $p > n$, there exists a function $\varphi \in \mathcal{C}^3(\Omega)$ such that $\varphi_{|\Gamma} = 0$ and $A' \in W^{1,p}(\Omega, \mathbb{R}^n)$, satisfying $A = A' + \nabla\varphi$, $A' \wedge \nu = 0$, and $divA' = 0$. Moreover, there exists a constant $C > 0$, such that*

$$\|A'\|_{W^{1,p}(\Omega)} \leq C\,\|curl\,A'\|_{L^p(\Omega)}. \tag{7.44}$$

Proof. Let φ be the solution of the following problem

$$\begin{cases} \Delta\varphi = \mathrm{div} A, & \text{in}\,\Omega \\ \varphi = 0, & \text{in}\,\Gamma. \end{cases} \tag{7.45}$$

Then, setting $A' = A - \nabla\varphi$, using the fact that $A_{|\Gamma} = \varphi_{|\Gamma} = 0$, one gets

$$A' \wedge \nu = A \wedge \nu - \nabla\varphi \wedge \nu = 0, \quad \text{and} \quad \mathrm{div} A' = 0.$$

In order to prove (7.44), we argue by contradiction. We assume that for all $k \geq 1$ there exists a non-null $\widetilde{A'}_k \in W^{1,p}(\Omega)$ such that

$$\|\widetilde{A'_k}\|_{W^{1,p}(\Omega)} \geq k\, \|\mathrm{curl}\, \widetilde{A'}_k\|_{L^p(\Omega)}. \tag{7.46}$$

We set $A'_k = \dfrac{\widetilde{A'}_k}{\|\widetilde{A'}_k\|_{W^{1,p}(\Omega)}}$. Then we have $\|A'_k\|_{W^{1,p}(\Omega)} = 1$ and $k\, \|\mathrm{curl}\, A'_k\|_{L^p(\Omega)} \leq 1$. In view of the weak compactness theorem, there exists a subsequence of $(A'_k)_k$ such that $A'_k \rightharpoonup A'$ in $W^{1,p}(\Omega)$. Using the fact that $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, we deduce that $A'_k \to A'$ in $L^p(\Omega)$. As a consequence, we have

$$\|A'\|_{W^{1,p}(\Omega)} = 1 \quad \text{and} \quad \|\mathrm{curl}\, A'\|_{L^p(\Omega)} = 0.$$

This entails that there exists $\eta \in W^{1,p}(\Omega)$ such that $A' = \nabla\eta$. Then, using the fact that div $A' = 0$ and $A' \wedge \nu = 0$, we deduce that there exists a constant $\lambda \in \mathbb{R}$ such that

$$\begin{cases} \Delta\eta = 0, & \text{in } \Omega \\ \eta = \lambda, & \text{in } \Gamma. \end{cases}$$

Finally, using the fact that Ω is a simply connected domain we conclude that $\eta = \lambda$ in $\overline{\Omega}$. This entails that $A' = 0$ and contradicts the fact that $\|A'\|_{W^{1,p}(\Omega)} = 1$. $\square$

As a consequence of Lemma 7.5.1, we have the following result

Lemma 7.5.2. *Let $\Omega \subset \mathbb{R}^n$ be a simply connected domain, and let $A \in \mathcal{C}^2(\Omega, \mathbb{R}^n)$*

such that $A_{|\Gamma} = 0$. If we further assume that div $A = 0$, then the following estimate

$$\|A\|_{W^{1,p}(\Omega)} \leq C\, \|curl\, A\|_{L^p(\Omega)},$$

holds true for some positive constant C which is independent of A.

In order to identify the magnetic field, we should normally apply the Hodge decomposition to $A = A_1 - A_2 = A' + \nabla\varphi$ and use (7.44) which holds for any $p > n$. But in this chapter, since u_0 is not frozen to zero, we don't have invariance under Gauge transformation, so will further assume that A is divergence free in such a way that the estimate (7.44) holds for $A' = A$.

For a fixed $y \in B(0,1)$, we consider solutions u_j to the Schrödinger equation of the form (7.30) with $\rho_j = \sigma\omega_j^* + y$, where $\xi \in \mathbb{R}^n$ and $\omega \in \mathbb{S}^{n-1}$ are as in the previous section, and w_j^*, $j = 1, 2$, are given by (7.31) and (7.32).

In contrast to the previous section, y is no longer equal to zero, as we need to estimate the Fourier transform of q with respect to x and t.

7.5.2 An identity for the electric potential

In this section, we establish a preliminary identity for the electric potential.

Lemma 7.5.3. *Let u_j be the solutions given by (7.30) for $j = 1, 2$. For all $\sigma \geq \sigma_0$ and $\xi \in \mathbb{R}^n$ such that $|\xi| < 2\sigma$, we have the following identity*

$$\int_Q q(x,t) u_2 \overline{u_1}\, dx\, dt = \int_Q q(x,t) e^{-i(2y.\xi t + x.\xi)}\, dx\, dt + P_1(\xi, y, \sigma) + P_2(\xi, y, \sigma),$$

where $P_1(\xi, y, \sigma)$ and $P_2(\xi, y, \sigma)$ satisfy the estimates

$$|P_1(\xi, y, \sigma)| \leq C \left(\|A\|_{L^\infty(\Omega)} + \frac{|\xi|}{\sigma} \right), \qquad |P_2(\xi, y, \sigma)| \leq \frac{C}{\sigma}.$$

Here σ_0 is as in Lemma 7.3.7 and C is independent of σ, y, and ξ.

Proof. In light of (7.31), (7.32) and (7.30), a direct calculation gives us

$$\begin{aligned} u_2\overline{u_1} &= e^{-i\left((\rho_2.\rho_2-\overline{\rho_1.\rho_1})t+x.(\rho_2-\overline{\rho_1})\right)}\left(e^{i(\phi_2-\overline{\phi_1})}+e^{-i\overline{\phi}_1}w_2+e^{i\phi_2}\overline{w_1}+w_2\overline{w}_1\right) \\ &= e^{-i(2y.\xi t+x.\xi)}e^{-i(\overline{\phi_1}-\phi_2)}+e^{-i(2y.\xi t+x.\xi)}\left(e^{-\overline{\phi_1}}w_2+e^{i\phi_2}\overline{w_1}+w_2\overline{w_1}\right) \end{aligned} \tag{7.47}$$

which yields

$$\int_Q q(x,t)u_2\overline{u_1}\,dx\,dt = \int_Q q(x,t)e^{-i(2y.\xi t+x.\xi)}\,dx\,dt + P_1(\xi,y,\sigma)+P_2(\xi,y,\sigma), \tag{7.48}$$

where we have set

$$\begin{aligned} P_1(\xi,y,\sigma) &= \int_Q q(x,t)e^{-i(2y\cdot\xi t+x\cdot\xi)}e^{-i\overline{\phi}_1}\left(e^{i\phi_2}-e^{i\overline{\phi}_1}\right)\,dx\,dt, \\ P_2(\xi,y,\sigma) &= \int_Q q(x,t)e^{(-i2y\cdot\xi t+x.\xi)}\left(e^{-i\overline{\phi}_1}w_2+e^{i\phi_2}\overline{w_1}+w_2\overline{w_1}\right)\,dx\,dt. \end{aligned}$$

