Periods and line arrangements: contributions to the Kontsevich-Zagier periods conjecture and to the Terao conjecture

DOCTORAL THESIS

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Introduction

The principal motivation of the present Ph.D subject is the study of certain interactions between number theory, algebraic geometry and dynamical systems. This Ph.D thesis is divided in two different parts: a first one about **periods of Kontsevich-Zagier** and another one about **logarithmic vector fields on line arrangements**. Each of these subjects are dominated by a main conjecture: **the Kontsevich-Zagier period conjecture** and **the Terao conjecture**, respectively. Moreover, a common idea is that a solution to these conjectures seems to lie in a very distant future, due principally to the absence of a tractable or clear strategy of proof. In both cases, we introduce a new point of view making connexion with different fields of research, which allows us to propose a new understanding of these conjectures, as well as new approaches toward their resolution.

The first part concerns a problem of **number theory**, for which we develop a **geometric approach** based on tools coming from **algebraic geometry** and **combinatorial geometry**. The periods of Kontsevich-Zagier are complex numbers expressed as values of integrals of a special form, where both the domain and the integrand are expressed using polynomials with rational coefficients. One of the principal problem of periods is to understand their algebraic relations as numbers. In the spirit of other classical conjectures, the Kontsevich-Zagier period conjecture affirms that any polynomial relation between periods can be obtained by linear relations between the integral representations expressed by classical rules of integral calculus.

The second part deals with the understanding of particular objects coming from **algebraic geometry** with a strong background in **combinatorial geometry**. Using tools from **dynamical systems theory**, we develop a **dynamical approach** for these objects. The logarithmic vector fields are an algebraic-analytic tool used to study sub-varieties and germs of analytic manifolds. We are concerned with the case of **line arrangements** in the affine or projective space. One is interested to study how the combinatorial data of the arrangement determines relations between its associated logarithmic vector fields. This problem is known as the Terao conjecture.

– Part I –
Contributions to the Kontsevich-Zagier periods conjecture

Historically, the **study of numbers** was based on the study of **geometrics relations**. In Ancient Greece, numbers were always considered to be associated to different measures of

geometric objects: lengths of segments and curves, areas and volumes. *Rational numbers* were obtained naturally as a *ratio* between lengths or areas. First examples of *irrational numbers* were obtained by the Pythagoreans constructing squares over a right triangle. In terms of the Pythagorean Theorem, if we impose simple arithmetic relations between the areas, we can obtain irrational lengths. If one looks for π, it seems impossible to dissociate this number from the length of the perimeter of the circumference, or the area of a circle.

Areas of polygons in the plane and volumes of tetrahedra are exhaustively studied in the famous geometric treatise *Euclid's Elements*. In fact, Euclid never defined the notion of length, area or volume. He never needed to associate a numerical value to an area. Euclid's Elements develop the so-called *geometric algebra* where terms were represented by sides of geometric objects, and the relations between them are explored in a geometric way. In particular, Euclid study geometrically congruent triangles, for which there exists a direct displacement in the plane moving a triangle into another one.

Following Euclid, a classical way to study polygons of the same area is via *scissor-congruences*: can we recover two polygons of the same area by given partitions of congruent pieces? An affirmative answer was given by Bolyai, Wallace and Gerwien in the eighteenth century. The same question for 3–polyhedra is exactly Hilbert's third problem, whose answer is negative in general but affirmative if we restrict to polyhedra with same volume and same *Dehn's invariant* [Deh01, Syd65], introduced by Dehn in 1900.

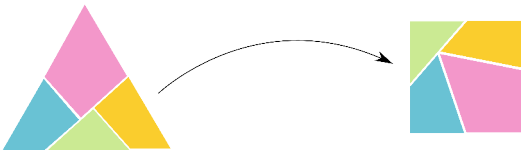

Figure 1: Scissor-congruence between two polygons of same area.

Progressively, this geometric vision of numbers shift to a more analytic and algebraic one, leading to great achievements like the proof of the transcendence of π, e, etc. A good overview of analytic number theory can be found in the classical textbook of Hardy and Wright [HW08].

In this thesis, considering a large class of numbers called *periods*, introduced by M. Kontsevich and D. Zagier in their seminal article [KZ01], we are able to **recover this interaction between numbers and geometry** using more rich geometric objects, called **semi-algebraic sets**. We study then periods and their **relations from a geometrical point of view**.

A *period* is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over real domains defined by polynomials with rational coefficients. Most of the important constants in mathematics and physics are periods, as well as algebraic numbers, π or logarithms of algebraic numbers. This vast class of numbers forms an algebra over the real algebraic numbers and posses an inherit constructible nature.

As the definition of this numbers is "simple", one would ask if the type of relations between them are also "simple". This idea is expressed in the so-called *Kontsevich-Zagier period conjecture*, which states that any relation between periods comes from linearity and adequate transformations on the integral, using classical operations of integral calculus. More specifically, if a real period admits two integral representations, then we can pass from one formulation to the other one using only three operations of integral calculus (called the *KZ–rules*): integral sums by domains or integrands, change of variables and the Stokes formula. When such a path

between different integrals can be found and is also *simple and explicit*, we refer to the notion of *accessible identities*. The previous conjecture is an "affine" and "more computational" version of a conjecture due to A. Grothendieck [Gro66]. A modern statement of *Grothendieck's period conjectured* is described in terms of *motives* [And04]. Another related question is to search an algorithm which allows to prove if two periods are equal of not.

In this thesis, we develop a **geometric approach of this numbers and its related problems** by reinterpreting them as **volumes of compact semi-algebraic domains**, focusing on obtaining **algorithmic and constructive methods**.

In order to obtain such an approximation, we first give an **algorithmic procedure to express any non-zero period** given by a certain integral form **as a volume of a (non-unique) compact semi-algebraic set**. We give then a **reinterpretation of the Kontsevich-Zagier period conjecture** in terms of a **geometrical problem**: if we have two compact semi-algebraic sets of same volume, then we can transform one to the other by a sequence of scissor-congruences and algebraic volume-preserving transformations, up to Cartesian products by unit intervals. Using triangulations of semi-algebraic sets, one can relate it to a **generalized Hilbert's third problem** for polyhedra equipped with piecewise algebraic forms, for which some partial results are already known. Following J. Wan [Wan11], a notion of **geometric complexity for periods** based on the minimal polynomial complexity of the semi-algebraic sets representing a period can be also derived. In the same spirit, another notion of **arithmetic complexity** is also introduced, coming from generalizations of complexity measures for multivariate polynomials.

The structure of this part consist in five chapters developed as follows.

We give a general presentation of periods in Chapter I, detailing the origin of this numbers as comparison data between algebraic and geometrical cohomologies of algebraic varieties. We introduce the basic tools to deal with semi-algebraic geometry in real algebraic coefficients. Then, we are able to present the class of periods of Kontsevich-Zagier, as well as some examples and their first properties based in geometrical methods. Straightaway, we introduce the principal problems asking for the type of relations between periods: the Kontsevich-Zagier period conjecture, illustrating it by some examples. We also explain other problems concerning *accessible identities* and the determination of an *equality algorithm* for periods. Finally, we discuss about related arithmetic problems and transcendence of periods. Then, we describe the work of M. Yoshinaga [Yos08] about periods and elementary computable numbers.

We detailed in Chapter II our principal result in order to construct our geometric approach: an algorithmic procedure to rely any non-zero period to the volume of a compact semi-algebraic set. This procedure is called the *semi-canonical reduction procedure for non-zero periods*. We first study the compactification of domains by defining the projective closure of a semi-algebraic set. Then, using resolution of singularities, we resolve geometrically the poles of the integrand placed in the boundary of the domains. We describe the way to give a partition of the domain by affine compact sets on the resolution variety. As a consequence, we prove that periods can be expressed as the difference of volumes of two compact semi-algebraic sets. A more computable method is given for integrals defined over the real plane. We conclude proving that we can express the difference of the volumes of two compact semi-algebraic as a single compact semi-algebraic set constructed algorithmically from the other two. Some examples illustrate how this reduction can be obtained.

Chapter III deals with possible reformulations of the Kontsevich-Zagier conjecture. We first discuss the use of the Stokes and Fubini's formula in the actual conjecture. Taking advantage of the semi-canonical reduction, we introduce a geometric Kontsevich-Zagier's problem for periods.

This problem is described in terms of scissor-congruences and volume-preserving transformations for compact semi-algebraic sets of same volume. Using triangulations of semi-algebraic sets, we formulate an equivalent problem for rational polyhedra: the generalized Hilbert's third problem for polyhedra. We discuss how this approach can be useful for the comprehension of the Kontsevich-Zagier period conjecture. Using known results in combinatorial geometry, we derive some partial results towards the generalized Hilbert's third problem.

We complete a degree theory for periods in Chapter IV, coming from the minimal dimension on the set of semi-canonical reductions of a period. Following this notion, a natural filtration of the algebra of periods is defined. We focus on a geometrical criterion of transcendence for periods in terms of degree. Then, we reproduce the main properties of the degree, as well as some results in transcendence and linear independence of complex numbers. We illustrate the main difficulties on the computation of the degree by taking powers of π. Supported by these examples, we discuss the effectiveness of this criterion to differentiate periods. In order to define a notion of complexity for periods, we first introduce a geometric complexity measure for semi-algebraic sets. We complete this notion proposing the logarithmic Mahler measure as an arithmetical notion of complexity. We show that this notion of complexity distinguish different transcendental periods of same degree.

Finally, we give some perspectives relatives to this part in Chapter V. In particular, we outline the *decidability* of the Kontsevich-Zagier conjecture by using Yoshinaga's work and the semi-canonical reduction for periods. We study the construction of a theoretical algorithm which takes a pair of periods as input: if the two periods are different numbers, the algorithm stops in a finite time indicating this fact; in the case when the two periods are equal, the algorithm stops in a finite time giving as output the passage between the two integral representations if and only if the Kontsevich-Zagier conjecture is true.

– Part II –
Contributions to Terao's conjecture on line arrangements

This part of this Ph.D concerns the **interaction between** two fields of study in mathematics which are usually disconnected: **algebraic geometry and dynamical systems**. An example of such a situation is given by works around **line arrangements**, which are finite collections of lines in the affine or projective plane, and **polynomial vector fields**. Indeed, since the seminal work of H. Terao [Ter80] this study incorporates a powerful tools based on a special type of vector fields, called *logarithmic vector fields*. In the other hand, a classical theme of dynamical systems is the study of differential polynomial systems via their algebraic or analytic invariant curves and their dynamical consequences, as for example *Darboux integrability* [Dar78].

Roughly speaking, the aim of this part is to promote a new **point of view on the theory of logarithmic vector fields using dynamical system theory**, and more generally, a sketch of a dynamical approach to geometry. Specifically, we look for a **dynamical approach on the study of the module of logarithmic vector fields for line arrangements**.

One of the main conjectures concerning line arrangements is the so-called *Terao's conjecture*, dealing with the interplay between polynomial relations of logarithmic vector fields and the combinatorial data of a line arrangement. Precisely, the Terao conjecture is concerned with the algebraic properties of the *module of logarithmic vector fields*, defined as the polynomial module of derivations which keep stable the ideal generated by the defining polynomial of the

arrangement. This conjecture can be formulated as follows: the freeness of the module of logarithmic vector fields is of combinatorial nature. Despite many works concerning the freeness of arrangements and the very simple criterion given by K. Saito, this notion is not well-understood. In particular, as pointed by P. Cartier in his Bourbaki's seminar in 1981 [Car81, p. 19], we do not know the exact geometrical interpretation of freeness.

The principal guideline of our work is **to determine the impacts of freeness and combinatorics on the dynamics of logarithmic vectors fields**.

In order to achieve this goal, we have first **interpreted logarithmic vector fields using** the notion of **dynamical invariance**. Doing so, natural questions about the relations between equilibrium points of logarithmic vector fields and singularities of a line arrangement appear, as well as the **possible global qualitative dynamical behavior** of such a vector fields and its **connexion with the combinatorial and geometric properties of the arrangement**. Specifically for affine line arrangement in the plane, where freeness is not an issue, only the role of the combinatorics is questionable. We then study the following:

Question. *Is the module of logarithmic vector fields of an affine line arrangement determined by its combinatorics?*

Our strategy to solve this problem is to fix a given line arrangement, more precisely a given combinatorics, and to determine what are the constraints induced on its module of logarithmic vector field. If we look for invariants sets, we study when a vector field possess a maximal collection of invariant lines including the arrangement. In this perspective, a special role is played by polynomial vector fields fixing infinitely many lines: *radial* and *parallel* vector fields (Theorem VIII.3.11). We prove that the minimal degree where such a vector fields appears in the module of logarithmic derivations depends on the combinatorics (Propositions VIII.3.11 and VIII.3.16). Thus, we are interested in studying the appearance of derivations by degree only fixing a finite number of lines besides the arrangement, called *finite derivations*. This rises to the notion of *minimal degree* of finite derivations. An idea is then to search for combinatorially equivalent arrangements which have different minimal degree of finite derivations (Theorem VIII.4.5). This search is made more tractable by a series of results relying combinatorial properties and the minimal degree, summarized in Corollaries VIII.3.10 and VIII.3.19. Then, we give a negative answer to the previous Question in Theorem VIII.1.3 by giving two pairs of affine line arrangements, each pair corresponding to different notions of combinatorics, for which the module of logarithmic vector field has different minimal degree.

This first result gives an strong support on our dynamical approach of the Terao conjecture.

This second part is composed of four chapters in the following way.

We introduce in Chapter VI the basic notions and tools on the study of hyperplane arrangements via logarithmic vector fields, focusing on affine line arrangements. We explain the principal geometrical properties as well as their relations with the notions of combinatorics for an arrangement. Our principal object of study, the module of logarithmic derivations of an arrangement, is then discussed: we recall its principal properties an its relation with logarithmic differential forms. This allows us to introduce *free arrangements* and then Terao's conjecture for central arrangements, as well as its sheafificated version in terms of logarithmic vector bundles for projective line arrangements.

Chapter VII is concerned to the dynamical interpretation of logarithmic vector fields, introduced by K. Saito [Sai80]. We first gives a dynamical interpretation of logarithmic vector fields in terms of dynamically invariant sets, discussing on the differences between analytic and algebraic

vector fields. This allows us to present a general dynamical approach to the study of geometrical objects based on the previous ideas. We center our approach in the study of configurations of curves considering algebraic logarithmic vector fields in the plane. This allows to reverse the classical point of view for two classical problems of real planar vector fields: the Dulac conjecture and the algebraic 16th Hilbert problem. Afterwards, some relations between logarithmic vector fields and specific dynamical notions of invariant algebraic curves of polynomial vector fields, as first integrals, Darboux integrability and Lie brackets are discussed. We conclude this chapter by dealing with the case of curves defined by homogeneous equations, giving relations in terms of logarithmic vector fields between affine line arrangements and central plane arrangements in $\mathbb{A}^3_k$ as well as projective line arrangements in $\mathbb{P}^2_k$.

We apply our dynamical approach in the case of affine line arrangements in Chapter VIII, via polynomial vector fields fixing the arrangement. A study by degree of derivation requires to introduce a filtration of the module of logarithmic vector fields, which is respected by the polynomial automorphisms of the affine plane. Following the dynamical point of view, we characterize vectors fields fixing infinitely many lines in $\mathbb{A}^2_k$. Studying the influence of the combinatorics of the arrangements over such logarithmic vector fields, we give a lower bound for the minimal degree of non-zero logarithmic derivations only fixing a finite set of lines, including the arrangement. These facts allows us to give two pair of counter-examples which show that the module of logarithmic vector field of an affine line arrangement is not determined by the weak or strong combinatorics. We analyze the combinatorics and the strata of the logarithmic filtration by creating a program in `Sage`, which is completely described in Appendix B. Finally, we study a quadratic formula depending on the degree for the ranks of the strata in the filtration, and how this formula is determined by the combinatorics of the line arrangement.

We conclude this part with Chapter IX, where we give a conclusion and we outline the perspectives and future work in the development on a dynamical approach for freeness of arrangements as well as the Terao conjecture.

Introduction (French)

La principale motivation de ce sujet de thèse est l'étude de certaines interactions entre la théorie des nombres, la géométrie algébrique et les systèmes dynamiques. Cette mémoire est divisé en deux parties : la première porte sur **les périodes de Kontsevich-Zagier** et la seconde sur **les champs de vecteurs logarithmiques des arrangements de droites**. Chacun de ces sujets est dominé par une conjecture principale : **la conjecture des périodes de Kontsevich-Zagier** et **la conjecture de Terao** respectivement. De plus, une idée commun entre ces conjectures est qu'elles ne semblent pas être sur le point d'être résolues, principalement dû au faite qu'il n'existe pas des stratégies de preuve. Dans chacun des cas, nous introduisons un nouveau point de vu reliant différents domaines de recherches, nous permettant ainsi une nouvelle compréhension des ces conjectures, ainsi qu'une nouvelle approche de résolution.

La première partie concerne un problème de **théorie des nombres**, pour lequel nous développons une **approche géométrique** basé sur des outils provenant de la **géométrie algébrique** et de la **combinatoire géométrique**. Les périodes de Kontsevich-Zagier sont des nombres complexes exprimés comme les valeurs d'intégrales d'un forme particulière, où le domaine et l'intégrande peuvent être exprimés par des polynômes à coefficients rationnels. L'un des principaux problèmes pour les périodes est de comprendre leurs relations algébriques en tant que nombres. Dans l'esprit des autres conjectures classiques, la conjecture des périodes de Kontsevich-Zagier affirme que toutes les relations polynomiales existant entre des périodes peuvent être obtenues à partir de relations linéaires provenant des règles classiques de calcul intégrale entre les représentations intégrales.

La seconde partie traite de la compréhension d'objets provenant de la **géométrie algébrique** avec une forte connexion avec la **combinatoire géométrique**. En utilisant des outils provenant de la **théorie des systèmes dynamiques**, nous avons développé une approche dynamique de ces objets. Les champs de vecteurs logarithmiques sont un outil algébro-analytique utilisé dans l'étude des sous-variétés et des germes dans des variétés analytiques. Nous nous sommes concentré sur le cas des **arrangements de droites** dans des espaces affines ou projectifs. On s'est plus particulièrement intéressé à comprendre comment la combinatoire d'un arrangement détermine les relations entre les champs de vecteurs logarithmiques associés. Ce problème est

connu sous le nom de conjecture de Terao.

– Partie I –
Contributions à la conjecture des périodes de Kontsevich-Zagier

Historiquement, **l'étude des nombres** est basée sur l'étude des **relations géométriques**. Dans la Grèce Antique, les nombres ont toujours été considéré comme associés à des mesures d'objets géométriques : la longueur d'un segment ou d'une courbe, des aires ou encore de volumes. Les nombres rationnels ont été obtenus de façon naturelle comme *ratio* entre des longueurs ou des aires. Les premiers exemples de *nombres irrationnels* ont été obtenus par les Pythagoriciens en construisant des carrés sur les côtés d'un triangle rectangle. Du point de vu du Théorème de Pythagore, si nous imposons des relations arithmétiques simples entre les aires, nous pouvons obtenir des longueurs irrationnelles. Si on regarde le nombre π, il semble impossible de le dissocier de la longueur du périmètre ou de l'aire d'un cercle.

L'aire des polygones dans le plan et le volume des tetraedres sont exhaustivement étudié dans le fameux traité de géométrique : *Les Éléments d'Euclide*. En fait, Euclide n'a jamais défini la notion de longueur d'aire ou de volume. Il n'a jamais eu besoin d'associer une valeur numérique à une aire. Les Éléments d'Euclide développent l'*algèbre géométrique* où les valeurs sont représentés à côtés des objets géométriques, et les relations sont explorer de façon géométrique. En particulier, Euclide étude les triangles congruents géométriquement, pour lesquels il existe un déplacement direct du plan déplaçant un triangle sur l'autre.

D'après Euclide, un voie classique pour étudier les polygones de même volume est via des opérations de congruences–ciseaux : peut-on recouvrir deux polygones de même aire avec des partitions de pièces congruentes ? Un réponse affirmative fut donné par par Bolyai, Wallace et Gerwien dans le dix-huitième siècle. Cette même question pour les 3–polyèdres correspond exactement au 3ème problème de Hilbert, dont la réponse est négative en générale, mais affirmative si on se restreint aux polyèdres possédant le même volume et le même *invariant de Dehn* [Deh01, Syd65], introduit par Dehn en 1900.

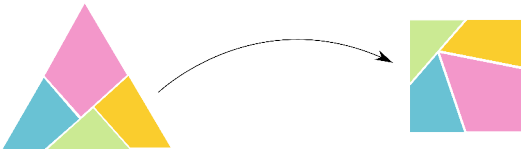

Figure 2: Opération de congruences–ciseaux entre deux polygones de même aire.

Progressivement, cette vision géométrique d'un nombre s'est changée en une vision plus analytique ou algébrique, permettant ainsi l'obtention de grands résultats tels la transcendance de π, e, etc. Une bonne présentation de la théorie analytique des nombres peut être trouvée dans le livre de Hardy et Wright [HW08].

Dans cette thèse, en considérant une large classe de nombres appelés *périodes*, introduites par M. Kontsevich et D. Zagier dans leur article fondateur [KZ01], nous sommes capables de **retrouver cette interaction entre nombres et géométrie** en utilisant des objets géométriques

plus riches, appelés **ensembles semi-algébriques**. Nous introduisons alors les périodes et **leurs relations avec le point de vu géométrique**.

Une *période* est un nombre complexe dont la partie réelle et la partie imaginaire peuvent être obtenues comme l'évaluation d'intégrales absolument convergentes de fonctions rationnelles à coefficients rationnels sur un domaine défini par des polynômes à coefficients rationnels. La plupart des constantes mathématiques et physique, tels les nombres algébriques, π ou encore les logarithmes de nombres algébriques sont des périodes. Cette classe de nombres très vaste forme une algèbre sur les nombres réels algébrique et possèdent une nature constructible inhérente provenant de leur définition.

Comme la définition de ces nombres est "simple", on peut se demander si les relations entre eux elle aussi "simple". Ce nous ramène à la *conjecture des périodes de Kontsevich-Zagier*, qui affirme que les relations entre les périodes proviennent de la linéarité et des transformations adéquates des intégrales, en utilisant des opérations classiques du calcul intégral. Plus particulièrement, si une période réelle admets deux représentations intégrales, alors elles sont reliées par une suite d'opérations nous permettant de passer d'une forme à l'autre en utilisant trois opérations usuelles du calcul différentiel (appelés les *KZ–règles*) : sommes d'intégrales par domaines ou par intégrandes, changement de variables et formule de Stokes. Lorsqu'un tel cheminement existe pour passer d'une forme à l'autre et qu'il est *simple et explicite*, on parle alors d'*identités accessibles*. La conjecture précédente est une version "affine" et "plus calculatoire" d'une conjecture due à A. Grothendieck [Gro66]. Un énoncé plus moderne de la *conjecture de Grothendieck pour les périodes* est décrite en terme de *motifs* [And04]. Un autre problème est de chercher un algorithme qui permettrait de déterminer si deux périodes sont égales ou non.

On développe une **approche géométrique de ces nombres et des problèmes associés** en les ré-interprétant comme des **volumes de domaines semi-algébriques**, en se concentrant sur le fait d'obtenir une **méthode algorithmique et constructive**.

Afin d'obtenir une telle approximation, nous présentons d'abord une **procédure algorithmique pour exprimer toute période non nulle** donnée par une certaine forme intégrale **comme un volume d'un ensemble semi-algébrique compact (non unique)**. Nous donnons ensuite une **réinterprétation de la conjecture des périodes de Kontsevich-Zagier** en termes d'un **problème géométrique** : si deux ensembles semi-algébriques compactes ont même volume, peut-on transformer l'un en l'autre en utilisant une suite d'opérations de congruences–ciseaux et de transformations algébriques préservant le volume ? En utilisant les triangulations des ensembles semi-algébriques, on peut obtenir une **généralisation du 3ème problème de Hilbert** pour les polyèdres munis de formes algébriques par morceaux, pour lesquels des résultats partiels sont connus. En suivant J. Wan [Wan11], une notion de **complexité géométrique pour les périodes** basée sur la complexité du polynôme minimal de l'ensemble semi-algébrique représentant la période peut être obtenue. Dans le même esprit, une autre notion de **complexité arithmétique** est aussi introduite, elle provient d'une généralisation des mesures de complexités pour les polynômes à plusieurs variables.

La structure de cette partie est formée de cinq chapitres comment présenté ci-dessous.

Nous donnons une présentation générale des périodes dans le Chapitre I, détaillant l'origine de ces nombres comme information de la comparaison entre les cohomologies algébrique et géométrique de variétés algébriques. Nous introduisons les outils basiques pour travailler avec la géométrie des ensembles semi-algébriques à coefficients algébriques réels. Ensuite, nous aurons les outils nécessaires à présenter la classe des périodes de Kontsevich-Zagier, ainsi que quelques exemples et leurs premières propriétés basées sur des méthodes géométriques. Ensuite,

nous introduisons le problème principal interrogeant le type de relations entre les périodes : la conjecture des périodes de Kontsevich-Zagier, que nous l'illustrons par quelques exemples. Nous expliquons également d'autres problèmes concernant l'*existence d'identités accessibles* et la détermination d'un *algorithme d'égalité* pour les périodes. Pour finir, nous discutons des problèmes arithmétiques et de transcendance des périodes. Enfin, nous décrivons les travaux de M. Yoshinaga [Yos08] sur les périodes et les nombres calculables élémentaires.

Nous détaillons au Chapitre II notre principal résultat dans le but de construire une approche géométrique : une procédure algorithmique reliant n'importe quelle période non nulle au volume d'un ensemble semi-algébrique compacte. Cette procédure est appelée *procédure de réduction semi-canonique* des périodes. Nous étudions tout d'abord la compactification des domaines en définissant une clôture projective d'un ensemble semi-algébrique. Ensuite, en utilisant des résolutions des singularités, nous résolvons géométriquement les pôles de l'intégrande sur le bord du domaine. Nous donnons alors la façon construire une partition du domaine par des ensembles affines compactes sur la variété de résolution. Comme conséquence, nous prouvons que les périodes peuvent être exprimées comme la différence de volumes de deux ensembles semi-algébriques compactes. Une méthode plus efficace est donnée pour les intégrales définies sur le plan réel. Nous concluons en prouvant que l'on peut exprimer cette différence de volumes de deux ensembles semi-algébriques compactes comme le volume d'un seul ensemble semi-algébrique compacte construit de façon algorithmique à partir des deux précédents. Quelques exemples de calculs de ces réductions seront aussi donnés.

Le Chapitre III est consacré à des possibles reformulations de la conjecture de Kontsevich-Zagier. Nous discutons tout d'abord de l'utilisation de la formule de Stokes et de la formule de Fubini dans la conjecture actuelle. En tirant partie de la réduction semi-canonique, nous introduisons un problème géométrique de Kontsevich-Zagier pour les périodes. Ce problème est décrit en termes d'opérations de congruences–ciseaux et de transformations préservant les volumes pour les ensembles semi-algébriques compactes de même volume. En utilisant la triangulation des ensembles semi-algébriques, nous formulons un problème équivalent pour des polyèdres rationnels : le 3ème problème de Hilbert généralisé pour les polyèdres. Nous discutons comment cette approche peut être utile pour une compréhension de la conjecture des périodes de Kontsevich-Zagier. En utilisant des résultats déjà connus de géométrie combinatoire, nous dérivons des résultats partiels sur le 3ème problème de Hilbert généralisé.

Nous complétons une théorie du degré pour les périodes dans le Chapitre IV, provenant de la dimension minimale parmi les réductions semi-canoniques d'une période. Cela induit une filtration naturelle sur l'ensemble des périodes. Nous nous intéressons ensuite à un critère géométrique de transcendance donnée par le degré. Ensuite, nous reproduisons les propriétés principales de degré, de la transcendance et de la dépendance linéaire. Nous illustrons les principales difficultés de calcul du degré en prenant pour exemples les puissances de π. Ils nous ouvriront la voie vers une discussion sur l'efficacité de ce critère de différenciation des périodes. Dans le but de définir une notion de complexité pour les périodes, nous introduirons d'abord une mesure de complexité géométrique pour les ensembles semi-algébriques. Nous complétons cette notion en proposant la mesure logarithmique de Mahler comme notion arithmétique de la complexité. Nous prouvons que cette notion de complexité permet de distinguer différentes périodes transcendantes de même degré.

Enfin, nous donnons les perspectives de cette première partie dans le Chapitre V. En particulier, nous analysons la *décidablilité* de la conjecture de Kontsevich-Zagier en utilisant les travaux de Yoshinaga et la réduction semi-canonique des périodes. Nous construisons un algorithme théorique prenant une paire de périodes en entré, si les deux périodes sont distinctes alors l'algorithme s'arrête en un temps fini en indiquant ce fait, et si les périodes sont égales l'algo-

rithme s'arrête également en un temps fini et retourne le passage entre les deux représentations intégrales si et seulement si la conjecture de Kontsevich-Zagier est vraie.

– Partie II –
Contributions à la conjecture de Terao sur les arrangements de droites

Cette partie de ma thèse concerne les interactions entre deux domaines d'étude habituellement disjoins en mathématiques : la **géométrie algébrique** et les **systèmes dynamiques**. Un exemple d'une telle situation est donné dans les travaux portant sur les arrangements de droites –définis comme des collections finies de droites dans le plan affine ou projectif– et les champs de vecteurs polynomiaux. En effet, les travaux fondamentaux de H. Terao [Ter80] ont introduit un outil puissant basé sur un genre particulier de champs de vecteurs, appelés les *champs de vecteurs logarithmiques*. D'un autre côté, un thème classique des systèmes dynamiques est l'étude des systèmes différentiels polynomiaux via leurs courbes invariantes (qu'elles soient algébriques ou analytiques) ainsi que leurs conséquences au niveau de la dynamiques, par exemple l'*intégrabilité au sens de Darboux* [Dar78].

À proprement parler, le but principal de cette partie est de promouvoir un nouveau **point de vu sur la théorie des champs de vecteurs logarithmiques en utilisant la théorie de systèmes dynamiques**, et plus généralement, d'esquisser une approche dynamique de la géométrie. Nous nous intéresserons plus spécifiquement à une **approche dynamique de l'étude du module des champs de vecteurs logarithmiques dans le cas des arrangements de droites**.

L'une des conjectures majeures concernant les arrangements de droites est la *conjecture de Terao*, annonçant un lien entre les relations polynomiales des champs de vecteurs logarithmiques et la combinatoire d'un arrangement de droites. Plus précisément, cette conjecture traite avec les propriétés algébriques du *module des champs de vecteurs logarithmiques*, définis comme le module des dérivations laissant stable l'idéal engendré par le polynôme de définition de l'arrangement. Cette conjecture peut être formulée de la façon suivante : la liberté du module des champs de vecteurs logarithmiques est déterminée par la combinatoire de l'arrangement. En d'épis du grand nombre de travaux portant sur la liberté des arrangements ainsi que l'existence d'un critère simple, donné par K. Saito, cette notion est encore mal comprise. En particulier, comme l'a pointé P. Cartier lors de son séminaire Bourbaki en 1981 [Car81, p. 19] : ≪ La signification géométrique de l'hypothèse de "liberté" reste obscure ≫.

Le principal fil directeur de notre travail est de **déterminer l'impacte de la liberté et de la combinatoire sur la dynamique des champs de vecteurs logarithmiques**.

Afin d'atteindre notre objectif, nous avons d'abord **interprété les champs de vecteurs logarithmiques** en utilisant la notion d'**invariance dynamique**. Une fois cela effectué, des questions naturelles sur les relations entre les points d'équilibre des champs de vecteurs logarithmiques et les singularités de l'arrangement sont apparues. De même nous nous sommes interrogés sur une **possible compréhension d'un comportement global et qualitatif du comportement dynamique** de tels champs de vecteurs ainsi que sa **connexion avec la combinatoire et les propriétés géométriques de l'arrangement**. Dans le cas plus spécifique des arrangements de droites dans le plan affine, où la liberté n'est pas mis en question, seul le rôle de la

combinatoire est intéressant à déterminer. Nous avons donc étudié la question suivante :

Question. *Le module des champs de vecteurs logarithmiques d'un arrangement de droites affine est-il déterminé par la combinatoire de l'arrangement ?*

Notre stratégie pour résoudre ce problème est de fixer un arrangement de droites donné, plus précisément sa combinatoire, et de déterminer les contraintes induites sur le module de champs de vecteurs logarithmiques. Si nous regardons les ensembles invariants, nous cherchons à savoir quand un champ de vecteurs possède une collection maximal de droites invariantes incluant notre arrangement fixé. Dans cette perspective, un rôle particulier est joué par les champs de vecteurs fixant une infinité de droites : les champs de vecteurs *radiaux* et *centraux* (Théorème VIII.3.11). Nous prouvons ensuite que le degré minimal pour lequel de tels champs de vecteurs apparaissent dans le module des champs de vecteurs logarithmiques est déterminé par la combinatoire (Propositions VIII.3.11 et VIII.3.16). Ensuite, nous nous intéressons à l'apparition, en fonction du degré, de champs de vecteurs ne fixant qu'un nombre fini de droite. Tels champs de vecteurs sont appelés champs de vecteurs finis. Cela nous amène donc à la notion de degré minimal des champs de vecteurs finis. L'idée est alors de chercher des arrangements combinatoirement équivalents ayant des degrés minimaux de champs de vecteurs finis différents (Théorème VIII.4.5). Cette recherche est facilité par une série de résultats reliant les propriétés combinatoires et le degré minimal, elle est résumé dans les Corollaires VIII.3.10 and VIII.3.19. Ainsi, nous obtenons une réponse négative à la question posée précédement avec le Théorème VIII.1.3 donnant deux paires d'arrangements de droites affines, chacune correspondant à des notions différentes de combinatoires, pour lesquelles le module des champs de vecteurs logarithmiques ont des degrés minimaux différents.

Ce premier résultat nous fourni un support solide pour une approche dynamique de la conjecture de Terao.

Cette deuxième partie est composée de quatre chapitres dans l'ordre suivant.

Nous introduisons au Chapitre VI les notions basiques ainsi que les outils nécessaires à l'étude des arrangements d'hyperplans via les champs de vecteurs logarithmiques. Nous expliquons les principales propriétés géométriques ainsi que leurs relations avec la notion de combinatoire d'un arrangement. Notre principal objet d'étude, le module des champs de vecteurs logarithmiques d'un arrangement, y est donc présenté : nous rappelons ces principales propriétés et ses relations avec les formes différentielles logarithmiques. Cela nous permet d'introduire la notion d'*arrangements libres*, et d'énoncer la conjecture de Terao pour les arrangements centraux, ainsi que sa version en langage de faisceaux exprimé en termes de fibrés vectoriels logarithmiques pour les arrangements projectifs de droites.

Le Chapitre VII concerne l'interprétation dynamique des champs de vecteurs logarithmiques introduits par K. Saito [Sai80]. Nous donnons tout d'abord une première interprétation en termes d'ensembles dynamiquement invariants, en discutant sur les différences entre champs de vecteurs analytiques ou algébriques. Cela nous permet donc de présenter une approche dynamique générale de l'étude d'objets géométriques basée sur l'idée précédente. Nous centrons alors notre approche sur l'étude des configurations de courbes en considérant les champs de vecteurs logarithmiques du plan. Cela nous permet d'inverser le point du vue classique de deux problèmes classiques des champs de vecteurs polynomiaux réels : la conjecture de Dulac et le 16ème problème de Hilbert. Après cela, quelques relations entre les champs de vecteurs logarithmiques et des notions spécifiques de dynamique des courbes algébriques invariantes par des champs de vecteurs polynomiaux (telles les intégrales premières, l'intégrabilité au sens de Darboux, et les crochets de Lie) sont discutés. Nous concluons ce chapitre en abordant le cas

des courbes définies par des équations homogènes, donnant alors une relation entre les arrangements de droites affine et les arrangements de plans centraux dans $\mathbb{A}^3_k$ ainsi que les arrangements projectifs de droites dans $\mathbb{P}^2_k$.

Nous appliquons notre approche dynamique au cas des arrangements de droites affines dans le Chapitre VIII, via les champs de vecteurs polynomiaux fixant l'arrangement. Une étude par le degré des champs de vecteurs demande l'introduction d'une filtration du module des champs de vecteurs logarithmiques qui est respectée par les automorphismes polynomiaux du plan affine. En suivant le point de vu dynamique, nous caractérisons les champs de vecteurs fixant une infinité de droites dans $\mathbb{A}^2_k$. En étudiant l'influence de la combinatoire d'un arrangement sur tels champs de vecteurs logarithmiques, nous donnons une borne inférieure pour le degré minimal des champs de vecteurs non nuls fixant uniquement un nombre fini de droites (dont notre arrangement). Cela nous permet alors de donner deux paires de contre-exemples montrant que le module des champs de vecteurs logarithmiques d'un arrangement affine de droites n'est pas déterminé par le combinatoire faible ou forte. Nous analysons la combinatoire et les espaces filtrés en créant un programme sur `Sage` décrit complètement dans l'Annexe B. Enfin, nous étudions une formule quadratique –dépendant du degré– pour le rang des strates dans la filtration, et comment cette formule est déterminée par la combinatoires de l'arrangement.

Nous finissons cette partie avec le Chapitre IX, dans lequel nous concluons notre étude des champs de vecteurs logarithmiques et donnons également les perspectives et les futurs travaux permettant le développement de cette approche dynamique de la notion de liberté d'un arrangement et donc de la conjecture de Terao.

Introducción (Spanish)

La principal motivación de la presente tesis es el estudio de ciertas interacciones entre teoría de números, geometría algebraica y sistemas dinámicos. Esta tesis está divida en dos partes diferentes: una primera sobre **periodos de Kontsevich-Zagier** y otra sobre **campos de vectores logarítmicos en configuraciones de rectas**. Cada uno de estos temas se encuentra dominado por una conjetura principal: **la conjetura de periodos de Kontsevich-Zagier** y la **conjetura de Terao**, respectivamente. Más aún, la idea comúnmente aceptada es que la solución a estas conjeturas yace en un futuro muy distante, debido principalmente a la ausencia de una estrategia de demostración tratable o clara. En ambos casos, introducimos un nuevo punto de vista realizando conexiones entre diferentes campos de investigación, lo cual nos permite proponer nuevas formas de comprender estas conjeturas, así como nuevos enfoques hacia su resolución.

La primera parte concierne un problema de **teoría de números**, para el cual desarrollamos un **enfoque geométrico** basado en herramientas provenientes de la **geometría algebraica** y la **geometría combinatoria**. Los periodos de Kontsevich-Zagier son números complejos que pueden ser expresados como valores de integrales de una forma especial, donde tanto el dominio como el integrando estás expresados por polinomios con coeficientes racionales. Uno de los principales problemas sobre los periodos es entender sus relaciones algebraicas como números. Siguiendo el espíritu de otras conjeturas clásicas, la conjectura de periodos de Kontsevich-Zagier afirma que cualquier relación polinómica entre periodos puede ser obtenida a través de relaciones lineales entre las distintas representaciones integrales, expresadas por reglas clásicas del cálculo integral.

La segunda parte trata sobre la comprensión de ciertos objetos provenientes de la **geometría algebraica** con un fuerte trasfondo en **geometría combinatoria**. Utilizando herramientas de **sistemas dinámicos**, desarrollamos un **enfoque dinámico** para tales objetos. Los campos de vectores logarítmicos son herramientas algebraico-analíticas utilizadas para el estudio de subvariedades y gérmenes de variedades analíticas. Nos centraremos en el caso de **configuraciones de rectas** en el plano afín o en el proyectivo. Estamos interesados en el estudio de cómo la información combinatoria de la configuración determina relaciones entre sus campos de vectores logarítmicos. Este problema es conocido como la conjetura de Terao.

– Parte I –
Contribuciones a la conjetura de periodos de Kontsevich-Zagier

Históricamente, el **estudio de números** se ha basado en el estudio de **relaciones geométricas**. En la Antigua Grecia, los números siempre se consideraban asociados a diferentes medidas de objetos geométricos: longitudes de segmentos y curvas, áreas y volúmenes. Los *números racionales* provenían naturalmente como la *la proporción* entre longitudes o áreas. Los primeros ejemplos de *números irracionales* se obtuvieron por los Pitagóricos construyendo cuadrados sobre un triángulo rectángulo. En términos del Teorema de Pitágoras, si se imponen relaciones aritméticas simples entre las áreas, se pueden obtener longitudes irracionales. Si uno mira el caso de π, parece imposible disociar este número de la longitud de una circunferencia o el área de un círculo.

Las áreas de los polígonos en el plano y los volúmenes de los tetraedros son exhaustivamente estudiados en el famoso tratado geométrico de *Los Elementos de Euclides*. De hecho, Euclides nunca define la noción de longitud, área o volumen, ya que nunca necesitó asociar un valor numérico a un área. Los elementos de Euclides desarrollan lo que se conoce como *álgebra geométrica*, donde los términos estaban representados por lados de objetos geométricos, y las relaciones entre ellos eran explorados de forma geométrica. En particular, Euclides estudia geométricamente triángulos congruentes, para los cuales existe un desplazamiento directo en el plano que lleva un triangulo en otro.

Siguiendo a Euclides, una forma clásica de estudiar polígonos de mismo área es a través de las *congruencias-tijera*: ¿podemos obtener dos polígonos de la misma área por particiones de piezas congruentes? Una respuesta afirmativa fue dada por Bolyai, Wallace et Gerwien en el siglo dieciocho. Esta misma cuestión para 3–poliedros enuncia exactamente el tercer problema de Hilbert, cuya respuesta es negativa en general, pero afirmativa si nos restringimos a poliedros poseyendo el mismo volumen y el mismo *invariante de Dehn* [Deh01, Syd65], introducido por Dehn en 1900.

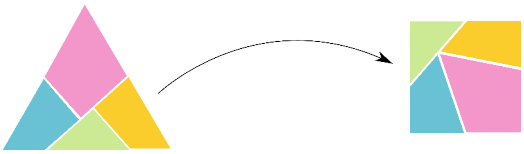

Figura 3: Congruencias-tijera entre dos polígonos de la misma área.

Progresivamente, está visión geométrica de los números fue desplazada por otras más algebraicas y analíticas, aportando grandes logros como la demostración de la transcendencia de π, e, etc. Una buena reseña de la teoría analítica de números se puede encontrar en el texto clásico de Hardy y Wright [HW08].

En esta tesis, considerando una amplia clase de números llamados *periodos* introducidos por M. Kontsevich y D. Zagier en su trabajo fundacional [KZ01], somos capaces de **recuperar esta interacción entre números y geometría** utilizando objetos más ricos, llamados *conjuntos semi-algebraicos*. Estudiamos entonces los periodos y sus **relaciones desde un punto de vista geométrico**.

Un *periodo* es un número complejo cuyas partes real e imaginaria son valores de integrales absolutamente convergentes de funciones racionales sobre dominios reales descritos por polinomios con coeficientes racionales. La gran mayoría de constantes importantes en matemáticas y

física son periodos, como los números algebraicos, π o los logaritmos de números algebraicos. Esta vasta clase de números forma un álgebra sobre los números reales algebraicos y posee una inherente naturaleza constructiva.

Como la definición de estos números es "simple", uno puede preguntarse si el tipo de relaciones entre ellos son también "simples". Esta idea es expresada en la llamada *conjetura de periodos de Kontsevich-Zagier*, la cual afirma que toda relación entre periodos proviene de la linealidad y de transformaciones adecuadas sobre sobre la integral, usando operaciones clásicas del cálculo integral. Más específicamente, si un periodo real admite dos representaciones integrales, entonces se puede pasar de una formulación a la otra utilizando solo tres operaciones del cálculo integral (llamadas las *reglas–KZ*): sumas integrales por dominios o integrandos, cambio de variables y la fórmula de Stokes. Cuando se puede encontrar este camino entre diferentes integrales y además es *simple y explícito*, nos referiremos a la noción de *identidades accesibles*. La anterior conjetura es una versión "afín" y "más computacional" de una conjetura de A. Grothendieck [Gro66]. El enunciado moderno de la *conjetura de periodos de Grothendieck* está expresado en términos de *motivos* [And04]. Otra cuestión relacionada es la de determinar un algoritmo que permita probar si dos periodos son iguales o no.

En esta tesis, desarrollamos un **enfoque geométrico de estos números y sus problemas relacionados**, reinterpretándolos como **volúmenes de dominios semi-algebraicos compactos**, centrándonos en obtener **métodos algorítmicos y constructivos**.

A fin de obtener esta aproximación, primero mostramos un **procedimiento algorítmico para expresar cualquier periodo no nulo** dado por una cierta forma integral **como el volumen de un conjunto semi-algebraico compacto (no-único)**. Damos entonces un **reinterpretación de la conjetura de periodos de Kontsevich-Zagier** en términos de un **problema geométrico**: si tenemos dos compactos semi-algebraicos de mismo volumen, entonces podemos transformar uno en el otro por una sucesión de congruencias-tijera y transformaciones algebraicas conservando el volumen, salvo productos cartesianos por intervalos unidad. Utilizando triangulaciones de conjuntos semi-algebraicos, uno puede relacionar este problema con un **tercer problema de Hilbert generalizado** para poliedros equipados con formas algebraicas por partes, para el cual existen algunos resultados parciales ya conocidos. Siguiendo a J. Wan [Wan11], se introduce la noción de **complejidad geométrica para periodos**, basado en la complejidad polinomial minimal del conjunto semi-algebraico que representa un periodo. En el mismo espíritu, otra noción de **complejidad aritmética** también es introducida, proveniente de generalizaciones de medidas de complejidad para polinomios multivariantes.

La estructura de esta parte consiste en cinco capítulos desarrollados como sigue.

Damos una presentación general en el Capítulo I, detallando el origen de estos números como información que surge entre la comparación de cohomologías algebraica y geométrica de variedades algebraicas. Introducimos las herramientas necesarias para trabajar con la geometría semi-algebraica en coeficientes reales algebraicos. De esta forma, podemos presentar la clase de periodos de Kontsevich-Zagier, así como algunos ejemplos y sus primeras propiedades basadas en métodos geométricos. Seguidamente, introducimos los principales problemas sobre el tipo de relaciones entre periodos: la conjetura de periodos de Kontsevich-Zagier, ilustrada por varios ejemplos. También explicamos otros problemas concernientes a las *identidades accesibles* y la determinación de un *algoritmo de igualdad* para periodos. En conclusión, discutimos sobre problemas relacionados de naturaleza aritmética y sobre transcendencia de periodos. Finalmente, describimos el trabajo de M. Yoshinaga [Yos08] sobre periodos y números elementales computables.

Detallamos en el Capítulo II nuestro principal resultado en la construcción de nuestro enfoque geométrico: un procedimiento algorítmico para asociar a cualquier periodo no nulo el volumen de un compacto semi-algebraico. Este procedimiento es denominado la *reducción semi-canónica para periodos no nulos*. Primero estudiamos la compactificación de dominios definiendo la clausura proyectiva de un conjunto semi-algebraico. Entonces, usando resolución de singularidades, resolvemos geométricamente los polos del integrando que yacen en la frontera del dominio. Describimos la forma de dar una partición de los dominios por compactos afines en la variedad de resolución. Como consecuencia, probamos que los periodos pueden ser expresados como la diferencia entre volúmenes de dos compactos semi-algebraicos. Un método más computable y eficiente viene dado en el caso de integrales en el plano. Concluimos probando que podemos expresar la diferencia entre volúmenes de dos compactos semi-algebraicos como un único compacto semi-algebraico construido algorítmicamente a partir de los otros dos. Ilustramos cómo esta reducción se puede obtener a partir de varios ejemplos.

El Capítulo III trata sobre posibles reformulaciones de la conjetura de Kontsevich-Zagier. Primero discutimos sobre el uso de las fórmulas de Stokes y Fubini en la conjetura original. Usando la reducción semi-canónica, introducimos un problema geométrico de Kontsevich-Zagier para periodos. Este problema es descrito en términos de congruencias-tijera y transformaciones preservando el volumen para semi-algebraicos compactos de mismo volumen. Utilizando triangulaciones semi-algebraicas, formulamos un problema equivalente para poliedros racionales: el tercer problema de Hilbert generalizado para poliedros. Analizamos cómo este enfoque puede ser útil para la comprensión de la conjetura de periodos de Kontsevich-Zagier. Utilizando resultados conocidos en geometría combinatoria, podemos deducir algunos resultados parciales hacia el tercer problema de Hilbert generalizado.

Completamos la teoría de grado para periodos en el Capítulo IV, proveniente de la dimensión minimal en el conjunto de reducciones semi-canónicas de un periodo. Siguiendo esta noción, podemos definir una filtración natural del álgebra de periodos. Nos centramos en un criterio geométrico para la transcendencia de periodos en términos del grado. Entonces, reproducimos las propiedades principales del grado, así como algunos resultados en transcendencia e independencia lineal de números complejos. Ilustramos los principales problemas en el cálculo del grado mediante potencias de π. Apoyados en estos ejemplos, discutimos la efectividad de este criterio para diferenciar periodos. A fin de definir un noción de complejidad para periodos, introducimos primero una medida de complejidad geométrica para conjuntos semi-algebraicos. Completamos esta noción proponiendo la medida de Mahler como noción aritmética de complejidad. Comprobamos que esta noción de complejidad distingue diferentes periodos trascendentes de mismo grado.

Finalmente, damos algunas perspectivas relativas a esta parte en el Capítulo V. En particular, presentamos la idea de *decidabilidad* de la conjetura de Kontsevich-Zagier utilizando el trabajo de Yoshinaga y la reducción semi-canónica para periodos. Estudiamos la construcción de un algoritmo teórico que toma un par de periodos como entrada: si los dos periodos son números diferentes, el algoritmo para en tiempo finito indicando este hecho; en el caso que los dos periodos sean iguales, el algoritmo para en tiempo finito dando como salida el paso entre las dos representaciones integrales si y sólo si la conjetura de Kontsevich-Zagier es cierta.

– Parte II –
Contribuciones a la conjetura de Terao sobre configuraciones de rectas

Esta parte de la tesis concierne la **interacción entre** dos campos de estudio en matemáticas

que suelen estar usualmente desconectados: **la geometría algebraica y los sistemas dinámicos**. Un ejemplo de esta situación se refleja en los trabajos sobre **configuraciones de rectas**, es decir colecciones finitas de rectas en el plano afín o proyectivo, y los **campos vectoriales polinómicos**. De hecho, desde el transcendental trabajo de H. Terao [Ter80], este estudio incorpora potentes herramientas basadas en un tipo especial de campos vectoriales, llamados *campos vectoriales logarítmicos*. Por otra parte, un tema clásico de investigación en sistemas dinámicos es el estudio de sistemas diferenciales polinómicos a través de sus curvas invariantes (algebraicas o analíticas) y sus consecuencias dinámicas, como por ejemplo la *integrabilidad de Darboux* [Dar78].

A grandes rasgos, la pretensión de esta parte es promover un nuevo **punto de vista en la teoría de campos vectoriales logarítmicos utilizando teoría de sistemas dinámicos**, y más generalmente, dar un esbozo de un enfoque dinámico a la geometría. Específicamente, buscamos un **enfoque dinámico en el estudio del módulo de campos vectoriales logarítmicos para configuraciones de rectas**.

Uno de los principales problemas concernientes a configuraciones de rectas es la llamada *conjetura de Terao*, que trata sobre la interacción entre las relaciones polinomiales de los campos vectoriales logarítmicos y la información combinatoria de la configuración de rectas. Concretamente, la conjetura de Terao concierne las relaciones algebraicas del *módulo de campos vectoriales logarítmicos*, que es definido como el módulo polinomial de derivaciones que dejan estable el ideal generado por el polinomio de definición de la configuración. Esta conjetura puede ser formulada como sigue: la libertad del módulo de campos vectoriales logarítmicos es de naturaleza combinatoria. A pesar de la gran cantidad de trabajos sobre la libertad de configuraciones de rectas y del criterio simple sobre libertad dado por K. Saito, esta noción no está aún completamente bien comprendida. En particular, como fue remarcado por P. Cartier en su seminario Bourbaki in 1981 [Car81, p. 19], no se conoce todavía una interpretación geométrica exacta de la libertad.

La principal guía directriz de nuestro trabajo es **determinar el impacto de la libertad y la combinatoria sobre la dinámica de los campos vectoriales logarítmicos**.

A fin de avanzar en esta meta, primeramente hemos **interpretado campos vectoriales logarítmicos usando** la noción de **invarianza dinámica**. De esta forma, aparecen naturalmente preguntas sobre las relaciones entre los puntos de equilibrio de campos vectoriales logarítmicos y las singularidades de la configuración de rectas, así como el **posible comportamiento dinámico global cualitativo** de un tal campo vectorial y sus **conexiones con las propiedades combinatorias y geométricas de la configuración**. Específicamente para configuraciones afines de rectas en el plano, donde la libertad está fuera de cuestión, solamente el rol de la combinatoria puede ser analizado. Estudiamos entonces lo siguiente:

Pregunta. *¿Está el módulo de campos vectoriales logarítmicos de una configuración afín de rectas determinado por la combinatoria?*

Nuestra estrategia para resolver este problema es fijar una configuración dada, más precisamente un combinatoria dada, y determinar cuáles son las restricciones inducidas sobre su modulo de campos de vectores logarítmicos. Si buscamos conjuntos invariantes, debemos estudiar cuándo un campo vectorial posee una colección maximal de rectas invariantes incluyendo la configuración. Siguiendo esta perspectiva, tomarán un rol especial los campos de vectores polinómicos fijando un número infinito de rectas: campos vectoriales *radiales* y *paralelos* (Teorema VIII.3.11). Probamos que el grado minimal para que un campo vectorial de estos tipos

aparezcan en el módulo de derivaciones logarítmicas está determinado por la combinatoria (Proposiciones VIII.3.11 y VIII.3.16). Por lo cual, estamos interesados en estudiar la aparición por grado de las derivaciones que solo fijan un número finito de rectas además de las de la configuración, denominadas *derivaciones finitas*. Esto origina la noción de *grado minimal* de derivaciones finitas. Una idea es, entonces, buscar configuraciones combinatoriamente equivalentes que tengan diferentes grados minimales de derivaciones finitas (Teorema VIII.4.5). Esta búsqueda se vuelve más asequible a través de una serie de resultados que relacionan las propiedades combinatorias y el grado minimal, resumidas en los Corolarios VIII.3.10 y VIII.3.19. De esta manera, damos una respuesta negativa a la pregunta precedente en el Teorema VIII.1.3 presentando dos pares de configuraciones afines de rectas, correspondiendo cada par a una noción diferente de combinatoria, en los cuales el modulo de campos vectoriales logarítmicos tiene distinto grado minimal.

Este resultado da un fuerte soporte a nuestro enfoque dinámico sobre la conjetura de Terao.

Esta segunda parte se divide en cuatro capítulos de la forma siguiente

Introducimos en el Capítulo VI las nociones básicas y herramientas de estudio de las configuraciones de hiperplanos a través de los campos vectoriales logarítmicos, focalizando en configuraciones afines de rectas. Explicamos las principales propiedades geométricas así como sus relaciones con las nociones de combinatoria de una configuración. Nuestro principal objeto de estudio, el módulo de derivaciones logarítmicas, es presentado: evocamos sus principales propiedades y su relación con las formas diferenciales logarítmicas. Esto nos permite introducir las *configuraciones libres* y por lo tanto la conjetura de Terao para configuraciones centrales, así como su versión en lenguaje de haces expresada en términos de fibrados vectoriales logarítmicos en el plano proyectivo complejo.

El Capítulo VII concierne la reinterpretación dinámica de los campos vectoriales logarítmicos, introducidos por K. Saito [Sai80]. Damos primero una interpretación dinámica de estos campos vectoriales en términos de conjuntos dinámicamente invariantes, analizando las diferencias entre campos vectoriales algebraicos y analíticos. Esto nos permite presentar un enfoque dinámico general al estudio de objetos geométricos basado en las ideas precedentes. Luego, centramos nuestro enfoque en el estudio de configuraciones de curvas considerando campos vectoriales logarítmicos algebraicos en el plano. Esto nos permite darle la vuelta al punto de vista clásico sobre dos problemas clásicos de campos vectoriales reales planos: la conjetura de Dulac y el decimosexto problema de Hilbert algebraico. Después, analizamos algunas relaciones entre campos vectoriales logarítmicos y nociones específicamente dinámicas de curvas algebraicas invariantes de campos vectoriales polinómicos, como integrales primeras, integrabilidad de Darboux y corchetes de Lie. Concluimos este capítulo tratando el caso de curvas definidas por ecuaciones homogéneas, mostrando relaciones en términos de campos vectoriales logarítmicos entre configuraciones afines de rectas y configuraciones centrales de planos en $\mathbb{A}^3_k$, así como configuraciones proyectivas de rectas en $\mathbb{P}^2_k$.

Aplicamos nuestro enfoque dinámico en el caos de configuraciones afines de rectas en el Capítulo VIII, a través de campos de vectores polinómicos fijando la configuración. El estudio por grado de las derivaciones requiere introducir una filtración del módulo de derivaciones logarítmicas, la cual es respetada por los automorfismos polinómicos del plano afín. Siguiendo el punto de vista dinámico, caracterizamos campos vectoriales que fijan un número infinito de rectas en $\mathbb{A}^2_k$. Estudiando la influencia de la combinatoria de las configuraciones sobre estos campos vectoriales logarítmicos, damos una cota inferior para el grado minimal de las derivaciones logarítmicas no nulas fijando únicamente un número finito de rectas, incluida la configuración. Este hecho nos permite dar dos pares de contraejemplos que muestran que el módulo de campos

vectoriales logarítmicos de una configuración afín de rectas no está determinada por la combinatoria fuerte o débil. Analizamos la combinatoria y los estratos de la filtración logarítmica creando un programa en `Sage`, el cual es descrito completamente en el Apéndice B. Finalmente, estudiamos una formula cuadrática dependiendo en el grado para los rangos de los estratos de la filtración, y también cómo esta fórmula está determinada por la combinatoria de la configuración de rectas.

Concluimos esta parte con el Capítulo IX, donde damos una idea de las perspectivas y el trabajo futuro en el desarrollo de este enfoque dinámico sobre la libertad de configuraciones así como sobre la conjetura de Terao.

Part I

Contributions to the Kontsevich-Zagier periods conjecture

Chapter I

Periods of Kontsevich-Zagier

In this chapter, we introduce the basic tools, fundamental results and actual problems related with the first part of our work: the class of periods of Kontsevich-Zagier.

We start with a brief general presentation of the modern number theory as well as the niche where periods take place, in its different forms and principal definitions. Straightaway, we introduce the first part of the mathematical machinery needed to define and understand periods: the ($\widetilde{\mathbb{Q}}$–)semi-algebraic sets and maps, as well as some indications about the use of these objects in our setting for periods. Then, following the foundational article [KZ01] of M. Kontsevich and D. Zagier, we are able to present periods in Section I.3, giving some first fundamental algebraic and geometrical properties as well as some examples. The set of *abstract periods* and its relations with our predefined definition for periods are presented at the end of this section.

We finish this chapter by presenting the fundamental problems stated for periods. In Section I.4, we discuss the two open problems coming from the fact that a period has different integral representations: the Kontsevich-Zagier period conjecture, and the existence of accessible identities and an equality algorithm for periods. We conclude with associated problems of arithmetic nature in Section I.5: the existence of a number which is not a period, the construction of an approximation theory for periods in the spirit of Liouville, the work of M. Yoshinaga in this direction ([Yos08]) and, finally, some related results and conjectures about transcendence for periods following [Wal06].

I.1 A general presentation

I.1.1 A brief history of numbers

The development and understanding of numbers through human history have been connected with the evolution of thinking and culture into the different civilizations in your way to interpret, describe and transform reality.

Ancient humans started with the numbers associated to our first primitive knowledge tools, counting and ordering. Described in a modern view, we are considering the set of *natural numbers*:

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

Related to the development of arithmetics in the different ancient civilizations, specially associated to the commerce, the set of *integer numbers* appears:

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

Rational numbers (from *ratio* between two numbers) are studied particularly by ancient Egyptian and Greek mathematicians by its relation with geometry and construction.

$$\mathbb{Q} = \left\{ \frac{p}{q} \;\middle|\; p, q \in \mathbb{Z}, p \neq 0, (p, q) = 1 \right\}$$

Greek mathematicians gives the first proof of existence of another types of numbers when they discover (or meet) and important class of the irrational numbers, the *square roots* of natural numbers and the constant π.

Taking the completion of $\mathbb{Q}$, we get the set of *real numbers* $\mathbb{R}$ as limit of convergent suites of rationals numbers.

After studying the solutions of polynomial equations, Italian mathematicians extends real numbers to the *complex plane* defining the *imaginary unit* $i = \sqrt{-1}$:

$$\mathbb{C} = \{a + i \cdot b \mid a, b \in \mathbb{R}\},$$

and completing the suite of elementary sets of numbers

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

I.1.2 Modern number theory

This classical sets of numbers are equipped with natural operations (sum and product) and they form the basis to generalize and consider the basic algebraic structures of sets: groups, rings, algebras and fields.

Algebraic number theory studies the properties and relations of these algebraic structures. The language of field extensions and Galois Theory are powerful tools to study finite extensions of $\mathbb{Q}$, called *algebraic number fields*, obtained by considering roots of polynomials with rational coefficients: the *algebraic numbers*. The *field of algebraic numbers* $\overline{\mathbb{Q}}$ is obtained by a generalization of rational numbers considering all the solutions of polynomial equalities with rational coefficients. Some well-known examples of algebraic numbers are the own rational numbers, n-roots of rational numbers, the golden ratio or the imaginary unit: $\sqrt{2}, \frac{1+\sqrt{5}}{2}, i \in \overline{\mathbb{Q}}$. Algebraic numbers have good properties, for example, $\overline{\mathbb{Q}}$ is a countable field (since rational numbers and polynomials are countable sets) and we have a graduation given by the degree of an algebraic number (defined as the degree of his minimal polynomial).

In the same way, *transcendental* numbers are complex numbers which are not algebraic. Almost all of real (and complex) numbers are transcendental, since the algebraic numbers form a countable subset of $\mathbb{C}$. Nevertheless, it is very difficult in general to prove if a given number is transcendental or not. Example of transcendental numbers are π, e, $\Gamma(1/3)$ or Liouville numbers.

The analytic number theory gives another way to study properties of numbers, distributions of primes and factorizations using techniques of complex analysis. Analytic and algebraic number theories are overlapped in many times to pose questions and give answers, as in p-adic analysis and p-adic field extensions or in the study of the *Riemann zeta function* and their generalizations: *multiple zeta functions* (also called *polyzeta functions*) and *Dedeking zeta functions*, for example. Liouville proved the existence of transcendental numbers in a first time at 1844, first proving a well-known result of approximation of algebraic numbers by rationals:

Theorem I.1.1 (Liouville,1844)**.** *Let θ be an irrational number and take $\frac{p}{q} \in \mathbb{Q}$ with $q > 0$, then:*

$$\theta \text{ is algebraic of degree } n \Longrightarrow \exists K(\theta) > 0 \text{ such that } \left| x - \frac{p}{q} \right| \geq \frac{K(\theta)}{q^n}$$

Finally, he constructed in 1855 a family of numbers which are not transcendental: the *Liouville numbers*, including the *Liouville constant*

$$\sum_{k=1}^{\infty} 10^{-k!} = 0,1100010000000000000000001000\ldots$$

Considering real and complex algebraic numbers, we complete the schema of rings and fields in the classical study of numbers:

$$\begin{array}{ccccccc} \mathbb{Z} & \subset & \mathbb{Q} & \subset & \overline{\mathbb{Q}} \cap \mathbb{R} & \subset & \overline{\mathbb{Q}} \\ & & & & \cap & & \cap \\ & & & & \mathbb{R} & \subset & \mathbb{C} \end{array}$$

I.1.3 Periods as a countable algebra over the algebraic numbers

We do not know too much about transcendental numbers, even if the set of real algebraic numbers is a countable subset of $\mathbb{R}$, there are few examples of transcendental numbers. Proving the transcendence of a number is not immediate. The transcendence of π was proved in 1882 by Von Lindemann and of e in 1873 by Hermite. The problem is in general that we can not describe and obtain properties of transcendental numbers in a simple way due to their actual definition as a set, i.e. the complementary of algebraic numbers in $\mathbb{C}$.

The usual strategy is to construct a increasing suite of subsets $(\mathcal{E}_i)_{i\geq 0}$ of $\mathbb{R}$ or $\mathbb{C}$ with relatively good properties containing the algebraic numbers

$$\overline{\mathbb{Q}} \subset \mathcal{E}_0 \subset \ldots \subset \mathcal{E}_i \subset \mathcal{E}_{i+1} \subset \ldots \subset \mathbb{C}$$

in a way that each $\mathcal{E}_i$ be easily definable, simple and manipulable. We are interested to obtain sets with some good algebraic structures and properties which can be checked extended from algebraic numbers.

In [KZ01], M. Kontsevich and M. Zagier introduce the class of numbers called *periods*:

> *A* period *is a complex number whose real and imaginary parts are values of absolutely convergent integral of rational functions with rational coefficients, over domains of a real affine space given by polynomial inequalities with rational coefficients.*

The set of periods of Kontsevich-Zagier is denoted by $\mathcal{P}_{\text{KZ}}$. In the same way, we also consider $\mathcal{P}_{\text{KZ}}^{\mathbb{R}} = \mathcal{P}_{\text{KZ}} \cap \mathbb{R}$ is the *set of real periods*.

As it is mentioned in [KZ01, p. 3]:

> *In the above definition one can replace "rational function" and "rational coefficients" by "algebraic function" and "algebraic coefficients" without changing the set of numbers which one obtains.*

In the present work, we will consider periods as in the original definition but taking coefficients in the *field of real algebraic numbers* $\widetilde{\mathbb{Q}} = \overline{\mathbb{Q}} \cap \mathbb{R}$ (see Definition I.3.1). A first discussion about this choice can be founded in Remark I.3.3.

It is easy to see that $\mathcal{P}_{\text{KZ}}$ contains the algebraic numbers and is a countable set also with a structure of $\overline{\mathbb{Q}}$-algebra given by linearity of integral and Fubini's formula (see Theorem I.3.8). Thus, we can extend our schema of relations

$$\begin{array}{ccccccc} \mathbb{Z} & \subset & \mathbb{Q} & \subset & \widetilde{\mathbb{Q}} & \subset & \overline{\mathbb{Q}} \\ & & & & \cap & & \cap \\ & & & & \mathcal{P}_{\text{KZ}}^{\mathbb{R}} & \subset & \mathcal{P}_{\text{KZ}} \\ & & & & \cap & & \cap \\ & & & & \mathbb{R} & \subset & \mathbb{C} \end{array}$$

The way that periods are defined which allows us to take many different approaches and fields to study them: algebra, geometry, analysis, measure theory, ... We are focusing in the spirit and techniques of algebraic geometry and algebraic analysis in order to develop our work.

Coming from a *computational* point of view for the study of real periods, we can consider another set of real numbers containing $\mathcal{P}_{\text{KZ}}^{\mathbb{R}}$, the *field of elementary numbers*, $\mathbb{R}_{(\mathsf{Elem})}$ related to the theory of computability by a Turing's machine (see [Tur36]).

$$\mathbb{Q} \subset \widetilde{\mathbb{Q}} \subset \mathcal{P}_{\text{KZ}}^{\mathbb{R}} \subset \mathbb{R}_{(\mathsf{Elem})} \subset \mathbb{R}$$

This field $\mathbb{R}_{(\mathsf{Elem})}$ is a countable subset of real numbers which can be identified with a particular type of limits of rational numbers. In [Yos08], Yoshinaga proves the inclusion of periods in the set of real elementary numbers and gives a first example of a real number which is not contained in $\mathcal{P}_{\text{KZ}}$ showing that this number is not elementary.

I.1.4 Periods and (co)homology of algebraic varieties

The principle of the philosophy of periods is that we are dealing with values and transformations of integrals *coming from algebraic geometry*. We get a closer type of periods defined in terms of algebraic geometry and described also in [KZ01], the *effective periods* $\mathcal{P}_{\mathsf{abs}}$.

Let X be a smooth algebraic variety and Y a closed subvariety of X, both of them defined over $\mathbb{Q}$. Fix an integer d, let $\omega \in \Omega^d(X)$ be a closed algebraic (thus rational) differential form with coefficients in $\mathbb{Q}$ such that $\omega_{|Y} = 0$, and let γ be a singular d–chain on the complex manifold $X(\mathbb{C})$ whose boundary lies on $Y(\mathbb{C})$. Then, we consider the evaluation of the integral

$$\int_\gamma \omega \in \mathbb{C}. \tag{I.1}$$

Beside this kind of integrals, we find the comparison isomorphism of different cohomology theories associated to algebraic varieties. Let $H_{\text{B}}^\bullet(X, Y; \mathbb{Q})$[1] and $H_{\text{dR}}^\bullet(X, Y; \mathbb{Q})$ be the Betti (or singular) and the algebraic de Rham cohomology groups of X relative to Y with coefficients in $\mathbb{Q}$. Both of them are finite-dimensional $\mathbb{Q}$–vector spaces. Using Poincaré duality over the Betti cohomology, we can define a pairing via the evaluations of integrals of the form (I.1):

$$H_{\text{dR}}^\bullet(X, Y; \mathbb{Q}) \times H_{\text{B}}^\bullet(X, Y; \mathbb{Q}) \longrightarrow \mathbb{C}$$

This pairing rises up into the comparison isomorphism between Betti and de Rham cohomologies after tensoring by $\mathbb{C}$:

$$\text{comp}_{\text{B,dR}} : H_{\text{dR}}^\bullet(X, Y; \mathbb{Q}) \otimes \mathbb{C} \xrightarrow{\simeq} H_{\text{B}}^\bullet(X, Y; \mathbb{Q}) \otimes \mathbb{C}$$

[1] We are performing an abuse of notation in the sense that the Betti cohomology is defined over the associated complex analytic manifolds $X(\mathbb{C})$ and $Y(\mathbb{C})$.

Taking $\mathbb{Q}$–basis $\{\omega_1, \ldots, \omega_s\}$ and $\{\gamma_1 \ldots, \gamma_s\}$ of $H^\bullet_{\mathrm{dR}}(X, Y; \mathbb{Q})$ and $H^{\mathrm{B}}_\bullet(X, Y; \mathbb{Q})$ respectively, we can construct the *period matrix* representing this isomorphism via the evaluation of integrals:

$$\Pi = \left(\int_{\gamma_i} \omega_j \right)_{i,j=1,\ldots,s}$$

The preceded matrix is a classical tool in the study of the Hodge structure of algebraic varieties and explains the choose of the name "periods" for this class of numbers.

The importance of periods as coefficients of the period matrix comes from the fact that the comparison isomorphism $\mathrm{comp}_{\mathrm{B,dR}}$ can not be induced by an isomorphism between $H^\bullet_{\mathrm{dR}}(X, Y; \mathbb{Q})$ and $H^\bullet_{\mathrm{B}}(X, Y; \mathbb{Q})$. Indeed, consider the punctured affine line $X = \mathbb{A}^1_{\mathbb{Q}} \setminus \{0\} = \operatorname{Spec} \mathbb{Q}[t, t^{-1}]$ and $Y = \emptyset$ in order to take absolute cohomology. The first de Rham cohomology group is one-dimensional, generated by the 1–form $\frac{\mathrm{d}t}{t}$. Looking at the complex manifold $X(\mathbb{C}) = \mathbb{C}^*$, it is easy to see that $H^\bullet_{\mathrm{B}}(\mathbb{C}^*; \mathbb{Q}) = \mathbb{Q}\gamma^*$, where γ^* is the dual of the unit circle in $\mathbb{C}^*$ counterclockwise oriented. Under the comparison isomorphism $H^\bullet_{\mathrm{dR}}(X; \mathbb{C}) \simeq H^\bullet_{\mathrm{B}}(X; \mathbb{C})$, the form $\frac{\mathrm{d}t}{t}$ is mapped to $2\pi i \gamma^*$, since the integral

$$\int_\gamma \frac{\mathrm{d}t}{t} = 2\pi i,$$

which is clearly a non-rational number.

An important subclass is formed by those periods which are defined by linear objects: the *Aomoto periods*. Let $\mathcal{L} = \{L_0, \ldots, L_n\}$ and $\mathcal{M} = \{M_0, \ldots, M_n\}$ be two non-concurrent ordered hyperplane arrangements in $\mathbb{P}^n_{\mathbb{C}}$. This kind of periods are given by values of absolute convergent integrals

$$\int_{\Delta_{\mathcal{M}}} \omega_{\mathcal{L}}$$

where:

- $\omega_{\mathcal{L}} = \sum_{i=0}^n (-1)^i \mathrm{d}f_0/f_0 \wedge \ldots \wedge \widehat{\mathrm{d}f_i/f_i} \wedge \ldots \wedge \mathrm{d}f_n/f_n$, choosing for each $L_i \in \mathcal{L}$ a defining linear form f_i.
- $\Delta_{\mathcal{M}} : \Delta^n \to \mathbb{P}^n_{\mathbb{C}}$ an embedding of the standard n-simplex that maps the interior of Δ^n to $\mathbb{P}^n_{\mathbb{C}} \setminus \mathcal{L}$, and the j-th face $\partial\Delta^n$ to the hyperplane $M_j \in \mathcal{M}$, for all $j = 0, \ldots, n$.

Important periods as $2\pi i$, logarithms of algebraic numbers, classical polylogarithms or multiple zeta values are examples of Aomoto periods.

I.2 Preliminaries in semi-algebraic geometry

Firstly, we need to introduce the notion and properties of *semi-algebraic sets*. As highlighted in the previous section, we are going to consider semi-algebraic sets in $\mathbb{R}^d$ defined by coefficients in the field of real algebraic numbers $\widetilde{\mathbb{Q}}$. We follow [BCR98] remind basic definitions and properties about semi-algebraic sets and functions defined with real algebraic coefficients.

I.2.1 Semi-algebraic sets

Definition I.2.1. A subset $S \subset \mathbb{R}^d$ is called $\widetilde{\mathbb{Q}}$–*semi-algebraic* if it can be described as

$$S = \bigcup_{i=1}^{s} \bigcap_{j=1}^{r_i} \{f_{i,j} *_{i,j} 0\} \tag{I.2}$$

where $f_{i,j} \in \widetilde{\mathbb{Q}}[x_1, \ldots, x_d]$ and $*_{i,j} \in \{=, >\}$ for $i = 1, \ldots, s$ and $j = 1, \ldots, r_i$.

Remark I.2.2. Note that for a fixed $\widetilde{\mathbb{Q}}$–semi-algebraic set S there exist many different representations of the form stated in the previous Definition. We denote by $\mathcal{R}_S$ the family of all representations (I.2) of a given $\widetilde{\mathbb{Q}}$–semi-algebraic set S.

As a first consequence of the definition, we get:

Property I.2.3. *The $\widetilde{\mathbb{Q}}$–semi-algebraic class is closed by finite unions, finite intersections and taking complements.*

Example I.2.4.

1. Any algebraic set in $\mathbb{R}^d$ defined using polynomials with coefficients in $\widetilde{\mathbb{Q}}$ is a $\widetilde{\mathbb{Q}}$–semi-algebraic set.

2. It is easy to see that, in particular, $\widetilde{\mathbb{Q}}$–semi-algebraic subsets of $\mathbb{R}$ are exactly finite unions of points and open intervals.

3. Open and closed d–dimensional balls of radius $\rho \in \widetilde{\mathbb{Q}}$ centered in points $x = (x_1, \ldots, x_d) \in \widetilde{\mathbb{Q}}^d \subset \mathbb{R}^d$
$$\mathbb{B}_x(\rho) = \left\{ (y_1, \ldots, y_d) \in \mathbb{R}^d \;\middle|\; \sum_{i=1}^{d} (y_i - x_i)^2 < \rho^2 \right\},$$
and
$$\overline{\mathbb{B}}_x(\rho) = \left\{ (y_1, \ldots, y_d) \in \mathbb{R}^d \;\middle|\; \sum_{i=1}^{d} (y_i - x_i)^2 \leq \rho^2 \right\}$$
are $\widetilde{\mathbb{Q}}$–semi-algebraic sets, as well as the d–hypercube
$$\mathbb{B}_x^{\infty}(\rho) = \{(y_1, \ldots, y_d) \mid \max_{i=1,\ldots,d} |y_i - x_i| < \rho\} \subset \mathbb{R}^d.$$

4. The graph of the exponential map $\{(x, y) \in \mathbb{R}^2 \mid y = e^x\}$ or the infinite staircase over the integers $\{(x, y) \in \mathbb{R}^2 \mid \exists n \in \mathbb{N} \text{ such that } y = nx\}$ are not $\widetilde{\mathbb{Q}}$–semi-algebraic sets.

The class of $\widetilde{\mathbb{Q}}$–semi-algebraic is also stable by projection:

Theorem I.2.5 (Tarski–Seidenberg)**.** *Let $S \subset \mathbb{R}^{d+1}$ be $\widetilde{\mathbb{Q}}$–semi-algebraic and $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ the projection of the space on the first d coordinates. Then $\pi(S)$ is a $\widetilde{\mathbb{Q}}$–semi-algebraic subset of $\mathbb{R}^d$.*

As a consequence of the previous Theorem, we obtain that the $\widetilde{\mathbb{Q}}$–semi-algebraic class is stable by the usual topological operations for sets are

Property I.2.6. *The $\widetilde{\mathbb{Q}}$–semi-algebraic class is stable by taking the interior, closure and boundary.*

Remark I.2.7. It is worth noticing that, in general, we can not obtain the closure (resp. the interior) of a $\widetilde{\mathbb{Q}}$–semi-algebraic set only by relaxing (resp. strengthen) inequalities involved in its description. For example, take $S = \{(x, y) \in \mathbb{R}^2 \mid x^3 - x^2 - y^2 > 0\}$, then $\overline{S} = \{(x, y) \in \mathbb{R}^2 \mid x^3 - x^2 - y^2 \geq 0\} \setminus \{0\}$ and can be described as $\overline{S} = \{(x, y) \in \mathbb{R}^2 \mid x^3 - x^2 - y^2 \geq 0, x \geq 1\}$. Take now $S = \{x \in \mathbb{R} \mid x^2 \geq 0\} = \mathbb{R}$, then $\{x \in \mathbb{R} \mid x^2 > 0\} = \mathbb{R}^*$ which clearly is not the interior of $\mathbb{R}$.

A particular simple case of $\widetilde{\mathbb{Q}}$–semi-algebraic sets are basic $\widetilde{\mathbb{Q}}$–semi-algebraic sets.

Definition I.2.8. A *basic open* (resp. *closed*) $\widetilde{\mathbb{Q}}$–subset of $\mathbb{R}^d$ is a set described on the form:

$$\{(x_1,\ldots,x_d)\in\mathbb{R}^d \mid f_1(x_1,\ldots,x_d) *_n 0,\ldots,f_n(x_1,\ldots,x_d) *_n 0\}$$

where $f_1,\ldots,f_n\in\widetilde{\mathbb{Q}}[x_1,\ldots,x_d]$ and $*_i =$ " $>''$ (resp. $*_i =$ " $\geq''$) for any $i=1\ldots,n$.

I.2.2 Semi-algebraic mappings

The Tarski–Seidenberg Theorem allows us to give a consistent notion of $\widetilde{\mathbb{Q}}$–semi-algebraic functions.

Definition I.2.9. Let $A\subset\mathbb{R}^m$ and $B\subset\mathbb{R}^n$ be two $\widetilde{\mathbb{Q}}$–semi-algebraic set. A mapping $f:A\to B$ is $\widetilde{\mathbb{Q}}$–*semi-algebraic* if its graph

$$\Gamma_f=\{(a,f(a))\in A\times B \mid a\in A\}$$

is $\widetilde{\mathbb{Q}}$–semi-algebraic in $\mathbb{R}^{m+n}$.

