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Spécialité Mathématiques appliquées

Étude asymptotique d'équations aux dérivées partielles de type diffusion non linéaire et inégalités fonctionnelles associées

présentée par
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À ma famille…

Table des matières

Liste de publications **ix**

Introduction **xi**
Étude des équations de diffusion xi
Un exemple d'équation de diffusion non linéaire xii
Inégalités de Sobolev et Hardy-Littlewood-Sobolev : dualités xiii
Le cas du laplacien classique xiv
L'équation de diffusion rapide comme outil xvii
Le cas du laplacien fractionnaire xix
La dimension deux et l'inégalité d'Onofri xxi
L'inégalité d'Onofri comme limite de l'inégalité de Sobolev xxii
Modèle Keller-Segel critique et équation de diffusion rapide xxii
Un résultat de rigidité xxv
Extension aux variétés riemanniennes xxvi
Étude de modèles de mouvement de foule xxviii
Références. xxxiii

I. Inégalités de Sobolev et Hardy-Littlewood-Sobolev **3**

II. Inégalités de Sobolev et HLS fractionnaires **33**

III. Inégalité d'Onofri **57**

IV. Inégalité d'Onofri sur des variétés riemanniennes **89**

V. Modèles de mouvement de foule **113**

Bibliographie générale **143**

Liste de publications

[1] **Stationary solutions of Keller-Segel type crowd motion and herding models : multiplicity and dynamical stability**
Jean Dolbeault, Gaspard Jankowiak, Peter A. Markowich
à paraître dans *Mathematics and Mechanics of Complex Systems*
arXiv:1305.1715

[2] **Sobolev and Hardy-Littlewood-Sobolev inequalities**
Jean Dolbeault, Gaspard Jankowiak
à paraître dans *Journal of Differential Equations*
arXiv:1312.2568

[3] **The Moser-Trudinger-Onofri inequality**
Jean Dolbeault, Maria J. Esteban, Gaspard Jankowiak
arXiv:1403.5042

[4] **Fractional Sobolev and Hardy-Littlewood-Sobolev inequalities**
Gaspard Jankowiak, Van Hoang Nguyen
arXiv:1404.1028

[5] **The Onofri inequality on two dimensional Riemannian manifolds**
Jean Dolbeault, Maria J. Esteban, Gaspard Jankowiak
arXiv:1404.7338

Introduction

Étude des équations de diffusion

Ce travail est consacré à l'étude du comportement en temps grand d'équations aux dérivées partielles de type parabolique. Plus particulièrement, on s'intéresse à des équations *non linéaires* de type *diffusion*, qui interviennent dans de nombreux modèles issus de la physique (par exemple l'équation des milieux poreux) ou de la biologie (par exemple le modèle de Patlak-Keller-Segel pour la chimiotaxie).

En plus des problèmes classiques de l'existence, de l'unicité ou de la régularité des solutions qui apparaissent dans l'étude des équations elliptiques, comme l'équation de Poisson, les équations d'évolution posent la question du *comportement asymptotique* des solutions, c'est-à-dire pour des temps très grands. Cette analyse est intéressante pour valider théoriquement les modèles et pour vérifier la pertinence des calculs numériques : si on laisse évoluer le système, à quoi doit-on s'attendre, à la fois qualitativement et quantitativement ? Est-ce que les solutions sont attirées par vers un profil limite ? Est-ce qu'elles explosent, et alors, à quelle vitesse ? Ou bien, a-t-on extinction ? Pour l'analyse mathématique, on peut souvent se ramener le problème au cas d'un profil limite par un changement de variable, dit *auto-similaire*. On est ensuite réduit à l'étude des états *stationnaires* d'une équation de type elliptique, et à montrer des résultats de convergence en temps.

Il est essentiel d'étudier les propriétés qualitatives de ces profils limite, comme leurs symétries mais aussi leur unicité : il est possible en effet que plusieurs tels profils existent simultanément. Il devient alors important du point de vue des applications de pouvoir caractériser leur *stabilité*. Pour les équations non linéaires, l'étude de stabilité est faite en deux temps. Par développement autour du profil considéré, on se ramène au cadre linéaire dans lequel on peut étudier la stabilité de l'opérateur d'évolution linéarisé, par exemple à l'aide de méthodes spectrales, et caractériser le noyau de l'opérateur linéarisé. Il faut ensuite prendre en compte les effets non linéaires, par exemple à l'aide d'une fonction de Lyapunov. Enfin, on peut pousser l'étude au niveau suivant en caractérisant les vitesses de retour vers l'équilibre et les modes associés. Cette information permet de savoir quelles propriétés de la donnée initiale vont déterminer le comportement de la solution lors de la relaxation vers l'équilibre.

À est toutes ces étapes, on est amené à *définir*, *estimer* et *comparer* des quantités intégrales

caractéristiques du système étudié d'où la nécessité d'avoir des inégalités fonctionnelles. Ces inégalités, si elles sont écrites de manière optimale, permettent non seulement d'avoir des estimations fines, mais aussi, par la connaissance du cas d'égalité, et donc des fonctions extrémales, d'obtenir une information d'une nature différente qui permet par exemple d'identifier de manière explicite les équilibres, et d'avoir des taux de convergences améliorés.

Dans les chapitres I et II on s'intéresse à une amélioration de l'inégalité de Sobolev à travers son inégalité duale, l'inégalité de Hardy-Littlewood-Sobolev, dans le cadre du laplacien ordinaire et du laplacien fractionnaire, respectivement. Le chapitre III est un passage en revue de l'inégalité d'Onofri, qui joue le rôle de l'inégalité de Sobolev pour la dimension 2. De nouveaux résultats sont apportés, dont certains sont étendus aux variétés riemanniennes au chapitre IV. Enfin, le chapitre V traite des états stationnaires de deux modèles paraboliques, utilisés pour l'étude du déplacement de foules et la modélisation en biologie (chimiotaxie).

Un exemple d'équation de diffusion non linéaire

L'équation suivante est une exemple fondamentale d'une équation de diffusion non linéaire ayant une structure mathématique riche. Elle revient régulièrement dans les travaux de cette thèse.

$$\frac{\partial}{\partial t} v(t,x) = \Delta v^m \,, \quad x \in \mathbb{R}^d \,, \; t > 0 \,. \tag{1}$$

On peut réécrire cette équation comme une équation de diffusion avec un coefficient de diffusion v^{m-1}. Le comportement des solutions est bien compris pour les exposants $m > 0$, en particulier grâce aux travaux de Vázquez [Váz07], et on va brièvement exposer quelques faits intéressants la concernant. Dans le cas $m = 1$ on retrouve simplement l'équation de la chaleur. Le cas $m > 1$ est connu sous le nom d'*équation des milieux poreux*, dont une des caractéristiques est que la solution pour une donnée initiale à support compact sera à support compact pour tout temps. Le coefficient de diffusion s'annule autour des zéros de la solution. La situation $m < 1$ est celle de la *diffusion rapide* : le coefficient de diffusion est singulier autour des zéros de la solution. Ce cas présente un changement de comportement des solutions autour de l'exposant critique $m_c = \frac{d-2}{d}$. Si $m > m_c$, la masse $M = \int v \, \mathrm{d}x$ est conservée au cours de l'évolution. Le flot a aussi un effet régularisant, en particulier la solution est bornée pour tout temps strictement positif. Au contraire, pour $m < m_c$, il y a perte de masse et la solution s'éteint en temps fini. Dans cette situation, il existe aussi des solutions non régulières. Ces différents cas de figure sont illustrés dans la Figure 1.

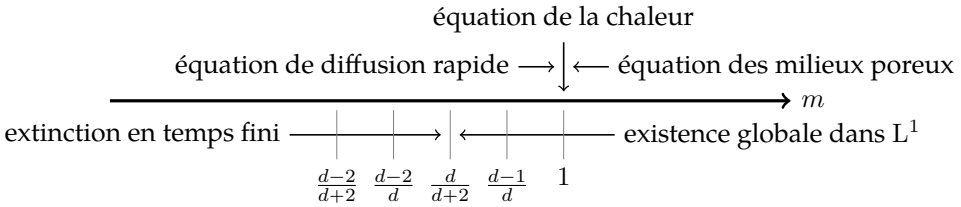


FIGURE 1 : Les différents régimes pour l'équation de diffusion non linéaire (1)

Dans tous ces cas de figure, comme rien de s'oppose à la diffusion, toute solution de (1) va converger ponctuellement vers 0. On peut faire un changement de variable du type

$$v(t,x) = \frac{1}{R(t)^d}\, u\left(\log R(t), \frac{x}{R(t)}\right),$$

où l'expression de $R(t)$ dépend de m et de m_c. Ceci nous donne l'équation suivante pour u,

$$\frac{\partial}{\partial t}u(t,x) = \Delta u^m + \nabla\cdot(x\,u)\,, \quad x\in\mathbb{R}^d\,,\ t>0\,, \tag{2}$$

qui est une version non linéaire de l'équation de *Fokker-Planck*. Grâce à la présence du terme de rappel $\nabla\cdot(xu)$, il existe des états stationnaires : les solutions de *Barenblatt–Prattle*. Elles s'écrivent

$$u_\infty(x) = \left(D - \frac{m-1}{2m}|x|^2\right)_+^{\frac{1}{m-1}},$$

où $(\,\cdot\,)_+$ est la partie positive, et D une constante fixée par la masse de la donnée initiale. On peut associer une fonctionnelle d'*entropie relative* à la deuxième équation, c'est-à-dire une quantité décroissante le long du flot, qui va mesurer une distance au profil asymptotique. Son introduction est due à Newman et Ralston [New84 ; Ral84].

$$\mathcal{E}[u] = \frac{1}{m-1}\int_{\mathbb{R}^d}\left[u^m - u_\infty^m - m\,u_\infty^{m-1}(u-u_\infty)\right]\,\mathrm{d}x\,.$$

Sa dérivée le long du flot est donnée par

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}[u] = -\left(\frac{m}{m-1}\right)^2\int_{\mathbb{R}^d} u\left|\nabla u^{m-1} - \nabla u_\infty^{m-1}\right|^2\,\mathrm{d}x =: -\mathcal{I}[u]\,,$$

où on appelle $\mathcal{I}$ l'*information de Fischer relative* associée à u. On peut alors montrer que pour $m\in\left(\frac{d-1}{d},1\right)$ on a l'inégalité

$$\mathcal{E}[u] \leqslant \frac{1}{2}\mathcal{I}[u]$$

en utilisant une famille d'inégalités de *Gagliardo-Nirenberg*, qui généralisent les inégalités de Sobolev. Elles s'écrivent

$$\|f\|_{\mathrm{L}^p(\mathbb{R}^d)} \leqslant C_{p,d}\|\nabla f\|^\theta_{\mathrm{L}^2(\mathbb{R}^d)}\|f\|^{1-\theta}_{\mathrm{L}^{p+1}(\mathbb{R}^d)}\,, \quad p\in\left(1,\tfrac{d}{d-2}\right), \tag{3}$$

où la meilleure constante $C_{p,d}$ et fonctions extrémales sont connues grâce à del Pino et Dolbeault [PD02]. Ceci nous donne la décroissance exponentielle de $\mathcal{F}$ vers 0, et donc, avec un peu plus de travail, la convergence de u vers u_∞ dans $\mathrm{L}^1(\mathbb{R}^d)$, voir [DT13]. On a bien stabilité des profils u_∞.

Ceci conclut l'illustration du schéma grossier dressé plus haut. La richesse de l'équation (1) va bien au-delà de ce qui pourrait être présenté dans cette introduction, et le lecteur intéressé pourra trouver un exposé plus complet dans [Váz07].

Inégalités de Sobolev et Hardy-Littlewood-Sobolev : dualités

Les inégalités intégrales du type Sobolev ont une longue histoire, dont les prémices remontent au XIXe siècle, dans le cadre de l'étude de problèmes de mécanique céleste. Par inégalité de type Sobolev, on entend les inégalités qui bornent la norme d'une fonction f par une norme impliquant une dérivée de f. Elles ont aujourd'hui une place centrale dans l'étude des équations aux dérivées partielles.

Le cas du laplacien classique

En 1976, Aubin [Aub76] et Talenti [Tal76] donnent la première preuve rigoureuse de l'inégalité de Sobolev optimale dans le cadre L^2. Il faut noter deux résultats antérieurs mais partiels dans cette direction, celui de Bliss [Bli30] dans le cadre des fonctions à symétrie radiales, puis de Rosen en 1971, dans le cas de la dimension 3, voir [Ros71].

En notant $2^* = \frac{2d}{d-2}$ l'exposant critique de Sobolev en dimension $d \geqslant 3$ et $\mathcal{D}$ la complétion de l'espace des fonctions régulières à support compact par la norme $\|\nabla \cdot \|_{\mathrm{L}^2}$, le résultat d'Aubin et Talenti est le suivant.

Théorème 1 (Inégalité de Sobolev optimale — Talenti, 1976 et Aubin, 1976)**.** *Soit $d \geqslant 3$. Pour toute fonction $u \in \mathcal{D}$, l'inégalité*

$$\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \leqslant \mathsf{S}_d \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\,, \tag{4}$$

est vérifiée, avec la constante d'Aubin-Talenti

$$\mathsf{S}_d = \frac{1}{\pi\, d\,(d-2)} \left(\frac{\Gamma(d)}{\Gamma\left(\frac{d}{2}\right)}\right)^{\frac{2}{d}}\,,$$

où Γ désigne la fonction gamma d'Euler. Le cas d'égalité est donné pour $u = u_\star$ à translation et dilatation près, où

$$u_\star(x) = (1+|x|^2)^{1-\frac{d}{2}}\,.$$

Plus tard, dans [Lie83], Lieb remarque que l'inégalité de Sobolev est *duale* d'une autre inégalité, celle de Hardy-Littlewood-Sobolev, qu'il réécrit donc de manière optimale, par réhaussement sur la sphère via la projection stéréographique inverse.

Théorème 2 (Inégalité de Hardy-Littlewood-Sobolev optimale — Lieb, 1983)**.** *Soit $d \geqslant 3$. Pour toute fonction $v \in \mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)$, l'inégalité*

$$\int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,\mathrm{d}x \leqslant \mathsf{S}_d \|v\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \tag{5}$$

est vérifiée. Le cas d'égalité est donné pour $v = v_\star = u_\star^{\frac{d+2}{d-2}}$ à translation et dilatation près, et donc

$$v_\star(x) = (1+|x|^2)^{-\left(1+\frac{d}{2}\right)}\,.$$

Les preuves des inégalités (4) et (5) reposent sur un résultat de Pólya et Szegő (voir [PS51]) affirmant que l'énergie de Dirichlet est décroissante par symétrisation de Schwarz. Ce fait est beaucoup utilisé pour réduire l'étude de problèmes variationnels au cadre radial. Soit A une partie mesurable de $\mathbb{R}^d$. On appelle alors *symétrisée de Schwarz* et on note $\mathring{A}$ la boule centrée à l'origine de même mesure que A. En utilisant la représentation *en couches* (« layer-cake ») d'une fonction f

$$|f(x)| = \int_0^\infty \chi_{\{|f|>t\}}(t)\,\mathrm{d}t\,,$$

on peut définir sa symétrisée de Schwarz $\mathring{f}$ par

$$\mathring{f}(x) = \int_0^\infty \chi_{\overbrace{\{|f|>t\}}^{\circ}}(t)\,\mathrm{d}t\,.$$

On a alors que $\mathring{f}$ est à symétrie radiale et décroissante et possède les propriétés suivantes :

$$\mathring{f} \geqslant 0\,, \qquad \|\mathring{f}\|_{\mathrm{L}^p} = \|f\|_{\mathrm{L}^p}\ \forall\, p\,, \qquad \|\nabla \mathring{f}\|_{\mathrm{L}^2} \leqslant \|\nabla f\|_{\mathrm{L}^2}\,,$$

sous réserve que ces quantités soient bien définies pour f. Ceci permet, entre autres, de ramener au cas radial l'étude des fonctions optimisantes de l'inégalité de Sobolev. Un résultat similaire existe pour le terme de convolution $(-\Delta)^{-1}v$ de l'inégalité de Hardy-Littlewood-Sobolev, l'inégalité de réarrangement de Riesz, qui peut d'ailleurs être utilisée pour la démonstration du résultat de Pólya et Szegő (voir [LL01]).

La dualité entre les inégalités de Sobolev et Hardy-Littlewood-Sobolev (abrégée *HLS*), a été étudiée par Lieb dans [Lie83]. La transformée de Legendre F^* d'une fonctionnelle convexe F est définie par

$$F^*[v] \coloneqq \sup_u \int_{\mathbb{R}^d} v\,u\,\mathrm{d}x - F[u]\,.$$

Il faut faire attention à l'ensemble dans lequel on prend u, mais par densité on peut se restreindre l'espace de minimisation à des fonctions régulières, à support compact. En considérant la transformée de Legendre des deux termes apparaissant dans (4) on obtient

$$\left(u \mapsto \frac{1}{2}\mathsf{S}_d \int_{\mathbb{R}^d} |\nabla u|^2\,\mathrm{d}x\right)^* = v \mapsto \frac{1}{2}\mathsf{S}_d^{-1} \int_{\mathbb{R}^d} v(-\Delta)^{-1}v\,\mathrm{d}x\,,$$
$$\left(u \mapsto \frac{1}{2}\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right)^* = v \mapsto \frac{1}{2}\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}\,.$$

Comme pour tout F_1, F_2 convexes, $F_1[u] \leqslant F_2[u]$ équivaut à $F_1^*[v] \geqslant F_2^*[v]$, les inégalités (4) et (5) sont équivalentes. Un des objectifs des travaux présentés ici a été de mieux comprendre les implications de cette dualité, notamment en utilisant certaines équations de diffusion non linéaires.

La recherche d'une amélioration quantitative de l'inégalité de Sobolev a déjà été l'objet de nombreux travaux, après que Brézis et Lieb [BL85] eurent posé la question de la *stabilité* de l'inégalité : si $\mathsf{S}_d\|\nabla u\|^2_{\mathrm{L}^2} - \|u\|_{\mathrm{L}^{2^*}}$ est petit, est-ce que u est proche d'une solution extrémale de type $u_\star$? Bianchi et Egnell apportent une réponse positive, et montrent dans [BE91] qu'il existe une constante $c > 0$ telle que

$$\mathsf{S}_d\|\nabla u\|^2_{\mathrm{L}^2} - \|u\|^2_{\mathrm{L}^{2^*}} \geqslant c \inf_{h\in\mathcal{M}} \|\nabla(u-h)\|^2_{\mathrm{L}^2}\,, \tag{6}$$

où

$$\mathcal{M} \coloneqq \left\{h : h(x) = \left(\alpha + \beta\,|x - x_0|^2\right)^{-\frac{d-2}{2}}\,,\ \alpha,\beta > 0\,, x_0 \in \mathbb{R}^d\right\}\,.$$

est la variété des fonctions optimales pour (4). Leur preuve se fait en deux temps, en écrivant que $u = h + \sigma f$ avec $\|u\|_{\mathrm{L}^{2^*}} = \|f\|_{\mathrm{L}^{2^*}} = 1$ et que h minimise le membre de droite dans (6). Si $\sigma \ll 1$ on peut linéariser et obtenir un trou spectral. Dans le cas contraire, le terme de gauche est borné inférieurement par une constante strictement positive, et par contradiction, on obtient l'inégalité, quitte à prendre c assez petit. Cette approche nous prive cependant d'une estimation explicite de c.

Dans le chapitre I, on affine un résultat de [Dol11] allant dans la même direction : une amélioration de l'inégalité de Sobolev avec un terme de reste correspondant à l'inégalité

de Hardy-Littlewood-Sobolev. Dans le but de préciser de résultat, introduisons $\mathcal{S}$ et $\mathcal{H}$, les quantités positives associées aux inégalités de *S*obolev et *H*ardy-Littlewood-Sobolev, respectivement :

$$\mathcal{S}[u] \coloneqq \mathsf{S}_d \|\nabla u\|_2^2 - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\,, \qquad \mathcal{H}[v] \coloneqq \mathsf{S}_d \|v\|^2_{\frac{2d}{d+2}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} v(-\Delta)^{-1} v \,\mathrm{d}x\,,$$

qui s'annulent sur les profils d'Aubin-Talenti $u_\star$ et $v_\star$, respectivement. Notre résultat est alors le suivant.

Théorème 3 (Dolbeault, *Jankowiak*, 2014 — [2, Th. 1])**.** *Pour $d \geqslant 3$, on a l'inégalité*

$$\mathcal{H}[v] \leqslant \mathsf{C}_d \, \|u\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \mathcal{S}[u] \tag{7}$$

si $v = u^{\frac{d+2}{d-2}}$, pour tout $u \in \mathcal{D}$. La constante optimale C_d est telle que

$$\frac{d}{d+4}\,\mathsf{S}_d \leqslant \mathsf{C}_d < \mathsf{S}_d\,.$$

L'idée centrale de la preuve consiste à développer une quantité quadratique bien choisie, méthode qu'on peut formaliser dans un cadre un peu plus général. On considère un espace de Hilbert H muni du produit scalaire $\langle\cdot,\,\cdot\rangle$ et des normes p définies par $\|u\|_p = \left\langle u^{\frac{p}{2}},\, u^{\frac{p}{2}}\right\rangle^{\frac{1}{p}}$. Pour tout opérateur $\mathcal{L}$ positif et auto-adjoint sur H pour lequel il existe p tel que

$$\langle u,\,\mathcal{L}u\rangle \geqslant \|u\|_p^2\,, \qquad \forall\, u \in \mathrm{H}\,,$$

on a aussi par transformée de Legendre

$$\left\langle v,\,\mathcal{L}^{-1}v\right\rangle \leqslant \|v\|^2_{p'}\,,$$

où p' est le conjugué de Hölder de p, c'est-à-dire $\frac{1}{p} + \frac{1}{p'} = 1$. Continuons avec un développement formel du carré

$$\begin{aligned} 0 \leqslant \|\mathcal{L}^{\frac12}(\alpha u - \mathcal{L}^{-1}v)\|_2^2 &= \alpha^2 \|\mathcal{L}^{\frac12}u\|_2^2 - 2\alpha\left\langle \mathcal{L}^{\frac12}u,\,\mathcal{L}^{\frac12}\mathcal{L}^{-1}v\right\rangle + \|\mathcal{L}^{\frac12}\mathcal{L}^{-1}v\|_2^2 \\ &= \alpha^2\,\langle u,\,\mathcal{L}u\rangle - 2\alpha\,\langle u,\,v\rangle + \left\langle v,\,\mathcal{L}^{-1}v\right\rangle\,. \end{aligned}$$

On peut alors choisir $v = u^{p-1}$, c'est-à-dire $u = v^{p'-1}$, ce qui donne

$$0 \leqslant \alpha^2\,\langle u,\,\mathcal{L}u\rangle - \alpha\|u\|_p^p - \left(\alpha\|v\|_{p'}^{p'} - \left\langle v,\,\mathcal{L}^{-1}v\right\rangle\right)\,,$$

puis avec $\alpha = \|v\|_{p'}^{2-p'} = \|u\|_p^{p-2}$

$$0 \leqslant \|u\|_p^{2(p-2)}\left(\langle u,\,\mathcal{L}u\rangle - \|u\|_p^2\right) - \left(\|v\|_{p'}^2 - \left\langle v,\,\mathcal{L}^{-1}v\right\rangle\right)\,.$$

En choisissant $\mathcal{L} = -\mathsf{S}_d\Delta$ et $p = 2^*$, on obtient une preuve de (7). Ceci nous donne aussi la borne supérieure $\mathsf{C}_d \leqslant \mathsf{S}_d$. À titre de remarque, cette méthode peut être adaptée à d'autres espaces que ceux de Sobolev, notamment les espaces à poids, pour obtenir un résultat similaire pour les inégalités d'Onofri et de Hardy-Littlewood-Sobolev logarithmique par exemple (voir plus loin).

À ce stade il est bon de rappeler que la meilleure constante C_d est caractérisée par

$$\frac{1}{\mathsf{C}_d} = \inf_{u\in\mathcal{D}} \mathcal{Q}[u]\,, \quad \text{où} \quad \mathcal{Q}[u] \coloneqq \inf_{u\in\mathcal{D}} \frac{\|u\|_{\mathrm{L}^{\frac{8}{d-2}}(\mathbb{R}^d)}\,\mathcal{S}[u]}{\mathcal{H}[u^{\frac{d+2}{d-2}}]}\,.$$

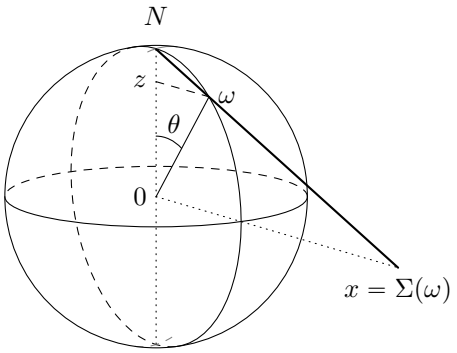


FIGURE 2 : Illustration de la projection stéréographique $\Sigma : \mathbb{S}^2\backslash\{N\} \to \mathbb{R}^2$. Elle n'est pas définie pour le pôle nord N, qui est formellement envoyé à l'infini.

La borne inférieure sur C_d est obtenue via des méthodes de linéarisation : puisqu'on a une forme indéterminée quand on considère le quotient écrit pour $u_\star$, il paraît naturel de développer $\mathcal{S}$ et $\mathcal{H}$ autour de ces profils. Ceci nous ramène à un problème de valeurs propres pour l'opérateur de Laplace, mais dans un espace de fonctions à poids. Ce poids est explicite et correspond en fait au *jacobien* de la projection stéréographique $(2/(1+|x|^2))^d$, et on a donc tout intérêt à réécrire le problème en fonction de $v(\omega) = (1-z)^{-\frac{d-2}{2}} u(\Sigma(x))$, où Σ est la projection stéréographique (voir aussi Figure 2) :

$$\Sigma^{-1} : \begin{array}{l} \mathbb{R}^d \to \mathbb{S}^d\backslash\{N\} \subset \mathbb{R}^{d+1} \\ x = (x_1, \dots, x_d) \mapsto \dfrac{1}{1+|x|^2}\big(|x|^2 - 1, 2x_1, \dots, 2x_d\big)\,, \end{array} \qquad r = |x|\,, \quad z = \frac{r^2-1}{r^2+1}\,.$$

Il ne reste qu'à considérer les valeurs propres de l'opérateur de Laplace-Beltrami sur la sphère, ce qui donne la borne $\frac{d}{d+4}\mathsf{S}_d \leqslant \mathsf{C}_d$.

L'équation de diffusion rapide comme outil

L'exclusion du cas $\mathsf{C}_d = \mathsf{S}_d$ est possible grâce à l'utilisation de l'équation de diffusion rapide présentée au début de cette introduction. Dans le cas sur-critique $m = \frac{d}{d+2}$, qui est tel que $m_c < m < \frac{d-1}{d}$, Carlen, Carrillo et Loss [CCL10] ont remarqué la chose suivante. Si v est une solution de (1), alors

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}[v(\cdot,t)] = -2\,\mathcal{G}[v(\cdot,t)^{\frac{d-1}{d+2}}]\,,$$

où

$$\mathcal{G}[f] \coloneqq \frac{d\,(d-2)}{(d-1)^2}\mathsf{S}_d\,\|f\|_{\mathrm{L}^{\frac{2d}{d-1}}}^{\frac{4}{d-1}}\,\|\nabla f\|_{\mathrm{L}^2}^2 - \|f\|_{\mathrm{L}^{\frac{2(d+1)}{d-1}}}^{\frac{2(d+1)}{d-1}}$$

est positive grâce à l'inégalité de Gagliardo-Nirenberg (3) écrite pour $p = \frac{d+1}{d-1}$. D'autre part, on connait le comportement asymptotique des solutions, qui est donné en variables autosimilaires par les profils $(1+|x|^2)^{-\left(1+\frac{d}{2}\right)}$, qui annulent $\mathcal{H}$. On a donc $\lim_{t\to\infty}\mathcal{H}[v(\cdot,t)] = 0$, et donc que $\mathcal{H}$ est positive pour toute donnée initiale : c'est une preuve de (5).

On peut faire le même calcul pour $m = \frac{d-2}{d+2} < m_c$, l'équation de diffusion correspond alors au flot géométrique de Yamabe. Dans ce cas, le comportement asymptotique est donné

dans [PS01]. Plus précisément, l'équation (1) admet des solutions v_T à variables séparées de la forme

$$v_T(t,x) = (T-t)^{\frac{d+2}{4}} \left(\frac{a\, k_d}{a^2 + |x-x_0|^2} \right)^{1+\frac{d}{2}} .$$

où T est le temps d'extinction, $k_d = (4\,d\,(d-2)/(d+2))^{\frac12}$. Les constantes $a \in \mathbb{R}$ et $x_0 \in \mathbb{R}^d$ traduisent l'invariance par translation et dilatation. Pour toute solution v de (1), il existe T, a et x_0 tels que

$$\sup_{x\in\mathbb{R}^d} \left(1+|x|^{d+2}\right) |v(t,x) - v_T(t,x)| \xrightarrow{t\to T} 0\,.$$

On a alors

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}[v(\cdot,t)] = -2\left(\int_{\mathbb{R}^d} v(\cdot,t)^{m+1}\,\mathrm{d}x\right)^{\frac{2}{d}} \mathcal{S}[u(\cdot,t)] \geqslant 0\,,$$

avec toujours $v = u^{\frac{d+2}{d-2}}$. De la même façon que précédemment, la positivité de $\mathcal{S}$ entraine la positivité de $\mathcal{H}$. On peut pousser plus loin en reprenant l'intégration le long du flot comme dans [Dol11], pour obtenir une inégalité non linéaire liant $\mathcal{S}$ et $\mathcal{H}$. On note $\mathcal{C} = \mathsf{C}_d/\mathsf{S}_d$ et on introduit

$$\mathsf{J}(\cdot,t) \coloneqq \int_{\mathbb{R}^d} v^{m+1}(t)\,\mathrm{d}x\,, \quad \text{qui est tel que} \quad \mathsf{J}' \coloneqq \frac{\mathrm{d}}{\mathrm{d}t}\mathsf{J} = -(m+1)\|\nabla v^m\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \leqslant 0\,.$$

Un calcul direct montre que

$$-\mathcal{H}'' \leqslant -\frac{\mathsf{J}'}{\mathsf{J}}\mathcal{H}'\,, \quad \text{et donc} \quad -\mathcal{H}' \leqslant \kappa_0\,\mathsf{J}\,,$$

où l'on a noté $\kappa_0 = -\mathcal{H}'(0)/\mathsf{J}(0)$. Comme J est décroissante, il existe donc une fonction positive $\mathsf{Y} : [0, \mathsf{J}(0)] \to \mathbb{R}$ telle que $\mathcal{H}(t) = \mathsf{Y}(\mathsf{J}(t))$. En dérivant par rapport à t on a

$$-\mathsf{Y}'(\mathsf{J})\,\mathsf{J}' = -\mathcal{H}' \leqslant \kappa_0\,\mathsf{J}\,,$$

Cette inéquation différentielle peut être intégrée pour obtenir

$$0 \leqslant \mathsf{S}_d\,\mathsf{J}^{1+\frac{2}{d}}\,\varphi\left(\mathsf{J}^{\frac{2}{d}-1}\,\mathcal{S}[u]\right) - \mathcal{H}[v]\,, \tag{8}$$

avec $\varphi(x) \coloneqq \sqrt{\mathcal{C}^2 + 2\,\mathcal{C}\,x} - \mathcal{C}$. Cette inégalité non linéaire est une amélioration de (7) si x est plus grand que $x_{\mathcal{C}} \coloneqq 2\frac{1-\mathcal{C}}{\mathcal{C}}$, ce qui correspond à $\phi(x) < x$, voir Figure 3. Comme $x_1 = 0$, on

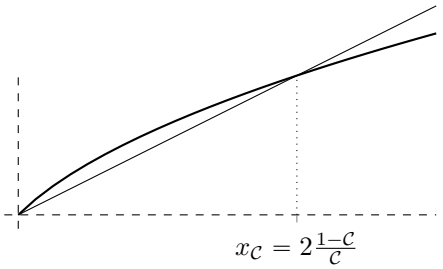


FIGURE 3 : Comparaison de $x \mapsto \phi(x)$ et $x \mapsto \mathcal{C}\,x$. Pour $\mathcal{C} < 1$, $x_{\mathcal{C}} > 0$

peut en fait exclure le cas $\mathcal{C} = 1$, c'est-à-dire $\mathsf{C}_d = \mathsf{S}_d$: supposons que ce soit le cas et considérons une suite minimisante (u_k) pour $\mathcal{Q}$. Par homogénéité, on peut imposer que $\mathsf{J}[u_k^{\frac{d+2}{d-2}}] = \mathsf{J}[u_\star^{\frac{d+2}{d-2}}]$. Deux cas sont alors possibles :

- $\lim_{k\to\infty} \mathcal{H}[u_k^r] > 0$ et $\lim_{k\to\infty} \mathcal{S}[u_k] > 0$, et donc

$$\begin{aligned} 0 = \lim_{k\to\infty} \left(\mathsf{S}_d \, \mathsf{J}_\star^{\frac{4}{d}} \mathcal{S}[u_k] - \mathcal{H}[u_k] \right) \\ = \lim_{k\to\infty} \left(\mathsf{S}_d \, \mathsf{J}_\star^{\frac{4}{d}} \mathcal{S}[u_k] - \mathsf{S}_d \, \mathsf{J}_\star^{1+\frac{2}{d}} \varphi \left(\mathsf{J}_\star^{\frac{2}{d}-1} \mathcal{S}[u_k] \right) \right) \\ + \lim_{k\to\infty} \left(\mathsf{S}_d \, \mathsf{J}_\star^{1+\frac{2}{d}} \varphi \left(\mathsf{J}_\star^{\frac{2}{d}-1} \mathcal{S}[u_k] \right) - \mathcal{H}[u_k] \right) . \end{aligned}$$

  Les deux limites sont positives, ce qui donne une contradiction.
- $\lim \mathcal{S}[u_k] = \lim \mathcal{H}[u_k^{\frac{d+2}{d-2}}] = 0$, auquel cas u_k converge vers un profil d'Aubin-Talenti et la limite de $\mathcal{Q}$ est donnée la linéarisation. Ce qui, après calcul, donne à nouveau une contradiction.

Le cas du laplacien fractionnaire

Le résultat de Lieb de 1983 est en fait plus général que l'énoncé donné dans le Théorème 2 et couvre les noyaux de la forme $x \mapsto |x|^{-\lambda}$, avec $0 < \lambda < n$. En choisissant $\lambda = d-2s$ on obtient le contrôle de puissances non entières du laplacien, c'est-à-dire de $(v, (-\Delta)^{-s} v)$, par $\|v\|^2_{\frac{2d}{d+2s}}$. Dans le cas particulier $s = 1$, on retrouve (5). Par dualité, on peut aussi généraliser l'inégalité de Sobolev aux dérivées fractionnaires.

Le calcul différentiel fractionnaire est relativement bien connu en mathématiques (voir par exemple [Ste70 ; Ada75]), mais les applications pour la modélisation de phénomènes physiques n'est qu'assez récente. Au début des années 2000, il a commencé à être utilisé pour rendre compte de phénomènes de diffusion *anormale*, c'est-à-dire pour lesquelles la distance caractéristique de diffusion est en $t^{\frac{\alpha}{2}}$ (voir [BG90]), avec $\alpha \neq 1$, le cas classique $\alpha = 1$ correspondant à l'équation de la chaleur. Ce type de diffusion apparait par exemple en physique lors de l'observation d'atomes ultra-froids [Sag+12] ou en biologie lors de l'étude de diffusion de particules à l'intérieur de cytoplasmes [Reg+13].

Il est naturel de vouloir étendre le résultat précédent au cadre fractionnaire, ce qui est fait en détail dans le chapitre II, pour $0 < s < \frac{d}{2}$. Le schéma de preuve reste le même que dans le cas $s = 1$ et contraste avec celui de [JX11], entièrement basé sur un flot de Yamabe fractionnaire. Notre méthode permet ici aussi de simplifier la preuve et de fournir de meilleures bornes sur les constantes. L'inégalité est établie par développement du carré, ce qui donne aussi la borne supérieure pour la constante optimale, la borne inférieure étant obtenue par linéarisation. L'identification des solutions extrémales pour le quotient linéarisé demande ici plus de travail : après projection stéréographique inverse pour reformuler le problème sur la sphère, il faut passer par la formule Funk-Hecke. Celle-ci affirme que, sur la sphère, les opérateurs de convolution dont le noyau ne dépend que de l'angle zénithal (angle θ de la Figure 2) commutent avec l'opérateur de Laplace-Beltrami. Elle donne aussi une expression des valeurs propres de tels opérateurs à l'aide des polynômes de Jacobi. Finalement, on obtient le Théorème 1 du chapitre II, qui n'est pas repris ici.

De manière indépendante, Carlen et Loss [CL92] et ensuite Beckner [Bec93] ont étudié les inégalités (4)-(5) réécrites sur la sphère $\mathbb{S}^d$. Ils se sont en particulier intéressés à la limite $s \to \frac{d}{2}$, ce qui correspond à $\lambda \to 0$ dans le potentiel newtonien. Par différentiation, l'inégalité de Sobolev donne l'inégalité de *Moser-Trudinger-Onofri* qui est étudiée en détail en dimension 2

dans les chapitres III et IV ; quant à l'inégalité de Hardy-Littlewood-Sobolev, elle devient l'inégalité de Hardy-Littlewood-Sobolev *logarithmique*.

En utilisant le même passage à la limite dans (7), on obtient un résultat similaire au Théorème 3 : le décifit dans l'inégalité de Moser-Trudinger-Onofri contrôle le déficit dans l'inégalité de Hardy-Littlewood-Sobolev logarithmique. En introduisant la dérivée logarithmique de la fonction Gamma, ainsi que l'entropie

$$\mathrm{Ent}_\sigma(f) = \int_{\mathbb{S}^d} f \log f \, \mathrm{d}\sigma - \left(\int_{\mathbb{S}^d} f \, \mathrm{d}\sigma \right) \log \left(\int_{\mathbb{S}^d} f \, \mathrm{d}\sigma \right), \qquad \text{et} \qquad \Psi(t) = \Gamma'(t)/\Gamma(t),$$

nous avons le théorème suivant.

Théorème 4 (*Jankowiak*, Nguyen, 2014 — [4, Th. 2]). *Soit $d \geqslant 2$ et $\mathrm{d}\sigma$ la mesure sur $\mathbb{S}^d$, normalisée à 1, induite par la mesure de Lebesgue sur $\mathbb{R}^{d+1}$. Il existe une constante C_d^e telle que pour toute fonction F définie sur $\mathbb{S}^d$ avec un développement en harmoniques sphériques $F = \sum_{k\geqslant 0} F_k$ l'inégalité suivante est satisfaite*

$$\begin{aligned} \mathrm{Ent}_\sigma(f) + \frac{d}{M} \iint\limits_{\mathbb{S}^d \times \mathbb{S}^d} f(\xi) \, \log|\xi - \eta| \, f(\eta) \, \mathrm{d}\sigma(\xi) \, \mathrm{d}\sigma(\eta) + \frac{Md}{2} \left(\Psi(d) - \Psi\left(\frac{d}{2}\right) - \log 4 \right) \\ \leqslant C_d^e \, M \left[\frac{1}{2d} \sum_{k\geqslant 1} \frac{\Gamma(k+d)}{\Gamma(d)\Gamma(k)} \int_{\mathbb{S}^d} |F_k|^2 \, \mathrm{d}\sigma + \int_{\mathbb{S}^d} F \, \mathrm{d}\sigma - \log \left(\int_{\mathbb{S}^d} e^F \mathrm{d}\sigma \right) \right], \end{aligned} \tag{9}$$

où $f = e^F$ et $M = \int_{\mathbb{S}^d} f \, \mathrm{d}\sigma$. De plus, en notant C_d^e la meilleure constante dans l'inégalité ci-dessus, on a

$$\frac{1}{d+1} \leqslant \mathsf{C}_d^e \leqslant 1. \tag{10}$$

On peut réécrire l'inégalité (9) en repassant dans l'espace euclidien, dans le cas particulier de la dimension 2, ce qui fait apparaître une variante probablement plus familière des inégalités d'Onofri et de Hardy-Littlewood-Sobolev logarithmique. On retrouve aussi le poids associé au jacobien de la projection stéréographique, $\mu(x) = \frac{1}{\pi}(1+|x|^2)^2$.

Corollaire 5 (*Jankowiak*, Nguyen, 2014 — [4, Cor. 3]). *Il existe une constante C_2^e telle que pour toute fonction $u \in \mathrm{L}^1(\mu)$ et $\nabla u \in \mathrm{L}^2(\mathbb{R}^2)$ l'inégalité suivante est vérifiée*

$$\begin{aligned} \int_{\mathbb{R}^2} v \log\left(\frac{v}{M}\right) \, \mathrm{d}x - \frac{4\pi}{M} \int_{\mathbb{R}^2} v \, (-\Delta)^{-1}(v) \, \mathrm{d}x + M(1 + \log \pi) \\ \leqslant C_2^e \, M \left[\frac{1}{16\pi} \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^2)} + \int_{\mathbb{R}^2} u \, \mathrm{d}\mu - \log \left(\int_{\mathbb{R}^2} e^u \, \mathrm{d}\mu \right) \right]. \end{aligned} \tag{11}$$

où $v = e^u \, \mu$ et $M = \int_{\mathbb{R}^2} v \, \mathrm{d}x$. De plus, en notant C_2^e la meilleure constante pour laquelle l'inégalité (11) est valide, on a

$$\frac{1}{3} \leqslant \mathsf{C}_2^e \leqslant 1.$$

Ce résultat peut être obtenu par un passage à la limite dans une inégalité similaire à (7) écrite pour l'inégalité de Caffarelli-Kohn-Nirenberg et son inégalité duale. Avec un calcul identique à celui de [DET08], on obtient une famille d'inégalités de type Onofri, dont un cas particulier est donné dans [2, Th. 2], qui est une version affaiblie du Corollaire 5.

La dimension deux et l'inégalité d'Onofri

Nous allons maintenant décrire quelques idées du chapitre III. Sortons du cadre fractionnaire pour revenir aux espaces de Sobolev classiques. Une inégalité du même type que (4) ne peut pas être vraie en dimension deux, comme on peut le voir par le changement d'échelle $u(x) = u_\lambda(\lambda x)$, qui laisse invariante la norme $\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^2)}$ mais pas $\|u\|^2_{\mathrm{L}^q(\mathbb{R}^2)}$, quel que soit le choix de q. En étudiant comment étendre (4) à des domaines bornés de $\mathbb{R}^2$, Trudinger [Tru67] et Pohozaev [Poh65] montrent que la bonne norme à considérer est de type $u \to \int e^{|u|^2}$. Plus tard, en se basant sur ce résultat et par symétrisation, Moser [Mos70, Theorem 2] obtient l'inégalité suivante, formulée sur la sphère $\mathbb{S}^2$

$$\frac{1}{4}\int_{\mathbb{S}^2} |\nabla u|^2 \,\mathrm{d}\sigma + c \geqslant \log \int_{\mathbb{S}^2} e^u \,\mathrm{d}\sigma - \int_{\mathbb{S}^2} u \,\mathrm{d}\sigma\,,$$

avec une constante c qui n'est pas connue. Elle est reprise et écrite pour la première fois sous forme optimale par le physicien Enrico Onofri en 1982 [Ono82] pour répondre à des questions issues de la théorie des cordes de Polyakov.

Théorème 6 (Inégalité d'Onofri — Onofri, 1982). *Si $\mathrm{d}\sigma$ est la mesure de probabilité induite par la mesure de Lebesgue sur $\mathbb{S}^2 \subset \mathbb{R}^3$, alors*

$$\frac{1}{4}\int_{\mathbb{S}^2} |\nabla v|^2 \,\mathrm{d}\sigma \geqslant \log\left(\int_{\mathbb{S}^2} e^v \,\mathrm{d}\sigma\right) - \int_{\mathbb{S}^2} v \,\mathrm{d}\sigma\,, \tag{12}$$

et l'égalité a lieu pour les constantes et leurs images par transformation conforme.

Par la projection stéréographique introduite p. xvii, on obtient une forme équivalente sur l'espace euclidien. Cette remarque a été faite pour la première fois dans [CL92, Theorem 1].

Théorème 7 (Inégalité d'Onofri euclidienne). *Si $d\mu = \mu(x)\,\mathrm{d}x$ avec $\mu(x) = \frac{1}{\pi}(1+|x|^2)^{-2}$, alors*

$$\frac{1}{16\pi}\int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x \geqslant \log\left(\int_{\mathbb{R}^2} e^u \,\mathrm{d}\mu\right) - \int_{\mathbb{R}^2} u \,\mathrm{d}\mu\,. \tag{13}$$

La preuve d'Onofri repose fortement sur l'invariance conforme de (12) et le cas d'égalité est identifié en étudiant l'équation d'Euler-Lagrange associée. On peut trouver plus de détail ce sur point dans [Hon86]. D'autres stratégies de preuves existent dans la littérature, par exemple en utilisant des méthodes de convexité comme l'a fait Ghigi dans [Ghi05], à partir de l'inégalité de Prékopa-Leindler.

Comme évoqué plus haut, l'inégalité d'Onofri est duale de l'inégalité de Hardy-Littlewood-Sobolev logarithmique, dérivée d'une part par Carlen et Loss [CL92], et d'autre part par Beckner [Bec93] via des inégalités d'interpolation sur la sphère dues à Bidaut-Véron et Véron [BVV91]

Théorème 8 (Inégalité de HLS logarithmique — Carlen et Loss, 1992 / Beckner, 1993).

$$\int_{\mathbb{R}^2} u\log u \,\mathrm{d}x - \frac{4\pi}{M}\int_{\mathbb{R}^2} u\,(-\Delta)^{-1}(u) \,\mathrm{d}x + M(1+\log\pi) \geqslant 0\,, \tag{14}$$

avec $M = \int_{\mathbb{R}^2} u \,\mathrm{d}x$. L'égalité a lieu pour $u = \mu$, à translations et changements d'échelle près.

En dimension 2, c'est donc l'inégalité de Moser-Trudinger-Onofri ou simplement inégalité d'Onofri qui joue le rôle de l'inégalité de Sobolev optimale. Cette affirmation est justifiée de manière plus profonde en considérant plusieurs limites d'inégalités fonctionnelles, exposées dans le chapitre III, p. 62. Ce chapitre offre un bref passage en revue des résultats connus sur cette inégalité, ainsi qu'un certain nombre de résultats nouveaux. Voici un exemple d'une telle justification, qui sera étendue au chapitre III.

L'inégalité d'Onofri comme limite de l'inégalité de Sobolev

Pour concrétiser le lien entre l'inégalité d'Onofri et l'inégalité de Sobolev, on peut en effet réécrire cette dernière pour des fonctions à symétrie radiale :

$$\left(\int_0^\infty f(r)^{2^*}\, r^{d-1}\,\mathrm{d}r\right)^{1-\frac{2}{d}} \leqslant \omega_d^{\frac{2}{d}}\,\mathsf{S}_d \int_0^\infty |f'(r)|^2\, r^{d-1}\,\mathrm{d}r\,, \tag{15}$$

où $\omega_d = 2\pi^{\frac{d+1}{2}}/\Gamma\left(\frac{d+1}{2}\right)$ est la surface de la sphère unité, et où l'égalité est atteinte pour

$$f_\star(r) = (1+r^2)^{1-\frac{d}{2}}\,.$$

De cette façon, d peut être vu comme un paramètre réel, que l'on va faire tendre vers 2 pour récupérer la forme radiale de l'inégalité d'Onofri. Considérons alors $f = f_\star(1+\frac{2-d}{2\,d}\,u)$. D'une part nous avons

$$\left(\int_0^\infty |f_\star\,(1+\tfrac{d-2}{2\,d}\,u)|^{\frac{2\,d}{d-2}}\, r^{d-1}\,\mathrm{d}r\right)^{\frac{d-2}{d}} - 1 \sim \frac{d-2}{2}\,\log\left(\int_0^\infty e^u\,\frac{r\,\mathrm{d}r}{(1+r^2)^2}\right).$$

nous avons aussi

$$\left|\mathbb{S}^{d-1}\right|^{\frac{2}{d}}\,\mathsf{S}_d \int_0^\infty |f'|^2\, r^{d-1}\,\mathrm{d}r \sim 1 + (d-2)\left[\frac{1}{8}\int_0^\infty |u'|^2\, r\,\mathrm{d}r + \int_0^\infty u\,\frac{2\,r\,\mathrm{d}r}{(1+r^2)^2}\right].$$

En prenant la limite $d \to 2$ et en ne retenant que les termes d'ordre $d-2$, nous obtenons

$$\frac{1}{8}\int_0^\infty |u'|^2\, r\,\mathrm{d}r + \int_0^\infty u\,\frac{2\,r\,\mathrm{d}r}{(1+r^2)^2} \geqslant \log\left(\int_0^\infty e^u\,\frac{2\,r\,\mathrm{d}r}{(1+r^2)^2}\right),$$

qui correspond exactement à l'inégalité (13) écrite pour une fonction u radiale. Bien que valable uniquement avec une hypothèse de symétrie forte, ce calcul montre clairement le lien entre les deux inégalités. En partant d'autres inégalités optimales, par exemple celle de Gagliardo-Nirenberg et en considérant des limites différentes, on peut retrouver (13) sans hypothèse de symétrie.

Modèle Keller-Segel critique et équation de diffusion rapide

Considérons le modèle de Patlak-Keller-Segel pour la chimiotaxie dans sa version elliptique. Celui-ci, introduit en 1970 par Keller et Segel [KS70], décrit l'agrégation d'amibes du type *Dictyostelium discoideum* via leur interaction un *chimioattractant*. Le déplacement de ces amibes est bien décrit par la combinaison d'un mouvement aléatoire (diffusion) et d'une attraction vers les zones denses en chimioattractant (dérive). Le chimioattractant est

sécrété par les amibes elles-mêmes, et subit lui aussi une diffusion. Dans le cas qui nous intéresse ici, on suppose que le temps caractéristique pour l'évolution de ce chimioattractant est bien plus petit que celui de l'évolution de la densité des amibes. On a donc un système parabolique–elliptique [1]. Si on note ρ la densité d'amibes et c la densité de chimioattractant, on a l'évolution suivante :

$$\begin{cases} \dfrac{\partial}{\partial t}\rho = \Delta\rho - \nabla\cdot(\rho\,\nabla c) \\ -\Delta c + \rho = 0 \end{cases}, \quad x\in\mathbb{R}^2\,,\ t>0\,. \tag{16}$$

Au cours de l'évolution, la masse $M = \int_{\mathbb{R}^2}\rho\,\mathrm{d}x$ est conservée, et le comportement qualitatif des solutions dépend de son positionnement par rapport à une valeur critique, dont la valeur exacte est donnée dans [DP04]. Si $M<8\pi$, la diffusion l'emporte : les solutions existent globalement en temps et restent bornées. À l'inverse, si $M>8\pi$ l'attraction prévaut : si la donnée initiale n'est pas trop étalée, plus précisément si son moment d'ordre deux est fini, alors il y a explosion en temps fini. Dans le cas sous-critique, l'énergie libre $\mathcal{F}_{\mathrm{PKS}}$, est décroissante le long du flot :

$$\mathcal{F}_{\mathrm{PKS}}[\rho] \coloneqq \int_{\mathbb{R}^2}\rho\log\rho\,\mathrm{d}x + \frac{1}{4\pi}\iint\limits_{R^2\times\mathbb{R}^2}\rho(x)\log|x-y|\rho(y)\,\mathrm{d}x\,\mathrm{d}y\,,$$

et, combinée avec l'inégalité de Hardy-Littlewood-Sobolev logarithmique (14) permet d'obtenir l'existence globale de solutions pour $M<8\pi$, en remarquant que

$$\begin{aligned}\mathcal{F}_{\mathrm{PKS}}[\rho] &= \frac{M}{8\pi}\left(\int_{\mathbb{R}^2}\rho\log\rho\,\mathrm{d}x + \frac{1}{4\pi}\iint\limits_{R^2\times\mathbb{R}^2}\rho(x)\log|x-y|\rho(y)\,\mathrm{d}x\,\mathrm{d}y\right)\\ &\quad+\left(1-\frac{M}{8\pi}\right)\int_{\mathbb{R}^2}\rho\log\rho\,\mathrm{d}x\\ &\geqslant -\frac{M}{8\pi}M(1+\log\pi)+\left(1-\frac{M}{8\pi}\right)\int_{\mathbb{R}^2}\rho\log\rho\,\mathrm{d}x\,,\end{aligned}$$

est bornée inférieurement grâce à un moment d'ordre deux. Dans la situation $M=8\pi$, le comportement asymptotique dépend du second moment de la donnée initiale. Si celui-ci est fini, il est prouvé dans [BCM08] que la solution converge vers une fonction de Dirac de masse 8π, en temps infini. Dans le cas contraire, Blanchet, Carlen et Carrillo montrent dans [BCC12, Theorem 1.7] que le profil limite est donné, à un changement d'échelle près, par le *profil limite de l'équation de diffusion rapide de même masse*, si on prend pour celle-ci $m=\frac{1}{2}$. Ces résultats sont affinés dans [CF13], où des estimations des taux de convergence sont donnés. La preuve de [BCC12] se base sur la structure de flot gradient des deux équations, qui les lie étroitement.

Revenons maintenant à l'équation de diffusion rapide (1), pour $0<m<1$, écrite en dimension deux. Par changement de variable, on peut se ramener à une équation de type Fokker-Planck non linéaire :

$$\frac{\partial u}{\partial t} + \nabla\cdot\left[u\,\left(\nabla u^{m-1} - 2\,x\right)\right] = 0\,. \tag{17}$$

1. Si on s'intéresse au cas où les échelles de temps sont comparables, on a un système parabolique–parabolique. Le comportement asymptotique est alors plus complexe, point qui sera abordé dans les chapitres IV et V.

Les profils limite de cette équation sont les profils de *Barenblatt* :

$$u_\infty(x) = (D + |x|^2)^{-\frac{1}{1-m}} ,$$

où D est tel que $\int u_\infty \, \mathrm{d}x = M := \int u \, \mathrm{d}x$. On peut dès lors considérer l'*entropie relative* $\mathcal{E}$ et l'*information de Fischer relative* $\mathcal{I}$:

$$\mathcal{E}_m[u] := \frac{1}{m-1} \int_{\mathbb{R}^2} \left[u^m - u_\infty^m - m\, u_\infty^{m-1} (u - u_\infty)\right] \mathrm{d}x \,, \quad \mathcal{I}_m[u] := \int_{\mathbb{R}^2} u \left|u^{m-1} - u_\infty^{m-1}\right|^2 \mathrm{d}x \,.$$

qui sont telles que $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_m[u] = -\mathcal{I}_m[u]$. De plus, si $m > \frac{1}{2}$ alors

$$\mathcal{E}_m[u] \leqslant \frac{1}{4}\mathcal{I}_m[u] \,, \tag{18}$$

où l'exclusion du cas $m = \frac{1}{2}$ vient de la non intégrabilité de u_∞^m et de $|x|^2 u_\infty$. Ce résultat est dû à del Pino et Dolbeault [PD02], qui adaptent la méthode de Bakry et Emery [BÉ84] au cadre non linéaire. Le résultat de [BCC12] repose sur la remarque que le flot de Keller-Segel (16) est un flot gradient pour $\mathcal{E}_{\frac{1}{2}}$. Ceci nous dit aussi, d'après les travaux de Matthes, McCann et Savaré [MMS09] que l'équation de diffusion rapide est un flot gradient pour $\mathcal{F}_{\mathrm{PKS}}$.

Dans le chapitre III, nous faisons l'observation suivante : on peut étendre l'inégalité d'entropie–production d'entropie (18) au cas $m = \frac{1}{2}$, et elle est alors équivalente à l'inégalité d'Onofri euclidienne (13). Prenons en effet u une fonction régulière telle que $u - u_\infty$ soit à support compact et considérons $w = \frac{u}{u_\infty}$. Il vient alors avec $m = \frac{1}{2}$

$$\begin{aligned} \mathrm{E}\,[w] &:= \mathcal{E}_{\frac{1}{2}}[w\, u_\infty] = \int_{\mathbb{R}^2} \frac{|\sqrt{u} - \sqrt{u_\infty}|^2}{\sqrt{u_\infty}} \, \mathrm{d}x = \int_{\mathbb{R}^2} \frac{\left|\sqrt{w} - 1\right|^2}{D + |x|^2} \, \mathrm{d}x \,, \\ \mathrm{I}\,[w] &:= \mathcal{I}_{\frac{1}{2}}[w\, u_\infty] = \int_{\mathbb{R}^2} u_\infty w \left|\nabla\left(u_\infty^{-\frac{1}{2}}(w^{-\frac{1}{2}} - 1)\right)\right|^2 \mathrm{d}x \,, \end{aligned}$$

où, comme le problème d'intégrabilité est levé, E et I sont tels que

$$\mathrm{E}\,[w] \leqslant \frac{1}{4}\mathrm{I}\,[w] \,. \tag{19}$$

En prenant $v = \log w$ on obtient pour v :

$$0 \leqslant \frac{1}{4}\mathrm{I}\,[w] - E[w] = \frac{1}{16}\int_{\mathbb{R}^2} |\nabla v|^2 \, \mathrm{d}x - D \int_{\mathbb{R}^2} \frac{e^v - 1 - v}{(D + |x|^2)^2} \, \mathrm{d}x \,.$$

On peut alors prendre $D = 1$, ce qui revient à requérir $M = \int_{\mathbb{R}^2} e^v \, \mathrm{d}\mu = 1$:

$$\frac{1}{16\pi}\int_{\mathbb{R}^2} |\nabla v|^2 \, \mathrm{d}x + \int_{\mathbb{R}^2} v \, \mathrm{d}\mu \geqslant 0 \,.$$

Sous contrainte de masse, cette inégalité est équivalente à l'inégalité d'Onofri euclidienne. Ceci est tout à fait cohérent : une solution de (16) dont le second moment est infini tend vers un minimiseur de $\mathcal{F}$, et donc un minimiseur de l'inégalité de HLS logarithmique (14), duale de l'inégalité d'Onofri.

Un résultat de rigidité

Il est possible de donner une autre preuve de l'inégalité optimale, par une méthode de *rigidité*. Considérons une version relaxée de la fonctionnelle associée à l'inégalité d'Onofri

$$\mathcal{G}_\lambda[v] \coloneqq \frac{1}{4}\int_{\mathbb{S}^2} |\nabla v|^2 \,\mathrm{d}\sigma + \lambda\left[\int_{\mathbb{S}^2} v\,\mathrm{d}\sigma - \log\left(\int_{\mathbb{S}^2} e^v \mathrm{d}\sigma\right)\right], \tag{20}$$

dont on sait qu'elle est positive pour tout $\lambda > 0$ d'après [OPS88]. Elle admet des minimiseurs dans $\mathrm{H}^1(\mathbb{S}^2)$ qui sont en fait à symétrie radiale d'après [GM13, Lemma 17.1.2]. Si v est un tel minimiseur, en écrivant $v(\omega) = f(z)$ où z désigne la hauteur (voir Figure 2 p. xvii), et en choisissant f telle que $\int_{-1}^1 e^f \,\mathrm{d}z = 2$, on a que f est solution de l'équation d'Euler-Lagrange

$$-\frac{1}{2}\,\mathcal{L}f + \lambda = \lambda\, e^f\,, \tag{21}$$

où $\mathcal{L}$ est l'opérateur *ultrasphérique* $\mathcal{L}f = \nu f'' + \nu' f'$ et $\nu(z) = 1 - z^2$, qui correspond à la projection de l'opérateur de Laplace-Beltrami sur l'axe vertical. Par un jeu d'intégrations par parties on obtient

Théorème 9 (Dolbeault, *Jankowiak*, Esteban — [3, Th. 20])**.** *Pour tout $\lambda \in (0,1)$, toute solution régulière de (21) est constante égale à 0. Si $\lambda = 1$, f est solution de l'équation différentielle $f'' = \frac{1}{2}\,|f'|^2$, elle est donc soit constante, soit telle que*

$$f(z) = C_1 - 2\,\log(C_2 - z)\,, \qquad \text{où} \qquad C_1 \in \mathbb{R}\,, C_2 > 1\,. \tag{22}$$

Cette fonction correspond à l'invariance conforme de (12), pour des fonctions ne dépendant que de z.

Voici un résumé de preuve : en multipliant (21) part $\mathcal{L}\big(e^{-f/2}\big)$ puis en intégrant par parties on obtient

$$0 = \frac{1}{4}\int_{-1}^1 \nu^2\,|f''|^2\,e^{-f/2}\,\mathrm{d}z - \frac{1}{8}\int_{-1}^1 \nu^2\,|f'|^2\,f''\,e^{-f/2}\,\mathrm{d}z + \frac{1}{2}\int_{-1}^1 \nu\,|f'|^2\,e^{-f/2}\,\mathrm{d}z - \frac{1}{2}\int_{-1}^1 \nu\,|f'|^2\,e^{f/2}\,\mathrm{d}z\,.$$

Le même calcul en multipliant par $\frac{\nu}{2}\,|f'|^2\,e^{-f/2}$ donne quant à lui

$$0 = \frac{1}{8}\int_{-1}^1 \nu^2\,|f'|^2\,f''\,e^{-f/2}\,\mathrm{d}z - \frac{1}{16}\int_{-1}^1 \nu^2\,|f'|^4\,e^{-f/2}\,\mathrm{d}z + \frac{\lambda}{2}\int_{-1}^1 \nu\,|f'|^2\,e^{-f/2}\,\mathrm{d}z - \frac{1}{2}\int_{-1}^1 \nu\,|f'|^2\,e^{f/2}\,\mathrm{d}z\,.$$

En soustrayant la seconde identité à la première, il vient

$$\frac{1}{8}\int_{-1}^1 \nu^2\,\left|f'' - \tfrac{1}{2}\,|f'|^2\right|^2\,e^{-f/2}\,\mathrm{d}z + \frac{1-\lambda}{4}\int_{-1}^1 \nu\,|f'|^2\,e^{-f/2}\,\mathrm{d}z = 0\,.$$

Cette méthode a été introduite par Gidas et Spruck [GS81] pour l'étude des solutions d'équations non linéaire elliptiques du type

$$-\Delta u = u^q\,, \quad x \in \mathfrak{M}\,, \tag{23}$$

où $\mathfrak{M}$ est une variété Riemannienne de dimension $d > 2$. Ils montrent en particulier que si $1 \leqslant q < \frac{d+2}{d-2}$, alors toute solution *positive et régulière* de (23) est identiquement nulle. Dans le cas critique $q = \frac{d+2}{d-2}$, les profils d'Aubin-Talenti $u_\star$ sont aussi admissibles (voir [Gid82 ; JL72]). Cette analyse a été généralisée par Bidaut-Véron et Véron dans [BVV91] à l'équation

$$-\Delta u - \lambda u + u^q = 0\,,$$

qui est, à une normalisation près de u, l'équation d'Euler-Lagrange correspondant au cas d'optimalité pour l'inégalité d'interpolation

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathfrak{M})} \geqslant \frac{\lambda}{p-2}\left[\|v\|^2_{\mathrm{L}^p(\mathfrak{M})} - \|v\|^2_{\mathrm{L}^2(\mathfrak{M})}\right]\,, \quad v \in \mathrm{H}^1(\mathfrak{M})\,,$$

grâce à une utilisation systématique de la formule de Bochner-Lichnerowicz-Weitzenböck, qui va être détaillée dans la suite.