Recalling that $\phi_j = N_{\omega_j^*}^{-1}(-\omega_j^*\cdot A_j)$, for $j=1,\,2$, we deduce from the definition of P_1 that

$$|P_1(\xi,y,\sigma)|\leq C\Big(\|e^{iN_{\omega_2^*}^{-1}(-\omega_2^*\cdot A_2)}-e^{iN_{\omega_2^*}^{-1}(-\omega_2^*\cdot A_1)}\|_{L^\infty(\Omega)}+\|e^{iN_{\omega_2^*}^{-1}(-\omega_2^*\cdot A_1)}-e^{iN_{\overline{\omega_1}^*}^{-1}(-\overline{\omega_1}^*\cdot A_1)}\|_{L^\infty(\Omega)}\Big),$$

with $C>0$ is depending on T, M, Ω and $\|A_1\|$. Using the continuity of $N_\omega^{-1}(-\omega\cdot A)$ with respect to ω (see Lemma 2.4 in **[47]**), we get that

$$\begin{aligned} |P_1(\xi,y,\sigma)| &\leq C\left(\|{N_{\omega_2}^*}^{-1}(-\omega_2^*.A_2)-N_{\omega_2^*}^{-1}(-\omega_2^*.A_1)\|_{L^\infty(\Omega)}+|\omega_2^*-\overline{\omega}_1^*|\right) \\ &\leq C\left(\|A\|_{L^\infty(\Omega)}+\frac{|\xi|}{\sigma}\right). \end{aligned}$$

On the other hand, from Cauchy Schwarz inequality, Lemma 7.3.3 and (7.24), we get

$$\begin{aligned} |P_2(\xi,y,\sigma)| &\leq C\left(\|w_2\|_{L^2(Q)}\|e^{-i\overline{\phi_1}}\|_{L^2(Q)}+\|e^{i\phi_2}\|_{L^2(Q)}\|\overline{w_1}\|_{L^2(Q)}+\|w_2\|_{L^2(Q)}\|\overline{w_1}\|_{L^2(\Omega)}\right) \\ &\leq \frac{C}{\sigma}. \end{aligned}$$

This completes the proof of Lemma 7.5.3. $\square$

7.5.3 An estimate for the Fourier transform of the electric potential

In view of relating the Fourier transform of the electric potential $q = q_1 - q_2$ to $\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2}$, we first establish the following auxiliary result

Lemma 7.5.4. *For any $\sigma \geq \sigma_0$ and $\xi \in \mathbb{R}^n$ such that $|\xi| < 2\sigma$, we have the following estimate*

$$|\widehat{q}(\xi, 2y.\xi)| \leq C\Big(e^{C\sigma}\|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\| + e^{C\sigma}\|d\alpha_{A_1} - d\alpha_{A_2}\|_{L^\infty(\Omega)} + \frac{|\xi|}{\sigma} + \frac{1}{\sigma}\Big),$$

for some C that is independent of $|\xi|$ and σ.

Proof. First, for $\sigma > \sigma_0$, Lemma 7.3.7 guarantees the existence of a geometrical optics solution u_2 of the form

$$u_2(x,t) = e^{-i((\rho_2.\rho_2)t + x.\rho_2)}(e^{i\phi_2(x)} + w_2(x,t)),$$

to the magnetic Schrödinger equation

$$\begin{cases} (i\partial_t + \Delta_{A_2} + q_2(x,t))u_2(x,t) = 0 & \text{in } Q, \\ u_2(x,0) = u_0 & \text{in } \Omega, \end{cases} \tag{7.49}$$

where ρ_2 is given by (7.32) and $w_2(x,t)$ satisfies

$$\sigma\|w_2\|_{H^2(0,T,H^1(\Omega))} + \|w_2\|_{L^2(0,T,H^2(\Omega))} \leq C; \tag{7.50}$$

Let us denote by $f_\sigma := u_{2|\Sigma}$. We consider a solution v to the following non homogeneous boundary value problem

$$\begin{cases} (i\partial_t + \Delta_{A_1} + q_1(x,t))v = 0 & \text{in } Q, \\ v(.,0) = u_2(.,0) = u_0 & \text{in } \Omega, \\ v = u_2 = f_\sigma & \text{on } \Sigma. \end{cases} \tag{7.51}$$

Denote $u = v - u_2$, then u is a solution to the following homogenous boundary value problem for the magnetic Schrödinger equation

$$\begin{cases} (i\partial_t + \Delta_{A_1} + q_1(x,t))u = 2iA\cdot\nabla u_2 + h(x,t)u_2 & \text{in } Q, \\ u(x,0) = 0 & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Sigma, \end{cases}$$

where we recall that

$$A = A_1 - A_2, \quad q = q_1 - q_2 \quad \text{and} \quad h = i\,\mathrm{div}A - (|A_1|^2 - |A_2|^2) + q.$$

On the other hand, we consider a solution u_1 of the magnetic Shrödinger equation (7.21) corresponding to the potentials A_1 and q_1, of the form

$$u_1(x,t) = e^{-i((\rho_1.\rho_1)t + x.\rho_1)}(e^{i\phi_1(x)} + w_1(x,t)),$$

where ρ_1 is given by (7.31) and $w_1(x,t)$ satisfies

$$\sigma\|w_1\|_{H^2(0,T,H^1(\Omega))} + \|w_1\|_{L^2(0,T,H^2(\Omega))} \leq C. \tag{7.52}$$

Integrating by parts and using the Green Formula (7.36), we get

$$\begin{aligned} \int_Q q(x,t)u_2\overline{u_1}\,dx\,dt &= i\int_\Omega(\Lambda^1_{A_2,q_2} - \Lambda^1_{A_1,q_1})(g)\overline{u_1}(.,T)\,dx - \int_\Sigma(\Lambda^1_{A_2,q_2} - \Lambda^2_{A_1,q_1})(g)\overline{u_1}\,d\sigma_x\,dt \\ &+ \int_Q iA(x)\cdot(\overline{u_1}\nabla u_2 - u_2\nabla\overline{u_1})\,dx\,dt - \int_Q(|A_1|^2 - |A_2|^2)u_2\overline{u_1}\,dx\,dt, \end{aligned}$$

where $g = (u_{2|t=0}, u_{2|\Sigma})$. To bring the Fourier transform of q out of the above identity, we extend q by zero outside the cylindrical domain Q, we use Lemma 7.5.3 and take to account that

$$\|u_2\overline{u_1}\|_{L^1(Q)} \leq C, \text{ and } \|\overline{u_1}\nabla u_2\|_{L^1(Q)} + \|u_2\nabla\overline{u_1}\|_{L^1(Q)} \leq C\sigma,$$