Property I.2.10. *Let $f:A\to B$ be a $\widetilde{\mathbb{Q}}$–semi-algebraic mapping.*

1. *The image and inverse image of $\widetilde{\mathbb{Q}}$–semi-algebraic sets by f are $\widetilde{\mathbb{Q}}$–semi-algebraic.*
2. *If $g:B\to C$ is a $\widetilde{\mathbb{Q}}$–semi-algebraic mapping, then the composition $g\circ f$ is $\widetilde{\mathbb{Q}}$–semi-algebraic.*
3. *The real valuated $\widetilde{\mathbb{Q}}$–semi-algebraic functions on a $\widetilde{\mathbb{Q}}$–semi-algebraic set A form a ring with sum and composition.*

Example I.2.11. As examples of functions defined over $\widetilde{\mathbb{Q}}$–semi-algebraic sets which are $\widetilde{\mathbb{Q}}$–semi-algebraic, we have:

1. Polynomial and rational functions.
2. Let $\{S_\alpha\}_{\{\alpha\in I\}}$ be a finite disjoint collection of $\widetilde{\mathbb{Q}}$–semi-algebraic sets in $\mathbb{R}^d$ and $\{\phi_\alpha\}_{\alpha\in I}$ a finite collection of $\widetilde{\mathbb{Q}}$–semi-algebraic maps $\phi_\alpha:S_\alpha\to\mathbb{R}^m$. Then the piecewise map $\Phi:\bigcup_{\alpha\in I}S_\alpha\to\mathbb{R}^m$ defined by $\Phi(x)=\phi_\alpha(x)$ if $x\in S_\alpha$, is $\widetilde{\mathbb{Q}}$–semi-algebraic.
3. The absolute value $|f|$ and the square-root $\sqrt{|f|}$ of a $\widetilde{\mathbb{Q}}$–semi-algebraic map f.
4. Let $\emptyset\neq A\subset\mathbb{R}^d$ be a $\widetilde{\mathbb{Q}}$–semi-algebraic set. The *distance function* $x\mapsto\operatorname{dist}(x,A)$ to A defined in $\mathbb{R}^d$ is a continuous $\widetilde{\mathbb{Q}}$–semi-algebraic map, vanishing in $\overline{A}$ and positive elsewhere.
5. If a $\widetilde{\mathbb{Q}}$–semi-algebraic function $f:U\to\mathbb{R}$ admits a partial derivative $\partial f/\partial x_\alpha$ over an open $\widetilde{\mathbb{Q}}$–semi-algebraic set U, then $\partial f/\partial x_\alpha$ is $\widetilde{\mathbb{Q}}$–semi-algebraic.

A particular interesting case of $\widetilde{\mathbb{Q}}$–semi-algebraic functions are $\widetilde{\mathbb{Q}}$–algebraic functions.

Definition I.2.12. Let $S\subset\mathbb{R}^d$ an open $\widetilde{\mathbb{Q}}$–semi-algebraic set. We say that a continuous map $f:S\to\mathbb{R}$ is $\widetilde{\mathbb{Q}}$–*algebraic* if it is algebraic over the field $\widetilde{\mathbb{Q}}(x_1,\ldots,x_d)$, i.e. if there exist a monic polynomial $P=y^n+a_1y^{n-1}+\ldots++a_{n-1}y+a_n\in\widetilde{\mathbb{Q}}(x_1,\ldots,x_d)[y]$ such that the algebraic functional equation

$$P(f)=f^n+a_1f^{n-1}+\ldots+a_{n-1}f+a_n=0$$

is verified in S.

Example I.2.13. The map $f(x)=1/\sqrt{1-x^2}$ is algebraic over $I=(-1,1)$, since

$$f(x)^2-\frac{1}{(1-x^2)}=0$$

is verified for any $x\in I$.

I.2.3 Detecting real roots by Sturm sequences

A classical problem for real polynomials is to give methods to detect and localize its real roots contained in an interval of $\mathbb{R}$. A useful approach to this problem is performed counting the roots using Sturm sequences.

Definition I.2.14. Let $P \in \widetilde{\mathbb{Q}}[x]$. The *Sturm sequence* of P is the sequence of polynomials $(P_0, \ldots, P_s)$ constructed recursively applying Euclid's algorithm to P and its derivative:

1. $P_0 = P$ and $P_1 = P'$, where P' is the derivative of P.
2. $P_i = P_{i-1}Q_i - P_{i-2}$, with $Q_i \in \widetilde{\mathbb{Q}}[x]$ and $\deg P_i < \deg P_{i-1}$ for $i = 2, \ldots, s$. Note that P_s is the greatest common divisor of P and P'.

Proposition I.2.15. *Let $[a, b] \subset \mathbb{R}$ be an interval and $P \in \widetilde{\mathbb{Q}}[x]$ be a polynomial with only simple roots on $[a, b]$ and $P(a)P(b) \neq 0$. Then, the Sturm sequence $(P_0, \ldots, P_s)$ of P verifies:*

1. *P_k has no zeros on $[a, b]$.*
2. *For any $i = 1, \ldots, s-1$, if there exists $c \in [a, b]$ such that $P_i(c) = 0$, then $P_{i-1}(c)P_{i+1}(c) < 0$.*
3. *If there exist $c \in (a, b)$ verifying $P_0(c) = 0$, then $\operatorname{sgn}(P_0(x)P_1(x)) = \operatorname{sgn}(x - c)$ in a neighborhood of c.*

Proposition I.2.16. *Let $a, b \in \mathbb{R}$ with $a < b$ and $P \in \widetilde{\mathbb{Q}}[x]$ be a polynomial such that $P(a)P(b) \neq 0$. Consider the Sturm sequence $(P_0, \ldots, P_s)$ of P and define the function $v_P : \mathbb{R} \to \mathbb{N}$ given by:*

$$v_P(x) = \#\{(i, j) \in \mathbb{N}^2 \mid 0 \leq i < j \leq s,\ P_i(x)P_j(x) < 0 \text{ and } P_k(x) = 0, \forall i < k < j\}. \tag{I.3}$$

Then, the number of roots of P in the open interval (a, b) is equal to $v_P(a) - v_P(b)$. In particular, if P has only simple roots on $[a, b]$, then $v_P(a) - v_P(b)$ is the number of simple roots of P in (a, b).

An estimation of a maximal interval containing any root of P is given by the coefficients of the polynomial.

Lemma I.2.17. *Let $P \in \widetilde{\mathbb{Q}}[x]$ be a polynomial of the form $P = a_n x^n + \ldots + a_0$ such that $a_n \neq 0$. Consider the constant given by*

$$M = 1 + \left|\frac{a_{n-1}}{a_n}\right| + \cdots + \left|\frac{a_0}{a_n}\right|.$$

Then P never vanishes on $[M, +\infty)$ (resp. $(-\infty, -M]$) and its sign is the sign of a_n (resp. $(-1)^n a_n$).

This method will be very useful later for localize graphs of algebraic functions over semi-algebraic sets.

Proposition I.2.18. *Let $(P_0, \ldots, P_s)$ be the Sturm sequence of a polynomial $P \in \widetilde{\mathbb{Q}}[x]$. Consider the function $v_P : \mathbb{R} \to \mathbb{N}$ defined by (I.3). Then, for any $k \in \mathbb{N}$,*

$$\{x \in \mathbb{R} \mid v_P(x) = k\}$$

is a $\widetilde{\mathbb{Q}}$–semi-algebraic set in $\mathbb{R}$.

Proof. Denote by Δ the finite set of integer points $\{(i,j) \in \mathbb{N}^2 \mid 0 \leq i < j \leq s\}$. For a fixed $k \in \mathbb{N}$, define $\Delta_k = \{S \subset \Delta \mid \#S = k\}$. Thus, by definition (I.3) of the function $v_P : [a,b] \to \mathbb{N}$, we can express:

$$\{x \in \mathbb{R} \mid v_P(x) \geq k\} = \bigcup_{S \in \Delta_k} \bigcap_{(i,j) \in S} \{x \in \mathbb{R} \mid P_i(x)P_j(x) < 0, P_{i+1}(x) = \cdots = P_{j-1}(x) = 0\},$$

which is a $\widetilde{\mathbb{Q}}$–semi-algebraic set. As we can write $\{x \in \mathbb{R} \mid v_P(x) = k\}$ as the difference between $\{x \in \mathbb{R} \mid v_P(x) \geq k\}$ and the complementary of $\{x \in \mathbb{R} \mid v_P(x) \geq k+1\}$, the result holds. □

I.2.4 Decomposition, connectedness, dimension and birational geometry

Every $\widetilde{\mathbb{Q}}$–semi-algebraic sets can be decomposed in an algorithmic way into simpler semi-algebraic subsets that are $\widetilde{\mathbb{Q}}$–semi-algebraic homeomorphic to open hypercubes $(0,1)^k$ for some $k \in \mathbb{N}$. This decomposition can be easily constructed by induction over the dimension of the ambient space. This give us a first notion of *dimension*.

Theorem I.2.19. *Let $\mathcal{F} = \{f_1, \ldots, f_n\} \subset \widetilde{\mathbb{Q}}[x_1, \ldots, x_d][y]$ be a finite family of polynomials. There exist:*

- *A finite partition of $\mathbb{R}^d$ by $\widetilde{\mathbb{Q}}$–semi-algebraic sets $\{A_1, \ldots, A_m\}$,*
- *For $i = 1, \ldots, m$, a finite number (possibly zero) of continuous $\widetilde{\mathbb{Q}}$–semi-algebraic functions $\xi_{i,1} < \cdots < \xi_{i,l_i} : A_i \to \mathbb{R}$,*

verifying that, for every $a = (a_1, \ldots, a_d) \in A_i$:

1. *$\{\xi_{i,1}(a), \ldots, \xi_{i,l_i}(a)\}$ is the set of roots of those polynomials among $f_1(a,y), \ldots, f_n(a,y)$ which are not identically zero.*
2. *The signs of of $f_1(a,y), \ldots, f_n(a,y)$ depend only on the signs of $\{y - \xi_{i,1}(a), \ldots, y - \xi_{i,l_i}(a)\}$.*

In particular, the graph of each $\xi_{i,j}$ is contained in the set of zeros of some $f_k \in \mathcal{F}$, with $k = k(i,j)$.

Definition I.2.20. Let $\mathcal{F} = \{f_1, \ldots, f_n\} \subset \widetilde{\mathbb{Q}}[x_1, \ldots, x_d][y]$ be a finite family of polynomials. A partition by $\widetilde{\mathbb{Q}}$–semi-algebraic sets $\{A_1, \ldots, A_m\}$ of $\mathbb{R}^d$, together with a family of continuous $\widetilde{\mathbb{Q}}$–semi-algebraic functions $\xi_{i,1} < \cdots < \xi_{i,l_i} : A_i \to \mathbb{R}$, described in Theorem I.2.19 is called a *slicing of $\mathcal{F}$*. If the sets in $\{A_1, \ldots, A_m\}$ are given by boolean combinations of sings conditions on a family of polynomials $\mathcal{G} = \{g_1, \ldots, g_h\} \subset \widetilde{\mathbb{Q}}[x_1, \ldots, x_d]$, we said that $\mathcal{G}$ *slice* $\mathcal{F}$.

Remark I.2.21. An improved result about stratification, cylindrical adapted decomposition and triangulation of $\widetilde{\mathbb{Q}}$–semi-algebraic sets is described and discussed in Chapter III.

Corollary I.2.22. *Every $\widetilde{\mathbb{Q}}$–semi-algebraic set of $\mathbb{R}^d$ is the disjoint union of a finite number of $\widetilde{\mathbb{Q}}$–semi-algebraic sets, each of them $\widetilde{\mathbb{Q}}$–semi-algebraically homeomorphic to an open hypercube $(0,1)^k$, for some $k \in \mathbb{N}$.*

The previous result introduces an interesting question in semi-algebraic topology: semi-algebraic connectedness.

Definition I.2.23. A $\widetilde{\mathbb{Q}}$–semi-algebraic set S of $\mathbb{R}^d$ is said to be *semi-algebraically connected* if for every pair of $\widetilde{\mathbb{Q}}$–semi-algebraic sets T_1 and T_2 closed in S, disjoint and satisfying $S = T_1 \cap T_2$, one has either $S = T_1$ or $S = T_2$.

As a consequence of Theorem I.2.19:

Theorem I.2.24. *Every $\widetilde{\mathbb{Q}}$–semi-algebraic set of $\mathbb{R}^d$ is the disjoint union of a finite number of semi-algebraically connected $\widetilde{\mathbb{Q}}$–semi-algebraic sets $C_1, \ldots, C_s$, each of them open and closed in S. The $C_1, \ldots, C_s$ are called the* semi-algebraically connected components *of S.*

Due to the semi-algebraic character of the class of objects we are interest, we study the relation with the Zariski topology in $\mathbb{R}^d$. For a family $\mathcal{F} \subset \widetilde{\mathbb{Q}}[x_1, \ldots, x_d]$, we define the *zero set of $\mathcal{F}$* as $\mathcal{Z}(\mathcal{F}) = \bigcap_{f \in \mathcal{F}} f^{-1}(0)$. Let S be $\widetilde{\mathbb{Q}}$–semi-algebraic subset of $\mathbb{R}^d$, we denote by $\mathcal{I}(S)$ the ideal of polynomials in $\widetilde{\mathbb{Q}}[x_1, \ldots, x_d]$ vanishing in S, i.e

$$\mathcal{I}(S) = \{f \in \widetilde{\mathbb{Q}}[x_1, \ldots, x_d] \mid f(x) = 0,\ \forall x \in S\}.$$

Definition I.2.25. The *Zariski closure* of S is defined as $\mathcal{Z}(\mathcal{I}(S))$.

The consistency between the euclidean topology and the Zariski topology in the semi-algebraic class is reflected in the intrinsic definition of the *dimension*.

Definition I.2.26. The *ring of regular functions of S*, denoted $\mathcal{P}(S)$, is defined as the quotient ring $\widetilde{\mathbb{Q}}[x_1, \ldots, x_d]/\mathcal{I}(S)$.

Definition I.2.27. The *dimension* of S, denoted $\dim S$, is the dimension of the ring $\mathcal{P}(S)$, i.e. the maximal length of chains of prime ideals of $\mathcal{P}(S)$.

Proposition I.2.28.

1. *The dimension of S coincides with the dimension of its closure $\overline{S}$ and its Zariski closure.*
2. *Let U be a non-empty open $\widetilde{\mathbb{Q}}$–semi-algebraic subset of $\mathbb{R}^d$. Then $\dim U = d$.*
3. *Let $S = S_1 \cup \ldots \cup S_k$ a finite union of $\widetilde{\mathbb{Q}}$–semi-algebraic sets. Then $\dim S = \max_{i=1,\ldots,k}\{\dim S_i\}$.*
4. *Let S and T two $\widetilde{\mathbb{Q}}$–semi-algebraic sets. Then $\dim(S \times T) = \dim S + \dim T$.*

We can extend the notion of $\widetilde{\mathbb{Q}}$–*semi-algebraic set* for a real algebraic variety X defined over $\widetilde{\mathbb{Q}}$ via the Zariski topology. We said that $S \subset X$ is $\widetilde{\mathbb{Q}}$–semi-algebraic if for any chart (U, φ) of X given by an open Zariski set $U \subset X$ and a regular birational map $\varphi : U \to \mathbb{R}^d$, $\varphi(S \cap U)$ is a $\widetilde{\mathbb{Q}}$–semi-algebraic subset of $\mathbb{R}^d$.

Example I.2.29. Let $\mathcal{C} \subset \mathbb{P}^2_{\mathbb{R}}$ a real algebraic curve in the projective plane defined over $\widetilde{\mathbb{Q}}$. Then, it is easy to see that any union of connected components of $\mathbb{P}^2_{\mathbb{R}} \setminus \mathcal{C}$ is a $\widetilde{\mathbb{Q}}$–semi-algebraic set of $\mathbb{P}^2_{\mathbb{R}}$.

Recall that two irreducible algebraic varieties are said $\widetilde{\mathbb{Q}}$–*birationally equivalent* if the fields of fractions of $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ are isomorphic over $\widetilde{\mathbb{Q}}$, and this is equivalent to the existence of a biregular isomorphism from a nonempty Zariski open subset of X onto a Zariki open subset of Y.

Remark I.2.30. Looking at our actual definition of periods as absolutely convergent integrals of $\widetilde{\mathbb{Q}}$–rational functions over $\widetilde{\mathbb{Q}}$–semi-algebraic domains, the $\widetilde{\mathbb{Q}}$–*birational geometry* will be a good choice for a general setting. In this case, we are dealing with transformations of the real affine defined outside a real divisor, for which the class of integrals of $\widetilde{\mathbb{Q}}$–rational functions over $\widetilde{\mathbb{Q}}$–semi-algebraic domains is stable. This setting incorporate a lot of geometrical intuition in order to study this kind of integrals.

I.2.5 Coefficients in $\widetilde{\mathbb{Q}}$ and the topology over the real numbers

Following [BCR98], the reader would notice that our definitions do not match exactly with the general setting for Real Algebraic Geometry.

The general theory is constructed over fields R which have similar algebraic properties of $\mathbb{R}$, the *real closed fields*, i.e. fields which admit a unique ordering, such that every positive element has a square root and every polynomial of odd degree has a root. Equivalently, R is a real closed field if and only if $R[i] = R[x]/(x^2+1)$ is an algebraically closed field.

Algebraic and semi-algebraic sets are constructed as subsets of R^d using polynomials in $R[x_1, \ldots, x_d]$, and equally for the maps. Nevertheless, our setting becomes consistent from the fact that $\widetilde{\mathbb{Q}}$ is a real closed field contained in $\mathbb{R}$ and by the Tarski-Seidenberg Principle used, which is stated further down in its geometric form. For a complete and justified review of the Tarski-Seidenberg Principle used by Transfer Tool between real closed extensions, see [BCR98, Chapter 5].

Let K be a real closed extension of R. For a semi-algebraic set $S \subset R^d$ given by a boolean combination $\mathcal{B}(x_1, \ldots, x_d)$ of signs conditions on polynomials on $R[x_1, \ldots, x_d]$, denote by S_K the subset of K^d given by $\{(x_1, \ldots, x_d) \in K^d \mid \mathcal{B}(x_1, \ldots, x_d)\}$.

Proposition I.2.31.

1. *S_K is semi-algebraic in K^d and depends only on the set S and not on the boolean combination $\mathcal{B}(x_1, \ldots, x_d)$.*
2. *The mapping $S \mapsto S_K$ preserves the boolean operators: finite unions, finite intersections and taking complements.*
3. *If $S \subset T$, then $S_K \subset T_K$. Hence $S = \emptyset$ implies $S_K = \emptyset$ and $S = T$ implies $S_K = T_K$.*

Theorem I.2.32 (Tarski-Seidenberg Principle [BCR98, Proposition 5.2.1])**.** *Let $\pi : R^{d+1} \to R^d$ and $\pi_K : K^{d+1} \to K^d$ the projections onto the space of the first d coordinates in R^{d+1} and K^{d+1}, respectively. Then $\pi(S)$ is semi-algebraic in R and $\pi_K(S_K) = (\pi(S))_K$.*

The Tarski-Seidenberg Principle has very important consequences from the point of view of model theory: the theory of real closed fields admits quantifier elimination in the language of ordered fields and this theory is model-complete.

Corollary I.2.33. *Let $(\mathcal{P})$ be a property whose formulation is constructed from expressions "$x \in A_i$", where A_i is a semi-algebraic subset of some R^{d_i}, using a finite number of conjunctions, disjunctions and negations, and a finite number of universal and existential quantificators over variables ranging over semi-algebraic subsets B_j of R^{d_j}. Let $(\mathcal{P}_K)$ be the property obtaining by replacing A_i and B_j occurring in $(\mathcal{P})$ by their extensions $(A_i)_K$ and $(B_j)_K$ to K. Then:*

$$(P) \text{ holds true in } R \iff (P_K) \text{ holds true in } K.$$

Making use of this fact, we can also well-extend semi-algebraic mappings.

Proposition I.2.34. *Let $f : A \to B$ be a $\widetilde{\mathbb{Q}}$–semi-algebraic mapping. Let C and D be two semi-algebraic subsets of A and B, respectively. Then*

1. *$(\Gamma_f)_K$ is the graph of a semi-algebraic mapping $f_K : A_K \to B_K$, i.e. $(\Gamma_f)_K = \Gamma_{f_K}$.*
2. *$(f(C))_K = f_K(C_K)$ and $(f^{-1}(D))_K = f_K^{-1}(D_K)$.*

3. *f is injective (resp. surjective) if and only if f_K is injective (resp. surjective).*

Finally, we can also assure that the extension behaves well with respect to topology:

Proposition I.2.35. *1. A semi-algebraic set S is open (resp. closed) in R^d if and only if S_K is open (resp. closed) in K^d. Moreover, $\overline{S}_K = \overline{S_K}$.*

2. A semi-algebraic map f is continuous if and only if f_K is continuous.

3. A semi-algebraic set S is closed and bounded in R^d if and only if S_K is closed and bounded in K^d.

Remark I.2.36. It is worth noticing that the topology of $\widetilde{\mathbb{Q}}^d$ is completely different from those of $\mathbb{R}^d$, since $\widetilde{\mathbb{Q}}$ is a totally disconnected field. Nevertheless, we can give a basis for the euclidean topology in $\mathbb{R}^d$ by balls of radius in $\widetilde{\mathbb{Q}}$, which are in fact extensions of $\widetilde{\mathbb{Q}}$–semi-algebraic sets (Example I.2.4).
Despite the previous fact, we need to take care with some topological concepts as the notion of *compactness*. By the Heine–Borel Theorem, we now that this notion is equivalent to "closed and bounded" for subsets of the affine real space, but this is not true in general for real closed fields. For example, the interval $[0,1]$ is not compact in $\widetilde{\mathbb{Q}}$, since we can construct an open cover of semi-algebraic subsets $\{[0,r)\cap(s,1] \mid r,s \in \widetilde{\mathbb{Q}} \text{ and } 0<r<\pi/4<s<1\}$ for which is impossible to extract a finite subcover.

Notation. From now on, let use simply *(semi-)algebraic sets* and *(semi-)algebraic maps* to refer to $\widetilde{\mathbb{Q}}$–(semi-)algebraic sets and maps defined in $\mathbb{R}^d$. We also consider that we work with the euclidean and Zariski topology of $\mathbb{R}^d$, and we use the terminology with refers to these topological spaces. We do the same for the measure theory in $\mathbb{R}^d$ given by objects defined with coefficients in $\widetilde{\mathbb{Q}}$.

Notation. We denote by $\mathcal{SA}^d_{\widetilde{\mathbb{Q}}}$ the class of d–dimensional semi-algebraic subsets in $\mathbb{R}^d$, i.e. $\widetilde{\mathbb{Q}}$–semi-algebraic sets with nonempty interior in the euclidean topology of $\mathbb{R}^d$.

I.3 Periods

We introduce the class of numbers called *periods*, presenting some of the different definitions and its relations appearing in the original paper of M. Kontsevich and D. Zagier of 2001, [KZ01].

I.3.1 Periods of Kontsevich-Zagier

Definition I.3.1. A *period of Kontsevich-Zagier* (also called *effective period*) is a complex number whose real and imaginary parts are values of absolutely convergent integral of rational functions over domains in a real affine space given by polynomial inequalities both with coefficients in $\widetilde{\mathbb{Q}}$, i.e. absolutely convergent integrals of the form

$$\mathcal{I}(S,P/Q) = \int_S \frac{P(x_1,\ldots,x_d)}{Q(x_1,\ldots,x_d)} \cdot \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d \tag{I.4}$$

where $S \subset \mathbb{R}^d$ is a d–dimensional $\widetilde{\mathbb{Q}}$–semi-algebraic set and $P,Q \in \widetilde{\mathbb{Q}}[x_1,\ldots,x_d]$ are coprimes.

We denote by $\mathcal{P}_{\mathrm{KZ}}$ the set of periods of Kontsevich-Zagier and by $\mathcal{P}^{\mathbb{R}}_{\mathrm{KZ}} = \mathcal{P}_{\mathrm{KZ}} \cap \mathbb{R}$ the set of real periods. This kind of numbers are constructible, in the sense of a period is associated

directly with a set of integrands and domains of integrations given by polynomials of real algebraic coefficients.

Remark I.3.2. As we have noticed in Section I.1, the original definition of periods in [KZ01, p. 3] involve polynomials and rational functions with coefficients in $\mathbb{Q}$, but the authors remark straightaway that we can replace use real algebraic coefficients and algebraic functions without changing the set of numbers we are describing.

We are interested to study periods from the point of view of top-dimensional differential forms and measure theory in $\mathbb{R}^d$, using techniques of real birational geometry, thus we place our setting in the study of rational functions. The choice of in $\widetilde{\mathbb{Q}}$ comes principally from two reasons:

- The use of powerful and algorithmic constructions and techniques of real algebraic geometry for real closed fields.
- The choice of $\mathbb{Q}$ as field of coefficients increase the *complexity* of the semi-algebraic domains in terms of its description, as we can see in the next Example. This consideration is fundamental in the study of a *degree theory* for periods, discussed in Chapter IV.

Example I.3.3. The set of periods $\mathcal{P}_{\text{KZ}}$ contains the algebraic numbers $\overline{\mathbb{Q}}$. Let $\alpha \in \widetilde{\mathbb{Q}}$ be positive, then from Definition I.3.1

$$\alpha = \int_{0<x<\alpha} 1 \, \mathrm{d}x \in \mathcal{P}_{\text{KZ}}^{\mathbb{R}}$$

In fact, we can see that the interval $(0, \alpha) = \{x \in \mathbb{R} \mid 0 < x < \alpha\}$ is also a $\mathbb{Q}$–semi-algebraic subset of $\mathbb{R}$. Taking $P_\alpha \in \mathbb{Q}[x]$ the minimal polynomial of α, we know that P_α have only simple roots (because it's minimal) so P_α has zero in α and it change his sign. There exist $q_1, q_2 \in \mathbb{Q}$ positives such that $q_1 < \alpha < q_2$ and P_α is strictly monotone in $[q_1, q_2]$ (we can find such numbers using, for example, a Sturm's sequence of P_α). Suppose P_α is strictly increasing in $[q_1, q_2]$ (in other case, we can consider $-P_\alpha$), we have

$$\{0 < x < \alpha\} = \{0 < x \leq q_1\} \cup [\{q_1 < x < q_2\} \cap \{P_\alpha(x) < 0\}]$$

Example I.3.4.

1. The transcendental number π can be written in well-know several ways:

$$\pi = \int_{\{x^2+y^2\leq 1\}} 1 \, \mathrm{d}x\mathrm{d}y = \int_{-\infty}^{\infty} \frac{1}{1+x^2} \, \mathrm{d}x = \int_{\{(1-x^2)y^2<1\}} \frac{\mathrm{d}x\mathrm{d}y}{2}.$$

2. Let $\alpha \in \widetilde{\mathbb{Q}}$ such that $\alpha > 1$. Then:

$$\log(\alpha) = \int_1^\alpha \frac{\mathrm{d}t}{t} = \int_{\substack{1<x<\alpha \\ 0<xy<1}} 1 \, \mathrm{d}x\mathrm{d}y.$$

3. Some special values of the Gamma function are periods. Let $p, q \in \mathbb{N}$, using the relation

between Gamma and Beta functions:

$$\begin{aligned}\Gamma\left(\frac{p}{q}\right)^q &= \Gamma(p)\prod_{i=1}^{q-1}\mathrm{B}\left((i-1)\frac{p}{q},\frac{p}{q}\right)\\ &= (p-1)!\prod_{i=1}^{q-1}\int_0^1 t_i^{\frac{(i-1)p-q}{q}}(1-t_i)^{\frac{p-q}{q}}\mathrm{d}t_i\\ &= (p-1)!\int_{[0,1]^{q-1}}\left(\prod_{i=1}^{q-1}t_i^{(i-1)p-q}(1-t_i)^{(p-q)}\right)^{\frac{1}{q}}\mathrm{d}t_1\cdots\mathrm{d}t_{q-1}\\ &= \int_S (p-1)!\ \mathrm{d}t_1\cdots\mathrm{d}t_{q-1}\mathrm{d}t\end{aligned}$$

where

$$S=\left\{(t_1,\ldots,t_{q-1},t)\in[0,1]^{q-1}\times\mathbb{R}\;\middle|\;0<t^q<\prod_{i=1}^{q-1}t_i^{(i-1)p-q}(1-t_i)^{(p-q)}\right\}.$$

Note that the Gamma function is in a fact a transcendental function.

4. The Goncharov integral:

$$\int_{[0,1]^n}\frac{\prod_{j=1}^n x_j^{r_j}(1-x_j)^{s_j}\mathrm{d}x_j}{\prod_{1\le i<j\le n}(1-x_i\cdots x_j)^{t_{i,j}}}\in\mathcal{P}_{\mathrm{KZ}}.$$

5. Elliptic integrals and Feynmann integrals are also periods.

Example I.3.5 (Multiple Zeta Values)**.** Other examples of periods are the Multiple zeta values (MZV):

$$\zeta(s_1,\ldots,s_k)=\sum_{n_1>n_2>\cdots>n_k>0}\frac{1}{n_1^{s_1}\cdots n_k^{s_k}}$$

for $s_1,\ldots,s_k$ positive integers and $s_1>1$. These numbers, properties and representations are studied in a combinatorial way by expressing these series as iterated integrals of two kind of very simple rational functions over simplexes. Define the 1–forms $\omega_0(t)=\mathrm{d}t/t$, $\omega_1(t)=\mathrm{d}t/(1-t)$ and the simplex $\Delta_d=\{(t_1,\ldots,t_d)\in\mathbb{R}^d\mid 1>t_1>\cdots>t_d>0\}$. Denoting $|\underline{s}|=s_1+\ldots+s_k$, then

$$\begin{aligned}\zeta(s_1,\ldots,s_k) &= \int_{1>t_1>\cdots>t_{s_1}>0}\omega_0^{s_1-1}\omega_1\int_{t_{s_1}>\cdots>t_{s_1+s_2}>0}\omega_0^{s_2-1}\omega_1\cdots\int_{t_{|\underline{s}|-s_k}>\ldots>t_{|\underline{s}|}>0}\omega_0^{s_k-1}\omega_1\\ &= \int_{\Delta_{|\underline{s}|}}\omega_0^{s_1-1}\omega_1\omega_0^{s_2-1}\omega_1\cdots\omega_0^{s_k-1}\omega_1,\end{aligned}$$

which an integral of a rational function over a semi-algebraic set, given both of them by linear factors. In fact, this is an example of Aomoto periods.

We can generalize this construction to show that *multiple polylogarithm values* in one variable of algebraic numbers are also periods. Let $x\in[0,1]$,

$$\mathrm{Li}_{\underline{s}}(x)=\sum_{n_1>n_2>\cdots>n_k>0}\frac{x^{n_1}}{n_1^{s_1}\cdots n_k^{s_k}}=\int_{\Delta_{|\underline{s}|}(x)}\omega_0^{s_1-1}\omega_1\omega_0^{s_2-1}\omega_1\cdots\omega_0^{s_k-1}\omega_1,$$

where $\Delta_d(x)=\{(t_1,\ldots,t_d)\in\mathbb{R}^d\mid x>t_1>\cdots>t_d>0\}$ for $d\ge 1$. Look at [Wal00b] for more details.

I.3.2 First algebraic and geometric properties

From Definition I.3.1, we can obtain a first geometric result relating volume of semi-algebraic sets and periods. This result is very useful in order to prove that $\mathcal{P}_{\text{KZ}}$ is stable by sums, difference and product. Part of these results follows from previous works of M. Monier [Mon02] and N. Koehl and P. Pichery [KP04].

Proposition I.3.6 (Semi-algebraic pair's representation). *Let $p \in \mathbb{R}$. Then $p \in \mathcal{P}_{\text{KZ}}^{\mathbb{R}}$ if and only there exist two disjoint semi-algebraic sets S_1 and S_2 in $\mathbb{R}^d$, for some $d \in \mathbb{N}$, such that*

$$p = \text{vol}_d(S_1) - \text{vol}_d(S_2)$$

where vol_d is the canonical volume map of $\mathbb{R}^d$ and S_1 and S_2 have finite volume.

Proof. Suppose $p \neq 0$, since the case $p = 0$ is trivial. By definition of real period, there exist a $S \in \mathcal{SA}_{\widetilde{\mathbb{Q}}}^d$ and $0 \neq P/Q \in \widetilde{\mathbb{Q}}(x_1, \ldots, x_n)$ for some $d \in \mathbb{N}$ such that

$$p = \mathcal{I}(S, P/Q) = \int_S \frac{P(x_1, \ldots, x_d)}{Q(x_1, \ldots, x_d)} \cdot \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d,$$

where $\mathcal{I}(S, P/Q)$ is absolutely convergent. According the sign of $P/Q(x_1, \ldots, x_n)$, we can give a disjoint semi-algebraic partition of S by $S^{\pm} = \left\{(x_1, \ldots, x_d) \in S \mid \text{sgn}(\frac{P}{Q}(x_1, \ldots, x_d)) = \pm 1\right\}$:

$$p = \mathcal{I}(S, P/Q) = \mathcal{I}(S^+, P/Q) - \mathcal{I}(S^-, -P/Q)$$

Note that the values of $\mathcal{I}(S^+, P/Q)$ and $\mathcal{I}(S^-, -P/Q)$ are non-negative real numbers, since the integral $\mathcal{I}(S, P/Q)$ is supposed to be absolutely convergent and both of integrand functions are non-negative over its corresponding domain. Considering integrals by the volume of the region delimited by $\pm P/Q$ over each domain $S^{\pm}$ we perform a change of variables over each integral obtaining:

$$\mathcal{I}(S^{\pm}, P/Q) = \int_{S_1} 1 \ \mathrm{d}x_1 \cdots \mathrm{d}x_d \mathrm{d}t - \int_{S_2} 1 \ \mathrm{d}x_1 \cdots \mathrm{d}x_d \mathrm{d}t$$

where

$$S_1 = \left\{ (x_1, \ldots, x_d, t) \in S \times \mathbb{R} \;\middle|\; t \geq 0, \ t \leq \frac{P}{Q}(x_1, \ldots, x_d) \right\},$$
$$S_2 = \left\{ (x_1, \ldots, x_d, t) \in S \times \mathbb{R} \;\middle|\; t \leq 0, \ t \geq \frac{P}{Q}(x_1, \ldots, x_d) \right\},$$

It remains to prove that S_1, S_2 are semi-algebraic sets. We define $H = t \cdot Q - P \in \widetilde{\mathbb{Q}}[t, x_1, \ldots, x_d]$, then $\{t < P/Q(x_1, \ldots, x_d)\}$ is expressed as the union of

$$\{H(t, x_1, \ldots, x_d) < 0\} \cap \{Q(x_1, \ldots, x_d) > 0\}$$

and

$$\{H(t, x_1, \ldots, x_d) > 0\} \cap \{Q(x_1, \ldots, x_d) < 0\}.$$

Thus $S_1 \in \mathcal{SA}_{\widetilde{\mathbb{Q}}}^{d+1}$ since semi-algebraic domains are stable by finite union and intersection. Analogously, $S_2 \in \mathcal{SA}_{\widetilde{\mathbb{Q}}}^{d+1}$.

Reciprocally, consider that $p \in \mathbb{R}$ can be described as

$$p = \int_{S_1} \mathrm{d}x_1 \cdots \mathrm{d}x_d - \int_{S_2} \mathrm{d}x_1 \cdots \mathrm{d}x_d$$

with S_1, S_2 given by the hypotheses of the statement. We can express:

$$\begin{aligned} p &= \int_{S_1} 1 \ \mathrm{d}x_1 \cdots \mathrm{d}x_d - \int_{S_2} 1 \ \mathrm{d}x_1 \cdots \mathrm{d}x_d \\ &= \int_{S_1} \left(\int_{-1}^{0} 2t \ \mathrm{d}t \right) \ \mathrm{d}x_1 \cdots \mathrm{d}x_d + \int_{S_2} \left(\int_{0}^{1} 2t \ \mathrm{d}t \right) \ \mathrm{d}x_1 \cdots \mathrm{d}x_d \end{aligned}$$

Since S_1 and S_2 have finite measure, we can use Fubini Theorem in order to obtain:

$$p = \int_{T_1 \cup T_2} 2t \ \mathrm{d}x_1 \cdots \mathrm{d}x_d \mathrm{d}t$$

where $T_1 = (0,1) \times S_1$ and $T_1 = (-1,0) \times S_2$ are disjoint semi-algebraic sets. Thus, $p \in \mathcal{P}_{\mathrm{KZ}}^{\mathbb{R}}$ by definition. □

Remark I.3.7. The previous Proposition is very useful in order to get a more geometric vision of periods. This result is improved in the next Chapter, focusing in the geometry properties and the algorithmic of the process taken in order to obtain such a representation.

Using Proposition I.3.6, we can easily prove that periods posses an algebra structure.

Theorem I.3.8. *The periods of Kontsevich-Zagier $\mathcal{P}_{\mathrm{KZ}}$ form a countable $\overline{\mathbb{Q}}$-algebra.*

Proof. We have already see that $\overline{\mathbb{Q}} \subset \mathcal{P}_{\mathrm{KZ}}$. The polynomial ring $\widetilde{\mathbb{Q}}[x_1, \ldots, x_d]$ is a countable set, then $\widetilde{\mathbb{Q}}(x_1, \ldots, x_d)$ and $\mathcal{SA}_{\widetilde{\mathbb{Q}}}^{d}$ are also countable set, for any $d \in \mathbb{N}$. In this way, $\mathcal{P}_{\mathrm{KZ}}^{\mathbb{R}}$ is countable and also $\mathcal{P}_{\mathrm{KZ}}$.

It suffices to prove that $\mathcal{P}_{\mathrm{KZ}}^{\mathbb{R}}$ is an algebra. The stability of product between two periods is given directly by Fubini Theorem. Let p_1 and p_2 be two real periods. By Proposition I.3.6, there exists S_1, S_2 and T_1, T_2 semi-algebraic subsets of $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, respectively and for some $d_1, d_2 \in \mathbb{N}$, such that

$$p_1 = \mathrm{vol}_{d_1}(S_1) - \mathrm{vol}_{d_1}(S_2) \quad \text{and} \quad p_2 = \mathrm{vol}_{d_2}(T_1) - \mathrm{vol}_{d_2}(T_2),$$

with each subset being of finite volume and $S_1 \cap S_2 = T_1 \cap T_2 = \emptyset$. Note that we can suppose that $d_1 = d_2 = d$, since it suffices to take the Cartesian product with a certain kth-power of the unity interval $[0,1]^k$ in order to semi-algebraic subsets with same volume lying on $\mathbb{R}^{d_1+k}$ or $\mathbb{R}^{d_2+k}$. In this way, we can express:

$$\begin{aligned} p_1 + p_2 &= \mathrm{vol}_d(S_1) + \mathrm{vol}_d(T_1) - \mathrm{vol}_d(S_2) - \mathrm{vol}_d(T_2) \\ &= \mathrm{vol}_d(S_1 \cup T_1) + \mathrm{vol}_d(S_1 \cap T_1) - \mathrm{vol}_d(S_2 \cup T_2) - \mathrm{vol}_d(S_2 \cap T_2) \end{aligned}$$

Define $A_i = S_i \cup T_i$ and $B_i = S_i \cap T_i$, for $i = 1, 2$. Taking the Cartesian product of previous semi-algebraic sets by disjoint intervals $I_1, I_2, I_3, I_4 \subset \mathbb{R}$ of length 1 and extrema in $\widetilde{\mathbb{Q}}$, we obtain:

$$p_1 + p_2 = \mathrm{vol}_d((A_1 \times I_1) \cup (B_1 \times I_2)) - \mathrm{vol}_d((A_2 \times I_3) \cup (B_2 \times I_4)) \in \mathcal{P}_{\mathrm{KZ}}.$$

□

Remark I.3.9. As well as the product of two integrals of type (I.4) can be easily expressed as an integral of the same kind using Fubini Theorem, the stability of $\mathcal{P}_{\mathrm{KZ}}$ by the sum is very hard to prove directly from Definition I.3.1.

Finally, we give an explicit proof of an assertion about the first definition of periods given in [KZ01, p.3]: in Definition I.3.1, *we can change "rational functions" by "algebraic functions" without change the set of numbers which one obtains.* We obtain this result by using Proposition I.3.6, following the same geometrical spirit of see periods as difference of volumes of semi-algebraic sets.

Proposition I.3.10. *Let $S \subset \mathbb{R}^d$ a semi-algebraic set and let $f : S \to \mathbb{R}$ be an algebraic function defined in the interior of S. If the integral*

$$\int_S f(x_1, \ldots, x_d) \cdot \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d$$

converges absolutely, then its value belongs to $\mathcal{P}_{\mathrm{KZ}}$.

Proof. Suppose that

$$\mathcal{I}(S, f) = \int_S f(x_1, \ldots, x_d) \cdot \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d$$

is a non-zero absolutely convergent integral. We construct a disjoint semi-algebraic partition of S according the sign of $f(x_1, \ldots, x_n)$ by $S^{\pm} = \{(x_1, \ldots, x_d) \in S \mid \operatorname{sgn} f(x_1, \ldots, x_d) = \pm 1\}$:

$$p = \mathcal{I}(S, P/Q) = \mathcal{I}(S^+, f) - \mathcal{I}(S^-, -f)$$

As before, the values of $\mathcal{I}(S^+, f)$ and $\mathcal{I}(S^-, -f)$ are non-negative real numbers.

As f is an algebraic function over the interior of S, there exist a monic polynomial $P = y^n + a_1 y^{n-1} + \ldots + + a_{n-1} y + a_n \in \widetilde{\mathbb{Q}}(x_1, \ldots, x_d)[y]$ such that

$$P_{(x_1,\ldots,x_d)}(f) = f(x_1, \ldots, x_d)^n + \sum_{i=1}^{n} a_i(x_1, \ldots, x_d) f(x_1, \ldots, x_d)^{n-i} = 0, \tag{I.5}$$

for any point $(x_1, \ldots, x_d) \in S$. Note that $a_i \neq 0$ in S, for any $i = 1, \ldots, n$. We go to express the previous integrals as volumes of subsets of $S \times \mathbb{R}$ whose boundary is contained in the semi-algebraic set $\{(x_1, \ldots, x_d, t) \in S \times \mathbb{R} \mid t = 0 \text{ or } t = P_{(x_1,\ldots,x_d)}(t)\}$. We prove that they are in fact semi-algebraic sets in $\mathbb{R} \times S$ using Sturm sequences to determine which is the branch of the algebraic set $\{t - P_{(x_1,\ldots,x_d)}(t) = 0\}$ which correspond to the graph of the algebraic function f.

Let $\Delta(x_1, \ldots, x_d)$ be the discriminant of $P_{(x_1,\ldots,x_d)}$. For $i = 1, 2$, define $X = S \cap \Delta^{-1}(0)$, which is a semi-algebraic set of dimension strictly smaller than S, thus a zero-measure set in $\mathbb{R}^d$.

Let C be a connected component of $S^+ \setminus X$. For any $(x_1, \ldots, x_d)$, $P_{(x_1,\ldots,x_d)}$ has only simple roots in C, which are all contained in the interval $[-M(x_1, \ldots, x_d), M(x_1, \ldots, x_d)]$ given in Lemma I.2.17 by

$$M(x_1, \ldots, x_d) = 1 + |a_{n_1}(x_1, \ldots, x_d)| + \cdots + |a_0(x_1, \ldots, x_d)|.$$

As a consequence, we can assume that $f(x_1, \ldots, x_d)$ if the kth–root of $P_{(x_1,\ldots,x_d)}$ in ascending order over $\mathbb{R}$, where k is independent of $(x_1, \ldots, x_d)$. In this way, the subset

$$V = \left\{(x_1, \ldots, x_d, t) \in C \times \mathbb{R}^+ \;\middle|\; t \leq f(x_1, \ldots, x_d)\right\}, \tag{I.6}$$

can be expressed as

$$\begin{aligned} V &= \{(x_1,\dots,x_d,t)\in C\times\mathbb{R}^+ \mid v_{P_{(x_1,\dots,x_d)}}(-M(x_1,\dots,x_d)) - v_{P_{(x_1,\dots,x_d)}}(t) < k\} \\ &= \bigcup_{i=0}^{k-1}\{(x_1,\dots,x_d,t)\in C\times\mathbb{R}^+ \mid v_{P_{(x_1,\dots,x_d)}}(-M(x_1,\dots,x_d)) - v_{P_{(x_1,\dots,x_d)}}(t) = i\} \\ &= \bigcup_{i=0}^{k-1}\bigcup_{j=i}^{s}\{(x_1,\dots,x_d,t)\in C\times\mathbb{R}^+ \mid v_{P_{(x_1,\dots,x_d)}}(-M(x_1,\dots,x_d)) = j,\ v_{P_{(x_1,\dots,x_d)}}(t) = i-j\} \end{aligned}$$

From Proposition I.2.18, V is a semi-algebraic set.

We know from Theorem I.2.24 that S^+ posses a finite number of semi-algebraically connected components $C_1,\dots,C_m$. For each C_i of these connected components, we can define the set V_i as in (I.6). Note that $V_i\cap V_j=\emptyset$ for any $i,j=1,\dots,m$. Thus, defining $W^+=V_1\cup\cdots\cup V_m$ and performing a transformation by using Fubini formula:

$$\mathcal{I}(S^+,f)=\sum_{i=1}^{m}\int_{C_i} f(x_1,\dots,x_d)\,\mathrm{d}x_1\cdots\mathrm{d}x_d=\sum_{i=1}^{m}\int_{W_i}1\,\mathrm{d}x_1\cdots\mathrm{d}x_d\mathrm{d}t=\mathrm{vol}_{d+1}(W^+).$$

Analogously, we can obtain a semi-algebraic set W^- in $\mathbb{R}^{d+1}$ such that $\mathcal{I}(S^-,-f)=\mathrm{vol}_{d+1}(W-)$. In this way,

$$\mathcal{I}(S,f)=\mathrm{vol}_{d+1}(W^+)-\mathrm{vol}_{d+1}(W^-)\in\mathcal{P}^{\mathbb{R}}_{\mathrm{KZ}},$$

from Proposition I.3.6. □

I.3.3 Abstract periods

In [KZ01], M. Kontsevich and D. Zagier give a definition of a (in principle) more general class of periods, related to the field of algebraic geometry.

Consider the quadruples (X,E,ω,γ) composed by the following data:

- X a smooth algebraic variety of dimension d defined over $\mathbb{Q}$.
- $E\subset X$ a divisor with normal crossings.
- $\omega\in\Omega^d(X)$ an algebraic differential form on X of top degree.
- $\gamma\in H_d(X(\mathbb{C}),E(\mathbb{C});\mathbb{Q})$ a (homology class of a) singular chain on the complex manifold $X(\mathbb{C})$ with boundary on the divisor $E(\mathbb{C})$.

Definition I.3.11. The space $\mathcal{P}_{\mathsf{abs}}$ of *abstract periods* is defined as a vector space over $\mathbb{Q}$ generated by the symbols $[(X,E,\omega,\gamma)]$ representing equivalence classes of quadruples as above, by an equivalence relation generated by the following relations:

1. **(Linearity)** $[(X,E,\omega,\gamma)]$ is $\mathbb{Q}$-linear in both ω and γ, i.e. for any $\lambda,\mu\in\mathbb{Q}$, $\omega_1,\omega_2\in\Omega^d(X)$ and $\gamma_1,\gamma_2\in H_d(X(\mathbb{C}),E(\mathbb{C});\mathbb{Q})$:

$$[(X,E,\lambda\omega_1+\mu\omega_2,\gamma)]=\lambda[(X,E,\omega_1,\gamma)]+\mu[(X,E,\omega_2,\gamma)]$$

and

$$[(X,E,\omega,\lambda\gamma_1+\mu\gamma_2)]=\lambda[(X,E,\omega,\gamma_1)]+\mu[(X,E,\omega,\gamma_2)].$$

2. **(Change of variables)** If $f : (X_1, E_1) \to (X_2, E_2)$ is a morphism of pairs defined over $\mathbb{Q}$, $\gamma_1 \in H_n(X_1(\mathbb{C}), E_1(\mathbb{C}); \mathbb{Q})$ and $\omega_2 \in \Omega^d(X_2)$, then
$$(X_1, E_1, f^*\omega_2, \gamma_1) \sim (X_2, E_2, \omega_2, f_*\gamma)$$

3. **(Stokes formula)** Denote by $\tilde{E}$ the normalization of E (i.e. locally, it is the disjoint union of irreducible components of E), the variety $\tilde{E}$ contains a normal crossing divisor $\tilde{E}_1$ coming from double points in E. If $\beta \in \Omega^{d-1}(X)$ and $\gamma \in H_d(X(\mathbb{C}), E(\mathbb{C}); \mathbb{Q})$, then
$$(X, E, \mathrm{d}\beta, \gamma) \sim (\tilde{E}, \tilde{E}_1, \beta_{|\tilde{E}}, \partial\gamma)$$
where $\partial : H_d(X(\mathbb{C}), E(\mathbb{C}); \mathbb{Q}) \to H_{d-1}(\tilde{E}(\mathbb{C}), \tilde{E}_1(\mathbb{C}); \mathbb{Q})$ is the boundary operator.

Remark I.3.12. The previous equivalence relations arise naturally in order to consider the values of the integrals defined by $\int_\gamma \omega$ and rise up into a conjecture for periods detailed in the next section.

We define the inner product between elements of the previous equivalence classes. For $[(X_i, E_i, \omega_i, \gamma_i)] \in \mathcal{P}_{\mathsf{abs}}$, take $p_i : X_1 \times X_2 \to X_i$ the natural projections for $i = 1, 2$, then we associate as the product of the elements above
$$[(X_1 \times X_2, (X_1 \times E_2) \cup (E_1 \times X_2), p_1^*\omega_1 \wedge p_2^*\omega_2, \gamma_1 \otimes \gamma_2)]$$
where $\gamma_1 \otimes \gamma_2$ is an element of
$$H_{d_1}(X_1(\mathbb{C}), E_1(\mathbb{C}); \mathbb{Q}) \otimes H_{d_2}(X_2(\mathbb{C}), E_2(\mathbb{C}); \mathbb{Q})$$
which is contained in $H_{d_1+d_2}((X_1 \times X_2)(\mathbb{C}), (X_1 \times E_2) \cup (E_1 \times X_2)(\mathbb{C}); \mathbb{Q})$.

Property I.3.13. *The $\mathbb{Q}$-vector space $\mathcal{P}_{\mathsf{abs}}$ form an algebra with the product described above.*

Consider now, the *evaluation morphism* of algebras $\mathrm{ev} : \mathcal{P}_{\mathsf{abs}} \to \mathbb{C}$ related to the integral defined by each quadruple:
$$[(X, E, \omega, \gamma)] \longmapsto \int_\gamma \omega$$

Theorem I.3.14 ([KZ01, Sec. 4, p. 31])**.** *The evaluation morphism* ev *is an epimorphism over $\mathcal{P}_{\mathrm{KZ}}$.*

Remark I.3.15. The equivalence notion given in last theorem comes of consider that we can always deform γ in a semi-algebraic chain. We can also replace $\mathbb{Q}$ by $\overline{\mathbb{Q}}$ without problem and we obtain similar results (see [KZ01]). But there is an imprecision in relation with possible singularities in the boundary of γ, which can compromise the convergence of the integral and the belonging in effective periods taking the evaluation morphism. Some well-know examples of periods, as multi-zeta values, are defined using integrals with poles on the boundary of the semi-algebraic domain. In [BB03], P. Belkale and P. Brosnan fix this imprecision in language of effective periods, using resolution of singularities in characteristic 0.

Theorem I.3.16 ([BB03, Thm. 2.6])**.** *Let X be an algebraic variety over $\widetilde{\mathbb{Q}}$, $F \subset X$ a reduced effective divisor and $\omega \in \Omega^n(X - F)$. If $\Delta \subset X(\mathbb{R})$ is a semi-algebraic domain with non-empty interior, then the integral $\int_\Delta \omega \in \mathcal{P}_{\mathrm{KZ}}^{\mathbb{R}}$ provided that it is absolutely convergent.*

Thus, $\mathcal{P}_{\mathrm{KZ}}$ is generated by integral evaluations of elements in $\mathcal{P}_{\mathsf{abs}}$. The structure described above for the effective periods permit us to give an equivalent formulation of Conjecture I.4.1 for the equality of two periods.

Conjecture I.3.17 ([KZ01, Chpt. 4, p. 31]). *The evaluation morphism $\mathcal{P}_{\mathsf{abs}} \to \mathcal{P}_{\mathrm{KZ}}$ is an isomorphism.*

Example I.3.18. Let $\alpha \in \overline{\mathbb{Q}}$, we go to see that $\alpha \in \operatorname{Im} \mathrm{ev}$. Take $p \in \mathbb{Q}[x]$ the minimal polynomial of α and consider $X = \{p = 0\}$. Trivially, α is a closed point in $X(\mathbb{C})$ and $\omega = x\,\mathrm{d}x \in \Omega^0(X(\mathbb{C}))$. Any closed point $x \in X(\mathbb{C})$ gives us an element of 0th class homology $H_0(X(\mathbb{C}); \mathbb{Q})$, then

$$\mathrm{ev}[X(\mathbb{C}), \varnothing, \omega, \{x\}] = \alpha.$$

Remark I.3.19. A fine proof and analysis of the affirmation that $\mathcal{P}_{\mathsf{abs}}$, $\mathcal{P}_{\mathrm{KZ}}$ and the periods of the comparison isomorphism are the same set of numbers is given by B. Friedrich in [Fri05].

I.4 Open problems for periods

The key-point of the philosophy of periods is that these numbers appears in integral form in the comparison between *algebraic/geometric data* of algebraic varieties. Staying in the setting of effective periods, M. Kontsevich and D. Zagier states three main problems in the study and comprehension of this new class of numbers (see [KZ01, pags. 6-8]): one relative to the type of relations between different integral forms of the same period, another relative to how to know algorithmically if two integral representations correspond to the same number, and finally, show a number which is not a period.

I.4.1 The Kontsevich-Zagier period conjecture

We presented in Section I.3 abstract periods $\mathcal{P}_{\mathsf{abs}}$ which are tuples of algebraic/geometrical data modulo some geometrical relations coming from the coefficients $\mathbb{Q}$ and the evaluation morphism. Conjecture I.3.17 states that any algebraic relation between periods are essentially of this nature. In the case of effective periods, these relations are reformulated by Kontsevich and Zagier using the classical operations in integral calculus.

Conjecture I.4.1 (Kontsevich-Zagier period conjecture). *If a real period admits two integral representations, then we can pass from one formulation to the other one utilizing only the next three rules:*

1. **(Sums)** *Let $S \in \mathcal{SA}^d_{\overline{\mathbb{Q}}}$ and f and algebraic function such that $\mathcal{I}(S, f)$ converges absolutely:*

   (a) For any g_1, g_2 algebraic functions in S such that $f = g_1 + g_2$,

   $$\int_S g_1 + g_2 = \int_S g_1 + \int_S g_2.$$

   (b) For any $S_1, S_2 \in \mathcal{SA}^d_{\overline{\mathbb{Q}}}$ such that $S = S_1 \bigsqcup S_2$,

   $$\int_{S_1 \cup S_2} f = \int_{S_1} f + \int_{S_2} f.$$

2. **(Change of variables)** *Let $S \in \mathcal{SA}^d_{\overline{\mathbb{Q}}}$ and f an algebraic map over S. Define $\omega = f \cdot \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d$. For any algebraic function φ of $\mathbb{R}^d$:*

   $$\int_S \omega = \int_{\varphi^{-1} S} \varphi^* \omega,$$

   where $\varphi^*\omega = ((f \circ \varphi) \cdot |J_\varphi|)\,(y_1, \ldots, y_d) \cdot \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_d$ is also a top-dimensional differential form given by an algebraic function.

3. **(Stokes's Formula)** *For $S \in \mathcal{SA}^d_{\widetilde{\mathbb{Q}}}$ an oriented sub-manifold of $\mathbb{R}^d$ and $\alpha \in \Omega^{d-1}(S)$ which has algebraic functions by coefficients, then:*

$$\int_S \mathrm{d}\alpha = \int_{\partial S} \alpha,$$

obtaining classical integrals by partitions and algebraic parametrization of ∂S.

In fact, this Conjecture by M. Kontsevich and D. Zagier is only explicitly formulated in the one dimensional case. In particular, the Stokes formula rule becomes the simple Newton-Leibniz formula:

$$\int_a^b f'(x)\mathrm{d}x = f(b) - f(a).$$

They left to the reader the rigorous formulation in the generalization of these rules to the multidimensional case, only indicating that in all these manipulations *"all functions and domains of integrations are algebraic with coefficients in $\overline{\mathbb{Q}}$"*. This remark has lead to the previous usual formulation in Conjecture I.4.1. However, replacing directly the one-dimensional rules by their classical multidimensional counterpart comes with a series of questions and problems. From the computational point of view, most of these operations are non trivial to manipulate: the Stokes formula is an example of this fact, because most of the times the partitions and the algebraic parametrization of the boundary are not easy to obtain.

Let illustrate the complexity of the Kontsevich-Zagier period conjecture with an example.

Example I.4.2. We have exhibit in Example I.3.4 two different well-known integral expressions of π. Using the rules given in the Kontsevich-Zagier conjecture, we can pass from one expression to the other by the sequence:

$$\pi = \int_{\{x^2+y^2\leq 1\}} 1 \ \mathrm{d}x\mathrm{d}y \overset{\text{(i)}}{=} \int_{-1}^{1} 2\sqrt{1-x^2}\ \mathrm{d}x \overset{\text{(ii)}}{=} \int_{-1}^{1} \frac{\mathrm{d}x}{\sqrt{1-x^2}} \overset{\text{(iii)}}{=} \int_{-\infty}^{\infty} \frac{1}{1+x^2}\ \mathrm{d}x,$$

where:

(i) Let $C = \{x^2 + y^2 \leq 1\}$ be the surface of the closed unity disk at the origin. If we choose an orientation in trigonometrical sense, by Stokes's formula and sums:

$$\pi = \int_C 1 \ \mathrm{d}x\mathrm{d}y = \int_{\partial C} x\ \mathrm{d}y = \int_{\partial C_1} x\ \mathrm{d}y + \int_{\partial C_2} x\ \mathrm{d}y$$

where ∂C_1 and ∂C_2 are semi-circles forming ∂C, in symmetry with the y-axis, containing points $(1,0)$ and $(-1,0)$ respectively and oriented in trigonometrical sense. Taking respective parameterizations $\{(\pm\sqrt{1-t^2}, t),\ -1 \leq t \leq 1\}$, we have:

$$\int_C 1 \ \mathrm{d}x\mathrm{d}y = \int_{-1}^{1} \sqrt{1-t^2}\ \mathrm{d}t + \int_{-1}^{1} \sqrt{1-t^2}\ \mathrm{d}t = \int_{-1}^{1} 2\sqrt{1-t^2}\ \mathrm{d}t$$

(ii) Decomposing by sums in the integrand and in the domain:

$$\begin{aligned}\int_{-1}^{1} 2\sqrt{1-t^2}\ \mathrm{d}t &= \int_{-1}^{1} \sqrt{1-t^2}\ \mathrm{d}t + \int_{-1}^{1} \sqrt{1-t^2}\ \mathrm{d}t \\ &= \int_{-1}^{1} \sqrt{1-t^2}\ \mathrm{d}t + \int_{-1}^{0} \sqrt{1-t^2}\ \mathrm{d}t + \int_{0}^{1} \sqrt{1-t^2}\ \mathrm{d}t\end{aligned}$$

Applying the changes of variables $\varphi_- : (-1,0) \to (-1,0)$ and $\varphi_+ : (0,1) \to (0,1)$ given by $\varphi_\pm(u) = \pm\sqrt{1-u^2}$ to the second and third integrals respectively, followed by the sums rules, we obtain:

$$\begin{aligned}\int_{-1}^{1} 2\sqrt{1-t^2}\,\mathrm{d}t &= \int_{-1}^{1} \sqrt{1-t^2}\,\mathrm{d}t + \int_{-1}^{0} \sqrt{1-t^2}\,\mathrm{d}t + \int_{0}^{1} \sqrt{1-t^2}\,\mathrm{d}t \\ &= \int_{-1}^{1} \sqrt{1-u^2}\,\mathrm{d}u + \int_{-1}^{0} \frac{u^2}{\sqrt{1-u^2}}\,\mathrm{d}u + \int_{0}^{1} \frac{u^2}{\sqrt{1-u^2}}\,\mathrm{d}u \\ &= \int_{-1}^{1} \frac{1-u^2}{\sqrt{1-u^2}}\,\mathrm{d}u + \int_{-1}^{1} \frac{u^2}{\sqrt{1-u^2}}\,\mathrm{d}u \\ &= \int_{-1}^{1} \frac{\mathrm{d}u}{\sqrt{1-u^2}}\end{aligned}$$

(iii) Finally, we perform a last change of variables $\psi_- : (-\infty,0) \to (-1,0)$ and $\psi_+ : (0,\infty) \to (0,1)$ given by

$$\psi_\pm(v) = \pm\frac{1}{\sqrt{1+v^2}} \quad \left(\Longrightarrow\ J_{\psi_\pm}(v) = \mp\frac{v}{(1+v^2)\sqrt{1+v^2}}\right).$$

Separating the domains by the sum rule in order to apply the change of variables:

$$\begin{aligned}\int_{-1}^{1} \frac{1}{\sqrt{1-u^2}} &= \int_{-1}^{0} \frac{\mathrm{d}u}{\sqrt{1-u^2}} + \int_{0}^{1} \frac{\mathrm{d}u}{\sqrt{1-u^2}} \\ &= \int_{-\infty}^{0} \sqrt{\frac{1+v^2}{v^2}} \cdot \frac{v\,\mathrm{d}v}{(1+v^2)\sqrt{1+v^2}} + \int_{0}^{+\infty} \sqrt{\frac{1+v^2}{v^2}} \cdot \frac{v\,\mathrm{d}v}{(1+v^2)\sqrt{1+v^2}} \\ &= \int_{-\infty}^{0} \frac{\mathrm{d}v}{1+v^2} + \int_{0}^{+\infty} \frac{\mathrm{d}v}{1+v^2} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}v}{1+v^2}.\end{aligned}$$

Notation. From now, we refer simply by *Kontsevich-Zagier's period conjecture* the Kontsevich-Zagier period conjecture, and we denote by *KZ–rules* the three operations described above in the Conjecture.

Remark I.4.3. The Kontsevich-Zagier's period conjecture must be understood in the following way: if we want to study the algebraic relations between periods and as $\mathcal{P}_{\mathrm{KZ}}$ is a $\overline{\mathbb{Q}}$–algebra, it suffices to study linear relations over $\overline{\mathbb{Q}}$ because they generate any relation in $\mathcal{P}_{\mathrm{KZ}}$. This conjecture is in the same spirit of the Hodge conjecture, which states that any Hodge cycle in a smooth complex projective manifold X is an *algebraic cycle*, i.e. a linear combination with rational coefficients of the cohomology classes of complex subvarieties of X.

Remark I.4.4. The work developed in the present manuscript focus on the change of variables and partitions of semi-algebraic sets in order to study periods of Kontsevich-Zagier and the Kontsevich-Zagier's period conjecture, in contrast with the work of J. Ayoub [Ayo15], who follows the philosophy of Motifs in order to prove a modified version of the conjecture, "hiding" the change of variables in his formulation (see [Ayo15, Remarque 1.5]).

I.4.2 Admissible identities and Equality algorithm

It is worth noticing that proving the Kontsevich-Zagier's period conjecture does not give automatically a way to find a path between two integral representations of a period: even if

we can prove that two integral representations of a period are related by a finite sequence of the KZ–rules, we are interested to give a *simple* and *explicit* path between these integrals, this give rise to the notion of *accessible identity*. Example I.4.2 exhibit this notion in the case of low dimensional integral representations.

A related problem, which is neither solved by the Kontsevich-Zagier's period conjecture, is to know if two periods are equal or not: can we give an algorithm which determine if two integrals representations of two periods determine the same number? can we base such an algorithm using the KZ–rules? In [KZ01, Problem 1, p. 7], this problem is stated as

Problem I.4.5. *Find an algorithm to determine whether or not two given numbers in $\mathcal{P}_{\text{KZ}}$ are equal.*

This problem is the analogue for periods of the *zero recognition problem for holonomic functions*.

In the other hand, we are continuously assuming that we are placed in the set of periods, and we "know" two given numbers in $\mathcal{P}_{\text{KZ}}$ by its explicit integral representations. Real numbers are described usually in a numerical form with a certain accuracy. How can recognize real numbers as periods? Can we give a way to construct a *simple* integral representation from a period described numerically? As in is original statement in [KZ01, Problem 2, p. 8]:

Problem I.4.6. *Find an algorithm to determine whether a given real number, know numerically to high accuracy, is equal (within that accuracy) to some simple period.*

The notion of "simplicity" of an integral representation of a period we are using stand on the description of the semi-algebraic domain, looking at the dimension and the complexity of the polynomials required on the description. The use of this notion is justified in Chapters II and IV.

I.5 A Liouville-like problem for periods

A natural question which reverse the previous problems is to find a number which does not belong to $\mathcal{P}_{\text{KZ}}$. These numbers exist, since $\mathcal{P}_{\text{KZ}}$ is a countable set. Thus as the case of algebraic numbers, most of the complex numbers are not periods. This gives as an analogue of *Liouville's problem*, i.e. construct an explicit example of a number which is not a period giving also a characterization for periods via rational approximation. This reflects the historical approach used by Liouville to resolve this problem for algebraic numbers in the 19th century (Theorem I.1.1).

I.5.1 Counterexample

It is conjectured [KZ01, p. 8] that some well-know transcendental constants as the Euler number e, the inverse $1/\pi$ and the Euler-Mascheroni constant

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log(n)\right) = 0.57721566490153286060...$$

are not periods. A possible approach to this problem is to determine a *complexity* for periods coming by its constructive nature. This approach is used by M. Yoshinaga in [Yos08] to determine that periods are *computable numbers* in the sense that a machine can *well-approximate* these

numbers by elementary functions. Then, using a Cantor-diagonal argument, he is able to exhibit a real number α which is not belonging to $\mathcal{P}_{\text{KZ}}$, for which he gives the first 30 digits of:

$$\frac{\alpha}{2} = 0.38883222177382464125624300958 1\ldots \notin \mathcal{P}_{\text{KZ}}.$$

The basic notions of elementary numbers and a idea of the construction of this number is detailed in the next Section.

I.5.2 About Yoshinaga's construction of a non-period number

We follow the presentation given by M. Yoshinaga in [Yos08].

Elementary computable real numbers

The notion of *computable numbers* was introduced by A. Turing in [Tur36]. This notion is a tentative to formalize the fact for a number to be *computable by a machine*.

Definition I.5.1. Let $\mathcal{E} \subset \mathbb{N}^{\mathbb{N}}$ be a class of functions. A positive real number $\alpha \in \mathbb{R}$ is said *$\mathcal{E}$-computable* if here exist $a(n), b(n), c(n) \in \mathcal{E}$ such that, for any integer $k > 0$:

$$\left| \frac{a(n)}{b(n)+1} - \alpha \right| < \frac{1}{k}, \quad \forall n \geq c(k). \tag{I.7}$$

We denote by $\mathbb{R}_{\mathcal{E}}$ the set of real $\mathcal{E}$-computable numbers.

We describe the central class of functions of study of periods: the elementary functions (Elem). Consider functions $f : \mathbb{N}^n \to \mathbb{N}$ of any number of arguments. We start with the simplest elementary functions.

Definition I.5.2. The class (Elem) of elementary functions is the smallest class of functions

(i) containing the initial functions:

- The *zero* function: $o(x) = 0$.
- The *successor* function: $s(x) = x + 1$.
- The *ith-projection* function: $\pi_i^d(x_1, \ldots, x_d) = x_i$.

as well as the sum $(+)$, the product $(\cdot)$ and the *modified subtraction*:

$$m(x, y) = x \dot{-} y := \begin{cases} x - y & \text{if } x \geq y \\ 0 & \text{if } x < y \end{cases},$$

(ii) is closed under composition and

(iii) is closed under bounded summation and product:

$$\sum_{0 \leq t \leq x} f(t, x_1, \ldots, x_n) \quad \text{and} \quad \prod_{0 \leq t \leq x} f(t, x_1, \ldots, x_n),$$

for any $x \in \mathbb{N}$.

Example I.5.3.

1. Constant functions are elementary, since $(s\circ \overset{k)}{\cdots} \circ s)(o(x)) = k$ for any $k > 0$.
2. The identity function is also elementary: $s(x) \dot{-} s(0) = x$.
3. $\operatorname{sgn} : \mathbb{N} \to \{0, 1\}$ is also elementary since $\operatorname{sgn}(x) = 1 \dot{-} (1 \dot{-} x)$.
4. It is easy to prove that the power $x^{y+1} = \prod_{k=0}^{y+1} x$, quotient $\lfloor \frac{x}{y+1} \rfloor$, logarithm $\lfloor \log_a b \rfloor$ and square root $\lfloor \sqrt{x} \rfloor$ functions are elementary.

Consider thus the set of *elementary reals numbers*, denoted by $\mathbb{R}_{(\mathsf{Elem})}$, as all real numbers which can be approximate by quotients of this functions in the sense of (I.7).

Proposition I.5.4. *The set $\mathbb{R}_{(\mathsf{Elem})}$ is countable and forms a subfield in $\mathbb{R}$.*

The countability of $\mathbb{R}_{(\mathsf{Elem})}$ comes from the fact that any real elementary number is the limit of a special sequence of rational numbers, given by quotients of elementary functions.

Definition I.5.5. A map $g : \mathbb{N} \to \mathbb{Q}$ is said *elementary* if there exists $a, b \in (\mathsf{Elem})$ such that

$$g(n) = \frac{a(n)}{b(n) + 1}, \quad \forall n \in \mathbb{N}$$

In addition, g is said *fast* if satisfies

$$|g(n) - g(n + 1)| < \frac{1}{7^{n+1}}, \ \forall n \in \mathbb{N}.$$

Lemma I.5.6. *The real number $\alpha \in \mathbb{R}$ is elementary if and only if there is an elementary fast map $g : \mathbb{N} \to \mathbb{Q}$ such that $\lim_{n\to\infty} g(n) = \alpha$.*

Using a Cantor-diagonal argument, we can enumerate explicitly any positive elementary map $g = g_e : \mathbb{N} \to \mathbb{Q}_{>0}$ by $e \in \mathbb{N}$ (see [Yos08, p. 8-9]). Obviously, we can not guarantee that $(g_e(n))_{n\geq 0}$ be a Cauchy sequence, but we can obtain a fast elementary sequence from the following transformation.

Definition I.5.7. Let $P \subset \mathbb{N}^n$ be an *elementary predicate*, i.e a subset such that the characteristic function χ_P is elementary. We define the *bounded minimizer* over the first component as $(\mu x_1 \leq n)(P) = \min\{t \mid \chi_P(t, x_2, \ldots, x_n) = 1 \ \wedge \ t = n\}$.

Thus, if P is an elementary predicate then the bounded minimizer is an elementary function over n. Now, for any elementary sequence $g : \mathbb{N} \to \mathbb{Q}$, we define the associated fast elementary sequence $\overline{g}$ as follows

$$\overline{g}(n) = \begin{cases} g(n) & \text{if } (\forall i < n)(|g(i) - g(i + 1)| < 7^{-i-1}) \\ g(n_0) & \text{in other case, where } n_0 := (\mu i < n)(|g(i) - g(i + 1)| \geq 7^{-i-1}) \end{cases}. \tag{I.8}$$

Proposition I.5.8. *g is fast if and only if $g = \overline{g}$.*

From Lemma I.5.6, any real elementary number is the limit of a fast elementary sequence. Thus, if we define the eth elementary number for any $e \in \mathbb{N}$:

$$\beta_e := \lim_{n\to\infty} \overline{g_e}(n),$$

then we can express $\{\beta_0, \beta_1, \ldots, \beta_e, \ldots\} = \mathbb{R}_{(\mathsf{Elem})}$. Thus, M. Yoshinaga construct an non-elementary real number $\alpha \in \mathbb{R}$ as limit of the sequence

$$\alpha_n = \frac{2\varepsilon_1}{3} + \frac{2\varepsilon_2}{3^2} + \frac{2\varepsilon_3}{3^3} + \ldots + \frac{2\varepsilon_n}{3^n} \tag{I.9}$$

for any $n \in \mathbb{N}$. We take $\alpha_0 = 0$ and we define inductively the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ as follows:

$$\varepsilon_{n+1} = \begin{cases} 1 & \text{if } \overline{g_n}(n) \leq \alpha_n + \frac{1}{2\cdot 3^n} \\ 0 & \text{if } \overline{g_n}(n) > \alpha_n + \frac{1}{2\cdot 3^n} \end{cases} .$$

Proposition I.5.9 ([Yos08, Proposition 17]). *Let $\alpha = \lim_{n\to\infty} \alpha_n$, then $\alpha \notin \mathbb{R}_{(\mathsf{Elem})}$.*

Periods as elementary numbers

Now, we present the idea to prove the principal result of M. Yoshinaga: he constructs an elementary approximation by rationals coming from a geometrical view of periods. His main result is then:

Theorem I.5.10 ([Yos08, Theorem 18]). *Real periods of Kontsevich-Zagier are elementary real numbers, i.e. $\mathcal{P}_{\mathrm{KZ}}^{\mathbb{R}} \subset \mathbb{R}_{(\mathsf{Elem})}$.*

As a first step to give a geometrical approximation, he uses Theorem I.3.16 of P. Belkale and P. Brosnan for abstract periods in order to deduce that periods are generated by volumes of bounded basic semi-algebraic sets defined over $\mathbb{Q}$ (see [Yos08, Lemma 24]). This comes from the evaluation map and Theorem I.3.14. Thus, the problem of studying approximation properties of periods is reduced to take volumes of basic compact semi-algebraic domains and ton consider approximations given by inner Riemann sums.

Let $K \subset \mathbb{R}^d$ be a basic compact semi-algebraic set which is expressed in the way:

$$K = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid g_1(x_1, \ldots, x_d) \geq 0, \ldots, g_s(x_1, \ldots, x_d) \geq 0\} \tag{I.10}$$

where $g_k \in \mathbb{Q}[x_1, \ldots, x_d]$. We suppose that K is contained in a large cube $[0, r]^d$ for some $r > 0$. Taking a positive integer n and $k_1, \ldots, k_d \in \mathbb{N}$, we define the small cube of size r/n:

$$C_n(k_1, \ldots, k_d) = \left[\frac{k_1 r}{n}, \frac{(k_1+1)r}{n}\right] \times \cdots \times \left[\frac{k_d r}{n}, \frac{(k_d+1)r}{n}\right]$$

Trivially, these cubes make a homogeneous partition of $[0, r]^d$. For any $n \in \mathbb{N}$, we take those cubes which are into K, i.e.

$$V_n = \bigcup_{C_n(k_1,\ldots,k_d)\subset K} C_n(k_1, \ldots, k_d).$$

In this way, $V_n \longrightarrow K$ when $n \to \infty$ and then

$$\mathrm{vol}_d(V_n) \xrightarrow[n\to\infty]{} \mathrm{vol}_d(K) \tag{I.11}$$

Using the equivalence of predicates in the language of ordered rings and the quantifier free formulas given by Tarski (see [Yos08, p. 14-16]) in order to describe the definition of V_n in terms of $g_1, \ldots, g_s$ and the cubes, we can prove:

Lemma I.5.11. *The sequence*

$$\begin{aligned} \phi : \mathbb{N} &\longrightarrow \mathbb{Q} \\ n &\longmapsto \mathrm{vol}_d(V_n) \end{aligned}$$

is elementary.

An estimate of the rate of convergence in last sequence can be given looking at the semi-algebraic nature of the boundary of K and via some tools and results relatives to concepts *Minkowski dimension* and *Minkowski content*, as well as an Uniformization theorem for analytic sets (see [Yos08, p. 16-18]).

Lemma I.5.12. *There exists a constant $L = L(K)$ which only depends of K such that*

$$|\operatorname{vol}_d(K) - \operatorname{vol}_d(V_n)| < \frac{4r\sqrt{d}L}{n},$$

for any $n > 0$.

From the previous lemma, we can find a constant $C \in \mathbb{N}^*$ such that $4rL\sqrt{d} \leq C$. If we set $c(k) = C \cdot k$ for any $k > 0$, clearly $c \in (\mathsf{Elem})$. So, we have

$$|\operatorname{vol}_d(K) - \operatorname{vol}_d(V_n)| < \frac{1}{k}, \quad \forall n \geq c(k)$$

and the result holds.

Nevertheless, some points in Yoshinaga's preprint [Yos08] have to be specified, especially those related to the geometry of semi-algebraic sets and the fact that the ring of periods is generated by volumes of open bounded semi-algebraic sets in any dimension (see [Yos08, Lemma 24]). Some of these issues are discussed and resolved in Chapter II.

I.5.3 Transcendence of periods

Transcendence, algebraic notions and approximation

As a first example of periods, we have seen that $\overline{\mathbb{Q}} \subset \mathcal{P}_{\text{KZ}}$, thus if a number is not a period it must be transcendental. In the other hand, we know period contains a lot of transcendental numbers as the powers of π, logarithms of algebraic numbers, values of elliptic integrals and some particular values of Gamma and Beta functions. What we know and what we can say about the transcendence in $\mathcal{P}_{\text{KZ}}$? Can we define or construct intrinsic notions of transcendence for periods?

For an algebraic number $\alpha \in \overline{\mathbb{Q}}$ with minimal polynomial $m_\alpha \in \mathbb{Z}[x]$, we can define well-known numerical algebraic notions associated to α as the *degree* $\deg(\alpha) = \deg m_\alpha = [\mathbb{Q}(\alpha) : \mathbb{Q}]$, the *height* $\operatorname{height}(\alpha) = \operatorname{height}(m_\alpha)$ given by the maximum of the absolute values of the coefficients of m_α, as well as the *Mahler measure*, the *house*,... These notions are defined from m_α and posses very strong relations with complexity, approximation by rationals or on the distribution of the solutions of algebraic equations in number fields (see [Wal00a, Chapter 3] for more details).

This question for periods is outlined in [KZ01, p.6]: can we give an analogous notions for periods? The authors give an example of D. Schanks a pair of different periods

$$\frac{\pi}{6}\sqrt{3502} \quad \text{and} \quad \log\left(2\prod_{j=1}^{4}\left(a_j + \sqrt{a_j^2 - 1}\right)\right)$$

with

$$a_1 = \frac{1071}{2} + 92\sqrt{34},\ a_2 = \frac{1553}{2} + 133\sqrt{34},\ a_3 = 429 + 304\sqrt{2},\ a_4 = \frac{627}{2} + 221\sqrt{2},$$

which agree numerically to more than 80 decimal digits. A first step wold be an analogous of notion of complexity for periods, in the spirit of Liouville inequality, Theorem I.1.1.

Question I.5.13. *Let $\eta \in \mathcal{P}_{\mathrm{KZ}} \setminus \overline{\mathbb{Q}}$. Does there exist a constant $c(\eta) > 0$ such that, for any $\frac{p}{q} \in \mathbb{Q}$ distinct form η, with $q \geq 2$, the lower bound*

$$\left|\frac{p}{q} - \eta\right| > \frac{1}{q^{c(\eta)}}$$

holds?

Remark that the study of this question implies to think about if a Liouville number can be a period. From a point of view of Diophantine approximation by algebraic numbers, can we define a good *transcendence measure* for periods? Equivalently, (see [Wal00a, Section 15.1.3, p. 559]):

Question I.5.14. *Let $\eta \in \mathcal{P}_{\mathrm{KZ}} \setminus \overline{\mathbb{Q}}$. Does there exist a constant $\kappa(\eta)$ such that, for any polynomial $P \in \mathbb{Z}[x]$, we have*

$$|P(\eta)| \geq H^{-\kappa(\eta) \deg P},$$

where $H \geq 2$ is an upper bound for $\mathrm{height}(P)$*?*

Complexity of Multiple Zeta Values and diophantine conjectures

Let illustrate the study of transcendence for a particular case of periods: Multiple Zeta Values. Denoting $\underline{s} = (s_1, \ldots, s_k) \in \mathbb{N}^k$ with $s_1 \geq 1$ and $p = |\underline{s}| = s_1 + \ldots + s_k$, we shown in Example I.3.5 that

$$\zeta(\underline{s}) = \sum_{n_1 > n_2 > \cdots > n_k > 0} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}} \tag{I.12}$$

$$= \int_{\Delta_p} \omega_0^{s_1 - 1} \omega_1 \omega_0^{s_2 - 1} \omega_1 \cdots \omega_0^{s_k - 1} \omega_1, \tag{I.13}$$

where $\Delta_p = \{(t_1, \ldots, t_p) \in \mathbb{R}^d \mid 1 > t_1 > \cdots > t_p > 0\}$ and $\omega_0(t) = \mathrm{d}t/t$, $\omega_1(t) = \mathrm{d}t/(1-t)$. For MZV, we call the *height* and the *weight* the numbers k and p, respectively.

For the simplest case of Euler's numbers ($k = 1$), the arithmetic of zeta values for even integers is completely determined, since

$$\zeta(2k) = \mathbb{Q}\pi^{2k}, \quad k \in \mathbb{N},$$

due to a Euler's formula involving Bernoulli numbers $B_{2k} \in \mathbb{Q}$. This is not the case for odd positive integers: R. Apéry proved in 1978 that $\zeta(3)$ is irrational but not a Liouville number and recently T. Rivoal showed that the $\mathbb{Q}$–vector space spanned by $\{\zeta(2k+1) \mid k \in \mathbb{N}\}$ has infinite dimension. Nevertheless, the main object of the transcendence study is to determine all algebraic relations between numbers. Conjecturally any Euler number is transcendental, and there is no polynomial relations between them, i.e.

