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Abstract

The thesis is concerned with various models arising from the study of the dynamics of the population facing a climate change. We aim at achieving two following goals : The first one is to extend original work of Berestycki, Diekmann, Nagelkerke, Zegeling [5] and later developments of Berestycki and Rossi [18, 19] ; the second one is to investigate the deeper mathematical aspects of this model and deal with the new problems where nonlocal and nonlinear diffusion are considered.

The Chapter 1 deals with the problem in an infinite cylindrical domain and in the whole space where the reaction term is (resp.) independent or periodically dependent on time. The novelty of this work is that we consider a global condition in term of the spectral theory to assume that the environment of the population is globally unfavorable at infinity instead of pointwise unfavorable near infinity as in [5, 18, 19]. We further study the concentration of the species in the cylindrical domain when the exterior domain is changed to be extremely unfavorable.

In the Chapter 2, we focus on conditioning the a sharp criterion for the existence, nonexistence and uniqueness of positive solution of fully semilinear elliptic equation. When the divergence of the drift term is zero, the existence of positive solution can be characterized by the amplitude of the drift term under some fair assumptions on the growth rate. The large time behavior of associated parabolic equation is considered, where we have to deal with the case of possibly unbounded coefficients.

The Chapter 3 extends the existence, nonexistence and uniqueness in the second chapter for a quasilinear equation involving p-laplacian operator. The main difficulty is that it seems hard to apply the strong maximum principle and thus we make use a variational approach to attain an important comparison principle.

In Chapter 4, we investigate three notion of generalized principal eigenvalues for nonlocal operators in bounded and unbounded domains (eventually $\mathbb{R}^N$). If the kernel is compactly supported, we can also prove the equivalence of these eigenvalues in unbounded domain. We consider the limits of the eigenvalues of the rescaling operator with respect to the diffusion. The results are very different depending on the rate of rescaling.

In Chapter 5, by the help of the results in Chapter 4, we consider the nonlocal evolution equation and prove that the solution of evolution equation converges to the unique stationary solution, whose existence is directly conditioned by the sign of the generalized principal eigenvalue. The convergences holds in $L^\infty(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$, $p > 0$. In the second part of this chapter, we further investigate the singular limits of the unique positive solution of the rescaling equations. We show that the unique solution of nonlocal equation either approximates the unique solution of local KPP type equation or approximates a solution of reaction-equation, which may not be unique.

Introduction générale

The thesis is devoted to the studies of the local and nonlocal reaction-diffusion equations, which arise from various models of ecology, medicine, neutron, and population dynamics involving KPP-type nonlinearity.

Reaction-diffusion equations with KPP nonlinearity have been the subject of intensive research after the pioneering work of Kolmogorov, Petrovsky, Piskounov, (KPP), 1937, [69], where the authors wanted to study long time dynamics of the biological populations modeled by

$$\partial_t u - u_{xx} = 1u - u^2 = f(u) \qquad \text{in } \mathbb{R}.$$

They showed that $u(t,x)$ converges to traveling wave solutions $u(t,x) = v(x-ct)$ of speed c (unknown) as $t \to \infty$ satisfying

$$v'' + cv' + f(u) = 0; \qquad v(-\infty) = 0; \qquad v(+\infty) = 1$$

if and only if $c \geq c^* = 2$ (or $2\sqrt{f'(0)}$). This convergence is uniform in the domain $\{x \leq ct\}$. Note that traveling wave solution is *not unique* since $v(x+a)$ is another solution for all $a \in \mathbb{R}$.

In 2005, Berestycki, Hamel, Nadirashvili [8] showed that $c \geq 2\sqrt{f'(0)}$ is sharp, conversely $v = 1$ is the unique nontrivial solution. Later, Berestycki, Hamel, Rossi [9] proved that the equation

$$a_{ij}(x)u_{ij} + b_i(x)u_i + f(x,u) = 0, \qquad \text{in } \mathbb{R}^N$$

admits at most a positive bounded solution, which will satisfy also $\inf_{\mathbb{R}^N} u > 0$ if

$$\liminf_{|x|\to\infty} (4\underline{\alpha}(x) f_s(x,0) - |b(x)|^2) > 0, \tag{1}$$

where $0 < \alpha_* \leq \underline{\alpha}(x) = \inf_{\substack{\xi\in\mathbb{R}^N \\ |\xi|=1}} a_{ij}(x)\xi_i\xi_j$. In these frameworks, the common hypothesis on f is

$$\liminf_{|x|\to\infty} f_s(x,0) > 0.$$

In this thesis, we mainly focus on the opposite assumption, where $f_s(x,0)$ may be negative, sign-changing and may decay fast up to Hardy potential.

The thesis is a collection of results from several independent topics related to the local and nonlocal models and new assumptions. Therefore, it will be presented by a chronological scheme, where each chapter is treated as a separate part of the thesis. The thesis consists of five chapters and is organized as follows :

the two first chapters study various aspects of the PDE models in cylindrical domain and the whole space. We further extend the first results for p-Laplacian operator in chapter 3. Finally, chapter 4 and 5, also the most abstract parts of the thesis, involve the studies of nonlocal equations, where we respectively present some developments of generalized principal eigenvalues for nonlocal linear operators and positive solutions for nonlinear equations.

I. Persistence versus extinction under a climate change in mixed environments.

Our first interest is to understand the dynamics of the population under the effect of a climate change, which is modeled by

$$u_t - \Delta u = f(t, x_1 - ct, y, u) \qquad t > 0,\, x \in \Omega, \tag{2}$$

where Ω is of cylindrical or partially periodic domain, f is of KPP-type and the scalar $c > 0$ is a given forced speed. This type of equation has first been introduced from the pioneering works of Berestycki, Diekmann, Nagelkerke and Zegeling [5], also of Potapov and Lewis [80], where the authors want to characterize the persistence versus extinction of the species who are sensitive to the climatic condition. More precisely, in [5], the authors already studied the time-independent version of Eq. (2) in one dimensional space

$$u_t - u_{xx} = f(x - ct, u) \qquad x \in \mathbb{R}. \tag{3}$$

A typical f considered in [5] is

$$f(x,s) = \begin{cases} -sm & x < 0 \text{ and } x > L \\ sm'\left(1 - \dfrac{s}{K}\right) & 0 \le x \le L, \end{cases} \tag{4}$$

for some positive constants m, m', L, K and more general assumption is given by

$$f_s(x, 0) \le -m, \qquad \forall |x| \ge R. \tag{5}$$

Higher dimensional versions was later studied by Berestycki and Rossi [18],[19], where (5) is also imposed. This assumption involved a completely unfavorable environment at infinity, i.e. the favorable zone has compact support. Note that, $f_s(x, 0)$ is understood as the initial per capita rate of growth.

In Chapter 1, we extend the model of (3) in two frameworks, for which condition (5) is no longer true. The first is for an infinite cylindrical domain with Neumann boundary condition :

$$\begin{cases} u_t - \Delta u = f(x_1 - ct, y, u) & t > 0,\, x \in \Omega \\ \partial_\nu u(t, x_1, y) = 0 & t > 0,\, x \in \partial\Omega, \end{cases} \tag{6}$$

where $\Omega = \mathbb{R} \times \omega$, ω is an open bounded and smooth domain in $\mathbb{R}^{N-1}$, ν denotes the exterior unit normal vector field to Ω. In this framework, the environments are assumed to be independent of time. Moreover, we are especially interested in considering the environments of mixed type, which are only assumed to be globally unfavorable at infinity.

To understand the model, one can think of the environment containing both favorable and unfavorable regions that extend all the way to infinity, namely $f_s(x_1, y, 0) > 0$ and $f_s(x_1, y, 0) < 0$ respectively as $x_1 \to \pm\infty$, depending on the location of y. The competitive and mutual influences between these regions play a major role in characterizing the persistence and extinction of the species in the whole domain. For example of such environments, we refer to the species in alpine ecosystem, their habitats are very complex,

heterogeneous or fragmentary and the climate change may give different effects on each patch of their environment. Our paper presents a new look on such environments.

In the second framework, we investigate another type of mixed environment with periodic dependence on y and t. More precisely, the equation is now of following type

$$u_t - \Delta u = f(t, x_1 - ct, y, u) \qquad t > 0,\ x = (x_1, y) \in \mathbb{R}^N,$$

where the nonlinearity reaction is assumed to be periodic in y and t.

We shall use a global condition in terms of spectral property to describe that the environment is globally unfavorable at infinity. More precisely, let L_1 and L_2 be as follows

$$L_1 u = a_{ij}(x)\partial_{ij}u(x) + b_i(x)u_i(x) + c(x)u, \qquad x \in \Omega$$

$$L_2\phi = \partial_t\phi - a_{ij}(t,y)\partial_{ij}\phi(t,y) - b_i(t,y)\phi_i(t,y) - c(t,y)\phi(t,y), \qquad (t,y) \in \mathbb{R}\times\mathbb{R}^{N-1},$$

with bounded coefficients, we define the following eigenvalues

$$\lambda_N(-L_1, \mathcal{O}) := \sup\{\lambda \in \mathbb{R} : \exists \phi \in W^{2,N}_{loc}(\mathcal{O}), \phi > 0, (L_1 + \lambda)\phi \le 0 \text{ a.e in } \mathcal{O},\ \partial_\nu\phi \ge 0 \text{ on } \partial\mathcal{O}\}.$$

$$\bar{\lambda}_1(L_2, \mathbb{R}\times\mathbb{R}^{N-1}) = \sup\{\lambda \in \mathbb{R} : \exists\ \phi > 0,\ \phi \in C^{1,2}_{t,y}(\mathbb{R}\times\mathbb{R}^{N-1}),\ \phi \text{ is T-periodic in } t \text{ and periodic in } y \text{ such that } (L_2 - \lambda)\phi \ge 0 \text{ in } \mathbb{R}\times\mathbb{R}^{N-1}\}.$$

Note that if $\mathcal{O}$ is bounded, then $\lambda_N(-L, \mathcal{O})$ coincides with λ_N, which is the classical Neumann principal eigenvalue. Assumption (5) is respectively replaced in the first and second framework by :

1) There exists a measurable bounded function $\mu : \omega \mapsto \mathbb{R}$ such that

$$\mu(y) = \limsup_{|x_1|\to\infty} f_s(x_1, y, 0) \qquad \text{and} \qquad \lambda_N(-\Delta_y - \mu(y), \omega) > 0.$$

2) There exists a function $\gamma(t,y) \in L^\infty(\mathbb{R}\times\mathbb{R}^{N-1})$, which is periodic in y and T-periodic in t such that

$$\gamma(t,y) = \limsup_{|x_1|\to\infty} f_s(t, x_1, y, 0) \qquad \text{and} \qquad \bar{\lambda}_1(\partial_t - \Delta_y - \gamma(t,y), \mathbb{R}\times\mathbb{R}^{N-1}) > 0.$$

We examine the criteria for persistence and extinction phenomena under these new conditions. The similar conclusions are also derived for problem (6) under Dirichlet boundary condition. Therefore, we are motivated to investigate the concentration of the species in the more favorable region. The problem is described as follows :

We consider equation (6) in the whole space $\mathbb{R}^N$ and study the limit of the sequence of traveling fronts with the reaction terms $F_n(x,s)$ such that their growth rates are negative outside the cylindrical domain Ω and tend to $-\infty$ as $n \to \infty$. These solutions solve the equations

$$\Delta U_n + c\partial_1 U_n + F_n(x, U_n) = 0, \quad x \in \mathbb{R}^N,$$

where $F_n(x,s) = f(x,s)$ for $x \in \overline{\Omega}$ and $\dfrac{\partial F_n}{\partial s}(x,0) \to -\infty$ as $n \to \infty$ locally uniformly in $\mathbb{R}^N \setminus \Omega$. Set

$$\lambda_D(-L, \mathcal{O}) := \sup\{\lambda \in \mathbb{R}, \exists \phi \in W^{2,N}_{loc}(\mathcal{O}), \phi > 0, (\mathcal{L} + \lambda)\phi \le 0 \text{ a.e in } \mathcal{O},\ \phi = 0 \text{ on } \partial\mathcal{O}\}.$$

$$\lambda_1(-L, \mathcal{O}) := \sup\{\lambda \in \mathbb{R}, \exists \phi \in W^{2,N}_{loc}(\mathcal{O}), \phi > 0, (\mathcal{L} + \lambda)\phi \le 0 \text{ a.e in } \mathcal{O}\}.$$

$$L\phi = \Delta\phi + c\partial_1\phi + f_s(x,0)\phi, \quad x \in \Omega; \qquad L_n\phi = \Delta\phi + c\partial_1\phi + \partial_s F_n(x,0)\phi, \quad x \in \mathbb{R}^N.$$
$$\lambda_D = \lambda_D(-L,\Omega) \qquad \lambda_n = \lambda_1(-L_n,\mathbb{R}^N).$$

If $\lambda_D < 0$, we prove the following limits :

$$\lim_{n\to\infty} \lambda_n = \lambda_D.$$
$$\lim_{n\to\infty} \|U_n - U_\infty\|_{L^\infty(\mathbb{R}^N)} = 0,$$

where $U_\infty \in W^{2,N}(\mathbb{R}^N)$ vanishes in Ω^c and coincides with the unique positive solution of the following equation

$$\begin{cases} \Delta U + c\partial_1 U + f(x,U) = 0 & x \in \Omega \\ U(x) = 0 & x \in \partial\Omega. \end{cases}$$

In the last result of Chapter 1, we prove the monotonicity and the symmetry breaking of the unique traveling front of Eq. (6).

II. Liouville-type result for semilinear elliptic equations with possibly vanishing and sign-changing potentials

In Chapter 2, we study conditions of Liouville-type result in the class of possibly unbounded solutions of the following semilinear elliptic equations

$$a_{ij}(x)\partial_{ij}u(x) + Kq(x)\cdot\nabla u(x) + f(x,u) = 0 \qquad \text{in } \mathbb{R}^N, \tag{7}$$

where f is of KPP-monostable type. The goal is to allow the medium $f_s(x,0)$ to be sign-changing and and it may behave like a slow decay or Hardy potential near infinity, namely $f_s(x,0) \le -C|x|^{-\alpha}$ in $\mathbb{R}^N \setminus B_{R_0}$ for some $C, R_0 > 0$ and $\alpha \in [0,2]$. This considerably extends (5) and partially concerns the case that (1) has not covered. Condition (1) was used in [9] to prove the existence of positive bounded solutions of (7). The solutions must have positive infima if exist. Coarsely, this condition implies

$$\liminf_{|x|\to\infty} f_s(x,0) > 0, \tag{8}$$

which is opposite to (5). In this chapter, we have the following targets :

• First, we extend the results of [5],[18] to a more general framework and study the new cases in which (1),(5) are not fulfilled. More precisely, we allow $f_s(x,0)$ to be sign-changing all the way as $|x| \to \infty$ and it may behave like a slow decay or Hardy potential near infinity, namely $f_s(x,0) \le -C|x|^{-\alpha}$ in $\mathbb{R}^N \setminus B_{R_0}$ for some $C, R_0 > 0$ and $\alpha \in [0,2]$. From the biological point of view, these new assumptions refer to the environments that can be unfavorable ($\alpha = 0$) or unfavorably neutral ($\alpha \in (0,2)$) or nearly neutral ($\alpha = 2$) mixing with favorable patches extending to infinity. Our results confirm that the unique persistence and extinction of the species can be conditioned by the sign of the generalized principal eigenvalue of the linearized operator of (7). This is a delicate conclusion since if (1) holds, we know from [9] that there exists a unique positive solution of (7), which has a positive infimum and therefore the species are persistent. To this aim, we will propose a global condition in term of spectral theory. The central role is to develop the notions of generalized principal eigenvalues in unbounded domains, which are inspired from [14],[20]. These generalizations are originally conjectured by Prof H. Berestycki (personal communication).

• Second, under the setting framework, we further investigate the influence of the drift term on the existence of nontrivial solutions of (7). Indeed, in the works [5], [7], [18], the authors implicitly showed that the existence of positive solution is strongly connected with the amplitude of the drift term. Here, we consider the drifts satisfying divergence-free and some fair growth conditions. By adequate compactness compensations, we partially extend the result of Berestycki-Hamel-Nadirashvili [7] on the limit of large drift for the first eigenvalues in bounded domain to the generalized principal eigenvalues in unbounded domain with no periodic assumption. This is in fact an intermediate result that we shall use to derive the existence of sharp threshold of the drift-amplitude, $K^\star$, such that positive solutions of (7) exist if and only if $K < K^\star$. Detailed explanation is postponed until stating the result.

• Third, the long time behaviors of

$$(|x|+1)^{-\alpha}\partial_t u = a_{ij}(x)\partial_{ij}u(x) + Kq(x)\cdot\nabla u + f(x,u) \qquad t\in\mathbb{R}^+, x\in\mathbb{R}^N$$

are lastly investigated. We prove that, if the initial datum decays enough, solutions of this equation converge as $t\to\infty$ in $L^p(\mathbb{R}^N)$- weighted norm to the unique positive stationary of (7). Here $0<p\leq\infty$.

We obtain an intriguing counterpart of these results between two cases $\alpha\in[0,2)$ and $\alpha=2$. The different effects due to the slow decay and Hardy potentials of $f_s(x,0)$ for $|x|$ large are fully understood. Our results also confirm that the Liouville-type result for KPP-type nonlinearity is strongly depended on the behavior near infinity of zero-order coefficient of the linearized operator about zero.

The central role is the generalized principal eigenvalues. Let us define it now.

Definition 0.0.1. Let $\alpha\in[0,2]$, the generalized principal eigenvalue associated with $\mathcal{L}$ in Ω is

$$\lambda_\alpha(-\mathcal{L},\Omega) := \sup\left\{\lambda\in\mathbb{R} : \exists h_\alpha\in C(\Omega), \lim_{x\in\Omega,|x|\to\infty}\frac{h_\alpha(x)}{|x|^{-\alpha}} = 1, \exists\phi\in\overline{\mathcal{D}}_\alpha, (\mathcal{L}+\lambda h_\alpha(x))\,\phi\leq 0 \text{ in } \Omega_{h_\alpha^+}\right\},$$

where $\Omega_{h^+}=\{x\in\Omega, h(x)\geq 0\}$ and $\overline{\mathcal{D}}_\alpha$ denotes the class of admissible test-functions

$$\overline{\mathcal{D}}_\alpha := \left\{\phi\in W^{2,N}_{loc}(\Omega), \phi>0 \text{ in } \Omega_{h^+}, \limsup_{x\in\Omega,|x|\to\infty}\frac{|\nabla\phi(x)||x|^{\frac{\alpha}{2}}}{\phi(x)}\leq\mathcal{D}_\alpha \text{ and } \lim_{x\in\Omega,|x|\to\infty}\frac{\ln\phi(x)}{|x|^{1-\frac{\alpha}{2}}}=0\right\},$$

for $\alpha\in[0,2)$ and for $\alpha=2$, $\overline{\mathcal{D}}_2$ is given by

$$\overline{\mathcal{D}}_2 := \left\{\phi\in W^{2,N}_{loc}(\Omega), \phi>0 \text{ in } \Omega_{h^+}, \limsup_{x\in\Omega,|x|\to\infty}\frac{|\nabla\phi(x)||x|}{\phi(x)}\leq\mathcal{D}_2 \text{ and } \lim_{x\in\Omega,|x|\to\infty}\frac{\ln\phi(x)}{\ln|x|}=0\right\}.$$

The following nice properties of the functions in $\overline{\mathcal{D}}_\alpha$ will be frequently used in the proofs :

$$\text{For } \alpha\in[0,2),\ \varphi\in\overline{\mathcal{D}}_\alpha, \qquad \forall\lambda\in\mathbb{R},\lambda>0, \qquad \lim_{|x|\to\infty}\varphi(x)e^{\lambda|x|^{1-\frac{\alpha}{2}}}=\infty \qquad \lim_{|x|\to\infty}\varphi(x)e^{-\lambda|x|^{1-\frac{\alpha}{2}}}=0.$$

$$\text{For } \alpha=2,\ \varphi\in\overline{\mathcal{D}}_2, \qquad \forall\lambda\in\mathbb{R},\lambda>0, \qquad \lim_{|x|\to\infty}\varphi(x)|x|^{\lambda}=\infty \qquad \lim_{|x|\to\infty}\varphi(x)|x|^{-\lambda}=0.$$

The following new assumptions are considered :

There exist $\alpha \in [0,2]$ and $\mu \in L^\infty(\mathbb{R}^N)$ satisfying

$$\limsup_{|x|\to\infty}\{|x|^\alpha(f_s(x,0)-\mu(x))\} \leq 0 \tag{9}$$

and one of the followings is true

$$\overline{\eta}_\alpha = \lambda_\alpha(-a_{ij}(x)\partial_{ij} - Kq(x).\nabla - \mu(x), \mathbb{R}^N) > 0,$$

$$\begin{cases} (a_{ij}) \text{ is the identity matrix, } I_N, \text{ div} q = 0 \text{ and there exists } \psi \in C^2_{loc}(\mathbb{R}^N) \text{ such that} \\ 2I_N \cdot \nabla\psi(x) = -q(x), \qquad \underline{\eta}_\alpha = \lambda_\alpha(-\Delta - \mu(x), \mathbb{R}^N) > 0. \end{cases} \tag{10}$$

We examine the Liouville type result in the class of functions ϕ satisfying

$$\limsup_{|x|\to\infty} \frac{\ln\phi}{|x|^{1-\frac{\alpha}{2}}} = 0 \text{ if } \alpha \in [0,2) \text{ and } \limsup_{|x|\to\infty} \frac{\ln\phi}{\ln|x|} = 0 \text{ if } \alpha = 2.$$

The existence of threshold value of drift is directly related to the set of first integrals. Let us define it now.

Definition 0.0.2. A function $z \neq 0$ is called a first integral of the vector fields $q \in \mathbb{R}^N$ if $z \in H^1(\mathbb{R}^N)$ and $q \cdot \nabla z = 0$ almost everywhere in $\mathbb{R}^N$. We denote the set of first integrals of q by

$$\mathcal{I}_q = \{z | z \in H^1(\mathbb{R}^N), z \neq 0, \text{ and } q \cdot \nabla z = 0 \text{ a.e in } \mathbb{R}^N\}.$$

We obtain the following result.

Theorem 0.0.3. *Let $\alpha \in [0,2)$ assumptions (9), (10) hold and q satisfy some appropriate growth rates. If one has*

$$\inf_{\phi\in H^1(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla\phi|^2 - f_s(x,0)\phi^2}{\int_{\mathbb{R}^N}\phi^2} < 0 < \inf_{\phi\in\mathcal{I}_q} \frac{\int_{\mathbb{R}^N} |\nabla\phi|^2 - f_s(x,0)\phi^2}{\int_{\mathbb{R}^N}\phi^2},$$

with the convention that the infimum is $+\infty$ if $\mathcal{I}_q$ being empty, then there exists a sharp threshold value $K^\star$ such that Eq. (7) possesses a positive solution if and only if $K < K^\star$.

Theorem 0.0.4. *Let $\alpha = 2$ the assumptions of Theorem* (0.0.3) *hold. Its conclusion remains true if*

$$\begin{cases} \mathcal{D}_2 \geq 1 \quad or \\ \mathcal{D}_2 < 1 \quad and \quad \underline{\eta}_2 > N(1-\mathcal{D}_2) - \frac{N^2}{4}. \end{cases} \tag{11}$$

We note that (11) becomes unconditioned when $N \geq 2$.

These results partially extend the result of Berestycki-Hamel-Nadirashvili [7] on the lower limit of the first eigenvalue for large drift in bounded domain to generalized principal eigenvalues in unbounded domain with no periodic assumption.

III. Liouville-type result for a quasilinear equation with possibly vanishing and sign-changing potentials

In this chapter, we pursue to understand whether the characterization on the existence, nonexistence and uniqueness for semilinear equations in chapter 2 are still true for the equation involving p-Laplacian and possibly vanishing potentials. In a joint work with Phuoc-Tai Nguyen, Technion Israel Technology Institute, during my visit, we successfully characterize the existence and uniqueness of the following equation

$$\mathcal{K}_a[u] + b(x)g(u) = 0 \qquad \text{in } \mathbb{R}^N \tag{12}$$

where $\mathcal{K}_a[u] = -\text{div}(|\nabla u|^{p-2}\nabla u) - a(x)u^{p-1}$, $p \geq 2$, $a, b \in L^\infty(\mathbb{R}^N)$, $b \not\equiv 0$ in $\mathbb{R}^N$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function. The following assumptions are made :

$$\exists \alpha \in [0, p], m > 0 \qquad \limsup_{|x|\to\infty} a(x)|x|^\alpha < -m \tag{C1}$$

$$\lim_{s\to 0} \frac{g(s)}{s^{p-1}} = 0 \quad \text{and} \quad \frac{g(s)}{s^{p-1}} \text{ is increasing on } (0, \infty). \tag{C2}$$

$$\exists s_0 > 0 \text{ such that } -a(x)s_0^{p-1} + b(x)g(s_0) \geq 0 \tag{C3}$$

Using the "global eigenvalue"

Definition 0.0.5. Let $\lambda_R = \lambda_{B_R}(a)$, we define

$$\lambda_\infty(a) = \lim_{R\to\infty} \lambda_R(a), \tag{13}$$

where $\lambda_R(a)$ is the first eigenvalue and first eigenfunction φ_R of $\mathcal{K}_a$ with Dirichlet boundary condition on Ω satisfying

$$\begin{cases} \mathcal{K}_a[\varphi_R] &= \lambda_R(a)\varphi_R^{p-1} & \text{in } B_R \\ \varphi_R &> 0 & \text{in } B_R \\ \varphi_R &= 0 & \text{on } \partial B_R. \end{cases} \tag{14}$$

we derive the following results :

Theorem 0.0.1. *Let $\alpha \in [0, p)$, $p \geq 2$, and conditions (C1)-(C2) hold, then Eq. (12) cannot possess more than one positive solution such that*

$$\limsup_{|x|\to\infty} \frac{u(x)}{e^{\overline{\omega}|x|^{1-\frac{\alpha}{p}}}} < \infty,$$

where $\overline{\omega} = \left(\frac{2m}{Np-N}\right)^{1/p}\left(1 - \frac{\alpha}{p}\right)$. Assume in addition that (C3) holds, then the unique positive solution is achieved if and only if $\lambda_\infty(a) < 0$. It has to decay exponentially as $|x| \to \infty$ when exists.

Theorem 0.0.2. *Let $\alpha = p$, $p \geq 2$, and conditions (C1)-(C2) hold. Assume that $m > N^p(Np + 3p + 2)$, then Eq. (12) cannot possess more than one positive solution such that*

$$\limsup_{|x|\to\infty} \frac{u(x)}{|x|^{\gamma_0}} < \infty,$$

where γ_0 is the unique positive solution of algebraic equation

$$(Np + 1)\gamma^p + (3p + 1)N\gamma^{p-1} - m = 0. \tag{15}$$

Assume in addition that (C3) holds, then the unique positive solution is achieved if and only if $\lambda_\infty(a) < 0$. It has to decay polynomially as $|x| \to \infty$ when exists.

The two last sections are devoted to the studies of the nonlocal models of the previous equations. These are also the most abstract parts of thesis, where will present some developments on the theory of the generalized principal eigenvalue for nonlocal operators in unbounded domains. Furthermore, we also investigate some asymptotic behavior of the rescaling equations and the connection between solutions of local and nonlocal equations, which are just recently emerged in some contemporary researches. For the sake of coherence, we will write a short introduction for each chapters here. More explanations are left for the details of chapter IV and V. These materials are based on a joint work with H. Berestycki and J. Coville.

The main goal is finding survival criteria for a species that has a long range dispersal strategy. As a model species, we can think as trees whose seed and pollens are disseminate on a long range. In ecology a commonly used model that integrate such long range dispersal [62, 66, 70] is the following :

$$\frac{\partial u}{\partial t}(t,x) = J \star u(t,x) - u + f(x,u(t,x)) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N, \tag{16}$$

where $u(t,x)$ is the density of the considered population, J is a dispersal kernel, $f(x,s)$ is a KPP type non-linearity describing the growth rate of the population. The possibility of a long range dispersal is well known in ecology, where numerous data are now available to support this assumptions [32, 33, 34, 99, 48]. In this setting the tail of the kernel is a measure of the frequency at which long dispersal events occur. A biological motivation for the use of the steady state

$$J \star u(x) - u(x) + f(x,u(x)) = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{17}$$

to describe the evolution of the population comes from the observation that the intrinsic variability in the capacity of the individuals to disperse generates, at the scale of a population, a long range dispersal of the population. The effect of such variability has been investigate in [64, 82] by means of the study of correlated random walks. In such framework, all individuals follow a simple random walks where the diffusion coefficient follows a probability law, it can be checked that the probability of the density of population will then follow an integro-differential equation [64, 82, 97] where a dispersal kernel J describes the probability to jump from one location to another.