and get

$$|\widehat{q}(\xi, 2y\cdot\xi)| \leq C\Big(\|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\|\|g\|_{H^2(\Omega)\times H^{2,1}(\Sigma)}\|\phi\|_{L^2(\Sigma)\times L^2(\Omega)} + C\sigma\|A\|_{L^\infty(\Omega)} + \frac{|\xi|}{\sigma} + \frac{1}{\sigma}\Big),$$

where $\phi = (\overline{u_1}_{|\Sigma}, \overline{u_1}_{|t=T})$. Now, bearing in mind that

$$\|g\|_{H^2(\Omega)\times H^{2,1}(\Sigma)} \leq Ce^{C\sigma}, \quad \text{and} \quad \|\phi\|_{L^2(\Sigma)\times L^2(\Omega)} \leq Ce^{C\sigma},$$

we get for all $\xi \in \mathbb{R}^n$ such that $|\xi| < 2\sigma$ and for all $y \in B(0,1)$,

$$|\widehat{q}(\xi, 2y.\xi)| \leq C\Big(e^{C\sigma}\|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\| + e^{C\sigma}\|A\|_{L^\infty(\Omega)} + \frac{|\xi|}{\sigma} + \frac{1}{\sigma}\Big). \tag{7.53}$$

Finally, using the fact that $\|A\|_{W^{1,\infty}(\Omega)} \leq C\|\mathrm{curl}\, A\|_{L^\infty(\Omega)}$, (see Lemma 7.5.2), we obtain the desired result. $\square$

We are now in position to estimate $\widehat{q}(\xi,\tau)$ for all (ξ,τ) in the following set

$$E_\alpha = \{(\xi,\tau) \in (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R},\ |\xi| < 2\alpha,\ |\tau| < 2|\xi|\},$$

for any fixed $0 < \alpha < \sigma$.

Lemma 7.5.5. *Suppose that the conditions of Lemma 7.5.4 are satisfied. Then we have for all $(\xi,\tau) \in E_\alpha$,*

$$|\widehat{q}(\xi,\tau)| \leq C\Big(e^{C\sigma}\|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\| + e^{C\sigma}\|d\alpha_{A_1} - d\alpha_{A_2}\|_{L^\infty(\Omega)} + \frac{\alpha}{\sigma} + \frac{1}{\sigma}\Big). \tag{7.54}$$

Here C is independent of $|\xi|$ and σ.

Proof. Fix $(\xi,\tau)\in E_\alpha$, and set $y=\frac{\tau}{2|\xi|^2}\cdot\xi$, in such away that $y\in B(0,1)$ and $2y\cdot\xi=\tau$. Since $\alpha<\sigma$ we have $|\xi|<2\alpha<2\sigma$. Hence, Lemma 7.5.4 yields the desired result. □

7.5.4 Stability estimate for the electric potential

We are now in position to state and prove the second main result of this chapter which lies in the stable determination of the time-dependnet electric potential q appearing in the magnetic Schrödinger equation (7.1).

Theorem 7.5.1. *Let $q_i\in\mathcal{Q}_M$, $A_i\in\mathcal{A}_\varepsilon$, for $i=1,2$. Assume that div $A_i=0$. Then there exist three constants $C>0$, and $m,\mu\in(0,1)$, such that we have*

$$\|q_1-q_2\|_{H^{-1}(Q)}\leq C\Phi_m(\eta),$$

where

$$\Phi_m(\eta)=\begin{cases}|\log|\log|\log\eta|^{\mu}||^{-1} & \text{if } \eta<m,\\ \\ \dfrac{1}{m}\,\eta & \text{if } \eta\geq m.\end{cases}$$

Here $\eta=\|\Lambda_{A_2,q_2}-\Lambda_{A_1,q_1}\|$, C depends on Ω, M, ε and T, and $\mathcal{A}_\varepsilon$ (resp., $\mathcal{Q}_M$) is given by (6.3) (resp., 6.4).

Proof. For fixed $0<\alpha<\sigma$, let us set $F_\alpha(\xi,\tau)=\widehat{q}(\alpha(\xi,\tau))$, for $(\xi,\tau)\in\mathbb{R}^{n+1}$. It is easily seen that F_α is analytic and we have for $\kappa\in(\mathbb{N}^n\cup\{0\})^{n+1}$

$$\begin{aligned}|\partial^\kappa F_\alpha(\xi,\tau)|=|\partial^\kappa\widehat{q}(\alpha(\xi,\tau))| &= \left|\partial^\kappa\int_{\mathbb{R}^{n+1}}q(x,t)e^{-\alpha(x,t).(\tau,\xi)}\,dx\,dt\right|\\ &= \left|\int_{\mathbb{R}^{n+1}}q(x,t)(-i)^{|\kappa|}\alpha^{|\kappa|}(x,t)^\kappa e^{-i\alpha(x,t).(\xi,\tau)}\,dx\,dt\right|.\end{aligned}$$

Hence one gets

$$|\partial^\kappa F_\alpha(\xi,\tau)|\leq\int_{\mathbb{R}^{n+1}}|q(x,t)|\alpha^{|\kappa|}(|x|^2+t^2)^{\frac{|\kappa|}{2}}\,dx\,dt\leq\|q\|_{L^1(Q)}\alpha^{|\kappa|}(2T^2)^{\frac{|\kappa|}{2}}\leq C\frac{|\kappa|!}{(T^{-1})^{|\kappa|}}e^{\alpha}.$$

Applying Theorem 3.2.1 on the set $\mathcal{O} = E_1 \cap B(0,1)$ with $M = Ce^{\alpha}$, $2\rho = T^{-1}$, we may find a constant $\gamma \in (0,1)$ such that we have

$$|F_\alpha(\xi,\tau)| = |\widehat{q}(\alpha(\xi,\tau))| \leq Ce^{\alpha(1-\gamma)} \|F_\alpha\|^{\gamma}_{L^\infty(\mathcal{O})}, \quad (\xi,\tau) \in B(0,1).$$

Now the idea is to estimate the Fourier transform of q in a suitable ball. Bearing in mind that $\alpha E_1 = E_\alpha$, we have for all $(\xi,\tau) \in B(0,\alpha)$,

$$\begin{aligned} |\widehat{q}(\xi,\tau)| = |F_\alpha(\alpha^{-1}(\xi,\tau)| &\leq Ce^{\alpha(1-\gamma)} \|F_\alpha\|^{\gamma}_{L^\infty(\mathcal{O})} \\ &\leq Ce^{\alpha(1-\gamma)} \|\widehat{q}\|^{\gamma}_{L^\infty(B(0,\alpha)\cap E_\alpha)} \\ &\leq Ce^{\alpha(1-\gamma)} \|\widehat{q}\|^{\gamma}_{L^\infty(E_\alpha)}. \end{aligned} \tag{7.55}$$