Conjecture I.5.15. *The numbers $\zeta(2), \zeta(3), \zeta(5), \zeta(7), \ldots$ are algebraically independent over $\mathbb{Q}$.*

The classic approach to the previous Conjecture is to study linear relations between MZV. The product of two series of the form (I.12) can be expressed as a $\mathbb{Z}$–linear combination of series describing MZV, giving a structure of algebra to the $\mathbb{Q}$–vector space generated by MZV, noted by $\mathfrak{Z}$, by a product derived by the series called *stuffle*.

In the other hand, it can be seen that the product of two integrals (I.13) can be also expresses as a linear combination of integrals expressing MZV, but these relations are no the same as in the series case. This fact gives another structure of algebra for $\mathfrak{Z}$ by a product called *shuffle*.

Finally, taking the difference between these linear combinations for the same MZV, we obtain new relations in $\mathfrak{Z}$, called *double-shuffle relations*. In this way, it is conjectured that any algebraic relation between MZV comes from double-shuffle relations.

We can give a graduation of MZV given by the weight

$$\mathfrak{Z} = \bigoplus_{p\geq 0} \mathfrak{Z}_p$$

where $\mathfrak{Z}_p$ is the $\mathbb{Q}$–subspace of $\mathbb{R}$ generated by $\{\zeta(\underline{s}) \mid p = |\underline{s}|\}$, with $\mathfrak{Z}_0 = \mathbb{Q}$ and $\mathfrak{Z}_1 = \{0\}$. We used these intrinsic notions of MZV to get a very rich linear structure for investigate these class of numbers and the arithmetic properties given by the linear relations. As examples of actual conjectures based in this work, we present:

Conjecture I.5.16 (Zagier)**.** *The numbers $d_p = \dim_{\mathbb{Q}} \mathfrak{Z}_p$ satisfy the recurrent relation*

$$d_p = d_{p-2} + d_{p-3}, \quad p \geq 3$$

with $d_0 = 1, d_1 = 0, d_2 = 1$.

Conjecture I.5.17 (Hoffman)**.** *A basis of $\mathfrak{Z}_p$ over $\mathbb{Q}$ is given by the numbers $\zeta(s_1, \ldots, s_k)$, where $p = s_1 + \ldots + s_k$ and $s_i \in \{2, 3\}$.*

Abelian integrals

In general, Diophantine methods are more efficient for determine linear relations rather than algebraic relations, as shows the following Baker's Theorem of linear interdependence for logarithmic of algebraic numbers.

Theorem I.5.18 (Baker)**.** *Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ be algebraic numbers. Suppose that $\alpha_i \neq 0$ and consider $\log \alpha_i$ a complex logarithm of α_i, for $i = 1, \ldots, n$. Then, the number*

$$\alpha_1 \log \beta_1 + \ldots + \alpha_n \log \beta_n$$

is either zero or transcendental.

The previous Theorem is the main tool in the study of the arithmetic nature of *Abelian integrals of genus zero*, i.e. integrals of complex variable over paths with algebraic or infinite limits.

Corollary I.5.19. *Let $P, Q \in \overline{\mathbb{Q}}[z]$ such that $\deg P < \deg Q$ and let γ be either a closed path, or a path whose limits are infinite or algebraic. If the integral*

$$\int_\gamma \frac{P(z)}{Q(z)}\, \mathrm{d}z$$

exist, then its value is either zero or transcendental.

In this case, we can obtain explicit lower bounds of these kind of integrals using measures of linear independence between logarithms in terms of the degrees and heights of P, Q and the algebraic numbers intervening in $\partial\gamma$, see [Wal00a, Chapter 6].

An analogue result can be obtained for the study of arithmetic of *elliptic integrals*, integrals of rational functions over paths lying in a Riemann surface of genus 1.

Theorem I.5.20 (Schneider). *The value of any elliptic integral of the first or the second kind with algebraic coefficients between algebraic limits is either zero or else transcendental.*

These kind of integrals are historically associated to the calculus of arc the length of an ellipse. In particular, last Theorem concludes that the perimeter of the ellipse with major and minus radius $a, b \in \widetilde{\mathbb{Q}}_{>0}$:

$$2\int_{-b}^{b}\sqrt{1+\frac{a^2x^2}{b^4-b^2x^2}}\,\mathrm{d}x$$

is a transcendental number, as well, as any nonzero arc length between algebraic points of the ellipse.

Similar results can be found for general Abelian integrals, connecting with some particular class of periods as values of the Gamma and Beta functions using its relating formula.

Theorem I.5.21 (Schneider). *Let $a, b \in \mathbb{Q} \setminus \mathbb{Z}$ such that $a + b \notin \mathbb{Z}$. Then the number:*

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=\int_0^1 x^{a-1}(1-x)^{b-1}\,\mathrm{d}x$$

is transcendental.

Even if modern number theory has notably advanced in the understanding of linear relations with algebraic coefficients between Abelian integrals, the comprehension of algebraic relations stay as a very difficult problem. In its modern statement given by Y. André in [And04, 7.5], the *Grothendieck period conjecture* predicts that *for periods coming from the comparison isomorphism $\mathrm{comp}_{B,dR}$ between the Betti and de Rham cohomologies over a field $k \subset \overline{\mathbb{Q}}$ of an abelian variety X, any polynomial relation between them come from the relations between algebraic cycles over a power of X*. This conjecture comes from the ideas given by the *philosophy of motives*.

A consequence of the Grothendieck period conjecture is for example the algebraic independence of $\{\zeta(2k+1) \mid k \in \mathbb{N}\}$ over $\mathbb{Q}$, stated in ConjectureI.5.15, see specifically [And04, Prop. 25.7.4.1, p. 237] and [And04, p. 231] for more details.

Conjugation of transcendental numbers and motivic Galois groups

Galois theory gives an exquisite and consistent study of the roots of a rational polynomial and the permutation group of these roots and heir conjugates. Can we assign *conjugates* of a transcendental number and a permutation group (Galois group) which permute them as in the case of algebraic numbers? One must expect that there exist an infinity of conjugates for a transcendental number.

In the case of π, we can not find a polynomial of rational coefficients admitting π as solution, but a power series of rational coefficients:

$$\prod_{n\in\mathbb{Z}^*}\left(1-\frac{x}{n\pi}\right)=\frac{\sin x}{x}=\sum_{k=0}^{+\infty}(-1)^k\frac{x^{2k}}{(2k+1)!}\in\mathbb{Q}[[x]].$$

This expression suggest that any non-zero integer multiple of π must be a conjugate of this number. If we want to construct a Galois group of permutations which in addition acts transitively in the conjugates, we need necessarily to consider any non-zero rational multiple of π. In this case, $\mathbb{Q}^\times$ as multiplicative group will be the Galois group of π.

The previous approach could not be good generalized for any complex number, as we can see in [And12, Sec. 4], because A. Hurwitz proved that *any complex number can be a zero of*

a power series in $\mathbb{Q}[[x]]$ *which defines an entire function of exponential growth.* These kind of power series forms and uncountably set and there is not canonical way to choose one to represent a general complex number.

The definition of periods allows us to introduce a geometric intuition in order to consider conjugates and Galois groups. As periods are fundamentally expressed by pairing geometric objects and analysis functions of algebraic nature over algebraic varieties, it is natural to consider the conjugates of a period as any number which can be obtained by "symmetries" of a *general universal homological object for which domains and differential forms can be seen as realizations of such an object.*

Initiated by Alexander Grothendieck in the sixties, the theory of motives (or *motifs*) try to unify the different cohomology theories associated to an algebraic variety: Betti, de Rham, ℓ–adic or crystalline...These cohomologies are viewed as realizations of an "universal" cohomology over $\mathbb{Q}$ for which the *motives* are the algebraic-geometrical"elementary particle", which can be decomposed and recombined by furnishing a category with similarly properties with representations of groups. The first lectures over the theory of motives are given by M. Demazure[Dem70] and S. L. Kleiman [Kle72].

The Galois group of periods will be the group of symmetries of motives respecting isomorphisms between its realizations as Betti and de Rham cohomologies in different coefficients other than the rationals. These groups of symmetry are called *motivic Galois groups*, which gives a generalization of usual Galois groups for systems of multivariate polynomials of rational coefficients. Any symmetry between motives in this group acts into the periods associated to this motive, giving the conjugates as the images of the period by the action of the motivic Galois group.

The Grothendieck period conjecture is essentially studied from the motivic point of view [And04, Chapters 7 and 23]: as a period can be written in different ways, any relation between periods comes from a relation between motives. As is noticed by Y. André [And04, 23.3.4, p 207] and further developed in [HMS14], this conjecture is the essence of the Kontsevich-Zagier's period conjecture for effective periods, which basically said that *the only relations between different ways to express an effective period come from linearity in the domain of integration and in the rational function, up to change of variables and Stokes.* J. Ayoub proved the equivalence between the two conjectures in [Ayo14, Cor. 32].

Chapter II

Semi-canonical reduction for periods

As we have already seen in the previous chapter, even if the notion of periods can be elementarily expressed, the exact nature of these numbers is not easy to determine. This is due to the fact that a period is constructed from data associated to different mathematical object: a domain of integration which is a geometrical data, and the differential form which comes from algebraic analysis. As a consequence, it is not easy to determine which properties are coming from the geometry and the interplay with the algebraic analysis encoded by the differential form.

In order to simplify the presentation, we can decide to transfer all the information about a given period over the geometry of the semi-algebraic domain, taking the canonical volume form as fixed differential form. In the other hand, we can do the same but ciphering the information over the differential form, fixing a simple domain. Choosing for example the unit complex polydisk as an integration domain, one must consider periods of the form

$$\int_{[0,1]^d} f(z)\mathrm{d}z,$$

where $f(z)$ must belongs to a good category of functions. This choice, taking for f holomorphic functions over the unit polydisk $[0,1]^d$, corresponds to the set of periods considered by J. Ayoub in his relative version of the Kontsevich-Zagier conjecture [Ayo14, Ayo15].

In this Chapter, we try to perform the opposite choice encoding all the information of a given period on its integration domain, i.e. looking for periods only from the geometrical side.

In the following, we give a positive answer to this problem giving a geometric counterpart to Ayoub's approach to periods. We present the one of the main results of Part I, conforming the fundamental result to develop our geometrical approach for periods and their related problems: the Semi-canonical Reduction theorem for periods, which states that any non-zero real period can be algorithmically expressed as the volume of a compact semi-algebraic set from any integral representation.

The plan of the Chapter is as follows: In Section II.2, we construct a compactification of semi-algebraic sets by the natural inclusion into the real projective space $\mathbb{P}^d_{\mathbb{R}}$ defining the *projective closure* of a semi-algebraic set and we resolve the poles at the boundary of the integral function using resolution of singularities in the same spirit as P. Belkale and P. Brosnan in [BB03, Proposition 4.2]. However, and contrary to [BB03], we focus on the constructibility of the resolution, as well as the way to give a partition of the domain by affine compact sets. As a consequence, we prove that periods can be expressed as the difference of the volumes of two compact semi-algebraic sets (see Corollary II.2.12).

Section II.3 deals specifically with the two dimensional case, for which an easiest and explicit

method is implemented based in geometrical properties of plane curve singularities and blow-ups in the plane.

In Section II.4, we complete the proof of our main result providing an explicit asymptotic method which allows us to write the difference of the volumes of two compact semi-algebraic sets K_1 and K_2 obtained in Corollary II.2.12 as the volume of a single compact semi-algebraic set constructed algorithmically from K_1 and K_2.

Examples of semi-canonical representations of periods are given in Section II.5. A pseudo-code explaining each of the algorithms are given in each Section.

This chapter arise from the author's preprint [VS15]: *"Periods of Kontsevich-Zagier I: A semi-canonical reduction"*, J. Viu-Sos, `arXiv:1509.01097 [math.NT]`, 2015.

II.1 A semi-canonical reduction

Let $\overline{\mathbb{Q}}$ and $\widetilde{\mathbb{Q}}$ be the field of complex and real algebraic numbers, respectively. Recall the Definition I.3.1 of periods of Kontsevich-Zagier as absolutely convergent integrals of the form

$$\mathcal{I}(S, P/Q) = \int_S \frac{P(x_1, \ldots, x_d)}{Q(x_1, \ldots, x_d)} \cdot \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d \tag{II.1}$$

where $S \subset \mathbb{R}^d$ is a d–dimensional $\widetilde{\mathbb{Q}}$–semi-algebraic set and $P, Q \in \widetilde{\mathbb{Q}}[x_1, \ldots, x_d]$ are coprimes.

Our purpose is to reduce any integral representation of a period by putting all the complexity of the period in the domain of integration in an algorithmic way and only using the KZ–rules. This will be used in Chapter III to formulate an alternative statement to the Kontsevich-Zagier conjecture (Conjecture I.4.1). The algorithmic component is important if we want also obtain a more geometric way to address Equality algorithm via accessible identities (Problems I.4.5 and I.4.6). More precisely, we construct an affine "good" object which naturally represents a given period and which can be calculated in a constructive way and respecting the three operations of the KZ–conjecture using classical tools in algebraic geometry, in particular *resolution of singularities*.

It must be noted that even if such a representation can be obtained case by case using elementary calculus, the resolution of singularities seems to be unavoidable in order to prove the existence of such a reduction in the general case. Indeed, using Proposition I.3.6, one can write explicitly any period as the difference of the volume of two semi-algebraic set. However, it is not usually easy to write this difference as the volume of a unique semi-algebraic set. This is done in Section II.4 in the compact case. The unboundedness is due to the presence of poles in the rational function defining the period, a classical way to deal with this problem is to resolve them by a sequence of blow-ups.

By an *algorithm*, we mean a finite sequence of operations which produces an *output* from a given *input*. Here, we distinguish between two types of algorithm: explicit and effective. An *explicit or constructive* algorithm is one for which each operation can be described explicitly. The word "explicit" does not mean that each operation can be effectively tested. An algorithm is called *effective* if it is explicit and each operation can be effectively implemented on a machine. An example of such algorithm is given by O. Villamayor [Vil89] for Hironaka's resolution of singularities.

Theorem II.1.1 (Semi-canonical reduction). *Let p be a non-zero real period given in a certain integral form (II.1) in $\mathbb{R}^d$. Then there exists an effective algorithm satisfying the KZ–rules such that $\mathcal{I}(S, P/Q)$ can be written as*

$$p = \mathrm{sgn}(p) \cdot \mathrm{vol}_k(K),$$

where $K \in \mathcal{SA}^k_{\widetilde{\mathbb{Q}}}$ *for* $0 < k \leq d+1$ *is a compact semi-algebraic set and* $\mathrm{vol}_k(\cdot)$ *is the canonical volume in* $\mathbb{R}^k$*. Such representation is called* a geometric semi-canonical representation of p.

Remark II.1.2. We can extend this theorem for the whole set of periods $\mathcal{P}_{\mathrm{KZ}} \subset \mathbb{C}$ considering representations of the real and imaginary part respectively.

We call *reduction algorithm* the algorithm of Theorem II.1.1. An explicit **pseudo-code** of this reduction is given in Algorithm 1 (see bellow). Procedures CompactifyDomain, ResolvePoles and VolumeFromDiffSA are explicitly described in Algorithm 2, Algorithm 3 and Algorithm 4 in Sections II.2 and II.4, respectively.

Algorithm 1 Semi-canonical form of $p \in \mathcal{P}^{\mathbb{R}}_{\mathrm{KZ}}$ given by an integral form $p = \mathcal{I}(S, P/Q)$.

Input: A semi-algebraic set S of maximal dimension and a rational function P/Q defined with coefficients in $\widetilde{\mathbb{Q}}$.

Output: A compact semi-algebraic K with same dimension of S such that $\mathrm{vol}(K) = \mathcal{I}(S, P/Q)$.

procedure SemiCanPeriod($S, P/Q$)
  ▷ Partition by sign of the integrand
  $S^+ \leftarrow \{\mathbf{x} \in S \mid 0 < P/Q(\mathbf{x})\}$
  $S^- \leftarrow \{\mathbf{x} \in S \mid P/Q(\mathbf{x}) < 0\}$
  ▷ Lists of triples $\left(S_i^{\pm}, P_i^{\pm}, Q_i^{\pm}\right)$ where $S_i^{\pm}$ is bounded
  $L^+ \leftarrow$ CompactifyDomain(S^+, P, Q)
  $L^- \leftarrow$ CompactifyDomain(S^-, P, Q)
  ▷ Lists of triples $(\widetilde{S_j}^{\pm}, \widetilde{P_j}^{\pm}, \widetilde{Q_j}^{\pm})$ with resolved poles at the boundary
  $\widetilde{L}^+, \widetilde{L}^- \leftarrow \{\}, \{\}$
  for $(S^+, P^+, Q^+) \in L^+$ and $(S^-, P^-, Q^-) \in L^-$ **do**
    $\widetilde{L}^+ \leftarrow \widetilde{L}^+ \cup$ ResolvePoles(S^+, P^+, Q^+)
    $\widetilde{L}^- \leftarrow \widetilde{L}^- \cup$ ResolvePoles(S^-, P^-, Q^-)
  ▷ We define the compact sets under the integrand
  $K^+, K^- \leftarrow \emptyset, \emptyset$
  for $(\widetilde{S}^+, \widetilde{P}^+, \widetilde{Q}^+) \in \widetilde{L}^+$ and $(\widetilde{S}^-, \widetilde{P}^-, \widetilde{Q}^-) \in \widetilde{L}^-$ **do**
    $K^+ \leftarrow K^+ \cup \{(\mathbf{x}, t) \in S^+ \times \mathbb{R} \mid 0 \leq t \leq P^+/Q^+(\mathbf{x})\}$
    $K^- \leftarrow K^- \cup \{(\mathbf{x}, t) \in S^- \times \mathbb{R} \mid P^-/Q^-(\mathbf{x}) \leq t \leq 0\}$
  ▷ We construct the compact set K from K^+ and K^- which volume is the difference of these sets
  if $\int_S P/Q > 0$ **then**
    $K \leftarrow$ VolumeFromDiffSA(K^+, K^-)
  else
    $K \leftarrow$ VolumeFromDiffSA(K^-, K^+)
  return K ▷ A compact semi-algebraic set K representing p

This kind of presentation was suggested by M. Kontsevich and D. Zagier in their original paper [KZ01, p. 3] and by M. Yoshinaga in [Yos08, p. 13] and finally assumed by J. Wan in [Wan11] in order to develop a *degree theory* for periods.

The word *semi-canonical* refers to the non-uniqueness of such a geometric compact semi-algebraic set in the reduction theorem. This follows from two phenomena:

- **Non-uniqueness of the dimension.** Given a period, we can obtain two representations in two different dimensions. For example, π^2 can be obtained as the 4–dimensional volume of the Cartesian product of two copies of the unit disk and the 3–dimensional volume of the set
$$S_1 = \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1, 0 \leq z((x^2 + y^2)^2 + 1) \leq 4\right\}.$$
- **Non-uniqueness in fixed dimension.** Looking for reduction in a fixed dimension, we can find two compact semi-algebraic sets with the same volume. For example, taking the 2–dimensional volume of the unity semi-disk and the 2–dimensional volume of
$$S_2 = \left\{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1, 0 < y(1 + x^2) < 1\right\},$$
we obtain $\pi/2$ in both cases.

The first issue can be fixed considering the *minimal dimension* for which a period admits such a representation. This leads to the notion of *degree of a period* introduced by J. Wan [Wan11]. For the second one, we can try to rigidify the situation, introducing more information on the nature of the compact semi-algebraic set representing a period, for example using the notion of *complexity of semi-algebraic sets* (see [BR90, sec. 4.5, p. 211]). Nevertheless, the indeterminacy of a canonical representation of a semi-algebraic set as conjunctions and disjunctions polynomial (in)equalities (see Remark I.2.2) will give an incidence on the future discussion about complexity in Chapter IV. Despite this ambiguity, this furnishes a convenient tool to manipulate and compare different periods. In particular, this gives a way to deal with the Kontsevich-Zagier period conjecture (see [CVS]).

The proof of Theorem II.1.1 is based in compactification of semi-algebraic sets and resolution of singularities. We have three main difficulties to overcome:

- The first is due to the framework of the KZ–conjecture, namely that one allows only operations and constructions authorized by the KZ–rules.
- The second one is to provide constructive methods at each step of the proof. This constraint is not contained in the formulation of the KZ–conjecture, but motivated by the problem of **accessible identities**, i.e. identities between periods which can be obtained by a construction algorithm (I.4.5). As a general rule in our procedures, we give partitions of semi-algebraic sets cutting off by hyperplanes, in order to not increase the complexity of the representation of the resulting semi-algebraic sets.
- The last one is more technical and it is related to the fact that we have to deal with compact semi-algebraic domains. Then we need to provide affine charts which guarantee local compacity during the resolution process. Note that the arithmetic nature of the objects is not an issue due to the behavior of the resolution of singularities theory [Hir64].

Remark II.1.3. A connexion between periods and volumes is known for sums of generalized harmonic series (see [BKC93]). However, the type of change of variables which are used does not belong to those authorized by the KZ–rules.

Remark II.1.4. In this chapter, all the algebraic varieties are considered over the field of real algebraic numbers. We construct our theory from the real point of view, but most of the results about resolution of singularities can be obtained using classical algebraic geometry over algebraically closed fields by complexification of the varieties.

Remark II.1.5. Recall that we consider that our closed domains of integration S are *regular*, i.e. the semi-algebraic set S coincides with the topological closure of its interior. We are also considering rational top-dimensional differential forms forgetting the orientation $\frac{P}{Q}(x_1,\ldots,x_d)\cdot|\mathrm{d}x_1\wedge\ldots\wedge\mathrm{d}x_d|$, i.e. integration of rational function over the Lebesgue measure over $\mathbb{R}^d$. With a slight abuse of notation, we will from now on use $\mathrm{d}x_1\wedge\ldots\wedge\mathrm{d}x_d$.

Remark II.1.6. Prof. T. Rivoal asked us if it is possible to detect zero period using the semi-canonical reduction. The answer is negative, because we need to suppose in Section II.4 that the volumes of the two compact semi-algebraic sets which express the period by their difference are not equal. In fact, this question is equivalent to find an Equality algorithm for periods.

II.2 Semi-algebraic compactification of domains and resolution of poles

The aim of this section is to explain how to obtain a representation of a period as integrals of well-defined rational functions over compact semi-algebraic sets, holding ambient dimension, and using partitions of domains and birational change of variables from another representation $\mathcal{I}(S,P/Q)$. We are interested to work with real semi-algebraic sets described by coefficients in $\widetilde{\mathbb{Q}}$, the field of real algebraic numbers.

II.2.1 Projective closure of semi-algebraic sets and compact domains

Recall the definitions and properties of semi-algebraic sets and maps given in Section I.2. We are interested in the study of semi-algebraic sets in their passage to the real projective space $\mathbb{P}^d_{\mathbb{R}}$.

Denote by $[x_0:\ldots:x_d]$ the coordinates in $\mathbb{P}^d_{\mathbb{R}}$ and define the projective hyperplanes $\mathcal{H}_{x_i}=\{x_i=0\}$. We consider the usual *atlas* of $\mathbb{P}^d_{\mathbb{R}}$ given by $\{(U_{x_i},\varphi_{x_i})\}_{i=0}^d$, described by open Zariski sets $U_{x_i}=\mathbb{P}^d_{\mathbb{R}}\setminus\mathcal{H}_{x_i}=\{x_i\neq 0\}$, and birational functions

$$\begin{array}{rccc}\varphi_{x_i}: & U_{x_i} & \longrightarrow & \mathbb{R}^d \\ & [x_0:\ldots:x_d] & \longmapsto & \left(\frac{x_0}{x_i},\ldots,\frac{x_{i-1}}{x_i},\frac{x_{i+1}}{x_i},\ldots,\frac{x_d}{x_i}\right)\end{array}$$

For any real homogeneous polynomial $F\in\mathbb{R}[x_0,\ldots,x_d]$, we define the *real projective variety of* F as $V_{\mathbb{P}_{\mathbb{R}}}(F)=\{[x_0:\ldots:x_d]\mid F(x_0,\ldots,x_d)=0\}$. Note that $\varphi_{x_i}(V_{\mathbb{P}_{\mathbb{R}}}(F)\cap U_{x_i})=V_{\mathbb{R}}(f_i)$ for all $i=0,\ldots,d$ where $f_i\in\mathbb{R}[x_0,\ldots,\hat{x_i},\ldots,x_d]$ is the *deshomogenization of* F *with respect to the variable* x_i, i.e. $f_i=F_{|x_i=1}$. Thus, we can express:

$$V_{\mathbb{P}_{\mathbb{R}}}(F)=\bigcup_{i=0}^{d}\varphi_{x_i}^{-1}V_{\mathbb{R}}(f_i).$$

Remark II.2.1. In the complex case, the projectivization of a curve via homogenization is a classical tool to study algebraic sets. For $f\in\mathbb{C}[x_1,\ldots,x_d]$, we define the *projective closure* of $V_{\mathbb{C}}(f)$ as the topological closure of its inclusion in $\mathbb{P}^d_{\mathbb{C}}$, and this set coincides with $V_{\mathbb{P}_{\mathbb{C}}}(F)$ where F is the homogenization of f. Note that this does not works in the real case by continuity of roots over algebraically closed fields: some extra points can appear in the real projective variety defined by homogenization, outside the topological closure. For example, if we take

$f(x,y) = y^4 + x^2 + 1$, clearly $V_{\mathbb{R}}(f) = \emptyset$ but the homogenization gives $F(x,y,z) = y^4 + x^2z^2 + z^4$ and $V_{\mathbb{P}_{\mathbb{R}}}(F) = \{[1:0:0]\}$. In general, for the real case,

$$\overline{\varphi_{x_i}^{-1} V_{\mathbb{R}}(f)} \subset V_{\mathbb{P}_{\mathbb{R}}}(F).$$

Remark II.2.2. Taking a semi-algebraic component S in the first chart U_{x_0} described by

$$S = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid p(x_1, \ldots, x_d) = 0,\ q_i(x_1, \ldots, x_d) > 0, i = 1, \ldots, n\},$$

its image in the other charts $\tilde{S}_j = \varphi_{x_j}\varphi_{x_0}^{-1}(S \setminus \{x_j \neq 0\})$ is also a semi-algebraic set and may be expressed in local coordinates $(t_0, \ldots, \hat{t_j}, \ldots, t_d) \in \mathbb{R}^d$ by

$$\tilde{S}_j = \left\{ t_0 \neq 0,\ t_0^{-d_p} P(t_0, \ldots, t_d)_{|t_j=1} = 0,\ t_0^{-d_i} Q_i(t_0, \ldots, t_d)_{|t_j=1} > 0, i = 1, \ldots, n \right\},$$

where P and $Q_1, \ldots, Q_n$ are the homogenizations of p and $q_1, \ldots, q_n$ respectively and $d_p = \deg p$, $d_i = \deg q_i$ for $i = 1, \ldots, n$.
It is easy to see that $\tilde{S}_j$ splits into two disjoints semi-algebraic sets $\tilde{S}_j^{\pm}$ where:

$$\tilde{S}_j^{+} = \left\{ t_0 > 0,\ P_{|t_j=1} = 0,\ Q_{i|t_j=1} > 0,\ i = 1, \ldots, n \right\},$$
$$\tilde{S}_j^{-} = \left\{ t_0 < 0,\ P_{|t_j=1} = 0,\ (-1)^{d_i} Q_{i|t_j=1} > 0,\ i = 1, \ldots, n \right\}.$$

Note that if S is not contained in $x_j = 0$, then either $\tilde{S}_j^{+}$ or $\tilde{S}_j^{-}$ is not an empty set.

We define the *projective closure* of a semi-algebraic set $S \subset \mathbb{R}^d$ by $\overline{\varphi_{x_0}^{-1} S}$, i.e. the topological closure of the inclusion of S into $\mathbb{P}_{\mathbb{R}}^d$ considering $\mathcal{H}_{x_0}$ as the hyperplane at infinity. Note that the restriction of this projective closure to any chart is a semi-algebraic set in the corresponding chart. Thus the projective closure of S is a compact semi-algebraic set in $\mathbb{P}_{\mathbb{R}}^d$, since the projective space is a compact variety.

Using the this notion, we decompose the integration domain into affine compact domains. We give a useful decomposition of the real projective space $\mathbb{P}_{\mathbb{R}}^d$ as the gluing of $d+1$ hypercubes through their opposite faces. Denote by $\mathbb{B}_o^{\infty}(r)$ (resp. $\overline{\mathbb{B}}_o^{\infty}(r)$) the open (resp. closed) hypercube in $\mathbb{R}^d$ centered at the origin of radius $r > 0$, i.e. $\mathbb{B}_o^{\infty}(r) = \{|x_i| < r\}$ (resp. $\overline{\mathbb{B}}_o^{\infty}(r) = \{|x_i| \leq r\}$).

Proposition II.2.3. *Let $\{C_i\}_{i=0}^d$ the family of compact semi-algebraic sets in $\mathbb{P}_{\mathbb{R}}^d$ defined by $C_i = \overline{\varphi_{x_0}^{-1} V_i}$ where V_i is the union of*

$$\bigcap_{\substack{j=1 \\ j \neq i}}^{d} \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_i - 1 \geq 0, x_i - x_j \geq 0, x_i + x_j \geq 0 \right\}$$

and

$$\bigcap_{\substack{j=1 \\ j \neq i}}^{d} \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_i + 1 \leq 0, x_i - x_j \leq 0, x_i + x_j \leq 0 \right\}.$$

for $1 \leq i \leq d$, and $C_0 = \overline{\varphi_{x_0}^{-1} \overline{\mathbb{B}}_o^{\infty}(1)}$. Then:

1. *$C_i \subset U_{x_i}$ and $\varphi_{x_i} C_i = \overline{\mathbb{B}}_o^{\infty}(1)$, for any $0 \leq i \leq d$.*
2. *$\bigcup_{i=0}^d C_i = \mathbb{P}_{\mathbb{R}}^d$.*

3. *The Zariski closure of* $\bigcup_{i,j=0}^{d}(C_i \cap C_j)$ *is the hyperplane arrangement* $\mathcal{A} = \{x_i^2 - x_j^2 = 0 \mid 0 \leq i < j \leq d\}$ *of* $\mathbb{P}^d_{\mathbb{R}}$.

Proof. For any $i \in \{0, \ldots, d\}$, it is clear that $\mathcal{H}_{x_i} \cap C_i = \emptyset$ by definition of C_i. Performing a change of charts $\varphi_{x_i}\varphi_{x_0}^{-1}$ in $\mathbb{P}^d_{\mathbb{R}}$ by taking $\mathcal{H}_{x_i}$ as hyperplane at infinity, we obtain

$$\varphi_{x_i}\varphi_{x_0}^{-1}V_i = \bigcap_{j=1}^{d} \{t_0 \neq 0, -1 \leq t_0 \leq 1, -1 \leq t_j \leq 1\},$$

in local coordinates $(t_0, \ldots, \hat{t}_i, \ldots, t_d)$ in $\mathbb{R}^d$. Taking the topological closure, we obtain $\overline{\varphi_{x_i}\varphi_{x_0}^{-1}V_i} = \varphi_{x_i}C_i = \overline{\mathbb{B}}_o^{\infty}(1)$.

It is easy to see that $\bigcup_{i=0}^{d} V_i = \mathbb{R}^d = \mathbb{P}^d_{\mathbb{R}} \setminus \mathcal{H}_{x_0}$, thus the topological closure of this partition gives us a partition of $\mathbb{P}^d_{\mathbb{R}}$. Finally, the intersection of two regions C_i and C_j is a $(d-1)$–dimensional semi-algebraic set contained in $\{x_i + x_j = 0\} \cup \{x_i - x_j = 0\}$, and this completes the proof. □

Using this family of semi-algebraic sets for predefined coordinates, we compactify our semi-algebraic domain of integration passing through the projective space by projective compactification and decomposing it using $\{C_i\}_{i=0}^{d}$.

Theorem II.2.4. *Let* $S \in \mathcal{SA}^d_{\widetilde{\mathbb{Q}}}$ *an open semi-algebraic set and* $\omega = P/Q \cdot \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d$ *with* $P/Q \in \widetilde{\mathbb{Q}}(x_1, \ldots, x_d)$ *such that the integral* $\mathcal{I}(S, P/Q)$ *converges absolutely. Then there exists a* $(d-1)$*–dimensional semi-algebraic set* $X \subset \mathbb{R}^d$*, a partition* $S = X \cup S_0 \cup \cdots \cup S_d$*, and a collection* $\{\varphi_i\}_{i=1}^{d}$ *of birational morphisms* $\varphi_i : \mathbb{R}^d \to \mathbb{R}^d$ *such that*

$$\int_S \omega = \sum_{i=0}^{d} \int_{\varphi_i^{-1}S_i} \varphi_i^*\omega,$$

where $\varphi_i^{-1}S_i$ *is bounded and* $\varphi_i^*\omega$ *is a rational* d*–form defined in the interior of* S_i *for any* $i = 0, \ldots, d$*. Moreover, this procedure is algorithmic and depends only on the representation of* S*.*

Proof. We give a proof of this theorem with an explicit construction: the change of charts in the projective space gets a way to obtain compact semi-algebraic sets. Define $S_0 = S \cap \mathbb{B}_o^{\infty}(1)$ and $\varphi_0 = \mathrm{Id}_{\mathbb{R}^d}$. For $i = 1, \ldots, d$, we fix a hyperplane of the form $\{x_i = 1\}$ for local coordinates $(x_1, \ldots, x_d)$ in $\mathbb{R}^d$ and we consider V_i the unbounded semi-algebraic region given in II.2.3. Defining $S_i = S \cap \mathring{V}_i$ and performing a change of charts $\varphi_{x_i}\varphi_{x_0}^{-1}$ in $\mathbb{P}^d_{\mathbb{R}}$ by taking $\mathcal{H}_{x_i}$ as hyperplane at infinity, we obtain

$$\varphi_{x_i}\varphi_{x_0}^{-1}S_i \subset \varphi_{x_i}C_i = \overline{\mathbb{B}}_o^{\infty}(1),$$

which is a bounded semi-algebraic set in local coordinates $(t_0, \ldots, \hat{t}_i, \ldots, t_d)$ in $\mathbb{R}^d$. Thus, the result holds. □

Corollary II.2.5. *Any period can be represented as a sum of absolutely convergent integrals of rational functions in* $\widetilde{\mathbb{Q}}(x_1, \ldots, x_d)$ *over compact semi-algebraic sets, obtained algorithmically and respecting the KZ–rules from another integral representation.*

Proof. It follows directly from Theorem II.2.4. □

Algorithm 2 Partition and compactification of domains.

Input: A semi-algebraic domain S and two polynomials P, Q.

Output: A list of triples (S_i, P_i, Q_i) where S_i is compact a semi-algebraic set and coprime polynomials P_i, Q_i such that $\mathcal{I}(S, P/Q) = \sum_i \mathcal{I}(S_i, P_i/Q_i)$.

procedure CompactifyDomain(S, P, Q)
  $d \leftarrow \dim S$
  $S_0 \leftarrow S \cap \{-1 \leq x_1 \leq 1, \ldots, -1 \leq x_d \leq 1\}$
  $L \leftarrow \{(S_0, P, Q)\}$
  for $i \leftarrow 1, \ldots, d$ **do**
    $V_i \leftarrow \bigcap_{j=1}^{d} \{x_i \geq 1, x_i \geq x_j, x_i \geq -x_j\} \cup \{x_i \leq -1, x_i \leq x_j, x_i \leq -x_j\}$
    $S_i \leftarrow S \cap V_i$
    $S_i \leftarrow$ Change of variables in S_i: $x_i = 1/x_0$, $x_j = x_j/x_0, \forall j \neq i$
    $P_i/Q_i \leftarrow$ Change of variables in P_i/Q_i: $x_i = 1/x_0$, $x_j = x_j/x_0, \forall j \neq i$
    $P_i/Q_i \leftarrow P_i/Q_i \times (1/x_0^{d+1})$ ▷ The Jacobian of the change of variables
    $L \leftarrow L \cup \{(S_i, P_i, Q_i)\}$
  return L

Due to potential poles at the boundary of the compact domains, we can not do a direct transformation to remove the differential form of the integral in order to encode all the complexity of a given period in the geometrical domain of integration. This will be done in the next Section using resolution of singularities.

II.2.2 Resolution of singularities and compactification

From Theorem II.2.4, we only consider bounded semi-algebraic domains in $\mathbb{R}^d$ for $\mathcal{I}(S, P/Q)$. It is easy to check that, for absolutely convergent integrals $\mathcal{I}(S, P/Q)$ with semi-algebraic domains defined in $\mathbb{R}$, the change of variables over the projective line $\mathbb{P}^1_{\mathbb{R}}$ removes automatically the pole of order 2 which appears in the boundary (see Example II.5.1). In higher dimension, we need to remove the possible poles in the boundary of our domain. We suppose that P/Q is not constant, otherwise we get our result by a linear change of variables in order to have the canonical d–differential form as integrand. We use resolution of singularities techniques in order to obtain integrands defined in the border of the semi-algebraic domain. In [Hir64], Hironaka proves his famous

Theorem II.2.6 (Embedded Resolution of Singularities)**.** *Given W_0 a smooth variety defined over a field of characteristic zero and X a closed reduced subvariety of W_0. There exists a finite sequence*

$$(W_0, X_0) \xleftarrow{\pi_1} (W_1, X_1 \cup E_1) \xleftarrow{\pi_2} (W_2, X_2 \cup E_1 \cup E_2) \ldots \xleftarrow{\pi_r} (W_r, X_r \cup E_1 \cup \ldots \cup E_r) \quad \text{(II.2)}$$

where:

1. *$W_{j-1} \xleftarrow{\pi_j} W_j$ are proper birational maps between smooth varieties, given by* blow-ups *over a smooth center $Z_{j-1} \subset Z_j$.*
2. *The composite $W_0 \xleftarrow{\pi} W_r$ is a proper birational map such that $W_0 \backslash \operatorname{Sing} X_0 \simeq W_r \backslash \bigcup_{i=1}^{r} E_i$.*
3. *The* strict transform $X_r = \overline{\pi^{-1}(X_0 \setminus \operatorname{Sing} X_0)}$ *is a regular subvariety and has normal crossings with the* exceptional hypersurface $\bigcup_{i=1}^{r} E_i$ *in W_r.*

Previous diagram represents a sequence of blow-ups of varieties. This process is *efficiently algorithmic* after the constructible proof of Villamayor [Vil89], who gives a way to choose the smooths centers to blow-up at each step. Villamayor's resolution of singularities algorithm was implemented by Bodnár and Schicho [BS00a], [BS00b], for algebraic computation software as `Maple` and `Singular` [DGPS14].

Remark II.2.7. Let $f \in \widetilde{\mathbb{Q}}[x_1, \ldots, x_d]$ be a non-constant polynomial and let $X = \{a \in \mathbb{R}^d \mid f(a) = 0\}$. Hironaka's desingularization theorem constructs proper birational map $\pi : W \to \mathbb{R}^d$ where W is a closed d–dimensional $\widetilde{\mathbb{Q}}$–subvariety of $\mathbb{R}^d \times \mathbb{P}^m_{\mathbb{R}}$ for some positive integer m, rising in an isomorphism $W \setminus \pi^{-1} \operatorname{Sing} X \simeq \mathbb{R}^d \setminus \operatorname{Sing} X$. An atlas of W is given by $\{V_i\}_{i=0}^m$, where any V_i is isomorphic to a $W_i = W \cap (\mathbb{R}^d \times U_{x_i})$ via ϕ_i, where $\{U_{x_i}\}_{i=0}^m$ is the usual atlas of $\mathbb{P}^m_{\mathbb{R}}$.

Considering the family of exceptional hypersurfaces $\{E_1, \ldots, E_r\}$ of the resolution and setting by E_0 the strict transform, there exist a collection of couples of positive integers $\{(N_i, \nu_i)\}_{i=0}^r$, called the *numerical data* of the resolution such that the divisors in W of the pull-back of f and the canonical differential d–form by π are of the form $\sum_{i=0}^r N_i E_i$ and $\sum_{i=0}^r (\nu_i - 1) E_i$, respectively. Thus, numbers N_i and $\nu_i - 1$ are the multiplicity of $f \circ \pi$ and $\pi^* \omega$ over E_i, for $i \in \{0, \ldots, r\}$. The property to have *normal crossings* for the family of smooth hypersurfaces $\{E_0, E_1, \ldots, E_r\}$ means that they are transversal at any point of their intersection, i.e. for any point $a \in W$ verifying $(f \circ \pi)(a) = 0$, there exist local coordinates $(y_1, \ldots, y_d)$ centered in a and $f_1, \ldots, f_r \in \widetilde{\mathbb{Q}}[y_1, \ldots, y_n]$ such that

1. E_i has local equation $f_i = 0$, for $0 \leq i \leq r$.
2. $(\mathrm{d}f_1)_{|_0}, \ldots, (\mathrm{d}f_r)_{|_0}$ are linearly independents.
3. There exists $g, h \in \widetilde{\mathbb{Q}}[y_1, \ldots, y_d]$ satisfying $g(0), h(0) \neq 0$ and

$$(f \circ \pi) = g \cdot \prod_{k=1}^{r} f_{i_k}^{N_{i_k}} \quad \text{and} \quad \pi^* \left(\bigwedge_{i=1}^{d} \mathrm{d}x_i \right) = h \cdot \prod_{k=1}^{r} f_{i_k}^{\nu_{i_k} - 1} \cdot \bigwedge_{i=1}^{d} \mathrm{d}y_i,$$

for some $1 \leq i_1, \ldots, i_r \leq d$.

In particular, locally near a we can express

$$(f \circ \pi) = \varepsilon \cdot \prod_{k=1}^{r} y_{i_k}^{N_{i_k}} \quad \text{and} \quad \pi^* \left(\bigwedge_{i=1}^{d} \mathrm{d}x_i \right) = \eta \cdot \prod_{k=1}^{r} y_{i_k}^{\nu_{i_k} - 1} \cdot \bigwedge_{i=1}^{d} \mathrm{d}y_i,$$

for some $1 \leq i_1, \ldots, i_r \leq d$ and ε, η real analytic functions with $\varepsilon(0), \eta(0) \neq 0$. See [Igu00, Chapters 3 and 11] or [Liu02, Chapter 8]) for more details.

Remark II.2.8. Since any connected algebraic variety W is covered by charts $\{(U_i, \varphi_i)\}_{i \in I}$ given by open Zariski sets and morphisms coming from ring morphisms and any non-trivial closed Zariski set has measure zero, the calculation of an integral in one chart U gives the complete value of the integral, i.e. $\int_D \omega = \int_{D_{|U}} \omega_{|U}$, for any measurable set $D \subset W$.

For a semi-algebraic set S and a top–dimensional differential rational form ω in a variety W, denote by $\partial_z S$ the Zariski closure of ∂S and by $Z(\omega)$ and $P(\omega)$ the real zero and pole locus of ω, respectively. Let $\mathcal{Z}$ be the Zariski closure of $Z(\omega) \cap P(\omega) \cap \partial S \subset \partial_z S$. It is worth noticing that the Zariski closure of $\partial(\pi^{-1} S)$ is a subvariety of $\pi^{-1} \partial_z S$. We use embedded resolution of singularities over $\mathcal{Z}$ to send the poles of the form in $\mathcal{I}(S, P/Q)$ "far away" from ∂S. It follows from the following geometric criterion for the convergence of rational integrals over semi-algebraic sets on $\mathbb{R}^d$:

Proposition II.2.9. *Let W_0 be a smooth real algebraic variety defined over $\widetilde{\mathbb{Q}}$. Let $S \subset W_0$ be a compact semi-algebraic set in W_0 and ω a top differential rational form in W_0.*
Then, the integral $\int_S \omega$ converges absolutely if and only if there exist a finite sequence of blow-ups $\pi = \pi_r \circ \cdots \circ \pi_1 : W_r \to W_0$ over smooth centers as in (II.2) such that $\widetilde{S} \cap P(\pi^*\omega) = \emptyset$, where $\widetilde{S}$ the strict transform of S.

Proof. Suppose that $\int_K \omega$ converges absolutely. Note that $P(\omega)$ does not intersect the interior of S in this case. Let $X = \partial_z S \cup Z(\omega) \cup P(\omega)$ be a $\widetilde{\mathbb{Q}}$–subvariety of W_0 and consider $\pi : W_r \to W_0$ and embedded resolution of X given by Theorem II.2.6. Let a be a point in $\partial\widetilde{S}$. Following Remark II.2.7, we know that there exits local coordinates $(y_1, \ldots, y_d)$ with $d = \dim W_0$ such that we can express $\int_{\widetilde{S}} \pi^*\omega$ for a sufficiently small $\epsilon > 0$ as

$$\int_{0<s_i y_i<\epsilon} \delta \cdot \prod_{k=1}^{r} y_{i_k}^{M_{i_k}}$$

for some $1 \leq i_1, \ldots, i_r \leq d$, with $M_{i_k} \in \mathbb{Z}$, δ a real analytic function which non-vanish at the origin, and a choice of signs $s_i = \pm 1$ such that $\{0 \leq s_i y_i < \epsilon\}$ is the local expression of $\widetilde{S}$ near a. It is clear that the preceded integral converges if and only if the exponents M_{i_k} are all non-negatives. This is equivalent to assert that $a \notin P(\pi^*\omega)$.

Reciprocal it trivial, since $\int_S \omega = \int_{\widetilde{S}} \pi^*\omega$ and $\pi^*\omega$ is well-defined over the compact set $\widetilde{S}$. □

A similar result can be found in [BB03, Proposition 4.2]. Note that, in our case $W_0 = \mathbb{R}^d$ and we do not need in general to give a complete embedded resolution of $X = \partial_z S \cup Z(\omega) \cup P(\omega)$ but only consider a finite sequence of blow-ups over smooth centers containing real points which separates $\partial_z \widetilde{S}$ and the pole locus of the pull-back of the differential form. In particular, this implies that $\mathcal{Z}$ can not be a hypersurface of $\mathbb{R}^d$ in the case of periods, because the integral $\mathcal{I}(S, P/Q)$ becomes divergent.

Remark II.2.10. Note that the integers $\{M_i\}_{i=0}^r$ which appears in Proof of Proposition II.2.9 can be expressed as

$$M_i = N_i^P - N_i^Q + \nu_i - 1,$$

where N_i^P and N_i^Q are the multiplicities of $P \circ \pi$ and $Q \circ \pi$, respectively, over the divisor E_i, for any $0 \leq i \leq r$.

Corollary II.2.11. *Any period can be represented as a sum of well-defined integrals of rational functions in $\widetilde{\mathbb{Q}}(x_1, \ldots, x_d)$ over compact semi-algebraic sets, obtained algorithmically respecting the KZ–rules from another integral representation.*

Proof. Let $\mathcal{I}(S, P/Q)$ be an absolute convergent integral over $\mathbb{R}^d$ and note $\omega = P/Q \cdot \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d$. By Corollary II.2.5, we can assume that the domain S is compact. Denote $X = \partial_z S \cup \{P = 0\} \cup \{Q = 0\}$, using Proposition II.2.9, there exist a morphism $\pi : W \to \mathbb{R}^d$ over a closed smooth d–dimensional $\widetilde{\mathbb{Q}}$–subvariety W of $\mathbb{R}^d \times \mathbb{P}^m_{\mathbb{R}}$ such that the pullback $\pi^*\omega$ is well-defined over the topological closure of $\pi^{-1}\mathring{S}$, denoted by $\widetilde{S}$. As π is proper and W is a closed set of $\mathbb{R}^d \times \mathbb{P}^m_{\mathbb{R}}$, then $\pi^{-1}S$ is compact in $\mathbb{R}^d \times \mathbb{P}^m_{\mathbb{R}}$ and this implies the compacity of $\widetilde{S}$ in $\mathbb{R}^d \times \mathbb{P}^m_{\mathbb{R}}$.

Let $\mathcal{C} = \{C_i\}_{i=0}^m$ be the closed partition of $\mathbb{P}^m_{\mathbb{R}}$ given in Proposition II.2.3. Define by $\widetilde{S}_i = \widetilde{S} \cap \left(\mathbb{R}^d \times C_i\right)$ and by $S_i = \pi\widetilde{S}_i$. It is clear that $S = \bigcup_{i=0}^m \widetilde{S}_i$ and that $\bigcap_{i=0}^m \widetilde{S}_i$ is a $(d-1)$–dimensional semi-algebraic set. Moreover, $\widetilde{S}_i$ is compact since the projection of $\widetilde{S}$ in $\mathbb{R}^d$ is compact. Let $\{(V_i, \phi_i)\}_{i=0}^m$ be the affine charts of W given by the desingularization composed by Zariski open sets such that $V_i \simeq W_i = W \cap (\mathbb{R}^d \times U_{x_i})$ via ϕ_i, for any U_{x_i} of the usual atlas

$\mathcal{U}$ of $\mathbb{P}^m_{\mathbb{R}}$ (see Remark II.2.7). By Proposition II.2.3, for any $C_i \in \mathcal{C}$ there is $U_i \in \mathcal{U}$ such that $C_i \subset U_i$. Thus, any $\widetilde{S}_i$ is contained in a W_i.

Following this decomposition and defining $\varphi_i = \pi \circ \phi_i$ a birational map in $\mathbb{R}^d$, we obtain a sequence of KZ–operations:

$$\mathcal{I}(S, P/Q) = \sum_{i=0}^{m} \int_{\varphi_i^{-1} S_i} \varphi_i^* \omega = \sum_{i=0}^{m} \mathcal{I}(T_i, P_i/Q_i)$$

where $T_i = \varphi_i^{-1} S_i \in \mathcal{SA}^d_{\widetilde{\mathbb{Q}}}$ is compact and $P_i, Q_i \in \widetilde{\mathbb{Q}}[x_1, \ldots, x_d]$ are coprime polynomials verifying that Q_i has not zero locus over T_i, for any $i = 0, \ldots, m$. □

Algorithm 3 Resolution of poles on the boundary.

Input: A compact semi-algebraic domain S and two polynomials P, Q.

Output: A list of triples $(\widetilde{S}_i, \widetilde{P}_i, \widetilde{Q}_i)$ where $\widetilde{S}_i$ is compact a semi-algebraic set and coprime polynomials $\widetilde{P}_i, \widetilde{Q}_i$ such that $\widetilde{Q}_i$ has not zeros in $\widetilde{S}_i$ and $\mathcal{I}(S, P/Q) = \sum_i \mathcal{I}(\widetilde{S}_i, \widetilde{P}_i/\widetilde{Q}_i)$.

1: **procedure** ResolvePoles(S, P, Q)
2: $d \leftarrow \dim S$
3: $X \leftarrow \partial_z S \cup \{P = 0\} \cup \{Q = 0\}$
4: $\{(V_i, \phi_i)\} \leftarrow$ The list of affine charts of the embedded resolution $\pi : W \to \mathbb{R}^d$ of X.
5: $L \leftarrow \{\}$
6: **for** $i \leftarrow 0, \ldots, m$ **do**
7: $\varphi_i \leftarrow \pi \circ \phi_i$
8: $\widetilde{S}_i \leftarrow \varphi_i^{-1} S \cap \phi_i^{-1} \left(W \cap (\mathbb{R}^d \times C_i)\right)$
9: $\widetilde{P}_i/\widetilde{Q}_i \leftarrow$ Change of variables in P_i/Q_i given by φ_i
10: $\widetilde{P}_i/\widetilde{Q}_i \leftarrow \widetilde{P}_i/\widetilde{Q}_i \times \mathrm{Jac}(\varphi_i)$ ▷ The Jacobian of the change of variables
11: $L \leftarrow L \cup \{(\widetilde{S}_i, \widetilde{P}_i, \widetilde{Q}_i)\}$
12: **return** L

Corollary II.2.12. *Let $p \in \mathcal{P}^{\mathbb{R}}_{\mathrm{KZ}}$ be expressed as an absolutely convergent integral of the form $\mathcal{I}(S, P/Q)$. Then p can be expressed as*

$$p = \mathrm{vol}_d(K_1) - \mathrm{vol}_d(K_2),$$

where K_1, K_2 are compact $(d+1)$-dimensional $\widetilde{\mathbb{Q}}$–semi-algebraic sets, algorithmically and respecting the KZ–rules from $\mathcal{I}(S, P/Q)$.

Proof. Suppose that $0 \neq p$. Up to zero measure sets, we can give a partition of S depending on the sign of the rational function $\frac{P}{Q}(x_1, \ldots, x_d)$ in $\mathbb{R}^d$:

$$\mathcal{I}(S, P/Q) = \mathcal{I}(S^+, P/Q) - \mathcal{I}(S^-, -P/Q)$$

where $S^{\pm} = \left\{(x_1, \ldots, x_d) \in S \mid \mathrm{sgn}(\frac{P}{Q}(x_1, \ldots, x_d)) = \pm 1\right\}$. Note that both integrals give finite positive numbers, since $\mathcal{I}(S, P/Q)$ is absolutely convergent. By Corollary II.2.11, we can express both integrals as:

$$\mathcal{I}(S^{\pm}, P/Q) = \sum_{i=1}^{n_{\pm}} \mathcal{I}(S_i^{\pm}, P_i^{\pm}/Q_i^{\pm})$$

where $S_i^{\pm} \in \mathcal{SA}^d_{\widetilde{\mathbb{Q}}}$ is compact and $P_i^{\pm}/Q_i^{\pm} \in \widetilde{\mathbb{Q}}(x_1,\dots,x_d)$ reduced and well-defined over $S_i^{\pm}$, for any $i = 1,\dots,n_{\pm}$. Note that $P_i^{\pm}/Q_i^{\pm}$ does not change of sign over $S_i^{\pm}$. Considering integrals by the volume of the region delimited by $P_i^{\pm}/Q_i^{\pm}$ we perform a change of variables over each integral obtaining:

$$\mathcal{I}(S^{\pm}, P/Q) = \sum_{i=1}^{n_{\pm}} \int_{K_i^{\pm}} 1 \, \mathrm{d}t\mathrm{d}x_1 \cdots \mathrm{d}x_d$$

where

$$K_i^+ = \left\{ (t, x_1, \dots, x_d) \in \mathbb{R}_+ \times S_i^+ \;\middle|\; t \leq \frac{P_i^+}{Q_i^+}(x_1,\dots,x_d) \right\}$$
$$K_i^- = \left\{ (t, x_1, \dots, x_d) \in \mathbb{R}_+ \times S_i^- \;\middle|\; t \geq \frac{P_i^-}{Q_i^-}(x_1,\dots,x_d) \right\},$$

which are compact sets. It remains to prove that $K_i^{\pm} \in \mathcal{SA}^{d+1}_{\widetilde{\mathbb{Q}}}$. We define $H_i^+ = t \cdot Q_i^+ - P_i^+ \in \widetilde{\mathbb{Q}}[t, x_1, \dots, x_d]$, then $\left\{ t < P_i^+/Q_i^+(x_1,\dots,x_d) \right\}$ is expressed as the union of

$$\{H_i^+(t, x_1, \dots, x_d) < 0\} \cap \{Q_i^+(x_1, \dots, x_d) > 0\}$$

and

$$\{H_i^+(t, x_1, \dots, x_d) > 0\} \cap \{Q_i^+(x_1, \dots, x_d) < 0\}.$$

Thus $K_i^+ \in \mathcal{SA}^{d+1}_{\widetilde{\mathbb{Q}}}$ since semi-algebraic domains are stable by finite union and intersection. Analogously, $K_i^- \in \mathcal{SA}^{d+1}_{\widetilde{\mathbb{Q}}}$. Since the sets $K_i^{\pm}$ are compact, there exist a sequence of $\widetilde{\mathbb{Q}}$–translations $\left(\phi_i^{\pm}\right)_{i=1}^{n_{\pm}}$ in $\mathbb{R}^{d+1}$ such that $\bigcap_{i=1}^{n_{\pm}} K_i^{\pm} = \emptyset$. Defining $K_1 = \bigcup_{i=1}^{n_+} K_i^+$ and $K_2 = \bigcup_{i=1}^{n_-} K_i^-$, the result holds. □

II.3 Explicit algorithmic reduction in $\mathbb{R}^2$

In the general case, despite the algorithmic character of resolution of singularities, the previous construction is hardly implementable for concrete examples. However, this is not the case for resolution of plane curve singularities since the singular locus of reduced plane curves is a finite set of points. Taking advantage of this fact, we exhibit an explicit algorithm to remove the poles at the boundary in the case of integrals defined over compact semi-algebraic domains in the plane, obtaining directly Corollaries II.2.11 and II.2.12.

Let $\partial_z S, P(\omega)$ and $\mathcal{Z}$ be as in Section II.2. In this case, $\partial_z S$ and $P(\omega)$ are real plane curves. The absolute convergence assumption for $\mathcal{I}(S, P/Q)$ guarantees that $\mathcal{Z}$ is a finite set of points.

Consider $\pi : \widehat{\mathbb{R}}^2_o \to \mathbb{R}$ the *blow-up of* $\mathbb{R}^2$ *at the origin* O, where

$$\widehat{\mathbb{R}}^2_o = \left\{ ((x,y), [u_1 : u_2]) \in \mathbb{R}^2 \times \mathbb{P}^1_{\mathbb{R}} \mid xu_2 - yu_1 = 0 \right\}.$$

Recall that $\widehat{\mathbb{R}}^2_o$ is a manifold covered by two charts $U_1 = \{u_1 \neq 0\}$ and $U_2 = \{u_2 \neq 0\}$ diffeomorphic to $\mathbb{R}^2$, mapping to the base $\mathbb{R}^2$ via

$$\phi_1 : \begin{array}{ccc} U_1 \simeq \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (s_1, t_1) & \longmapsto & (s_1, s_1 t_1) \end{array} \quad \text{and} \quad \phi_2 : \begin{array}{ccc} U_2 \simeq \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (s_2, t_2) & \longmapsto & (s_2 t_2, t_2) \end{array},$$

in local coordinates (s_1, t_1) and (s_2, t_2) of U_1 and U_2, respectively. Denote by $E = \pi^{-1}O$ the *exceptional divisor*, note that $\widehat{\mathbb{R}}^2_o \setminus E \overset{\pi}{\simeq} \mathbb{R}^2 \setminus O$, i.e. $\phi_{1|\{s_1 \neq 0\}}$ and $\phi_{2|\{t_2 \neq 0\}}$ are diffeomorphisms. For an algebraic set $X \subset \mathbb{R}^2$, we define its *strict transform*, denoted by $\widetilde{X}$, as the Zariski closure of $\pi^{-1}(X \setminus O)$. In general, we define by $\pi : \widehat{\mathbb{R}}^2_p \to \mathbb{R}$ the *blow-up of* $\mathbb{R}^2$ *at the point* $p \in \mathbb{R}^2$.

Remark II.3.1. In the complex case, the strict transform of an algebraic set X coincides with the topological closure of $\pi^{-1}(X \setminus p)$. This property is not longer true in the real case. For example, let C be a real curve with one component given by the zero locus of $f(x, y) = x^2(y + x)(y^2 + x^4) + y^5$. If we take local coordinates (s, t) in the first chart of the blow up, then:

$$(f \circ \phi_1) = s^5 \left((t + 1)(t^2 + s^2) + t^5\right).$$

Outside the exceptional divisor $\{s = 0\}$ of multiplicity 5, we can see that the origin is an isolated point of the Zariski closure of $\pi^{-1}(C \setminus O)$, which corresponds to the intersection locus of two complex conjugated branches of $\widetilde{C}$.

Definition II.3.2. Let $A \subset \mathbb{R}^2$, we define the *τ–strict transform* of A, denoted by $\widetilde{A}^{\tau}$, as the topological closure of $\pi^{-1}(A \setminus p)$.

This notion will be useful in order to distinguish and control the points we are interested to resolve in the pole locus: those which stay in our semi-algebraic domain's boundary at each birational transformation.

Property II.3.3. *Let $X \subset \mathbb{R}^2$ be an algebraic set. Then $\widetilde{X}^{\tau}$ is a union of connected components of $\widetilde{X}$.*

Embedded resolution of singularities of curves in the affine plane is obtained by a sequence of blow-ups of the singular points. In addition, in dimension 2, there exists a *minimal* embedded resolution of singularities, i.e. a desingularization $W \to \mathbb{R}^2$ such that any other desingularization $W' \to \mathbb{R}^2$ factors with it: $W' \to W \to \mathbb{R}^2$ (see [Lip78] and [Liu02, Section 9.3.4]).

II.3.1 Local compacity and tangent cone

The exceptional divisor E is isomorphic to the projective line. This transformation "separates" the lines passing by the origin, which become transversed to E in the blow-up variety and we obtain a bijection between the points of $\mathbb{P}^1_{\mathbb{R}}$ and the pencil of lines passing through the origin. This transformation is represented in the local case in Figure II.1.

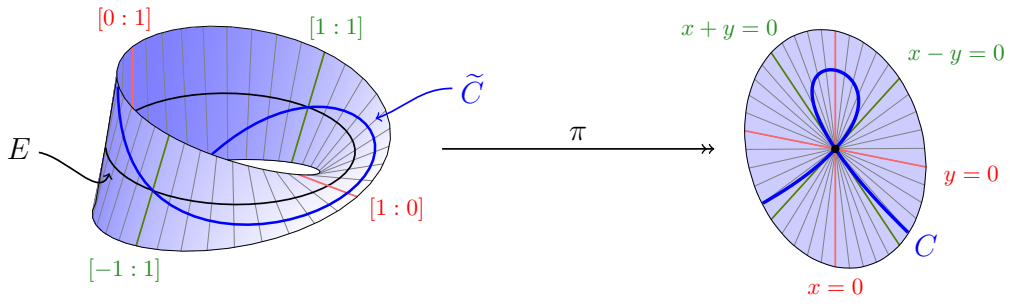

Figure II.1: Local blow-up of $C = \{x^2 + y^2(y - 1) = 0\}$ at the origin.

For a reduced polynomial f of degree n and a point $p = (p_1, p_2) \in \mathbb{R}^2$, we consider the Taylor expansion of f about $p = (p_1, p_2) \in \mathbb{R}^2$ expressed in homogeneous components, i.e. $f = f_{(0)} + \ldots + f_{(n)}$ where

$$f_{(j)}(x, y) = \sum_{i=0}^{j} a_{i,j-i}(x - p_1)^i (y - p_2)^{j-i}$$

We define the *algebraic tangent cone* of $C = f^{-1}(0)$ at p as the zero set $T_p(C) = f_{(k)}^{-1}(0)$ where $k = \min\{j \geq 0 \mid f_{(j)} \neq 0\}$ is the *order* of f in p. Note that the algebraic tangent cone of a curve is always decomposable as a union of lines in the complex plane, but not over the reals. The algebraic tangent cone coincides with the tangent space in the $\mathcal{C}^\infty$ sense over a nonsingular point of a real algebraic curve (see [BCR98, Sec. 3]). Lines belonging to the algebraic tangent cone at a point p in a curve can be characterized in the blow-up at p.

Lemma II.3.4. *Let $f \in \widetilde{\mathbb{Q}}[x, y]$ be a reduced polynomial and $C = f^{-1}(0)$ a real algebraic curve. A line L belongs to $T_p(C)$ if and only if $\widetilde{C}^\tau \cap \widetilde{L} \cap E \neq \emptyset$.*

Proof. Without loss of generality, assume that p is the origin, and L is given by the equation $x - \alpha y = 0$, for some $\alpha \in \mathbb{R}$. Expressing f in homogeneous components:

$$f(x, y) = f_{(k)}(x, y) + f_{(k+1)}(x, y) + \ldots + f_{(n)}(x, y)$$

where $f_{(k)}(x, y) \neq 0$. Taking local coordinates (s, t) in the second chart of the blow-up, it is easy to see:

$$(f \circ \phi_2)(s, t) = t^k \left(f_{(k)}(s, 1) + t f_{(k+1)}(s, 1) + \ldots + t^{n-k} f_{(n)}(s, 1) \right) = t^k \tilde{f}(s, t)$$

In this chart, $\widetilde{L}$ is given by $s - \alpha = 0$. The points in $\pi^{-1}(C \setminus p)$ over this chart verify the equation $\tilde{f}(s, t) = 0$. In this setting, $L \in T_p(C)$ is equivalent to say that s divides $f_d(s, t)$. Let $((s_n, t_n))_{n \in \mathbb{N}}$ a sequence of points contained in $\pi^{-1}(C \setminus p)$ such that their image by π converges to the origin, i.e. if t_n tends to zero. If (s_n, t_n) converges to $(s, 0) \in E \cap U_2$, by argument of continuity $0 = \tilde{f}(s, 0) = f_{(k)}(s, 1)$. Then, $\widetilde{C}^\tau \cap \widetilde{L} \cap E \cap U_2 = \{(\alpha, 0)\}$ if and only if $s - \alpha$ divides $f_{(k)}(s, 1)$. □

Note that any line contained in the algebraic tangent cone of a real algebraic curve as above is defined by algebraic real coefficients. For a point $p \in Z$, our main objective is to separate the boundary of S from the pole locus $P(\omega)$ at p by a finite sequence of blow-ups. In order to hold compact domains in our integrals at some affine chart, we need to take charts in the blow-up with respect to a line which does not belongs to the algebraic tangent cone at p of the Zariski closure of ∂S. We consider in general $T_p(\partial_z S)$ at any point $p \in Z$ with the purpose to give a global procedure. Remark that $T_p(\partial_z S)$ contains at least one line since S is an open semi-algebraic set and the defining polynomial of $\partial_z S$ change of sign locally at p.

Proposition II.3.5. *Let $p \in \partial S$ and suppose that there exists a line L such that $\overline{S} \cap L = \{p\}$. If $L \notin T_p(\partial_z S)$ then there exist a Zariski open $U \subset \widehat{\mathbb{R}}^2$ such that $\widetilde{S}^\tau \cap U$ is compact.*

Proof. As the map $\pi : \widehat{\mathbb{R}}^2 \to \mathbb{R}$ becomes an isomorphism outside the exceptional divisor, i.e. $\widehat{\mathbb{R}}^2 \setminus E \overset{\pi}{\simeq} \mathbb{R}^2 \setminus p$, it is clear that $\overline{\pi^{-1} S} = \overline{\pi^{-1}(\overline{S} \setminus p)}$. This closed set is contained in $\pi^{-1}\overline{S}$, which is compact in $\widehat{\mathbb{R}}^2$ since π is a proper map, so $\overline{\pi^{-1} S}$ is also compact in the blow-up of the real plane. Taking $V = \widehat{\mathbb{R}}^2 \setminus E$, we have $\widetilde{L} \cap \widetilde{S}^\tau \cap V = \emptyset$ since $\overline{S} \cap L = \{p\}$. Also, by Lemma II.3.4, $L \notin T_p(\partial_z S)$ is equivalent to say that $\widetilde{L} \cap \widetilde{S}^\tau \cap E = \emptyset$. Thus, defining $U = \widehat{\mathbb{R}}^2 \setminus \widetilde{L}$ we have that $\overline{\pi^{-1} S} \subset U$ and the result holds. □

Remark II.3.6. Lemma II.3.4 and Proposition II.3.5 can be interpreted geometrically as follows. For a point p of a real algebraic plane curve C, the algebraic tangent cone contains the *geometric tangent cone*, i.e. the limits of all *secant rays* which originates from p and pass through a sequence of points $(p_n)_{n\in\mathbb{N}} \subset C \setminus p$ converging to p. These generalizations of tangent spaces were introduced by Whitney in [Whi65a]–[Whi65b] to study the singularities of real and complex analytic varieties. As $T_p(C)$ is of algebraic nature, it codifies much more information that the geometric tangent cone, specially in the real plane where we can detect algebraically the tangent cone of two complex conjugate branches which intersect at p.
Lemma II.3.4 implies that $T_p(\partial_z S)$ is a discrete set, and the union of the set of secant lines of ∂S at p with $T_p(\partial_z S)$ forms a closed set in $E \simeq \mathbb{P}^1_{\mathbb{R}}$ identifying each line $L_{[\alpha:\beta]} : \alpha x + \beta y + \gamma = 0$ with a point $[\alpha : \beta] \in \mathbb{P}^1_{\mathbb{R}}$. Then, under the hypotheses of Proposition II.3.5, if we found a line L such that $\overline{S} \cap L = \{p\}$ and $L \notin T_p(\partial_z S)$, then there exists an open cone $V \subset \mathbb{R}^2$ centered at p containing L such that any line L' in V is not in the algebraic tangent cone $L \notin T_p(\partial_z S)$.
As a consequence, we can always choose lines with algebraic coefficients which respect taking charts at each blow-up. Moreover, as S is a bounded set, there exists an open subcone $V' \subset V$ containing L such that any line L' in V' verifies that $\overline{S} \cap L' = \{p\}$.

Theorem II.3.7. *Let an open bounded $S \in \mathcal{SA}^d_{\widetilde{\mathbb{Q}}}$ and $\omega = P/Q \cdot \mathrm{d}x \wedge \mathrm{d}y$ with $P/Q \in \widetilde{\mathbb{Q}}(x, y)$ such that the integral $\mathcal{I}(S, P/Q)$ converges absolutely. Then there exist a 1–dimensional semi-algebraic set $X \subset \mathbb{R}^2$, a finite disjoint partition $S = X \cup S_0 \cup \cdots \cup S_n$, and a collection $\{\varphi_i\}_{i=1}^n$ of birational morphisms $\varphi_i : \mathbb{R}^2 \setminus X \to \mathbb{R}^2 \setminus X$ such that*

$$\int_S \omega = \sum_{i=0}^{n} \int_{\psi_i^{-1} S_i} \psi_i^* \omega$$

where $\psi_i^{-1} S_i$ is bounded and $\psi_i^* \omega$ is a rational 2–form defined in $\overline{S}_i$ for any $i = 0, \ldots, n$. Moreover, this process is algorithmic and depends only of the representation of S.

Corollary II.3.8. *Any period expressed as $\mathcal{I}(S, P/Q)$ in dimension 2 can be represented as a finite sum of absolutely convergent integrals of a rational functions in $\widetilde{\mathbb{Q}}(x, y)$ over compact semi-algebraic sets, obtained algorithmically and respecting the KZ–rules from $\mathcal{I}(S, P/Q)$.*

II.3.2 Algorithmic and proof of Theorem II.3.7

In the case of $d = 2$, we deal with absolute convergent integrals of the form

$$\mathcal{I}(S, P/Q) = \int_S \frac{P(x, y)}{Q(x, y)} \cdot \mathrm{d}x \wedge \mathrm{d}y$$

By Theorem II.2.4, we can suppose that S is compact. Denote by X_Q the pole locus of $\mathcal{I}(S, P/Q)$ in this case.

Choosing an order in the set of points $\mathcal{Z}$, we construct a procedure of resolution of poles in the boundary of S, by a successive use of birational maps over special partitions of S by intersection of semi-plans. In general, for a point $p \in Z$ we give a partition $S = X \cap (S \setminus X)$, choosing X a 1–dimensional semi-algebraic set as follows:

- If $T_p(\partial_z S)$ contains $n \geq 2$ lines: let $X = T_p(X) \cap S$, and $S = X \cup S_1 \cup \ldots \cup S_n$ such that $S_i \neq \emptyset$, for any $i = 1, \ldots, n$.

- If $T_p(\partial_z S)$ only contains one line: consider $N_p(\partial_z S)$ the normal space of $\partial_z S$ at p and let $X = (T_p(X) \cup N_p(\partial_z S)) \cap S$. We obtain a partition $S = X \cup S_1 \cup S_2$. In this case, $T_p(\partial_z S)$ is in fact the tangent space of $\partial_z S$ at p and we create a cone using $N_p(\partial_z S)$. Note that this case contains when p is smooth in $\partial_z S$.

For any $i = 1, \ldots, n$, let V_i^p be the open cone centered at p such that ∂V_i^p is the Zariski closure of X and $S_i \subset V_i^p$. Choosing a line $L_i \not\subset \overline{V_i^p}$ defined by real algebraic coefficients, we are in the hypotheses of Proposition II.3.5 and we can explicitely choose a chart (U_i, φ_i) in the blow-up $\pi : \widehat{\mathbb{R}}_p^2 \to \mathbb{R}$ such that L_i coincides with the exceptional divisor in U_i, φ_i is an diffeomorphism of $\mathbb{R}^2 \setminus L_i$, and $\varphi_i^{-1} S_i$ is a bounded set in $\mathbb{R}^2$. We obtain:

$$\begin{aligned}\mathcal{I}(S, P/Q) &= \sum_{i=1}^{n} \int_{\pi_i^{-1} S_i} \pi_i^* \left(\frac{P(x,y)}{Q(x,y)} \cdot \mathrm{d}x \wedge \mathrm{d}y \right) = \sum_{i=1}^{n} \int_{\varphi_i^{-1} S_i} \varphi_i^* \left(\frac{P(x,y)}{Q(x,y)} \cdot \mathrm{d}x \wedge \mathrm{d}y \right) \\ &= \sum_{i=1}^{n} \int_{\varphi_i^{-1} S_i} \frac{\widetilde{P}_i(s,t)}{\widetilde{Q}_i(s,t)} \cdot \mathrm{d}s \wedge \mathrm{d}t,\end{aligned}$$

where $\widetilde{P}_i$ and $\widetilde{Q}_i$ are coprime polynomials over $\widetilde{\mathbb{Q}}$.

Remark II.3.9. A simple case is obtained when $S \setminus p$ is contained in an open semi-plane whose boundary is a line L defined by real algebraic coefficients and such that $p \in L$ and $L \not\subset T_p(\partial_z S)$. Moreover, if in addition $T_p(X_Q) = \{L\}$, then taking charts to respect the line L in the blow-up of p, the possible intersection point between the boundary of the τ–strict transform of S and the new pole divisor will be outside the affine chart.

In order to apply this procedure inductively:

Initiation: Define $\mathcal{Z}^{(0)} = \mathcal{Z} = \{p_1, \ldots, p_{n_0}\}$ and $S^{(0)} = S$. We choose $p_1 \in \mathcal{Z}^{(0)}$ and we construct a 1-dimensional semi-algebraic set X_1 and partition with respect this point as before. We obtain:

$$S = X_1 \cup \bigcup_{i_1=1}^{n_1} S_{i_1},$$

and a sequence of lines $(L_{i_1})_{i_1=1}^{n_1}$ and diffeomorphisms $(\varphi_{i_1})_{i_1=1}^{n_1}$ of $\mathbb{R}^2 \setminus L_{i_1}$ coming from taking charts in he blow-ups $\pi_i : \widehat{\mathbb{R}}_p^2 \to \mathbb{R}$ such that $\widetilde{S}_{i_1} = \varphi_{i_1}^{-1} S_{i_1}$ is a bounded set in $\mathbb{R}^2$. We define the new sets of poles for each $\widetilde{S}_{i_1}$:

$$\mathcal{Z}^{(i_1)} = \partial \widetilde{S}_{i_1} \cap V(\widetilde{Q}_{i_1}), \quad i_1 = 1, \ldots, n_1.$$

Repeating this process at each $\mathcal{I}(\varphi_{i_1}^{-1} S_{i_1}, \widetilde{P}_{i_1}/\widetilde{Q}_{i_1})$, we construct the partitions:

$$\widetilde{S}_{i_1} = X_2 \cup \bigcup_{i_2=1}^{n_2} S_{i_1 i_2}, \quad i_1 = 1, \ldots, n_1.$$

and a sequence of lines $(L_{i_1 i_2})_{i_2=1}^{n_2}$ and diffeomorphisms $(\varphi_{i_1 i_2})_{i_2=1}^{n_2}$ of $\mathbb{R}^2 \setminus L_{i_2}$ such that $\widetilde{S}_{i_1 i_2} = \varphi_{i_1}^{-1} S_{i_1}$ are bounded sets. In this way,

$$\begin{aligned}\mathcal{I}(S, P/Q) &= \sum_{i_1=1}^{n_1} \int_{\varphi_{i_1}^{-1} S_{i_1}} \frac{\widetilde{P}_{i_1}(s_1,t_1)}{\widetilde{Q}_{i_1}(s_1,t_1)} \cdot \mathrm{d}s_1 \wedge \mathrm{d}t_1 \\ &= \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \int_{\widetilde{S}_{i_1 i_2}} \frac{\widetilde{P}_{i_1 i_2}(s_2,t_2)}{\widetilde{Q}_{i_1 i_2}(s_2,t_2)} \cdot \mathrm{d}s_2 \wedge \mathrm{d}t_2.\end{aligned}$$

Thus, we define:

$$\mathcal{Z}^{(i_1 i_2)} = \partial \widetilde{S}_{i_1 i_2} \cap V(\widetilde{Q}_{i_1 i_2}), \quad i_2 = 1, \ldots, n_2.$$

Induction: Let $\mathcal{I}(S, P/Q)$ expressed as

$$\mathcal{I}(S, P/Q) = \sum_{i_1=1}^{n_1} \cdots \sum_{i_k=1}^{n_k} \int_{\widetilde{S}_{i_1 \cdots i_k}} \frac{\widetilde{P}_{i_1 \cdots i_k}(s_k, t_k)}{\widetilde{Q}_{i_1 \cdots i_k}(s_k, t_k)} \cdot \mathrm{d}s_k \wedge \mathrm{d}t_k$$

and

$$\mathcal{Z}^{(i_1 \cdots i_k)} = \partial \widetilde{S}_{i_1 \cdots i_k} \cap V(\widetilde{Q}_{i_1 \cdots i_k})$$

Repeating this process at each $\mathcal{I}(\widetilde{S}_{i_1 \cdots i_k}, \widetilde{P}_{i_1 \cdots i_k}/\widetilde{Q}_{i_1 \cdots i_k})$, we construct the partitions:

$$\widetilde{S}_{i_1 \cdots i_k} = X_{k+1} \cup \bigcup_{i_{k+1}=1}^{n_{k+1}} S_{i_1 \cdots i_k i_{k+1}}, \quad i_k = 1, \ldots, n_k$$

and a sequence of lines $(L_{i_1 \cdots i_k i_{k+1}})_{i_{k+1}=1}^{n_{k+1}}$ and diffeomorphisms $(\varphi_{i_1 \cdots i_k i_{k+1}})_{i_{k+1}=1}^{n_{k+1}}$ of $\mathbb{R}^2 \setminus L_{i_1 \cdots i_k i_{k+1}}$ such that $\widetilde{S}_{i_1 \cdots i_k i_{k+1}} = \varphi^{-1}_{i_1 \cdots i_k i_{k+1}} S_{i_1 \cdots i_k i_{k+1}}$ are bounded sets.

$$\begin{aligned}
\mathcal{I}(S, P/Q) &= \sum_{i_1=1}^{n_1} \cdots \sum_{i_k=1}^{n_k} \int_{\widetilde{S}_{i_1 \cdots i_k}} \frac{\widetilde{P}_{i_1 \cdots i_k}(s_k, t_k)}{\widetilde{Q}_{i_1 \cdots i_k}(s_k, t_k)} \cdot \mathrm{d}s_k \wedge \mathrm{d}t_k \\
&= \sum_{i_1=1}^{n_1} \cdots \sum_{i_k=1}^{n_k} \sum_{i_{k+1}=1}^{n_{k+1}} \int_{\widetilde{S}_{i_1 \cdots i_k i_{k+1}}} \frac{\widetilde{P}_{i_1 \cdots i_k i_{k+1}}(s_{k+1}, t_{k+1})}{\widetilde{Q}_{i_1 \cdots i_k i_{k+1}}(s_{k+1}, t_{k+1})} \cdot \mathrm{d}s_{k+1} \wedge \mathrm{d}t_{k+1}.
\end{aligned}$$

Finally, we define:

$$\mathcal{Z}^{(i_1 \cdots i_k i_{k+1})} = \partial \widetilde{S}_{i_1 \cdots i_k i_{k+1}} \cap V(\widetilde{Q}_{i_1 \cdots i_k i_{k+1}}), \quad i_{k+1} = 1, \ldots, n_{k+1}.$$

Lemma II.3.10. *There exist a positive integer $N > 0$ such that $\mathcal{Z}^{(i_1 i_2 \cdots i_N)} = \emptyset$, for any $i_1, \ldots, i_N \in \mathbb{N}$.*

Proof. This result holds directly from Proposition II.2.9. □

Previous Lemma concludes that the induction procedure stops after a finite number of steps, and Theorem II.3.7 holds.