For compactly supported dispersal kernels J, we derive an optimal survival criteria. We prove that a positive stationary solution of (17) exists if and only if the principal eigenvalue

$$J \star \varphi - \varphi + \partial_s f(x,0)\varphi + \lambda_p \varphi = 0 \qquad x \in \mathbb{R}^N, \tag{18}$$

is negative. In addition, for any continuous non-negative initial data that is bounded or integrable, we establish the long time behavior of $u(t,x)$. We also analyse the impact of the size of the support of the dispersal kernel on the persistence criteria. We exhibit situations where the dispersal strategy has "no impact" on the survival of the species and situations where the slowest dispersal strategy is not any more an Ecological Stable Strategy. Some generalisations of the survival criteria are also discussed for fat-tailed kernels. The proofs of these results essentially rely on some news properties of the principal eigenvalue λ_p in $\mathbb{R}^N$ that we derive in the first part of the paper. In particular, we prove the equivalence of three definitions of the principal eigenvalue and identify some asymptotic limits of λ_p with respect to some rescaling of the dispersal operator.

Furthermore we investigate the asymptotic behavior of the solutions u_ε of rescaling equations

$$\frac{1}{\varepsilon^m}(J_\varepsilon \star u_\varepsilon(x) - u_\varepsilon(x)) + a(x)u_\varepsilon - u_\varepsilon^2 = 0 \quad \text{in} \quad \mathbb{R}^N, \tag{19}$$

as $\varepsilon \to 0$ and $\varepsilon \to \infty$, where $m = 0$, $0 < m < 2$, $m = 2$ and $J_\varepsilon(x) = \frac{1}{\varepsilon^N} J\left(\frac{x}{N}\right)$. The results are very different between theses cases. In particular for the case $m = 2$, we derive an interesting connection between the local and nonlocal solution, where we prove that the unique positive solution u_ε of (19) approximates in $L^2_{loc}(\mathbb{R}^N)$ norm the unique positive solution of the elliptic equation

$$\frac{K_{2,N} D_2(J)}{2} \Delta v + v(a(x) - v) = 0 \quad \text{in} \quad \mathbb{R}^N,$$

with $D_2(J) := \int_{\mathbb{R}^N} J(z) z^2 \, dz$ and $K_{2,N} := \frac{1}{|S^{N-1}|} \int_{S^{N-1}} (\sigma . e_1)^2 \, d\sigma = \frac{1}{N}$.

The central role is the spectral theory of the nonlocal operator. Here, in the first part of this work, we will study carefully three notions of eigenvalues, which may be also useful for other various researches.

IV. The generalized principal eigenvalues for nonlocal operators

We consider the following generalized principal eigenvalues :

Definition 0.0.6. Let $\mathcal{L}_\Omega[\varphi] = \int_\Omega J(x-y)\varphi(y)dy - \varphi(x)$ and $a(x) \in L^\infty(\Omega)$. We set

$$\lambda_p(\mathcal{L}_\Omega + a(x)) := \sup\{\lambda \in \mathbb{R} \,|\, \exists \varphi > 0, \varphi \in C(\Omega), \text{ s.t } \mathcal{L}_\Omega[\varphi] + a(x)\varphi + \lambda\varphi \leq 0 \text{ in } \Omega\}$$

$$\lambda'_p(\mathcal{L}_\Omega + a(x)) := \inf\{\lambda \in \mathbb{R} \,|\, \exists \varphi > 0, \varphi \in C(\Omega) \cap L^\infty(\Omega), \text{ s.t } \mathcal{L}_\Omega[\varphi] + a(x)\varphi + \lambda\varphi \geq 0 \text{ in } \Omega\}.$$

and the variational characterization

Definition 0.0.7. $\mathcal{L}_\Omega[\varphi] = \int_\Omega J(x-y)\varphi(y)dy - \varphi(x)$ and $a(x) \in L^\infty(\Omega)$. Assume further that J is radially symmetric. We set

$$\lambda_v(\mathcal{L}_\Omega + a(x)) := \inf_{\varphi \in L^2(\Omega), \varphi \not\equiv 0} \frac{\int_\Omega \int_\Omega J(x-y)(\varphi(x) - \varphi(y))^2 \, dxdy - \int_\Omega (a(x) - k(x))\varphi^2(x) \, dx}{\|\varphi\|^2_{L^2(\Omega)}}.$$

where $k(x) := \int_\Omega J(x-y) \, dy$.

The main achievements on the relation of these eigenvalues are :

i) If Ω is bounded

$$\lambda_p(\mathcal{L}_\Omega + a(x)) = \lambda'_p(\mathcal{L}_\Omega + a(x)) = \lambda_v(\mathcal{L}_\Omega + a(x)).$$

ii) If $\Omega = \mathbb{R}^N$ and J is compactly supported, we also have the same conclusion as i).

For the later, we investigate some asymptotics of the eigenvalue of the following rescaling operators. Set

$$\mathcal{M}_{\varepsilon,m}[\varphi] = \frac{1}{\varepsilon^m}(J_\varepsilon \star \varphi - \varphi),$$

we obtain the following result

Theorem 0.0.8. *Let J be compactly supported. Assume $a(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ then we have*

- *When* $0 < m \leq 2$ $\quad \lim_{\varepsilon\to+\infty} \lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)) = -\sup_{\mathbb{R}^N} a(x)$
- *When* $0 \leq m < 2$, $\quad \lim_{\varepsilon\to 0} \lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)) = -\sup_{\mathbb{R}^N} a(x)$
- *When* $m = 0$, $\quad \lim_{\varepsilon\to+\infty} \lambda_p(\mathcal{M}_{\varepsilon} + a(x)) = 1 - \sup_{\mathbb{R}^N} a(x)$

  In this case, if $a(x)$ *is radially symmetric non increasing and* $\sup_{\mathbb{R}^N} a(x) < 1$ *then there is a sharp threshold* ε_0*, in the sense that* $\lambda_p(\mathcal{M}_{\varepsilon,m} + a(x))$ *is nonincreasing with respect to* ε *and it is negative if and only if* $\varepsilon < \varepsilon_0$*.*
- *When* $m = 2$ *and* $a(x)$ *is globally Lipschitz, then*

$$\lim_{\varepsilon\to 0} \lambda_p(\mathcal{M}_{\varepsilon,2} + a(x)) = \lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right)$$

  where

$$\lambda_1\left(K_{2,N}D_2(J)\Delta + a(x)\right) := \inf_{\varphi\in H_0^1(\mathbb{R}^N),\varphi\not\equiv 0} \frac{K_{2,N}D_2(J)}{2}\frac{\int_{\mathbb{R}^N}|\nabla\varphi|^2(x)\,dx}{\|\varphi\|_2^2} - \frac{\int_{\mathbb{R}^N} a(x)\varphi^2(x)\,dx}{\|\varphi\|_2^2}.$$

In the second part of this work, we investigate the positive solution of nonlinear equations (16)-(17).

V. The positive solutions of nonlocal equations with KPP nonlinearity

As previously mentioned, we will investigate the existence, nonexistence and uniqueness of Eq. (17) and the large time behavior of Eq. (16) in $L^\infty(\mathbb{R}^N)$ and $L^1(\mathbb{R}^N)$ norms. We further characterize the limits of solutions of (19) as $\varepsilon \to \infty$ and $\varepsilon \to 0$ in three case $m = 0$, $0 < m < 2$, $m = 2$. One of the important results that we achieved up to now is

Theorem 0.0.9. *Assume that* J *is compactly supported and* $f(x,s) = a(x)s - s^2$ *and let* $m = 2$*. Then there exists* $\varepsilon_1 \in (0,\infty)$ *so that for all* $\varepsilon \geq \varepsilon^*$ *there exists a positive solution* u_ε *to* (19)*. Moreover,*

$$\lim_{\varepsilon\to+\infty} \|u_\varepsilon - a^+\|_\infty = 0, \qquad \lim_{\varepsilon\to+\infty} \|u_\varepsilon - a^+\|_{L^2(\mathbb{R}^N)} = 0.$$

In addition, the following dichotomy holds

- *When* $\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right) < 0$*, there exists* $\varepsilon_0 \in (0,\infty)$ *so that for all* $\varepsilon \leq \varepsilon_0$ *there exists a positive solution to* (19) *and*

$$u_\varepsilon \to v, \quad \text{in} \quad L^2_{loc}(\mathbb{R}^N),$$

  where v *is the unique bounded non-trivial solution to*

$$\frac{K_{2,N}D_2(J)}{2}\Delta v + v(a(x) - v) = 0 \quad \text{in} \quad \mathbb{R}^N.$$

- *When* $\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right) > 0$*, then there* $\varepsilon_0 \in (0,\infty)$ *so that for all* $\varepsilon \leq \varepsilon_0$ *there exists no positive solution to* (19)*.*

Unlike the elliptic PDE case, due to the lack of a regularising effect of the diffusion operator, the above spectral problem may not have a solution in any reasonable space of functions i.e $(L^p(\Omega), C(\Omega))$[41, 37, 68]. As a consequence, even in bounded domain, simple sharp survival criteria are already quite involved to obtain. Another difficulty inherent to the study of nonlocal equations (18) in unbounded domain concerns the lack of "reasonable" *a priori* estimates for the solution thus making standard approximations difficult to use in most cases.

Chapitre 1

Persistence versus extinction under a climate change in mixed environments

1.1 Introduction and main results

1.1.1 Introduction and definitions

In a pioneering paper [5], Berestycki et al. studied the influence of climate change (global warming) on the population dynamics of biological species, who are strongly sensitive to temperature conditions. The authors proposed a mathematical model in $\mathbb{R}$, which is formulated as a reaction-diffusion equation with a forced speed c :

$$u_t - u_{xx} = f(x - ct, u) \qquad x \in \mathbb{R}. \tag{1.1}$$

Here c is the speed of the climate change. A typical f considered in [5] is

$$f(x,s) = \begin{cases} -sm & \text{for } x < 0 \text{ and } x > L \\ sm'\left(1 - \dfrac{s}{K}\right) & \text{for } 0 \le x \le L, \end{cases} \tag{1.2}$$

for some positive constants m, m', L, K. This nonlinearity expresses that the environment is unfavorable outside a compact set $[0, L]$ and favorable inside. The higher dimensional versions with more general type of f were studied later in [18], [19]. Beside that a similar model was also considered in the context of competing species by Potapov and Lewis [80], where the authors investigated the co-existence of two species under the effect of climate change and moving range boundaries on habitat invasibility. All these papers assume that the environments are completely unfavorable near infinity, i.e the favorable zone has compact support. More precisely, there exist $R, m > 0$ such that

$$f_s(x, 0) \le -m, \qquad \forall |x| \ge R. \tag{1.3}$$

Note that $f_s(x, 0)$ is understood as the initial per capita rate of growth.

The main purpose of this paper is to study the criterion for persistence and extinction of species in more general frameworks than the ones considered in these previous works [5],[18],[19],[80] and further provide some applications of this theory. Our aim is to deal with the new cases for which condition (1.3) is no longer true. We extend the model of (1.2) in two frameworks. The first one is for an infinite cylindrical

domain with Neumann boundary condition:

$$\begin{cases} u_t - \Delta u = f(x_1 - ct, y, u) & t > 0,\ x \in \Omega \\ \partial_\nu u(t, x_1, y) = 0 & t > 0,\ x \in \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega = \mathbb{R} \times \omega$, ω is an open bounded and smooth domain in $\mathbb{R}^{N-1}$, ν denotes the exterior unit normal vector field to Ω. In this framework, the environments are assumed to be independent of time. We are especially interested in considering environments of mixed type, which are only assumed to be globally unfavorable at infinity.

One can think of the environment containing both favorable and unfavorable regions that extend all the way to infinity, namely $f_s(x_1, y, 0) > 0$ and $f_s(x_1, y, 0) < 0$ respectively as $x_1 \to \pm\infty$, depending on the location of y. The competitive and mutual influence between these regions play a major role in characterizing the persistence and extinction of the species in the whole domain.

Mathematically, we will use a global condition in terms of spectral property to describe that the environment is globally unfavorable at infinity. The more detailed explanations of this condition will be given in subsection 1.2.1.

In the second framework, we investigate another type of mixed environment with periodic dependence on y and t. More precisely, the equation is now of following type

$$u_t - \Delta u = f(t, x_1 - ct, y, u) \qquad t > 0,\ x = (x_1, y) \in \mathbb{R}^N,$$

where the nonlinearity reaction is assumed to be periodic in y and t. The time-periodic dependent reaction has been previously investigated in various frameworks, the interested reader are referred to [18],[51],[76],[77],[79]. The main difference of the present work with respect to these papers is that here f is not assumed to be periodic in x_1-direction but be shifted with the forced speed c, which can be seen as an effect of climate change. This has been considered in [18] for an environment pointwise unfavorable at infinity. Our extension here is to consider $f_s(t, x_1, y, 0)$ to be sign-changing depending on the location of $y \in \mathbb{R}^{N-1}$ at the time t. We only require f to satisfy a global condition as $x_1 \to \pm\infty$. The additional difficulties are due to the fact that we do not a priori require the solutions to be periodic in y nor in t and also we do not impose any boundary conditions as $x_1 \to \pm\infty$. The time-periodic dependence of reaction term can be thought of as representation of a seasonal dependence of environment.

We further investigate the concentration of the species in regions surrounded by highly hostile environments. More precisely, our aim is to describe the dynamics of the species in the first framework not only in the cylindrical domain Ω but in the whole space $\mathbb{R}^N$ under the assumption that the environment outside Ω becomes more and more unfavorable. From the biological point of view one may wonder whether the species still survives if some parts of the environment becomes extremely unfavorable. This question can be addressed by solving the following mathematical problem. We consider equation (1.4) in the whole space $\mathbb{R}^N$ and study the limit of the sequence of traveling fronts with a reaction term $F_n(x, s)$ such that their growth rates are negative outside the cylindrical domain Ω and tend to $-\infty$ as $n \to \infty$. These solutions solve the equations

$$\Delta U_n + c\partial_1 U_n + F_n(x, U_n) = 0, \quad x \in \mathbb{R}^N,$$

where $F_n(x, s) = f(x, s)$ for $x \in \overline{\Omega}$ and $\dfrac{\partial F_n}{\partial s}(x, 0) \to -\infty$ as $n \to \infty$ locally uniformly in $\mathbb{R}^N \setminus \Omega$. If the species survives, we aim to characterize the limit. This is the object of section 4. Very recently, Guo and Hamel [63] have studied the similar problem on the periodic and not necessarily connected domains without the effect of climate change, namely when $c = 0$. From a different point of view, here we consider

the concentration of the species facing a climate change in an infinite cylindrical domain Ω when the exterior domain $\mathbb{R}^N \setminus \Omega$ becomes extremely unfavorable. To this aim, we first need to ascertain the existence and uniqueness of traveling front for problem (1.4) with Dirichlet boundary condition on $\partial\Omega$. The lack of compactness of Ω as well as the presence of $c \neq 0$ and the fact that near infinity of Ω, the environment contains both favorable and unfavorable regions are the main difficulties to be overcome. Here, we will make use of some recent advances of spectral theory in [20].

Finally, the last result is devoted to the study of symmetry breaking of the fronts in Ω. The main reason leading to the symmetry breaking is the difference of asymptotic behaviors near $\pm\infty$. To be more clear, due to the spectral theory for cylindrical domains developed by Berestycki-Nirenberg in [13], under some fair assumptions on the growth rate of $f_s(x_1, y, 0)$ as $x_1 \to \pm\infty$, we can find the exact behaviors of the unique solution of (1.4) as x_1 tends to $\pm\infty$ depending on c. By conditioning that these behaviors are different, we obtain the asymmetry of the solution. In particular, assuming $c \neq 0$, we will see that the asymmetry holds when $f_s(x_1, y, 0)$ converges fast enough to the same negative constant as $x_1 \to \pm\infty$.

In the remainder of this section, we give notations and definitions that are used in the paper. The set Ω denotes an infinite straight cylindrical domain $\Omega = \mathbb{R} \times \omega$, where ω is an open bounded and smooth domain in $\mathbb{R}^{N-1}$. We use the notation $x = (x_1, y) \in \mathbb{R} \times \omega$ for the points in Ω and denote :

$$\Omega^+ = \{x \in \Omega, x_1 \geq 0, y \in \omega\} \quad ; \quad \Omega^- = \{x \in \Omega, x_1 \leq 0, y \in \omega\};$$

$$\Omega_r = \{x \in \Omega, -r < x_1 < r, y \in \omega\}.$$

Let $\mathcal{O} \subset \mathbb{R}^N$ and L be a uniformly elliptic operator with coefficients bounded on $\mathcal{O}$ defined by

$$Lu = a_{ij}(x)\partial_{ij}u(x) + b_i(x)u_i(x) + c(x)u.$$

If $\mathcal{O}$ is smooth and bounded, it is classical that L admits a unique eigenvalue $-\lambda_D$ (respectively $-\lambda_N$) and a unique (up to multiplication) eigenfunction with Dirichlet (respectively Neumann) boundary condition i.e :

$$\begin{cases} L\varphi = -\lambda_D\varphi & x \in \mathcal{O} \\ \varphi = 0 & x \in \partial\mathcal{O}. \end{cases} \qquad \begin{cases} L\varphi = -\lambda_N\varphi & x \in \mathcal{O} \\ \partial_\nu\varphi = 0 & x \in \partial\mathcal{O}. \end{cases}$$

As is known, the principal eigenpair (eigenvalue and eigenfunction) for an associated elliptic operator plays an important role in deriving persistence results and long time dynamics. In 1994, Berestycki, Nirenberg and Varadhan [12] gave a very simple and general definition of the principal eigenvalue of L for general domains whose boundaries are not necessarily smooth and later Berestycki, Hamel and Rossi [9] used this approach to define generalized principal eigenvalues in unbounded domains. More precisely now allowing $\mathcal{O}$ to be a smooth and possibly unbounded domain, they defined the generalized Neumann principal eigenvalue as follow

$$\lambda_N(-L, \mathcal{O}) := \sup\{\lambda \in \mathbb{R} : \exists \phi \in W^{2,N}_{loc}(\mathcal{O}), \phi > 0, (\mathcal{L} + \lambda)\phi \leq 0 \text{ a.e in } \mathcal{O}, \partial_\nu\phi \geq 0 \text{ on } \partial\mathcal{O}\}. \tag{1.5}$$

When $\mathcal{O}$ is bounded, these two notions coincide : $\lambda_N(-L, \mathcal{O}) = \lambda_N$. We adopt this definition in our paper. Under the assumption a_{ij}, b_i, $c \in L^\infty(\mathcal{O})$, it is easily seen that $\lambda_N(-\mathcal{L}, \mathcal{O})$ is well defined. For related definition and more properties of generalized eigenvalues, the reader is referred to [20].

1.1.2 Hypotheses and main results

The cylindrical environment without time dependence

The function $f(x_1, y, s) : \mathbb{R} \times \omega \times [0, +\infty) \mapsto \mathbb{R}$ is assumed to be continuous in x_1, measurable in y, and locally Lipschitz continuous in s. In addition, the map $s \mapsto f(x, s)$ is of class $C^1(0, s_0)$ for some positive constant s_0, uniformly in x. We assume that $f(x, 0) = 0$, $\forall x \in \Omega$.

As we will see, the dynamics is controlled by traveling fronts. Thus, we look for the solutions of Eq. (1.4) of the type $u(t, x) = U(x_1 - ct, y)$, which are called traveling front solutions with forced speed c. Such solutions are given by the equation :

$$\begin{cases} \Delta U + c\partial_1 U + f(x, U) = 0 & x \in \Omega \\ \partial_\nu U = 0 & x \in \partial\Omega \\ U > 0 \text{ in } \Omega \\ U \text{ is bounded.} \end{cases} \tag{1.6}$$

In the results below, we will require the following hypotheses on f :

$$\exists S > 0 \text{ such that } f(x, s) \leq 0 \text{ for } s \geq S, \forall x \in \Omega, \tag{1.7}$$

$$\begin{gathered} s \to f(x, s)/s \text{ is nonincreasing a.e in } \Omega \text{ and there exist } D \subset \Omega, |D| > 0 \\ \text{such that it is strictly decreasing in } D. \end{gathered} \tag{1.8}$$

Both of these conditions are classical in the context of population dynamics. The first condition means that there is a maximum carrying capacity effect : when the population density is very large, the death rate is higher than the birth rate and the population decreases. The second condition means the intrinsic growth rate decreases when the population density is increasing. This is due to the intraspecific competition for resources.

As has been already mentioned, we are looking for a condition that applies to mixed environments. We assume that there exists a measurable bounded function $\mu : \omega \mapsto \mathbb{R}$ such that

$$\mu(y) = \limsup_{|x_1| \to \infty} f_s(x_1, y, 0) \qquad \text{and} \qquad \lambda_\mu := \lambda_N(-\Delta_y - \mu(y), \omega) > 0. \tag{1.9}$$

Condition (1.9) means that in the mixed environment is globally unfavorable at infinity in the direction of x_1. This generalizes the condition

$$f_s(x_1, y, 0) \leq -m < 0 \qquad \text{for } |x_1| \text{ large enough, } y \in \omega, \tag{1.10}$$

which is used in [19]. Indeed, if $\mu(y) \leq -m < 0$ one gets $\lambda_N(-\Delta_y - \mu(y), \omega) \geq m > 0$. Our generalization here aims at allowing $f_s(x_1, y, 0)$ to change sign when $|x_1|$ is large. An illustration of condition (1.9) will be given in Section 2.1.

We are now ready to state the main results regarding this framework

The existence and uniqueness of traveling front

The existence and uniqueness results are directly conditioned by the amplitude of the speed of climate change and the sign of the principal eigenvalue $\lambda_0 := \lambda_N(-\mathcal{L}_0, \Omega)$, where

$$\mathcal{L}_0 \varphi = \Delta\varphi + f_s(x, 0)\varphi.$$

Theorem 1.1.1. *Assume that (1.7)-(1.9) hold. Then there exists a unique critical speed c^* such that (1.4) admits a traveling front solution, that is a solution of (1.6) if and only if $0 \leq c < c^*$. Moreover, when the front exists, it is unique.*

This theorem yields an analogue to the results obtained in [5],[19],[80]. Indeed, under the assumption of type (1.9), one should not expect too many times the same thing. We also point out that the uniqueness of (1.6) is achieved in the class of positive bounded solutions without necessarily prescribing the boundary condition as $x_1 \to \pm\infty$.

Definition of the critical speed c^*

By using the Liouville transformation $V(x_1, y) := U(x_1, y)e^{\frac{c}{2}x_1}$, problem (1.6) is equivalent to

$$\begin{cases} \Delta V + f(x_1, y, V(x_1, y)e^{-\frac{c}{2}x_1})e^{\frac{c}{2}x_1} - \dfrac{c^2}{4}V = 0 & x \in \Omega \\ \partial_\nu V = 0 & x \in \partial\Omega \\ V > 0 \text{ in } \Omega & \\ V(x_1, y)e^{-\frac{c}{2}x_1} \text{ is bounded.} & \end{cases} \tag{1.11}$$

Linearizing this equation about 0, one gets a self-adjoint operator :

$$\widetilde{\mathcal{L}}w := \Delta w + (f_s(x, 0) - c^2/4)w.$$

We set $\mathcal{L}_0\varphi = \Delta\varphi + f_s(x, 0)\varphi$ and $\lambda_0 := \lambda_N(-\mathcal{L}_0, \Omega)$ is the generalized Neumann principal eigenvalue of $\mathcal{L}_0$ in Ω. Since $f_s(x, 0)$ is bounded, λ_0 is well defined and finite. We are led to

Definition 1.1.2. We define the critical speed by

$$c^* := 2\sqrt{-\lambda_0} \qquad \text{if } \lambda_0 < 0 \tag{1.12}$$

Proposition 1.1.3. *The eigenvalue $\lambda_N(-\Delta - c\partial_1 - f_s(x, 0), \Omega) < 0$ iff $0 \leq c < c^*$.*

Proof. Let $\mathcal{L} = \Delta + c\partial_1 + f_s(x, 0)$. Since we do not assume the test-function of (1.5) to be bounded. It immediately follows from the definition (1.5) that $\lambda_N(-\mathcal{L}, \Omega) = \lambda_N(-\widetilde{\mathcal{L}}, \Omega) = \lambda_0 + \frac{c^2}{4}$. □

The next two results deal with the long time dynamics of the evolution equation (1.4) in $L^\infty(\Omega)$ and $L^1(\Omega)$.

Long time dynamics

Theorem 1.1.4. *Let $u(t, x)$ be the solution of (1.4) with initial condition $u(0, x) \in L^\infty(\Omega)$, which is nonnegative and not identically equal to zero. Assume that (1.7) − (1.9) hold.*

i) If $c \geq c^*$, then

$$\lim_{t\to\infty} \|u(t, x)\|_{\infty,\Omega} = 0;$$

ii) if $0 \leq c < c^*$ then

$$\lim_{t\to\infty} \|(u(t, x_1, y) - U(x_1 - ct, y))\|_{\infty,\Omega} = 0,$$

where U is the unique solution of (1.6) and $\|\cdot\|_{\infty,\Omega}$ denotes the sup-norm on Ω.

This theorem means that, a species cannot keep pace with a climate change if its speed is too large. This theorem generalizes the results in [5], [19]. Note that, in [5], [19], condition (1.10) was actually used in the proofs, in particular, to derive the exponential behavior at infinity. Although our approaches are similar to those in [5], [19], new difficulties arise from the non-constant unfavorable characterization at infinity, especially to obtain the comparison principle.

The next result is concerned with the $L^1(\Omega)$ convergence of the traveling fronts. This result describes the long time dynamics of the total population.

Theorem 1.1.5. *Let $u(t,x)$ be the solution of (1.4) with initial condition $u(0,x) \in L^\infty(\Omega) \cap L^1(\Omega)$, which is nonnegative and not identically equal to zero. Assume that (1.7) − (1.9) are satisfied then the same conclusions as in Theorem 1.1.4 hold with the $L^1(\Omega)$ norm besides $L^\infty(\Omega)$ norm.*

The partially periodic environment with time dependence

We now consider problem (1.4) in partially periodic environments with seasonal dependence. Namely, the reaction term f now depends periodically in the time variable and (1.4) becomes

$$u_t - \Delta u = f(t, x_1 - ct, y, u) \qquad t \in \mathbb{R},\ x = (x_1, y) \in \mathbb{R}^N, \tag{1.13}$$

where $c > 0$ is the given forced speed and f is now assumed to be periodic in y. More precisely, we say that the environment is partially periodic in y and depends seasonally on time if :

1) $\forall i \in \{2, ..., N\}$, there exist the constants $L_2, ..., L_N$ such that

$$f(t, x + L_i e_i, s) = f(t, x, s) \quad \forall t \in \mathbb{R}, s \in \mathbb{R}, x \in \mathbb{R}^N$$

where $\{e_1, ...e_N\}$ denotes the unit normal orthogonal basis of $\mathbb{R}^N$.

2) There exists $T > 0$, such that

$$f(t + T, x, s) = f(t, x, s) \quad \forall t \in \mathbb{R}, s \in \mathbb{R}, x \in \mathbb{R}^N.$$

We assume in addition that $f(t, x, 0) = 0$, f is C^1 with respect to s, f and f_s are Holder-continuous with respect to t and x, precisely

$$\forall s > 0, \quad f(\cdot, \cdot, s), f_s(\cdot, \cdot, 0) \in C^{\frac{\alpha}{2}, \alpha}_{t,x}(\mathbb{R} \times \mathbb{R}^N),$$

where $C^{\frac{\alpha}{2}, \alpha}_{t,x}(\mathcal{I} \times \mathcal{H})$, $\mathcal{I} \subset \mathbb{R}$, $\mathcal{H} \subset \mathbb{R}^N$ denotes the space of functions $\phi(t,x)$ such that $\phi(\cdot, x) \in C^{\frac{\alpha}{2}}(\mathcal{I})$ and $\phi(t, \cdot) \in C^\alpha(\mathcal{H})$ uniformly with respect to t and x respectively.

We are interested in pulsating fronts of (1.13), namely the solutions of the form $u(t,x) = U(t, x_1 - ct, y) > 0$. They are obtained from the equation

$$\begin{cases} U_t = \Delta U + c\partial_1 U + f(t, x, U) & t \in \mathbb{R}, x \in \mathbb{R}^N \\ U \text{ is bounded.} \end{cases} \tag{1.14}$$

Note that $U(t, x_1, y)$ is not a priori assumed to be periodic in t nor in y.