Extension aux variétés riemanniennes

Le résultat de rigidité obtenu pour la sphère s'étend au cadre des variétés riemanniennes de dimension 2. Voici maintenant une présentation des résultats du chapitre IV. Dans [Aub79], Aubin montre que sur une variété riemannienne compacte $(\mathfrak{M}, g)$, pour tout $\varepsilon > 0$, il existe $C_\varepsilon > 0$ tel que

$$\int_{\mathfrak{M}} e^u \,\mathrm{d}v_g \leqslant C_\varepsilon e^{(\frac{1}{16\pi}+\varepsilon)\|\nabla u\|^2_{\mathrm{L}^2(\mathfrak{M})}}\,, \qquad \forall\, u \in \mathrm{H}^1(\mathfrak{M})\,,\ \int_{\mathfrak{M}} u \ \text{ tel que } \mathrm{d}v_g = 0\,.$$

Le problème de la *première meilleure constante*, dont la valeur ici est $\frac{1}{16\pi}$, est de savoir si on peut prendre $\varepsilon = 0$ dans l'inégalité ci-dessus, tout en gardant C_ε finie. Une réponse affirmative est apportée par Cherrier dans [Che79]. En dimension supérieure, le problème est ensuite résolu par Aubin [Aub79] sur la sphère et par Faget [Fag08 ; Fag06] sur une variété riemannienne générale. En passant au logarithme, et en considérant $u - \int_{\mathfrak{M}} u \,\mathrm{d}v_g$, on en déduit qu'il existe une constante C telle que

$$\frac{1}{16\pi}\|\nabla u\|^2_{\mathrm{L}^2(\mathfrak{M})} + \int_{\mathfrak{M}} u \,\mathrm{d}v_g - \log \int_{\mathfrak{M}} e^u \,\mathrm{d}v_g \geqslant C\,, \qquad \forall\, u \in \mathrm{H}^1(\mathfrak{M})\,.$$

Pour obtenir une inégalité de type Onofri, nous nous sommes posé le problème de la valeur de la *seconde meilleure constante*, c'est-à-dire comment choisir λ pour que

$$\frac{1}{4}\|\nabla u\|^2_{\mathrm{L}^2(\mathfrak{M})} + \lambda \int_{\mathfrak{M}} u \,\mathrm{d}v_g - \lambda \log \int_{\mathfrak{M}} e^u \,\mathrm{d}v_g \geqslant 0\,. \tag{24}$$

Notre résultat est basé sur la méthode de rigidité. On note $\mathfrak{R}$ le tenseur de Ricci sur $\mathfrak{M}$ et Δ_g l'opérateur de Laplace-Beltrami. On définit aussi deux opérateurs sans trace :

$$\mathrm{L}_g u \coloneqq \mathrm{H}_g u - \frac{g}{d}\,\Delta_g u\,, \qquad \text{et} \qquad \mathrm{M}_g u \coloneqq \nabla u \otimes \nabla u - \frac{g}{d}\,|\nabla u|^2\,,$$

et avec ces notations, on a

Théorème 10 (Dolbeault, *Jankowiak*, Esteban, 2014 — [5, Th. 1])**.** *Supposons que $d = 2$ et que $\lambda_\star > 0$. Si u est une solution régulière de*

$$-\frac{1}{2}\,\Delta_g u + \lambda = e^u\,, \tag{25}$$

alors u est constante si $\lambda \in (0, \lambda_\star)$, avec

$$\lambda_\star := \inf_{u \in \mathrm{H}^2(\mathfrak{M}) \setminus \{0\}} \frac{\displaystyle\int_{\mathfrak{M}} \Big[\| \mathrm{L}_g u - \tfrac{1}{2} \mathrm{M}_g u \|^2 + \mathfrak{R}(\nabla u, \nabla u) \Big] e^{-u/2} \, \mathrm{d}v_g}{\displaystyle\int_{\mathfrak{M}} |\nabla u|^2 \, e^{-u/2} \, \mathrm{d}v_g} \,. \tag{26}$$

La preuve, un peu longue mais pas particulièrement technique, repose sur des intégrations par parties et la formule de Bochner-Lichnerowicz-Weitzenböck :

$$\frac{1}{2} \Delta |\nabla u|^2 = \|\mathrm{H}_g u\|^2 + \nabla(\Delta_g u) \cdot \nabla u + \mathfrak{R}(\nabla u, \nabla u) \,.$$

Une conséquence du Théorème 10 est l'estimation suivante.

Corollaire 11 (Dolbeault, *Jankowiak*, Esteban, 2014 — [5, Cor. 24])**.** *Si $d = 2$ et si λ_1 désigne la plus petite valeur propre de Δ_g, alors (24) est vraie pour $\lambda = \Lambda := \min\{4\,\pi, \lambda_\star\}$. De plus, si Λ est strictement plus petit que $\lambda_1/2$, alors la constante optimale dans (24) strictement plus grande que Λ.*

La question de l'estimation de λ dans (24) a déjà été abordée par Fontenas dans [Fon97] dans le formalisme plus abstrait du carré du champ, avec une contrainte de courbure positive. Son approche se base sur l'inégalité de *courbure-dimension*, introduite par Bakry et Émery pour l'étude de processus de diffusion linéaires. Un bon exposé de leur méthode est fait dans [Bak94, Chapitre 6]. Bakry obtient ainsi une inégalité de type Sobolev sur la variété, puis dérive (24) par passage à la limite. De manière surprenante cette méthode n'entraine pas de perte d'information si la contrainte sur la courbure est prise en compte (voir la remarque 26, p. 97).

La méthode que nous présentons peut-être étendue au cas de l'espace *plat* à poids, poids qui prennent le rôle joué par la courbure de Ricci sur les variétés. Nous cherchons donc en quelque sorte à généraliser l'inégalité d'Onofri (13) écrite sur $\mathbb{R}^2$. Plus précisément, nous étudions les mesures $d\mu = \mu(x) \, \mathrm{d}x$ pour lesquelles on peut trouver $\lambda > 0$ telle que l'inégalité

$$\frac{1}{16\,\pi} \int_{\mathbb{R}^2} |\nabla u|^2 \, \mathrm{d}x \geqslant \lambda \left[\log \left(\int_{\mathbb{R}^2} e^u \, \mathrm{d}\mu \right) - \int_{\mathbb{R}^2} u \, \mathrm{d}\mu \right] , \tag{27}$$

soit vérifiée. Rappelons que le choix de $\mu = \frac{1}{\pi}(1 + |x|^2)^{-2}$ donne (13). Comme une telle inégalité est invariante par ajout d'une constante, on considère des minimiseurs sous la contrainte $\int_{\mathbb{R}^2} e^u \, \mathrm{d}\mu = 0$. Ceux-ci, quand ils existent, sont solutions de l'équation d'Euler-Lagrange

$$-\frac{1}{8\,\pi} \Delta u + \lambda\,\mu - \lambda\, e^u\, \mu = 0 \,. \tag{28}$$

En suivant la même méthode que précédemment, on arrive à un critère similaire à celui du Théorème 10. Du point de vue des applications, il n'est pas vraiment gênant de restreindre aux mesures à densité, à symétrie radiale et décroissantes, et dans ce cas on peut donner le résultat suivant.

Théorème 12 (Dolbeault, *Jankowiak*, Esteban, 2014 — [5, Th. 2])**.** *Supposons que μ est une fonction à symétrie radiale et notons*

$$\Lambda_\star := \inf_{x \in \mathbb{R}^2} \frac{(-\Delta) \log \mu}{8\,\pi\,\mu} \,.$$

Alors toute solution radiale de (28) *est constante si* $\lambda \in (0, \Lambda_\star)$. *De plus, si l'égalité dans* (27) *avec* $\lambda = \Lambda_\star$ *est atteinte parmi les fonctions radiales, alors l'inégalité* (27) *est vraie pour* $\lambda = \Lambda_\star$.

Les inégalités du type (27) peuvent être utiles pour étudier la stabilité d'états stationnaires, comme c'est fait pour le modèle de Keller-Segel parabolique–elliptique dans [CD12]. Cependant, le deuxième critère soulève la question délicate de la *brisure de symétrie* dans (27), à laquelle il n'est pas toujours facile d'apporter une réponse. À titre d'exemple, on peut prendre le modèle de Keller-Segel parabolique–parabolique

$$\begin{cases} \dfrac{\partial}{\partial t}\rho = \Delta\rho - \nabla(\rho\nabla c)\,, \\ \tau\dfrac{\partial}{\partial t}c = \Delta c + \rho\,. \end{cases}$$

D'après les résultats de [BCD11], on sait que ce système admet des solutions auto-similaires à symétrie radiale, qui peuvent s'écrire

$$\begin{cases} u = M\dfrac{e^{v-\frac14|x|^2}}{\int_{\mathbb{R}^2} e^{v-\frac14|x|^2}\,\mathrm{d}x}\,, \\ -\Delta v = \dfrac{\tau}{2}x\cdot\nabla v + u \end{cases} \qquad \text{où} \quad M = \int_{\mathbb{R}^2}\rho\,\mathrm{d}x = \int_{\mathbb{R}^2} u\,\mathrm{d}x\,.$$

Un autre fait intéressant est que si τ est assez grand, il existe aussi plusieurs solutions de même masse, pour les masses plus grandes que 8π. Si on veut s'intéresser à l'inégalité (27) écrite avec $\mu = \frac{u}{M}$, on considère $\Lambda_\star$ qui s'écrit

$$\Lambda_\star = \inf_{x\in\mathbb{R}^2}\frac{M}{8\pi}\frac{(-\Delta)\log u}{u} = \frac{M}{8\pi} + \inf_{x\in\mathbb{R}^2} M\frac{\frac{\tau}{2}x\cdot\nabla v + 1}{u}\,.$$

Ceci nous donne un critère qui n'est pas nécessairement optimal, mais qui est facile à évaluer numériquement. La méthode suppose de savoir que les minimiseurs de (27) sont des fonctions radiales, ce qui est connu pour les modèles simples de la chimiotaxie, mais peut être plus problématique, par exemple pour des poids qui ne seraient pas monotones.

Étude de modèles de mouvement de foule

Le dernier chapitre de cette thèse est consacré à l'étude de deux systèmes paraboliques, proches de celui de Keller-Segel. Ils diffèrent des études précédentes puisqu'ils sont formulés sur un domaine borné avec des conditions de Neumann homogènes au bord. La masse ne peut plus partir à l'infini, et il n'y a plus besoin de faire de changement de variable auto-similaire. Les phénomènes de concentration sont aussi absents de ces modèles, le terme attractif étant borné. La difficulté ici vient de la non convexité des fonctionnelles sous-jacentes au flot, en particulier de la fonction de Lypapunov dans le cas du deuxième modèle. Le comportement asymptotique a donc une structure plus riche, ce qui se traduit par l'existence d'états stationnaires multiples dont il s'agit la stabilité.

Sur un ouvert borné $\Omega \subset \mathbb{R}^d$ on considère

$$\begin{cases} \dfrac{\partial}{\partial t}\rho = \Delta\rho - \nabla\cdot(\rho(1-\rho)\nabla D) \\ \dfrac{\partial}{\partial t}D = \kappa\,\Delta D - \delta D + g(\rho) \end{cases}, \qquad g(\rho) = \begin{cases} \rho(1-\rho) & \text{modèle 1,} \\ \rho & \text{modèle 2,} \end{cases} \tag{29}$$

avec conditions de flux nul sur $\partial\Omega$, de telle sorte que la masse $M \coloneqq \int_\Omega \rho \, \mathrm{d}x$ soit conservée.

Le premier modèle est la limite macroscopique [BMP11] d'un automate cellulaire modélisant la circulation piétonne [KS02], où ρ représente la densité de piétons et D la stratégie de déplacement ; elle encode la mémoire du système. L'évolution de ρ se fait à la fois par diffusion et par transport dans la direction de D croissant. La densité maximale est 1, ce qui est traduit par l'expression du terme de sensibilité $\rho(1-\rho)$ devant ∇D. Dans l'équation pour D, la diffusion traduit l'incertitude sur la perception de l'environnement et le terme d'amortissement $-\delta D$ la perte d'information au cours du temps. Le terme source non linéaire $\rho(1-\rho)$ rend compte de la façon dont est construite la stratégie de déplacement : on préfère les zones de densité intermédiaires. C'est ce terme qui est source de difficulté.

Le second modèle, proposé par Painter et Hillen [HP01], est une variante du modèle de Keller-Segel dite *avec prévention d'entassement* (« overcrowding »), cette appellation venant du terme de sensibilité $\rho(1-\rho)$ qui limite la densité à 1. Son interprétation est identique à celle du modèle de Keller-Segel parabolique–elliptique donné p. xxii, ρ devenant la densité d'amibes et D la densité de chimioattractant. Contrairement au modèle 1, on peut contruire une fonction de Lypapunov, clé pour l'étude du comportement asymptotique, en se basant sur la méthode de Calvez et Corrias [CC08] :

$$\mathcal{L}[\rho, D] \coloneqq \int_\Omega \rho \log \rho + (1-\rho)\log(1-\rho) - \rho D \, \mathrm{d}x + \frac{\kappa}{2}\int_\Omega |\nabla D|^2 \, \mathrm{d}x + \frac{\delta}{2}\int_\Omega D^2 \, \mathrm{d}x \, .$$

Pour toute valeur des paramètres κ, δ et $0 \leqslant M \leqslant |\Omega|$, on peut trouver ρ et D constants sur Ω qui constituent un état stationnaire pour (29). Pour obtenir des solutions non triviales il faut considérer de petites valeurs de κ et δ, pour lesquelles apparaissent des profils en plateau, aussi appelés *mesas* (voir Figure 4), auxquels nous nous sommes intéressés. Pour

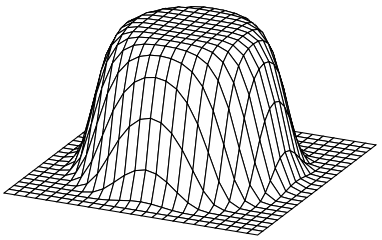


FIGURE 4 : Une *mesa* dans $\mathbb{R}^2$, illustration du type de profil étudiés ici.

caractériser les états stationnaires, on peut exprimer la densité ρ comme fonction de D :

$$\rho = \frac{1}{1 + e^{\phi_0 - D}} \, ,$$

où la constante d'intégration ϕ_0 est entièrement déterminée par le paramètre de masse M. C'est aussi la relation qu'on obtient si on fixe D et qu'on minimise $\mathcal{L}$; ϕ_0 apparaît alors comme le multiplicateur de Lagrange associé à la contrainte de masse. On est donc ramené à une équation non linéaire pour $\phi = D - \phi_0$:

$$\begin{cases} -\kappa\Delta\phi + \delta(\phi + \phi_0) - F'(\phi) = 0 \, , \ x \in \Omega \\ \qquad\qquad\qquad\qquad \nabla\phi \cdot \nu = 0 \, , \ x \in \partial\Omega \end{cases} \quad \text{avec} \quad F'(\phi) = \begin{cases} \frac{e^{-\phi}}{(1+e^{-\phi})^2} & \text{modèle 1} \\ \frac{1}{1+e^{-\phi}} & \text{modèle 2.} \end{cases} \tag{30}$$

Il paraît ici assez naturel d'utiliser ϕ_0 pour paramétrer l'ensemble des solutions, alors que le paramètre intéressant pour l'étude de la stabilité est la masse M. La relation entre M et ϕ_0 n'est en fait pas simple, ce qui complexifie l'analyse et traduit la richesse de ces modèles.

Pour étudier les états stationnaires, on introduit la fonctionnelle

$$\mathcal{E}_{\phi_0}[\phi] \coloneqq \frac{\kappa}{2}\int_\Omega |\nabla\phi|^2 \,\mathrm{d}x + \frac{\delta}{2}\int_\Omega (\phi-\phi_0)^2 \,\mathrm{d}x - \int_\Omega F(\phi)\,\mathrm{d}x\,,$$

dont les points critiques sont solutions de (30). L'existence de points critiques ne présente pas de difficultés particulières, contrairement à l'obtention de leur propriétés qualitatives. Par exemple, le fait de travailler sur un domaine borné avec des conditions au bord de type Neumann rend inutilisables certaines méthodes classiques comme la symétrisation de Schwarz, et il faut faire appel à d'autres résultats, comme lui de Lopes [Lop96]. Dans le cas où Ω est une boule, on obtient le résultat suivant, illustré par le diagramme de bifurcation de la Figure 5, obtenu numériquement.

Théorème 13 (Dolbeault, *Jankowiak*, Markowich, 2013 — [1, Th. 21])**.** *Si Ω est une boule, on a les propriétés suivantes pour les solutions à symétrie radiale :*

(i) Les solutions stationnaires non constantes n'existent que pour M dans un intervalle $I \subsetneq (0,|\Omega|)$.

(ii) Les solutions constantes sont instables dans un sous-intervalle de I.

(iii) Il existe un sous-intervalle de I dans lequel seules les solutions stationnaires non constantes sont stables. Il est donné par une condition de petitesse sur $\kappa\lambda_1+\delta$, où λ_1 est la première valeur propre non nulle pour l'opérateur $-\Delta$ avec conditions de Neumann homogènes au bord.

*(iv) Pour toute valeur de la masse, les solutions stationnaires variationnellemement stables de basse énergie sont soit monotones (*mesas*) soit constantes. Dans le case du modèle 2, il existe un intervalle de masse pour lequel les solutions monotones attirent toutes les solutions du problème d'évolution de basse énergie.*

On s'intéresse ensuite à la caractérisation de la stabilité des états stationnaires, qui nécessite de considérer les quantités linéarisées associées à $\mathcal{E}_{\phi_0}$ et $\mathcal{L}$, pour un état stationnaire (ρ, D).

$$\lim_{\varepsilon\to0}\frac{\mathcal{E}_{\phi_0}[\phi+\varepsilon\varphi]-\mathcal{E}_{\phi_0}[\phi]}{2\varepsilon^2} = \int_\Omega \varphi(E_\phi\varphi)\,,\qquad \text{où}\qquad \mathrm{E}_\phi \coloneqq -\kappa\Delta+\delta+F''(\phi)\,,$$

$$\mathrm{L}_\phi[u,v] \coloneqq \lim_{\varepsilon\to0}\frac{\mathcal{L}[\rho+\varepsilon u, D+\varepsilon v]-\mathcal{L}[\rho,D]}{2\varepsilon^2} = \int_\Omega\left(\frac{u^2}{2\rho(1-\rho)}-uv\right)+\frac{\kappa}{2}\int_\Omega|\nabla v|^2+\frac{\delta}{2}\int_\Omega v^2.$$

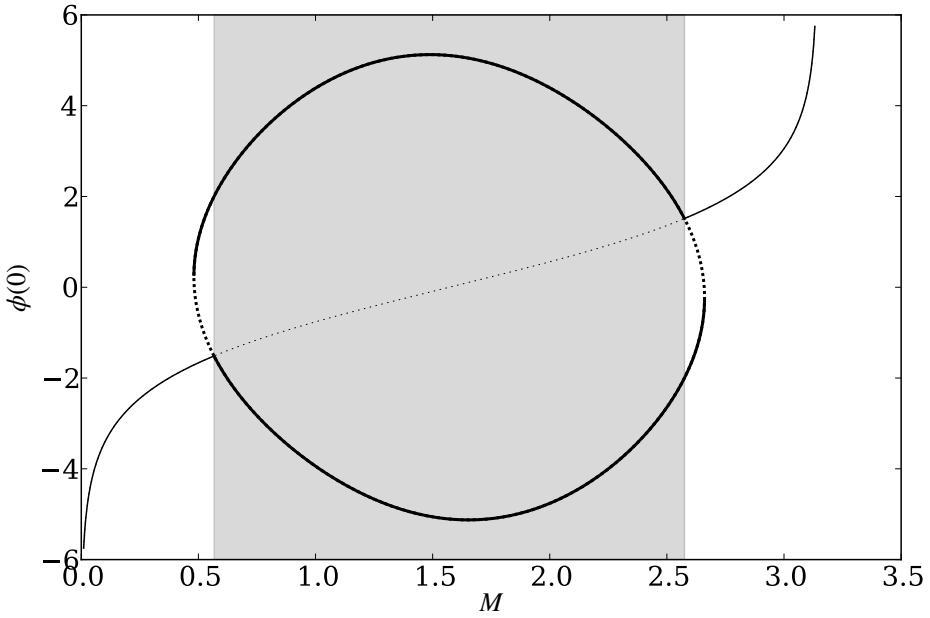
On peut alors considérer deux notions de stabilité :

— la stabilité au sens de la fonctionnelle $\mathcal{E}_{\phi_0}$. À ϕ_0 fixé, on dit qu'un état (ρ, D) est stable si c'est un minimum local pour $\mathcal{E}_{\phi_0}$, ce qui est donné par le signe de

$$\Lambda \coloneqq \inf_{\substack{v\neq0\\ \int_\Omega v\rho(1-\rho)=0}} \frac{\int_\Omega v(\mathrm{E}_\phi v)\,\mathrm{d}x}{\int_\Omega v^2\,\mathrm{d}x}\,,$$

on parle alors de *stabilité variationnelle*.

FIGURE 5 : Diagramme de bifurcation pour la valeur de ϕ en 0 en fonction de M, pour le modèle 2, en dimension 2. La branche de solutions stationnaires constantes est représentée en trait fin, celle des solutions stationnaires monotones en trait gras. Les portions de courbe en pointillés correspondent aux zones d'instabilité. La zone grisée correspond à l'intervalle d'instabilité pour les constantes.



— la stabilité au sens de l'opérateur d'évolution : on dit qu'un état (ρ, D) est stable si le spectre de l'opérateur linéarisé autour de (ρ, D) est négatif et on parle alors de *stabilité dynamique*. Pour le modèle 2, on peut relier la stabilité dynamique au fait que (ρ, D) soit un minimum local pour $\mathcal{L}$, situation qui est caractérisée par le signe de

$$\Lambda_1 \coloneqq 2 \inf_{\substack{\int_\Omega u=0 \\ \int_\Omega v^2=1}} \mathrm{L}_D[u,v]\,.$$

Notre résultat principal concerne le modèle 2 et utilise une idée présentée dans [CD12] pour le modèle de Keller-Segel parabolique-elliptique : en linéarisant la fonctionnelle de Lyapunov autour d'un minimum local, on obtient un produit scalaire pour lequel l'opérateur d'évolution linéarisé est auto-adjoint. Cette remarque importante permet de lier les deux notions de stabilité.

Théorème 14 (Dolbeault, *Jankowiak*, Markowich, 2013 — [1, Th. 46]). *Soit $M > 0$ et (ρ, D) une solution stationnaire de* (29) pour le modèle 2 *telle que $\int_\Omega \rho = M$, et soit $\phi = D - \phi_0$ telle que $\int_\Omega \frac{1}{1+e^{-\phi}} = M$.*

(i) Il ne peut y avoir ni instabilité dynamique ni instabilité variationnelle si (ρ, D) est un minimum local de $\mathcal{L}$ sous la contrainte de masse. De manière équivalente, il ne peut y avoir instabilité si ϕ est un minimum local de $\mathcal{E}$.

(ii) *Si (ρ, D) est un minimum local de $\mathcal{L}$ sous la contrainte de masse, la solution (u, v) au problème d'évolution linéarisé converge vers $(0, 0)$.*

(iii) *Stabilité dynamique et stabilité variationnelle sont équivalentes.*

La preuve de l'équivalence entre les deux notions de stabilité se base sur l'étude des deux équations d'Euler-Lagrange associées aux problèmes de minimisations définissant Λ et Λ_1. On arrive à montrer que si $\min(\Lambda, \Lambda_1) < \delta$, alors $\Lambda = \Lambda_1$, ce qui nous donne le résultat attendu.

Ces résultats théoriques sont complétés par une analyse numérique des deux problèmes, dans le cadre radial et sur la sphère unité, en dimension 1 et 2. Les solutions stationnaires sont obtenues numériquement pour tout ϕ_0 à partir de l'équation (30) par une méthode de tir paramétrée par $\phi(0)$: on ajuste la valeur de $\phi(0)$ pour obtenir $\phi'(1) = 0$. Il y a plusieurs difficultés. D'abord la petitesse des paramètres κ et δ, qui se traduit par une très forte variabilité de $\phi'(1)$ par rapport à $\phi(0)$. Il faut ensuite prendre en compte la contrainte $\int_\Omega \rho = M$ lors des calculs de stabilité, qui sont donc fait en deux temps : on passe d'abord par une diagonalisation de l'opérateur linéarisé et ensuite par une minimisation du quotient de Rayleigh sous la contrainte de masse.

En ce qui concerne le modèle 1, les deux notions de stabilité se sont *a priori* pas équivalentes. Un indice est donné pas l'étude numérique du spectre de l'opérateur linéarisé, présentée dans la Figure V.6, p. 136. On voit que l'intervalle de stabilité variationnelle est plus petit que l'intervalle de stabilité dynamique. Ceci suggère qu'il n'existe pas de fonctionnelle de Lyapunov pour le modèle 1.

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Chapitre I

Inégalités de Sobolev et Hardy-Littlewood-Sobolev

Ce chapitre reprend en grande partie l'article « Sobolev and Hardy-Littlewood-Sobolev inequalities », écrit en collaboration avec Jean Dolbeault. Il est à paraître dans *Journal of Differential Equations*.

Résumé

Ce chapitre est consacré à une amélioration de l'inégalité de Sobolev. Le terme de reste qui apparaît fait intervenir l'inégalité duale, c'est-à-dire l'inégalité de Hardy-Littlewood-Sobolev. Dans le cas de la dimension deux, un résultat similaire est obtenu pour l'inégalité d'Onofri, via passage à la limite. L'analyse s'attache particulièrement aux constantes de proportionnalité intervenant dans ces améliorations. Nous fournissons des bornes basées des méthodes de complétion du carré et de linéarisation autour des solutions extrémales d'Aubin-Talenti. L'utilisation d'un flot de type diffusion non linéaire permet d'affiner encore ces estimations.

Structure

1 Introduction 4
2 A completion of the square and consequences 7
3 Linearization 12
4 Improved inequalities and nonlinear flows 15
5 Caffarelli-Kohn-Nirenberg inequalities and duality 19
A Some useful formulae 23
B Poincaré inequality and stereographic projection 24
C Flow on the sphere and consequences 25
D Symmetry in Caffarelli-Kohn-Nirenberg inequalities 26
References 27

1 Introduction

E. Carlen, J.A. Carrillo and M. Loss noticed in [CCL10] that Hardy-Littlewood-Sobolev inequalities in dimension $d \geqslant 3$ can be deduced from some special Gagliardo-Nirenberg inequalities using a fast diffusion equation. Sobolev's inequalities and Hardy-Littlewood-Sobolev inequalities are dual. A fundamental reference for this issue is E.H. Lieb's paper [Lie83]. This duality has also been investigated using a fast diffusion flow in [Dol11]. Although [CCL10] has motivated [Dol11], the two approaches are so far unrelated. Actually [Dol11] is closely connected with the approach by Legendre's duality developed in [Lie83]. We shall take advantage of this fact in the present paper and also use of the flow introduced in [Dol11].

For any $d \geqslant 3$, the space $\mathcal{D}^{1,2}(\mathbb{R}^d)$ is defined as the completion of smooth solutions with compact support w.r.t. the norm

$$w \mapsto \|w\| := \left(\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right)^{1/2} ,$$

where $2^* := \frac{2\,d}{d-2}$. The Sobolev inequality in $\mathbb{R}^d$ is

$$\mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geqslant 0 \quad \forall\, u \in \mathcal{D}^{1,2}(\mathbb{R}^d) \,, \tag{1}$$

where the best constant, or Aubin-Talenti constant, is given by

$$\mathsf{S}_d = \frac{1}{\pi\, d\, (d-2)} \left(\frac{\Gamma(d)}{\Gamma\left(\frac{d}{2}\right)} \right)^{\frac{2}{d}}$$

(see A for details). The optimal Hardy-Littlewood-Sobolev inequality

$$\mathsf{S}_d \, \|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} v \, (-\Delta)^{-1} \, v \, dx \geqslant 0 \quad \forall\, v \in \mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d) \tag{2}$$

involves the same best constant S_d, as a result of the duality method of [Lie83]. When $d \geqslant 5$, using a well chosen flow, it has been established in [Dol11] that the l.h.s. in (1) is actually bounded from below by the l.h.s. in (2), multiplied by some positive proportionality constant. In our first result, we will remove the technical restriction $d \geqslant 5$ and cover all dimensions $d \geqslant 3$. An elementary use of the duality method – in fact a simple completion of the square method – provides a simple upper bound on the optimal proportionality constant in any dimension.

Theorem 1. *For any $d \geqslant 3$, if $q = \frac{d+2}{d-2}$ the inequality*

$$\mathsf{S}_d \, \|u^q\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} u^q \, (-\Delta)^{-1} \, u^q \, dx \\ \leqslant \mathsf{C}_d \, \|u\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right] \tag{3}$$

holds for any $u \in \mathcal{D}^{1,2}(\mathbb{R}^d)$ where the optimal proportionality constant C_d is such that

$$\frac{d}{d+4} \, \mathsf{S}_d \leqslant \mathsf{C}_d < \mathsf{S}_d \,.$$

Inequality (3) is obtained with C_d replaced by S_d by expanding a well chosen square in Section 2. The lower bound on C_d follows from an expansion of both sides of the inequality around the Aubin-Talenti functions, which are optimal for Sobolev and Hardy-Littlewood-Sobolev inequalities (see Section 2 for more details), and spectral estimates that will be studied in Section 3 : see Corollary 6. The computation based on the flow as was done in [Dol11] can be optimized to get an improved inequality compared to (3), far from the Aubin-Talenti functions : see Theorem 9 in Section 4. As a consequence, we also prove the strict inequality $\mathsf{C}_d < \mathsf{S}_d$.

In dimension $d = 2$, consider the probability measure $d\mu$ defined by

$$d\mu(x) := \mu(x)\,dx \quad \text{with} \quad \mu(x) := \frac{1}{\pi\,(1+|x|^2)^2} \quad \forall\, x \in \mathbb{R}^2.$$

The Euclidean version of Onofri's inequality [Ono82]

$$\frac{1}{16\,\pi}\int_{\mathbb{R}^2} |\nabla f|^2\,dx - \log\left(\int_{\mathbb{R}^2} e^{\,f}\,d\mu\right) + \int_{\mathbb{R}^2} f\,d\mu \geqslant 0 \quad \forall\, f \in \mathcal{D}(\mathbb{R}^2) \tag{4}$$

plays the role of Sobolev's inequality in higher dimensions. Here the inequality is written for smooth and compactly supported functions in $\mathcal{D}(\mathbb{R}^2)$, but can be extended to the appropriate Orlicz space which corresponds to functions such that both sides of the inequality are finite.

This inequality is dual of the logarithmic Hardy-Littlewood-Sobolev inequality that can be written as follows : for any $g \in \mathrm{L}^1_+(\mathbb{R}^2)$ with $M = \int_{\mathbb{R}^2} g\;dx$, such that $g\log g$, $(1+\log|x|^2)\,g \in \mathrm{L}^1(\mathbb{R}^2)$, we have

$$\int_{\mathbb{R}^2} g\,\log\left(\frac{g}{M}\right)dx - \frac{4\,\pi}{M}\int_{\mathbb{R}^2} g\,(-\Delta)^{-1}\,g\,dx + M\,(1+\log\pi) \geqslant 0 \tag{5}$$

with

$$\int_{\mathbb{R}^2} g\,(-\Delta)^{-1}\,g\,dx = -\frac{1}{2\,\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} g(x)\,g(y)\,\log|x-y|\,dx\,dy\,.$$

Then, in dimension $d = 2$, we have an analogue of Theorem 1, which goes as follows.

Theorem 2. *The inequality*

$$\int_{\mathbb{R}^2} g\,\log\left(\frac{g}{M}\right)dx - \frac{4\,\pi}{M}\int_{\mathbb{R}^2} g\,(-\Delta)^{-1}\,g\,dx + M\,(1+\log\pi) \leqslant M\left[\frac{1}{16\,\pi}\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^2)} + \int_{\mathbb{R}^2} f\,d\mu - \log M\right] \tag{6}$$

holds for any function $f \in \mathcal{D}(\mathbb{R}^2)$ such that $M = \int_{\mathbb{R}^2} e^{\,f}\,d\mu$ and $g = e^{\,f}\,\mu$.

Using for instance [AL89] or [CL92, Lemma 2] (also see [LL01, chapter 3–4]), it is known that optimality is achieved in (1), (2), (4) or (5) when the problem is reduced to radially symmetric functions. However, no such result applies when considering a difference of the terms in two such inequalities, like in (3) or (6). Optimality therefore requires a special treatment. In Section 2, we shall use the *completion of the square method* to establish the inequalities (without optimality) under an assumption of radial symmetry in case of Theorem 2. For radial functions, Theorem 1 can indeed be written with $d > 2$ considered as a real parameter

and Theorem 2 corresponds, in this setting, to the limit case as $d \to 2_+$. To handle the general case (without radial symmetry assumption), a more general setting is required. In Section 5, we extend the results established for Sobolev inequalities to weighted spaces and obtain an improved version of the Caffarelli-Kohn-Nirenberg inequalities (see Theorem 15). Playing with weights is equivalent to varying d or taking limits with respect to d, except that no symmetry assumption is required. This allows to complete the proof of Theorem 2.

Technical results regarding the computation of the constants, a weighted Poincaré inequality and the stereographic projection, the extension of the flow method of [Dol11] to the case of the dimensions $d = 3$ and $d = 4$, and symmetry results for Caffarelli-Kohn-Nirenberg inequalities have been collected in various appendices.

At this point, we emphasize that Theorems 15 and 16, which are used as intermediate steps in the proof of Theorem 2 are slightly more general than, respectively, Theorems 1 and 2, except for the issue of the optimal value of the proportionality constant, which has not been studied. It is likely that the method used for Sobolev's inequality can be adapted, but since weights break the translation invariance, some care should be given to this question, which is of independent interest and known to raise a number of difficulties of its own (see for instance [DE12]). The question of a lower estimate of the proportionality constant in (6) in connection with a larger family of Onofri type inequalities is currently being studied, see [Jan14].

Let us conclude this introduction by a brief review of the literature. To establish the inequalities, our approach is based on a completion of the square method which accounts for duality issues. Linearization (spectral estimates) and estimates based on a nonlinear flow are used for optimality issues. Although some of these methods have been widely used in the literature, for instance in the context of Hardy inequalities (see [BDE08] and references therein), it seems that they have not been fully exploited yet in the case of the functional inequalities considered in this paper. The main tool in [Dol11] is a flow of fast diffusion type, which has been considered earlier in [PS01]. In dimension $d = 2$, we may refer to various papers (see for instance [DP99 ; DP95 ; DS10]) in connection with Ricci's flow for properties of the solutions of the corresponding evolution equation.

Many papers have been devoted to the asymptotic behaviour near extinction of the solutions of nonlinear flows, in bounded domains (see for instance [BH80 ; GK02 ; SV94 ; BGV12]) or in the whole space (see [Kin93 ; PZ95 ; GP97] and references therein). In particular, the Cauchy-Schwarz inequality has been repeatedly used, for instance in [BH80 ; SV94], and turns out to be a key tool in the main result of [Dol11], as well as the solution *with separation of variables*, which is related to the Aubin-Talenti optimal function for (1).

Getting improved versions of Sobolev's inequality is a question which has attracted lots of attention. See [BL85] in the bounded domain case and [BN83] for an earlier related paper. However, in [BL85], H. Brezis and E. Lieb also raised the question of measuring the distance to the manifold of optimal functions in the case of the Euclidean space. A few years later, G. Bianchi and H. Egnell gave an answer in [BE91] using the concentration-compactness method, with no explicit value of the constant. Since then, considerable efforts have been devoted to obtain quantitative improvements of Sobolev's inequality. On the whole Euclidean space, nice estimates based on rearrangements have been obtained in [Cia+09] and we refer to [Cia09] for an interesting review of various related results. The method there is in

some sense constructive, but it hard to figure what is the practical value of the constant. As in [Dol11] our approach involves much weaker notions of distances to optimal functions, but on the other hand offers clear-cut estimates. Moreover, it provides an interesting way of obtaining global estimates based on a linearization around Aubin-Talenti optimal functions.

2 A completion of the square and consequences

Before proving the main results of this paper, let us explain in which sense Sobolev's inequality and the Hardy-Littlewood-Sobolev inequality, or Onofri's inequality and the logarithmic Hardy-Littlewood-Sobolev inequality, for instance, are *dual inequalities*.

To a convex functional F, we may associate the functional F^* defined by Legendre's duality as

$$F^*[v] := \sup\left(\int_{\mathbb{R}^d} u\,v\,dx - F[u]\right).$$

For instance, to $F_1[u] = \frac{1}{2}\,\|u\|^2_{\mathrm{L}^p(\mathbb{R}^d)}$ defined on $\mathrm{L}^p(\mathbb{R}^d)$, we henceforth associate $F_1^*[v] = \frac{1}{2}\,\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}$ on $\mathrm{L}^q(\mathbb{R}^d)$ where p and q are Hölder conjugate exponents : $1/p + 1/q = 1$. The supremum can be taken for instance on all functions in $\mathrm{L}^p(\mathbb{R}^d)$, or, by density, on the smaller space of the functions $u \in \mathrm{L}^p(\mathbb{R}^d)$ such that $\nabla u \in \mathrm{L}^2(\mathbb{R}^d)$. Similarly, to $F_2[u] = \frac{1}{2}\,\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$, we associate $F_2^*[v] = \frac{1}{2}\,\mathsf{S}_d^{-1}\int_{\mathbb{R}^d} v\,(-\Delta)^{-1}\,v\,dx$ where $(-\Delta)^{-1}\,v = G_d * v$ with $G_d(x) = \frac{1}{d-2}\,|\mathbb{S}^{d-1}|^{-1}\,|x|^{2-d}$, when $d \geqslant 3$, and $G_2(x) = -\frac{1}{2\pi}\,\log|x|$. As a straightforward consequence of Legendre's duality, if we have a functional inequality of the form $F_1[u] \leqslant F_2[u]$, then we have the *dual inequality* $F_1^*[v] \geqslant F_2^*[v]$. In this sense, (1) and (2) are dual of each other, as it has been noticed in [Lie83]. Also notice that Inequality (2) is a consequence of Inequality (1).

In this paper, we go one step further and establish that

$$F_1^*[u] - F_2^*[u] \leqslant \mathsf{C}\,(F_2[u] - F_1[u]) \tag{7}$$

for some positive constant C, at least under some normalization condition (or up to a multiplicative term which is required for simple homogeneity reasons). Such an inequality has been established in [Dol11, Theorem 1.2] when $d \geqslant 5$. Here we extend it to any $d \geqslant 3$ and get and improved value for the constant C.

It turns out that the proof can be reduced to the completion of a square. Let us explain how the method applies in case of Theorem 1, and how Theorem 2 can be seen as a limit of Theorem 1 in case of radial functions.

Proof of Theorem 1, part 1 : the completion of a square.
Integrations by parts show that

$$\int_{\mathbb{R}^d} |\nabla(-\Delta)^{-1}\,v|^2\,dx = \int_{\mathbb{R}^d} v\,(-\Delta)^{-1}\,v\,dx$$

and, if $v = u^q$ with $q = \frac{d+2}{d-2}$,

$$\int_{\mathbb{R}^d} \nabla u\cdot\nabla(-\Delta)^{-1}\,v\,dx = \int_{\mathbb{R}^d} u\,v\,dx = \int_{\mathbb{R}^d} u^{2^*}\,dx\,.$$

Hence the expansion of the square

$$0 \leqslant \int_{\mathbb{R}^d} \left| \mathsf{S}_d \, \|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^{\frac{4}{d-2}} \nabla u - \nabla (-\Delta)^{-1} \, v \right|^2 dx$$

shows that

$$0 \leqslant \mathsf{S}_d \, \|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^{\frac{8}{d-2}} \left[\mathsf{S}_d \, \|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^d)}^2 - \|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^2 \right] - \left[\mathsf{S}_d \, \|u^q\|_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}^2 - \int_{\mathbb{R}^d} u^q \, (-\Delta)^{-1} \, u^q \, dx \right].$$

Equality is achieved if and only if

$$\mathsf{S}_d \, \|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^{\frac{4}{d-2}} \, u = (-\Delta)^{-1} \, v = (-\Delta)^{-1} \, u^q \, ,$$

that is, if and only if u solves

$$-\Delta u = \frac{1}{\mathsf{S}_d} \, \|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^{-\frac{4}{d-2}} \, u^q \, ,$$

which means that u is an Aubin-Talenti function, optimal for (1). This completes the proof of Theorem 1, up to the optimality of the proportionality constant, for which we know that

$$\mathsf{C}_d = \mathcal{C} \, \mathsf{S}_d \quad \text{with} \quad \mathcal{C} \leqslant 1 \, . \tag{8}$$

Incidentally, this also proves that v is optimal for (2). □

As a first step towards the proof of Theorem 2, let us start with a result for radial functions. If d is a positive integer, we can define

$$\mathsf{s}_d \coloneqq \mathsf{S}_d \, |\mathbb{S}^{d-1}|^{\frac{2}{d}}$$

and get

$$\mathsf{s}_d = \frac{4}{d \, (d-2)} \left(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi} \, \Gamma\left(\frac{d}{2}\right)} \right)^{\frac{2}{d}} . \tag{9}$$

Using this last expression allows us to consider d as a real parameter.

Lemma 3. *Assume that $d \in \mathbb{R}$ and $d > 2$. Then*

$$0 \leqslant \mathsf{s}_d \left(\int_0^\infty u^{\frac{2\,d}{d-2}} \, r^{d-1} \, dr \right)^{1+\frac{2}{d}} - \int_0^\infty u^{\frac{d+2}{d-2}} \left((-\Delta)^{-1} u^{\frac{d+2}{d-2}} \right) r^{d-1} \, dr \leqslant \mathsf{c}_d \left(\int_0^\infty u^{\frac{2\,d}{d-2}} \, r^{d-1} \, dr \right)^{\frac{4}{d}} \left[\mathsf{s}_d \int_0^\infty |u'|^2 \, r^{d-1} \, dr - \left(\int_0^\infty u^{\frac{2\,d}{d-2}} \, r^{d-1} \, dr \right)^{\frac{d-2}{d}} \right]$$

holds for any radial function $u \in \mathcal{D}^{1,2}(\mathbb{R}^d)$ with optimal constant $\mathsf{c}_d \leqslant \mathsf{s}_d$.

Here we use the notation $(-\Delta)^{-1} \, v = w$ to express the fact that w is the solution to $w'' + \frac{d-1}{r} \, w' + v = 0$, that is,

$$(-\Delta)^{-1} \, v \, (r) = \int_r^\infty s^{1-d} \int_0^s v(t) \, t^{d-1} \, dt \, ds \quad \forall \, r > 0 \, . \tag{10}$$

Démonstration. In the case of a radially symmetric function u, and with the standard abuse of notations that amounts to identify $u(x)$ with $u(r)$, $r = |x|$, Inequality (1) can be written as

$$\mathsf{s}_d \int_0^\infty |u'|^2 \, r^{d-1} \, dr \geqslant \left(\int_0^\infty |u|^{\frac{2\,d}{d-2}} \, r^{d-1} \, dr \right)^{1-\frac{2}{d}} . \tag{11}$$

However, if u is considered as a function of one real variable r, then the inequality also holds for any *real parameter* $d \in (2, \infty)$ and is equivalent to the one-dimensional Gagliardo-Nirenberg inequality

$$\mathsf{s}_d \left(\int_{\mathbb{R}} |w'|^2 \, dt + \tfrac{1}{4} \, (d-2)^2 \int_{\mathbb{R}} |w|^2 \, dt \right) \geqslant \left(\int_{\mathbb{R}} |w|^{\frac{2\,d}{d-2}} \, dt \right)^{1-\frac{2}{d}}$$

as can be shown using the Emden-Fowler transformation

$$u(r) = (2\,r)^{-\frac{d-2}{2}} \, w(t) \,, \quad t = -\log r \,. \tag{12}$$

The corresponding optimal function is, up to a multiplication by a constant, given by

$$w_\star(t) = (\cosh t)^{-\frac{d-2}{2}} \qquad \forall \, t \in \mathbb{R} \,,$$

which solves the Euler-Lagrange equation

$$-(p-2)^2 \, w'' + 4\, w - 2\,p\,|w|^{p-2} \, w = 0 \,.$$

for any real number $d > 2$ and the optimal function for (11) is

$$u_\star(r) = (2\,r)^{-\frac{d-2}{2}} \, w_\star(-\log r) = \left(1 + r^2\right)^{-\frac{d-2}{2}}$$

up to translations, multiplication by a constant and scalings. This establishes (9). See A for details on the computation of s_d. The reader is in particular invited to check that the expression of s_d is consistent with the one of S_d given in the introduction.

Next we apply Legendre's transform to (11) and get a Hardy-Littlewood-Sobolev inequality that reads

$$\int_0^\infty v \, (-\Delta)^{-1} \, v \, r^{d-1} \, dr \leqslant \mathsf{s}_d \left(\int_0^\infty v^{\frac{2\,d}{d+2}} \, r^{d-1} \, dr \right)^{1+\frac{d}{2}} \tag{13}$$

for any $d > 2$. Inequality (13) holds on the functional space which is obtained by completion of the space of smooth compactly supported radial functions with respect to the norm defined by the r.h.s. in (13). Inequality (13) is the first inequality of Lemma 3.

Finally, we apply the completion of the square method. By expanding

$$0 \leqslant \int_0^\infty \left| \, a \, u' - \left((-\Delta)^{-1} v \right)' \, \right|^2 \, r^{d-1} \, dr$$

with $a = \mathsf{s}_d \left(\int_0^\infty u^{\frac{2\,d}{d-2}} \, r^{d-1} \, dr \right)^{\frac{2}{d}}$ and $v = u^{\frac{d-2}{d+2}}$, we establish the second inequality of Lemma 3 (with optimal constant $\mathsf{c}_d \leqslant \mathsf{s}_d$). □

Now let us turn our attention to the case $d = 2$ and to Theorem 2. Using the fact that d in Lemma 3 is a real parameter, we can simply consider the limit of the inequalities as $d \to 2_+$.

Corollary 4. *For any function $f \in \mathrm{L}^1(\mathbb{R}^+;\, r\,dr)$ such that $f' \in \mathrm{L}^2(\mathbb{R}^+;\, r\,dr)$ and $M = \int_0^\infty e^f\,(1+r^2)^{-2}\,2\,r\,dr$, we have the inequality*

$$0 \leqslant \int_0^\infty e^f\,\log\left(\frac{e^f}{M\,(1+r^2)^2}\right)\,\frac{2\,r\,dr}{(1+r^2)^2} - \frac{2}{M}\int_0^\infty \frac{e^f}{(1+r^2)^2}\,(-\Delta)^{-1}\left(\frac{e^f}{(1+r^2)^2}\right)\,2\,r\,dr + M \leqslant M\left[\frac{1}{8}\int_0^\infty |f'|^2\,r\,dr + \int_0^\infty f\,\frac{2\,r\,dr}{(1+r^2)^2} - \log\left(\int_0^\infty e^f\,\frac{2\,r\,dr}{(1+r^2)^2}\right)\right]. \quad (14)$$

Here again $(-\Delta)^{-1}$ is defined by (10), but it coincides with the inverse of $-\Delta$ acting on radial functions.

Démonstration. We may pass to the limit in (11) written in terms of

$$u(r) = u_\star(r)\left(1 + \tfrac{d-2}{2\,d}\,f\right)$$

to get the radial version of Onofri's inequality for f. By expanding the expression of $|u'|^2$ we get

$$u'^2 = u_\star'^2 + \frac{d-2}{d}\,u_\star'\,(u_\star\,f)' + \left(\frac{d-2}{2\,d}\right)^2\left(u_\star'\,f + u_\star\,f'\right)^2.$$

Using the fact that $\lim_{d\to 2_+}(d-2)\,\mathsf{s}_d = 1$,

$$\mathsf{s}_d = \frac{1}{d-2} + \frac{1}{2} - \frac{1}{2}\,\log 2 + o(1) \quad \text{as} \quad d \to 2_+\,,$$

and

$$\lim_{d\to 2_+}\frac{1}{d-2}\int_0^\infty |u_\star'|^2\,r^{d-1}\,dr = 1\,,$$

$$\frac{1}{d-2}\int_0^\infty |u_\star'|^2\,r^{d-1}\,dr - 1 \sim -\frac{1}{2}\,(d-2)\,,$$

$$\lim_{d\to 2_+}\frac{1}{d-2}\int_0^\infty u_\star'\,(u_\star\,f)'\,r^{d-1}\,dr = \int_0^\infty f\,\frac{2\,r\,dr}{(1+r^2)^2}\,,$$

$$\lim_{d\to 2_+}\frac{1}{4\,d^2}\int_0^\infty |f'|^2\,u_\star^2\,r^{d-1}\,dr = \frac{1}{16}\int_0^\infty |f'|^2\,r\,dr\,,$$

and finally

$$\lim_{d\to 2_+}\int_0^\infty |u_\star\,(1+\tfrac{d-2}{2\,d}\,f)|^{\frac{2\,d}{d-2}}\,r^{d-1}\,dr = \int_0^\infty e^f\,\frac{r\,dr}{(1+r^2)^2}\,,$$

so that, as $d \to 2_+$,

$$\left(\int_0^\infty |u_\star\,(1+\tfrac{d-2}{2\,d}\,f)|^{\frac{2\,d}{d-2}}\,r^{d-1}\,dr\right)^{\frac{d-2}{d}} - 1 \sim \frac{d-2}{2}\,\log\left(\int_0^\infty e^f\,\frac{r\,dr}{(1+r^2)^2}\right).$$

By keeping only the highest order terms, which are of the order of $(d-2)$, and passing to the limit as $d \to 2_+$ in (11), we obtain that

$$\frac{1}{8}\int_0^\infty |f'|^2\,r\,dr + \int_0^\infty f\,\frac{2\,r\,dr}{(1+r^2)^2} \geqslant \log\left(\int_0^\infty e^f\,\frac{2\,r\,dr}{(1+r^2)^2}\right),$$

which is Onofri's inequality written for radial functions.

Similarly, we can pass to the limit as $d \to 2_+$ in (13). Let v be a compactly supported smooth radial function, considered as a function of $r \in [0,\infty)$ and let us compute the limit as $d \to 2_+$ of

$$\mathsf{h}(d) \coloneqq \left(\int_0^\infty v^{\frac{2\,d}{d+2}}\, r^{d-1}\, dr \right)^{1+\frac{2}{d}} - \frac{1}{\mathsf{s}_d} \int_0^\infty v\, k_d[v]\, r^{d-1}\, dr$$

where $k_d[v] \coloneqq (-\Delta)^{-1}\, v$ is given by (10) for any $d \geqslant 2$. If $d > 2$, since

$$\begin{aligned}
(2-d) \int_0^\infty v(r)\, k_d[v](r)\, r^{d-1}\, dr & \\
&= (2-d) \int_0^\infty v(r)\, r^{d-1} \int_r^\infty s^{1-d} \int_0^s v(t)\, t^{d-1}\, dt\, ds\, dr \\
&= (2-d) \int_0^\infty r^{1-d} \left(\int_0^r v(t)\, t^{d-1}\, dt \right)^2 dr \\
&= -2 \int_0^\infty r\, v(r) \int_0^r v(t)\, t^{d-1}\, dt\, dr
\end{aligned}$$

we see that $\lim_{d\to 2_+} \mathsf{h}(d) = 0$ since

$$2 \int_0^\infty r\, v(r) \int_0^r v(t)\, t\, dt\, dr = \left(\int_0^\infty r\, v(r)\, dr \right)^2 .$$

Let us compute the $O(d-2)$ term. With the above expression, it is now easy to check that

$$\begin{aligned}
&\lim_{d\to 2_+} \frac{\mathsf{h}(d)}{d-2} \\
&= \frac{1}{2} \int_0^\infty v\, r\, dr \int_0^\infty v\, \log\left(\frac{v}{\int_0^\infty v\, r\, dr} \right) r\, dr - \frac{\log 2 - 1}{2} \left(\int_0^\infty r\, v(r)\, dr \right)^2 \\
&\quad + 2 \int_0^\infty v\, r\, dr \int_0^\infty v(r)\, r \log r\, dr - 2 \int_0^\infty r\, v(r) \int_0^r v(t)\, t \log t\, dt\, dr \\
&= \frac{1}{2} \int_0^\infty v\, r\, dr \int_0^\infty v\, \log\left(\frac{v}{\int_0^\infty v\, r\, dr} \right) r\, dr - \frac{\log 2 - 1}{2} \left(\int_0^\infty r\, v(r)\, dr \right)^2 \\
&\quad + 2 \int_0^\infty v\, r\, dr \int_r^\infty v(t)\, t \log t\, dt
\end{aligned}$$

since $\frac{1}{(d-2)\,\mathsf{s}_d} \sim 1 + \frac{d-2}{2}\,(\log 2 - 1)$. A computation corresponding to $d = 2$ similar to the one done above for $d > 2$ shows that, when $d = 2$,

$$\begin{aligned}
\int_0^\infty v\, k_2[v]\, r\, dr &= \int_0^\infty v(r)\, r \int_r^\infty \frac{1}{s} \int_0^s v(t)\, t\, dt\, ds\, dr \\
&= \int_0^\infty \frac{1}{r} \left(\int_0^r v(t)\, t\, dt \right)^2 dr \\
&= -2 \int_0^\infty r\, \log r\, v(r) \int_0^r v(t)\, t\, dt\, dr\,,
\end{aligned}$$

thus proving that

$$\begin{aligned}
\lim_{d\to 2_+} \frac{\mathsf{h}(d)}{d-2} = \frac{1}{2} \int_0^\infty v\, r\, dr \int_0^\infty v\, \log\left(\frac{v}{\int_0^\infty v\, r\, dr} \right) r\, dr - \int_0^\infty v\, k_2[v]\, r^{d-1}\, dr \\
- \frac{1}{2}\,(\log 2 - 1) \left(\int_0^\infty r\, v(r)\, dr \right)^2 .
\end{aligned}$$

Now let us consider as above the limit

$$u^{\frac{d+2}{d-2}} = (1+r^2)^{-\frac{d+2}{2}} \left(1 + \tfrac{d-2}{2\,d}\, f\right)^{\frac{d+2}{d-2}} \to (1+r^2)^{-2}\, e^f =: g$$

as $d \to 2$. This concludes the proof of Corollary 4 by passing to the limit in the inequalities of Lemma 3 and taking $v = g$. □

Proof of Theorem 2 : a passage to the limit in the radial case. If we consider g as a function on $\mathbb{R}^2 \ni x$ with $r = |x|$, this means that

$$\lim_{d\to 2_+} \frac{\mathsf{h}(d)}{d-2} = \frac{1}{2}\int_{\mathbb{R}^2} g\,dx \int_{\mathbb{R}^2} g\,\log\left(\frac{g}{\int_{\mathbb{R}^2} g\,dx}\right) dx - 2\,\pi \int_{\mathbb{R}^2} g\,(-\Delta)^{-1}\,g\,dx + \frac{1}{2}\,(1+\log\pi)\left(\int_{\mathbb{R}^2} g\,dx\right)^2$$

which precisely corresponds to the terms involved in (5), up to a factor $\frac{1}{2}\,M = \frac{1}{2}\int_{\mathbb{R}^2} g\,dx$. The proof in the non-radial case will be provided at the end of Section 5. □

3 Linearization

In the previous section, we have proved that the optimal constant C_d in (3) is such that $\mathsf{C}_d \leqslant \mathsf{S}_d$. Let us prove that $\mathsf{C}_d \geqslant \frac{d}{d+4}\,\mathsf{S}_d$ using a special sequence of test functions. Let $\mathcal{F}$ and $\mathcal{G}$ be the positive integral quantities associated with, respectively, the Sobolev and Hardy-Littlewood-Sobolev inequalities :

$$\mathcal{F}[u] := \mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\,,$$

$$\mathcal{G}[v] := \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} v\,(-\Delta)^{-1}\,v\,dx\,.$$

Since that, for the Aubin-Talenti extremal function $u_\star$, we have $\mathcal{F}[u_\star] = \mathcal{G}[u_\star^q] = 0$, so that $u_\star$ gives a case of equality for (3), a natural question to ask is whether the infimum of $\mathcal{F}[u]/\mathcal{G}[u^q]$, under an appropriate normalization of $\|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$, is achieved as a perturbation of the $u_\star$.

Recall that $u_\star$ is the Aubin-Talenti extremal function

$$u_\star(x) := (1+|x|^2)^{-\frac{d-2}{2}} \quad \forall\, x \in \mathbb{R}^d\,.$$

With a slight abuse of notations, we use the same notation as in Section 2. We may notice that $u_\star$ solves

$$-\Delta u_\star = d\,(d-2)\,u_\star^{\frac{d+2}{d-2}}$$

which allows to compute the optimal Sobolev constant as

$$\mathsf{S}_d = \frac{1}{d\,(d-2)}\left(\int_{\mathbb{R}^d} u_\star^{2^*}\,dx\right)^{-\frac{2}{d}} \tag{15}$$

using (12). See A for details. This shows that

$$\frac{1}{\mathsf{S}_d}\,\mathcal{F}[u] = \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - d\,(d-2)\left(\int_{\mathbb{R}^d} u^{2^*}\,dx\right)^{1-\frac{2}{d}}\left(\int_{\mathbb{R}^d} u_\star^{2^*}\,dx\right)^{\frac{2}{d}}.$$

The goal of this section is to perform a linearization. By expanding $\mathcal{F}[u_\varepsilon]$ with $u_\varepsilon = u_\star + \varepsilon f$, for some f such that $\int_{\mathbb{R}^d} \frac{f\,u_\star}{(1+|x|^2)^2}\,dx = 0$ at order two in terms of ε, we get that

$$\frac{1}{\mathsf{S}_d}\,\mathcal{F}[u_\varepsilon] = \varepsilon^2\,\mathsf{F}[f] + o(\varepsilon^2)$$

where

$$\mathsf{F}[f] := \int_{\mathbb{R}^d} |\nabla f|^2\,dx - d\,(d+2)\int_{\mathbb{R}^d} \frac{|f|^2}{(1+|x|^2)^2}\,dx\,.$$

According to Lemma 17 (see B), we know that

$$\mathsf{F}[f] \geqslant 4\,(d+2)\int_{\mathbb{R}^d} \frac{|f|^2}{(1+|x|^2)^2}\,dx$$

for any $f \in \mathcal{D}^{1,2}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \frac{f\,f_i}{(1+|x|^2)^2}\,dx = 0 \quad \forall\, i = 0\,,\ 1\,,\ 2\,,\ \dots d+1\,, \tag{16}$$

where

$$f_0 := u_\star\,, \quad f_i(x) = \frac{x_i}{1+|x|^2}\,u_\star(x) \quad \text{and} \quad f_{d+1}(x) := \frac{1-|x|^2}{1+|x|^2}\,u_\star(x)\,.$$

Notice for later use that

$$-\,\Delta f_0 = d\,(d-2)\,\frac{f_0}{(1+|x|^2)^2}$$

and

$$-\,\Delta f_i = d\,(d+2)\,\frac{f_i}{(1+|x|^2)^2} \quad \forall\, i = 1\,,\ 2\,,\ \dots d+1\,.$$

Also notice that

$$\int_{\mathbb{R}^d} \frac{f_i\,f_j}{(1+|x|^2)^2}\,dx = 0$$

for any $i, j = 0, 1, \dots d+1$, $j \neq i$.

Similarly, we can consider the functional $\mathcal{G}$ as given above, associated with the Hardy-Littlewood-Sobolev inequality, and whose minimum $\mathcal{G}[v_\star] = 0$ is achieved by $v_\star := u_\star^q$, $q = \frac{d+2}{d-2}$. Consistently with the above computations, let $v_\varepsilon := (u_\star + \varepsilon\,f)^q = v_\star\left(1 + \varepsilon\,\frac{f}{u_\star}\right)^q$ where f is such that $\int_{\mathbb{R}^d} \frac{f\,f_0}{(1+|x|^2)^2}\,dx = 0$. By expanding $\mathcal{G}[v_\varepsilon]$ at order two in terms of ε, we get that

$$\mathcal{G}[v_\varepsilon] = \varepsilon^2\,\left(\frac{d+2}{d-2}\right)^2\,\mathsf{G}[f] + o(\varepsilon^2)$$

where

$$\mathsf{G}[f] := \frac{1}{d\,(d+2)}\int_{\mathbb{R}^d} \frac{|f|^2}{(1+|x|^2)^2}\,dx - \int_{\mathbb{R}^d} \frac{f}{(1+|x|^2)^2}\,(-\Delta)^{-1}\left(\frac{f}{(1+|x|^2)^2}\right)dx\,.$$

Lemma 5. $\mathrm{Ker}(\mathsf{F}) = \mathrm{Ker}(\mathsf{G})$.

It is straightforward to check that the kernel is generated by f_i with $i = 1, 2, \dots d, d+1$. Details are left to the reader. Next, by Legendre duality we find that

$$\frac{1}{2}\int_{\mathbb{R}^d} \frac{|g|^2}{(1+|x|^2)^2}\,dx = \sup_f \left(\int_{\mathbb{R}^d} \frac{f\,g}{(1+|x|^2)^2}\,dx - \frac{1}{2}\int_{\mathbb{R}^d} \frac{|f|^2}{(1+|x|^2)^2}\,dx\right)\,,$$

$$\frac{1}{2}\int_{\mathbb{R}^d}\frac{g}{(1+|x|^2)^2}\,(-\Delta)^{-1}\left(\frac{g}{(1+|x|^2)^2}\right)dx \\ = \sup_f\left(\int_{\mathbb{R}^d}\frac{f\,g}{(1+|x|^2)^2}\,dx-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla f|^2\,dx\right).$$

Here the supremum is taken for all f satisfying the orthogonality conditions (16). It is then straightforward to see that duality holds if g is restricted to functions satisfying (16) as well. Consider indeed an optimal function f subject to (16). There are Lagrange multipliers $\mu_i\in\mathbb{R}$ such that

$$g-f-\sum_{i=0}^{d+1}\mu_i\,f_i=0$$

and after multiplying by $f\,(1+|x|^2)^{-2}$, an integration shows that

$$\int_{\mathbb{R}^d}\frac{f\,g}{(1+|x|^2)^2}\,dx=\int_{\mathbb{R}^d}\frac{|f|^2}{(1+|x|^2)^2}\,dx$$

using the fact that f satisfies (16). On the other hand, if g satisfies (16), after multiplying by $g\,(1+|x|^2)^{-2}$, an integration gives

$$\int_{\mathbb{R}^d}\frac{|g|^2}{(1+|x|^2)^2}\,dx=\int_{\mathbb{R}^d}\frac{f\,g}{(1+|x|^2)^2}\,dx\,,$$

which establishes the first identity of duality. As for the second identity, the optimal function satisfies the Euler-Lagrange equation

$$\frac{g}{(1+|x|^2)^2}+\Delta\,f=\sum_{i=0}^{d+1}\mu_i\,\frac{f_i}{(1+|x|^2)^2}$$

for some Lagrange multipliers that we again denote by μ_i. By multiplying by f and $(-\Delta)^{-1}\big(g\,(1+|x|^2)^{-2}\big)$, we find that

$$\int_{\mathbb{R}^d}\frac{f\,g}{(1+|x|^2)^2}\,dx=\int_{\mathbb{R}^d}|\nabla f|^2\,dx \\ \int_{\mathbb{R}^d}\frac{g}{(1+|x|^2)^2}\,(-\Delta)^{-1}\left(\frac{g}{(1+|x|^2)^2}\right)dx=\int_{\mathbb{R}^d}\frac{f\,g}{(1+|x|^2)^2}\,dx$$

where we have used the fact that

$$\int_{\mathbb{R}^d}\frac{f_i}{(1+|x|^2)^2}\,(-\Delta)^{-1}\left(\frac{g}{(1+|x|^2)^2}\right)dx \\ =\int_{\mathbb{R}^d}\frac{g}{(1+|x|^2)^2}\,(-\Delta)^{-1}\left(\frac{f_i}{(1+|x|^2)^2}\right)dx=0$$

because $(-\Delta)^{-1}\big(f_i\,(1+|x|^2)^{-2}\big)$ is proportional to f_i. As a straightforward consequence, the dual form of Lemma 17 then reads as follows.