The next step of the proof is to get an estimate linking the coefficient q to the measurement $\Lambda_{A_1,q_1} - \Lambda_{A_2,q_2}$. To do that we first decompose the $H^{-1}(\mathbb{R}^{n+1})$ norm of q as follows

$$\begin{aligned} \|q\|^{\frac{2}{\gamma}}_{H^{-1}(\mathbb{R}^{n+1})} &= \left(\int_{|(\xi,\tau)|<\alpha} < (\xi,\tau) >^{-2} |\widehat{q}(\xi,\tau)|^2 \, d\xi \, d\tau + \int_{|(\xi,\tau)|\geq\alpha} < (\xi,\tau) >^{-2} |\widehat{q}(\xi,\tau)|^2 \, d\tau \, d\xi \right)^{\frac{1}{\gamma}} \\ &\leq C \left(\alpha^{n+1} \|\widehat{q}\|^2_{L^\infty(B(0,\alpha))} + \alpha^{-2} \|q\|^2_{L^2(\mathbb{R}^{n+1})} \right)^{\frac{1}{\gamma}}. \end{aligned}$$

It follows from (7.55) and Lemma 7.5.5, that

$$\|q\|^{\frac{2}{\gamma}}_{H^{-1}(\mathbb{R}^{n+1})} \leq C \Big[\alpha^{\frac{n+1}{\gamma}} e^{\frac{2\alpha(1-\gamma)}{\gamma}} \left(e^{C\sigma}\eta^2 + e^{C\sigma} \|d\alpha_{A_1} - d\alpha_{A_2}\|^2_{L^\infty(\Omega)} + \frac{\alpha^2}{\sigma^2} + \frac{1}{\sigma^2} \right) + \frac{1}{\alpha^{\frac{2}{\gamma}}} \Big], \tag{7.56}$$

where we have set $\eta = \|\Lambda_{A_2,q_2} - \Lambda_{A_1,q_1}\|$. In light of Theorem 7.4.1, one gets

$$\|q\|^{\frac{2}{\gamma}}_{H^{-1}(\mathbb{R}^{n+1})} \leq C \Big[\alpha^{\frac{n+1}{\gamma}} e^{\frac{2\alpha(1-\gamma)}{\gamma}} \left(e^{C\sigma}\eta^2 + e^{c\sigma}\eta^s + e^{C\sigma} |\log\eta|^{-2\gamma s} + \frac{\alpha^2}{\sigma^2} + \frac{1}{\sigma^2} \right) + \frac{1}{\alpha^{\frac{2}{\gamma}}} \Big]. \tag{7.57}$$

The above statements are valid provided σ is sufficiently large. Then, we choose α so large that $\sigma = \alpha^{\frac{2\gamma+n+3}{2\gamma}} e^{\frac{\alpha(1-\gamma)}{\gamma}}$, and hence $\alpha^{\frac{2\gamma+n+1}{\gamma}} e^{\frac{2\alpha(1-\gamma)}{\gamma}} \sigma^{-2} = \alpha^{\frac{-2}{\gamma}}$, so the estimate (7.57) yields

$$\|q\|_{H^{-1}(\mathbb{R}^{n+1})}^{\frac{2}{\gamma}} \leq C\Big[e^{Ce^{N\alpha}}(\eta^2+\eta^s+|\log\eta|^{-2\gamma s})+\alpha^{\frac{-2}{\gamma}}\Big], \tag{7.58}$$

where N depends on γ and n. Thus, if $\eta\in(0,1)$, we have

$$\|q\|_{H^{-1}(\mathbb{R}^{n+1})}^{\frac{2}{\gamma}} \leq C\Big(e^{Ce^{N\alpha}}|\log\eta|^{-2\gamma s}+\alpha^{\frac{-2}{\gamma}}\Big). \tag{7.59}$$

Finally, if η is small enough, taking $\alpha=\frac{1}{N}\log\Big(\log|\log\eta|^{\frac{\gamma s}{C}}\Big)$, we get from (7.59) that

$$\|q\|_{H^{-1}(\mathbb{R}^{n+1})}^{\frac{2}{\gamma}} \leq C\Big[|\log\eta|^{-\gamma s}+\Big[\log\Big(\log|\log\eta|^{\frac{\gamma s}{c}}\Big)\Big]^{-\frac{2}{\gamma}}\Big].$$

This completes the proof of Theorem 7.5.1;

□

CHAPTER 8

Determination of coefficients in a magnetic Schrödinger equation from a finite number of measurements

The contents of this chapter are collected in a papaer that will appear in *Inverse Problems*.

8.1 Introduction

In this chapter, we study the following inverse problem: Given $T > 0$ and a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, with smooth boundary Γ, we want to determine simultaneously the divergence free magnetic potential of the form $a(x,t) := \chi(t)a(x)$ and the electric potential $q(x,t) := \beta(t)q(x)$ appearing in the following equation

$$\begin{cases} (-i\partial_t + H_a(t) + q(x,t))u(x,t) = 0 & \text{in } Q = \Omega \times (0,T), \\ u(x, \frac{T}{2}) = u_0(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Sigma = \Gamma \times (0,T), \end{cases} \tag{8.1}$$

where $H_a(t) := (i\nabla + \chi(t)a)^2$ denotes the time-dependent Hamiltonian, associated to the magnetic potential vector $\chi(t)a(x)$. Here $a = (a_1, ..., a_n) \in A := H^1(\Omega)^n \cap \{a \in L^\infty(\Omega, \mathbb{R}^n),\ \nabla \cdot a = 0\}$, and $q \in L^\infty(\Omega)$ are unknown real valued functions. Moreover, the functions $\beta, \chi \in \mathcal{C}^3(0,T;\mathbb{R})$ are assumed to be known functions satisfying

$$\chi(\frac{T}{2}) = \beta(\frac{T}{2}) = 0,\ \chi'(\frac{T}{2}) \neq 0,\ \beta'(\frac{T}{2}) \neq 0. \tag{8.2}$$

We denote by Γ^+ an open subset of Γ satisfying an appropriate geometrical condition given in Section 6.2.2 and by $\Sigma^+ := \Gamma^+ \times (0,T)$.

The inverse problem we investigate in this chapter, is to know whether the knowledge of a finite number of Neumann measurements $\partial_\nu u_{|\Sigma^+} := (\nabla u \cdot \nu)_{|\Sigma^+}$ uniquely determines $a(x)$ and $q(x)$ simultaneously. Here $\nu(x)$ denotes the unit outward normal to Γ at x.

To our knowledge, there is a few results on the recovery of coefficients appearing in a Schrödinger equation, from a finite number of boundary measurements. By a method based essentially on an appropriate Carleman estimate, Baudouin and Puel [**3**] showed that the electric potential in the Shcrödinger equation can be stably recovered from a single boundary measurement. In [**18**] Cristofol and Soccorsi proved a Lipschitz stability in recovering the magnetic field in the Schrödinger equation from a finite number of observations, measured on a subboundary for different choices of the initial condition u_0.