Remark II.3.11. Another way to proceed is to "isolate" the pole locus at each step. Consider a partition of the domain

$$S = S' \cup \bigcup_{p \in \mathcal{Z}} S \cap \mathbb{B}_\varepsilon(p)$$

for a sufficient small $\varepsilon \in \widetilde{\mathbb{Q}}_{>0}$, localizing the problem over the poles in the boundary and applying the procedure previously explained at each $S \cap \mathbb{B}_\varepsilon(p)$.

II.4 Difference of two semi-algebraic sets and volumes

We finish the proof of Theorem II.1.1 giving an algorithmic construction of a compact semi-algebraic set from the difference of two ones, obtained in Corollary II.2.12.

II.4.1 Partition by Riemann sums

We assume that p is positive and $0 < \mathrm{vol}_d(K_2) < \mathrm{vol}_d(K_1)$, without loss of generality. The aim of this part is to prove that we can construct a third compact semi-algebraic set K from K_1 and K_2 such that $p = \mathrm{vol}_d(K)$. We use an approximation by inner and outer Riemann sums, following the procedure described in [Yos08, sec. 3.4].

As K_1 and K_2 are compact then bounded, suppose that there exists a positive integer $r > 0$ such that both of them are contained in the cube $[0, r]^d$. We construct a partition of both semi-algebraic sets using rational cubes. Let n be a positive integer and define the family of cubes subdividing $[0, r]^d$:

$$C_n(k_1, \ldots, k_d) = \left[\frac{k_1}{n}r, \frac{k_1+1}{n}r\right] \times \ldots \times \left[\frac{k_d}{n}r, \frac{k_d+1}{n}r\right]$$

where $0 \leq k_1, \ldots, k_d \leq n$ are integers. Denote by $\mathring{C}_n(k_1, \ldots, k_d)$ the interior of the previously defined cube.

For any $n \in \mathbb{N}$, we give a partition of $[0, r]^d$ composed by cubes of size $(r/n)^d$. Consider those which intersect K_1 and K_2,

$$\hat{\Delta}_n^{(i)} = \{(k_1, \ldots k_d) \in \{0, \ldots, n\}^d \mid C_n(k_1, \ldots k_d) \cap K_i \neq \varnothing\},$$

and those which are contained in our semi-algebraic sets

$$\check{\Delta}_n^{(i)} = \{(k_1, \ldots k_d) \in \{0, \ldots, n\}^d \mid C_n(k_1, \ldots k_d) \subset K_i\}.$$

Denote by $\hat{\delta}_i(n)$ and $\check{\delta}_i(n)$ respectively the cardinal of $\hat{\Delta}_n^{(i)}$ and $\check{\Delta}_n^{(i)}$, for any $n \in \mathbb{N}$. The compact semi-algebraic sets K_1 and K_2 are Borel sets, thus:

$$\lim_{n\to\infty} \hat{\delta}_i(n) \cdot \left(\frac{r}{n}\right)^d = \lim_{n\to\infty} \check{\delta}_i(n) \cdot \left(\frac{r}{n}\right)^d = \mathrm{vol}_d(K_i), \qquad i = 1, 2. \tag{II.3}$$

Lemma II.4.1. *There exists a positive integer n_0 such that for any $N \geq n_0$ we have $\hat{\delta}_2(N) < \hat{\delta}_1(N)$ and $\check{\delta}_2(N) < \check{\delta}_1(N)$.*

Proof. If we consider the volume covered by the cubes defined by the elements of $\hat{\Delta}_n^{(i)}$, we have for any n:

$$0 < \mathrm{vol}_d(K_i) \leq \hat{\delta}_i(n) \cdot \left(\frac{r}{n}\right)^d, \quad i = 1, 2.$$

We deduce from (II.3) that there exists a positive integer $\hat{n}_0$ such that, for any $N \geq \hat{n}_0$,

$$0 < \mathrm{vol}_d(K_2) \leq \hat{\delta}_2(N) \cdot \left(\frac{r}{N}\right)^d < \mathrm{vol}_d(K_1) \leq \hat{\delta}_1(N) \cdot \left(\frac{r}{N}\right)^d.$$

Then, we have

$$\hat{\delta}_2(N) \cdot \left(\frac{r}{N}\right)^d < \hat{\delta}_1(N) \cdot \left(\frac{r}{N}\right)^d.$$

The same argument is also valid for $\check{\Delta}_n^{(i)}$ by inner approximations to obtain an analogous $\check{n}_0$. Taking $n_0 = \max\{\hat{n}_0, \check{n}_0\}$, the result holds. □

Lemma II.4.2. *There exists a positive integer n_0 such that for any $N \geq n_0$ we have $\hat{\delta}_2(N) \leq \check{\delta}_1(N)$.*

Proof. We decompose, for any $n \in \mathbb{N}$:

$$\check{\delta}_1(n) - \hat{\delta}_2(n) = (\hat{\delta}_1(n) - \hat{\delta}_2(n)) - (\hat{\delta}_1(n) - \check{\delta}_1(n)).$$

Multiplying by $\left(\frac{r}{n}\right)^d$ and taking limits, we obtain:

$$\lim_{n\to\infty} (\hat{\delta}_1(n) - \hat{\delta}_2(n)) \left(\frac{r}{n}\right)^d = \mathrm{vol}_d(K_1) - \mathrm{vol}_d(K_2) = p$$
$$\lim_{n\to\infty} (\check{\delta}_1(n) - \hat{\delta}_1(n)) \left(\frac{r}{n}\right)^d = \mathrm{vol}_d(K_1) - \mathrm{vol}_d(K_1) = 0$$

Note that $p > 0$ and $\hat{\delta}_1(n) - \check{\delta}_1(n) \geq 0$ for any $n \in \mathbb{N}$. Furthermore, $\hat{\delta}_1(n) - \hat{\delta}_2(n) > 0$ for n sufficiently large by Lemma II.4.1. We have:

$$\forall \varepsilon_0 > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall N > n_0 : \ (\hat{\delta}_1(N) - \check{\delta}_1(N)) \left(\frac{r}{N}\right)^d < \varepsilon_0$$

and

$$\forall \varepsilon_1 > 0, \exists n_1 \in \mathbb{N} \text{ s.t. } \forall N > n_1 : \ \left|(\hat{\delta}_1(N) - \hat{\delta}_2(N)) \left(\frac{r}{N}\right)^d - p\right| < \varepsilon_1.$$

Taking $\varepsilon_1 = 1$ and $\varepsilon_0 = C - \varepsilon_1 = C - 1$, there exists $n_2 \in \mathbb{N}$ such that $\forall N > n_2$:

$$0 \leq (\hat{\delta}_1(N) - \check{\delta}_1(N)) \left(\frac{r}{N}\right)^d < C - 1 < (\hat{\delta}_1(N) - \hat{\delta}_2(N)) \left(\frac{r}{N}\right)^d.$$

Then, $\hat{\delta}_2(N) \leq \check{\delta}_1(N)$ for any $N > n_2$ and the result holds. □

II.4.2 Construction of the difference set

We construct $K \in \mathcal{SA}^d_{\mathbb{Q}}$, a compact set such that $|p| = \mathrm{vol}_d(K)$ from K_1 and K_2. The basic idea of this construction is to use inner and outer Riemann approximation by cubes in K_1 and K_2, respectively. Taking a sufficiently small rational size of cubes, we can give a re-ordination of cubes such that the outer cubes of K_2 can be translated into the inner cubes of K_1, which is assumed to have a bigger volume.

By Lemma II.4.2, we know that there exists $n_0 \in \mathbb{N}$ such that $\hat{\delta}_2(n_0) \leq \check{\delta}_1(n_0)$. Consider the wire net in $[0, r]^d$ defined by the boundary of all cubes in the partition:

$$W = \bigcup_{(k_1,\ldots,k_d)\in\{0,\ldots,n_0\}^d} \left\{(x_1, \ldots, x_d) \in [0, r]^d \mid x_i = \frac{k_i}{n_0} r, \ 1 \leq i \leq d\right\}.$$

and removes this zero measure subset in $[0, r]^d$:

$$H = [0, r]^d \setminus W = \bigcup_{(k_1,\ldots,k_d)\in\{0,\ldots,n_0\}^d} \mathring{C}_{n_0}(k_1, \ldots, k_d).$$

Thus, there exists a $\sigma = (\sigma_1, \ldots, \sigma_d) \in \Sigma(\{0, \ldots, n_0\}^d)$ such that, if we consider the induced bijective map

$$\begin{array}{rccc} \psi_\sigma : & \{0, \ldots, n_0\}^d & \longrightarrow & \{0, \ldots, n_0\}^d \\ & (k_1, \ldots, k_d) & \longmapsto & (\sigma_1(k_1), \ldots, \sigma_d(k_d)) \end{array},$$

then

1. $\psi_\sigma(\hat{\Delta}^{(2)}_{n_0}) \subset \check{\Delta}^{(1)}_{n_0}$.
2. $\psi_\sigma = \mathrm{Id}$ in $\{0,\dots,n_0\}^d \setminus \hat{\Delta}^{(2)}_{n_0}$.

Lemma II.4.3. *There exist a semi-algebraic map $\Psi : H \to H$ such that Ψ preserves the volume and $\Psi(H \cap K_2) \subset (H \cap K_1)$.*

Proof. The map ψ_σ induces a bijective map $\Psi : H \to H$ which sends a point $(x_i)_{i=1}^d$ contained in some $\mathring{C}_{n_0}(k_1,\dots,k_d)$ to the point

$$(x_i - k_i + \sigma_i(k_i))_{i=1}^d \in \mathring{C}_{n_0}(\sigma_1(k_1),\dots,\sigma_d(k_d)).$$

This map makes a re-organization of the open cubes in the partition of $[0,r]^d$ by translations following σ and it is easy to see that it is semi-algebraic. This is clearly a volume preserving map and the fact that $\psi_\sigma(\hat{\Delta}^{(2)}_{n_0}) \subset \check{\Delta}^{(1)}_{n_0}$ gives us the last property.

□

Finally, we can define K as the closure over $\mathbb{R}^d$ of $(H \cap K_1) \setminus \Psi(H \cap K_2)$ and we have proved Theorem II.1.1.

Remark II.4.4. Note that the previously described process which constructs the new compact semi-algebraic set K from K_1 and K_2 is completely algorithmic and respects the KZ–rules.

Algorithm 4 Construction of a compact semi-algebraic set from the difference of other two.

Input: Two compact semi-algebraic sets K_1, K_2 of maximal dimension d such that $\mathrm{vol}_d(K_2) < \mathrm{vol}_d(K_1) < +\infty$.

Output: A compact semi-algebraic K such that $\dim K = d$ and $\mathrm{vol}_d(K) = \mathrm{vol}_d(K_1) - \mathrm{vol}_d(K_2)$.

procedure VolumeFromDiffSA(K_1, K_2)
  $r \leftarrow \min\{n \in \mathbb{N} \mid K_1 \cup K_2 \subset [0,n]^d\}$
  $\Delta_1 \leftarrow \{\}$, $\Delta_2 \leftarrow \{\}$
  $\delta_1 \leftarrow 0$, $\delta_2 \leftarrow 1$
  $n \leftarrow 1$
  while $\delta_1 < \delta_2$ **do**
    for $(k_1,\dots,k_d) \in \{0,\dots,n\}^d$ **do**
      $\mathring{C}_n(k_1,\dots,k_d) \leftarrow \left(\frac{k_1}{n}r, \frac{k_1+1}{n}r\right) \times \dots \times \left(\frac{k_d}{n}r, \frac{k_d+1}{n}r\right)$
      if $\mathring{C}_n(k_1,\dots,k_d) \subset K_1$ **then**
        $\Delta_1 \leftarrow \Delta_1 \cup \{\mathring{C}_n(k_1,\dots,k_d)\}$
      else if $\mathring{C}_n(k_1,\dots,k_d) \cap K_2 \neq \emptyset$ **then**
        $\Delta_2 \leftarrow \Delta_2 \cup \{\mathring{C}_n(k_1,\dots,k_d)\}$
      $\delta_1 \leftarrow \#\Delta_1$, $\delta_2 \leftarrow \#\Delta_2$
  $K \leftarrow K_1$
  for $k \leftarrow 1,\dots,\delta_2$ **do** ▷ Elimination
    $D \leftarrow K_2 \cap \Delta_2[k]$
    $D \leftarrow$ Change of variables in D: $\tilde{x_i} = x_i - k_i + k'_i, \forall x_i$, where $(k'_1,\dots,k'_d) = \Delta_1[k]$
    $K \leftarrow K \setminus D$
  return $\overline{K}$

II.5 Some examples of semi-canonical reduction

We present some examples of the effective reduction algorithm described in the previous Sections, starting from different integral representations of π and π^2. These examples gives representations of the main problem's difficulties.

II.5.1 A basic example: π

Example II.5.1. A classical way to write π as an integral is:

$$\mathcal{I}\left(\mathbb{R}, 1/(1+x^2)\right) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{1+x^2}.$$

Following our procedure in order to obtain π as the volume of a semi-algebraic set from $\mathcal{I}\left(\mathbb{R}, 1/(1+x^2)\right)$, we decompose the real line in three pieces using the point arrangement $\mathcal{A} = \{\{x=-1\}, \{x=1\}\}$ of $\mathbb{R}$:

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{1+x^2} = \int_{-1}^{1} \frac{\mathrm{d}x}{1+x^2} + \int_{S} \frac{\mathrm{d}x}{1+x^2},$$

where $S = \{x^2-1>0\}$ is a unbounded semi-algebraic set. Consider now the canonical inclusion of S into the second chart $U_y = \{[x:y] \mid y \neq 0\}$ of the projective line $\mathbb{P}^1_{\mathbb{R}}$. The change of charts with the first chart $U_x = \{[x:y] \mid x \neq 0\}$ gives as a diffeomorphism ϕ of $\mathbb{R}^*$ expressed by $\phi(y) = 1/y$, where $|\operatorname{Jac}(\phi)(y)| = 1/y^2$ and $\phi^{-1}S = \{y \neq 0, 1-y^2>0\} = (-1,1)\setminus\{0\}$. Then:

$$\int_S \frac{\mathrm{d}x}{1+x^2} = \int_{\phi^{-1}S} \phi^*\left(\frac{\mathrm{d}x}{1+x^2}\right) = \int_{(-1,1)\setminus\{0\}} \frac{y^2}{1+y^2}\cdot\frac{1}{y^2}\mathrm{d}y = \int_{-1}^{1}\frac{\mathrm{d}y}{1+y^2}.$$

Thus, using partitions and rational change of variables given by ϕ, we express:

$$\mathcal{I}\left(\mathbb{R}, 1/(1+x^2)\right) = \int_{-1}^{1} \frac{\mathrm{d}x}{1+x^2} + \int_{S} \frac{\mathrm{d}x}{1+x^2} = \int_{-1}^{1} \frac{\mathrm{d}x}{1+x^2} + \int_{-1}^{1} \frac{\mathrm{d}y}{1+y^2}.$$

Taking the area under the graph in both integrals and after a symmetry across the horizontal axis in the second integral, we obtain:

$$\begin{aligned}
\pi &= \int_{\left\{\begin{array}{c} -1\le x\le 1 \\ 0\le z(1+x^2)\le 1\end{array}\right\}} \mathrm{d}x\mathrm{d}z + \int_{\left\{\begin{array}{c} -1\le y\le 1 \\ 0\le u(1+y^2)\le 1\end{array}\right\}} \mathrm{d}y\mathrm{d}u \\
&= \int_{\left\{\begin{array}{c} -1\le x\le 1 \\ 0\le z(1+x^2)\le 1\end{array}\right\}} \mathrm{d}x\mathrm{d}z + \int_{\left\{\begin{array}{c} -1\le y\le 1 \\ -1\le u(1+y^2)\le 0\end{array}\right\}} \mathrm{d}u\mathrm{d}y \\
&= \mathrm{vol}_2\left(\left\{\begin{array}{c} -1\le x\le 1 \\ -1\le z(1+x^2)\le 1\end{array}\right\}\right).
\end{aligned}$$

This semi-canonical reduction for π is represented in Figure II.2.

Example II.5.2. Let revisiting the previous example, seeing a part of our integral described directly as an area of an unbounded two dimensional semi-algebraic set:

$$\frac{\pi}{4} = \int_1^{\infty} \frac{1}{1+x^2}\mathrm{d}x = \int_D \mathrm{d}x\mathrm{d}y$$

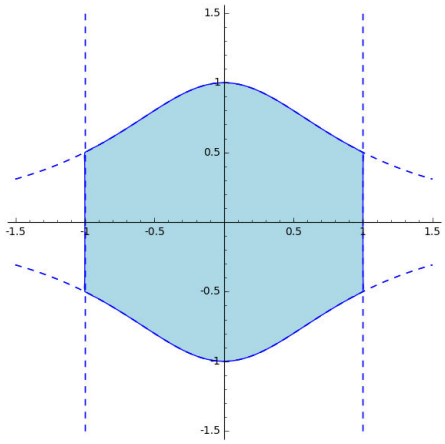

Figure II.2: A semi-canonical reduction for π as a 2-dimensional volume of $K = \{-1 \leq x \leq 1, -1 \leq z(1+x^2) \leq 1\}$.

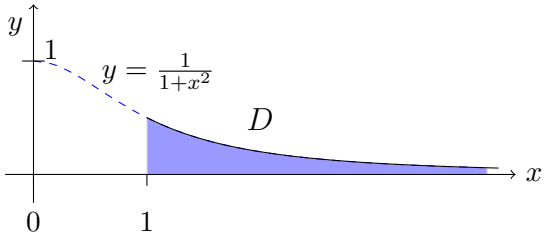

Figure II.3: The unbounded set $D = \{x > 1, 0 < y(1+x^2) < 1\}$.

with $D = \{x > 1, 0 < y(1+x^2) < 1\} \subset \mathbb{R}^2$ (see Figure II.3).

We consider the inclusion of D in $U_z = \{[x : y : z] \mid z \neq 0\} \subset \mathbb{P}^2_{\mathbb{R}}$ in order to obtain a compact domain. Composing the change of charts taking the line $\{x = 0\} \subset \mathbb{P}^2_{\mathbb{R}}$ as line at infinity, we obtain a diffeomorphism φ of $\mathbb{R}^2$ minus a line given by $\varphi(x_1, y_1) = (1/x_1, y_1/x_1)$, with associated jacobian determinant $|\operatorname{Jac}(\varphi)|(x, y) = \frac{1}{x_1^3}$. Thus,

$$\begin{aligned} D_1 = \varphi^{-1}D &= \left\{ \frac{1}{x_1} > 1,\ 0 < \frac{y_1}{x_1}\left(1 + \frac{1}{x_1^2}\right) < 1 \right\} \\ &= \left\{0 < x_1 < 1,\ 0 < y_1(1+x_1^2) < x_1^3\right\} \\ &= \left\{0 < x_1 < 1,\ 0 < y_1,\ 0 < x_1^3 - y_1(1+x_1^2)\right\}, \end{aligned}$$

which is a bounded set. We obtain:

$$\mathcal{I}(D, 1) = \int_D \mathrm{d}x\mathrm{d}y = \int_{D_1} \frac{\mathrm{d}x_1\mathrm{d}y_1}{x_1^3}.$$

Looking at the closure of D_1, the jacobian gives us a pole of order 3 at the origin.

We are going to decrease the order of this pole (which is the intersection multiplicity of the curve $y_1(1+x_1^2) = x_1^3$ with the coordinate axis) by a sequence of blow-ups at the origin. The tangent cone of the Zariski closure of ∂D_1 at the origin is given by the line $y_1 = 0$. After a first blow-up seeing the first chart by $\phi(x_2, y_2) = (x_2, x_2y_2)$, we obtain that $\mathcal{I}\left(D_1, 1/x_1^3\right) = \mathcal{I}\left(D_2, 1/x_2^2\right)$, where $D_2 = \left\{0 < x_2 < 1,\ \ 0 < y_2,\ 0 < x_2^2 - y_2(1+x_2^2)\right\}$. As we are in the same situation as before, we repeat the process one more time and we obtain $\mathcal{I}\left(D_2, 1/x_2^2\right) = \mathcal{I}\left(D_3, 1/x_3\right)$, where

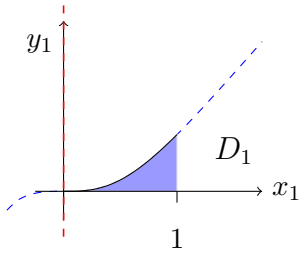

Figure II.4: The domain $D_1 = \{0 < x_1 < 1,\ 0 < y_1,\ 0 < x_1^3 - y_1(1 + x_1^2)\}$ and the pole locus of the integrand function (red).

$D_3 = \{0 < x_3 < 1,\ \ 0 < y_3,\ 0 < -x_3^2 y_3 + x_3 - y_3\}$. We still have a pole of order one at the origin, and $T_0(\partial D_3) = \{x_3 - y_3 = 0\}$, so we can first blowing-up seeing again the first chart one last time obtaining the 2-dimensional volume of $D_4 = \{0 < x_4 < 1,\ \ 0 < y_4,\ 0 < -x_4^2 y_4 - y_4 + 1\}$. This procedure is pictured in Figure II.5.

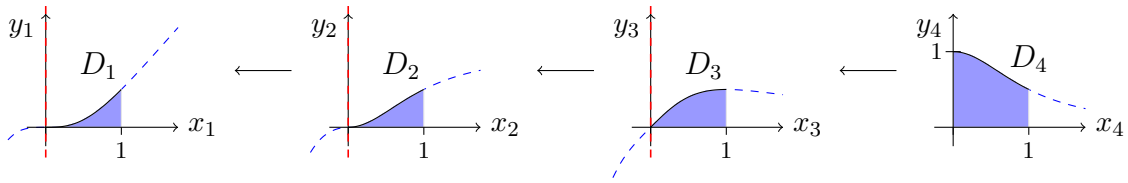

Figure II.5: Desingularization from D_1 to $D_4 = \{0 < x_4 < 1,\ \ 0 < y_4,\ 0 < -x_4^2 y_4 - y_4 + 1\}$.

Note that we have just obtained a quarter of the semi-canonical reduction for the previous example, because we have obtained the constant function 1 as integrand after removing the poles. This is not in general the case: starting from the volume of an unbounded semi-algebraic set, our algorithm produces a compact semi-algebraic set of one dimension because we take the volume under a non-constant integrand function.

Example II.5.3 (Another expression for π)**.** Consider the period $\nu \in \mathcal{P}_{\mathrm{KZ}}^{\mathbb{R}}$ described by the volume of a unbounded two dimensional semi-algebraic set:

$$\eta = \int_S \mathrm{d}x\mathrm{d}y \quad \text{where} \quad S = \{x^4 y^2 - x + 1 < 0\}.$$

As before, composing the change of charts taking the line $\{x = 0\} \subset \mathbb{P}^2_{\mathbb{R}}$ as line at infinity, we obtain a diffeomorphism φ of $\mathbb{R}^2$ minus a line which contribute which a pole of order 3 over the new line at infinity:

$$\int_S \mathrm{d}x\mathrm{d}y = \int_{\varphi^{-1}S} \frac{\mathrm{d}y\mathrm{d}z}{z^3} \quad \text{where} \quad \varphi^{-1}S = \{z^6 + y^2 - z^2 < 0\}.$$

Note that $S_0 = \varphi^{-1}S$ is contained in the upper semi-plane (see Figure II.6) and $T_0(\partial S_0) = \{y^2 = 0\}$. Composing two blow-ups at the origin and taking the second chart, we transform the integral by a diffeomorphism $\phi(y_2, z_2) = (y_2 z_2^2, z_2)$ of $\mathbb{R}^2 \setminus \{z = 0\}$ giving:

$$\int_{S_0} \frac{\mathrm{d}y\mathrm{d}z}{z^3} = \int_{S_2} \frac{\mathrm{d}y_2\mathrm{d}z_2}{z_2}$$

over the domain $S_2 = \{y_2^2 + z_2^2 + -z_2 < 0\}$. At this step, we notice that the boundary of S_2

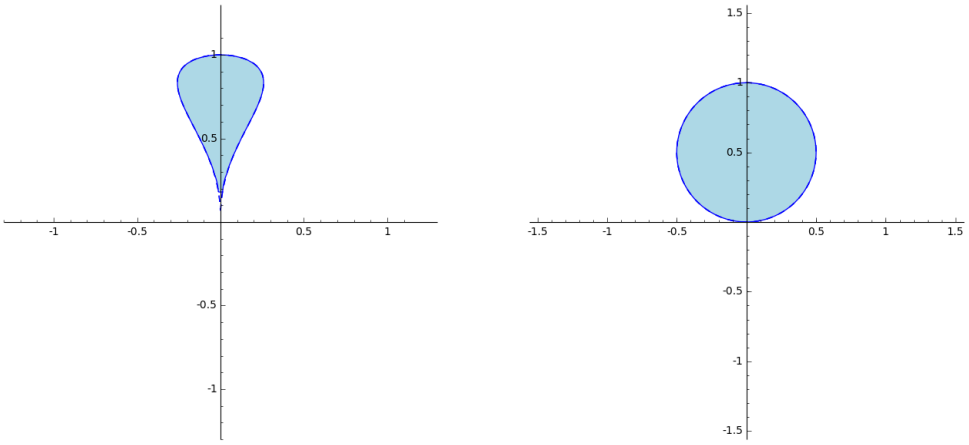

Figure II.6: Domains $S_0 = \varphi^{-1}S = \{z^6 + y^2 - z^2 < 0\}$ (left), and $S_2 = \left\{y_2^2 + z_2^2 + -z_2 < 0\right\}$ (right).

is in fact a smooth variety whose tangent line at the origin is $z = 0$. Taking any chart in the blow-up, the strict transform of S_2 loss compacity. We do a partition of our domain in two pieces separated by the tangent and normal lines of ∂S_2 at the origin, which correspond to the coordinate axis. Thus, $S_3 = X \cup S_3^1 \cup S_3^2$, and it is easy to verify that $\mathcal{I}\left(S_3^1, 1/z_3\right) = \mathcal{I}\left(S_3^2, 1/z_3\right)$ by symmetry. Looking at the first piece, we remark that $S_3^1 \subset \{z_3 + y_3 > 0\}$. Composing the isometry in the plane which sends the line $z_3 = 0$ into $z_3 + y_3 = 0$ and the blow-up at the origin taking the first chart:

$$\int_{S_3^1} \frac{\mathrm{d}y_3\mathrm{d}z_3}{z_3} = \int_{S_4^1} \frac{\sqrt{2}\mathrm{d}y_4\mathrm{d}z_4}{1 + z_4}$$

with $S_4^1 = \left\{y_4 > 0, 1 - z_4 > 0, -y_4 z_4^2 - y_4 + \frac{\sqrt{2}}{2}(z_4 + 1)\right\}$, pictured in Figure II.7. It remains to resolve the pole of order 1 at $(0, -1)$, where the tangent cone has equations centered at the origin $2y_4 - \frac{2}{2}z_4 = 0$. As S_4^1 is contained in the semi-plane $\{z_4 + 1 > 0\}$, we take the chart with respect to the line bounding it to achieve our resolution of the integrand:

$$\int_{S_4^1} \frac{\sqrt{2}\mathrm{d}y_4\mathrm{d}z_4}{1 + z_4} = \int_{S_5^1} \sqrt{2}\mathrm{d}y_5\mathrm{d}z_5,$$

with $S_5^1 = \left\{y_5 > 0, -1 < z_5 < 1, y_5(1 + z_5^2) < \sqrt{2}/2\right\}$ (Figure II.7). Repeating this process with S_3^2, we obtain an identical piece S_5^2 symmetric to the OZ–axis. In fact, it is worth noticing that after a linear change of variables $y_5' = \sqrt{2}/2y_5$ in the union of $S_5 = S_5^1 \cap S_5^2$, we obtain up to isometry the same semi-canonical reduction as in Example II.5.2, thus $\eta = \pi$.

II.5.2 Multiple Zeta Values

We have previously introduced multi-zeta values $\zeta(s_1, \ldots, s_k)$ as examples of real periods. Usually, this numbers are described as iterated integrals which can be expressed as the integral of a rational function which depends on the tuple $(s_1, \ldots, s_k)$ over a simplex $\triangle$ of dimension $k + 1$. For an exhaustive introduction to MZV, see [Wal00b].

Example II.5.4 ($\zeta(2)$)**.** In the case of $\zeta(2) = \sum_{n=1}^{+\infty} 1/n^2 = \pi^2/6$, we know that it can be expressed as

$$\frac{\pi^2}{6} = \int_{\triangle} \frac{\mathrm{d}x\mathrm{d}y}{(1 - x)y}$$

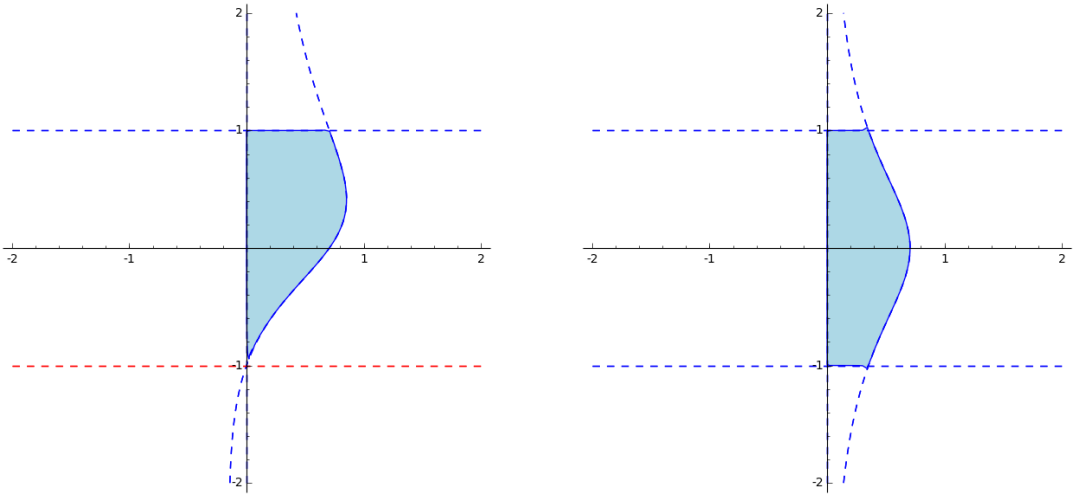

Figure II.7: Domains $S_4^1 = \left\{y_4 > 0, 1 - z_4 > 0, -y_4 z_4^2 - y_4 + \frac{\sqrt{2}}{2}(z_4 + 1)\right\}$ with the pole locus in red (left) and $S_5^1 = \left\{y_5 > 0, -1 < z_5 < 1, y_5(1 + z_5^2) < \frac{\sqrt{2}}{2}\right\}$ (right).

over the open simplex $\triangle = \{0 < x < y < 1\}$. The denominator of the integral function gives two poles in $\partial\triangle$ at the origin and at $(1, 1)$. The tangent cone of $\partial\triangle$ at a point $p \in \partial\triangle$ is given by the lines which contains the facets involving p. After a first blow-up at the origin, and taking the second chart $\phi(x_1, y_1) = (x_1 y_1, y_1)$:

$$\int_{\triangle} \frac{\mathrm{d}x\mathrm{d}y}{(1-x)y} = \int_{\square} \frac{\mathrm{d}x_1\mathrm{d}y_1}{1 - x_1 y_1},$$

where $\square = \phi^{-1}\triangle = \{-1 < x, y < 1\}$ (Figure II.8). We need to resolve the last pole at $(1, 1)$.

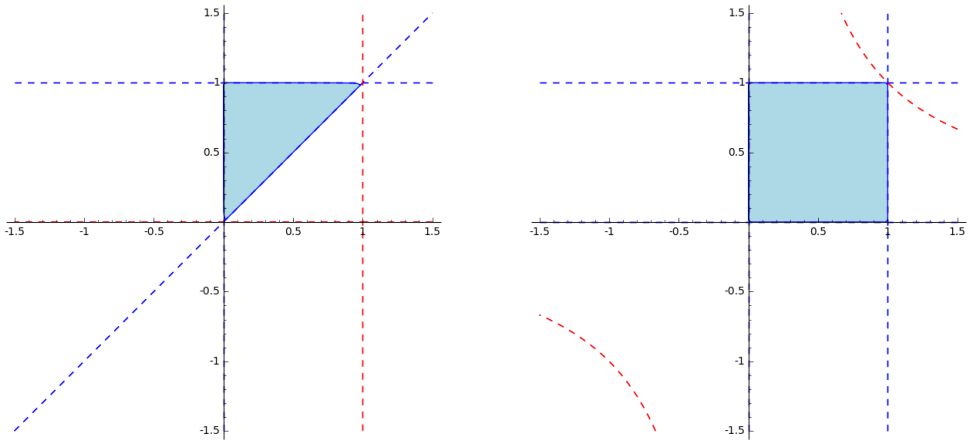

Figure II.8: Domains $\triangle = \{0 < x < y < 1\}$ (left), and $\square = \{-1 < x, y < 1\}$ (right).

The tangent cone of $\partial\square$ at this point are exactly the translated coordinate axis, we will take coordinates in the blow-up to respect to the line $L : x_1 + y_1 - 2 = 0$. We can construct such a map ϕ composing the blow-up at the origin with the isometry which sends the origin to $(1, 1)$ and the line $\{y_1 = 0\}$ to $x_1 + y_1 - 2 = 0$. So, ϕ is an isomorphism between $\mathbb{R}^2 \setminus \{x_2 = 0\}$ and

$\mathbb{R}^2 \setminus L$ for which:

$$\int_{\square} \frac{\mathrm{d}x_1\mathrm{d}y_1}{1-x_1y_1} = \int_T \frac{2\mathrm{d}x_2\mathrm{d}y_2}{-x_2y_2^2+x_2+2\sqrt{2}}$$

with $T = \left\{x<0, -1<y_2<1, -x_2y_2+x_2+\sqrt{2}>0, x_2y_2+x+\sqrt{2}>0\right\}$, without poles of the integral denominator at its boundary (Figure II.9). The integrand function $f(x_2,y_2) = 2/(-x_2y_2^2+x_2+2\sqrt{2})$ does not change of sign over T, then taking the volume of the area under the hypersurface $f=0$:

$$T_f = \left\{(x,y,z)\in T\times\mathbb{R} \mid z>0, 2+z(xy^2-x-2\sqrt{2})>0\right\}$$

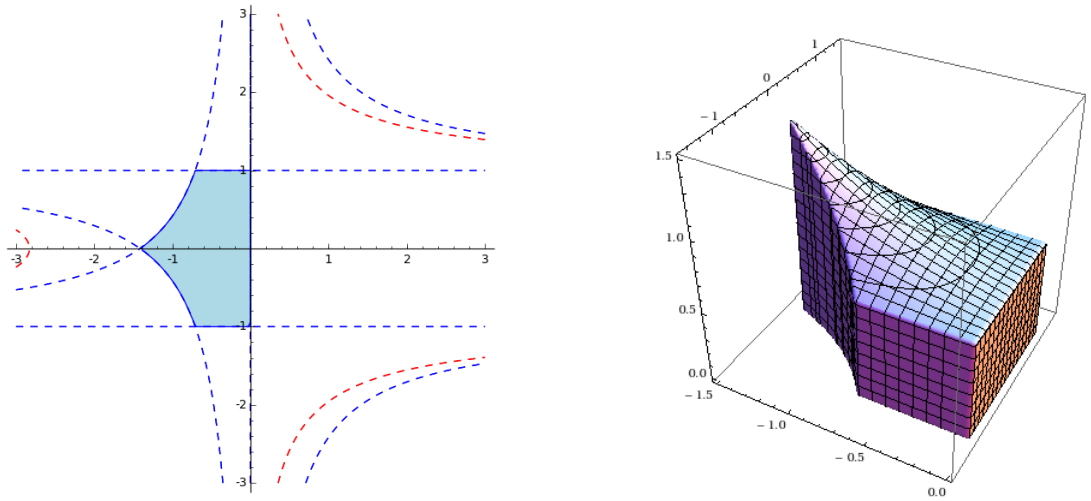

Figure II.9: Domains $T = \left\{x<0, -1<y_2<1, -x_2y_2+x_2+\sqrt{2}>0, x_2y_2+x+\sqrt{2}>0\right\}$ with the pole locus (left) and T_f (right).

Chapter III

About the Kontsevich-Zagier periods conjecture

A common idea around the Kontsevich-Zagier period conjecture is that even for their authors *"this problem looks completely intractable and may remain so for many years"* (see also [Wal06, And12, Ayo15]). This is due to the fact that no strategy of proof is up to our knowledge sketched.

The aim of this Chapter is to discuss possible reformulations of this conjecture which suggest a possible scheme of proof. The point of view obtained from the semi-canonical reduction (Theorem II.1.1) suggests to understand the KZ–rules as operations of geometrical nature between volumes. Doing so, we can interpret the Kontsevich-Zagier period conjecture as a kind of generalized Hilbert's third problem. This problem has a long history, we refer to [Car86] for a classical overview of the subject. Following the existing strategies dealing with analogues of Hilbert's third problem, we propose a scheme of proof for the *geometric Kontsevich-Zagier problem*.

We introduce in Section III.1 a first discussion about the nature of the Stokes formula in the original formulation of the Kontsevich-Zagier period conjecture, as well as its possible replacement by the *Fubini* and *Newton-Leibniz* formula as allowed operations in the conjecture. Then we present a geometric problem asking for a minimal set of relations between volumes of compact semi-algebraic sets, which follows the spirit of the Kontsevich-Zagier period conjecture.

Based in the fact that any compact semi-algebraic set is triangulable, we are able to give in Section III.2 an analogue to the previous problem in piecewise linear geometry: the generalized Hilbert's third problem for rational polyhedra. In this case, the rules of transformation between rational polyhedra are of more combinatorial nature. Some known partial results of generalized Hilbert's third problem are given in Section III.3, especially in the case of different polyhedra possessing the same canonical volume. In particular, we emphasize the case of convex polyhedra.

Finally, we discuss in Section III.4 how the previously introduced results can give a first schema to prove the Kontsevich-Zagier conjecture.

III.1 A reformulation of the Kontsevich-Zagier period conjecture

Our aim is to present a modified form of the Kontsevich-Zagier period conjecture (Conjecture I.4.1) which avoid some of the problems stated in Section I.4. We are motivated by the following:

- Any operation should be simple and natural from the point of view of integral calculus.
- Easy to implement and manipulate.

III.1.1 Discussion about Stokes formula versus Fubini's theorem

Sums by integrand functions or domains and change of variables are natural and induces explicit formulas between integrals. In the other hand, computation using Stokes's formula requires an exhaustive analysis in order to determine partitions and parameterizations of the boundary. Our main concern is then to find an alternative to the use of Stokes's formula, which lead more tractable manipulations.

How to determine such an alternative to the Stokes formula? An idea is to recover basic operations of integral calculus satisfying our previous constraints which enter in the proof of the Stokes formula. In the classical case, the *Fubini theorem* as well as the *Newton-Leibniz formula* (i.e. *Fundamental theorem of calculus*) are important ingredients of the proof [Spi79, p. 253-254].

Thus, a first natural tentative would be to replace the Stokes formula by Fubini's theorem and the Newton-Leibniz formula. This choice is also motivated by geometric considerations related to the semi-canonical reduction for periods. Indeed, Fubini's theorem is a convenient tool to bring down the dimension of the semi-canonical reduction, which play an important role in the discussions about complexity of periods in the next Chapter. However, this strategy can not be achieved for at least two reasons. First, primitive of algebraic functions are in general transcendental functions. As a consequence, the integral representations obtained by Fubini's theorem are out of the algebraic class. This is illustrated in the following expression:

$$\int_{P\times Q} f(x,y)\ \mathrm{d}x\mathrm{d}y = \int_P \left[\int_Q f(x,y)\ \mathrm{d}x\right] \mathrm{d}y \tag{III.1}$$

For example:

$$\begin{aligned}\int_{\left\{\substack{0<x<1\\1<y<x+1}\right\}} \frac{x}{y}\ \mathrm{d}x\mathrm{d}y &= \int_0^1 \left(\int_1^{1+x} \frac{x}{y}\ \mathrm{d}y\right) \mathrm{d}x \\ &= \int_0^1 x\log(x+1)\ \mathrm{d}x \\ &= \frac{1}{4}.\end{aligned}$$

Remark III.1.1. In [KZ01, p. 5], Kontsevich and Zagier says that an integral of a transcendental function can be a period *"by accident"*, giving as example

$$\int_0^1 \frac{x}{\log\frac{1}{1-x}}\ \mathrm{d}x = \log 2.$$

However, the previous discussion shows that many periods will be possible to write as a integral of a transcendental function by using Fubini's theorem.

Secondly, in the case where Fubini's theorem gives an integral in the algebraic class, for example when $f(x,y) = g(x)h(y)$ in III.1, this induces in fact quadratic relations between periods.

III.1.2 A geometric Kontsevich-Zagier's problem for periods

Our idea is to take advantage of the geometric representation obtained in Chapter II in order to formulate a geometric problem in the spirit of the Kontsevich-Zagier conjecture in terms of volumes of compact semi-algebraic sets. It is worth noticing that the rules which appear in the Kontsevich-Zagier conjecture does not respect the representation of periods as volumes.

Let $\mathcal{CSA}^d_{\widetilde{\mathbb{Q}}}$ be the set of compact d–dimensional semi-algebraic sets in $\mathbb{R}^d$. In the following, we study the set of relations in $\mathcal{CSA}^d_{\widetilde{\mathbb{Q}}}$ which are possible to obtain with the following geometrical operations.

- (**Semi-algebraic scissors congruences**) For any pair $K, K' \in \mathcal{CSA}^d_{\widetilde{\mathbb{Q}}}$ such that $\operatorname{codim}(K \cap K') \geq 1$:
$$[K \cup K'] = [K] + [K']$$
- (**Algebraic volume-preserving maps**) Let U be open semi-algebraic sets in $\mathbb{R}^d$ and let $f : U \to \mathbb{R}^d$ be a volume-preserving algebraic map. For any $K \in \mathcal{CSA}^d_{\widetilde{\mathbb{Q}}}$ contained in U:
$$[K] = [f(K)]$$

We can extend the relations between compact semi-algebraic sets of codimension zero defined in different real spaces. A natural connexion between them comes from Fubini's theorem:

- (**Flattening relation**) For any $K \in \mathcal{CSA}^d_{\widetilde{\mathbb{Q}}}$ and $K' \in \mathcal{CSA}^{d-n}_{\widetilde{\mathbb{Q}}}$ verifying that $K = K' \times I^n$, where $I = [0, 1]$, then:
$$[K] = [K' \times I^n] = [K']$$

All these relation make sense over the set $\mathcal{CSA}_{\widetilde{\mathbb{Q}}} = \bigcup_{d=0}^{\infty} \mathcal{CSA}^d_{\widetilde{\mathbb{Q}}}$. Let $(\mathcal{K}(\mathcal{CSA}_{\widetilde{\mathbb{Q}}}), +)$ be the free abelian group equipped with the sum modulo the previous geometric relations, simply denoted by $\mathcal{K}(\mathcal{CSA}_{\widetilde{\mathbb{Q}}})$.

Problem III.1.2 (A geometric Kontsevich-Zagier's problem for periods)**.** *Let K_1, K_2 be two compact semi-algebraic sets in $\mathbb{R}^d$ such that $\operatorname{vol}_d(K_1) = \operatorname{vol}_d(K_2)$. Can we transform K_1 into K_2 only using the geometric operations: semi-algebraic scissors congruences, algebraic volume-preserving maps and flattening relations? Equivalently, are $[K_1]$ and $[K_2]$ equal in $\mathcal{K}(\mathcal{CSA}_{\widetilde{\mathbb{Q}}})$?*

This problem is reminiscent of two classical problems about *cut-and-glue* in geometry: Hilbert's third problem and Tarski's circle-squaring problem.

The Hilbert's third problem asks for scissor-congruence of polyhedron of the same volume. The 3–dimensional case was solved by Dehn [Deh01, Syd65] in 1900 introducing its *Dehn invariant* to distinguish scissor-congruent polyhedrons. The main difference with our problem in terms of *cut-and-glue* is that we work in semi-algebraic class (with coefficients in $\widetilde{\mathbb{Q}}$) which is a weaker scissor-congruence constrain than in original Hilbert's problem, where one must stay in the polyhedral class. Also, the class of transformations allowed for each piece is larger than classical isometries on the affine space.

The Tarski's circle-squaring problem is the equivalent question about scissor-congruence between a square and a disk of same area, but without restriction on the class of decompositions. This problem was solved by M. Laczkovich [Lac90] in 1990, proving that there exists a decomposition of the circle by non-measurable sets covering the square only using translations. In Chapter IV, we prove that this problem has no solution in our setting, and this is due to the

fact that the semi-algebraic class has strong measurable properties, as well as constraints of arithmetic nature.

Remark III.1.3. The study of the *flattering relation* is related with the characterization of compact semi-algebraic sets which can be trivialized by lines up to cut-and-glue operations. This problem will have a fundamental importance in the next Chapter concerning *degree theory and transcendence of periods*.

Remark III.1.4. As has already mentioned, Ayoub's approach of the Kontsevich-Zagier conjecture consists in putting all the complexity in the differential form. Following our strategy to formulate our geometric Kontsevich-Zagier problem, Ayoub's formulation invite to look for a problem only dealing with differential forms. This is precisely what he has obtained in [Ayo15, Conj. 1.1].

Remark III.1.5. Note that to give an affirmative answer to the geometric Kontsevich-Zagier problem implies in fact to prove the Kontsevich-Zagier period conjecture. A negative answer will be also interesting to obtain in order to determine possible obstructions.

Remark III.1.6. The construction of $\mathcal{K}(\mathcal{CSA}_{\widetilde{\mathbb{Q}}})$ presents the geometrical relations in the spirit of Grothendieck groups. Following Cartier [Car86, p. 268] presentation of Dehn's invariant, it would be interesting to look for an analogue of this invariant for $\mathcal{K}(\mathcal{CSA}_{\widetilde{\mathbb{Q}}})$.

Before going to an approach of the previous problem, we consider an analogous problem in Piecewise Linear (PL) case. The interest of this new problem is twofold. First, we can make a connexion between the PL and semi-algebraic cases. Second, we can take advantages of know results in the PL class.

III.2 From Semi-algebraic to Piecewise Linear geometry

Up to now, we have not used all the good properties of semi-algebraic sets. In particular, it is well-know that compact semi-algebraic sets admit "good" triangulations, which allows us to obtain a representation in the PL category. Using this result, called *Semi-algebraic Triangulation Theorem*, we make a connexion between the geometric Kontsevich-Zagier problem and a PL version.

III.2.1 Semi-algebraic triangulations

Let $a_0, \ldots, a_k$ be $k+1$ points affinely independent in $\mathbb{R}^d$. Recall that a *k–simplex* $[a_0, \ldots, a_k]$ is the set of points $p \in \mathbb{R}^d$ such that there exist non-negative $\lambda_0, \ldots, \lambda_k \in \mathbb{R}$ verifying $\sum_{i=1}^k \lambda_i = 1$ and $p = \sum_{i=1}^k \lambda_i a_i$. In this case, the numbers $(\lambda_0, \ldots, \lambda_k)$ are called the *barycentric coordinates* of p. For any non-empty subset $\{a_{i_0}, \ldots, a_{i_\ell}\} \subset \{a_0, \ldots, a_k\}$, the ℓ–simplex $[a_{i_0}, \ldots, a_{i_\ell}]$ is called a *ℓ–face* of $[a_0, \ldots, a_k]$. If σ is a simplex, we denote by $\mathring{\sigma}$ the *open simplex*, composed bu the points of σ whose barycentric coordinates are all positive.

A *finite simplicial complex* of $\mathbb{R}^d$ is a collection of simplices $\mathcal{K} = (\sigma_i)_{i=1,\ldots,m}$ verifying that the faces of every σ_i belongs to $\mathcal{K}$ and such that, for every $1 \leq i < j \leq m$, $\sigma_i \cap \sigma_j$ is either empty or a common face of σ_i and σ_j. The *realization* of the complex is $|\mathcal{K}| = \bigcup_{i=1}^m \sigma_i$. Note that the open simplices $\mathring{\sigma}_i$ form a partition of $|\mathcal{K}|$.

Following [BCR98, Sec. 9.2], we have:

Theorem III.2.1 (Semi-algebraic Triangulation Theorem)**.** *Every compact semi-algebraic set K in $\mathbb{R}^d$ is* semi-algebraically triangulable*, ie.e there exists a finite simplicial complex $\mathcal{K} = (\sigma)_{i=1,\dots,p}$ and a semi-algebraic homeomorphism $\Phi : |\mathcal{K}| \to K$.*

It is important to notice that the triangulation procedure is algorithmic (see [BPR06, Sec. 5.7, p. 183]) and moreover, that any operation used in this procedure belong to the KZ–rules. In fact, this triangulation can be chosen such that the triangulation $\Phi : |\mathcal{K}| \to K$ is a C^1–map, see [OS15].

Lemma III.2.2. *For any compact $K \in \mathcal{SA}^d_{\widetilde{\mathbb{Q}}}$ in $\mathbb{R}^d$, there exist a rational polyhedra P in $\mathbb{R}^d$ and a piecewise algebraic volume form ω such that*

$$\mathrm{vol}_d(K) = \int_P \omega$$

Moreover, the construction of P and ω from K is algorithmic and the passage between the two integrals respect the KZ–rules.

III.2.2 A PL version of the geometric Kontsevich-Zagier problem

The previous Lemma III.2.2 leads us to formulate an analogous version of the geometric Kontsevich-Zagier problem in terms of PL geometry.

Problem III.2.3 (Generalized Hilbert's third problem for rational polyhedra)**.** *Let P_1 and P_2 be two rational polyhedra in $\mathbb{R}^d$ equipped respectively by two piecewise algebraic volume forms ω_1 and ω_2 such that $\int_{P_1} \omega_1 = \int_{P_2} \omega_2$. Can we pass from an integral to the other one only by rational polyhedron scissors-congruence and rational polyhedra transformations?*

Due to the similarity between the two problems, it seems that we have not gain any advantage from the new formulation. Nevertheless, this problem is now a part of discrete and polyhedral geometry, for which many powerful algorithmic methods already exist to deal with scissor-congruences. We refer to the book of I. Pak [Pak15] for an overview on the state of art of discrete and polyhedral geometry with combinatorial methods. In the next section, we give some hints that the previous problem has a positive answer. The distance to a complete proof is related to the fact that we are using some classical results for which the given proof is unsatisfying.

III.3 About volume of rational polyhedra, scissor-congruence and mappings

In this Section, we show some known partial results in the problem of constructing transformations between polyhedra equipped with volume form which have the same volume.

III.3.1 Canonical volume form

A general result

Before discussing the general case, a first simplification of Problem III.2.3 is to consider the polyhedra equipped with the canonical volume form. In that case, we have found an interesting result discussed by A. Henriques and I. Pak in [HP04] which gives a priori a positive answer to the problem.

The strategy is first to look to this problem with convex polytopes and then to reduce the general case using appropriate decompositions by convex parts.

Theorem III.3.1 ([Pak15, Example 18.3, p. 171]). *Let $P, Q \subset \mathbb{R}^d$ be two rational polyhedra of equal volume, i.e. $\mathrm{vol}_d(P) = \mathrm{vol}_d(Q)$. Then one can pass from P to Q only using rational polyhedron scissors-congruence and rational polyhedra transformations.*

The main ingredient of the proof is the following result:

Theorem III.3.2 ([HP04, Theorem 2]). *Let $P, Q \subset \mathbb{R}^d$ be two convex rational polyhedra of equal volume, i.e. $\mathrm{vol}_d(P) = \mathrm{vol}_d(Q)$. Then there exists a one-to-one rational map $f : P \to Q$, which is continuous, piecewise linear and volume-preserving.*

The previous result is more that what we need, because no decomposition of the polyhedra is used. However, one of the essentially points of the proof is based on the *Moser's theorem* [Mos65] about volume-preserving maps for manifolds, which is not explicit. In fact, a combinatorial proof of Moser's Theorem for convex polyhedra is an open problem, see [HP04, Sec. 8.7, p. 17]. Pak refers to continuous, piecewise linear and volume-preserving maps as *Monge maps* (see [Pak15, Sec. 18.1, p. 170]).

The illustration of Theorem III.3.1 for the case of convex plane polygons in the following. The main point is that in this case the Monge maps are explicitly constructed and moreover we see directly how these maps are rational. We then discuss how Theorem III.3.1 in the non-convex case can be derived from Theorem III.3.2.

Convex polygons

In the plane case, Pak describes a simple way to obtain a Monge maps between two convex polygons of same area [Pak15, Ex. 18.2, p. 170]. The idea is to reduce any convex polygon into a triangle, by a recursively series of continuous, piecewise linear and volume-preserving transformations displacing vertex in order to reduce the number of faces of the polygon.

Let P an polygon of $n > 3$ faces and consider a vertex $v \in P$. Take the triangle (uvw) formed by u and its neighbors u and w, which is contained in P. Let z be the second neighboring vertex of w other than v. We go to transform the triangle (uvw) into another one $(uv'w)$ by shifting v along the parallel line to (uw) passing through v, such that the new vertex v' lies in the line (wz). This induces a global transformation keeping the rest of the polygon $P \setminus (uvw) = P \setminus (uv'w)$ unchanged (see Figure III.1). Remark that we have obtained a polygon of $(n-1)$ faces and that this transformation is continuous and piecewise linear. In fact, the triangles (uvw) and $(uv'w)$ has the same area since we are not modifying neither the base of the triangle (lying in (uw)) or the height (because v and v' lyes in the same parallel line to (uw)). Thus it is also a volume-preserving map.

It is worth noticing that the previously described transformation respects the rationality of the polygon. Repeating this procedure $n-3$ times, we can transform P on to a triangle $\triangle_1$ of same area by a Monge map $\zeta_1 : P \to \triangle_1$.

Let Q be a polygon with same area of P. Repeating the previous procedure on Q, we obtain a triangle $\triangle_2$ a Monge map $\zeta_2 : Q \to \triangle_2$. As $\triangle_1$ and $\triangle_2$ has the same area by hypothesis, there exist a volume-preserving linear map $\phi : \triangle_1 \to \triangle_2$. Thus, by composing the functions we obtain a Monge map $\zeta = \zeta_2^{-1} \circ \phi \circ \zeta_1 : P \to Q$ between P and Q.

Remark III.3.3. Note that the previous procedure is not unique and depends on many choices during the process.

Non-convex case

In the non-convex case, the idea is to decompose each polyhedra in the same number of convex pieces $\{P_1, \ldots, P_n\}$ and $\{Q_1, \ldots, Q_n\}$ such that $\mathrm{vol}_d(P_i) = \mathrm{vol}_d(Q_i)$ for any $i = 1, \ldots, n$.

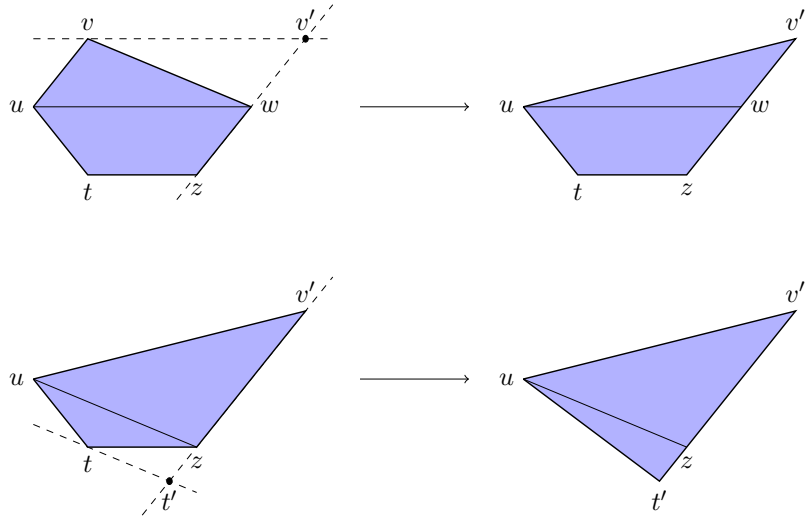

Figure III.1: Illustration of the Monge map described by Pak.

Such a subdivision is detailed in [HP04, Sec. 1, p. 2]. Moreover, such a decomposition can be made in order to preserve the rationality. Applying the previous Theorem at each pair of convex pieces, the result holds. Note that this proof makes an essential use of the two allowed rules: scissor-congruence and decomposition.

Remark III.3.4. It is worth noticing that the non-constructive character of the previous results comes entirely from the convex case in dimension strictly greater than two.

Now, consider the original case, where the polyhedra P and Q are equipped with two piecewise algebraic volume forms ω_1 and ω_2 verifying $\int_{P_1} \omega_1 = \int_{P_2} \omega_2$.

III.3.2 General case

Up to our knowledge, the following result due to G. Kuperberg in [Kup96, Thm. 3] is the most general result as long as non-canonical volume forms are considered. This Theorem is more suitable stated in [HP04, Theorem 1] with respect to our problem.

Theorem III.3.5. *Let $M_1, M_2 \subset \mathbb{R}^d$ be two PL–manifolds, possibly with boundary, which are PL–homeomorphic and equipped with piecewise constant volume forms ω_1 and ω_2. Suppose that M_1 and M_2 have equal volume, i.e. $\int_{M_1} \omega_1 = \int_{M_2} \omega_2$. Then there exists a volume-preserving PL–homeomorphism $f : M_1 \to M_2$, in particular $f^*(\omega_2) = \omega_1$.*

The proof of this Theorem by G. Kuperberg and by I. Pak and A. Henriques use different methods.

Remark III.3.6. In fact, Pak and Henriques only give a sketch of proof of this Theorem, saying that its follows more or less directly from the canonical volume form case. However, certain parts need to be detailed, in particular the one dealing with two simplicial decomposition for P and Q by $\{P_1, \ldots, P_n\}$ and $\{Q_1, \ldots, Q_n\}$ such that $\int_{P_i} \omega_1 = \int_{Q_i} \omega_2$ for any $i = 1, \ldots, n$.

Looking at the hypotheses of Theorem III.3.5, we first remark that they are very strong. As long as we are looking for *global* continuous, piecewise-linear, volume-preserving maps, one can not avoid to have the PL–homeomorphism condition. With respect to our problem, we can not ensure that this condition is satisfied in general. Moreover, the volume forms which follows from

our construction are a priori non piecewise constant.

The first condition can be removed by restricting our attention on convex polyhedra. Indeed, we have the following result:

Lemma III.3.7 ([HP04, Lemma 1.1])**.** *For any two convex polyhedra $P, Q \subset \mathbb{R}^d$ there exist a PL–homeomorphism $f : P \to Q$.*

In this case, we obtain a more satisfying result:

Theorem III.3.8. *Let $P, Q \subset \mathbb{R}^d$ be two convex polyhedra equipped with piecewise constant volume forms ω_1 and ω_2. Suppose that P and Q have equal volume, i.e. $\int_P \omega_1 = \int_Q \omega_2$. Then there exists a volume-preserving PL–homeomorphism $f : P \to Q$, in particular $f^*(\omega_2) = \omega_1$.*

III.4 Conclusion

Even if Theorem III.3.8 is exactly in the spirit of our problem, the gap between this result and a complete answer to Problem III.2.3 is large due to the following reasons:

- We think that the more difficult problem is to extend the previous Theorem for volume form in the algebraic class.
- We would like to avoid the convexity assumption. In this case, the situation seems more tractable following the same strategy as for canonical volume forms. Indeed, our problem does not ask for a global volume-preserving PL–homeomorphism by allowing scissors-congruences between the polyhedra: it seems reasonable that we could give decompositions of non-convex polyhedra in convex parts with a predefined volume.

Due to the connexion between Problem III.2.3 and the geometric Kontsevich-Zagier problem, the previous points are exactly key point to solve in order to prove the geometric Kontsevich-Zagier problem.

Chapter IV

Degree theory, complexity and transcendence of periods

Along this Chapter, we discuss two natural notions associated to periods coming from the semi-canonical representation obtain in the previous Chapter and which can be informally formulated as follows:

- Degree of periods: the minimal dimension of a compact semi-algebraic set for a semi-canonical representation.
- Geometric complexity of periods: measure the complexity of the underlying compact semi-algebraic set.
- Arithmetic complexity of periods: this notion uses the logarithmic Mahler measure for multivariate polynomial in order to generalize the classical notion of height for algebraic numbers.

These three notions will be rigorously defined in Section IV.1 and IV.2. They can be used to classify periods. This answer a question raised by M. Kontsevich and D. Zagier in [KZ01] where they look for a good notion of "simplicity" which can be the analogue of the height for algebraic numbers. They propose as a joke to consider "the amount of ink or the number of TEX keystrokes required to write down the integral". But, the main application of these results will be the possibility to give a transcendence criterion of geometric origin. Up to our knowledge, this is the first time that a transcendence characterization is based on a geometrical argument. We refer to the review article of M. Waldschmidt [Wal06].

The plan of this Chapter is the following: in Section IV.1, we introduce the notion of degree for a period completing a previous work of J. Wan [Wan11]. This induces a natural filtration of the set of periods. We derive the main properties of the degree and in particular the fact that a period is transcendent if and only if it has degree at least two. We then discuss the effectiveness of this criterion and we illustrate the main difficulties in computing the degree with the case of powers of π.

In Section IV.2 we introduce the notions of complexity for a period which mixes a geometric complexity measure for semi-algebraic set and an arithmetical data using the logarithmic Mahler measure. Inside a given strata of the filtration by degree, this complexity allows us to distinguish between different transcendental periods. We illustrate the difficulties related to this notion in the case of π and π^2.

IV.1 Degree of periods

The Semi-canonical Reduction Theorem II.1.1 allows us to construct a filtration on the $\overline{\mathbb{Q}}$–algebra $\mathcal{P}_{\text{KZ}}$ via a notion of *degree* for periods initiated by J. Wan [Wan11].

IV.1.1 A notion of degree coming from dimension of volumes

Definition IV.1.1. Let $p \in \mathcal{P}_{\text{KZ}}^{\mathbb{R}}$. Define the *degree of* p, noted by $\deg(p)$, as the minimum positive integer $d \in \mathbb{N}$ such that there exist a compact semi-algebraic set $K \in \mathcal{SA}_{\overline{\mathbb{Q}}}^{d}$ such that we can express

$$|p| = \text{vol}_d(K).$$

For any period $p = p_1 + ip_2 \in \mathcal{P}_{\text{KZ}}$ with $p_1, p_2 \in \mathcal{P}_{\text{KZ}}^{\mathbb{R}}$, we define $\deg(p) = \max\{\deg(p_1), \deg(p_2)\}$. We can extend this definition to all $\mathbb{C}$ imposing naturally $\deg(p) = \infty$ for any $p \in \mathbb{C} \setminus \mathcal{P}_{\text{KZ}}$.

This notion is natural with respect to the presentation of periods as volume of compact semi-algebraic sets. But, despite its apparent simplicity, the effective computation of the degree of a given period is very difficult. Let us look for some illustrative examples:

The number π can be usually seen as the volume of the unit disk $D_1 = \{x^2 + y^2 \leq 1\}$ in the plane:

$$\pi = \int_{D_1} \mathrm{d}x_1 \mathrm{d}x_2 \tag{IV.1}$$

The fact that π can not be of degree one is related to the fact that this number is transcendent. In fact, this result is general and give a geometric characterization of transcendental periods (see Section ?). As we know, this question is not trivial and gives some hints that the computation of the degree will not be an easy task.

Another example is provided by π^2. In that case, we know that the degree is at most 4. But, some simple computations shows that we have

$$\pi^2 = \int_{S_3} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \tag{IV.2}$$

where $S_3 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \leq 1, 0 \leq x_3((x_1^2 + x_2^2)^2 + 1) \leq 4\right\}$.

As a consequence, π^2 is of degree at most 3. The natural question is of course to know if π^2 is of degree 2. In fact, we conjecture the following:

Conjecture IV.1.2. *For any* $n \in \mathbb{N}$, $\deg \pi^n = n + 1$.

How to solve this question ? We have at least two steps which are related to our formulation of the KZ conjecture. We now discuss the general case.

A first obstruction for a period to be of degree $d-1$ when we have a representation of degree d is to know if the underlying compact semi-algebraic set can be transformed using a global change of variable in a set of the form $I \times A$ where I is the unit interval and A is a compact semi-algebraic set of dimension $d-1$.

In order to give an idea of the difficulty of such a question, we can weaken our regularity assumption by looking for the same question in the smooth case. In this case, the answer is related to Whitney-Stiefel differential forms and classical results in K–theory. In the case of π,

this problem related to the classical circle squaring problem for the disk. Up to our knowledge, the semi-algebraic version of the circled squaring problem (even with real coefficients) is not solved, see [BS].

Assume that this first obstruction is fulfilled then we must prove that the previous result can not be obtain by allowing partition of the original domain. Again returning to π, this problem is related to the famous Tarski's problem for the disk which corresponds to the search for a finite partition of the unit disk which produces a square up to isometries. This problem was solved only recently by Laczkovich in 1990. In our case, the class of transformation is bigger but our constraints on the partition are stronger. For example, the partition exhibited by Laczkovich consist of non measurable sets.

IV.1.2 First results and considerations about transcendence of periods

A geometrical criterion of transcendence

Coming directly form the definition of degree, J. Wan sketches some principal properties of the degree of periods. One of the most significant results is the geometric characterization of transcendental periods by degree.

Proposition IV.1.3 (Geometric transcendence criterion for periods)**.** *Let $p \in \mathcal{P}_{\mathrm{KZ}}$ be a period. Then:*

1. $\deg(p) = 0$ *if and only if* $p = 0$.
2. $\deg(p) = 1$ *if and only* $p \in \overline{\mathbb{Q}}^{\times}$.

Proof. Note that it suffices to consider real periods. Let K be a compact semi-algebraic set of dimension 1. As we have already remarked in ExampleI.2.4, K is a finite disjoint union of bounded closed intervals and points, then the volume of K is expressed as the finite sum of (finite) volumes of each of these intervals.

Let $I = [a, b] = \{Q_1 \geq 0, \ldots, Q_k \geq 0\}$ be any of this intervals, where Q_i is an univariate polynomial with real algebraic coefficients for any $i = 1, \ldots, k$. Thus, the volume of I is $b-a$, but a, b are roots of some polynomials in $\{Q_1, \ldots, Q_k\}$ and hence they are real algebraic numbers. Thus the volume of K is also an algebraic number.

Reciprocally, any non-zero real algebraic number $\alpha \in \widetilde{\mathbb{Q}}$ can be written up to sign as the length of the interval $I_\alpha = [0, \alpha]$. □

As a simple consequence of the previous Proposition we obtain a sufficient and necessary condition for a period to be transcendental: *a period $p \in \mathcal{P}_{\mathrm{KZ}}$ is transcendental if and only if* $\deg(p)$ *is greater than or equal to* 2.

The case of definite integrals of rational functions

In [Wan11], J. Wan obtain an explicit expression for periods of degree ≥ 2 in the case of definite integrals of rational functions. He then observe that the characterization of periods of degree 2 between them seems to be a difficult task. In this Section, a complete answer to this problem is obtain using Van der Poorten results [VdP71] about the arithmetic nature of definite integrals of rational functions in complex variable. By the way, it proves again that the computation of the degree of a given period in the general case is not easy.

Indeed, we have:

Theorem IV.1.4 ([Wan11, Thm. 4.1]). *Let $p \in \mathcal{P}_{\mathrm{KZ}}^{\mathbb{R}}$ be a real period such that p can be expressed as*

$$p = \int_S \frac{P(x)}{Q(x)}\,\mathrm{d}x$$

with $P, Q \in \mathbb{Q}[x]$ and $S \in \mathcal{SA}^1_{\widetilde{\mathbb{Q}}}$. Then

$$p = a \arctan \xi + b \log \eta + c,$$

for certain $a, b, c, \xi, \eta \in \overline{\mathbb{Q}}$.

Proof. We know that any rational function can be decompose as linear combinations with algebraic coefficients of four types of functions:

$$\frac{1}{x-a},\ \frac{1}{(x-a)^n},\ \frac{x+d}{x^2+bx+c},\ \frac{x+d}{(x^2+bx+c)^n}$$

with $a, b, c, d \in \overline{\mathbb{Q}}$ and $n > 1$. Then, using the integration formulas of basic calculus, the result holds. □

To deduce the arithmetical nature of these numbers from the explicit expression is complicated. Indeed, it is related to proving some algebraic independence between logarithm of algebraic numbers and value of arctan at algebraic points. However, we have the following theorem more in the spirit of periods :

Theorem IV.1.5 ([VdP71]). *Let $P, Q \in \overline{\mathbb{Q}}[z]$ be coprime polynomials. Denote by $\alpha_1, \ldots, \alpha_n$ the distinct zeros of $Q(z)$ and by $r_1, \ldots, r_n$ the residues respectively at the poles of the rational function $P(z)/Q(z)$. Further let Γ be some contour in the complex plane for which the definite integral*

$$\int_\Gamma \frac{P(z)}{Q(z)}\mathrm{d}z,$$

exits, and suppose that Γ is either closed, or has endpoints which are algebraic or infinite. Then the definite integral is algebraic if and only if

$$\int_\Gamma \left(\sum_{k=1}^{n} \frac{r_k}{z-\alpha_k} \right) \mathrm{d}z = 0.$$

This result is a consequence of Baker's Lemma (see Theorem I.5.18) about the $\overline{\mathbb{Q}}$–linear independence of logarithms of algebraic numbers. From the previous Chapter, this result can be formulated using the semi-canonical reduction of a period and then directly as a geometric properties of the underlying semi-algebraic set representing a given period. As an example, let us consider

$$\int_0^1 \frac{\mathrm{d}x}{1+x^3} = \frac{1}{3}\left(\log 2 + \frac{\pi}{\sqrt{3}}\right),$$

whose arithmetic nature was asked by C. Siegel in its famous monograph about transcendental numbers [Sie49, p. 97] of 1949. By the previous Theorem, we know that this number is transcendental. Now, this integral can be written as

$$\mathrm{vol}_2\left(\left\{ \begin{array}{l} 0 \le x \le 1,\ y \ge 0 \\ 1 - y(1+x^3) \ge 0 \end{array} \right\}\right) = \frac{1}{3}\left(\log 2 + \frac{\pi}{\sqrt{3}}\right)$$

From the geometric criterion for the transcendence of periods, the previously defined semi-algebraic set can not be transformed into a rectangle up to volume-preserving algebraic transformations and semi-algebraic partitions. We do not any method to deduce this kind of result from the data of the semi-canonical reduction set.

There do not exist similar theorems in the multidimensional case, or replacing the rational function in the integrand by algebraic functions. This difficulty is related directly with the determination of the degree of a period from a semi-canonical reduction representation.

IV.1.3 Filtration and first consequences about transcendence

The main result of this Section is that degree allows to define a filtration and an ultrametric of the $\widetilde{\mathbb{Q}}$–algebra of periods. We have:

Theorem IV.1.6. *The degree of periods defines a filtration of the $\widetilde{\mathbb{Q}}$–algebra of periods.*

Proposition IV.1.7. *The map*

$$\begin{aligned} d : \mathcal{P}_{\mathrm{KZ}} \times \mathcal{P}_{\mathrm{KZ}} &\longrightarrow \quad \mathbb{Z} \\ (p_1, p_2) &\longmapsto \deg(p_1 - p_2) \end{aligned}$$

defines an ultrametric over $\mathcal{P}_{\mathrm{KZ}}$.

The proofs of these results are based on the following fact: the degree of periods satisfies good properties with respect to addition and product of periods.

Proposition IV.1.8. *If $p_1, p_2 \in \mathcal{P}_{\mathrm{KZ}}$, then:*

1. $\deg(p_1 p_2) \leq \deg(p_1) + \deg(p_2)$.
2. $\deg(p_1 + p_2) \leq \max\{\deg(p_1), \deg(p_2)\}$.

In addition, non-zero algebraic numbers acts trivially on the degree under product.

Proposition IV.1.9. *Let $p \in \mathcal{P}_{\mathrm{KZ}}$. If $\alpha \in \overline{\mathbb{Q}}^{\times}$, then $\deg(\alpha p) = \deg(p)$.*

Using the previously defined ultrametric for $\mathcal{P}_{\mathrm{KZ}}$, we construct the ball of radius $k \in \mathbb{N}$ centered at the origin in the $\overline{\mathbb{Q}}$-algebra of periods:

$$[\mathcal{P}_{\mathrm{KZ}}]_k = \{p \in \mathcal{P}_{\mathrm{KZ}} \mid \deg(p) \leq k\}$$

Following Proposition IV.1.8, we deduce easily that $[\mathcal{P}_{\mathrm{KZ}}]_k$ is an additive group but not a ring in general (except $[\mathcal{P}_{\mathrm{KZ}}]_1 = \overline{\mathbb{Q}}$). Moreover, from the previous results $[\mathcal{P}_{\mathrm{KZ}}]_k$ is a $\overline{\mathbb{Q}}$-module for any $k \in \mathbb{N}$ and it has a good enough properties to define a filtrations for periods. Let $k, l \in \mathbb{N}$, then:

- $[\mathcal{P}_{\mathrm{KZ}}]_k + [\mathcal{P}_{\mathrm{KZ}}]_l = \{p_k + p_l \mid p_k \in [\mathcal{P}_{\mathrm{KZ}}]_k, p_l \in [\mathcal{P}_{\mathrm{KZ}}]_l\} \subset [\mathcal{P}_{\mathrm{KZ}}]_{\max(k,l)}$.
- $[\mathcal{P}_{\mathrm{KZ}}]_k \cdot [\mathcal{P}_{\mathrm{KZ}}]_l = \{p_k p_l \mid p_k \in [\mathcal{P}_{\mathrm{KZ}}]_k, p_l \in [\mathcal{P}_{\mathrm{KZ}}]_l\} \subset [\mathcal{P}_{\mathrm{KZ}}]_{k+l}$.

Imposing $[\mathcal{P}_{\mathrm{KZ}}]_k = 0$ for $k < 0$, we obtain a filtered $\overline{\mathbb{Q}}$–algebra induced by the degree of periods:

$$\mathcal{P}_{\mathrm{KZ}} = \bigcup_{k \in \mathbb{Z}} [\mathcal{P}_{\mathrm{KZ}}]_k \tag{IV.3}$$

Proof of Proposition IV.1.8. The cases $p_1 = 0$, $p_2 = 0$ and $p_1 = -p_2$ are trivial.

We consider first real periods. Suppose that $\deg(p_1) = k \leq l = \deg(p_2)$, thus there exists two compact sets $K_1 \in \mathcal{SA}^k_{\widetilde{\mathbb{Q}}}$ in $\mathbb{R}^k$ and $K_2 \in \mathcal{SA}^l_{\widetilde{\mathbb{Q}}}$ in $\mathbb{R}^l$ such that:

$$p_1 = \operatorname{sgn}(p_1)\operatorname{vol}_k(K_1) \quad \text{and} \quad p_2 = \operatorname{sgn}(p_2)\operatorname{vol}_l(K_2).$$

Up to translation using coefficients in $\widetilde{\mathbb{Q}}$, we can assume that K_1 and $\pi_k K_2$ are disjoints, where $\pi_k : \mathbb{R}^d \to \mathbb{R}^l$ is the projection on the first k components of $\mathbb{R}^l$.

1. Using Fubini, we can express:

$$p_1 p_2 = \operatorname{sgn}(p_1)\operatorname{sgn}(p_2)\operatorname{vol}_k(K_1)\operatorname{vol}_l(K_2) = \operatorname{sgn}(p_1 p_2)\operatorname{vol}_{k+l}(K_1 \times K_2).$$

   Thus, $\deg(p_1 p_2) \leq \deg(p_1) + \deg p_2$.

2. As $k \leq l$, we can define the hyper-cylinder $K_1 \times [0,1]^{l-k}$. Then:

$$\begin{aligned} p_1 + p_2 &= \operatorname{sgn}(p_1)\operatorname{vol}_k(K_1) + \operatorname{sgn}(p_2)\operatorname{vol}_l(K_2) \\ &= \operatorname{sgn}(p_1)\operatorname{vol}_l(K_1 \times [0,1]^{l-k}) + \operatorname{sgn}(p_2)\operatorname{vol}_l(K_2) \end{aligned}$$

   Separating by cases:

   - If $\operatorname{sgn}(p_1) = \operatorname{sgn}(p_2)$:

$$|p_1 + p_2| = \operatorname{vol}_l(K_1 \times [0,1]^{l-k}) + \operatorname{vol}_l(K_2) = \operatorname{vol}_l\left((K_1 \times [0,1]^{l-k}) \cup K_2\right),$$

     since K_1 and $\pi_k K_2$ are disjoint semi-algebraic sets.

   - If $\operatorname{sgn}(p_1) = -\operatorname{sgn}(p_2)$: assume that $\operatorname{vol}_l(K_1 \times [0,1]^{l-k}) < \operatorname{vol}_l(K_2)$. Using the algorithmic procedure described in Section II.4, we can construct a compact semi-algebraic set $K \in \mathcal{SA}^l_{\widetilde{\mathbb{Q}}}$ in $\mathbb{R}^l$ such that

$$\operatorname{vol}_l(K) = \operatorname{vol}_l(K_2) - \operatorname{vol}_l(K_1 \times [0,1]^{l-k}).$$

     which implies that $|p_1 + p_2| = \operatorname{vol}_l(K)$.

   Then $\deg(p_1 + p_2) \leq \max\{\deg(p_1), \deg(p_2)\}$.

In the complex case, let $p_1 = a_1 + ib_1$ and $p_2 = a_2 + ib_2$ be complex periods with $a_1, a_2, b_1, b_2 \in \mathcal{P}^{\mathbb{R}}_{\mathrm{KZ}}$. Using the properties four real periods, we can obtain:

$$\begin{aligned} \deg(p_1 p_2) &= \deg(a_1 a_2 - b_1 b_2 + i(a_1 b_2 + a_2 b_1)) \\ &= \max\{\deg(a_1 a_2 - b_1 b_2), \deg(a_1 b_2 + a_2 b_1)\} \\ &\leq \max\{\max\{\deg(a_1 a_2), \deg(b_1 b_2)\}, \max\{\deg(a_1 b_2), \deg(a_2 b_1)\}\} \\ &= \max\{\deg(a_1 a_2), \deg(b_1 b_2), \deg(a_1 b_2), \deg(a_2 b_1)\} \\ &\leq \max\{\deg(a_1) + \deg(a_2), \deg(b_1) + \deg(b_2), \deg(a_1) + \deg(b_2), \\ &\qquad \deg(a_2) + \deg(b_1)\} \\ &= \deg(p_1) + \deg(p_2), \end{aligned}$$

and also:

$$\begin{aligned}\deg(p_1+p_2) &= \max\{\deg(a_1+a_2),\deg(b_1+b_2)\}\\ &\leq \max\{\max\{\deg(a_1,\deg(a_2)\},\max\{\deg(b_1),\deg(b_2)\}\}\\ &= \max\{\deg(a_1),\deg(a_2),\deg(b_1),\deg(b_2)\}\\ &= \max\{\max\{\deg(a_1,\deg(b_1)\},\max\{\deg(a_2),\deg(b_2)\}\}\\ &= \max\{\deg(p_1),\deg(p_2)\}.\end{aligned}$$

□

Proof of Proposition IV.1.9. If $p \in \mathcal{P}_{\text{KZ}}^{\mathbb{R}}$, we construct an algebraic change of variables by a change of sign (if $\alpha < 0$) on the integrals and a homothecy of ratio $\sqrt[d]{|\alpha|} \in \overline{\mathbb{Q}}$ (where $d = \deg(p)$) over the compact domains in the integration. In the complex case, let $\alpha = \alpha_1 + i\alpha_2$ and $p = p_1 + ip_2$ with $\alpha_1, \alpha_2 \in \overline{\mathbb{Q}}$ et $p_1, p_1 \in \mathcal{P}_{\text{KZ}}^{\mathbb{R}}$:

$$\begin{aligned}\deg(\alpha p) &= \deg(\alpha_1 p_1 - \alpha_2 p_2 + i(\alpha_1 p_2 + \alpha_2 p_1))\\ &= \max\{\deg(\alpha_1 p_1 - \alpha_2 p_2),\deg(\alpha_1 p_2 + \alpha_2 p_1)\}\\ &\leq \max\{\max\{\deg(\alpha_1 p_1),\alpha_2 p_2)\},\max\{\deg(\alpha_1 p_2),\deg(\alpha_2 p_1)\}\}\\ &= \max\{\deg(\alpha_1 p_1),\deg(\alpha_2 p_2),\deg(\alpha_1 p_2),\deg(\alpha_2 p_1)\}\\ &\overset{\alpha\neq 0}{=} \max\{\deg(p_1),\deg(p_2)\}\\ &= \deg(p).\end{aligned}$$

But, this implies in addition:

$$\deg(p) = \deg\left(\frac{1}{\alpha}\alpha p\right) \leq \deg(\alpha p)$$

and then we find the desired equality. □

IV.1.4 Further properties on degree, transcendence and linear independence

Proposition IV.1.10. *If $\alpha \in \overline{\mathbb{Q}}$ and $p \in \mathcal{P}_{\text{KZ}}$ are non-zero such that $\alpha+p \neq 0$, then* $\deg(\alpha+p) = \deg(p)$.