Corollary 6. *For any g satisfying the orthogonality conditions (16), we have*

$$\int_{\mathbb{R}^d}\frac{g}{(1+|x|^2)^2}\,(-\Delta)^{-1}\left(\frac{g}{(1+|x|^2)^2}\right)dx\leqslant\frac{1}{(d+2)\,(d+4)}\int_{\mathbb{R}^d}\frac{g^2}{(1+|x|^2)^2}\,dx\,.$$

Moreover, if f obeys to (16), then we have

$$\frac{4}{d\,(d+2)\,(d+4)}\int_{\mathbb{R}^d}\frac{f^2}{(1+|x|^2)^2}\,dx\leqslant\mathsf{G}[f]\leqslant\frac{1}{d\,(d+2)^2\,(d+4)}\,\mathsf{F}[f]$$

and equalities are achieved in $\mathrm{L}^2(\mathbb{R}^d,(1+|x|^2)^{-2}\,dx)$.

Démonstration. The first inequality follows from the above considerations on duality and the second one from the definition of G, using

$$\frac{4}{d\,(d+2)\,(d+4)} = \frac{1}{d\,(d+2)} - \frac{1}{(d+2)\,(d+4)}\,.$$

To establish the last inequality, we can decompose f on $(f_k)_k$, the stereographic projection of the spherical harmonics associated to eigenvalues $\lambda_k = k\,(k+d-1)$ with $k \geqslant 2$, so as to meet condition (16). See B for more details. The corresponding eigenvalues for the Laplacian operator on the Euclidean space are $\mu_k = 4\,\lambda_k + d\,(d-2)$, so that $-\Delta f_k = \mu_k\,f_k\,(1+|x|^2)^{-2}$, with $\|f_k\|_{L^2(\mathbb{R}^d,\,(1+|x|^2)^{-2}\,dx)} = 1$. By writing $f = \sum_{k\geqslant 2} a_k\,f_k$ we have

$$\mathsf{F}[f] = \sum_{k\geqslant 2} c_k\,, \quad \text{with} \quad c_k := a_k^2\,(\mu_k - \mu_1)\,,$$

$$\mathsf{G}[f] = \sum_{k\geqslant 2} d_k\,, \quad \text{with} \quad d_k := a_k^2\left(\frac{1}{\mu_1} - \frac{1}{\mu_k}\right)\,,$$

with $c_k = \mu_1\,\mu_k\,d_k \leqslant \mu_1\,\mu_2\,d_k$ since $(\mu_k)_k$ is increasing in k. This yields

$$\frac{\mathsf{F}[f]}{\mathsf{G}[f]} \leqslant \mu_1\,\mu_2 = d\,(d+2)^2\,(d+4)\,,$$

with equality for $f = f_2$. □

As a consequence of Corollary 6 and (15), we have found that

$$\frac{1}{\mathcal{C}} := \frac{\mathsf{S}_d}{\mathsf{C}_d} = \inf_{\mathcal{G}[u^q]\neq 0} \frac{\|u\|_{\mathrm{L}^{2*}(\mathbb{R}^d)}^{\frac{8}{d-2}}\,\mathsf{S}_{\mathsf{d}}\,\mathcal{F}[u]}{\mathcal{G}[u^q]} \leqslant \frac{1}{d^2\,(d+2)^2}\,\inf_f \frac{\mathsf{F}[f]}{\mathsf{G}[f]} = \frac{d+4}{d}\,, \tag{17}$$

where the last infimum is taken on the set of all non-trivial functions in $\mathrm{L}^2(\mathbb{R}^d,\,(1+|x|^2)^{-2}\,dx)$ satisfying (16). This establishes the lower bound in (3).

Remark 7. *One may hope to get a better estimate by considering the case $f \in \mathrm{Ker}(\mathsf{F}) = \mathrm{Ker}(\mathsf{G})$ and expanding $\mathcal{F}$ and $\mathcal{G}$ to the fourth order in ε but, interestingly, this yields exactly the same lower bound on C_d as the linearization shown above.*

4 Improved inequalities and nonlinear flows

In Section 3, the basic strategy was based on the completion of a square. The initial approach for the improvement of Sobolev inequalities in [Dol11] was based on a fast diffusion flow. Let us give some details and explain how even better results can be obtained using a combination of the two approaches.

Let us start with a summary of the method of [Dol11]. It will be convenient to define the functionals

$$\mathsf{J}_d[v] := \int_{\mathbb{R}^d} v^{\frac{2\,d}{d+2}}\,dx \quad \text{and} \quad \mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}\,.$$

Consider a positive solution v of the *fast diffusion* equation

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t>0\,, \quad x\in\mathbb{R}^d\,, \quad m = \frac{d-2}{d+2} \tag{18}$$

and define the functions

$$\mathsf{J}(t) \coloneqq \mathsf{J}_d[v(t,\cdot)] \quad \text{and} \quad \mathsf{H}(t) \coloneqq \mathsf{H}_d[v(t,\cdot)]\,.$$

We shall denote by J_0 and H_0 the corresponding initial values. Elementary computations show that

$$\mathsf{J}' = -\,(m+1)\,\|\nabla v^m\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \leqslant -\,\frac{m+1}{\mathsf{S}_d}\,\mathsf{J}^{1-\frac{2}{d}} = -\,\frac{2\,d}{d+2}\,\frac{1}{\mathsf{S}_d}\,\mathsf{J}^{1-\frac{2}{d}}\,, \tag{19}$$

where the inequality is a consequence of Sobolev's inequality. Hence v has a finite extinction time $T > 0$ and since

$$\mathsf{J}(t)^{\frac{2}{d}} \leqslant \mathsf{J}_0^{\frac{2}{d}} - \frac{4}{d+2}\,\frac{t}{\mathsf{S}_d}\,,$$

we find that

$$T \leqslant \frac{d+2}{4}\,\mathsf{S}_d\,\mathsf{J}_0^{\frac{2}{d}}\,.$$

We notice that H is nonpositive because of the Hardy-Littlewood-Sobolev inequality and by applying the flow of (18), we get that

$$\frac{1}{2}\,\mathsf{J}^{-\frac{2}{d}}\,\mathsf{H}' = \mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2*}(\mathbb{R}^d)} \quad \text{with} \quad u = v^{\frac{d-2}{d+2}}\,.$$

The right hand side is nonnegative because of Sobolev's inequality. One more derivation with respect to t gives that

$$\mathsf{H}'' = \frac{\mathsf{J}'}{\mathsf{J}}\,\mathsf{H}' - 4\,m\,\mathsf{S}_d\,\mathsf{J}^{\frac{2}{d}}\,\mathsf{K} \tag{20}$$

where $\mathsf{K} \coloneqq \int_{\mathbb{R}^d} v^{m-1}\,|(-\Delta)v^m - \Lambda\,v|^2\,dx$ and $\Lambda \coloneqq -\frac{d+2}{2\,d}\,\frac{\mathsf{J}'}{\mathsf{J}}$. This identity makes sense in dimension $d \geqslant 5$, because, close to the extinction time, v behaves like the Aubin-Talenti functions. The reader is invited to check that all terms are finite when expanding the square in K and can refer to [Dol11] for more details. It turns out that the following estimate is also true if $d = 3$ or $d = 4$.

Lemma 8. *Assume that $d \geqslant 3$. With above notations, we have*

$$\frac{\mathsf{H}''}{\mathsf{H}'} \leqslant \frac{\mathsf{J}'}{\mathsf{J}}\,.$$

The main idea is that even if each of the above integrals is infinite, there are cancellations in low dimensions. To clarify this computation, it is much easier to get rid of the time-dependence corresponding to the *solution with separation of variables* and use the inverse stereographic projection to recast the problem on the sphere. The sketch of the proof of this lemma will be given in C.

A straightforward consequence is the fact that

$$\frac{\mathsf{H}''}{\mathsf{H}'} \leqslant -\,\kappa \quad \text{with} \quad \kappa \coloneqq \frac{2\,d}{d+2}\,\frac{\mathsf{J}_0^{-\frac{2}{d}}}{\mathsf{S}_d}$$

where the last inequality is a consequence of (19). Two integrations with respect to t show that

$$-\,\mathsf{H}_0 \leqslant \frac{1}{\kappa}\,\mathsf{H}_0'\,(1 - e^{-\kappa\,T}) \leqslant \frac{1}{2}\,\mathcal{C}\,\mathsf{S}_d\,\mathsf{J}_0^{\frac{2}{d}}\,\mathsf{H}_0' \quad \text{with} \quad \mathcal{C} = \frac{d+2}{d}\,(1 - e^{-d/2})\,,$$

which is the main result of [Dol11] (when $d \geqslant 5$), namely

$$-\mathsf{H}_0 \leqslant \mathcal{C}\,\mathsf{S}_d\,\mathsf{J}_0^{\frac{4}{d}}\left[\mathsf{S}_d\,\|\nabla u_0\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u_0\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right] \quad \text{with} \quad u_0 = v_0^{\frac{d-2}{d+2}}\,.$$

Since this inequality holds for any initial datum $u_0 = u$, we have indeed shown that

$$-\mathsf{H}_d[v] \leqslant \mathcal{C}\,\mathsf{S}_d\,\mathsf{J}_d[v]^{\frac{4}{d}}\left[\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right]$$
$$\forall\, u \in \mathcal{D}^{1,2}(\mathbb{R}^d)\,,\ v = u^{\frac{d+2}{d-2}}\,.$$

It is straightforward to check that our result of Theorem 1 is an improvement, not only because the restriction $d \geqslant 5$ is removed, but also because the inequality holds with $\frac{d}{d+4} \leqslant \mathcal{C} < 1 < \frac{d+2}{d}\,(1 - e^{-d/2})$. In other words, the result of Theorem 1 is equivalent to

$$-\mathsf{H}_0 \leqslant \frac{1}{2}\,\mathcal{C}\,\mathsf{S}_d\,\mathsf{J}_0^{\frac{2}{d}}\,\mathsf{H}_0' \quad \text{with} \quad \mathcal{C} = \frac{d}{d+4}\,. \tag{21}$$

Up to now, we have not established yet the fact that $\mathcal{C} < 1$. This is what we are now going to do.

Now let us reinject in the flow method described above our improved inequality of Theorem 1, which can also be written as

$$\mathcal{C}\,\mathsf{S}_d\,\mathsf{J}^{\frac{4}{d}}\left[\frac{d+2}{2\,d}\,\mathsf{S}_d\,\mathsf{J}' + \mathsf{J}^{1-\frac{2}{d}}\right] - \mathsf{H} \leqslant 0 \tag{22}$$

if v is still a positive solution of (18). From Lemma 8, we deduce that

$$\mathsf{H}' \leqslant \kappa_0\,\mathsf{J} \quad \text{with} \quad \kappa_0 := \frac{\mathsf{H}_0'}{\mathsf{J}_0}\,.$$

Since $t \mapsto \mathsf{J}(t)$ is monotone decreasing, there exists a function Y such that

$$\mathsf{H}(t) = -\,\mathsf{Y}(\mathsf{J}(t)) \quad \forall\, t \in [0,T)\,.$$

Differentiating with respect to t, we find that

$$-\,\mathsf{Y}'(\mathsf{J})\,\mathsf{J}' = \mathsf{H}' \leqslant \kappa_0\,\mathsf{J}$$

and, by inserting this expression in (22), we arrive at

$$\mathcal{C}\left(-\,\frac{d+2}{2\,d}\,\kappa_0\,\mathsf{S}_d^2\,\frac{\mathsf{J}^{1+\frac{4}{d}}}{\mathsf{Y}'} + \mathsf{S}_d\,\mathsf{J}^{1+\frac{2}{d}}\right) + \mathsf{Y} \leqslant 0\,.$$

Summarizing, we end up by considering the differential inequality

$$\mathsf{Y}'\left(\mathcal{C}\,\mathsf{S}_d\,s^{1+\frac{2}{d}} + \mathsf{Y}\right) \leqslant \frac{d+2}{2\,d}\,\mathcal{C}\,\kappa_0\,\mathsf{S}_d^2\,s^{1+\frac{4}{d}}\,, \quad \mathsf{Y}(0) = 0\,, \quad \mathsf{Y}(\mathsf{J}_0) = -\,\mathsf{H}_0 \tag{23}$$

on the interval $[0, \mathsf{J}_0] \ni s$. It is then possible to obtain estimates as follows. On the one hand we know that

$$\mathsf{Y}' \leqslant \frac{d+2}{2\,d}\,\kappa_0\,\mathsf{S}_d\,s^{\frac{2}{d}}$$

and, hence,

$$\mathsf{Y}(s) \leqslant \frac{1}{2}\,\kappa_0\,\mathsf{S}_d\,s^{1+\frac{2}{d}} \quad \forall\, s \in [0, \mathsf{J}_0]\,.$$

On the other hand, after integrating by parts on the interval $[0, \mathsf{J}_0]$, we get

$$\frac{1}{2}\,\mathsf{H}_0^2 - \mathcal{C}\,\mathsf{S}_d\,\mathsf{J}_0^{1+\frac{2}{d}}\,\mathsf{H}_0 \leqslant \frac{1}{4}\,\mathcal{C}\,\kappa_0\,\mathsf{S}_d^2\,\mathsf{J}_0^{2+\frac{4}{d}} + \frac{d+2}{d}\,\mathcal{C}\,\mathsf{S}_d\int_0^{\mathsf{J}_0} s^{\frac{2}{d}}\,\mathsf{Y}(s)\,ds\,.$$

Using the above estimate, we find that

$$\frac{d+2}{d}\,\mathsf{S}_d\int_0^{\mathsf{J}_0} s^{\frac{2}{d}}\,\mathsf{Y}(s)\,ds \leqslant \frac{1}{4}\,\mathsf{J}_0^{2+\frac{4}{d}}\,,$$

and finally

$$\frac{1}{2}\,\mathsf{H}_0^2 - \mathcal{C}\,\mathsf{S}_d\,\mathsf{J}_0^{1+\frac{2}{d}}\,\mathsf{H}_0 \leqslant \frac{1}{2}\,\mathcal{C}\kappa_0\,\mathsf{S}_d^2\,\mathsf{J}_0^{2+\frac{4}{d}}\,.$$

This is a strict improvement of (21) when $\mathcal{C} = 1$ since (21) is then equivalent to

$$-\,\mathsf{S}_d\,\mathsf{J}_0^{1+\frac{2}{d}}\,\mathsf{H}_0 \leqslant \frac{1}{2}\,\mathcal{C}\kappa_0\,\mathsf{S}_d^2\,\mathsf{J}_0^{2+\frac{4}{d}}\,.$$

However, it is a strict improvement of (21) if $\mathcal{C} < 1$ only when $|\mathsf{H}_0| = -\,\mathsf{H}_0$ is large enough (we will come back to this point in Remarks 10 and 11). Altogether, we have shown an improved inequality that can be stated as follows.

Theorem 9. *Assume that $d \geqslant 3$. Then we have*

$$0 \leqslant \mathsf{H}_d[v] + \mathsf{S}_d\,\mathsf{J}_d[v]^{1+\frac{2}{d}}\,\varphi\left(\mathsf{J}_d[v]^{\frac{2}{d}-1}\,\left[\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right]\right)$$
$$\forall\, u \in \mathcal{D}^{1,2}(\mathbb{R}^d)\,,\ v = u^{\frac{d+2}{d-2}}$$

where $\varphi(x) := \sqrt{\mathcal{C}^2 + 2\,\mathcal{C}\,x} - \mathcal{C}$ for any $x \geqslant 0$.

Démonstration. We have shown that $y^2 + 2\,\mathcal{C}\,y - \mathcal{C}\,\kappa_0 \leqslant 0$ with $y = -\,\mathsf{H}_0/(\mathsf{S}_d\,\mathsf{J}_0^{1+\frac{2}{d}}) \geqslant 0$. This proves that $y \leqslant \sqrt{\mathcal{C}^2 + \mathcal{C}\kappa_0} - \mathcal{C}$, which proves that

$$-\,\mathsf{H}_0 \leqslant \mathsf{S}_d\,\mathsf{J}_0^{1+\frac{2}{d}}\left(\sqrt{\mathcal{C}^2 + \mathcal{C}\,\kappa_0} - \mathcal{C}\right)$$

after recalling that

$$\frac{1}{2}\,\kappa_0 = \frac{\mathsf{H}_0'}{\mathsf{J}_0} = \mathsf{J}_d[v_0]^{\frac{2}{d}-1}\,\left[\mathsf{S}_d\,\|\nabla u_0\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u_0\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right]\,.$$

□

Remark 10. *We may observe that $x \mapsto x - \varphi(x)$ is a convex nonnegative function which is equal to 0 if and only if $x = 0$. Moreover, we have*

$$\varphi(x) \leqslant x \quad \forall\, x \geqslant 0$$

with equality if and only if $x = 0$. However, one can notice that

$$\varphi(x) \leqslant \mathcal{C}\,x \quad \Longleftrightarrow \quad x \geqslant 2\,\frac{1-\mathcal{C}}{\mathcal{C}}\,.$$

Remark 11. *A more careful analysis of* (23) *shows that*

$$\mathsf{Y}(s) \leqslant \tfrac{1}{2}\left(\sqrt{1+\tfrac{2\,\kappa_0}{\mathcal{C}}}-1\right)\mathcal{C}\,\mathsf{S}_d\,s^{1+\frac{2}{d}}\,,$$

which shows that the inequality of Theorem 9 holds with the improved function

$$\varphi(x) := \sqrt{\mathcal{C}^2+\mathcal{C}\,x+\tfrac{1}{2}\,\mathcal{C}^2\left(\sqrt{1+\tfrac{4\,x}{\mathcal{C}}}-1\right)}-\mathcal{C}$$

but again the reader is invited to check that $\varphi(x)\leqslant x$ *for any* $x\geqslant 0$ *and* $\lim_{x\to 0_+}\varphi(x)/x=1$.

Corollary 12. *With the above notations, we have* $\mathcal{C}<1$.

Démonstration. Assume by contradiction that $\mathcal{C}=1$. With the notations of Section 3, let us consider a minimizing sequence $(u_n)_{n\in\mathbb{N}}$ for the functional $u\mapsto\frac{\mathcal{F}[u]}{\mathcal{G}[u^q]}$ but assume that $\mathsf{J}_d[u_n^q]=\mathsf{J}_d[u_\star^q]=:\mathsf{J}_\star$ for any $n\in\mathbb{N}$. This condition is not restrictive because of the homogeneity of the inequality. It implies that $(\mathcal{G}[u_n^q])_{n\in\mathbb{N}}$ is bounded.

If $\lim_{n\to\infty}\mathcal{G}[u_n^q]>0$, then we also have $\mathsf{L}:=\lim_{n\to\infty}\mathcal{F}[u_n]>0$, at least up to the extraction of a subsequence. As a consequence we find that

$$\begin{aligned}0=\lim_{n\to\infty}\left(\mathsf{S}_d\,\mathsf{J}_\star^{\frac{4}{d}}\,\mathcal{F}[u_n]-\mathcal{G}[u_n^q]\right)\\ =\mathsf{S}_d\lim_{n\to\infty}\left[\mathsf{J}_\star^{\frac{4}{d}}\,\mathcal{F}[u_n]-\mathsf{J}_\star^{1+\frac{2}{d}}\,\varphi\left(\mathsf{J}_\star^{\frac{2}{d}-1}\,\mathcal{F}[u_n]\right)\right]\\ +\lim_{n\to\infty}\left[\mathsf{S}_d\,\mathsf{J}_\star^{1+\frac{2}{d}}\,\varphi\left(\mathsf{J}_\star^{\frac{2}{d}-1}\,\mathcal{F}[u_n]\right)-\mathcal{G}[u_n^q]\right],\end{aligned}$$

a contradiction since the last term is nonnegative by Theorem 9 and, as observed in Remark 10, $\mathsf{J}_\star^{4/d}\,\mathcal{F}[u_n]-\mathsf{J}_\star^{1+2/d}\,\varphi\big(\mathsf{J}_\star^{2/d-1}\,\mathcal{F}[u_n]\big)$ is positive unless $\mathcal{F}[u_n]=0$.

Hence we know that $\mathsf{L}=\lim_{n\to\infty}\mathcal{F}[u_n]=0$ and $\lim_{n\to\infty}\mathcal{G}[u_n^q]=0$. According to the caracterisation of minimizers of $\mathcal{G}$ by Lieb [Lie83, Theorem 3.1], we know that up to translations and dilations, u_k converges to $u_\star$. Thus there exists f_k such that $u_k=u_\star+f_k$ with $f_k\to 0$, and then

$$\frac{1}{\mathcal{C}}=\frac{\mathsf{S}_d}{\mathsf{C}_d}=\lim_{k\to\infty}\frac{1}{d^2\,(d+2)^2}\,\frac{\mathsf{F}[f_k]}{\mathsf{G}[f_k]}\geqslant\frac{d+4}{d}\,.$$

This shows that $\mathcal{C}\leqslant\frac{d}{d+4}$, a contradiction. □

We may observe that $\mathcal{C}<1$ means $\mathsf{C}_d<\mathsf{S}_d$. This completes the the proof of Theorem 1.

5 Caffarelli-Kohn-Nirenberg inequalities and duality

Let $2^*:=\infty$ if $d=1$ or 2, $2^*:=2\,d/(d-2)$ if $d\geqslant 3$ and $a_c:=(d-2)/2$. Consider the space $\mathcal{D}_a^{1,2}(\mathbb{R}^d)$ obtained by completion of $\mathcal{D}(\mathbb{R}^d\backslash\{0\})$ with respect to the norm $u\mapsto\|\,|x|^{-a}\,\nabla u\,\|_{\mathrm{L}^2(\mathbb{R}^d)}^2$. In this section, we shall consider the Caffarelli-Kohn-Nirenberg inequalities

$$\left(\int_{\mathbb{R}^d}\frac{|u|^p}{|x|^{bp}}\,dx\right)^{\frac{2}{p}}\leqslant\mathsf{C}_{a,b}\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{|x|^{2a}}\,dx \tag{24}$$

These inequalities generalize to $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$ the Sobolev inequality (1) and in particular the exponent p is given in terms of a and b by

$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

as can be checked by a simple scaling argument. A precise statements on the range of validity of (24) goes as follows.

Lemma 13. *[CKN84] Let $d \geqslant 1$. For any $p \in [2, 2^*]$ if $d \geqslant 3$ or $p \in [2, 2^*)$ if $d = 1$ or 2, there exists a positive constant $\mathsf{C}_{a,b}$ such that (24) holds if a, b and p are related by $b = a - a_c + d/p$, with the restrictions $a < a_c$, $a \leqslant b \leqslant a+1$ if $d \geqslant 3$, $a < b \leqslant a+1$ if $d = 2$ and $a + 1/2 < b \leqslant a+1$ if $d = 1$.*

At least for radial solutions in $\mathbb{R}^d$, weights can be used to work as in Section 2 as if the dimension d was replaced by the *dimension* $(d-2a)$. We will apply this heuristic idea to the case $d = 2$ and $a < 0$, $a \to 0$ in order to prove Theorem 2. See D for symmetry results for optimal functions in (24).

On $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$, let us define the functionals

$$\mathsf{F}_1[u] := \frac12 \left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{bp}}\,dx \right)^{\frac2p} \quad \text{and} \quad \mathsf{F}_2[u] := \frac12\,\mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2a}}\,dx$$

so that Inequality (24) amounts to $\mathsf{F}_1[u] \leqslant \mathsf{F}_2[u]$. Assume that $\langle\cdot,\cdot\rangle$ denotes the natural scalar product on $\mathrm{L}^2\big(\mathbb{R}^d, |x|^{-2a}\,dx\big)$, that is,

$$\langle u, v\rangle := \int_{\mathbb{R}^d} \frac{u\,v}{|x|^{2a}}\,dx$$

and denote by $\|u\| = \langle u, u\rangle^{1/2}$ the corresponding norm. Consider the operators

$$\mathsf{A}_a\,u := \nabla u\,, \quad \mathsf{A}_a^*\,w := -\nabla\cdot w + 2a\,\frac{x}{|x|^2}\cdot w$$
$$\text{and} \quad \mathsf{L}_a\,u := \mathsf{A}_a^*\,\mathsf{A}_a\,u = -\,\Delta u + 2a\,\frac{x}{|x|^2}\cdot\nabla u$$

defined for u and w respectively in $\mathrm{L}^2\big(\mathbb{R}^d, |x|^{-2a}\,dx\big)$ and $\mathrm{L}^2\big(\mathbb{R}^d, |x|^{-2a}\,dx\big)^d$. Elementary integrations by parts show that

$$\langle u, \mathsf{L}_a\,u\rangle = \langle \mathsf{A}_a\,u, \mathsf{A}_a\,u\rangle = \|\mathsf{A}_a\,u\|^2 = \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2a}}\,dx\,.$$

If we define the Legendre dual of F_i by $\mathsf{F}_i^*[v] = \sup_{u\in\mathcal{D}^{1,2}_a(\mathbb{R}^d)} \big(\langle u, v\rangle - \mathsf{F}_i[u]\big)$, then it is clear that we formally have the inequality $\mathsf{F}_2^*[v] \leqslant \mathsf{F}_1^*[v]$ for any $v \in \mathrm{L}^q(\mathbb{R}^d, |x|^{-(2a-b)\,q}\,dx) \cap \mathsf{L}_a(\mathcal{D}^{1,2}_a(\mathbb{R}^d))$, where q is Hölder's conjugate of p, *i.e.*

$$\frac1p + \frac1q = 1\,.$$

Using the invertibility of L_a, we indeed observe that

$$\mathsf{F}_2^*[v] = \langle u, v\rangle - \mathsf{F}_2[u] \quad \text{with} \quad v = \mathsf{C}_{a,b}\,\mathsf{L}_a\,u \iff u = \frac{1}{\mathsf{C}_{a,b}}\,\mathsf{L}_a^{-1}\,v\,,$$

hence proving that

$$\mathsf{F}_2^*[v] = \frac{1}{2\,\mathsf{C}_{a,b}}\,\langle v, \mathsf{L}_a^{-1}\,v\rangle\,.$$

Similarly, we get that $\mathsf{F}_1^*[v] = \langle u, v\rangle - \mathsf{F}_1[u]$ with

$$|x|^{-\,2a}\,v = \kappa^{2-p}\,|x|^{-\,bp}\,u^{p-1} \tag{25}$$

and

$$\kappa = \left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{bp}}\,dx\right)^{\frac{1}{p}} = \langle u, v\rangle = \left(\int_{\mathbb{R}^d} \frac{|v|^q}{|x|^{(2a-b)\,q}}\,dx\right)^{\frac{1}{q}}\,,$$

that is

$$\mathsf{F}_1^*[v] = \frac{1}{2}\left(\int_{\mathbb{R}^d} \frac{|v|^q}{|x|^{(2a-b)\,q}}\,dx\right)^{\frac{2}{q}}\,.$$

This proves the following result.

Lemma 14. *With the above notations and under the same assumptions as in Lemma 13, we have*

$$\frac{1}{\mathsf{C}_{a,b}}\,\langle v, \mathsf{L}_a^{-1}\,v\rangle \leqslant \left(\int_{\mathbb{R}^d} \frac{|v|^q}{|x|^{(2a-b)\,q}}\,dx\right)^{\frac{2}{q}}$$
$$\forall\, v \in \mathrm{L}^q(\mathbb{R}^d, |x|^{-(2a-b)\,q}\,dx) \cap \mathsf{L}_a(\mathcal{D}_a^{1,2}(\mathbb{R}^d))\,.$$

The next step is based on the completion of the square. Let us compute

$$\begin{aligned}\|\mathsf{A}_a\,u - \lambda\,\mathsf{A}_a\,\mathsf{L}_a^{-1}\,v\|^2 &= \|\mathsf{A}_a\,u\|^2 - 2\,\lambda\,\langle \mathsf{A}_a\,u, \mathsf{A}_a\,\mathsf{L}_a^{-1}\,v\rangle + \lambda^2\,\langle \mathsf{A}_a\,\mathsf{L}_a^{-1}\,v, \mathsf{A}_a\,\mathsf{L}_a^{-1}\,v\rangle\\ &= \|\mathsf{A}_a\,u\|^2 - 2\,\lambda\,\langle u, v\rangle + \lambda^2\,\langle v, \mathsf{L}_a^{-1}\,v\rangle\,.\end{aligned}$$

With the choice $\lambda = 1/\mathsf{C}_{a,b}$ and v given by (25), we have proved the following

Theorem 15. *Under the assumptions of Lemma 13 and with the above notations, for any $u \in \mathcal{D}_a^{1,2}(\mathbb{R}^d)$ and any $v \in \mathrm{L}^q(\mathbb{R}^d, |x|^{-(2a-b)\,q}\,dx) \cap \mathsf{L}_a(\mathcal{D}_a^{1,2}(\mathbb{R}^d))$ we have*

$$0 \leqslant \left(\int_{\mathbb{R}^d} \frac{|v|^q}{|x|^{(2a-b)\,q}}\,dx\right)^{\frac{2}{q}} - \frac{1}{\mathsf{C}_{a,b}}\,\langle v, \mathsf{L}_a^{-1}\,v\rangle$$
$$\leqslant \mathsf{C}_{a,b}\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2a}}\,dx - \left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{bp}}\,dx\right)^{\frac{2}{p}}$$

if u and v are related by (25), if a, b and p are such that $b = a - a_c + d/p$ and verify the conditions of Lemma 13, and if $q = p/(p-1)$.

If, instead of (25), we simply require that

$$|x|^{-\,2a}\,v = |x|^{-\,bp}\,u^{p-1}\,,$$

then the inequality becomes

$$0 \leqslant \mathsf{C}_{a,b}\left(\int_{\mathbb{R}^d} \frac{|v|^q}{|x|^{(2a-b)\,q}}\,dx\right)^{\frac{2}{q}} - \langle v, \mathsf{L}_a^{-1}\,v\rangle$$
$$\leqslant \mathsf{C}_{a,b}\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{bp}}\,dx\right)^{\frac{2}{p}(p-2)}\left[\mathsf{C}_{a,b}\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2a}}\,dx - \left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{bp}}\,dx\right)^{\frac{2}{p}}\right]$$

Hence Theorem 15 generalizes Theorem 1, which is recovered in the special case $a = b = 0$, $d \geqslant 3$. Because of the positivity of the l.h.s. due to Lemma 14, the inequality in Theorem 15 is an improvement of the Caffarelli-Kohn-Nirenberg inequality (24). It can also be seen as an interpolation result, namely

$$2\left(\int_{\mathbb{R}^d}\frac{|v|^q}{|x|^{(2a-b)\,q}}\,dx\right)^{\frac{2}{q}} = 2\left(\int_{\mathbb{R}^d}\frac{|u|^p}{|x|^{bp}}\,dx\right)^{\frac{2}{p}} \leqslant \mathsf{C}_{a,b}\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{|x|^{2a}}\,dx + \frac{1}{\mathsf{C}_{a,b}}\,\langle v, \mathsf{L}_a^{-1}\,v\rangle$$

whenever u and v are related by (25). The explicit value of $\mathsf{C}_{a,b}$ is not known unless equality in (24) is achieved by radial functions, that is when *symmetry* holds. See Proposition 19 in D for some symmetry results. Now, as in [DET08], we may investigate the limit $(a, b) \to (0, 0)$ with $b = \alpha\, a/(1+\alpha)$ in order to investigate the Onofri limit case. A key observation is that optimality in (24) is achieved by radial functions for any $\alpha \in (-1, 0)$ and $a < 0$, $|a|$ small enough. In that range $\mathsf{C}_{a,b}$ is known and given by (28).

Proof of Theorem 2 (continued). Theorem 2 has been established for radial functions in Section 2. Now we investigate the general case. We shall restrict our purpose to the case of dimension $d = 2$. For any $\alpha \in (-1, 0)$, let us denote by $d\mu_\alpha$ the probability measure on $\mathbb{R}^2$ defined by $d\mu_\alpha \coloneqq \mu_\alpha\,dx$ where

$$\mu_\alpha \coloneqq \frac{1+\alpha}{\pi}\,\frac{|x|^{2\,\alpha}}{(1+|x|^{2\,(1+\alpha)})^2}\,.$$

It has been established in [DET08] that

$$\log\left(\int_{\mathbb{R}^2} e^{\,u}\,d\mu_\alpha\right) - \int_{\mathbb{R}^2} u\,d\mu_\alpha \leqslant \frac{1}{16\,\pi\,(1+\alpha)}\int_{\mathbb{R}^2}|\nabla u|^2\,dx \quad \forall\, u \in \mathcal{D}(\mathbb{R}^2)\,, \tag{26}$$

where $\mathcal{D}(\mathbb{R}^2)$ is the space of smooth functions with compact support. By density with respect to the natural norm defined by each of the inequalities, the result also holds on the corresponding Orlicz space.

We adopt the strategy of [DET08, Section 2.3] to pass to the limit in (24) as $(a, b) \to (0, 0)$ with $b = \frac{\alpha}{\alpha+1}\,a$. Let

$$a_\varepsilon = -\frac{\varepsilon}{1-\varepsilon}\,(\alpha+1)\,,\quad b_\varepsilon = a_\varepsilon + \varepsilon,\quad p_\varepsilon = \frac{2}{\varepsilon}\,,$$

and

$$u_\varepsilon(x) = \left(1+|x|^{2\,(\alpha+1)}\right)^{-\frac{\varepsilon}{1-\varepsilon}}\,,$$

assuming that u_ε is an optimal function for (24), define

$$\kappa_\varepsilon = \int_{\mathbb{R}^2}\left[\frac{u_\varepsilon}{|x|^{a_\varepsilon+\varepsilon}}\right]^{2/\varepsilon}dx = \int_{\mathbb{R}^2}\frac{|x|^{2\,\alpha}}{\left(1+|x|^{2\,(1+\alpha)}\right)^2}\,\frac{u_\varepsilon^2}{|x|^{2a_\varepsilon}}\,dx = \frac{\pi}{\alpha+1}\,\frac{\Gamma\left(\frac{1}{1-\varepsilon}\right)^2}{\Gamma\left(\frac{2}{1-\varepsilon}\right)}\,,$$

$$\lambda_\varepsilon = \int_{\mathbb{R}^2}\left[\frac{|\nabla u_\varepsilon|}{|x|^a}\right]^2 dx = 4\,a_\varepsilon^2\int_{\mathbb{R}^2}\frac{|x|^{2\,(2\,\alpha+1-a_\varepsilon)}}{\left(1+|x|^{2\,(1+\alpha)}\right)^{\frac{2}{1-\varepsilon}}}\,dx = 4\,\pi\,\frac{|a_\varepsilon|}{1-\varepsilon}\,\frac{\Gamma\left(\frac{1}{1-\varepsilon}\right)^2}{\Gamma\left(\frac{2}{1-\varepsilon}\right)}\,.$$

Then $w_\varepsilon = (1 + \frac{1}{2}\,\varepsilon\,u)\,u_\varepsilon$ is such that

$$\lim_{\varepsilon\to 0_+} \frac{1}{\kappa_\varepsilon} \int_{\mathbb{R}^2} \frac{|w_\varepsilon|^{p_\varepsilon}}{|x|^{b_\varepsilon p_\varepsilon}}\,dx = \int_{\mathbb{R}^2} e^u\,d\mu_\alpha\,,$$
$$\lim_{\varepsilon\to 0_+} \frac{1}{\varepsilon}\left[\frac{1}{\lambda_\varepsilon}\int_{\mathbb{R}^2} \frac{|\nabla w_\varepsilon|^2}{|x|^{2a_\varepsilon}}\,dx - 1\right] = \int_{\mathbb{R}^2} u\,d\mu_\alpha + \frac{1}{16\,(1+\alpha)\,\pi}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^2)}\,.$$

Hence we can recover (26) by passing to the limit in (24) as $\varepsilon \to 0_+$. On the other hand, if we pass to the limit in the inequality stated in Theorem 15, we arrive at the following result, for any $\alpha \in (-1, 0)$.

Theorem 16. *Let $\alpha \in (-1, 0]$. With the above notations, we have*

$$0 \leqslant \int_{\mathbb{R}^2} v\,\log\left(\frac{v}{\mu_\alpha}\right)dx - 4\,\pi\,(1+\alpha)\int_{\mathbb{R}^2}(v-\mu_\alpha)\,(-\Delta)^{-1}\,(v-\mu_\alpha)\,dx$$
$$\leqslant \frac{1}{16\,\pi\,(1+\alpha)}\int_{\mathbb{R}^2}|\nabla u|^2\,dx - \log\left(\int_{\mathbb{R}^2} e^{\,u}\,d\mu_\alpha\right) + \int_{\mathbb{R}^2} u\,d\mu_\alpha$$

for any $u \in \mathcal{D}$, where u and v are related by

$$v = \frac{e^u\,\mu_\alpha}{\int_{\mathbb{R}^2} e^u\,d\mu_\alpha}\,.$$

The case $\alpha = 0$ is achieved by taking the limit as $\alpha \to 0_-$. Since $-\Delta \log \mu_\alpha = 8\,\pi\,(1+\alpha)\,\mu_\alpha$ holds for any $\alpha \in (-1, 0]$, the proof of Theorem 2 is now completed, with $\mu = \mu_0$. □

A Some useful formulae

We recall that

$$f(q) := \int_{\mathbb{R}} \frac{dt}{(\cosh t)^q} = \frac{\sqrt{\pi}\,\Gamma(\frac{q}{2})}{\Gamma(\frac{q+1}{2})}$$

for any $q > 0$. An integration by parts shows that $f(q+2) = \frac{q}{q+1}\,f(q)$. The following formulae are reproduced with no change from [Pin+10] (also see [DEL12 ; DE14]). The function $w(t) := (\cosh t)^{-\frac{2}{p-2}}$ solves

$$-(p-2)^2\,w'' + 4\,w - 2\,p\,w^{p-1} = 0$$

and we can define

$$\mathsf{I}_q := \int_{\mathbb{R}} |w(t)|^q\,dt \quad \text{and} \quad \mathsf{J}_2 := \int_{\mathbb{R}} |w'(t)|^2\,dt\,.$$

Using the function f, we can compute $\mathsf{I}_2 = f\big(\frac{4}{p-2}\big)$, $\mathsf{I}_p = f\big(\frac{2\,p}{p-2}\big) = f\big(\frac{4}{p-2}+2\big)$ and get the relations

$$\mathsf{I}_2 = \frac{\sqrt{\pi}\,\Gamma\big(\frac{2}{p-2}\big)}{\Gamma\big(\frac{p+2}{2\,(p-2)}\big)}\,,\quad \mathsf{I}_p = \frac{4\,\mathsf{I}_2}{p+2} = \frac{4\,\sqrt{\pi}\,\Gamma\left(\frac{2}{p-2}\right)}{(p+2)\,\Gamma\left(\frac{p+2}{2\,(p-2)}\right)}\,,\quad \mathsf{J}_2 = \frac{4\,\mathsf{I}_2}{(p+2)\,(p-2)}\,.$$

In particular, this establishes (9), namely

$$\mathsf{s}_d = \frac{\mathsf{I}_p^{1-\frac{2}{d}}}{\mathsf{J}_2 + \frac{1}{4}\,(d-2)^2\,\mathsf{I}_2}\,,\quad \text{with } p = \frac{2\,d}{d-2}$$

for any $d > 2$. The expression of the optimal constant in Sobolev's inequality (1) : $\mathsf{S}_d = \mathsf{s}_d\,|\mathbb{S}^{d-1}|^{-2/d}$, where

$$|\mathbb{S}^{d-1}| = \frac{2\,\pi^{d/2}}{\Gamma(d/2)}$$

denotes the volume of the unit sphere, for any integer $d \geqslant 3$, follows from the duplication formula

$$2^{d-1}\,\Gamma\left(\tfrac{d}{2}\right)\,\Gamma\left(\tfrac{d+1}{2}\right) = \sqrt{\pi}\,\Gamma(d)$$

according for instance to [AS64]. See [DEL13a, Appendix B.4] for further details.

B Poincaré inequality and stereographic projection

On $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, consider the coordinates $\omega = (\rho\,\phi, z) \in \mathbb{R}^d \times \mathbb{R}$ such that $\rho^2 + z^2 = 1$, $z \in [-1, 1]$, $\rho \geqslant 0$ and $\phi \in \mathbb{S}^{d-1}$, and define the *stereographic projection* $\Sigma : \mathbb{S}^d \backslash \{\mathrm{N}\} \to \mathbb{R}^d$ by $\Sigma(\omega) = x = r\,\phi$ and

$$z = \frac{r^2-1}{r^2+1} = 1 - \frac{2}{r^2+1}\,, \quad \rho = \frac{2\,r}{r^2+1}\,.$$

The *North Pole* N corresponds to $z = 1$ (and is formally sent at infinity) while the *equator* (corresponding to $z = 0$) is sent onto the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$. Now we can transform any function v on $\mathbb{S}^d$ into a function u on $\mathbb{R}^d$ using

$$v(\omega) = \left(\frac{r}{\rho}\right)^{\frac{d-2}{2}}\,u(x) = \left(\tfrac{r^2+1}{2}\right)^{\frac{d-2}{2}}\,u(x) = (1-z)^{-\frac{d-2}{2}}\,u(x)\,.$$

A standard computation shows that

$$\int_{\mathbb{S}^d} |\nabla v|^2\,d\omega + \frac{1}{4}\,d\,(d-2)\int_{\mathbb{S}^d} |v|^2\,d\omega = \int_{\mathbb{R}^d} |\nabla u|^2\,dx$$

and

$$\int_{\mathbb{S}^d} |v|^q\,d\omega = \int_{\mathbb{R}^d} |u|^q\,\left(\tfrac{2}{1+|x|^2}\right)^{d-(d-2)\frac{q}{2}}\,dx\,.$$

On $\mathbb{S}^d$, the kernel of the Laplace-Beltrami operator is generated by the constants and the lowest positive eigenvalue is $\lambda_1 = d$. The corresponding eigenspace is generated by $v_0(\omega) = 1$ and $v_i(\omega) = \omega_i$, $i = 1, 2, \ldots d+1$. All eigenvalues of the Laplace-Beltrami operator are given by the formula

$$\lambda_k = k\,(k+d-1) \quad \forall\,k \in \mathbb{N}$$

according to [BGM71]. We still denote by $u_\star$ the Aubin-Talenti extremal function

$$u_\star(x) \coloneqq (1+|x|^2)^{-\frac{d-2}{2}} \quad \forall\,x \in \mathbb{R}^d\,.$$

Using the inverse stereographic projection, the reader is invited to check that Sobolev's inequality is equivalent to the inequality

$$\frac{4}{d\,(d-2)}\int_{\mathbb{S}^d} |\nabla v|^2\,d\omega + \int_{\mathbb{S}^d} |v|^2\,d\omega \geqslant |\mathbb{S}^d|^{\frac{2}{d}}\,\left(\int_{\mathbb{S}^d} |v|^{\frac{2\,d}{d-2}}\,d\omega\right)^{\frac{d-2}{d}}$$

so that the Aubin-Talenti extremal function is transformed into a constant function on the sphere and incidentally this shows that

$$\mathsf{S}_d = \frac{4}{d\,(d-2)}\,|\mathbb{S}^d|^{-\frac{2}{d}}\,.$$

With these preliminaries on the Laplace-Beltrami operator and the stereographic projection in hand, we can now state the counterpart on $\mathbb{R}^d$ of the Poincaré inequality on $\mathbb{S}^d$.

Lemma 17. *For any function $f \in \mathcal{D}^{1,2}(\mathbb{R}^d)$ such that*

$$\int_{\mathbb{R}^d} f\,\frac{u_\star}{(1+|x|^2)^2}\,dx = 0\,,\quad \int_{\mathbb{R}^d} f\,\frac{(1-|x|^2)\,u_\star}{(1+|x|^2)^3}\,dx = 0\,,$$
$$\text{and}\quad \int_{\mathbb{R}^d} f\,\frac{x_i\,u_\star}{(1+|x|^2)^3}\,dx = 0\quad \forall\, i = 1\,,\ 2\,,\ \dots d$$

the following inequality holds

$$\int_{\mathbb{R}^d} |\nabla f|^2\,dx \geqslant (d+2)\,(d+4)\int_{\mathbb{R}^d} \frac{f^2}{(1+|x|^2)^2}\,dx\,.$$

Démonstration. On the sphere we know that

$$\int_{\mathbb{S}^d} |\nabla v|^2\,d\omega + \frac{1}{4}\,d\,(d-2)\int_{\mathbb{S}^d} v^2\,d\omega \geqslant \left(\lambda_2 + \frac{1}{4}\,d\,(d-2)\right)\int_{\mathbb{S}^d} v^2\,d\omega$$
$$= \frac{1}{4}\,(d+2)(d+4)\int_{\mathbb{S}^d} v^2\,d\omega$$

if v is orthogonal to v_i for any $i = 0, 1, \dots d+1$. The conclusion follows from the stereographic projection. □

C Flow on the sphere and consequences

We recall that Equation (18) admits special solutions with *separation of variables* given by

$$v_\star(t,x) = \lambda^{(d+2)/2}\,(T-t)^{\frac{d+2}{4}}\,(u_\star((x-x_0)/\lambda))^{\frac{d+2}{d-2}} \tag{27}$$

where $u_\star(x) := (1+|x|^2)^{-(d-2)/2}$ is the Aubin-Talenti extremal function, $x \in \mathbb{R}^d$ and $0 < t < T$. Such a solution is generic near the extinction time T, in the following sense.

Lemma 18. *[PS01 ; Váz06]. For any solution v of (18) with nonnegative, not identically zero initial datum $v_0 \in \mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, there exists $T > 0$, $\lambda > 0$, $c > 0$ and $x_0 \in \mathbb{R}^d$ such that $v(t,\cdot) \not\equiv 0$ for any $t \in (0,T)$ and*

$$\lim_{t\to T_-} (T-t)^{-\frac{d+2}{4}}\,\sup_{x\in\mathbb{R}^d} (1+|x|^2)^{\frac{d+2}{2}}\left|\frac{v(t,x)}{v_\star(t,x)} - c\right| = 0$$

if $v_\star$ is defined by (27).

If v solves the *fast diffusion* equation (18) on $\mathbb{R}^d$, then we may use the inverse stereographic projection (see B) to define the function w on $\mathbb{S}^d$ such that

$$v(t,x) = e^{-\frac{d+2}{4}\,\tau}\left(\tfrac{2}{1+r^2}\right)^{\frac{d+2}{2}}\,w(\tau,y)$$

where $\tau = -\log(T-t)$, $r = |x|$ and $y = \left(\frac{2\,x}{1+r^2}, \frac{1-r^2}{1+r^2}\right) \in \mathbb{S}^d \subset \mathbb{R}^d \times \mathbb{R}$.

With no loss of generality, assume that $c = \lambda = 1$ and $x_0 = 0$. According to Lemma 18, w uniformly converges as $\tau \to \infty$ to 1 on $\mathbb{S}^d$. Let $d\sigma_d$ denote the measure induced on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ by Lebesgue's measure on $\mathbb{R}^{d+1}$. We may then write

$$\mathsf{J}(t) = e^{-\frac{d}{2}\tau} \int_{\mathbb{S}^d} w^{\frac{2\,d}{d+2}}\, d\sigma_d$$

and

$$\int_{\mathbb{R}^d} |\nabla u^{\frac{d-2}{d+2}}|^2\, dx = e^{-\frac{d-2}{2}\tau} \left(\int_{\mathbb{S}^d} \left|\nabla w^{\frac{d-2}{d+2}}\right|^2 d\sigma_d + \frac{1}{4}\, d\,(d-2) \int_{\mathbb{S}^d} \left|w^{\frac{d-2}{d+2}}\right|^2 d\sigma_d \right)$$

with $\tau = -\log(T-t)$, so that $\frac{d\tau}{dt} = e^{\tau}$. Hence w solves

$$w_\tau = \mathcal{L}\, w^{\frac{d-2}{d+2}} - \frac{1}{4}\, d\,(d-2)\, w^{\frac{d-2}{d+2}} + \frac{1}{4}\,(d+2)\, w$$

where $\mathcal{L}$ denotes the Laplace-Beltrami operator on the sphere $\mathbb{S}^d$, and

$$\frac{d}{dt}\mathsf{J} = -\,\frac{2\,d}{d+2}\, e^{-\frac{d-2}{2}\,\tau} \left(\int_{\mathbb{S}^d} \left|\nabla w^{\frac{d-2}{d+2}}\right|^2 d\sigma_d + \frac{1}{4}\, d\,(d-2) \int_{\mathbb{S}^d} \left|w^{\frac{d-2}{d+2}}\right|^2 d\sigma_d \right),$$

$$\frac{d}{dt} \int_{\mathbb{R}^d} |\nabla u^{\frac{d-2}{d+2}}|^2\, dx = -\,2\,\frac{d-2}{d+2} \int_{\mathbb{S}^d} \left(\mathcal{L}\, w^{\frac{d-2}{d+2}} - \frac{1}{4}\, d\,(d-2)\, w^{\frac{d-2}{d+2}} \right)^2 w^{-\frac{4}{d+2}}\, d\sigma_d\,.$$

Using the Cauchy-Schwarz inequality, that is, by writing that

$$\begin{aligned}
&\left[\int_{\mathbb{S}^d} \left|\nabla w^{\frac{d-2}{d+2}}\right|^2 d\sigma_d + \frac{1}{4}\, d\,(d-2) \int_{\mathbb{S}^d} \left|w^{\frac{d-2}{d+2}}\right|^2 d\sigma_d \right]^2 \\
&\qquad = \left[\int_{\mathbb{S}^d} \left(\mathcal{L}\, w^{\frac{d-2}{d+2}} - \frac{1}{4}\, d\,(d-2)\, w^{\frac{d-2}{d+2}} \right) w^{-\frac{2}{d+2}}\, w^{\frac{d}{d+2}}\, d\sigma_d \right]^2 \\
&\qquad\qquad \leqslant \int_{\mathbb{S}^d} \left(\mathcal{L}\, w^{\frac{d-2}{d+2}} - \frac{1}{4}\, d\,(d-2)\, w^{\frac{d-2}{d+2}} \right)^2 w^{-\frac{4}{d+2}}\, d\sigma_d \int_{\mathbb{S}^d} w^{\frac{2\,d}{d+2}}\, d\sigma_d\,,
\end{aligned}$$

we conclude that

$$\mathsf{Q} = \mathsf{J}^{\frac{2}{d}-1} \int_{\mathbb{S}^d} \left|\nabla w^{\frac{d-2}{d+2}}\right|^2 d\sigma_d$$

is monotone decreasing, and hence

$$\mathsf{H}'' = \frac{\mathsf{J}'}{\mathsf{J}}\,\mathsf{H}' + 2\,\mathsf{J}\,\mathsf{S}_d\,\mathsf{Q}' \leqslant \frac{\mathsf{J}'}{\mathsf{J}}\,\mathsf{H}'\,.$$

This establishes the proof of Lemma 8 for any $d \geqslant 3$.

D Symmetry in Caffarelli-Kohn-Nirenberg inequalities

In this Appendix, we recall some known results concerning *symmetry* and *symmetry breaking* in the Caffarelli-Kohn-Nirenberg inequalities (24).

Proposition 19. *Assume that $d \geqslant 2$. There exists a continuous function $\alpha : (2, 2^*) \to (-\infty, 0)$ such that $\lim_{p \to 2^*} \alpha(p) = 0$ for which the equality case in (24) is not achieved among radial functions if $a < \alpha(p)$ while for $a < \alpha(p)$ equality is achieved by*

$$u_\star(x) := \left(1 + |x|^{\frac{2}{\delta}\,(a_c - a)}\right)^{-\delta} \qquad \forall\, x \in \mathbb{R}^d$$

where $\delta = \frac{a_c + b - a}{1 + a - b}$. Moreover the function α has the following properties

(i) *For any* $p \in (2, 2^*)$, $\alpha(p) \geqslant a_c - 2\sqrt{\frac{d-1}{p^2-4}}$.

(ii) *For any* $p \in \left(2, 2\,\frac{d^2-d+1}{d^2-3\,d+3}\right)$, $\alpha(p) \leqslant a_c - \frac{1}{2}\sqrt{\frac{(d-1)\,(6-p)}{p-2}}$.

(iii) *If* $d = 2$, $\lim_{p\to 2^*} \beta(p)/\alpha(p) = 0$ *where* $\beta(p) := \alpha(p) - a_c + d/p$.

This result summarizes a list of partial results that have been obtained in various papers. Existence of optimal functions has been dealt with in [CW01], while Condition (i) in Proposition 19 has been established in [FS03]. See [Dol+09] for the existence of the curve $p \mapsto \alpha(p)$, [Dol+11 ; DE12] for various results on symmetry in a larger class of inequalities, and [DEL12] for Property (ii) in Proposition 19. Numerical computations of the branches of non-radial optimal functions and formal asymptotic expansions at the bifurcation point have been collected in [DE13 ; DE14]. The paper [DET08] deals with the special case of dimension $d = 2$ and contains Property (iii) in Proposition 19, which can be rephrased as follows : the region of radial symmetry contains the region corresponding to $a \geqslant \alpha(p)$ and $b \geqslant \beta(p)$, and the parametric curve $p \mapsto (\alpha(p), \beta(p))$ converges to 0 as $p \to 2^* = \infty$ tangentially to the axis $b = 0$. For completeness, let us mention that [Bet+99, Theorem 3.1] covers the case $a > a_c - d/p$ also we will not use it. Finally, let us observe that in the *symmetric case*, the expression of $\mathsf{C}_{a,b}$ can be computed explicitly in terms of the Γ function as

$$\mathsf{C}_{a,b} = |\mathbb{S}^{d-1}|^{\frac{p-2}{p}} \left[\frac{(a-a_c)^2\,(p-2)^2}{p+2}\right]^{\frac{p-2}{2\,p}} \left[\frac{p+2}{2\,p\,(a-a_c)^2}\right] \left[\frac{4}{p+2}\right]^{\frac{6-p}{2\,p}} \left[\frac{\Gamma\left(\frac{2}{p-2}+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{2}{p-2}\right)}\right]^{\frac{p-2}{p}} \tag{28}$$

where the volume of the unit sphere is given by $|\mathbb{S}^{d-1}| = 2\,\pi^{\frac{d}{2}}/\Gamma\left(\frac{d}{2}\right)$.

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Chapitre II

Inégalités de Sobolev et Hardy-Littlewood-Sobolev fractionnaires

Ce chapitre reprend en grande partie l'article « Fractional Sobolev and Hardy-Littlewood-Sobolev inequalities », écrit en collaboration avec Van Hoang Nguyen [1].

Résumé

Ce chapitre prolonge le précédent, et s'attache à l'amélioration de Sobolev fractionnaire par un terme de reste impliquant l'inégalité de Hardy-Littlewood-Sobolev. En réutilisant certaines méthodes de preuve, nous offrons une preuve simplifiée d'un résulat de Jin et Xiong et des meilleurs estimations sur la constante optimales. Par un passage à la limite différent, nous dérivons un résultat semblable pour l'inégalité de Moser-Trudinger-Onofri sur la sphère, et, par projection stéréographique, pour l'inégalité d'Onofri euclidienne.

Structure

1 Introduction 34
2 Results 35
3 Upper bound on the best constant via an expansion of the square 39
4 Lower bound via linearization 40
5 Improved Moser-Trudinger-Onofri inequality via endpoint differentiation 47
6 Fractional fast diffusion flow. 50
References. 53

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1 Introduction

The sharp Sobolev inequality and the Hardy-Littlewood-Sobolev inequality are *dual* inequalities. This has been brought to light first by Lieb [Lie83] using the *Legendre transform*. Later, Carlen, Carrillo, and Loss [CCL10] showed that the Hardy-Littlewood-Sobolev inequality can also be related to a particular Gagliardo-Nirenberg interpolation inequality via a fast diffusion equation. Since the sharp Sobolev inequality is in fact an endpoint in a familly of sharp Gargliardo-Nirenberg inequalities [PD02], this eventually led to Dolbeault [Dol11] pointing out that a Yamabe type flow is related with the duality between the sharp Sobolev inequality, and the Hardy-Littlewood-Sobolev inequality. Still relying on that flow, he proved an enhanced Sobolev inequality, with a remainder term involving the Hardy-Littlewood-Sobolev inequality and also provided an estimate on the best multiplicative constant. This was soon extended to the setting of the fractional Laplacian operator by Jin and Xiong [JX11]. This approach heavily relies on the use of the fast diffusion equation, which introduces technical restrictions on the dimension or the exponent of the Laplacian operator. A simpler proof is provided in [DJ14], which lifts some of these restrictions, and provides better estimates on the best constant.

Let us now go into more details. The sharp fractional Sobolev inequality states (see *e.g.* [Swa92 ; CT04 ; Lie83]) that

$$\left(\int_{\mathbb{R}^n} |u(x)|^q dx\right)^{\frac{2}{q}} \leqslant S_{n,s}\|u\|_s^2 \qquad \text{for all } u \in \dot{W}^s(\mathbb{R}^n), \tag{1.1}$$

where $0 < s < \frac{n}{2}$, $q = \frac{2n}{n-2s}$, and the best constant $S_{n,s}$ is given by

$$S_{n,s} = \frac{\Gamma\left(\frac{n-2s}{2}\right)}{2^{2s}\pi^s\,\Gamma\left(\frac{n+2s}{2}\right)}\left(\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)}\right)^{\frac{2s}{n}}. \tag{1.2}$$

Moreover, equality in (1.1) holds if and only if $u(x) = c\,u_*\left(\frac{x-x_0}{t}\right)$ for some $c \in \mathbb{R}$, $t > 0$, $x_0 \in \mathbb{R}^n$ and where

$$u_*(x) = (1+|x|^2)^{-(\frac{n}{2}-s)}$$

is an Aubin-Talenti type extremal function.

The best constant $S_{n,s}$ has been computed first in the special cases $s = 1$ and $n = 3$ by Rosen [Ros71], and later for $s = 1$ and $n \geqslant 3$ by Aubin [Aub76] and Talenti [Tal76] independently. For general $0 < s < \frac{n}{2}$, this best constant has been given by Lieb [Lie83] by computing the sharp constant in the sharp Hardy-Littlewood-Sobolev inequality,

$$\left|\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{f(x)f(y)}{|x-y|^\lambda}dxdy\right| \leqslant \pi^{\frac{\lambda}{2}}\frac{\Gamma\left(\frac{n-\lambda}{2}\right)}{\Gamma\left(n-\frac{\lambda}{2}\right)}\left(\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)}\right)^{1-\frac{\lambda}{n}}\|f\|^2_{L^p(\mathbb{R}^n)}, \tag{1.3}$$

where $0 < \lambda < n$ and $p = \frac{2n}{2n-\lambda}$. There is equality in (1.3) if and only if $f(x) = c\,H_\lambda(\frac{x-x_0}{t})$ where

$$H_\lambda(x) = (1+|x|^2)^{-(n-\frac{\lambda}{2})},$$

with $c \in \mathbb{R}$, $t > 0$, and $x_0 \in \mathbb{R}^n$. For $0 < s < \frac{n}{2}$, $2^{-2s}\pi^{-\frac{n}{2}}\frac{\Gamma((n-2s)/2)}{\Gamma(s)}\frac{1}{|x|^{n-2s}}$ is the Green's function of $(-\Delta)^s$, so that the inequality (1.3) can be rewritten in the following equivalent

form, by taking $\lambda = n - 2s$

$$\left| \int_{\mathbb{R}^n} f\,(-\Delta)^{-s}(f)\,dx \right| \leqslant S_{n,s} \|f\|^2_{L^p(\mathbb{R}^n)}\,. \tag{1.4}$$

The sharp Hardy-Littlewood-Sobolev inequality was first proved by Lieb based on a rearrangement argument (see [Lie83]). Recently, Frank and Lieb (see [FL12a]) have given a new and rearrangement-free proof of this inequality. Their method was also used to prove the sharp Hardy-Littlewood-Sobolev inequality in the Heisenberg group (see [FL12b]). See also [CCL10 ; FL10] for the other rearrangement-free proofs for some special cases of the sharp Hardy-Littlewood-Sobolev inequality.

Using duality, Jin and Xiong state in [JX11, Theorem 1.4] that when $0 < s < 1$, $n \geqslant 2$, and $n > 4\,s$, there exists a constant $C_{n,s}$ such that the following inequality

$$S_{n,s}\|u^r\|^2_{L^{\frac{2n}{n+2s}}(\mathbb{R}^n)} - \int_{\mathbb{R}^n} u^r\,(-\Delta)^{-s}u^r\,dx \\ \leqslant C_{n,s}\|u\|^{\frac{8s}{n-2s}}_{L^{\frac{2d}{d-2s}}(\mathbb{R}^n)} \left[S_{n,s}\|u\|^2_s - \|u\|^2_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} \right], \tag{1.5}$$

holds for any positive $u \in \dot{W}^s(\mathbb{R}^n)$, where $r = \frac{n+2s}{n-2s}$. Moreover, the best value $C^*_{n,s}$ for the constant $C_{n,s}$ is such that $C^*_{n,s} \leqslant \frac{n+2s}{n}\left(1 - e^{-\frac{n}{2s}}\right) S_{n,s}$. This adapts to the fractional setting the original result of Dolbeault [Dol11, Theorem 1.2] which was restricted to the case $s = 1$.

In (1.5), the left-hand side is positive by the Hardy-Littlewood-Sobolev inequality (1.4), and the right-hand side is positive by Sobolev inequality (1.1), so this is an improvement of the Sobolev inequality.

The strong condition on the dimension required for (1.5) stems from the heavy reliance on a fast diffusion flow to achieve these results. Although the constraint on n can be removed by lifting the flow to the sphere, Dolbeault and Jankowiak propose in [DJ14] a new, simpler proof that brings a number of benefits in the case $s = 1$: the role of duality is made more explicit, and it holds for any $n \geqslant 3$.

The aim of this paper is to extend and unify these results in the fractional setting. We provide a better estimate on the best constant and by taking limits in s, we also derive an improved Moser-Trudinger-Onofri inequality, and recover the Onofri inequality for $n = 2$. Our paper is organized as follows : in Section 2 we detail our results, both in the Sobolev (Theorem 1) and Moser-Trudinger-Onofri (Theorem 2) settings. Sections 3 and 4 are dedicated to the proof of our main theorem using a completion of the square and linearization techniques, respectively. Next we provide a proof of Theorem 2 in Section 5, by taking the limit $s \to \frac{n}{2}$. Finally, in Section 6, we complete the proof of Theorem 1 using a fractional nonlinear diffusion flow.