Here, we require the principal eigenvalue of the linearized operator of Eq. (1.14). More generally, we consider the operators of the form :

$$\mathcal{L}u = \partial_t u - a_{ij}(t,x)\partial_{ij} u(t,x) - b_i(t,x)u_i(t,x) - c(t,x)u(t,x), \qquad x = (x_1, y) \in \mathbb{R}^N, \tag{1.15}$$

where a_{ij}, b_i, c_i are T-periodic in t and periodic in y with the same period. To define the generalized principal eigenvalue of $\mathcal{L}$, we assume that the coefficients satisfy the regularity condition as mentioned above and the matrix $(a_{ij}(t,x))$ is uniformly elliptic, namely $a_{ij}, b_i, c_i \in C^{\frac{\alpha}{2},\alpha}_{t,x}(\mathbb{R}\times\mathbb{R}^N)$ and there exist some positive constants E_1, E_2 such that for all $\xi \in \mathbb{R}^N$ and $(t,x) \in \mathbb{R}\times\mathbb{R}^N$ such that

$$E_1|\xi|^2 \leq \sum_{1\leq i\leq j\leq N} a_{ij}(t,x)\xi_i\xi_j \leq E_2|\xi|^2.$$

Definition 1.1.6. Let $\mathcal{O} \subset \mathbb{R}$ and $\mathcal{Q} = \{(t,x) = (t,x_1,y) \in \mathbb{R}\times\mathcal{O}\times\mathbb{R}^{N-1}\}$, the generalized principal eigenvalue of $\mathcal{L}$ in $\mathcal{Q}$ is defined by:

$$\tilde{\lambda}_1(\mathcal{L},\mathcal{Q}) = \sup\Big\{\lambda\in\mathbb{R} : \exists\ \phi>0,\ \phi\in C^{1,2}_{t,x}(\mathcal{Q}),\ \phi \text{ is T-periodic in } t \text{ and periodic in } y \text{ such that } (\mathcal{L}-\lambda)\phi \geq 0 \text{ in } \mathcal{Q}\Big\}. \tag{1.16}$$

By assuming in addition that $a_{ij}, b_i, c_i \in L^\infty(\mathbb{R}^{N+1})$, one can take $\lambda = -\sup_{\mathbb{R}\times\mathbb{R}^N} f_s(t,x,0)$ and 1 as a test function to see that $\tilde{\lambda}_1$ is well-defined and $-\sup_{\mathbb{R}\times\mathbb{R}^N} f_s(t,x,0) \leq \tilde{\lambda}_1$. We point out that this definition does not make sense if we do not require that the test functions to be periodic in t. Indeed, since $(\mathcal{L}-\lambda)(\phi e^{\alpha t}) = (\mathcal{L}+\alpha-\lambda)(\phi e^{\alpha t})$, $\forall\alpha\in\mathbb{R}$, if we do not force the periodicity in t, it would yield $\tilde{\lambda}_1 = \tilde{\lambda}_1+\alpha$ for all α. This kind of eigenvalue seems analogous to the ones introduced by Berestycki and Rossi [18] and Nadin [76]. However, the difference is that here we force the test functions to be periodic in t and y but not in x_1 while in [18], the test functions are not periodic in any direction of $x = (x_1, y)$ and in [76], the test functions must be periodic in both t and $x = (x_1, y)$.

We further need the two following conditions that are similar to (1.7),(1.8), but take into account the time-periodic dependence of f :

$$\exists S>0 \text{ such that } f(t,x,s) \leq 0 \text{ for } s\geq S,\ \forall t\in\mathbb{R}, x\in\mathbb{R}^N, \tag{1.17}$$

$$s \to f(t,x,s)/s \text{ is nonincreasing and for all } t_0\in[0,T) \text{ there exist } D\subset(-\infty,t_0)\times\mathbb{R}^N, \ |D|>0 \text{ such that it is strictly decreasing in } D. \tag{1.18}$$

Suppose that a parabolic operator $\widetilde{\mathcal{L}}$ is defined on $\mathbb{R}\times\mathbb{R}^{N-1}$ and has the form

$$\widetilde{\mathcal{L}}\phi = \partial_t\phi - a_{ij}(t,y)\partial_{ij}\phi(t,y) - b_i(t,y)\phi_i(t,y) - c(t,y)\phi(t,y), \qquad y\in\mathbb{R}^{N-1}.$$

Under the assumptions that $a_{ij}(t,y), b_i(t,y), c_i(t,y) \in L^\infty(\mathbb{R}\times\mathbb{R}^{N-1})$ and the matrix $a_{ij}(t,x)$ satisfies the uniform elliptic condition, we define the generalized space-time periodic eigenvalue of $\widetilde{\mathcal{L}}$:

$$\bar{\lambda}_1(\widetilde{\mathcal{L}},\mathbb{R}\times\mathbb{R}^{N-1}) = \sup\{\lambda\in\mathbb{R} : \exists\ \phi>0,\ \phi\in C^{1,2}_{t,y}(\mathbb{R}\times\mathbb{R}^{N-1}),\ \phi \text{ is T-periodic in } t \text{ and periodic in } y \text{ such that } (\widetilde{\mathcal{L}}-\lambda)\phi\geq 0 \text{ in } \mathbb{R}\times\mathbb{R}^{N-1}\}.$$

It is worth mentioning that, Nadin [76] has shown the existence and uniqueness of the space-time periodic eigenpair (λ_p, φ_p) of the eigenvalue problem

$$\begin{cases} \widetilde{\mathcal{L}}\varphi_p = \lambda_p\varphi_p \\ \varphi_p > 0 \\ \varphi_p(.,.+T) = \varphi_p \\ \varphi_p(.+L_ie_i,.) = \varphi_p. \end{cases}$$

For more details, we refer the reader to the Theorems 2.7 in [76]. Obviously, $\lambda_p \leq \bar{\lambda}_1(\widetilde{\mathcal{L}}, \mathbb{R}\times\mathbb{R}^{N-1})$.

Using this notion, we assume that there exists a function $\gamma(t,y) \in L^\infty(\mathbb{R}\times\mathbb{R}^{N-1})$, which is periodic in y and T-periodic in t such that

$$\gamma(t,y) = \limsup_{|x_1|\to\infty} f_s(t,x_1,y,0) \qquad \text{and} \qquad \bar{\lambda}_1(\partial_t - \Delta_y - \gamma(t,y), \mathbb{R}\times\mathbb{R}^{N-1}) > 0. \tag{1.19}$$

Condition (1.19) yields the characterization of the environment expressing that it is globally unfavorable at infinity. The new difficulties of this problem arise since we deal with the solution in the unbounded domain (whole space) without a-priori assuming that the solutions are periodic in y nor in t. Moreover, the monotonicity in time of solutions of parabolic operators starting by a stationary sub (or super) solution no longer holds.

Let us call $\mathcal{P}\varphi = \partial_t\varphi - \Delta\varphi - c\partial_1\varphi - f_s(t,x,0)\varphi$ the linearized operator associated with (1.14). In the sequel, we will briefly denote by $\tilde{\lambda}_1 = \tilde{\lambda}_1(\mathcal{P}, \mathbb{R}\times\mathbb{R}^{N+1})$. We are now able to state the results of this section :

Theorem 1.1.7. *Assume that* (1.17) − (1.19) *hold, then there exists a positive pulsating front* $U \in C^{1,2}(\mathbb{R}\times\mathbb{R}^N)$ *of (1.14) if and only if* $\tilde{\lambda}_1 < 0$. *If it exists, it is unique, T-periodic in t, periodic in y and decays exponentially in* $|x_1|$, *uniformly in y and t.*

One of the interesting points of this theorem is the loss of compactness since f is not periodic in x_1 and the solution is not a priori assumed to be periodic in y nor in t. We will prove that the uniqueness of (1.14) still holds in this larger class of solution, that is the class of nonnegative bounded solutions. As pointed out in section 1.5 of [77], one cannot expect to show a general uniqueness in the class of nonnegative bounded solutions, even when the coefficients of (1.14) are periodic in $x = (x_1, y)$ and in t under condition $\tilde{\lambda}_1 < 0$ only. Some extra assumptions are needed. Here the uniqueness holds due to assumption (1.19), which is a key ingredient to derive the exponential behavior of solutions of (1.14) as $x_1 \to \pm\infty$.

Theorem 1.1.8. *Let* $u(t,x)$ *be the solution of (1.13) with nonnegative initial datum* $u_0(x) \in L^\infty(\mathbb{R}^N)$ *and not identically equal to* 0. *Assume that (1.17)-(1.19) hold.*

i) If $\tilde{\lambda}_1 \geq 0$ *then*

$$\lim_{t\to\infty} u(t,x) = 0,$$

uniformly in $x \in \mathbb{R}^N$.

ii) if $\tilde{\lambda}_1 < 0$ *then*

$$\lim_{t\to\infty} (u(t,x_1,y) - U(t,x_1-ct,y)) = 0,$$

where U is the unique solution of (1.14), *uniformly in* x_1, *locally uniformly in y. If, in addition,* u_0 *is periodic in y or satisfies*

$$\forall r > 0, \qquad \inf_{|x_1|<r, y\in\mathbb{R}^{N-1}} u_0(x_1,y) > 0, \tag{1.20}$$

then above convergence is uniform also in y.

The fact that the convergence holds uniformly in x_1 is a consequence of the exponential decay as $x_1 \to \pm\infty$, which can be derived from (1.19).

Organization of the paper. We divide the rest of the paper into four sections. Section 2 deals with problem (1.4) in the cylindrical domain Ω without time dependence. Section 3 investigates problem (1.13) in partially periodic domain with time dependence. In section 4, we first study the concentration of the species in Ω of Section 2, when the exterior domain becomes extremely unfavorable and further prove the symmetry breaking of the fronts. Finally, some auxiliary results are contained in the Appendix.

1.2 The cylindrical environment without time dependence

1.2.1 An illustration

Before proving the main results of this section, let us provide an illustration of how theorems (1.1.1)-(1.1.5) apply and why condition (1.9) is useful to describe the heterogeneity of habitat of a species facing a climate change.

We consider $\Omega = \mathbb{R} \times (0,1)$ and for $\alpha \in (0,1), L > 0$ the family $f_{\alpha,L}(x_1, y, s) = (\rho_L(x_1) + \mu_\alpha(y))s - s^2$, where

$$\rho_L(x_1) = \begin{cases} 2 & \text{on } [-L, L] \\ -2 & \text{outside } [-L, L], \end{cases} \qquad \mu_\alpha(y) = \begin{cases} 1 & \text{on } [0, \alpha] \\ -1 & \text{on } [\alpha, 1]. \end{cases}$$

These nonlinearities are discontinuous. However, f_s is well defined a.e and all our results apply for zero order coefficient in L^∞ (see [18],[19] for further discussion of this extension). For every $\alpha \in (0,1)$, let $(\lambda_\alpha, \phi_\alpha)$ be the (unique) eigenpair of the Neumann eigenvalue problem

$$\begin{cases} -\phi_\alpha'' - \mu_\alpha(y)\phi_\alpha = \lambda_\alpha \phi_\alpha & \text{in } (0,1) \\ \partial_\nu \phi_\alpha = 0 & \text{at 0 and 1.} \end{cases}$$

Dividing the equation by ϕ_α and integrating by part, we get

$$-\int_0^1 \frac{\phi_\alpha'^2}{\phi_\alpha^2} dy - \int_0^1 \mu_\alpha(y) dy = \lambda_\alpha.$$

Hence, $\lambda_\alpha \leq -\int_0^1 \mu_\alpha(y)dy = -(2\alpha - 1)$. It is also known that λ_α is decreasing with respect to α and that $\alpha \mapsto \lambda_\alpha$ is continuous. Since $\lambda_0 = 1$ we see that there exists a unique $\overline{\alpha}$ such that $\lambda_{\overline{\alpha}} = 0$ and $\lambda_\alpha < 0$ iff $\alpha > \overline{\alpha}$.

For $\alpha > \overline{\alpha}$, $\lambda_\alpha < 0$, it is well-known that there exists a unique positive solution (Berestycki,[3]) of

$$\begin{cases} -p_\alpha'' - \mu_\alpha(y)p_\alpha + p_\alpha^2 = 0 & \text{in } (0,1) \\ \partial_\nu p_\alpha = 0 & \text{at 0 and 1.} \end{cases}$$

In this case, the environment is globally favorable at infinity and there is always persistence. Indeed, it can be shown in this case that $u(t, x_1, y) \to p_\alpha(y)$ as $t \to \infty$ pointwise in Ω. Under the condition $\alpha < \overline{\alpha}$, the environment is globally unfavorable at infinity. Then, for fixed c not too large, for instance $c = 1$, we claim that there exists a unique L^* such that the persistence holds iff $L > L^*$.

To prove this, let us call $\mathcal{Q}_L[\phi] = \phi'' + \phi' + \partial_s f_{\alpha,L}(x_1, y, 0)\phi$, where $\partial_s f_{\alpha,L}(x_1, y, 0) = \rho_L(x_1) + \mu_\alpha(y)$ defined on Ω. Since Ω is unbounded, we cannot define the classical eigenvalue of $\mathcal{Q}_L$ on Ω. We make use of the definition (1.5). Let us call $\lambda_L = \lambda_N(-\mathcal{Q}_L, \Omega)$ and $\lambda_L' = \lambda_N(-\phi'' - \partial_s f_{\alpha,L}(x_1, y, 0)\phi, \Omega)$. By Propsition 1.1.3, one has

$$\lambda_L = \lambda_L' + \frac{1}{4}.$$

Since ρ_L is increasing with respect to L, λ_L is decreasing with respect to L. Moreover, the map $L \mapsto \lambda_L$ is continuous on $[0, \infty]$. Indeed, for any $L \in [0, \infty]$, let $\{L_n\} \in [0, \infty]$ be an arbitrary sequence converging to L, we see that $\|\partial_s f_{\alpha,L_n}(x_1, y, 0) - \partial_s f_{\alpha,L}(x_1, y, 0)\|_{L^\infty(\Omega)} \to 0$ as $n \to \infty$. Arguing as in the proof of Proposition 9.2 part (ii) [20], we get $\lim_{n\to\infty} \lambda_{L_n} = \lambda_L$. It is worth noting that since Ω is a smooth domain with Neumann boundary condition, we can apply the Harnack inequality up to the boundary (see Berestycki-Caffarelli-Nirenberg [4]) as in proof of Proposition 9.2 part (ii) [20]. Moreover, since $\partial_s f_{\alpha,0}(x_1, y, 0) = -2 + \mu_\alpha(y) \leq -1$

and $\partial_s f_{\alpha,\infty}(x_1, y, 0) = 2 + \mu_\alpha(y) \geq 1$, by taking 1 as a test-function, we have $\lambda_0 = \lambda_0' + 1/4 \geq 5/4$ and $\lambda_\infty = \lambda_\infty' + 1/4 \leq -3/4$. The claim is proved.

As we will see in the next section, the equation

$$\begin{cases} q_\alpha'' + cq' + f_{\alpha,L}(x_1, y, q) = 0 & \text{in } \Omega \\ \partial_\nu q_\alpha = 0 & \text{on } \partial\Omega \end{cases} \tag{1.21}$$

admits a unique positive solution if and only if $\lambda_L < 0$. From this result, it follows that for $L > L^*$, as $t \to \infty$, $u(t, x_1, y)$ converges to the unique positive solution $q(x_1, y)$ of (1.21) and $u(t, x_1, y)$ converges to zero for $L \leq L^*$. Thus L^* is the threshold value.

Therefore, the problem is more subtle when $\alpha < \overline{\alpha}$. Our theory applies in this case. Even if the persistence is known, the non-persistence and uniqueness are still delicate questions.

1.2.2 The existence and uniqueness of the front

To achieve the existence and uniqueness of Eq. (1.6), the key property is the exponential decay of solution. This estimate is the object of the next section

Exponential decay

Proposition 1.2.1. *Let U be a nonnegative bounded solution of (1.6). Assume that (1.8)-(1.9) hold, then for all $0 < \alpha < \alpha^*$ with $\alpha^* = \frac{-c+\sqrt{c^2+4\lambda_\mu}}{2}$, there exists a positive constant $C(\alpha)$ such that*

$$U(x_1, y) + |\nabla U(x_1, y)| \leq C(\alpha) e^{-\alpha|x_1|}.$$

We point out that this proposition generalizes Proposition 3 of [19]. Moreover, our proof is somewhat simpler than the proof of Berestycki and Rossi in that we do not use the Liouville transformation.

Proof. Consider $\varphi \in W^{2,p}(\omega)$ the eigenfunction associated with λ_μ

$$\begin{cases} \Delta\varphi + \mu(y)\varphi + \lambda_\mu \varphi = 0 & \text{in } \omega \\ \partial_\nu \varphi(y) = 0 & \text{on } \partial\omega \end{cases} \tag{1.22}$$

where $\nu = \nu(y)$ is the outward unit normal on ω. As is well-known by the Hopf lemma, $\inf_{\overline{\omega}} \varphi > 0$.

For any $\delta \in (0, \lambda_\mu)$, let $\mathcal{L}w = \Delta w + c\partial_1 w + (\mu(y) + \delta)w$ be defined in Ω. Due to (1.8)-(1.9), one has

$$\Delta U + c\partial_1 U + f_s(x, 0)U \geq 0 \qquad \text{in } \Omega.$$

and there exists $R = R(\delta) > 0$ such that

$$f_s(x_1, y, 0) \leq \mu(y) + \delta \quad \text{in } \Omega \setminus \Omega_R,$$

therefore $\mathcal{L}U \geq 0$ in $\Omega \setminus \Omega_R$. For any $p > 0$, set

$$w_p(x_1, y) = e^{(R+p)(\tau-\alpha)} e^{\alpha|x_1|}\varphi(y) + e^{R(\tau+\alpha)} e^{-\alpha|x_1|}\varphi(y) = C_1 e^{\alpha|x_1|}\varphi(y) + C_2 e^{-\alpha|x_1|}\varphi(y),$$

where $R, \tau, \alpha > 0$ will be chosen. Direct computation shows that

$$\begin{aligned}\frac{\mathcal{L}w_p}{\varphi} &= C_1\left(\alpha^2 + \frac{\Delta_y\varphi}{\varphi} + c\alpha\frac{x_1}{|x_1|} + \mu(y) + \delta\right)e^{\alpha|x_1|} + \\ &\qquad + C_2\left(\alpha^2 + \frac{\Delta_y\varphi}{\varphi} - c\alpha\frac{x_1}{|x_1|} + \mu(y) + \delta\right)e^{-\alpha|x_1|} \\ &\leq (\alpha^2 + c\alpha - \lambda_\mu + \delta)(C_1e^{\alpha|x_1|} + C_2e^{-\alpha|x_1|}) \qquad (1.23)\end{aligned}$$

For w_p to be a supersolution of $\mathcal{L}$ in $\Omega \setminus \Omega_R$, it suffices to take

$$\alpha = \alpha(\delta) = \frac{-c + \sqrt{c^2 + 4\lambda_\mu - 4\delta}}{2} > 0.$$

Clearly, $\alpha(\delta)$ is decreasing with respect to $\delta \in (0, \lambda_\mu)$. Choosing $\tau = \alpha/2$ and R large enough , we have the following estimates on $\partial(\Omega_{R+p} \setminus \Omega_R)$

$$\begin{cases} w_p(x_1, y) \geq e^{R\tau}\varphi(y) \geq e^{R\tau}\inf_\omega \varphi \geq U(x_1, y) & \text{as } |x_1| = R, y \in \omega \\ w_p(x_1, y) \geq e^{(R+p)\tau}\varphi(y) \geq e^{(R+p)\tau}\inf_\omega \varphi \geq U(x_1, y) & \text{as } |x_1| = R + p, y \in \omega. \end{cases}$$

Fix α, τ and R, we set $z(x_1, y) = \frac{w_p(x_1,y) - U(x_1,y)}{\varphi(y)}$. Obviously, $\mathcal{L}(z\varphi) = \mathcal{L}w_p - \mathcal{L}U \leq 0$. Routine computation yields

$$\mathcal{L}_1[z] = \frac{\mathcal{L}[z\varphi]}{\varphi} = \Delta z + 2\frac{\nabla_y\varphi}{\varphi}\cdot\nabla_y z + c\partial_1 z + \frac{\Delta\varphi + (\mu(y) + \delta)\varphi}{\varphi}z \leq 0$$

Observe that $z \geq 0$ when $|x_1| \in \{R, R + p\}$, $y \in \omega$ and $\partial_\nu z = 0$ on $\partial\omega$. Moreover, the zero order coefficient of $\mathcal{L}_1$ is negative. Hence, by the maximum principle, we get that

$$U(x_1, y) \leq w_p(x_1, y) = e^{-(R+p)\alpha/2}e^{\alpha|x_1|}\varphi(y) + e^{3R\alpha/2}e^{-\alpha|x_1|}\varphi(y) \qquad \text{in } \Omega_{R+p} \setminus \Omega_R.$$

Letting $p \to \infty$, we obtain

$$U(x_1, y) \leq e^{3R\frac{-c+\sqrt{c^2+4\lambda_\mu-4\delta}}{4}}e^{-\frac{-c+\sqrt{c^2+4\lambda_\mu-4\delta}}{2}|x_1|}\varphi(y) \qquad \text{in } \Omega.$$

Set $\alpha^* = \frac{-c+\sqrt{c^2+4\lambda_\mu}}{2}$, one sees that when δ goes to 0, α is arbitrarily close to α^*. Since φ is bounded, obviously for all $0 < \alpha < \alpha^*$ we can choose $C(\alpha)$ (possibly changed if necessary) such that

$$U(x_1, y) \leq C(\alpha)e^{-\alpha|x_1|} \qquad \text{in } \Omega.$$

This implies that $U(x_1, y)$ decays exponentially as $|x_1| \to \infty$, uniformly in y. On the other hand, since U is a solution of (1.6), we can use L^p estimate for the Neumann problem with $p > N$ and Harnack inequality up to the boundary (see Berestycki-Caffarelli-Nirenberg [4]) to derive

$$\|\nabla U\|_{L^\infty(B_1(x))} \leq C_1\|U\|_{W^{2,p}(B_1(x))} \leq C_2\|U\|_{L^\infty(B_2(x))} \leq C_3U(x).$$

These inequalities end the proof. □

Proof of Theorem 1.1.1

Let us first consider the case $c < c^*$. By Proposition 1.1.3, we know that

$$\tilde{\lambda}_1 := \lambda_N(-\Delta - c\partial_1 - f_s(x_1, y, 0), \Omega) < 0.$$

Thanks to Proposition 1, [19], we have the limit $\lim_{R\to\infty} \lambda_R = \tilde{\lambda}_1 < 0$, where λ_R is the unique eigenvalue of problem :

$$\begin{cases} -\Delta\varphi_R - c\partial_1\varphi_R - f_s(x,0)\varphi_R = \lambda_R\varphi_R & x \in \Omega_R \\ \varphi_R(x) > 0 & x \in \Omega_R \\ \partial_\nu\varphi_R(x_1, y) = 0 & |x_1| < R, y \in \partial\omega \\ \varphi_R(\pm R, y) = 0 & y \in \omega. \end{cases}$$

Moreover there exists an eigenfunction $\varphi_\infty \in W^{2,N}(\Omega)$ associated with $\tilde{\lambda}_1$. Fix $R > 0$ large enough such that $\lambda_R < 0$, we define $\phi(x)$ as following :

$$\phi(x) = \begin{cases} \varphi_R(x) & x \in \Omega_R \\ 0 & \text{otherwise.} \end{cases}$$

Since $f(x, s)$ is of class $C^1[0, s_0]$ with respect to s, for $\varepsilon > 0$ small enough, we see that

$$\Delta(\varepsilon\phi) + c\partial_1(\varepsilon\phi) + f(x_1, y, \varepsilon\phi) = \varepsilon\phi\left[-\lambda_R + \frac{f(x_1, y, \varepsilon\phi)}{\varepsilon\phi} - f_s(x_1, y, 0)\right] > 0.$$

Hence, $\varepsilon\phi$ is a subsolution of Eq. (1.6). Since ϕ is compactly supported, we can choose ε small such that $\varepsilon \sup\phi \le S$, where S is a super solution of Eq. (1.6) given by (1.7). Therefore, by the classical iteration method, there exists a nonnegative solution U satisfying $\varepsilon\phi \le U \le S$. Furthermore, thanks to the strong maximum principle, U is strictly positive .

The nonexistence and uniqueness are direct consequences of the following comparison principle. Let U and V be respectively super and subsolutions of (1.6). We will now show that $V(x) \le U(x)$ in Ω. Indeed, by condition (1.9), there exists an eigenpair (λ_μ, φ) of (1.22), where φ satisfies $\inf_{\overline{\omega}} \varphi > 0$ due to the Hopf lemma. On the other hand, Proposition 1.2.1 implies that V decays exponentially as $|x_1| \to \infty$, uniformly in y, therefore, for any $\varepsilon > 0$, there exist $R(\varepsilon) > 0$ such that $V(x_1, y) \le \varepsilon\varphi(y)$ in $\Omega \setminus \Omega_{R(\varepsilon)}$. Then, the set

$$K_\varepsilon := \{k > 0 : kU \ge V - \varepsilon\varphi \text{ in } \overline{\Omega}\},$$

is nonempty. Let us call $k(\varepsilon) := \inf K_\varepsilon$. Obviously, the function $k(\varepsilon) : \mathbb{R}^+ \to \mathbb{R}$ is nonincreasing. Assume by a contradiction

$$k^* = \lim_{\varepsilon\to 0^+} k(\varepsilon) > 1.$$

Take $0 < \varepsilon < \sup_\Omega V / \sup_\omega \varphi$, we have $k(\varepsilon) > 0$, $k(\varepsilon)U - V + \varepsilon\varphi \ge 0$. By the definition of $k(\varepsilon)$, there exists a sequence $(x^\varepsilon_{1,n}, y^\varepsilon_n)$ in $\overline{\Omega}$ such that

$$\left(k(\varepsilon) - \frac{1}{n}\right) U(x^\varepsilon_{1,n}, y^\varepsilon_n) < V(x^\varepsilon_{1,n}, y^\varepsilon_n) - \varepsilon\varphi(y^\varepsilon_n).$$

Fix $\varepsilon > 0$, we have $(x^\varepsilon_{1,n}, y^\varepsilon_n) \in \Omega_{R(\varepsilon)}$ for n large enough, therefore $(x^\varepsilon_{1,n}, y^\varepsilon_n)$ converges up to subsequence to some $(x_1(\varepsilon), y(\varepsilon)) \in \Omega_{R(\varepsilon)}$. This limiting point must satisfy :

$$(k(\varepsilon)U - V + \varepsilon\varphi)(x_1(\varepsilon), y(\varepsilon)) = 0. \tag{1.24}$$

Without loss of generality, we may assume $\lim_{\varepsilon\to 0^+} y(\varepsilon) = y_0 \in \overline{\omega}$. The case that there exists x_0 such that $|x_0| = \liminf_{\varepsilon\to 0^+} |x_1(\varepsilon)| < \infty$ is ruled out. Indeed, from (2.15), $k^* < \infty$, the function $W = k^*U - V$ is nonnegative and vanishes at (x_0, y_0). Since f is Lipschitz continuous with respect to second variable and $k^* > 1$, we have

$$-\Delta W - c\partial_1 W \geq k^* f(x,U) - f(x,V) \geq f(x,k^*U) - f(x,V) \geq z(x)W,$$

for some function $z(x) \in L^\infty_{loc}(\Omega)$. Thanks to condition (1.9), this inequality holds strictly in $D \subset \Omega$, with $|D| > 0$. The strong maximum principle implies that W cannot achieve a minimum value in the interior of Ω. This means $y_0 \in \partial\omega$, but the Hopf lemma yields another contradiction: $\partial_\nu W(x_0, y_0) < 0$.

It remains to consider the case $\lim_{\varepsilon\to 0^+} |x_1(\varepsilon)| = \infty$. Set $W^\varepsilon = k(\varepsilon)U - V + \varepsilon\varphi$, we have $W^\varepsilon \geq 0$ and W^ε vanishes at $(x_1(\varepsilon), y(\varepsilon))$. Thus there exists $r > 0$ such that $k(\varepsilon)U < V$ in $B_r(x_1(\varepsilon), y(\varepsilon)) \cap \Omega$. For ε small enough, $k(\varepsilon) > 1$, we derive from (1.8) for $B_r(x_1(\varepsilon), y(\varepsilon)) \cap \Omega$:

$$\begin{aligned}(\Delta + c\partial_1)W^\varepsilon &\leq f(x,V) - k(\varepsilon)f(x,U) - (\mu(y) + \lambda_\mu)\varepsilon\varphi \leq f(x,V) - f(x,k(\varepsilon)U) - (\mu(y) + \lambda_\mu)\varepsilon\varphi \\ &\leq -\frac{f(x,k(\varepsilon)U)}{k(\varepsilon)U}(k(\varepsilon)U - V + \varepsilon\varphi) - \frac{\lambda_\mu}{2}\varepsilon\varphi - \left(\frac{\lambda_\mu}{2} + \mu(y) - \frac{f(x_1,y,k(\varepsilon)U)}{k(\varepsilon)U}\right)\varepsilon\varphi.\end{aligned} \tag{1.25}$$

Take $0 < \varepsilon \ll 1$, then $|x_1(\varepsilon)|\gamma 1$, we have

$$\frac{f(x_1,y,k(\varepsilon)U)}{k(\varepsilon)U} < \mu(y) + \frac{\lambda_\mu}{2}, \qquad \forall (x_1,y) \in B_r(x_1(\varepsilon), y(\varepsilon)) \cap \Omega,$$

choosing r smaller if necessary. Since $\lambda > 0$, we get from (2.46)

$$-\Delta W^\varepsilon - c\partial_1 W^\varepsilon - \varrho(x)W^\varepsilon > \frac{\lambda_\mu}{2}\varepsilon\varphi > 0 \quad \text{in } B_r(x_1(\varepsilon), y(\varepsilon)) \cap \Omega,$$

where $\varrho(x) = \frac{f(x,k(\varepsilon)\overline{U})}{k(\varepsilon)\overline{U}}$ is bounded. The strong maximum principle asserts that $(x_1(\varepsilon), y(\varepsilon))$ cannot be an interior point of Ω. Hence $(x_1(\varepsilon), y(\varepsilon)) \in \partial\Omega$, but then the Hopf lemma yields another contradiction $\partial_\nu W^\varepsilon(x_1(\varepsilon), y(\varepsilon)) < 0$.