In the present chapter, we improve the two above mentionned results by showing that the electric potential and the magnetic field can be stably and simultaneously recovered from a finite number of boundary observations of the solution. As a matter of fact, the method of derivation of the stability estimate given in this chapter is different for the one of [**18**][Theorem 1.1], as second order time-derivatives of the solution only are used.

Let us introduce some notations. Let us denote by $\mathcal{V}$, an arbitrary neighborhood of the boundary Γ. For $M > 0$, and $(a_0, q_0) \in A \times L^\infty(\Omega)$, we recall the admissible set of the unknown coefficients a and q:

$$\mathcal{S}_M(a_0, q_0) := \{(a, q) \in A \times L^\infty(\Omega), \text{such that } a = a_0, \text{ and } q = q_0 \text{ in } \mathcal{V}\}.$$

8.2 Well posedness

Before dealing with the inverse problem under consideration, we need first to justify the existence of a unique solution of (8.1). To this end, we introduce the space $\mathcal{H}_1(\Omega) := H_0^1(\Omega)$ equipped with the scalar product

$$< (-\Delta + 1)^{1/2} u, (-\Delta + 1)^{1/2} v >_{L^2(\Omega)}, \qquad \text{for any } u,\ v \in \mathcal{H}_1(\Omega),$$

and denote by $\mathcal{H}_2 := H^2(\Omega) \cap H_0^1(\Omega)$ equipped with the scalar product

$$< (-\Delta + 1) u, (-\Delta + 1) v >_{L^2(\Omega)} \qquad \text{for any } u,\ v \in \mathcal{H}_2(\Omega).$$

Here and below $< \cdot, \cdot >_{L^2(\Omega)}$ denotes the usual scalar product in $L^2(\Omega)$. Then, we have the following theorem:

Theorem 8.2.1. *Let $a \in A$, $q \in L^\infty(\Omega, \mathbb{R})$, $\chi \in \mathcal{C}^3(0, T; \mathbb{R})$ and $\beta \in \mathcal{C}^3(0, T; \mathbb{R})$. Then, for every u_0 satisfying $\Delta^k u_0 \in \mathcal{H}_2$, $k = 0, 1, 2$, and for any $f \in W^2(0, T; \mathcal{H}_2(\Omega))$,* *there exists a unique solution*

$$u \in \mathcal{C}^2(0,T;\mathcal{H}_2) \cap \mathcal{C}^3(0,T;\mathcal{H}_0),$$

to the equation

$$\begin{cases} (-i\partial_t + H_a(t) + \beta(t)q(x))u(x,t) = f, & \text{in } Q = \Omega \times (0,T), \\ u(x,\frac{T}{2}) = u_0(x) & \text{in } \Omega. \end{cases} \tag{8.3}$$

Moreover, there exists a constant $C > 0$ such that

$$\|\partial_t^j u(.,t)\|_{\mathcal{H}_1(\Omega)} \leq C \sum_{k=0}^{j} \|\Delta^k u_0\|_{\mathcal{H}_1(\Omega)}, \qquad j = 0,1,2, \qquad t \in (0,T). \tag{8.4}$$

Proof. Since $H_a(t)$ is a self adjoint operator in $L^2(\Omega)$, associated with the sesquilinear form

$$u \mapsto \|(i\nabla + \chi(t)a)\|^2_{L^2(\Omega)^n}, \quad u \in H_0^1(\Omega),$$

then, it holds true (see. e. g. [**18**]) that the domain of $H_a(t)$ is

$$\mathcal{D}(H_a(t)) = H_0^1(\Omega) \cap H^2(\Omega).$$

Further, as $q \in L^\infty(\Omega)$ for all $t \in (0,T)$, we deduce from the Kato-Rellich Theorem that

$$\mathcal{D}(H_a(t) + \beta(t)q) = \mathcal{D}(H_a(t)).$$

From [**33**] there exists a family of unitary operotors $(U(t,s))_{0 \leq s,t \leq T}$ in $\mathcal{H}_0$ satisfying the following statements:

1. $U(s,s) = Id$, the identity mapping in $\mathcal{H}_0$,
2. $U(t,s)\,\mathcal{D}(H_a(s) + \beta(s)q) \subset \mathcal{D}(H_a(t) + \beta(t)q),\ t,s \in [0,T]$.
3. For all $\phi \in \mathcal{D}(H_a(s) + \beta(s)q)$, the mapping $t \mapsto U(t,s)x$, is continuously differentiable in $[0,T]$ and satisfies

$$-i\partial_t U(t,s)\,\phi + (H_a(t)+\beta(t)q(x))U(t,s)\phi = 0,$$

therefore, arguing as in **[18]**[Section 2], we check that

$$u(\,\cdot\,,t) = U(t,\frac{T}{2})u_0 + i\int_{\frac{T}{2}}^{t} U(t,s)f(s)\,ds,$$

is solution to (8.3) and satisfies the estimate 8.4. We complete the proof of this theorem.. $\square$

8.3 Simultaneous stable determination of the electric and the magnetic potentials

In this section, we derive the stability estimate for a and q appearing in the magnetic Schrödinger equation (8.1). Here and henceforth the symbol "$'$" stands for the differentiation with respect to t.

8.3.1 Linearization

Let u_j, for $j=1,\,2$, be solutions to

$$\begin{cases} (-i\partial_t + H_{a_j}(t)+\beta(t)q_j)u_j = 0 & \text{in } \ Q,\\ u_j(\,\cdot\,,\frac{T}{2}) = u_0(x) & \text{in } \ \Omega,\\ u_j = 0 & \text{on } \ \Sigma. \end{cases} \tag{8.5}$$

Then, $u=u_1-u_2$ is a solution to the following boundary value problem

$$\begin{cases} (-i\partial_t + H_{a_1}+\beta(t)q_1)u = f & \text{in } \ Q,\\ u(\,\cdot\,,\frac{T}{2}) = 0 & \text{in } \ \Omega,\\ u = 0 & \text{on } \ \Sigma, \end{cases} \tag{8.6}$$

where $f = \chi(a_1 - a_2) \cdot (-2i\nabla - \chi(a_1 + a_2))u_2 - \beta(q_1 - q_2)u_2$. By differentiating (8.6), we get

$$\begin{cases} (-i\partial_t + H_{a_1} + \beta(t)q_1)v = g := f' - H'_{a_1}u - \beta' q_1 u & \text{in } Q, \\ v(\,\cdot\,, \frac{T}{2}) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \Sigma, \end{cases} \tag{8.7}$$

with $v = \partial_t u$. Thus, $w = \partial_t v$ is a solution to

$$\begin{cases} (-i\partial_t + H_{a_1}(t) + \beta(t)q_1)w = h := f'' - 2(H'_{a_1} + \beta' q_1)v - (H''_{a_1} + \beta'' q_1)u & \text{in } Q, \\ w(\,\cdot\,, \frac{T}{2}) = 2\chi'(\frac{T}{2})(a_1 - a_2)(x) \cdot \nabla u_0 - i\beta'(\frac{T}{2})(q_1 - q_2)(x)u_0 & \text{in } \Omega, \\ w = 0 & \text{on } \Sigma. \end{cases} \tag{8.8}$$

8.3.2 Preliminary estimates

We start by stating a powerful tool introduced by A. L. Bughkeim and M. V. Klibanov in [14]

Lemma 8.3.1. *Let η be given by (6.7). Then, there exists a positive constant $\kappa > 0$, depending only on T, such that we have*

$$\int_0^T \int_\Omega e^{-2s\eta(x,t)} \left| \int_{\frac{T}{2}}^t p(x,\xi)\, d\xi \right|^2 dx\, dt \leq \frac{\kappa}{s} \|e^{-s/\eta} p\|^2_{L^2(Q)},$$

for every $p \in L^2(Q_T)$.