2 Results

Let us first introduce notation. First recall the definition of the homogeneous Sobolev space $\dot{W}^s(\mathbb{R}^n)$ with $s \in \mathbb{R}$. A Borel function $u\colon \mathbb{R}^n \to \mathbb{R}$ is said to vanish at the infinity if the Lebesgue measure of $\{x \in \mathbb{R}^n \,:\, |u(x)| > t\}$ is finite for all $t > 0$. For $s \in \mathbb{R}$, we define the fractional Laplace operator $(-\Delta)^s u$ by the distributional function whose Fourier transform

is $|\xi|^{2s}\hat{u}(\xi)$, where $\hat{u}$ is the Fourier transform of u. For a test function u in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, $\hat{u}$ is defined as $\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} u(x)dx$. From the Plancherel-Parseval identity, we have $\|u\|_{L^2(\mathbb{R}^n)} = (2\pi)^{-\frac{n}{2}}\|\hat{u}\|_{L^2(\mathbb{R}^n)}$. We know that the Fourier transform is extended to a bijection from the space of the tempered distributions to itself. Then $\dot{W}^s(\mathbb{R}^n)$ is defined to be the space of all tempered distributions u which vanishes at the infinity and $(-\Delta)^{\frac{s}{2}}u \in L^2(\mathbb{R}^n)$. For $u \in \dot{W}^s(\mathbb{R}^n)$, we define

$$\|u\|_s^2 := \|(-\Delta)^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^n)}^2 = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |\xi|^{2s}|\hat{u}(\xi)|^2\, d\xi = \int_{\mathbb{R}^n} u(x)\,(-\Delta)^s u(x)dx\,.$$

With these notations, our main result is the following

Theorem 1. *Let $n \geqslant 2$, $0 < s < \frac{n}{2}$, and denote $r = \frac{n+2s}{n-2s}$*

(i) There exists a positive constant $C_{n,s}$ for which the following inequality

$$S_{n,s}\|u^r\|^2_{L^{\frac{2n}{n+2s}}(\mathbb{R}^n)} - \int_{\mathbb{R}^n} u^r\,(-\Delta)^{-s}u^r dx \leqslant C_{n,s}\|u\|^{\frac{8s}{n-2s}}_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)}\left(S_{n,s}\|u\|_s^2 - \|u\|^2_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)}\right) \tag{2.1}$$

holds for any positive $u \in \dot{W}^s(\mathbb{R}^n)$.

(ii) Let $C^*_{n,s}$ be the best constant in (2.1). It is such that

$$\frac{n-2s+2}{n+2s+2}\,S_{n,s} \leqslant C^*_{n,s} \leqslant S_{n,s}\,. \tag{2.2}$$

Additionally, in the case $0 < s < 1$ we know that :

$$C^*_{n,s} < S_{n,s}\,. \tag{2.3}$$

Theorem 1 contains both the result of Dolbeault and Jankowiak [DJ14, Theorem 1] in the case $n \geqslant 3$ and $s = 1$ and the one of Jin and Xiong [JX11, Theorem 4.1] in the case $s \in (0,1)$, $n \geqslant 2$ and $n > 4s$ for positive u. The proof of Jin and Xiong is based on a fractional fast diffusion flow and some estimates on the extinction profiles. They also provide the upper bound $C^*_{n,s} \leqslant \frac{n+2s}{n}(1-e^{-\frac{n}{2s}})$, a bound which is larger that 1 when $n > 4s$, so that Theorem 1 not only extends the result of Jin and Xiong to all $n \geqslant 2$ and $s \in (0, \frac{n}{2})$, but also improve the constant $C_{n,s}$ on the right-hand side of (2.1).

Before continuing, we introduce the logarithmic derivative of the Euler Gamma function $\Psi(a) = (\log\Gamma(a))'$ for $a > 0$, and also define $\mathcal{H}_k$, the space spanned by k-homogeneous harmonic polynomials on $\mathbb{R}^{n+1}$ restricted to S^n. In the following, $d\sigma$ denotes the normalized surface area measure on S^n induced by the Lebesgue measure on $\mathbb{R}^{n+1}$.

In the spirit of [Bec93 ; CL92], we consider the limit $s \to \frac{n}{2}$ and obtain an inequality between the functionals associated with the *Moser-Trudinger-Onofri* and the *logarithmic* Hardy-Littlewood-Sobolev inequalities. Details will be given below, but let us first state our result.

Theorem 2. *There exists a positive constant C_n such that for any real-valued function F defined on S^n with an expansion on spherical harmonics $F = \sum_{k\geqslant 0} F_k$ where $F_k \in \mathcal{H}_k$, then the following*

inequality holds :

$$C_n\left(\int_{S^n} e^F d\sigma\right)^2\left[\frac{1}{2n}\sum_{k\geqslant 1}\frac{\Gamma(k+n)}{\Gamma(n)\Gamma(k)}\int_{S^n}|F_k|^2 d\sigma+\int_{S^n} F d\sigma-\log\left(\int_{S^n} e^F d\sigma\right)\right]$$
$$\geqslant n\iint\limits_{S^n\times S^n} e^{F(\xi)}\,\log|\xi-\eta|\,e^{F(\eta)} d\sigma(\xi)d\sigma(\eta)$$
$$+\left(\int_{S^n} e^F d\sigma\right)^2\left[\frac{n}{2}\left(\Psi(n)-\Psi\left(\frac{n}{2}\right)-\log 4\right)+\frac{\mathrm{Ent}_\sigma(e^F)}{\int_{S^n} e^F d\sigma}\right], \qquad (2.4)$$

where $\mathrm{Ent}_\sigma(f)=\int_{S^n} f\log f\ d\sigma-(\int_{S^n} f\ d\sigma)\log(\int_{S^n} f\ d\sigma)$. *Moreover, if* C_n^* *denotes the best constant for which the above inequality holds, then*

$$\frac{1}{n+1}\leqslant C_n^*\leqslant 1. \qquad (2.5)$$

The inequality (2.5) is proved in the same way as item *(ii)* in Theorem 1. We will expand both sides of the inequality (2.4) around the function $F\equiv 0$ which is an optimal function for the Moser-Trudinger-Onofri inequality (2.9).

A direct consequence of Theorem 2 written for $n=2$ is an improved version of the Euclidean Onofri inequality with a remainder term involving the two dimensional logarithmic Hardy-Littlewood-Sobolev inequality. We will use the following notation

$$d\mu(x)=\mu(x)dx,\qquad \mu(x)=\frac{1}{\pi\,(1+|x|^2)^2},\qquad x\in\mathbb{R}^2.$$

Corollary 3. *There exists a positive constant* C_2 *such that for any* $f\in L^1(\mu)$ *and* $\nabla f\in L^2(\mathbb{R}^2)$, *the following inequality holds :*

$$C_2\left(\int_{\mathbb{R}^2} e^f d\mu\right)^2\left[\frac{1}{16\pi}\|\nabla f\|^2_{L^2(\mathbb{R}^2}+\int_{\mathbb{R}^2} f d\mu-\log\left(\int_{\mathbb{R}^2} e^f d\mu\right)\right]$$
$$\geqslant\left(\int_{\mathbb{R}^2} e^f d\mu\right)^2\left(1+\log\pi+\int_{\mathbb{R}^2}\frac{e^f\mu}{\int_{\mathbb{R}^2} e^f d\mu}\log\left(\frac{e^f\mu}{\int_{\mathbb{R}^2} e^f d\mu}\right)dx\right)$$
$$-4\pi\int_{\mathbb{R}^2} e^{f(x)}\mu(x)\,(-\Delta)^{-1}(e^f\mu)(x)\,dx. \qquad (2.6)$$

Moreover, if C_2^* *denotes the best constant for which the inequality* (2.6) *holds, then*

$$\frac{1}{3}\leqslant C_2^*\leqslant 1.$$

As above, the right-hand side of (2.6) is nonnegative by the logarithmic Hardy-Littlewood-Sobolev inequality since Green's function of $-\Delta$ in $\mathbb{R}^2$ is given by $-\frac{1}{2\pi}\log(|x|)$. The inequality (2.6) is a straightforward consequence of (2.4) since $\Psi(2)-\Psi(1)=1$, and the fact that if $f(x)=F(\mathcal{S}(x))$ with $\mathcal{S}$ is the stereographic projection from $\mathbb{R}^2$ to S^2, then

$$\int_{\mathbb{R}^2}|\nabla f(x)|^2dx=4\pi\int_{S^2}|\nabla F|^2 d\sigma.$$

Another proof of Corollary 3 is provided in Theorem 2 of [DJ14] by using a completely different method. More precisely, Dolbeault and Jankowiak use the square method to obtain

an improved version of the Caffarelli-Kohn-Nirenberg inequalities on the weighted spaces, and then take a limit to get (2.6).

The proof of (2.1) is similar to the one of Dolbeault and Jankowiak [DJ14] which is based on the duality between the Sobolev and Hardy-Littlewood-Sobolev inequalities, in fact a simple expansion of a square integral functional. The first inequality in (2.2) is proved by expanding both sides of (2.1) around the function $(1+|x|^2)^{-\frac{n-2s}{2}}$ which is an extremal function for the fractional Sobolev inequality, and thus is a zero of both the left-hand side and right-hand side. To solve the linearized problem, we recast it to the unit sphere S^n using the stereographic projection, and then identify the minimizers using the Funk-Hecke theorem (see [Erd+81, Sec. 11.4]). The Funk-Hecke theorem gives a decomposition of $L^2(S^n)$ into the orthogonal summation of the spaces $\mathcal{H}_l$'s, that is

$$L^2(S^n) = \bigoplus_{l=0}^{\infty} \mathcal{H}_l, \tag{2.7}$$

Moreover, the integral operators on S^n whose kernels have the form $K(\langle \omega, \eta \rangle)$ are diagonal with respect to this decomposition and their eigenvalues can be computed explicitly by using the Gegenbauer polynomials (see [AS64, Chapter 22]).

By using stereographic projection, we can lift the sharp Hardy-Littlewood-Sobolev inequality (1.3) to the conformally equivalent setting of the sphere S^n as follows

$$\left| \iint_{S^n \times S^n} \frac{F(\xi)F(\eta)}{|\xi-\eta|^{\lambda}} d\sigma(\xi) d\sigma(\eta) \right| \leqslant B_\lambda \left(\int_{S^n} |F(\xi)|^p d\sigma(\xi) \right)^{\frac{2}{p}}, \tag{2.8}$$

with

$$B_\lambda = 2^{-\lambda} \frac{\Gamma\left(\frac{n-\lambda}{2}\right)}{\Gamma\left(n-\frac{\lambda}{2}\right)} \frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)}, \quad p = \frac{2n}{2n-\lambda},$$

and $d\sigma$ is the normalized surface area measure on S^n. Note that the distance $|\cdot|$ is the distance in $\mathbb{R}^{n+1}$, not the geodesic distance on S^n. Some geometric and probabilistic informations can be obtained from this inequality through endpoint differentiation arguments (see [Bec93]). Carlen and Loss, but also Beckner considered the limit case of (2.8) when $\lambda = 0$ while studying the two dimensional limit of the Sobolev interpolation inequality on the sphere, pioneered by Bidaut-Véron and Véron in [BVV91, Corollary 6.2]. In this limit, they proved the following Moser-Trudinger-Onofri inequality. For any real valued function F defined on S^n with an expansion $F = \sum_{k\geqslant 0} F_k$, where $F_k \in \mathcal{H}_k$, the following holds

$$\log\left(\int_{S^n} e^{F(\xi)} d\sigma(\xi) \right) \leqslant \int_{S^n} F(\xi) d\sigma(\xi) + \frac{1}{2n} \sum_{k\geqslant 1} \frac{\Gamma(n+k)}{\Gamma(n)\Gamma(k)} \int_{S^n} |Y_k(\xi)|^2 d\sigma(\xi). \tag{2.9}$$

Moreover, equality holds in (2.9) if and only if

$$F(\xi) = -n \log|1 - \langle \xi, \zeta \rangle| + C,$$

for some $|\zeta| < 1$ and $C \in \mathbb{R}$.

When $n = 2$, the inequality (2.9) becomes the classical Onofri inequality on S^2 (see [Mos70 ; Ono82]). Under the stereographic projection, this inequality is equivalent to the

following inequality

$$\log\left(\int_{\mathbb{R}^2} e^{g(x)}d\mu(x)\right) - \int_{\mathbb{R}^2} g(x)d\mu(x) \leqslant \frac{1}{16\pi}\int_{\mathbb{R}^2} |\nabla g(x)|^2 dx \tag{2.10}$$

for any $g \in L^1(\mu)$ and $\nabla g \in L^2(\mathbb{R}^2)$.

The Onofri inequality (2.10) plays the role of Sobolev inequality in two dimensions, see for example [DEJ14] for a thorough review and justification of this statement. This inequality has several extensions, for instance to higher dimensions, which are out of the scope of this paper.

Just like the dual of the fractional Sobolev inequality is the Hardy-Littlewood-Sobolev inequality, the Legendre dual of (2.9) is the logarithmic Hardy-Littlewood-Sobolev inequality, first written in [CL92] and [Bec93]. It states that for nonnegative function F such that $\int_{S^n} F d\sigma = 1$,

$$\begin{aligned} -n \iint_{S^n\times S^n} F(\xi)\log|\xi-\eta|\,F(\eta)d\sigma(\xi)d\sigma(\eta) \\ \leqslant \frac{n}{2}\left(\Psi(n) - \Psi\left(\frac{n}{2}\right) - \log 4\right) + \int_{S^n} F\log F\, d\sigma\,, \end{aligned} \tag{2.11}$$

where we recall $\Psi(a) = (\log\Gamma(a))'$. We remark that the appearance of the logarithmic kernel $-2\log|\xi-\eta|$ is quite natural since it is Green's function on S^2. We can rewrite inequality (2.11) in two dimensions and on the Euclidean space, and get that for any nonnegative function $f \in L^1(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} f(x)dx = 1$, with $f\log f$ and $(1+\log|x|^2)f$ in $L^1(\mathbb{R}^2)$, we have

$$\int_{\mathbb{R}^2} f\log f dx + 2\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\log|x-y|\,f(y)\,dx\,dy + (1+\log\pi) \geqslant 0. \tag{2.12}$$

This more common version of the logarithmic Hardy-Littlewood-Sobolev inequality is the Legendre dual of the Onofri inequality (2.10). It has already seen a number of applications, *e.g.* in chemotaxis models [CC08].

In this paper, we take a step towards unification of the results of [Dol11 ; DJ14 ; JX11]. However, a number of questions remain unanswered. The restriction $0 < s < 1$ in (2.3) comes from the representation of the fractional Laplace operator, is this purely technical ? To extend this part of the result to Theorem 2, it would make sens to consider a fractional logarithmic diffusion flow. However, this raises difficulties which are already presented in [Dol11, Proposition 3.4], so we cannot exclude the case $C_n^* = 1$ yet. Finally, the computation of the exact value of $C_{n,s}^*$ is still open and probably requires new tools.

3 Upper bound on the best constant via an expansion of the square

In this section, we give a proof of Theorem 1 by the completion of the square method.

Proof of Theorem 1. By a density argument, it suffices to prove the inequality (2.1) for any positive smooth function u which belongs to Schwartz space on $\mathbb{R}^n$. For such functions, integration by parts gives us

$$\int_{\mathbb{R}^n} |\nabla(-\Delta)^{-\frac{1+s}{2}}v|^2\,dx = \int_{\mathbb{R}^n} v(-\Delta)^{-s}v\,dx,$$

and, if $v = u^r$ with $r = \frac{n+2s}{n-2s}$,

$$\int_{\mathbb{R}^n} \nabla(-\Delta)^{\frac{s-1}{2}} u \, \nabla(-\Delta)^{-\frac{1+s}{2}} v \, dx = \int_{\mathbb{R}^n} u(x)v(x) \, dx = \int_{\mathbb{R}^n} u(x)^q \, dx,$$

where $q = \frac{2n}{n-2s}$. Using these equalities, we have

$$\begin{aligned} 0 &\leqslant \int_{\mathbb{R}^n} \left| S_{n,s} \|u\|_{L^q(\mathbb{R}^n)}^{\frac{4s}{n-2s}} \nabla(-\Delta)^{\frac{s-1}{s}} u - \nabla(-\Delta)^{-\frac{1+s}{2}} v \right|^2 dx \\ &= S_{n,s}^2 \|u\|_{L^q(\mathbb{R}^n)}^{\frac{8s}{n-2s}} \|u\|_s^2 - 2S_{n,s} \|u\|_{L^q(\mathbb{R}^n)}^{\frac{4s}{n-2s}} \int_{\mathbb{R}^n} u(x)^q \, dx + \int_{\mathbb{R}^n} u^r (-\Delta)^{-s} u^r \, dx. \end{aligned} \tag{3.1}$$

Further, since $q = pr$, we have $\|u\|_{L^q(\mathbb{R}^n)}^q = \|u\|_{L^{pr}(\mathbb{R}^n)}^q = \|u^r\|_{L^p(\mathbb{R}^n)}^p$. This shows that

$$\|u\|_{L^q(\mathbb{R}^n)}^{\frac{4s}{n-2s}} \int_{\mathbb{R}^n} u(x)^q \, dx = \|u^r\|_{L^p(\mathbb{R}^n)}^{p\frac{q-2}{q}} \|u^r\|_{L^p(\mathbb{R}^n)}^p = \|u^r\|_{L^p(\mathbb{R}^n)}^2.$$

Since the left hand side of (3.1) is nonnegative, it implies

$$S_{n,s} \|u^r\|_{L^p(\mathbb{R}^n)}^2 - \int_{\mathbb{R}^n} u^r (-\Delta)^{-s} u^r \, dx \leqslant S_{n,s} \|u\|_{L^q(\mathbb{R}^n)}^{\frac{8s}{n-2s}} \left(S_{n,s} \|u\|_s^2 - \|u\|_{L^q(\mathbb{R}^n)}^2 \right).$$

This is exactly (2.1) with $C_{n,s} = S_{n,s}$. □

4 Lower bound via linearization

Let us start this section by briefly recalling some facts about the stereographic projection from the Euclidean space $\mathbb{R}^n$ to the unit sphere S^n. Denote $N = (0, \cdots, 0, 1) \in \mathbb{R}^{n+1}$ the *north pole* of S^n and consider the map $\mathcal{S} \colon \mathbb{R}^n \mapsto S^n \backslash \{N\}$ defined by

$$\mathcal{S}(x) = \left(\frac{2x}{1+|x|^2}, \frac{|x|^2 - 1}{1+|x|^2} \right),$$

the Jacobian of $\mathcal{S}$ is then given by

$$J_{\mathcal{S}}(x) = \left(\frac{2}{1+|x|^2} \right)^n.$$

If F is an integrable function on S^n then $F(\mathcal{S}(x)) J_{\mathcal{S}}(x) \in L^1(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} F(\mathcal{S}(x)) J_{\mathcal{S}}(x) \, dx = \int_{S^n} F(\omega) d\omega,$$

where $d\omega$ is the unnormalized surface area measure on S^n induced by the Lebesgue measure on $\mathbb{R}^n$. The inverse of $\mathcal{S}$ is given by $\mathcal{S}^{-1}(\omega) = \left(\frac{\omega_1}{1-\omega_{n+1}}, \cdots, \frac{\omega_n}{1-\omega_{n+1}} \right)$ with Jacobian $J_{\mathcal{S}^{-1}}(\omega) = (1-\omega_{n+1})^{-n}$, where $\omega = (\omega_1, \omega_2, \cdots, \omega_{n+1}) \in S^n \backslash \{N\}$. Given $f \in \dot{W}^s(\mathbb{R}^n)$ and $q = \frac{2n}{n-2s}$, we define the new function F on S^n by

$$F(\omega) = f(\mathcal{S}^{-1}(\omega)) J_{\mathcal{S}^{-1}}(\omega)^{\frac{1}{q}}. \tag{4.1}$$

Then we have

$$\int_{\mathbb{R}^n} \frac{f(x)^2}{(1+|x|^2)^{2s}} \, dx = 2^{-2s} \int_{S^n} F(\omega)^2 \, d\omega, \tag{4.2}$$

and

$$\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{f(x)^2}{(1+|x|^2)^{2s}}\,|x-y|^{-n+2s}\,\frac{f(y)^2}{(1+|y|^2)^{2s}}\,dx\,dy \\ = 2^{-4s}\iint\limits_{S^n\times S^n} F(\omega)\,|\omega-\eta|^{-n+2s}\,F(\eta)\,d\omega\,d\eta. \quad (4.3)$$

Equality (4.3) is derived from the fact that

$$|\mathcal{S}(x)-\mathcal{S}(y)|^2 = \frac{2}{1+|x|^2}\,|x-y|^2\,\frac{2}{1+|y|^2}.$$

Next, we prove inequality (2.2). For this purpose, let us denote $\mathcal{F}$ and $\mathcal{G}$ the positive functionals associated with the Sobolev and Hardy-Littlewood-Sobolev inequalities, respectively :

$$\mathcal{F}[u] = S_{n,s}\|u\|_s^2 - \|u\|^2_{L^q(\mathbb{R}^n)}, \qquad u\in\dot{W}^s(\mathbb{R}^n)\,,$$
$$\mathcal{G}[v] = S_{n,s}\|v\|^2_{L^p(\mathbb{R}^n)} - \int_{\mathbb{R}^n} v(-\Delta)^{-s}v\,dx, \qquad v\in L^p(\mathbb{R}^n)\,,$$

and recall that $\mathcal{F}[u_*]=0$ and $\mathcal{G}[u_*^r]=0$. The inequality of Theorem 1 thus reads

$$C_{n,s}\,\|u\|^{\frac{8s}{n-2s}}_{L^q(\mathbb{R}^n)}\mathcal{F}[u] \geqslant \mathcal{G}[u^r]\,,$$

and we are interested in a lower bound for

$$C^*_{n,s} = \sup_{u\in\dot{W}^s}\frac{\mathcal{G}[u^r]}{\|u\|^{\frac{8s}{n-2s}}_{L^q(\mathbb{R}^n)}\mathcal{F}[u]}\,.$$

Consider now $u=u_*+\epsilon f$ where f is smooth and compactly supported such that

$$\int_{\mathbb{R}^n}\frac{u_*(x)\,f(x)}{(1+|x|^2)^{2s}}\,dx = 0\,. \quad (4.4)$$

By using the fact that u_* is a critical point of $\mathcal{F}$ and as such solves

$$(-\Delta)^s u_*(x) = \frac{2^{2s}\Gamma\left(\frac{n+2s}{2}\right)}{\Gamma\left(\frac{n-2s}{2}\right)}\,u_*(x)^r = \frac{2^{2s}\Gamma\left(\frac{n+2s}{2}\right)}{\Gamma\left(\frac{n-2s}{2}\right)}\,\frac{u_*(x)}{(1+|x|^2)^{2s}}\,, \quad (4.5)$$

we in fact have the following.

Proposition 4. *With the above notation and f satisfying (4.4),*

$$\frac{\mathcal{F}[u_\epsilon]}{S_{n,s}} = \epsilon^2\left(\|f\|_s^2 - \frac{2^{2s}\Gamma\left(\frac{n+2s+2}{2}\right)}{\Gamma\left(\frac{n-2s+2}{2}\right)}\int_{\mathbb{R}^n}\frac{f(x)^2}{(1+|x|^2)^{2s}}\,dx\right) + o(\epsilon^2). \quad (4.6)$$

Démonstration. By a direct computation, we have

$$\frac{d}{d\epsilon}\left(\mathcal{F}[u_\epsilon]\right)_{\epsilon=0} = 2S_{n,s}\int_{\mathbb{R}^n} f(-\Delta)^s u_*\,dx - 2\left(\int_{\mathbb{R}^n} u_*^q\,dx\right)^{\frac{2}{q}-1}\int_{\mathbb{R}^n} u_*^{q-1}f\,dx = 0,$$

here, we use the fact that $(-\Delta)^s u_*$ and u_*^{q-1} are proportional to $u_*(x)(1+|x|^2)^{-2s}$. Taking the second derivative of $\mathcal{F}[u_\epsilon]$ at $\epsilon = 0$, we obtain

$$\begin{aligned}\frac{d^2}{d\epsilon^2}\left(\mathcal{F}[u_\epsilon]\right)_{\epsilon=0} &= 2S_{n,s}\|f\|_s^2 - 2(q-1)\left(\int_{\mathbb{R}^n} u_*^q\,dx\right)^{\frac{2}{q}-1}\int_{\mathbb{R}^n} u_*^{q-2}f^2\,dx \\ &= 2S_{n,s}\left(\|f\|_s^2 - \frac{2^{2s}\Gamma\left(\frac{n+2s+2}{2}\right)}{\Gamma\left(\frac{n-2s+2}{2}\right)}\int_{\mathbb{R}^n}\frac{f(x)^2}{(1+|x|^2)^{2s}}\,dx\right).\end{aligned}$$

Since $\mathcal{F}[u_*] = 0$, using Taylor's expansion, we get (4.6). □

Let us denote

$$\mathrm{F}[f] = \|f\|_s^2 - \frac{2^{2s}\Gamma\left(\frac{n+2s+2}{2}\right)}{\Gamma\left(\frac{n-2s+2}{2}\right)}\int_{\mathbb{R}^n}\frac{f(x)^2}{(1+|x|^2)^{2s}}\,dx.$$

Now, we introduce the new functions

$$f_0(x) = u_*(x), \quad f_i(x) = \frac{2x_i}{1+|x|^2}u_*(x),\ i=1,\cdots,n, \quad f_{n+1}(x) = \frac{|x|^2-1}{1+|x|^2}u_*(x).$$

We remark that

$$f_i(x) = -\frac{2}{n-2s}\,\partial_{x_i}u_*(x) \qquad i=1,\cdots,n,$$

and

$$f_{n+1}(x) = -\frac{2}{n-2s}\,\partial_\lambda\left(\lambda^{-(s-\frac{n}{2})}u_*(\lambda x)\right)_{\lambda=1}.$$

Using these relations and (4.5), we get

Lemma 5. *The following assertions hold :*

$$(-\Delta)^s f_0(x) = \frac{2^{2s}\Gamma\left(\frac{n+2s}{2}\right)}{\Gamma\left(\frac{n-2s}{2}\right)}\frac{f_0(x)}{(1+|x|^2)^{2s}}, \tag{4.7}$$

$$(-\Delta)^s f_i(x) = \frac{2^{2s}\Gamma\left(\frac{n+2s+2}{2}\right)}{\Gamma\left(\frac{n-2s+2}{2}\right)}\frac{f_i(x)}{(1+|x|^2)^{2s}}, \quad i=1,\cdots,n+1. \tag{4.8}$$

We also notice that

$$\int_{\mathbb{R}^n}\frac{f_i(x)f_j(x)}{(1+|x|^2)^{2s}}\,dx = 0, \quad i,j=0,1,\cdots,n+1, \quad i\neq j.$$

Next, we consider the other functional $\mathcal{G}$ associated with the Hardy-Littlewood-Sobolev inequality as defined above.

Proposition 6. *With the above notation and f satisfying (4.4), we have*

$$\mathcal{G}[(u_*+\epsilon f)^r] = \epsilon^2\left(\frac{n+2s}{n-2s}\right)^2\mathrm{G}[f] + o(\epsilon^2), \tag{4.9}$$

where

$$\mathrm{G}[f] = \frac{\Gamma(\frac{n-2s+2}{2})}{2^{2s}\Gamma(\frac{n+2s+2}{2})}\int_{\mathbb{R}^n}\frac{f(x)^2}{(1+|x|^2)^{2s}}\,dx - \int_{\mathbb{R}^n}\frac{f(x)}{(1+|x|^2)^{2s}}(-\Delta)^{-s}\left(\frac{f(x)}{(1+|x|^2)^{2s}}\right)dx.$$

Démonstration. First, u_*^r solves the following integral equation which is the Euler-Lagrange equation associated with $\mathcal{G}$:

$$(-\Delta)^{-s} u_*^r = \frac{\Gamma\left(\frac{n-2s}{2}\right)}{2^{2s}\Gamma\left(\frac{n+2s}{2}\right)} u_*. \tag{4.10}$$

Then

$$\frac{d}{d\epsilon}\left(\mathcal{G}[(u_* + \epsilon f)^r]\right)_{\epsilon=0} = \frac{2S_{n,s}q}{p}\left(\int_{\mathbb{R}^n} u_*^q \, dx\right)^{\frac{2}{p}-1} \int_{\mathbb{R}^n} u_*^{q-1} f \, dx \\ - 2r \int_{\mathbb{R}^n} u_*^{r-1} f (-\Delta)^{-s} u_*^r \, dx = 0,$$

since u_*^{q-1} and $u_*^{r-1}(-\Delta)^{-s}u_*^r$ are proportional to $u_*(x)(1+|x|^2)^{-2s}$. By taking the second derivative, we get

$$\begin{aligned}\frac{d^2}{d\epsilon^2}\left(\mathcal{G}[(u_* + \epsilon f)^r]\right)_{\epsilon=0} &= \frac{2S_{n,s}q(q-1)}{p}\left(\int_{\mathbb{R}^n} u_*^q \, dx\right)^{\frac{2}{p}-1} \int_{\mathbb{R}^n} u_*^{q-2} f^2 \, dx \\ &\quad - 2r(r-1)\int_{\mathbb{R}^n} u_*^{r-2} f^2 (-\Delta)^{-s} u_*^r \, dx \\ &\quad - 2r^2 \int_{\mathbb{R}^n} u_*^{r-1} f (-\Delta)^{-s}(u_*^{r-1} f) \, dx \\ &= 2r^2 \Bigg[\frac{\Gamma\left(\frac{n-2s+2}{2}\right)}{2^{2s}\Gamma\left(\frac{n+2s+2}{2}\right)} \int_{\mathbb{R}^n} \frac{f(x)^2}{(1+|x|^2)^{2s}} \, dx \\ &\quad - \int_{\mathbb{R}^n} \frac{f(x)}{(1+|x|^2)^{2s}} (-\Delta)^{-s}\left(\frac{f(x)}{(1+|x|^2)^{2s}}\right) dx \Bigg].\end{aligned}$$

This concludes the proof. □

Next, by Legendre duality, we have

Lemma 7. *Suppose that g satisfies the following conditions :*

$$\int_{\mathbb{R}^n} \frac{g(x) f_i(x)}{(1+|x|^2)^{2s}} \, dx = 0, \quad i = 1, \cdots, n+1. \tag{4.11}$$

Then

$$\frac{1}{2}\int_{\mathbb{R}^n} \frac{g(x)^2}{(1+|x|^2)^{2s}} \, dx = \sup_f \left(\int_{\mathbb{R}^n} \frac{f(x)g(x)}{(1+|x|^2)^{2s}} \, dx - \frac{1}{2}\int_{\mathbb{R}^n} \frac{f(x)^2}{(1+|x|^2)^{2s}} \, dx \right),$$

and

$$\frac{1}{2}\int_{\mathbb{R}^n} \frac{g(x)}{(1+|x|^2)^{2s}} (-\Delta)^{-s}\left(\frac{g(x)}{(1+|x|^2)^{2s}}\right) dx = \sup_f \left(\int_{\mathbb{R}^n} \frac{f(x)g(x)}{(1+|x|^2)^{2s}} \, dx - \frac{1}{2}\|f\|_s^2 \right),$$

where supremum is taken over the functions f satisfying the conditions (4.11).

Démonstration. The proof of this proposition is elementary and is completely similar with the one of the dual formulas in [DJ14]. □

Given $f \in \dot{W}^s(\mathbb{R}^n)$, we consider the function F defined by (4.1) and its decomposition on spherical harmonics

$$F(\omega) = \sum_{k=0}^{\infty} F_k(\omega), \tag{4.12}$$

where $F_k \in \mathcal{H}_k$. Using the Funk-Hecke theorem and the dual principle for $\|\cdot\|_s$, we obtain the following.

Lemma 8. *With f and F taken as in (4.1)-(4.12), we have*

$$\|f\|_s^2 = \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{2k+n+2s}{2}\right)}{\Gamma\left(\frac{2k+n-2s}{2}\right)} \int_{S^n} F_k(\omega)^2 d\omega\,. \tag{4.13}$$

Démonstration. We have

$$\begin{aligned}\|f\|_s^2 &= \sup_g \left(2\int_{\mathbb{R}^n} f(x)\, g(x)\, dx - \int_{\mathbb{R}^n} g(x)\, (-\Delta)^{-s} g(x)\, dx\right)\\ &= \sup_g \left(2\int_{\mathbb{R}^n} f(x)\, g(x)\, dx - \frac{\Gamma\left(\frac{n-2s}{2}\right)}{\pi^{n/2}\, 2^{2s}\, \Gamma(s)} \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} g(x)|x-y|^{-n+2s} g(y)\, dx\, dy\right).\end{aligned}$$

Defining the function G on S^n by

$$G(\omega) = g(\mathcal{S}^{-1}(\omega)) J_{\mathcal{S}^{-1}}(\omega)^{\frac{1}{p}}, \qquad p = \frac{2n}{n+2s},$$

and considering its decomposition $G = \sum_{k=0}^{\infty} G_k$, $G_k \in \mathcal{H}_k$, we then have

$$\begin{aligned}&2\int_{\mathbb{R}^n} f(x)\, g(x)\, dx - \frac{\Gamma\left(\frac{n-2s}{2}\right)}{\pi^{n/2}\, 2^{2s}\, \Gamma(s)} \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} g(x)|x-y|^{-n+2s} g(y)\, dx\, dy\\ &\quad = 2\int_{S^n} F(\omega)\, G(\omega) d\omega - \frac{\Gamma\left(\frac{n-2s}{2}\right)}{\pi^{n/2}\, 2^{2s}\, \Gamma(s)} \iint\limits_{S^n\times S^n} G(\omega)|\omega-\eta|^{-n+2s} G(\eta)\, d\omega\, d\eta.\end{aligned}$$

Since $|\omega-\eta|^{-n+2s} = 2^{-\frac{n-2s}{2}}(1-\langle\omega,\eta\rangle)^{-\frac{n-2s}{2}}$, by [FL12b, Propostion 5.2] the integral operator with kernel $\frac{\Gamma\left(\frac{n-2s}{2}\right)}{\pi^{n/2}\, 2^{2s}\, \Gamma(s)}|\omega-\eta|^{-n+2s}$ is diagonal with respect to the decomposition (2.7), and its eigenvalues are given by (see [FL12b, Corollary 5.3])

$$\gamma_k = \frac{\Gamma\left(\frac{2k+n-2s}{2}\right)}{\Gamma\left(\frac{2k+n+2s}{2}\right)}, \qquad k = 0, 1, 2\cdots. \tag{4.14}$$

This implies that

$$\begin{aligned}&2\int_{\mathbb{R}^n} f(x)\, g(x)\, dx - \frac{\Gamma\left(\frac{n-2s}{2}\right)}{\pi^{n/2}\, 2^{2s}\, \Gamma(s)} \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} g(x)|x-y|^{-n+2s} g(y)\, dx\, dy\\ &\qquad = \sum_{k=0}^{\infty}\left(2\int_{S^n} F_k(\omega)\, G_k(\omega) d\omega - \gamma_k \int_{S^n} G_k(\omega)^2 d\omega\right)\\ &\qquad \leqslant \sum_{k=0}^{\infty} \frac{1}{\gamma_k}\int_{S^n} F_k(\omega)^2 d\omega.\end{aligned}$$

□

As a consequence, if f satisfies the conditions (4.11), then f satisfies the following Poincaré type inequality :

$$\|f\|_s^2 \geqslant \frac{2^{2s}\Gamma\left(\frac{n+2s+4}{2}\right)}{\Gamma\left(\frac{n-2s+4}{2}\right)} \int_{\mathbb{R}^n} \frac{f(x)^2}{(1+|x|^2)^{2s}}\, dx. \tag{4.15}$$

Indeed, using the stereographic projection, we have

$$\int_{S^n} F(\omega) d\omega = \int_{\mathbb{R}^n} f(x) f_0(x) (1+|x|^2)^{-2s}\, dx = 0,$$

and

$$\int_{S^n} F(\omega)\, \omega_i\, d\omega = \int_{\mathbb{R}^n} f(x)\, f_i(x)\, (1+|x|^2)^{-2s}\, dx = 0, \qquad i = 1, 2, \cdots, n+1.$$

This shows that $F_0 = F_1 = 0$ in the decomposition (4.12) of F, then

$$\begin{aligned} \|f\|_s^2 &\geqslant \frac{\Gamma\left(\frac{n+2s+4}{2}\right)}{\Gamma\left(\frac{n-2s+4}{2}\right)} \int_{S^n} F(\omega)^2 d\omega \\ &= \frac{2^{2s}\Gamma\left(\frac{n+2s+4}{2}\right)}{\Gamma\left(\frac{n-2s+4}{2}\right)} \int_{\mathbb{R}^n} \frac{f(x)^2}{(1+|x|^2)^{2s}}\, dx. \end{aligned}$$

To sum up, we have

Proposition 9.

(i) If $f \in \dot{W}^s(\mathbb{R}^n)$ satisfies the conditions (4.11) then

$$\mathrm{F}[f] \geqslant \frac{4s}{n-2s+2} \frac{2^{2s}\Gamma\left(\frac{n+2s+2}{2}\right)}{\Gamma\left(\frac{n-2s+2}{2}\right)} \int_{\mathbb{R}^n} \frac{f(x)^2}{(1+|x|^2)^{2s}}\, dx. \tag{4.16}$$

(ii) If g satisfies the conditions (4.11) then

$$\mathrm{G}[g] \geqslant \frac{4s}{n+2s+2} \frac{\Gamma\left(\frac{n-2s+2}{2}\right)}{2^{2s}\Gamma\left(\frac{n+2s+2}{2}\right)} \int_{\mathbb{R}^n} \frac{g(x)^2}{(1+|x|^2)^{2s}}\, dx. \tag{4.17}$$

Démonstration. Item (i) follows immediately from the definition of $\mathrm{F}[f]$ and (4.15), while for (ii), from (4.15) and Corollary 7, we have

$$\int_{\mathbb{R}^n} \frac{g(x)}{(1+|x|^2)^{2s}} (-\Delta)^{-s} \left(\frac{g(x)}{(1+|x|^2)^{2s}} \right) dx \leqslant \frac{\Gamma\left(\frac{n-2s+4}{2}\right)}{2^{2s}\Gamma\left(\frac{n+2s+4}{2}\right)} \int_{\mathbb{R}^n} \frac{g(x)^2}{(1+|x|^2)^{2s}}\, dx.$$

Using the definition of $\mathrm{G}[g]$, we obtain (4.17). □

Corollary 10. *If $f \in \dot{W}^s(\mathbb{R}^n)$ and satisfies the conditions (4.11), then*

$$\mathrm{G}[f] \leqslant 2^{-4s} \frac{n-2s+2}{n+2s+2} \left(\frac{\Gamma\left(\frac{n-2s+2}{2}\right)}{\Gamma\left(\frac{n+2s+2}{2}\right)} \right)^2 \mathrm{F}[f], \tag{4.18}$$

and equality holds if and only if the function F defined by (4.1) belongs to $\mathcal{H}_2$.

Démonstration. Considering the function F defined by (4.1) and its decomposition $F = \sum_{k=2}^{\infty} F_k$, we know that

$$\|f\|_s^2 = \sum_{k\geqslant 2} \frac{1}{\gamma_k} \int_{S^n} F_k(\omega)^2 d\omega,$$

where γ_k is given by (4.14). Using equality (4.2), we also have

$$\int_{\mathbb{R}^n} \frac{f(x)^2}{(1+|x|^2)^{2s}}\, dx = 2^{-2s} \int_{S^n} F(\omega)^2 d\omega = 2^{-2s} \sum_{k=2}^{\infty} \int_{S^n} F_k(\omega)^2 d\omega.$$

From these equalities, we get

$$\begin{aligned} \mathrm{F}[f] &= \sum_{k=2}^{\infty} \left(\frac{1}{\gamma_k} - \frac{\Gamma\left(\frac{n+2s+2}{2}\right)}{\Gamma\left(\frac{n-2s+2}{2}\right)} \right) \int_{S^n} F_k(\omega)^2 d\omega \\ &= \sum_{k=2}^{\infty} \alpha_k \int_{S^n} F_k(\omega)^2 d\omega, \end{aligned} \tag{4.19}$$

with

$$\alpha_k = \frac{\Gamma\left(\frac{n+2s+2k}{2}\right)\Gamma\left(\frac{n-2s+2}{2}\right) - \Gamma\left(\frac{n-2s+2k}{2}\right)\Gamma\left(\frac{n+2s+2}{2}\right)}{\Gamma\left(\frac{n-2s+2k}{2}\right)\Gamma\left(\frac{n-2s+2}{2}\right)}.$$

Denote $g(x) = f(x)(1+|x|^2)^{-2s}$. Using the integral expression of $(-\Delta)^{-s}$ and equality (4.3),

$$\begin{aligned} \int_{\mathbb{R}^n} g(x)(-\Delta)^{-s} g(x)\, dx &= 2^{-4s} \frac{\Gamma\left(\frac{n-2s}{2}\right)}{\pi^{n/2}\, 2^{2s}\, \Gamma(s)} \iint_{S^n \times S^n} F(\omega)|\omega-\eta|^{-n+2s} F(\eta)\, d\omega\, d\eta \\ &= 2^{-4s} \sum_{k\geqslant 2} \gamma_k \int_{S^n} F_k(\omega)^2 d\omega. \end{aligned}$$

Therefore, we get

$$\begin{aligned} \mathrm{G}[f] &= \sum_{k=1}^{\infty} \left(\frac{\Gamma\left(\frac{n-2s+2}{2}\right)}{2^{4s}\Gamma\left(\frac{n+2s+2}{2}\right)} - \frac{\gamma_k}{2^{4s}} \right) \int_{S^n} F_k(\omega)^2 d\omega \\ &= \frac{1}{2^{4s}} \sum_{k=2}^{\infty} \beta_k \int_{S^n} F_k(\omega)^2 d\omega, \end{aligned} \tag{4.20}$$

with

$$\beta_k = \frac{\Gamma\left(\frac{n+2s+2k}{2}\right)\Gamma\left(\frac{n-2s+2}{2}\right) - \Gamma\left(\frac{n-2s+2k}{2}\right)\Gamma\left(\frac{n+2s+2}{2}\right)}{\Gamma\left(\frac{n+2s+2k}{2}\right)\Gamma\left(\frac{n+2s+2}{2}\right)}.$$

We have $\alpha_k, \beta_k > 0$ for all $k \geqslant 2$. Moreover, we can prove that

$$\frac{\beta_k}{\alpha_k} \leqslant \frac{\beta_2}{\alpha_2} = \frac{n-2s+2}{n+2s+2} \left(\frac{\Gamma\left(\frac{n-2s+2}{2}\right)}{\Gamma\left(\frac{n+2s+2}{2}\right)} \right)^2, \quad \text{for all } k \geqslant 2,$$

and equality holds if $k = 2$. From this inequality, we have

$$\mathrm{G}[f] = \frac{1}{2^{4s}} \sum_{k=2}^{\infty} \beta_k \int_{S^n} F_k(\omega)^2 d\omega \leqslant 2^{-4s} \frac{n-2s+2}{n+2s+2} \left(\frac{\Gamma\left(\frac{n-2s+2}{2}\right)}{\Gamma\left(\frac{n+2s+2}{2}\right)} \right)^2 \mathrm{F}[f].$$

This proves the inequality (4.18). Additionally, we see from the proof that equality in (4.18) occurs if and only if $\int_{S^n} F_k(\omega)^2 d\sigma(\omega) = 0$ for all $k \geqslant 3$, hence $F \in \mathcal{H}_2$. □

As a consequence, we have

$$\sup_f \frac{G(f)}{F(f)} = 2^{-4s}\,\frac{n-2s+2}{n+2s+2}\left(\frac{\Gamma\left(\frac{n-2s+2}{2}\right)}{\Gamma\left(\frac{n+2s+2}{2}\right)}\right)^2, \tag{4.21}$$

where supremum is taken over $f \in \dot{W}^s(\mathbb{R}^n)$, $f \neq 0$, and f satisfying the conditions (4.11).

We can now prove the first inequality in (2.2) of Theorem 1.

Proof of (2.2). For all $f \in \dot{W}^s(\mathbb{R}^n)$, $f \neq 0$ and f satisfies the conditions (4.11), denote $u_\epsilon = u_* + \epsilon f$, then

$$C^*_{n,s}\|u_\epsilon\|^{\frac{8s}{n-2s}}_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} \geqslant \frac{\mathcal{G}[u^r_\epsilon]}{\mathcal{F}[u_\epsilon]}.$$

Let $\epsilon \to 0^+$, we get

$$C^*_{n,s} \geqslant \frac{1}{\|u_*\|^{\frac{8s}{n-2s}}_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} S_{n,s}}\left(\frac{n+2s}{n-2s}\right)^2 \frac{G(f)}{F(f)}$$

Taking supremum over $f \in \dot{W}^s(\mathbb{R}^n)$, $f \neq 0$, and f satisfying the conditions (4.11), using (4.21) and the fact that

$$\int_{\mathbb{R}^n} u_*(x)^{\frac{2n}{n-2s}}\,dx = \int_{\mathbb{R}^n}(1+|x|^2)^{-n}\,dx = \pi^{\frac{n}{2}}\,\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)},$$

we get

$$C^*_{n,s} \geqslant \frac{n-2s+2}{n+2s+2} S_{n,s}$$

as desired. □

5 Improved Moser-Trudinger-Onofri inequality via endpoint differentiation

This section is dedicated to the proof of Theorem 2. By an approximation argument, it suffices to prove the inequality (2.4) for bounded functions. We first prove for functions F such that $\int_{S^n} F(\xi)d\xi = 0$. We define a new function u on $\mathbb{R}^n$ by

$$u(x) = \left(1 + \frac{n-2s}{2n}F(\mathcal{S}(x))\right) J_{\mathcal{S}}(x)^{-(s-\frac{n}{2})}. \tag{5.1}$$

Since F is bounded, then u is positive when s is close enough to $\frac{n}{2}$. Considering the expansion of F in terms of spherical harmonics $F = \sum_{k\geqslant 1} F_k$ with $F_k \in \mathcal{H}_k$, it follows from Lemma 8 that

$$\|u\|_s^2 = |S^n|\frac{\Gamma(\frac{n+2s}{2})}{\Gamma(\frac{n-2s}{2})} + |S^n|\frac{(n-2s)^2}{4n^2}\sum_{k\geqslant 1}\frac{\Gamma(\frac{2k+n+2s}{2})}{\Gamma(\frac{2k+n-2s}{2})}\int_{S^n} F_k^2 d\sigma.$$

Using the stereographic projection, we get

$$\|u\|^2_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} = |S^n|^{\frac{n-2s}{n}}\left(\int_{S^n}\left(1+\frac{n-2s}{2n}F\right)^{\frac{2n}{n-2s}} d\sigma\right)^{\frac{n-2s}{n}}.$$

For simplicity, we denote $t = \frac{n-2s}{2n}$, then

$$\begin{aligned} S_{n,s}\|u\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)}^{\frac{8s}{n-2s}} \left(S_{n,s}\|u\|_s^2 - \|u\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)}^2 \right) \\ = |S^n| \frac{\Gamma(nt)}{\Gamma(n(1-t))} \left[\left(\int_{S^n} (1+tF)^{\frac{1}{t}} \, d\sigma \right)^{2-4t} - \left(\int_{S^n} (1+tF)^{\frac{1}{t}} \, d\sigma \right)^{2-2t} \right] \\ + |S^n| \frac{t^2 \Gamma(nt)^2}{\Gamma(n(1-t))^2} \left[\sum_{k \geqslant 1} \frac{\Gamma(k+n(1-t))}{\Gamma(k+nt)} \int_{S^n} F_k^2 d\sigma \right] \left(\int_{S^n} (1+tF)^{\frac{1}{t}} \, d\sigma \right)^{2-4t} . \end{aligned} \tag{5.2}$$

Since $\Gamma(nt) \sim 1/(nt)$ when $t \to 0^+$, by taking $t \to 0^+$ (or $s \to \frac{n}{2}$) in (5.2), we obtain

$$\begin{aligned} \lim_{s \to \frac{n}{2}} \left[S_{n,s}\|u\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)}^{\frac{8s}{n-2s}} \left(S_{n,s}\|u\|_s^2 - \|u\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)}^2 \right) \right] \\ = -\frac{2|S^n|}{n\Gamma(n)} \left(\int_{S^n} e^F d\sigma \right)^2 \log \left(\int_{S^n} e^F d\sigma \right) \\ + \frac{|S^n|}{n^2 \Gamma(n)^2} \left[\sum_{k \geqslant 1} \frac{\Gamma(k+n)}{\Gamma(k)} \int_{S^n} F_k^2 d\sigma \right] \left(\int_{S^n} e^F d\sigma \right)^2 . \end{aligned} \tag{5.3}$$

We also have

$$S_{n,s}\|u^{\frac{n+2s}{n-2s}}\|_{L^{\frac{2n}{n+2s}}(\mathbb{R}^n)}^2 = \frac{|S^n|\Gamma(nt)}{\Gamma(n(1-t))} \left(\int_{S^n} (1+tF)^{\frac{1}{t}} d\sigma \right)^{2-2t} \tag{5.4}$$

$$\begin{aligned} &\int_{\mathbb{R}^n} u^{\frac{n+2s}{n-2s}} (-\Delta)^{-s} u^{\frac{n+2s}{n-2s}} \, dx \\ &= \frac{|S^n|^2 \Gamma(nt)}{4^s \pi^{\frac{n}{2}} \Gamma(s)} \iint_{S^n \times S^n} \frac{(1+tF(\xi))^{\frac{1-t}{t}} (1+tF(\eta))^{\frac{1-t}{t}}}{|\xi - \eta|^{2nt}} \, d\sigma(\xi) \, d\sigma(\eta) \\ &= \frac{|S^n|\Gamma(n-nt)}{4^s \pi^{\frac{n}{2}} \Gamma(s)} \left(\int_{S^n} (1+tF)^{\frac{1-t}{t}} d\sigma \right)^2 \\ &\quad + \frac{|S^n|^2 \Gamma(nt)}{4^s \pi^{\frac{n}{2}} \Gamma(s)} \iint_{S^n \times S^n} \frac{(1+tF(\xi))^{\frac{1-t}{t}} (1+tF(\eta))^{\frac{1-t}{t}}}{\left(|\xi - \eta|^{-2nt} - 1 \right)^{-1}} \, d\sigma(\xi) \, d\sigma(\eta) . \end{aligned} \tag{5.5}$$

Letting $s \to \frac{n}{2}$ (*i.e.* $t \to 0$) in (5.4)-(5.5), we obtain

$$\begin{aligned} \lim_{s \to \frac{n}{2}} \left[S_{n,s}\|u^{\frac{n+2s}{n-2s}}\|_{L^{\frac{2n}{n+2s}}(\mathbb{R}^n)}^2 - \int_{\mathbb{R}^n} u^{\frac{n+2s}{n-2s}} (-\Delta)^{-s} u^{\frac{n+2s}{n-2s}} \, dx \right] \\ = \frac{|S^n|}{\Gamma(n)} \left(\int_{S^n} e^F d\sigma \right)^2 \left(\Psi(n) - \Psi\left(\frac{n}{2}\right) - \log 4 + \frac{2}{n} \frac{\mathrm{Ent}_\sigma(e^F)}{\int_{S^n} e^F d\sigma} \right) \\ + \frac{|S^n|}{\Gamma(n)} \iint_{S^n \times S^n} e^{F(\xi)} \log\left(|\xi - \eta|^2\right) e^{F(\eta)} d\sigma(\xi) d\sigma(\eta) , \end{aligned} \tag{5.6}$$

where $\mathrm{Ent}_\sigma(f) = \int_{S^n} f \log f \, d\sigma - (\int_{S^n} f \, d\sigma) \log(\int_{S^n} f \, d\sigma)$.

Now, applying the inequality (2.1) to function u defined by (5.1), then letting $s \to \frac{n}{2}$, and using the equalities (5.3) and (5.6), we obtain

$$\begin{aligned}\left(\int_{S^n} e^F d\sigma\right)^2 &\left[\frac{1}{2n}\sum_{k\geqslant 1}\frac{\Gamma(k+n)}{\Gamma(n)\Gamma(k)}\int_{S^n}|F_k|^2 d\sigma - \log\left(\int_{S^n} e^F d\sigma\right)\right] \\ &\geqslant \frac{n}{2}\iint\limits_{S^n\times S^n} e^{F(\xi)}\,\log\left(|\xi-\eta|^2\right)\,e^{F(\eta)} d\sigma(\xi)d\sigma(\eta) \\ &\quad + \left(\int_{S^n} e^F d\sigma\right)^2\left[\frac{n}{2}\left(\Psi(n) - \Psi\left(\frac{n}{2}\right) - \log 4\right) + \frac{\mathrm{Ent}_\sigma(e^F)}{\int_{S^n} e^F d\sigma}\right]. \end{aligned} \tag{5.7}$$

For any bounded function F, applying (5.7) to function $F - \int_{S^n} F d\sigma$, we obtain (2.4) with $C_n = 1$.

The above proof shows that $C_n^* \leqslant 1$. Let us now prove $C_n^* \geqslant \frac{1}{n+1}$. Indeed, for any function F such that $\int_{S^n} F d\sigma = 0$. Considering an expansion of F by $F = \sum_{k\geqslant 1} F_k$, with $F_k \in \mathcal{H}_k$ and applying inequality (2.4) to the function ϵF with $\epsilon > 0$, we get

$$\begin{aligned} C_n^*\left(\int_{S^n} e^{\epsilon F} d\sigma\right)^2 &\left[\frac{\epsilon^2}{2n}\sum_{k\geqslant 1}\frac{\Gamma(k+n)}{\Gamma(n)\Gamma(k)}\int_{S^n}|F_k|^2 d\sigma - \log\left(\int_{S^n} e^{\epsilon F} d\sigma\right)\right] \\ &\geqslant \frac{n}{2}\iint\limits_{S^n\times S^n} e^{\epsilon F(\xi)}\,\log\left(|\xi-\eta|^2\right)\,e^{\epsilon F(\eta)} d\sigma(\xi)d\sigma(\eta) \\ &\quad + \left(\int_{S^n} e^{\epsilon F} d\sigma\right)^2\left[\frac{n}{2}\left(\Psi(n) - \Psi\left(\frac{n}{2}\right) - \log 4\right) + \frac{\mathrm{Ent}_\sigma(e^{\epsilon F})}{\int_{S^n} e^{\epsilon F} d\sigma}\right]. \end{aligned} \tag{5.8}$$

When ϵ is small, we have

$$\int_{S^n} e^{\epsilon F} d\sigma = 1 + \frac{\epsilon^2}{2}\int_{S^n}|F|^2 d\sigma + o(\epsilon^2),$$

$$\mathrm{Ent}_\sigma(e^{\epsilon F}) = \frac{\epsilon^2}{2}\int_{S^n}|F|^2 d\sigma + o(\epsilon^2).$$

Moreover, since

$$\int_{S^n}\log\left(|\xi-\eta|^2\right) d\sigma(\eta) = -\left(\Psi(n) - \Psi\left(\frac{n}{2}\right) - \log 4\right) =: A(n),$$

then

$$\begin{aligned}&\iint\limits_{S^n\times S^n} e^{\epsilon F(\xi)}\,\log\left(|\xi-\eta|^2\right)\,e^{\epsilon F(\eta)} d\sigma(\xi)d\sigma(\eta) \\ &\qquad = A(n) + \epsilon^2 A(n)\int_{S^n}|F|^2 d\sigma - \epsilon^2\sum_{k\geqslant 1}\frac{\Gamma(n)\Gamma(k)}{\Gamma(n+k)}\int_{S^n}|F_k|^2 d\sigma + o(\epsilon^2).\end{aligned}$$

Substituting these above estimates into (5.8), we obtain

$$\begin{aligned}\frac{\epsilon^2}{2}C_n^*\sum_{k\geqslant 2}\left(\frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)} - 1\right)\int_{S^n}|F_k|^2 d\sigma + o(\epsilon^2) \\ \geqslant \frac{\epsilon^2}{2}\sum_{k\geqslant 2}\left(1 - \frac{\Gamma(n+1)\Gamma(k)}{\Gamma(n+k)}\right)\int_{S^n}|F_k|^2 d\sigma + o(\epsilon^2),\end{aligned}$$

since $\Gamma(n+1) = n\,\Gamma(n)\,\Gamma(1)$. If $F_k \neq 0$ for some $k \geqslant 2$, then dividing both sides by $\frac{\epsilon^2}{2}$ and letting $\epsilon \to 0$, we get

$$C_n^* \geqslant \frac{\sum_{k\geqslant 2}\left(1-\frac{\Gamma(n+1)\Gamma(k)}{\Gamma(n+k)}\right)\int_{S^n}|F_k|^2 d\sigma}{\sum_{k\geqslant 2}\left(\frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}-1\right)\int_{S^n}|F_k|^2 d\sigma}.$$

Taking supremum over $F = \sum_{k\geqslant 1} F_k$, $F_k \neq 0$ for some $k \geqslant 2$, we obtain

$$C_n^* \geqslant \sup\left\{\frac{\sum_{k\geqslant 2}\left(1-\frac{\Gamma(n+1)\Gamma(k)}{\Gamma(n+k)}\right)\int_{S^n}|F_k|^2 d\sigma}{\sum_{k\geqslant 2}\left(\frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}-1\right)\int_{S^n}|F_k|^2 d\sigma} \,:\, F=\sum_{k\geqslant 1}F_k,\ F_k\neq 0 \text{ for some } k\geqslant 2\right\}$$
$$= \frac{1}{n+1}.$$

This completes the proof of Theorem 2.

6 Fractional fast diffusion flow

At this point, we know using the expansion of the square that $C^*_{n,s} \leqslant S_{n,s}$, so that if we define

$$\mathcal{C} = \frac{C^*_{n,s}}{S_{n,s}}\,,$$

we know $\mathcal{C} \leqslant 1$. In this section we will show that in fact $\mathcal{C} < 1$ when $0 < s < 1$. This condition is enforced throughout this section. With the notations above, we consider the following fractional fast diffusion equation :

$$\begin{aligned} &\partial_t v + (-\Delta)^s v^m = 0\,, \quad t>0\,, \quad x\in\mathbb{R}^n\,, \quad m=\frac{1}{r}=\frac{n-2s}{n+2s}\,. \\ &v(0)=v_0\,. \end{aligned} \tag{6.1}$$

which is well posed for $v_0 \in L^1 \bigcap L^\ell$ for some $\ell > \frac{2n}{n+2s}$ according to [Pab+12, Theorem 2.3]. We will take initial datum v with sufficient decay at infinity, *e.g.* in the Schwartz space.

Let us define

$$\mathcal{G}_0 = \mathcal{G}[v_0] \quad \mathsf{J}[v(t)] = \int_{\mathbb{R}^n} v^p = \int_{\mathbb{R}^n} u^q\,, \quad \mathsf{J}_0 \coloneqq \mathsf{J}[v_0]\,.$$

which is such that

$$\mathsf{J}' \coloneqq \frac{d}{dt}\mathsf{J} = -p\int_{\mathbb{R}^n}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\,,$$

We can now consider the evolution along the flow of the functional $\mathcal{G}$ associated to the Hardy-Littlewood-Sobolev inequality. An easy computation gives

$$-\mathcal{G}'[v] = 2\left(\int_{\mathbb{R}^n} v^{\frac{2n}{n+2s}}\right)^{\frac{2s}{n}}\mathcal{F}[v^m] = 2\,\mathsf{J}^{\frac{2s}{n}}\,\mathcal{F}[u]\,,$$

which is nonnegative according to the fractional Sobolev inequality (1.1). Hence, $-\mathcal{G}[v]$ is nondecreasing and stationary only when u is an extremal function for (1.1). This and the

following computations are a straightforward extension of those done in [Dol11]. Going one step further, we compute

$$-\mathcal{G}'' = -\frac{\mathsf{J}'}{\mathsf{J}}\,\mathcal{G}' - 4\,m\,S_{n,s}\,\mathsf{J}^{\frac{2s}{n}}\mathsf{K}\,,$$

with $\mathsf{K} = \int v^{m-1}\,|(-\Delta)^s v^m - \Lambda\,v|^2$, $\Lambda = \frac{n+2s}{2n}\frac{\mathsf{J}'}{\mathsf{J}}$. Then, using the fact that $\mathcal{G}' \leqslant 0$, we have the following :

Lemma 11. *With the above notation and assuming* $0 < s < 1$,

$$\frac{\mathcal{G}''}{\mathcal{G}'} \leqslant \frac{\mathsf{J}'}{\mathsf{J}}\,.$$

Using Lemma 11 and (1.1), we have

$$-\mathcal{G}' \leqslant \kappa_0\,\mathsf{J} \quad \text{with} \quad \kappa_0 := \frac{-\mathcal{G}'(0)}{\mathsf{J}_0}$$

Since J is nonincreasing in time, there exists $\mathsf{Y} : [0, \mathsf{J}_0] \to \mathbb{R}$ such that

$$\mathcal{G}(t) = \mathsf{Y}(\mathsf{J}(t))\,.$$

Differentiating with respect to t gives

$$-\mathsf{Y}'(\mathsf{J})\,\mathsf{J}' = -\,\mathcal{G}' \leqslant \kappa_0\,\mathsf{J}\,,$$

then, substituting J' in the inequality of Theorem 1 (ii) we get

$$\mathcal{C}\left(-\frac{\kappa_0}{p}\,S_{n,s}^2\,\frac{\mathsf{J}^{1+\frac{4s}{n}}}{\mathsf{Y}'} + S_{n,s}\,\mathsf{J}^{1+\frac{2s}{n}}\right) + \mathsf{Y} \leqslant 0\,.$$

With $\mathsf{Y}' = \frac{d}{dz}\mathsf{Y}$, we end up with the following differential inequality for Y :

$$\mathsf{Y}'\left(\mathcal{C}\,S_{n,s}\,z^{1+\frac{2s}{n}} + Y\right) \leqslant \mathcal{C}\frac{\kappa_0}{p}\,S_{n,s}^2\,z^{1+\frac{4s}{n}}\,, \quad \mathsf{Y}(0) = 0\,, \quad \mathsf{Y}(\mathsf{J}_0) = \mathcal{G}(0)\,. \tag{6.2}$$

We have the following estimates. On the one hand

$$\mathsf{Y}' \leqslant \frac{p}{\kappa_0}\,S_{n,s}\,z^{\frac{2s}{n}}$$

and, hence,

$$\mathsf{Y}(z) \leqslant \frac{1}{2}\,\kappa_0\,S_{n,s}\,z^{1+\frac{2s}{n}} \quad \forall\, z \in [0, \mathsf{J}_0]\,.$$

On the other hand, after integrating by parts on the interval $[0, \mathsf{J}_0]$, we get

$$\frac{1}{2}\,\mathcal{G}(0)^2 + \mathcal{C}\,S_{n,s}\,\mathsf{J}_0^{1+\frac{2s}{n}}\,\mathcal{G}(0) \leqslant \frac{1}{4}\,\mathcal{C}\,\kappa_0\,S_{n,s}^2\,\mathsf{J}_0^{2+\frac{4s}{n}} + \frac{n+2s}{n}\,\mathcal{C}\,S_{n,s}\int_0^{\mathsf{J}_0} z^{\frac{2s}{n}}\,\mathsf{Y}(z)\,dz\,.$$

Using the above estimate, we find that

$$\frac{2}{p}\,S_{n,s}\int_0^{\mathsf{J}_0} z^{\frac{2s}{n}}\,\mathsf{Y}(z)\,dz \leqslant \frac{1}{4}\,\mathsf{J}_0^{2+\frac{4s}{n}}\,,$$

and finally

$$\frac{1}{2}\,\mathcal{G}_0^2 - \mathcal{C}\,S_{n,s}\,\mathsf{J}_0^{1+\frac{2s}{n}}\,\mathcal{G}_0 \leqslant \frac{1}{2}\,\mathcal{C}\kappa_0\,S_{n,s}^2\,\mathsf{J}_0^{2+\frac{4s}{n}}\,.$$

Altogether, we have shown an improved inequality that can be stated as follows.