Proof. We can obtain by a sequence of inequalities:

$$\begin{aligned}\deg(p) = \deg(-\alpha+\alpha+p) &\leq \max\{1,\deg(\alpha+p)\} \overset{\alpha+p\neq 0}{=} \deg(\alpha+p)\\ &\leq \max\{1,\deg(p)\} = \deg(p).\end{aligned}$$

□

Theorem IV.1.11. *Let p_1, p_2 be transcendental periods. If* $\deg(p_1) \neq \deg(p_2)$ *then p_1+p_2 and p_1/p_2 are transcendental numbers.*

Proof. Suppose $\deg(p_1) < \deg(p_2)$. We have:

$$\begin{aligned}\deg(p_2) = \deg(-p_1+p_1+p_2) &\leq \max\{\deg(p1),\deg(p_1+p_2)\} = \deg(p_1+p_2)\\ &\leq \max\{\deg(p_1),\deg(p_2)\} = \deg(p_2).\end{aligned}$$

Thus $\deg(p_1+p_2) = \deg(p_2)$ if and only if p_1+p_2 is transcendental too.

Suppose now that $p_1/p_2 = a \in \overline{\mathbb{Q}}$, then $\deg(p_1) = \deg(ap_2) = \deg(p_2)$. Thus, we get a contradiction. □

More generally,

Theorem IV.1.12. *Let p_1, p_2 be transcendental complex numbers. If $\deg(p_1) \neq \deg(p_2)$ then p_1 and p_2 are linearly independents over $\overline{\mathbb{Q}}$.*

Proof. Suppose $\deg(p_1) < \deg(p_2)$. If $\deg(p_2) = \infty$, then p_1 is a period but not p_2 and the result is trivial.
If both of them are periods, suppose $ap_1 + bp_2 = c$ with $a, b, c \in \overline{\mathbb{Q}}$ such that a, b non-zero numbers. Thus,

$$\deg(p_1) = \deg\left(\frac{c}{a} - \frac{c}{b}p_2\right) = \deg(p_2)$$

and we get a contradiction. □

IV.2 Notions of complexities for periods

In Problem I.4.6, M. Kontsevich and D. Zagier talk about "simplicity" of periods, suggesting the idea to define a notion of complexity for periods. Following our geometrical approach focusing on periods as volumes of semi-algebraic sets, we discuss the idea of define a notion of complexity of periods looking at the complexity of the semi-algebraic sets on the semi-canonical reduction. These notions allows to refine the filtration by degree of periods.

IV.3 Reminder about some different notions of complexity

IV.3.1 Complexity for semi-algebraic sets

Following [BR90, chpts. 2-4], we define a usual notion of *complexity* for semi-algebraic sets. Recall that any subset $S \subset \mathbb{R}^d$ is called semi-algebraic if it is can be represented as

$$S = \bigcup_{i=1}^{s} \bigcap_{j=1}^{r_i} \{f_{i,j} *_{i,j} 0\} \tag{R}$$

where $f_{i,j} \in \widetilde{\mathbb{Q}}[x_1, \ldots, x_d]$ and $*_{i,j} \in \{=, >\}$ for $i = 1, \ldots, s$ and $j = 1, \ldots, r_i$.

Definition IV.3.1. Let $S \subset \mathbb{R}^d$ be a semi-algebraic set. Consider a representation (R) of S. The *complexity of the representation* (R) is defined by the triplet $(d, P(R), C(R))$, composed by the following data:

- The dimension of the ambient space $\mathbb{R}^d$.
- The number of conditions in (R), i.e.

$$P(R) = \sum_{i=1}^{s} r_i.$$

- The maximal degree of polynomials defining the conditions in (R), i.e.

$$C(R) = \sup_{\substack{i=1,\ldots,s \\ j=1,\ldots,r_i}} \deg f_{i,j}.$$

Following this notion, for any $(d, r, c) \in \mathbb{N}$ we define by $\mathcal{SA}_{\widetilde{\mathbb{Q}}}(d, r, s)$ the collection of semi-algebraic sets $S \subset \mathbb{R}^d$ admitting a representation (R) with $P(R) \leq r$ and $C(R) \leq c$.

Definition IV.3.2. We define the *complexity of the semi-algebraic set* $S \subset \mathbb{R}^d$ is defined by the triplet (d, r, c), where (r, c) is the smaller tuple following the lexicographic order such that there exist a representation (R) with $P(R) = r$ and $C(R) = c$.

Remark IV.3.3. Between the different notions of complexity for semi-algebraic sets defined in [BR90], it is remarked that the complexity in Definition IV.3.1 posses the *geometric property*, i.e. for any affine transformation ϕ of $\mathbb{R}^d$ the complexity of $\phi(S)$ can be *effectively* calculated from the complexity of (S) and the dimension of the ambient space (see [Definition 4.6.2][BR90]).

IV.3.2 Complexity notions for multivariate polynomials

For algebraic numbers, the notions of complexity comes from those of univariate polynomials thought the minimal polynomial of an algebraic number, as the degree, the height, the Mahler measure...We can generalize these approach for transcendental periods looking at complexity invariants of multivariate polynomials.

Definition IV.3.4. Let $P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ a complex Laurent polynomial, the *(logarithmic) Malher measure* of P is defined to be the average of $\log|P|$ over the real d–torus in $\mathbb{C}^d$, i.e.

$$\mu(P) = \int_0^1 \cdots \int_0^1 \log\left|P\left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_d}\right)\right| \mathrm{d}t_1 \cdots \mathrm{d}t_d$$

IV.3.3 Complexity of periods

A geometrical complexity notion for periods

Following our geometrical approach, it is natural to attach a notion of complexity for a period associated to a minimal complexity over the possible semi-canonical reduction of this period.

Definition IV.3.5. Let $p \in \mathcal{P}_{\mathrm{KZ}}$ a real period. We define the *geometrical complexity* of p as the minimal tuple $(d, r, c) \in \mathbb{N}$ with respect the lexicographic order such that there exist a compact semi-algebraic set $S \in \mathcal{SA}_{\widetilde{\mathbb{Q}}}(d, r, c)$ of complexity (d, r, c) verifying $|p| = \mathrm{vol}_d(S)$.

This notion is naturally extended to the whole $\overline{\mathbb{Q}}$–algebra of periods defining the geometrical complexity of a $p \in \mathcal{P}_{\mathrm{KZ}}$ as the maximum of the geometrical complexities of its real and imaginary part. We denote by $\mathcal{P}_{\mathrm{KZ}}(d, r, c)$ the subset of *periods of geometrical complexity* (d, r, s).

In the case of transcendental periods, we must have a degree greater than or equal to two by the geometric criterion of transcendence. Moreover, its geometrical complexity if of type (d, r, c) with $d, c \geq 2$. As a consequence:

Proposition IV.3.6. *The minimal geometrical complexity which can be reach by a transcendental period is* $(2, 1, 2)$.

From the previous result, the number π is of minimal geometrical complexity $(2, 1, 2)$, as we have seen in IV.1. We recover again the special status of π as a particular real number and the circle as a fundamental geometric object. In the same way, looking for periods which can represented by volumes in $\mathcal{SA}_{\widetilde{\mathbb{Q}}}(d, 1, 2)$, the volume of the unit d–dimensional sphere occupies a special place, and as a consequence the powers of π. Precisely, we have that the volume of the unit d–sphere is given by:

$$\mathrm{vol}_d(\mathbb{B}^d) = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2} + 1)},$$

which gives the following formulas for even and odd dimensions:

$$\mathrm{vol}_{2k}(\mathbb{B}^{2k}) = \frac{\pi^k}{k!} \quad \text{and} \quad \mathrm{vol}_{2k+1}(\mathbb{B}^{2k+1}) = \frac{2^{k+1}\pi^k}{(2k+1)!!},$$

for any $k \in \mathbb{N}$. Which gives for the first dimensions:

$$\left(\mathrm{vol}_d(\mathbb{B}^d)\right)_{d\in\mathbb{N}} = \left(2,\ \pi,\ \frac{4}{3}\pi,\ \frac{1}{2}\pi^2,\ \frac{8}{15}\pi^2,\ \frac{1}{6}\pi^3,\ \frac{16}{105}\pi^3,\ \frac{1}{24}\pi^4, \dots\right).$$

Remark that it does not mean that π^k belongs to $\mathcal{P}_{\mathrm{KZ}}(2k, 1, 2)$. As an example, we have seen in IV.2 that π^2 admits a representation as volume of a compact semi-algebraic set in dimension 3 of complexity $(3, 3, 5)$, thus $\pi^2 \in \mathcal{P}_{\mathrm{KZ}}(d, r, c)$ where $2 \leq d \leq 3$. In fact, J. Oesterlé has obtained a representation of π^2 with lower complexity $(3, 3, 4)$ as

$$\pi^2 = \int_T \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \tag{IV.4}$$

where $T = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 2x_1^2 \leq 1, x_3^2(1 - x_1^2) \leq 4(1 - x_2^2)\right\}$. Indeed, following Fubini formula and usual change of variables:

$$\begin{aligned}
\int_{\left\{\substack{2x_1^2\leq 1 \\ x_3^2(1-x_1^2)\leq 4(1-x_2^2)}\right\}} \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 &= 4\left(\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{\mathrm{d}x}{\sqrt{1-x^2}}\right)\cdot\left(\int_{-1}^{1}\sqrt{1-y^2}\ \mathrm{d}y\right)\\
&= 4\left(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \mathrm{d}u\right)\cdot\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 v\ \mathrm{d}v\right)\\
&= 4\frac{\pi}{2}\frac{\pi}{2}\\
&= \pi^2.
\end{aligned}$$

Arithmetic and complexity of periods

To distinguish between algebraic numbers which corresponds to periods of degree 1, one must usually use informations coming the arithmetical side, as for example the height of the minimal polynomial defining the number. Our idea is to complete the geometrical complexity of a period introduced in the previous Section, using some arithmetical information on the set of multivariate polynomials defining a a geometric semi-canonical representation whose complexity is the geometrical complexity of the period. In the following, we decide to use the logarithmic Mahler measure for multivariate polynomials in order to define an arithmetic complexity of periods.

Let S be in $\mathcal{SA}_{\widetilde{\mathbb{Q}}}(d, r, c)$. Consider a representation (R) of S, we define by $\mu(R)$ the maximum of the logarithmic Mahler measure of each polynomial in R. We call *logarithmic Mahler measure* of S, noted by $\mu(S)$, the infimum of the logarithmic Mahler measure of representations in $\mathcal{R}_S$ of complexity (d, r, c).

Definition IV.3.7. Let p be a period of geometric complexity (d, r, c). Define by μ the infimum of the logarithmic Mahler measures $\mu(S)$, where S is a semi-canonical representation of complexity (d, r, c). In this case, we say that p is *a period of geometric complexity* (d, r, c) *with arithmetic complexity* μ.

The main drawback of the arithmetic part of the complexity for periods is that the logarithmic Malher measure of a Laurent series is also a period (see [KZ01, pag. 5]).

Remark IV.3.8. An idea to avoid the previous problem is to try to formulate an equivalent notion for arithmetic complexity using a variant of the *Bombieri norm* (see [BG06]), which provides an alternative to the logarithmic Mahler measure for homogeneous multivariate polynomials. A possibility is to homogenize the polynomials of the representation ad then take the minimum of the Bombieri norms of each homogeneous polynomial. But, as remarked by Prof. Éric Gaudron, the choice of a good arithmetic complexity will be determine the type of Diophantine results one wants to prove.

Chapter V

Perspectives on periods of Kontsevich-Zagier

V.1 Conclusions and perspectives

Throughout this part about periods of Kontsevich-Zagier, we have presented our work following two main motivations. First, to clarify the construction and definition of periods given in the foundational article [KZ01], as well as its related open problems: the Kontsevich-Zagier period conjecture, the equality algorithm and the problem of accessible identities. Secondly, to introduce a geometrical approach for periods based on volumes of compact semi-algebraic sets, using methods and constructions as algorithmic as possible. The semi-canonical reduction Theorem, stated in Chapter II is one of the key-points of this approach. This has allowed us to introduce geometrical problems associated to the Kontsevich-Zagier period conjecture in Chapter III. Finally in Chapter IV, we have rigorously defined a notion of degree for periods initiated by J. Wan in [Wan11] and we have extended it in order to develop a geometrical complexity of periods. A geometrical arithmetic notion of complexity was also discussed.

Effective reduction algorithm in arbitrary dimension

In the two dimensional case, we have obtained an efficient algorithmic method due to the simplicity of the centers at each blow-up and the possibility to control the compacity of our semi-algebraic domain during the resolution process (see Proposition II.3.5). In the general case, this procedure can be certainly extended choosing refined decompositions of semi-algebraic sets around a good choice of centers. Moreover, it would be interesting to study strategies to perform resolution of poles for which we can *control* or even *minimize* the geometrical complexity of the final semi-canonical reduction.

Exponential periods

Professor M. Waldschmidt asked us for the possible extension of our result in the case of *exponential periods*, which are number that can be written as an absolutely convergent integral of the product of an algebraic function with the exponential of an algebraic function over a semi-algebraic set where all polynomials appearing in the integral have algebraic coefficients. A typical example is

$$\sqrt{\pi} = \int_{-\infty}^{+\infty} e^{-x^2} \mathrm{d}x.$$

It seems possible, using the same techniques, to find a reduction of exponential periods considering the exponential part as a volume form and generalizing our procedure over the non-exponential part, i.e. a reduction of the form

$$\int_K e^{g(x_1,\ldots,x_d)} \cdot \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d,$$

where $K \subset \mathbb{R}^d$ is a compact semi-algebraic set and $g \in \widetilde{\mathbb{Q}}(x_1, \ldots, x_d)$.

Another possible approach could be to give a "good" class of affine geometrical objects representing exponential periods as volumes, in analogy of the semi-canonical reduction.

Approximation of periods

The semi-canonical reduction suggests to derive rational or algebraic approximation of a period by computing the volume of a geometric approximation of the reduction set. The reason why such an approximation can be of interest is that approximations of bounded semi-algebraic sets satisfy particular constraints coming from the semi-algebraic class. In particular, we do not expect a too good approximation of periods of degree at least two by rational numbers. From the geometrical point of view, it must be related to the fact that the curvature of semi-algebraic functions is bounded by below. This must prevent for example the Liouvilleś numbers to be periods which is coherent with a conjecture of M. Waldschmidt.

A constructive proof of Moser's volume form theorem

As noticed in Chapter III, the non-algorithmic nature of our approach to the geometric Kontsevich-Zagier problem is related to the fact that Moser's theorem for volume form is non constructive. An interesting problem is then to look for such kind of result for algebraic volume forms.

Toward the complexity of periods

For fixed geometric complexity (d, r, c), there exists a result that gives finite semi-algebraic models for complexity (d, r, c) up to semi-algebraic homeomorphism [BR90, Thm. 2.8.3, p 108]. Can we have a similar result for $\widetilde{\mathbb{Q}}$–semi-algebraic sets up to partitions and volume-preserving isomorphisms ?

In complexity terms, a first interesting family of periods to study is those of complexity $(2, 1, c)$, for $c \geq 2$. A natural approach would be to study and characterize the family of real curves whose complements contains compact regions. Examples of these curves are the circle, the *nodal cubic*, *Folium of Descartes*, or the curve $C : y^2 = x(x^2 - 1)$, which one of its real connected components is compact.

Decidability of the Kontsevich-Zagier conjecture

We sketch an actual work in progress performed with M. Yoshinaga, which is finally left out of this manuscript: the study of the *decidability* of the Kontsevich-Zagier period conjecture.

The idea is to describe a theoretical algorithm which takes a pair of periods $\mathcal{I}(S_1, P_1/Q_1)$ and $\mathcal{I}(S_2, P_2/Q_2)$ as input: if the two periods are different numbers, the algorithm stops in a finite time indicating this fact; in the case when the two periods are equal, the algorithm stops in a finite time giving as output the passage between the two integral representations if and only if the Kontsevich-Zagier conjecture is true.

We sketch the construction of such an algorithm in the following. Given two periods by its integral representation, we start two different algorithms which run in parallel:

- **(Geometrical equality test)** Suppose that both periods are of same sign (easily tested by integral numerical approximation). After performing a semi-canonical reduction for both periods, we can assume that we obtain two semi-canonical reductions of the same dimension. As is explained in Section I.5.2, we can construct a elementary sequence of rational for the periods based in approximation by cubes. Then, using Yoshinaga's geometrical approximation result (Lemma I.5.12), we can determine a geometrical constant which gives a stop criterion in the sequence if the periods are different.

- **(Identity determination)** We have seen that the countability of $\mathcal{P}_{\mathrm{KZ}}$ follows from countability of multivariate polynomials with coefficients in $\widetilde{\mathbb{Q}}$. In the same way, any of the KZ–rules are represented by a list representing algebraic transformations and partitions or disjoint unions by semi-algebraic sets. By Cantor's diagonal arguments, we can run over a countable set which contains any finite sum of integrals representing the first of the periods obtained by applying successively the KZ–rules to $\mathcal{I}(S_1, P_1/Q_1)$. If we find $\mathcal{I}(S_2, P_2/Q_2)$ in this list, then the algorithm stops.

Finally, remark that the second algorithm always stops if both periods are equal and the Kontsevich-Zagier conjecture is true.

V.2 Conclusions et perspectives (French)

Au travers de cette partie sur les périodes de Kontsevich-Zagier, nous avons présenté nos travaux en suivant deux motivations principales. Tout d'abord avec la volonté de clarifier la construction et la définition de périodes donnée dans l'article fondateur [KZ01], ainsi que les problèmes reliés : la conjecture des périodes de Kontsevich-Zagier, l'algorithme d'égalité des périodes et le problème des identités accessibles. Mais aussi en souhaitant introduire une approche géométrique des périodes basée sur les volumes des ensembles semi-algébriques compactes, et ce en utilisant des méthodes aussi constructibles et algorithmiques que possible. Le théorème de réduction semi-canonique énoncé au Chapitre II est l'une des clés centrales de cette approche. Cela nous a permit d'introduire au Chapitre III, des problèmes géométriques associés à la conjecture des périodes de Kontsevich-Zagier. Finalement dans le Chapitre IV, nous avons définit rigoureusement la notion de degré pour périodes, initié par J. Wan dans [Wan11], et nous l'avons étendu a fin de développer une complexité géométrique de périodes. Une notion géométrique-arithmétique de complexité est aussi analysé.

Algorithme de réduction effective en dimension arbitraire

Dans le cas de dimension deux, nous avons obtenu un algorithme efficient grâce à la simplicité des centres dans chaque éclatement et la possibilité de contrôler la compacité des domaines semi-algébriques pendant le procédé de résolution (cf Proposition II.3.5). Dans le cas général, cette procédure peut certainement être étendue en choisissant un raffinement des décompositions des ensembles semi-algébriques avec un bon choix des centres. De plus, il serait intéressant d'étudier les différentes stratégies pour résoudre les pôles pour lesquels on peut *contrôler* ou sinon *minimiser* la complexité géométrique de la réduction semi-canonique.

Périodes exponentielles

Le Professeur M. Waldschmidt nous a demandé si une extension de notre résultat au cas des périodes exponentielles est possible. Ces nombres sont définies comme des intégrales absolument convergentes du produit d'une fonction algébrique et de l'exponentielle d'un fonction algébrique sur un ensemble semi-algébrique, où tous les polynômes apparaissant dans l'intégrale sont à coefficients algébriques. Un exemple typique est :

$$\sqrt{\pi} = \int_{-\infty}^{+\infty} e^{-x^2} \mathrm{d}x.$$

Il semble que cette extension soit possible en utilisant des techniques similaires pour trouver une réduction des périodes exponentielles en considérant la partie exponentielle comme une forme volume et en généralisant notre procédure pour la partie non-exponentielle, i.e. une réduction de la forme :

$$\int_K e^{g(x_1,\ldots,x_d)} \cdot \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d,$$

où $K \subset \mathbb{R}^d$ est un ensemble semi-algébrique compacte et $g \in \widetilde{\mathbb{Q}}(x_1, \ldots, x_d)$.

Une autre approche possible serait de donner une "'bonne"' classe des objets géométriques affines représentant les périodes exponentielles comme volumes, en analogie avec la réduction semi-canonique.

Approximation des périodes

La réduction semi-canonique suggère de dériver des approximations rationnelles ou algébriques d'une période en calculant le volume d'un approximation géométrique de l'ensemble de réduction. La raison qu'une telle approximation peut être intéressante est que les approximations des ensembles semi-algébriques bornés satisfont des contraintes particulières provenant de la classe des semi-algébriques. Plus précisément, on ne cherche pas une bonne approximation des périodes de degré au moins deux par des rationnels. Du point de vu géométrique, cela est relié au fait que la courbure des fonctions semi-algébriques est bornée par le bas. Ce qui doit empêcher par exemple aux nombres de Liouville d'être des périodes. Ce qui serait cohérent avec la conjecture de M. Waldschmidt.

Une preuve constructive du théorème de Moser sur les formes volumes

Comme remarqué au Chapitre III, la nature non algorithmique de notre approche du problème de Kontsevich-Zagier géométrique est relié au fait que le théorème de Moser pour les formes volumes est non constructif. Un problème intéressant serait alors de regarder ce genre de résultat avec des formes volumes algébriques.

Vers la complexité des périodes

Pour une complexité géométrique fixée (d, r, c), il existe un résultat donnant un nombre fini de modèles semi-algébriques pour la complexité (d, r, c) à homéomorphisme semi-algébrique près. Peut-on obtenir des résultats similaires pour des ensembles $\widetilde{\mathbb{Q}}$-semi-algébriques à isomorphisme près préservant les partitions et le volume ?

En terme de complexité, une première famille intéressante de périodes à étudier est celle de complexité $(2, 1, c)$, pour $c \geq 2$. Une approche naturelle serait d'étudier et de caractériser la famille des courbes réelles dont le complémentaire contient des régions compactes. Des exemples

de telles courbes sont le cercle, la *cubique nodale*, le *Folium de Descartes* ou encore la courbe $C : y^2 = x(x^2 - 1)$, qui a l'une de ces composantes réelles compacte.

Décidabilité de la conjecture de Kontsevich-Zagier

Nous décrivons ici un travail actuellement en cours de progression réalisé en collaboration avec M. Yoshinaga, qui n'a finalement pas fait partie de ce manuscrit : l'étude de la *décidabilité* la conjecture des périodes de Kontsevich-Zagier.

L'idée est de décrire un algorithme théorique qui prend comme entré une paire de périodes $\mathcal{I}(S_1, P_1/Q_1)$ and $\mathcal{I}(S_2, P_2/Q_2)$, et si les deux périodes sont différentes, l'algorithme s'arrête en un temps fini et indique ce fait ; et si les deux périodes sont égales, alors l'algorithme s'arrête également en un temps fini est indique le passage entre les deux représentations intégrales si et seulement si la conjecture de Kontsevich-Zagier est vraie.

Nous décrivons rapidement la construction d'un tel algorithme dans la suite. Prenons deux périodes données par leurs représentations intégrales, nous entamons deux différents algorithmes qui s'effectuent en parallèle :

- **Test géométrique d'égalité :** Supposons que les deux périodes ont le même signe (facilement testable par approximation numérique d'intégrale). Après avoir effectué une réduction semi-canonique pour les deux périodes, on peut supposer que l'on obtient deux semi-algébriques réductions de même dimension. Comme il est expliqué dans la Section I.5.2, on peut construire une suite élémentaire de rationnel pour les périodes basée sur l'approximation par des cubes. Ensuite en utilisant le résultat d'approximation géométrique de Yoshinaga (Lemme I.5.12), on peut déterminer une constante géométrique qui donne un critère d'arrêt de la suite si les périodes sont distinctes.

- **Identity détermination :** On a vu que la dénombrabilité de $\mathcal{P}_{\text{KZ}}$ provient de la dénombrabilité des polynômes multivariés à coefficients dans $\widetilde{\mathbb{Q}}$. Dans le même esprit, toutes les KZ–règles sont représentées par une liste des transformations algébriques et les partitions ou unions disjoins par des ensembles semi-algébriques. En utilisant un argument à la Cantor, on peut parcourir un ensemble dénombrable contenant chacune des sommes finies d'intégrales représentant la première des périodes obtenues en appliquant successivement les KZ–règles à $\mathcal{I}(S_1, P_1/Q_1)$. Si nous trouvons $\mathcal{I}(S_2, P_2/Q_2)$ dans la liste, alors l'algorithme s'arrête.

Pour finir, remarquons que le second algorithme s'arrête toujours si les deux périodes sont égales et que la conjecture de Kontsevich-Zagier est vraie.

V.3 Conclusiones y perspectivas (Spanish)

A través de esta parte sobre periodos de Kontsevich-Zagier, hemos presentado nuestro trabajo siguiendo dos principales motivaciones. En primer lugar, aclarar la construcción y la definición de los periodos presentados en el artículo inicial [KZ01], así como problemas relacionados: la conjetura de periodos de Kontsevich-Zagier, el algoritmo de igualdad y el problema de las identidades accesibles. En segundo lugar, introducir un enfoque geométrico para periodos basada en volúmenes de conjuntos semi-algebraicos compactos, usando métodos y construcciones lo más algorítmicas posible. El Teorema de reducción semi-canónica, enunciado en el Capítulo II, es uno de los puntos clave de nuestro propuesta. Esto nos ha permitido introducir problemas más geométricos relacionados con la conjetura de periodos de Kontsevich-Zagier en el Capítulo III. Por último, en el Capítulo IV, hemos definido de forma rigurosa una idea de grado para periodos

iniciada por J. Wan [Wan11] y la hemos ampliado para desarrollar una complejidad geométrica de periodos. También hemos analizado una idea de complejidad aritmético-geométrica.

Un algoritmo de reducción efectivo en dimensión arbitraria

En el caso de dimensión dos, hemos obtenido un método algorítmico eficaz debido a la simplicidad de los centros de cada explosión y a la posibilidad de controlar la compacidad de nuestro dominio semi-algebraico en el proceso de resolución (véase Proposición II.3.5). En el caso general, este proceso podría ciertamente extenderse a la elección de descomposiciones refinadas de conjuntos semi-algebraicos en torno a una buena elección de centros. Además, podría ser interesante estudiar estrategias para llevar a cabo resoluciones de polos para los cuales pudiésemos *controlar* e incluso *minimizar* la complejidad geométrica de la reducción semi-canónica final.

Periodos exponenciales

El Profesor M. Waldschmidt nos preguntó sobre la posible extensión de nuestro resultado en el caso de los *periodos exponenciales*, que son números que se pueden escribir como integral absolutamente convergente del producto de una función algebraica con la exponencial de otra función algebraica sobre un conjunto semi-algebraico en la que todos los polinomios de la integral tienen coeficientes algebraicos. Un ejemplo viene dado por

$$\sqrt{\pi} = \int_{-\infty}^{+\infty} e^{-x^2} \mathrm{d}x.$$

Parece posible, usando las mismas técnicas, hallar una reducción para periodos exponenciales si se considera la parte exponencial como una forma de volumen y si generalizamos nuestro procedimiento sobre la parte exponencial no exponencial, es decir una reducción de la forma

$$\int_K e^{g(x_1,\ldots,x_d)} \cdot \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d,$$

donde $K \subset \mathbb{R}^d$ es un compacto semi-algebraico y $g \in \widetilde{\mathbb{Q}}\,(x_1, \ldots, x_d)$.

Otro posible enfoque podría ser dar una "buena" clase de objetos geométricos afines que representen periodos exponenciales como volúmenes, en analogía con la reducción semi-canónica.

Aproximación de periodos

La reducción semi-canónica sugiere que se podría obtenir aproximaciones racionales o algebraicas de un periodo por medio de volúmenes de aproximaciones geométricas del conjunto de reducción semi-canónica. La razón por la que este enfoque puede ser interesante es que las aproximaciones de los conjuntos semi-algebraicos acotados satisfacen limitaciones particulares provenientes de la clase semi-algebraica. En particular, no esperamos una aproximación demasiado buena para periodos de grado al menos dos por números racionales. Desde el punto de vista de la geometría, esto debe estar relacionado con el hecho de que la curvatura de las funciones semi-algebraicas está acotada inferiormente. Esto debería evitar, por ejemplo, que los números de Liouville sean periodos, lo que es coherente con una conjetura enunciada por M. Waldschmidt.

Una prueba constructiva del Teorema de formas volumen de Moser

Tal y como ha sido advertido en el Capítulo III, la naturaleza no algorítmica de nuestro enfoque sobre el problema geométrico de Kontsevich-Zagier está relacionada con el hecho de que el Teorema de formas volumen de Moser no es constructivo. Así, un problema interesante sería el de buscar este tipo de resultado constructivo para formas volumen algebraicas.

Hacia la complejidad de periodos

Para una complejidad geométrica fijada (d, r, c), existe un resultado que proporciona un número finito de modelos semi-algebraicos para la complejidad (d, r, c) salvo homeomorfismo semi-algebraico [BR90, Thm. 2.8.3, p 108] ¿Podemos hallar un resultado similar para los conjuntos $\widetilde{\mathbb{Q}}$–semi-algebraicos salvo particiones e isomorfismos que conservan el volumen?

En términos de complejidad, una primera familia de periodos interesante para estudiar son aquellas de complejidad $(2, 1, c)$, para $c \geq 2$. Un enfoque natural sería estudiar y caracterizar la familia de curvas reales cuyos complementos contienen regiones compactas. Ejemplos de estas curvas son el círculo, la *cúbica nodal*, el *Folium de Descartes* o la curva $C : y^2 = x(x^2 - 1)$, quien posee una componente conexa compacta.

Decidabilidad de la conjetura de Kontsevich-Zagier

Describimos brevemente un trabajo en desarrollo llevado a cabo con M. Yoshinaga, el cual ha quedado finalmente fuera de este manuscrito: el estudio de la *decidibilidad* de la conjetura de periodos de Kontsevich-Zagier.

El objetivo es describir un algoritmo teórico que admite un par de periodos, $\mathcal{I}(S_1, P_1/Q_1)$ y $\mathcal{I}(S_2, P_2/Q_2)$, como entrada: si los dos periodos son números diferentes, el algoritmo se para en tiempo finito indicando este hecho; en el caso en el que los periodos son iguales, el algoritmo se para en tiempo finito dando como resultado el paso entre dos representaciones integrales si y solo si la conjetura de Kontsevich-Zagier es cierta.

Describimos seguidamente la idea de la construcción de este algoritmo. Dados dos periodos por su representación integral, lanzamos dos algoritmos diferentes que operan en paralelo:

- **(Test de igualdad geométrica)** Supongamos que ambos periodos tienen el mismo signo (hecho fácilmente verificable por aproximación numérica de la integral). Después de aplicar una reducción semi-canónica en ambos periodos, podemos suponer que obtenemos dos reducciones semi-canónicas de la misma dimensión. Como se ha explicado en la Sección I.5.2, podemos construir una sucesión elemental racional para periodos basada en la aproximación por cubos. Entonces, usando el resultado de la aproximación geométrica de Yoshinaga (Lema I.5.12), podemos determinar una constante geométrica que proporciona una norma de parada en la secuencia si los periodos son diferentes.

- **(Determinación de la identidad)** Hemos visto que la numerabilidad de Pkz proviene de la numerabilidad de polinomios multivariantes con coeficientes en $\widetilde{\mathbb{Q}}$. Del mismo modo, las reglas–KZ pueden ser representadas por una lista que describiendo transformaciones algebraicas y particiones o uniones disjuntas de conjuntos semi-algebraicos. Usando argumentos diagonales de Cantor, podemos recorrer un conjunto numerable que contenga cualquier suma finita de integrales representando el primero de los periodos obtenidas aplicando sucesivamente las reglas–KZ a $\mathcal{I}(S_1, P_1/Q_1)$. Si hallamos $\mathcal{I}(S_2, P_2/Q_2)$ en esta lista, el algoritmo se detiene.

Finalmente, notemos que el segundo algoritmo siempre se detiene si ambos periodos son iguales y la conjetura de Kontsevich-Zagier es cierta.

Part II

Contributions to the Terao conjecture on line arrangements

Chapter VI

Line arrangements and Terao's conjecture

In this first chapter dedicated to the second part of our work, we introduce the central tools and objects in our study of line arrangements via logarithmic vector fields and its relation with the combinatorics. This chapter is based in [OT92] and [Sai80].

Firstly, we give a general presentation on hyperplane arrangements, focusing on affine and projective line arrangements, as well as some elementary results to understand the notion of combinatorics of an arrangement. A weaker notion of combinatorics is also presented.

In Section VI.2, we present a fundamental tool related to the study of hyperplane arrangements: the module of logarithmic derivations of an arrangement. Using this module, we introduce the concept of freeness for hyperplane arrangements. The dual notion of the this module of derivations, the module of logarithmic forms with poles over the arrangement, is also described.

Finally, we present the general statement of Terao's conjecture which asks about the combinatoriality of freeness for arrangements. We display this conjecture in the usual setting of central hyperplane arrangements throughout Section VI.3, in order to introduce the problem before developing our point of view on this conjecture for affine and projective line arrangements.

VI.1 Line arrangements

VI.1.1 Definitions and basic notions

Let k be an arbitrary field and let V be a k–vector space of dimension d. Consider the *projectivization* $\mathbb{P}(V) = (V \setminus 0)/\sim$, where

$$v \sim w \iff \exists \lambda \in k^* \text{ such that } w = \lambda v.$$

Definition VI.1.1. An *affine* (resp. *projective*) *hyperplane arrangement* $\mathcal{A}$ is a finite collection of different hyperplanes in V (resp. $\mathbb{P}(V)$), i.e. $\mathcal{A} = \{H_1, \ldots, H_n\}$ where H_i is an affine (resp. projective) subspace of dimension $d-1$ and $H_i \neq H_j$, for any $i, j = 1, \ldots, n$.

Definition VI.1.2. A hyperplane arrangement is said to be *central* if $\bigcap_{H\in\mathcal{A}} H \neq \emptyset$. In the case of affine central hyperplane arrangements, we can consider that $\bigcap_{H\in\mathcal{A}} H$ contains the origin in V, up to translation.

We denote by Φ_d the empty arrangement. Consider $S = \mathrm{Sym}(V^*)$ the symmetric algebra of the dual space $V^* = \mathrm{Hom}_k(V, k)$. Choosing a basis $\{e_1, \ldots, e_d\}$ in V, we obtain a dual basis $\{x_1, \ldots, x_d\}$ for V^*, so $x_i(e_j) = \delta_{ij}$. Then, we can identify $S \simeq k[x_1, \ldots, x_d]$.

For an affine hyperplane arrangement $\mathcal{A}$, we can define an affine form $\alpha_H \in k \oplus V^*$ such that $H = \alpha_H^{-1}(0)$ for any $H \in \mathcal{A}$. In the case of central or projective arrangements, such a form is linear, i.e. $\alpha_H \in V^*$.

Definition VI.1.3. The *defining polynomial* of an affine (resp. projective) hyperplane arrangement is given by

$$\mathcal{Q}_{\mathcal{A}} = \prod_{H \in \mathcal{A}} \alpha_H,$$

where α_H is an affine (resp. a linear) form in $k[x_1, \ldots, x_d]$ such that $H = \alpha_H^{-1}(0)$.

Remark VI.1.4. The affine or linear forms α_H and the defining polynomial $\mathcal{Q}_{\mathcal{A}}$ are defined up to unity in k.

Notation VI.1.5. From now, we denote by $\mathbb{A}_k^d$ and $\mathbb{P}_k^d$ the affine and projective d–dimensional space over a k, respectively.

In this thesis, we are focusing on *line arrangements*, i.e. a finite collection of different lines

$$\mathcal{A} = \{L_1, \ldots, L_n\}$$

in $\mathbb{A}_k^2$ or $\mathbb{P}_k^2$. We consider in general that $k = \mathbb{R}$ or $\mathbb{C}$, and the polynomial algebras $S \simeq k[x, y]$ in the affine case, and $S \simeq k[x, y, z]$ in the projective case.

We denote by $\mathrm{Sing}\,\mathcal{A}$ the *set of singular points* of the line arrangement $\mathcal{A}$, given by any non-empty intersection $L \cap L'$ of different lines $L, L' \in \mathcal{A}$. For a point $P \in \mathrm{Sing}\,\mathcal{A}$, we define the sub-arrangement

$$\mathcal{A}_P = \{L \in \mathcal{A} \mid P \in L\}$$

of lines in $\mathcal{A}$ passing through P. The *multiplicity* of P in $\mathcal{A}$ is the positive integer $|\mathcal{A}_P|$.

Remark VI.1.6.

1. We can *complexify* real line arrangements $\mathcal{A}$ in $\mathbb{A}_{\mathbb{R}}^2$ (resp. $\mathbb{P}_{\mathbb{R}}^2$) via tensorizing $\mathbb{A}_{\mathbb{R}}^2 \otimes \mathbb{C}$ (resp. $\mathbb{P}_{\mathbb{R}}^2 \otimes \mathbb{C}$).

2. If $\mathcal{A}$ is a complexified real line arrangement, there exists a change of variables such that any line $L \in \mathcal{A}$ has a defining polynomial in $\mathbb{R}[x, y]$. This implies that $\mathcal{Q}_{\mathcal{A}} \in \mathbb{R}[x, y]$, in particular. Nevertheless, a real defining polynomial does not define a complexified real arrangement in general: take for example the complex arrangement defined by $\mathcal{Q} = xy(x^2 + y^2)$.

Example VI.1.7. Define the line arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively by $\mathcal{Q}_{\mathcal{A}_1} = y(x-2y)(2x-y)(2x+y-1)$ and $\mathcal{Q}_{\mathcal{A}_2} = z(x^2-y^2)(x^2-z^2)(y^2-z^2)$, (see Figure VI.1). The second arrangement $\mathcal{A}_2$ can be seen either as a projective line arrangement or a central hyperplane arrangement in $\mathbb{A}_{\mathbb{R}}^3$. In order to obtain a projective picture of $\mathcal{A}_2$, we project over a plane the intersection of the hyperplane arrangement with the upper semi-sphere in $\mathbb{A}_{\mathbb{R}}^3$. In this picture, we can see the intersection of plane $\{z = 0\}$ as the line at infinity L_∞.

Let $\mathcal{A} = \{L_1, \ldots, L_n\}$ be an affine line arrangement, we can define the *coning* $\mathrm{c}\mathcal{A}$ in $\mathbb{A}_k^3$ (resp. the *projectivization* $\widetilde{\mathcal{A}} = \mathcal{A} \cup L_\infty$ in $\mathbb{P}_k^2$). In both cases, we associate to the induced central plane (resp. projective line) arrangement the linear form $\mathcal{Q}_{\mathrm{c}\mathcal{A}}$ (resp. $\mathcal{Q}_{\widetilde{\mathcal{A}}}$) in $k[x, y, z]$ given by the

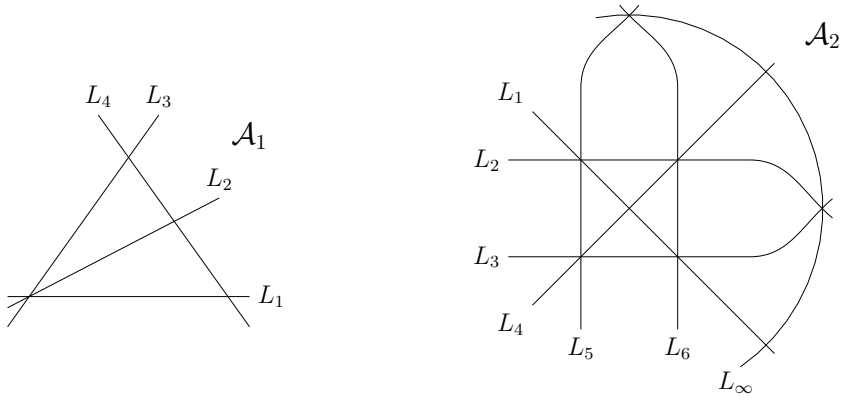

Figure VI.1: Line arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$.

homogenization $z\bar{\mathcal{Q}}_{\mathcal{A}} = z^{|\mathcal{A}|+1}\mathcal{Q}_{\mathcal{A}}(x/z, y/z) \in k[x, y, z]$. Remark that we are adding the plane $z = 0$ (resp. the line at infinity $L_\infty : z = 0$) contained in the new arrangement.

In the same same way, we can define the *deconing* $\mathrm{d}\mathcal{A}$ of a central plane arrangement in $\mathbb{A}^3_k$ with respect to the plane $H_\infty = \{z = 0\}$ as the affine arrangement in $\mathbb{A}^2_k$ with defining polynomial $\mathcal{Q}_{\mathrm{d}\mathcal{A}} = \mathcal{Q}_{\mathcal{A}|z=1} \in k[x, y]$. Similarly, if $\mathcal{A}$ is a projective line arrangement in $\mathbb{P}^2_k$, the *affine part* $\mathcal{A}^{\mathrm{aff}}$ with respect to the line $L_\infty = \{z = 0\}$ is the image of $\mathcal{A}$ in the affine chart $U_z = \{z \neq 0\} \simeq \mathbb{A}^2_k$. Again, $\mathcal{Q}_{\mathcal{A}^{\mathrm{aff}}} = \mathcal{Q}_{\mathcal{A}|z=1} \in k[x, y]$.

VI.1.2 Combinatorics

The most efficient way to encode the combinatorial data of a line arrangement is using the subset relations of inclusion into lines. More generally, we can define the *intersection semi-lattice* of a hyperplane arrangement [OT92, Chpt. 2].

Definition VI.1.8. Define the *intersection poset* of $\mathcal{A}$ as the set

$$L(\mathcal{A}) = \{\emptyset \neq L \cap L' \mid L, L' \in \mathcal{A}\} \cup V$$

partially ordered by the reverse inclusion:

$$Y \leq X \iff X \subset Y, \quad \forall X, Y \in L(\mathcal{A}).$$

Remark VI.1.9.

1. Note that the singularities of $\mathcal{A}$ are maximal elements in $L(\mathcal{A})$, while V is the only minimal element. Following the previous notation, the fact that $p \in \operatorname{Sing}\mathcal{A}$ is contained in a line L is denoted by $L \leq p$.

2. We can give a structure of *geometric semi-lattice* for $L(\mathcal{A})$ by a *rank* function $r : L(\mathcal{A}) \to \mathbb{N} \cup \{0\}$ defined by $r(X) = \operatorname{codim} X$, $X \in L(\mathcal{A})$. In the case of central arrangements, $L(\mathcal{A})$ becomes a *geometrical lattice*. See [OT92, Sec. 2.1] for more details.

Definition VI.1.10. We said that two arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ are *combinatorially equivalent* if $L(\mathcal{A}_1) \simeq L(\mathcal{A}_2)$, i.e. there exist an order preserving bijection $\Phi : L(\mathcal{A}_1) \to L(\mathcal{A}_2)$.

Example VI.1.11. Taking $\mathcal{A}_1$ as in Example VI.1.7, its intersection poset $L(\mathcal{A})$ is represented in Figure VI.2.

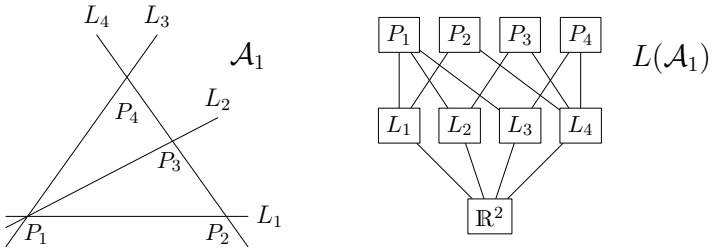

Figure VI.2: Affine line arrangement $\mathcal{A}_1$ and its associated poset $L(\mathcal{A}_1)$.

Definition VI.1.12. Let $\mathcal{A}$ be a hyperplane arrangement. We define the *Möbius function* as the map $\mu : L(\mathcal{A}) \times L(\mathcal{A}) \to \mathbb{Z}$ verifying the following conditions:

$$\begin{array}{rl} \mu(X,X) = 1, & \text{for any } X \in L(\mathcal{A}), \\ \sum_{X \leq Z \leq Y} \mu(X,Z) = 0, & \text{for any } X, Y, Z \in L(\mathcal{A}) \text{ such that } X < Y, \\ \mu(X,Y) = 0, & \text{otherwise.} \end{array}$$

Definition VI.1.13. Define $\mu(X) = \mu(V, X)$, for any $X \in L(\mathcal{A})$. We define the *characteristic polynomial* of $\mathcal{A}$ by

$$\chi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X}$$

Remark VI.1.14. In the case of $\mathcal{A}$ is a line arrangement, the function μ is easily computable, since

- $\mu(\mathcal{A}) = 1$.
- $\mu(L) = -1$, for any $L \in \mathcal{A}$.
- $\mu(P) = |\mathcal{A}_P| - 1$, for any $P \in \operatorname{Sing} \mathcal{A}$.

Thus,

$$\chi(\mathcal{A}, t) = t^2 - |\mathcal{A}| \cdot t - |\operatorname{Sing} \mathcal{A}| + \sum_{P \in \operatorname{Sing} \mathcal{A}} |\mathcal{A}_P|.$$

An usual problem of interest is the study of the influence of the combinatorics on the geometry or topology of hyperplane arrangements in $\mathbb{A}^d_k$ or $\mathbb{P}^d_k$ (for $k = \mathbb{R}, \mathbb{C}$ or finite fields $\mathbb{F}_q$). In the complex case, it has been proved that the integral cohomology of the exterior $E_{\mathcal{A}} = \mathbb{A}^d_{\mathbb{C}} \setminus \mathcal{A}$ of a hyperplane arrangement $\mathcal{A}$ is combinatorial [OS80]. Nevertheless, the topology of $E_{\mathcal{A}}$ is not combinatorial, G. Rybnikov constructs in [Ryb11] a two combinatorially equivalent complex line arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ for which $\pi_1(\mathbb{P}^2_{\mathbb{C}} \setminus \mathcal{A}_1) \not\simeq \pi_1(\mathbb{P}^2_{\mathbb{C}} \setminus \mathcal{A}_2)$.

VI.1.3 Weak combinatorics

In the classical study of line arrangements, the intersection poset is usually called the *(strong) combinatorics*. A softer notion of combinatorics is also usually used, considering the following combinatorial data of an affine arrangement $\mathcal{A}$:

- The number of lines $|\mathcal{A}| = n$.
- The sequence $\mathcal{S}_{\mathcal{A}} = (s_2, s_3, \ldots, s_n)$, where s_k denotes the number of singular points of multiplicity k in $\mathcal{A}$, for $k = 2, \ldots, n$.

- The sequence $\mathcal{P}_{\mathcal{A}} = (p_2, p_3, \ldots, p_n)$, where p_k denotes the number of families of exactly $(k-1)$ parallel lines in $\mathcal{A}$, for $k = 2, \ldots, n$.

Then the tuple $(|\mathcal{A}|, \mathcal{S}_{\mathcal{A}}, \mathcal{P}_{\mathcal{A}})$ is called the *weak combinatorics* of $\mathcal{A}$. For projective line arrangements, this notions is reduced to the pair $(|\mathcal{A}|, \mathcal{S}_{\mathcal{A}})$.

Proposition VI.1.15. *Let $\mathcal{A}$ be an affine line arrangement and $\widetilde{\mathcal{A}}$ its projectivization. If $\mathcal{A}$ has weak combinatorics $(|\mathcal{A}|, \mathcal{S}_{\mathcal{A}}, \mathcal{P}_{\mathcal{A}})$, then $\widetilde{\mathcal{A}}$ has weak combinatorics $(|\widetilde{\mathcal{A}}|, \mathcal{S}_{\widetilde{\mathcal{A}}})$ where $|\widetilde{\mathcal{A}}| = |\mathcal{A}| + 1$ and $t_k = s_k + p_k \in \mathcal{S}_{\widetilde{\mathcal{A}}}$ for any $k = 1, \ldots, n$.*

VI.2 Module of logarithmic derivations

The *module of logarithmic derivations* for divisors of complex analytic varieties was introduced by K. Saito [Sai80] for the study of the Gauss-Manin connection, as a generalization of a previous work of P. Deligne [Del70] for normal crossing divisors. This module seems to contain a lot of geometrical and local or global topological information of the divisor. We consider the global application of this module to the study of hyperplane arrangements, initiating by K. Saito and H. Terao from the algebraic setting.

VI.2.1 Derivations and freeness

Let V be a k–vectorial space of dimension d and $S = \mathrm{Sym}(V^*)$ the symmetric algebra of the dual space V^*. Remember that $S \simeq k[x_1, \ldots, x_d]$ for a dual basis $\{x_1, \ldots, x_d\}$ of V^*.

Definition VI.2.1. Let $\mathrm{Der}_k(S)$ be the set of k–linear maps $\chi : S \to S$ such that

$$\chi(fg) = f\chi(g) + g\chi(f) \qquad \text{for } f, g \in S$$

Proposition VI.2.2.

1. *$\mathrm{Der}_k(S)$ is a S–module of rank d.*
2. *Any k–linear map $\phi : V^* \to S$ can be extended uniquely into a derivation $\chi_\phi : S \to S$. In particular, for any $v \in V$ there exists a unique $D_v \in \mathrm{Der}_k(S)$ such that $D_v(\alpha) = \alpha(v)$ for $\alpha \in V^*$.*
3. *Let $\{e_1, \ldots, e_d\}$ a basis of V and $\{x_1 \ldots, x_d\}$ the dual basis. Then $\{D_1, \ldots, D_d\}$ forms a basis of $\mathrm{Der}_k(S)$, where $\partial_{x_i} = D_{e_i} = \partial/\partial x_i$ are the usual derivations.*

Definition VI.2.3. For any $f \in S$, we define the S-submodule of $\mathrm{Der}_k(S)$:

$$\mathcal{D}(f) = \{\chi \in \mathrm{Der}_k(S) \mid \chi(f) \in \mathcal{I}_f\}$$

where $\mathcal{I}_f$ is the ideal generated by f.

Definition VI.2.4. The *module of logarithmic derivations of $\mathcal{A}$* (also called *module of $\mathcal{A}$-derivations*) is defined by $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{Q}_{\mathcal{A}})$.

Proposition VI.2.5.

1. *The Euler derivation $\chi_E = \sum x_i \partial_{x_i}$ is in $\mathcal{D}(\mathcal{A})$ for any central arrangement $\mathcal{A}$.*
2. $\mathcal{D}(\mathcal{A}) = \bigcap_{H \in \mathcal{A}} \mathcal{D}(\alpha_H) = \{\chi \in \mathrm{Der}_k(S) \mid \chi(\alpha_H) \in \mathcal{I}_{\alpha_H}, \forall H \in \mathcal{A}\}$.

3. Let $\mathcal{A}_1, \mathcal{A}_2$ two arrangements in V such that $\mathcal{A}_1 \subset \mathcal{A}_2$. Then $\mathcal{D}(\mathcal{A}_1) \supset \mathcal{D}(\mathcal{A}_2)$.

Lemma VI.2.6. *We have the following inclusion of modules:*

$$\mathcal{Q}_{\mathcal{A}} \operatorname{Der}_k(S) \subset \mathcal{D}(\mathcal{A}) \subset \operatorname{Der}_k(S)$$

Definition VI.2.7. If $\chi \in \operatorname{Der}_k(S)$, then $\chi = \sum \chi(x_i)\partial_{x_i}$ written in coordinates. Given $\chi_1, \ldots, \chi_d \in \operatorname{Der}_k(S)$, define the *coefficient matrix*:

$$M(\chi_1, \ldots, \chi_d) = \begin{bmatrix} \chi_1(x_1) & \cdots & \chi_d(x_1) \\ \vdots & & \vdots \\ \chi_1(x_d) & \cdots & \chi_d(x_d) \end{bmatrix}$$

Proposition VI.2.8. *If $\chi_1, \ldots, \chi_d \in \mathcal{D}(\mathcal{A})$, then $\det M(\chi_1, \ldots, \chi_d) \in \mathcal{I}_{\mathcal{Q}}$.*

A special case of arrangements are those for which the module of logarithmic derivations is *free*. In this case, there is a lot of information concerning $\mathcal{D}(\mathcal{A})$ which comes from the combinatorics. We give the notion of *freeness* as well as the most important properties of free arrangements, which allows us to introduce Terao's conjecture of freeness for central hyperplane arrangements. Look at [OT92, Chpt. 4] for a more extended dissertation about freeness of hyperplane arrangements.

Definition VI.2.9. An arrangement $\mathcal{A}$ is called *free* if $\mathcal{D}(\mathcal{A})$ is a free module over S.

There exists a very practical criterion in order to check freeness given by K. Saito:

Theorem VI.2.10 (Saito's criterion)**.** *Given $\chi_1, \ldots, \chi_d \in \mathcal{D}(\mathcal{A})$, then the following are equivalent:*

1. *$\det M(\chi_1, \ldots, \chi_d) \doteq \mathcal{Q}_{\mathcal{A}}$, where "$\doteq$" means* up to unit *in S.*
2. *The derivations $\chi_1, \ldots, \chi_d$ form a basis for $\mathcal{D}(\mathcal{A})$ over S.*

Example VI.2.11. Take $\mathcal{A}_1$ with $\mathcal{Q}_{\mathcal{A}_1} = y(x-2y)(2x-y)(2x+y-1)$ of Example VI.1.7, and consider the derivations of $\mathcal{D}(\mathcal{A})$ given by $\chi_1 = x(2x+y-1)\partial_x + y(2x+y-1)\partial_y$ and $\chi_2 = (10xy^2 - xy - 2y^2 2)\partial_x + (10y^3 + 2xy - 6y^2)\partial_y$. Then $\mathcal{D}(\mathcal{A}_1)$ is a free $k[x,y]$–module whose basis is $\{\chi_1, \chi_2\}$, since:

$$\det \begin{pmatrix} x(2x+y-1) & 10xy^2 - xy - 2y^2 \\ y(2x+y-1) & 10y^3 + 2xy - 6y^2 \end{pmatrix} = y(x-2y)(2x-y)(2x+y-1).$$

VI.2.2 Logarithmic differential forms

Let $\{x_1, \ldots, x_d\}$ be a basis for V^* and take its symmetric algebra $S \simeq k[x_1, \ldots, x_d]$. Denote by S' the quotient field of S. For a non-negative integer p, we define the S–module of *algebraic p–forms* in V by

$$\Omega^p[V] = \bigoplus_{1 \le i_1 < \cdots < i_p \le d} S \cdot \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}.$$

Note that $\Omega^0[V] = S$ and $\Omega^p[V] = 0$ if $p > d$. In the same way we define the S–module of *algebraic p–forms* in V by

$$\Omega^p(V) = \Omega^p[V] \otimes_S S' = \bigoplus_{1 \le i_1 < \cdots < i_p \le d} S' \cdot \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}.$$

Definition VI.2.12. Let $\mathcal{A}$ be a hyperplane arrangement. The module $\Omega^p(\mathcal{A})$ of *logarithmic p–forms with poles along $\mathcal{A}$* is defined as

$$\Omega^p(\mathcal{A}) = \{\omega \in \Omega^p(V) \mid \mathcal{Q}_{\mathcal{A}}\omega \in \Omega^p[V] \text{ and } \mathcal{Q}_{\mathcal{A}}\mathrm{d}\omega \in \Omega^{p+1}[V]\}.$$

Lemma VI.2.13. *We have the following inclusion of modules:*

$$\Omega^p[V] \subset \Omega^p(\mathcal{A}) \subset \frac{1}{\mathcal{Q}_{\mathcal{A}}}\Omega^p[V],$$

for any $0 \leq p \leq d$.

We are specially interested in logarithmic 1–forms because we are focusing on line arrangements and also due to its duality with logarithmic derivations of an arrangement. Consider the *interior product* of derivations on forms given by

$$\begin{array}{rccc} \langle \cdot, \cdot \rangle : & \mathrm{Der}_k(S) \times \Omega^1(V) & \longrightarrow & \Omega^0(V) = S' \\ & (\chi, \omega) & \longmapsto & \langle \chi, \omega \rangle \end{array}$$

verifying $\langle \chi, \mathrm{d}x_i \rangle = \chi(x_i)$ for any $i = 1, \ldots, d$. Thus, we can construct [Sai80, secs. 1.5–1.7] a well-defined non-degenerated S-bilinear pairing

$$\begin{array}{rccc} \langle \cdot, \cdot \rangle : & \mathcal{D}(\mathcal{A}) \times \Omega^1(\mathcal{A}) & \longrightarrow & S \\ & \left(\chi, \frac{\omega}{\mathcal{Q}_{\mathcal{A}}}\right) & \longmapsto & \frac{\langle \chi, \omega \rangle}{\mathcal{Q}_{\mathcal{A}}} \end{array}$$

Theorem VI.2.14. *The S–modules $\mathcal{D}(\mathcal{A})$ and $\Omega^1(\mathcal{A})$ are duals of each other, i.e. $\mathcal{D}(\mathcal{A})^* \simeq \Omega^1(\mathcal{A})$ and $\Omega^1(\mathcal{A})^* \simeq \mathcal{D}(\mathcal{A})$.*

Corollary VI.2.15. *The S–modules $\mathcal{D}(\mathcal{A})$ and $\Omega^1(\mathcal{A})$ are reflexive.*

An analogous study of freeness of arrangements can be made looking at the freeness on $\Omega^1(\mathcal{A})$, as a consequence of last Theorem. The study of basis in $\Omega^1(\mathcal{A})$ can be made using an equivalent Saito's Criterion for logarithmic forms [OT92, Proposition 4.80, pag. 130]. In the case of central or projective arrangements, we get a graduation of $\Omega^1(\mathcal{A})$ by decomposition on homogeneous components based in rational fractions degree.

VI.3 Terao's conjecture

VI.3.1 General statement for central arrangements

We introduce Terao's conjecture for the general setting of central hyperplane arrangements. Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be central in a k-vector space V of dimension d, and $S = \mathrm{Sym}(V^*)$ as before.

Let S_p be the k–vector subspace of S consisting of 0 and homogeneous polynomials of degree p. Define $S_p = 0$ for $p < 0$, then:

$$S = \bigoplus_{p \in \mathbb{Z}} S_p$$

is a graded k–algebra.

Definition VI.3.1. A non-zero $\chi \in \mathrm{Der}_k(S)$ is *homogeneous of (polynomial) degree p* if $\chi = \sum f_i \partial_{x_i}$ and $f_i \in S_p, \forall i = 1, \ldots, d$. In this case we write $\deg \chi = p$.

With this degree, $\mathrm{Der}_k(S)$ is a graded module

$$\mathrm{Der}_k(S) = \bigoplus_{p\in\mathbb{Z}} \mathrm{Der}_k(S)_p$$

where $\mathrm{Der}_k(S)_p = \{\sum f_i\partial_{x_i} \mid f_i \in S_p, \forall i = 1,\dots,d\}$.

Proposition VI.3.2. *Let $\mathcal{D}(\mathcal{A})_k = \mathcal{D}(\mathcal{A}) \cap \mathrm{Der}_k(S)_p$. Then*

$$\mathcal{D}(\mathcal{A}) = \bigoplus_{p\in\mathbb{Z}} \mathcal{D}(\mathcal{A})_p$$

and $\mathcal{D}(\mathcal{A})$ is a graded S–module of $\mathrm{Der}_k(S)$.

Proposition VI.3.3. *Let $\mathcal{A}$ be non-empty, and consider the Euler derivation $\chi_E = \sum_{i=1}^d x_i\partial_{x_i}$. Then, for any non-empty $\mathcal{A}' \subset \mathcal{A}$ we can express*

$$\mathcal{D}(\mathcal{A}) = \langle\chi_E\rangle \oplus \mathrm{Ann}(\mathcal{A}'),$$

where $\mathrm{Ann}(\mathcal{A}') = \{\chi \in \mathcal{D}(\mathcal{A}) \mid \chi(\mathcal{Q}_{\mathcal{A}'}) = 0\}$ is a graded submodule.

Proof. As $\mathcal{A}$ is central, then any sub-arrangement $\mathcal{A}'$ is also central and $\chi_E(\mathcal{Q}_{\mathcal{A}'}) = |\mathcal{A}'|\mathcal{Q}_{\mathcal{A}'}$.

For any $\chi \in \mathcal{D}(\mathcal{A})$, we define the derivation

$$\chi - \frac{\chi(\mathcal{Q}_{\mathcal{A}'})}{|\mathcal{A}'|\mathcal{Q}_{\mathcal{A}'}}\chi_E,$$

which is easy to see that it belongs to $\mathrm{Ann}(\mathcal{A}')$. We have also that $\langle\chi_E\rangle \cap \mathrm{Ann}(\mathcal{A}') = 0$, thus the result holds. □

Two particular interesting cases of the previous decomposition appear when $\mathcal{A}' = \{H_0\}$ for some fixed $H_0 \in \mathcal{A}$, and $\mathcal{A}' = \mathcal{A}$.

Proposition VI.3.4. *If $\mathcal{A}$ is free, then $\mathcal{D}(\mathcal{A})$ has a basis consisting of p homogeneous elements.*

Proof. This result is proved in [OT92, Proposition 4.18, pag. 108]. □

Remark VI.3.5.

1. Let $\chi_1,\dots,\chi_d \in \mathcal{D}(\mathcal{A})$ homogeneous and linearly independent over S. K. Saito shows that

$$\sum_{i=1}^d \deg\chi_i \geq |\mathcal{A}|.$$

   Using the previous Proposition with the Saito's Criterion, it comes a numerical characterization of freeness over the degrees: $\chi_1,\dots,\chi_d$ form a basis of $\mathcal{D}(\mathcal{A})$ if and only if

$$\sum_{i=1}^d \deg\chi_i = |\mathcal{A}|.$$

2. For a free arrangement $\mathcal{A}$ with homogeneous basis $\chi_1,\dots,\chi_d$ we can define the *exponents of $\mathcal{A}$* as the ordered set

$$\exp\mathcal{A} = \{\deg\chi_1,\dots,\deg\chi_d\},$$

   depending only on $\mathcal{A}$ ([Ter80, sec. 2]).

3. Note that for a nonempty free arrangement we have $1 \in \exp \mathcal{A}$, since we can give a decomposition using the Euler derivation $\chi_E = \sum_{i=1}^{d} x_i \partial_{x_i}$, as we have seen in Proposition VI.3.3: if Γ is a minimal system of generators of $\mathrm{Ann}(\mathcal{A}')$, for any non-empty $\mathcal{A}' \subset \mathcal{A}$, then $\{\chi_E\} \cup \Gamma$ is the minimal system of generators for $\mathcal{D}(\mathcal{A}) = \langle \chi_E \rangle \oplus \mathrm{Ann}(\mathcal{A}')$. This set becomes a homogeneous basis for $\mathcal{D}(\mathcal{A})$ when $\mathcal{A}$ is free.

Remark VI.3.6. The freeness condition can be also studied using logarithmic differential forms of $\mathcal{A}$, since $\mathcal{D}(\mathcal{A})$ is free if and only if $\Omega^1(\mathcal{A})$ is free, by Theorem VI.2.14. In the case of central arrangements, $\Omega^1(\mathcal{A})$ is also a graded module, by the degree of homogeneous decomposition over the components.

Example VI.3.7. Consider $\mathcal{A}_2$ in Example VI.1.7 viewed as a central hyperplane arrangement in $\mathbb{A}^3_k$, given by $\mathcal{Q}_{\mathcal{A}_2} = z(x^2 - y^2)(x^2 - z^2)(y^2 - z^2)$. Then $\mathcal{A}_2$ is free arrangement with basis

$$\chi_E, \quad \chi_2 = yz^2\partial_x + xz^2\partial_y + xyz\partial_z, \quad \chi_3 = x^3\partial_x + y^3\partial_y + z^3\partial_z$$

We can check that

$$\det \begin{pmatrix} x & yz^2 & x^3 \\ y & xz^2 & y^3 \\ z & xyz & z^3 \end{pmatrix} = z(x^2 - y^2)(x^2 - z^2)(y^2 - z^2),$$

and we have $\exp \mathcal{A}_2 = \{1, 3, 3\}$. In fact, it is more easy to see that $7 = |\mathcal{A}| = 1 + \deg \chi_2 + \deg \chi_3$ in order to check that these derivations are a basis of $\mathcal{D}(\mathcal{A})$.

In general, $\mathcal{D}(\mathcal{A})$ is not determined as graded module by the combinatorics of $\mathcal{A}$, but H. Terao proves that in the case of free arrangements, the sequence $\exp \mathcal{A}$ is determined by the combinatorics $L(\mathcal{A})$, via the Factorization Theorem:

Theorem VI.3.8 ([Ter81])**.** *If $\mathcal{A}$ is free of exponents $\exp \mathcal{A} = \{b_1, \ldots, b_d\}$, then*

$$\chi(\mathcal{A}, t) = \prod_{i=1}^{d} (t - b_i),$$

where $\chi(\mathcal{A}, t)$ is the characteristic polynomial of $\mathcal{A}$.

The characteristic polynomial $\chi(A, t)$ is of combinatorial nature, i.e. determined by $L(\mathcal{A})$. Thus, as a consequence of the previous Theorem, we obtain:

Corollary VI.3.9. *Let $\mathcal{A}$ and $\mathcal{A}'$ be two central hyperplane arrangements such that $L(\mathcal{A}) \simeq L(\mathcal{A}')$. If $\mathcal{A}$ and $\mathcal{A}'$ are free, then $\mathcal{D}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A}')$ are isomorphic graded modules.*

Thus, the *freeness condition* guarantees that the combinatorics determines the module of logarithmic derivations of a central hyperplane arrangement $\mathcal{A}$ respecting the graduation. The Terao's conjecture [OT92, Conjecture 4.138] suggests that even the freeness condition is itself a combinatorially determined.

Conjecture VI.3.10 (Terao's conjecture)**.** *Let $\mathcal{A}$ and $\mathcal{A}'$ be two central hyperplane arrangements such that $L(\mathcal{A}) \simeq L(\mathcal{A}')$. If $\mathcal{A}$ is free then $\mathcal{A}'$ is also free.*

Thus, it is not the geometry but only the combinatorics of the arrangement which determines $\mathcal{D}(\mathcal{A})$ as free graded module.

VI.3.2 Vector bundles on projective plane

We are also interested in the particular formulation of Terao's conjecture for projective line arrangements. In this setting, Terao's conjecture is studied as a *splitting problem of vector bundles* over $\mathbb{P}^2_k$.

Let $\mathcal{O}_{\mathbb{P}^2_k}$ the structural sheaf of the projective plane, the sheafification of $k[x,y,z]$ whose sections in the local chart $U_x = \{x \neq 0\}$ are $\mathcal{O}_{\mathbb{P}^2_k}(U_x) = (k[x,y,z,1/x])_0$, where $(-)_d$ denotes the degree d component of the graded module (resp. for U_y and U_z). Following this notion, $\mathcal{O}_{\mathbb{P}^2_k}(d)$ is the rank one $\mathcal{O}_{\mathbb{P}^2_k}$–module corresponding to the sheafification of $(k[x,y,z])_d$, for $k \in \mathbb{Z}$.

Let $\mathcal{A}$ a projective line arrangement. The graded sub-module of derivations $\mathcal{D}_0(\mathcal{A}) = \mathrm{Ann}(\mathcal{Q}_\mathcal{A})$ is a $k[x,y,z]$–module, whose sheafification is the rank two vector bundle $\mathcal{T}_\mathcal{A}$ called *bundle of logarithmic vector fields among* $\mathcal{A}$. Since any $\chi \in \mathcal{D}_0(\mathcal{A})$ verify $\chi(\mathcal{Q}_\mathcal{A}) = 0$, thus $\mathcal{T}_\mathcal{A}$ is the kernel of the jacobian map $\nabla\mathcal{Q}_\mathcal{A} = (\partial_x\mathcal{Q}_\mathcal{A}, \partial_y\mathcal{Q}_\mathcal{A}, \partial_z\mathcal{Q}_\mathcal{A})$, i.e.

$$\begin{array}{ccc} \mathcal{O}^3_{\mathbb{P}^2_k} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2_k}(|\mathcal{A}|-1) \\ u & \longmapsto & u \cdot \nabla\mathcal{Q}_\mathcal{A} \end{array}$$

A non-zero section in $\Gamma(\mathbb{P}^2_k, \mathcal{T}_\mathcal{A}(d))$, for some shift d, correspond to a derivation

$$\chi_d = P_d\partial_x + Q_d\partial_y + R_d\partial_z \in \mathcal{D}_0(\mathcal{A}),$$

for some homogeneous polynomials $P_d, Q_d, R_d \in k[x,y,z]$ of degree d.

In this case, a projective line arrangement $\widetilde{\mathcal{A}}$ is called *free* if $\mathcal{T}_{\widetilde{\mathcal{A}}}$ splits, i.e.

$$\mathcal{T}_{\widetilde{\mathcal{A}}} \simeq \mathcal{O}_{\mathbb{P}^2_k}(-a) \oplus \mathcal{O}_{\mathbb{P}^2_k}(-b),$$

The previous splitting of $\mathcal{T}_{\widetilde{\mathcal{A}}}$ is equivalent to say that the graded sub-module of derivations $\mathcal{D}_0(\widetilde{\mathcal{A}}) = \mathrm{Ann}(\mathcal{Q}_{\widetilde{\mathcal{A}}})$ is a free $k[x,y,z]$–module generated by two derivations χ_a and χ_b of degree a et b, respectively. The numbers $a \leq b \in \mathbb{Z}$ verifies numerical conditions coming for the combinatorics and relied on the Chern's classes of the vector field (see [FV12b]). These derivations can be seen as global sections of $\mathcal{T}_{\widetilde{\mathcal{A}}}(a)$ and $\mathcal{T}_{\widetilde{\mathcal{A}}}(b)$.

Chapter VII

Dynamical approach of logarithmic vector fields

The aim of this Chapter is to give a dynamical interpretation of an algebro-analytic tool introduced by K. Saito [Sai80] in the eighties in his study of divisors in a complex manifold. Doing so, we will introduce a very general dynamical approach to the study of geometrical objects which has many different realizations in the context of algebraic geometry and dynamical systems. In particular, we will be interested in the study of configurations of algebraic curves and line arrangements.

After a heuristic introduction of our principal ideas and aims, we introduce in Section VII.2 the logarithmic vector fields. We show a dynamical fundamental meaning of this tool in algebraic/analytic geometry by the notion of invariant sets of vector fields, discussing the differences between their analytic and algebraic nature. This allows us to introduce a detailed explanation about our dynamical approach to geometry in Section VII.3.

Section VII.4 deals more specifically with algebraic logarithmic vector fields in real and complex plane, we discuss from our point of view two classical problems of real planar vector fields: the Dulac conjecture and the algebraic 16th Hilbert problem. As the simplest case of configurations of curves, the case of line arrangements is introduced.

In Section VII.5, we give relations between the notion of logarithmic vector fields and some classical notions and tools in the study of invariant algebraic curves of polynomial vector fields, as first integrals, Darboux integrability and Lie brackets.

The special case of curves defined by homogeneous equations is discussed in Section VII.6. In particular, we are interested in the relations in terms of logarithmic vector fields between affine line arrangements and central plane arrangements in $\mathbb{A}^3_k$ as well as projective line arrangements in $\mathbb{P}^2_k$.

We conclude in Section VII.7 making the direction of our future studies from the point of view presented in this Chapter.

VII.1 Introduction

Informally, our idea is to study the embedding of a given sufficiently regular geometrical object, denoted O, by the set of vector fields in the ambient space letting O dynamically invariant and denoted $\mathcal{D}(O)$ in the following. Heuristically, one can think to this point of view as trying to reconstruct the geometry of O by immersing it in a river and studying any possible flow traversing it. This flow will have singularities and particularities directly related to the geometry

and topology of the underlying set.

By restricting the class of vector fields to analytic or algebraic vector fields, depending on the nature of the object O, one recover in an informal way the classical notion of logarithmic vector fields. However, this point of view has the advantage to be transposable to many different domains and we will discuss some of them in the following. In particular, we will use this approach to deal with two classical problems in dynamical systems, precisely polynomial differential equations: the Dulac conjecture and the Hilbert 16th problem. In each case, these problems ask for a given polynomial vector fields if some particular geometric configurations can be obtained: accumulation near a polycycle for the Dulac conjecture and limit cycles for the Hilbert 16th problem. The direct approach to such problems is in general very hard and lead to some intractable formal computations as long as high degree are considered.

Reversing the problematic, one can try to use the benefit of the algebraic or geometric tools developed in the context of logarithmic vector fields but extended to these new configurations.

On the contrary, we can use dynamical systems results in order to understand the set of logarithmic vector fields for a given object O. We will give an example of such study in the case of line arrangements in the plane in the next Chapter. It must be said that, up to our knowledge, there exists no study of the dynamics generated by logarithmic vector fields. The usual point of view deals with algebraic properties (freeness as module) or geometric via *logarithmic bundles*.

VII.2 Logarithmic vector fields: a dynamical view point

We introduce the definition of logarithmic vector fields originally given by Saito in [Sai80]:

Definition VII.2.1. Let X be a d–complex manifold with structural sheaf $\mathcal{O}_X$, and let $D \subset X$ be a hypersurface. A holomorphic vector field χ in X is said to be *logarithmic* if it satisfies that for any point $p \in D$ the derivation $\chi(h_p)$ of the local equation $h_p = 0$ for D belongs to the ideal $(h_p)\mathcal{O}_{X,p}$.

Even if the Saito's definition is restricted to complex analytic manifolds, one can imagine an extension of the previous definitions to the case of submanifolds $D = f^{-1}(0)$ defined by sufficiently smooth functions.

VII.2.1 Invariant sets of vector fields

For a given vector field χ and a given point x_0 in X, we denote by $\phi_t(x_0)$ the solution of the differential equation associated to χ starting at x_0 at time $t = 0$. This defines a map $\phi_t : X \to X$ called the *flow* generated by χ. We assume in the following that χ is *complete*, i.e. that the flow is defined for $t \in \mathbb{R}$. From a qualitative point of view, this is important to localize *dynamically invariant subsets* of X. Precisely, see [Wig03, Def. 3.0.3, p. 28].

Definition VII.2.2. Let D be a subset of X and χ be a vector field on X. We said that χ possesses D as *invariant set* if for all $x_0 \in D$, we have $\phi_t(x_0) \in D$ for all $t \in \mathbb{R}$.

Dynamical invariance has a simple geometric characterization (see [Wig03, Sec. 3.2A, p. 39]):

Lemma VII.2.3. *A hypersurface D is invariant under a vector field χ if χ is tangent to D at every smooth point of D.*

It is important to notice that these notions are defined for $\mathcal{C}^r$–vector fields, for any $1 \leq r \leq \infty$ or $r = \omega$. In the analytic or algebraic class, we will see a more practical characterization of the invariance.

VII.2.2 Dynamical meaning of logarithmic vector fields

The previous result induces a dynamical interpretation of logarithmic vector fields:

Proposition VII.2.4. *The hypersurface D is an invariant set under its module of logarithmic vector fields.*

Proof. Saito gives an equivalence between the fact for a vector field to be logarithmic and that it is tangent to any smooth point of D in [Sai80, Def. 1.4–(i), p. 267]. By Lemma VII.2.3, the result holds. □

It is worth noticing that the previous result can still holds for smooth submanifolds defined by the zero set of a smooth function and vector fields χ satisfying $\chi(f) = Kf$, where K is a function in the same class. Indeed, for any point p we have $\chi(f)_p = \chi(p) \cdot (\nabla f)(p) = K(p)f(p)$, and when we restricts over D we obtain the tangency condition.

Hence, the notion of logarithmic vector fields of a hypersurface D corresponds to vector fields letting D dynamically invariant. However, in the algebraic complex case, we have more strong relations. Indeed, if D is defined by a global reduced polynomial equation, the set of polynomial vector fields possessing D as dynamically invariant set corresponds to logarithmic vector fields. This is a deep difference between the smooth case and the algebraic or analytical case, explaining the important role of logarithmic vector fields in complex geometry.

Remark VII.2.5. Previous discussion is not longer true in analytic or algebraic real geometry, due to the fact the ordering of $\mathbb{R}$ gives an obstruction to identify geometric properties in algebraic sets from reduced algebraic equations. For example, if we consider the polynomial vector field $\chi = x^2\partial_x + y^2\partial_y$ in $\mathbb{A}^2_{\mathbb{R}}$, the origin is a invariant set by χ, but if we take $f = x^2 + y^2$ as defining equation of the origin, then $\chi(f) = 2(x^3 + y^3) = 2(x + y)(x^2 - xy + y^2)$, but this polynomial can not be expressed as product of f and a real polynomial.

In the following, we give a series of problems which are related to this point of view and what can be done with it.

VII.2.3 Analytic versus algebraic logarithmic vector fields

As already noted, the logarithmic vector fields introduced by K. Saito are initially defined in the analytic category. However, it seems reasonable for a given geometrical set $D \subset S$ to pay attention to a given class of vector fields which is mainly related to the initial class of the geometric object. For example, in the case of analytic subset of an analytic manifold it seems reasonable to look for analytic (holomorphic) vector fields and in the case of algebraic set to algebraic (i.e. polynomial) vector fields. This leads to the corresponding notion of analytic or algebraic logarithmic vector fields.

The choice of such a class is in general not trivial since some phenomenon, which are naturally studied in the algebraic class are in fact analytic phenomenon. We will give an example of such a problem in the part concerning the *Dulac conjecture* (see [Éca92, p. 4]) and the *algebraic 16th Hilbert problem*.

In the contrary, for arrangements it was proved that we can pass from analytic to algebraic considerations (see [OT92, p. 15]).

Remark VII.2.6. In fact, Orlik and Terao in [OT92, p. 15] refers to an article of Terao [Ter83] for a proof. However, we have been unable to find a discussion on this problem in the cited article. It would be interesting to have a complete proof of this result.

VII.3 A dynamical approach to geometry

The previous construction lead to a *dynamical system approach to geometry* which is up to our knowledge new. Indeed, for an arbitrary geometric object O in $\mathbb{R}^d$ or $\mathbb{C}^d$, one can study the set denoted $\mathcal{D}(O)$ of vector fields for which O is an invariant set. The regularity of O must be of course fixed. In the analytic case, we recover the notion of logarithmic vector fields introduced by Saito, and in the algebraic case the one introduced by Terao. Heuristically, the idea is to see the geometric or topological properties of a given object from the dynamical behavior of the flow generated by vector fields in $\mathcal{D}(O)$ on O and its complement. We then obtain information on the complement of O in the ambient space.

Other dynamical systems approaches to geometry already exist:

- For Riemannian manifolds a natural dynamics is provided by the *geodesic flow*. The dynamics is indeed strongly related to the topology and geometry of the manifold via the curvature. The geodesic flow is of course intrinsic and we loose this aspect by considering our objects as embedded in a given ambient space. However, this non-canonicity is replaced by the fact that we have access to a priori more informations on the object, in particular the complement.
- Another example is given by the connexion between *integrable connexion* and divisors as for example *Gauss-Manin connexion* and divisors with normal crossings. In this case, we obtain a dynamics which is only connected to the object and not directly related to.

A heuristic image can be given supporting our dynamical approach to geometry. The idea is to consider a given geometric object in the ambient three dimensional space $\mathbb{R}^3$. In order to distinguish between different geometries, we can look at the flow which is induced by the presence of this object in a three dimensional fluid with a uniform flow. The geometric object will make appeared some particular behaviors like the presence of an island inside a river. We know that the regularity and topology of the object is important. The set of logarithmic vector fields captures these informations.

Of course, the previous description is informal but the dynamical view point to the study of geometry seems to give interesting results and ideas. We illustrate some of them in the following.

VII.4 Configuration of algebraic curves and logarithmic vector fields

We consider a polynomial vector field defined for $(x, y) \in \mathbb{R}^2$ by

$$\chi = P\partial_x + Q\partial_y, \tag{VII.1}$$

where $P, Q \in \mathbb{R}[x, y]$. The associated polynomial differential equation is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = P(x, y) \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = Q(x, y). \tag{VII.2}$$

A polynomial vector field χ is said of *degree* d if $\max\{\deg P, \deg Q\} = d$.

The dynamical behavior of the flow $\phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ associated to polynomial differential systems is a classical area of study of the qualitative theory of differential equations initiated by

Poincaré in 1882 (see [Poi87a], [Poi87b], [Poi87c]). Classically, central objects in the qualitative study are first integrals and invariant sets, as for example equilibrium points and their stability properties. We refer to [PdM82] for an introduction to qualitative theory of dynamical systems.