We have proved that $k^* = \lim_{\varepsilon\to 0^+} k(\varepsilon) \leq 1$. Letting $\varepsilon \to 0^+$, we derive

$$V \leq \lim_{\varepsilon\to 0^+} (k(\varepsilon)U + \varepsilon\varphi) \leq U \qquad \text{in } \Omega.$$

The uniqueness of Eq. (1.6) is obviously achieved by exchanging the roles of U and V. We end the proof by showing the nonexistence when $c \geq c^*$. Assume by contradiction that (1.6) possesses a positive solution U when $c \geq c^*$. One has $\tilde{\lambda}_1 = \lambda_N(-\Delta - c\partial_1 - f_s(x_1,y,0), \Omega) \geq 0$. Let φ_∞ be a generalized principal eigenfunction with Neumann boundary condition associated with $\tilde{\lambda}_1$. Without loss of generality, we may assume that $0 < \varphi_\infty(0) < U(0)$. We derive, from (1.8), that

$$-\Delta\varphi_\infty - c\partial_1\varphi_\infty = (f_s(x_1,y,0) + \tilde{\lambda}_1)\varphi_\infty \geq f(x_1,y,\varphi_\infty) \qquad \text{in } \Omega$$

By Proposition 1.2.1, U decays exponentially as $|x_1| \to \infty$ uniformly in y. The above comparison principle implies that $U(x) \leq \varphi_\infty(x)$ for all $x \in \Omega$. This contradiction ends the proof.

Remark 1. The globally unfavorable characterization of the environment near infinity plays the key role to derive the uniqueness of (1.6). Indeed, if the environment is favorable at infinity, that is $\lambda_\mu < 0$ (see (1.9)), from section 2.1, we see that there exists a positive solution $p(y)$ at infinity. If f is homogeneous in the traveling direction, i.e independent of x_1, then for all $a \in \mathbb{R}$, $U(x_1 + a, y)$ are also solutions of (1.6) and thus the uniqueness does not hold.

1.2.3 Long time dynamics

In order to study the long time dynamics of Eq. (1.4), we first prove the following Liouville type theorem for entire solutions (solutions for all $t \in \mathbb{R}$). Consider the evolution problem in the cylindrical domain with Neumann boundary condition

$$\begin{cases} \partial_t u^* = \Delta u^* + c\partial_1 u^* + f(x, u^*) & t \in \mathbb{R},\ x \in \Omega \\ \partial_\nu u^* = 0 & t \in \mathbb{R},\ x \in \partial\Omega, \end{cases} \tag{1.26}$$

we have the following auxiliary result

Theorem 1.2.2. *Assume that conditions* (1.7) – (1.9) *are satisfied, then Eq. (1.26) admits a positive bounded entire solution only if $c < c^*$, where c^* is defined in Proposition 1.1.3. Conversely, if $c < c^*$ and there exist a sequence $(t_n) \in \mathbb{R}$ as $n \to \infty$ and a point $x_0 \in \overline{\Omega}$ such that*

$$\lim_{n\to\infty} t_n = +\infty, \qquad \liminf_{n\to\infty} u^*(-t_n, x_0) > 0, \tag{1.27}$$

then $u^*(t, x) \equiv U(x)$, where $U(x)$ is the unique solution of (1.6), *given by Theorem 1.1.1.*

Proof. Set $S^* = \max\{S, \|u^*\|_{L^\infty(\Omega)}\}$, where S is the positive constant given in (1.7), obviously S^* is a super solution of stationary equation of Eq. (1.26). Let $v(t, x)$ be the solution of (1.26) starting by $v(0, x) = S^*$, the parabolic maximum principle and standard estimates imply that v is nonincreasing in t and converges locally uniformly in Ω to a stationary solution $V(x)$ of (1.26). That $V(x)$ solves Eq. (1.6). For any $h \in \mathbb{R}$, we define $v_h(t, x) = v(t - h, x)$. This function is a solution of (1.26) in $(h, +\infty) \times \Omega$ and satisfies $v_h(h, x) = S^* \geq u^*(h, x)$. The parabolic comparison principle thus yields

$$0 \leq u^*(t, x) \leq \lim_{h\to-\infty} v_h(t, x) = V(x) \qquad \forall t \in \mathbb{R}, x \in \Omega. \tag{1.28}$$

We consider separately two different cases:

Case 1. $c \geq c^*$.

Theorem 1.1.1 asserts that the stationary equation of Eq. (1.26) only has zero-solution. Namely, $V(x) \equiv 0$ in Ω. Therefore, the necessary condition for existence of nontrivial entire solution of (1.26) is $c < c^*$.

Case 2. $c < c^*$ and (1.27) holds.

Theorem 1.1.1 again asserts that the stationary equation of Eq. (1.26) admits a unique positive solution U. We will prove that $u^*(t, x) \equiv U(x)$. Assume by contradiction that there exists $x_0 \in \Omega$ such that $u^*(t, x_0) \neq U(x_0)$. We will reach a contradiction by proving the following claim.

Claim. There exist $\varepsilon \in (0, 1]$ and $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, one has $\varepsilon U(x) \leq u^*(-t_n, x)$.

Assume for a moment that this claim holds true, the concluding argumentation goes as follows. Thanks to (1.8), for any $\varepsilon \in (0, 1]$, εU is a subsolution of stationary equation of Eq. (1.26). Let $w(t, x)$ be a solution

of (1.26) with initial condition $w(0,x) = \varepsilon U(x)$ and $w_n(t,x) = w(t+t_n,x)$. We know, by the standard parabolic estimates, that as $t \to \infty$, $w(t,x)$ is nondecreasing, bounded from above by S^* and converges locally uniformly in Ω to the unique stationary solution $W(x)$ of Eq. (1.26). The strict positivity of W is derived from the condition $c < c^*$. By the way of setting, one has $w_n(-t_n,x) = \varepsilon U(x) \leq u^*(-t_n,x)$. The parabolic comparison principle implies that $w_n(t,x) \leq u^*(t,x)$ in $(-t_n,+\infty)\times\Omega$. Therefore, by letting $n \to \infty$, one has

$$u^*(t,x) \geq \lim_{n\to\infty} w_n(t,x) = W(x) \qquad \text{locally in } \mathbb{R}\times\Omega.$$

Combining this inequality with (1.28), we obtain $W(x) \leq u^*(t,x) \leq V(x)$, $\forall t \in \mathbb{R}$, $\forall x \in \Omega$. The uniqueness result of Theorem 1.1.1 yields $u^* \equiv W \equiv V$.

It remains to prove the claim. Assume by contradiction that for all $\varepsilon \in (0,1]$ and for all $n_0 \in \mathbb{N}$ there exist $n(\varepsilon) > n_0$ and $x_{n(\varepsilon)} \in \Omega$ so that $\varepsilon U(x(n_\varepsilon)) \geq u^*(-t_{n(\varepsilon)}, x_{n(\varepsilon)})$. Since U is bounded, choosing a sequence $\varepsilon_k \to 0$ as $k \to \infty$, by a diagonal extraction, one finds sequences $(t_k) \in \mathbb{R}^+$ and $(x_k) \in \Omega$ such that $t_k \to +\infty$ and $u^*(-t_k, x_k) \to 0$ as $k \to \infty$. We set

$$\tilde{u}_k(t,x) = u^*(t+t_k, x+x_k).$$

Obviously, $\tilde{u}_k(t,x)$ is bounded from above by S^* and satisfies the equation

$$\begin{cases} \partial_t \tilde{u}_k = \Delta \tilde{u}_k + c\partial_1 \tilde{u}_k + f(x+x_k, \tilde{u}_k) & t \in \mathbb{R},\ x \in \Omega, \\ \partial_\nu \tilde{u}_k = 0 & t \in \mathbb{R},\ x \in \partial\Omega \end{cases}$$

By standard parabolic estimates, we get $\tilde{u}_k \to \tilde{u}_\infty$ (up to subsequences) as $k \to \infty$. Thanks to the Lipschitz continuity of $f(x,s)$ with respect to s, there exists a negative constant $-M$ so that $\tilde{u}_\infty$ satisfies the equation

$$\begin{cases} \partial_t \tilde{u}_\infty \geq \Delta \tilde{u}_\infty + c\partial_1 \tilde{u}_\infty - M\tilde{u}_\infty & t \in \mathbb{R},\ x \in \Omega \\ \partial_\nu \tilde{u}_\infty = 0 & t \in \mathbb{R},\ x \in \partial\Omega. \end{cases}$$

Moreover, $\tilde{u}_\infty(0,0,0) = 0$. The strong maximum principle implies that $\tilde{u}_\infty(t,x) = 0$, $\forall t \leq 0, x \in \Omega$. Choosing $t = -2t_k$, we get $\lim_{k\to\infty} u^*(-t_k,x) = 0$, $\forall x \in \Omega$. This contradicts assumption (1.27) and thus completes the proof. □

Remark 2. In this proof, we have used different arguments from the ones of Berestycki and Rossi, Lemma 3.4 [19]. More precisely, we choose a solution of Eq. (1.26), $w(t,x)$ starting by a subsolution of stationary equation $\varepsilon U(x)$, which is not necessarily compactly supported but bounded. On the other hand, we reach the contradiction by showing that $\lim_{k\to\infty} u^*(-t_k,x) = 0$, $\forall x \in \Omega$, which differs from the way to show that for all $r > 0$, $\liminf_{n\to\infty, x\in\Omega_r} u^*(t_n,x) > 0$ in [19].

We are now ready to prove Theorem 1.1.4 to derive long time behavior of solution of (1.4) in $L^\infty(\Omega)$.

Proof of Theorem 1.1.4. Let $S' := \max\{S, \|u_0\|_{L^\infty(\Omega)}\}$, where S is the positive constant in (1.7). Then 0 and S' are respectively sub and super solution of (1.4). It follows from [72], by the standard theory of semilinear parabolic equations, that there exists a unique (weak) solution to (1.4) satisfying $0 \leq u \leq S'$ with initial condition $u_0(x)$. We deduce, from the parabolic strong maximum principle, that $u(t,x) > 0$ $\forall t > 0, x \in \overline{\Omega}$ (by extending $u(t,x)$ to larger cylinder to make the "corner" smooth). The locally long time behavior of u follows by applying directly Theorem 1.2.2 and the standard parabolic estimates. Actually, one sets $\tilde{u}(t,x_1,y) = u(t,x_1+ct,y)$. The solution of this type satisfies $\tilde{u}(0,x) = u_0(x)$ and satisfies

$$\begin{cases} \partial_t \tilde{u} = \Delta \tilde{u} + c\partial_1 \tilde{u} + f(x,\tilde{u}) & t \in \mathbb{R},\ x \in \Omega \\ \partial_\nu \tilde{u} = 0 & t \in \mathbb{R},\ x \in \partial\Omega. \end{cases} \tag{1.29}$$

To apply Theorem 1.2.2, we only need to verify condition (1.27) when $c < c^*$. Indeed, the first case $c \geq c^*$ is easily seen. Let (t_n) be a sequence such that $t_n \to +\infty$ as $n \to \infty$, we infer, by the parabolic estimates and embedding theorems, that the sequence $\tilde{u}(t + t_n, x)$ converges (up to subsequences) to some nonnegative bounded solution $u^*(t, x)$ of Eq. (1.29) as $n \to \infty$ locally in Ω. By Theorem 1.2.2, this limit is identically equal to 0 when $c \geq c^*$. Consider the case $c < c^*$, we necessarily verify condition (1.27). Let U be the unique solution of stationary solution of (1.29) and (t_n) be such that $t_n \to -\infty$ as $n \to \infty$. Fix $R > 0$, the Hopf lemma implies that $\inf_{\overline{\Omega}_R} \tilde{u}(1, x) > 0$. For $\varepsilon > 0$, the function εU is a subsolution to stationary equation of (1.29) when $\varepsilon \leq 1$. Take ε small enough such that $\varepsilon U \leq \tilde{u}(1, x)$ and in $\overline{\Omega}_R$. Hence $(t, x) \mapsto \varepsilon U(x)$ is a subsolution to (1.29) in $\mathbb{R} \times \Omega_R$. The parabolic comparison yields $\varepsilon U(x) \leq \tilde{u}(t+1, x)$ for $t > 0$ and $x \in \Omega_R$. As a consequence

$$\inf_{t \in \mathbb{R}} u^*(t, 0, y_0) \geq U(0, y_0) > 0, \qquad \text{for some } y_0 \in \omega.$$

It remains to show that the convergences hold uniformly in Ω. Assume by contradiction that

$$\lim_{t \to \infty} \tilde{u}(t, x) = U(x)$$

is not uniform in $x \in \Omega$. This means that there exist $\varepsilon > 0$, $(t_n) \in \mathbb{R}^+$ and $(x_{1,n}, y_n) \in \Omega$ such that

$$\lim_{t \to \infty} t_n = \infty, \qquad |\tilde{u}(t_n, x_{1,n}, y_n) - U(x_{1,n}, y_n)| \geq \varepsilon \quad \forall n \in \mathbb{N}.$$

Since $y_n \in \omega$, which is bounded, one may assume that y_n converges (up to subsequences) to $\zeta \in \overline{\omega}$. The locally uniform convergences yields $\lim_{n \to \infty} |x_{1,n}| = \infty$, therefore $\lim_{n \to \infty} U(x_{1,n}, y_n) = 0$ in both cases $c \geq c^*$ and $c < c^*$. Then we get

$$\liminf_{n \to \infty} \tilde{u}(t_n, x_{1,n}, y_n) \geq \varepsilon.$$

The standard parabolic estimates and compact injections again imply that $\tilde{u}(t + t_n, x_1 + x_{1,n}, y_n)$ converges (up to subsequences) to $\tilde{u}_\infty(t, x_1, \zeta)$ uniformly in $(-\rho, \rho) \times \Omega_\rho$, for any $\rho > 0$. In particular, $\tilde{u}_\infty$ satisfies $\tilde{u}_\infty(0, 0, \zeta) \geq \varepsilon$ and satisfies the following equation

$$\begin{cases} \partial_t \tilde{u}_\infty \leq \Delta \tilde{u}_\infty + c\partial_1 \tilde{u}_\infty + \mu(y)\tilde{u}_\infty & t \in \mathbb{R},\ x \in \Omega \\ \partial_\nu \tilde{u}_\infty = 0 & t \in \mathbb{R},\ x \in \partial\Omega. \end{cases} \tag{1.30}$$

By condition (1.9), there exists an eigenpair (λ_μ, φ) of Eq. (1.22) satisfying $\lambda_\mu > 0$. Setting $\omega(t, y) = S'' e^{-\lambda_\mu (t+h)} \varphi(y)$, we have

$$\partial_t \omega - \Delta \omega - c\partial_1 \omega - \mu(y)\omega \geq 0.$$

We know, by the Hopf lemma, that $\inf_\omega \varphi(y) > 0$. Hence, the function $W(t, x) = \tilde{u}_\infty(t, x) - \omega(t, y)$ satisfies $W(-h, x) \leq 0$ for S'' large enough. Let us call $\mathcal{L}_1 = \partial_t - \Delta - c\partial_1 - \mu(y)$, then

$$\begin{cases} \mathcal{L}_1 W \leq 0 & t \in \mathbb{R},\ x \in \Omega \\ \partial_\nu W \leq 0 & t \in \mathbb{R},\ x \in \partial\Omega. \end{cases}$$

Set $W(t, x) = z(t, x)\varphi(y)$ we have $\partial_\nu z \leq 0$ on $\partial\Omega$, $z(-h, x) \leq 0$ $\forall x \in \Omega$ and z satisfies

$$0 \geq \frac{\mathcal{L}_1 W}{\varphi} = z_t - \Delta z - \frac{2}{\varphi} \nabla_y \varphi \cdot \nabla_y z - c\partial_1 z + \lambda_\mu z.$$

Since $\lambda_\mu > 0$, the parabolic maximum principle implies that $z \leq 0$ in $(-\infty, -h) \times \overline{\Omega}$. As a consequence, one gets

$$0 < \varepsilon \leq \tilde{u}_\infty(0, 0, \zeta) \leq \lim_{h \to -\infty} \omega(0, \zeta) = 0.$$

This contradiction concludes the proof. □

The next result concerns the long time behavior of the solution of Eq. (1.4) in $L^1(\Omega)$. The central difficulty is to deal with the case $f_s(x,0)$ being sign-changing as $|x|$ large. To overcome it, we decompose solution of (1.4) into the sum of two integrable functions. The following lemma plays a key role.

Lemma 1.2.3. *Let $w(t,x)$ be a nonnegative bounded solution of*

$$\begin{cases} \partial_t w = \Delta w + c\partial_1 w + \zeta(t,x)w, & t>0,\ x\in\Omega \\ \partial_\nu w(t,x) = 0, & t>0,\ x\in\partial\Omega \end{cases} \tag{1.31}$$

with initial function $w(0,\cdot) = w_0 \in L^1(\Omega)\cap L^\infty(\Omega)$. We assume, in addition, that $\lim_{t\to\infty} w(t,x) = 0$, pointwise in $x\in\Omega$, $\zeta(x)\in L^\infty(\Omega)$ and that there exists $\mu\in L^\infty(\omega)$ satisfying

$$\mu(y) = \lim_{R\to\infty} \sup_{\substack{t>0\\ |x_1|\geq R}} \zeta(t,x_1,y), \quad \textit{and} \quad \lambda_N(-\Delta_y - \mu(y),\omega) > 0. \tag{1.32}$$

Then, there holds

$$\lim_{t\to\infty} \|w(t,x)\|_{L^1(\Omega)} = 0.$$

Proof of Lemma 1.2.3. From Eq. (1.31), for any $\delta>0$, we have

$$\partial_t w - \Delta w - c\partial_w - (\mu(y)+\delta)w = (\zeta(t,x) - \mu(y) - \delta)w.$$

Let us call

$$P := \partial_t - \Delta - c\partial_1 - \mu(y) - \delta, \qquad g(t,x) := (\zeta(t,x) - \mu(y) - \delta)w(t,x).$$

Then, we infer, from the superposition principle, that $w = w_1 + w_2$, where (w_1, w_2) is the solution of the system

$$\begin{cases} Pw_1 = 0 & t>0,\ x\in\Omega \\ \partial_\nu w_1 = 0 & t>0,\ x\in\partial\Omega, \end{cases} \qquad \begin{cases} Pw_2 = g(t,x) & t>0,\ x\in\Omega \\ \partial_\nu w_2 = 0 & t>0,\ x\in\partial\Omega, \end{cases}$$

with the initial condition $(w_1,w_2)(0,x) = (w_0(x),0)$. From condition (1.32), for any $\delta>0$, there exist $R>0$, such that

$$\zeta(t,x_1,y) \leq \mu(y) + \delta, \qquad \forall t>0, \forall (x_1,y)\in\Omega\setminus\Omega_R.$$

and there exists an eigenpair (λ_μ,φ) of Eq. (1.22) satisfying $\lambda_\mu > 0$.

Letting $\delta < \lambda_\mu$, we set $v_1(t,x) = e^{(\lambda_\mu-\delta)t}w_1(t,x)/\varphi(y)$. Then v_1 satisfies the equation

$$\begin{cases} \partial_t v_1 - \Delta_x v_1 - 2\nabla_y v_1 \cdot \dfrac{\nabla_y\varphi}{\varphi} - c\partial_1 v_1 \leq 0. & x\in\Omega. \\ \partial_\nu v_1 \geq 0 & x\in\partial\Omega. \end{cases} \tag{1.33}$$

By the Hopf lemma, we know that $\inf_{y\in\omega}\varphi > 0$, then $\|v_1(0,\cdot)\|_{L^\infty(\Omega)} \leq \|w_0\|_{L^\infty(\Omega)}/\inf_\omega\varphi$. Then, the parabolic maximum principle yields $\|v_1\|_{L^\infty(\Omega)} \leq \|w_0\|_{L^\infty(\Omega)}/\inf_\omega\varphi$. It follows immediately that $\|w_1(t,\cdot)\|_{L^\infty(\Omega)} \to 0$ as $t\to\infty$. On the other hand, set $v_1^r(t,\rho,y) = \int_{-r}^r v_1(t,x_1+\rho,y)dx_1$, we obtain

$$\partial_t v_1^r - \Delta v_1^r - 2\frac{\nabla_y\varphi}{\varphi}\cdot\nabla_y v_1^r(t,\rho,y) - c\partial_\rho v_1^r \leq 0 \qquad (\rho,y)\in\Omega. \tag{1.34}$$

Since $w_0(0,\cdot) \in L^1(\Omega) \cap L^\infty(\Omega)$, $v_1^r(0,\rho,y)$ is well-defined a.e on Ω, $\forall r > 0$. Moreover, there exists a constant M such that $v_1^r(0,\rho,y) \leq M$, a.e in Ω, $\forall r > 0$, M is a supersolution of Eq. (1.34). Then, we infer from the parabolic comparison principle [72], that

$$v_1^r(t,\cdot) \leq M, \quad \text{a.e in } \Omega, \forall t > 0, \forall r > 0.$$

Therefore, $\|v_1(t,\cdot)\|_{L^1(\Omega)} = \lim_{r\to\infty} \int_\omega \int_{-r}^{r} v_1(t, x_1+\rho, y) dx_1 dy \leq M|\omega|$, $\forall t > 0$. As a consequence, we get

$$\lim_{t\to\infty} \|w_1(t,\cdot)\|_{L^\infty(\Omega)} = \lim_{t\to\infty} \|w_1(t,\cdot)\|_{L^1(\Omega)} = 0.$$

On the other hand, by assumption, w is bounded, then w_2 is bounded and so it is integrable on any compact set. The same argumentation as of the Proposition 1.2.1 enables us to find supersolution of the problem satisfied by w_2 of the form $\xi(x_1,y) = Ce^{-\tau|x_1|}\varphi(y)$ such that $P(\xi) \geq 0 \geq g(t,x)$ in $\Omega \setminus \Omega_R$. We have $w_2(0,x) = 0 < \xi(x)$, $\partial_\nu w_2 \leq \partial_\nu \xi$ for $y \in \partial\omega$ and the fact that $\inf_\omega \varphi > 0$ allows us to find a constant C such that $w_2(\pm R, y) \leq Ce^{-\tau R}\varphi(y)$, $\forall y \in \omega$. Therefore, the parabolic maximum principle implies that $w_2(t,x) \leq \xi(x)$, $\forall x \in \Omega \setminus \overline{\Omega}_R$. Since w_2 is bounded, one can choose C large enough so that $w_2(t,x) \leq \xi(x)$, $\forall x \in \Omega$. Moreover,

$$\forall x \in \Omega \qquad \lim_{t\to\infty} w_2^+(t,x) = \lim_{t\to\infty} (w - w_1)^+(t,x) = 0.$$

Hence $0 \leq w = w_1 + w_2 \leq w_1 + w_2^+$, which is integrable on Ω. It follows from Lebesgue's dominated convergence theorem that $\lim_{t\to\infty} \|w(t,x)\|_{L^1(\Omega)} = 0$ because $\lim_{t\to\infty} w(t,x) = 0$ for $x \in \Omega$. We thus conclude the proof. □

We are now in a position to prove Theorem 1.1.5

Proof of Theorem 1.1.5. The proof is a direct consequence of Lemma 1.2.3. Let u be the solution of (1.4) with $u(0,x) = u_0(x) \in L^\infty(\Omega) \cap L^1(\Omega)$. The function $\tilde{u}(t,x_1,y) := u(t,x_1+ct,y)$ satisfies Eq. (1.31) with the same initial condition u_0. Let W be defined as following :

$$W(x) = \begin{cases} 0 & \text{if } c \geq c^* \\ U(x) & \text{if } c < c^*, \end{cases} \tag{1.35}$$

where $U(x)$ is the unique positive solution of Eq. (1.6) when $c < c^*$.

Let $\overline{u}$, $\underline{u}$ be respectively the solutions of (1.31) with initial conditions $\overline{u}(0,x) = \max\{u_0(x), W(x)\}$ and $\underline{u}(0,x) = \min\{u_0(x), W(x)\}$. We know, from Theorem 1.1.4 that the functions $\overline{u}(t,x_1-ct,y)$ and $\underline{u}(t,x_1-ct,y)$ converge to $W(x)$ as $t \to \infty$, uniformly with respect to $x \in \Omega$. Moreover, the parabolic maximum principle yields

$$\forall t > 0, x \in \Omega \quad \overline{u}(t,x) \geq \max\{\tilde{u}(t,x), W(x)\} \quad \underline{u}(t,x) \leq \min\{\tilde{u}(t,x), W(x)\}.$$

Therefore, the functions $\overline{w}(t,x) := \overline{u}(t,x) - W(x)$ and $\underline{w}(t,x) := W(x) - \underline{u}(t,x)$ is nonnegative bounded solution of Eq. (1.31) with

$$\overline{\zeta}(t,x) = \frac{f(x,\overline{u}) - f(x,W)}{\overline{u} - W} \quad ; \quad \underline{\zeta}(t,x) = \frac{f(x,W) - f(x,\underline{u})}{W - \underline{u}}.$$

Thanks to condition (1.8), one easily sees that $\overline{\zeta}$ and $\underline{\zeta}$ are less than $f_s(x,0)$. Thanks to condition (1.9), $\overline{\zeta}$ and $\underline{\zeta}$ satisfy (1.32). The initial conditions $\overline{w}(0,x), \underline{w}(0,x) \in L^1(\Omega)$ allow one to apply Lemma 1.2.3 to derive

$$\lim_{t\to\infty} \|\overline{u} - W\|_{L^1(\Omega)} = 0 \quad ; \quad \lim_{t\to\infty} \|W - \underline{u}\|_{L^1(\Omega)} = 0.$$

This completes the proof because $\underline{u} \leq \tilde{u} \leq \overline{u}$. □

The next section is of independent interest. We are concerned with the existence, uniqueness, long time behavior of pulsating fronts, which are T-periodic in t and periodic in y.

1.3 The partially periodic environment with time dependence

Before proving the main results, let us introduce some new definitions and preliminary results that are needed in this section.

Proposition 1.3.1. *Let $\mathcal{O}_r = \mathbb{R} \times (-r, r) \times \mathbb{R}^{N-1}$, then for any $r > 0$, there exists a unique real number λ_p such that the eigenvalue problem*

$$\begin{cases} \mathcal{L}\chi_r = \lambda_p(r)\chi_r & \text{a.e in } \mathcal{O}_r \\ \chi_r = 0 & \text{on } \partial\mathcal{O}_r \\ \chi_r \text{ is periodic both in } y \text{ and } t \end{cases}$$

admits a positive solution $\chi_r(t,x) \in C^{1,2}_{t,x}(\mathbb{R} \times (-r,r) \times \mathbb{R}^{N-1})$, where $\mathcal{L}$ is the parabolic operator of the form defined in (1.15). The function χ_r (unique up to a multiplication) is called the principal eigenfunction associated with eigenvalue $\lambda_p(r)$ of $\mathcal{L}$ in $\mathcal{O}_r$.

Proof. The existence and uniqueness of principal eigenvalue for time periodic operator with Dirichlet boundary condition have been derived in [65]. For the framework of space-time periodic operator, one can refer to the work of Nadin [76] and the proof of this theorem is essentially similar to the proof of Theorem 2.7, in [76]. We omit the proof here. □

Proposition 1.3.2. *There holds $\lambda_p(r) \to \lambda_p$ as $r \to \infty$, strictly decreasing in r, where $\lambda_p = \tilde{\lambda}_1(\mathcal{L}, \mathbb{R} \times \mathbb{R}^N)$ defined in (1.15). Moreover, there exists an eigenfunction $\chi \in C^{1,2}_{t,x}(\mathbb{R} \times \mathbb{R}^N)$ associated with $\tilde{\lambda}_1$ such that $\mathcal{L}\chi = \tilde{\lambda}_1\chi$ a.e in $\mathbb{R}^{N+1}$.*

Proof. Assume by contradiction that there exist $0 < r_1 < r_2$ but $\lambda_p(r_1) \geq \lambda_p(r_2)$. Thanks to the periodicity in y and t and the boundedness of domain in x_1, there exist $\alpha > 0$ and $x_0 \in \mathcal{O}_{r_1}$ such that $\alpha\chi_{r_1} \geq \chi_{r_2}$ and $\alpha\chi_{r_1}(x_0) = \chi_{r_2}(x_0)$. It follows immediately that

$$\mathcal{L}(\alpha\chi_{r_1} - \chi_{r_2}) \geq \lambda_p(r_1)(\alpha\chi_{r_1} - \chi_{r_2}) \qquad \text{a.e } x \in \mathcal{O}_{r_1}.$$

The parabolic strong maximum principle implies $\alpha\chi_{r_1} \equiv \chi_{r_2}$, contradiction. Moreover, it is easily seen, by strong maximum principle, that $\lambda_p(r) > -\sup_{\mathcal{O}_r} f_s(t,x,0) \geq -\sup_{\mathbb{R}\times\mathbb{R}^N} f_s(t,x,0)$, $\forall r > 0$. This implies that the limit $\lambda_p' = \lim_{r\to\infty} \lambda_p(r)$ does exist and bounded from below by λ_p.