Theorem 12. *Assume that $0 < s < 1$. Then we have*

$$0 \leqslant S_{n,s}\, \mathsf{J}^{1+\frac{2s}{n}}\, \varphi\left(\mathsf{J}^{\frac{2s}{n}-1}\, \mathcal{F}[u]\right) - \mathcal{G}[v]\,, \quad \forall\, u \in \dot{W}^s(\mathbb{R}^n)\,,\ v = u^r \tag{6.3}$$

where $\varphi(x) := \sqrt{\mathcal{C}^2 + 2\,\mathcal{C}\,x} - \mathcal{C}$ for any $x \geqslant 0$.

Démonstration. We have shown that for $u \in \mathcal{S}$, $y^2 + 2\,\mathcal{C}\,y - \mathcal{C}\,\kappa_0 \leqslant 0$ with $y = \mathcal{G}_0/(S_{n,s}\,\mathsf{J}_0^{1+\frac{2s}{n}}) \geqslant 0$. This proves that $y \leqslant \sqrt{\mathcal{C}^2 + \mathcal{C}\kappa_0} - \mathcal{C}$, which proves that

$$\mathcal{G}_0 \leqslant S_{n,s}\, \mathsf{J}_0^{1+\frac{2s}{n}} \left(\sqrt{\mathcal{C}^2 + \mathcal{C}\,\kappa_0} - \mathcal{C}\right)$$

after recalling that

$$\frac{1}{2}\,\kappa_0 = -\frac{\mathcal{G}_0'}{\mathsf{J}_0} = \mathsf{J}^{\frac{2s}{n}-1}\, \mathcal{F}[u]\,.$$

Arguing by density, we recover the results for $u \in \dot{W}^s(\mathbb{R}^n)$. □

Remark 20. *We may observe that $x \mapsto x - \varphi(x)$ is a convex nonnegative function which is equal to 0 if and only if $x = 0$. Moreover, we have*

$$\varphi(x) \leqslant x \quad \forall\, x \geqslant 0$$

with equality if and only if $x = 0$. However, one can notice that

$$\varphi(x) \leqslant \mathcal{C}\,x \quad \Longleftrightarrow \quad x \geqslant 2\,\frac{1-\mathcal{C}}{\mathcal{C}}\,.$$

We recall that (6.1) admits special solutions with separation of variables given by

$$v_*(t,x) = \lambda^{-(n+2s)/2}\,(T-t)^{\frac{n+2s}{4s}}\, u_*^{\frac{n+2s}{n-2s}}\left(\tfrac{x-x_0}{\lambda}\right) \tag{6.4}$$

where $u_*(x) := (1+|x|^2)^{-\frac{n-2s}{2}}$ is an Aubin-Talenti type extremal function, $x \in \mathbb{R}^n$ and $0 < t < T$. Such a solution is generic near the extinction time T, see [JX11, Theorem 1.3].

Corollary 13. *With the above notations, $\mathcal{C} < 1$.*

Démonstration. Argue by contradiction and suppose $\mathcal{C} = 1$. Let (u_k) be a minimizing sequence for the quotient $u \mapsto \frac{\mathcal{F}[u]}{\mathcal{G}[u^r]}$. Thanks to homogeneity, we can assume that $\mathsf{J}[u_k] = \mathsf{J}_* = \mathsf{J}[u_*]$ with J_* fixed, so that in fact $\mathcal{G}[u_k^r]$ is a bounded sequence. There are two possibilities. Either $\lim_{k\to\infty} \mathcal{G}[u_k^r] > 0$, and then, up to a subsequence, $\lim_{k\to\infty} \mathcal{F}[u_k] > 0$, and then

$$\begin{aligned} 0 &= \lim_{k\to\infty} \left(S_{n,s}\, \mathsf{J}_*^{\frac{4s}{n}}\, \mathcal{F}[u_k] - \mathcal{G}[u_k]\right) \\ &= \lim_{k\to\infty} \left(S_{n,s}\, \mathsf{J}_*^{\frac{4s}{n}}\, \mathcal{F}[u_k] - S_{n,s}\, \mathsf{J}_*^{1+\frac{2s}{n}}\, \varphi\left(\mathsf{J}_*^{\frac{2s}{n}-1}\, \mathcal{F}[u_k]\right)\right) \\ &\qquad + \lim_{k\to\infty} \left(S_{n,s}\, \mathsf{J}_*^{1+\frac{2s}{n}}\, \varphi\left(\mathsf{J}_*^{\frac{2s}{n}-1}\, \mathcal{F}[u_k]\right) - \mathcal{G}[u_k]\right)\,. \end{aligned}$$

The last term is nonnegative by Theorem 12, and since $\lim_{k\to\infty} \mathcal{F}[u_k] > 0$, the first term is positive because of the properties of φ, see Remark 20. This is a contradiction, so in fact we have $\lim_{k\to\infty} G[u_k^r] = \lim_{k\to\infty} \mathcal{F}[u_k] = 0$. Since $\mathsf{J}[u_k] = \mathsf{J}_*$, $v_k = u_k^r$ maximizes

$$\left\{ \int_{\mathbb{R}^n} v(-\Delta)^{-s} v\, dx \ :\ \|v\|_{\frac{2n}{n+2s}} = \mathsf{J}_* \right\}$$

According to [Lie83, Theorem 3.1], up to translations and dilations, v_k converges to $v_* = u^r_*$, and then the limit of the quotient $\frac{\mathcal{F}[u_k]}{\mathcal{G}[u_k^r]}$ is given by the linearization around the Aubin-Talenti profiles. That is

$$\frac{1}{S_{n,s}} = \lim_{k\to\infty} \frac{\mathcal{F}[u_k]}{\mathcal{G}[u_k^r]} \geqslant \frac{n+2s+2}{n-2s+2}\frac{1}{S_{n,s}},$$

which is a contradiction. Thus, $C^*_{n,s} < S_{n,s}$. □

Inequality (2.3) holds by Corollary 13, and the proof of Theorem 1 is complete.

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Chapitre III

Inégalité d'Onofri

Ce chapitre reprend en grande partie l'article « The Moser-Trudinger-Onofri Inequality », écrit en collaboration avec Jean Dolbeault et Maria J. Esteban.

Résumé

Ce chapitre traite de résultats sur l'inégalité de Moser-Trudinger-Onofri (ou inégalité d'Onofri). En dimension deux, cette inégalité joue un rôle identique à l'inégalité de Sobolev critique pour les dimensions supérieures. Pour justifier cela, nous la retrouvons à partir de différents passages à la limite. Nous passons aussi en revue certains résultats connus, ainsi que quelques remarques élémentaires.

Nous présentons aussi de nouveaux résultats. Nous donnons une preuve de l'inégalité par transport de masse, dans le case radial. Ceci est cohérent avec des résultats équivalents pour l'inégalité de Sobolev. Nous rappelons aussi la dualité entre l'inégalité d'Onofri et l'inégalité de Hardy-Littlewood-Sobolev logarithmique. Dans le cadre des équations de diffusion rapide, nous montrons que l'inégalité d'Onofri apparaît comme inégalité d'entropie–production d'entropie. Nous terminons par une preuve basée sur des méthodes de rigidité.

Structure

1 Introduction . . . 58
2 A review of the literature . . . 58
3 Preliminaries . . . 62
4 Mass Transportation . . . 69
5 An improved inequality based on Legendre's duality and logarithmic diffusion . . . 73
6 An improved inequality based on entropy–entropy production and fast diffusion . . . 75
7 Rigidity (or *carré du champ*) methods and adapted nonlinear diffusion equations . . . 78
References . . . 81

1 Introduction

In this paper we consider the Moser-Trudinger-Onofri inequality, or *Onofri inequality*, for brevity. This inequality takes any of the three following forms, which are all equivalent.

▷ The Euclidean Onofri inequality :

$$\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx\geqslant\log\left(\int_{\mathbb{R}^2}e^u\,d\mu\right)-\int_{\mathbb{R}^2}u\,d\mu\,. \tag{1.1}$$

Here $d\mu=\mu(x)\,dx$ denotes the probability measure defined by $\mu(x)=\frac{1}{\pi}\,(1+|x|^2)^{-2}$, $x\in\mathbb{R}^2$.

▷ The Onofri inequality on the two-dimensional sphere $\mathbb{S}^2$:

$$\frac{1}{4}\int_{\mathbb{S}^2}|\nabla v|^2\,d\sigma\geqslant\log\left(\int_{\mathbb{S}^2}e^v\,d\sigma\right)-\int_{\mathbb{S}^2}v\,d\sigma\,. \tag{1.2}$$

Here $d\sigma$ denotes the uniform probability measure, that is, the measure induced by Lebesgue's measure on the unit sphere $\mathbb{S}^2\subset\mathbb{R}^3$ divided by a 4π factor.

▷ The Onofri inequality on the two-dimensional cylinder $\mathcal{C}=\mathbb{S}^1\times\mathbb{R}$:

$$\frac{1}{16\,\pi}\int_{\mathcal{C}}|\nabla w|^2\,dy\geqslant\log\left(\int_{\mathcal{C}}e^w\,\nu\,dy\right)-\int_{\mathcal{C}}w\,\nu\,dy\,. \tag{1.3}$$

Here $y=(\theta,s)\in\mathcal{C}=\mathbb{S}^1\times\mathbb{R}$ and $\nu(y)=\frac{1}{4\pi}\,(\cosh s)^{-2}$ is a weight.

These three inequalities are equivalent. Indeed, on $\mathbb{S}^2\subset\mathbb{R}^3$, let us consider the coordinates $(\omega,z)\in\mathbb{R}^2\times\mathbb{R}$ such that $|\omega|^2+z^2=1$ and $z\in[-1,1]$. Let $\rho:=|\omega|$ and define the *stereographic projection* $\Sigma:\mathbb{S}^2\backslash\{\mathrm{N}\}\to\mathbb{R}^2$ by $\Sigma(\omega)=x=r\,\omega/\rho$ and

$$z=\frac{r^2-1}{r^2+1}=1-\frac{2}{r^2+1}\,,\quad\rho=\frac{2\,r}{r^2+1}\,.$$

The *North Pole* N corresponds to $z=1$ (and is formally sent at infinity) while the *equator* (corresponding to $z=0$) is sent onto the unit sphere $\mathbb{S}^1\subset\mathbb{R}^2$. Whereas on the cylinder $\mathcal{C}$, we can consider the *Emden-Fowler transformation* using the coordinates $\theta=x/|x|=\omega\rho$ and $s=-\log r=-\log|x|$. The functions u, v and w in (1.1), (1.2) and (1.3) are then related by

$$u(x)=v(\omega,z)=w(\theta,s)\,.$$

2 A review of the literature

Inequality (1.2) has been established in [Mos70] without a sharp constant, based on the Moser-Trudinger inequality which was itself proved in [Tru67 ; Mos70], and in [Ono82] with a sharp constant. For this reason it is sometimes called the *Moser-Trudinger-Onofri inequality* in the literature. The result of E. Onofri strongly relies on a paper of T. Aubin, [Aub79], which contains a number of results of existence for inequalities of Onofri type on manifolds (with unknown optimal constants). Also based on the Moser-Trudinger inequality, one has to mention [OPS88] which connects Inequality (1.2) with the Lebedev-Milin inequalities.

Concerning the other equivalent forms of the inequality, we may refer to [DET08] for (1.3) while it is more or less a standard result that (1.1) is equivalent to (1.2) ; an important

result concerning this last point is the paper of E. Carlen and M. Loss, [CL92], which will be considered in more detail in Section 5. Along the same line of thought, one has to mention [Bec93], which also is based on the Funk-Hecke formula for the dual inequality, as was E. Lieb's work on Hardy-Littlewood-Sobolev inequalities on the sphere, [Lie83].

The optimal function can be identified using the associated Euler-Lagrange equation, see [Hon86, Lemma 3.1] which provides details that are somewhat lacking in Onofri's original paper. We may also refer to [DET09, Theorem 12] for multiplicity results for a slightly more general equation than the one associated with (1.1).

Another strategy can be found in the existing literature. In [Ghi05], A. Ghigi provides a proof based on the Prékopa-Leindler inequality, which is also explained in full detail in the book [GM13, Chapters 16-18] of N. Ghoussoub and A. Moradifam. Let us mention that the book contains much more material and tackles the issues of improved inequalities under additional constraints, a question that was raised in [Aub79] and later studied in [CY88 ; CY87 ; GL10].

Symmetrization, which allows to prove that optimality in (1.1), (1.2) or (1.3) is achieved among functions which are respectively radial (on the Euclidean space), or depend only on the azimuthal angle (the latitude, on the sphere), or only on the coordinate along the axis (of the cylinder) are an essential tool to reduce the complexity of the problem. For brevity, we shall refer to the *symmetric case* when the function set is reduced to one of the above cases. Symmetrization results are widespread in the mathematical literature, so we shall only quote a few key papers. A standard reference is the paper of [BT76] and in particular [BT76, Theorem 2] which is a key result for establishing the Hardy-Littlewood-Sobolev inequalities on the sphere and its limiting case, the logarithmic Hardy-Littlewood-Sobolev inequality. By duality and by considering the optimality case, one gets a symmetry result for the Onofri inequality, that can be found for instance in [CL92]. It is also standard that the kinetic energy (Dirichlet integral) is decreased by symmetrization (a standard reference in the Euclidean case can be found in [LL01, Lemma 7.17] ; also see [BZ88, p. 154]) and the adaptation to the sphere is straightforward. Historically, this was known much earlier and one can for instance quote [Mos70] (without any justification) and [Aub76, Lemma 1 and 2, p. 586]. This is precisely stated in the context of the Onofri inequality on $\mathbb{S}^2$ in [Ghi05, Lemma 1], which itself refers to [Bae94, Corollary 3 p. 60] and [Kaw85]. A detailed statement can be found in [GM13, Lemma 17.1.2]. *Competing symmetries* is another aspect of symmetrization that we will not study in this paper and for which we refer to [CL92].

In [Rub08a], Y.A. Rubinstein gives a proof of the Onofri inequality that does not use symmetrization/rearrangement arguments. Also see [Rub08b] and in particular [Rub08b, Corollary 10.12] which contains a reinforced version of the inequality. In [CY87, Remark (1), page 217], there is another proof which does not rely on symmetry, based on a result in [Her70]. Another proof that went rather unnoticed is used in the paper of E. Fontenas [Fon97]. This approach is based on the so-called Γ_2 or *carré du champ* method. In the symmetric case the problem can be reduced to an inequality involving the ultraspherical operator that we will consider in Section 7 : see (3.9), with $\lambda = 1$. As far as we know, the first observation concerning this equivalent formulation can be found in [Ben93], although no justification of the symmetrization appears in this paper. In a series of recent papers, [Dol+14 ; Dol+13b ; DEL13a ; Dol+13a ; Dol+13c ; DEL13b] two of the authors have clarified the link

that connects the *carré du champ* method with rigidity results that can be found in [BVV91] and earlier papers. Even better, their method involves a nonlinear flow which produces a remainder term, which will be considered in Section 7.

Spherical harmonics play a crucial but hidden role, so we shall not insist on them and refer to [Bec93] and, in the symmetric case, to [GM13, Chapter 16] for further details. As quoted in [GM13], other variations on the Onofri-Moser-Trudinger inequality were given in [AT00 ; CC86 ; Flu92 ; MP89 ; CY88 ; CY87]. The question of dimensions higher than $d = 2$ is an entire topic by itself and one can refer to [Bec93 ; BFM07 ; Oki08 ; PD13] for some results in this direction. Various extensions of the Moser-Trudinger and Moser-Trudinger-Onofri inequalities have been proposed, which are out of the scope of this paper ; let us simply mention [LL13] as a contribution in this direction and refer the interested reader to references therein.

In this paper, we will neither cover issues related to conformal invariance, that were central in [Ono82], nor motivations arising from differential geometry. The reader interested in understanding how Onofri's inequality is related to the problem of prescribing the Gaussian curvature on $\mathbb{S}^2$ is invited to refer to [Cha87, Section 3] for an introductory survey, and to [CY88 ; CY87 ; CY03] for more details.

Onofri's inequality also has important applications, for instance in chemotaxis : see [GZ98 ; CC08] in the case of the Keller-Segel model.

As a conclusion of this review, we can list the main tools that we have found in the literature :

(T1) Existence by variational methods,

(T2) Symmetrization techniques which allow to reduce the problem for (1.1) to radial functions,

(T3) Identification of the solutions to the Euler-Lagrange equations (among radially symmetric functions),

(T4) Duality with the logarithmic Hardy-Littlewood-Sobolev inequality and study of the logarithmic Hardy-Littlewood-Sobolev inequality based on spherical harmonics and the Funk-Hecke formula,

(T5) Convexity methods related with the Prékopa-Leindler inequality,

(T6) Γ_2 or *carré du champ* methods,

(T7) Limiting procedures based on other functional inequalities.

With these tools we may try to summarize the strategies of proof that have been developed. The approach of E. Onofri is based on (T1)+(T2)+(T3), while (T4), (T5), (T6) and (T7) have been used in four independent and alternative strategies of proofs. None of them is elementary, in the sense that they rely on fundamental, deep or rather technical intermediate results.

In this paper, we intend to give new methods which, although not being elementary, are slightly simpler, or open new lines of thought. They also provide various additional terms which are all improvements. Several of them are based on the use of nonlinear flows, which, as far as we know, have not been really considered up to now, except in [Dol11 ; DJ14]. They

borrow some key issues from at least one of the above mentioned tools (T1-7) or enlarge the framework.

1. *Limiting procedures* based on other functional inequalities than Onofri's one, as in (T7), will be considered in Section 3. Six cases are studied, none of them being entirely new, but we thought that it was quite interesting to collect them. They also justify why we claim that "the Onofri inequality plays in dimension two a role similar to the Sobolev inequality in higher dimensions." Other preliminary results (linearization, and (T2) : symmetry results) will be considered in Sections 3 and 3.

2. Section 4 is devoted to a *mass transportation* approach of Onofri's inequality. Because of (T5), it was to be expected that such a technique would apply, at least formally (see Section 4). A rigorous proof is established in the symmetric case in Section 4 and the consistency with a mass transportation approach of Sobolev's inequalities is shown in Section 4. We have not found any result in this direction in the existing literature. (T2) is needed for a rigorous proof.

3. In Section 5, we will come back to *duality methods*, and get a first improvement to the standard Onofri inequality based on a simple *expansion of a square*. This has of course to do with (T4) and (T5) but Proposition 17 is, as far as we know, a new result. We also introduce the super-fast (or logarithmic) diffusion, which has striking properties in relation with Onofri's inequality and duality, but we have not been able to obtain an improvement of the inequality as it has been done in the case of Sobolev's inequalities in [DJ14].

4. In Section 6, we observe that in dimension $d = 2$, the Onofri inequality is the natural functional inequality associated with the *entropy–entropy production method for the fast diffusion equation with exponent* $m = 1/2$. It is remarkable that no singular limit has to be taken. Moreover, the entropy–entropy production method provides an integral remainder term which is new.

5. In the last section (Section 7), we establish *rigidity results*. Existence of optimal functions is granted by (T1). Our results are equivalent to whose obtained with Γ_2 or *carré du champ* methods. This had already been noticed in the literature (but the equivalence of the two methods has never been really emphasized as it should have been). For the sake of simplicity, we start by a proof in the symmetric case in Section 7. However, our method does not *a priori* require (T2) and directly provides essential properties for (T3), that is, the uniqueness of the solutions up to conformal invariance (for the critical value of a parameter, which corresponds to the first bifurcation point from the branch of the trivial constant solutions). Not only this point is remarkable, but we are also able to exhibit a nonlinear flow (in Section 7) which unifies the various approaches and provides a new integral remainder term. Our main results in this perspective are collected in Section 7.

3 Preliminaries

Onofri's inequality as a limit of various interpolation inequalities

Onofri's inequality appears as an endpoint of various families of interpolation inequalities and corresponds to a critical case in dimension $d = 2$ exactly like Sobolev's inequality when $d \geqslant 3$. This is why one can claim that it plays in dimension two a role similar to the Sobolev inequality in higher dimensions. Let us give some six examples of such limits, which are probably the easiest way of proving Onofri's inequality.

Onofri's inequality as a limit of interpolation inequalities on $\mathbb{S}^2$

On the sphere $\mathbb{S}^2$, one can derive the Onofri inequality from a family of interpolation inequalities on $\mathbb{S}^2$. We start from

$$\frac{q-2}{2}\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{S}^2)} + \|f\|^2_{\mathrm{L}^2(\mathbb{S}^2)} \geqslant \|f\|^2_{\mathrm{L}^q(\mathbb{S}^2)}\,, \tag{3.1}$$

which holds for any $f \in \mathrm{H}^1(\mathbb{S}^2)$. See [BVV91 ; Bec93 ; Dol+13b]. Proceeding as in [Bec93] (also see [DET08]), we choose $q = 2\,(1+t)$, $f = 1 + \frac{1}{2\,t}\,v$, for any positive t and use (3.1). This gives

$$\left(\frac{1}{4\,t}\int_{\mathbb{S}^2}|\nabla v|^2\,d\sigma + 1 + \frac{1}{t}\int_{\mathbb{S}^2} v\,d\sigma + \frac{1}{4\,t^2}\int_{\mathbb{S}^2}|v|^2\,d\sigma\right)^{1+t} \geqslant \int_{\mathbb{S}^2}\left|1+\frac{1}{2\,t}\,v\right|^{2\,(1+t)} d\sigma\,.$$

By taking the limit $t \to \infty$, we recover (1.2).

Onofri's inequality as a limit of Gagliardo-Nirenberg inequalities

Consider the following sub-family of Gagliardo-Nirenberg inequalities

$$\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \leqslant \mathsf{C}_{p,d}\,\|\nabla f\|^{\theta}_{\mathrm{L}^2(\mathbb{R}^d)}\,\|f\|^{1-\theta}_{\mathrm{L}^{p+1}(\mathbb{R}^d)}\,, \tag{3.2}$$

with $\theta = \theta(p) := \frac{p-1}{p}\,\frac{d}{d+2-p\,(d-2)}$, $1 < p \leqslant \frac{d}{d-2}$ if $d \geqslant 3$ and $1 < p < \infty$ if $d = 2$. Such an inequality holds for any smooth function f with sufficient decay at infinity and, by density, for any function $f \in \mathrm{L}^{p+1}(\mathbb{R}^d)$ such that ∇f is square integrable. We shall assume that $\mathsf{C}_{p,d}$ is the best possible constant. In [PD02], it has been established that equality holds in (3.2) if $f = F_p$ with

$$F_p(x) = (1+|x|^2)^{-\frac{1}{p-1}} \quad \forall\, x \in \mathbb{R}^d\,, \tag{3.3}$$

and that all extremal functions are equal to F_p up to multiplication by a constant, a translation and a scaling. If $d \geqslant 3$, the limit case $p = d/(d-2)$ corresponds to Sobolev's inequality and one recovers the results of T. Aubin and G. Talenti in [Aub76 ; Tal76], with $\theta = 1$: the optimal functions for it are, up to scalings, translations and multiplications by a constant, all equal to $F_{d/(d-2)}(x) = (1+|x|^2)^{-(d-2)/2}$, and

$$\mathsf{S}_d = \left(\mathsf{C}_{d/(d-2),\,d}\right)^2.$$

We can recover the Euclidean Onofri inequality as the limit case $d = 2$, $p \to \infty$ in the above family of inequalities, in the following way :

Proposition 14. *[Dol11] Assume that $u \in \mathcal{D}(\mathbb{R}^2)$ is such that $\int_{\mathbb{R}^2} u\,d\mu = 0$ and let*

$$f_p := F_p\left(1+\frac{u}{2\,p}\right)\,,$$

where F_p is defined by (3.3). Then we have

$$1 \leqslant \lim_{p\to\infty} \mathsf{C}_{p,2}\,\frac{\|\nabla f_p\|^{\theta(p)}_{\mathrm{L}^2(\mathbb{R}^2)}\,\|f_p\|^{1-\theta(p)}_{\mathrm{L}^{p+1}(\mathbb{R}^2)}}{\|f_p\|_{\mathrm{L}^{2p}(\mathbb{R}^2)}} = \frac{e^{\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx}}{\int_{\mathbb{R}^2} e^{\,u}\,d\mu}\,.$$

We recall that $\mu(x) := \frac{1}{\pi}\,(1+|x|^2)^{-2}$, and $d\mu(x) := \mu(x)\,dx$.

Démonstration. For completeness, let us give a short proof. We can rewrite (3.2) as

$$\frac{\int_{\mathbb{R}^2}|f|^{2p}\,dx}{\int_{\mathbb{R}^2}|F_p|^{2p}\,dx} \leqslant \left(\frac{\int_{\mathbb{R}^2}|\nabla f|^2\,dx}{\int_{\mathbb{R}^2}|\nabla F_p|^2\,dx}\right)^{\frac{p-1}{2}}\,\frac{\int_{\mathbb{R}^2}|f|^{p+1}\,dx}{\int_{\mathbb{R}^2}|F_p|^{p+1}\,dx}$$

and observe that, with $f = f_p$, we have :
(i) $\lim_{p\to\infty}\int_{\mathbb{R}^2}|F_p|^{2p}\,dx = \int_{\mathbb{R}^2}\frac{1}{(1+|x|^2)^2}\,dx = \pi$ and

$$\lim_{p\to\infty}\int_{\mathbb{R}^2}|f_p|^{2p}\,dx = \int_{\mathbb{R}^2}F_p^{2p}\,(1+\tfrac{u}{2p})^{2p}\,dx = \int_{\mathbb{R}^2}\frac{e^u}{(1+|x|^2)^2}\,dx\,,$$

so that $\int_{\mathbb{R}^2}|f_p|^{2p}\,dx/\int_{\mathbb{R}^2}|F_p|^{2p}\,dx$ converges to $\int_{\mathbb{R}^2}e^{\,u}\,d\mu$ as $p\to\infty$.
(ii) $\int_{\mathbb{R}^2}|F_p|^{p+1}\,dx = (p-1)\,\pi/2$, $\lim_{p\to\infty}\int_{\mathbb{R}^2}|f_p|^{p+1}\,dx = \infty$, but

$$\lim_{p\to\infty}\frac{\int_{\mathbb{R}^2}|f_p|^{p+1}\,dx}{\int_{\mathbb{R}^2}|F_p|^{p+1}\,dx} = 1\,.$$

(iii) Expanding the square and integrating by parts, we find that

$$\int_{\mathbb{R}^2}|\nabla f_p|^2\,dx = \frac{1}{4p^2}\int_{\mathbb{R}^2}F_p^2\,|\nabla u|^2\,dx - \int_{\mathbb{R}^2}(1+\tfrac{u}{2p})^2\,F_p\,\Delta F_p\,dx$$
$$= \frac{1}{4p^2}\int_{\mathbb{R}^2}|\nabla u|^2\,dx + \frac{2\pi}{p+1} + o(p^{-2})\,.$$

Here we have used $\int_{\mathbb{R}^2}|\nabla F_p|^2\,dx = \frac{2\pi}{p+1}$ and the condition $\int_{\mathbb{R}^2}u\,d\mu = 0$ in order to discard one additional term of the order of p^{-2}. On the other hand, we find that

$$\left(\frac{\int_{\mathbb{R}^2}|\nabla f_p|^2\,dx}{\int_{\mathbb{R}^2}|\nabla F_p|^2\,dx}\right)^{\frac{p-1}{2}} \sim \left(1+\frac{p+1}{8\,\pi\,p^2}\int_{\mathbb{R}^2}|\nabla u|^2\,dx\right)^{\frac{p-1}{2}} \sim e^{\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx}$$

as $p\to\infty$. Collecting these estimates concludes the proof. □

Onofri's inequality as a limit of Sobolev inequalities

Another way to derive Onofri's inequality is to consider the usual optimal Sobolev inequalities in $\mathbb{R}^2$, written for an $\mathrm{L}^p(\mathbb{R}^2)$ norm of the gradient, for an arbitrary $p\in(1,2)$. This method is inspired by [PD13], which is devoted to inequalities in exponential form in dimensions $d\geqslant 2$. See in particular [PD13, Example 1.2]. In the special case $p\in(1,2)$, $d=2$, let us consider the Sobolev inequality

$$\|f\|^p_{\mathrm{L}^{\frac{2\,p}{2-p}}(\mathbb{R}^2)} \leqslant \mathsf{C}_p\,\|\nabla f\|^p_{\mathrm{L}^p(\mathbb{R}^2)} \quad \forall\, f\in\mathcal{D}(\mathbb{R}^2)\,, \tag{3.4}$$

where equality is achieved by the Aubin-Talenti extremal profile

$$f_\star(x) = \left(1 + |x|^{\frac{p}{p-1}}\right)^{-\frac{2-p}{p}} \qquad \forall\, x \in \mathbb{R}^2\,.$$

The extremal functions were already known from the celebrated papers by T. Aubin and G. Talenti, [Aub76 ; Tal76]. See also [Bli30 ; Ros71] for earlier related computations, which provided the value of some of the best constants. It is easy to check that $f_\star$ solves

$$-\Delta_p f_\star = 2\left(\frac{2-p}{p-1}\right)^{p-1} f_\star^{\frac{2\,p}{2-p}-1}\,,$$

hence

$$\|\nabla f_\star\|^p_{\mathrm{L}^p(\mathbb{R}^2)} = \frac{1}{\mathsf{C}_p}\,\|f_\star\|^p_{\mathrm{L}^{\frac{2\,p}{2-p}}(\mathbb{R}^2)} = 2\left(\frac{2-p}{p-1}\right)^{p-1} \|f_\star\|^{\frac{2\,p}{2-p}}_{\mathrm{L}^{\frac{2\,p}{2-p}}(\mathbb{R}^2)}\,,$$

so that the optimal constant is

$$\mathsf{C}_p = \frac{1}{2}\left(\frac{p-1}{2-p}\right)^{p-1}\left(\frac{p^2\,\sin(2\,\pi/p)}{2\,(p-1)\,(p-2)\,\pi^2}\right)^{\frac{p}{2}}\,.$$

We can study the limit $p \to 2_-$ in order to recover the Onofri inequality by considering $f = f_\star\left(1 + \frac{2-p}{2\,p}\,u\right)$, where u is a given smooth, compactly supported function, and $\varepsilon = \frac{2-p}{2\,p}$. A direct computation gives

$$\lim_{p\to 2_-} \int_{\mathbb{R}^2} f^{\frac{2\,p}{2-p}}\,dx = \int_{\mathbb{R}^2} \frac{e^u}{(1+|x|^2)^2}\,dx = \pi \int_{\mathbb{R}^2} e^u\,d\mu\,,$$

and

$$\int_{\mathbb{R}^2} |\nabla f|^p\,dx = 2\,\pi\,(2-p)\left[1 + \tfrac{2-p}{2}\int_{\mathbb{R}^2} u\,d\mu\right] + (\tfrac{2-p}{2\,p})^p \int_{\mathbb{R}^2} |\nabla u|^2\,dx + o((2-p)^2)\,,$$

as $p \to 2_-$. By taking the logarithm of both sides of (3.4), we get

$$\begin{aligned}\frac{2-p}{2}\,\log\left(\int_{\mathbb{R}^2} e^u\,d\mu\right) &\sim \frac{2-p}{2}\,\log\left(\frac{\int_{\mathbb{R}^2} f^{\frac{2\,p}{2-p}}\,dx}{\int_{\mathbb{R}^2} f_\star^{\frac{2\,p}{2-p}}\,dx}\right)\\ &\leqslant \log\left(\frac{\int_{\mathbb{R}^2} |\nabla f|^p\,dx}{\int_{\mathbb{R}^2} |\nabla f_\star|^p\,dx}\right)\\ &= \log\left(1 + \tfrac{2-p}{2}\int_{\mathbb{R}^2} u\,d\mu + \tfrac{2-p}{32\,\pi}\int_{\mathbb{R}^2} |\nabla u|^2\,dx + o(2-p)\right)\end{aligned}$$

Gathering the terms of order $2-p$, we recover the Euclidean Onofri inequality by passing to the limit $p \to 2_-$.

The radial Onofri inequality as a limit when $d \to 2$

Although this approach is restricted to radially symmetric functions, one of the most striking way to justify the fact that the Onofri inequality plays in dimension two a role similar to the Sobolev inequality in higher dimensions goes as follows. To start with, one can

consider the Sobolev inequality applied to radially symmetric functions only. The dimension d can now be considered as a real parameter. Then, by taking the limit $d \to 2$, one can recover a weaker (*i.e.* for radial functions only) version of the Onofri inequality. The details of the computation, taken from [DJ14], follow.

Consider the radial Sobolev inequality

$$\mathsf{s}_d \int_0^\infty |f'|^2 \, r^{d-1} \, dr \geqslant \left(\int_0^\infty |f|^{\frac{2\,d}{d-2}} \, r^{d-1} \, dr \right)^{1-\frac{2}{d}} , \tag{3.5}$$

with optimal constant

$$\mathsf{s}_d = \frac{4}{d\,(d-2)} \left(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d}{2}\right)} \right)^{\frac{2}{d}} .$$

We may pass to the limit in (3.5) with the choice

$$f(r) = f_\star(r) \left(1 + \tfrac{d-2}{2\,d}\, u\right) ,$$

where $f_\star(r) = (1+r^2)^{-\frac{d-2}{2}}$ gives the equality case in (3.5), to get the radial version of Onofri's inequality for u. By expanding the expression of $|f'|^2$ we get

$$f'^2 = f_\star'^2 + \frac{d-2}{d}\, f_\star' \, (f_\star\, u)' + \left(\frac{d-2}{2\,d}\right)^2 \left(f_\star'\, u + f_\star\, u'\right)^2 .$$

We have

$$\lim_{d \to 2_+} \int_0^\infty |f_\star \, (1 + \tfrac{d-2}{2\,d}\, u)|^{\frac{2\,d}{d-2}} \, r^{d-1} \, dr = \int_0^\infty e^u \, \frac{r\, dr}{(1+r^2)^2} ,$$

so that, as $d \to 2_+$,

$$\left(\int_0^\infty |f_\star \, (1 + \tfrac{d-2}{2\,d}\, u)|^{\frac{2\,d}{d-2}} \, r^{d-1} \, dr \right)^{\frac{d-2}{d}} - 1 \sim \frac{d-2}{2} \, \log \left(\int_0^\infty e^u \, \frac{r\, dr}{(1+r^2)^2} \right) .$$

Also, using the fact that

$$\mathsf{s}_d = \frac{1}{d-2} + \frac{1}{2} - \frac{1}{2} \log 2 + o(1) \quad \text{as} \quad d \to 2_+ ,$$

we have

$$\mathsf{s}_d \int_0^\infty |f'|^2 \, r^{d-1} \, dr \sim 1 + (d-2) \left[\frac{1}{8} \int_0^\infty |u'|^2 \, r \, dr + \int_0^\infty u \, \frac{2\, r\, dr}{(1+r^2)^2} \right] .$$

By keeping only the highest order terms, which are of the order of $(d-2)$, and passing to the limit as $d \to 2_+$ in (3.5), we obtain that

$$\frac{1}{8} \int_0^\infty |u'|^2 \, r \, dr + \int_0^\infty u \, \frac{2\, r\, dr}{(1+r^2)^2} \geqslant \log \left(\int_0^\infty e^u \, \frac{2\, r\, dr}{(1+r^2)^2} \right) ,$$

which is Onofri's inequality written for radial functions.

Onofri's inequality as a limit of Caffarelli-Kohn-Nirenberg inequalities

Onofri's inequality can be obtained as the limit in a familly of Caffarelli-Kohn-Nirenberg inequalities, as was first done in [DET08].

Let $2^* := \infty$ if $d = 1$ or 2, $2^* := 2\,d/(d-2)$ if $d \geqslant 3$ and $a_c := (d-2)/2$. Consider the space $\mathcal{D}_a^{1,2}(\mathbb{R}^d)$ obtained by completion of $\mathcal{D}(\mathbb{R}^d\backslash\{0\})$ with respect to the norm $u \mapsto \|\,|x|^{-a}\,\nabla u\,\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$. In this section, we shall consider the Caffarelli-Kohn-Nirenberg inequalities

$$\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{bp}}\,dx\right)^{\frac{2}{p}} \leqslant \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2a}}\,dx\,. \tag{3.6}$$

These inequalities generalize to $\mathcal{D}_a^{1,2}(\mathbb{R}^d)$ the Sobolev inequality and in particular the exponent p is given in terms of a and b by

$$p = \frac{2\,d}{d-2+2\,(b-a)}\,,$$

as can be checked by a simple scaling argument. A precise statement on the range of validity of (3.6) goes as follows.

Lemma 15. *[CKN84] Let $d \geqslant 1$. For any $p \in [2, 2^*]$ if $d \geqslant 3$ or $p \in [2, 2^*)$ if $d = 1$ or 2, there exists a positive constant $\mathsf{C}_{a,b}$ such that (3.6) holds if a, b and p are related by $b = a - a_c + d/p$, with the restrictions $a < a_c$, $a \leqslant b \leqslant a+1$ if $d \geqslant 3$, $a < b \leqslant a+1$ if $d = 2$ and $a + 1/2 < b \leqslant a+1$ if $d = 1$.*

We shall restrict our purpose to the case of dimension $d = 2$. For any $\alpha \in (-1, 0)$, let us denote by $d\mu_\alpha$ the probability measure on $\mathbb{R}^2$ defined by $d\mu_\alpha := \mu_\alpha\,dx$ where

$$\mu_\alpha := \frac{1+\alpha}{\pi}\,\frac{|x|^{2\,\alpha}}{(1+|x|^{2\,(1+\alpha)})^2}\,.$$

It has been established in [DET08] that

$$\log\left(\int_{\mathbb{R}^2} e^{\,u}\,d\mu_\alpha\right) - \int_{\mathbb{R}^2} u\,d\mu_\alpha \leqslant \frac{1}{16\,\pi\,(1+\alpha)} \int_{\mathbb{R}^2} |\nabla u|^2\,dx \quad \forall\, u \in \mathcal{D}(\mathbb{R}^2)\,, \tag{3.7}$$

where $\mathcal{D}(\mathbb{R}^2)$ is the space of smooth functions with compact support. By density with respect to the natural norm defined by each of the inequalities, the result also holds on the corresponding Orlicz space.

We adopt the strategy of [DET08, Section 2.3] to pass to the limit in (3.6) as $(a, b) \to (0, 0)$ with $b = \frac{\alpha}{\alpha+1}\,a$. Let

$$a_\varepsilon = -\frac{\varepsilon}{1-\varepsilon}\,(\alpha+1)\,, \quad b_\varepsilon = a_\varepsilon + \varepsilon, \quad p_\varepsilon = \frac{2}{\varepsilon}\,,$$

and

$$u_\varepsilon(x) = \left(1+|x|^{2\,(\alpha+1)}\right)^{-\frac{\varepsilon}{1-\varepsilon}}\,.$$

Assuming that u_ε is an optimal function for (3.6), define

$$\kappa_\varepsilon = \int_{\mathbb{R}^2} \left[\frac{u_\varepsilon}{|x|^{a_\varepsilon+\varepsilon}}\right]^{2/\varepsilon} dx = \int_{\mathbb{R}^2} \frac{|x|^{2\,\alpha}}{\left(1+|x|^{2\,(1+\alpha)}\right)^2}\,\frac{u_\varepsilon^2}{|x|^{2a_\varepsilon}}\,dx = \frac{\pi}{\alpha+1}\,\frac{\Gamma\left(\frac{1}{1-\varepsilon}\right)^2}{\Gamma\left(\frac{2}{1-\varepsilon}\right)}\,,$$

$$\lambda_\varepsilon = \int_{\mathbb{R}^2} \left[\frac{|\nabla u_\varepsilon|}{|x|^a}\right]^2 dx = 4\,a_\varepsilon^2 \int_{\mathbb{R}^2} \frac{|x|^{2\,(2\,\alpha+1-a_\varepsilon)}}{\left(1+|x|^{2\,(1+\alpha)}\right)^{\frac{2}{1-\varepsilon}}}\,dx = 4\,\pi\,\frac{|a_\varepsilon|}{1-\varepsilon}\,\frac{\Gamma\left(\frac{1}{1-\varepsilon}\right)^2}{\Gamma\left(\frac{2}{1-\varepsilon}\right)}\,.$$

Then $w_\varepsilon = (1+\frac12\,\varepsilon\,u)\,u_\varepsilon$ is such that

$$\lim_{\varepsilon\to 0_+} \frac{1}{\kappa_\varepsilon}\int_{\mathbb{R}^2} \frac{|w_\varepsilon|^{p_\varepsilon}}{|x|^{b_\varepsilon p_\varepsilon}}\,dx = \int_{\mathbb{R}^2} e^u\,d\mu_\alpha\,,$$
$$\lim_{\varepsilon\to 0_+} \frac{1}{\varepsilon}\left[\frac{1}{\lambda_\varepsilon}\int_{\mathbb{R}^2} \frac{|\nabla w_\varepsilon|^2}{|x|^{2a_\varepsilon}}\,dx - 1\right] = \int_{\mathbb{R}^2} u\,d\mu_\alpha + \frac{1}{16\,(1+\alpha)\,\pi}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^2)}\,.$$

Limits of some Gagliardo-Nirenberg inequalities on the line

Onofri's inequality on the cylinder, (1.3) can also be recovered by a limiting process, in the symmetric case. As far as we know, this method for proving the inequality is new, but a clever use of the Emden-Fowler transformation and of the results based on the Caffarelli-Kohn-Nirenberg inequalities shows that this was to be expected. See [DET08] for more considerations in this direction.

Consider the Gagliardo-Nirenberg inequalities on the line

$$\|f\|_{\mathrm{L}^p(\mathbb{R})} \leqslant \mathsf{C}^p_{\mathrm{GN}}\,\|f'\|^\theta_{\mathrm{L}^2(\mathbb{R})}\,\|f\|^{1-\theta}_{\mathrm{L}^2(\mathbb{R})} \quad \forall\, f\in \mathrm{H}^1(\mathbb{R})\,,$$

with $\theta = \frac{p-2}{2\,p}$, $p>2$. Equality is achieved by the function

$$f_\star(x) := (\cosh s)^{-\frac{2}{p-1}} \quad \forall\, s\in\mathbb{R}\,.$$

See [Dol+13a] for details. By taking the logarithm of both sides of the inequality, we find that

$$\frac{2}{p}\,\log\left(\frac{\int_{\mathbb{R}} f^p\,ds}{\int_{\mathbb{R}} f_\star^p\,ds}\right) \leqslant \theta\,\log\left(\frac{\int_{\mathbb{R}} |f'|^2\,ds}{\int_{\mathbb{R}} |f_\star'|^2\,ds}\right) + (1-\theta)\,\log\left(\frac{\int_{\mathbb{R}} f^2\,ds}{\int_{\mathbb{R}} f_\star^2\,ds}\right)$$

and elementary computations show that as $p\to+\infty$, $f_\star^p \to 2\,\xi$ and $-f_\star\,f_\star'' \sim \frac{4}{p}\,\xi$ with $\xi(s) := \frac12\,(\cosh s)^{-2}$. If we take $f = f_\star\,(1+w/p)$, we have

$$\lim_{p\to\infty}\int_{\mathbb{R}} f^p\,ds = 2\int_{\mathbb{R}} e^w\,\xi\,ds\,,$$
$$\lim_{p\to\infty}\log\left(\frac{\int_{\mathbb{R}} f^p\,ds}{\int_{\mathbb{R}} f_\star^p\,ds}\right) = \log\left(\int_{\mathbb{R}} e^w\,\xi\,ds\right)\,.$$

We can also compute

$$\int_{\mathbb{R}} |f_\star|^2\,ds = \frac{p-1}{2} + 2\,\log 2 + O\left(\tfrac1p\right)$$

and

$$\int_{\mathbb{R}} |f|^2\,ds = \int_{\mathbb{R}} |f_\star|^2\,(1+\tfrac2p\,w+\tfrac{1}{p^2}\,w^2)\,ds = \frac{p-1}{2} + 2\,\log 2 + O\left(\tfrac1p\right)$$

as $p\to+\infty$, so that

$$\frac{\int_{\mathbb{R}} f^2\,ds}{\int_{\mathbb{R}} f_\star^2\,ds} - 1 = O\left(\tfrac{1}{p^2}\right) \quad\text{and}\quad \lim_{p\to\infty} p\,\log\left(\frac{\int_{\mathbb{R}} f^2\,ds}{\int_{\mathbb{R}} f_\star^2\,ds}\right) = 0\,.$$

For the last term, we observe that, pointwise,

$$-f_\star\,f_\star'' \sim \frac{2}{p}\,\frac{1}{(\cosh s)^2}$$

and

$$\int_{\mathbb{R}} |f_\star'|^2 \, ds = -\int_{\mathbb{R}} f_\star \, f_\star'' \, ds = \frac{2}{p} + O\big(\tfrac{1}{p^2}\big) \quad \text{as} \quad p \to +\infty \, .$$

Passing to the limit as $p \to +\infty$, we get that

$$\begin{aligned} \int_{\mathbb{R}} |f'|^2 \, ds &= \frac{1}{p^2} \int_{\mathbb{R}} f_\star^2 \, |w'|^2 \, ds - \int_{\mathbb{R}} f_\star \, f_\star'' \left(1 + \frac{w}{p}\right)^2 ds \\ &= \frac{1}{p^2} \int_{\mathbb{R}} |w'|^2 \, ds + \frac{2}{p} \left(1 + \frac{4}{p} \int_{\mathbb{R}} w \, \xi \, ds\right) + o\big(\tfrac{1}{p^2}\big) \, , \end{aligned}$$

and finally

$$\log\left(\frac{\int_{\mathbb{R}} |f'|^2 \, ds}{\int_{\mathbb{R}} |f_\star'|^2 \, ds}\right) \sim \frac{1}{p} \left(4 \int_{\mathbb{R}} w \, \xi \, ds + \frac{1}{2} \int_{\mathbb{R}} |w'|^2 \, ds\right) + o\big(\tfrac{1}{p}\big) \, .$$

Collecting terms, we find that

$$\frac{1}{8} \int_{\mathbb{R}} |w'|^2 \, ds \geqslant \log\left(\int_{\mathbb{R}} e^w \, \xi \, ds\right) - \int_{\mathbb{R}} w \, \xi \, ds \, .$$

Linearization and optimal constant

Consider (1.2) and define

$$\mathcal{I}_\lambda := \inf_{\substack{v \in \mathrm{H}^1(\mathbb{S}^2) \\ \int_{\mathbb{S}^2} v \, d\sigma > 0}} \mathcal{Q}_\lambda[v] \quad \text{with} \quad \mathcal{Q}_\lambda[v] := \frac{\frac{1}{4} \int_{\mathbb{S}^2} |\nabla v|^2 \, d\sigma + \lambda \int_{\mathbb{S}^2} v \, d\sigma}{\log\left(\int_{\mathbb{S}^2} e^v \, d\sigma\right)} \, .$$

By Jensen's inequality, $\log\left(\int_{\mathbb{S}^2} e^v \, d\sigma\right) \geqslant \int_{\mathbb{S}^2} v \, d\sigma > 0$, so that $\mathcal{I}_\lambda$ is well defined and nonnegative for any $\lambda > 0$. Since constant functions are admissible, we also know that

$$\mathcal{I}_\lambda \leqslant \lambda \, ,$$

for any $\lambda > 0$. Moreover, since $\lambda \mapsto \mathcal{Q}_\lambda[v]$ is affine, we know that $\lambda \mapsto \mathcal{I}_\lambda$ is concave and continuous. Assume now that $\int_{\mathbb{S}^2} v \, d\sigma = 0$ and for any $c > 0$, let us consider

$$\mathcal{Q}_\lambda[v + c] = \frac{\frac{1}{4} \int_{\mathbb{S}^2} |\nabla v|^2 \, d\sigma + \lambda \, c}{c + \log\left(\int_{\mathbb{S}^2} e^v \, d\sigma\right)} \geqslant \frac{\log\left(\int_{\mathbb{S}^2} e^v \, d\sigma\right) + \lambda \, c}{c + \log\left(\int_{\mathbb{S}^2} e^v \, d\sigma\right)} \, , \tag{3.8}$$

where the inequality follows from (1.2). It is clear that for such functions v,

$$\begin{aligned} &\lim_{c \to +\infty} \mathcal{Q}_\lambda[v + c] = \lambda \, , \\ &\lim_{c \to 0_+} \mathcal{Q}_\lambda[v + c] = \frac{\int_{\mathbb{S}^2} |\nabla v|^2 \, d\sigma}{\log\left(\int_{\mathbb{S}^2} e^v \, d\sigma\right)} = \mathcal{Q}_\lambda[v] \, . \end{aligned}$$

If $\lambda < 1$, using (3.8), we can write that for all $c > 0$,

$$\mathcal{Q}_\lambda[v + c] \geqslant \lambda + (1 - \lambda) \, \frac{\log\left(\int_{\mathbb{S}^2} e^v \, d\sigma\right)}{c + \log\left(\int_{\mathbb{S}^2} e^v \, d\sigma\right)} \geqslant \lambda \, ,$$

thus proving that $\mathcal{I}_\lambda = \lambda$ is optimal when $\lambda < 1$.

When $\lambda \geqslant 1$, we may take $v = \varepsilon \, \phi$, where ϕ is an eigenfunction of the Laplace-Beltrami operator $-\Delta_{\mathbb{S}^2}$ on the sphere $\mathbb{S}^2$, such that $-\Delta_{\mathbb{S}^2} \phi = 2 \, \phi$ and take the limit as $\varepsilon \to 0_+$, so that

$\int_{\mathbb{S}^2} |\nabla v|^2 \, d\sigma = \varepsilon^2 \int_{\mathbb{S}^2} |\nabla \phi|^2 \, d\sigma = 2\,\varepsilon^2 \int_{\mathbb{S}^2} |\phi|^2 \, d\sigma$ and $\log\left(\int_{\mathbb{S}^2} e^v \, d\sigma\right) = \log\left(1 + \frac{1}{2}\,\varepsilon^2 \int_{\mathbb{S}^2} \phi^2 \, d\sigma + o(\varepsilon^2)\right)$. Collecting terms, we get that

$$\lim_{\varepsilon \to 0_+} \mathcal{Q}_\lambda[\varepsilon\,\phi] = 1\,.$$

Altogether, we have found that

$$\mathcal{I}_\lambda = \min\{\lambda, 1\} \quad \forall\, \lambda > 0\,.$$

Symmetrization results

For the sake of completeness, let us state a result of symmetry. Consider the functional

$$\mathcal{G}_\lambda[v] := \frac{1}{4} \int_{\mathbb{S}^2} |\nabla v|^2 \, d\sigma + \lambda \int_{\mathbb{S}^2} v \, d\sigma - \log\left(\int_{\mathbb{S}^2} e^v \, d\sigma\right)\,,$$

and denote by $\mathrm{H}^1_*(\mathbb{S}^2)$ the function in $\mathrm{H}^1(\mathbb{S}^2)$ which depend only on the azimuthal angle (latitude), that we shall denote by $\theta \in [0, \pi]$.

Proposition 16. *For any $\lambda > 0$,*

$$\inf_{v \in \mathrm{H}^1(\mathbb{S}^2)} \mathcal{G}_\lambda[v] = \inf_{v \in \mathrm{H}^1_*(\mathbb{S}^2)} \mathcal{G}_\lambda[v]\,.$$

We refer to [GM13, Lemma 17.1.2] for a proof of the symmetry result and to Section 2 for further historical references.

Hence, for any function $v \in \mathrm{H}^1(\mathbb{S}^2)$, the inequality $\mathcal{G}_1[v] \geqslant 0$ reads

$$\frac{1}{8} \int_0^\pi |v'(\theta)|^2 \, \sin\theta \, d\theta + \frac{1}{2} \int_0^\pi v(\theta) \, \sin\theta \, d\theta \geqslant \log\left(\frac{1}{2} \int_0^\pi e^v \, \sin\theta \, d\theta\right)\,.$$

The change of variables $z = \cos\theta$, $v(\theta) = f(z)$ allows to rewrite this inequality as

$$\frac{1}{8} \int_{-1}^1 |f'|^2 \, \nu \, dz + \frac{1}{2} \int_{-1}^1 f \, dz \geqslant \log\left(\frac{1}{2} \int_{-1}^1 e^f \, dz\right)\,, \tag{3.9}$$

where $\nu(x) := 1 - z^2$. Let us define the *ultraspherical* operator $\mathcal{L}$ by $\langle f_1, \mathcal{L}\, f_2\rangle = -\int_{-1}^1 f_1' \, f_2' \, \nu \, dz$ where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on $\mathrm{L}^2(-1, 1; dz)$. Explicitly we have :

$$\mathcal{L}\, f := (1 - z^2)\, f'' - 2\, z\, f' = \nu\, f'' + \nu'\, f'$$

and (3.9) simply means

$$-\frac{1}{8} \langle f, \mathcal{L}\, f\rangle + \frac{1}{2} \int_{-1}^1 f \, \nu \, dz \geqslant \log\left(\frac{1}{2} \int_{-1}^1 e^f \, \nu \, dz\right)\,.$$

4 Mass Transportation

Since Onofri's inequality appears as a limit case of Sobolev's inequalities which can be proved by mass transportation according to [CENV04], it makes a lot of sense to look for a proof based on such techniques. Let us start by recalling some known results.

Assume that F and G are two probability distributions on $\mathbb{R}^2$ and consider the convex function ϕ such that G is the *push-forward* of F through $\nabla\phi$

$$\nabla\phi_* F = G\,,$$

where $\nabla\phi$ is the *Brenier map* and ϕ solves the Monge-Ampère equation

$$F = G(\nabla\phi)\det\left(\mathrm{Hess}(\phi)\right) \quad \text{on } \mathbb{R}^d\,. \tag{4.1}$$

See [McC95] for details. Here $d = 2$ but to emphasize the role of the dimension, we will keep it as a parameter for a while. The Monge-Ampère equation (4.1) holds in the $F\,dx$ sense almost everywhere according to [McC97, Remark 4.5], as discussed in [CENV04]. By now this strategy is standard and we shall refer to [Vil09] for background material and technical issues that will be omitted here. We can for instance assume that F and G are smooth functions and argue by density afterwards.

Formal approach

Let us start by a formal computation. Using (4.1), since

$$G(\nabla\phi)^{-\frac1d} = F^{-\frac1d}\det\left(\mathrm{Hess}(\phi)\right)^{\frac1d} \leqslant \frac1d\,F^{-\frac1d}\,\Delta\phi$$

by the arithmetic-geometric inequality, we get the estimate

$$\int_{\mathbb{R}^d} G(y)^{1-\frac1d}\,dy = \int_{\mathbb{R}^d} G(\nabla\phi)^{1-\frac1d}\det\left(\mathrm{Hess}(\phi)\right)\,dx \leqslant \frac1d\int_{\mathbb{R}^d} F^{1-\frac1d}(x)\,\Delta\phi\,dx \tag{4.2}$$

using the change of variables $y = \nabla\phi(x)$ and (4.1). Assume that

$$G(y) = \mu(y) = \frac{1}{\pi\,(1+|y|^2)^2} \quad \forall\, y\in\mathbb{R}^d$$

and

$$F = \mu\,e^u\,.$$

With $d = 2$, we obtain

$$\begin{aligned}4\int_{\mathbb{R}^2}\sqrt{\mu}\,dx = 2\,d\int_{\mathbb{R}^d} G(y)^{1-\frac1d}\,dy\\ = 2\int_{\mathbb{R}^d} F^{1-\frac1d}(x)\,\Delta\phi\,dx = -\int_{\mathbb{R}^2}\nabla\log F\cdot\sqrt{F}\,\nabla\phi\,dx\\ = -\int_{\mathbb{R}^2}(\nabla\log\mu+\nabla u)\cdot\sqrt{F}\,\nabla\phi\,dx\,,\end{aligned}$$

which can be estimated using the Cauchy-Schwarz inequality by

$$\begin{aligned}16\left(\int_{\mathbb{R}^2}\sqrt{\mu}\,dx\right)^2 = \left(\int_{\mathbb{R}^2}(\nabla\log\mu+\nabla u)\cdot\sqrt{F}\,\nabla\phi\,dx\right)^2\\ \leqslant \int_{\mathbb{R}^2}|\nabla u+\nabla\log\mu|^2\,dx\int_{\mathbb{R}^2}F\,|\nabla\phi|^2\,dx\,.\end{aligned}$$

If we expand the square, that is, if we write

$$\begin{aligned}\int_{\mathbb{R}^2}|\nabla u+\nabla\log\mu|^2\,dx\\ = \int_{\mathbb{R}^2}|\nabla u|^2\,dx - 2\int_{\mathbb{R}^2}u\,\Delta\log\mu\,dx + \int_{\mathbb{R}^2}|\nabla\log\mu|^2\,dx\,,\end{aligned}$$

after recalling that

$$-\Delta \log \mu = 8\,\pi\,\mu\ ,$$

and after undoing the change of variables $y = \nabla\phi(x)$, so that we get

$$\int_{\mathbb{R}^2} F\,|\nabla\phi|^2\,dx = \int_{\mathbb{R}^2} G(y)\,|y|^2\,dx = \int_{\mathbb{R}^2} \mu\,|y|^2\,dx\,,$$

we end up, after collecting the terms, with

$$\frac{16\left(\int_{\mathbb{R}^2}\sqrt{\mu}\;dx\right)^2}{\int_{\mathbb{R}^2}\mu\,|y|^2\,dx} - \int_{\mathbb{R}^2}|\nabla\log\mu|^2\,dx \leqslant \int_{\mathbb{R}^2}|\nabla u|^2\,dx + 16\,\pi\int_{\mathbb{R}^2} u\,d\mu\,.$$

Still at a formal level, we may observe that

$$\begin{aligned} 16\left(\int_{\mathbb{R}^2}\sqrt{\mu}\;dx\right)^2 &= \left(-\,2\int_{\mathbb{R}^2} y\cdot\nabla\sqrt{\mu}\;dx\right)^2 \\ &= \left(\int_{\mathbb{R}^2} y\,\sqrt{\mu}\cdot\nabla\log\mu\;dx\right)^2 \\ &\leqslant \int_{\mathbb{R}^2}\mu\,|y|^2\,dx\int_{\mathbb{R}^2}|\nabla\log\mu|^2\,dx \end{aligned}$$

by the Cauchy-Schwarz inequality. This would prove Onofri's inequality since $\log\left(\int_{\mathbb{R}^2} e^u\,d\mu\right) = \log\left(\int_{\mathbb{R}^2} F\,dx\right) = 0$, if $y\mapsto\sqrt{\mu}$, $y\mapsto\mu\,|y|^2$ and $y\mapsto|\nabla\log\mu|^2$ were integrable, but this is not the case. As we shall see in the next section, this issue can be solved by working on balls.

The radially symmetric case

When F and G are assumed to depend only on $r = |x|$, so that we may write that $|y| = s = \varphi(r)$, then (4.1) becomes

$$(G\circ\varphi')\left(\frac{\varphi'}{r}\right)^{d-1}\varphi'' = F$$

what allows to compute φ' using

$$\int_0^{\varphi'(R)} G(s)\;s^{d-1}\,ds = \int_0^R (G\circ\varphi')\left(\frac{\varphi'}{r}\right)^{d-1}\varphi''\;r^{d-1}\,dr = \int_0^R F(r)\;r^{d-1}\,dr\,.$$

With a straightforward abuse of notation we shall indifferently write that F is a function of x or of r and G a function of y or s.

The proof is similar to the one in Section 4 except that all integrals can be restricted to a ball B_R of radius $R>0$ with center at the origin. Assume that $G = \mu/Z_R$, $F = e^u\,\mu/Z_R$ where $Z_R = \int_{B_R}\mu\,dx$ and u has compact support inside the ball B_R. An easy computation shows that

$$Z_R = \frac{R^2}{1+R^2}\quad\forall\,R>0\,.$$

We shall also assume that u is normalized so that $\int_{B_R} F\,dx = 1$.

All computations are now done on B_R. The only differences with Section 4 arise from the integrations by parts, so we have to handle two additional terms :

$$\int_{B_R} F^{1-\frac{1}{d}}(x)\,\Delta\phi\,dx + \frac{1}{2}\int_{B_R}\nabla\log F\cdot\sqrt{F}\,\nabla\phi\,dx \\ = \pi\,R\,\sqrt{F(R)}\,\varphi'(R) = \pi\,R\,\sqrt{\mu(R)/Z_R}\,\varphi'(R)$$

and

$$2\int_{B_R}\nabla u\cdot\nabla\log\mu\,dx + 2\int_{B_R} u\,\Delta\log\mu\,dx = 4\,\pi\,R\,(\log\mu)'(R)\,u(R) = 0\,.$$

If we fix u (smooth, with compact support) and let $R\to\infty$, then it is clear that none of these two terms plays a role. Notice that there exists a constant κ such that

$$\frac{(\varphi'(R))^2}{1+(\varphi'(R))^2} = \frac{R^2}{1+R^2} + \kappa$$

for large values of R, and hence $\varphi'(R)\sim R$. Hence,

$$\lim_{R\to\infty}\pi\,R\,\sqrt{\mu(R)/Z_R}\,\varphi'(R) = \sqrt{\pi}\,.$$

After collecting the terms, we obtain

$$\frac{16\left(\int_{B_R}\sqrt{\mu}\,dy-\sqrt{\pi}\right)^2}{\int_{B_R}\mu\,|y|^2\,dy} - \int_{B_R}|\nabla\log\mu|^2\,dy + o(1) \\ \leqslant \int_{\mathbb{R}^2}|\nabla u|^2\,dx - 16\,\pi\int_{\mathbb{R}^2} u\,d\mu$$

as $R\to\infty$. Now the Cauchy-Schwarz inequality has to be written as

$$16\left(\int_{B_R}\sqrt{\mu}\,dy-\sqrt{\pi}\right)^2 = \left(-2\int_{B_R} y\cdot\nabla\sqrt{\mu}\,dy\right)^2 \\ \leqslant \int_{B_R}\mu\,|y|^2\,dy\int_{B_R}|\nabla\log\mu|^2\,dy\,.$$

This establishes the result in the radial case.