Hence, the classical approach in dynamical systems is to consider a fixed vector field and to search its invariants sets. The dynamical approach to geometry is exactly the reverse operation: we fix a given set of analytic or algebraic curves and we look for the conditions that must satisfies a vector field in order to have these curves as dynamically invariant sets.

VII.4.1 The Dulac conjecture

In 1900, Hilbert [Hil00] presented his list of 23 problems of mathematics at the Paris conference of the International Congress of Mathematicians. The sixteenth problem of this list, named *Problem of the topology of algebraic curves and surfaces*, asked about relative position of real algebraic curves, surfaces and limit cycles of polynomial vector fields of a fixed degree d. These questions implies the determination of an upper bound for these algebraic objects as a function of the degree d. A *limit cycle* γ of χ is an analytic periodic orbit which is isolated, i.e. there exist an open set $U \subset \mathbb{R}^2$ such that γ is the only periodic orbit of χ contained in U.

The finiteness of limit cycles of polynomial vector field in the plane, known as the *Dulac's conjecture*, was proved independently by Ilyashenko and Écalle ([Ily91], [Éca92], see [Yoc88] for more details). However, these proofs deal with sophisticated analytic methods and tools for vector fields in analytic class. We have here a first example of the phenomenon which is discussed in Section VII.2.3. Indeed, as noticed by J. Écalle in [Éca92, p. 4, 7] the finiteness of limit cycles is not an algebraic phenomenon but an analytical one. As a consequence, the study of the module of logarithmic derivations for a given family of analytic limit cycles in the algebraic setting is perhaps not sufficiently rich.

VII.4.2 The algebraic 16th Hilbert problem

Limit cycles are not algebraic in general, see [GGT07] for explicit examples. It is natural to consider a simplification of the Hilbert's 16th problem, by focusing on the algebraic class (see [LRS10]). An *algebraic limit cycle* of χ is a limit cycle contained in an algebraic curve in $\mathbb{R}^2$ which is invariant by χ.

Problem VII.4.1 (**Algebraic Hilbert's 16th problem**). *Is there a bound C on the number of algebraic limit cycles of (VII.1) such that $C \leq d^q$ for $q > 0$?*

A possible approach to the algebraic Hilbert's 16th problem is to reverse the point of view by characterizing polynomial vector fields leaving invariant a given finite family of algebraic limit cycles. More generally, one can look for polynomial vector fields leaving invariant a given finite *configuration of algebraic curves*. A strong combinatorial background appears in this context, relative to the number and multiplicity of curves and singularities. Note that the study of the finiteness requirement in the configuration is essential to understand the Dulac's conjecture as a consequence of the algebraic setting.

VII.4.3 Line arrangements

A first step in this direction, as in [AGL98] or [ZY98], is to study the maximal number of lines for a given polynomial vector field. Thus, it is completely natural to explore the relation between polynomial vector fields in the plane and the *line arrangement* fixed by them.

VII.5 Algebraic invariant curves, integrability and Lie brackets

In this Section, we discuss several classical results relating existence of algebraic invariant curves and dynamical properties of polynomial vector fields, in particular, integrability. This gives a strong support to the relation between integrability and algebraic properties of logarithmic vector fields.

VII.5.1 Algebraic logarithmic vector fields and Darboux integrability

Let $\mathcal{A}$ be an arrangement and $\mathcal{Q}_{\mathcal{A}}$ a defining polynomial. Note that the set $\mathcal{D}(\mathcal{A})$ can be equivalently defined as

$$\mathcal{D}(\mathcal{A}) = \{\chi \in \mathrm{Der}_k(k[x,y]) \mid \exists K \in k[x,y],\ \chi(\mathcal{Q}_{\mathcal{A}}) = K\mathcal{Q}_{\mathcal{A}}\}. \tag{VII.3}$$

This definition makes a direct connexion with what is known as *Darboux polynomials* in the study of polynomial vector fields in the complex plane. They were introduced by G. Darboux in his seminal work about rational first integrals of algebraic differential equations in 1878 [Dar78]. Precisely, we have:

Definition VII.5.1. Let χ be a polynomial vector field. A polynomial $f \in \mathbb{C}[x,y] \setminus \mathbb{C}$ is a *Darboux polynomial* for χ if f divides $\chi(f)$ in $\mathbb{C}[x,y]$. The polynomial K such that $\chi(f) = Kf$ is called the *cofactor* of the Darboux polynomial f.

Using this definition, we deduce that logarithmic vector fields associated to a given line arrangement $\mathcal{A}$ correspond to algebraic vector fields having $\mathcal{Q}_{\mathcal{A}}$ as a Darboux polynomial.

Classical results on Darboux polynomial can be used to go further on the properties of the module $\mathcal{D}(\mathcal{A})$. Indeed, we have [DLA06]:

Proposition VII.5.2. *Let $f \in \mathbb{C}[x,y]$ and $f = f_1 f_2$ be a factorization in $\mathbb{C}[x,y]$ with f_1 and f_2 coprime. Then f is a Darboux polynomial for χ if and only if f_1 and f_2 are Darboux polynomial for χ. Moreover, in this case, if K, K_1 and K_2 denote the cofactor of f, f_1 and f_2 respectively, then we have $K = K_1 + K_2$.*

A consequence is that if we are considering a family of algebraic curves f_i and that we consider the derivation module associated to $f = f_1 \cdots f_n$ then the polynomial let the full set $f = 0$ invariant but also all its components $f_i = 0$. This property can be used to characterize the polynomial vector fields belonging $\mathcal{D}(\mathcal{A})$ writing separately the invariance of each lines.

The previously described interpretation seems to be useless. However, Darboux polynomial gives strong informations on the dynamics of the associated vector fields in particular with respect to its *integrability* [IY08, Def. 11.1, p. 180]. A vector field χ is *integrable* if it possesses a *first integral*, i.e. a differentiable function $F : \mathbb{C}^2 \to \mathbb{C}$ constant on the solutions of the differential equation associated to χ, equivalently $\chi(F) = 0$.

Theorem VII.5.3 (Darboux)**.** *Let χ be a polynomial vector field of degree d. Assume that χ possesses m algebraic invariant curves $\{f_i = 0\}$ for $f_i \in \mathbb{C}[x,y]$, with associated cofactor $K_i \in \mathbb{C}[x,y]$, $i = 1, \ldots, m$. Then:*

1. *There exist m non simultaneously zero complex numbers $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ such that $\sum_{i=1}^m \lambda_i K_i = 0$ if and only if $f_1^{\lambda_1} \cdots f_m^{\lambda_m}$ is a first integral of χ.*
2. *If $m \geq \frac{d(d+1)}{2} + 1$, then there exist $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ non simultaneously zero such that $\sum_{i=1}^m \lambda_i K_i = 0$.*

In the case of line arrangements, the previous result concludes that the number of lines constrains the Darboux integrability of the logarithmic vector fields in a fixed degree. We can go further in this analysis trying to understand the role of the weak combinatorics of the arrangement from a dynamical point of view. This is done in the next Chapter.

VII.5.2 Lie algebraic structure and integrability of logarithmic vector fields

A usual action on vector fields is the *Lie bracket* denoted $[\cdot,\cdot]$ and defined for two arbitrary vector fields χ_1 and χ_2 by

$$[\chi_1,\chi_2] = \chi_1 \circ \chi_2 - \chi_2 \circ \chi_1,$$

where $\circ$ is the usual composition of differential operators. By definition of the logarithmic vector fields, we have the following important result, already noted by Saito in his first article on the subject (see [Sai80],ii) p. 208):

Theorem VII.5.4 (Lie structure)**.** *The module of logarithmic vector fields $\mathcal{D}(\mathcal{A})$ endowed with the Lie bracket $[\cdot,\cdot]$ is a Lie algebra.*

The proof is a simple computation. Let χ_1 and χ_2 be two logarithmic vector fields. Then, there exists two cofactor K_1 and K_2 such that $\chi_1(Q_{\mathcal{A}}) = K_1 Q_{\mathcal{A}}$ and $\chi_2(Q_{\mathcal{A}}) = K_2 Q_{\mathcal{A}}$. The Lie bracket $[\chi_1,\chi_2]$ then satisfies

$$\begin{array}{rcl} [\chi_1,\chi_2](Q_{\mathcal{A}}) & = & \chi_1\left[\chi_2(Q_{\mathcal{A}})\right] - \chi_2\left[\chi_1(Q_{\mathcal{A}})\right], \\ & = & \chi_1(K_2 Q_{\mathcal{A}}) - \chi_2(K_1 Q_{\mathcal{A}}). \end{array}$$

As χ_1 and χ_2 satisfies the Leibniz rule, we have

$$\begin{array}{rcl} \chi_1(K_1 Q_{\mathcal{A}}) & = & \chi_1(K_2) Q_{\mathcal{A}} + K_2 \chi_1(Q_{\mathcal{A}}), \\ & = & \left[\chi_1(K_2) + K_2 K_1\right](Q_{\mathcal{A}}), \end{array}$$

and an analogous formula for $\chi_2(K_1 Q_{\mathcal{A}})$. Replacing in the Lie bracket expression, we obtain

$$[\chi_1,\chi_2](Q_{\mathcal{A}}) = \left[\chi_1(K_2) - \chi_2(K_1)\right](Q_{\mathcal{A}}),$$

which concludes the proof.

Up to our knowledge, the dynamical information coming from the Lie structure is not really used in the study of logarithmic vector fields. However, the relation between freeness and the Lie structure of formal logarithmic derivations for divisors has been extensively studied in [GS06].

Indeed, the Lie bracket of two vector fields encodes the relation between the flow generated by each of them. Precisely, a given vector field χ on a manifold M generates a *local flow* denoted by $\phi_\chi(t,x)$ which satisfies for all $x \in M$, $\phi_\chi(0,x) = x$ and the semi-group property (see [Olv93, p. 27]). The vector field χ corresponds to the *infinitesimal generator* of the action generated by the flow. For a singular point $x_0 \in M$ of a vector field $\chi = P\partial_x + Q\partial_y$, we can study the local behavior at x_0 of the flow of χ using the *linear part* of χ at x_0:

$$D\chi(x_0) = \begin{pmatrix} \frac{\partial P}{\partial_x}(x_0) & \frac{\partial P}{\partial_y}(x_0) \\ \frac{\partial Q}{\partial_x}(x_0) & \frac{\partial Q}{\partial_y}(x_0) \end{pmatrix}$$

We say that x_0 is a *non-degenerate singular point* of χ if $\det D\chi(x_0) \neq 0$.

The Lie bracket can then be interpreted dynamically as a particular *correspondence* between the flow of two given vector fields. Precisely, we have (see [Olv93, Thm. 1.33, p. 35]) that the Lie bracket corresponds to the infinitesimal generator of the map

$$\psi(t,x) = \left(\phi_{\chi_2,-t} \circ \phi_{\chi_1,-t} \circ \phi_{\chi_2,t} \circ \phi_{\chi_1,t}\right)(x),$$

which gives

$$[\chi_1, \chi_2]_x = \frac{\mathrm{d}}{\mathrm{d}t}\psi(t,x)_{|t=0}.$$

An easy consequence, is the following Theorem (see [Olv93, Thm. 1.34, p. 36]):

Theorem VII.5.5. *Let χ_1 and χ_2 be two vector fields. The flow of χ_1 and χ_2 commutes if and only if $[\chi_1, \chi_2] = 0$.*

As a consequence, we have a strong algebraic property of the Lie bracket of two vector fields which has important consequences on the dynamical behavior of the vector fields. More on this subject can be found in [Sab97]. This is in particular the case when a given vector field χ_1 admits a commutation relation of the form $[\chi_1, \chi_2] = \mu(x,y)\chi_1$.

Theorem VII.5.6. *Let χ_1 be a polynomial vector field admitting a non-degenerate singular point at the origin. Assume that there exist a polynomial vector field χ_2 which can be written as the sum of the Euler vector field χ_E and a polynomial vector field with monomials of order greater than two such that $[\chi_1, \chi_2] = \mu\chi_1$, where $\mu : \mathbb{C}^2 \to \mathbb{C}$ is an analytic function satisfying $\mu(0,0) = 0$. Then χ_1 is integrable.*

See [GG06] for a proof and an extended study of the consequences of commutation relations. In the case of non-free arrangements, we are waiting for commutation relation between elements of the module of logarithmic vector fields. In such a case, we can guess that there exist connexions between integrability and freeness.

VII.6 About the homogeneous case

If $f \in \mathbb{C}[x_1, \ldots, x_d]$ is a square-free homogeneous polynomial, then $\mathcal{D}(f)$ contains the Euler vector field χ_E. As a consequence, one can study the graded sub-module $\mathcal{D}_0(f) = \mathrm{Ann}(f)$, i.e. the polynomial vector fields χ admitting f as first integral. Considering the irreducible factorization of $f = f_1 \cdots f_m$, we are in fact considering vector fields with m first integrals, one for each irreducible component.

Note that in the 2–dimensional case $\mathcal{D}_0(f)$ is generated by $\nabla f^\perp = (\partial_y f)\partial_x - (\partial_x f)\partial_y$. In the 3–dimensional case, we can construct another element of $\mathcal{D}_0(f)$ using the wedge product between χ_E and $\nabla f^\perp$.

VII.6.1 Relation between affine and homogeneous case

In next chapter, we introduce a first dynamical study focusing in affine line arrangements in the affine plane. Our purpose is to rely this approach to study *central plane arrangements* in $\mathbb{A}^3_k$ (resp. *projective line arrangements* in $\mathbb{P}^2_k$) via the *coning* $\mathrm{c}\mathcal{A}$ (resp. the *projectivization* $\widetilde{\mathcal{A}} = \mathcal{A} \cup L_\infty$), introduced in Section VI.1.

Module of logarithmic vector fields and central arrangements

For central plane arrangements $\mathcal{A}$ in $\mathbb{A}^3_k$, we have set up the study of the graded module $\mathcal{D}(\mathcal{A})$ in Section VI.3. Take $H_0 : z = 0$ and suppose that $H_0 \in \mathcal{A}$. Using Proposition VI.3.3, we can see that:

$$\mathcal{D}(\mathcal{A}) = \langle \chi_E \rangle \oplus \mathcal{D}_z(\mathcal{A}) = \langle \chi_E \rangle \oplus \mathcal{D}_0(\mathcal{A}),$$

where $\chi_E = x\partial_x + y\partial_y + z\partial_z$ is the Euler derivation and the modules

$$\begin{aligned}\mathcal{D}_z(\mathcal{A}) &= \mathrm{Ann}(\{z = 0\}) = \{\chi \in \mathcal{D}(\mathcal{A}) \mid \chi(z) = 0\} \\ &= \{P\partial_x + Q\partial_y + R\partial_z \in \mathcal{D}(\mathcal{A}) \mid R = 0\},\end{aligned}$$

and

$$\mathcal{D}_0(\mathcal{A}) = \mathrm{Ann}(\mathcal{A}) = \{\chi \in \mathcal{D}(\mathcal{A}) \mid \chi(\mathcal{Q}_\mathcal{A}) = 0\},$$

are isomorphic graded submodules. In particular, if $(\mathcal{D}_z(\mathcal{A}))_p$ and $(\mathcal{D}_0(\mathcal{A}))_p$ are the k–vector spaces of homogeneous derivations of degree p in each module, then $\dim(\mathcal{D}_z(\mathcal{A}))_p = \dim(\mathcal{D}_0(\mathcal{A}))_p$ for any $p \geq 0$.

Let $\mathcal{A} = \{L_1, \ldots, L_n\}$ be an affine line arrangement and consider its coning $\mathrm{c}\mathcal{A}$ in $\mathbb{A}^3_k$, where $\mathcal{Q}_{\mathrm{c}\mathcal{A}} = z\mathcal{Q}_\mathcal{A} \in k[x, y, z]$. A natural approach to the dynamics of $\mathrm{c}\mathcal{A}$ would be via the homogenization of the module of logarithmic of derivations of $\mathcal{A}$.

Let $\chi = P\partial_x + Q\partial_y \in \mathcal{D}(\mathcal{A})$ be a derivation in $\mathbb{A}^2_k$ of degree $d = \max\{\deg P, \deg Q\} \in \mathbb{N}$. We define an homogeneous derivation of degree d as

$$\hat{\chi} = z^d \cdot (P(x/z, y/z)\partial_x + Q(x/z, y/z)\partial_y)\,.$$

Let $\alpha_L = ax + by + c$ an affine form associated to a line $L \in \mathcal{A}$. As, $\chi \in \mathcal{D}(\mathcal{A}) = \bigcap_{L\in\mathcal{A}} \mathcal{D}(\alpha_L)$, this vector fields verifies that $\chi(\alpha_L) = K_L\alpha_L$ where K_L is a polynomial of degree $d-1$. Denoting $\bar{f} = z^{\deg f} f(x/z, y/z)$ the homogenization in the variable z of a polynomial $f \in k[x, y]$, we remark that that

$$\begin{aligned}\hat{\chi}(\bar{\alpha}_L) &= z^d \cdot (P(x/z, y/z)\partial_x\bar{\alpha}_L + Q(x/z, y/z)\partial_y\bar{\alpha}_L) \\ &= z^d \cdot (aP(x/z, y/z) + bQ(x/z, y/z)) \\ &= z^d \cdot K_L(x/z, y/z)\alpha_L(x/z, y/z) \\ &= \bar{K}_L\bar{\alpha}_L,\end{aligned}$$

since $\deg K_L = d - 1$. As $\partial_x(z\bar{\mathcal{Q}}_\mathcal{A}) = z\partial_x(\bar{\mathcal{Q}}_\mathcal{A})$ (resp. for ∂_y), for some $K \in k[x, y]$ we have:

$$\hat{\chi}(\mathcal{Q}_{\mathrm{c}\mathcal{A}}) = \hat{\chi}(z\mathcal{Q}_\mathcal{A}) = z\hat{\chi}(\mathcal{Q}_\mathcal{A}) = z\bar{K}\mathcal{Q}_\mathcal{A} = \bar{K}\mathcal{Q}_{\mathrm{c}\mathcal{A}}.$$

Reciprocally, we can obtain χ from $\hat{\chi}$ by deshomogenization, i.e. $\chi = \hat{\chi}_{|z=1}$. In fact, we have proved that:

Proposition VII.6.1. *The derivation χ of degree d belongs to $\mathcal{D}(\mathcal{A})$ if and only the homogeneous derivation $\hat{\chi}$ of degree d belongs to $\mathcal{D}_z(\mathrm{c}\mathcal{A})$.*

Similarly, we can construct a homogeneous derivation in $\mathcal{D}_0(\mathrm{c}\mathcal{A})$ from $\hat{\chi}$. Define:

$$\bar{\chi} = \left((n+1)\bar{P} - x\bar{K}\right)\partial_x + \left((n+1)\bar{Q} - y\bar{K}\right)\partial_y - z\bar{K}\partial_z \in \mathcal{D}_0(\mathrm{c}\mathcal{A}), \tag{VII.4}$$

since, using Euler's formula:

$$\bar{\chi}(\mathcal{Q}_{\mathrm{c}\mathcal{A}}) = \bar{K}\mathcal{Q}_{\mathrm{c}\mathcal{A}} = \frac{\bar{K}}{n+1}\chi_E(\mathcal{Q}_{\mathrm{c}\mathcal{A}}) \iff \bar{\chi}(\mathcal{Q}_{\mathrm{c}\mathcal{A}}) = 0.$$

This case is dynamically interesting as we have seen before, but the previous equation shows explicitly the role of the cofactor K inside the derivation.

Finally, we can see that our construction respects linear dependences between derivations.

Proposition VII.6.2. *Let $\{\chi_1, \ldots, \chi_r\} \subset \mathcal{D}(\mathcal{A})$ be a family of derivations of degree d, i.e. $d = \max\{\deg P_i, \deg Q_i\}$ for any $\chi_i = \{P_i\partial_x + Q_i\partial_y\}$. Then, the following properties are equivalents:*

1. *$\{\chi_1, \ldots, \chi_r\}$ are linearly independents.*
2. *$\{\bar{\chi}_1, \ldots, \bar{\chi}_r\}$ are linearly independents.*
3. *$\{\hat{\chi}_1, \ldots, \hat{\chi}_r\}$ are linearly independents.*

Proof. By construction using homogenization, we see that the linear independence condition in $\{\chi_1, \ldots, \chi_r\}$ and $\{\bar{\chi}_1, \ldots, \bar{\chi}_r\}$ are equivalents. Finally, as $\mathcal{D}_z(\mathcal{A})$ and $\mathcal{D}_0(\mathcal{A})$ are isomorphic graded modules, the result holds. □

Logarithmic bundles of projective line arrangements

As the polynomials of the coning and the projectivization of an affine line arrangement are equal up to unit in $k[x, y, z]$, we can use the previous construction to obtain sections of the logarithmic bundle of a line arrangement in $\mathbb{P}^2_k$ via sheafification of $\mathcal{D}_0(\mathrm{c}\mathcal{A})$.

Let $\mathcal{A} = \{L_1, \ldots, L_n\}$ be an affine line arrangement and consider its projectivization $\widetilde{\mathcal{A}} = \mathcal{A} \cup L_\infty$ in $\mathbb{P}^2_k$ previously described. Assume that $\chi = P\partial_x + Q\partial_y \in \mathcal{D}(\mathcal{A})$ in $\mathbb{A}^2_k$ is a minimal derivation of degree $d = \max\{\deg P, \deg Q\} \in \mathbb{N}$. Then, the logarithmic vector field $\chi = P\partial_x + Q\partial_z$ induces a section s_χ of $\mathcal{T}_{\widetilde{\mathcal{A}}}(d)$ corresponding to the homogeneous derivation $\bar{\chi}$ in (VII.4).

From a dynamical point of view, we can look at the relation between affine and projective line arrangements. Consider the singular points of $\chi = P\partial_x + Q\partial_y$, which are the points $p \in \mathbb{A}^2_k$ such that $P(p) = Q(p) = 0$. Assuming that $K(p) \neq 0$, one can see that the affine point $p = (u, v) \in \mathbb{A}^2_k$ is a singular point of χ if and only if $\bar{p} = [u : v : 1] \in \mathbb{P}^2_k$ is a fixed point of the projective meromorphic function given by the components of induced homogeneous $\bar{\chi}$:

$$\begin{array}{rccc} F: & \mathbb{P}^2_k & \longrightarrow & \mathbb{P}^2_k \\ & [x : y : z] & \longmapsto & [(n+1)\bar{P} - x\bar{K} : (n+1)\bar{Q} - y\bar{K} \ : -z\bar{K}] \end{array}$$

Note that we have supposed that the cofactor does not vanish in the singular point in order to have this result. This reinforces the idea that a complete understanding of the cofactor is necessary to study the dynamics. We refer to [IY08, Chp. 5] for more results about the dynamics of polynomial vector fields in $\mathbb{P}^2_{\mathbb{C}}$.

VII.7 Conclusion

The dynamical approach proposed in this Chapter lead us to give a new interpretation of classical algebraic objects like logarithmic vector fields. In the case of arrangements, the previous discussion induces questions on the dynamical behavior of logarithmic vector fields. In particular, at a more speculative level, it seems possible to look for a dynamical interpretation of freeness of the module of logarithmic vector fields.

In next Chapter, we begin this program in the case of line arrangements in the affine plane.

Chapter VIII

Dynamics of polynomial vector fields and Terao's conjecture in the affine plane

Let $\mathcal{A}$ be an affine line arrangement. The module of logarithmic derivations $\mathcal{D}(\mathcal{A})$ is usually studied from an algebraic point of view, but we have seen that it also has a dynamical interpretation: $\mathcal{D}(\mathcal{A})$ can be identified with the set of polynomial vector fields in $\mathbb{A}^2_k$ possessing $\mathcal{A}$ as invariant set. We use this point of view to study $\mathcal{D}(\mathcal{A})$ in the real or complex plane.

Focusing on arrangements and polynomial vector fields in the affine plane, we first outline a study of logarithmic derivations by degree introducing a filtration of $\mathcal{D}(\mathcal{A})$, in contrast with the graduation for central line arrangements.

Following a dynamical point of view, we give in Section VIII.3 a characterization of vectors fields in $\mathcal{D}(\mathcal{A})$ fixing infinite many lines in $\mathbb{A}^2_k$. We also study how the maximal multiplicity of singularities and maximal number of parallel lines in $\mathcal{A}$ give a lower bound for $d_f(\mathcal{A})$: the minimal degree of non-zero logarithmic derivations of $\mathcal{A}$ only fixing a finite set of lines.

We prove in Section VIII.4 that $d_f(\mathcal{A})$ is not determined by the number of lines and singularities counted by multiplicity (i.e. *weak combinatorics*) or by the intersection poset (i.e. *strong combinatorics*) of the line arrangement $\mathcal{A}$, using two explicit counter-examples constructed from Ziegler and Pappus arrangements.

Finally, we study in Section VIII.5 a quadratic formula for the ranks in the filtration of $\mathcal{D}(\mathcal{A})$, and how this formula is determined by the combinatorics of the line arrangement.

This chapter is developed over a joint work with B. Guerville-Ballé and a part of the results can be found in [GBVS15a]: *"Combinatorics of line arrangements and dynamics of polynomial vector fields"*, B. Guerville-Ballé and J. Viu-Sos, `arXiv:1412.0137 [math.DS]`, 2015.

VIII.1 Introduction

The influence of combinatorics of line arrangements over the properties of their realizations into different ambient spaces (affine and projective planes over fields of any characteristic) is largely studied, e.g. [Arn69], [OS80], [Ryb11]. Our work consists in the study of the relationship between $\mathcal{A}$, the poset $L(\mathcal{A})$, and the module $\mathcal{D}(\mathcal{A})$, in order to understand the influence of the combinatorial structure on the minimal degree of vector fields in $\mathcal{D}(\mathcal{A})$ and their corresponding dynamics in the real or complex plane.

As a first step, a characterization of the polynomial vector fields admitting only a finite number of invariant lines is required. Then we investigate the minimal degree $d_f(\mathcal{A})$ of logarithmic

vector fields of this kind. We obtain lower bounds of this minimal degree depending only on the combinatorics of $\mathcal{A}$. Finally, we prove that even if $d_f(\mathcal{A})$ admits a combinatorial lower bound, it is not determined by the intersection poset $L(\mathcal{A})$.

This approach contrasts with the classical ones given in dynamical systems in the study of invariant lines in systems of low fixed degree by Llibre et al. and Xiang ([LV06],[ZY98]). We are also far away from a more algebraic point of view, as in the works of Abe, Vallès and Faenzi ([FV12a],[FV12b],[AFV14]), in terms of logarithmic bundles on the complex projective plane.

Following this approach, we expect to give an interpretation in the real or complex plane of Terao's conjecture (Conjecture VI.3.10), which asks about the combinatoriality of freeness for arrangements. It is difficult to identify the specific role of the freeness condition in relation with the combinatoric information of the arrangement in the conjecture. In the case of affine line arrangements in the real or complex plane, we have a strong constraint: any module of logarithmic derivations of an affine line arrangement is free.

Theorem VIII.1.1 (Quillen-Suslin ([Qui76],[Sus76])**.** *Let k be a field of characteristic zero. Any finitely-presented projective module over $k[X_1, \ldots, X_d]$ is free.*

Corollary VIII.1.2. *Let $\mathcal{A}$ be an affine line arrangement in $\mathbb{A}^2_k$. Then $\mathcal{D}(\mathcal{A})$ is a free module.*

This comes from the fact that a reflexive module is projective over two variables, and $\mathcal{D}(\mathcal{A}) = \Omega^1(\mathcal{A})^* = \mathcal{D}(\mathcal{A})^{**}$ as we have seen in Theorem VI.2.14. Thus $\mathcal{D}(\mathcal{A})$ is projective then free.

In this way, a natural question arises: *is the module of logarithmic derivations of an affine line arrangement $\mathcal{A}$ determined by $L(\mathcal{A})$, as in the case of free central hyperplane arrangements?* In this Chapter, we give a negative answer to these question by our main result:

Theorem VIII.1.3. *The module of logarithmic derivations of $\mathcal{A}$ in the real of complex plane is not determined by the combinatorial information of the line arrangement.*

The proof of this theorem is based on the result announced in [GBVS15b]:

Theorem VIII.1.4. *There exists two real line arrangements $\mathcal{Z}_1$ and $\mathcal{Z}_2$ such that $L(\mathcal{Z}_1) \simeq L(\mathcal{Z}_2)$ and $\mathcal{D}(\mathcal{Z}_1) \not\simeq \mathcal{D}(\mathcal{Z}_2)$.*

VIII.2 Logarithmic vector fields in the plane

VIII.2.1 Planar vector fields and logarithmic derivations

Let V be a 2-dimensional k–vector space, for $k = \mathbb{R}$ or $\mathbb{C}$. Let $S = \mathrm{Sym}(V^*)$ be the symmetric algebra of the dual space of V. Taking $\{x, y\}$ the dual basis of the canonical one in V, we may identify S with $k[x, y]$.

Let $\mathcal{A}$ be an affine line arrangement in $\mathbb{A}^2_k$. Recall that the defining polynomial of $\mathcal{A}$ is given by $\mathcal{Q}_\mathcal{A} = \prod_{L\in\mathcal{A}} \alpha_L$, where $\alpha_L = ax + by + c$ are affine forms such that $L = \alpha_L^{-1}(0)$. Let $\mathrm{Der}_k(k[x, y])$ be the algebra of k-derivations of $k[x, y]$. Taking Definition VI.2.4 in the plane, the *module of logarithmic derivations of* $\mathcal{A}$ is the $k[x, y]$-module defined by:

$$\mathcal{D}(\mathcal{A}) = \{\chi \in \mathrm{Der}_k(k[x, y]) \mid \chi\mathcal{Q}_\mathcal{A} \in \mathcal{I}_{\mathcal{Q}_\mathcal{A}}\}$$

where $\mathcal{I}_{\mathcal{Q}_\mathcal{A}}$ is the ideal generated by $\mathcal{Q}_\mathcal{A}$. We also have that $\mathcal{D}(\mathcal{A}) = \bigcap_{L\in\mathcal{A}}\{\chi \in \mathrm{Der}_k(k[x, y]) \mid \chi\alpha_L \in \mathcal{I}_{\alpha_L}\}$, where $\mathcal{I}_{\alpha_L}$ is the ideal generated by α_L.

Consider a planar polynomial differential system defined for $(x, y) \in \mathbb{A}_k^2$ by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = P(x, y) \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = Q(x, y)$$

where $P, Q \in k[x, y]$. This globally defined autonomous system is associated to a polynomial vector field in the plane given by

$$\chi = P\partial_x + Q\partial_y \tag{VIII.1}$$

Following the language of dynamical systems, a polynomial vector field χ is considered of *degree* d if $\max\{\deg P, \deg Q\} = d$.

Since $\mathrm{Der}_k(k[x, y])$ is in correspondence with polynomials vector fields on the plane, we obtain a dynamical interpretation of $\mathcal{D}(\mathcal{A})$:

Lemma VIII.2.1. *Let $\mathcal{A}$ be an arrangement and $\chi \in \mathrm{Der}_k(k[x, y])$. Then $\chi \in \mathcal{D}(\mathcal{A})$ if and only if $\mathcal{A}$ is invariant by χ.*

Remark VIII.2.2. A set $X \subset \mathbb{A}_k^2$ is invariant by χ if $\phi_t(X) \subset X$, for every $t \in \mathbb{R}$, where $\phi_t : \mathbb{A}_k^2 \to \mathbb{A}_k^2$ is the flow associated to χ at instant t.

In the case of real line arrangements, the required condition for a derivation to belong to $\mathcal{D}(\mathcal{A})$ is equivalent to the definition of algebraic invariant sets in complex dynamical systems: a complex algebraic curve $\mathcal{C} = \{f = 0\}$ is invariant by a polynomial vector field χ if there exists $K \in \mathbb{C}[x, y]$ such that $\chi f = Kf$ (see [DLA06]).

VIII.2.2 Automorphisms of the plane and filtrations of $\mathcal{D}(\mathcal{A})$

As we are studying affine line arrangements which are no necessarily central, $\mathcal{D}(\mathcal{A})$ is no more a graded module by the degree of homogeneous components of the logarithmic derivations. Thus, we introduce the notion of *filtration by degree* for $\mathcal{D}(\mathcal{A})$.

Consider $k[x, y]_d$ the k–vector subspace of $k[x, y]$ consisting of 0 and polynomials of degree at most d. Define $k[x, y]_d = 0$ for $d < 0$. Then

$$k[x, y] = \bigcup_{d \in \mathbb{Z}} k[x, y]_d$$

is a filtered k–algebra.

The notion of degree of polynomial vector fields gives a natural filtration of the module of derivations.

$$\mathrm{Der}_k(k[x, y]) = \bigcup_{d \in \mathbb{Z}} \mathrm{Der}_k(k[x, y]_d),$$

where $\mathrm{Der}_k(k[x, y]_d) = \{P\partial_x + Q\partial_y \mid \deg P, \deg Q \leq d\}$. Note that $\mathrm{Der}_k(k[x, y]_d) = 0$, for $d < 0$. Restricting to the module of derivations, we obtain an *ascending filtration* of $\mathcal{D}(\mathcal{A})$ by the vector spaces

$$\mathcal{F}_d\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}) \cap \mathrm{Der}_k(k[x, y]_d)$$

Definition VIII.2.3. We denote by $\mathcal{D}_d(\mathcal{A}) = \mathcal{F}_d\mathcal{D}(\mathcal{A}) \setminus \mathcal{F}_{d-1}\mathcal{D}(\mathcal{A})$ the set of *polynomial vector fields of degree d fixing $\mathcal{A}$*.

Our purpose is to study the module of logarithmic derivations of $\mathcal{A}$ by this filtration. Let $\mathrm{Aut}[\mathbb{A}_k^2]$ be the group of algebraic automorphisms of the plane.

Proposition VIII.2.4. *Let $\mathcal{A}$ be an affine line arrangement such that $|\operatorname{Sing}\mathcal{A}| \geq 1$, and let $\phi \in \mathrm{Aut}[\mathbb{A}_k^2]$. Then $\phi^{-1}(\mathcal{A})$ is a line arrangement if and only if ϕ is an affine transformation.*

Proof. First, we remark that any affine automorphism of $\mathbb{A}^2_k$ transforms lines into lines, thus line arrangements into line arrangements.

Let $\phi = (f, g) \in \mathrm{Aut}[\mathbb{A}^2_k]$ such that $\phi^{-1}(\mathcal{A})$ is a line arrangement. Up to affine transformation, we can consider that one of the singularities in $\mathrm{Sing}\,\mathcal{A}$ is the origin and that there exist two lines $L_1, L_2 \in \mathcal{A}$ of the form:

$$L_1 : x = 0 \quad \text{and} \quad L_2 : y = 0.$$

Thus, we obtain that $\phi^{-1}(L_1) = \{f(x, y) = 0\}$ and $\phi^{-1}(L_2) = \{g(x, y) = 0\}$ are both lines. Then $\deg(f) = \deg(g) = 1$. □

Remark VIII.2.5. In the statement of the previous Proposition, note that the assumption of $|\,\mathrm{Sing}\,\mathcal{A}| \geq 1$ is fundamental. If we consider $\mathcal{A}$ the line arrangement composed by three parallel lines $L_1 : x - 1 = 0$, $L_2 : x = 0$ and $L_3 : x + 1 = 0$, it is clear that an automorphism in $\mathbb{A}^2_k$ given by $(x, y) \mapsto (x, y + x^2)$ preserves the arrangement $\mathcal{A}$.

Thus, any k–algebraic automorphisms of $k[x, y]$ which respects geometrically line arrangements must respect the degree of polynomials in $k[x, y]$. This fact justifies the study of $\mathcal{D}(\mathcal{A})$ by filtration of degree of the polynomial components, which is respected by affine transformations in $\mathbb{A}^2_k$. Furthermore, as we seen in the last Chapter, the following inclusion of modules holds:

$$\mathcal{Q}_{\mathcal{A}}\,\mathrm{Der}_k(k[x, y]) \subset \mathcal{D}(\mathcal{A}) \subset \mathrm{Der}_k(k[x, y])$$

In this way, in a first time we are interested to study $\mathcal{D}_d(\mathcal{A})$ for $0 < d < |\mathcal{A}|$.

VIII.2.3 Geometry of logarithmic vector fields

We begin with a necessary and sufficient condition on a line to be invariant by a polynomial vector field.

Proposition VIII.2.6. *Let L be a line of $\mathbb{A}^2_k$ defined by the equation $f = \alpha x + \beta y + \gamma = 0$, and let $\chi = P(x, y)\partial_x + Q(x, y)\partial_y$ be a polynomial vector field on $\mathbb{A}^2_k$. The line L is invariant for χ if and only if we are in one of the following cases:*

1. *$\beta = 0$ and $P(-\gamma/\alpha, y) = 0$,*
2. *$\beta \neq 0$ and $\alpha P(\beta y, -\alpha y - \gamma/\beta) + \beta Q(\beta y, -\alpha y - \gamma/\beta) = 0$.*

Proof. It is easy to check that the vertical line $L_0 = \{x = 0\}$ is invariant by χ if and only if $P(0, y) = 0$. In order to obtain the result, we make an affine transformation of the plane $\varphi : \mathbb{A}^2_k \to \mathbb{A}^2_k$ such that L is sent on the line L_0. The vector field χ can be seen as a section of the tangent bundle $T(\mathbb{A}^2_k)$ and we denote by χ_φ the pushforward of χ by φ.

$$\begin{array}{ccc} T(\mathbb{A}^2_k) & \xrightarrow{\varphi_*} & T(\mathbb{A}^2_k) \\ \chi \uparrow\downarrow & & \uparrow\downarrow \chi_\varphi \\ \mathbb{A}^2_k & \xrightarrow{\varphi} & \mathbb{A}^2_k \end{array}$$

Hence, $L = \{f = 0\}$ is invariant by χ if and only if $L_0 = \{x = 0\}$ is invariant by χ_φ.

Assume $\beta = 0$, thus L is vertical and only a translation $\varphi^{-1}(x, y) = (x - \gamma/\alpha, y)$ is needed:

$$\chi_\varphi = P(x - \gamma/\alpha, y)\partial_x + Q(x - \gamma/\alpha, y)\partial_y$$

If $\beta \neq 0$, we consider $\varphi^{-1}(x, y) = (\alpha x + \beta y, \beta x - \alpha y - \gamma/\beta)$ and we obtain:

$$\begin{aligned}\chi_\varphi = [\alpha P(\alpha x + \beta y, \beta x - \alpha y - \gamma/\beta) + \beta Q(\alpha x + \beta y, \beta x - \alpha y - \gamma/\beta)]\,\partial_x + \\ [\beta P(\alpha x + \beta y, \beta x - \alpha y - \gamma/\beta) - \alpha Q(\alpha x + \beta y, \beta x - \alpha y - \gamma/\beta)]\,\partial_y.\end{aligned}$$

Clearly, L_0 is invariant by χ_φ if and only if the coordinate of ∂_x in $\chi_\varphi(0, y)$ is zero. This implies the result. □

Remark VIII.2.7. Considering the vector field $\chi = P(x, y)\partial_x + Q(x, y)\partial_y$ of degree d defined by generic polynomials

$$P(x, y) = \sum_{i+j\leq d} a_{i,j}x^i y^j \quad \text{and} \quad Q(x, y) = \sum_{i+j\leq d} b_{i,j}x^i y^j, \tag{VIII.2}$$

with coefficients in k, we can express the LHS of the equation of Proposition VIII.2.6 case (2), as a univariate polynomial in $k[y]$ in terms of P and Q:

$$R(y) = P(\beta y, -\alpha y - \gamma/\beta) + \beta Q(\beta y, -\alpha y - \gamma/\beta)$$

Thus, in the case $\beta \neq 0$ we can verify that the equation $R(y) = 0$ is equivalent to the system composed by:

$$0 = \text{Coeff}_{y^m} R(y) = \sum_{k=0}^{d-m}\sum_{l=0}^{m}(\alpha a_{m-l,k+l} + \beta b_{m-l,k+l}) \cdot \binom{k+l}{k} \cdot (-1)^{k+l}\alpha^l\beta^{m-k-l}\gamma^k \tag{E_m}$$

for every $0 \leq m \leq \deg R$. A similar condition can be obtained for case (1).

Consider $\mathcal{C}(d)$ the k-linear space of coefficients of a pair of polynomials of degree less or equal than d, plotted in equation (VIII.2). We have $\mathcal{C}(d) = k^{(d+1)(d+2)/2} \oplus k^{(d+1)(d+2)/2} \simeq k^{(d+1)(d+2)}$. Fixing a line arrangement $\mathcal{A}$, we get by Proposition VIII.2.6 and Remark VIII.2.7 that the equations defining $\mathcal{F}_d\mathcal{D}(\mathcal{A})$ are linear in the coefficients of P and Q, thus we can compute $\mathcal{F}_d\mathcal{D}(\mathcal{A})$ as kernel of a linear map $\Phi : \mathcal{C}(d) \to k^{n(d+1)}$, where $|\mathcal{A}| = n$.

Theorem VIII.2.8 (Structure of polynomial vector fields). *Let $\mathcal{A}$ be a line arrangement. For each $d \in \mathbb{N}$, $\mathcal{F}_d\mathcal{D}(\mathcal{A})$ is a linear sub-space of the space of coefficients $\mathcal{C}(d)$.*

VIII.3 Finiteness of derivations and combinatorial data

VIII.3.1 Finiteness of fixed families of lines

In order to efficiently characterize line arrangements as invariant sets of a polynomial vector field, the first step is to obtain conditions on the finiteness of the family of invariant lines under a vector field. This leads us to the notion of maximal line arrangement fixed by a polynomial vector field.

Definition VIII.3.1. Let χ be a polynomial vector field in the plane. We said that a line arrangement $\mathcal{A}$ is *maximal fixed by* χ if any line $L \subset \mathbb{A}^2_k$ invariant by χ belongs to $\mathcal{A}$.

Remark VIII.3.2. The notion of line arrangement is taken considering a finite collection of lines. Thus, there exist polynomial vector fields in the plane for which there not exists a maximal line arrangement fixed by them: the null vector field is a trivial example, as well as a "radial" vector field $\chi_c = x\partial_x + y\partial_y$ or a "parallel" vector field $\chi_p = (x+1)\partial_y$, pictured in Figure VIII.1.In Theorem VIII.3.11 we prove that the derivations which have not maximal fixed arrangements are essentially of these types.

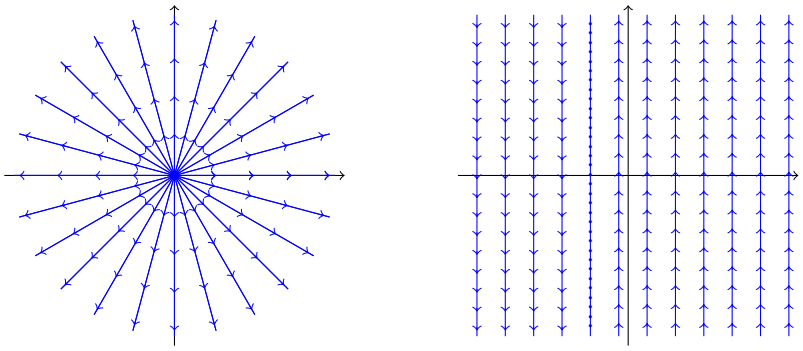

Figure VIII.1: Phase portraits of $\chi_c = x\partial_x + y\partial_y$ and $\chi_p = (x+1)\partial_y$, respectively.

Definition VIII.3.3. We said that χ *fixes only a finite set of lines* if there exists a maximal arrangement fixed by χ. Conversely, we said that χ *fixes infinitely many lines* if there is no such maximal line arrangement.

We consider the partition $\mathcal{D}(\mathcal{A}) = \mathcal{D}^{\infty}(\mathcal{A}) \sqcup \mathcal{D}^{f}(\mathcal{A})$ following this notion, where $\mathcal{D}^{f}(\mathcal{A})$ and $\mathcal{D}^{\infty}(\mathcal{A})$ are the sets of elements in $\mathcal{D}(\mathcal{A})$ fixing only a finite set of lines and fixing infinitely many lines, respectively. We are interested in the study of this notion in the filtration by degree previously defined, denoting $\mathcal{D}^{\infty}_{d}(\mathcal{A}) = \mathcal{D}_{d}(\mathcal{A}) \cap \mathcal{D}^{\infty}(\mathcal{A})$ and $\mathcal{D}^{f}_{d}(\mathcal{A}) = \mathcal{D}_{d}(\mathcal{A}) \cap \mathcal{D}^{f}(\mathcal{A})$, for $d \in \mathbb{N}$. We denote by $d_f(A)$ the minimal degree $d \in \mathbb{N}$ for which $\mathcal{D}^{f}_{d}(\mathcal{A})$ is not empty.

Remark VIII.3.4. The understanding of $\mathcal{D}^{\infty}_{d}(\mathcal{A})$ and $\mathcal{D}^{f}_{d}(\mathcal{A})$ is precisely what is needed to obtain an intrinsic formulation in the article of Llibre at al. [AGL98]: the polynomial vector field under consideration in any statement belongs to $\mathcal{D}^{f}_{d}(\mathcal{A})$ by hypothesis (see for example Proposition 6 or Theorem 7 in [AGL98]).

Property VIII.3.5. *The sets $\mathcal{D}^{\infty}(\mathcal{A})$ and $\mathcal{D}^{f}(\mathcal{A})$ are closed by multiplication of non-zero elements of $k[x,y]$, i.e. for any $F \in k[x,y] \setminus 0$*

$$F \cdot \mathcal{D}^{\infty}(\mathcal{A}) \subset \mathcal{D}^{\infty}(\mathcal{A}) \quad \textit{and} \quad F \cdot \mathcal{D}^{f}(\mathcal{A}) \subset \mathcal{D}^{f}(\mathcal{A}).$$

Proof. Let $\chi \in \mathcal{D}(\mathcal{A})$ and $F \in k[x,y]$ be a non-zero polynomial of degree n. If α_L is an affine form of a line L fixed by χ, then the condition $\chi(\alpha_L) \in \mathcal{I}_{\alpha_L}$ implies $F\chi(\alpha_L) \in \mathcal{I}_{\alpha_L}$. So any line fixed by χ is trivially fixed by χ.

Reciprocally, consider L a line which is not fixed by χ but fixed by $F\chi$. Since $\chi(\alpha_L) \notin \mathcal{I}_{\alpha_L}$ and $F\chi(\alpha_L) \in \mathcal{I}_{\alpha_L}$, we deduce that α_L divides F.

Thus, if χ admit a fixed maximal line arrangement $\mathcal{A}' \supset \mathcal{A}$ but we assume that $F\chi$ fixes infinitely many lines, there exist a family of $d+1$ different lines $\{L_1, \ldots, L_{n+1}\} \not\subset \mathcal{A}'$ fixed by $F\chi$. Following the previous argument, any affine factor $\alpha_{L_1}, \ldots, \alpha_{L_{d+1}}$ divides F. Thus $\prod_{i=1}^{d+1} \alpha_{L_i}$ also divides F since the lines are different. This gives a contradiction. □

In order to determine the elements of $\mathcal{D}^{\infty}(\mathcal{A})$, we introduce a geometrical characterization for vector fields with fix an infinity family of lines.

Definition VIII.3.6. A non-null vector field χ is said to be *radial* if there is a point $(x_0, y_0) \in \mathbb{A}_k^2$ such that $(x - x_0, y - y_0)$ and $\big(P(x,y), Q(x,y)\big)$ are collinear vectors, for any $(x,y) \in \mathbb{A}_k^2$. Otherwise, χ is said to be *parallel* if there is a $\vec{v} \in \mathbb{A}_k^2$ such that $\vec{v}$ and $\big(P(x,y), Q(x,y)\big)$ are collinear vectors, for any $(x,y) \in \mathbb{A}_k^2$.

Remark VIII.3.7. There is no vector field simultaneously radial and parallel.

Let us present a first result relating the combinatorics of an arrangement and the nature of the vector fields in $\mathcal{D}_d(\mathcal{A})$. Let $m(\mathcal{A})$ be the maximal multiplicity of singular points of $\mathcal{A}$, and let $p(\mathcal{A})$ be the maximal number of parallel lines of the same slope in $\mathcal{A}$. Note that the triplet $(|\mathcal{A}|, m(\mathcal{A}), p(\mathcal{A}))$ is combinatorial, i.e. can be obtained from the weak combinatorics of $\mathcal{A}$.

Theorem VIII.3.8. *Let $\mathcal{A}$ be a line arrangement and define $\nu_\infty(\mathcal{A}) = \max\{m(\mathcal{A}) - 1, p(\mathcal{A})\}$. If $d < \nu_\infty(\mathcal{A})$, then $\mathcal{D}_d(\mathcal{A})$ and $\mathcal{D}_d^\infty(\mathcal{A})$ are equal.*

This theorem holds directly from the following result:

Proposition VIII.3.9. *Let $\mathcal{A}$ be an arrangement and suppose that there exist a $\chi \in \mathcal{D}_d(\mathcal{A})$.*

1. *If $d < m(\mathcal{A}) - 1$ then χ is a radial vector field and $\chi \in \mathcal{D}_d^\infty(\mathcal{A})$.*
2. *If $d < p(\mathcal{A})$ then χ is a parallel vector field and $\chi \in \mathcal{D}_d^\infty(\mathcal{A})$.*

Proof. We decompose this proof in two cases.

First, suppose $d + 1 < m(\mathcal{A})$. Up to an affine transformation, we may assume that the singular point $p \in \mathbb{A}_k^2$ of multiplicity at least $d+2$ of $\mathcal{A}$ is the origin and that there is no vertical lines in $\mathcal{A}$. Let $L_i : \alpha_i x + y = 0$ be the $d + 2$ lines passing by point p. Proposition VIII.2.6 implies that for all $i = 1, \ldots, d + 2$, we have

$$\alpha_i P(y, -\alpha_i y) + Q(y, -\alpha_i y) = \sum_{n=0}^{d} \Big(\sum_{j=0}^{n} \big(\alpha_i a_{n-j,j} + b_{n-j,j}\big)(-\alpha_i)^j \Big) y^n = 0,$$

which is equivalent to the system of $(d + 2)(d + 1)$ equations defined, for any $n = 0, \ldots, d$ and $i = 1, \ldots, d + 1$, by

$$\sum_{j=0}^{n} \big(\alpha_i a_{n-j,j} + b_{n-j,j}\big)(-\alpha_i)^j = 0 \qquad (Eq_{(n,j)})$$

We regroup them in $d + 1$ systems S_n formed by the $d + 2$ equations (indexed by i). These equations are polynomial of degree $n + 1$ in α_i. We denote by c_k the coefficient of α^k, that is $c_0 = b_{n,0}$, $c_{n+1} = a_{0,n}$ and $c_k = (-1)^k b_{n-k,k}(-1)^{k+1} a_{n-k+1,k-1}$ for $k = 1, \ldots, n$. If we restrict the system S_n to their $n+2$ first equations, then we remark that the square system in c_k obtained is in fact a Vandermonde system. Since all the α_i are distinct then the system admits a unique solution $c_k = 0$. This implies that $a_{0,n} = 0$, $b_{n,0} = 0$ and $a_{k,d-k} = b_{k-1,d-k+1}$ for $k = 1, \ldots, d$. Thus we have $yP(x,y) = xQ(x,y)$, which is a radial vector field.

In a second case, assume that $d < p(\mathcal{A})$ hence $\mathcal{A}$ has at least $d + 1$ parallel lines. Then, without lost of generality, we may assume that these lines are vertical. Let $y_0 \in k$, from Proposition VIII.2.6 we have that $P(x, y_0) = 0$ for $d + 1$ different values of x. Since P is a polynomial of degree less or equal than d, then $P(x,y) = 0$ and χ fixes all the vertical lines. □

Following this study of the appearance of elements in $\mathcal{D}(\mathcal{A})$ by degree, we can give a first bound for $d_f(\mathcal{A})$ in terms of combinatorics of the line arrangement.

Corollary VIII.3.10. *Let $\mathcal{A}$ be a line arrangement in $\mathbb{A}_k^2$, then $d_f(\mathcal{A}) \geq \nu_\infty(\mathcal{A})$.*

VIII.3.2 Characterization of elements in $\mathcal{D}^{\infty}(\mathcal{A})$

In Definition VIII.3.6, we have introduced some classes of vector fields fixing an infinity of lines, defined from a geometric point of view. We prove that any element of $\mathcal{D}^{\infty}(\mathcal{A})$ is essentially of this kind of vector fields.

Theorem VIII.3.11. *Let χ be a polynomial vector field fixing an infinity of lines, then χ is either null, radial or parallel.*

Remark VIII.3.12. The nature of the previous Theorem is purely algebraic. A fixed line which is not in a projective pencil defined by a radial or parallel affine vector field necessarily corresponds to a root in the polynomial coefficients of the vector field after a suitable projection. In the analytic category, we can find analytic vector fields which possess a countably infinite set of invariant lines, an example of those vector fields is represented in Figure VIII.2.

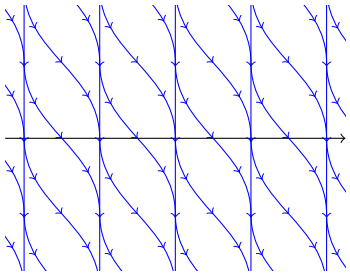

Figure VIII.2: Phase portrait of the analytic vector field $\chi = \sin^2(x)\partial_x - \partial_y$.

The proof of the previous Theorem is based on the following lemma, about the number of singular points in a collection of a countable infinity of lines.

Lemma VIII.3.13. *Let $\mathcal{A}_{\infty} = \{L_1, L_2, L_3, \ldots\}$ be an infinite countable collection of distinct lines in the plane, then we have:*

$$\#\operatorname{Sing}(\mathcal{A}_{\infty}) \in \{0, 1, \infty\}.$$

Proof. We decompose the proof by cases:

1. If all the lines of $\mathcal{A}_{\infty}$ are parallel, then $\#\operatorname{Sing}(\mathcal{A}_{\infty}) = 0$.
2. If all the lines of $\mathcal{A}_{\infty}$ are concurrent, then $\#\operatorname{Sing}(\mathcal{A}_{\infty}) = 1$.
3. If $\#\operatorname{Sing}(\mathcal{A}_{\infty}) \geq 2$, we prove by recurrence that:

$$\forall n \geq 2, \exists k \in \mathbb{N}^*, \text{ s.t. } \#\operatorname{Sing}(\mathcal{A}_k) \geq n,$$

where $\mathcal{A}_i = \{L_1, L_2, \ldots, L_i\}$. It is obviously true for $n = 2$. Assume that it is true for rank n. Since $\mathcal{A}_n \subset \mathcal{A}_{n+1}$, then $\#\operatorname{Sing}(\mathcal{A}_{n+1}) \geq \#\operatorname{Sing}(\mathcal{A}_n)$, with equality if $L_{n+1} \cap \mathcal{A}_n \subset \operatorname{Sing}(\mathcal{A}_n)$ (in other terms if L_{n+1} only passes through singular points of $\mathcal{A}_n$). Since there is only a finite number of alignment of points of $\operatorname{Sing}(\mathcal{A}_n)$ then there is an integer k such that $L_{n+k} \cap \mathcal{A}_n \nsubseteq \operatorname{Sing}(\mathcal{A}_n)$. We obtain:

$$\#\operatorname{Sing}(\mathcal{A}_{n+k}) > \#\operatorname{Sing}(\mathcal{A}_n).$$

□

Proof of Theorem VIII.3.11. Let $P(x,y)$ and $Q(x,y)$ be such that $\chi = P\partial_x + Q\partial_y$. We define $\mathcal{A}_\infty = \{L_1, L_2, L_3, \ldots\}$ the set (or a subset) of different lines fixed by χ, and we denote by α_i the equation of L_i. In all what follows, we assume that we are not in the first case (i.e. $(P,Q) \neq (0,0)$).

The vector field χ fixes only a finite number of lines of $\mathcal{A}_\infty$ point by point. Indeed, L_i is fixed point by point by χ if and only if $\alpha_i \mid P$ and $\alpha_i \mid Q$. Since P and Q are polynomials then they have finite degree, and only a finite number of α_i can divide them. Assume that these lines are $L_1, \ldots, L_k$.

Denote by $\chi' = P'\partial_x + Q'\partial_y$ the derivation of components $P' = P/(\alpha_1 \cdots \alpha_k)$ and $Q' = Q/(\alpha_1 \cdots \alpha_k)$. It is clear that χ and χ' are collinear vector fields. In this way, if χ' is radial (resp. parallel) then χ is radial (resp. parallel). By construction, the set of points fixed by χ' (i.e. the common zeros of P' and Q') contains the intersection points of $\mathcal{A}'_\infty = \mathcal{A} \setminus \{L_1, \ldots, L_k\}$. By Lemma VIII.3.13 we have 3 possible cases:

1. $\#\operatorname{Sing}(\mathcal{A}'_\infty) = 0$, then all the lines of $\mathcal{A}'_\infty$ are parallel. By Proposition VIII.3.9 χ' is a parallel vector field.

2. $\#\operatorname{Sing}(\mathcal{A}'_\infty) = 1$, then all the lines of $\mathcal{A}'_\infty$ are concurrent. By Proposition VIII.3.9 χ' is a radial vector field.

3. $\#\operatorname{Sing}(\mathcal{A}'_\infty) = \infty$, then the polynomial P' and Q' have an infinity of zero, which is impossible since P' and Q' are not simultaneously null.

□

VIII.3.3 Influence of the combinatorics in $\mathcal{D}_d^\infty(\mathcal{A})$: a minimal bound

The dynamical/geometrical characterization of elements in $\mathcal{D}^\infty(\mathcal{A})$ obtained in Theorem VIII.3.11 allows us to identify and construct them explicitly. Using this, we determine combinatorially the minimal degree from which $\mathcal{D}^\infty(\mathcal{A})$ is not empty.

Theorem VIII.3.14. *Let $\mathcal{A}$ be a line arrangement and define $\nu_f(\mathcal{A}) = \min\{|\mathcal{A}| - m(\mathcal{A}) + 1, |\mathcal{A}| - p(\mathcal{A})\}$. Then, $0 < d < \nu_f(\mathcal{A})$ if and only if $\mathcal{D}_d(\mathcal{A})$ and $\mathcal{D}_d^f(\mathcal{A})$ are equal.*

In order to prove this result, we study the minimal degree of radial or parallel vector fields fixing $\mathcal{A}$ (presented in Proposition VIII.3.15 and Proposition VIII.3.16), to show that we are able to construct explicit elements of $\mathcal{D}^\infty(\mathcal{A})$ of degree greater than or equal to $\nu_f(\mathcal{A})$.

Proposition VIII.3.15. *There exist a radial vector field in $\mathcal{D}_d(\mathcal{A})$ if and only if $d \geq |\mathcal{A}| - m(\mathcal{A}) + 1$. Moreover, any radial vector field of minimal degree in $\mathcal{D}(\mathcal{A})$ is of the form*

$$\chi_p = [(x - p_1)\partial_x + (y - p_2)\partial_y] \cdot \prod_{\substack{L \in \mathcal{A} \\ L \not\ni p}} \alpha_L,$$

for some $p = (p_1, p_2) \in \operatorname{Sing}\mathcal{A}$ of multiplicity $m(\mathcal{A})$.

Proof. Let $\chi = P\partial_x + Q\partial_y \in \mathcal{D}_d(\mathcal{A})$ be a radial vector field. A line $L = \alpha_L^{-1}(0)$ is invariant by χ if we are in one of the following cases:

1. L passes through the center of the vector field.

2. α_L divides both P and Q.

The second condition is the most expensive in terms of degree. To minimize this condition, we maximize the first one. Without loss of generality, we may assume that the origin is a singular point of maximal multiplicity. Consider $\mathcal{A}' \subset \mathcal{A}$ the sub-arrangement composed by lines of $\mathcal{A}$ which does not pass by the origin, we have $P(x,y) = \mathcal{Q}_{\mathcal{A}'}p(x,y)$, and $Q(x,y) = \mathcal{Q}_{\mathcal{A}'}q(x,y)$, with p and q such that $yp - xq = 0$. The only polynomials of minimal degree verifying this condition are $p(x,y) = x$ and $q(x,y) = y$. Hence, the result holds. □

Proposition VIII.3.16. *There exist a parallel vector field in $\mathcal{D}_d(\mathcal{A})$ if and only if $d \geq |\mathcal{A}| - p(\mathcal{A})$. Moreover, any parallel vector field of minimal degree in $\mathcal{D}(\mathcal{A})$ is of the form*

$$\chi_{\mathcal{P}} = [\lambda\partial_x + \mu\partial_y] \cdot \prod_{\substack{L\in\mathcal{A}\\ L\notin\mathcal{P}}} \alpha_L,$$

for some $\mathcal{P} \subset \mathcal{A}$ be family of exactly $p(\mathcal{A})$ parallel lines of direction vector $\vec{v} = (\lambda, \mu)$.

Proof. Let $\chi = P\partial_x + Q\partial_y \in \mathcal{D}_d(\mathcal{A})$ be a parallel vector field. A line $L = \alpha_L^{-1}(0)$ is invariant by χ if:

1. L is parallel to χ,
2. α_L divides both P and Q.

Once again, the second condition is the most expensive in terms of degree and we maximize the first one in degree. Consider $\mathcal{A}' \subset \mathcal{A}$ the sub-arrangement composed by lines of $\mathcal{A}$ which are not parallel to the vector field, then $\mathcal{Q}_{\mathcal{A}'}$ divides both P and Q. Thus, the vector field $\chi' = \mathcal{Q}_{\mathcal{A}'}(\partial_x + \partial_y)$ is a vector field of minimal degree fixing $\mathcal{A}$, collinear to the vector $(1,1)$, and the result holds. □

Remark VIII.3.17. Note that parallelism in an affine line arrangement is of combinatorial nature. Let $L_0 \in \mathcal{A}$ be a fixed line, then

$$L \in \mathcal{A} \text{ is parallel to } L_0 \iff \nexists p \in \operatorname{Sing}\mathcal{A} : \ L_0, L \leq p$$

Corollary VIII.3.18. *Let $\mathcal{A}$ be a line arrangement and define $\nu_f(\mathcal{A}) = \min\{|\mathcal{A}| - m(\mathcal{A}) + 1, |\mathcal{A}| - p(\mathcal{A})\}$. Then, $d \geq \nu_f(\mathcal{A})$ if and only if $\mathcal{D}_d^{\infty}(\mathcal{A}) \neq \emptyset$.*

Proof of Theorem VIII.3.14. Let $\chi \in \mathcal{D}_d(\mathcal{A})$ with $0 < d < \nu_f(\mathcal{A})$, then χ does not satisfy the hypotheses of Proposition VIII.3.15 and Proposition VIII.3.16. Thus χ is neither radial nor parallel. Since $d \neq 0$, Theorem VIII.3.11 implies that $\chi \notin \mathcal{D}_d^{\infty}(\mathcal{A})$, and then $\chi \in \mathcal{D}_d^{f}(\mathcal{A})$. □

Using Theorem VIII.3.8 and Theorem VIII.3.14, we obtain the following corollary.

Corollary VIII.3.19. *Let $\mathcal{A}$ be a line arrangement. Let $\nu(\mathcal{A}) = \min\{\nu_{\infty}(\mathcal{A}), \nu_f(\mathcal{A})\}$, if $0 < d < \nu(\mathcal{A})$ then $\mathcal{D}_d(\mathcal{A}) = \emptyset$.*

Note that these minimal bound for the existence of logarithmic vector fields of an arrangement is completely combinatorial and it is obtained following a dynamical point of view, something which is not easy to obtain from a purely algebraic geometric setting.

VIII.4 Non combinatoriallity of the minimal finite derivations

Using the results obtained in Section VIII.3, we prove explicitly that $d_f(\mathcal{A})$ is not determined by the number of lines and singular points counted with multiplicities and, as a more strongest result, by the combinatorial information. For that, we consider two explicit counterexamples of line arrangements. As a first pair, we consider the realizations of configurations $(9_3)_1$ and $(9_3)_2$ described in [HCV52, p. 102], called the *Pappus* and *non-Pappus* arrangements and denoted by $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively (see [Suc01]). Both arrangements have the same weak combinatorics (*i.e.* they share the same number of singularities for each multiplicity). The second pair correspond to Ziegler's arrangement $\mathcal{Z}_1$ (see [Zie89]) and $\mathcal{Z}_2$ a small deformation of $\mathcal{Z}_1$, with same strong combinatorics, *i.e.* $L(\mathcal{Z}_1) \simeq L(\mathcal{Z}_2)$.

Remark VIII.4.1. These examples are constructed as the affine parts of the projective arrangements previously described, choosing a line of the arrangement as line at infinity.

VIII.4.1 Dependency of weak combinatorics

The result presented here is a weaker restrictive case of Theorem VIII.4.5, as a first step to explore the relation between the minimal degree of derivations in $\mathcal{D}(\mathcal{A})$ and the combinatorics of $\mathcal{A}$.

Theorem VIII.4.2. *The minimal degree $d_f(\mathcal{A})$ of a finite polynomial vector field fixing $\mathcal{A}$ is not determined by the number of lines and singular points counted with multiplicities of $\mathcal{A}$.*

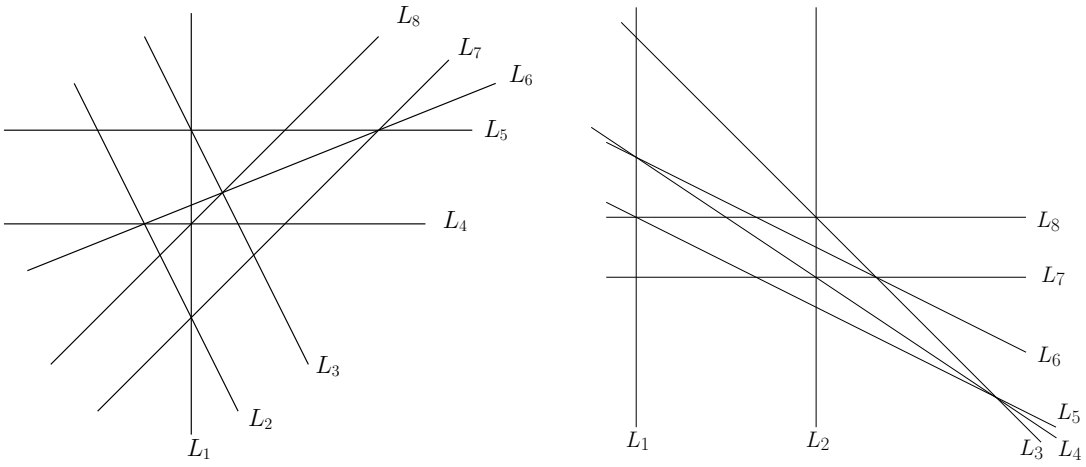

Figure VIII.3: The arrangements $\mathcal{P}_1$ and $\mathcal{P}_2$

In order to prove this theorem, we consider two line arrangements in the plane pictured in Figure VIII.3 $\mathcal{P}_1$ (Pappus arrangement) and $\mathcal{P}_2$ (non-Pappus arrangement) defined respectively by:

$$\begin{aligned} \mathcal{P}_1 &: xy(x-y)(y-1)(x-y-1)(2x+y+1)(2x+y-1)(2x-5y+1) \\ \mathcal{P}_2 &: xy(x+y)(y+1)(x+3)(x+2y+1)(x+2y+3)(2x+3y+3) \end{aligned}$$

These two arrangements have the same weak combinatorics: 8 lines, 6 triple points and 7 double points.

Proposition VIII.4.3. *The arrangements $\mathcal{P}_1$ and $\mathcal{P}_2$ have not the same combinatorial data,* i.e. $L(\mathcal{P}_1) \not\simeq L(\mathcal{P}_2)$.

Proof. If we look for lines which possess three triple points and a double point, the only lines in $\mathcal{P}_1$ of this condition are L_1 and L_6 whose intersection is the common double point, whereas in the line arrangement $\mathcal{P}_2$ we found L_3 and L_4 with $L_3 \cap L_4 \cap L_5$ a triple point. □

Finally, Theorem VIII.4.2 holds from the following result.

Proposition VIII.4.4.

1. *For any $i \in \{1,2,3\}$, we have $\mathcal{D}_i(\mathcal{P}_1) = \emptyset$; and $\mathcal{D}_4(\mathcal{P}_1) \neq \emptyset$,*
2. *For any $i \in \{1,2,3,4\}$, we have $\mathcal{D}_i(\mathcal{P}_2) = \emptyset$; and $\mathcal{D}_5(\mathcal{P}_2) \neq \emptyset$.*

The proof of this proposition is given in Section VIII.4.3.

Proof of Theorem VIII.4.2. From last proposition, we deduce that $d_f(\mathcal{P}_1) \geq 4$ and $d_f(\mathcal{P}_2) \geq 5$. Using Theorem VIII.3.14, we have that $\mathcal{D}_i(\mathcal{P}_1) = \mathcal{D}_i^f(\mathcal{P}_1)$ and $\mathcal{D}_i(\mathcal{P}_2) = \mathcal{D}_i^f(\mathcal{P}_2)$, for any $i = 1,2,3,4,5,6$. Hence, we obtain that $d_f(\mathcal{P}_1) = 4$ and $d_f(\mathcal{P}_2) = 5$. This concludes the proof. □

VIII.4.2 Dependency of strong combinatorics

The main result of this chapter is the following:

Theorem VIII.4.5. *The minimal degree $d_f(\mathcal{A})$ of a finite polynomial vectors fields fixing $\mathcal{A}$ is not determined by the combinatorial information of $\mathcal{A}$.*

In order to prove this theorem, consider $\mathcal{Z}_1$ be the affine image of Ziegler arrangement [Zie89], pictured in Figure VIII.4. This arrangement verifies a very strong geometric condition: the six triple points of the projective image of $\mathcal{Z}_1$ (considering an additional line in the arrangement: the line at infinity) are contained in a conic $\mathcal{C}$. Hence, we construct a line arrangement $\mathcal{Z}_2$ as a small rational perturbation of Ziegler arrangement, displacing the triple point $L_1 \cap L_3 \cap L_7$ outside of the conic and preserving the combinatorial data. They are both formed by 8 lines with 4 triples points, 14 doubles points and three pairs of parallel lines. Consider the following equations for $\mathcal{Z}_1$ and $\mathcal{Z}_2$:

$$\begin{array}{rcl} \mathcal{Z}_1 & : & Z(x,y)(9x-2y+3)(11x+2y+1)(5x+5y-2) \\ \mathcal{Z}_2 & : & Z(x,y)(21x-4y+7)(19x+4y+1)(10x+10y-5) \end{array}$$

where $Z(x,y) = y(2x+2y+1)(3x+y+1)(8x-y+4)(9x+3y-1)$.

Proposition VIII.4.6. *The arrangements $\mathcal{Z}_1$ and $\mathcal{Z}_2$ have the same combinatorial information, i.e. $L(\mathcal{Z}_1) \simeq L(\mathcal{Z}_2)$.*

We complete the proof with the following result, discussed in Section VIII.4.3.

Proposition VIII.4.7.

1. *For any $i \in \{1,2,3,4\}$, we have $\mathcal{D}_i(\mathcal{Z}_1) = \emptyset$; and $\mathcal{D}_5(\mathcal{Z}_1) \neq \emptyset$,*
2. *For any $i \in \{1,2,3,4,5\}$, we have $\mathcal{D}_i(\mathcal{Z}_2) = \emptyset$; and $\mathcal{D}_6(\mathcal{Z}_2) \neq \emptyset$,*

Proof of Theorem VIII.4.5. As a consequence of last proposition, $d_f(\mathcal{Z}_1) \geq 5$ and $d_f(\mathcal{Z}_2) \geq 6$. But by Theorem VIII.3.14, we have that $\mathcal{D}_i(\mathcal{Z}_j) = \mathcal{D}_i^f(\mathcal{Z}_j)$ for any $i = 1,2,3,4,5$ and $j = 1,2$. Then $d_f(\mathcal{Z}_1) = 5$, which is different of $d_f(\mathcal{Z}_2)$ since it is greater than 6. □

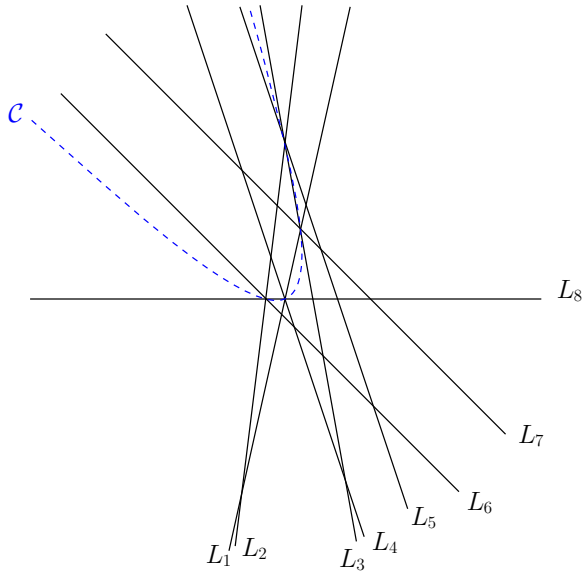

Figure VIII.4: The Ziegler arrangement $\mathcal{Z}_1$ with the conic $\mathcal{C} = \{6x^2 + 2y^2 + 5x + 8xy + 1 = 0\}$.

VIII.4.3 Proof of Propositions VIII.4.4 and VIII.4.7

Following Proposition VIII.2.6 and Theorem VIII.2.8, and using equations (E_m) in Remark VIII.2.7, the proof of both results is obtained by constructing the matrices $M_{\mathcal{P}_1}, M_{\mathcal{P}_2}, M_{\mathcal{Z}_1}$ and $M_{\mathcal{Z}_2}$ for which $\mathcal{F}_4\mathcal{D}(\mathcal{P}_1), \mathcal{F}_5\mathcal{D}(\mathcal{P}_2), \mathcal{F}_5\mathcal{D}(\mathcal{Z}_1)$ and $\mathcal{F}_6\mathcal{D}(\mathcal{Z}_2)$ are the kernels in the respective space of coefficients.

It is easy to see that these matrices have $n(d+1)$ rows and $(d+1)(d+2)$ columns, where n is the number of lines in each line arrangement and d is the degree in the filtration $\mathcal{F}_d$.

In order to construct and analyze these matrices, we use a set of functions programming over `Sage` [S+14], to obtain that $\mathcal{D}_4(\mathcal{P}_1) = \mathcal{F}_4\mathcal{D}(\mathcal{P}_1) \setminus \{0\}$, $\mathcal{D}_5(\mathcal{P}_2) = \mathcal{F}_5\mathcal{D}(\mathcal{P}_2) \setminus \{0\}$, $\mathcal{D}_5(\mathcal{Z}_1) = \mathcal{F}_5\mathcal{D}(\mathcal{Z}_1) \setminus \{0\}$ and $\mathcal{D}_6(\mathcal{Z}_2) = \mathcal{F}_6\mathcal{D}(\mathcal{Z}_2) \setminus \{0\}$.

The matrices and subspaces for the preceded filtrations are detailed in Appendix A. The complete code source `line_arrangements_vector_fields.sage` with the different functions written in `Sage` can be found in Appendix B.