Let us argue that there exists an eigenfunction $\chi \in C^{1,2}_{t,x}(\mathbb{R} \times \mathbb{R}^N)$ which is periodic in y and t such that $\mathcal{L}\chi = \lambda_p'\chi$. Indeed, thanks to the periodicity in y and t, one can apply the Harnack inequality for the family $(\chi(r))_r > 0$ with normalization $\chi_r(0,0) = 1$ to derive that it is uniformly bounded on any the compact set of $\mathbb{R} \times \mathbb{R}^N$. Then, the standard parabolic estimates imply that $\chi_r \to \chi$, as $r \to \infty$ locally uniformly in $\mathbb{R} \times \mathbb{R}^N$ and χ satisfies $\mathcal{L}\chi = \lambda_p'\chi$. Moreover, χ is strictly positive, periodic in y and t and satisfies $\chi(0,0) = 1$ by the strong maximum principle. Lastly, taking χ as a test super solution for λ_p, one finds $\lambda_p' = \lambda_p$. □

We are now able to prove Theorem 1.1.7.

Proof of Theorem 1.1.7. Recall that $\tilde{\lambda}_1 = \tilde{\lambda}_1(\mathcal{P}, \mathbb{R}\times\mathbb{R}^N)$. We first consider the case that $\tilde{\lambda}_1 < 0$. It then follows from Proposition 1.3.2 that for $r > 0$ large enough, $\lambda_p(r) < 0$, where $\lambda_p(r)$ is the space-time periodic principal eigenvalue on $\mathcal{O}_r$, with Dirichlet boundary condition, which is defined in Proposition 1.3.1. Let χ_r be an eigenfunction associated to $\lambda_p(r)$ in $\mathcal{O}_r$, we define the function :

$$\phi(t,x) = \begin{cases} \eta\chi_r(t,x) & x \in \mathcal{O}_r \\ 0 & \text{otherwise.} \end{cases}$$

For $\eta \in \mathbb{R}$ small enough, one obtains immediately that

$$\partial_t\phi - \Delta\phi - c\partial_1\phi - f(t,x,\phi) = (f_s(t,x,0) + \lambda_p(r))\phi_R - f(t,x,\phi) < 0.$$

That is, ϕ is a subsolution of Eq. (1.14) while the constant S given in (1.17) is a super solution of Eq. (1.14). Let us consider the solution u of Eq. (1.14) with the initial condition $u(0,x) = \phi(0,x)$. The standard parabolic theory and maximum principle imply that there exists such a solution for any $t > 0$ and satisfies $\phi(t,x) \le u(t,x) \le S$, $\forall (t,x) \in \mathbb{R}\times\mathbb{R}^{N+1}$. In particular, $\phi(T,x) \le u(T,x)$, where T is the period of ϕ and f with respect to t. Consider the function $u(t+T,x)$; it is also a solution of Eq. (1.14) with initial condition $u(T,x) \ge u(0,x)$, then $u(t+T,x) \ge u(t,x)$. In particular, $u(2T,x) \ge u(T,x)$. By induction, one sees that the sequence $u_n(t,x) = u(t+nT,x)$ is nondereasing in n and uniformly bounded by S. Therefore, $u_n(t,x)$ converges pointwise to a bounded function $U(t,x)$ such that U is T-periodic in t, $\phi \le U \le S$, U solves Eq. (1.14). The partial periodicity in y of solution follows from the construction.

Let us postpone for a moment the proof of the necessary condition to prove the uniqueness of the solution. We emphasize that the uniqueness of (1.14) is proved to hold in the class of positive bounded solutions without a-priori assuming to be periodic in y nor in t.

Assume by contradiction that $\overline{U}$ and $\underline{U}$ are two positive bounded solutions of (1.14), Theorem A.2, Appendix, yields

$$\lim_{|x_1|\to\infty} \overline{U}(t,x_1,y) = \lim_{|x_1|\to\infty} \underline{U}(t,x_1,y) = 0,$$

uniformly in y and t. By condition (1.19), there exists a pair (λ,ϕ) such that $\lambda > 0, \phi > 0$, ϕ is periodic in both y and t and $\partial_t\phi - \Delta\phi - \gamma(t,y)\phi \ge \lambda\phi$ in $\mathbb{R}\times\mathbb{R}^{N-1}$. Thanks to the periodicity, we have $\inf_{\mathbb{R}^N} \varphi(t,y) > 0$. For any $\varepsilon > 0$, there exists $R(\varepsilon) > 0$ such that

$$\underline{U}(t,x_1,y) \le \varepsilon\varphi(t,y), \qquad \forall\, |x_1| \ge R(\varepsilon), y \in \mathbb{R}^{N-1}, t \in \mathbb{R}, \tag{1.36}$$

and therefore, the set

$$K_\varepsilon := \{k > 0 : k\overline{U} \ge \underline{U} - \varepsilon\varphi \text{ in } \mathbb{R}\times\mathbb{R}^N\}$$

is nonempty. Set $k(\varepsilon) := \inf K_\varepsilon$. Obviously, the function $k(\varepsilon) : \mathbb{R}^+ \to \mathbb{R}$ is nonincreasing. Assume by way of contradiction that

$$k^* = \lim_{\varepsilon\to 0^+} k(\varepsilon) > 1$$

Take $0 < \varepsilon < \sup_{\mathbb{R}^{N+1}} \underline{U}/\varphi$, we see that $k(\varepsilon) > 0$, $k(\varepsilon)\overline{U} - \underline{U} + \varepsilon\varphi \ge 0$. The definition of $k(\varepsilon)$ yields that there exists a sequence $(t_n^\varepsilon, x_{1,n}^\varepsilon, y_n^\varepsilon)$ in $\overline{\Omega}$ such that

$$\left(k(\varepsilon) - \frac{1}{n}\right)\overline{U}(t_n^\varepsilon, x_{1,n}^\varepsilon, y_n^\varepsilon) < \underline{U}(t_n^\varepsilon, x_{1,n}^\varepsilon, y_n^\varepsilon) - \varepsilon\varphi(t_n^\varepsilon, y_n^\varepsilon). \tag{1.37}$$

From (2.27), we have $(t_n^\varepsilon, x_{1,n}^\varepsilon, y_n^\varepsilon) \in \mathcal{O}_{R(\varepsilon)}$ for n large enough. Taking the sequences (τ_n^ε) and (z_n^ε) such that $t_n^\varepsilon - \tau_n^\varepsilon \in [0,T)$ and $y_n^\varepsilon - z_n^\varepsilon \in [0,L_2) \times ... \times [0,L_N)$. For any $\varepsilon > 0$, one sees that $\overline{U}_n^\varepsilon(t,x_1,y) = \overline{U}(t+\tau_n^\varepsilon, x_1, y+z_n^\varepsilon)$ and $\underline{U}_n^\varepsilon(t,x_1,y) = \underline{U}(t+\tau_n^\varepsilon, x_1, y+z_n^\varepsilon)$ are solutions of the following equation

$$\partial_t U_n^\varepsilon - \Delta U_n^\varepsilon - c\partial_1 U_n^\varepsilon = f(t+\tau_n^\varepsilon, x_1, y+z_n^\varepsilon, U_n^\varepsilon).$$

Using the priori estimates of solutions (Theorem A.2, Appendix), we deduce that as $n \to \infty$, up to extractions, $\overline{U}_n^\varepsilon \to \overline{U}_\infty^\varepsilon$ and $\underline{U}_n^\varepsilon \to \underline{U}_\infty^\varepsilon$ locally uniformly in $\mathbb{R}^{N+1}$. Moreover, since f is periodic in y and T-periodic in t, there exist τ_∞^ε, z_∞^ε such that by the standard parabolic estimates, we know that $\overline{U}_\infty^\varepsilon$ and $\underline{U}_\infty^\varepsilon$ are the solutions of :

$$\partial_t U_\infty^\varepsilon - \Delta U_\infty^\varepsilon - c\partial_1 U_\infty^\varepsilon = f(t+\tau_\infty^\varepsilon, x_1, y+z_\infty^\varepsilon, U_\infty^\varepsilon).$$

By passing to the limit in (1.37), $W_\infty^\varepsilon = k(\varepsilon)\overline{U}_\infty^\varepsilon - \underline{U}_\infty^\varepsilon + \varepsilon\varphi_\infty^\varepsilon \geq 0$, where $\varphi_\infty^\varepsilon(t,y) = \lim_{n\to\infty}\varphi(t+\tau_\infty^\varepsilon, y+z_n^\varepsilon)$ satisfying $\partial_t\phi_\infty^\varepsilon - \Delta\phi_\infty^\varepsilon - \gamma_\infty^\varepsilon(t,y)\phi_\infty^\varepsilon \geq \lambda\phi_\infty^\varepsilon$, with $\gamma_\infty^\varepsilon(t,y) = \gamma(t+\tau_\infty^\varepsilon, y+z_\infty^\varepsilon)$, ϕ_∞^ε is periodic in y and t, and $\lambda > 0$. Moreover, there exist $(t(\varepsilon), x_1(\varepsilon), y(\varepsilon))$ such that

$$(k(\varepsilon)\overline{U}_\infty^\varepsilon - \underline{U}_\infty^\varepsilon + \varepsilon\varphi_\infty^\varepsilon)(t(\varepsilon), x_1(\varepsilon), y(\varepsilon)) = 0. \tag{1.38}$$

Note that $t(\varepsilon) \in [0,T)$ and $y(\varepsilon) \in [0,L_2) \times ... \times [0,L_N)$ are bounded with respect to ε. The case that $\liminf_{\varepsilon\to 0^+} |x_1(\varepsilon)| < \infty$ is ruled out. Indeed, if $\liminf_{\varepsilon\to 0^+} |x_1(\varepsilon)| < \infty$, there exists a sequence $(\varepsilon_n) \to 0$ as $n \to \infty$ such that $(t(\varepsilon_n), x_1(\varepsilon_n), y(\varepsilon_n)) \to (t_0, x_0, y_0)$ as $n \to \infty$, up to subsequences. Moreover, by the partial periodicity in y and the periodicity in t of f, the standard parabolic estimates yield that $\overline{U}_\infty^{\varepsilon_n} \to \overline{U}_\infty^0$ and $\underline{U}_\infty^{\varepsilon_n} \to \underline{U}_\infty^0$ locally uniformly in $\mathbb{R}^{N+1}$ satisfying the following equation

$$\partial_t U_\infty^0 - \Delta U_\infty^0 - c\partial_1 U_\infty^0 = f(t+t_\infty^0, x_1, y+z_\infty^0, U_\infty^0).$$

for some $t_\infty^0 \in [0,T)$ and $z_\infty^0 \in [0,L_2)\times...\times[0,L_N)$. From (2.28) $k^* < \infty$, then the function $W = k^*\overline{U}_\infty^0 - \underline{U}_\infty^0$ is nonnegative and vanishes at (t_0,x_0,y_0). Since $k^* > 1$, The Lipschitz continuity of f with respect to s and condition (1.18) yield

$$\begin{aligned}
\partial_t W - \Delta W - c\partial_1 W &\geq k^* f(t+t_\infty^0, x_1, y+z_\infty^0, \overline{U}_\infty^0) - f(t+t_\infty^0, x_1, y+z_\infty^0, \underline{U}_\infty^0) \\
&\geq f(t+t_\infty^0, x_1, y+z_\infty^0, k^*\overline{U}_\infty^0) - f(t+t_\infty^0, x_1, y+z_\infty^0, \underline{U}_\infty^0) \\
&\geq z(t+t_\infty^0, x_1, y+z_\infty^0)W,
\end{aligned} \tag{1.39}$$

where $z(t,x) \in L^\infty_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^N)$. Hence, the parabolic strong maximum principle implies $W = 0$ in $(-\infty,t_0)\times\mathbb{R}^N$. This is a contradiction because from condition (1.18), the inequality (1.39) hold strictly in some $D \subset (-\infty,t_0)\times\mathbb{R}^N$ with $|D| > 0$. It remains to consider the case $\lim_{\varepsilon\to 0^+} |x_1(\varepsilon)| = \infty$. We have shown that $W_\infty^\varepsilon \geq 0$ and W_∞^ε vanishes at $(t(\varepsilon), x_1(\varepsilon), y(\varepsilon))$, we infer that there exists a neighborhood $\mathcal{O}$ of $(t(\varepsilon), x_1(\varepsilon), y(\varepsilon))$ such that $k(\varepsilon)\overline{U} < \underline{U}$ in $\mathcal{O}$, shrinking $\mathcal{O}$ if necessary. Since $k^* > 1$, for ε small enough, $k(\varepsilon) > 1$, we derive from (1.18) for $x \in \mathcal{O}$

$$\begin{aligned}
\partial_t W_\infty^\varepsilon &- \Delta W_\infty^\varepsilon - c\partial_1 W_\infty^\varepsilon \\
&\geq k(\varepsilon) f(t+\tau_\infty^\varepsilon, x_1, y+z_\infty^\varepsilon, \overline{U}_\infty^\varepsilon) - f(t+\tau_\infty^\varepsilon, x_1, y+z_\infty^\varepsilon, \underline{U}_\infty^\varepsilon) + (\gamma(t+\tau_\infty^\varepsilon, y+z_\infty^\varepsilon)+\lambda)\varepsilon\varphi_\infty^\varepsilon \\
&\geq f(t+\tau_\infty^\varepsilon, x_1, y+z_\infty^\varepsilon, k(\varepsilon)\overline{U}_\infty^\varepsilon) - f(t+\tau_\infty^\varepsilon, x_1, y+z_\infty^\varepsilon, \underline{U}_\infty^\varepsilon) + (\gamma(t+\tau_\infty^\varepsilon, y+z_\infty^\varepsilon)+\lambda)\varepsilon\varphi_\infty^\varepsilon \\
&\geq \frac{f(t+\tau_\infty^\varepsilon, x_1, y+z_\infty^\varepsilon, k(\varepsilon)\overline{U}_\infty^\varepsilon)}{k(\varepsilon)\overline{U}_\infty^\varepsilon}(k(\varepsilon)\overline{U}_\infty^\varepsilon - \underline{U}_\infty^\varepsilon + \varepsilon\varphi_\infty^\varepsilon) + \frac{\lambda}{2}\varepsilon\varphi_\infty^\varepsilon \\
&+ \left(\frac{\lambda}{2} + \gamma(t+\tau_\infty^\varepsilon, y+z_\infty^\varepsilon) - \frac{f(t+\tau_\infty^\varepsilon, x_1, y+z_\infty^\varepsilon, k(\varepsilon)\overline{U}_\infty^\varepsilon)}{k(\varepsilon)\overline{U}_\infty^\varepsilon}\right)\varepsilon\varphi_\infty^\varepsilon.
\end{aligned} \tag{1.40}$$

Condition (1.19) yields, for ε small enough, that

$$\frac{f(t+\tau^\varepsilon_\infty, x_1, y+z^\varepsilon_\infty, k(\varepsilon)\overline{U}^\varepsilon_\infty)}{k(\varepsilon)\overline{U}^\varepsilon_\infty} < \gamma^\varepsilon_\infty(t,y) + \frac{\lambda}{2}, \qquad \forall (x_1,y)\in\mathcal{O}.$$

Then, it follows from (1.40) that

$$\partial_t W^\varepsilon_\infty - \Delta W^\varepsilon_\infty - c\partial_1 W^\varepsilon_\infty - \varrho(t+\tau^\varepsilon_\infty, x_1, y+z^\varepsilon_\infty)W^\varepsilon_\infty > \frac{\lambda}{2}\varepsilon\varphi^\varepsilon_\infty > 0 \quad \text{in } \mathcal{O},$$

where $\varrho(t,x) = \frac{f(t,x,k(\varepsilon)\overline{U})}{k(\varepsilon)\overline{U}}$ is bounded. This is a contradiction because the strong maximum principle asserts that $W^\varepsilon(t,x_1,y)=0$ in $\mathcal{O}$. As a consequence, we have proved that $k^* = \lim\limits_{\varepsilon\to 0^+} k(\varepsilon) \le 1$. Therefore

$$\underline{U} \le \lim_{\varepsilon\to 0^+}(k(\varepsilon)\overline{U} + \varepsilon\varphi) \le \overline{U} \quad \text{in } \mathbb{R}^{N+1}.$$

We derive the uniqueness due to the equivalence of $\overline{U}$ and $\underline{U}$. Note that we do not use the periodicity of y and t of solution in the proof of uniqueness. This thus infers that any positive bounded solutions of Eq. (1.14) must be periodic in y and T-periodic in t.

To conclude the proof of Theorem 1.1.7, it only remains to prove the necessary condition. Assume by contradiction that $\tilde{\lambda}_1 \ge 0$ and Eq. (1.14) admits a solution U, which is T-periodic in t but not necessarily periodic in y. Let χ be a principal eigenfunction associated with $\tilde{\lambda}_1$ (Proposition 1.3.2) with normalization $\chi(0,0) < U(0,0)$. Then

$$\partial_t\chi - \Delta\chi - c\partial_1\chi - f(t,x,\chi) = \tilde{\lambda}_1\chi + f_s(t,x,0)\chi - f(t,x,\chi) \ge 0.$$

Arguing similarly as the proof of uniqueness, we achieve the contradiction : $U \le \chi$ in $\mathbb{R}^{N+1}$. □

Before investigating the long time behavior, we point out that the monotonicity in time of solutions starting by a stationary sub (or super) solution of parabolic operator with time-dependent coefficients no longer holds. In addition, the boundedness of initial datum does not suffice to guarantee that the solutions of Eq. (1.13) converge uniformly to the unique solution of Eq. (1.14) as $t\to\infty$. However, thanks to the periodicity in t of solutions, which obtained by the uniqueness, we will have the locally uniform convergence and under some extra conditions (part (ii), Theorem 1.1.8) we can actually derive the uniform convergence as $t\to\infty$.

Proof of Theorem 1.1.8. Set $S' := \max\{S, \|u_0\|_{L^\infty(\Omega)}\}$, S is the positive constant given in (1.17). Then, the function $\tilde{u}(t,x) = u(t, x_1+ct, y)$ satisfies $0 < \tilde{u} \le S'$ in $\mathbb{R}^+\times\mathbb{R}^N$ and solves

$$\partial_t\tilde{u} = \Delta\tilde{u} + c\partial_1\tilde{u} + f(t,x,\tilde{u}) \quad t>0, x\in\mathbb{R}^N, \tag{1.41}$$

with initial condition $\tilde{u}(0,u) = u_0(x)$. Let w be the solution to (1.41) with initial condition $w_0(x) = S'$. Clearly, the constant S' is T-periodic in t and periodic in y. Arguing as the proof of Theorem 1.1.7, we deduce that the sequence $w_n(t,x) = w(t+nT,x)$ is nonincreasing and converges locally uniformly to $W(t,x)$, which is a solution of

$$\partial_t W - \Delta W - c\partial_1 W - f(t,x,W) = 0 \quad \forall t>0,\, x\in\mathbb{R}^N. \tag{1.42}$$

Moreover, $W(t,x)$ is T periodic in t and periodic in y. Then

$$\forall r>0, \quad \lim_{t\to\infty}\sup_{x\in\mathcal{O}_r}(\tilde{u}(t,x)-W(t,x))\leq\lim_{t\to\infty}\sup_{x\in\mathcal{O}_r}(w(t,x)-W(t,x))=0.$$

If $\tilde{\lambda}\geq 0$, then $W\equiv 0$ in $\mathbb{R}\times\mathbb{R}^N$. Therefore, $\tilde{u}(t,x)\to 0$ as $t\to\infty$ locally uniformly with respect to $x\in\mathbb{R}^N$. This convergence is uniform in $x\in\mathbb{R}^N$ due to the following claim.

Claim.

$$\lim_{\min(t,|x_1|)\to\infty}\tilde{u}(t,x_1,y)=0 \quad \text{uniformly in } y\in\mathbb{R}^{N-1}. \tag{1.43}$$

Let us postpone for a moment the proof of claim to consider the case $\tilde{\lambda}_1<0$. From Propositions 1.3.1 and 1.3.2, there exists $\rho>0$ such that $\lambda_p(\rho)<0$. Let $\chi_\rho(t,x)$ is an associated principal eigenfunction of $\lambda_p(\rho)$, for $\gamma>0$ small enough, one sees that the function

$$V(t,x)=\begin{cases}\gamma\chi_\rho(t,x) & x\in\mathcal{O}_\rho\\ 0 & \text{otherwise}\end{cases}$$

is a subsolution of Eq. (1.42). Then if (1.20) holds, there exist γ small enough such that $V(0,x)\leq u_0(x)$. Alternatively, $u_0(x)$ is periodic in y, then $\tilde{u}(t,x)$ is strictly positive, periodic in y and T-periodic in t, then the parabolic strong maximum principle yields $V(T,x)\leq\tilde{u}(T,x)$. In both case, we always can define $\tilde{v}(t,x)$ is such that $\tilde{v}(0,x)=V(0,x)$ or $\tilde{v}(T,x)=V(T,x)$. It follows immediately by parabolic maximum principle that $\tilde{v}(t,x)\leq u(t,x)$, $\forall t>T, x\in\mathbb{R}^N$. Arguing similarly as the proof of Theorem 1.1.7, we deduce that the sequence $v_n(t,x)=\tilde{v}(t+nT,x)$ is nondecreasing and converges locally uniformly to $P(t,x)$, which is a solution of

$$\partial_t P-\Delta P-c\partial_1 P-f(t,x,P)=0 \quad \forall t>0,\, x\in\mathbb{R}^N.$$

Moreover, P is strictly positive, periodic in y and T-periodic in t. The uniqueness of Theorem 1.1.7 implies that $W=P=U$ in $\mathbb{R}^{N+1}$, which is a solution of Eq. (1.14). Assume by contradiction that this convergence is not uniform in x, this means that there exist $\varepsilon>0$ and a sequence $(t_n,x_{1,n},y_n)\in\mathbb{R}^+\times\mathbb{R}^N$ such that

$$\lim_{n\to\infty}t_n=\infty, \quad \forall n\in\mathbb{N}, \quad |\tilde{u}(t_n,x_{1,n},y_n)-U(x_{1,n},y_n)|\geq\varepsilon. \tag{1.44}$$

Due to the locally uniform convergence of $\tilde{u}$, necessarily, the sequence $(x_{1,n})$ is unbounded. We get, by the a priori estimates of U, that $U(x_{1,n},y_n)\to 0$ as $n\to\infty$, uniformly in y. But from (1.44), this inference contradicts the claim (1.43). Thus, to conclude the proof, it remains to prove the Claim 1.43.

Let us call $(t_n,x_{1,n},y_n)\in\mathbb{R}^+\times\mathbb{R}^N$ be a sequence such that the claim (1.43) is not true :

$$\lim_{n\to\infty}t_n=\lim_{n\to\infty}x_{1,n}=\infty, \quad \forall n\in\mathbb{N}, \quad \liminf_{n\to\infty}\tilde{u}(t_n,x_{1,n},y_n)\geq\varepsilon \quad \text{for some } \varepsilon>0.$$

For any $n\in\mathbb{N}$, we define the functions $\tilde{u}_n(t,x)=\tilde{u}(t+t_n,x_1+x_{1,n},y+y_n)$. It holds that $0\leq\tilde{u}_n\leq S'$ and

$$\partial_t\tilde{u}_n=\Delta\tilde{u}_n+c\partial_1\tilde{u}_n+f(t+t_n,x_1+x_{1,n},y+y_n,\tilde{u}_n) \qquad t>-t^n,\, x\in\mathbb{R}^N.$$

Since f is periodic in y and T-periodic in t, we can assume without loss of generality that $t_n\to t_0$ and $y_n\to y_0$ (up to a subsequence) as $n\to\infty$. Thanks to (1.18)-(1.19), we deduce, by the parabolic estimates and embedding theorems that $\tilde{u}_n$ converges (up to a subsequence) to $\tilde{u}_\infty$ locally uniformly in $\mathbb{R}\times\mathbb{R}^N$ satisfying

$$\partial_t\tilde{u}_\infty\leq\Delta\tilde{u}_\infty+c\partial_1\tilde{u}_\infty+\gamma(t+t_0,y+y_0)\tilde{u}_\infty \qquad \forall t\in\mathbb{R},\, x\in\mathbb{R}^N$$

and $\tilde{u}_\infty(0,0) \geq \varepsilon$. Moreover, condition (1.19) implies that there exists a pair (λ, ϕ) such that $\partial_t\phi - \Delta\phi - \gamma(t+t_0, y+y_0)\phi \geq \lambda\phi$, ϕ is periodic in y and t, and $\lambda > 0$. Let us define the function $\omega(t,x) = S'\phi(t,y)e^{-\lambda(t-h)}$ we have

$$\partial_t\omega \geq \Delta\omega + c\partial_1\omega + \gamma(t+t_0, y+y_0)\omega \qquad (t,x) \in \mathbb{R}^{N+1}.$$

Then, set $\widetilde{\mathcal{L}} = \partial_t - \Delta - c\partial_1 - \gamma(t+t_0, y+y_0)$, $W(t,x) = \omega - \tilde{u}_\infty$ and $W(t,x) = z(t,x)\phi(t,y)$, we obtain

$$0 \leq \frac{\widetilde{\mathcal{L}}W}{\phi} = z_t - \Delta z - \nabla_y z \cdot \frac{\nabla_y \phi}{\phi} + \lambda z.$$

Since the periodicity yields $\inf_{\mathbb{R}^N} \phi(t,y) > 0$ and $\tilde{u} \leq S$, we can enlarge S' so that $z(h,x) \geq 0$ in $\mathbb{R}^N$. The parabolic comparison principle yields $z(t,x) \geq 0$, $\forall t \leq h, x \in \mathbb{R}^N$. As a consequence

$$\varepsilon \leq \tilde{u}(0,0) = S'\phi(0,0)e^{\lambda h}.$$

Letting h to $-\infty$, we get a contradiction. This concludes the proof.

□

1.4 Further results and applications

1.4.1 Similarity of the problem with Dirichlet boundary condition

In this subsection, we aim at proving an analogous result of Theorem 1.1.1, where Dirichlet instead of Neumann condition is imposed on the boundary of Ω. This is to prepare for the main goal in the next subsection. To this end, let us define the generalized Dirichlet principal eigenvalue

$$\lambda_D(-\mathcal{L}, \Omega) := \sup\{\lambda \in \mathbb{R} : \exists \phi \in W^{2,N}_{loc}(\Omega), \phi > 0, (\mathcal{L}+\lambda)\phi \leq 0 \text{ a.e in } \Omega \text{ and } \phi = 0 \text{ on } \partial\Omega\}. \tag{1.45}$$

Note that, if Ω is bounded, λ_D coincides with the classical Dirichlet eigenvalue ([20]). Similar to assumption (1.9), we assume that there exists a measurable bounded function $\mu : \omega \to \mathbb{R}$ such that

$$\mu(y) = \limsup_{|x_1|\to\infty} f_s(x_1, y, 0) \qquad \text{and} \qquad \lambda_D(-\Delta_y - \mu(y), \omega) > 0. \tag{1.46}$$

Then, we obtain an analogue of Theorem 1.1.1 as following

Theorem 1.4.1. *Assume that conditions (1.7)-(1.8) and (1.46) are satisfied. Then there exists a unique critical speed c^* such that the equation*

$$\begin{cases} \Delta U + c\partial_1 U + f(x,U) = 0 & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega \\ U > 0 \text{ in } \Omega \\ U \text{ is bounded.} \end{cases} \tag{1.47}$$

admits a solution if and only if $0 \leq c < c^*$. Moreover, the solution is unique when it exists.

Proof. The proof of this theorem is essentially similar to the proof of Theorem 1.1.1. However, there are significant differences to be outlined here:

i) Existence.