Proof. For $\gamma \in (0,1)$, we define the integral I_γ as folows

$$I_\gamma := \int_\gamma^{T-\gamma} \int_\Omega e^{-2s\eta(x,t)} \left| \int_{\frac{T}{2}}^t p(x,\xi)\, d\xi \right|^2 dx\, dt.$$

From Cauchy-Schwarz inequality, one can easily see that

$$I_\gamma \leq \int_\gamma^{T-\gamma} \int_\Omega e^{-2\eta(x,t)} \left(\int_{\frac{T}{2}}^t |p(x,\xi)|^2\, d\xi \right) \left(t - \frac{T}{2} \right) dx\, dt.$$

Then, in light of (6.7) we get

$$I_\gamma \leq \int_\gamma^{T-\gamma}\int_\Omega e^{-2s\eta(x,t)}\Big(\partial_t\eta(x,t)e^{-2s\eta(x,t)}\Big)\Big(\int_{\frac{T}{2}}^t |p(x,\xi)|^2\,d\xi\Big)\,\frac{t^2(T-t)^2}{2(\alpha-e^{\lambda\psi(x)})}\,dx\,dt.$$

By integrating by parts with respect to t and using the fact that there exists $\alpha_0>0$ such that $\alpha-e^{\lambda_{\psi(x)}}>\alpha_0$, we get

$$I_\gamma \leq \frac{T^4}{2s\alpha_0}\Big(\int_\Omega\int_\gamma^{T-\gamma} e^{-2s\eta(x,t)}\Big|p(x,t)\Big|^2\,dx\,dt+\int_\Omega \tilde{p}(x,\gamma)-\tilde{p}(x,T-\gamma)\,dx\Big), \quad (8.9)$$

where $\tilde{p}(x,t) := e^{-2s\eta(x,t)}\int_{\frac{T}{2}}^t |p(x,\xi)|^2\,d\xi$. Bearing in mind that $\int_\Omega \tilde{p}(x,\tau)\,dx \leq e^{-2s\frac{\alpha_0}{\gamma(T-\gamma)}}\|p\|^2_{L^2(Q)}$ for $\tau=\gamma$ or $T-\gamma$, we obtain by taking the limit as $\gamma\to 0$ in (8.9) the desired result. $\square$

We turn now to establishing the coming statement with the aid of Proposition 6.2.1 and the above lemma.

Lemma 8.3.2. *There exists $s_1>0$ such that for any $s\geq s_1$, we have the following estimate*

$$\begin{aligned} s^3\|e^{-s\eta}w\|^2_{L^2(Q)}+\|P_1e^{-s\eta}w\|^2_{L^2(Q)} \leq C\Big(&\|e^{-s\eta}(a_1-a_2)\|^2_{L^2(Q)^n}+\|e^{-s\eta}(q_1-q_2)\|^2_{L^2(Q)}\\ &+s\,\|e^{-s\eta}\theta^{1/2}(\partial_\nu\psi)^{1/2}\partial_\nu w\|^2_{L^2(\Sigma^+)}\Big),\end{aligned}$$

where C is a positive constant independent of s.

Proof. By applying Proposition 6.2.1 to the solution w, we find a constant $C>0$ such that

$$\begin{aligned} s^3\|e^{-s\eta}w\|^2_{L^2(Q)}&+\|P_1e^{-s\eta}w\|^2_{L^2(Q)}+s\|e^{-s\eta}\nabla w\|^2_{L^2(Q)}\\ &\leq C\Big(\|e^{-s\eta}L\,w\|^2_{L^2(Q)}+s\|e^{-s\eta}\theta^{1/2}(\partial_\nu\psi)^{1/2}\partial_\nu w\|^2_{L^2(\Sigma^+)}\Big),\quad s>s_0,\end{aligned} \quad (8.10)$$

where $Lw(x,t)=(-h(x,t)+2i\chi(t)a_1(x)\cdot\nabla+\chi^2(t)a_1^2(x)+\beta(t)q_1(x))w(x,t)$. Here

$h(x,t)$ is given by the following identity

$$h(x,t) = -2(H'_{a_1} + \beta' q_1)v - (H''_{a_1}(t) + \beta''(t) q_1)u + f_1(q_1 - q_2)(x) + f_2(a_1 - a_2)(x),$$

where

$$f_1(x,t) = \beta'' u_2 + 2\beta' \partial_t u_2 + \beta \partial_t^2 u_2,$$

and

$$\begin{aligned} f_2(x,t) &= \chi''\Big(-2i\nabla - \chi(a_1+a_2) - 2\chi'^2(a_1+a_2) - \chi\chi''(a_1+a_2\Big)u_2 \\ &\quad +2\chi'\Big(-2i\nabla - \chi(a_1+a_2) - 2\chi\chi'(a_1+a_2)\Big)\partial_t u_2 \\ &\quad +\chi\Big(-2i\nabla - \chi(a_1-a_2)\Big)\partial_t^2 u_2. \end{aligned}$$

In view of (8.4), we have $f_j \in \mathcal{C}^0([0,T]; L^\infty(\Omega))$ for $j = 1, 2$. Moreover, it is easy to see that $H'_{a_1} + \beta' q_1$ and $H''_{a_1} + \beta'' q_1$ are bounded operators from $L^2(0,T; H^1(\Omega))$ into $L^2(Q)$. Thus, there exists $C > 0$, independent of s, such that we have

$$\begin{aligned} & s^3 \|e^{-s\eta} w\|^2_{L^2(Q)} + \|P_1 e^{-s\eta} w\|^2_{L^2(Q)} + s\|e^{-s\eta}\nabla w\|^2_{L^2(Q)} \\ &\quad \leq C\Big(\|e^{-s\eta}(a_1 - a_2)\|^2_{L^2(Q)^n} + \|e^{-s\eta}(q_1 - q_2)\|^2_{L^2(Q)} + s\|e^{-s\eta}\theta^{1/2}(\partial_\nu\psi)^{1/2}\partial_\nu w\|^2_{L^2(\Sigma^+)} \\ &\qquad + \sum_{\rho = u,v,w} (\|e^{-s\eta}\rho\|^2_{L^2(Q_T)} + \|e^{-s\eta}\nabla\rho\|^2_{L^2(Q)})\Big). \end{aligned}$$