Mass transportation for approximating critical Sobolev inequalities

Inspired by the limit of Section 3, we can indeed obtain Onofri's inequality as a limiting process of critical Sobolev inequalities involving mass transportation. Let us recall the method of [CENV04]. Let us consider the case where $p<d=2$,

$$F = f^{\frac{d\,p}{d-p}}$$

and G are two probability measures, $p' = p/(p-1)$ is the Hölder conjugate exponent of p and consider the critical Sobolev inequality

$$\|f\|^p_{\mathrm{L}^{\frac{2\,p}{2-p}}(\mathbb{R}^d)} \leqslant \mathsf{C}_{p,d}\,\|\nabla f\|^p_{\mathrm{L}^p(\mathbb{R}^d)} \quad \forall\, f\in\mathcal{D}(\mathbb{R}^d)\,.$$

This inequality generalizes the one in Section 3 which corresponds to $d = 2$ and in particular we have $\mathsf{C}_{p,2} = \mathsf{C}_p$. Starting from (4.2), the proof by mass transportation goes as follows. An integration by parts shows that

$$\int_{\mathbb{R}^d} G^{1-\frac{1}{d}}\,dy \leqslant -\frac{p\,(d-1)}{d\,(d-p)}\int_{\mathbb{R}^d} \nabla(F^{\frac{1}{p}-\frac{1}{d}})\cdot F^{\frac{1}{p'}}\,\nabla\phi\,dx \\ \leqslant \frac{p\,(d-1)}{d\,(d-p)}\,\|\nabla f\|_{\mathrm{L}^p(\mathbb{R}^d)}\left(\int_{\mathbb{R}^d} F\,|\nabla\phi|^{p'}\,dx\right)^{1/p'}\,,$$

where the last line relies on Hölder's inequality and the fact that $F^{\frac{1}{p}-\frac{1}{d}} = f$. The conclusion of the proof arises from the fact that $\int_{\mathbb{R}^d} F\,|\nabla\phi|^{p'}\,dx = \int_{\mathbb{R}^d} G\,|y|^{p'}\,dy$. It allows to characterize $\mathsf{C}_{p,d}$ by

$$\mathsf{C}_{p,d} = \frac{p\,(d-1)}{d\,(d-p)}\,\inf\frac{\left(\int_{\mathbb{R}^d} G\,|y|^{p'}\,dy\right)^{1/p'}}{\int_{\mathbb{R}^d} G^{1-\frac{1}{d}}\,dy}\,,$$

where the infimum is taken on all positive probability measures and is achieved by $G = f_\star^{\frac{d\,p}{d-p}}$. Here $f_\star(x) = (1+|x|^{p'})^{-(d-p)/p}$ is the optimal Aubin-Talenti function.

If we specialize in the case $d = 2$ and consider $f = f_\star\left(1+\frac{2-p}{2\,p}\,(u-\bar u)\right)$, where $\bar u$ is adjusted so that $\|f\|_{\mathrm{L}^{\frac{2\,p}{2-p}}(\mathbb{R}^2)} = 1$, then we recover Onofri's inequality by passing to the limit as $p \to 2^-$. Moreover, we may notice that $\nabla(F^{\frac{1}{p}-\frac{1}{d}})\cdot F^{\frac{1}{p'}}\,\nabla\phi$ formally approaches $\nabla\log F\cdot\sqrt{F}\,\nabla\phi$, so that the mass transportation method for critical Sobolev inequalities is consistent with the formal computation of Section 4.

5 An improved inequality based on Legendre's duality and the logarithmic diffusion or super-fast diffusion equation

In [DJ14, Theorem 2], it has been shown that

$$\int_{\mathbb{R}^2} f\,\log\left(\frac{f}{M}\right)dx - \frac{4\,\pi}{M}\int_{\mathbb{R}^2} f\,(-\Delta)^{-1}\,f\,dx + M\,(1+\log\pi) \\ \leqslant M\left[\frac{1}{16\,\pi}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^2)} + \int_{\mathbb{R}^2} u\,d\mu - \log M\right] \tag{5.1}$$

holds for any function $u \in \mathcal{D}(\mathbb{R}^2)$ such that $M = \int_{\mathbb{R}^2} e^{\,u}\,d\mu$ and $f = e^u\mu$. The l.h.s. in (5.1) is nonnegative by the logarithmic Hardy-Littlewood-Sobolev type inequality according to [CL92, Theorem 1] (also see [Bec93, Theorem 2]). The inequality (5.1) is proven by simply expanding the square

$$0 \leqslant \int_{\mathbb{R}^2}\left|\frac{1}{8\,\pi}\,\nabla u + \kappa\,\nabla\,(-\Delta)^{-1}(v-\mu)\right|^2 dx\,,$$

for some constant κ to be appropriately chosen. Alternatively, we may work on the sphere. Let us expand the square

$$0 \leqslant \int_{\mathbb{S}^2}\left|\frac{1}{2}\,\nabla(u-\bar u) + \frac{1}{\bar v}\,\nabla\,(-\Delta)^{-1}(v-\bar v)\right|^2 d\sigma\,.$$

It is then straightforward to see that

$$\begin{aligned}\frac{1}{4}\int_{\mathbb{S}^2}|\nabla u|^2\,d\sigma+\int_{\mathbb{S}^2}u\,d\sigma-\log\left(\int_{\mathbb{S}^2}e^u\,d\sigma\right)&\\ +\frac{1}{\bar v^2}\int_{\mathbb{S}^2}(v-\bar v)\,(-\Delta)^{-1}(v-\bar v)\,d\sigma-\frac{1}{\bar v}\int_{\mathbb{S}^2}v\,\log\left(\frac{v}{\bar v}\right)d\sigma&\\ \geqslant\frac{2}{\bar v}\int_{\mathbb{S}^2}(u-\bar u)\,(v-\bar v)\,d\sigma&\\ +\int_{\mathbb{S}^2}u\,d\sigma-\log\left(\int_{\mathbb{S}^2}e^u\,d\sigma\right)-\frac{1}{\bar v}\int_{\mathbb{S}^2}v\,\log\left(\frac{v}{\bar v}\right)d\sigma&\\ =:\mathcal{R}[u,v]\,.&\end{aligned}$$

Here we assume that

$$\bar u:=\log\left(\int_{\mathbb{S}^2}e^u\,d\sigma\right)\quad\text{and}\quad\bar v:=\int_{\mathbb{S}^2}v\,d\sigma\,.$$

With the choice

$$v=e^u\,,\quad\bar v=e^{\bar u}\,,$$

the reader is invited to check that $\mathcal{R}[u,v]=0$. Altogether, we have shown that

$$\begin{aligned}\frac{1}{4}\int_{\mathbb{S}^2}|\nabla u|^2\,d\sigma+\int_{\mathbb{S}^2}u\,d\sigma-\log\left(\int_{\mathbb{S}^2}e^u\,d\sigma\right)&\\ \geqslant\int_{\mathbb{S}^2}f\,\log f\,d\sigma-\int_{\mathbb{S}^2}(f-1)\,(-\Delta)^{-1}(f-1)\,d\sigma\,,&\end{aligned}$$

with $f:=e^u/\int_{\mathbb{S}^2}e^u\,d\sigma$. This inequality is exactly equivalent to (5.1). Notice that the r.h.s. is nonnegative by the logarithmic Hardy-Littlewood-Sobolev inequality, which is the dual inequality of Onofri's. See [CL92 ; Dol11 ; DJ14] for details.

Keeping track of the square, we arrive at the following identity.

Proposition 17. *For any $u\in\mathrm{H}^1(\mathbb{S}^2)$, we have*

$$\begin{aligned}\frac{1}{4}\int_{\mathbb{S}^2}|\nabla u|^2\,d\sigma+\int_{\mathbb{S}^2}u\,d\sigma-\log\left(\int_{\mathbb{S}^2}e^u\,d\sigma\right)&\\ =\int_{\mathbb{S}^2}f\,\log f\,d\sigma-\int_{\mathbb{S}^2}(f-1)\,(-\Delta)^{-1}(f-1)\,d\sigma&\\ +\int_{\mathbb{S}^2}\left|\frac{1}{2}\,\nabla u+\nabla\,(-\Delta)^{-1}(f-1)\right|^2d\sigma\,,&\end{aligned}$$

with $f:=e^u/\int_{\mathbb{S}^2}e^u\,d\sigma$.

It is an open question to get an improved inequality compared to (5.1) using a flow, as was done in [DJ14] for Sobolev and Hardy-Littlewood-Sobolev inequalities. We may for instance consider the logarithmic diffusion equation, which is also called the super-fast diffusion equation, on the two-dimensional sphere $\mathbb{S}^2$

$$\frac{\partial f}{\partial t}=\Delta_{\mathbb{S}^2}\log f\,,\tag{5.2}$$

where $\Delta_{\mathbb{S}^2}$ denotes the Laplace-Beltrami operator on $\mathbb{S}^2$. In dimension $d = 2$ Eq. (5.2) plays a role which is the analogue of the Yamabe flow in dimensions $d \geqslant 3$ or, to be precise, to the equation $\frac{\partial f}{\partial t} = \Delta_{\mathbb{S}^2} f^{\frac{d-2}{d+2}}$. See [Dol11 ; DJ14] for details. The flow defined by (5.2) does not give straightforward estimates although we may notice that

$$\mathsf{H} := \int_{\mathbb{S}^2} f \, \log f \, d\sigma - \int_{\mathbb{S}^2} (f-1)\,(-\Delta)^{-1}(f-1)\, d\sigma$$

is such that, if $f = e^{u/2}$ is a solution to (5.2) such that $\int_{\mathbb{S}^2} f \, d\sigma = 1$, then

$$\frac{d\mathsf{H}}{dt} = -\left[\int_{\mathbb{S}^2} |\nabla u|^2 \, d\sigma + \int_{\mathbb{S}^2} u \, d\sigma - \int_{\mathbb{S}^2} u \, e^{u/2} \, d\sigma\right] \\ \leqslant -\left[\int_{\mathbb{S}^2} |\nabla u|^2 \, d\sigma + \int_{\mathbb{S}^2} u \, d\sigma - \log\left(\int_{\mathbb{S}^2} e^u \, d\sigma\right)\right]$$

because $\int_{\mathbb{S}^2} u \, e^{u/2} \, d\sigma \leqslant \log\left(\int_{\mathbb{S}^2} e^u \, d\sigma\right)$ according to [Dol11, Proposition 3.1].

6 An improved inequality based on the entropy–entropy production method and the fast diffusion equation

In $\mathbb{R}^2$ we consider the fast diffusion equation written in self-similar variables

$$\frac{\partial v}{\partial t} + \nabla \cdot \left[v \, \left(\nabla v^{m-1} - 2\, x\right)\right] = 0 \,, \tag{6.1}$$

where the parameter m is taken in the interval $[1/2, 1)$. According to [PD02], the mass $M = \int_{\mathbb{R}^2} v \, dx$ is independent of t. Stationary solutions are the so-called *Barenblatt profiles*

$$v_\infty(x) := \left(D + |x|^2\right)^{\frac{1}{m-1}} \,,$$

where D is a positive parameter which is uniquely determined by the mass condition $M = \int_{\mathbb{R}^2} v_\infty \, dx$. The *relative entropy* is defined by

$$\mathcal{E}[v] := \frac{1}{m-1} \int_{\mathbb{R}^2} \left[v^m - v_\infty^m - m\, v_\infty^{m-1}\,(v - v_\infty)\right] dx \,.$$

According to [PD02], it is a Lyapunov functional, since

$$\frac{d}{dt}\mathcal{E}[v] = -\,\mathcal{I}[v] \,,$$

where $\mathcal{I}$ is the *relative Fisher information* defined by

$$\mathcal{I}[v] := \int_{\mathbb{R}^2} v \, |v^{m-1} - v_\infty^{m-1}|^2 \, dx \,,$$

and for $m > \frac{1}{2}$ the inequality

$$\mathcal{E}[v] \leqslant \frac{1}{4}\,\mathcal{I}[v] \tag{6.2}$$

is equivalent to a Gagliardo-Nirenberg inequality written with an optimal constant according to [PD02]. Note that for $m = 1/2$, $v_\infty(x) := \left(D + |x|^2\right)^{-2}$ and so $v_\infty^m \notin \mathrm{L}^1(\mathbb{R}^2)$ and $|x|^2 \, v_\infty \notin \mathrm{L}^1(\mathbb{R}^2)$.

However, we may consider $w = v/v_\infty$ at least for a function v such that $v - v_\infty$ is compactly supported, take the limit $m \to 1/2$ and argue by density to prove that

$$\mathcal{E}[w\,v_\infty] =: E[w] = \int_{\mathbb{R}^2} \frac{|\sqrt{w}-1|^2}{D+|x|^2}\,dx \leqslant \frac{1}{4}\,I[w]\,,$$

where

$$I[w] \coloneqq \mathcal{I}[w\,v_\infty] = \int_{\mathbb{R}^2} v_\infty\,w\,\left|\nabla\left(v_\infty^{m-1}\,(w^{m-1}-1)\right)\right|^2\,dx$$

can be rewritten as

$$\begin{aligned}
I[w] &= \int_{\mathbb{R}^2} \frac{w}{(D+|x|^2)^2}\,\left|\nabla\left(v_\infty^{m-1}\,(w^{m-1}-1)\right)\right|^2\,dx\\
&= \int_{\mathbb{R}^2} \frac{w}{(D+|x|^2)^2}\,\left|\nabla\left((D+|x|^2)\,(w^{-1/2}-1)\right)\right|^2\,dx\\
&= \int_{\mathbb{R}^2} \frac{1}{(D+|x|^2)^2}\,\left|2\,x\,(1-\sqrt{w}\,) - \frac{1}{2}\,(D+|x|^2)\,\nabla\log w\right|^2\,dx\\
&= \int_{\mathbb{R}^2} \frac{4\,|x|^2\,(1-\sqrt{w}\,)^2}{(D+|x|^2)^2}\,dx + \frac{1}{4}\int_{\mathbb{R}^2}|\nabla\log w|^2\,dx\\
&\qquad\qquad - 2\int_{\mathbb{R}^2} x\cdot\frac{\nabla\log w + 2\,\nabla(1-\sqrt{w}\,)}{D+|x|^2}\,dx\\
&= \int_{\mathbb{R}^2} \frac{4\,|x|^2\,(1-\sqrt{w}\,)^2}{(D+|x|^2)^2}\,dx + \frac{1}{4}\int_{\mathbb{R}^2}|\nabla\log w|^2\,dx\\
&\qquad\qquad + 4\,D\int_{\mathbb{R}^2}\frac{\log w + 2\,(1-\sqrt{w}\,)}{(D+|x|^2)^2}\,dx\,,
\end{aligned}$$

where we performed an integration by parts in the last line. Collecting terms and letting $u = \log w$, we arrive at

$$\begin{aligned}
\frac{1}{4}\,I[w] - E[w] &= -\,D\int_{\mathbb{R}^2}\frac{(1-\sqrt{w}\,)^2}{(D+|x|^2)^2}\,dx + \frac{1}{16}\int_{\mathbb{R}^2}|\nabla\log w|^2\,dx\\
&\qquad\qquad + D\int_{\mathbb{R}^2}\frac{\log w - 2\,(\sqrt{w}-1)}{(D+|x|^2)^2}\,dx\\
&= -\,D\int_{\mathbb{R}^2}\frac{(1-e^{u/2})^2}{(D+|x|^2)^2}\,dx + \frac{1}{16}\int_{\mathbb{R}^2}|\nabla u|^2\,dx\\
&\qquad\qquad + D\int_{\mathbb{R}^2}\frac{u - 2\,(e^{u/2}-1)}{(D+|x|^2)^2}\,dx\\
&= \frac{1}{16}\int_{\mathbb{R}^2}|\nabla u|^2\,dx - D\int_{\mathbb{R}^2}\frac{e^u-1-u}{(D+|x|^2)^2}\,dx\,,
\end{aligned}$$

and thus prove that (6.2) written for $m = 1/2$ shows that the r.h.s. in the above identity is nonnegative. As a special case consider $D = 1$ and define $d\mu = \mu(x)\,dx$ where $\mu(x) = \frac{1}{\pi}\,(1+|x|^2)^{-2}$. Inequality (6.2) can therefore be written as

$$\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx \geqslant \int_{\mathbb{R}^2} e^u\,d\mu - 1 - \int_{\mathbb{R}^2} u\,d\mu\,.$$

Since $z - 1 \geqslant \log z$ for any $z > 0$, this inequality implies the Onofri inequality (1.1), namely,

$$\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx \geqslant \log\left(\int_{\mathbb{R}^2} e^u\,d\mu\right) - \int_{\mathbb{R}^2} u\,d\mu\,.$$

The two inequalities are actually equivalent since the first one is not invariant under a shift by a given constant : if we replace u by $u+c$ with c such that

$$\int_{\mathbb{R}^2} e^u\,d\mu - 1 - \int_{\mathbb{R}^2} u\,d\mu \geqslant e^c \int_{\mathbb{R}^2} e^u\,d\mu - 1 - \int_{\mathbb{R}^2} u\,d\mu - c\,,$$

and minimize the r.h.s. with respect to c, we get that $c = -\log\left(\int_{\mathbb{R}^2} e^u\,d\mu\right)$ and recover the standard form (1.1) of Onofri's inequality.

Various methods are available for proving (6.2). The Bakry-Emery method, or *carré du champ* method, has been developed in [BÉ84 ; Arn+01] in the linear case and later extended to nonlinear diffusions in [PD02 ; CT00 ; Car+01] using a *relative entropy* which appears first in [New84 ; Ral84]. This *entropy–entropy production* method has the advantage of providing an integral remainder term. Here we adopt a setting that can be found in [DT13].

Let us consider a solution v to (6.1) and define

$$z(x,t) \coloneqq \nabla v^{m-1} - 2\,x\,,$$

so that (6.1) can be rewritten for any $m \in [\frac{1}{2}, 1)$ as

$$\frac{\partial v}{\partial t} + \nabla\cdot(v\,z) = 0\,.$$

A tedious computation shows that

$$\frac{d}{dt}\int_{\mathbb{R}^2} v\,|z|^2\,dx + 4\int_{\mathbb{R}^2} v\,|z|^2\,dx = -2\,\frac{1-m}{m}\,\mathcal{R}[v,z]\,,$$

with

$$R[v,z] \coloneqq \int_{\mathbb{R}^2} v^m\,\Big[\,|\nabla z|^2 - (1-m)\,(\nabla\cdot z)^2\Big]\,dx\,, \tag{6.3}$$

where $|\nabla z|^2 = \sum_{i,j=1,2}(\frac{\partial z_i}{\partial x_j})^2$ and $\nabla\cdot z = \sum_{i=1,2}\frac{\partial z_i}{\partial x_i}$. Summarizing, when $m=\frac{1}{2}$, we have shown that

$$\frac{1}{4}\,I[w(t=0,\cdot)] - E[w(t=0,\cdot)] = 2\int_0^\infty \mathcal{R}[v(t,\cdot),z(t,\cdot)]\,dt\,.$$

Proposition 18. *If we denote by v the solution to (6.1) with initial datum*

$$v_{|t=0} = \frac{e^u}{(1+|x|^2)^2}\,,$$

then we have the identity

$$\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx + \int_{\mathbb{R}^2} u\,d\mu - \log\left(\int_{\mathbb{R}^2} e^u\,d\mu\right) = 2\int_0^\infty \mathcal{R}[v,z]\,dt\,,$$

with $\mathcal{R}$ defined by (6.3) and $z(t,x) = \nabla v^{-1/2}(t,x) - 2\,x$.

Notice that the kernel of $\mathcal{R}$ is spanned by all Barenblatt profiles, which are the stationary solutions of (6.1) (one has to take into account the invariances : multiplication by a constant, translation and dilation). This has to do with the conformal transformation on the sphere : see Theorem 20 and [GM13, Section 17.3] for more details.

As a straightforward consequence of Propostion 18, we have the

Corollary 19. *With the notations of Section 3 we have*

$$\mathcal{I}_1 = 1\,.$$

Moreover any minimizing sequence converges to a function in the kernel of $\mathcal{R}$.

The fact that Onofri's inequality is intimately related with the fast diffusion equation (6.1) with $m = 1/2$ sheds a new light on the role played by this equation for the dual inequality, the logarithmic Hardy-Littlewood-Sobolev inequality, which has been studied in [CCL10] and applied to the critical parabolic-elliptic Keller-Segel model in [CF13 ; BCC12].

7 Rigidity (or *carré du champ*) methods and adapted nonlinear diffusion equations

By *rigidity method* we refer to a method which has been popularized in [GS81] and optimized later in [BVV91]. We will first consider the symmetric case in which computations can be done along the lines of [Dol+14] and are easy. Then we will introduce flows as in [Dol+14] (for Sobolev's inequality and interpolation inequalities in the subcritical range), still in the symmetric case. The main advantage is that the flow produces an integral remainder term which is, as far as we know, a new result in the case of Onofri's inequality.

The integrations by parts of the rigidity method can be encoded in the Γ_2 or *carré du champ* methods, thus providing the same results. In the case of Onofri's inequality, this has been observed by É. Fontenas in [Fon98, Theorem 2] (actually, without symmetry).

A striking observation is indeed that no symmetry is required : the rigidity computations and the flow can be used in the general case, as was done in [DEL13b], and produce an integral remainder term, which is our last new result.

Rigidity method in the symmetric case

As shown for instance in [OPS88] the functional

$$\mathcal{G}_\lambda[v] := \frac{1}{4}\int_{\mathbb{S}^2} |\nabla v|^2\,d\sigma + \lambda\left[\int_{\mathbb{S}^2} v\,d\sigma - \log\left(\int_{\mathbb{S}^2} e^v\,d\sigma\right)\right]$$

is nonnegative for all $\lambda > 0$ and it can be minimized in $\mathrm{H}^1(\mathbb{S}^2)$ and, up to the addition of a constant, any minimizer satisfies the Euler-Lagrange equation

$$-\frac{1}{2}\,\Delta v + \lambda = \lambda\, e^v \quad \text{on} \quad S^2\,. \tag{7.1}$$

According to Proposition 16, minimizing $\mathcal{G}_\lambda$ amounts to minimizing

$$\mathsf{G}_\lambda[f] := \frac{1}{8}\int_{-1}^1 |f'|^2\,\nu\,dz + \frac{\lambda}{2}\int_{-1}^1 f\,dz \geqslant \lambda\,\log\left(\frac{1}{2}\int_{-1}^1 e^f\,dz\right)\,,$$

and (3.9) can be reduced to the fact that the minimum of G_1 is achieved by constant functions. For the same reasons as above, G_λ has a minimum which solves the Euler-Lagrange equation

$$-\frac{1}{2}\,\mathcal{L}\,f + \lambda = 2\,\lambda\,\frac{e^f}{\int_{-1}^1 e^f\,dz}\,,$$

where $\mathcal{L}\, f := \nu\, f'' + \nu'\, f'$ and $\nu(z) = 1 - z^2$. Up to the addition of a constant, we may choose f such that $\int_{-1}^1 e^f\, dz = 2$ and hence solves

$$-\frac{1}{2}\,\mathcal{L}\, f + \lambda = \lambda\, e^f\,. \tag{7.2}$$

Theorem 20. *For any $\lambda \in (0,1)$, (7.2) has a unique smooth solution f, which is the constant function*

$$f = 0\,.$$

As a consequence, if f is a critical point of the functional G_λ, then f is a constant function for any $\lambda \in (0,1)$, while for $\lambda = 1$, f has to satisfy the differential equation $f'' = \frac{1}{2}\,|f'|^2$ and is either a constant, or such that

$$f(z) = C_1 - 2\,\log(C_2 - z)\,, \tag{7.3}$$

for some constants $C_1 \in \mathbb{R}$ and $C_2 > 1$.

Let us define

$$\mathsf{R}_\lambda[f] := \frac{1}{8}\int_{-1}^1 \nu^2\,\left|f'' - \tfrac{1}{2}\,|f'|^2\right|^2\, e^{-f/2}\, dz + \frac{1-\lambda}{4}\int_{-1}^1 \nu\,|f'|^2\, e^{-f/2}\, dz\,. \tag{7.4}$$

The proof is a straightforward consequence of the following lemma.

Lemma 21. *If f solves (7.2), then*

$$\mathsf{R}_\lambda[f] = 0\,.$$

Démonstration. The ultraspherical operator does not commute with the derivation with respect to z :

$$(\mathcal{L}\, f)' = \mathcal{L}\, f' - 2\, z\, f'' - 2\, f'\,,$$

where $f' = \frac{df}{dz}$. After multiplying (7.2) by $\mathcal{L}\left(e^{-f/2}\right)$ and integrating by parts, we get

$$\begin{aligned} 0 &= \int_{-1}^1 \left(-\tfrac{1}{2}\,\mathcal{L}\, f + \lambda - e^f\right)\, \mathcal{L}\left(e^{-f/2}\right)\, dz \\ &= \frac{1}{4}\int_{-1}^1 \nu^2\,|f''|^2\, e^{-f/2}\, dz - \frac{1}{8}\int_{-1}^1 \nu^2\,|f'|^2\, f''\, e^{-f/2}\, dz \\ &\qquad + \frac{1}{2}\int_{-1}^1 \nu\,|f'|^2\, e^{-f/2}\, dz - \frac{1}{2}\int_{-1}^1 \nu\,|f'|^2\, e^{f/2}\, dz\,. \end{aligned}$$

Similarly, after multiplying (7.2) by $\frac{\nu}{2}\,|f'|^2\, e^{-f/2}$ and integrating by parts, we get

$$\begin{aligned} 0 &= \int_{-1}^1 \left(-\tfrac{1}{2}\,\mathcal{L}\, f + \lambda - e^f\right)\, \left(\tfrac{\nu}{2}\,|f'|^2\, e^{-f/2}\right)\, dz \\ &= \frac{1}{8}\int_{-1}^1 \nu^2\,|f'|^2\, f''\, e^{-f/2}\, dz - \frac{1}{16}\int_{-1}^1 \nu^2\,|f'|^4\, e^{-f/2}\, dz \\ &\qquad + \frac{\lambda}{2}\int_{-1}^1 \nu\,|f'|^2\, e^{-f/2}\, dz - \frac{1}{2}\int_{-1}^1 \nu\,|f'|^2\, e^{f/2}\, dz\,. \end{aligned}$$

Subtracting the second identity from the first one establishes the first part of the theorem. If $\lambda \in (0,1)$, then f has to be a constant. If $\lambda = 1$, there are other solutions, because of the conformal transformations : see for instance [GM13, Section 17.3] for more details. In our case, all solutions of the differential equation $f'' = \frac{1}{2}\,|f'|^2$ that are not constant are given by (7.3). □

A nonlinear flow method in the symmetric case

Consider the nonlinear evolution equation

$$\frac{\partial g}{\partial t} = \mathcal{L}\,(e^{-g/2}) - \tfrac{\nu}{2}\,|g'|^2\,e^{-g/2}\,. \tag{7.5}$$

Proposition 22. *Assume that g is a solution to (7.5) with initial datum $f \in \mathrm{L}^1(-1,1;dz)$ such that $\int_{-1}^1 |f'|^2\,\nu\,dz$ is finite and $\int_{-1}^1 e^f\,dz = 1$. Then for any $\lambda \in (0,1]$ we have*

$$\mathsf{G}_\lambda[f] \geqslant \int_0^\infty \mathsf{R}_\lambda[g(t,\cdot)]\,dt\,,$$

where R_λ is defined in (7.4).

Démonstration. A standard regularization method allows to reduce the evolution problem to the case of smooth bounded functions, at least in a finite time interval. Then a simple computation shows that

$$\frac{d}{dt}\mathsf{G}_\lambda[g(t,\cdot)] = -\frac{1}{2}\int_{-1}^1 \left(-\tfrac{1}{2}\,\mathcal{L}\,g + \lambda - \lambda\,e^g\right)\,\frac{\partial g}{\partial t}\,dz = -\mathsf{R}_\lambda[g(t,\cdot)]\,.$$

We may then argue by continuation. Because $\mathsf{G}_\lambda[g(t,\cdot)]$ is bounded from below, $\mathsf{R}_\lambda[g(t,\cdot)]$ is integrable with respect to $t \in [0,\infty)$. Hence, as $t \to \infty$, g converges to a constant if $\lambda < 1$, or the conformal transformation of a constant if $\lambda = 1$ and therefore $\lim_{t\to\infty} \mathsf{G}_\lambda[g(t,\cdot)] = 0$. The result holds with equality after integrating on $[0,\infty) \ni t$. For a general initial datum without smoothness assumption we conclude by density and get an inequality instead of an equality by lower semi-continuity. □

For a general function $v \in \mathrm{H}^1(\mathbb{S}^2)$, if we denote by v_* the symmetrized function which depends only on θ (see [GM13, Section 17.1] for more details) and denote by f the function such that $f(\cos\theta) = v_*(\theta)$, then it follows from Propositions 16 and 22 that

$$\mathcal{G}_\lambda[v] \geqslant \int_0^\infty \mathsf{R}_\lambda[g(t,\cdot)]\,dt\,,$$

where g is the solution to (7.5) with initial datum f. However, we do not need any symmetrization step, as we shall see in the next section.

A nonlinear flow method in the general case

On $\mathbb{S}^2$ let us consider the nonlinear evolution equation

$$\frac{\partial f}{\partial t} = \Delta_{\mathbb{S}^2}\,(e^{-f/2}) - \tfrac{1}{2}\,|\nabla f|^2\,e^{-f/2}\,, \tag{7.6}$$

where $\Delta_{\mathbb{S}^2}$ denotes the Laplace-Beltrami operator. Let us define

$$\mathcal{R}_\lambda[f] := \frac{1}{2}\int_{\mathbb{S}^2} \|\mathrm{L}_{\mathbb{S}^2} f - \frac{1}{2}\,\mathrm{M}_{\mathbb{S}^2} f\|^2\,e^{-f/2}\,d\sigma + \frac{1}{2}\,(1-\lambda)\int_{\mathbb{S}^2} |\nabla f|^2\,e^{-f/2}\,d\sigma\,,$$

where

$$\mathrm{L}_{\mathbb{S}^2} f := \mathrm{Hess}_{\mathbb{S}^2}\,f - \frac{1}{2}\,\Delta_{\mathbb{S}^2} f\,\mathrm{Id} \quad\text{and}\quad \mathrm{M}_{\mathbb{S}^2} f := \nabla f\otimes\nabla f - \frac{1}{2}\,|\nabla f|^2\,\mathrm{Id}\,.$$

This definition of $\mathcal{R}_\lambda$ generalizes the definition of R_λ given in Section 7 in the symmetric case. We refer to [DEL13b] for more detailed considerations, and to [Dol+14] for considerations and improvements of the method that are specific to the sphere $\mathbb{S}^2$.

Theorem 23. *Assume that f is a solution to (7.6) with initial datum $v - \log\left(\int_{\mathbb{S}^2} e^v \, d\sigma\right)$, where $v \in \mathrm{L}^1(\mathbb{S}^2)$ is such that $\nabla v \in \mathrm{L}^2(\mathbb{S}^2)$. Then for any $\lambda \in (0,1]$ we have*

$$\mathcal{G}_\lambda[v] \geqslant \int_0^\infty \mathcal{R}_\lambda[f(t,\cdot)]\, dt\,.$$

Démonstration. With no restriction, we may assume that $\int_{\mathbb{S}^2} e^v \, d\sigma = 1$ and it is then straightforward to see that $\int_{\mathbb{S}^2} e^{f(t,\cdot)} \, d\sigma = 1$ for any $t > 0$. Next we compute

$$\frac{d}{dt}\mathcal{G}_\lambda[f] = \int_{\mathbb{S}^2} \left(-\tfrac{1}{2}\,\Delta_{\mathbb{S}^2} f + \lambda\right)\left(\Delta_{\mathbb{S}^2}\left(e^{-f/2}\right) - \tfrac{1}{2}\,|\nabla f|^2\, e^{-f/2}\right) d\sigma = -\,\mathcal{R}_\lambda[f]$$

in the same spirit as in [DEL13b]. □

As a concluding remark, let us notice that the *carré du champ* method is not limited to the case of $\mathbb{S}^2$ but also applies to two-dimensional Riemannian manifolds : see for instance [Fon97]. The use of the flow defined by (7.6) gives an additional integral remainder term, in the spirit of what has been done in [DEL13b]. This is however out of the scope of the present paper.

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Chapitre IV

Inégalité d'Onofri sur des variétés riemanniennes

Ce chapitre reprend en grande partie l'article « Rigidity results for semilinear elliptic equation with exponential nonlinearities and Moser-Trudinger-Onofri inequalities on two-dimensional Riemannian manifolds », écrit en collaboration avec Jean Dolbeault et Maria J. Esteban.

Résumé

Cet article est dédié à l'inégalité de Moser-Trudinger-Onofri pour les variétés riemanniennes régulières, compactes et connexes. Nous établissons un résultat de rigidité pour l'équation d'Euler-Lagrange et nosu déduisons une estimation de la meilleure constante dans l'inégalité pour les variétés fermées de dimensions 2. Par rapport aux résultats existants, nous apportons un critère non local qui est bien adapté aux méthodes variationnelles. Nous introduisons aussi un flot non linéaire qui nous permet d'attaquer les questions d'optimalité à travers l'obtention d'un terme de reste. Comme application importante, nous étudions le cas non compact de l'équation de Moser-Trudinger-Onofri dans l'espace euclidien de dimension 2, avec poids. Nous partons du poids issu de la projection stéréographique, puis nous généralisons pour obtenir d'autres résultat intéressants, notamment dans le cadre du modèle de Keller-Segel pour la chimiotaxie.

Structure

1 Proof of Theorem 1 92
2 Proof of Corollary 24 98
3 The case $d = 1$ 99
4 Weighted Moser-Trudinger-Onofri inequalities on the two-dimensional Euclidean space 100
References 106

In this paper we assume that $(\mathfrak{M}, g)$ is a smooth compact connected Riemannian manifold of dimension $d \geqslant 1$, without boundary. We denote by Δ_g the Laplace-Beltrami operator on $\mathfrak{M}$. For simplicity, we assume that the volume of $\mathfrak{M}$, is chosen equal to 1 and use the notation dv_g for the volume element. We shall also denote by $\mathfrak{R}$ the Ricci tensor, by $\mathrm{H}_g u$ the Hessian of u and by

$$\mathrm{L}_g u \coloneqq \mathrm{H}_g u - \frac{g}{d}\,\Delta_g u$$

the trace free Hessian. Let us denote by $\mathrm{M}_g u$ the trace free tensor

$$\mathrm{M}_g u \coloneqq \nabla u \otimes \nabla u - \frac{g}{d}\,|\nabla u|^2\,.$$

We define

$$\lambda_\star \coloneqq \inf_{u\in \mathrm{H}^2(\mathfrak{M})\setminus\{0\}} \frac{\displaystyle\int_{\mathfrak{M}} \left[\,\|\,\mathrm{L}_g u - \tfrac{1}{2}\,\mathrm{M}_g u\,\|^2 + \mathfrak{R}(\nabla u, \nabla u)\right] e^{-u/2}\,dv_g}{\displaystyle\int_{\mathfrak{M}} |\nabla u|^2\,e^{-u/2}\,dv_g}\,. \tag{0.1}$$

If $A = \big(a_{ij}\big)_{i,j=1,2}$ and $B = \big(b_{ij}\big)_{i,j=1,2}$ are two matrices, then we use the convention that $A : B = \sum_{i,j=1,2} a_{ij}\,b_{ij}$ and $\|A\|^2 = A : A$. I_2 denotes the 2×2 identity matrix.

Theorem 1. *Assume that $d = 2$ and $\lambda_\star > 0$. If u is a smooth solution to*

$$-\,\frac{1}{2}\,\Delta_g u + \lambda = e^u\,, \tag{0.2}$$

then u is a constant function if $\lambda \in (0, \lambda_\star)$.

Next, let us consider the Moser-Trudinger-Onofri inequality on $\mathfrak{M}$ written as

$$\frac{1}{4}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathfrak{M})} + \lambda \int_{\mathfrak{M}} u\,dv_g \geqslant \lambda\,\log\left(\int_{\mathfrak{M}} e^u\,dv_g\right) \quad \forall\, u \in \mathrm{H}^1(\mathfrak{M})\,, \tag{0.3}$$

for some constant $\lambda > 0$. Let us denote by λ_1 the first positive eigenvalue of $-\,\Delta_g$.

Corollary 24. *If $d = 2$, then (0.3) holds with $\lambda = \Lambda \coloneqq \min\{4\,\pi, \lambda_\star\}$. Moreover, if Λ is strictly smaller than $\lambda_1/2$, then the optimal constant in (0.3) is strictly larger than Λ.*

In the case of the normalized sphere, $\lambda_\star = 4\,\pi$ is optimal but (0.2) has non-constant solutions because of the conformal invariance : see [GM13 ; DEJ14] for a review of the Moser-Trudinger-Onofri inequality on the sphere, and references therein. The interested reader is invited to refer to the historical papers [Tru67 ; Mos70 ; Ono82] and to [Fag08 ; Fag06] for recent results on functionals related to the inequality, that have been obtained by variational methods. These two papers solve the question, in any dimension, of whether the *first best constant* can be reached. This is equivalent to showing that the difference of the two terms in (0.3) is bounded from below. Earlier results have been obtained by T. Aubin in [Aub79], in the case of the sphere $\mathbb{S}^n$, and P. Cherrier in [Che79] for general 2-manifolds. The present paper focuses on the value of the *second best constant*, defined as the largest value of λ such that (0.3) holds. The value of the *first best constant* is of little concern to us, as it appears as the $\frac{1}{4}$ coefficient in front of $\|\nabla u\|^2_{\mathrm{L}^2(\mathfrak{M})}$ and can be factored out into λ. The method of Z. Faget relies on a blow-up analysis which is reminiscent for $d = 2$ of [NT98]. It

generalizes some results contained in [Aub82, Theorem 2.50 page 68]. Other references of general interest in the context of the Moser-Trudinger-Onofri inequality are [CL92 ; Bec93 ; Hon86 ; OPS88 ; GM13]. A review of results related with the the Moser-Trudinger-Onofri inequality in the case $\mathfrak{M} = \mathbb{S}^2$ can be found in [DEJ14]. Let us mention that in [Ghi05], A. Ghigi provides a proof based on the Prékopa-Leindler inequality and that many more details can be found in the book [GM13, Chapters 16-18] of N. Ghoussoub and A. Moradifam. In the context of Einstein-Kähler geometry another proof appears in [Rub08a, Theorem 5.2] (also see for instance [TZ00 ; Pho+08] and [BB11] for recent results on Kählerian manifolds). In [Rub08a], Y.A. Rubinstein gives a proof of the Onofri inequality on $\mathbb{S}^2$ that does not use symmetrization/rearrangement arguments. Also see [Rub08b] and in particular [Rub08b, Corollary 10.12] which contains a reinforced version of the inequality. We shall refer to [Cha04] for background material in this direction. The reader interested in understanding how the Moser-Trudinger-Onofri inequality is related to the problem of prescribing the Gaussian curvature on $\mathbb{S}^2$ is invited to refer to [Cha87, Section 3] for an introductory survey, and to [CY88 ; CY87 ; CY03] for more details. More references will be given within the text, whenever needed.

At this point, we should emphasize that in most of the literature the Moser-Trudinger-Onofri inequality in dimension $d = 2$ is not written as in (0.3), but in the form

$$e^{\mu_2 \|\nabla u\|^2_{\mathrm{L}^2(\mathfrak{M})}} \geqslant \mathrm{C} \int_{\mathfrak{M}} e^u \, dv_g$$

for all functions $u \in \mathrm{H}^1(\mathfrak{M})$ such that $\int_{\mathfrak{M}} u \, dv_g = 0$, for some constant C which is in general non-explicit. In dimension $d = 2$, the optimal constant is $\mu_2 = \frac{1}{16\,\pi}$. This amounts to write that the functional

$$u \mapsto \mu_2 \, \|\nabla u\|^2_{\mathrm{L}^2(\mathfrak{M})} + \int_{\mathfrak{M}} u \, dv_g - \log \left(\int_{\mathfrak{M}} e^u \, dv_g \right)$$

is bounded from below by $\log \mathrm{C}$. The issue of the *first best constant* is to prove that μ_2 cannot be replaced by a smaller constant unless the functional is unbounded from below. This problem is not the same as Inequality (0.3), except when $\mathrm{C} = 1$ and $\lambda = 1$. E. Onofri proved in [Ono82] that this is the case, with optimal values for both C and λ, when $\mathfrak{M} = \mathbb{S}^2$, up to a factor $4\,\pi$ that comes from the normalization of $\mathrm{vol}_g(\mathfrak{M})$. Except for the sphere we are aware of only one occurrence in the literature of the form (0.3) of the inequality, that has been derived by E. Fontenas in [Fon97, Théorème 2] under more restrictive conditions on $\mathfrak{M}$. This result will be commented in more detail in Remark 26.

The proof of Theorem 1 is a *rigidity* method inspired by the one of [BVV91] for the equation

$$-\Delta_g u + \lambda\, u = u^{p-1} \,,$$

which is the Euler-Lagrange equation corresponding to the optimality case in the interpolation inequality

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathfrak{M})} \geqslant \frac{\lambda}{p-2} \left[\|v\|^2_{\mathrm{L}^p(\mathfrak{M})} - \|v\|^2_{\mathrm{L}^2(\mathfrak{M})} \right] \quad \forall\, v \in \mathrm{H}^1(\mathfrak{M}) \,. \tag{0.4}$$

See [BVV91 ; LV95 ; BL96 ; Dem08 ; Dol+14] for further results on this problem and [Aub82 ; Heb99] for general accounts on Sobolev's inequality on Riemannian manifolds. Concerning spectral issues, a standard textbook is [BGM71].

The case of the exponential nonlinearity in (0.2) has been much less considered in the literature, except when $\mathfrak{M}$ is the two dimensional sphere $\mathbb{S}^2$. Let us mention the uniqueness result of [CL91] for (0.2) with $\lambda = 1$. In [Fon97], and in [Ben93] in the case of the ultraspherical operator, the result is achieved by considering the interpolation inequalities (0.4) and then, as in [BVV91] or [Bec93] (in the case of the sphere), by taking the limit as $p \to \infty$. Here we consider a direct approach, based on rigidity methods and an associated nonlinear flow. As far as we know, this is an entirely new approach which has the interest of providing explicit estimates on the optimal constant in (0.3).

One may wonder if rigidity results can be achieved for dimensions $d > 2$ by our method. We will give a negative answer in Section 1. Corollary 24 is established in Section 2 using a nonlinear flow that has already been considered on the sphere in [DEJ14]. The case $d = 1$ is very simple and will be considered for completeness in Section 3. An important application of our method is the case of the Euclidean space with weights, with applications to chemotaxis. Section 4 is devoted to this issue with a main result in this direction stated in Theorem 2, that raises difficult questions of symmetry breaking.

1 Proof of Theorem 1

In this section we consider a smooth solution to (0.2) and perform a computation to prove the rigidity result of Theorem 1. There is no *a priori* reason to assume that $d = 2$ and so we shall do the computations for any dimension $d \geqslant 1$, which raises no special additional difficulties. However, due to restrictions that are inherent to the method and will be explicitely exposed, only $d = 2$ can be covered. On several occasions, one has to divide by $(d-1)$, so the case $d = 1$ has to be excluded and will be handled directly in Section 3.

In the case of (0.4), it is well known (see [LV95 ; BL96 ; DEL13b]) that an interpolation depending on a parameter $\theta \in (0,1)$ between an estimate based on the Ricci curvature and another one based on the first eigenvalue of the Laplace-Beltrami operator can be used to obtain some improvements. Here we apply the same technique and realize in the end that only $\theta = 1$ is admissible in dimension $d = 2$. However, when d is considered as a real parameter in the range $(1,2)$, it is possible to optimize on θ when $0 \leqslant \theta \leqslant 1$. We will comment this and possible improvements at the end of this section.

Preliminaries

A simple expansion of the square shows that

$$\|\mathrm{H}_g u\|^2 = \|\mathrm{L}_g u\|^2 + \frac{1}{d}\,(\Delta_g u)^2\,.$$

The Bochner-Lichnerovicz-Weitzenböck formula asserts that

$$\frac{1}{2}\,\Delta\,|\nabla u|^2 = \|\mathrm{H}_g u\|^2 + \nabla(\Delta_g u)\cdot\nabla u + \mathfrak{R}(\nabla u,\nabla u)$$

where $\mathfrak{R}$ denotes the Ricci tensor and, as a consequence,

$$\frac{1}{2}\,\Delta\,|\nabla u|^2 = \|\mathrm{L}_g u\|^2 + \frac{1}{d}\,(\Delta_g u)^2 + \nabla(\Delta_g u)\cdot\nabla u + \mathfrak{R}(\nabla u,\nabla u)\,. \tag{1.1}$$

An identity based on integrations by parts

Using integrations by parts, we may notice that

$$\begin{aligned}
&-\int_{\mathfrak{M}} \Delta_g u\,|\nabla u|^2\,e^{-u/2}\,dv_g \\
&= -\frac{1}{2}\int_{\mathfrak{M}} |\nabla u|^4\,e^{-u/2}\,dv_g + 2\int_{\mathfrak{M}} \mathrm{H}_g u : \nabla u\otimes\nabla u\,e^{-u/2}\,dv_g \\
&= -\frac{1}{2}\int_{\mathfrak{M}} |\nabla u|^4\,e^{-u/2}\,dv_g + 2\int_{\mathfrak{M}} \big(\mathrm{L}_g u + \frac{g}{d}\,\Delta_g u\big) : \nabla u\otimes\nabla u\,e^{-u/2}\,dv_g \\
&= -\frac{1}{2}\int_{\mathfrak{M}} |\nabla u|^4\,e^{-u/2}\,dv_g + 2\int_{\mathfrak{M}} \mathrm{L}_g u : \mathrm{M}_g u\,e^{-u/2}\,dv_g \\
&\qquad\qquad + \frac{2}{d}\int_{\mathfrak{M}} \Delta_g u\,|\nabla u|^2\,e^{-u/2}\,dv_g\,,
\end{aligned}$$

with $\mathrm{M}_g u \coloneqq \nabla u\otimes\nabla u - \frac{g}{d}\,|\nabla u|^2$, which proves that

$$\begin{aligned}
&\frac{d+2}{d}\int_{\mathfrak{M}} \Delta_g u\,|\nabla u|^2\,e^{-u/2}\,dv_g \\
&\qquad\qquad = \frac{1}{2}\int_{\mathfrak{M}} |\nabla u|^4\,e^{-u/2}\,dv_g - 2\int_{\mathfrak{M}} \mathrm{L}_g u : \mathrm{M}_g u\,e^{-u/2}\,dv_g
\end{aligned}$$

and finally

$$\begin{aligned}
&\int_{\mathfrak{M}} \Delta_g u\,|\nabla u|^2\,e^{-u/2}\,dv_g \\
&\qquad\qquad = \frac{1}{2}\,\frac{d}{d+2}\int_{\mathfrak{M}} |\nabla u|^4\,e^{-u/2}\,dv_g - \frac{2\,d}{d+2}\int_{\mathfrak{M}} \mathrm{L}_g u : \mathrm{M}_g u\,e^{-u/2}\,dv_g\,. \qquad (1.2)
\end{aligned}$$

An identity based on the Bochner-Lichnerovicz-Weitzenböck formula

By expanding $\Delta_g(e^{-u/2}) = (\frac{1}{4}\,|\nabla u|^2 - \frac{1}{2}\,\Delta_g u)\,e^{-u/2}$, we have that

$$\int_{\mathfrak{M}} |\nabla u|^2\,\Delta_g(e^{-u/2})\,dv_g = \frac{1}{4}\int_{\mathfrak{M}} |\nabla u|^4\,e^{-u/2}\,dv_g - \frac{1}{2}\int_{\mathfrak{M}} \Delta_g u\,|\nabla u|^2\,e^{-u/2}\,dv_g$$

so that, if we multiply (1.1) by $e^{-u/2}$ and integrate by parts, then we get

$$\begin{aligned}
&\frac{1}{8}\int_{\mathfrak{M}} |\nabla u|^4\,e^{-u/2}\,dv_g - \frac{1}{4}\int_{\mathfrak{M}} \Delta_g u\,|\nabla u|^2\,e^{-u/2}\,dv_g \\
&\quad = \int_{\mathfrak{M}} \|\mathrm{L}_g u\|^2\,e^{-u/2}\,dv_g + \frac{1}{d}\int_{\mathfrak{M}} (\Delta_g u)^2\,e^{-u/2}\,dv_g \\
&\qquad + \int_{\mathfrak{M}} \big(\nabla(\Delta_g u)\cdot\nabla u\big)\,e^{-u/2}\,dv_g + \int_{\mathfrak{M}} \mathfrak{R}(\nabla u,\nabla u)\,e^{-u/2}\,dv_g \\
&\qquad = \int_{\mathfrak{M}} \|\mathrm{L}_g u\|^2\,e^{-u/2}\,dv_g - \frac{d-1}{d}\int_{\mathfrak{M}} (\Delta_g u)^2\,e^{-u/2}\,dv_g \\
&\qquad\qquad + \frac{1}{2}\int_{\mathfrak{M}} \Delta_g u\,|\nabla u|^2\,e^{-u/2}\,dv_g + \int_{\mathfrak{M}} \mathfrak{R}(\nabla u,\nabla u)\,e^{-u/2}\,dv_g\,,
\end{aligned}$$

from which we deduce that

$$\int_{\mathfrak{M}} (\Delta_g u)^2\, e^{-u/2}\, dv_g = \frac{3}{4}\,\frac{d}{d-1}\int_{\mathfrak{M}} \Delta_g u\, |\nabla u|^2\, e^{-u/2}\, dv_g - \frac{1}{8}\,\frac{d}{d-1}\int_{\mathfrak{M}} |\nabla u|^4\, e^{-u/2}\, dv_g + \frac{d}{d-1}\int_{\mathfrak{M}} \|\mathrm{L}_g u\|^2\, e^{-u/2}\, dv_g + \frac{d}{d-1}\int_{\mathfrak{M}} \Re(\nabla u, \nabla u)\, e^{-u/2}\, dv_g\,. \quad (1.3)$$

A Poincaré inequality

Since

$$4\,\Delta_g(e^{-u/4}) = \frac{1}{4}\,|\nabla u|^2\, e^{-u/4} - \Delta_g u\, e^{-u/4}\,,$$

we get that

$$16\int_{\mathfrak{M}} |\Delta_g(e^{-u/4})|^2\, dv_g = \frac{1}{16}\int_{\mathfrak{M}} |\nabla u|^4\, e^{-u/2}\, dv_g - \frac{1}{2}\int_{\mathfrak{M}} |\nabla u|^2\, \Delta_g u\, e^{-u/2}\, dv_g + \int_{\mathfrak{M}} (\Delta_g u)^2\, e^{-u/2}\, dv_g\,.$$

On the other hand, a Poincaré inequality applied to $\nabla(e^{-u/4})$, as in [DEL13b, Lemma 7], shows that

$$\int_{\mathfrak{M}} |\Delta_g(e^{-u/4})|^2\, dv_g \geqslant \lambda_1 \int_{\mathfrak{M}} |\nabla(e^{-u/4})|^2\, dv_g = \frac{\lambda_1}{16}\int_{\mathfrak{M}} |\nabla u|^2\, e^{-u/2}\, dv_g\,,$$

so that

$$\int_{\mathfrak{M}} (\Delta_g u)^2\, e^{-u/2}\, dv_g \geqslant \lambda_1 \int_{\mathfrak{M}} |\nabla u|^2\, e^{-u/2}\, dv_g - \frac{1}{16}\int_{\mathfrak{M}} |\nabla u|^4\, e^{-u/2}\, dv_g + \frac{1}{2}\int_{\mathfrak{M}} |\nabla u|^2\, \Delta_g u\, e^{-u/2}\, dv_g\,. \quad (1.4)$$

An identity based on the equation

By expanding $\Delta_g(e^{-u/2}) = (\frac{1}{4}\,|\nabla u|^2 - \frac{1}{2}\,\Delta_g u)\, e^{-u/2}$, we have that

$$\int_{\mathfrak{M}} (-\tfrac{1}{2}\,\Delta_g u)\, \Delta_g(e^{-u/2})\, dv_g = \frac{1}{4}\int_{\mathfrak{M}} (\Delta_g u)^2\, e^{-u/2}\, dv_g - \frac{1}{8}\int_{\mathfrak{M}} \Delta_g u\, |\nabla u|^2\, e^{-u/2}\, dv_g$$

so that, if we multiply (0.2) by $\Delta_g(e^{-u/2}) - \frac{1}{2}\,|\nabla u|^2\, e^{-u/2}$, then we get

$$\frac{1}{4}\int_{\mathfrak{M}} (\Delta_g u)^2\, e^{-u/2}\, dv_g + \frac{1}{8}\int_{\mathfrak{M}} \Delta_g u\, |\nabla u|^2\, e^{-u/2}\, dv_g - \frac{\lambda}{2}\int_{\mathfrak{M}} |\nabla u|^2\, e^{-u/2}\, dv_g = 0\,.$$

Let us introduce the parameter $\theta \leqslant 1$, to be chosen later, and use (1.4) and (1.3) to estimate $\frac{1-\theta}{4}\int_{\mathfrak{M}} (\Delta_g u)^2\, e^{-u/2}\, dv_g$ and $\frac{\theta}{4}\int_{\mathfrak{M}} (\Delta_g u)^2\, e^{-u/2}\, dv_g$ respectively. We get

$$\begin{aligned}
\frac{1-\theta}{4}\Big[\lambda_1\int_{\mathfrak{M}} |\nabla u|^2\, e^{-u/2}\, dv_g - \frac{1}{16}\int_{\mathfrak{M}} |\nabla u|^4\, e^{-u/2}\, dv_g \\
+ \frac{1}{2}\int_{\mathfrak{M}} |\nabla u|^2\, \Delta_g u\, e^{-u/2}\, dv_g\Big] \\
+ \frac{\theta}{4}\Big[\frac{3}{4}\,\frac{d}{d-1}\int_{\mathfrak{M}} \Delta_g u\, |\nabla u|^2\, e^{-u/2}\, dv_g - \frac{1}{8}\,\frac{d}{d-1}\int_{\mathfrak{M}} |\nabla u|^4\, e^{-u/2}\, dv_g \\
+ \frac{d}{d-1}\int_{\mathfrak{M}} \|\mathrm{L}_g u\|^2\, e^{-u/2}\, dv_g + \frac{d}{d-1}\int_{\mathfrak{M}} \mathfrak{R}(\nabla u, \nabla u)\, e^{-u/2}\, dv_g\Big] \\
+ \frac{1}{8}\int_{\mathfrak{M}} \Delta_g u\, |\nabla u|^2\, e^{-u/2}\, dv_g - \frac{\lambda}{2}\int_{\mathfrak{M}} |\nabla u|^2\, e^{-u/2}\, dv_g \leqslant 0\,.
\end{aligned}$$

Collecting terms, we get

$$\begin{aligned}
\frac{\theta}{4}\,\frac{d}{d-1}\Big[\int_{\mathfrak{M}} \|\mathrm{L}_g u\|^2\, e^{-u/2}\, dv_g + \int_{\mathfrak{M}} \mathfrak{R}(\nabla u, \nabla u)\, e^{-u/2}\, dv_g\Big] \\
- \frac{1}{64}\left(1-\theta+2\,\theta\,\frac{d}{d-1}\right)\int_{\mathfrak{M}} |\nabla u|^4\, e^{-u/2}\, dv_g \\
+ \left(\frac{1-\theta}{8} + \frac{3\,\theta}{16}\,\frac{d}{d-1} + \frac{1}{8}\right)\int_{\mathfrak{M}} \Delta_g u\, |\nabla u|^2\, e^{-u/2}\, dv_g \\
+ \left(\frac{1-\theta}{4}\,\lambda_1 - \frac{\lambda}{2}\right)\int_{\mathfrak{M}} |\nabla u|^2\, e^{-u/2}\, dv_g \leqslant 0
\end{aligned}$$

and can now use (1.2) to obtain

$$\begin{aligned}
\frac{\theta}{4}\,\frac{d}{d-1}\Big[\int_{\mathfrak{M}} \|\mathrm{L}_g u\|^2\, e^{-u/2}\, dv_g + \int_{\mathfrak{M}} \mathfrak{R}(\nabla u, \nabla u)\, e^{-u/2}\, dv_g\Big] \\
- \frac{1}{64}\left(1-\theta+2\,\theta\,\frac{d}{d-1}\right)\int_{\mathfrak{M}} |\nabla u|^4\, e^{-u/2}\, dv_g \\
+ \left(\frac{1-\theta}{8} + \frac{3\,\theta}{16}\,\frac{d}{d-1} + \frac{1}{8}\right)\Big[\frac{1}{2}\,\frac{d}{d+2}\int_{\mathfrak{M}} |\nabla u|^4\, e^{-u/2}\, dv_g \\
- \frac{2\,d}{d+2}\int_{\mathfrak{M}} \mathrm{L}_g u : \mathrm{M}_g u\, e^{-u/2}\, dv_g\Big] \\
+ \left(\frac{1-\theta}{4}\,\lambda_1 - \frac{\lambda}{2}\right)\int_{\mathfrak{M}} |\nabla u|^2\, e^{-u/2}\, dv_g \leqslant 0\,.
\end{aligned}$$

Recall that $\mathrm{M}_g u$ denotes the trace free tensor

$$\mathrm{M}_g u \coloneqq \nabla u \otimes \nabla u - \frac{g}{d}\,|\nabla u|^2\,.$$

We observe that

$$\|\mathrm{M}_g u\|^2 = \Big\|\nabla u \otimes \nabla u - \frac{g}{d}\,|\nabla u|^2\Big\|^2 = \left(1-\frac{1}{d}\right)|\nabla u|^4\,.$$

Altogether we end up with

$$\int_{\mathfrak{M}} \Big(\mathsf{a} \, \|\mathrm{L}_g u\|^2 + \mathsf{b} \, (\mathrm{L}_g u : \mathrm{M}_g u) + \mathsf{c} \, \|\mathrm{M}_g u\|^2 \Big) \, e^{-u/2} \, dv_g \\ + \frac{\theta}{4} \, \frac{d}{d-1} \int_{\mathfrak{M}} \Re(\nabla u, \nabla u) \, e^{-u/2} \, dv_g \\ + \left(\frac{1-\theta}{4} \, \lambda_1 - \frac{\lambda}{2} \right) \int_{\mathfrak{M}} |\nabla u|^2 \, e^{-u/2} \, dv_g \leqslant 0$$

with

$$\mathsf{a} = \frac{\theta}{4} \, \frac{d}{d-1} \, , \\ \mathsf{b} = - \left(\frac{1-\theta}{8} + \frac{3\,\theta}{16} \, \frac{d}{d-1} + \frac{1}{8} \right) \frac{2\,d}{d+2} \, , \\ \mathsf{c} = \left[\left(\frac{1-\theta}{8} + \frac{3\,\theta}{16} \, \frac{d}{d-1} + \frac{1}{8} \right) \frac{1}{2} \, \frac{d}{d+2} - \frac{1}{64} \left(1 - \theta + 2\,\theta \, \frac{d}{d-1} \right) \right] \frac{d}{d-1} \, .$$

Remark 25. *By the Lichnerowicz' theorem (see [Lic58] or [DEL13b, Section 2])*

$$\frac{d}{d-1} \int_{\mathfrak{M}} \Re(\nabla u, \nabla u) \, e^{-u/2} \, dv_g \leqslant \int_{\mathfrak{M}} |\nabla u|^2 \, e^{-u/2} \, dv_g$$

so that the largest possible value of λ for which we a priori *know that*

$$\frac{\theta}{4} \, \frac{d}{d-1} \int_{\mathfrak{M}} \Re(\nabla u, \nabla u) \, e^{-u/2} \, dv_g + \left(\tfrac{1}{4} \, \lambda_1 \, (1-\theta) - \lambda \right) \int_{\mathfrak{M}} |\nabla u|^2 \, e^{-u/2} \, dv_g$$

is nonnegative corresponds to the smallest possible value of θ, i.e. *$\theta = \theta_0(d)$.*

Discussion

A simple but tedious commutation shows that the discriminant $\delta \coloneqq \mathsf{b}^2 - 4\,\mathsf{a}\,\mathsf{c}$ has the sign of

$$16 \, (d-1)^2 - (6-d) \, (d+2) \, \theta \, .$$

If we denote by $\theta_0 = \theta_0(d)$ the value of θ for which $\delta = 0$, then we have

$$\theta_0 = \frac{16 \, (d-1)^2}{(6-d) \, (d+2)} \, .$$

Altogether we can rewrite our estimate as

$$\mathsf{a} \int_{\mathfrak{M}} \left\| \mathrm{L}_g u + \tfrac{\mathsf{b}}{2\,\mathsf{a}} \, \mathrm{M}_g u \right\|^2 \, e^{-u/2} \, dv_g + \left(\mathsf{c} - \tfrac{\mathsf{b}^2}{4\,\mathsf{a}} \right) \int_{\mathfrak{M}} \|\mathrm{M}_g u\|^2 \, e^{-u/2} \, dv_g \\ + \, 4 \, (1-\theta) \int_{\mathfrak{M}} |\Delta_g (e^{-u/4})|^2 \, dv_g + \frac{\theta}{4} \, \frac{d}{d-1} \int_{\mathfrak{M}} \Re(\nabla u, \nabla u) \, e^{-u/2} \, dv_g \\ - \frac{\lambda}{2} \int_{\mathfrak{M}} |\nabla u|^2 \, e^{-u/2} \, dv_g = 0$$

and use the Poincaré inequality, that is,

$$4 \int_{\mathfrak{M}} |\Delta_g (e^{-u/4})|^2 \, dv_g \geqslant \frac{1}{4} \, \lambda_1 \int_{\mathfrak{M}} |\nabla u|^2 \, e^{-u/2} \, dv_g \, ,$$

to establish that

$$0 \geqslant \mathsf{a} \int_{\mathfrak{M}} \left\| \mathrm{L}_g u + \tfrac{\mathsf{b}}{2\,\mathsf{a}}\, \mathrm{M}_g u \right\|^2 e^{-u/2}\, dv_g - \tfrac{\delta}{4\,\mathsf{a}} \int_{\mathfrak{M}} \|\mathrm{M}_g u\|^2\, e^{-u/2}\, dv_g + \left(\tfrac{1}{4}\,\lambda_1\,(1-\theta) - \tfrac{\lambda}{2}\right) \int_{\mathfrak{M}} |\nabla u|^2\, e^{-u/2}\, dv_g + \tfrac{\theta}{4}\,\tfrac{d}{d-1} \int_{\mathfrak{M}} \mathfrak{R}(\nabla u, \nabla u)\, e^{-u/2}\, dv_g\,. \quad (1.5)$$

Our goal is to show that u has to be a constant, that is, $\int_{\mathfrak{M}} |\nabla u|^2\, e^{-u/2}\, dv_g = 0$. We assume that $d \geqslant 2$ is an integer. The discriminant δ is nonpositive if and only if $d < 6$ and $\theta \geqslant \theta_0(d)$. This is compatible with the condition $\theta \leqslant 1$ only if $d = 2$; in that case, $\theta = \theta_0(2) = 1$ and, as a consequence of (1.5), we get that

$$0 \geqslant \int_{\mathfrak{M}} \|\mathrm{L}_g u - \tfrac{1}{2}\,\mathrm{M}_g u\|^2\, e^{-u/2}\, dv_g + \int_{\mathfrak{M}} \mathfrak{R}(\nabla u, \nabla u)\, e^{-u/2}\, dv_g - \lambda \int_{\mathfrak{M}} |\nabla u|^2\, e^{-u/2}\, dv_g \geqslant (\lambda_\star - \lambda) \int_{\mathfrak{M}} |\nabla u|^2\, e^{-u/2}\, dv_g\,.$$

Hence we have shown that $\nabla u \equiv 0$ for any $\lambda < \lambda_\star$, which concludes the proof of Theorem 1.