Since the problem is set up with the Dirichlet boundary condition, the Proposition 1, in [19], cannot be applied. However, the Dirichlet boundary condition allows us to use Theorem 1.9, [20] to prove the existence of Eq. (1.47). Indeed, let us call $\lambda_D = \lambda_D(-\Delta - c\partial_1 - f_s(x,0), \Omega)$ and let c^* be defined as Definition 1.2, then the Proposition 1.3 yields that $\lambda_D < 0$ iff $0 \leq c < c^*$. Let (λ_R, φ_R) be the Dirichlet principal eigenvalue and eigenfunction of the problem

$$\begin{cases} -\Delta\varphi_R - c\partial_1\varphi_R - f_s(x,0)\varphi_R = \lambda_R\varphi_R & x \in \Omega_R \\ \varphi_R(x_1, y) = 0 & |x_1| < R, y \in \partial\omega \\ \varphi_R(\pm R, y) = 0 & y \in \omega, \end{cases}$$

we deduce, by Theorem 1.9 in [20], that $\lambda_D = \lim_{R\to\infty} \lambda_R$. Note that Theorem 1.9 in [20] also deals with the case of nonsmooth domain as the set Ω_R of ours. Then the existence of Eq. (1.47) can be obtained in the same way with Theorem 1.1.1.

ii) Nonexistence and Uniqueness.

Let $\tilde{U}(x) = U(x)e^{\frac{c}{2}x_1}$, then U solves Eq. (1.47) if and only if $\tilde{U}$ solves the equation

$$\begin{cases} \Delta\tilde{U} + f(x_1, y, \tilde{U}(x_1, y)e^{-\frac{c}{2}x_1})e^{\frac{c}{2}x_1} - \dfrac{c^2}{4}\tilde{U} = 0 & x \in \Omega \\ \tilde{U} = 0 & x \in \partial\Omega \\ \tilde{U} > 0 \text{ in } \Omega \\ \tilde{U}(x_1, y)e^{-\frac{c}{2}x_1} \text{ is bounded.} \end{cases} \tag{1.48}$$

The argument of Theorem 1.1.1 for the nonexistence result can be applied if one can prove that $\tilde{U}$ decays exponentially as $|x_1| \to \infty$. Moreover, if $\tilde{U}$ decays exponentially, the uniqueness can be obtained by using variational argument as Theorem 2.3 in [18]. Note that the sliding argument as of Theorem 1.1.1 does not work in this case due to the lack of the Hopf lemma under Dirichlet condition. We end the proof by showing that $\tilde{U}$ really decays exponentially.

From assumption (1.46), we know that for any $\delta > 0$ there exists $R = R(\delta) > 0$ such that $f_s(x_1, y, 0) \leq \mu(y) + \delta$ when $|x_1| \geq R$. Since ω is bounded, $\lambda_D(-\Delta_y - \mu(y), \omega)$ coincides with the classical Dirichlet principal eigenvalue in ω, says λ. There exists an eigenfunction $\phi \in L^\infty(\omega)$, associated with λ, positive in ω, such that

$$\Delta_y\phi + \mu(y)\phi + \lambda\phi = 0 \quad \text{in } \omega \qquad ; \qquad \phi = 0 \quad \text{on } \partial\omega.$$

One can actually find a function $\tilde{\phi} > 0$ in $\overline{\omega}$ such that $\Delta_y\tilde{\phi} + (\tilde{\mu}(y) + \lambda)\tilde{\phi} = 0$ in ω, with $\tilde{\mu}(y)$ sufficiently close to $\mu(y)$ and $\sup_{\overline{\omega}} |\tilde{\mu} - \mu| < \tilde{\delta}$, which is chosen later. Set $\mathcal{L} + \delta = \Delta + \mu(y) - c^2/4 + \delta$, conditions (1.8) and (1.46) yield that for any $\delta > 0$ there exists $R = R(\delta) > 0$ such that $(\mathcal{L} + \delta)\tilde{U} \geq 0$ in $\Omega \setminus \Omega_R$. On the other hand, we set $w(x) = C\theta_a(x_1)\tilde{\phi}(y)$, where θ_a is the solution of equation

$$\begin{cases} \theta_a'' = (\kappa + \delta)\theta_a & \text{in } (R, R + a) \\ \theta_a(R) = Ce^{\sqrt{\kappa}R} \\ \theta_a(R + a) = Ce^{\sqrt{\kappa}(R+a)}, \end{cases}$$

with $\kappa = \lambda/2 + c^2/4 - 2\delta$ and $C = \sup_\Omega \tilde{U}(x)e^{-\frac{c}{2}x_1} / \inf_{\overline{\omega}} \tilde{\phi}$. Choosing $\tilde{\delta} < \lambda/2$, direct computation yields $(\mathcal{L} + \delta)w \leq 0$ in $\overline{\Omega \setminus \Omega_R}$. The same argumentation of Proposition 1.2.1 enables us to conclude that $U(x_1, y) \leq Ce^{-(\sqrt{\kappa} - c/2)|x_1|}\tilde{\phi}(y)$ in Ω. This concludes the proof of theorem. □

By this preliminary, we are ready to present the main result of this section.

1.4.2 Concentration of species in the more favorable region

Based on the characterizations of the persistence and extinction of the species in the cylindrical domain Ω under Dirichlet boundary condition, we study the behavior of the species when a part of their habitat changes to be extremely unfavorable. More precisely, we consider Eq. (1.47) in the whole space $\mathbb{R}^N$ with two disjointed regions : the cylindrical domain Ω as in Subsection 4.1 and its complement $\Omega^c = \mathbb{R}^N \setminus \Omega$. We shall prove that the species concentrate in the more favorable zone, Ω, and the annihilation occurs in the dead zone Ω^c if the death rate in Ω^c becomes extremely high. Our goal is to characterize the limit of the sequence $U_n(x)$, which are solutions of the equations

$$\begin{cases} \Delta U_n + c\partial_1 U_n + F_n(x, U_n) = 0 \quad x \in \mathbb{R}^N, \\ U_n > 0 \text{ and bounded in } \mathbb{R}^N. \end{cases} \tag{1.49}$$

For any n, the nonlinearities $F_n(x, s)$ are assumed to be continuous with respect to x and of class C^1 with respect to s, $F_n(x, 0) = 0$, $\forall x \in \mathbb{R}^N$. Moreover, $F_n(x, s)$ are assumed to satisfy

$$\exists S > 0 \text{ such that } F_n(x, s) \leq 0 \text{ for } s \geq S, \forall x \in \mathbb{R}^N, \tag{1.50}$$

$$\begin{gathered} s \to F_n(x, s)/s \text{ is nonincreasing a.e in } \mathbb{R}^N \text{ and there exist } D \subset \mathbb{R}^N, |D| > 0 \\ \text{such that it is strictly decreasing in } D. \end{gathered} \tag{1.51}$$

Let $f(x, s) : \Omega \times [0, +\infty) \mapsto \mathbb{R}$ satisfy (1.7), (1.8), (1.46), $\rho_n(x) = \frac{\partial F_n}{\partial s}(x, 0)$, we assume further that:

$$\begin{cases} F_n(x, s) = f(x, s) & x \in \Omega,\ s \in \mathbb{R}^+ \text{ for all } n \in \mathbb{N} \\ F_n(x, s) \text{ and } \rho_n(x) \text{ are nonincreasing in } n & \forall (x, s) \in \overline{\Omega} \times \mathbb{R}^+ \\ \rho_n(x) \to -\infty \text{ as } n \to \infty & \text{locally uniformly in } \Omega^c. \end{cases} \tag{1.52}$$

Before stating the result, let us briefly explain the meaning of this condition and of our achievement. This condition means that the environment of the species outside Ω is unfavorable and it becomes extremely unfavorable as $n \to \infty$. Our result confirms that no species can persist outside Ω under such condition as $n \to \infty$. As is proved in Subsection 4.1 that the species is persistent in Ω if and only if $0 \leq c < c^*$. In the following result, we will see that as $n \to \infty$ the species can only persist in Ω and it is immediately mortal outside Ω. Moreover, we will prove that the limit as $n \to \infty$ coincides with the unique solution of Eq. (1.47) in Ω and zero in Ω^c.

We derive the following result :

Theorem 1.4.2. *Let $U_n(x)$ be the sequence of of traveling front solution of Eq. (1.49) with $F_n(x, s)$ satisfies (1.50)-(1.52). If $c < c^*$ with c^* is given in Theorem 1.4.1, then the following limit holds*

$$U_n(x) \to U_\infty(x) \quad \text{as } n \to \infty,$$

uniformly for $x \in \mathbb{R}^N$, where $U_\infty \in W^{2,N}(\mathbb{R}^N)$ is nonnegative, vanishing in Ω^c and coincides with the unique positive solution of the following equation

$$\begin{cases} \Delta U + c\partial_1 U + f(x, U) = 0 & x \in \Omega \\ U(x) = 0 & x \in \partial\Omega. \end{cases} \tag{1.53}$$

Proof. Some arguments in the proof are inspired from [63].

Let us call $\mathcal{L}_n = \Delta + c\partial_1 + \rho_n(x)$, defined in $\mathbb{R}^N$, and the generalized principal eigenvalue of $\mathcal{L}_n$ as following

$$\lambda_n = \sup\{\lambda \in \mathbb{R} : \exists \phi \in W^{2,N}_{loc}(\mathbb{R}^N), \phi > 0, (\mathcal{L}_n + \lambda)\phi \leq 0 \text{ a.e in } \mathbb{R}^N\}.$$

We also denote by λ_D the generalized Dirichlet principal eigenvalue of the operator $\Delta + c\partial_1 + f_s(x,0)$ in Ω defined by (1.45). We have the following lemma:

Lemma 1.4.3. *There holds that λ_n converges increasingly to λ_D.*

Let us postpone the proof of this lemma for a moment to continue the proof of theorem. As is known, $c < c^*$ if and only if $\lambda_D < 0$. By Lemma 1.4.3 and assumption, we have $\lambda_n < \lambda_D < 0$. Then Theorem 1.1 [18] yields that the equation

$$\begin{cases} \Delta U_n + c\partial_1 U_n + F_n(x, U_n) = 0 & x \in \mathbb{R}^N \\ 0 \leq U_n \leq \bar{S} & x \in \mathbb{R}^N. \end{cases} \tag{1.54}$$

admits a strictly positive solution U_n. Let $V_n(x_1, y) = U_n(x_1, y)e^{\frac{c}{2}x_1}$, then U_n is a solution of Eq. (1.54) if and only if V_n is a solution of

$$\begin{cases} \Delta V_n + F_n(x_1, y, V_n(x_1, y)e^{-\frac{c}{2}x_1})e^{\frac{c}{2}x_1} - \dfrac{c^2}{4}V_n = 0 \quad x \in \mathbb{R}^N \\ V_n(x_1, y)e^{-\frac{c}{2}x_1} \text{ is bounded.} \end{cases} \tag{1.55}$$

For any $n \in \mathbb{N}$, one has $\limsup_{|x|\in\Omega^c, |x|\to\infty} \rho_n(x) < 0$, The Proposition 4, [18] yields that $V_n(x)$ decays exponentially for $x \in \mathbb{R}^N \setminus \Omega$ and since $F_n(x,s) = f(x,s)$ satisfies condition (1.46) in Ω, Theorem 1.4.1 yields that $V_n(x)$ also decays exponentially for $x \in \Omega$. Thanks to condition (1.51), we derive, by Theorem 1.1 of [18], that $U_n(x)$ is unique. Moreover, since F_n is nonincreasing, for $m, k \in \mathbb{N}$, $k \geq m$, one has

$$\Delta U_m + c\partial_1 U_m + F_k(x, U_m) = -F_m(x, U_m) + F_k(x, U_m) \leq 0, \quad x \in \mathbb{R}^N.$$

Thus U_m is a supersolution of equation satisfied by U_k. One can apply the comparison principle, Theorem 2.3 [18], to imply that $U_k \leq U_m$ in $\mathbb{R}^N$ for $k \geq m$. Then U_n is nonincreasing with respect to n and converges pointwise to a nonnegative function $U_\infty \leq S$. We will prove now that $U_\infty = 0$ in Ω^c.

From above arguments, $V_n(x)$ decays exponentially as $|x| \to \infty$. Multiplying $V_n(x)$ to Eq. (1.55), we derive, by applying the Stokes formula, that

$$\int_{\mathbb{R}^N} \nabla V_n \cdot \nabla V_n = \int_{\mathbb{R}^N} F_n(x_1, y, V_n e^{-\frac{c}{2}x_1})e^{\frac{c}{2}x_1}V_n - \frac{c^2}{4}V_n^2 \leq \int_{\mathbb{R}^N} F_0(x_1, y, V_n e^{-\frac{c}{2}x_1})e^{\frac{c}{2}x_1}V_n \leq$$
$$\leq \max_{x\in\mathbb{R}^N} \partial_s F_0(x,0) \int_{\mathbb{R}^N} V_0^2(x) \leq M < \infty.$$

This implies, by Lesbesgue monotone convergence theorem, that the sequence V_n converges monotonically to some $V_\infty \in H^1(\mathbb{R}^N)$ as $n \to \infty$, weakly in $H^1(\mathbb{R}^N)$ and strongly in $L^2(\mathbb{R}^N)$. Moreover, taking an

arbitrary compact set $K \subset \Omega^c$, one gets

$$-(\max_K \rho_n)\int_K V_n^2 \leq -\int_K \rho_n V_n^2 \leq -\int_K F_n(x, V_n e^{-\frac{c}{2}x_1})e^{\frac{c}{2}x_1}V_n =$$
$$-\int_{\mathbb{R}^N} |\nabla V_n|^2 - \int_{\mathbb{R}^N} \frac{c^2}{4}V_n + \int_{\mathbb{R}^N\setminus K} F_n(x, V_n e^{-\frac{c}{2}x_1})e^{\frac{c}{2}x_1}V_n \leq \int_{\mathbb{R}^N\setminus K} F_n(x, V_n e^{-\frac{c}{2}x_1})e^{\frac{c}{2}x_1}V_n \leq M$$

Then, from assumption (1.52), we have $\max_K \rho_n \to -\infty$, $\forall K \subset \Omega^c$, whence $V_\infty = 0$ for all compact set in Ω^c. This implies $V_\infty = 0$ a.e in Ω^c or in the other words $U_\infty = 0$ a.e in Ω^c. As a consequence, the restriction of U_∞ in Ω belongs to $H_0^1(\Omega)$. Moreover, since $F_n(x,s) = f(x,s)$ in Ω, we have

$$\Delta U_n + c\partial_1 U_n + f(x, U_n) = 0 \qquad x \in \Omega.$$

The standard elliptic estimates yield that $U_n \to U_\infty$ as $n \to \infty$ locally uniformly in Ω and moreover U_∞ is a solution of the same equation in Ω in the weak $H_0^1(\Omega)$ sense. Thanks to Theorem 1.4.1, we know that U_∞ is unique and $U_\infty(x) \to 0$ as $|x_1| \to \infty$, uniformly in $y \in \omega$. Assume by contradiction that the convergence $U_n \to U_\infty$ as $n \to \infty$ is not uniform, one finds a positive constant $\varepsilon > 0$ and a sequence $(x_k) \in \mathbb{R}^N$ such that for some $n_0 \in \mathbb{N}$,

$$|U_n(x_k) - U_\infty(x_k)| \geq \varepsilon, \qquad \forall n \geq n_0.$$

Since the convergence is already locally uniform in $\mathbb{R}^N$, we deduce that $|x_n| \to \infty$ as $n \to \infty$. Since U_n is nonincreasing with respect to n, we deduce that for some $0 < \varepsilon_1 < \varepsilon$, one has

$$U_{n_0}(x_k) \geq \varepsilon_1 + U_\infty(x_k).$$

This is a contradiction since we know that $U_\infty(x), U_{n_0}(x) \to 0$ as $|x| \to \infty$.

Lastly, to conclude the proof, it remains to prove Lemma 1.4.3, $\lambda_n \nearrow \lambda_D$. To this end, we first show that $\lambda_n < \lambda_D$, $\forall n \in \mathbb{N}$. Since $\rho_n(x)$ is nonincreasing in n, one sees that λ_n is nondecreasing in n. Assume by contradiction that $\lambda_n \geq \lambda_D$ for some n. Let us denote by φ_n and φ respectively the principal eigenfunctions associated with λ_n and λ_D, it holds that :

$$\Delta\varphi_n + c\partial_1\varphi_n + f_s(x,0)\varphi_n = -\lambda_n\varphi_n \leq -\lambda_D\varphi_n, \quad x \in \Omega,$$

and $\varphi_n > 0$ in $\overline{\Omega}$. Note that the existence of a positive eigenfunction associated with the generalized principal eigenvalue λ_n in unbounded domain is given in [20]. Because φ_n is a supersolution of equation satisfied by φ, if there exists $0 < \kappa < +\infty$ such that $\kappa\varphi \leq \varphi_n$ in Ω, one can enlarge κ until $\kappa\varphi$ touches φ_n from below at some point. The strong maximum principle implies $\kappa\varphi \equiv \varphi_n$ in Ω, which is impossible because $\varphi = 0$ on $\partial\Omega$. In other words, $\sup\{\kappa \in (0, +\infty], \kappa\varphi \leq \varphi_n \text{ in } \Omega\} = +\infty$. This yields another contradiction since $\varphi > 0$ in Ω. As a result, $\lambda_n < \lambda_D$, $\forall n \in \mathbb{N}$. Next, we aim to show the limit $\lim_{n\to\infty}\lambda_n = \lambda_D$. Since λ_n is nondecreasing and bounded from above, there exists $\lambda_\infty = \lim_{n\to\infty}\lambda_n \leq \lambda_D$. We shall prove that $\lambda_\infty = \lambda_D$.

Observe that, by the transformation $\tilde{\varphi}_n = \varphi_n e^{\frac{c}{2}x_1}$, we see that $\tilde{\varphi}_n$ satisfies the equation

$$\Delta\tilde{\varphi}_n + \rho_n(x)\tilde{\varphi}_n - \frac{c^2}{4}\tilde{\varphi}_n + \lambda_n\tilde{\varphi}_n = 0, \qquad x \in \mathbb{R}^N. \tag{1.56}$$

Let us show that $\tilde{\varphi}_n$ decays exponentially. Indeed, as in the proof of Theorem 1.4.1, let $\tilde{\phi}$ be the function such that $\inf_{\overline{\omega}}\tilde{\phi} > 0$ and $\Delta_y\tilde{\phi} + \tilde{\mu}(y)\tilde{\phi} = 0$ in ω, with $\tilde{\mu}(y)$ sufficiently close to $\mu(y) - \lambda$, where $\lambda = \lambda_D(-\Delta_y -$

$\mu(y), \omega) > 0$. Then, one sees that, $(\lambda_n, \tilde{\varphi}_n)$ is the principal eigenpair of the operator $\tilde{\mathcal{L}}_n = \Delta + \rho_n(x) - c^2/4$ if and only if (λ_n, ϕ_n), with $\phi_n = \tilde{\varphi}_n/\tilde{\phi}$, is the principal eigenpair of the following operator

$$\Delta + \frac{2\nabla\tilde{\phi}\cdot\nabla}{\tilde{\phi}} + c\partial_1 + \frac{\rho_n(x)\tilde{\phi} - \mu(y)\tilde{\phi} - \lambda\tilde{\phi}}{\tilde{\phi}}.$$

Beside that, by assumptions (1.46) and (1.52), one has

$$\limsup_{|x|\to\infty} \frac{\rho_n(x)\tilde{\phi} - \mu(y)\tilde{\phi} - \lambda\tilde{\phi}}{\tilde{\phi}} < 0 < -\lambda_n. \tag{1.57}$$

Hence, applying the Proposition 1.11 [20], we know that λ_n is simple, moreover ϕ_n is unique (up to multiplications) and decays exponentially as $|x| \to \infty$. It follows immediately that $\tilde{\varphi}_n$ also decays exponentially.

On the other hand, since $\lambda_n < \lambda_D < 0$ and $\rho_n(x)$ is nonincreasing in n, one has

$$\Delta\tilde{\varphi}_n + \rho_0(x)\tilde{\varphi}_n \geq 0.$$

From above, $\tilde{\varphi}_n$ is bounded and $\tilde{\varphi}_n$ solves linear equation (1.56), we can normalize φ_n in such the way $\sup_{\mathbb{R}^N}\varphi_n = 1$. Thanks to conditions (1.46) and $\limsup_{x\in\Omega^c,|x|\to\infty}\rho_0(x) < 0$, the same argumentation of Proposition 8.6 in [20] may be applied to derive that there exists an exponential decay function $\overline{\varphi}$ depending only on $\rho_0(x)$ such that $\tilde{\varphi}_n \leq \overline{\varphi}$ in $\mathbb{R}^N$. From the equation (1.56), one has

$$\int_{\mathbb{R}^N} |\nabla\tilde{\varphi}_n|^2 \leq \int_{\mathbb{R}^N} \rho_n(x)\tilde{\varphi}_n^2(x) + \lambda_n \int_{\mathbb{R}^N} \tilde{\varphi}_n^2(x) \leq \int_{\mathbb{R}^N} \rho_0(x)\overline{\varphi}^2(x) < \infty \tag{1.58}$$

This implies that there exists $\varphi_\infty \in H^1(\mathbb{R}^N)$ such that $\tilde{\varphi}_n$ converges up to subsequence to φ_∞ weakly in $H^1(\mathbb{R}^N)$ and strongly in $L^2(K)$ for all compact set $K \subset \mathbb{R}^N$. For any compact set $K \subset \Omega^c$, we derive from (1.58)

$$-\max_K \rho_n \int_K \tilde{\varphi}_n^2 dx \leq -\int_K \rho_n\tilde{\varphi}_n^2 dx \leq -\int_{\mathbb{R}^N} |\nabla\tilde{\varphi}_n|^2 + \int_{\mathbb{R}^N\setminus K} \rho_n\tilde{\varphi}_n^2 dx \leq \sup_{\mathbb{R}^N}\rho_0 \int_{\mathbb{R}^N} \overline{\varphi}^2 dx.$$

Since, from (1.52) for all $K \subset \Omega^c$, $-\max_K \rho_n \to \infty$ as $n \to \infty$, we have $\varphi_\infty = 0$ a.e in K and then a.e in Ω^c. Lastly, again from (1.56), one has

$$\int_{\mathbb{R}^N} |\nabla\tilde{\varphi}_n|^2 \leq \int_{\mathbb{R}^N} \rho_n(x)\tilde{\varphi}_n^2(x) + \lambda_n - \frac{c^2}{4} \leq \int_{\mathbb{R}^N} \rho_0(x)\tilde{\varphi}_n^2(x)dx + \lambda_\infty - \frac{c^2}{4}.$$

Since $\tilde{\varphi}_n \leq \bar{\varphi}$, we derive, by Lebesgue dominated convergence theorem that

$$\int_{\mathbb{R}^N} \rho_0\tilde{\varphi}_n^2(x)dx \to \int_{\mathbb{R}^N} \rho_0\varphi_\infty^2(x)dx = \int_\Omega f_s(x,0)\varphi_\infty^2(x)dx.$$

Whence, the lower semicontinuity property yields

$$\int_\Omega |\nabla\varphi_\infty|^2 = \int_{\mathbb{R}^N} |\nabla\varphi_\infty|^2 \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla\tilde{\varphi}_n|^2 \leq \int_\Omega f_s(x,0)\varphi_\infty^2(x)dx + \lambda_\infty - \frac{c^2}{4} \tag{1.59}$$

By the Liouville transformation, λ_D is the principal eigenvalue of a self-adjoint operator. We know from [20] that it has a variational structure.

$$\lambda_D = \inf_{w\in C_c^1(\Omega), \|w\|_{L^2(\Omega)}=1} \int_\Omega |\nabla w|^2 - f_s(x,0)w^2 dx + \frac{c^2}{4}.$$

Since $C_c^1(\Omega)$ is dense in $H_0^1(\Omega)$ and $\varphi_\infty \in H_0^1(\Omega)$, there exists a sequence $w_n \in C_c^1(\Omega)$ of $L^2(\Omega)$-norm equal to 1 converges to φ_∞ in $H^1(\Omega)$. Combining with (1.59), we derive

$$\lambda_D \leq \int_\Omega |\nabla w_n|^2 - f_s(x,0)w_n^2 dx + \frac{c^2}{4} \to \int_\Omega |\nabla \varphi_\infty|^2 - f_s(x,0)\varphi_\infty^2 dx + \frac{c^2}{4} \leq \lambda_\infty. \tag{1.60}$$

Eventually, we obtain $\lambda_D = \lambda_\infty$. This completes the proof. □

Remark 3. In the proof, we have proved a result, which is stronger than what we really need. In fact, to obtain the conclusion of Theorem 1.4.2, one only needs to prove $\lambda_n < \lambda_D$. However, by proving Lemma 1.4.3, we obtain a more interesting result on the convergence of the eigenvalues and eigenfunctions. This indeed makes Theorem 1.4.2 more transparent and more interesting.

Remark 4. Inequality (1.60) in fact implies that

$$\lambda_D = \inf_{w \in H_0^1(\Omega), \|w\|_{L^2(\Omega)}=1} \int_\Omega |\nabla w|^2 - f_s(x,0)w^2 dx + \frac{c^2}{4}.$$

By assumption (1.46) and the same arguments as (1.56)-(1.57), we can actually prove that λ_D is simple. This proof is referred to Proposition 1.11 [20]. In this case, due to the density of $C_c^1(\Omega)$ in $H_0^1(\Omega)$, the infima of variational characterization of λ_D taken over $H_0^1(\Omega)$ and $C_c^1(\Omega)$ are equivalent and equal to the unique λ_D. This fact is not true in general, especially when (1.46) does not hold. Here, thanks to linear structure of Eq. (1.56), we can normalize $\tilde{\varphi}_n$ of the sup-norm equal 1. Therefore, we can find a uniform exponential decay, $\overline{\varphi} \geq \tilde{\varphi}_n$, depending only on $\rho_0(x)$, which helps us to compensate the lack of compactness of Ω and $\mathbb{R}^N$.

The last result is concerned with a further qualitative property of the fronts of Eq. (1.6), namely the symmetry breaking in x_1 axis. To this aim, the monotonicity and exact asymptotic behavior of the fronts play the crucial role. From Proposition 1.2.1, we know that the fronts $U(x_1, y)$ decay exponentially as $x \to \pm\infty$. Therefore, natural questions may arise, which are the right conditions such that the fronts are monotone when $|x_1|$ large enough and whether they are symmetric in x_1 axis. These questions are addressed in the following by studying the asymptotic behavior of solutions as $x_1 \to \pm\infty$.

1.4.3 Symmetry breaking of the fronts

Theorem 1.4.4. *Let U be a traveling front solution of Eq. (1.6) with f is such that*

$$\begin{cases} |f_s(x_1, y, 0) - \alpha(y)| = O(e^{px_1}) \quad \text{as } x_1 \to -\infty, \text{ and} \quad \lambda_\alpha = \lambda_N(-\Delta_y - \alpha(y), \omega) > 0 \\ |f_s(x_1, y, 0) - \beta(y)| = O(e^{-qx_1}) \quad \text{as } x_1 \to +\infty, \text{ and} \quad \lambda_\beta = \lambda_N(-\Delta_y - \beta(y), \omega) > 0 \end{cases} \tag{1.61}$$

uniformly in $y \in \omega$, for some $\alpha, \beta \in \mathcal{L}^\infty(\omega)$, $p, q > 0$. We assume further that $s \to f(x,s) \in C^{1,r}(0,\delta)$ for some $r, \delta > 0$. Then, U is asymmetric if

$$\lambda_\beta \neq \lambda_\alpha + c^2 - 2c\sqrt{\lambda_\alpha + \frac{c^2}{4}},$$

where c is the given forced speed of traveling front.