Therefore, since $\nabla^k u(\,\cdot\,,t) = \int_{\frac{T}{2}}^t \nabla^k v(\,\cdot\,,\tau)\,d\tau$ and $\nabla^k v(\,\cdot\,,t) = \int_{\frac{T}{2}}^t \nabla^k w(\,\cdot\,,\tau)\,d\tau$, for $k = 0, 1$, we deduce from Lemma 8.3.1 that

$$\begin{aligned} & s^3 \|e^{-s\eta} w\|^2_{L^2(Q)} + \|P_1 e^{-s\eta} w\|^2_{L^2(Q)} + s\|e^{-s\eta}\nabla w\|^2_{L^2(Q)} \leq C\Big(\|e^{-s\eta}(a_1 - a_2)\|^2_{L^2(Q)^n} \\ & + \|e^{-s\eta}(q_1 - q_2)\|^2_{L^2(Q)} + s\|e^{-s\eta}\theta^{1/2}(\partial_\nu\psi)^{1/2}\partial_\nu w\|^2_{L^2(\Sigma^+)} + \|e^{-s\eta} w\|^2_{L^2(Q_T)} + \|e^{-s\eta}\nabla w\|_{L^2(Q)}\Big), \end{aligned}$$

for any $s \geq s_0$. Thus, by taking s sufficiently large, we obtain

$$\begin{aligned} s^3 \|e^{-s\eta} w\|^2_{L^2(Q)} + \|P_1 e^{-s\eta} w\|^2_{L^2(Q)} &\leq C\Big(\|e^{-s\eta}(a_1 - a_2)\|^2_{L^2(Q)} + \|e^{-s\eta}(q_1 - q_2)\|^2_{L^2(Q)} \\ &\quad + s\,\|e^{-s\eta}\theta^{1/2}(\partial_\nu\psi)^{1/2}\partial_\nu w\|^2_{L^2(\Sigma^+)}\Big). \end{aligned}$$

This completes the proof of the Lemma. $\square$

8.3.3 The stability estimate

We are now in position to finish the proof of our main result which can be stated as follows

Theorem 8.3.1. *Let $M>0$, let χ and β be as in Theorem 8.2.1 and satisfy 8.2. Let (a_j,q_j), $j=1,2$ be in $\mathcal{S}_M(a_0,q_0)$, where (a_0,q_0) are the same as above. Then, there exists $n+1$ initial conditions $u_{0,k}$, $k=0,...,n$, such that we have*

$$\|a_1-a_2\|_{L^2(\Omega)}+\|q_1-q_2\|_{L^2(\Omega)}\leq C\Big(\sum_{k=0}^{n}\|\partial_\nu\partial_t^2u_{1,k}-\partial_\nu\partial_t^2u_{2,k}\|^2_{L^2(0,T;L^2(\Gamma^+))}\Big).$$

Here $C>0$ is a constant depending only on Ω, T, χ and β and $u_{j,k}$, $j=1,\,2$, is the solution of (8.1) where $u_{0,k}$ is substituted for u_0.

Proof. Putting $\phi(x,t)=e^{-s\eta(x,t)}w(x,t)$ and using the fact that $\phi(x,0)=0$, we get

$$\|\phi(\,\cdot\,,\frac{T}{2})\|^2_{L^2(\Omega)}=\int_0^{\frac{T}{2}}\int_\Omega\partial_t|\phi(x,t)|^2\,dx\,dt=2\,\Re\Big(\int_0^{\frac{T}{2}}\int_\Omega\partial_t\phi(x,t)\overline{\phi(x,t)}\,dx\,dt\Big).$$

Hence, from the Green formula and (6.8) one can see that

$$\begin{aligned}\|\phi(\,\cdot\,,\tfrac{T}{2})\|^2_{L^2(\Omega)} &= 2\,\Im\Big(\int_0^{\frac{T}{2}}\int_\Omega(i\partial_t+\Delta+s^2|\nabla\eta(x,t)|^2)\phi(x,t)\overline{\phi(x,t)}\,dx\,dt\Big)\\ &= 2\,\Im\Big(\int_0^{\frac{T}{2}}\int_\Omega P_1\phi(x,t)\overline{\phi(x,t)}\,dx\,dt\Big).\end{aligned}$$

Therefore, we get from the Cauchy-Schwarz inequality that

$$\begin{aligned}\|\phi(\,\cdot\,,\frac{T}{2})\|^2_{L^2(\Omega)} &\leq 2\|P_1\phi\|^2_{L^2(Q)}\|\phi\|^2_{L^2(Q)} \\ &\leq s^{-3/2}\Big(s^3\|e^{-s\eta}w\|^2_{L^2(Q)} + \|P_1 e^{-s\eta}w\|^2_{L^2(Q)}\Big), \quad s>0. \end{aligned} \tag{8.11}$$

Then, by Lemma 8.3.2, we obtain for all $s \geq s_2$

$$\|\phi(\,\cdot\,,\frac{T}{2})\|^2_{L^2(\Omega)} = 4\chi'(\frac{T}{2})^2\|e^{-s\eta(\,\cdot\,,\frac{T}{2})}(a_1-a_2)\cdot\nabla u_0\|^2_{L^2(\Omega)^n} + \beta'(\frac{T}{2})^2\|e^{-s\eta(\,\cdot\,,\frac{T}{2})}(q_1-q_2)u_0\|^2_{L^2(\Omega)}$$
$$\leq Cs^{-3/2}\Big(\|e^{-s\eta}(a_1-a_2)\|^2_{L^2(Q)^n} + \|e^{-s\eta}(q_1-q_2)\|^2_{L^2(Q)} + s\,\|e^{-s\eta}\theta^{1/2}(\partial_\nu\psi)^{1/2}\partial_\nu w\|^2_{L^2(\Sigma^+)}\Big).$$

Let us now choose the initial conditions u_0 as follows. Pick $\omega \subseteq \Omega$ such that $\omega \supset \Omega\setminus\mathcal{V}$. Then, we choose $u_0 \in C^6_0(\Omega)$ such that $u_0(x) = 1$ for any $x \in \omega$. Taking into account that $q_1 - q_2$ and $a_1 - a_2$ vanish in $\mathcal{V}$ and that $\eta(x, \frac{T}{2}) \leq \eta(x,t)$ for all $x \in \Omega$, we deduce from the last inequality that

$$\begin{aligned}C_1\|e^{-s\eta(\,\cdot\,,\frac{T}{2})}(q_1-q_2)\|^2_{L^2(\Omega)} &\leq Cs^{-3/2}\Big(\|e^{-s\eta(\,\cdot\,,\frac{T}{2})}(a_1-a_2)\|^2_{L^2(\Omega)^n} \\ &\quad + \|e^{-s\eta(\,\cdot\,,\frac{T}{2})}(q_1-q_2)\|^2_{L^2(\Omega)} + s\,\|\partial_\nu w\|^2_{L^2(\Sigma^+)}\Big)\end{aligned} \tag{8.12}$$