Sage

```
sage: reset()

sage: load "line_arrangements_vector_fields.sage":

sage: R.<x,y>=QQ[]; R

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```

We define the $\mathcal{P}_1, \mathcal{P}_2, \mathcal{Z}_1$ and $\mathcal{Z}_2$ by the 3–tuple in $\mathbb{Q}$ representing the coefficients of the affine form defining each line.

Sage
```
sage: P1=[(1,0,0),(0,1,0),(1,-1,0),(0,1,-1),(1,-1,-1),(2,1,1),(2,1,-1),(2,-5,
1)] # Pappus

sage: P2=[(1,0,0),(0,1,0),(1,1,0),(0,1,1),(1,0,3),(1,2,1),(1,2,3),(2,3,3)] #N
on-Pappus

sage: Z1=[(0,1,0),(2,2,1),(3,1,1),(8,-1,4),(9,3,-1),(9,-2,3),(11,2,1),(5,5,-2
)] #Ziegler

sage: Z2=[(0,1,0),(2,2,1),(3,1,1),(8,-1,4),(9,3,-1),(21,-4,7),(19,4,1),(10,10
,-5)] #Non-Ziegler
```

We use the function `basis_vector_field_from_arrangement(R,A,d)`, which gives the basis of $\mathcal{F}_d\mathcal{D}(\mathcal{A})$.

Sage
```
sage: V=basis_vector_field_from_arrangement(R,P1,3); print V; print len(V)

[]
0
sage: V=basis_vector_field_from_arrangement(R,P1,4); print V; print len(V)

[(x^4 - x^3*y - 1/4*x^2*y^2 + 5/2*x*y^3 - x^3 - 1/2*x^2*y - 3/4*x*y^2 - 1/4*x^2
+ 1/4*x, 2*x^2*y^2 - x*y^3 + 5/4*y^4 - 2*x^2*y - 1/4*y^3 + x*y - 5/4*y^2 + 1/4*
y)]
1
```

Sage
```
sage: V=basis_vector_field_from_arrangement(R,P2,4); print V; print len(V)

[]
0
sage: V=basis_vector_field_from_arrangement(R,P2,5); print len(V)

3
```

We obtain then that $\dim \mathcal{F}_3\mathcal{D}(\mathcal{P}_1) = 0$ and $\dim \mathcal{F}_4\mathcal{D}(\mathcal{P}_1) = 1$, but $\mathcal{F}_4\mathcal{D}(\mathcal{P}_2) = 0$ and $\mathcal{F}_5\mathcal{D}(\mathcal{P}_2) = 3$. Repeating this process for the Ziegler and non-Ziegler arrangement:

Sage
```
sage: V=basis_vector_field_from_arrangement(R,Z1,4); print len(V)

0
sage: V=basis_vector_field_from_arrangement(R,Z1,5); print len(V)

1
```

Sage
```
sage: V=basis_vector_field_from_arrangement(R,Z2,5); print len(V)

0
sage: V=basis_vector_field_from_arrangement(R,Z2,6); print len(V)

6
```

Thus, we conclude that $\dim \mathcal{F}_4\mathcal{D}(\mathcal{Z}_1) = 0$ and $\dim \mathcal{F}_5\mathcal{D}(\mathcal{Z}_1) = 1$, but $\dim \mathcal{F}_5\mathcal{D}(\mathcal{Z}_2) = 0$ and $\dim \mathcal{F}_6\mathcal{D}(\mathcal{Z}_2) = 6$.

VIII.5 A quadratic growth for ranks in the filtration

We have determined in the previous Section that the minimal degree of a logarithmic vector field of a line arrangement is not determined by the combinatorics. We are now interested in the study of the dimension of the k–linear space $\mathcal{F}_d\mathcal{D}(\mathcal{A})$. Is this dimension combinatorially determined for a suitable choice of d? Is there a relation between the values of the sequence $(\nu_d)_{d\in\mathbb{N}}$ where $\nu_d = \dim \mathcal{F}_d\mathcal{D}(\mathcal{A})$?

VIII.5.1 A recursive relation for matrix conditions

Let $\mathcal{A} = \{L_1, \ldots, L_n\}$ be a line arrangement in $\mathbb{A}^2_k$. Assume that no line of $\mathcal{A}$ is vertical or horizontal after fixing a system of coordinates in $\mathbb{A}^2_k$, that is, any line of the arrangements is of the form $L_i : \alpha_i x + y + \gamma_i = 0$, with $\alpha_i \neq 0$. Let $d > 0$ be a positive integer, and consider the basis $\mathscr{B}_d$ of the space of coefficients $\mathcal{C}(d)$ induced by the lexicographical order:

$$\mathcal{C}(d) = k^{2(d+1)} \oplus k^{2d} \oplus \cdots \oplus k^4 \oplus k^2$$

where, for any $j = 0, \ldots, d$, we are fixing an ordering in the basis expressed as:

$$k^{2(d-j+1)} \simeq k\langle a_{(j,0)}, a_{(j,1)}, \ldots, a_{(j,d-j)}\rangle \oplus k\langle b_{(j,0)}, b_{(j,1)}, \ldots, b_{(j,d-j)}\rangle \tag{VIII.3}$$

In Remark VIII.2.7, we have seen that

$$\mathcal{F}_d\mathcal{D}(\mathcal{A}) = \bigcap_{i=1}^{n} \mathcal{F}_d\mathcal{D}(L_i) = \bigcap_{i=1}^{n}\bigcap_{m=0}^{d} \ker E_{i,m}, \tag{VIII.4}$$

where $E_{i,m} : \mathcal{C}(d) \to \mathbb{R}$ are linear forms over the k-linear space of coefficients $\mathcal{C}(d) \simeq k^{(d+1)(d+2)}$ defined by:

$$E_{i,m} = \sum_{k=0}^{d-m}\sum_{l=0}^{m} (\alpha_i a_{(m-l,k+l)} + b_{(m-l,k+l)}) \binom{k+l}{k} (-1)^{k+l} \alpha_i^l \gamma_i^k, \tag{VIII.5}$$

for $0 \leq m \leq d$ and $i = 1, \ldots, n$. Thus, for each line we obtain equations of the form:

$$\begin{aligned}
m = 0: \quad E_{i,0} = & \sum_{k=0}^{d} (\alpha_i a_{(0,k)} + b_{(0,k)})(-1)^k \gamma_i^k, \\
m = 1: \quad E_{i,1} = & \sum_{k=0}^{d-1} (\alpha_i a_{(1,k)} + b_{(1,k)})(-1)^k \gamma_i^k + \\
& \sum_{k=0}^{d-1} (\alpha_i a_{(0,k+1)} + b_{(0,k+1)} \tbinom{k+1}{k})(-1)^{k+1} \alpha_i \gamma_i^k, \\
m = 2: \quad E_{i,1} = & \sum_{k=0}^{d-2} (\alpha_i a_{(2,k)} + b_{(2,k)})(-1)^k \gamma_i^k + \\
& \sum_{k=0}^{d-2} (\alpha_i a_{(1,k+1)} + b_{(1,k+1)} \tbinom{k+1}{k})(-1)^{k+1} \alpha_i \gamma_i^k + \\
& \sum_{k=0}^{d-2} (\alpha_i a_{(0,k+2)} + b_{(0,k+2)} \tbinom{k+2}{k})(-1)^{k+2} \alpha_i^2 \gamma_i^k. \\
& \vdots
\end{aligned}$$

Lemma VIII.5.1. *For $i = 1, \ldots, n$ and $m = 0, \ldots, d$, consider the linear forms $E_{i,m}$ in (VIII.5) given in coordinates over $\mathscr{B}$ in $\mathcal{C}(d)$. Then, for any $j, h \in \mathbb{N}$ such that $j + h \leq d$ we have*

$$\operatorname{Coeff}(b_{(j,h)}, E_{i,m}) = \begin{cases} (-1)^h \tbinom{h}{m-j} \alpha_i^{m-j} \gamma_i^{j+h-m}, & \text{si } j + h \geq m \\ 0, & \text{otherwise} \end{cases},$$

and

$$\operatorname{Coeff}(a_{(j,h)}, E_{i,m}) = \alpha_i \cdot \operatorname{Coeff}(b_{(j,h)}, E_{i,m})$$

Proof. Using an extensive change of index in order to determine the coefficients of $a_{j,h}$ and $b_{j,h}$ at each linear form $E_{i,m}$ in (VIII.5) :

$$E_{i,m} = \sum_{l=0}^{m} \sum_{k=0}^{d-m} (\alpha_i a_{(m-l,k+l)} + b_{(m-l,k+l)}) \binom{k+l}{k} (-1)^{k+l} \alpha_i^l \gamma_i^k.$$

Taking $h = k + l$ in a first time, we have that $k = 0$ implies $h = l$, and $k = d - m$ implies $h = d - m + l$. Then:

$$E_{i,m} = \sum_{l=0}^{m} \sum_{h=l}^{d-m+l} (\alpha_i a_{(m-l,h)} + b_{(m-l,h)}) \binom{h}{h-l} (-1)^h \alpha_i^l \gamma_i^{h-l}.$$

Finally, if we take $j = m - l$, then:

$$\begin{aligned}
E_{i,m} &= \sum_{j=0}^{m} \sum_{h=m-j}^{d-j} (\alpha_i a_{(j,h)} + b_{(j,h)}) \binom{h}{h-m+j} (-1)^h \alpha_i^{m-j} \gamma_i^{h-m+j} \\
&= \sum_{j=0}^{m} \sum_{h=m-j}^{d-j} (\alpha_i a_{(j,h)} + b_{(j,h)}) \binom{h}{m-j} (-1)^h \alpha_i^{m-j} \gamma_i^{h+j-m}.
\end{aligned}$$

Thus, we can give an explicit expression of the coefficients $a_{j,h}$ in $E_{i,m}$, for $m = 0, \ldots, d$ and $j, h \in \mathbb{N}$ such that $j + h \leq d$, and the result holds. □

Considering the linear map $\Phi : \mathcal{C}(d) \to k^{n(d+1)}$ for which $\mathcal{F}_d\mathcal{D}(\mathcal{A}) \simeq \ker \Phi$, we can construct an explicit matrix $M_d(\mathcal{A})$ representing Φ in basis $\mathscr{B}_d$ by using the previous Lemma.

For $i, j = 0, \ldots, d$, we construct the matrices $N_d^{i,j} \in \mathrm{Mat}_k(n, 2(d-j+1))$ representing the n linear equations (each of them associated to each line $L \in \mathcal{A}$) projected into the $(2(d-j+1))$–sub-space given by the sub-basis (VIII.3) of $\mathscr{B}_d$.

If $i \geq j$, we define $N_d^{i,j} = \left(A_d^{i,j} \;\middle|\; B_d^{i,j}\right)$ expressed by column matrices $A_d^{i,j}, B_d^{i,j} \in \mathrm{Mat}_k(n, d-j+1)$, defined by

$$A_d^{i,j} = \left(\begin{array}{ccc|ccc} 0 & \cdots & 0 & \binom{i-j}{i-j}(-1)^{i-j}\alpha_1^{i-j+1} & \cdots & \binom{d-j}{i-j}(-1)^{d-j}\alpha_1^{i-j+1}\gamma_1^{d-i} \\ \vdots & & \vdots & \vdots & & \vdots \\ \underbrace{0 & \cdots & 0}_{i-j} & \binom{i-j}{i-j}(-1)^{i-j}\alpha_n^{i-j+1} & \cdots & \binom{d-j}{i-j}(-1)^{d-j}\alpha_n^{i-j+1}\gamma_n^{d-i} \end{array}\right)$$

and

$$B_d^{i,j} = \left(\begin{array}{ccc|ccc} 0 & \cdots & 0 & \binom{i-j}{i-j}(-1)^{i-j}\alpha_1^{i-j} & \cdots & \binom{d-j}{i-j}(-1)^{d-j}\alpha_1^{i-j}\gamma_1^{d-i} \\ \vdots & & \vdots & \vdots & & \vdots \\ \underbrace{0 & \cdots & 0}_{i-j} & \binom{i-j}{i-j}(-1)^{i-j}\alpha_n^{i-j} & \cdots & \binom{d-j}{i-j}(-1)^{d-j}\alpha_n^{i-j}\gamma_n^{d-i} \end{array}\right)$$

If $i < j$, then $N_d^{i,j}$ is the zero matrix $\mathcal{O}_{n,2(d-j+1)}$. Note that in the case of equality of the indexes, for any $i = 0, \ldots, d$:

$$N_d^{i,i} = \left(\begin{array}{cccc|cccc} \alpha_1 & -\alpha_1\gamma_1 & \alpha_1\gamma_1^2 & \cdots & (-1)^{d-i}\alpha_1\gamma_1^{d-i} & 1 & -\gamma_1 & \gamma_1^2 & \cdots & (-1)^{d-i}\gamma_1^{d-i} \\ \alpha_2 & -\alpha_2\gamma_2 & \alpha_2\gamma_2^2 & \cdots & (-1)^{d-i}\alpha_2\gamma_2^{d-i} & 1 & -\gamma_2 & \gamma_2^2 & \cdots & (-1)^{d-i}\gamma_2^{d-i} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \alpha_n & -\alpha_n\gamma_n & \alpha_n\gamma_n^2 & \cdots & (-1)^{d-i}\alpha_n\gamma_n^{d-i} & 1 & -\gamma_n & \gamma_n^2 & \cdots & (-1)^{d-i}\gamma_n^{d-i} \end{array}\right)$$

Remark VIII.5.2. If we denote by $\mathrm{VDM}_i(-\gamma_1, \ldots, -\gamma_n)$ the Vandermonde matrix in which the lines are composed by the first $(d-i)$–powers in the geometrical progression of $(-\gamma_k)$ for $k = 0, \ldots, n$, then the matrix we can express by blocks

$$N_d^{i,i} = \left(\mathrm{Diag}(\alpha_1, \ldots, \alpha_n)\,\mathrm{VDM}_d(-\gamma_1, \ldots, -\gamma_n) \;\middle|\; \mathrm{VDM}_d(-\gamma_1, \ldots, -\gamma_n)\right)$$

By construction of the previous matrices, we can express the matrix associated to the system defined by equations VIII.5 in the basis $\mathscr{B}(d)$ as a triangular by blocks:

$$M_d(\mathcal{A}) = \begin{pmatrix} N_d^{0,0} & & & \\ N_d^{1,0} & N_d^{1,1} & & \\ \vdots & \vdots & \ddots & \\ N_d^{d,0} & N_d^{d,1} & \cdots & N_d^{d,d} \end{pmatrix} \in \mathrm{Mat}(n(d+1), (d+1)(d+2)). \tag{VIII.6}$$

Remark that $M_d(\mathcal{A})$ is a non-square matrix, since each $N_d^{i,j}$ is in $\mathrm{Mat}_k(n, 2(d-j+1))$.

Proposition VIII.5.3. *We have that* $\mathcal{F}_d\mathcal{D}(\mathcal{A}) \simeq \ker M_d(\mathcal{A})$. *In particular,*

$$\dim_k \mathcal{F}_d\mathcal{D}(\mathcal{A}) = (d+1)(d+2) - \operatorname{rank} M_d(\mathcal{A}).$$

Proof. This result comes directly from (VIII.4) and the construction of matrix $M_d(\mathcal{A})$. □

We relies the matrix $M_{d-1}(\mathcal{A})$ with $M_d(\mathcal{A})$ in the following.

Proposition VIII.5.4. *Let* $d > 0$ *be a positive integer. For any* $i, j = 0, \ldots, d-1$, *we have that* $N_{d-1}^{i,j} = N_d^{i+1,j+1}$. *In particular,* $M_{d-1}(\mathcal{A})$ *is a sub-matrix of* $M_d(\mathcal{A})$*:*

$$M_d(\mathcal{A}) = \left(\begin{array}{c|c} N_d^{0,0} & \\ \hline N_d^{1,0} & \\ \vdots & M_{d-1}(\mathcal{A}) \\ N_d^{d,0} & \end{array}\right) \tag{VIII.7}$$

Proof. Clearly, $N_{d-1}^{i,j}$ and $N_d^{i+1,j+1}$ are both in $\operatorname{Mat}_k(n, 2(d-j))$. The column matrices

$$A_d^{i+1,j+1} = \left(\begin{array}{ccc|ccc} 0 & \cdots & 0 & \binom{i-j}{i-j}(-1)^{i-j}\alpha_1^{i-j+1} & \cdots & \binom{d-j}{i-j}(-1)^{d-j}\alpha_1^{i-j}\gamma_1^{d-i-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ \underbrace{0 \quad \cdots \quad 0}_{i-j} & & & \binom{i-j}{i-j}(-1)^{i-j}\alpha_n^{i-j+1} & \cdots & \binom{d-j}{i-j}(-1)^{d-j}\alpha_n^{i-j+1}\gamma_n^{d-i-1} \end{array}\right)$$

and $A_{d-1}^{i,j}$ are equal by definition. Analogously, we can prove that $B_d^{i+1,j+1} = B_{d-1}^{i,j}$. □

Lemma VIII.5.5. *The matrix* $M_d(\mathcal{A})$ *is congruent with a matrix of the form*

$$\left(\begin{array}{c|c} N_d^{0,0} & \\ \hline S_d & \\ & \widetilde{M}_{d-1}(\mathcal{A}) \\ & \end{array}\right)$$

where $S \in \operatorname{Mat}_k(d, 2(d+1))$ *is of the form*

$$S_d = \left(\begin{array}{ccccc|ccccc} 0 & \alpha_1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_1 & 0 & 0 & 0 & \cdots & 1 \end{array}\right),$$

and $\widetilde{M}_{d-1}(\mathcal{A})$ *is congruent to* $M_{d-1}(\mathcal{A})$.

Proof. Looking at the decomposition (VIII.7) in Proposition VIII.5.4, we note that the matrices $N_d^{i,0} = \left(A_d^{i,0} \;\middle|\; B_d^{i,0}\right)$ are of the form:

$$A_d^{i,0} = \left(\begin{array}{ccc|ccc} 0 & \cdots & 0 & (-1)^i\alpha_1^{i+1} & \binom{i+1}{i}(-1)^i\alpha_1^{i+1}\gamma_1 & \cdots & \binom{d}{i}(-1)^d\alpha_1^{i+1}\gamma_1^{d-i} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \underbrace{0 & \cdots & 0}_{i} & (-1)^i\alpha_n^{i+1} & \binom{i+1}{i}(-1)^{i-j}\alpha_n^{i+1}\gamma_n & \cdots & \binom{d}{i}(-1)^d\alpha_n^{i+1}\gamma_n^{d-i} \end{array}\right)$$

and

$$B_d^{i,0} = \left(\begin{array}{ccc|ccc} 0 & \cdots & 0 & (-1)^i\alpha_1^{i} & \binom{i+1}{i}(-1)^i\alpha_1^{i}\gamma_1 & \cdots & \binom{d}{i}(-1)^d\alpha_1^{i}\gamma_1^{d-i} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \underbrace{0 & \cdots & 0}_{i} & (-1)^i\alpha_n^{i} & \binom{i+1}{i}(-1)^i\alpha_n^{i}\gamma_n & \cdots & \binom{d}{i}(-1)^d\alpha_n^{i}\gamma_n^{d-i} \end{array}\right)$$

In particular,

$$N_d^{d,0} = \left(\begin{array}{ccc|ccc} 0 & \cdots & 0 & (-1)^d\alpha_1^{d+1} & 0 & \cdots & 0 & (-1)^d\alpha_1^{d} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \underbrace{0 & \cdots & 0}_{d} & (-1)^d\alpha_n^{d+1} & \underbrace{0 & \cdots & 0}_{d} & (-1)^d\alpha_n^{d} \end{array}\right)$$

By row operations from $N_d^{d,0}$, we can obtain zeros in the dth and $(2d+2)$th columns of any $N_d^{d-1,0}, \ldots, N_d^{1,0}$: for any $i = 1, \ldots d-1$ and $h = 1, \ldots, n$, we perform a row operation in $M_d(\mathcal{A})$ of the form

$$L_{i,h} \longleftarrow L_{i,h} - \binom{d}{i}\left(\frac{\gamma_h}{\alpha_h}\right)^{d-i} L_{0,h},$$

where $L_{i,h}$ is the row in $M_d(\mathcal{A})$ containing the hth–row of the sub-matrix $N_d^{i,0}$. We obtain a congruent matrix of $M_d(\mathcal{A})$ of the form

$$\left(\begin{array}{c|c} N_d^{0,0} & \\ \hline \widetilde{N}_d^{1,0} & \\ \vdots & \widetilde{M} \\ \widetilde{N}_d^{d,0} & \end{array}\right)$$

where $\widetilde{M}$ is congruent with $M_{d-1}(\mathcal{A})$ and now

$$\widetilde{N}_d^{d-1,0} = \left(\begin{array}{ccc|ccc} 0 & \cdots & 0 & (-1)^{d-1}\alpha_1^{d} & 0 & 0 & \cdots & 0 & (-1)^{d-1}\alpha_1^{d-1} & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \underbrace{0 & \cdots & 0}_{d-1} & (-1)^{d-1}\alpha_n^{d} & 0 & \underbrace{0 & \cdots & 0}_{d-1} & (-1)^{d-1}\alpha_n^{d-1} & 0 \end{array}\right)$$

Following the same procedure, we can obtain zeros in the $(d-1)$th and $(2d)$th columns of any $\widetilde{N}_d^{d-2,0},\ldots,\widetilde{N}_d^{1,0}$. Repeating the previous process recursively, we obtain a congruent matrix to $M_d(\mathcal{A})$ of the form

$$\mathscr{M}_d = \left(\begin{array}{c|c} N_d^{0,0} & \\ \hline S_d^1 & \\ \vdots & \mathscr{M}_{d-1} \\ S_d^d & \end{array}\right)$$

where $\mathscr{M}_{d-1}$ is congruent to $M_{d-1}(\mathcal{A})$ and

$$S_d^{i,0} = \left(\begin{array}{ccccccc|ccccccc} 0 & \cdots & 0 & (-1)^i\alpha_1^{i+1} & 0 & \cdots & 0 & 0 & \cdots & 0 & (-1)^i\alpha_1^i & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \underbrace{0 \cdots 0}_{i} & & & (-1)^i\alpha_n^{i+1} & \underbrace{0 \cdots 0}_{d-i} & & & \underbrace{0 \cdots 0}_{i} & & & (-1)^i\alpha_n^i & \underbrace{0 \cdots 0}_{d-i} & & \end{array}\right),$$

for any $i = 1,\ldots,d$. Now, we can normalize the rows of $\mathscr{M}$ containing the hth row of the sub-matrix $S_d^{i,0}$ in order to obtain sub-matrices:

$$\widetilde{S}_d^{i,0} = \left(\begin{array}{ccccccc|ccccccc} 0 & \cdots & 0 & \alpha_1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \underbrace{0 \cdots 0}_{i} & & & \alpha_n & \underbrace{0 \cdots 0}_{d-i} & & & \underbrace{0 \cdots 0}_{i} & & & 1 & \underbrace{0 \cdots 0}_{d-i} & & \end{array}\right).$$

By row operations in order to void the elements non-containing a α_1 followed by permutations of rows in order to obtain S_d, the result holds. □

Consider the sequence $(\nu_d)_{d\in\mathbb{N}}$ defined by the ranks $\nu_d = \dim \mathcal{F}_d\mathcal{D}(\mathcal{A})$ of the strata in the filtration. Proposition VIII.5.3 and previous Lemma suggest that the sequence $(\nu_d)_{d\in\mathbb{N}}$ follows a quadratic growth from a suitable $d_0 > 0$, since the new linear independence relations in $M_d(\mathcal{A})$ will depend necessarily from the columns of $N_d^{0,0}$, which is of size $nd \times 2(d+1)$. So, the rank of $M_d(\mathcal{A})$ will be the addition of $\operatorname{rank} M_{d-1}(\mathcal{A})$ and a number depending in d but controlled by a degree one polynomial in d.

Question VIII.5.6. *Let $\mathcal{A}$ an affine line arrangement. Is there a positive integer $d_0 > 0$ and $b, c \in \mathbb{Z}$ such that*

$$\dim \mathcal{F}_d\mathcal{D}(\mathcal{A}) = d^2 + bd + c, \quad \forall d \geq d_0,$$

where d_0, b and c depends only of the combinatorics of $\mathcal{A}$?

VIII.5.2 Some computations on the filtration

We look at some simple examples of line arrangements in order to determine the ranks of the ascending filtration of the module of logarithmic derivations. We start with the line arrangement $\mathcal{A}_1$ defined in Example VI.1.7 by $\mathcal{Q}_{\mathcal{A}} = y(x-2y)(2x-y)(2x+y-1)$, which possesses 4 lines, 3 double points, a triple point, and it has not parallel lines. By using our functions in `Sage`, we determine $\dim \mathcal{F}_d\mathcal{D}(\mathcal{A}_1)$ up to degree 12:

Sage
```
sage: A1=[(0,1,0), (1,-2,0), (2,-1,0),(2,1,-1)]

sage: dmax=12

sage: L=map(lambda d: len(basis_vector_field_from_arrangement(R,A1,d)),
range(1,dmax+1)); print L

[0, 1, 5, 11, 19, 29, 41, 55, 71, 89, 109, 131]
```

We remark that the non-zero terms of the previous sequence verify the quadratic expression:

$$\dim \mathcal{F}_d\mathcal{D}(\mathcal{A}_1) = d^2 - d - 1, \quad d = 2, 3, \ldots, 12.$$

Consider now the *Ceva's arrangement* $\mathcal{C}$ (pictured in Figure VIII.5) and defined by $\mathcal{Q}_{\mathcal{C}} = xy(\sqrt{3}x - y + \sqrt{3})(\sqrt{3}x + y - \sqrt{3})(\sqrt{3}x + 3y - \sqrt{3})(\sqrt{3}x - 3y + \sqrt{3})$. This arrangement possesses 6 lines, 3 double points, 4 triple points, and it does not possess parallel lines. Following the same procedure as before over the number filed $\mathbb{Q}(\sqrt{3})$:

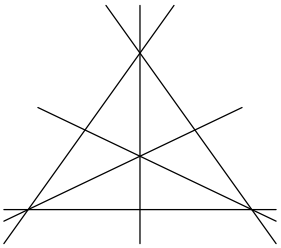

Figure VIII.5: Ceva's arrangement.

Sage
```
sage: var('t')

sage: K.<sqrt3> = NumberField(t^2-3,embedding=1.73);

sage: R.<x,y>=K[]; R

Multivariate Polynomial Ring in x, y over Number Field in sqrt3 with
defining polynomial t^2 - 3
sage: ceva=[(1,0,0), (0,1,0), (sqrt3,-1,sqrt3), (sqrt3,1,-sqrt3),
(sqrt3,3,-sqrt3), (sqrt3,-3,sqrt3)]

sage: L=map(lambda d: len(basis_vector_field_from_arrangement(R,ceva,d)),
range(1,dmax+1)); print L

[0, 0, 1, 4, 10, 18, 28, 40, 54, 70, 88, 108]
```

In this case, the non-zero terms of the previous sequence verify the quadratic expression:

$$\dim \mathcal{F}_d\mathcal{D}(\mathcal{A}_1) = d^2 - 3d, \quad d = 4, 5, \ldots, 12.$$

By computations over other examples of line arrangements, we have remarked that the ranks in the ascending filtration of the module of logarithmic derivations verify a quadratic expression starting for some term of degree smaller than the number of lines.

We can naturally ask if the coefficients of such a quadratic formula depends on the combinatorics of the line arrangement. To illustrate this question, we look at the ranks of the ascending filtration of the module of logarithmic derivations for previous arrangements $\mathcal{P}_1$, $\mathcal{P}_2$, sharing the same weak combinatorics, and $\mathcal{Z}_1$, $\mathcal{Z}_2$ for which their intersection posets are isomorphic :

Sage
```
sage: L=map(lambda d: len(basis_vector_field_from_arrangement(R,P1,d)),
range(1,dmax+1)); print L

[0, 0, 0, 1, 3, 9, 17, 27, 39, 53, 69, 87]
sage: L=map(lambda d: len(basis_vector_field_from_arrangement(R,P2,d)),
range(1,dmax+1)); print L

[0, 0, 0, 0, 3, 9, 17, 27, 39, 53, 69, 87]
```

Sage
```
sage: L=map(lambda d: len(basis_vector_field_from_arrangement(R,Z1,d)),
range(1,dmax+1)); print L

[0, 0, 0, 0, 1, 6, 14, 24, 36, 50, 66, 84]
sage: L=map(lambda d: len(basis_vector_field_from_arrangement(R,Z2,d)),
range(1,dmax+1)); print L

[0, 0, 0, 0, 0, 6, 14, 24, 36, 50, 66, 84]
```

We remark that, at each pair, the values of the ranks of the corresponding filtration of degree d verify the quadratic relations:

$$\dim \mathcal{F}_d\mathcal{D}(\mathcal{P}_1) = \dim \mathcal{F}_d\mathcal{D}(\mathcal{P}_2) = d^2 - 5d + 3, \quad d = 5, 6, \ldots, 12,$$

and

$$\dim \mathcal{F}_d\mathcal{D}(\mathcal{Z}_1) = \dim \mathcal{F}_d\mathcal{D}(\mathcal{Z}_2) = d^2 - 5d, \quad d = 6, \ldots, 12$$

We give some examples of line arrangements with a singularity of "big" multiplicity, in order to study the influence of different combinatorics in the quadratic growth in the filtration. Let $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4$ and $\mathcal{B}_5$ the line arrangements defined by

$$Q_{\mathcal{B}_1} = xy(x+y)(x-y)(2x+y)(x+2y)(2x-y)(x-2y),$$

$$Q_{\mathcal{B}_2} = (y+1)Q_{\mathcal{B}_1}, \quad Q_{\mathcal{B}_3} = (y-1)Q_{\mathcal{B}_2}, \quad Q_{\mathcal{B}_4} = (x^2-1)Q_{\mathcal{B}_3}, \quad Q_{\mathcal{B}_5} = \frac{Q_{\mathcal{B}_4}}{xy},$$

which are pictured in Figure VIII.6.

Sage
```
sage: B1=[(1,0,0), (0,1,0), (1,1,0), (1,-1,0), (2,1,0), (1,2,0), (2,-1,0),
(-1,2,0)]

sage: L=map(lambda d: len(basis_vector_field_from_arrangement(R,B1,d)),
range(1,dmax+1)); print L

[1, 3, 6, 10, 15, 21, 29, 39, 51, 65, 81, 99]
```

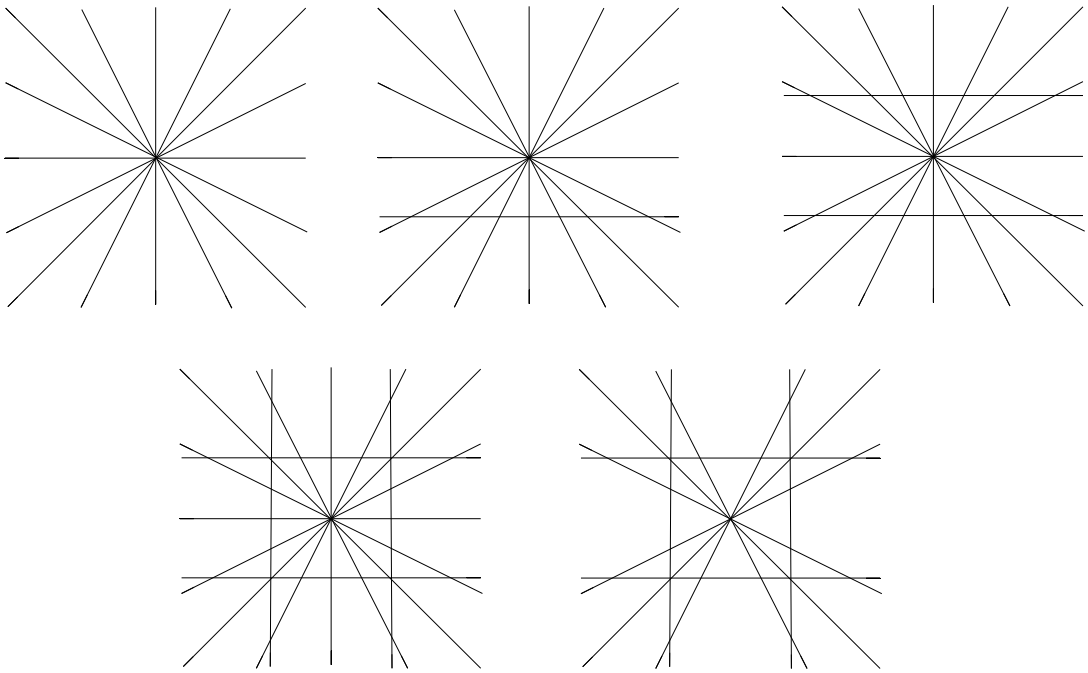

Figure VIII.6: Line arrangements $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ (up) and $\mathcal{B}_4$, $\mathcal{B}_5$ (down).

Sage
```
sage: B2=[(1,0,0), (0,1,0), (1,1,0), (1,-1,0), (2,1,0), (1,2,0), (2,-1,0),
(-1,2,0), (0,1,1)]

sage: L=map(lambda d: len(basis_vector_field_from_arrangement(R,B2,d)),
range(1,dmax+1)); print L

[0, 1, 3, 6, 10, 15, 22, 31, 42, 55, 70, 87]
```

Sage
```
sage: B3=[(1,0,0), (0,1,0), (1,1,0), (1,-1,0), (2,1,0), (1,2,0), (2,-1,0),
(-1,2,0), (0,1,1), (0,1,-1)]

sage: L=map(lambda d: len(basis_vector_field_from_arrangement(R,B3,d)),
range(1,dmax+1)); print L

[0, 0, 1, 3, 6, 10, 16, 24, 34, 46, 60, 76]
```

Sage
```
sage: B4=[(1,0,0), (0,1,0), (1,1,0), (1,-1,0), (2,1,0), (1,2,0), (2,-1,0),
(-1,2,0), (0,1,1), (0,1,-1), (1,0,1), (1,0,-1)]

sage: L=map(lambda d: len(basis_vector_field_from_arrangement(R,B4,d)),
range(1,dmax+1)); print L

[0, 0, 0, 0, 1, 3, 7, 13, 21, 31, 43, 57]
```

Sage
```
sage: B5=[(1,1,0), (1,-1,0), (2,1,0), (1,2,0), (2,-1,0), (-1,2,0), (0,1,1),
(0,1,-1), (1,0,1), (1,0,-1)]

sage: L=map(lambda d: len(basis_vector_field_from_arrangement(R,B5,d)),
range(1,dmax+1)); print L

[0, 0, 0, 0, 1, 3, 8, 16, 26, 38, 52, 68]
```

Summarizing these results, we can express the dimensions of the filtration as quadratic expressions from a suitable minimal degree, for each arrangement.

$$\begin{aligned}
\dim \mathcal{F}_d\mathcal{D}(\mathcal{B}_1) &= d^2 - 5d + 15, & d &= 5, \dots, 12\\
\dim \mathcal{F}_d\mathcal{D}(\mathcal{B}_2) &= d^2 - 6d + 15, & d &= 5, \dots, 12\\
\dim \mathcal{F}_d\mathcal{D}(\mathcal{B}_3) &= d^2 - 7d + 16, & d &= 5, \dots, 12\\
\dim \mathcal{F}_d\mathcal{D}(\mathcal{B}_4) &= d^2 - 9d + 21, & d &= 5, \dots, 12\\
\dim \mathcal{F}_d\mathcal{D}(\mathcal{B}_5) &= d^2 - 7d + 8, & d &= 7, \dots, 12
\end{aligned}$$

We remark that the linear term of the expression is always 3 minus the number of lines. In addition, it seem that singularities of multiplicity greater than two affects to the constant term as well as parallel lines. The families of exactly k parallel lines, for $k \geq 2$, can be viewed in the projective completion $\mathbb{P}^2_k = \mathbb{A}^2_k \cup L_\infty$ as singularities of multiplicity $k+1$ when the k parallel lines meet at the line of infinity L_∞.

Remark VIII.5.7. From Proposition VIII.5.3, we can develop

$$\begin{aligned}
\dim \mathcal{F}_d\mathcal{D}(\mathcal{A}) &= (d+1)(d+2) - \operatorname{rank} M_d(\mathcal{A})\\
&= (d+1)(d+2) - |\mathcal{A}|(d+1) + \operatorname{corank} M_d(\mathcal{A})\\
&= d^2 + (3 - |\mathcal{A}|)d + 2 - |\mathcal{A}| + \operatorname{corank} M_d(\mathcal{A}), \qquad \text{(VIII.8)}
\end{aligned}$$

where $3 - |\mathcal{A}|$ correspond with the coefficient of monomial in degree one of the quadratic formula found in the examples.

We can ask Question VIII.5.6 following the following result.

Theorem VIII.5.8. *Consider the weak combinatorics of an affine line arrangement $\mathcal{A}$, given by*

- *The number of lines $|\mathcal{A}| = n$.*
- *The sequence $\mathcal{S}_\mathcal{A} = (s_2, s_3, \dots, s_n)$, where s_k denotes the number of singular points of multiplicity k in $\mathcal{A}$, for $k = 2, \dots, n$.*
- *The sequence $\mathcal{P}_\mathcal{A} = (p_2, p_3, \dots, p_n)$, where p_k denotes the number of families of exactly $(k-1)$ parallel lines in $\mathcal{A}$, for $k = 2, \dots, n$.*

There exists a positive integer d_0 such that for any $\forall d \geq d_0$:

$$\dim \mathcal{F}_d\mathcal{D}(\mathcal{A}) = d^2 + (3-n)d + \left[2 - n + \sum_{k=3}^{n} \binom{k-1}{2}(s_k + p_k)\right].$$

Prof. J. Vallès has proposed a sheaf-theoretic proof of the previous Theorem based in the following relation between filtrations spaces and the vector spaces of global sections of the logarithmic bundle $\mathcal{T}_{\widetilde{\mathcal{A}}}(d))$ in $\mathbb{P}^2_k$:

Proposition VIII.5.9. *Let $\mathcal{A}$ be an affine line arrangement and consider $\widetilde{\mathcal{A}}$ its projectivization in $\mathbb{P}^2_k$. Then, for any $d \in \mathbb{N}$:*

$$\dim \mathcal{F}_d\mathcal{D}(\mathcal{A}) = \dim \Gamma(\mathbb{P}^2_k, \mathcal{T}_{\widetilde{\mathcal{A}}}(d)).$$

Proof. Let $\mathrm{c}\mathcal{A}$ be the coning of $\mathcal{A}$ in $\mathbb{A}^3_K K$. We have described in Section VII.6 a bijective procedure to obtain homogeneous derivations of $\mathcal{D}_z(\mathrm{c}\mathcal{A})$ of degree d from derivations of $\mathcal{F}_d\mathcal{D}(\mathcal{A})$. Using Proposition VII.6.2 and the fact that $\mathcal{D}_0(\mathrm{c}\mathcal{A})$ and $\mathcal{D}_z(\mathrm{c}\mathcal{A})$ are isomorphic graded modules, we obtain that:

$$\dim \mathcal{F}_d\mathcal{D}(\mathcal{A}) = \dim(\mathcal{D}_z(\mathrm{c}\mathcal{A}))_d = \dim(\mathcal{D}_0(\mathrm{c}\mathcal{A}))_d.$$

By construction, $\dim \Gamma(\mathbb{P}^2_k, \mathcal{T}_{\widetilde{\mathcal{A}}}(d)) = \dim \mathcal{D}_0(\mathrm{c}\mathcal{A})$, and the result holds. □

Proof of Theorem VIII.5.8. Let $\widetilde{\mathcal{A}}$ be the projectivization of $\mathcal{A}$ in $\mathbb{P}^2_k$. By a classical result of J. P. Serre [Liu02, Thm. 3.2, p. 195], we known that the p–cohomology for $p \geq 1$ of the twisted logarithmic bundle $\mathcal{T}_{\widetilde{\mathcal{A}}}(d)$ over projective line arrangements vanishes for any sufficiently large d, i.e. there exist a $d_0 \gg 0$ such that

$$H^p(\mathbb{P}^2_k, \mathcal{T}_{\widetilde{\mathcal{A}}}(d)) = 0, \quad \forall d \geq d_0,$$

for any $p \geq 1$. Thus, computing the Euler-Poincaré characteristic of $\mathcal{T}_{\widetilde{\mathcal{A}}}(d)$:

$$\chi(\mathcal{T}_{\widetilde{\mathcal{A}}}(d)) = \dim H^0(\mathbb{P}^2_k, \mathcal{T}_{\widetilde{\mathcal{A}}}(d)) - \dim H^1(\mathbb{P}^2_k, \mathcal{T}_{\widetilde{\mathcal{A}}}(d)) + \dim H^2(\mathbb{P}^2_k, \mathcal{T}_{\widetilde{\mathcal{A}}}(d)).$$

We obtain that, for any $d \geq d_0$:

$$\chi(\mathcal{T}_{\widetilde{\mathcal{A}}}(d)) = \dim H^0(\mathbb{P}^2_k, \mathcal{T}_{\widetilde{\mathcal{A}}}(d)) = \dim \Gamma(\mathbb{P}^2_k, \mathcal{T}_{\widetilde{\mathcal{A}}}(d)).$$

In the other hand, using the Riemann-Roch Theorem, $\chi(\mathcal{T}_{\widetilde{\mathcal{A}}}(d))$ can be obtained in terms of the Chern classes $c_1(\mathcal{T}_{\widetilde{\mathcal{A}}}(d))$ and $c_2(\mathcal{T}_{\widetilde{\mathcal{A}}}(d))$ of the twisted logarithmic bundle $\mathcal{T}_{\widetilde{\mathcal{A}}}(d)$ (see [Har78, p. 242]):

$$\chi(\mathcal{T}_{\widetilde{\mathcal{A}}}(d)) = \frac{1}{2}(c_1(\mathcal{T}_{\widetilde{\mathcal{A}}}(d))^2 - 2c_2(\mathcal{T}_{\widetilde{\mathcal{A}}}(d)) + 3c_1(\mathcal{T}_{\widetilde{\mathcal{A}}}(d)) + 4).$$

We can describe the Chern classes of $\mathcal{T}_{\widetilde{\mathcal{A}}}(d) = \mathcal{T}_{\widetilde{\mathcal{A}}} \otimes \mathcal{O}_{\mathbb{P}^2_k}(d)$ by those of $\mathcal{T}_{\widetilde{\mathcal{A}}}$. Since $\mathcal{T}_{\widetilde{\mathcal{A}}}$ and $\mathcal{O}_{\mathbb{P}^2_k}(d)$ are respectively a rank two and a rank one vector bundles, we have that $c_1(\mathcal{T}_{\widetilde{\mathcal{A}}}(d)) = c_1(\mathcal{T}_{\widetilde{\mathcal{A}}}) + 2d$ and $c_2(\mathcal{T}_{\widetilde{\mathcal{A}}}(d)) = c_2 + 2dc_1 + d^2$, where $c_1 = c_1(\mathcal{T}_{\widetilde{\mathcal{A}}})$ and $c_2 = c_2(\mathcal{T}_{\widetilde{\mathcal{A}}})$ (see [Rub14, pag. 24]). This gives $\dim \Gamma(\mathbb{P}^2_k, \mathcal{T}_{\widetilde{\mathcal{A}}}(d))$ described as a quadratic polynomial in d:

$$\dim \Gamma(\mathbb{P}^2_k, \mathcal{T}_{\widetilde{\mathcal{A}}}(d)) = \frac{1}{2}(2d^2 + 2(c_1 + 3)d + c_1^2 + 3c_1 - 2c_2 + 4). \tag{VIII.9}$$

Moreover, these Chern classes are determined by the weak combinatorics of the projective arrangement [FV12b, Remark 2.2]:

$$c_1 = 1 - |\widetilde{\mathcal{A}}| \quad \text{and} \quad c_2 = \binom{|\widetilde{\mathcal{A}}| - 1}{2} - \sum_{k \geq 3} \binom{k-1}{2} t_k,$$

where t_k is the number of singular points in $\widetilde{\mathcal{A}}$ of multiplicity k. In this way, we obtain from (VIII.9):

$$\dim \Gamma(\mathbb{P}^2_k, \mathcal{T}_{\widetilde{\mathcal{A}}}(d)) = d^2 + (4 - |\widetilde{\mathcal{A}}|)d + \left[3 - |\widetilde{\mathcal{A}}| + \sum_{k\geq 3} \binom{k-1}{2} t_k\right].$$

Finally, we express the weak combinatorics of $\widetilde{\mathcal{A}}$ from those of $\mathcal{A}$ by Proposition VI.1.15: $|\widetilde{\mathcal{A}}| = |\mathcal{A}| + 1$ and $t_k = s_k + p_k$ for any $k \geq 2$. Thus, the result holds. □

Using the previous computations on the matrices of conditions of each $\mathcal{F}_d(\mathcal{A})$, we can look for a direct proof of the previous Theorem without using the formalism of sheaves in projective plane, for any field of characteristic zero, and deep result on coherent sheaves given by Serre. In particular, it reduces to the computation of corank $M_d(\mathcal{A})$. Precisely, we can formulate an equivalent statement of this Theorem using (VIII.8):

Corollary VIII.5.10. *Consider the weak combinatorics of an affine line arrangement $\mathcal{A}$ as before and the matrices $M_d(\mathcal{A})$ defined in (VIII.6). There exists a positive integer d_0 such that for any $\forall d \geq d_0$:*

$$\operatorname{corank} M_d(\mathcal{A}) = 2 - n + \sum_{k=3}^{n} \binom{k-1}{2}(s_k + p_k).$$

Remark VIII.5.11. If we look at the previous examples, we observe that d_0 is strictly smaller than the number of lines of each arrangement.

Question VIII.5.12. *Is the positive integer d_0 combinatorially determined, i.e. $d_0 = d_0(L(\mathcal{A}))$?*

Chapter IX

Perspectives on logarithmic vector fields for line arrangements

IX.1 Conclusions and perspectives

The results of this part show the classical approach on the study of logarithmic vector fields in algebraic geometry can be enriched with informations of dynamical origin and reciprocally. In particular, the result concerning affine line arrangements can be considered as a first step in our approach to the Terao conjecture about freeness of combinatorially equivalent line arrangements in $\mathbb{P}^2_k$ from a dynamical point of view.

In [Car81, p.19], P. Cartier states that the geometrical interpretation of the freeness condition for a line arrangement is "obscure". His comment relies on the fact that **freeness does not seems to be related to any geometrical particularities** in the simple case of simplicial line arrangements classified by Grünbaum [Grü09]. Our previous approach suggest to **look for a dynamical interpretation of freeness**. The understanding of $d_f(\mathcal{A})$, and more generally of $\mathcal{D}^f_d(\mathcal{A})$, is a first necessary step in this dynamical approach. The next step will be to dynamically characterize free arrangements.

Here, we give a non-exhaustive list of possible extensions of the present work.

Freeness and dynamics

Line arrangements in the affine plane are always free, as we have seen in Section VIII.1. Our purpose is to introduce the dynamical approach for central plane arrangements in $\mathbb{A}^3_k$ (resp. projective line arrangements in $\mathbb{P}^2_k$) via the coning $\mathrm{c}\mathcal{A}$ (resp. the projectivization $\widetilde{\mathcal{A}} = \mathcal{A} \cup L_\infty$), as we have detailed in Section VII.6. In particular, we have observed significant differences for positions of equilibrium points of logarithmic vector fields in minimal degree between the free and non-free case of projective line arrangements. Indeed, in the free case it seems that any section of minimal degree induces logarithmic derivations which possess equilibrium points only on the arrangement. Thus, it would be interesting to study the possible constraints coming from freeness on the nature and positions of equilibrium points of logarithmic vector fields.

A complete proof of the quadratic formula

We give in Section VIII.5 a first matrix analysis of the linear application which determines each stratum $\mathcal{F}_d\mathcal{D}(\mathcal{A})$, remarking that the sequence $(\nu_d)_{d\in\mathbb{N}}$ defined by the ranks $\nu_d =$

$\dim \mathcal{F}_d\mathcal{D}(\mathcal{A})$ verifies a quadratic expression depending on the degree d. This formula is proved using deep results in algebraic geometry, as well as the fact that it is determined by the combinatorics of the arrangement, from a certain $d_0 \geq 0$. On the other side, a direct proof can certainly be derived from elementary computations on the matrix introduced to study $\mathcal{F}_d\mathcal{D}(\mathcal{A})$.

We show that this formula is equivalent to prove that the corank of the matrix $M_d(\mathcal{A})$ defined in (VIII.6) becomes constant for d greater than a certain d_0. From Proposition VIII.5.4 and Lemma VIII.5.5, it would be sufficient to study the sub-matrix $N_d^{0,0}$, which contains two Vandermonde's matrices depending in the coefficients of the lines, whose must reflect the combinatorial conditions of the arrangement. In this way, the minimal $d_0 \geq 0$ and the formula for the corank of $M_d(\mathcal{A})$ could be determined, as well as the combinatorial or geometrical influence of $\mathcal{A}$ on d_0.

From the previous discussion about projectivization and freeness, it would be interesting to determine if the first terms in the sequence $(\nu_d)_{d\in\mathbb{N}}$ gives information about the freeness of its projectivization $\widetilde{\mathcal{A}}$.

A functional library for line/hyperplane arrangements in `Sage`

In the development of Sections VIII.4 and VIII.5, we have extensively performed computations and analysis using the open-source mathematical software `Sage`. The automation of these kinds of computations for examples of line arrangements have raised into computational functions. We describe these functions in our code `line_arrangements_vector_fields.sage`, described in Appendix B. Specifically, we have taken advantage of the good functionalities which `Sage` has for working in polynomials and modules with coefficients in number fields.

Our intention is firstly to debug this code, and then to extend the procedures to any polynomials with coefficients in any field: as algebraic real and complex numbers, or numbers in floating point. Secondly, we would like to develop a complete library/module for *hyperplane arrangements* over fields of any characteristic on future versions of `Sage`, adding functions to compute combinatorial, algebraic and geometric invariants of hyperplane arrangements.

Polynomial and modular computations in `Sage` are based in an interface for the free computer algebra system `Singular`, which provides a good computational background for commutative algebra, algebraic geometry and singularity theory. Thus, we can also extend the computational analysis of the module of logarithmic vector fields of central and projective arrangements, taking advantage of computational resolution of modules and calculus of algebraic invariants.

IX.2 Conclusions et perspectives (French)

Les résultats de cette partie montrent que l'approche classique des champs de vecteurs logarithmiques peut être enrichie par des informations d'origines dynamiques et réciproquement. En particulier, les résultats concernant les arrangements affines de droites peuvent être considérés comme une première étape de notre approche de la conjecture de Terao –annonçant la combinatorialité de la liberté d'un arrangement– d'un point de vu dynamique.

Dans [Car81, p.19], P. Cartier dit : ≪ La signification géométrique de l'hypothèse de "liberté" reste obscure ≫. Son commentaire repose sur le fait que la notion de liberté ne semble pas reliée à une particularité géométrique dans le cas simple des arrangements de droites simpliciaux classifiés par Grünbaum [Grü09]. Notre approche précédente suggère de chercher une interprétation dynamique de la notion de liberté. La compréhension de $d_f(\mathcal{A})$, et plus généralement de $\mathcal{D}_d^f(\mathcal{A})$ est une première étape nécessaire de cette approche dynamique. La suite sera de donner un

caractérisation dynamique de la notion de liberté d'un arrangement.

Nous donnons maintenant une liste non-exhaustive de possible continuation des travaux présentés dans ce manuscrit.

Liberté et dynamique

Les arrangements de droites dans le plan affine sont toujours libres, comme nous l'avons vu dans la Section VIII.1. Notre but est d'introduire notre approche dynamique pour les arrangements de plans centraux dans $\mathbb{A}^3_k$ (resp. les arrangements de droites projectifs dans $\mathbb{P}^2_k$) via la conification $\mathrm{c}\mathcal{A}$ (resp. la projectivisation $\widetilde{\mathcal{A}} = \mathcal{A} \cup L_\infty$), comme nous l'avons détaillé dans la Section VII.6. En particulier, nous avons observé des différences significatives pour la position des points d'équilibres des champs de vecteurs logarithmiques en degré minimal entre le cas non-libre et le cas libre d'un arrangement projectif de droites. En effet, dans le cas libre, il semble que toute section de degré minimal induise un champs de vecteurs polynomial ayant ses points d'équilibre uniquement sur l'arrangment. Ainsi il serait intéressant d'étudier la possible contrainte provenant de la liberté sur la nature et la position des points d'équilibre des champs de vecteurs logarithmiques.

Une preuve complète de la formule quadratique

Nous donnons dans la Section VIII.5 une première matrice provenant des applications linéraires définissant l'arrangement, et déterminant les différentes espaces de filtration $\mathcal{F}_d\mathcal{D}(\mathcal{A})$. Et ce, en remarquant que cette séquence $(\nu_d)_{d\in\mathbb{N}}$ définie par les rangs $\nu_d = \dim \mathcal{F}_d\mathcal{D}(\mathcal{A})$ vérifie une expression quadratique dépendant du degré d et de la combinatoire faible. Cette formule est prouvée en utilisant de résultats profonds de géométrie algébrique, ainsi que le fait qu'elle est déterminé par la combinatoire de l'arrangement, à partir d'un certain degré $d_0 \geq 0$. D'un autre côté, une preuve directe peut certainement être déduite de calculs élémentaires sur la matrice introduite dans l'étude de $\mathcal{F}_d\mathcal{D}(\mathcal{A})$.

Nous avons prouvé que cette formule est équivalente au fait que le co-rang de la matrice $M_d(\mathcal{A})$ définie dans (VIII.6) reste constant lorsque d est plus grand qu'un certain d_0. A partir de la Proposition VIII.5.4 et du Lemme VIII.5.5, il serait suffisant d'étudier la sous-matrice $N_d^{0,0}$ (qui contient deux matrices de Vandermonde dépendant des coefficients des équations des droites) qui doit refléter les conditions combinatoires de l'arrangement. Dans ce sens, le degré minimal d_0 et la formule du co-rang de $M_d(\mathcal{A})$ pourraient être déterminés, ainsi que l'influence combinatoire ou géométrique de $\mathcal{A}$ sur d_0.

A partir de la discussion précédente sur la projectivisation et la liberté, il serait intéressant de déterminer si les premiers termes de la séquence $(\nu_d)_{d\in\mathbb{N}}$ donnent des information sur la liberté de l'arrangement projectivisé $\widetilde{\mathcal{A}}$.

Une libraire fonctionnelle `Sage` pour les arrangements de droites/hyperplans

Dans le développement des Sections VIII.4 et VIII.5, nous avons beaucoup utilisé de calculs et d'analyse utilisant le logiciel mathématiques open-source `Sage`. L'automatisation de ce type de calculs pour des examples d'arrangemnts de droites a été traduite en fonctions de calcul formel. Nous donnons ces fonctions dans notre code `line_arrangements_vector_fields.sage`, qui sont décrites dans l'Annexe B. Nous avons en particulier tiré un avantage des bonnes fonctionnalités que `Sage` a pour travailler avec des polynômes et des modules à coefficients dans un corps de nombres.

Notre intention premier est de débugguer ce code, et de l'étendre à des polynômes à coefficients dans n'importe quel corps comme : les nombres algébriques réels ou complexes ou les nombres à virgule flottantes. Ensuite, nous voudrions developper dans les futures version de `Sage` une librairie complète pour les arrangements d'hyperplans définis sur un corps de caractéristique quelconque, et ce en ajoutant des fonctions calculant des invariants/propriétés combinatoires, algébriques ou géométriques des arrangements d'hyperplans.

Les calculs polynomiaux et modulaires dans `Sage` sont basés sur l'interface libre de calcul algébrique `Singular`, qui fourni un bon support pour l'algèbre commutative, la géométrie algébrique et la théorie des singularités. Ainsi, nous pouvons étendre l'analyse des calculs pour les module des champs de vecteurs logarithmiques des arrangements projectifs ou centraux, en tirant partie des avantages des résolutions des modules et des calculs des invariants algébriques.

IX.3 Conclusiones y perspectivas (Spanish)

Los resultados de esta parte muestran que el enfoque clásico en el estudio de campos vectoriales logarítmicos en geometría algebraica se puede enriquecer con informaciones de origen dinámico y recíprocamente. En particular, el resultado con respecto a una configuración de rectas afines se puede considerar un primer paso en nuestro enfoque sobre la Conjetura de Terao – que trata sobre la libertad de configuraciones de rectas combinatorialmente equivalentes en $\mathbb{P}^2_k$ – desde un punto de vista más dinámico.

En [Car81, p.19], P. Cartier enuncia que la interpretación geométrica de la condición de libertad para configuraciones de rectas es "oscura". Este comentario proviene del hecho de que la **libertad no parece estar relacionada con ninguna particularidad geométrica** en el caso simple de configuraciones de rectas simpliciales, clasificadas por Grünbaum [Grü09]. El enfoque previamente descrito sugiere **buscar una interpretación dinámica de la libertad**. El conocimiento de $d_f(\mathcal{A})$, y más generalmente de $\mathcal{D}^f_d(\mathcal{A})$, es un primer paso necesario en este enfoque dinámico. El siguiente paso será de caracterizar dinámicamente configuraciones libres.

Proponemos una lista no exhaustiva de posibles ampliaciones del presente trabajo.

Libertad y dinámica

Las configuraciones de rectas en el plano afín son siempre libres, como hemos visto en la Sección VIII.1. Nuestro objetivo es introducir el enfoque dinámico para configuraciones de planos centrales en $\mathbb{A}^3_k$ (respectivamente configuraciones de rectas proyectivas en $\mathbb{P}^2_k$) a través del conificado $\mathrm{c}\mathcal{A}$ (respectivamente, la proyectivización $\widetilde{\mathcal{A}} = \mathcal{A} \cup L_\infty$), como hemos detallado en la Sección VII.6. Concretamente, hemos observado diferencias significantes entre el caso libre y no libre de las posiciones de puntos de equilibrio de campos vectoriales logarítmicos de grado mínimo de configuraciones de rectas proyectivas. De hecho, en el caso libre parece que cualquier sección de grado mínimo induce derivaciones logarítmicas con puntos de equilibrio yaciendo únicamente sobre la configuración. De este modo, sería interesante estudiar las posibles restricciones procedentes de la libertad sobre la naturaleza y las posiciones de puntos de equilibrio de campos vectoriales logarítmicos.

Una prueba completa de la fórmula cuadrática

En la Sección VIII.5 ofrecemos un primer análisis matricial de la aplicación lineal que determina cada estrato $\mathcal{F}_d\mathcal{D}(\mathcal{A})$, destacando que la sucesión $(\nu_d)_{d\in\mathbb{N}}$ definida por los rangos

$\nu_d = \dim \mathcal{F}_d\mathcal{D}(\mathcal{A})$ verifica una expresión cuadrática que depende del grado d. Esta fórmula puede ser probada usando profundos resultados de geometría algebraica, así como el hecho de que está determinada por la combinatoria de las configuraciones, desde un cierto $d_0 \geq 0$. Por otra parte, una prueba directa podría derivarse de cómputos elementales sobre la matriz previamente introducida para estudiar $\mathcal{F}_d\mathcal{D}(\mathcal{A})$.

Mostramos que probar dicha fórmula es equivalente a probar que el corango de la matriz $M_d(\mathcal{A})$ definida en (VIII.6) es constante para todo d superior a un cierto d_0. A partir de la Proposición VIII.5.4 y el Lema VIII.5.5, sería suficiente con estudiar la submatriz $N_d^{0,0}$, quien contiene dos matrices de Vandermonde que dependen de los coeficientes de las rectas, que deben reflejar las condiciones combinatorias de las configuraciones. De esta forma, el mínimo $d_0 \geq 0$ y la fórmula para el corango de $M_d(\mathcal{A})$ podrían determinarse, así como la influencia combinatoria o geométrica de $\mathcal{A}$ sobre un tal d_0.

Siguiendo la discusión precedente sobre proyectivización y libertad, sería interesante determinar si los primeros términos de la sucesión $(\nu_d)_{d\in\mathbb{N}}$ ofrecen información sobre la libertad de su proyectivización $\widetilde{\mathcal{A}}$.

Una librería funcional para configuraciones de rectas/hiperplanos en `Sage`

En el desarrollo de las Secciones VIII.4 y VIII.5, hemos realizado una gran cantidad de cálculos y análisis utilizando el sistema algebraico computacional de código abierto `Sage`. La automatización de este tipo de cálculos para ejemplos de configuraciones de rectas han terminado por desarrollar funciones computacionales. Estas funciones son desarrolladas en nuestro código `line_arrangements_vector_fields.sage`, descrito explícitamente en el Apéndice B. Más específicamente, se han aprovechado las buenas funcionalidades que `Sage` posee para el trabajo con polinomios y módulos polinómicos con coeficientes en cuerpos de números.

Nuestra intención es, en primer lugar, de depurar este código, para entonces extender las funciones a polinomios con coeficientes en cualquier cuerpo: como números algebraicos reales y complejos, o números en coma flotante. Seguidamente, nos gustaría desarrollar una librería o módulo completos para *configuraciones de hiperplanos* sobre cuerpos de cualquier característica para futuras versiones de `Sage`, añadiendo funciones para el cálculo de invariantes combinatorios, algebraicos y geométricos de configuraciones de hiperplanos.

La computación sobre polinomios y módulos en `Sage` está basada en una interfaz para el sistema algebraico computacional libre `Singular`, que provee un buen trasfondo computacional para álgebra conmutativa, geometría algebraica y teoría de singularidades. De esta forma, también podremos extender este análisis computacional del módulo de campos vectoriales logarítmicos a configuraciones centrales y proyectivas, aprovechando la computación de resoluciones de módulos y del cálculo de invariantes algebraicos.

APPENDICES

Computations and codes

Appendix A

Computations of filtrations for Pappus, non-Pappus, Ziegler and non-Ziegler arrangements

In order to give a complete proof of Proposition VIII.4.4 and Proposition VIII.4.7, we present the explicit calculations to obtain the matrices $M_{\mathcal{P}_1}$, $M_{\mathcal{P}_2}$, $M_{\mathcal{Z}_1}$ and $M_{\mathcal{Z}_2}$ for which $\mathcal{F}_4\mathcal{D}(\mathcal{P}_1)$, $\mathcal{F}_5\mathcal{D}(\mathcal{P}_2)$, $\mathcal{F}_5\mathcal{D}(\mathcal{Z}_1)$ and $\mathcal{F}_6\mathcal{D}(\mathcal{Z}_2)$ are the kernels in the respective space of coefficients. As these matrices have $n(d+1)$ rows and $(d+1)(d+2)$ columns, where n is the number of lines in each line arrangement and d is the degree in the filtration $\mathcal{F}_d$, we describe them decomposed by columns in the last pages of the present document. These matrix represents linear maps $\mathcal{C}(d) \to \mathbb{R}^{n(d+1)}$, taking the basis in the space of coefficients in inverse degree lexicographic order, i.e.

$$\mathscr{B}_{\mathcal{C}(d)} = \{a_{d0}, a_{d-1,1}, \ldots, a_{0d}, a_{d-1,0}, a_{d-2,1}, \ldots, a_{00}, b_{d0}, b_{d-1,1}, \ldots, b_{0d}, b_{d-1,0}, b_{d-2,1}, \ldots, b_{00}\}$$

where $\chi \in \mathrm{Der}_{\mathbb{R}}(\mathbb{R}[x,y])$ with $P = \sum_{i,j=0}^{i+j\leq d} a_{ij}x^iy^j$ and $Q = \sum_{i,j=0}^{i+j\leq d} b_{ij}x^iy^j$.

The details of the computations are given in the following pages.

- **Pappus**: $M_{\mathcal{P}_1} \in \mathscr{M}_{40\times 30}(\mathbb{R})$ has rank 29 and $\ker M_{\mathcal{P}_1} = \mathbb{R}\langle(4,-4,-1,10,0,-4,-2,-3,0,-1,0,0,1,0,0,0,0,8,-4,5,0,-8,0,-1,0,4,-5,0,1,0)\rangle$, then:

$$\mathcal{F}_4\mathcal{D}(\mathcal{P}_1) = \mathbb{R}\langle(4x^4 - 4x^3y - x^2y^2 + 10xy^3 - 4x^3 - 2x^2y - 3xy^2 - x^2 + x)\partial_x + (8x^2y^2 - 4xy^3 + 5y^4 - 8x^2y - y^3 + 4xy - 5y^2 + y)\partial_y\rangle$$

- **Non-Pappus**: $M_{\mathcal{P}_2} = \left[M^{(1)}_{\mathcal{P}_2} \mid M^{(2)}_{\mathcal{P}_2}\right] \in \mathscr{M}_{48\times 42}(\mathbb{R})$ has rank 39. If we compute the kernel with `Sage`:

$$\begin{aligned}\ker M_{\mathcal{P}_2} = \mathbb{R}\langle &\left(1,0,0,-5,0,0,-\frac{5}{6},-\frac{40}{3},-\frac{110}{3},-15,0,-\frac{155}{6},-85,-110,0,-\frac{105}{2},-135,0,-\frac{57}{2},0,0,0,0,-\frac{65}{2},-\frac{265}{2},-190,-84,0,-\frac{65}{2},-\frac{1475}{6},-\frac{1025}{2},-290,0,-\frac{340}{3},\right.\\ &\left.-\frac{815}{2},-345,0,-85,-\frac{335}{2},0,-\frac{57}{2},0\right),\\ &\left(0,1,0,2,0,0,\frac{8}{3},\frac{29}{3},\frac{52}{3},6,0,\frac{44}{3},41,52,0,24,63,0,12,0,0,0,0,13,54,80,36,0,13,\frac{299}{3},215,124,0,\frac{137}{3},167,147,0,32,71,0,12,0\right),\\ &\left(0,0,1,0,0,0,-\frac{2}{3},-\frac{2}{3},-\frac{4}{3},0,0,-\frac{11}{3},-8,-13,0,-6,-18,0,-3,0,0,0,0,-4,-17,-26,-12,0,-4,-\frac{92}{3},-68,-40,0,-\frac{41}{3},-50,-45,0,-8,-20,0,-3,0\right)\rangle\end{aligned}$$

We obtain a 3–dimensional vector space:

$$\begin{aligned}\mathcal{F}_5\mathcal{D}(\mathcal{P}_2) = \mathbb{R}\langle &\left(x^5 - 5x^2y^3 - \frac{5}{6}x^4 - \frac{40}{3}x^3y - \frac{110}{3}x^2y^2 - 15xy^3 - \frac{155}{6}x^3 - 85x^2y - 110xy^2 - \frac{105}{2}x^2 - 135xy - \frac{57}{2}x\right)\partial_x +\\ &\left(-\frac{65}{2}x^3y^2 - \frac{265}{2}x^2y^3 - 190xy^4 - 84y^5 - \frac{65}{2}x^3y - \frac{1475}{6}x^2y^2 - \frac{1025}{2}xy^3 - 290y^4 - \frac{340}{3}x^2y - \frac{815}{2}xy^2 - 345y^3 - 85xy - \frac{335}{2}y^2 - \frac{57}{2}y\right)\partial_y,\\ &\left(x^4y + 2x^2y^3 + \frac{8}{3}x^4 + \frac{29}{3}x^3y + \frac{52}{3}x^2y^2 + 6xy^3 + \frac{44}{3}x^3 + 41x^2y + 52xy^2 + 24x^2 + 63xy + 12x\right)\partial_x +\\ &\left(13x^3y^2 + 54x^2y^3 + 80xy^4 + 36y^5 + 13x^3y + \frac{299}{3}x^2y^2 + 215xy^3 + 124y^4 + \frac{137}{3}x^2y + 167xy^2 + 147y^3 + 32xy + 71y^2 + 12y\right)\partial_y,\\ &\left(x^3y^2 - \frac{2}{3}x^4 - \frac{2}{3}x^3y - \frac{4}{3}x^2y^2 - \frac{11}{3}x^3 - 8x^2y - 13xy^2 - 6x^2 - 18xy - 3x\right)\partial_x +\\ &\left(-4x^3y^2 - 17x^2y^3 - 26xy^4 - 12y^5 - 4x^3y - \frac{92}{3}x^2y^2 - 68xy^3 - 40y^4 - \frac{41}{3}x^2y - 50xy^2 - 45y^3 - 8xy - 20y^2 - 3y\right)\partial_y\rangle\end{aligned}$$

Note that, if we look at the corresponding coefficients associated of monomials of maximal degree $\{a_{50}, a_{41}, a_{32}, a_{23}, a_{14}, a_{05}\}$, it is easy to see that they form a family of independently linear vectors. This shows that they are not polynomial vector fields of degree 4 in $\mathcal{F}_5\mathcal{D}(\mathcal{P}_2)$.

- **Ziegler**: $M_{\mathcal{Z}_1} = \left[M^{(1)}_{\mathcal{Z}_1} \mid M^{(2)}_{\mathcal{Z}_1}\right] \in \mathscr{M}_{48\times 42}(\mathbb{R})$ has rank 41. Then, we have:

$$\begin{aligned}\ker M_{\mathcal{Z}_1} = \mathbb{R}\langle &\left(1,-\frac{553}{165},-\frac{2753}{1485},\frac{1}{9},\frac{4}{495},-\frac{8}{1485},\frac{409}{990},-\frac{3221}{990},-\frac{293}{165},-\frac{14}{165},\frac{2}{165},-\frac{551}{2970},-\frac{301}{330},-\frac{40}{99},-\frac{1}{27},-\frac{67}{990},-\frac{17}{330},-\frac{2}{135},\frac{1}{330},\frac{1}{198},\frac{1}{1485},0,\right.\\ &\left.\frac{21}{5},\frac{191}{495},-\frac{53}{33},-\frac{1}{5},-\frac{4}{495},0,\frac{1553}{495},\frac{127}{330},-\frac{887}{990},-\frac{13}{99},0,\frac{35}{33},\frac{29}{90},\frac{7}{330},0,\frac{31}{165},\frac{17}{495},0,-\frac{2}{495},0\right)\rangle\end{aligned}$$

$$\mathcal{F}_5\mathcal{D}(\mathcal{Z}_1) = \mathbb{R}\langle \left(x^5 - \frac{553}{165}x^4y - \frac{2753}{1485}x^3y^2 + \frac{1}{9}x^2y^3 + \frac{4}{495}xy^4 - \frac{8}{1485}y^5 + \frac{409}{990}x^4 - \frac{3221}{990}x^3y - \frac{293}{165}x^2y^2 - \frac{14}{165}xy^3 + \frac{2}{165}y^4 - \frac{551}{2970}x^3 - \frac{301}{330}x^2y - \frac{40}{99}xy^2 - \frac{1}{27}y^3 - \frac{67}{990}x^2 - \frac{17}{330}xy - \frac{2}{135}y^2 + \frac{1}{330}x + \frac{1}{198}y + \frac{1}{1485}\right)\partial_x + \left(\frac{21}{5}x^4y + \frac{191}{495}x^3y^2 - \frac{53}{33}x^2y^3 - \frac{1}{5}xy^4 - \frac{4}{495}y^5 + \frac{1553}{495}x^3y + \frac{127}{330}x^2y^2 - \frac{887}{990}xy^3 - \frac{13}{99}y^4 + \frac{35}{33}x^2y + \frac{29}{90}xy^2 + \frac{7}{330}y^3 + \frac{31}{165}xy + \frac{17}{495}y^2 - \frac{2}{495}y\right)\partial_y\rangle$$

• **Non-Ziegler**: $M_{\mathcal{Z}_2} = \left[M^{(1)}_{\mathcal{Z}_2} \mid M^{(2)}_{\mathcal{Z}_2} \mid M^{(3)}_{\mathcal{Z}_2} \mid M^{(4)}_{\mathcal{Z}_2}\right] \in \mathscr{M}_{56\times 56}(\mathbb{R})$ has rank 50. As in $\ker M_{\mathcal{P}_2}$, note that if we look at the corresponding coefficients associated of monomials of maximal degree $\{a_{60}, a_{51}, \ldots, a_{15}, a_{06}\}$, they form a family of independently linear vectors. This shows that they are not polynomial vector fields of degree 5 in $\mathcal{F}_6\mathcal{D}(\mathcal{Z}_2)$.

$$\ker M_{\mathcal{Z}_2} = \mathbb{R}\langle \left(1, 0, 0, 0, 0, 0, -\frac{4888}{3955513}, -\frac{539597342087}{297612798120}, \frac{8062854169481}{892838394360}, \frac{258118708997}{48700276056}, -\frac{265101936793}{892838394360}, -\frac{230953906}{2480106651}, \frac{5811061096}{334814397885}, -\frac{758215404767}{892838394360}, \frac{46806536310367}{5357030366160}, \frac{2731333250111}{535703036616}, \frac{35273178832}{111604799295}, -\frac{19124744689}{334814397885}, \frac{360158652101}{714270715488}, \frac{1643511637459}{669628795770}, \frac{508877880523}{446419197180}, \frac{340276246003}{2678515183080}, \frac{3789940805}{24630024672}, \frac{90136185523}{595225596240}, \frac{4548323371}{133925759154}, -\frac{45500308561}{3571353577440}, -\frac{250757939}{24801066510}, -\frac{1211300879}{1190451192480}, 0, \frac{12485601}{3955513}, \frac{13403687}{3955513}, \frac{87837833}{71199234}, -\frac{897283}{11866539}, -\frac{636013}{7911026}, -\frac{69692}{35599617}, 0, -\frac{15544170977}{1740425720}, \frac{153526385251}{178567678872}, \frac{1499386057931}{297612798120}, \frac{99713814197}{178567678872}, -\frac{3400413481}{74403199530}, 0, -\frac{2923922163241}{357135357744}, -\frac{1105655891567}{892838394360}, \frac{6818728061963}{2678515183080}, \frac{1233840981521}{2678515183080}, 0, -\frac{650328921481}{198408532080}, -\frac{391960305599}{324668507040}, -\frac{38122688161}{334814397885}, 0, -\frac{3496777122907}{5357030366160}, -\frac{279792733201}{2142812146464}, 0, \frac{9201428807}{1071406073232}, 0\right),$$

$$\left(0, 1, 0, 0, 0, 0, 0, \frac{272}{136397}, \frac{41457696}{7501835}, -\frac{275296057}{13503303}, -\frac{452871683}{36827190}, \frac{88996553}{135033030}, \frac{5062634}{22505505}, -\frac{7647008}{202549545}, \frac{34184416}{22505505}, -\frac{16587716459}{810198180}, -\frac{38408574637}{3240792720}, -\frac{100544453}{135033030}, \frac{5478397}{40509909}, -\frac{20547016}{13503303}, -\frac{1184995246}{202549545}, -\frac{96240919}{36008808}, -\frac{60694924}{202549545}, -\frac{5273128}{13503303}, -\frac{99295513}{270066060}, -\frac{267259589}{3240792720}, \frac{2363816}{67516515}, \frac{1125907}{45011010}, \frac{181832}{67516515}, 0, -\frac{355908}{136397}, -\frac{927173}{272794}, -\frac{760919}{409191}, \frac{12675}{272794}, \frac{16803}{136397}, \frac{1888}{409191}, 0, \frac{106054214}{4501101}, \frac{123586781}{135033030}, -\frac{161769517}{15003670}, -\frac{84967537}{67516515}, \frac{145531}{1500367}, 0, \frac{2619378923}{135033030}, \frac{3738813317}{1080264240}, -\frac{4573226513}{810198180}, -\frac{429333967}{405099090}, 0, \frac{678745501}{90022020}, \frac{27699053}{9820584}, \frac{230718071}{810198180}, 0, \frac{298776086}{202549545}, \frac{980109359}{3240792720}, 0, -\frac{20250839}{810198180}, 0\right),$$

$$\left(0, 0, 1, 0, 0, 0, -\frac{16888}{3955513}, -\frac{17816738919}{1740425720}, \frac{208721984549}{5221277160}, \frac{11408158051}{474661560}, -\frac{1201970449}{1044255432}, -\frac{276275248}{652659645}, \frac{15604504}{217553215}, -\frac{14690995249}{5221277160}, \frac{1216017502027}{31327662960}, \frac{175849938307}{7831915740}, \frac{933158359}{652659645}, -\frac{165848707}{652659645}, \frac{35320903471}{12531065184}, \frac{85875314303}{7831915740}, \frac{19648214813}{3915957870}, \frac{588730321}{1044255432}, \frac{312574367}{432105696}, \frac{7107723829}{10442554320}, \frac{2415065063}{15663831480}, -\frac{4063466771}{62655325920}, -\frac{123523751}{2610638580}, -\frac{312574367}{62655325920}, 0, \frac{15695064}{3955513}, \frac{29387025}{3955513}, \frac{37476151}{7911026}, \frac{162133}{3955513}, -\frac{1990257}{7911026}, -\frac{49772}{3955513}, 0, -\frac{78494215863}{1740425720}, -\frac{6686695921}{5221277160}, \frac{21907610201}{1044255432}, \frac{13065903209}{5221277160}, -\frac{77793059}{435106430}, 0, -\frac{385824482141}{10442554320}, -\frac{8707229783}{1305319290}, \frac{166027033561}{15663831480}, \frac{31353127921}{15663831480}, 0, -\frac{149217261529}{10442554320}, -\frac{10145990311}{1898646240}, -\frac{94545197}{174042572}, 0, -\frac{86550930347}{31327662960}, -\frac{35716728337}{62655325920}, 0, \frac{510323233}{10442554320}, 0\right)\rangle$$

We obtain a 6–dimensional linear subspace,

$$\mathcal{F}_5\mathcal{D}(\mathcal{P}_2) = \mathbb{R}\langle \left(x^6 - \frac{4888}{3955513}y^6 - \frac{539597342087}{297612798120}x^5 + \frac{8062854169481}{892838394360}x^4y + \frac{258118708997}{48700276056}x^3y^2 - \frac{265101936793}{892838394360}x^2y^3 - \frac{230953906}{2480106651}xy^4 + \frac{5811061096}{334814397885}y^5 - \frac{758215404767}{892838394360}x^4 + \frac{46806536310367}{5357030366160}x^3y + \frac{2731333250111}{535703036616}x^2y^2 + \frac{35273178832}{111604799295}xy^3 - \frac{19124744689}{334814397885}y^4 + \frac{360158652101}{714270715488}x^3 + \frac{1643511637459}{669628795770}x^2y + \frac{508877880523}{446419197180}xy^2 + \frac{340276246003}{2678515183080}y^3 + \frac{3789940805}{24630024672}x^2 + \frac{90136185523}{595225596240}xy + \frac{4548323371}{133925759154}y^2 - \frac{45500308561}{3571353577440}x - \frac{250757939}{24801066510}y - \frac{1211300879}{1190451192480}\right)\partial_x +$$

$$\left(\frac{12485601}{3955513}x^5y + \frac{13403687}{3955513}x^4y^2 + \frac{87837833}{71199234}x^3y^3 - \frac{897283}{11866539}x^2y^4 - \frac{636013}{7911026}xy^5 - \frac{69692}{35599617}y^6 - \frac{15544170977}{1740425720}x^4y + \frac{153526385251}{178567678872}x^3y^2 + \frac{1499386057931}{297612798120}x^2y^3 + \frac{99713814197}{178567678872}xy^4 - \frac{3400413481}{74403199530}y^5 - \frac{2923922163241}{357135357744}x^3y - \frac{1105655891567}{892838394360}x^2y^2 + \frac{6818728061963}{2678515183080}xy^3 + \frac{1233840981521}{2678515183080}y^4 - \frac{650328921481}{198408532080}x^2y - \frac{391960305599}{324668507040}xy^2 - \frac{38122688161}{334814397885}y^3 - \frac{3496777122907}{5357030366160}xy - \frac{279792733201}{2142812146464}y^2 + \frac{9201428807}{1071406073232}y\right)\partial_y,$$

$$\left(x^5y + \frac{272}{136397}y^6 + \frac{41457696}{7501835}x^5 - \frac{275296057}{13503303}x^4y - \frac{452871683}{36827190}x^3y^2 + \frac{88996553}{135033030}x^2y^3 + \frac{5062634}{22505505}xy^4 - \frac{7647008}{202549545}y^5 + \frac{34184416}{22505505}x^4 - \frac{16587716459}{810198180}x^3y - \frac{38408574637}{3240792720}x^2y^2 - \frac{100544453}{135033030}xy^3 + \frac{5478397}{40509909}y^4 - \frac{20547016}{13503303}x^3 - \frac{1184995246}{202549545}x^2y - \frac{96240919}{36008808}xy^2 - \frac{60694924}{202549545}y^3 - \frac{5273128}{13503303}x^2 - \frac{99295513}{270066060}xy - \frac{267259589}{3240792720}y^2 + \frac{2363816}{67516515}x + \frac{1125907}{45011010}y + \frac{181832}{67516515}\right)\partial_x +$$

$$\left(-\frac{355908}{136397}x^5y - \frac{927173}{272794}x^4y^2 - \frac{760919}{409191}x^3y^3 + \frac{12675}{272794}x^2y^4 + \frac{16803}{136397}xy^5 + \frac{1888}{409191}y^6 + \frac{106054214}{4501101}x^4y + \frac{123586781}{135033030}x^3y^2 - \frac{161769517}{15003670}x^2y^3 - \frac{84967537}{67516515}xy^4 + \frac{145531}{1500367}y^5 + \frac{2619378923}{135033030}x^3y + \frac{3738813317}{1080264240}x^2y^2 - \frac{4573226513}{810198180}xy^3 - \frac{429333967}{405099090}y^4 + \frac{678745501}{90022020}x^2y + \frac{27699053}{9820584}xy^2 + \frac{230718071}{810198180}y^3 + \frac{298776086}{202549545}xy + \frac{980109359}{3240792720}y^2 - \frac{20250839}{810198180}y\right)\partial_y,$$

$$\left(x^4y^2 - \frac{16888}{3955513}y^6 - \frac{17816738919}{1740425720}x^5 + \frac{208721984549}{5221277160}x^4y + \frac{11408158051}{474661560}x^3y^2 - \frac{1201970449}{1044255432}x^2y^3 - \frac{276275248}{652659645}xy^4 + \frac{15604504}{217553215}y^5 - \frac{14690995249}{5221277160}x^4 + \frac{1216017502027}{31327662960}x^3y + \frac{175849938307}{7831915740}x^2y^2 + \frac{933158359}{652659645}xy^3 - \frac{165848707}{652659645}y^4 + \frac{35320903471}{12531065184}x^3 + \frac{85875314303}{7831915740}x^2y + \frac{19648214813}{3915957870}xy^2 + \frac{588730321}{1044255432}y^3 + \frac{312574367}{432105696}x^2 + \frac{7107723829}{10442554320}xy + \frac{2415065063}{15663831480}y^2 - \frac{4063466771}{62655325920}x - \frac{123523751}{2610638580}y - \frac{312574367}{62655325920}\right)\partial_x +$$

$$\left(\frac{15695064}{3955513}x^5y + \frac{29387025}{3955513}x^4y^2 + \frac{37476151}{7911026}x^3y^3 + \frac{162133}{3955513}x^2y^4 - \frac{1990257}{7911026}xy^5 - \frac{49772}{3955513}y^6 - \frac{78494215863}{1740425720}x^4y - \frac{6686695921}{5221277160}x^3y^2 + \frac{21907610201}{1044255432}x^2y^3 + \frac{13065903209}{5221277160}xy^4 - \frac{77793059}{435106430}y^5 - \frac{385824482141}{10442554320}x^3y - \frac{8707229783}{1305319290}x^2y^2 + \frac{166027033561}{15663831480}xy^3 + \frac{31353127921}{15663831480}y^4 - \frac{149217261529}{10442554320}x^2y - \frac{10145990311}{1898646240}xy^2 - \frac{94545197}{174042572}y^3 - \frac{86550930347}{31327662960}xy - \frac{35716728337}{62655325920}y^2 + \frac{510323233}{10442554320}y\right)\partial_y\rangle$$

$$M_{\mathcal{P}_1} = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -2 & -3 & -4 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 3 & 6 & 0 & -1 & -2 & -3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -3 & -6 & 0 & 1 & 2 & 3 & -1 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -4 & 0 & 0 & 1 & 3 & 0 & -1 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & -1 & -3 & 0 & 1 & 2 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & -1 \\
2 & -4 & 8 & -16 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 4 & -8 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & -8 & 24 & -64 & -2 & 4 & -8 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 12 & -32 & -1 & 2 & -4 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & -12 & 48 & 0 & -2 & 8 & -24 & 2 & -4 & 8 & 0 & 0 & 0 & 0 & 0 & 1 & -6 & 24 & 0 & -1 & 4 & -12 & 1 & -2 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & -16 & 0 & 0 & -2 & 12 & 0 & 2 & -8 & -2 & 4 & 0 & 0 & 0 & 0 & 1 & -8 & 0 & 0 & -1 & 6 & 0 & 1 & -4 & -1 & 2 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & -2 & 0 & 0 & 2 & 0 & -2 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 1 \\
2 & -4 & 8 & -16 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 4 & -8 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 8 & -24 & 64 & -2 & 4 & -8 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -12 & 32 & -1 & 2 & -4 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & -12 & 48 & 0 & 2 & -8 & 24 & 2 & -4 & 8 & 0 & 0 & 0 & 0 & 0 & 1 & -6 & 24 & 0 & 1 & -4 & 12 & 1 & -2 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 16 & 0 & 0 & -2 & 12 & 0 & -2 & 8 & -2 & 4 & 0 & 0 & 0 & 0 & -1 & 8 & 0 & 0 & -1 & 6 & 0 & -1 & 4 & -1 & 2 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
1250 & 500 & 200 & 80 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3125 & -1250 & -500 & -200 & -80 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 50 & 40 & 24 & \frac{64}{5} & 250 & 100 & 40 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -125 & -100 & -60 & -32 & -625 & -250 & -100 & -40 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & \frac{12}{5} & \frac{48}{25} & 0 & 10 & 8 & \frac{24}{5} & 50 & 20 & 8 & 0 & 0 & 0 & 0 & 0 & -5 & -6 & -\frac{24}{5} & 0 & -25 & -20 & -12 & -125 & -50 & -20 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2}{25} & \frac{16}{125} & 0 & 0 & \frac{2}{5} & \frac{12}{25} & 0 & 2 & \frac{8}{5} & 10 & 4 & 0 & 0 & 0 & 0 & -\frac{1}{5} & -\frac{8}{25} & 0 & 0 & -1 & -\frac{6}{5} & 0 & -5 & -4 & -25 & -10 & 0 \\
0 & 0 & 0 & 0 & \frac{2}{625} & 0 & 0 & 0 & \frac{2}{125} & 0 & 0 & \frac{2}{25} & 0 & \frac{2}{5} & 2 & 0 & 0 & 0 & 0 & -\frac{1}{125} & 0 & 0 & 0 & -\frac{1}{25} & 0 & 0 & -\frac{1}{5} & 0 & -1 & -5
\end{pmatrix}$$

$$M_{\mathcal{P}_2}^{(1)} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -27 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 81 & 0 & 0 & 0 & 0 & 0 & -27 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & -3 & 0 & 0 & 1 & 0 \\
-243 & 0 & 0 & 0 & 0 & 0 & 81 & 0 & 0 & 0 & 0 & -27 & 0 & 0 & 0 & 9 & 0 & 0 & -3 & 0 & 1 \\
-32 & 16 & -8 & 4 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -8 & 8 & -6 & 4 & $-\frac{5}{2}$ & 16 & -8 & 4 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 3 & -3 & $\frac{5}{2}$ & 0 & 4 & -4 & 3 & -2 & -8 & 4 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & $-\frac{1}{2}$ & 1 & $-\frac{5}{4}$ & 0 & 0 & 1 & $-\frac{3}{2}$ & $\frac{3}{2}$ & 0 & -2 & 2 & $-\frac{3}{2}$ & 4 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & $-\frac{1}{8}$ & $\frac{5}{16}$ & 0 & 0 & 0 & $\frac{1}{4}$ & $-\frac{1}{2}$ & 0 & 0 & $-\frac{1}{2}$ & $\frac{3}{4}$ & 0 & 1 & -1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & $-\frac{1}{32}$ & 0 & 0 & 0 & 0 & $\frac{1}{16}$ & 0 & 0 & 0 & $-\frac{1}{8}$ & 0 & 0 & $\frac{1}{4}$ & 0 & $-\frac{1}{2}$ & 1 \\
-32 & 16 & -8 & 4 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -24 & 24 & -18 & 12 & $-\frac{15}{2}$ & 16 & -8 & 4 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -18 & 27 & -27 & $\frac{45}{2}$ & 0 & 12 & -12 & 9 & -6 & -8 & 4 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & $-\frac{27}{2}$ & 27 & $-\frac{135}{4}$ & 0 & 0 & 9 & $-\frac{27}{2}$ & $\frac{27}{2}$ & 0 & -6 & 6 & $-\frac{9}{2}$ & 4 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & $-\frac{81}{8}$ & $\frac{405}{16}$ & 0 & 0 & 0 & $\frac{27}{4}$ & $-\frac{27}{2}$ & 0 & 0 & $-\frac{9}{2}$ & $\frac{27}{4}$ & 0 & 3 & -3 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & $-\frac{243}{32}$ & 0 & 0 & 0 & 0 & $\frac{81}{16}$ & 0 & 0 & 0 & $-\frac{27}{8}$ & 0 & 0 & $\frac{9}{4}$ & 0 & $-\frac{3}{2}$ & 1 \\
-486 & 324 & -216 & 144 & -96 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -162 & 216 & -216 & 192 & -160 & 162 & -108 & 72 & -48 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -54 & 108 & -144 & 160 & 0 & 54 & -72 & 72 & -64 & -54 & 36 & -24 & 16 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -18 & 48 & -80 & 0 & 0 & 18 & -36 & 48 & 0 & -18 & 24 & -24 & 18 & -12 & 8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -6 & 20 & 0 & 0 & 0 & 6 & -16 & 0 & 0 & -6 & 12 & 0 & 6 & -8 & -6 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & -2 & 0 & 0 & 2 & 0 & -2 & 2
\end{pmatrix}$$

$$M^{(2)}_{\mathcal{P}_2} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-64 & 32 & -16 & 8 & -4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -16 & 16 & -12 & 8 & -5 & 32 & -16 & 8 & -4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -4 & 6 & -6 & 5 & 0 & 8 & -8 & 6 & -4 & -16 & 8 & -4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & $-\frac{5}{2}$ & 0 & 0 & 2 & -3 & 3 & 0 & -4 & 4 & -3 & 8 & -4 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & $-\frac{1}{4}$ & $\frac{5}{8}$ & 0 & 0 & 0 & $\frac{1}{2}$ & -1 & 0 & 0 & -1 & $\frac{3}{2}$ & 0 & 2 & -2 & -4 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & $-\frac{1}{16}$ & 0 & 0 & 0 & 0 & $\frac{1}{8}$ & 0 & 0 & 0 & $-\frac{1}{4}$ & 0 & 0 & $\frac{1}{2}$ & 0 & -1 & 2 & 0 \\
-64 & 32 & -16 & 8 & -4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -48 & 48 & -36 & 24 & -15 & 32 & -16 & 8 & -4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -36 & 54 & -54 & 45 & 0 & 24 & -24 & 18 & -12 & -16 & 8 & -4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -27 & 54 & $-\frac{135}{2}$ & 0 & 0 & 18 & -27 & 27 & 0 & -12 & 12 & -9 & 8 & -4 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & $-\frac{81}{4}$ & $\frac{405}{8}$ & 0 & 0 & 0 & $\frac{27}{2}$ & -27 & 0 & 0 & -9 & $\frac{27}{2}$ & 0 & 6 & -6 & -4 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & $-\frac{243}{16}$ & 0 & 0 & 0 & 0 & $\frac{81}{8}$ & 0 & 0 & 0 & $-\frac{27}{4}$ & 0 & 0 & $\frac{9}{2}$ & 0 & -3 & 2 & 0 \\
-729 & 486 & -324 & 216 & -144 & 96 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -243 & 324 & -324 & 288 & -240 & 243 & -162 & 108 & -72 & 48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -81 & 162 & -216 & 240 & 0 & 81 & -108 & 108 & -96 & -81 & 54 & -36 & 24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -27 & 72 & -120 & 0 & 0 & 27 & -54 & 72 & 0 & -27 & 36 & -36 & 27 & -18 & 12 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -9 & 30 & 0 & 0 & 0 & 9 & -24 & 0 & 0 & -9 & 18 & 0 & 9 & -12 & -9 & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & -3 & 0 & 0 & 3 & 0 & -3 & 3 & 0
\end{pmatrix}$$

$$M^{(1)}_{\mathcal{Z}_1} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-64 & 64 & -64 & 64 & -64 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -16 & 32 & -48 & 64 & -80 & 32 & -32 & 32 & -32 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -4 & 12 & -24 & 40 & 0 & 8 & -16 & 24 & -32 & -16 & 16 & -16 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 4 & -10 & 0 & 0 & 2 & -6 & 12 & 0 & -4 & 8 & -12 & 8 & -8 & 8 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & $-\frac{1}{4}$ & $\frac{5}{4}$ & 0 & 0 & 0 & $\frac{1}{2}$ & -2 & 0 & 0 & -1 & 3 & 0 & 2 & -4 & -4 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & $-\frac{1}{16}$ & 0 & 0 & 0 & 0 & $\frac{1}{8}$ & 0 & 0 & 0 & $-\frac{1}{4}$ & 0 & 0 & $\frac{1}{2}$ & 0 & -1 & 2 & 0 \\
-3 & 9 & -27 & 81 & -243 & 729 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 18 & -81 & 324 & -1215 & 3 & -9 & 27 & -81 & 243 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -3 & 27 & -162 & 810 & 0 & 3 & -18 & 81 & -324 & -3 & 9 & -27 & 81 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -3 & 36 & -270 & 0 & 0 & 3 & -27 & 162 & 0 & -3 & 18 & -81 & 3 & -9 & 27 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -3 & 45 & 0 & 0 & 0 & 3 & -36 & 0 & 0 & -3 & 27 & 0 & 3 & -18 & -3 & 9 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & -3 & 0 & 0 & 3 & 0 & -3 & 3 & 0 \\
8 & 64 & 512 & 4096 & 32768 & 262144 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 32 & 512 & 6144 & 65536 & 655360 & 8 & 64 & 512 & 4096 & 32768 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 128 & 3072 & 49152 & 655360 & 0 & 32 & 512 & 6144 & 65536 & 8 & 64 & 512 & 4096 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 512 & 16384 & 327680 & 0 & 0 & 128 & 3072 & 49152 & 0 & 32 & 512 & 6144 & 8 & 64 & 512 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2048 & 81920 & 0 & 0 & 0 & 512 & 16384 & 0 & 0 & 128 & 3072 & 0 & 32 & 512 & 8 & 64 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 8192 & 0 & 0 & 0 & 0 & 2048 & 0 & 0 & 0 & 512 & 0 & 0 & 128 & 0 & 32 & 8 & 0 \\
-2187 & 6561 & -19683 & 59049 & -177147 & 531441 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 243 & -1458 & 6561 & -26244 & 98415 & 729 & -2187 & 6561 & -19683 & 59049 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -27 & 243 & -1458 & 7290 & 0 & -81 & 486 & -2187 & 8748 & -243 & 729 & -2187 & 6561 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & -36 & 270 & 0 & 0 & 9 & -81 & 486 & 0 & 27 & -162 & 729 & 81 & -243 & 729 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & $-\frac{1}{3}$ & 5 & 0 & 0 & 0 & -1 & 12 & 0 & 0 & -3 & 27 & 0 & -9 & 54 & -27 & 81 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & $\frac{1}{27}$ & 0 & 0 & 0 & 0 & $\frac{1}{9}$ & 0 & 0 & 0 & $\frac{1}{3}$ & 0 & 0 & 1 & 0 & 3 & 9 & 0 \\
288 & 1296 & 5832 & 26244 & 118098 & 531441 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 216 & 1944 & 13122 & 78732 & $\frac{885735}{2}$ & 144 & 648 & 2916 & 13122 & 59049 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 162 & 2187 & 19683 & $\frac{295245}{2}$ & 0 & 108 & 972 & 6561 & 39366 & 72 & 324 & 1458 & 6561 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & $\frac{243}{2}$ & 2187 & $\frac{98415}{4}$ & 0 & 0 & 81 & $\frac{2187}{2}$ & $\frac{19683}{2}$ & 0 & 54 & 486 & $\frac{6561}{2}$ & 36 & 162 & 729 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & $\frac{729}{8}$ & $\frac{32805}{16}$ & 0 & 0 & 0 & $\frac{243}{4}$ & $\frac{2187}{2}$ & 0 & 0 & $\frac{81}{2}$ & $\frac{2187}{4}$ & 0 & 27 & 243 & 18 & 81 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & $\frac{2187}{32}$ & 0 & 0 & 0 & 0 & $\frac{729}{16}$ & 0 & 0 & 0 & $\frac{243}{8}$ & 0 & 0 & $\frac{81}{4}$ & 0 & $\frac{27}{2}$ & 9 & 0 \\
-352 & 1936 & -10648 & 58564 & -322102 & 1771561 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -88 & 968 & -7986 & 58564 & $-\frac{805255}{2}$ & 176 & -968 & 5324 & -29282 & 161051 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -22 & 363 & -3993 & $\frac{73205}{2}$ & 0 & 44 & -484 & 3993 & -29282 & -88 & 484 & -2662 & 14641 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & $-\frac{11}{2}$ & 121 & $-\frac{6655}{4}$ & 0 & 0 & 11 & $-\frac{363}{2}$ & $\frac{3993}{2}$ & 0 & -22 & 242 & $-\frac{3993}{2}$ & 44 & -242 & 1331 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & $-\frac{11}{8}$ & $\frac{605}{16}$ & 0 & 0 & 0 & $\frac{11}{4}$ & $-\frac{121}{2}$ & 0 & 0 & $-\frac{11}{2}$ & $\frac{363}{4}$ & 0 & 11 & -121 & -22 & 121 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & $-\frac{11}{32}$ & 0 & 0 & 0 & 0 & $\frac{11}{16}$ & 0 & 0 & 0 & $-\frac{11}{8}$ & 0 & 0 & $\frac{11}{4}$ & 0 & $-\frac{11}{2}$ & 11 & 0 \\
-15625 & 15625 & -15625 & 15625 & -15625 & 15625 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1250 & -2500 & 3750 & -5000 & 6250 & 3125 & -3125 & 3125 & -3125 & 3125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -100 & 300 & -600 & 1000 & 0 & -250 & 500 & -750 & 1000 & -625 & 625 & -625 & 625 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 8 & -32 & 80 & 0 & 0 & 20 & -60 & 120 & 0 & 50 & -100 & 150 & 125 & -125 & 125 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & $-\frac{16}{25}$ & $\frac{16}{5}$ & 0 & 0 & 0 & $-\frac{8}{5}$ & $\frac{32}{5}$ & 0 & 0 & -4 & 12 & 0 & -10 & 20 & -25 & 25 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & $\frac{32}{625}$ & 0 & 0 & 0 & 0 & $\frac{16}{125}$ & 0 & 0 & 0 & $\frac{8}{25}$ & 0 & 0 & $\frac{4}{5}$ & 0 & 2 & 5 & 0
\end{pmatrix}$$

$$M^{(2)}_{\mathcal{Z}_1} = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-64 & 64 & -64 & 64 & -64 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -16 & 32 & -48 & 64 & -80 & 32 & -32 & 32 & -32 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -4 & 12 & -24 & 40 & 0 & 8 & -16 & 24 & -32 & -16 & 16 & -16 & 16 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 4 & -10 & 0 & 0 & 2 & -6 & 12 & 0 & -4 & 8 & -12 & 8 & -8 & 8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & $-\frac{1}{4}$ & $\frac{5}{4}$ & 0 & 0 & 0 & $\frac{1}{2}$ & -2 & 0 & 0 & -1 & 3 & 0 & 2 & -4 & -4 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & $-\frac{1}{16}$ & 0 & 0 & 0 & 0 & $\frac{1}{8}$ & 0 & 0 & 0 & $-\frac{1}{4}$ & 0 & 0 & $\frac{1}{2}$ & 0 & -1 & 2 \\
-1 & 3 & -9 & 27 & -81 & 243 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 6 & -27 & 108 & -405 & 1 & -3 & 9 & -27 & 81 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 9 & -54 & 270 & 0 & 1 & -6 & 27 & -108 & -1 & 3 & -9 & 27 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 12 & -90 & 0 & 0 & 1 & -9 & 54 & 0 & -1 & 6 & -27 & 1 & -3 & 9 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 15 & 0 & 0 & 0 & 1 & -12 & 0 & 0 & -1 & 9 & 0 & 1 & -6 & -1 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 1 \\
-1 & -8 & -64 & -512 & -4096 & -32768 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -4 & -64 & -768 & -8192 & -81920 & -1 & -8 & -64 & -512 & -4096 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -16 & -384 & -6144 & -81920 & 0 & -4 & -64 & -768 & -8192 & -1 & -8 & -64 & -512 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -64 & -2048 & -40960 & 0 & 0 & -16 & -384 & -6144 & 0 & -4 & -64 & -768 & -1 & -8 & -64 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -256 & -10240 & 0 & 0 & 0 & -64 & -2048 & 0 & 0 & -16 & -384 & 0 & -4 & -64 & -1 & -8 & 0 \\
0 & 0 & 0 & 0 & 0 & -1024 & 0 & 0 & 0 & 0 & -256 & 0 & 0 & 0 & -64 & 0 & 0 & -16 & 0 & -4 & -1 \\
-729 & 2187 & -6561 & 19683 & -59049 & 177147 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 81 & -486 & 2187 & -8748 & 32805 & 243 & -729 & 2187 & -6561 & 19683 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -9 & 81 & -486 & 2430 & 0 & -27 & 162 & -729 & 2916 & -81 & 243 & -729 & 2187 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -12 & 90 & 0 & 0 & 3 & -27 & 162 & 0 & 9 & -54 & 243 & 27 & -81 & 243 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & $-\frac{1}{9}$ & $\frac{5}{3}$ & 0 & 0 & 0 & $-\frac{1}{3}$ & 4 & 0 & 0 & -1 & 9 & 0 & -3 & 18 & -9 & 27 & 0 \\
0 & 0 & 0 & 0 & 0 & $\frac{1}{81}$ & 0 & 0 & 0 & 0 & $\frac{1}{27}$ & 0 & 0 & 0 & $\frac{1}{9}$ & 0 & 0 & $\frac{1}{3}$ & 0 & 1 & 3 \\
-64 & -288 & -1296 & -5832 & -26244 & -118098 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -48 & -432 & -2916 & -17496 & -98415 & -32 & -144 & -648 & -2916 & -13122 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -36 & -486 & -4374 & -32805 & 0 & -24 & -216 & -1458 & -8748 & -16 & -72 & -324 & -1458 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -27 & -486 & $-\frac{10935}{2}$ & 0 & 0 & -18 & -243 & -2187 & 0 & -12 & -108 & -729 & -8 & -36 & -162 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & $-\frac{81}{4}$ & $-\frac{3645}{8}$ & 0 & 0 & 0 & $-\frac{27}{2}$ & -243 & 0 & 0 & -9 & $-\frac{243}{2}$ & 0 & -6 & -54 & -4 & -18 & 0 \\
0 & 0 & 0 & 0 & 0 & $-\frac{243}{16}$ & 0 & 0 & 0 & 0 & $-\frac{81}{8}$ & 0 & 0 & 0 & $-\frac{27}{4}$ & 0 & 0 & $-\frac{9}{2}$ & 0 & -3 & -2 \\
-64 & 352 & -1936 & 10648 & -58564 & 322102 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -16 & 176 & -1452 & 10648 & -73205 & 32 & -176 & 968 & -5324 & 29282 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -4 & 66 & -726 & 6655 & 0 & 8 & -88 & 726 & -5324 & -16 & 88 & -484 & 2662 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 22 & $-\frac{605}{2}$ & 0 & 0 & 2 & -33 & 363 & 0 & -4 & 44 & -363 & 8 & -44 & 242 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & $-\frac{1}{4}$ & $\frac{55}{8}$ & 0 & 0 & 0 & $\frac{1}{2}$ & -11 & 0 & 0 & -1 & $\frac{33}{2}$ & 0 & 2 & -22 & -4 & 22 & 0 \\
0 & 0 & 0 & 0 & 0 & $-\frac{1}{16}$ & 0 & 0 & 0 & 0 & $\frac{1}{8}$ & 0 & 0 & 0 & $-\frac{1}{4}$ & 0 & 0 & $\frac{1}{2}$ & 0 & -1 & 2 \\
-15625 & 15625 & -15625 & 15625 & -15625 & 15625 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1250 & -2500 & 3750 & -5000 & 6250 & 3125 & -3125 & 3125 & -3125 & 3125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -100 & 300 & -600 & 1000 & 0 & -250 & 500 & -750 & 1000 & -625 & 625 & -625 & 625 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 8 & -32 & 80 & 0 & 0 & 20 & -60 & 120 & 0 & 50 & -100 & 150 & 125 & -125 & 125 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & $-\frac{16}{25}$ & $\frac{16}{5}$ & 0 & 0 & 0 & $-\frac{8}{5}$ & $\frac{32}{5}$ & 0 & 0 & -4 & 12 & 0 & -10 & 20 & -25 & 25 & 0 \\
0 & 0 & 0 & 0 & 0 & $\frac{32}{625}$ & 0 & 0 & 0 & 0 & $\frac{16}{125}$ & 0 & 0 & 0 & $\frac{8}{25}$ & 0 & 0 & $\frac{4}{5}$ & 0 & 2 & 5
\end{pmatrix}$$

$$M^{(1)}_{\mathcal{Z}_2} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
128 & -128 & 128 & -128 & 128 & -128 & 128 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 32 & -64 & 96 & -128 & 160 & -192 & -64 & 64 & -64 & 64 & -64 & 64 & 0 \\
0 & 0 & 8 & -24 & 48 & -80 & 120 & 0 & -16 & 32 & -48 & 64 & -80 & 32 \\
0 & 0 & 0 & 2 & -8 & 20 & -40 & 0 & 0 & -4 & 12 & -24 & 40 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{5}{2} & \frac{15}{2} & 0 & 0 & 0 & -1 & 4 & -10 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{8} & -\frac{3}{4} & 0 & 0 & 0 & 0 & -\frac{1}{4} & \frac{5}{4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{32} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{16} & 0 \\
3 & -9 & 27 & -81 & 243 & -729 & 2187 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & -18 & 81 & -324 & 1215 & -4374 & -3 & 9 & -27 & 81 & -243 & 729 & 0 \\
0 & 0 & 3 & -27 & 162 & -810 & 3645 & 0 & -3 & 18 & -81 & 324 & -1215 & 3 \\
0 & 0 & 0 & 3 & -36 & 270 & -1620 & 0 & 0 & -3 & 27 & -162 & 810 & 0 \\
0 & 0 & 0 & 0 & 3 & -45 & 405 & 0 & 0 & 0 & -3 & 36 & -270 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & -54 & 0 & 0 & 0 & 0 & -3 & 45 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & -3 & 0 \\
8 & 64 & 512 & 4096 & 32768 & 262144 & 2097152 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 32 & 512 & 6144 & 65536 & 655360 & 6291456 & 8 & 64 & 512 & 4096 & 32768 & 262144 & 0 \\
0 & 0 & 128 & 3072 & 49152 & 655360 & 7864320 & 0 & 32 & 512 & 6144 & 65536 & 655360 & 8 \\
0 & 0 & 0 & 512 & 16384 & 327680 & 5242880 & 0 & 0 & 128 & 3072 & 49152 & 655360 & 0 \\
0 & 0 & 0 & 0 & 2048 & 81920 & 1966080 & 0 & 0 & 0 & 512 & 16384 & 327680 & 0 \\
0 & 0 & 0 & 0 & 0 & 8192 & 393216 & 0 & 0 & 0 & 0 & 2048 & 81920 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 32768 & 0 & 0 & 0 & 0 & 0 & 8192 & 0 \\
6561 & -19683 & 59049 & -177147 & 531441 & -1594323 & 4782969 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -729 & 4374 & -19683 & 78732 & -295245 & 1062882 & -2187 & 6561 & -19683 & 59049 & -177147 & 531441 & 0 \\
0 & 0 & 81 & -729 & 4374 & -21870 & 98415 & 0 & 243 & -1458 & 6561 & -26244 & 98415 & 729 \\
0 & 0 & 0 & -9 & 108 & -810 & 4860 & 0 & 0 & -27 & 243 & -1458 & 7290 & 0 \\
0 & 0 & 0 & 0 & 1 & -15 & 135 & 0 & 0 & 0 & 3 & -36 & 270 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{9} & 2 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{81} & 0 & 0 & 0 & 0 & 0 & \frac{1}{27} & 0 \\
86016 & 451584 & 2370816 & 12446784 & 65345616 & 343064484 & 1801088541 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 37632 & 395136 & 3111696 & 21781872 & 142943535 & \frac{1801088541}{2} & 21504 & 112896 & 592704 & 3111696 & 16336404 & 85766121 & 0 \\
0 & 0 & 16464 & 259308 & 2722734 & \frac{47647845}{2} & \frac{300181423 5}{16} & 0 & 9408 & 98784 & 777924 & 5445468 & \frac{142943535}{4} & 5376 \\
0 & 0 & 0 & 7203 & 151263 & \frac{15882615}{8} & \frac{333534915}{16} & 0 & 0 & 4116 & 64827 & \frac{1361367}{2} & \frac{47647845}{8} & 0 \\
0 & 0 & 0 & 0 & \frac{50421}{16} & \frac{5294205}{64} & \frac{333534915}{256} & 0 & 0 & 0 & \frac{7203}{4} & \frac{151263}{4} & \frac{15882615}{32} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{352947}{256} & \frac{22235661}{512} & 0 & 0 & 0 & 0 & \frac{50421}{64} & \frac{5294205}{256} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{2470629}{4096} & 0 & 0 & 0 & 0 & 0 & \frac{352947}{1024} & 0 \\
77824 & -369664 & 1755904 & -8340544 & 39617584 & -188183524 & 893871739 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4864 & -46208 & 329232 & -2085136 & 12380495 & -\frac{141137643}{2} & -19456 & 92416 & -438976 & 2085136 & -9904396 & 47045881 & 0 \\
0 & 0 & 304 & -4332 & 41154 & -\frac{651605}{2} & \frac{37141485}{16} & 0 & -1216 & 11552 & -82308 & 521284 & -\frac{12380495}{4} & 4864 \\
0 & 0 & 0 & 19 & -361 & \frac{34295}{8} & -\frac{651605}{16} & 0 & 0 & -76 & 1083 & -\frac{20577}{2} & \frac{651605}{8} & 0 \\
0 & 0 & 0 & 0 & \frac{19}{16} & -\frac{1805}{64} & \frac{102885}{256} & 0 & 0 & 0 & -\frac{19}{4} & \frac{361}{4} & -\frac{34295}{32} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{19}{256} & -\frac{1083}{512} & 0 & 0 & 0 & 0 & -\frac{19}{64} & \frac{1805}{256} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{19}{4096} & 0 & 0 & 0 & 0 & 0 & -\frac{19}{1024} & 0 \\
10000000 & -10000000 & 10000000 & -10000000 & 10000000 & -10000000 & 10000000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -500000 & 1000000 & -1500000 & 2000000 & -2500000 & 3000000 & -1000000 & 1000000 & -1000000 & 1000000 & -1000000 & 1000000 & 0 \\
0 & 0 & 25000 & -75000 & 150000 & -250000 & 375000 & 0 & 50000 & -100000 & 150000 & -200000 & 250000 & 100000 \\
0 & 0 & 0 & -1250 & 5000 & -12500 & 25000 & 0 & 0 & -2500 & 7500 & -15000 & 25000 & 0 \\
0 & 0 & 0 & 0 & \frac{125}{2} & -\frac{625}{2} & \frac{1875}{2} & 0 & 0 & 0 & 125 & -500 & 1250 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{25}{8} & \frac{75}{4} & 0 & 0 & 0 & 0 & -\frac{25}{4} & \frac{125}{4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{32} & 0 & 0 & 0 & 0 & 0 & \frac{5}{16} & 0
\end{pmatrix}$$

$$M^{(2)}_{\mathcal{Z}_2} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-32 & 32 & -32 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & -16 & 24 & -32 & -16 & 16 & -16 & 16 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & -6 & 12 & 0 & -4 & 8 & -12 & 8 & -8 & 8 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & -2 & 0 & 0 & -1 & 3 & 0 & 2 & -4 & -4 & 4 & 0 \\
0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & -1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-9 & 27 & -81 & 243 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & -18 & 81 & -324 & -3 & 9 & -27 & 81 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & -27 & 162 & 0 & -3 & 18 & -81 & 3 & -9 & 27 & 0 & 0 & 0 \\
0 & 0 & 3 & -36 & 0 & 0 & -3 & 27 & 0 & 3 & -18 & -3 & 9 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 & 0 & -3 & 0 & 0 & 3 & 0 & -3 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
64 & 512 & 4096 & 32768 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
32 & 512 & 6144 & 65536 & 8 & 64 & 512 & 4096 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 128 & 3072 & 49152 & 0 & 32 & 512 & 6144 & 8 & 64 & 512 & 0 & 0 & 0 \\
0 & 0 & 512 & 16384 & 0 & 0 & 128 & 3072 & 0 & 32 & 512 & 8 & 64 & 0 \\
0 & 0 & 0 & 2048 & 0 & 0 & 0 & 512 & 0 & 0 & 128 & 0 & 32 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2187 & 6561 & -19683 & 59049 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-81 & 486 & -2187 & 8748 & -243 & 729 & -2187 & 6561 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 9 & -81 & 486 & 0 & 27 & -162 & 729 & 81 & -243 & 729 & 0 & 0 & 0 \\
0 & 0 & -1 & 12 & 0 & 0 & -3 & 27 & 0 & -9 & 54 & -27 & 81 & 0 \\
0 & 0 & 0 & \frac{1}{9} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 1 & 0 & 3 & 9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
28224 & 148176 & 777924 & 4084101 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2352 & 24696 & 194481 & 1361367 & 1344 & 7056 & 37044 & 194481 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1029 & \frac{64827}{4} & \frac{1361367}{8} & 0 & 588 & 6174 & \frac{194481}{4} & 336 & 1764 & 9261 & 0 & 0 & 0 \\
0 & 0 & \frac{7203}{16} & \frac{151263}{16} & 0 & 0 & \frac{1029}{4} & \frac{64827}{16} & 0 & 147 & \frac{3087}{2} & 84 & 441 & 0 \\
0 & 0 & 0 & \frac{50421}{256} & 0 & 0 & 0 & \frac{7203}{64} & 0 & 0 & \frac{1029}{16} & 0 & \frac{147}{4} & 21 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-23104 & 109744 & -521284 & 2476099 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
304 & -2888 & 20577 & -130321 & -1216 & 5776 & -27436 & 130321 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 19 & -\frac{1083}{4} & \frac{20577}{8} & 0 & -76 & 722 & -\frac{20577}{4} & 304 & -1444 & 6859 & 0 & 0 & 0 \\
0 & 0 & \frac{19}{16} & -\frac{361}{16} & 0 & 0 & -\frac{19}{4} & \frac{1083}{16} & 0 & 19 & -\frac{361}{2} & -76 & 361 & 0 \\
0 & 0 & 0 & \frac{19}{256} & 0 & 0 & 0 & -\frac{19}{64} & 0 & 0 & \frac{19}{16} & 0 & -\frac{19}{4} & 19 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-100000 & 100000 & -100000 & 100000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-5000 & 10000 & -15000 & 20000 & -10000 & 10000 & -10000 & 10000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 250 & -750 & 1500 & 0 & 500 & -1000 & 1500 & 1000 & -1000 & 1000 & 0 & 0 & 0 \\
0 & 0 & -\frac{25}{2} & 50 & 0 & 0 & -25 & 75 & 0 & -50 & 100 & -100 & 100 & 0 \\
0 & 0 & 0 & \frac{5}{8} & 0 & 0 & 0 & \frac{5}{4} & 0 & 0 & \frac{5}{2} & 0 & 5 & 10
\end{pmatrix}$$

$$M^{(3)}_{\mathcal{Z}_2} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
128 & -128 & 128 & -128 & 128 & -128 & 128 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 32 & -64 & 96 & -128 & 160 & -192 & -64 & 64 & -64 & 64 & -64 & 64 & 0 \\
0 & 0 & 8 & -24 & 48 & -80 & 120 & 0 & -16 & 32 & -48 & 64 & -80 & 32 \\
0 & 0 & 0 & 2 & -8 & 20 & -40 & 0 & 0 & -4 & 12 & -24 & 40 & 0 \\
0 & 0 & 0 & 0 & $\frac{1}{2}$ & $-\frac{5}{2}$ & $\frac{15}{2}$ & 0 & 0 & 0 & -1 & 4 & -10 & 0 \\
0 & 0 & 0 & 0 & 0 & $\frac{1}{8}$ & $-\frac{3}{4}$ & 0 & 0 & 0 & 0 & $-\frac{1}{4}$ & $\frac{5}{4}$ & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & $\frac{1}{32}$ & 0 & 0 & 0 & 0 & 0 & $-\frac{1}{16}$ & 0 \\
1 & -3 & 9 & -27 & 81 & -243 & 729 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -6 & 27 & -108 & 405 & -1458 & -1 & 3 & -9 & 27 & -81 & 243 & 0 \\
0 & 0 & 1 & -9 & 54 & -270 & 1215 & 0 & -1 & 6 & -27 & 108 & -405 & 1 \\
0 & 0 & 0 & 1 & -12 & 90 & -540 & 0 & 0 & -1 & 9 & -54 & 270 & 0 \\
0 & 0 & 0 & 0 & 1 & -15 & 135 & 0 & 0 & 0 & -1 & 12 & -90 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -18 & 0 & 0 & 0 & 0 & -1 & 15 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
-1 & -8 & -64 & -512 & -4096 & -32768 & -262144 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -4 & -64 & -768 & -8192 & -81920 & -786432 & -1 & -8 & -64 & -512 & -4096 & -32768 & 0 \\
0 & 0 & -16 & -384 & -6144 & -81920 & -983040 & 0 & -4 & -64 & -768 & -8192 & -81920 & -1 \\
0 & 0 & 0 & -64 & -2048 & -40960 & -655360 & 0 & 0 & -16 & -384 & -6144 & -81920 & 0 \\
0 & 0 & 0 & 0 & -256 & -10240 & -245760 & 0 & 0 & 0 & -64 & -2048 & -40960 & 0 \\
0 & 0 & 0 & 0 & 0 & -1024 & -49152 & 0 & 0 & 0 & 0 & -256 & -10240 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -4096 & 0 & 0 & 0 & 0 & 0 & -1024 & 0 \\
2187 & -6561 & 19683 & -59049 & 177147 & -531441 & 1594323 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -243 & 1458 & -6561 & 26244 & -98415 & 354294 & -729 & 2187 & -6561 & 19683 & -59049 & 177147 & 0 \\
0 & 0 & 27 & -243 & 1458 & -7290 & 32805 & 0 & 81 & -486 & 2187 & -8748 & 32805 & 243 \\
0 & 0 & 0 & -3 & 36 & -270 & 1620 & 0 & 0 & -9 & 81 & -486 & 2430 & 0 \\
0 & 0 & 0 & 0 & $\frac{1}{3}$ & -5 & 45 & 0 & 0 & 0 & 1 & -12 & 90 & 0 \\
0 & 0 & 0 & 0 & 0 & $-\frac{1}{27}$ & $\frac{2}{3}$ & 0 & 0 & 0 & 0 & $-\frac{1}{9}$ & $\frac{5}{3}$ & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & $\frac{1}{243}$ & 0 & 0 & 0 & 0 & 0 & $\frac{1}{81}$ & 0 \\
-16384 & -86016 & -451584 & -2370816 & -12446784 & -65345616 & -343064484 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -7168 & -75264 & -592704 & -4148928 & -27227340 & -171532242 & -4096 & -21504 & -112896 & -592704 & -3111696 & -16336404 & 0 \\
0 & 0 & -3136 & -49392 & -518616 & -4537890 & $-\frac{142943535}{4}$ & 0 & -1792 & -18816 & -148176 & -1037232 & -6806835 & -1024 \\
0 & 0 & 0 & -1372 & -28812 & $-\frac{756315}{2}$ & $-\frac{15882615}{4}$ & 0 & 0 & -784 & -12348 & -129654 & $-\frac{2268945}{2}$ & 0 \\
0 & 0 & 0 & 0 & $-\frac{2401}{4}$ & $-\frac{252105}{16}$ & $-\frac{15882615}{64}$ & 0 & 0 & 0 & -343 & -7203 & $-\frac{756315}{8}$ & 0 \\
0 & 0 & 0 & 0 & 0 & $-\frac{16807}{64}$ & $-\frac{1058841}{128}$ & 0 & 0 & 0 & 0 & $-\frac{2401}{16}$ & $-\frac{252105}{64}$ & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & $-\frac{117649}{1024}$ & 0 & 0 & 0 & 0 & 0 & $-\frac{16807}{256}$ & 0 \\
16384 & -77824 & 369664 & -1755904 & 8340544 & -39617584 & 188183524 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1024 & -9728 & 69312 & -438976 & 2606420 & -14856594 & -4096 & 19456 & -92416 & 438976 & -2085136 & 9904396 & 0 \\
0 & 0 & 64 & -912 & 8664 & -68590 & $\frac{1954815}{4}$ & 0 & -256 & 2432 & -17328 & 109744 & -651605 & 1024 \\
0 & 0 & 0 & 4 & -76 & $\frac{1805}{2}$ & $-\frac{34295}{4}$ & 0 & 0 & -16 & 228 & -2166 & $\frac{34295}{2}$ & 0 \\
0 & 0 & 0 & 0 & $\frac{1}{4}$ & $-\frac{95}{16}$ & $\frac{5415}{64}$ & 0 & 0 & 0 & -1 & 19 & $-\frac{1805}{8}$ & 0 \\
0 & 0 & 0 & 0 & 0 & $\frac{1}{64}$ & $-\frac{57}{128}$ & 0 & 0 & 0 & 0 & $-\frac{1}{16}$ & $\frac{95}{64}$ & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & $\frac{1}{1024}$ & 0 & 0 & 0 & 0 & 0 & $-\frac{1}{256}$ & 0 \\
10000000 & -10000000 & 10000000 & -10000000 & 10000000 & -10000000 & 10000000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -500000 & 1000000 & -1500000 & 2000000 & -2500000 & 3000000 & -1000000 & 1000000 & -1000000 & 1000000 & -1000000 & 1000000 & 0 \\
0 & 0 & 25000 & -75000 & 150000 & -250000 & 375000 & 0 & 50000 & -100000 & 150000 & -200000 & 250000 & 100000 \\
0 & 0 & 0 & -1250 & 5000 & -12500 & 25000 & 0 & 0 & -2500 & 7500 & -15000 & 25000 & 0 \\
0 & 0 & 0 & 0 & $\frac{125}{2}$ & $-\frac{625}{2}$ & $\frac{1875}{2}$ & 0 & 0 & 0 & 125 & -500 & 1250 & 0 \\
0 & 0 & 0 & 0 & 0 & $-\frac{25}{8}$ & $\frac{75}{4}$ & 0 & 0 & 0 & 0 & $-\frac{25}{4}$ & $\frac{125}{4}$ & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & $\frac{5}{32}$ & 0 & 0 & 0 & 0 & 0 & $\frac{5}{16}$ & 0
\end{pmatrix}$$

$$M^{(4)}_{\mathcal{Z}_2} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-32 & 32 & -32 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & -16 & 24 & -32 & -16 & 16 & -16 & 16 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & -6 & 12 & 0 & -4 & 8 & -12 & 8 & -8 & 8 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & -2 & 0 & 0 & -1 & 3 & 0 & 2 & -4 & -4 & 4 & 0 \\
0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & -1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & 9 & -27 & 81 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -6 & 27 & -108 & -1 & 3 & -9 & 27 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -9 & 54 & 0 & -1 & 6 & -27 & 1 & -3 & 9 & 0 & 0 & 0 \\
0 & 0 & 1 & -12 & 0 & 0 & -1 & 9 & 0 & 1 & -6 & -1 & 3 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-8 & -64 & -512 & -4096 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4 & -64 & -768 & -8192 & -1 & -8 & -64 & -512 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -16 & -384 & -6144 & 0 & -4 & -64 & -768 & -1 & -8 & -64 & 0 & 0 & 0 \\
0 & 0 & -64 & -2048 & 0 & 0 & -16 & -384 & 0 & -4 & -64 & -1 & -8 & 0 \\
0 & 0 & 0 & -256 & 0 & 0 & 0 & -64 & 0 & 0 & -16 & 0 & -4 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-729 & 2187 & -6561 & 19683 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-27 & 162 & -729 & 2916 & -81 & 243 & -729 & 2187 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & -27 & 162 & 0 & 9 & -54 & 243 & 27 & -81 & 243 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{3} & 4 & 0 & 0 & -1 & 9 & 0 & -3 & 18 & -9 & 27 & 0 \\
0 & 0 & 0 & \frac{1}{27} & 0 & 0 & 0 & \frac{1}{9} & 0 & 0 & \frac{1}{3} & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-5376 & -28224 & -148176 & -777924 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-448 & -4704 & -37044 & -259308 & -256 & -1344 & -7056 & -37044 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -196 & -3087 & -\frac{64827}{2} & 0 & -112 & -1176 & -9261 & -64 & -336 & -1764 & 0 & 0 & 0 \\
0 & 0 & -\frac{343}{4} & -\frac{7203}{4} & 0 & 0 & -49 & -\frac{3087}{4} & 0 & -28 & -294 & -16 & -84 & 0 \\
0 & 0 & 0 & -\frac{2401}{64} & 0 & 0 & 0 & -\frac{343}{16} & 0 & 0 & -\frac{49}{4} & 0 & -7 & -4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4864 & 23104 & -109744 & 521284 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
64 & -608 & 4332 & -27436 & -256 & 1216 & -5776 & 27436 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & -57 & \frac{1083}{2} & 0 & -16 & 152 & -1083 & 64 & -304 & 1444 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{4} & -\frac{19}{4} & 0 & 0 & -1 & \frac{57}{4} & 0 & 4 & -38 & -16 & 76 & 0 \\
0 & 0 & 0 & \frac{1}{64} & 0 & 0 & 0 & -\frac{1}{16} & 0 & 0 & \frac{1}{4} & 0 & -1 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-100000 & 100000 & -100000 & 100000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-5000 & 10000 & -15000 & 20000 & -10000 & 10000 & -10000 & 10000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 250 & -750 & 1500 & 0 & 500 & -1000 & 1500 & 1000 & -1000 & 1000 & 0 & 0 & 0 \\
0 & 0 & -\frac{25}{2} & 50 & 0 & 0 & -25 & 75 & 0 & -50 & 100 & -100 & 100 & 0 \\
0 & 0 & 0 & \frac{5}{8} & 0 & 0 & 0 & \frac{5}{4} & 0 & 0 & \frac{5}{2} & 0 & 5 & 10
\end{pmatrix}$$

Appendix B

Code in Sage to compute filtrations of the module of logarithmic derivations of an affine line arrangement

This library of functions is written in Sage v.6.7. We are aware that Sage has some computational issues with polynomials defined over floating real or complex numbers, but in this work we are only considering line arrangements defined over number fields. The complete code can be found in the author's web:

http://jviusos.perso.univ-pau.fr/resources/sage/line_arrangements_vector_fields.sage

Sage