Remark 26. *In order to compare our results with the results deduced from the* curvature-dimension *method, we can consider the case where $\mathfrak{R}$ is uniformly bounded from below by some positive constant ρ and formally assume that d takes real values. This can be made precise for instance in the setting of the ultra-spherical operator (see for instance [Ben93]), with exactly the same conditions as above. See [Fon97] and [DEJ14, Section 7.1] for more details. If $1 < d \leqslant 2$, we find that rigidity holds if*

$$\lambda \leqslant \max_{\theta \in [\theta_0(d),1]} \left(\frac{1}{2}\,\lambda_1\,(1-\theta) + \frac{\theta}{2}\,\frac{d}{d-1}\,\rho\right) = \frac{1}{2}\,\lambda_1\,(1-\theta_0(d)) + \frac{\theta_0(d)}{2}\,\frac{d}{d-1}\,\rho$$

according to Remark 25. Let $x = \frac{d}{d-1}\,\frac{\rho}{\lambda_1} \leqslant 1$. We have found that that rigidity holds if

$$2\,\frac{\lambda}{\lambda_1} \leqslant 1 - \theta_0(d) + \theta_0(d)\,x =: f_1(x)$$

Quite surprisingly, a better condition has been obtained in [Fon97, Théorème 2], when $1 < d < 2$, which amounts to

$$2\,\frac{\lambda}{\lambda_1} \leqslant d\,(2-d) + (d-1)^2\,x =: f_2(x)\,,$$

by taking the limit as $p \to \infty$ in (0.4). We may indeed check that $f_2(x) - f_1(x) = \frac{(d-1)^2\,(d-2)^2}{(6-d)\,(d+2)}\,(1-x) \geqslant 0$

Without assuming the positivity of ρ, one gets a similar result with our approach. In the range $d \in (1,2)$, our computations show that rigidity holds for any λ at most equal to the infimum on $u \in \mathrm{H}^2(\mathfrak{M}) \backslash \{0\}$ of

$$2 \int_{\mathfrak{M}} \left[\mathsf{a}\,\left\|\mathrm{L}_g u + \tfrac{\mathsf{b}}{2\,\mathsf{a}}\,\mathrm{M}_g u\right\|^2 + 4\,(1-\theta)\,|\Delta_g(e^{-u/4})|^2 + \tfrac{\theta}{4}\,\tfrac{d}{d-1}\,\mathfrak{R}(\nabla u, \nabla u)\right] e^{-u/2}\, dv_g$$

under the condition that $\int_{\mathfrak{M}} |\nabla u|^2\, e^{-u/2}\, dv_g = 1$, $\mathsf{a} = \frac{4\,d\,(d-1)}{(6-d)\,(d+2)}$, $\mathsf{b} = -\frac{d\,(3\,d+2)}{2\,(6-d)\,(d+2)}$ and $\theta = \theta_0(d)$. However, in the same spirit as above, a passage to the limit as $p \to \infty$ in the inequality obtained in [DEL13b, Theorem 4] gives a better result.

Let us emphasize that these considerations are essentially formal because d is restricted to the interval $(1,2)$ but can be entirely justified, as it is currently done in the curvature-dimension *approach.*

2 Proof of Corollary 24

Let us minimize the functional

$$\mathcal{F}_\lambda[u] := \frac{1}{4}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathfrak{M})} + \lambda \int_{\mathfrak{M}} u\,dv_g - \lambda\,\log\left(\int_{\mathfrak{M}} e^u\,dv_g\right)$$

on $\mathrm{H}^1(\mathfrak{M})$. According to [Fag06 ; Fag08], $\mathcal{F}_\lambda[u]$ has no minimizer if $\lambda > 4\,\pi$. Let us assume that

$$\lambda < 4\,\pi$$

(we shall take care of the equality case later). It is then standard that there is a well-defined minimizer. If it is smooth and $\lambda < \lambda_\star$, we can apply the result of Theorem 1. Then the minimizer u can be any constant, for instance $u \equiv 1$, so that

$$\mathcal{F}_\lambda[u] \geqslant \mathcal{F}_\lambda[1] = 0 \quad \forall\, u \in \mathrm{H}^1(\mathfrak{M})\,.$$

Notice that we can get rid of any smoothness requirement by considering the flow below. By passing to the limit as $\lambda \nearrow \lambda_\star$, we get that the inequality also holds true if $\lambda = \lambda_\star$.

Using a Taylor expansion of

$$\mathcal{F}_\lambda[1+\varepsilon\,\varphi] = \left[\frac{1}{4}\,\|\nabla\varphi\|^2_{\mathrm{L}^2(\mathfrak{M})} + \frac{\lambda}{2}\int_{\mathfrak{M}} \varphi^2\,dv_g\right]\varepsilon^2 + o(\varepsilon^2)\,,$$

as $\varepsilon \to 0$, where φ is an eigenfunction associated with the first positive eigenvalue λ_1 of $-\,\Delta_g$, it is straightforward to see that the best constant λ in (0.3) is such that

$$\lambda \leqslant \frac{\lambda_1}{2}\,.$$

To complete the proof of Corollary 24, it remains to consider the case $\lambda_\star < \lambda_1/2$ and show that the optimal constant cannot be equal to $\lambda_\star$. This can be done in the same spirit as in [DEL13b, Corollary 2]. Let us consider the evolution equation defined by

$$\frac{\partial f}{\partial t} = \Delta_g(e^{-f/2}) - \tfrac{1}{2}\,|\nabla f|^2\,e^{-f/2}\,, \tag{2.1}$$

with initial datum $u \in \mathrm{H}^1(\mathfrak{M})$. Let us define

$$\mathcal{G}_\lambda[f] := \int_{\mathfrak{M}} \|\,\mathrm{L}_g f - \tfrac{1}{2}\,\mathrm{M}_g f\,\|^2\,e^{-f/2}\,dv_g + \int_{\mathfrak{M}} \mathfrak{R}(\nabla f,\nabla f)\,e^{-f/2}\,dv_g - \lambda\int_{\mathfrak{M}} |\nabla f|^2\,e^{-f/2}\,dv_g\,.$$

Then for any $\lambda \leqslant \lambda_\star$ we have

$$\frac{d}{dt}\mathcal{F}_\lambda[f(t,\cdot)] = \int_{\mathfrak{M}} \left(-\tfrac{1}{2}\,\Delta_g f + \lambda\right)\left(\Delta_g(e^{-f/2}) - \tfrac{1}{2}\,|\nabla f|^2\,e^{-f/2}\right)dv_g = -\,\mathcal{G}_\lambda[f(t,\cdot)]$$

Since $\mathcal{F}_\lambda$ is nonnegative and $\lim_{t\to\infty}\mathcal{F}_\lambda[f(t,\cdot)] = 0$, we obtain that

$$\mathcal{F}_\lambda[u] \geqslant \int_0^\infty \mathcal{G}_\lambda[f(t,\cdot)]\,dt$$

for any solution f to (2.1) with initial datum $u \in \mathrm{L}^1(\mathfrak{M})$ is such that $\nabla u \in \mathrm{L}^2(\mathfrak{M})$. We have an equality if the solution is smooth for any $t \geqslant 0$. Otherwise we have to regularize and then pass to the limit so that, with full generality, we can only expect for an inequality.

Remark 27. *One has to mention that the sphere $\mathfrak{M} = \mathbb{S}^2$ is an important case of application of our method, for which other types of remainder terms can be produced. See [DEJ14] for more details. It has to be noted that on $\mathbb{S}^2$ we have $\lambda_\star = \rho = \lambda_1/2 = 1$. As noted by many authors,* e.g., *in [Bec93 ; Ben93 ; Fon97 ; DEJ14] (also see references in [DEJ14]), the Onofri inequality is a limit case of various Sobolev type inequalities, for which similar methods have been developed : see [Dol+14] for a review and some recent results.*

3 The case $d = 1$

Since we assume that $\mathfrak{M}$ is compact, connected, and without boundary, we only have to consider the case of the circle. Hence we identify $\mathfrak{M}$ with the 1-periodic interval $[0, 1) \approx \mathbb{R}/\mathbb{Z} \approx \mathbb{S}^1$. Consider a solution of the ordinary differential equation

$$-\frac{1}{2}\,u'' + \lambda - e^u = 0 \tag{3.1}$$

with periodic boundary conditions. If we multiply the equation by $(e^{-u/2})'' - \frac{1}{2}\,|u'|^2\,e^{-u/2}$, then we get that

$$\int_0^1 \left(\tfrac{1}{4}\,|u''|^2 + \tfrac{1}{8}\,|u'|^2\,u'' - \tfrac{\lambda}{2}\,|u'|^2\right) e^{-u/2}\,dx = 0\,.$$

The middle term is easy to handle using one integration by parts :

$$\int_0^1 |u'|^2\,u''\,e^{-u/2}\,dx = \tfrac{1}{6}\int_0^1 |u'|^4\,e^{-u/2}\,dx\,. \tag{3.2}$$

Hence we have

$$\int_0^1 \left(\tfrac{1}{4}\,|u''|^2 + \tfrac{1}{48}\,|u'|^4 - \tfrac{\lambda}{2}\,|u'|^2\right) e^{-u/2}\,dx = 0\,. \tag{3.3}$$

On the other hand,

$$\int_0^1 \left|\left(e^{-u/4}\right)''\right|^2 dx \geqslant 4\,\pi^2 \int_0^1 \left|\left(e^{-u/4}\right)'\right|^2 dx\,,$$

where $4\,\pi^2$ is the first positive eigenvalue of $-\frac{d^2}{dx^2}$ on the periodic interval of length 1. From (3.2) we derive

$$\int_0^1 \left(|u''|^2 - \tfrac{1}{48}\,|u'|^4\right)\,e^{-u/2}\,dx - 4\,\pi^2 \int_0^1 |u'|^2\,e^{-u/2}\,dx \geqslant 0\,. \tag{3.4}$$

Combining (3.3) and (3.4), we get

$$\frac{5}{96}\int_0^1 |u'|^4\,e^{-u/2}\,dx + (2\pi^2 - \lambda)\int_0^1 |u'|^2\,e^{-u/2}\,dx \leqslant 0\,.$$

Hence we have proven the following result.

Proposition 28. *Assume that $d = 1$. With the above notations, if u is a smooth solution to (3.1) on $\mathbb{S}^1 \approx [0, 1)$, then u is a constant function for any $\lambda \in (0, 2\,\pi^2)$.*

Exactly as in the case of a manifold of dimension two, a variational approach allows to deduce a Moser-Trudinger-Onofri inequality.

Corollary 29. *If $d = 1$, then the following inequality holds on $\mathbb{S}^1 \approx [0,1)$:*

$$\frac{1}{8\,\pi^2}\int_0^1 |u'|^2\,dx + \int_0^1 u\,dx \geqslant \log\left(\int_0^1 e^u\,dx\right) \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^1)\,.$$

Moreover $8\,\pi^2$ is the optimal constant.

The only difference with Corollary 24 is that we can identify the optimal constant in the inequality by considering $u = 1+\varepsilon\,\varphi$ and by taking the limit as $\varepsilon \to 0$, with $\varphi(x) = \cos(2\pi\,x)$.

4 Weighted Moser-Trudinger-Onofri inequalities on the two-dimensional Euclidean space

The Euclidean Onofri inequality (see [CL92 ; DEJ14]) can be deduced from (0.3) when $\mathfrak{M} = \mathbb{S}^2$ using the stereographic projection and reads

$$\frac{1}{16\,\pi}\int_{\mathbb{R}^2} |\nabla u|^2\,dx \geqslant \log\left(\int_{\mathbb{R}^2} e^u\,d\mu\right) - \int_{\mathbb{R}^2} u\,d\mu\,. \tag{4.1}$$

Here $d\mu = \mu(x)\,dx$ denotes the probability measure defined by $\mu(x) = \frac{1}{\pi}\,(1+|x|^2)^{-2}$, $x \in \mathbb{R}^2$, and the inequality holds for any function $u \in \mathrm{L}^1(\mathbb{R}^2, d\mu)$ such that $\nabla u \in \mathrm{L}^2(\mathbb{R}^2)$. The constant $16\,\pi$ is optimal as can be shown by considering the inequality on $\mathbb{S}^2$ and comparing with the value given when expanding around a constant, as was done in Section 2.

In this section, our goal is to give sufficient conditions on a general probability measure μ so that the inequality

$$\frac{1}{16\,\pi}\int_{\mathbb{R}^2} |\nabla u|^2\,dx \geqslant \lambda\left[\log\left(\int_{\mathbb{R}^2} e^u\,d\mu\right) - \int_{\mathbb{R}^2} u\,d\mu\right] \tag{4.2}$$

holds for some $\lambda > 0$ and get an estimate of the optimal value of λ. Here $d\mu = \mu\,dx$ is a probability measure with density μ with respect to the Lebesgue measure. All our computations are done without symmetry assumption, and our final estimate is (4.15). In practical applications (see Examples 1–4) the function μ is radially symmetric *and* one has to assume that λ is in a range for which the solution to (4.2), or at least the optimal function for (4.1), is radially symmetric. This delicate issue of *symmetry breaking* will be illustrated in Example 3.

Since (4.2) does not change when adding some constant to u, we can look for minimizers satisfying the constraint $\int_{\mathbb{R}^2} e^u\,d\mu = 1$. These solve the Euler-Lagrange equation

$$-\frac{1}{8\,\pi}\,\Delta u + \lambda\,\mu - \lambda\,e^u\,\mu = 0\,. \tag{4.3}$$

We can multiply each term of (4.3) by $\frac{1}{\mu}\,\Delta(e^{-\frac{u}{2}}) - \frac{1}{2\,\mu}\,|\nabla u|^2\,e^{-\frac{u}{2}}$ and integrate, which gives the following identities

$$-\int_{\mathbb{R}^2} \Delta u\,\Delta(e^{-u/2})\,\frac{1}{\mu}\,dx = \int_{\mathbb{R}^2} \Delta u\left(\Delta u - \frac{1}{2}\,|\nabla u|^2\right) e^{-u/2}\,\frac{1}{2\,\mu}\,dx\,,$$

$$\int_{\mathbb{R}^2} \mu\,\frac{1}{\mu}\,\Delta(e^{-u/2})\,dx = 0\,,$$

$$\int_{\mathbb{R}^2} e^u\mu\left(\frac{1}{\mu}\,\Delta(e^{-\frac{u}{2}}) - \frac{1}{2\,\mu}\,|\nabla u|^2\,e^{-\frac{u}{2}}\right) dx = 0\,.$$

Defining $\nu \coloneqq e^{-u/2}/\mu = e^{-u/2-g}$ with $g \coloneqq \log \mu$ and $d\nu \coloneqq \nu\, dx$ we have

$$\mathcal{I}[u] = 2\int_{\mathbb{R}^2} (\Delta u)^2\, d\nu + \int_{\mathbb{R}^2} \Delta u\, |\nabla u|^2\, d\nu - 16\,\pi\,\lambda \int_{\mathbb{R}^2} |\nabla u|^2\, e^{-u/2}\, dx = 0\,.$$

Let us introduce some notations, which are consistent with the ones on manifolds. Let us denote by $\mathrm{H}u = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)_{i,j=1,2}$ the Hessian of u, $\mathrm{L}u = \mathrm{H}u - \frac{1}{2}\,\Delta u\,\mathrm{I}_2$ is the trace free Hessian and $\mathrm{M}u \coloneqq \nabla u \otimes \nabla u - \frac{1}{2}\,|\nabla u|^2\,\mathrm{I}_2$, where $\nabla u \otimes \nabla u = \left(\frac{\partial u}{\partial x_i}\,\frac{\partial u}{\partial x_j}\right)_{i,j=1,2}$. For the convenience of the reader, we split the computations in four steps.

1) Let us start with some preliminary computations. An integration by parts shows that

$$\begin{aligned}
&2\int_{\mathbb{R}^2} \Delta u\, \nabla u\cdot\nabla g\, d\nu - \int_{\mathbb{R}^2} |\nabla u|^2\, \nabla u\cdot\nabla g\, d\nu \\
&\qquad = -\,2\int_{\mathbb{R}^2} \mathrm{H}u : (\nabla u\otimes\nabla g)\, d\nu - \,2\int_{\mathbb{R}^2} (\mathrm{H}g - \nabla g\otimes\nabla g) : (\nabla u\otimes\nabla u)\, d\nu \\
&\quad = -\,2\int_{\mathbb{R}^2} \mathrm{H}u : (\nabla u\otimes\nabla g)\, d\nu - \,2\int_{\mathbb{R}^2} \mathrm{H}g : (\nabla u\otimes\nabla u)\, d\nu + 2\int_{\mathbb{R}^2} (\nabla u\cdot\nabla g)^2\, d\nu\,. \qquad (4.4)
\end{aligned}$$

By expanding $\mathrm{L}u - \frac{1}{2}\,\mathrm{M}u$, we also get that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left(\mathrm{L}u - \frac{1}{2}\,\mathrm{M}u\right) : (\nabla u\otimes\nabla g)\, d\nu \\
&\qquad = \int_{\mathbb{R}^2} \mathrm{H}u : (\nabla u\otimes\nabla g)\, d\nu - \frac{1}{2}\int_{\mathbb{R}^2} \Delta u\, \nabla u\cdot\nabla g\, d\nu - \frac{1}{4}\int_{\mathbb{R}^2} |\nabla u|^2\, \nabla u\cdot\nabla g\, d\nu\,. \qquad (4.5)
\end{aligned}$$

Recalling the definition of $\nu = e^{-\frac{1}{2}u-g}$, we also find that

$$\begin{aligned}
-\,\frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\, \nabla u\cdot\nabla g\, d\nu &= \int_{\mathbb{R}^2} |\nabla u|^2\, \nabla g\, e^{-g}\cdot\nabla(e^{-\frac{1}{2}u})\, dx \\
&= -\,2\int_{\mathbb{R}^2} \mathrm{H}u : (\nabla u\otimes\nabla g)\, d\nu + \int_{\mathbb{R}^2} |\nabla u|^2\, (|\nabla g|^2 - \Delta g)\, d\nu\,. \qquad (4.6)
\end{aligned}$$

Equations (4.4), (4.5) and (4.6) allow us to eliminate

$$\int_{\mathbb{R}^2} \Delta u\, \nabla u\cdot\nabla g\, d\nu\,, \quad \int_{\mathbb{R}^2} |\nabla u|^2\, \nabla u\cdot\nabla g\, d\nu \quad \text{and} \quad \int_{\mathbb{R}^2} \mathrm{H}u : (\nabla u\otimes\nabla g)\, d\nu$$

in terms of the other quantities :

$$\begin{aligned}
\int_{\mathbb{R}^2} \Delta u\, \nabla u\cdot\nabla g\, d\nu &= \int_{\mathbb{R}^2} |\nabla u|^2\, (|\nabla g|^2 - \Delta g)\, d\nu \\
&\quad - 2\int_{\mathbb{R}^2} \left(\mathrm{L}u - \frac{1}{2}\,\mathrm{M}u\right) : (\nabla u\otimes\nabla g)\, d\nu\,, \qquad (4.7)\\
\int_{\mathbb{R}^2} |\nabla u|^2\, \nabla u\cdot\nabla g\, d\nu &= 4\int_{\mathbb{R}^2} \mathrm{H}g : (\nabla u\otimes\nabla u)\, d\nu - 4\int_{\mathbb{R}^2} (\nabla u\cdot\nabla g)^2\, d\nu \\
&\quad - 8\int_{\mathbb{R}^2} \left(\mathrm{L}u - \frac{1}{2}\,\mathrm{M}u\right) : (\nabla u\otimes\nabla g)\, d\nu \qquad (4.8)\\
&\quad + 6\int_{\mathbb{R}^2} |\nabla u|^2\, (|\nabla g|^2 - \Delta g)\, d\nu\,,\\
\int_{\mathbb{R}^2} \mathrm{H}u : (\nabla u\otimes\nabla g)\, d\nu &= \int_{\mathbb{R}^2} \mathrm{H}g : (\nabla u\otimes\nabla u)\, d\nu - \int_{\mathbb{R}^2} (\nabla u\cdot\nabla g)^2\, d\nu \\
&\quad - 2\int_{\mathbb{R}^2} \left(\mathrm{L}u - \frac{1}{2}\,\mathrm{M}u\right) : (\nabla u\otimes\nabla g)\, d\nu \qquad (4.9)\\
&\quad + 2\int_{\mathbb{R}^2} |\nabla u|^2\, (|\nabla g|^2 - \Delta g)\, d\nu\,.
\end{aligned}$$

Moreover, the reader is invited to check that

$$\|\mathrm{L}u\|^2 = \|\mathrm{H}u\|^2 - \frac{1}{2}\,(\Delta u)^2 \qquad (4.10)$$

and

$$\|\mathrm{M}u\|^2 = \frac{1}{2}\,|\nabla u|^4\,. \qquad (4.11)$$

2) On the one hand, integrating the second term in the expression of $\mathcal{I}[u]$ by parts gives

$$\begin{aligned}
&\int_{\mathbb{R}^2} \Delta u\, |\nabla u|^2\, d\nu\\
&= -\int_{\mathbb{R}^2} \nabla u\cdot\nabla\left(|\nabla u|^2\,\nu\right) dx\\
&= -\int_{\mathbb{R}^2} \nabla u\cdot\left(2\,\mathrm{H}u\,\nabla u - \frac{1}{2}\,\nabla u\,|\nabla u|^2 - |\nabla u|^2\,\nabla g\right) d\nu\\
&= -\int_{\mathbb{R}^2} \left(2\,\mathrm{H}u : \nabla u\otimes\nabla u - \frac{1}{2}\,|\nabla u|^4 - |\nabla u|^2\,(\nabla u\cdot\nabla g)\right) d\nu\\
&= -\int_{\mathbb{R}^2} \left(2\,\mathrm{L}u : \nabla u\otimes\nabla u + \Delta u\,|\nabla u|^2 - \frac{1}{2}\,|\nabla u|^4 - |\nabla u|^2\,(\nabla u\cdot\nabla g)\right) d\nu\\
&= -\int_{\mathbb{R}^2} \left(2\,\mathrm{L}u : \mathrm{M}u + \Delta u\,|\nabla u|^2 - \frac{1}{2}\,|\nabla u|^4 - |\nabla u|^2\,(\nabla u\cdot\nabla g)\right) d\nu\,,
\end{aligned}$$

that is

$$\int_{\mathbb{R}^2} \Delta u\,|\nabla u|^2\, d\nu = -\int_{\mathbb{R}^2} \left(\mathrm{L}u : \mathrm{M}u - \frac{1}{4}\,|\nabla u|^4 - \frac{1}{2}\,|\nabla u|^2\,(\nabla u\cdot\nabla g)\right) d\nu\,.$$

According to (4.8) and (4.11), we obtain

$$\int_{\mathbb{R}^2} \Delta u \, |\nabla u|^2 \, d\nu = -\int_{\mathbb{R}^2} \mathrm{L}u : \mathrm{M}u \, d\nu + \frac{1}{2}\int_{\mathbb{R}^2} \|\mathrm{M}u\|^2 \, d\nu + 2\int_{\mathbb{R}^2} \mathrm{H}g : (\nabla u \otimes \nabla u) \, d\nu$$
$$- 2\int_{\mathbb{R}^2} (\nabla u \cdot \nabla g)^2 \, d\nu - 4\int_{\mathbb{R}^2} \left(\mathrm{L}u - \frac{1}{2}\mathrm{M}u\right) : (\nabla u \otimes \nabla g) \, d\nu$$
$$+ 3\int_{\mathbb{R}^2} |\nabla u|^2 \, (|\nabla g|^2 - \Delta g) \, d\nu \,. \quad (4.12)$$

3) On the other hand, integrating by parts twice yields

$$\int_{\mathbb{R}^2} \Delta \, |\nabla u|^2 \, d\nu = \int_{\mathbb{R}^2} |\nabla u|^2 \, \Delta(e^{-u/2}) \, \frac{dx}{\mu} + \int_{\mathbb{R}^2} |\nabla u|^2 \, \left(|\nabla g|^2 - \Delta g\right) d\nu$$
$$+ \int_{\mathbb{R}^2} |\nabla u|^2 \, (\nabla u \cdot \nabla g) \, d\nu \,,$$
$$= -\frac{1}{2}\int_{\mathbb{R}^2} \Delta u \, |\nabla u|^2 \, d\nu + \frac{1}{4}\int_{\mathbb{R}^2} |\nabla u|^4 \, d\nu$$
$$+ \int_{\mathbb{R}^2} |\nabla u|^2 \, \left(|\nabla g|^2 - \Delta g\right) d\nu + \int_{\mathbb{R}^2} |\nabla u|^2 \, (\nabla u \cdot \nabla g) \, d\nu \,. \quad (4.13)$$

Integrating by parts again we have that

$$\int_{\mathbb{R}^2} \nabla \Delta u \cdot \nabla u \, d\nu = -\int_{\mathbb{R}^2} \Delta u \left(\Delta u - \frac{1}{2}\,|\nabla u|^2 - (\nabla u \cdot \nabla g)\right) d\nu \,,$$

which we can use along with the Bochner-Lichnerovicz-Weitzenböck formula on $\mathbb{R}^2$ (with Ricci tensor identically equal to 0),

$$\Delta \, |\nabla u|^2 = 2 \, \|\mathrm{L}u\|^2 + (\Delta u)^2 + 2 \, \nabla \Delta u \cdot \nabla u \,,$$

to get

$$\int_{\mathbb{R}^2} \Delta \, |\nabla u|^2 \, d\nu = \int_{\mathbb{R}^2} \left(2 \, \|\mathrm{L}u\|^2 - (\Delta u)^2 + \Delta u \, |\nabla u|^2 + 2 \, \Delta u \, (\nabla u \cdot \nabla g)\right) d\nu \,.$$

Combined with (4.13) this proves that

$$\int_{\mathbb{R}^2} (\Delta u)^2 \, d\nu = 2\int_{\mathbb{R}^2} \|\mathrm{L}u\|^2 \, d\nu + \frac{3}{2}\int_{\mathbb{R}^2} \Delta u \, |\nabla u|^2 \, d\nu - \frac{1}{4}\int_{\mathbb{R}^2} |\nabla u|^4 \, d\nu$$
$$- \int_{\mathbb{R}^2} |\nabla u|^2 \, \left(|\nabla g|^2 - \Delta g\right) d\nu + 2\int_{\mathbb{R}^2} \Delta u \, (\nabla u \cdot \nabla g) \, d\nu - \int_{\mathbb{R}^2} |\nabla u|^2 \, (\nabla u \cdot \nabla g) \, d\nu \,.$$

Using (4.7), (4.8), (4.11) and (4.12), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} (\Delta u)^2 \, d\nu \\
&\quad = \; 2 \int_{\mathbb{R}^2} \|\mathrm{L}u\|^2 \, d\nu \\
&\qquad - \frac{3}{2} \int_{\mathbb{R}^2} \mathrm{L}u : \mathrm{M}u \, d\nu + \frac{3}{4} \int_{\mathbb{R}^2} \|\mathrm{M}u\|^2 \, d\nu + 3 \int_{\mathbb{R}^2} \mathrm{H}g : (\nabla u \otimes \nabla u) \, d\nu \\
&\qquad\qquad - 3 \int_{\mathbb{R}^2} (\nabla u \cdot \nabla g)^2 \, d\nu - 6 \int_{\mathbb{R}^2} \left(\mathrm{L}u - \frac{1}{2} \mathrm{M}u \right) : (\nabla u \otimes \nabla g) \, d\nu \\
&\qquad\qquad + \frac{9}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \, (|\nabla g|^2 - \Delta g) \, d\nu \\
&\qquad - \frac{1}{2} \int_{\mathbb{R}^2} \|\mathrm{M}u\|^2 \, d\nu - \int_{\mathbb{R}^2} |\nabla u|^2 \, \big(|\nabla g|^2 - \Delta g\big) \, d\nu \\
&\qquad + 2 \int_{\mathbb{R}^2} |\nabla u|^2 \, (|\nabla g|^2 - \Delta g) \, d\nu - 4 \int_{\mathbb{R}^2} \left(\mathrm{L}u - \frac{1}{2} \mathrm{M}u \right) : (\nabla u \otimes \nabla g) \, d\nu \\
&\qquad - 4 \int_{\mathbb{R}^2} \mathrm{H}g : (\nabla u \otimes \nabla u) \, d\nu + 4 \int_{\mathbb{R}^2} (\nabla u \cdot \nabla g)^2 \, d\nu \\
&\qquad\qquad + 8 \int_{\mathbb{R}^2} \left(\mathrm{L}u - \frac{1}{2} \mathrm{M}u \right) : (\nabla u \otimes \nabla g) \, d\nu \\
&\qquad\qquad - 6 \int_{\mathbb{R}^2} |\nabla u|^2 \, \big(|\nabla g|^2 - \Delta g\big) \, d\nu \, .
\end{aligned}$$

Collecting terms, we arrive at

$$\begin{aligned}
\int_{\mathbb{R}^2} (\Delta u)^2 \, d\nu = 2 \int_{\mathbb{R}^2} \|\mathrm{L}u\|^2 \, d\nu - \frac{3}{2} \int_{\mathbb{R}^2} \mathrm{L}u : \mathrm{M}u \, d\nu + \frac{1}{4} \int_{\mathbb{R}^2} \|\mathrm{M}u\|^2 \, d\nu \\
- 2 \int_{\mathbb{R}^2} \left(\mathrm{L}u - \frac{1}{2} \mathrm{M}u \right) : (\nabla u \otimes \nabla g) \, d\nu \\
- \int_{\mathbb{R}^2} \mathrm{H}g : (\nabla u \otimes \nabla u) \, d\nu + \int_{\mathbb{R}^2} (\nabla u \cdot \nabla g)^2 \, d\nu \\
- \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \, (|\nabla g|^2 - \Delta g) \, d\nu \, . \qquad (4.14)
\end{aligned}$$

4) By reinjecting (4.12) and (4.14) in the expression of $\mathcal{I}$, we get that

$$\begin{aligned}
0 = \mathcal{I}[u] = 4 \int_{\mathbb{R}^2} \|\mathrm{L}u\|^2 \, d\nu - 4 \int_{\mathbb{R}^2} \mathrm{L}u : \mathrm{M}u \, d\nu + \int_{\mathbb{R}^2} \|\mathrm{M}u\|^2 \, d\nu \\
- 8 \int_{\mathbb{R}^2} \left(\mathrm{L}u - \frac{1}{2} \mathrm{M}u \right) : (\nabla u \otimes \nabla g) \, d\nu \\
+ 2 \int_{\mathbb{R}^2} |\nabla u|^2 \, (|\nabla g|^2 - \Delta g) \, d\nu - 16 \, \pi \, \lambda \int_{\mathbb{R}^2} |\nabla u|^2 \, e^{-u/2} \, dx \, .
\end{aligned}$$

Since $\|\nabla u \otimes \nabla g\|^2 = |\nabla u|^2 \, |\nabla g|^2$, then we get for the corresponding trace free quantity $\mathrm{N}u \coloneqq \nabla u \otimes \nabla g - \frac{1}{2} (\nabla u \cdot \nabla g) \, \mathrm{I}_2$ that

$$\|\mathrm{N}u\|^2 = |\nabla u|^2 \, |\nabla g|^2 - \frac{1}{2} (\nabla u \cdot \nabla g)^2$$

and hence obtain the identity

$$0 = 4\int_{\mathbb{R}^2} \left\| \mathrm{L}u - \frac{1}{2}\,\mathrm{M}u - \mathrm{N}u \right\|^2 d\nu \\ - 2\int_{\mathbb{R}^2} \left[\left(\Delta g + |\nabla g|^2 - (\nabla g \cdot \omega)^2 \right) e^{-g} + 8\,\pi\,\lambda \right] |\nabla u|^2\, e^{-u/2}\, dx \quad (4.15)$$

where $\omega \coloneqq \nabla u/|\nabla u|$. If we assume that $\int_{\mathbb{R}^2} |\nabla u|^2\, dx\, e^{-u/2} \neq 0$ and define

$$\Lambda \coloneqq \frac{1}{8\,\pi}\, \frac{\int_{\mathbb{R}^2} \left[(\nabla u \cdot \nabla g)^2 - (\Delta g + |\nabla g|^2)\, |\nabla u|^2 \right] e^{-u/2-g}\, dx}{\int_{\mathbb{R}^2} |\nabla u|^2\, e^{-u/2}\, dx},$$

then we get a contradiction if $\lambda < \Lambda$. To keep maximal generality, one could even include the term $\int_{\mathbb{R}^2} \left\| \mathrm{L}u - \frac{1}{2}\,\mathrm{M}u - \mathrm{N}u \right\|^2 d\nu$ in the definition of Λ, as it was done in the case of manifolds. However, this is a quite complicated criterion to verify since it involves the solution to (4.3) itself. Hence it makes sense to consider the simpler case where μ has radial symmetry. In that case it is also known from [GNN81] that u is radially symmetric if μ is a monotone non-increasing function of $|x|$. Let

$$\Lambda_\star \coloneqq -\frac{1}{8\,\pi} \inf_{x \in \mathbb{R}^2} \left(\Delta g\, e^{-g} \right) = \inf_{x \in \mathbb{R}^2} \frac{-\,\Delta \log \mu}{8\,\pi\,\mu}\,.$$

Theorem 2. *Assume that μ is a radially symmetric function. Then any radially symmetric solution to (4.3) is a constant if $\lambda < \Lambda_\star$ and the inequality (4.2) holds with $\lambda = \Lambda_\star$ if equality is achieved among radial functions.*

Example 1. *The Euclidean Onofri inequality corresponds to*

$$\mu(x) = \frac{1}{\pi\,(1 + |x|^2)^2} \quad \forall\, x \in \mathbb{R}^2\,.$$

Since $-\Delta \log \mu = 8\,\pi\,\mu$, it is known that $\Lambda_\star = 1$ is the optimal constant. See [CL92] for further details. Let us notice that the analysis of the equation

$$\mathrm{L}u - \frac{1}{2}\,\mathrm{M}u - \mathrm{N}u = 0$$

in the case $\Lambda_\star = 1$ provides a proof of the uniqueness of the radial solution to (4.3), which is alternative to the result of [Bil+06].

Example 2. *It is straightforward to deduce a perturbation result from Theorem 2, that goes as follows. Let*

$$\mu(x) = \frac{e^{-h(x)}}{Z\,(1 + |x|^2)^2} \quad \forall\, x \in \mathbb{R}^2\,,$$

where h is a radial function and Z a normalization constant so that μ is a probability measure. We shall assume that h has a bounded variation and is such that $|x|^4\,\Delta h$ is bounded from below. Then we have the estimate

$$\inf_{x \in \mathbb{R}^2} \frac{-\,\Delta \log \mu}{8\,\pi\,\mu} \geqslant e^{-\mathrm{Var}(h)} \left[1 + \frac{1}{8} \inf_{x \in \mathbb{R}^2} (1 + |x|^2)^2\, \Delta h \right].$$

Example 3. *The subcritical Onofri inequality has been studied in [CD12]. It plays an important role for the study of the subcritical Keller-Segel model and its asymptotics for large times, and goes as follows. Let $\mu = n/M$ where n is given as the unique (up to constants) radial solution to*

$$-\Delta c = n = M\,\frac{e^{c-\frac12|x|^2}}{\int_{\mathbb{R}^2} e^{c-\frac12|x|^2}\,dx}$$

and the mass M is taken in the interval $M \in (0, 8\,\pi)$. According to [GNN81], n is radially symmetric as a consequence of the moving plane method. It is straightforward to check that

$$\inf_{x\in\mathbb{R}^2} \frac{-\Delta\log\mu}{8\,\pi\,\mu} = \frac{M}{8\,\pi} + \inf_{x\in\mathbb{R}^2} \frac{1}{4\,\pi\,\mu}$$

but the symmetry of the solution to (4.2) is not true for $\lambda > \frac{M}{8\,\pi}$ and $\frac{M}{8\,\pi}$ turns out to be the value of the optimal constant in (4.2). See [CD ; CD12] for further details.

Example 4. *The parabolic-parabolic Keller-Segel model has global in time solutions with mass larger than $8\,\pi$ for some values of its parameters, according to [BCD11]. The stationary solutions in self-similar variables can be written as*

$$-\Delta c = \varepsilon\,x\cdot\nabla c + n \quad \textit{with} \quad n = M\,\frac{e^{c-\frac12|x|^2}}{\int_{\mathbb{R}^2} e^{c-\frac12|x|^2}\,dx}$$

(where $\varepsilon > 0$ is a given parameter) and have been shown to be radially symmetric in [NSY02]. To prove that a weighted Onofri inequality holds with $\mu = n/M$, it is therefore sufficient to establish the range of $\lambda \in (0, \Lambda_\star)$ such that the minimizer of $u \mapsto \int_{\mathbb{R}^2} |\nabla u|^2\,dx - 16\,\pi\,\lambda\left[\log\left(\int_{\mathbb{R}^2} e^u\,d\mu\right) - \int_{\mathbb{R}^2} u\,d\mu\right]$ is radially symmetric, where

$$\inf_{x\in\mathbb{R}^2} \frac{-\Delta\log\mu}{8\,\pi\,\mu} = \frac{M}{8\,\pi} + \inf_{x\in\mathbb{R}^2} \frac{\varepsilon\,x\cdot\nabla c + 2}{8\,\pi\,\mu}\,.$$

Such symmetry breaking issues are however known to be difficult : see for instance [DE14] for a discussion of a related problem.

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Chapitre V

Modèles de mouvement de foule

Ce chapitre reprend en grande partie l'article « Stationary solutions of Keller-Segel type crowd motion and herding models : multiplicity and dynamical stability », écrit en collaboration avec Jean Dolbeault et Peter Markowich. Il est à paraître dans *Mathematics and Mechanics of Complex Systems*.

Résumé

Ce chapitre concerne l'étude de deux models d'aggrégation et de mouvement de foule. Chacun des deux modèles est du type *Keller-Segel* et donc constitué d'un système de deux équations paraboliques. La première régit l'évolution de la densité de personne, l'autre celle d'un potentiel de type champ moyen. Ce dernier joue le rôle de terme mémoire. Nous classifions toutes les solutions stationnaires à symétrie radiale et prouvons certains résultats de multiplicité. En caractérisant ces solutions comme points critiques d'une fonctionnelle d'énergie, nous établissons quelques une de leur propriétés qualitatives. Dans ce cadre, il est possible d'associer une notion de stabilité variationelle à ces états stationaires.

D'autre part, nous étudions la stabilité dynamique à travers l'analyse des propriétés spectrales de l'opérateur d'évolution linéarisé autour de ces solutions. La mise en évidence d'une fonctionnelle de Lyapunov associée au second problème nous permet de relier les deux notions de stabilité. Même dans ce cas, il est possible de choisir les paramètres de sorte que plusieurs solutions stables coexistent pour une même valeur du paramètre de masse. L'analyse des propriétés des solutions est faite à la fois théoriquement et via le calcul numérique.

Structure

1 Introduction 114
2 Radial stationary solutions 117
3 Unconstrained energy and constant solutions 119
4 Energy minimizers under mass constraint 124
5 A Lyapunov functional. 126
6 The linearized evolution operator 127
7 Numerical results 133
8 Concluding remarks 135
References. 138

1 Introduction

The Keller-Segel model in chemotaxis has attracted lots of attention over the last years. However, most of the theoretical results have been obtained either in a parabolic-elliptic setting or when the coefficients, like the chemosensitivity coefficient, are independent of the solution. Models used in biology usually involve coefficients which depend on the solution itself, thus making the problems far more nonlinear, and also far less understood. The crowd motion and herding models considered here are two problems in the same class, where the main additional features compared to the standard version of the Keller-Segel model are the limitation (prevention of overcrowding) of the drift for the mass density in both models, and the limitation of the source in the equation for the chemo-attractant in one of the two models. Such limitations have important consequences : there is multiplicity of solutions for a given mass, in certain regimes ; plateau-like solutions have an interesting pattern for modeling issues ; the flux limitation forbids concentration and guarantees nice properties of the solutions, but also raises non-trivial stability issues concerning the set of stationary solutions, which we investigate numerically. The two models can be considered as test cases for the understanding of a very large class of parabolic-parabolic systems with the property of having several attractors. The fact that radial solutions are bounded and can be fully parametrized in relatively simple terms makes the study tractable. Most of the difficulties come from the complicated dependence of the solutions on the total mass, which is the crucial parameter in the two cases. Numerically, the difficulty comes from the parameters of the model which have to be chosen in ranges that make the problem rather stiff.

Description of the models

In this paper, we shall consider herding and crowd motion models describing the evolution of a density ρ of individuals subject to a drift ∇D and confined to a bounded, open set $\Omega \subset \mathbb{R}^d$. The evolution equation for ρ is given by

$$\partial_t \rho = \Delta \rho - \nabla \cdot \big(\rho \, (1 - \rho) \, \nabla D \big) \tag{1.1}$$

where ρ_t stands for the derivative of ρ with respect to time t and the $\rho \, (1 - \rho)$ includes the prevention of overcrowding term. For an isolated system, it makes sense to introduce a no-flux boundary condition, that is

$$\big(\nabla \rho - \rho \, (1 - \rho) \, \nabla D \big) \cdot \nu = 0 \quad \text{on} \quad \partial \, \Omega \tag{1.2}$$

which guarantees the conservation of the number of individuals (or conservation of mass), namely that

$$\int_\Omega \rho \ \, \mathrm{d}x = M \tag{1.3}$$

is independent of t. In the models considered in this paper, we shall assume that the potential D solves a parabolic equation

$$\partial_t D = \kappa \, \Delta D - \delta \, D + g(\rho) \tag{1.4}$$

and is subject to homogeneous Neumann boundary conditions

$$\nabla D \cdot \nu = 0 \quad \text{on} \quad \partial \, \Omega \, . \tag{1.5}$$

We restrict our purpose either to *Model (I)* when

$$g(\rho) = \rho\,(1-\rho) \tag{1.6}$$

or to *Model (II)* when

$$g(\rho) = \rho\,. \tag{1.7}$$

In this paper, our purpose is to characterize stationary solutions and determine their qualitative properties.

Motivations

Human crowd motion models are motivated by the will to prevent stampedes in public places mainly by implementing a better design of walkways. Most crowd motion models do not convey herding effects well enough, that is, loosely speaking, when people bunch up and try to move in the same direction, as typically occurs in emergency situations.

In an effort to improve herding and crowd motion models, M. Burger *et al.* in [BMP11] have derived Model (I) and Model (II) as the continuous limit of a microscopic cellular automaton model introduced in [KS02]. It takes the form a parabolic-parabolic system for the density of people ρ and for the field D, where D is a mean field potential which carries the herding effects. Basically, people are subject to random motion, with a preference for moving in the direction others are following. Random effects are taken into account by a diffusion, while a drift is created by the potential D, which accounts for locations that are or were previously occupied. To account for the packing of the people, empty spaces are preferred, which explains the role of the $(1-\rho)$ term in front of the drift, with 1 being the maximal density. Such a correction is refered to as *prevention of overcrowding* in the mathematical literature.

Both quantities ρ and D undergo diffusion which happens much faster for ρ, this point being reflected by the fact that the constant κ is assumed to be small. The potential D decays in time with rate $\delta > 0$ and increases proportionally to the density ρ, but only if the density is not too high in case of Model (I) : this is taken into account by the source term $g(\rho)$ given either by (1.6) or by (1.7). As we shall see in this paper, interesting phenomena occur when δ is also taken small.

In many aspects, these models are quite similar to the Keller-Segel model used in chemotaxis. *Prevention of overcrowding* has already been considered in several papers, either in the parabolic-elliptic case in [Bur+10 ; BDSS08] or in the parabolic-parabolic case in [DR08] (with a diffusion dominated large time asymptotics) and [Bur+10] (where, additionally, the case of several species and cross-diffusion was taken in to account). In these papers the emphasis was put on the asymptotic behaviors, with a discussion of the possible asymptotic states and behaviors depending on the nonlinearities in [BDDS06] and a study of plateau-like quasi-stationary solutions and their motion in [BDSS08]. This of course makes sense when the domain is the entire space, but a classification of the stationary solutions in bounded domains and in particular plateau-like solutions is still needed, as it is strongly suggested by [BMP11] that such solutions have interesting properties, for instance in terms of stability.

Because of the $(1-\rho)$ factor in front of the drift, the transport term vanishes in our models as ρ approaches 1, so that for any initial data bounded by 1, the density remains bounded by 1. Hence blow-up, which is a major difficulty for the analysis of the usual Keller-Segel system for masses over 8π (*cf.* for instance [BDP06]), does not occur here. In contrast with the parabolic-elliptic Keller-Segel model with prevention of overcrowding studied in [Bur+10], Models (I) and (II) are based on a system of coupled parabolic equations. This has interesting consequences for the evolution problem as, *e.g.*, it introduces memory effects. It also has various consequences for the dynamical stability of the stationary states. In Model (I), the source term in the equation for D involves $\rho\,(1-\rho)$ instead of ρ. Such a nonlinear source term introduces additional difficulties as, for instance, no Lyapunov functional is known up to now.

Main results

Let us summarize some of the main results of this paper, in case of Models (I) and (II), when Ω is a ball, as far as radial nonnegative stationary solutions are concerned. As we shall see below the stationary solutions of interest are either constants or monotone functions, which are then plateau-like.

Theorem 21. *Let Ω be a ball and consider solutions of Models (I) and (II) subject to boundary conditions (1.2) and (1.5). Then the masses of the radial nonnegative stationary solutions as defined by (1.3) range between 0 and $|\Omega|$ and we have :*

(i) Non constant stationary solutions exists only for M in a strict sub-interval $(0, |\Omega|)$,

(ii) Constant solutions are variationally and dynamically unstable in a strictly smaller subinterval,

(iii) There is a range of masses in which only non-constant stationary solutions are stable, given by the condition that $\kappa\,\lambda_1 + \delta$ is small enough, where λ_1 denotes the lowest positive eigenvalue of $-\Delta$ in Ω subject to Neumann homogeneous boundary conditions,

(iv) For any given mass, variationally stable stationary solutions with low energy are either monotone or constant ; in case of Model (II), monotone, plateau-like solutions are then stable and attract all low energy solutions of the evolution problem in a certain a range of masses.

Much more can be said on stationary solutions, as we shall see below and some of our results are not restricted to radial solutions on a ball. The natural parameter for the solutions of Models (I) and (II) is M, but it is much easier to parametrize the set of solutions by an associated Lagrange multiplier : see Section 2. In particular, stationary solutions are then critical points of an energy defined in Section 3, and there is a notion of *variational stability* associated to this energy. Taking into account the mass constraint, as it is done in Section 4, makes the problem more difficult. To study the evolution problem, one can rely on a Lyapunov functional introduced in Section 5, but only in case of Model (II). *Dynamical stability* is studied through the spectrum of the linearized evolution operator in Section 6 and the interplay between notions of variational and dynamical stability is also studied in details. How to harmonize the two points of view on stability is a question that Model (I) and Model (II) share with all parabolic-parabolic models of chemotaxis. In case of Model (II), results are summarized in Theorem 46. The issue of the stability of monotone – constant or non

constant – solutions is a subtle question and most of this paper is devoted to this point. Precise definitions of variational and dynamical stability will be given later on.

Numerical results go beyond what can be proved rigorously. Because we use the parametrization by the Lagrange multiplier, we are able to compute *all* radial solutions. In practice, we shall focus on the role of constant and monotone plateau-like solutions. A list of detailed qualitative results is provided at the beginning of Section 7. Theoretical and numerical results are discussed in Section 8.

Some references

The two models considered in this paper have been introduced in [BMP11] at the PDE level. Considerations on the stability of constant solutions can be found there as well. Models (I) and (II) involve a system of two parabolic equations, like the so-called parabolic-parabolic Keller-Segel system for which we primarily refer to [CC08]. In such a model, stationary solutions have to be replaced by self-similar solutions, which also have multiplicity properties (see [BCD11]). How the parabolic-parabolic model is related to the parabolic-elliptic case has been studied in [BB09 ; CC08]. The parabolic-elliptic counterpart of Model (I) is known : for plateau solutions and the coarsening of the plateaus, we refer to [BDSS08] (also see [BDDS06 ; Bur+10] ; related models can be found in the literature under the name of Keller-Segel model with logistic sensitivity or congestion models).

One of the technical but crucial issues for a complete classification of all solutions is how to parametrize the set of solutions. Because Lyapunov or energy functionals are not convex, this is a by far more difficult issue than in the repulsive case, for which we refer to [DMU01]. The lack of convexity makes it difficult to justify but, at a formal level, the evolution equations in Model (II) can be interpreted as gradient flows with respect to some metric involving a Wasserstein distance (see [CL] in case of the Keller-Segel model and [BL13] for a more general setting ; also see [LM12] for an earlier result in the same spirit). To be precise, one has to consider the Wasserstein distance for ρ and a L^2 distance for D as in [CC12]. The difficulty comes from the fact that the Lyapunov functional is not displacement convex (see for instance [BCC08] and subsequent papers in the parabolic-elliptic case of the Keller-Segel system). Using methods introduced in [MMS09], this may eventually be overcome but is still open at the moment, as far as we know.

2 Radial stationary solutions

A parametrization of all radial stationary solutions

Any stationary solution of (1.1) solves

$$\nabla\rho - \rho\,(1-\rho)\,\nabla D = 0 \quad \text{on} \quad \Omega\,,$$

which means

$$\rho = \frac{1}{1+e^{-\phi}} \tag{2.1}$$

where $\phi = D - \phi_0$ and $\phi_0 \in \mathbb{R}$ is an integration constant determined by the mass constraint (1.3) : ϕ_0 is the unique real number such that

$$\int_\Omega \frac{1}{1+e^{\phi_0 - D}}\ \mathrm{d}x = M\ . \tag{2.2}$$

Taking into account boundary conditions (1.5), Eq. (1.4) now amounts to

$$-\ \kappa\,\Delta\phi + \delta\,(\phi+\phi_0) - f(\phi) = 0 \quad \text{on} \quad \Omega \tag{2.3}$$

with boundary conditions

$$\nabla\phi\cdot\nu = 0 \quad \text{on} \quad \partial\,\Omega\ . \tag{2.4}$$

The functions f and F are defined by $f = F'$ and

$$F(\phi) = \rho = \frac{1}{1+e^{-\phi}} \quad \text{and} \quad f(\phi) = \rho\,(1-\rho) = \frac{e^{-\phi}}{(1+e^{-\phi})^2} \quad \text{in case of Model (I)}\ ,$$

$$F(\phi) = \log(1+e^{\phi}) \quad \text{and} \quad f(\phi) = \rho = \frac{1}{1+e^{-\phi}} \quad \text{in case of Model (II)}\ .$$

The crucial observation for our numerical computation is based on the following result.

Proposition 22. *If Ω is the unit ball in $\mathbb{R}^d$, $d \geqslant 2$, all radial solutions of (2.3)-(2.4) with f as above are smooth and can be found by solving the shooting problem*

$$-\kappa\,\left(\varphi_a'' + \tfrac{d-1}{r}\,\varphi_a'\right) + \delta\,(\varphi_a+\phi_0) - f(\varphi_a) = 0\ , \quad \varphi_a'(0) = 0\ , \quad \varphi_a(0) = a$$

as a function of the parameter $a \in \mathbb{R}$. The shooting criterion is : $\varphi_a'(1) = 0$.

If $d = 1$, all solutions in $\Omega = (0,1)$ are given by the above ODE.

Démonstration. The proof presents no difficulty and is left to the reader. □

Constant solutions

Determining ϕ such that $\delta\,(\phi+\phi_0) - f(\phi) = 0$, that is

$$k(\phi) \coloneqq \frac{1}{\delta}\,f(\phi) - \phi = \phi_0\ , \tag{2.5}$$

exactly amounts to determine the (possibly multivalued) function $\phi_0 \mapsto k^{-1}(\phi_0)$. The following result is not restricted to the special case of f as defined in Model (I) or Model (II).

Lemma 23. *Let $\delta > 0$. Assume that $f \in C^1(\mathbb{R})$ is bounded and such that $\lim_{\phi\to\pm\infty} f'(\phi) = 0$. Then the function $\phi \mapsto k'(\phi) = \frac{1}{\delta}\,f'(\phi) - 1$ has $2\,\ell$ zeros for some $\ell \in \mathbb{N}$ and the equation (2.5) has at most $2\,\ell+1$ solutions. Moreover, for $|\phi_0|$ large enough, (2.5) has one and only one solution, which is such that ρ given by (2.1) converges to 0 as $\phi_0 \to +\infty$ and to 1 as $\phi_0 \to -\infty$.*

If $|\phi_0|$ is large, we observe that $k(\phi) \sim -\phi$. Other properties are elementary consequences of the intermediate values theorem and left to the reader. A plot is shown in Figure V.1.

With $f = F'$ and f corresponding either to Model (I) or (II), all assumptions of Lemma 23 are satisfied with $\ell = 0$ or 1. For later purpose, let us define

$$\phi_-(\phi_0) \coloneqq \min k^{-1}(\phi_0) \quad \text{and} \quad \phi_+(\phi_0) \coloneqq \max k^{-1}(\phi_0)$$

and emphasize that $\phi_\pm$ depend on ϕ_0. The set $k^{-1}(\phi_0)$ is reduced to a point if and only if $\phi_-(\phi_0) = \phi_+(\phi_0)$. From Lemma 23, we also know that

$$\phi_0^- := \inf\{\phi_0 \in \mathbb{R} \,:\, \phi_-(\phi_0) < \phi_+(\phi_0)\} \quad \text{and} \quad \phi_0^+ := \sup\{\phi_0 \in \mathbb{R} \,:\, \phi_-(\phi_0) < \phi_+(\phi_0)\}$$

are both finite.

Instead of parametrizing solutions by ϕ_0, it is interesting to think in terms of mass. Here is a first result (see Fig. V.2) in this direction, which follows from the property that $k'(\phi_\pm(\phi_0)) < 0$ for any $\phi_0 \in \mathbb{R}$.

Lemma 24. *Under the assumptions of Lemma 23, $\phi_0 \mapsto \phi_\pm(\phi_0)$ is monotone decreasing, and the corresponding masses are also monotone decreasing as a function of ϕ_0.*

The proof is elementary and left to the reader. If ϕ is a constant solution, it is a monotone increasing function of the mass according to (2.1). Hence the mass of a constant extremal solution $\phi = \phi_\pm(\phi_0)$ is a monotone decreasing function of ϕ_0. Moreover, we have

$$f'(\phi) = \rho\,(1-\rho)\,h(\rho)$$

with ρ given by (2.1), $h(\rho) = 1 - 2\,\rho$ in case of Model (I) and $h(\rho) = 1$ in case of Model (II). A simple computation shows that $\mathsf{m} := \max_{\rho\in[0,1]} \rho\,(1-\rho)\,h(\rho)$ is equal to $1/6\,\sqrt{3}$ and $1/4$ in case of Models (I) and (II) respectively. As a consequence, with the notations of Lemma 23, $\ell = 0$ if either $\delta \geqslant \mathsf{m}$ or $\delta < \mathsf{m}$ and $\phi_0 \in \mathbb{R}\backslash\big(\phi_0^-, \phi_0^+\big)$. If $\delta < \mathsf{m}$ we find that $\ell = 1$ if $\phi_0 \in \big(\phi_0^-, \phi_0^+\big)$: there are exactly 3 constant solutions.

In the case of Model (I) or (II), the (unique) constant solution taking values in $(\phi_-(\phi_0), \phi_+(\phi_0))$ is monotone increasing as a function of ϕ_0 (when it exists), thus defining a range of masses in which Theorem 21 (iii) holds, as we shall see below.

3 Unconstrained energy and constant solutions

In this section we consider the problem for fixed ϕ_0. On the space $\mathrm{H}^1(\Omega)$, let us define the *energy functional* by

$$\mathcal{E}_{\phi_0}[\phi] := \tfrac{\kappa}{2} \int_\Omega |\nabla\phi|^2 \,\mathrm{d}x + \tfrac{\delta}{2} \int_\Omega |\phi+\phi_0|^2 \,\mathrm{d}x - \int_\Omega F(\phi) \,\mathrm{d}x\,. \tag{3.1}$$

It is clear from (1.3) that stationary solutions of Model (I) and Model (II) are critical points of $\mathcal{E}_{\phi_0}$ (see below Lemma 25) for some given Lagrange multiplier ϕ_0. Moreover, for a given ϕ_0, we know how to compute all radial solutions as explained in Section 2. Hence we shall first fix ϕ_0, study the symmetry of the minimizers of $\mathcal{E}_{\phi_0}$ and clarify the role of constant solutions.

Critical points

Lemma 25. *Assume that F is Lipschitz continuous and Ω is bounded with $C^{1,\alpha}$ boundary for some $\alpha > 0$. With ϕ_0 kept constant, ϕ is a solution of (2.3)–(2.4) if and only if it is a critical point of $\mathcal{E}_{\phi_0}$ in $\mathrm{H}^1(\Omega)$.*

It is straightforward to check that $\mathcal{E}_{\phi_0}$ has a minimizer for any given ϕ, but such a minimizer is actually constant as we shall see below in Corollary 28. Non constant solutions are therefore not minimizers of $\mathcal{E}_{\phi_0}$, for fixed ϕ_0. The regularity of the solution of (2.3)–(2.4) depends on the regularity of F, but when it is smooth as in the case of Models (I) and (II), the standard elliptic theory applies and ϕ is smooth up to the boundary. We refer for instance to [Bre10] for a standard reference book. Details are left to the reader and we shall assume without further notice that solutions are smooth from now on.

Notice that our original problem is not set with ϕ_0 fixed, but with mass constraint (1.3). Understanding how results for a given ϕ_0 can be recast into problems with M fixed is a major source of difficulties and will be studied in particular in Section 4.

Linearized energy functional

Consider the linearized energy functional

$$\lim_{\varepsilon\to 0} \frac{\mathcal{E}_{\phi_0}[\phi+\varepsilon\,\psi]-\mathcal{E}_{\phi_0}[\phi]}{2\,\varepsilon^2} = \int_\Omega \psi\,(\mathrm{E}_\phi\,\psi)\;\mathrm{d}x$$

where ϕ is a stationary solution, $\psi\in \mathrm{H}^2(\Omega)$ and $\mathrm{E}_\phi\,\psi := -\,\kappa\,\Delta\psi+\delta\,\psi-F''(\phi)\,\psi$. Notice that with ρ given by (2.1), we have

$$\mathrm{E}_\phi\,\psi = -\,\kappa\,\Delta\psi+\delta\,\psi-\rho\,(1-\rho)\,h(\rho)\,\psi \tag{3.2}$$

with $h(\rho)=1-2\,\rho$ in case of Model (I) and $h(\rho)=1$ in case of Model (II).

Stability and instability of constant solutions

Denote by $(\lambda_n)_{n\in\mathbb{N}}$ the sequence of all eigenvalues of $-\Delta$ with homogeneous Neumann boundary conditions, counted with multiplicity. The eigenspace corresponding to $\lambda_0=0$ is generated by the constants. Three constant solutions co-exist when constant solutions ϕ take their values in $k\circ(k')^{-1}(0,+\infty)$, that is when

$$\delta-\rho\,(1-\rho)\,h(\rho)<0\;.$$

A constant solution $(\rho, D=\phi+\phi_0)$ is *variationally* unstable if E_ϕ has a negative eigenvalue, that is if

$$\kappa\,\lambda_1+\delta-\rho\,(1-\rho)\,h(\rho)<0\;. \tag{3.3}$$

When such a condition is satisfied, the constant solution ϕ cannot be a local minimizer of $\mathcal{E}_{\phi_0}$. *Dynamical* stability of the constant solutions with respect to the evolution governed by (1.1)–(1.5) will be studied in Section 6 : in case of constant solutions, such an instability is also determined by (3.3), as we shall see in Proposition 38.

Condition (3.3) is never satisfied if $\kappa\,\lambda_1+\delta\geqslant \mathsf{m}:=\max_{\rho\in[0,1]}\rho\,(1-\rho)\,h(\rho)$. Otherwise, this condition determines a strict subinterval of $(0,1)$ in terms of ρ, and hence an interval in ϕ. This proves Theorem 21 (ii). A slightly more precise statement goes as follows.

Lemma 26. *Let $\delta>0$. The set of values of ϕ_0 for which there are constant solutions of (2.3) which satisfy (3.3) with ρ given by (2.1) is contained in $\big(\phi_0^-,\phi_0^+\big)$. Moreover, if there exists a constant, variationally unstable solution, then there is also a constant, variationally stable solution of (2.3) for the same value of ϕ_0, but with lower energy.*

The proof of Lemma 26 requires some additional observations. It will be completed in Section 3.

Numerical range

We shall postpone the proof of Lemma 26 to the end of the next section. Cases of numerical interest studied in this paper are the following.

1. In dimension $d = 1$ with $\Omega = (0,1)$, the first unstable mode is generated by $x \mapsto \cos(\pi\, x)$ and corresponds to $\lambda_1 = \pi^2 \approx 9.87$.
2. In dimension $d = 2$, the first positive critical point of the first Bessel function of the first kind J_0, that is $r_0 := \min\{r > 0 \,:\, J_0'(r) = 0\}$, is such that $r_0 \approx 3.83$ so that $\lambda_{0,1} = r_0^2 \approx 14.68$ is an eigenvalue associated to the eigenspace generated by $r \mapsto J_0(r\, r_0)$. Applied to (3.3), this determines the range of *radial variational instability*. Recall that J_0 is the solution of $J_0'' + \frac{1}{r}\, J_0' + J_0 = 0$.

   We may notice that non-radial instability actually occurs in a larger range, since the first positive critical point of the second Bessel function of the first kind J_1, that is $r_1 := \min\{r > 0 \,:\, J_1'(r) = 0\}$, is such that $r_1 \approx 1.84$ so that $\lambda_{1,0} = r_1^2 \approx 3.39$ is an eigenvalue associated to the eigenspace generated by $r \mapsto J_1(r\, r_1)$, and $\lambda_1 = \lambda_{1,0} < \lambda_{0,1}$. Applied to (3.3), this determines the range of *variational instability*. Recall that J_1 is the solution of $J_1'' + \frac{1}{r}\, J_1' - \frac{1}{r^2}\, J_1' + J_1 = 0$.

Let us notice that the values of $\max_{\rho\in[0,1]} \rho\,(1-\rho)\, h(\rho)$ are in practice also rather small, namely $1/6\sqrt{3} \approx 0.096$ and $1/4 = 0.25$ in case of Models (I) and (II) respectively, which in practice, in view of the values of λ_1, makes the numerical computations rather stiff. In this paper we are interested in the qualitative behavior of the solutions and the role of the dimension, but not so much in the role of the surrounding geometry and hence we shall restrict our study to radial solutions. One of the advantages of dealing only with radial solutions is that we can use accurate numerical packages for solving ODEs and rely on shooting methods, thus getting a precise description of the solution set. Taking into account the effects of the geometry is another challenge but is, in our opinion, secondary compared to establishing all qualitative properties that can be inferred from our numerical computations. Another reason for restricting our study to radially symmetric functions is Proposition 22 : using the shooting method, we have the guarantee to describe all solutions, with additional informations like the knowledge of the range in which to adjust the shooting parameter, as a consequence of the observations of Section 2 (see also Proposition 29). Within the framework of radial solutions, we can henceforth give a thorough description of the set of solutions, that is clearly out of reach in more general geometries. However, as far as we deal with theoretical results, we will not assume any special symmetry of the solutions unless necessary.

In practice, numerical computations of this paper are done with $\delta = 10^{-3}$ and κ ranging from 5×10^{-4} to 10^{-2}. Such small values are dictated by (3.3). They are also compatible with the computations and modeling considerations that can be found in [BMP11]. See Fig. V.2 for a plot corresponding to a rather generic diagram representing constant solutions for Model (I) in dimension $d = 1$. Numerically, our interest lies in the non-constant radial solutions that bifurcate from the constant solutions ϕ at threshold values for condition (3.3), that is

for values of ϕ_0 such that $\kappa\,\lambda_1 + \delta - \rho\,(1-\rho)\,h(\rho) = 0$ with $\lambda_1 = \pi^2$ in dimension $d = 1$, and $\lambda_1 = \lambda_{0,1}$ when $d = 2$. We shall take ϕ_0 as the bifurcation parameter and compute the mass of the solution only afterwards, thus arriving at a simple parametrization of all solutions. Our main results are therefore a complete description of branches of solutions bifurcating from constant ones and giving rise to *plateau* solutions. See Fig. V.3 for some plots of the solutions. We notice that in the range considered for the parameters, the transition from high to low values is not too sharp. The numerical study will be confined to radial monotone solutions, but we will briefly explain in Section 4 (at least when $d = 1$) what can be expected for solution with several plateaus. Concerning stability issues, decomposition on appropriate basis sets will be required, as will be explained in Section 7.