Proof. We investigate at first the precise asymptotic behavior of solution of Eq.(1.6) on the branch Ω^-, By analogy, we derive also the asymptotic behavior on the branch Ω^+. According to Proposition 1.2.1 and Theorem A.1, Appendix, for any $\delta > 0$, we have shown that

$$C_{2,\delta} e^{\kappa_\delta x_1} \leq U(x_1, y) \leq C_{1,\delta} e^{\tau_\delta x_1} \qquad \forall (x_1, y) \in \Omega^-, \tag{1.62}$$

where $\kappa_\delta = \sqrt{\lambda_\alpha + \delta + \frac{c^2}{4}} - \frac{c}{2}$ and $\tau_\delta = \sqrt{\lambda_\alpha - \delta + \frac{c^2}{4}} - \frac{c}{2}$. Since ω is bounded, we refer to [12], that there exists a unique (up to a multiplication) eigenfunction φ with Neumann boundary condition associated to λ_α:

$$\begin{cases} -(\Delta_y + \alpha(y))\varphi = \lambda_\alpha \varphi & \text{in } \omega, \\ \partial_\nu \varphi(y) = 0 & \text{on } \partial\omega. \end{cases}$$

We rewrite Eq. (1.6) as in the following form

$$\begin{cases} \mathcal{M}U = -\Delta U - c\partial_1 U - \alpha(y)U = H(x_1, y) \\ H_f(x_1, y) = f(x_1, y, U) - \alpha(y)U \end{cases}$$

By the regularity condition $s \to f(x, s) \in C^{1,r}(0, \delta)$ and condition (1.61), we have

$$\begin{aligned} |H_f(x_1, y)| &\leq |f(x_1, y, U) - f_s(x_1, y, 0)U| + |f_s(x_1, y, 0)U - \alpha(y)U| \\ &\leq C_1 e^{(r+1)\tau_\delta x_1} + C_2 e^{(p+\tau_\delta)x_1} \leq C_3 e^{m(\tau_\delta)x_1} \quad \text{as } x \to -\infty, \end{aligned} \tag{1.63}$$

where $m(\tau_\delta) = \min\{(r+1)\tau_\delta, p + \tau_\delta\}$. By Theorem 4.3 of [13], we can write U as follow

$$U = u^0(x_1, y) + u^*(x_1, y),$$

where (u^0, u^*) is a solution of system

$$\begin{cases} \mathcal{M}u^0 = 0 & x \in \Omega^- \\ \partial_\nu u^0 = 0 & \partial\Omega^- \end{cases} \qquad \begin{cases} \mathcal{M}u^* = H_f(x_1, y) & x \in \Omega^- \\ \partial_\nu u^* = 0 & \partial\Omega^-. \end{cases} \tag{1.64}$$

Moreover, u^0 has a precisely exponential asymptotic behavior as $x_1 \to -\infty$, namely there exist $\lambda > 0$ and $\psi(x_1, y) = (-x_1)^k \psi_k(y) + ... + \psi_0(y) \not\equiv 0$ such that

$$\begin{cases} u^0(x_1, y) = e^{\lambda x_1}\psi(x_1, y) + O(e^{\lambda x_1}) \\ \nabla u^0(x_1, y) = \nabla(e^{\lambda x_1}\psi(x_1, y)) + O(e^{\lambda x_1}), \end{cases} \tag{1.65}$$

and for any $\varepsilon > 0$, u^* satisfies the inequality

$$|u^*(x_1, y)| + |\nabla u^*(x_1, y)| \leq C_{\varepsilon,\delta} e^{(m(\tau_\delta) - \varepsilon)x_1} \qquad \text{for some } C_\varepsilon > 0. \tag{1.66}$$

Let us define

$$\tau_0 = \sup\{\tau : \exists C_\tau \text{ such that } u(x_1, y) \leq C_\tau e^{\tau x_1} \text{ in } \Omega^-\}.$$

Inequalities (1.62) yields $\kappa_\delta \leq \tau_0 \leq \tau_\delta$, for any $\delta > 0$, thus τ_0 is indeed a real number. We want to prove that $\tau_0 = \sqrt{\lambda_\alpha + \frac{c^2}{4}} - \frac{c}{2}$. Taking $\tau < \tau_0$, then $0 \leq u(x_1, y) \leq C_\tau e^{\tau x_1}$ and moreover $|\nabla u(x_1, y)| \leq C'_\tau e^{\tau x_1}$

by the Harnack inequality. Proceeding as (1.63), we get : $|H_f(x_1,y)| \le C_4 e^{m(\tau)x_1}$, where $m(\tau) = \min\{(r+1)\tau, p+\tau\} > \tau$. As a result of (1.66), we have

$$|u^*(x_1,y)| + |\nabla u^*(x_1,y)| \le D_\tau e^{\frac{(\tau+m(\tau))x_1}{2}}.$$

One sees that as $\tau \nearrow \tau_0$, $\dfrac{\tau+m(\tau)}{2} \nearrow \dfrac{\tau_0+m(\tau_0)}{2} > \tau_0$. Therefore, there exist $\varepsilon > 0$ and $C_\varepsilon > 0$ such that

$$|u^*(x_1,y)| + |\nabla u^*(x_1,y)| \le C_\varepsilon e^{(\tau_0+\varepsilon)x_1} \quad \text{in } \Omega^-. \tag{1.67}$$

It follows immediately that $\forall \tau < \tau_0$

$$|u^0(x_1,y)| \le |u(x_1,y)| + |u^*(x_1,y)| \le C_\tau e^{\tau x_1} + C_\varepsilon e^{(\tau_0+\varepsilon)x_1} \le (C_\tau + C_\varepsilon) e^{\tau x_1} \quad \text{in } \Omega^-.$$

On the other hand, for δ small enough

$$u_0(x_1,y) = u(x_1,y) - u^*(x_1,y) \ge C_{2,\delta} e^{(\sqrt{\lambda_\alpha+\delta+\frac{c^2}{4}}-\frac{c}{2})x_1} - C_\varepsilon e^{(\tau_0+\varepsilon)x_1} \ge C_{3,\delta} e^{(\sqrt{\lambda_\alpha+\delta+\frac{c^2}{4}}-\frac{c}{2})x_1}.$$

Applying Theorem 4.2 of [13] to u^0, we deduce that there is exactly one positive constant λ such that $\tau \le \lambda \le \sqrt{\lambda_\alpha+\delta+\frac{c^2}{4}}-\frac{c}{2}$, $\forall \tau < \tau_0$ and (1.65) holds for a suitable exponential solution $w(x_1,y) = e^{\lambda x_1}\psi(y)$. From (1.67), λ cannot be strictly bigger than τ_0, therefore we must have $\lambda = \tau_0 > 0$. Since $u > 0$, we deduce $\psi_k > 0$ and thus Theorem 2.4 of [13] yields that $\psi(y)$ is a solution of

$$\begin{cases} -(\Delta_y + \alpha(y))\psi = (\lambda^2 + c\lambda)\psi & \text{in } \omega \\ \psi_\nu = 0 & \text{on } \partial\omega. \end{cases} \tag{1.68}$$

Since $\lambda_\alpha > 0$, Theorem 2.1 of [13] implies that (1.68) possesses exactly one positive principal eigenvalue, that is $\lambda = \dfrac{-c+\sqrt{c^2+4\lambda_\alpha}}{2} = \tau_0$. We obtain the precisely asymptotic behavior of $U(x_1,y)$ as $x_1 \to -\infty$.

By analogy, we obtain the precisely exponential behavior of $U(x_1,y)$ as $x_1 \to +\infty$. It is precisely exponentially asymptotic as $x_1 \to +\infty$ with the exponent $\lambda' = \dfrac{-c-\sqrt{c^2+4\lambda_\beta}}{2}$. As a consequence, we have proved that

$$U(x_1,y) \sim C_1 e^{-(\frac{c+\sqrt{c^2+4\lambda_\beta}}{2})x_1} \quad \text{as } x \to +\infty; \quad U(x_1,y) \sim C_2 e^{(\frac{-c+\sqrt{c^2+4\lambda_\alpha}}{2})x_1} \quad \text{as } x \to -\infty,$$

uniformly in y. This result, in particular, implies that $U(x_1,y)$ is increasing in $(-\infty,-R)\times\omega$ and decreasing in $(R_1,\infty)\times\omega$ for R, R_1 large enough. To achieve the symmetry in x_1, necessarily, we have

$$\frac{c+\sqrt{c^2+4\lambda_\beta}}{2} = \frac{-c+\sqrt{c^2+4\lambda_\alpha}}{2} \quad \Longleftrightarrow \quad \lambda_\beta = \lambda_\alpha + c^2 - 2c\sqrt{\lambda_\alpha+\frac{c^2}{4}}.$$

In other words U is asymmetric if $\lambda_\beta \ne \lambda_\alpha + c^2 - 2c\sqrt{\lambda_\alpha+\dfrac{c^2}{4}}$. □

Remark 5. we see that if $c \ne 0$, the asymmetry holds even when $\lambda_\beta = \lambda_\alpha$. The drift term is therefore the main inducement that makes the front asymmetric. However, we do not know that whether the front is symmetric when

$$\lambda_\beta = \lambda_\alpha + c^2 - 2c\sqrt{\lambda_\alpha+\frac{c^2}{4}}.$$

The answer of this question requires more involved analysis. We state this as an open question. Our result applies, in particular, to show that the asymmetry holds when condition (1.61) becomes as (1.2), namely $\lambda_\beta = \lambda_\alpha = m > 0$ and

$$|f_s(x_1, y, 0) + m| = O(e^{-p|x_1|}) \quad \text{as } |x_1| \to \infty, \text{ for some } m, p > 0.$$

Remark 6. We point out that the assumption of the exponential convergences $f_s(x_1, y, 0) \to \alpha(y)$ and $f_s(x_1, y, 0) \to \beta(y)$ in (1.61) is important. Indeed, if (1.61) does not hold, the precise exponential behavior of u^0 satisfying Eq. (1.64) may not be true as (1.65) in general. For instance, in one dimensional space, $\beta(y) \equiv -1/2$, if $f_s(x, 0)$ converges slowly to $-1/2$, we can take $w(x) = xe^{-x}$, which is a solution of

$$w'' + \frac{1}{2}w' + g(x)w = 0 \quad \text{in } \mathbb{R} \setminus (2, -\infty), \qquad g(x) = -\frac{1}{2} + \frac{1}{2x} \to -\frac{1}{2} \quad \text{as } x \to +\infty.$$

1.5 Appendix

Theorem 1.5.1. *Let U be a traveling front solution of (1.6). Assume that (1.9) holds and f is such that*

$$\liminf_{x_1 \to \pm\infty} f_s(x_1, y, 0) \geq \alpha_\pm(y) \quad \textit{and} \quad \lambda_{\alpha_\pm} = \lambda_N(-\Delta_y - \alpha_\pm(y), \omega) > 0, \tag{1.69}$$

for some functions $\alpha_\pm \in \mathcal{L}^\infty(\omega)$. Then, for any $\delta > 0$, there exist $A_\pm > 0$ and $\tau_{\alpha_\pm} \geq \sqrt{\lambda_{\alpha_\pm} + \delta + \frac{c^2}{4}}$ such that

$$U(x_1, y) \geq A_- e^{(\tau_{\alpha_-} - \frac{c}{2})x_1} \quad \forall (x_1, y) \in \Omega^- \quad \textit{and} \quad U(x_1, y) \geq A_+ e^{-(\tau_{\alpha_+} + \frac{c}{2})x_1} \quad \forall (x_1, y) \in \Omega^+.$$

Proof. Since (1.9) holds, from Proposition (1.2.1), we know that $U(x_1, y)$ decays exponentially as $|x_1| \to \infty$. Let us denote $I = \{\alpha_-, \alpha_+\}$. We know from [12] that there exist the principal eigenfunctions φ_i associated with λ_i such that for $i \in I$

$$\begin{cases} -\Delta\varphi_i - i(y)\varphi_i = \lambda_i\varphi_i & \text{in } \omega \\ \partial_\nu \varphi_i = 0 & \text{on } \partial\omega. \end{cases}$$

For $i \in I$, $\delta > 0$, we set $\widetilde{\mathcal{L}}_i = \Delta_x + c\partial_1 + i(y) - \delta$. By assumption (1.69), there exists $R = R(\delta) > 0$ such that $\widetilde{\mathcal{L}}_{\alpha_-} U \leq 0$ in $\Omega^- \setminus \Omega_R$ and $\widetilde{\mathcal{L}}_{\alpha_+} U \leq 0$ in $\Omega^+ \setminus \Omega_R$. Define the functions

$$\omega_{\alpha_-}(x) = e^{\tau_{\alpha_-} x_1}\varphi_{\alpha_-}(y) \qquad \text{and} \qquad \omega_{\alpha_+}(x) = e^{-\tau_{\alpha_+} x_1}\varphi_{\alpha_+}(y),$$

direct computation yields

$$\widetilde{\mathcal{L}}_{\alpha_-}\omega_{\alpha_-} = \left(\tau_{\alpha_-}^2 + c\tau_{\alpha_-} - \lambda_{\alpha_-} - \delta\right)\omega_{\alpha_-} \geq 0 \text{ in } \Omega^- \setminus \Omega_R \text{ if } \tau_{\alpha_-} \geq \frac{\sqrt{c^2 + 4(\lambda_{\alpha_-} + \delta)} - c}{2};$$

$$\widetilde{\mathcal{L}}_{\alpha_-}\omega_{\alpha_+} = \left(\tau_{\alpha_+}^2 - c\tau_{\alpha_+} - \lambda_{\alpha_+} - \delta\right)\omega_{\alpha_+} \geq 0 \text{ in } \Omega^+ \setminus \Omega_R \text{ if } \tau_{\alpha_+} \geq \frac{\sqrt{c^2 + 4(\lambda_{\alpha_+} + \delta)} + c}{2}.$$

The strong maximum principle and the Hopf lemma yield $\inf_{y\in\omega} \varphi_i(y) > 0$, $\inf_{y\in\omega} U(-R, y) > 0$ and $\inf_{y\in\omega} U(R, y) > 0$. Therefore, we can choose the positive constants $C_{\delta,i}$ small enough such that the functions satisfy $W_{\alpha_-}(x_1, y) = U(x_1, y) - C_{\delta,\alpha_-}\omega_{\alpha_-}(x_1, y) \geq 0$ and $W_{\alpha_+}(x_1, y) = U(x_1, y) - C_{\delta,\alpha_+}\omega_{\alpha_+}(x_1, y) \geq 0$

for $x_1 = -R, y \in \omega$. They have the Neumann boundary conditions $\partial_\nu W_i = 0$ on $\partial\Omega$ and satisfy the inequalities $\widetilde{\mathcal{L}}_i W_i \leq 0$ in $\Omega^\pm \setminus \Omega_R$. Set $z_i(x_1, y) = W_i(x_1, y)/\varphi_i(y)$, we get $\partial_\nu z_i = 0$ on $\partial\Omega$ and

$$0 \geq \frac{\widetilde{\mathcal{L}}_{\alpha_-} W_{\alpha_-}}{\varphi_{\alpha_-}} = \Delta z_{\alpha_-} + c\partial_1 z_{\alpha_-} + 2\frac{\nabla_y \varphi_{\alpha_-}}{\varphi_{\alpha_-}}.\nabla_y z_{\alpha_-} - \left(\lambda_{\alpha_-} + \delta\right) z_{\alpha_-} \quad x_1 < -R, \forall y \in \omega,$$

$$0 \geq \frac{\widetilde{\mathcal{L}}_{\alpha_+} W_{\alpha_+}}{\varphi_{\alpha_+}} = \Delta z_{\alpha_+} + c\partial z_{\alpha_+} + 2\frac{\nabla_y \varphi_{\alpha_+}}{\varphi_{\alpha_+}}.\nabla_y z_{\alpha_+} - \left(\lambda_{\alpha_+} + \delta\right) z_{\alpha_+}, \quad x_1 > R, \forall y \in \omega.$$

Since $z_{\alpha_-}(x_1, y) \to 0$ as $x_1 \to -\infty$, $z_{\alpha_+}(x_1, y) \to 0$ as $x_1 \to +\infty$, and zero-order coefficients of elliptic-operators with respect to z_i are negative, the weak maximum principle is applied to derive $z_i \geq 0$ in $\Omega^\pm \setminus \Omega_R$. As a consequence, there exist $\tau_i \geq \sqrt{\lambda_i + \delta + \dfrac{c^2}{4}}$ such that

$$\begin{cases} C_{\delta,\alpha_+} e^{-(\tau_{\alpha_+} + \frac{c}{2})x_1} \varphi_{\alpha_+}(y) \leq U(x_1, y) & \text{in } \Omega^+ \setminus \Omega_R. \\ C_{\delta,\alpha_-} e^{(\tau_{\alpha_-} - \frac{c}{2})x_1} \varphi_{\alpha_-}(y) \leq U(x_1, y) & \text{in } \Omega^- \setminus \Omega_R. \end{cases} \tag{1.70}$$

By the Harnack inequality, one has $\inf_{|x_1| \leq R} U(x_1, y) > 0$, we deduce that $C_{\delta,i}$ indeed can be chosen such that the inequality (1.70) holds respectively in $\Omega^\pm$. The proof is complete. □

Theorem 1.5.2. *Let $U \in W^{1,2}_{N+1,loc}(\mathbb{R} \times \mathbb{R}^N)$ be a solution of (1.14), where f is such that condition (1.19) holds true. Then there exist two positive constants k and ε such that*

$$\forall (t, x) \in \mathbb{R} \times \mathbb{R}^N \qquad U(t, x_1, y) \leq k e^{-\varepsilon |x_1|}.$$

Proof. We only need to prove that the statement holds for $x_1 \geq 0$ and by analogy we also derive the result for $x_1 \leq 0$. Using the transformation $V(t, x_1, y) = U(t, x_1, y) e^{\frac{c}{2} x_1}$, we see that $V(t, x)$ is periodic in y, T-periodic in t and satisfies the following equation

$$\begin{cases} V_t = \Delta V + f(t, x, V e^{-\frac{c}{2}x_1}) e^{\frac{c}{2}x_1} - \dfrac{c^2}{4} V & t \in \mathbb{R}, x \in \mathbb{R}^N \\ V e^{-\frac{c}{2}x_1} \text{ is bounded.} \end{cases}$$

For any $R > 0$, $\delta > 0$, we denote $\mathcal{Q}_R = \mathbb{R} \times [0, R] \times \mathbb{R}^{N-1}$ and set $\widetilde{\mathcal{L}} = \partial_t - \Delta - \gamma(t, y) - \delta + \dfrac{c^2}{4}$. From condition (1.19), there exist $R = R(\delta) > 0$ such that $\widetilde{\mathcal{L}} V \leq 0$, $\forall (t, x) \in \mathcal{Q}_R$. Moreover, since $\lambda_\gamma = \bar{\lambda}_1(\partial_t - \Delta_y - \gamma(t, y), \mathbb{R} \times \mathbb{R}^{N-1}) > 0$, we refer to Theorems 2.7, [76] that there exists a unique space-time periodic eigenpair $(\lambda_\gamma, \varphi)$ satisfying

$$\begin{cases} \widetilde{\mathcal{L}} \varphi = \lambda_\gamma \varphi \\ \varphi > 0 \\ \varphi(., . + T) = \varphi \\ \varphi(. + L_i e_i, .) = \varphi \end{cases} \qquad \text{for } i \in \{1, N-1\}.$$

Fix $\tau \in \mathbb{R}$ and define the function $\omega(t, x) = \theta_a(x_1) \varphi(t, y) e^{(\tau - t)\delta}$, where $\theta_a : [R, R + a] \to \mathbb{R}$ is the solution of

$$\begin{cases} \theta_a'' = (\kappa + \delta)\theta_a & \text{in } (R, R + a) \\ \theta_a(R) = C e^{\sqrt{\kappa} R} \\ \theta_a(R + a) = C e^{\sqrt{\kappa}(R + a)}, \end{cases}$$

where $C = \sup_{\mathbb{R}\times\mathbb{R}^N} V(t,x)e^{-\frac{c}{2}x_1}/\inf_{\mathbb{R}\times\mathbb{R}^{N-1}}\varphi(t,y)$, $\kappa > 0$ would be chosen later. Note that $\inf_{\mathbb{R}\times\mathbb{R}^{N-1}}\varphi(t,y) > 0$ thanks to the periodicity of φ in y and t. Direct calculation yields

$$\theta_a(\rho) = C(e^{(\sqrt{\kappa}+\sqrt{\kappa+\delta})R})\left(1 - \frac{e^{\sqrt{\kappa}a} - e^{-\sqrt{\kappa+\delta}a}}{e^{\sqrt{\kappa+\delta}a} - e^{-\sqrt{\kappa+\delta}a}}\right)e^{-\sqrt{\kappa+\delta}\rho} + \\ + C(e^{(\sqrt{\kappa}-\sqrt{\kappa+\delta})R})\frac{-e^{-\sqrt{\kappa+\delta}a} + e^{\sqrt{\kappa}a}}{e^{\sqrt{\kappa+\delta}a} - e^{-\sqrt{\kappa+\delta}a}}e^{\sqrt{\kappa+\delta}\rho}.$$

On the other hand, we have $\widetilde{\mathcal{L}}\omega = \left(-\kappa - 3\delta + \lambda_\gamma + c^2/4\right)\omega \geq 0$ if and only if $\kappa \leq \lambda_\gamma + c^2/4 - 3\delta$. Moreover, it is easily seen that $\omega(t,x_1,y) \geq V(t,x_1,y)$ for $t \leq \tau, x_1 \in \{R, R+a\}$. Taking $0 < \delta < \lambda_\gamma/3$ we can choose $\kappa \in \left(c^2/4, \lambda_\gamma + c^2/4 - 3\delta\right)$. Let $W(t,x) = \omega(t,x) - V(t,x) = z(t,x)\varphi(t,y)$, one sees that $z(t,x) \geq 0$ for $t \leq \tau$, $x_1 \in \{R, R+a\}, y \in \mathbb{R}^{N-1}$. Since $\sup_{t\in\mathbb{R}, x\in(R,R+a)\times\mathbb{R}^{N-1}} V(t,x) < +\infty$ and $\inf_{t\leq\tau, x\in(R,R+a)\times\mathbb{R}^{N-1}} \omega(t,x) > 0$, there exists $t_0(a) \ll \tau$, which may depend on a and sufficiently close to $-\infty$ such that $z(t_0(a),x) \geq 0$ for $x \in (R, R+a)\times\mathbb{R}^{N-1}$. In addition that, we have

$$0 \leq \frac{\widetilde{\mathcal{L}}W}{\varphi} = \partial_t z - \Delta z - 2\nabla z.\frac{\nabla\varphi}{\varphi} + \left(\lambda_\gamma - \delta + \frac{c^2}{4}\right)z.$$

Since the zero order coefficient of parabolic operator with respect to z is positive, we deduce from the parabolic weak maximum principle that $z(t,x) \geq 0$ in $(t_0(a),\tau)\times\mathcal{Q}_{R+a}\setminus\mathcal{Q}_R$ for every $a > 0$. Finally, the classical parabolic regularity implies that $V(\tau,x) \leq \omega(\tau,x)$ for $x \in \mathcal{Q}_{R+a}\setminus\mathcal{Q}_R$. Therefore,

$$U(\tau,x) \leq \lim_{a\to+\infty}\theta_a(x_1)\varphi(\tau,y)e^{-\frac{c}{2}x_1} = C(e^{(\sqrt{\kappa}+\sqrt{\kappa+\delta})R})e^{-(\sqrt{\kappa+\delta}+\frac{c}{2})x_1}.$$

The arbitrariness of τ enables us to conclude the proof. □

Remark 7. In the proof of this theorem, we need not assume that the solution U is periodic in y and T-periodic in t, but the local regularity of solutions plays an important role. On the other hand, as seen from above, it is possible to choose $\kappa = \lambda_\gamma + c^2/4 - 3\delta$ to obtain

$$U(\tau,x) \leq C_1 e^{-(\sqrt{\lambda_\gamma + \frac{c^2}{4} - 2\delta} + \frac{c}{2})x_1} \quad \text{for } \tau \in \mathbb{R}, x_1 \geq R, y \in \mathbb{R}^{N-1}$$

Using the same arguments, we derive that there exist $C_2 > 0$:

$$U(\tau,x) \leq C_2 e^{(\sqrt{\lambda_\gamma + \frac{c^2}{4} - 2\delta} - \frac{c}{2})x_1} \quad \text{for } \tau \in \mathbb{R}, x_1 \leq -R, y \in \mathbb{R}^{N-1}.$$

Since U is bounded, one can choose C_1, C_2 large enough such that these inequalities hold in $\mathbb{R}^{N+1}$.

Chapitre 2

Liouville-type result for semilinear elliptic equations with possibly vanishing and sign-changing potentials

2.1 Introduction and Main Results

2.1.1 Introduction

The paper is devoted to characterizing the set of positive solutions of

$$a_{ij}(x)\partial_{ij}u(x) + Kq(x)\cdot\nabla u(x) + f(x,u) = 0, \qquad x \in \mathbb{R}^N, \tag{2.1}$$

and large time behavior of

$$(|x|+1)^{-\alpha}\partial_t u = a_{ij}(x)\partial_{ij}u(x) + Kq(x)\cdot\nabla u + f(x,u) \qquad t \in \mathbb{R}^+, x \in \mathbb{R}^N, \tag{2.2}$$

where $\alpha \in [0,2]$ and $K \geq 0$.

In a pioneering work, Berestycki, Dickmann, Nagelkerke and Zegeling [5] have proposed a model in $\mathbb{R}$ to study the impact of climate change on the population dynamics of species, who are strongly sensitive to temperature conditions. More precisely, they considered the equation

$$u_t - u_{xx} = f(x-ct,u) \qquad t \in \mathbb{R}, x \in \mathbb{R}, \tag{2.3}$$

where $c > 0$ is a given forced speed. A typical f is

$$f(x,s) = \begin{cases} -sm & x < 0 \text{ and } x > L \\ sm'\left(1 - \dfrac{s}{K}\right) & 0 \leq x \leq L, \end{cases} \tag{2.4}$$

for some positive constants m, m', K, L. This nonlinearity has the property

$$f_s(x,0) = \begin{cases} -m < 0 & x < 0 \text{ and } x > L \\ m' > 0 & 0 \leq x \leq L. \end{cases} \tag{2.5}$$

Condition (2.5) means that the environment of the species is unfavorable outside a prescribed compact set. Ecologically speaking, the environments satisfying (2.5) are surrounded by unprosperous vicinities, where the source for persistence is so scarce or the climatic condition is unpropitious. Here we say that the environment is favorable (respectively unfavorable and neutral) at the point x_0 if $f_s(x_0, 0) > 0$ (respectively $f_s(x_0, 0) < 0$ and $f_s(x_0, 0) = 0$). The authors in [5] showed that $u(t, x)$ converges to $U(x)$, which is the unique positive solution of

$$U'' + cU' + f(x, U) = 0 \qquad \text{in } \mathbb{R}$$

uniformly as $t \to \infty$. Thus, to understand this dynamics, the characterization of the set of nontrivial stationary solutions plays the central role. Higher dimensional version of this model was later studied in [18], where condition (2.5) is relaxed to

$$\limsup_{|x|\to\infty} f_s(x, 0) < 0. \tag{2.6}$$

These equations are the particular cases of (2.1)-(2.2) where $\alpha = 0$ and q is a constant vector in $\mathbb{R}^N$. The use of condition (2.6) is to derive the exponential decay of solutions. The interested readers are also referred to the works of Potapov, Lewis [80] and Roques et al. [89], where similar models were investigated. On a different framework, Berestycki-Hamel-Rossi also proved a Liouville-type result for positive bounded solutions if the following condition holds

$$\liminf_{|x|\to\infty} (4\underline{a}(x) f_s(x, 0) - K|q(x)|^2) > 0, \tag{2.7}$$

where $0 < a_* \leq \underline{a}(x) = \inf_{\substack{\xi\in\mathbb{R}^N \\ |\xi|=1}} a_{ij}(x)\xi_i\xi_j$. Actually, it is proved in [9] that the solution is unique and has positive infimum if exists. To this paper, we pursue the following goals :

• First, we extend the results of [5],[18] to a more general framework and study the new cases in which (2.6),(2.7) are not fulfilled. More precisely, we allow $f_s(x, 0)$ to be sign-changing all the way as $|x| \to \infty$ and it may behave like a slow decay or Hardy potential near infinity, namely $f_s(x, 0) \leq -C|x|^{-\alpha}$ in $\mathbb{R}^N \setminus B_{R_0}$ for some $C, R_0 > 0$ and $\alpha \in [0, 2]$. From the biological point of view, these new assumptions refer to the environments that can be unfavorable ($\alpha = 0$) or unfavorably neutral ($\alpha \in (0, 2)$) or nearly neutral ($\alpha = 2$) mixing with favorable patches extending to infinity. Our results confirm that the unique persistence and extinction of the species can be conditioned by the sign of the generalized principal eigenvalue of the linearized operator of (2.1). This is a delicate conclusion since if (2.7) holds, we know from [9] that there exists a unique positive solution of (2.1), which has a positive infimum and therefore the species are persistent. To this aim, we will propose a global condition in term of spectral theory. The central role is to develop the notions of generalized principal eigenvalues in unbounded domains, which are inspired from [14],[20]. These generalizations are originally conjectured by Prof H. Berestycki (personal communication). Precise description is given in Section 1.2.

• Second, under the setting framework, we further investigate the influence of the drift term on the existence of nontrivial solutions of Eq. (2.1). Indeed, in the works [5], [7], [18], the authors implicitly showed that the existence of positive solution is strongly connected with the amplitude of the drift term. Here, we consider the drifts satisfying divergence-free and some fair growth conditions. By adequate compactness compensations, we partially extend the result of Berestycki-Hamel-Nadirashvili [7] on the limit of large drift for the first eigenvalues in bounded domain to the generalized principal eigenvalues in unbounded

domain with no periodic assumption. This is in fact the intermediate result that we shall use to derive the existence of sharp threshold of the drift-amplitude, $K^\star$, such that positive solutions of (2.1) exist if and only if $K < K^\star$. Detailed explanation is postponed until stating the result.

• Third, the long time behavior of Eq. (2.2) are lastly investigated. We prove that, if the initial datum decays enough, solutions of Eq. (2.2) converge in $L^p(\mathbb{R}^N)$- weighted norm to the unique positive stationary of Eq. (2.1). Here $0 < p \leq \infty$.