```
sage: reset()
sage: load "line_arrangements_vector_fields.sage"

New functions for Arrangements declared: dual_point_to_eq, def_poly, eq2points,
nlines, nslopes_and_parallels, singularities, nsing_multiplicities, info_arrang
ement, plot_arrangement.

New functions for Vector Fields declared: matrix_eqs_vector_field, basis_vector
_field_from_arrangement, line_is_invariant, curve_is_invariant, check_finitenes
s_base, check_infiniteness_list, check_finiteness_degree, singular_locus, linea
r_system, show_linear_system_by_points, plot_A_der.

New (very) useful functions: basis_monomials, generic_pols, pol_cond_to_matrix,
ordering_tuple, solve_to_points.
```

B.1 Line arrangements

```
def dual_point_to_eq(R,v,proj=False):
    """
    INPUT:
        - A polynomial ring ``R``.
        - A 3-tuple `v=(a,b,c)` of elements of the base ring of ``R``.

    OUTPUT:
        - A linear form `ax+by+c` where `x,y` are the generators of ``R``.
        - If ``proj=True`` (by default, ``False``): A linear form `ax+by+cz` \
        where `x,y,z` are the generators of ``R``.
    """
    if proj: vgens=vector(R,list(R.gens()))
    else: vgens=vector(R,list(R.gens())+[1])
    eq=vgens.dot_product(vector(v))
    return eq

def def_poly(R,A,in_list=False,proj=False):
    """
    INPUT:
        - A polynomial ring ``R`` for defining polynomials.
        - A list `A=[(a_1,b_1,c_1),\ldots,(a_n,b_n,c_n)]` representing a \
        line arrangement `\mathcal{A}=\{\mathcal{l}_1,\ldots,\mathcal{l}_n\}`\
        whose elements are described in the dual projective plane \
        (`(a_i,b_i,c_i)\leftrightarrow L_i: a_ix+b_iy+c_i=0`).
        - If ``proj=True`` (by default, ``False``): `A` is consider as \
        a projective line arrangement.

    OUTPUT:
        - A product of affine/projective factors `\prod_{i=1}^nf_i` in ``R`` \
        representing the line arrangement \
        `\mathcal{A}=\{\mathcal{l}_1,\ldots,\mathcal{l}_n\}`.
        - If ``list=True`` (by default ``False``), returns a list with all \
        the equations.
    """
    if proj: vgens=vector(R,list(R.gens()))
    else: vgens=vector(R,list(R.gens())+[1])
    eq_lines=map(lambda l: dual_point_to_eq(R,l,proj), A)
    if in_list==True: return eq_lines
    else: return prod(eq_lines)

def eq2points(R,P1,P2):
    """
    Enter 2 points to obtain the line which relates them in dual \
    coordinates, i.e. `(a:b:c)`.
    """
    return vector(R,(P2[1]-P1[1], P1[0]-P2[0],
                     (P2[0]-P1[0])*P1[1]-(P2[1]-P1[1])*P1[0]))
```

```
def nlines(A):
    """
    Returns the number of different lines contained in ``A``.
    """
    n=len(Set(A))
    if n is not len(A):
        print "Warning: There are repeated lines in the arrangement."
    return n

def nslopes_and_parallels(A,info=False):
    """
    Returns a tuple `(s,p)` where:
        - `s` is the number of different slopes contained in ``A``.\n
        - `p` is the maximum number of parallel lines contained in ``A``.\n
    If ``info=True``: detailed information is printed.
    """
    L=[]
    for l in A:
        if l[1]==0: L = L + [oo]
        else: L = L + [-l[0]/l[1]]
    slopes=Set(L)
    p=max(map(lambda s: L.count(s), slopes))
    if info: print 'Slopes and their lines:  ' + \
                    str(map(lambda s: (s,L.count(s)), slopes))
    return (len(slopes), p)

def singularities(R,A):
    """
    Returns a list with the singular points of the arrangement ``A`` \
    (with coordinates in ``R``).
    """
    x=R.gens()
    Aeqs=def_poly(R,A,in_list=True)
    Sing=[]
    n=nlines(A)
    for i in range(n):
        for j in range(i,n):
            p=R.ideal([Aeqs[i],Aeqs[j]]).groebner_basis()
            if p[0].coefficient(x[0]) is not 0 and len(p)!=1:
                Sing.append((-p[0].constant_coefficient() / \
                            p[0].coefficient(x[0]),
                            -p[1].constant_coefficient()/ \
                            p[1].coefficient(x[1])))
            elif len(p)!=1:
                Sing.append((-p[0].constant_coefficient()/ \
                            p[0].coefficient(x[1]),
                            -p[1].constant_coefficient()/ \
                            p[1].coefficient(x[0])))
            elif len(p)!=1 and(p[0].degree()>1 or p[1].degree()>1):
                print "Warning: multiple solutions!"
    return list(Set(Sing))
```

```
def nsing_multiplicities(R,A):
    """
    Returns a list of tuples `[(1,n_1),\ldot,(k,n_k)]` where `k` is the \
    maximal multiplicity of the singular points of the arrangement ``A`` \
    and `n_i` are the number of singular points of multiplicity `i`.
    """
    n = nlines(A); Aeqs=def_poly(R,A,in_list=True)
    sing = singularities(R,A)
    L=map(lambda i: vector((i,0)), range(2,n+1))
    for s in sing:
        k = len(filter(lambda p: p(s)==0, Aeqs))
        L[k-2] = L[k-2] + vector((0,1))
    while L[-1][1]==0: L.remove(L[-1])
    #L = filter(lambda v: v[1]!=0, L)
    return L
```

```
def info_arrangement(R,A,plot=True):
    """
    INPUT:
        - A ring ``R`` of defining polynomials.
        - A list `A=[(a_1,b_1,c_1),\ldots,(a_n,b_n,c_n)]` representing a \
        line arrangement `\mathcal{A}=\{\mathcal{l}_1,\ldots,\mathcal{l}_n\}`\
        whose elements are described in the dual projective plane \
        (`(a_i,b_i,c_i)\leftrightarrow L_i: a_ix+b_iy+c_i=0`).
        - A positive integer``d``.

    OUTPUT:
        - Prints a list of basic informations of ``A`` as the number of \
        lines, number of singularities, number of slopes, maximal number of \
        parallels.
        - Shows a plot of the arrangement containing of the singular points.
    """
    sing=singularities(R,A)
    print "Number of lines: " + str(nlines(A))
    print "Number of singularities: " + str(len(sing)) + '=' + \
                                          str(nsing_multiplicities(R,A))
    sandp=nslopes_and_parallels(A,info=True)
    print "Number of slopes: " + str(sandp[0])
    print "Maximal number of parallels: " + str(sandp[1])
    if plot:
        T=R.base_ring()
        x0=min(map(lambda s: T(s[0]), sing))-1/2
        x1=max(map(lambda s: T(s[0]), sing))+1/2
        y0=min(map(lambda s: T(s[1]), sing))-1/2
        y1=max(map(lambda s: T(s[1]), sing))+1/2
        show(plot_arrangement(R,A,(x0,x1),(y0,y1),color='red'))
```

B.2 Logarithmic vector fields by filtration

```
def matrix_eqs_vector_field(R,A,d,info=False):
    """
    INPUT:
        - A ring ``R`` of defining polynomials.
        - A list `A=[(a_1,b_1,c_1),\ldots,(a_n,b_n,c_n)]` representing a \
        line arrangement `\mathcal{A}=\{\mathcal{l}_1,\ldots,\mathcal{l}_n\}`\
        whose elements are described in the dual projective plane \
        (`(a_i,b_i,c_i)\leftrightarrow L_i: a_ix+b_iy+c_i=0`).
        - A positive integer``d``.

    OUTPUT:
        - A matrix of conditions for with a polynomial vector field in the \
        plane `\chi=P(x,y)\partial_x + Q(x,y)\partial_y`, where \
        `P,Q\in R` such that `\mathop{deg}P,\mathop{deg}Q\leq d`, and \
        `\mathcal{A}` is invariant by `\chi`. Each row of the matrix \
        represents a linear equation into with respect to coefficients \
        `\{a_{i,j}, b_{k,l}\}` of `P` and `Q` in generic form, i.e. \
        `P=\sum_{i,j=0}^{i+j\leq d}a_{i,j}\cdot x^i y^j` and \
        `Q=\sum_{k,l=0}^{k+l\leq d}b_{k,l}\cdot x^k y^l`. To obtain the \
        ordered set of basis coefficients, take ``factor=True`` \
        (by default ``False``).
    """
    [S1,P]=generic_pols(R,d,'a'); [S2,Q]=generic_pols(S1,d,'b',extension=True)
    eqs=[]
    for l in A:
        if l[1]==0:
            X=S2(-l[2]/l[0]); Y=S2(y)
        else:
            X=S2(-l[1]*y); Y=S2(l[0]*y-l[2]/l[1])
        P1 = l[0]*P(X,Y) + l[1]*Q(X,Y)
        eqs = eqs + P1.coefficients()
    S=S2.base_ring(); Sgens = (S2.base_ring()).gens()
    L=[]
    for e in eqs:
        L = L + map(lambda x: (S(e).numerator()).coefficient({x:1})/ \
                                S(e).denominator(), Sgens)
    M = matrix(S.base_ring(),len(eqs),(d+1)*(d+2),L)
    #matrix composed by eqs by rows and the 2x(d+1)*(d+2)/2 coefficients
    #of the polynomials
    if info==True: print S2.base_ring().gens()
    return M
```

```
def basis_vector_field_from_arrangement(R,A,d,factor=False):
    """
    INPUT:
        - A ring ``R`` of defining polynomials.
        - A list `A=[(a_1,b_1,c_1),\ldots,(a_n,b_n,c_n)]` representing a \
        line arrangement `\mathcal{A}=\{\mathcal{l}_1,\ldots,\mathcal{l}_n\}`\
        whose elements are described in the dual projective plane \
        (``(a_i,b_i,c_i)\leftrightarrow L_i: a_ix+b_iy+c_i=0`).
        - A positive integer``d``.

    OUTPUT:
        - A list contained a basis of polynomial vector fields in the plane \
        `\chi=P(x,y)\partial_x + Q(x,y)\partial_y`, where \
        `P,Q\in R` such that `\mathop{deg}P,\mathop{deg}Q\leq d`, and \
        `\mathcal{A}` is invariant by `\chi`.
        - If ``factor=True`` (by default ``False``): polynomials are given\
        in factorized form.
```

```
def line_is_invariant(R,L,V):
    """
    Returns if the line ``L`` written in dual coordinates `(a:b:c)` is \
    invariant by the vector field represented by `V=[P(x,y),Q(x,y)]`.
    """
    f=dual_point_to_eq(R,L)
    Vf= L[0]*V[0]+L[1]*V[1]
    return f.divides(Vf)
```

```
def curve_is_invariant(R,f,V):
    """
    Returns a boolean with the information of the invariance by the vector \
    field ``V`` of the algebraic curve defined by `\mathcal{C}=\{f=0\}`.
    """
    x=R.gens()
    Vf=V[0]*f.derivative(x[0])+V[1]*f.derivative(x[1])
    return f.divides(Vf)
```

```
def fixes_infinity_lines(R,V,(x0,y0)=(0,0)):
    """
    Test if a vector field `V` fixes a pencil of lines centered in `(x0,y0)` \
    (by default the origin (0,0)) or an infinity parallel lines.
    """
    Rgens= R.gens()
    if (Rgens[1]-y0)*V[0]-(Rgens[0]-x0)*V[1]==0: return True
    if V[1]==0: return True
    else:
        f=V[0]/V[1]
        if f.derivative(Rgens[0])==0 and f.derivative(Rgens[1])==0:
            return True
    return False
```

```
def check_infiniteness_base(W,S,(x0,y0)=(0,0)):
    """
    Checks if a subspace of vector fields `W` parametrized by coordinates \
    explicited in `S` fixes a pencil of lines centered in `(x0,y0)` \
    (by default the origin (0,0)) or an infinity parallel lines.
    """
    L=[]
    for i in S:
        for j in S:
            s='r'+str(j)+'=0'; exec s
        s='r'+str(i)+'=1'; exec s
        L = L + [(s,fixes_infinity_lines(R,V,(x0,y0)))]
    return L
```

```
def check_infiniteness_list(R,B,points):
    """
    Checks if a subspace of vector fields given by a list``B`` \
    fixes a pencil of lines centered in any point `(x0,y0)` contained in the \
    list ``points``, or an infinity parallel lines.\n
    Returns a list of booleans corresponding to each element of the basis.
    """
    L=[]
    for p in points:
        L = L + [ [p] + map(lambda V: fixes_infinity_lines(R,V,p),B) ]
    return L
```

```
def check_infiniteness_degree(R,A,d):
    """
    Checks if the basis of the subspace of vector fields of degree ``d`` \
    fixes a pencil of lines centered in the singularities of ``A`` or an \
    infinity parallel lines.\n
    Returns a list of booleans corresponding to each element of the basis.
    """
    B = basis_vector_field_from_arrangement(R,A,d)
    L = check_infiniteness_list(R,B,singularities(R,A))
    return L
```

```
def singular_locus(R,V):
    """
    Returns a list with the singular REAL points of the vector field \
    (with coordinates in ``R``).
    """
    if R.base_ring()!=QQ:
        print
        "Warning: this function don't work correctly over number fields."
    x=R.gens()
    L=solve([SR(V[0]), SR(V[1])], SR(x[0]), SR(x[1]))
    J = solve_to_points(R,L)
    return J
```

```
def linear_system(R,V,(x0,y0),jordan=True):
    """
    Returns the linearized vector field of ``V`` defined in the ring of/
    polynomials ``R`` in the point ``(x0,y0)``.\\
    If ``jordan=True`` (by default), the linearized system is given \
    directly in the jordan form.
    """
    Rbase=R.base_ring(); Rgens=R.gens()
    L=V[0].gradient() + V[1].gradient(); L
    M=matrix(R,2,L)
    N=M.subs({Rgens[0]:x0, Rgens[1]:y0}).change_ring(Rbase)
    if jordan==True: N=N.jordan_form(subdivide=False)
    return N
```

```
def show_linear_system_by_points(R,V,S,jordan=True):
    """
    Prints (typed by LateX) a relation between the linearized vector field \
    of ``V`` defined in the ring of polynomials ``R`` in the points \
    contained in ``S``.\\
    If ``jordan=True`` (by default), the linearized system is given directly \
    in the Jordan form.
    """
    s='['
    for p in SV: s = s + latex(p) + '\\to' + \
                         latex(linear_system(R,V,p,jordan)) + ', '
    s = s + ']'
    show(s)
```

```
def homogenize_vf(S,V):
    """
    Returns the homogenization of the polynomial vector field ``V`` in the \
    ring of homogeneous coordinates ``S``.
    """
    Rgens=list(V[0].parent().gens())
    Sgens=list(S.gens())
    z=(Set(Sgens).difference(Set(Rgens)))[0]
    d=max([V[0].degree(),V[1].degree()])
    W=vector(S,(z^d*S(V[0])(Rgens[0]/z,Rgens[1]/z,1),
                z^d*S(V[1])(Rgens[0]/z,Rgens[1]/z,1),0))
    return W
```

```
def extended_vf(S,V,A):
    """
    For a polynomial vector field in the plane \
    `\chi=P(x,y)\partial_x + Q(x,y)\partial_y` (represented by ``V``) \
    fixing the arrangement ``\mathcal{A}``, returns the associated \
    derivation `\\bar{\chi}` fixing the cone of ``\mathcal{A}``, defined by
    .. MATH::
```

B.3 Plotting

```
def plot_arrangement(R,A,(x0,x1),(y0,y1),color='red'):
    """
    Plots a real arrangement ``A`` in the box ``(x0,x1),(y0,y1)``.
    """
    Rgens=R.gens()
    return implicit_plot(R(def_poly(R, A)),(Rgens[0],x0,x1),
            (Rgens[1],y0,y1),color=color,plot_points=1000)
```

```
def plot_A_der(R,A,V,(x0,x1),(y0,y1)):
    """
    Plots a real arrangement ``A`` and the vector field ``V`` in the box \
    ``(x0,x1),(y0,y1)``.
    """
    Rgens=R.gens()
    return plot_vector_field((V[0],V[1]), (Rgens[0],x0,x1),
        (Rgens[1],y0,y1),color='blue') + plot_arrangement(R,A,(x0,x1),(y0,y1))
```

B.4 Extra functions

```
def basis_monomials(R,deg):
    """
    Returns an ordered list of monomials up to degree ``deg``.
    """
    rbase=R.base_ring()
    rgens=R.gens(); ngens=len(rgens)    #variables of R
    exps=filter(lambda e: sum(e)<=deg,Tuples(range(deg+1),ngens).list())
    #Exponents of monomials
    exps=ordering_tuple(R,map(lambda e: tuple(e), exps))
    #Tuple is ordered with respect to the ordering of R
    mons=[]
    for e in exps:
        m = 1
        for i in range(ngens):
            m = m*rgens[i]^e[i]  #associated monomial
        mons.append(m)
    return mons
```

```
def generic_pols(R,deg,parm='a',extension=False):
    """
    This function allows us to work with generic polynomials in ``R`` \
    of degree ``deg`` parametrized by coefficients `a_{i_1,\ldots,i_n}`\
    (``parm``='a', by default).\n
    If ``extension=True``, we suppose that the coefficient field has already \
    parameters and we do a symbolic field extension with the new ones.
    Returns a list ``[S,p]`` where:
        - ``S`` is the new polynomial ring with same variables of ``R`` and \
        extended coefficients `a_{i_1,\ldots,i_n}`.
        - ``p`` is a generic polynomial \
        `p=\sum_{i_1,\ldots,i_n=0}^{i_1+\ldots+i_n\leq deg}a_{i_1,\ldots,i_n}\
        \cdot x_1^{i_1}\ldots x_n^{i_n}`.
    """
    rbase=R.base_ring()
    rgens=R.gens(); ngens=len(rgens)    #variables of R
    exps=filter(lambda e: sum(e)<=deg,Tuples(range(deg+1),ngens).list())
    #Exponents of monomials
    exps=ordering_tuple(R,map(lambda e: tuple(e), exps))
    #Tuple is ordered with respect to the ordering of R
    parms, mons=[],[]
    for e in exps:
        s = str(e[0]); m = rgens[0]^e[0]
        for i in range(1,ngens):
            s = s +'_' + str(e[i])    #parameter
            m = m*rgens[i]^e[i]  #associated monomial
        parms.append(parm + s); mons.append(m)
    if extension is True:
    #If the coefficient field has already parameters, we do a field extension
        parms_ext = map(lambda x: str(x), rbase.gens()) + parms
        S=PolynomialRing(
          FractionField(PolynomialRing(rbase.base_ring(), parms_ext)),rgens)
    else:
        S=PolynomialRing(FractionField(PolynomialRing(rbase, parms)),rgens)
    gens=map(lambda p: S(p), parms)
    p=S(vector(S,gens)*vector(S,mons))
    return [S,p]
```

```
def pol_cond_to_matrix(R,F,d):
    """
    INPUT:
        - A polynomial equation `F=F(P,Q)=0` in two generic polynomials \
        `P=\sum_{i_1,\ldots,i_n=0}^{i_1+\ldots+i_n\leq d}a_{i_1,\ldots,i_n}\
        \cdot x_1^{i_1}\ldots x_n^{i_n}` and \
        `Q=\sum_{i_1,\ldots,i_n=0}^{i_1+\ldots+i_n\leq d}b_{i_1,\ldots,i_n}\
        \cdot x_1^{i_1}\ldots x_n^{i_n}` of degree `d`.

    OUTPUT:
        - A matrix which entries in each row correspond to the coefficients \
        in `(a_{ij},b_{ij})` for each coefficient in \
        `x_1^{i_1}\ldots x_n^{i_n}` in `F`.
    """
    Rgens = (R.base_ring()).gens()
    eqs=F.coefficients()
    L=[]
    for e in eqs:
        en=e.numerator()
        L = L + map(lambda v: en.coefficient({v:1}), Rgens)
    M = matrix(R.base_ring(),len(eqs),(d+1)*(d+2),L)
    return M
```

```
def solve_to_points(R,L):
    """
    Traduces an expression returned by the function``solve``\
    representing points into a list REAL of solutions in ``R``.
    """
    Sol=[]; iSol=[]; oSol=[]
    for e in L:
        if len(e)==2:
            s=[str(e[0].rhs()), str(e[1].rhs())]
            if 'I' in s[0] or 'I' in s[1]: iSol.append((e[0].rhs(),
                                                        e[0].rhs()))
            else:
                try: Sol.append((R(s[0]), R(s[1])))
                except Exception as err: oSol.append((e[0].rhs(),
                                                      e[0].rhs()))
        else: oSol.append(str(e))
    if len(iSol)!=0: print "Warning! Complex solutions: " + str(iSol)
    if len(oSol)!=0: print "Warning! Other solutions: " + str(oSol)
    return Sol
```

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NOTATIONS

General sets, spaces and maps

$\mathbb{N}$	set of natural numbers,	p. 3
$\mathbb{Z}$	ring of integer numbers,	p. 3
$\mathbb{Q}$	field of rational numbers,	p. 4
$\widetilde{\mathbb{Q}}$	field of real algebraic numbers,	p. 5,7
$\overline{\mathbb{Q}}$	field of algebraic numbers,	p. 4
$\mathbb{R}$	field of real numbers,	p. 4
$\mathbb{C}$	field of complex numbers,	p. 4
sgn	sign of a real number,	p. 36
$\mathbb{B}_x(\rho), \overline{\mathbb{B}}_x(\rho)$	open and closed d–dimensional balls of radius ρ centered at x,	p. 8
$\mathbb{B}_x^\infty(\rho), \overline{\mathbb{B}}_x^\infty(\rho)$	open and closed d–dimensional hypercube of radius ρ centered at x,	p. 8
vol_d	canonical volume map in $\mathbb{R}^d$,	p. 17
V^*	dual of a k–vector space V,	p. 92
$\mathbb{P}(V)$	projectivization of a k–vector space V,	p. 91
$\mathbb{A}_k^d$	affine d–dimensional space over a field k,	p. 92
$\mathbb{P}_k^d$	projective d–dimensional space over a field k,	p. 92
O	origin of the affine space,	p. 46
$X(\mathbb{C})$	complex analytic manifold associated to the variety X,	p. 6
$H_{\mathrm{B}}^\bullet(X, Y; \mathbb{Q})$	Betti (or singular) cohomology groups of X relative to Y with coefficients in $\mathbb{Q}$,	p. 6
$H_{\mathrm{dR}}^\bullet(X, Y; \mathbb{Q})$	algebraic de Rham cohomology groups of X relative to Y with coefficients in $\mathbb{Q}$,	p. 6
$\mathrm{comp}_{\mathrm{B,dR}}$	comparison isomorphism between Betti and de Rham cohomologies,	p. 6
$\Omega^p[V]$	module of algebraic p–forms in a vector space V,	p. 96

$\Omega^p(V)$	module of rational p–forms in a vector space V,	p. 96

$\widetilde{\mathbb{Q}}$–semi-algebraic geometry

$\mathcal{R}_S$	family of all representations of a semi-algebraic set S,	p. 8
Γ_f	semi-algebraic graph of a semi-algebraic map f,	p. 9
$\mathcal{Z}(\mathcal{F})$	zero set of a family $\mathcal{F} \subset \widetilde{\mathbb{Q}}[x_1, \ldots, x_d]$,	p. 12
$\mathcal{I}(S)$	ideal of polynomials in $\widetilde{\mathbb{Q}}[x_1, \ldots, x_d]$ vanishing over a semi-algebraic set $S \subset \mathbb{R}^d$,	p. 12
$\mathcal{P}(S)$	ring of regular functions of S,	p. 12
$\dim S$	dimension of a semi-algebraic set S,	p. 12
$\mathcal{SA}^d_{\widetilde{\mathbb{Q}}}$	class of d–dimensional semi-algebraic subsets in $\mathbb{R}^d$,	p. 14
$\mathrm{Sing}\, X$	singular locus of a variety X,	p. 42
$\partial_z S$	Zariski closure of the boundary of a semi-algebraic set S,	p. 43
$Z(\omega)$	real zero locus of a rational differential form ω,	p. 43
$P(\omega)$	real pole locus of a rational differential form ω,	p. 43
$\widehat{\mathbb{R}}^2_p$	blow-up of $\mathbb{R}^2$ at the point $p \in \mathbb{R}^2$,	p. 47
$\widetilde{A}^\tau$	τ–strict transform of A,	p. 47
$T_p(C)$	algebraic tangent cone of an algebraic curve C at a point $p \in \mathbb{R}^2$,	p. 48
$C_n(k_1, \ldots, k_d)$	cube subdividing $[0, r]^d$ of volume $(r/n)^d$,	p. 52
$\mathring{C}_n(k_1, \ldots, k_d)$	open cube subdividing $[0, r]^d$ of volume $(r/n)^d$,	p. 52
$\mathcal{SA}_{\widetilde{\mathbb{Q}}}(d, r, s)$	collection of semi-algebraic sets $S \subset \mathbb{R}^d$ admitting a representation of complexity smaller than (d, r, s),	p. 76

Periods

$\mathcal{P}_{\mathrm{KZ}}$	algebra of periods of Kontsevich-Zagier,	p. 5, 14
$\mathcal{P}^{\mathbb{R}}_{\mathrm{KZ}}$	algebra of real periods of Kontsevich-Zagier,	p. 5, 14
$\mathcal{I}(S, P/Q)$	integral of a rational map $P/Q \in \widetilde{\mathbb{Q}}[x_1, \ldots, x_d]$ over a semi-algebraic set S,	p. 14, 36
$\zeta(\underline{s})$	multiple zeta value of coefficients $\underline{s} = (s_1, \ldots, s_k)$,	p. 16, 30
$\mathrm{Li}_{\underline{s}}(x)$	multiple polylogarithm values in one variable of coefficients $\underline{s} = (s_1, \ldots, s_k)$,	p. 16
$\mathcal{P}_{\mathsf{abs}}$	algebra of abstract periods,	p. 20
$\mathbb{R}_{(\mathsf{Elem})}$	field of elementary reals numbers,	p. 27
$\deg(p)$	degree of a complex number,	p. 70
$[\mathcal{P}_{\mathrm{KZ}}]_k$	$\overline{\mathbb{Q}}$-module of periods of degree smaller than or equal to k,	p. 73

Hyperplane and line arrangements

$\mathcal{A}$	hyperplane arrangement,	p. 91
Φ_d	empty arrangement in dimension d,	p. 92
$\mathcal{Q}_{\mathcal{A}}$	defining polynomial of a hyperplane arrangement $\mathcal{A}$,	p. 92
$L(\mathcal{A})$	intersection poset of $\mathcal{A}$,	p. 93
$\operatorname{Sing}\mathcal{A}$	set of singular points of a line arrangement $\mathcal{A}$,	p. 92
$\lvert\mathcal{A}_P\rvert$	multiplicity of a singular point P of a line arrangement $\mathcal{A}$,	p. 92
$\mathcal{S}_{\mathcal{A}}$	an ordered sequence $(s_2, s_3, \ldots, s_n)$, where s_k denotes the number of singular points of multiplicity k of a line arrangement $\mathcal{A}$,	p. 94
$\mathcal{P}_{\mathcal{A}}$	an ordered sequence $(p_2, p_3, \ldots, p_n)$, where p_k denotes the number of families of exactly $(k-1)$ parallel lines of a line arrangement $\mathcal{A}$,	p. 95
$m(\mathcal{A})$	maximal multiplicity of singular points of a line arrangement $\mathcal{A}$,	p. 117
$p(\mathcal{A})$	maximal number of parallel lines of the same slope in an affine line arrangement $\mathcal{A}$,	p. 117
$\mathrm{c}\mathcal{A}$	coning in $\mathbb{A}^3_k$ of an affine line arrangement $\mathcal{A}$,	p. 92
$\widetilde{\mathcal{A}}$	projectivization in $\mathbb{P}^2_k$ of an affine line arrangement $\mathcal{A}$,	p. 92

Module of logarithmic derivations

$\operatorname{Der}_k(S)$	module of derivations of a polynomial algebra S,	p. 95
$\mathcal{D}(\mathcal{A})$	module of logarithmic derivations of a hyperplane arrangement $\mathcal{A}$,	p. 95
χ_E	Euler derivation $\sum_i x_i\partial_{x_i}$,	p. 95
$\Omega^p(\mathcal{A})$	logarithmic p–forms with poles along a hyperplane arrangement $\mathcal{A}$,	p. 97
$\mathcal{F}_d\mathcal{D}(\mathcal{A})$	d–filtration set of the filtration of $\mathcal{D}(\mathcal{A})$ by degree,	p. 113
$\mathcal{D}_d(\mathcal{A})$	derivations of degree d in $\mathcal{D}(\mathcal{A})$,	p. 113
$\mathcal{D}^\infty(\mathcal{A})$	sets of elements in $\mathcal{D}(\mathcal{A})$ fixing infinity many lines,	p. 116
$\mathcal{D}^f(\mathcal{A})$	sets of elements in $\mathcal{D}(\mathcal{A})$ fixing only a finite set of lines,	p. 116
$\mathcal{D}^\infty_d(\mathcal{A})$	derivations of degree d in $\mathcal{D}^\infty(\mathcal{A})$,	p. 116
$\mathcal{D}^f_d(\mathcal{A})$	derivations of degree d in $\mathcal{D}^f(\mathcal{A})$,	p. 116
$d_f(\mathcal{A})$	minimal degree d for which $\mathcal{D}^f_d(\mathcal{A})$ is not empty,	p. 116

Abstract

The principal motivation of the present Ph.D subject is the study of certain interactions between number theory, algebraic geometry and dynamical systems. This thesis is composed by two different parts: a first one about **periods of Kontsevich-Zagier** and another one about **logarithmic vector fields on line arrangements**.

The first part concerns a problem of **number theory**, for which we develop a **geometrical approach** based on tools coming from **algebraic geometry** and **combinatorial geometry**. Introduced by M. Kontsevich and D. Zagier in 2001, *periods* are complex numbers expressed as values of integrals of a special form, where both the domain and the integrand are expressed using polynomials with rational coefficients. The **Kontsevich-Zagier period conjecture** affirms that any polynomial relation between periods can be obtained by linear relations between their integral representations, expressed by classical rules of integral calculus.

Using resolution of singularities, we introduce a *semi-canonical reduction* for periods focusing on give constructible and algorithmic methods respecting the classical rules of integral transformations: we prove that any non-zero real period, represented by an integral, can be expressed up to sign as the volume of a compact semi-algebraic set.

The semi-canonical reduction permit a reformulation of the Kontsevich-Zagier conjecture in terms of volume-preserving change of variables between compact semi-algebraic sets. Via triangulations and methods of PL–geometry, we study the obstructions of this approach as a generalization of the Third Hilbert Problem.

We complete the works of J. Wan to develop a *degree theory* for periods based on the minimality of the ambient space needed to obtain such a compact reduction, this gives a first geometric notion of transcendence of periods. We extend this study introducing notions of *geometric and arithmetic complexities* for periods based in the minimal polynomial complexity among the semi-canonical reductions of a period.

The second part deals with the understanding of particular objects coming from **algebraic geometry** with a strong background in **combinatorial geometry**, for which we develop a **dynamical approach**. The logarithmic vector fields are an algebraic-analytic tool used to study sub-varieties and germs of analytic manifolds. We are concerned with the case of **line arrangements** in the affine or projective space. One is interested to study how the combinatorial data of the arrangement determines relations between its associated logarithmic vector fields: this problem is known as the **Terao conjecture**.

We study the module of logarithmic vector fields of an affine line arrangement by the filtration induced by the degree of the polynomial components. We determine that there exist only two types of non-trivial polynomial vector fields fixing an infinitely many lines. Then, we describe the influence of the combinatorics of the arrangement on the expected minimal degree for these kind of vector fields. We prove that the combinatorics do not determine the minimal degree of the logarithmic vector fields of an affine line arrangement, giving two pair of counter-examples, each pair corresponding to a different notion of combinatorics. We determine that the dimension of the filtered spaces follows a quadratic growth from a certain degree, depending only on the combinatorics of the arrangements. We illustrate these formula by computations over some examples. In order to study computationally these filtration, we develop a library of functions in the mathematical software `Sage`.

Keywords: periods, number theory, transcendence, algebraic geometry, singularity theory, line arrangements, dynamical systems, algorithmic, algebraic computation.

Résumé

La principale motivation de ce sujet de thèse est l'étude de certaines interactions entre la théorie des nombres, la géométrie algébrique et les systèmes dynamiques. Cette mémoire est divisé en deux parties : la première porte sur **les périodes de Kontsevich-Zagier** et la seconde sur **les champs de vecteurs logarithmiques des arrangements de droites**.

La première partie concerne un problème de **théorie des nombres**, pour laquel nous développons une **approche géométrique** basé sur des outils provenant de la **géométrie algébrique** et de la **géométrique combinatoire**. Introduites par M. Kontsevich et D. Zagier en 2001, les *périodes* sont des nombres complexes obtenus comme valeurs des intégrales d'une forme particulier, où le domaine et l'intégrande s'expriment par des polynômes avec coefficients rationnels. La **conjecture de périodes de Kontsevich-Zagier** affirme que n'importe quelle relation polynomiale entre périodes peut s'obtenir par des relations linéaires entre différentes représentations intégrales, exprimées par des règles classiques du calcul intégrale.

En utilisant des résolutions de singularités, on introduit une *réduction semi-canonique* de périodes en se concentrant sur le fait d'obtenir une méthode algorithmique et constructive respectant les règles classiques de transformation intégrale : nous prouvons que n'importe quelle période non nulle, représentée par une certaine intégrale, peut être exprimée sauf signe comme le volume d'un ensemble semi-algébrique compact.

La réduction semi-canonique permet une reformulation de la conjecture de périodes de Kontsevich-Zagier en termes de changement de variables préservant le volume entre ensembles semi-algébriques compacts. Via des triangulations et méthodes de la géométrie–PL, nous étudions les obstructions de cette approche comme la généralisation du 3ème Problème de Hilbert.

Nous complétons les travaux de J. Wan dans le développement d'une *théorie du degré* pour les périodes, basée sur la dimension minimale de l'espace ambiance nécessaire pour obtenir une telle réduction compacte, en donnant une première notion géométrique sur la transcendance de périodes. Nous étendons cet étude en introduisant des notions de *complexité géométrique et arithmétique* pour le périodes basées sur la complexité polynomiale minimale parmi les réductions semi-canoniques d'une période.

La seconde partie s'occupe de la compréhension d'objets provenant de la **géométrie algébrique** avec une forte connexion avec la **géométrique combinatoire**, pour lesquels nous avons développé une **approche dynamique**. Les champs de vecteurs logarithmiques sont un outils algébro-analytique utilisés dans l'étude des sous-variétés et des germes dans des variétés analytiques. Nous nous sommes concentré sur le cas des **arrangements de droites** dans des espaces affines ou projectifs. On s'est plus particulièrement intéressé à comprendre comment la combinatoire d'un arrangement détermine les relations entre les champs de vecteurs logarithmiques associés : ce problème est connu sous le nom de **conjecture de Terao**.

Nous étudions le module des champs de vecteurs logarithmiques d'un arrangement de droites affin en utilisant la filtration induite par le degré des composantes polynomiales. Nous déterminons qu'il n'existent que deux types de champs de vecteurs polynomiaux qui fixent une infinité de droites. Ensuite, nous décrivons l'influence de la combinatoire de l'arrangement de droites sur le degré minimal attendu pour ce type de champs de vecteurs. Nous prouvons que la combinatoire ne détermine pas le degré minimal des champs de vecteurs logarithmiques d'un arrangement de droites affin, en présentant deux pairs de contre-exemples, chaque qu'un d'eux correspondant à une notion différente de combinatoire. Nous déterminons que la dimension des espaces de filtration suit une croissance quadratique à partir d'un certain degré, en dépendant uniquement de la combinatoire de l'arrangement. A fin d'étudier de façon calculatoire une telle filtration, nous développons une librairie de fonctions sur le software de calcul formel `Sage`.

Mots clé : périodes, théorie de nombres, transcendance, géométrie algébrique, théorie de singularités, arrangements de droites, systèmes dynamiques, algorithmique, calcul formel.

Resumen

La principal motivación de la presente tesis es el estudio de ciertas interacciones entre teoría de números, geometría algebraica y sistemas dinámicos. Este manuscrito está compuesto por dos partes diferenciadas: una primera sobre **periodos de Kontsevich-Zagier** y otra sobre **campos vectoriales logarítmicos sobre configuraciones de rectas**.

La primera parte concierne un problema de **teoría de números**, para el cual desarrollamos un **enfoque geométrico** basado en herramientas provenientes de la **geometría algebraica** y la **geometría combinatoria**. Introducidos por M. Kontsevich y D. Zagier en 2001, los *periodos* son números complejos obtenidos como valores de integrales de una forma particular, donde el dominio y el integrando pueden expresarse utilizando polinomios con coeficientes racionales. La **conjetura de periodos de Kontsevich-Zagier** afirma que cualquier relación polinómica entre periodos puede ser obtenida a través de relaciones lineales entre las distintas representaciones integrales, expresadas por reglas clásicas del cálculo integral.

Utilizando resolución de singularidades, introducimos la *reducción semi-canónica* de periodos centrándonos en dar métodos constructivos y algorítmicos respetando las reglas clásicas de transformaciones integrales: demostramos que cualquier periodo real no nulo, representado por una integral, puede ser expresado salvo signo como el volumen de un conjunto semi-algebraico compacto.

La reducción semi-canónica permite una reformulación de la conjetura de Kontsevich-Zagier en términos de cambios de variables preservando el volumen entre conjuntos semi-algebraicos compactos. A través de triangulaciones y métodos de geometría–PL, estudiamos obstrucciones de este enfoque como una generalización del Tercer Problema de Hilbert.

Completamos los trabajos de J. Wan en el desarrollo de una *teoría de grado* para periodos basado en la dimensión minimal del espacio ambiente necesario para obtener una reducción compacta de este tipo, dando una primera noción geométrica de transcendencia de un periodo. Extendemos este estudio introduciendo nociones de *complejidades geométricas y aritméticas* basadas en la mínima complejidad polinomial entre las reducciones semi-canónicas de un periodo.

La segunda parte trata sobre la comprensión de ciertos objetos provenientes de la **geometría algebraica** con un fuerte trasfondo en **geometría combinatoria**, para los cuales desarrollamos un **enfoque dinámico**. Los campos de vectores logarítmicos son herramientas algebraico-analíticas utilizadas para el estudio de sub-variedades y gérmenes de variedades analíticas. Nos centraremos en el caso de **configuraciones de rectas** en el plano afín o en el proyectivo. Estamos interesados en el estudio de cómo la información combinatoria de la configuración determina relaciones entre sus campos de vectores logarítmicos: este problema es conocido como la **conjetura de Terao**.

Estudiamos el módulo de campos de vectores logarítmicos de una configuración de rectas afines utilizando la filtración inducida por el grado de las componentes polinomiales. Determinamos que existen únicamente dos tipos de campos de vectores polinómicos fijando una infinidad de rectas. Luego, describimos la influencia de la combinatoria de la configuración en el grado minimal esperado para este tipo de campos de vectores. Demostramos que la combinatoria no determina el grado minimal de los campos de vectores logarítmicos de una configuración de rectas afín, presentando dos pares de contra-ejemplos, donde cada par corresponde a una noción diferente de combinatoria. Determinamos que la dimensión de los espacios de filtración sigue un crecimiento cuadrático a partir de un cierto grado, dependiendo únicamente de la combinatoria de la configuración. Con el fin de estudiar computacionalmente la citada filtración, desarrollamos una librería de funciones sobre el software matemático `Sage`.

Palabras clave: periodos, teoría de números, transcendencia, geometría algebraica, teoría de singularidades, configuraciones de rectas, sistemas dinámicos, algorítmica, computación algebraica.