Here we used the fact that $\theta e^{-2s\eta}$ and $\partial_\nu\psi$ are bounded on Σ^+. Next, we select n initial conditions $u_{0,k} \in C^6_0(\Omega)$, for $k = 1, ..., n$, such that $u_{0,k} = x_k$ on ω. Then, we get in a similar way

$$\begin{aligned}C_2\|e^{-s\eta(\,\cdot\,,\frac{T}{2})}(a_1-a_2)_k\|^2_{L^2(\Omega)} &\leq Cs^{-3/2}\Big(\|e^{-s\eta(\,\cdot\,,\frac{T}{2})}(a_1-a_2)\|^2_{L^2(\Omega)^n} \\ &\quad + \|e^{-s\eta(\,\cdot\,,\frac{T}{2})}(q_1-q_2)\|^2_{L^2(\Omega)} + s\,\|\partial_\nu w\|^2_{L^2(\Sigma^+)}\Big),\end{aligned} \tag{8.13}$$

where $(a_1-a_2)_k$ denotes the k^{th} component of $a_1 - a_2$. Summing up (8.12) with (8.13) for $k = 1, .., n$, we get for $s \geq s_2$,

$$C_1\|e^{-s\eta(\,\cdot\,,\frac{T}{2})}(q_1-q_2)\|^2_{L^2(\Omega)} + C_2\sum_{j=1}^{n}\|e^{-s\eta(\,\cdot\,,\frac{T}{2})}(a_1-a_2)_j\|^2_{L^2(\Omega)}$$

$$\leq C\,(n+1)\,s^{-3/2}\Big(\|e^{-s\eta(\,\cdot\,,0)}(a_1-a_2)\|^2_{L^2(\Omega)^n}+\|e^{-s\eta(\,\cdot\,,\frac{T}{2})}(q_1-q_2)\|^2_{L^2(\Omega)}+s\,\|\partial_\nu w\|^2_{L^2(\Sigma^+)}\Big).$$

Thus, there exists $s_3>0$ such that for $s\geq s_3$, we have

$$\|e^{-s\eta(\,\cdot\,,\frac{T}{2})}(q_1-q_2)\|^2_{L^2(\Omega)}+\|e^{-s\eta(\,\cdot\,,\frac{T}{2})}(a_1-a_2)\|^2_{L^2(\Omega)^n}\leq Cs^{-1/2}\|\partial_\nu\partial_t^2(u_1-u_2)\|^2_{L^2(\Sigma^+)}.$$

Finally, from the inequality $e^{-s\eta(\,\cdot\,,\frac{T}{2})}\geq e^{-4s\frac{\alpha-1}{T^2}}>0$, we get the desired result. $\square$

We finish by giving some Remarks:

8.4 Concluding remarks

- Notice that as in [**33**] we impose the initial condition $u(\,\cdot\,,\frac{T}{2})=u_0$ in (8.1) at $t=\frac{T}{2}$ and not $t=0$. This allows us to "symmetrize" the solution u to (8.1) around $t=\frac{T}{2}$, and consequently to apply the Carleman estimate of Proposition 6.2.1 to u over $(0,T)$.

- The assumption $\nabla\cdot a=0$ is purely technical and does not restrict the generality of Theorem 8.3.1. Indeed, it is well known that the magnetic potential is not meaningful in physics. The physical relevant quantity is the "two-form" $da=\partial\wedge a$, which coïncides with the magnetic field curl a when $n=3$. Actually, given the "magnetic field" b, we can always choose a divergence free a such that we have $b=da$. This amounts to substituting $a+\nabla\psi$ for a, where $\psi\in H^1(\Omega)$ is solution to the system

$$\begin{cases}-\Delta\psi=\nabla\cdot a & \text{in }\ \Omega,\\ \psi=0 & \text{in }\ \partial\Omega.\end{cases}$$

- As in [**18**] we enforce homogeneous Dirichlet-boundary conditions to (8.1). These homogeneous Dirichlet conditions impose that q be known in the vicinity $\mathcal{V}$ of the boundary $\partial\Omega$. Nevertheless, this condition can be removed upon selecting suitable non-homogeneous Dirichlet boundary conditions on $\partial\Omega$ as in [**3**]

- Similarly, we can remove the assumption that a be known on $\mathcal{V}$ by selecting the initial conditions $u_{0,k}$, for $k = 1, ..., n$, as in [**18**][Theorem 1.1]. Nevertheless, the set of initial conditions cannot be defined explicitly, so we rather stick with the formulation of Theorem 8.3.1 given in this chapter. Nevertheless, in order to avoid the inadequate expense of the site of this chapter, we shall not go further into details in this matter.

- Evidently, it can be checked that if $a_1 = a_2$ then, the electric potential can be Lipschitz stably retrieved from one boundary observation of the solution. This extends the result of Baudouin and Puel [**3**] to the case of a magnetic Laplacian.
  Similarly, if $q_1 = q_2$, we can determine the divergence free magnetic potential for n boundary observations of the solution, which generalises the result of [**18**]

- It is worth mentionning that the stability estimate of Theorem 8.3.1 determines $n + 1$ unknown functions $(a_1, ..., a_n)$ and q from the knowledge of $n + 1$ boundary observations over the time-space $(0, T)$.

- We stress out that the conditions (8.2) imposed on the functions χ and β are essential in order to solve the inverse problem under study in this work. Indeed they allow us to recover the information on a and q from the knowledge of the "initial" condition of the second order derivative of the linearized system associated with 8.1 (see the second line in 8.8).

Conclusion

In this thesis, we were interested in studying the uniqueness and the stability issues for two types of inverse problems concerning mostly coefficients that are depending not only on the space variable but also on the time variable. Our primary focus was to analyse inverse coefficients problems for non-autonomous hyperbolic equations. We deepened the concept of local non-uniqueness in the analysis of inverse problems related to time-dependent hyperbolic operators and we gave a better description of the non-uniquness cloacking area. Moreover, we developped logarithmic stability estimates for the determinations of some coefficients appearing in wave equations. The same type of analysis was carried out for non-autonomous magnetic systems, and we were able to stably recover some coefficients appearing in magnetic Schrödinger equations.

As perspectives, these results can be extended and generalized to the framework of time-fractional partial differential equations.

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[P1] I. Ben Aïcha , *Stability estimate for a hyperbolic inverse problem with time-dependent coefficient*, Inverse problems, 31,12, pp.0266-5611, November,2015.

[P2] I. Ben Aïcha , *Stability estimate for an inverse problem for the Schrödinger equation in a magnetic field with time-dependent coefficient*. hal-01350770.

[P3] M. Bellassoued, I. Ben Aïcha , *Uniqueness for an inverse problem for a dissipative wave equation with time dependent coefficient*. hal-01312954, 2016.

[P4] M. Bellassoued, I. Ben Aïcha , *Stable determination outside a cloaking region of two time-dependent coefficients in an hyperbolic equation from Dirichlet to Neumann map*. arXiv:1605.03466, 2016.

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