Qualitative properties of the stationary solutions

Lemma 27. *Let Ω be a bounded open set in $\mathbb{R}^d$ with C^2 boundary and assume that $k : \mathbb{R} \to \mathbb{R}$ is Lipschitz continous with*

$$\liminf_{u\to-\infty} k(u) > 0 \quad \textit{and} \quad \limsup_{u\to+\infty} k(u) < 0\,.$$

Assume that all zeros of k are isolated and denote them by $u_1 < u_2 < \ldots < u_N$ for some $N \geqslant 1$. Then any solution of class C^2 of $\Delta u + k(u) = 0$ in Ω, $\nabla u \cdot x = 0$ on $\partial\Omega$ takes values in $[u_1, u_N]$.

Démonstration. Let $x^* \in \overline{\Omega}$ be a maximum point of u. We know that $-\Delta u(x^*) = k(u(x^*)) \geqslant 0$, even if $x^* \in \partial\,\Omega$ because of the boundary conditions. By assumption, we find that $u(x) \leqslant u(x^*) \leqslant u_N$ for any $x \in \overline{\Omega}$. Similarly, one can prove that $u \geqslant u_1$. □

Applying Lemma 27 to (2.3), (2.4) has straightforward but interesting consequences.

Corollary 28. *Under the assumptions of Lemma 23, for any given $\phi_0 \in \mathbb{R}$, if ϕ is a solution of (2.3)–(2.4), then we have that*

$$\phi_-(\phi_0) \leqslant \phi(x) \leqslant \phi_+(\phi_0) \quad \forall\, x \in \Omega\,.$$

The minimum of $\mathcal{E}_{\phi_0}$ is achieved by a constant function. Moreover, if (2.5) has only one solution ϕ, then (2.3)–(2.4) also has only one solution, which is constant and $\phi \equiv \phi_- = \phi_+$.

Démonstration. We simply observe that, according to the definition (3.1), we have

$$\mathcal{E}_{\phi_0}[\phi] \geqslant \tfrac{\delta}{2}\int_\Omega |\phi + \phi_0|^2 \;\mathrm{d}x - \int_\Omega F(\phi)\;\mathrm{d}x$$

and critical points of $\phi \mapsto \frac{\delta}{2}|\phi+\phi_0|^2 - F(\phi)$ are precisely the constant solutions of (2.5) with $f = F'$. □

In the cases which are numerically studied in this paper, there is an additional property which is of particular interest.

Proposition 29. *Consider either Model (I) or Model (II). Then there exists a constant unstable solution only if $\phi_0 \in \left(\phi_0^-, \phi_0^+\right)$.*

Démonstration. This is an easy consequence of the properties of $f = F'$. Details are left to the reader. □

A monotonicity result

For a given $\phi_0 \in \mathbb{R}$, non-monotone radial functions always have higher energy $\mathcal{E}_{\phi_0}$ than radial monotone functions. We can state this observation as a slightly more general result, as follows.

Proposition 30. *Assume that Ω is the unit ball in $\mathbb{R}^d$, $d \geqslant 2$, and let $G \in W^{1,\infty}(\Omega)$. Then the functional $\mathcal{G}[\phi] := \frac{1}{2}\int_\Omega |\nabla\phi|^2 \ \mathrm{d}x - \int_\Omega G(\phi) \ \mathrm{d}x$ is bounded from below and for any radial non monotone function $\phi \in C^2(\Omega)$ satisfying (2.4), with a finite number of critical points, there exists a radial monotone function $\tilde\phi$ which satisfies (2.4), coincides with ϕ on a neighborhood of 0 such that $\mathcal{G}[\tilde\phi] < \mathcal{G}[\phi]$.*

Démonstration. With a slight abuse of notations, we consider ϕ as a function of $r = |x| \in [0,1]$ and assume that it is solution of

$$\phi'' + G'(\phi) = -\frac{d-1}{r}\,\phi' \,.$$

Multiplying by ϕ', we find that

$$\frac{d}{dr}\left(\frac{1}{2}\,\phi'^2 + G(\phi)\right) = -\frac{d-1}{r}\,\phi'^2 < 0\,.$$

Unless ϕ is constant, assume that for some $r_0 \in (0,1)$ we have $\phi'(r_0) = 0$, and let $G_0 := G(\phi(r_0))$. Integrating on (r_0, r), $r > r_0$, we find that

$$\frac{1}{2}\,\phi'^2 + G(\phi) < G_0 \quad \text{on } (r_0,1)\,,$$

and then $\frac{1}{2}\,\phi'^2 - G(\phi) > \phi'^2 - G_0 > -G_0$ on (r_0, R). Hence we have that

$$\frac{\mathcal{G}[\phi]}{|\Omega|} = \int_0^{r_0}\left(\frac{1}{2}\,\phi'^2 - G(\phi)\right) r^{d-1} \ \mathrm{d}r + \int_{r_0}^1 \left(\frac{1}{2}\,\phi'^2 - G(\phi)\right) r^{d-1} \ \mathrm{d}r > \frac{\mathcal{G}[\tilde\phi]}{|\Omega|}$$

where $\tilde\phi$ defined by $\tilde\phi \equiv \phi$ on $(0, r_0)$ and $\tilde\phi \equiv \phi(r_0)$ on $(r_0, 1)$. This concludes the proof. □

Proposition 30 shows at ODE level why radial minimizers of the functional $\mathcal{G}$ have to be monotone. It is also preparatory for Lemma 36.

Proof of Lemma 26. A constant solution which satisfies (3.3) cannot be a global minimizer for ϕ_0 fixed. According to Corollary 28, there exists another constant solution under the assumptions of Lemma 26, which incidentally proves that $\phi_-(\phi_0) < \phi_+(\phi_0)$ with the notations of Section 2. The fact that there is a constant stable solution with an energy lower than the energy of the unstable one is a consequence of Proposition 30. □

Summarizing, for a given $\phi_0 \in \mathbb{R}$, only constant solutions are to be considered for the minimization of $\mathcal{E}_{\phi_0}$. However, the relevant problem in terms of modeling is the problem with mass constraint, at least in view of the evolution problem, and it is not as straightforward as the problem with a fixed Lagrange multiplier.

4 Energy minimizers under mass constraint

Existence and qualitative properties of minimizers

In this section, we assume that $M > 0$ is fixed and consider $\phi_0^M[D] = \phi_0$ uniquely determined by (2.2). Let us define the functional

$$D \mapsto \mathcal{F}_M[D] \coloneqq \tfrac{\kappa}{2}\int_\Omega |\nabla D|^2 \ \mathrm{d}x + \tfrac{\delta}{2}\int_\Omega |D|^2 \ \mathrm{d}x - \int_\Omega F\left(D - \phi_0^M[D]\right) \ \mathrm{d}x \ .$$

In such a case, ϕ_0 can be seen as a Lagrange multiplier associated to the mass constraint and $\mathcal{F}_M[D] = \mathcal{E}_{\phi_0}[D - \phi_0]$.

Proposition 31. *Assume that F is a continuous function with a subcritical growth. If Ω is bounded with $C^{1,\alpha}$ boundary for some $\alpha > 0$, then for any $M > 0$, the functional $\mathcal{F}_M$ has at least one minimizer $D = \phi + \phi_0$ with $\phi_0 = \phi_0^M[D]$ in $H^1(\Omega)$, which is such that $\mathcal{F}_M[D] = \mathcal{E}_{\phi_0}[\phi]$, and D is of class $C^\infty(\Omega)$ if F is of class C^∞.*

Démonstration. It is straightforward to check that $\mathcal{F}_M$ has at least one minimizer in $H^1(\Omega)$ because any minimizing sequence converges up to the extraction of subsequences to a minimum $D = \phi + \phi_0$ by compactness and lower semi-continuity. Then ϕ is a critical point of $\mathcal{E}_{\phi_0}$ and regularity is a standard result of elliptic theory (see *e.g.* [Bre10]) and bootstrapping methods. □

In Model (I) and (II), we respectively have $|F(\phi)| < 1$ and $F(\phi) \in [0, \log(2) + \max(0, \phi)]$ so the assumptions of Proposition 31 are satisfied. Notice that it is not implied anymore that minimizers of $\mathcal{F}_M$ under mass constraint are constant functions and hence they might not be minimizers of $\mathcal{E}_{\phi_0}$.

Lemma 32. *The mass of the density associated to non-constant solutions of (2.3)-(2.4) is bounded away from 0 and $|\Omega|$.*

Démonstration. Any non-constant solution of (2.3)-(2.4) has mass $M = \int_\Omega \frac{1}{1+e^{-\phi}} \ \mathrm{d}x$ associated to its density, according to (2.1) and (2.2). Corollary 28 gives the bounds

$$M_-(\phi_0) \coloneqq \frac{|\Omega|}{1 + e^{-\phi_+(\phi_0)}} \leqslant M \leqslant \frac{|\Omega|}{1 + e^{-\phi_-(\phi_0)}} =: M_+(\phi_0) \ .$$

Let $M^{(-)} \coloneqq \min\left\{M_-(\phi_0) \ : \ \phi_0 \in \left(\phi_0^-, \phi_0^+\right)\right\}$ and $M^{(+)} \coloneqq \max\left\{M_+(\phi_0) \ : \ \phi_0 \in \left(\phi_0^-, \phi_0^+\right)\right\}$. Since $\phi_0 \mapsto M_\pm(\phi_0)$ is a continuous function on $\mathbb{R}$, we know from Lemma 26 that $\left(M^{(-)}, M^{(+)}\right)$ is compactly included in $(0, |\Omega|)$. From Lemma 24, we deduce that $M^{(\pm)} = M_\pm(\phi_0^\mp)$. □

Notice that Lemma 32 proves Theorem 21 (i).

Corollary 33. *With above notations, we have $0 < M^{(-)} \leqslant M^{(+)} < 1$ and minimizers of $\mathcal{F}_M$ are constant functions if $M \in (0, M^{(-)}) \cup (M^{(+)}, 1)$. There is a subinterval of $(M^{(-)}, M^{(+)})$ in which minimizers of $\mathcal{F}_M$ are non constant functions.*

Whether minimizers of $\mathcal{F}_M$ are constant solutions or not for some $M \in (M^{(-)}, (M^{(+)})$ will be investigated numerically. For small masses, or masses close to the maximal mass $|\Omega|$ corresponding to the limit density $\rho = 1$, we can state one more result.

Corollary 34. *Under the assumptions of Lemma 23, with $M^{(\pm)} \in (0, |\Omega|)$ defined as above, there is one and only one solution ϕ of (1.1)–(1.5) with mass $M \in (0, M^{(-)}) \cup (M^{(+)}, |\Omega|)$, and this solution is constant, given by $\phi = -\log\left(\frac{|\Omega|}{M} - 1\right)$.*

A partial symmetry result

Lemma 35. *Assume that $d = 2$. If Ω is a disk, minimizers of $\mathcal{F}_M$ are symmetric under reflection with respect to a line which contains the origin.*

Démonstration. The proof of this lemma is inspired by [Lop96]. Assume that Ω is the unit disk centered at the origin and denote by (x_1, x_2) cartesian coordinates in $\mathbb{R}^2$. Let us also define the open upper half-disk $\Omega_+ \coloneqq \{x \in \Omega \,:\, x = (x_1, x_2)\,,\ x_1 > 0\}$. If ϕ is a minimizer of $\mathcal{F}_M$, we define $\tilde{\phi}$ by $\tilde{\phi}(x_1, x_2) = \phi(|x_1|, x_2)$, so that $\tilde{\phi}$ is symmetric with respect to the line $x_1 = 0$. Up to a rotation, we can assume that Ω_+ accounts for exactly half of the mass, *i.e.* $\int_{\Omega_+}(1 + e^{-\phi})^{-1}\,\mathrm{d}x = M/2$, so that $\int_{\Omega}(1 + e^{-\tilde{\phi}})^{-1}\,\mathrm{d}x = M$. Then, up to a reflection, we can assume that Ω_+ accounts for at most half of the the value of $\mathcal{F}_M$:

$$\frac{\kappa}{2}\int_{\Omega_+} |\nabla\phi|^2\ \mathrm{d}x + \frac{\delta}{2}\int_{\Omega_+} |\phi + \phi_0|^2\ \mathrm{d}x - \int_{\Omega_+} F(\phi)\ \mathrm{d}x \leqslant \frac{1}{2}\,\mathcal{F}_M[\phi]\,.$$

It is then clear that $\tilde{\phi}$ is a minimizer of $\mathcal{F}_M$ such that the mass constraint (1.3) is satisfied. As such, $\tilde{\phi}$ also solves the Euler-Lagrange equations, with same Lagrange multiplier ϕ_0 because ϕ and $\tilde{\phi}$ coincide on Ω_+. Then $w \coloneqq \phi - \tilde{\phi}$ solves the equation

$$-\kappa\,\Delta w + h\,w = 0 \quad \text{with} \quad h \coloneqq \frac{\delta\,(\phi - \tilde{\phi}) + F'(\tilde{\phi}) - F'(\phi)}{\phi - \tilde{\phi}}$$

on Ω. Since $F \in C^\infty$ and ϕ, $\tilde{\phi}$ are continuous, h is bounded. According to [Hor69, Theorem 8.9.1], Hörmander's uniqueness principle applies. Since $w \equiv 0$ on Ω_+, we actually have $w \equiv 0$ on the entire disk Ω, and so $\phi = \tilde{\phi}$. □

In higher dimensions, when Ω is a ball, the method can be extended and shows the symmetry of the solutions with respect to hyperplanes. Thus proving a result of so-called Schwarz foliated symmetry. The method also applies to the functional $\mathcal{E}_{\phi_0}$ with fixed ϕ_0 and shows that a minimizer is radially symmetric, but this is useless as we already know that the minimum is achieved among constant solutions.

One dimensional minimizers are monotone

One-dimensional stationary solution solve an autonomous ODE. This has several interesting consequences.

Lemma 36. *Let $d = 1$ and $M > 0$. Then minimizers of $\mathcal{F}_M$ are monotone, either increasing or decreasing.*

Démonstration. Assume that ϕ is a minimizer of $\mathcal{F}_M$ and $\Omega = (0, 1)$. If ϕ is not monotone, it has a finite number of extremal points $0 = r_0 < r_1 \ldots < r_N = 1$ for some $N > 1$. By uniqueness of the solution of the initial value problem with $\phi(r_i)$ given and $\phi'(r_i) = 0$ we

conclude that $\phi(r_i - s) = \phi(r_i + s)$ as long as $0 \leqslant r_i - s$ and $r_i + s \leqslant 1$, so that $r_i = \frac{i}{N}$, that is ϕ is $\frac{1}{N}$-periodic. With $\tilde\phi(r) \coloneqq \phi(r/N)$, $r \in (0,1)$, we find that

$$\int_0^1 |\tilde\phi'|^2 \, \mathrm{d}r = \frac{1}{N} \int_0^{1/N} |\phi'|^2 \, \mathrm{d}r = \frac{1}{N^2} \int_0^1 |\phi'|^2 \, \mathrm{d}r < \int_0^1 |\phi'|^2 \, \mathrm{d}r \ ,$$

thus proving that $\mathcal{E}_{\phi_0}[\tilde\phi] < \mathcal{E}_{\phi_0}[\phi]$ while $\int_0^1 (1+e^{-\tilde\phi}) \, \mathrm{d}r = \int_0^1 (1+e^{-\phi}) \, \mathrm{d}r$, a contradiction. □

From the scaling in the above proof, it is now clear that *all* non-monotone one-dimensional solutions can be built from monotone ones by symmetrizing them with respect to their critical points, duplicating and scaling them. The intuitive idea is simple but giving detailed statements is unnecessarily complicated, so we will focus on monotone, or *one-plateau* solutions.

5 A Lyapunov functional

In case of Model (II), let us consider the functional

$$\mathcal{L}[\rho, D] \coloneqq \int_\Omega \left[\rho \, \log \rho + (1-\rho) \, \log(1-\rho) - \rho \, D \right] \, \mathrm{d}x + \frac{\kappa}{2} \int_\Omega |\nabla D|^2 \, \mathrm{d}x + \frac{\delta}{2} \int_\Omega D^2 \, \mathrm{d}x \ .$$

Proposition 37. *The functional $\mathcal{L}$ is a* Lyapunov functional *for Model (II) and if (ρ, D) is a solution of (1.1)–(1.5) and (1.7), then*

$$\frac{d}{dt} \mathcal{L}[\rho(t,\cdot), D(t,\cdot)] = - \int_\Omega \frac{|\nabla \rho - \rho \, (1-\rho) \, \nabla D|^2}{\rho \, (1-\rho)} \, \mathrm{d}x - \int_\Omega |- \kappa \, \Delta D + \delta \, D - \rho|^2 \, \mathrm{d}x \leqslant 0 \ .$$

As a consequence, any critical point of $\mathcal{L}$ under the mass constraint (1.3) is a stationary solution of (1.1)–(1.5) and (1.7), and any solution converges to a stationary solution. If Ω is a ball and if the initial datum is radial, then the limit is a radial stationary solution.

Démonstration. An elementary computation shows that

$$\frac{d}{dt} \mathcal{L}[\rho(t,\cdot), D(t,\cdot)] = - \int_\Omega \left[\log\left(\frac{\rho}{1-\rho}\right) - D \right] D_t \, \mathrm{d}x - \int_\Omega (-\kappa \, \Delta D + \delta \, D - \rho) \, D_t \, \mathrm{d}x$$

and the expression of $\frac{d}{dt} \mathcal{L}[\rho(t,\cdot), D(t,\cdot)]$ follows from (1.1)–(1.5). Let $\rho_n(t,x) \coloneqq \rho(t+n,x)$ and $D_n(t,x) \coloneqq D(t+n,x)$. Since $\mathcal{L}$ is bounded from below, we have that

$$\lim_{n\to\infty} \int_0^1 \left(\int_\Omega \frac{|\nabla \rho_n - \rho_n \, (1-\rho_n) \, \nabla D_n|^2}{\rho_n \, (1-\rho_n)} \, \mathrm{d}x + \int_\Omega |- \kappa \, \Delta D_n + \delta \, D_n - \rho_n|^2 \, \mathrm{d}x \right) dt = 0 \ ,$$

which proves that (ρ_n, D_n) strongly converges to a stationary solution. Other details of the proof are left to the reader. □

Proposition 38. *Let $M > 0$ and consider Model (II). For any $D \in \mathrm{H}^1(\Omega)$, let ϕ_0 be the unique real number determined by the mass constraint (2.2). Then for any nonnegative $\rho \in \mathrm{L}^1(\Omega)$ satisfying the mass constraint (1.3), we have*

$$\mathcal{L}[\rho, D] \geqslant \mathcal{E}_{\phi_0}[D - \phi_0] \ ,$$

and equality holds if and only if ρ is given by (2.1), i.e. $\rho = 1/(1+e^{-\phi})$, with $\phi = D - \phi_0$. As a consequence, for any minimizer (ρ, D) of $\mathcal{L}$ satisfying (1.3), ρ is given by (2.1) with $\phi = D - \phi_0$, ϕ_0 satisfying the mass constraint (2.2), and ϕ is a minimizer of $\mathcal{E}_{\phi_0}[\phi] = \mathcal{L}[\rho, D]$.

Démonstration. We only need to notice that the minimum of $\mathcal{L}[\rho, D]$ with respect to ρ under the mass constraint (1.3) satisfies

$$\log\left(\frac{\rho}{1-\rho}\right) = D - \phi_0 \ .$$

The completion of the proof follows from elementary computations which are left to the reader. □

6 The linearized evolution operator

Dynamical instability of constant solutions

Assume that (ρ, D) is a stationary solution of (1.1)–(1.5). Because of (2.1) and (2.2), the solution is fully determined by D. Let us consider a time dependent perturbed solution of the form $(\rho + \varepsilon\, u, D + \varepsilon\, v)$. Up to higher order terms, u and v are solutions of the linearized system

$$\begin{cases} u_t = \nabla \cdot \big(\nabla u - (1-2\,\rho)\, u\, \nabla D - \rho\,(1-\rho)\, \nabla v\big)\,, \\ v_t = \kappa\, \Delta v - \delta\, v + h(\rho)\, u \end{cases} \tag{6.1}$$

with $h(\rho) = 1 - 2\,\rho$ in case of Model (I) and $h(\rho) = 1$ in case of Model (II). For later use, we introduce the notation H_D for the linear operator corresponding to the right hand side, so that we shall write

$$(u,v)_t = \mathrm{H}_D\,(u,v) = \begin{pmatrix} \mathrm{H}_D^{(1)}\,(u,v) \\ \mathrm{H}_D^{(2)}\,(u,v) \end{pmatrix} \quad \text{with} \quad \begin{cases} \mathrm{H}_D^{(1)}\,(u,v) = \nabla \cdot \Big[\rho\,(1-\rho)\,\nabla\big(\frac{u}{\rho\,(1-\rho)} - v\big)\Big]\,, \\ \mathrm{H}_D^{(2)}\,(u,v) = \kappa\,\Delta v - \delta\, v + h(\rho)\, u\ . \end{cases}$$

Dynamical instability of constant solutions can be studied along the lines of [BMP11]. Let us state a slightly more general result. We are interested in finding the lowest possible μ in the eigenvalue problem

$$-\ \mathrm{H}_D\,(u,v) = \mu\,(u,v)\ , \tag{6.2}$$

where $\mathrm{H}_D^{(1)}\,(u,v)$ now takes a simplified form, using the fact that ρ is a constant :

$$\mathrm{H}_D^{(1)}\,(u,v) = \Delta u - \rho\,(1-\rho)\,\Delta v\ .$$

The condition $\mu < 0$ provides a dynamically unstable mode. As in Section 3, let us denote by $(\lambda_n)_{n\in\mathbb{N}}$ the sequence of all eigenvalues of $-\Delta$ with homogeneous Neumann boundary conditions, counted with multiplicity, and by $(\phi_n)_{n\in\mathbb{N}}$ an associated sequence of eigenfunctions. If $u = \sum_{n\in\mathbb{N}} \alpha_n\, \phi_n$ and $v = \sum_{n\in\mathbb{N}} \beta_n\, \phi_n$, Problem (6.2) can be decomposed into

$$\begin{aligned} -\lambda_n\,\alpha_n + \rho\,(1-\rho)\,\lambda_n\,\beta_n &= -\,\mu_n\,\alpha_n \\ -\kappa\,\lambda_n\,\beta_n - \delta\,\beta_n + h(\rho)\,\alpha_n &= -\,\mu_n\,\beta_n \end{aligned}$$

for any $n \in \mathbb{N}$, that is

$$\begin{aligned} (\mu_n - \lambda_n)\,\alpha_n + \rho\,(1-\rho)\,\lambda_n\,\beta_n &= 0 \\ h(\rho)\,\alpha_n + (\mu_n - \kappa\,\lambda_n - \delta)\,\beta_n &= 0 \end{aligned}$$

which has non-trivial solutions α_n and β_n if and only if the discriminant condition

$$(\mu_n - \lambda_n)\,(\mu_n - \kappa\,\lambda_n - \delta) - \rho\,(1-\rho)\,h(\rho)\,\lambda_n = 0$$

is satisfied. This determines μ_n for any $n \in \mathbb{N}$, and the spectrum of H_D is then given by $(\mu_n)_{n\in\mathbb{N}}$. Collecting these observations, we can state the following result.

Proposition 39. *With the above notations,* $\inf_{n\geqslant 1} \mu_n < 0$ *if and only if* (3.3) *holds.*

Démonstration. The discriminant condition can be written as

$$\mu_n^2 - [(\kappa+1)\,\lambda_n + \delta]\;\mu_n + \lambda_n\;[\kappa\,\lambda_n + \delta - \rho\,(1-\rho)\,h(\rho)] = 0$$

so that there is a negative root if $\lambda_n\,(\kappa\,\lambda_n + \delta - \rho\,(1-\rho)\,h(\rho)) < 0$. Since $(\lambda_n)_{n\in\mathbb{N}}$ is nondecreasing and $\lambda_0 = 0$, there is at least one negative eigenvalue for (6.2) if the above condition is satisfied with $n = 1$. This concludes the proof. □

In other words, dynamical instability of the constant solutions implies their variational instability. As we shall see numerically, variational and dynamical instability are not anymore equivalent for plateau-like solutions.

Notice that $\lambda_0 = 0$ has anyway to be excluded, as it corresponds to the direction generated by constants. Because of (3.3) we can ensure that the perturbation has zero average. This will be further discussed below, in the general case of a stationary solution.

Variational criterion

In the case of Model (II), we can look at the Lyapunov functional $\mathcal{L}$ and linearize it around a stationary solution (ρ, D). Let

$$\mathrm{L}_D[u,v] \coloneqq \lim_{\varepsilon\to 0} \frac{\mathcal{L}[\rho+\varepsilon\,u, D+\varepsilon\,v] - \mathcal{L}[\rho,D]}{2\,\varepsilon^2}\;.$$

A simple computation shows that

$$\mathrm{L}_D[u,v] = \int_\Omega \left(\frac{u^2}{2\,\rho\,(1-\rho)} - u\,v\right)\,\mathrm{d}x + \frac{\kappa}{2}\int_\Omega |\nabla v|^2\,\mathrm{d}x + \frac{\delta}{2}\int_\Omega v^2\,\mathrm{d}x\;.$$

With E_ϕ defined by (3.2), let

$$\Lambda \coloneqq \inf_{\substack{\int_\Omega v\,\rho\,(1-\rho)\,\mathrm{d}x = 0 \\ v\not\equiv 0}} \frac{\int_\Omega v\,(\mathrm{E}_\phi\,v)\,\mathrm{d}x}{\int_\Omega v^2\,\mathrm{d}x} \tag{6.3}$$

with $\phi = D - \phi_0$, and ϕ_0 satisfying (2.2).

Lemma 40. *Let* $M > 0$ *and consider Model (II) only. If* (ρ, D) *satisfies* (1.3) *and is such that* ρ *is given by* (2.1) *with* $\phi = D - \phi_0$ *and* ϕ_0 *is determined by* (2.2)*, then*

$$\Lambda = 2 \inf_{\substack{\int_\Omega v\,\rho\,(1-\rho)\,\mathrm{d}x = 0 \\ \int_\Omega v^2\,\mathrm{d}x = 1}} \mathrm{L}_D[u,v]\;.$$

As a consequence, if (ρ, D) *is a local minimizer of* $\mathcal{L}$ *under the mass constraint* (1.3)*, then* Λ *is nonnegative.*

Démonstration. We notice that $\mathrm{L}_D[u,v] = \int_\Omega v\,(\mathrm{E}_\phi\,v)\;\mathrm{d}x$ holds true as soon as $u = v\,\rho\,(1-\rho)$ with ρ given by (2.1). In particular, this is the case if (ρ, D) is a local minimizer of $\mathcal{L}$. With v fixed, an optimization of $\mathrm{L}_D[u,v]$ with respect to u shows that $u = v\,\rho\,(1-\rho)$. When (ρ, D) is a local minimizer of $\mathcal{L}$, it is straightforward to check that $\mathrm{L}_D[u,v]$ cannot be negative. □

Entropy–entropy production

Along the linearized flow (6.1), we have

$$\frac{d}{dt}\mathrm{L}_D[u(t,\cdot),v(t,\cdot)] = -\,2\,\mathrm{I}_D[u(t,\cdot),v(t,\cdot)] \tag{6.4}$$

where

$$\mathrm{I}_D[u,v] := \frac{1}{2}\int_\Omega \rho\,(1-\rho)\left|\nabla\Big(\frac{u}{\rho\,(1-\rho)} - v\Big)\right|^2\;\mathrm{d}x + \frac{1}{2}\int_\Omega \big|-\,\kappa\,\Delta v + \delta\,v - u\big|^2\;\mathrm{d}x\;.$$

Let us define the bilinear form

$$\big\langle (u_1,v_1),(u_2,v_2)\big\rangle_D = \int_\Omega \left(\frac{u_1\,u_2}{\rho\,(1-\rho)} - (u_1\,v_2 + u_2\,v_1)\right)\;\mathrm{d}x + \kappa\int_\Omega \nabla v_1\cdot\nabla v_2\;\mathrm{d}x + \delta\int_\Omega v_1\,v_2\;\mathrm{d}x\;,$$

which is such that

$$2\,\mathrm{L}_D[u,v] = \big\langle (u,v),(u,v)\big\rangle_D\;.$$

Lemma 41. *Consider Model (II) only and assume that (ρ, D) is a local minimizer of $\mathcal{L}$ under the mass constraint (1.3). On the orthogonal of the kernel of E_ϕ with $\phi = D - \phi_0$, ϕ_0 satisfying (2.2), $\langle\cdot,\cdot\rangle_D$ is a scalar product and H_D is a self-adjoint operator with respect to $\langle\cdot,\cdot\rangle_D$. Moreover, if (u,v) is a solution of (6.1), then*

$$\frac{d}{dt}\mathrm{L}_D[u,v] = -\,2\,\mathrm{I}_D[u,v] = \big\langle (u,v),\mathrm{H}_D\,(u,v)\big\rangle_D \leqslant 0\;.$$

As a consequence, on the orthogonal of the kernel of E_ϕ, $(0,0)$ is the unique stationary solution of (6.1) and any solution with initial datum in the orthogonal of the kernel converges to $(0,0)$.

With a slight abuse of notations, we have denoted by the *kernel of* E_ϕ the set $\{(u,v)\;:\;v\in \mathrm{Ker}(\mathrm{E}_\phi)\}$.

Démonstration. The positivity of I_D is a consequence of the definition and self-adjointness results from the computation

$$\begin{aligned}-\,\big\langle (u_1,v_1),\mathrm{H}_D\,(u_2,v_2)\big\rangle_D = {}& \int_\Omega \rho\,(1-\rho)\,\nabla\Big(\frac{u_1}{\rho\,(1-\rho)} - v_1\Big)\cdot\nabla\Big(\frac{u_2}{\rho\,(1-\rho)} - v_2\Big)\;\mathrm{d}x\\ & + \int_\Omega \big(-\,\kappa\,\Delta v_1 + \delta\,v_1 - u_1\big)\,\big(-\,\kappa\,\Delta v_2 + \delta\,v_2 - u_2\big)\;\mathrm{d}x\;.\end{aligned}$$

□

Note that one has to take special care of the kernel of $\mathrm{H_D}$. If (ρ_M, D_M) is a stationary solution of (1.1)–(1.4) depending diffentiably on the mass parameter M, it is always possible to diffentiate ρ_M and D_M with respect to M and get a non trivial element in the kernel of $\mathrm{H_D}$. However, it is not guaranteed that this element generates the kernel of E_ϕ and although not

observed numerically, it cannot be excluded that secondary bifurcations occur on branches of plateau-like solutions.

If (ρ, D) is a stationary solution of (1.1)–(1.5), we can of course still consider $\mathrm{I}_D[u, v]$ and its sign determines whether (ρ, D) is dynamically stable or not. In this paper we are interested in the evolution according to the nonlinear flow given by (1.1)–(1.5). The fundamental property of mass conservation (1.3) can still be observed at the level of the linearized equations (6.1). The reader is invited to check that any classical solution of (6.1) is indeed such that

$$\frac{d}{dt}\int_\Omega u(t,x)\ \mathrm{d}x = 0$$

and it makes sense to impose that $\int_\Omega u\ \mathrm{d}x = 0$ at $t = 0$. If we linearize the problem at a stationary solution given by (2.1), it also makes sense to consider the constraint $\int_\Omega v\,\rho\,(1-\rho)\ \mathrm{d}x = 0$.

Dynamic criterion

After these preliminary observations, we can define two notions of stability. We shall say that a critical point ϕ of $\mathcal{E}_{\phi_0}$ is *variationally stable* (resp. unstable) if and only if $\Lambda > 0$ (resp. $\Lambda < 0$) where Λ is defined by (6.3). Alternatively, we shall say that a stationary solution (ρ, D) of (1.1)–(1.5) is *dynamically stable* (resp. unstable) if and only if

$$\inf_{\substack{\int_\Omega u\ \mathrm{d}x = 0\\ \int_\Omega v^2\ \mathrm{d}x = 1}} \mathrm{L}_D[u, v]$$

is positive (resp. negative) in case of Model (II). The operator H_D being self-adjoint, dynamical stability means *variational stability* of $\mathcal{L}$ on the product space, once mass constraints are taken into account. Most of the remainder of this section is devoted to this issue.

For Model (I) we can extend the notion of dynamical stability (resp. instability) by requesting that $\inf\big\{\mathrm{Re}(\lambda)\ :\ \lambda \in \mathrm{Spectrum}(\mathrm{H}_D)\big\}$ is positive (resp. negative). However, in the case of Model (I), notions of dynamical and variational instability are not so well related, as we shall see in Section 7.

Let us start by the following observation. In the case of Models (I) and (II), the kernel of the operator E_ϕ associated to the linearized energy functional and defined by (3.2) determines a subspace of the kernel of H_D.

Lemma 42. *Let $\phi_0 \in \mathbb{R}$ and assume that ϕ is a critical point of $\mathcal{E}_{\phi_0}$. Then ρ given by (2.1) and $D = \phi + \phi_0$ provides a stationary solution of (1.1)–(1.5). If v is in the kernel of E_ϕ, then (u, v) is in the kernel of H_D if $u = \rho\,(1-\rho)\,v$.*

Démonstration. Using (3.2), it is straightforward to check that $0 = \mathrm{E}_\phi\, v = \mathrm{H}_D^{(2)}\,(u, v)$ if $v \in \mathrm{Ker}(\mathrm{E}_\phi)$. Then

$$\mathrm{H}_D^{(1)}\,(u, v) = \nabla\cdot\left[\rho\,(1-\rho)\,\nabla\Big(\frac{u}{\rho\,(1-\rho)} - v\Big)\right] = 0$$

because of the special choice $u = \rho\,(1-\rho)\,v$. □

Since (1.1) preserves the mass, it makes sense to impose $\int_\Omega u \ \mathrm{d}x = 0$. This also suggest to consider the constraint $\int_\Omega \rho\,(1-\rho)\,v \ \mathrm{d}x = 0$, which has already been taken into account in (6.3). Let us give some more precise statements, in the case of Model (II). First we can state a more precise version of Lemma 40. Let us define

$$\Lambda_1 \coloneqq 2 \inf_{\substack{\int_\Omega u \ \mathrm{d}x = 0 \\ \int_\Omega v^2 \ \mathrm{d}x = 1}} \mathrm{L}_D[u,v] \,.$$

Lemma 43. *Let $M > 0$. Consider Model (II) only and assume that (ρ, D) is a critical point of $\mathcal{L}$ under the mass constraint (1.3). With $\phi = D - \phi_0$ where ϕ_0 is the unique real number determined by (2.2), consider Λ defined by (6.3). Then we have $\Lambda_1 \leqslant \Lambda$. If either $\Lambda < \delta$ or $\Lambda_1 < \delta$, then we have $\Lambda = \Lambda_1$.*

Démonstration. If (ρ, D) is a critical point of $\mathcal{L}$, the analysis of Section 2 shows that ρ is given by (2.1) with $\phi = D - \phi_0$ and ϕ_0 determined by (2.2). Consider first the minimization problem

$$\inf_{\substack{\int_\Omega v\,\rho\,(1-\rho) \ \mathrm{d}x = 0 \\ \int_\Omega v^2 \ \mathrm{d}x = 1}} \mathrm{L}_D[u,v] \,.$$

As in Lemma 40, the optimization with respect to u shows that $u = v\,\rho\,(1-\rho)$ and it is then straightforward to get that $2\,\mathrm{L}_D[u,v] = \int_\Omega v\,(\mathrm{E}_\phi\,v) \ \mathrm{d}x = \Lambda$. Additionally, we know that v solves the Euler-Lagrange equation

$$\mathrm{E}_\phi\,v = -\,\kappa\,\Delta v + \delta\,v - v\,\rho\,(1-\rho) = \Lambda\,v - \mu\,\rho\,(1-\rho) \tag{6.5}$$

for some Lagrange multiplier μ and we have $\int_\Omega u \ \mathrm{d}x = \int_\Omega v\,\rho\,(1-\rho) \ \mathrm{d}x = 0$. This proves that $\Lambda_1 \leqslant \Lambda$.

Now, consider a minimizer (u, v) for Λ_1. We find that

$$u = (v - \bar{v})\,\rho\,(1-\rho) \quad \text{with} \quad \bar{v} \coloneqq \frac{\int_\Omega v\,\rho\,(1-\rho) \ \mathrm{d}x}{\int_\Omega \rho\,(1-\rho) \ \mathrm{d}x} \,.$$

Moreover v solves the Euler-Lagrange equation

$$-\,\kappa\,\Delta v + \delta\,v - v\,\rho\,(1-\rho) = \Lambda_1\,v - \bar{v}\,\rho\,(1-\rho) \,. \tag{6.6}$$

Hence we have found that $2\,\mathrm{L}_D[u,v] = \kappa \int_\Omega |\nabla v|^2 \ \mathrm{d}x + \delta \int_\Omega v^2 \ \mathrm{d}x - \int_\Omega \rho\,(1-\rho)\,|v-\bar{v}|^2 \ \mathrm{d}x$, so that

$$\begin{aligned}
\Lambda_1 - \delta &= \inf_{v \neq 0} \frac{\kappa \int_\Omega |\nabla v|^2 \ \mathrm{d}x - \int_\Omega \rho\,(1-\rho)\,|v-\bar{v}|^2 \ \mathrm{d}x}{\int_\Omega v^2 \ \mathrm{d}x} \\
&= \inf_{\substack{\int_\Omega v\,\rho\,(1-\rho) \ \mathrm{d}x = 0 \\ v \not\equiv 0\,,\ c \in \mathbb{R}}} \frac{\kappa \int_\Omega |\nabla v|^2 \ \mathrm{d}x - \int_\Omega \rho\,(1-\rho)\,v^2 \ \mathrm{d}x}{\int_\Omega v^2 \ \mathrm{d}x + c^2} \\
&= \inf_{\substack{\int_\Omega v\,\rho\,(1-\rho) \ \mathrm{d}x = 0 \\ v \not\equiv 0}} \frac{\kappa \int_\Omega |\nabla v|^2 \ \mathrm{d}x - \int_\Omega \rho\,(1-\rho)\,v^2 \ \mathrm{d}x}{\int_\Omega v^2 \ \mathrm{d}x} \,,
\end{aligned}$$

where the last equality holds under the condition that either $\Lambda < \delta$ or $\Lambda_1 < \delta$. Hence we have shown that $\Lambda_1 - \delta = \Lambda - \delta$, which concludes the proof. □

Remark 44. *With no constraint, it is straightforward to check that δ is an eigenvalue of H_D, and $(u, v) = (0, 1)$ an eigenfunction. Hence, as soon as $\Lambda_1 < \delta$, we have that $\int_\Omega u \, \mathrm{d}x = 0$ if (u, v) is a minimizer for Λ_1, because of (6.6). This justifies why the condition of either $\Lambda < \delta$ or $\Lambda_1 < \delta$ enters in the statement of Lemma 43.*

In the case of Model (II), we can get a bound on the growth of the unstable mode.

Corollary 45. *Consider Model (II) only and assume that (ρ, D) is a critical point of $\mathcal{L}$ under the mass constraint (1.3). If Λ defined by (6.3) is negative, then we have*

$$\inf_{\substack{\int_\Omega u \, \mathrm{d}x = 0 \\ \int_\Omega v^2 \, \mathrm{d}x = 1}} \frac{\mathrm{I}_D[u, v]}{\mathrm{L}_D[u, v]} \leqslant \Lambda$$

and the growth rate of the most unstable mode of (6.1) is at least $2\,|\Lambda|$.

Démonstration. Consider a function v given by (6.5) with $\int_\Omega v\,\rho\,(1-\rho) \, \mathrm{d}x = 0$, $\int_\Omega v^2 \, \mathrm{d}x = 1$, $u = v\,\rho\,(1-\rho)$ and take (u, v) as a test function. Then

$$\frac{\mathrm{I}_D[u, v]}{\mathrm{L}_D[u, v]} = \frac{\int_\Omega (\mathrm{E}_\phi\, v)^2 \, \mathrm{d}x}{\int_\Omega v\,(\mathrm{E}_\phi\, v) \, \mathrm{d}x} = \frac{\int_\Omega (\Lambda\, v - \mu\,\rho\,(1-\rho))^2 \, \mathrm{d}x}{\int_\Omega (\Lambda\, v - \mu\,\rho\,(1-\rho))\, v \, \mathrm{d}x} = \Lambda + \frac{\mu^2}{\Lambda} \int_\Omega \rho^2\,(1-\rho)^2 \, \mathrm{d}x \leqslant \Lambda\,.$$

Using (6.4), if $\mathrm{L}_D[u, v]$ is negative, then we get $\frac{d}{dt}\mathrm{L}_D[u, v] \leqslant -2\,\Lambda\,\mathrm{L}_D[u, v]$, thus proving that $\mathrm{L}_D[u, v](t) \leqslant \mathrm{L}_D[u, v](0)\, e^{2\,|\Lambda|\,t}$ for any $t \geqslant 0$. □

The result of Corollary 45 on the most unstable mode can be rephrased in terms of standard norms. By definition of L_D, we get that $\int_\Omega (u^2 + v^2) \, \mathrm{d}x \geqslant 2 \int_\Omega u\, v \, \mathrm{d}x \geqslant 2\,|\mathrm{L}_D[u, v]| \geqslant 2\,|\mathrm{L}_D[u, v](0)|\, e^{2\,|\Lambda|\,t}$ for any $t \geqslant 0$.

Summarizing, we have shown the following result.

Theorem 46. *Let $M > 0$ and consider the case of Model (II). Assume that (ρ, D) is a stationary solution of (1.1)–(1.5) such that (1.3) is satisfied and let $\phi = D - \phi_0$ with ϕ_0 satisfying (2.2). Then the following properties hold true.*

- *(i) Neither dynamical instability nor variational instability can occur if (ρ, D) is a local minimizer of $\mathcal{L}$ under the mass constraint (1.3) or, equivalently, if ϕ is a local minimizer of $\mathcal{E}_{\phi_0}$ such that (1.3) and (2.1) hold.*
- *(ii) If (ρ, D) is a local minimizer of $\mathcal{L}$ under the mass constraint (1.3), then any solution (u, v) of (6.1) converges towards $(0, 0)$ when the initial datum is assumed to be in the orthogonal of the kernel of H_D and with sufficiently low energy.*
- *(iii) Dynamical stability implies variational stability.*
- *(iv) Variational instability and dynamical instability are equivalent and, with above notations, $\Lambda_1 = \Lambda$.*

On the contrary, no clear relation between variational and dynamical (in)stability is known in the case of Model (I), except the result of Lemma 42, which is not so easy to use from a numerical point of view.

7 Numerical results

Let us summarize our findings on radial stationary solutions of (1.1)–(1.5), with parameters δ and κ in the range discussed in Section 3, when Ω is the unit ball in $\mathbb{R}^d$, with $d = 1$ or $d = 2$. Our results deal with either Model (I) or Model (II), defined respectively by (1.6) and (1.7), as follows :

(i) We compute the branches of monotone, non-constant, radial solutions that bifurcate from constant solutions for the two models, in dimensions $d = 1$ and $d = 2$.

(ii) We study variational and dynamical stability of these solutions. The two notions coincide for Model (II), which is partially explained with the help of the Lyapunov functional.

(iii) Dynamical stability holds up to the turning point of the branch when it is parametrized by the mass for Model (II) in dimensions $d = 1$ and $d = 2$. This is also true in dimension $d = 1$ for Model (I).

(iv) In dimension $d = 1$, the variational stability of the branch of monotone, non-constant solutions is more restrictive than the dynamical stability in case of Model (I).

Before entering in the details, let us observe that bifurcation diagrams are more complicated in dimension $d = 2$ than for $d = 1$, and that the lack of a Lyapunov functional makes the study of Model (I) significantly more difficult.

All computations are based on the shooting method presented in Proposition 22. This allows us to find all radially symmetric stationary solutions, as the range of parameter a for which solutions exist is bounded according to Corollary 28. Hence we are left with a single ordinary differential equation, which can be solved using standard numerical methods. Because of the smallness of the parameter κ, the shooting criterion $\varphi'_a(1) = 0$ has a rather stiff dependence on a. This makes directly finding all zeros of the criterion for a given ϕ_0 difficult, so in practice we use perturbation and continuation methods to parametrize the whole branch of monotone, plateau-like solutions.

The computation of the spectrum of the linearized evolution operator (6.1) is done using a basis of cosines, normalized and scaled to meet the boundary conditions. This allows for fast decomposition of the coefficients by FFT. In the case $d = 2$, such a basis in not orthogonal, which is taken into account using a mass matrix during diagonalization. In cases where the constraints cannot be enforced directly at the basis level, a Rayleigh quotient minimization step is done, on the orthogonal of the constrained space.

Numerical computations have been made entirely using the `NumPy` and `SciPy` Python libraries, freely available from `http://scipy.org`. These make use of reference numerical libraries `LAPACK` and `odepack`.

We start by considering constant solutions and make use of the notations of Section 2.

Let us comment on the plots of Fig. V.2.

1 First turning point : $\phi_0 = \phi_0^-$, on the branch of constant solutions : for lower values of ϕ_0, there is only one constant solution $\phi \equiv \phi(0)$, which converges to $+\infty$ as $\phi_0 \to -\infty$.

2 and 3 Non constant solutions bifurcate from constant solutions, which are unstable in the corresponding interval for ϕ_0. The solutions of the two branches corres-

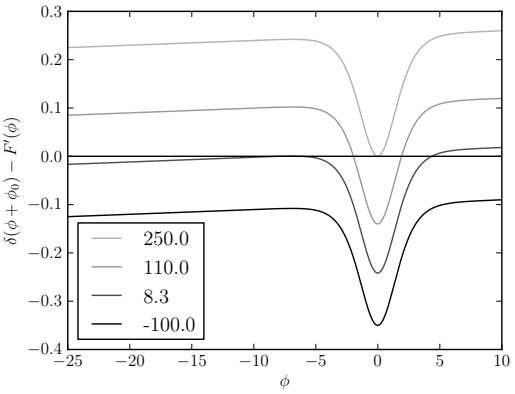

Figure V.1 : Plot of $\phi \mapsto \delta(\phi+\phi_0) - f(\phi) = \delta(\phi+\phi_0) - F'(\phi) = \delta(\phi_0 - k(\phi))$ for various values of ϕ_0. Each zero of the function provides a constant stationary solution of (1.1)–(1.5). The plot shown here corresponds to Model (I), with $\delta = 10^{-3}$.

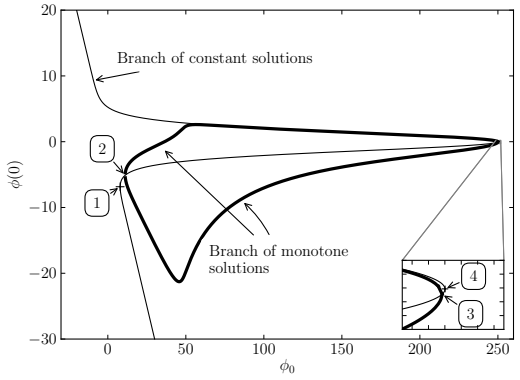

Figure V.2 : Parametrization by ϕ_0 of the branches of solutions in case of Model (I), $d = 1$, with $\delta = 10^{-3}$, $\kappa = 5 \times 10^{-4}$ and $\Omega = (0,1)$. There are either one or three constant solutions for a given value of ϕ_0. Strictly monotone solutions correspond to the bold curve. Notice that on the upper part of the graph the two branches are close but distinct.

pond to monotone solutions, either increasing or decreasing, and always bounded from above and from below by constant solutions.

4 Second turning point : $\phi_0 = \phi_0^+$, on the branch of constant solutions : for higher values of ϕ_0, there is only one constant solution $\phi \equiv \phi(0)$, which converges to $-\infty$ as $\phi_0 \to +\infty$.

The dependence of plateau-like solutions on parameters ϕ_0 and κ is shown in Fig. V.3.
Next we consider monotone, plateau-like solutions. In Figs. V.4 and V.5, the shaded region corresponds to masses for which constant solutions are unstable.

Dynamical and variational stability criteria and their interplay are a tricky issue, especially in case of Model (I) in which we have no theoretical framework to relate the two notions. See Fig. V.6.

Stationary solutions are critical points of $\mathcal{E}_{\phi_0}$. It is therefore interesting to determine whether they are minima or not, either for fixed values of ϕ_0 or for fixed values of M, which

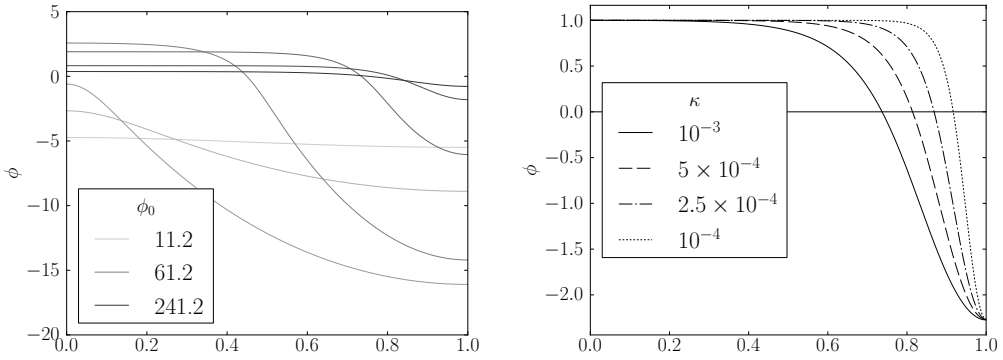

FIGURE V.3 : In the case of Model (I), $d = 1$, $\delta = 10^{-3}$, we consider various profiles for $x \mapsto \phi(x)$ with $x \in (0,1) = \Omega$ either (left) as ϕ_0 varies and $\kappa = 5 \times 10^{-4}$, or (right) as κ varies, with $\phi(0) = 1$.

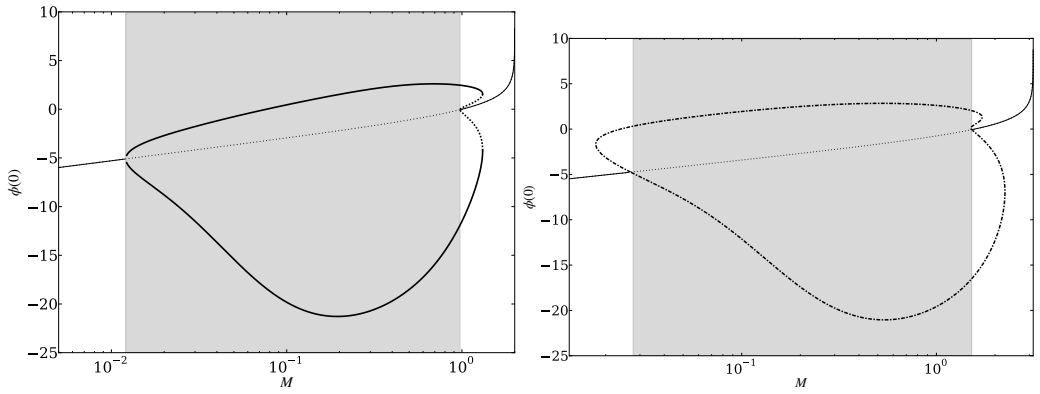

FIGURE V.4 : Model (I), $\kappa = 5 \times 10^{-4}$, $\delta = 10^{-3}$. Thin lines represent constant solutions and bold ones the plateau-like solutions. For readability purposes we use a logarithmic scale for the mass. *Left :* $d = 1$. The dotted part of each branch shows where solutions are dynamically unstable. *Right :* $d = 2$.

makes more sense from the dynamical point of view. However, only in the case of Model (II) minimizers of $\mathcal{E}_{\phi_0}$ are also minimizers of $\mathcal{L}$ and therefore dynamically and variationally stable. See Figs. V.7 and V.8.

Finally in case of Model (II), we can check that dynamical and variational stability are compatible, see Fig. V.9.

8 Concluding remarks

Model (II) is the (formal) gradient flow of the Lyapunov functional $\mathcal{L}$ with respect to a distance corresponding to Wasserstein's distance for ρ and an L^2 distance for D (see [BL13 ; CL ; LM12] for further considerations in this direction). Critical points of $\mathcal{L}$ are stationary solutions for the system, they attract all solutions of the evolution equation and the infimum of $\mathcal{L}$ is achieved by a monotone function, which is therefore either a plateau solution or a constant solution. When $d = 1$ numerics, at least for the values of the parameters we have

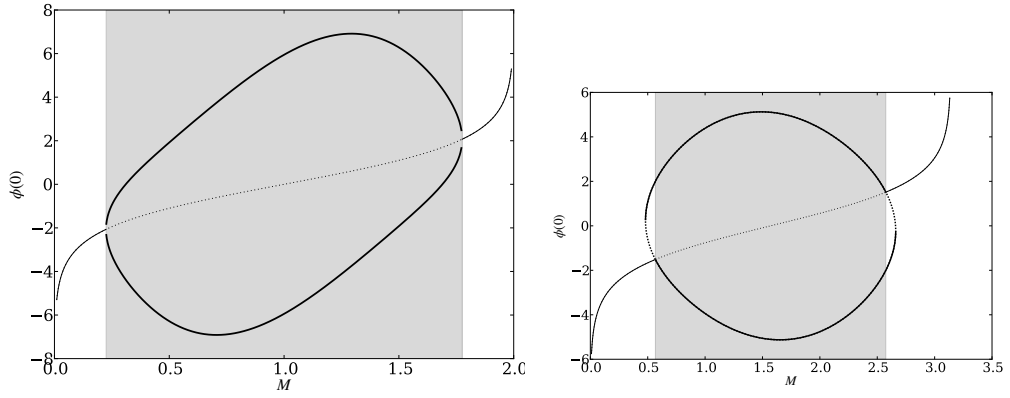

Figure V.5 : Model (II), $\kappa = 10^{-2}$, $\delta = 10^{-3}$: thin lines represent constant solutions and bold ones the plateau-like solutions. The dotted part of each branch shows where solutions are dynamically unstable. *Left :* $d = 1$. *Right :* $d = 2$.

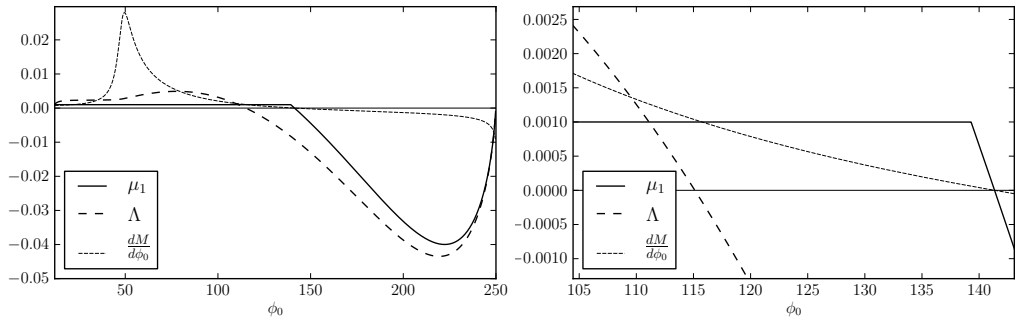

Figure V.6 : Model (I), $d = 1$. We numerically compare the criteria for variational and dynamical instability along the branch of monotone, non-constant solutions. When $dM/d\phi_0$ changes sign, this means that the branch has a turning point when plotted in terms of M. We observe that this turning point corresponds to the loss of dynamical stability, while variational stability is lost for smaller values of ϕ_0 along the branch : see in particular the enlargement (right). Here μ_1 corresponds to the lowest value of $\mathrm{Re}(\langle (u,v), -\mathrm{H}_D\,(u,v)\rangle)$ under the constraints $\langle (u,v),(u,v)\rangle = 1$ and $\int_\Omega u \ \mathrm{d}x = 0$, and $\langle \cdot,\cdot\rangle$ denotes the standard scalar product.

considered, show that plateau solutions exist only in the range in which constant solutions are unstable and are uniquely defined in terms of the mass, but when $d = 2$, the range for dynamically stable plateau solutions is larger than the range (in terms of the mass) of constant unstable solutions under radial perturbation. Infima of $\mathcal{L}$ and $\mathcal{E}_{\phi_0}$ actually coincide. Consistently with our analysis, we find that the linearized evolution operator around minimizing solutions has only positive eigenvalues. Moreover, this operator is self-adjoint in the norm corresponding to the quadratic form given by the second variation of $\mathcal{L}$ around a minimizer. Hence, when $d = 2$, we observe the existence of multiple stable (under radial perturbations) stationary solutions.

In case of **Model (I)**, no Lyapunov functional is available, to our knowledge. Still, all stationary states are characterized as critical points of $\mathcal{E}_{\phi_0}$ and obtained (as long as they are radially symmetric) using our shooting method. In dimension $d = 1$, the structure of the set of solutions is not as simple as in Model (II), and this can be explained by the frustration

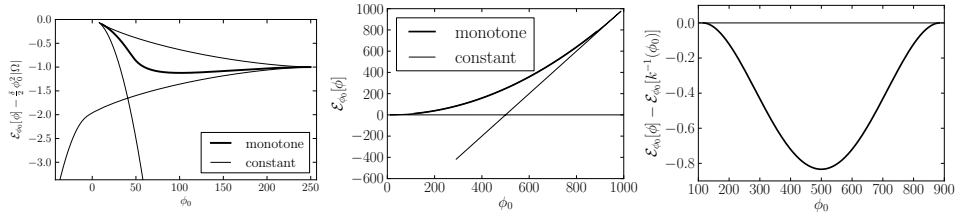

FIGURE V.7 : The energy is represented as a function of ϕ_0 for constant and monotone (either increasing or decreasing) solutions. Here we assume $d = 1$. *Left :* Model (I), the energy $\mathcal{E}_{\phi_0}$ is shifted by $\frac{\delta}{2}\,\phi_0^2\,|\Omega|$. *Center :* Model (II). Non constant solutions (upper curve) are undistinguishable from a branch of constant solutions. *Right :* Details for Model (II) : difference of the energies of the constant and non-constant solutions (under appropriate restrictions on ϕ_0).

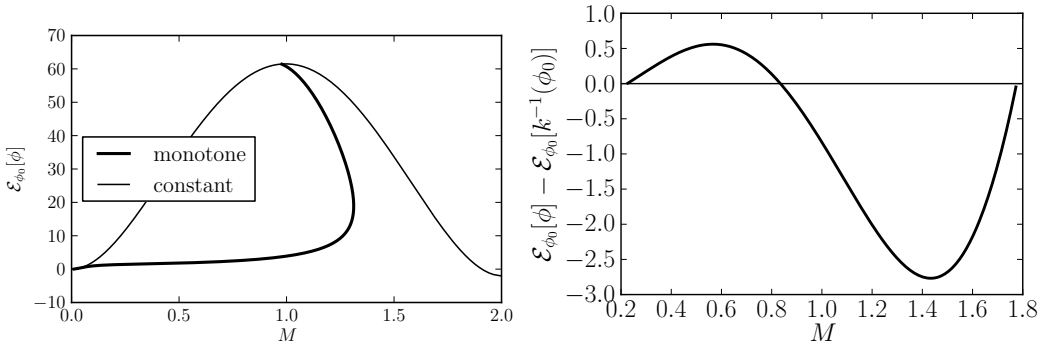

FIGURE V.8 : For any given mass, there is exactly one constant solution. Hence minimizers of $\mathcal{F}_M[D] = \mathcal{E}_{\phi_0}[\phi + \phi_0]$ with $\phi_0 = \phi_0^M[D]$ for masses M in a certain range are not constant. *Left :* Model (I), $d = 1$. *Right :* Model (II), $d = 1$. These minimizers are also minimizers of the Lyapunov functional and therefore dynamically stable (ϕ_0 is restricted to an appropriate range).

due to the $\rho\,(1-\rho)$ term in the equation for D. Numerically, when $d = 1$, we observe that monotone plateau solutions are uniquely defined and dynamically stable in the range where constant solutions are dynamically unstable. However, when $d = 1$, we also have a range in which both types of solutions are dynamically stable, which means that the system has no global attractor. We do not even know whether stationary solutions attract all solutions of the evolution problem or not.

To give a simple picture of the physics involved in the two models of crowd modeling studied in this paper, we may use the following image. The potential D defines the *strategy* of the individuals. It takes into account the source term (the density ρ in case of Model (II) and $\rho\,(1-\rho)$ in case of Model (I)) to determine a preferred direction. Because it is governed by a parabolic equation, it takes the value of the source term into account not only at instant t, but also in the past, which means that there is a memory effect. Of course, recent past receives a larger weight, and actually two mechanisms are at work to update the system : a local damping, with time scale determined by δ and a diffusion term (position of the source term gets lost on the long time range), with a time scale governed by κ. Both coefficients being small, the time scale (that is, the memory of the system) is long compared to the time scale for ρ.

As far as ρ is concerned, the diffusion accounts for random effects while the drift is

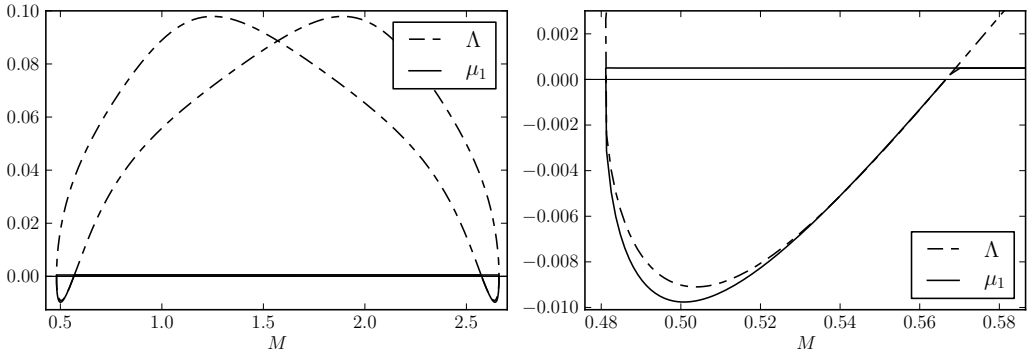

Figure V.9 : Model (II), $d = 2$. *Left :* Solution of $M \mapsto \Lambda$, where, for each M, we compute the two monotone plateau-like solutions, and then Λ according to (6.3). Hence $\Lambda < 0$ means that the solution is variationally unstable under the mass constraint. *Right :* Detail is shown. Here μ_1 corresponds to the lowest value of $\langle (u,v), -\mathrm{H}_D\,(u,v)\rangle$ under the constraints $\langle (u,v),(u,v)\rangle = 1$ and $\int_\Omega u\ \mathrm{d}x = 0$.

tempered by some *tactical* term, which tries to avoid densely populated areas, and is taken into account by the mean of the $(1-\rho)$ term in the drift.

In case of Model (II) the strategy defined by the source term is simple : individuals want to aggregate in high ρ densities. In case of Model (I) the strategy is different, as the system tends to favor regions with intermediate densities, typically ρ of the order of $1/2$. Of course, this is antagonist with the trend to concentrate in regions where D is large and introduces some frustration in the system. At a very qualitative level, this is an explanation for the fact that multiplicity of dynamically stable stationary state occurs in Model (I) even when $d = 1$.

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