We obtain an intriguing counterpart of these results between two cases $\alpha \in [0, 2)$ and $\alpha = 2$. The different effects due to the slow decay and Hardy potentials of $f_s(x, 0)$ for $|x|$ large are fully understood. Our results also confirm that the Liouville-type result for KPP-type nonlinearity is strongly depended on the behavior near infinity of zero-order coefficient of the linearized operator about zero.

2.1.2 The central role : generalized principal eigenvalues

In bounded domains, one of the classical approaches ([3],[6]) to investigate the existence and uniqueness for equations like Eq. (2.1), is to look at the first eigenvalue of the linearized operator of the nonlinear equation. Thanks to the compactness of domain and prescribed boundary condition (Dirichlet or Neumann), the existence and simplicity of the first eigenvalue is well-established by the Krein-Rutman theory (see [23]). However, if the domain is not smooth and possibly unbounded, the Krein-Rutman theory cannot be applied directly and in fact the first eigenvalue may not exist in general. Therefore, appropriate eigenvalues should be looked for to overcome the existing difficulties. The notion of generalized principal eigenvalue was first introduced in the celebrated work of Berestycki-Nirenberg-Varadhan [12] and has been widely used after in many researches of reaction-diffusion equations ([10],[14],[18],[76],[77],[90]). We also refer to the early works of Pinchover [100],[101] and López-Gómez [71] on the eigenvalues for indefinite-weight elliptic operators. More recently, Berestycki et al. [9], [14], [20] have deeply generalized this notion for unbounded domains in various aspects. In this paper, we use similar approach of these papers to define a new generalized principal eigenvalue. To this aim, let us provide first some basic assumptions.

In Eq. (2.1), we denote by (a_{ij}) with $i, j \in \{1, ..., N\}$ and $q(x) = (q_1(x), ..., q_N(x))$ respectively the matrix $N \times N$ and the N-dimensional drift term. The matrix (a_{ij}) is assumed to be symmetric but need not be uniformly elliptic. Throughout the paper, we assume that the functions a_{ij}, q_i are continuous and bounded in $\mathbb{R}^N$, the reaction term $f(x, s) : \mathbb{R}^N \times [0, +\infty) \mapsto \mathbb{R}$ belongs to $C^{0,\gamma}_{loc}(\mathbb{R}^N)$ with respect to x, locally in s for some $\gamma \in (0, 1)$ and $C^{1,\gamma}(\mathbb{R}^N)$ uniformly with respect to $s \in [0, \delta]$ for some $\delta > 0$. We further assume that $f(x, 0) = 0$ for all $x \in \mathbb{R}^N$.

Throughout this paper, we denote the linearized operator of Eq. (2.1) about 0 by

$$\mathcal{L}[\phi] = a_{ij}\partial_{ij}\phi + Kq(x) \cdot \nabla\phi + f_s(x, 0)\phi, \tag{2.8}$$

and sometimes we use $\mathcal{L}_K[\phi]$ to emphasize the dependence on K.

Let us recall the generalized principal eigenvalue in [9],[12],[20] that are used in this paper

$$\lambda_1(-\mathcal{L}, \Omega) := \sup\{\lambda \in \mathbb{R} : \exists \phi \in W^{2,N}_{loc}(\Omega), \phi > 0, (\mathcal{L} + \lambda)\phi \leq 0 \text{ a.e in } \Omega\}. \tag{2.9}$$

$$\lambda'_1(-\mathcal{L}, \Omega) := \inf\{\lambda \in \mathbb{R} : \exists \phi \in W^{2,N}_{loc}(\Omega), \phi > 0, \phi \in L^\infty(\Omega), (\mathcal{L} + \lambda)\phi \geq 0 \text{ a.e in } \Omega\}. \tag{2.10}$$

$$\lambda''_1(-\mathcal{L}, \Omega) := \sup\{\lambda \in \mathbb{R} : \exists \phi \in W^{2,N}_{loc}(\Omega), \inf_\Omega \phi > 0, (\mathcal{L} + \lambda)\phi \leq 0 \text{ a.e in } \Omega\}. \tag{2.11}$$

and we define the new generalized principal eigenvalues as follows.

Definition 2.1.1. Let $\alpha \in [0,2]$, the generalized principal eigenvalue associated with $\mathcal{L}$ in Ω is

$$\lambda_\alpha(-\mathcal{L},\Omega) := \sup\left\{\lambda \in \mathbb{R} : \exists h_\alpha \in C(\Omega), \lim_{x\in\Omega,|x|\to\infty} \frac{h_\alpha(x)}{|x|^{-\alpha}} = 1, \exists \phi \in \overline{\mathcal{D}}_\alpha, \left(\mathcal{L}+\lambda|x|^{-\alpha}\right)\phi \le 0 \text{ in } \Omega_{h_\alpha^+}\right\}, \tag{2.12}$$

where $\Omega_{h^+} = \{x \in \Omega, h(x) \ge 0\}$ and $\overline{\mathcal{D}}_\alpha$ denotes the class of admissible test-functions

$$\overline{\mathcal{D}}_\alpha := \left\{\phi \in W^{2,N}_{loc}(\Omega), \phi > 0 \text{ in } \Omega_{h^+}, \limsup_{x\in\Omega,|x|\to\infty} \frac{|\nabla\phi(x)||x|^{\frac{\alpha}{2}}}{\phi(x)} \le \mathcal{D}_\alpha \text{ and } \lim_{x\in\Omega,|x|\to\infty} \frac{\ln\phi(x)}{|x|^{1-\frac{\alpha}{2}}} = 0\right\},$$

for $\alpha \in [0,2)$ and for $\alpha = 2$, $\overline{\mathcal{D}}_2$ is given by

$$\overline{\mathcal{D}}_2 := \left\{\phi \in W^{2,N}_{loc}(\Omega), \phi > 0 \text{ in } \Omega_{h^+}, \limsup_{x\in\Omega,|x|\to\infty} \frac{|\nabla\phi(x)||x|}{\phi(x)} \le \mathcal{D}_2 \text{ and } \lim_{x\in\Omega,|x|\to\infty} \frac{\ln\phi(x)}{\ln|x|} = 0\right\}.$$

In principle, we define two different notions of generalized principal eigenvalues for $\alpha \in [0,2)$ and $\alpha = 2$, where the test-functions are assigned different growth conditions near infinity. Intrinsically, the difference of growth conditions are due to the different decays of fundamental solutions for the linear equations with slow decay and Hardy potentials. Further explanation of these eigenvalues are given in section 2. Note that, if Ω is bounded, all of growth conditions are superfluous and obviously $\lambda_\alpha(-\mathcal{L},\Omega) = \lambda_1(-\mathcal{L},\Omega)$. More generally, we refer the interested readers to the work of López-Gómez [71] for a careful investigation on the existence of weighted-eigenvalue in bounded domain. We are now ready to state main results

2.1.3 Hypothesis and Main Results

(H1) Hypothesis 1 : There exist $\alpha \in [0,2]$ and $\mu \in L^\infty(\mathbb{R}^N)$ satisfying

$$(|x|+1)^\alpha f_s(x,0), (|x|+1)^\alpha \mu(x) \in L^\infty(\mathbb{R}^N) \qquad \limsup_{|x|\to\infty}\{|x|^\alpha(f_s(x,0)-\mu(x))\} \le 0 \tag{H1a}$$

and one of the followings is true

$$\overline{\eta}_\alpha = \lambda_\alpha(-a_{ij}(x)\partial_{ij} - Kq(x).\nabla - \mu(x), \mathbb{R}^N) > 0, \tag{H1b}$$

$$\begin{cases} (a_{ij}) \text{ is the identity matrix, } \mathrm{div} q = 0 \text{ and there exists } \psi \in C^2_{loc}(\mathbb{R}^N) \text{ such that} \\ 2\nabla\psi(x) = -q(x), \qquad \underline{\eta}_\alpha = \lambda_\alpha(-\Delta - \mu(x), \mathbb{R}^N) > 0. \end{cases} \tag{H1c}$$

Remark 8. The particular case $f_s(x,0) \sim -m(|x|+1)^\alpha$ for some $m > 0, \alpha \in [0,2]$ as $|x| \to \infty$ is also covered by (H1b), where we can take $\mu(x) = -m(|x|+1)^\alpha$ and 1 to be a test-function for $\overline{\eta}_\alpha$. Note that, in this case, q is not necessarily involved in (H1b).

(H2) Hypothesis 2 :

$$s \to \frac{f(x,s)}{s} \text{ is nonincreasing for } x \in \mathbb{R}^N$$
$$\text{and it is strictly decreasing in some set } D \subset \mathbb{R}^N, \text{ with } |D| > 0.$$

(H3) Hypothesis 3 : For fixed K, there exists a function $0 < S_K(x) \in W^{2,N}_{loc}(\mathbb{R}^N)$ satisfying

$$a_{ij}(x)\partial_{ij}S_K + Kq(x)\cdot\nabla S_K(x) + f(x,S_K) \leq 0 \text{ a.e in } \mathbb{R}^N,$$

such that

$$\lim_{|x|\to\infty} \frac{\ln S_K(x)}{|x|^{1-\frac{\alpha}{2}}} = 0 \text{ if } \alpha \in [0,2) \text{ and } \lim_{|x|\to\infty} \frac{\ln S_K(x)}{\ln|x|} = 0 \text{ if } \alpha = 2.$$

Existence and Uniqueness of Eq. (2.1) (Liouville-type result)

Theorem 2.1.2. *Let $\alpha \in [0,2)$, $\theta \in \mathbb{R}^+$, $\overline{\Lambda} = \max_{ij}\|a_{ij}\|_\infty < \infty$ and Hypotheses 2 hold. Eq. (2.1) cannot possess more than one positive solution satisfying*

$$\limsup_{|x|\to\infty} [\ln u(x)]|x|^{\frac{\alpha}{2}-1} \leq 0. \tag{2.13}$$

If Hypothesis 3 is besides assumed then Eq. (2.1) possesses a unique positive solution iff $\lambda_1(-\mathcal{L},\mathbb{R}^N) < 0$. Furthermore, for all $\gamma \in (0,\gamma^\star)$, one finds $\varphi \in \overline{\mathcal{D}}_\alpha$, $C(\gamma), R(\gamma) > 0$ such that

$$u(x) \leq C(\gamma)e^{\gamma|x|^{1-\frac{\alpha}{2}}}\varphi(x) \quad \textit{for} \quad |x| \geq R(\gamma),$$

where $\gamma^\star$ is explicitly given as follows

(i) *if (H1a), (H1b) hold and $\limsup_{|x|\to\infty}|q(x)||x|^{\frac{\alpha}{2}} \leq \theta$, then $\gamma^\star = \overline{\gamma}$, which is the unique positive solution of*

$$\gamma^2\left(1-\frac{\alpha}{2}\right)^2 N^2 + 2\gamma\left(1-\frac{\alpha}{2}\right)N\mathcal{D}_\alpha + \gamma\left(1-\frac{\alpha}{2}\right)\frac{K}{\overline{\Lambda}}\theta - \frac{\overline{\eta}_\alpha}{\overline{\Lambda}} = 0; \tag{2.14}$$

(ii) *if (H1a), (H1c) hold, in addition that*

$$\liminf_{|x|\to\infty}|\nabla\psi(x)||x|^{\frac{\alpha}{2}} = \theta, \quad \limsup_{|x|\to\infty}\psi(x)|x|^{\frac{\alpha}{2}-1} = \frac{2\theta}{2-\alpha} \quad \textit{and} \quad \theta < \frac{\underline{\eta}_\alpha}{2K\mathcal{D}_\alpha}, \tag{2.15}$$

then

$$\gamma^\star = \underline{\gamma} = \gamma^+ - \frac{2K\theta}{2-\alpha} \qquad \textit{with} \qquad \gamma^+ = \frac{-2\mathcal{D}_\alpha + 2\sqrt{\mathcal{D}_\alpha^2 + \underline{\eta}_\alpha + K^2\theta^2}}{2-\alpha}. \tag{2.16}$$

In particular, u decays exponentially when it exists.

Observe that, when $\alpha = 2$, the characterizations (2.14), (2.15), (2.16) are not well-determined, for instance, from (2.15), ψ blows up. In fact, another phenomenon takes place, we are thus led to

Theorem 2.1.3. *Let $\alpha = 2$, $\theta \in \mathbb{R}^+$, $\overline{\Lambda} = \max_{ij}\|a_{ij}\|_\infty < \infty$ and Hypotheses 2 hold. Eq. (2.1) cannot possess more than one positive solution satisfying*

$$\limsup_{|x|\to\infty} \frac{\ln u(x)}{\ln|x|} \leq 0. \tag{2.17}$$

If Hypothesis 3 is besides assumed then Eq. (2.1) possesses a unique positive solution iff $\lambda_1(-\mathcal{L},\mathbb{R}^N) < 0$. Furthermore, for all $\beta \in (0,\sigma^\star)$, one finds $\varphi \in \overline{\mathcal{D}}_2, C(\beta), R(\beta) > 0$ such that

$$u(x) \leq C(\beta)|x|^\beta \quad \textit{for} \quad |x| \geq R(\beta),$$

where $\sigma^\star$ is explicitly given as follows

(i) *if (H1a), (H1b) hold and* $\limsup_{|x|\to\infty}|q(x)||x|\le\theta$, *then*

$$\sigma^\star=\overline{\sigma}=\frac{-b+\sqrt{b^2-4ac}}{2a},\quad a=\overline{\Lambda}N,\quad b=(3N+2\mathcal{D}_2)\overline{\Lambda}+K\theta,\quad c=-\overline{\eta}_2. \tag{2.18}$$

(ii) *if (H1a), (H1c) hold, in addition that*

$$\liminf_{|x|\to\infty}|\nabla\psi(x)||x|=\theta,\quad \limsup_{|x|\to\infty}\frac{\psi(x)}{(\ln|x|)^2}=\theta\quad \textit{and}\quad \theta+\frac{\theta}{2\mathcal{D}_2}(N-2)<\frac{\underline{\eta}_2}{2K\mathcal{D}_2}, \tag{2.19}$$

then

$$\sigma^\star=\underline{\sigma}=\beta_2^+-K\theta>0\quad \textit{with}\quad \beta_1^+=\frac{(N-2+2\mathcal{D}_2)+\sqrt{(N-2+2\mathcal{D}_2)^2+4\underline{\eta}_2+4K^2\theta^2}}{2}. \tag{2.20}$$

In particular, u decays polynomially when it exists.

Remark 9. In the next result, we will apply the particular case $\theta=0$ of part ii) to prove Theorem (2.1.5) and (2.1.6). Let us provide some examples of q that fulfill (H1a), (H1c) and one of two conditions (2.15) and (2.19) in this case.

Let $N\ge 2$, $\theta=0$ and ψ be the fundamental solutions of Laplace equations $\Delta\psi_1(x)=0$ in $\mathbb{R}^N$. It is known that ψ_1 has the explicit forms

$$\psi_1(x)=\begin{cases}\ln|x| & N=2\\ \frac{1}{|x|^{N-2}} & N\ge 3.\end{cases}$$

Let $\underline{R}>0$ to be chosen as small as we want and ψ_2,ψ_3 be as follows

$$\psi_2(x)=\begin{cases}\ln(|x|) & |x|\ge 1\\ -\ln(|x|) & |x|\in(\underline{R},1)\\ -\ln(\underline{R}) & |x|\le\underline{R}.\end{cases}\quad\text{for } N=2\,;\quad \psi_3(x)=\begin{cases}1/|x|^{N-2} & |x|\ge 1\\ 1/\underline{R}^{N-2} & |x|\le\underline{R}.\end{cases}\quad\text{for } N\ge 3$$

Obviously, ψ_2 and ψ_2+1 (respectively) ψ_3 and ψ_3+1 are continuous (not smooth) solutions of $\Delta\phi=0$ in $\mathbb{R}^N$ (resp. $N=2$ and $N\ge 3$). Hence, by the classical iteration method and the regularity of Laplace equation, there exists a solution $\psi\in C^2_{loc}(\mathbb{R}^N)$ of $\Delta\psi=0$ in $\mathbb{R}^N$ satisfying $\psi_2\le\psi\le\psi_2+1$ (reps.) $\psi_3\le\psi\le\psi_3+1$ in $\mathbb{R}^N$. Note that ψ need not be unique up to multiplicative and additive constants.

We choose $q(x)=\nabla\psi(x)$. Obviously, all component $q_i\in C^1_{loc}(\mathbb{R}^N)$, $i=\{1,..,N\}$, div$q=0$ and

$$\limsup_{|x|\to\infty}(\ln|x|)|x|^{\frac{\alpha}{2}-1}=0,\qquad \limsup_{|x|\to\infty}|x|^{\frac{\alpha}{2}-1}/|x|^{N-2}=0\qquad 0\le\alpha<2$$

$$\limsup_{|x|\to\infty}\frac{\ln|x|}{(\ln|x|)^2}=0,\qquad \limsup_{|x|\to\infty}\frac{1}{(\ln|x|)^2|x|^{N-2}}=0\qquad \alpha=2.$$

These verify (H1c), (2.15) and (2.19). Moreover, if $\theta\neq 0$, another important class for example of q is the class of constant vectors in $\mathbb{R}^N$, for instance, $q=\overrightarrow{c}$ when $\psi(x)=\overrightarrow{c}.x$, $x\in\mathbb{R}^N$. Very recently, the constant vector fields are used in the study of the impact of climate change (see [5, 18]). Theorem (2.1.2) and (2.1.3) extend the existence/nonexistence and uniqueness results in [5, 18] even in the case $\alpha=0$ and q is a constant vector.

Sharp threshold of drift

Next, we prove the existence of sharp threshold of drift term for which Eq. (2.1) possesses at least one positive solution. This result is in fact a consequence of the necessary and sufficient conditions for the existence of Eq. (2.1), which are directly conditioned by the sign of the eigenvalue $\lambda_1(-\mathcal{L}_K, \mathbb{R}^N)$ and the continuous monotonicity of $\lambda_1(-\mathcal{L}_K, \mathbb{R}^N)$ with respect to K. The goal is to understand the sign-dependence of $\lambda_1(-\mathcal{L}_K, \mathbb{R}^N)$ on the large and small K. Here, the interesting difficulties are due to the presence of zero's order coefficient of linearized operator of Eq. (2.1), $f_s(x, 0)$, which allows to change sign and possibly decays fast up to Hardy-potential as $|x| \to \infty$. In [7], Berestycki-Hamel-Nadirashvili have already studied the limit of the first eigenvalue for the large drift in bounded domains with Dirichlet (Neuman) boundary condition or periodic domains. However, the problem changes drastically when the domain becomes unbounded without periodic assumption. The new difficulties have been successfully overcome by imposing additional assumptions (H1a) and (H1c). This bears adequate (but different between $\alpha \in [0, 2)$ and $\alpha = 2$) decays for the principal eigenfunction associated with the generalized principal eigenvalue $\lambda_1(-\mathcal{L}_K, \mathbb{R}^N)$ to compensate for the lack of compactness of domain. In the sequel, we will prove that the limit of the eigenvalues as $K \to \infty$ depends on the existence of first integrals of the drift term. Let us now introduce this notion

Definition 2.1.4. A function $z \neq 0$ is called a first integral of the vector fields $q \in \mathbb{R}^N$ if $z \in H^1(\mathbb{R}^N)$ and $q \cdot \nabla z = 0$ almost everywhere in $\mathbb{R}^N$. We denote the set of first integrals of q by

$$\mathcal{I}_q = \{z | z \in H^1(\mathbb{R}^N), z \neq 0 \text{ and } q \cdot \nabla z = 0 \text{ a.e in } \mathbb{R}^N\}. \tag{2.21}$$

We obtain the following results:

Theorem 2.1.5. *Let $\alpha \in [0, 2)$ and assumptions (H1a), (H1c) (2.15) with $\theta = 0$ be fulfilled. If the followings hold*

$$\inf_{\phi \in H^1(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla \phi|^2 - f_s(x, 0)\phi^2}{\int_{\mathbb{R}^N} \phi^2} < 0 < \inf_{\phi \in \mathcal{I}_q} \frac{\int_{\mathbb{R}^N} |\nabla \phi|^2 - f_s(x, 0)\phi^2}{\int_{\mathbb{R}^N} \phi^2}, \tag{2.22}$$

with the convention that the infimum is $+\infty$ if $\mathcal{I}_q$ being empty. Then there exists a sharp threshold $K^\star$ such that Eq. (2.1) possesses a positive solution if and only if $K < K^\star$.

Theorem 2.1.6. *Let $\alpha = 2$ and (H1a), (H1c), (2.19) with $\theta = 0$, (2.22) hold. The conclusion of theorem (2.1.5) remains true if*

$$\begin{cases} \mathcal{D}_2 \geq 1 & or \\ \mathcal{D}_2 < 1 & and \quad \underline{\eta}_2 > N(1 - \mathcal{D}_2) - \frac{N^2}{4}. \end{cases} \tag{2.23}$$

Remark 10. We note that (2.23) becomes unconditioned when $N \geq 2$. This condition is indeed used to guarantee that the eigenfunction associated with λ_K belongs to $L^2(\mathbb{R}^N)$. This is unnecessary for the case $\alpha \in [0, 2)$ in Theorem (2.1.5) since, in that case, we will prove that the eigenfunctions decay exponentially .

Long time behavior in L^∞ and L^p norms

Theorem 2.1.7. *Let $\alpha \in [0, 2)$, $u(t, x)$ be the solution of (2.2) starting with nonnegative initial datum $u_0 \not\equiv 0$ and a_{ij}, q satisfy the growth conditions as in Theorem (2.1.2). Assume that Hypotheses 2, 3 and one of the followings are true*

i) (H1a), (H1b) and $\limsup_{|x| \to \infty} \frac{\ln u_0(x)}{|x|^{1-\frac{\alpha}{2}}} < -\gamma_0$, for some $\gamma_0 \in (0, \gamma^\star)$, where $\gamma^\star = \overline{\gamma}$ is the positive solution of (2.14).

ii) (H1a), (H1c), (2.19) and $\limsup_{|x|\to\infty} \frac{\ln u_0(x)}{|x|^{1-\frac{\alpha}{2}}} < -\gamma_0 - \frac{4K\theta}{2-\alpha}$*, for some* $\gamma_0 \in (0, \gamma^\star)$*, where* $\gamma^\star = \underline{\gamma}$ *is given in (2.16).*

If one has

$$\liminf_{|x|\to\infty} \sup_{t>0} \frac{u(t,x)}{e^{\gamma_0|x|^{1-\frac{\alpha}{2}}}} < \infty, \tag{2.24}$$

then for all $0 < p \leq \infty$

$$\lim_{t\to\infty} \| |u(t,x) - W(x)| e^{\gamma_0|x|^{1-\frac{\alpha}{2}}} \|_{L^\infty(\mathbb{R}^N)} = 0, \tag{2.25}$$

where $W(x)$ *is either identical to zero or the unique positive solution of Eq.* (2.1) *corresponding to* $\lambda_1(-\mathcal{L}, \mathbb{R}^N) \geq 0$ *or* $\lambda_1(-\mathcal{L}, \mathbb{R}^N) < 0$*. If* $u_0/S_K \in L^\infty(\mathbb{R}^N)$*, above convergences hold locally on any compact set of* $\mathbb{R}^N$ *.*

Theorem 2.1.8. *Let* $\alpha = 2$*,* $u(t,x)$ *be the solution of (2.2) starting with nonnegative initial datum* $u_0 \not\equiv 0$ *and* a_{ij}, q *satisfy the growth conditions as in Theorem (2.1.3). Assume that Hypotheses 2, 3 and one of the followings are true*

i) (H1a), (H1b) and $\limsup_{|x|\to\infty} \frac{\ln u_0(x)}{\ln|x|} < -\sigma_0$*, for some* $\sigma_0 \in (0, \sigma^\star)$*, where* $\sigma^\star = \overline{\sigma}$ *is the positive solution of (2.18).*

ii) (H1a), (H1c), (2.15) and $\limsup_{|x|\to\infty} \frac{\ln u_0(x)}{\ln|x|} < -\sigma_0 - 2K\theta$*, for some* $\sigma_0 \in (0, \sigma^\star)$*, where* $\sigma^\star = \underline{\sigma}$ *is given in (2.20).*

If one has

$$\liminf_{|x|\to\infty} \sup_{t>0} \frac{u(t,x)}{|x|^{\sigma_0}} < \infty, \tag{2.26}$$

then,

$$\lim_{t\to\infty} \| |u(t,x) - W(x)| (|x|+1)^{\sigma_0} \|_{L^\infty(\mathbb{R}^N)} = 0, \tag{2.27}$$

moreover, if $p \in (0, \infty)$ *is such that* $\sigma_0 \in (N/p, \sigma^\star)$*, then*

$$\lim_{t\to\infty} \| |u(t,x) - W(x)| (|x|+1)^{\sigma_0} \|_{L^p(\mathbb{R}^N)} = 0, \tag{2.28}$$

where $W(x)$ *is either identical to zero or the unique positive solution of Eq.* (2.1) *corresponding to* $\lambda_1(-\mathcal{L}, \mathbb{R}^N) \geq 0$ *or* $\lambda_1(-\mathcal{L}, \mathbb{R}^N) < 0$*. If* $u_0/S_K \in L^\infty(\mathbb{R}^N)$*, above convergences hold locally on any compact set of* $\mathbb{R}^N$ *.*

2.2 Further discussion on the hypotheses and results

2.2.1 Meaning of the hypotheses and results

In this section, we discuss about the meaning of the generalized principal eigenvalues in definition (2.1.1) and of the main results presented above.

Let us recall first the similar generalized eigenvalue used in [14], that is

$$\overline{\lambda}_1(\mathcal{L}, (R,\infty)) = \inf\{\lambda | \exists \phi \in \mathcal{A}_R \text{ such that } (\mathcal{L} - \lambda)\phi \leq 0 \text{ in } (R,\infty)\}, \tag{2.29}$$

where if $|R| < \infty$, $\mathcal{A}_R$ is the set of admissible test-functions over (R,∞):

$$\mathcal{A}_R = \left\{ \phi \in C^1([R,\infty)) \cap C^2((R,\infty)), \phi > 0 \text{ in } [R,\infty), \phi'/\phi \in L^\infty((R,\infty)), \lim_{x\to+\infty} \frac{\ln \phi(x)}{x} = 0 \right\}$$

and $\mathcal{A}_{-\infty}$ is the set of admissible test-functions over $\mathbb{R}$:

$$\mathcal{A}_{-\infty} = \Big\{\phi \in C^2(\mathbb{R}), \phi > 0 \text{ in } \mathbb{R}, \phi'/\phi \in L^\infty(\mathbb{R}), \lim_{|x|\to+\infty} \frac{\ln \phi(x)}{|x|} = 0\}.$$

Obviously, $-\overline{\lambda}_1(\mathcal{L}, (-\infty, \infty)) \leq \lambda_0(-\mathcal{L}, \mathbb{R})$. The authors of [14] define the eigenvalues in either half or whole $\mathbb{R}$. In definition (2.1.1), we distinguish different notions by the rate of decay of the zero's order coefficient. We realize that, for all $\alpha \in [0, 2)$, similar notions can be extended in the same manner as (2.29) but not for $\alpha = 2$. The reason is that if $\alpha = 2$, the growth rate $\lim_{|x|\to\infty} \frac{\ln \phi(x)}{|x|^{1-\frac{\alpha}{2}}} = 0$ may not be proper for any supersolution of the operators with Hardy potentials. Necessarily, a reasonable growth rate should be looked for to deal with the case. We impose $\lim_{|x|\to\infty} \frac{\ln \phi(x)}{\ln |x|} = 0$.

The eigenvalue λ_α also differs from λ_1, λ_1', and λ_1'' of Berestycki, Rossi [20] in the sense that we impose the milder constraint on the test-functions $\lim_{|x|\to\infty} \frac{\ln \phi(x)}{|x|^{1-\frac{\alpha}{2}}} = 0$ for $\alpha \in [0, 2)$ and $\lim_{|x|\to\infty} \frac{\ln \phi(x)}{\ln |x|} = 0$ for $\alpha = 2$ instead of asking some constant bounds from above or below on the test-functions. For $\alpha = 0$, we refer to [14] for the construction of an eigenfunctions in class of random stationary ergodic coefficients. The eigenfunction is not bounded but satisfies the prescribed growth condition. Further properties of this eigenvalue can also be found in [14]. If one restricts the set of admissible test-functions, for instance, to be

$$\widetilde{\mathcal{A}}_{-\infty} = \{\phi \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \inf_{\mathbb{R}^N} \phi > 0\}$$

then the corresponding generalized principal eigenvalues are not equal in general. Therefore, the class of random stationary ergodic coefficients emphasizes that it is very important to use the milder growth assumption for admissible test-functions.

The set of admissible test-functions $\overline{\mathcal{D}}_\alpha$ is quite large. Obviously, all positive constants belong to $\overline{\mathcal{D}}_\alpha$. For $\alpha \in [0, 2)$, $\overline{\mathcal{D}}_\alpha$ contains all of the functions that grow at most exponentially sublinear and polynomially near infinity, for instance, any function having exact asymptotic behavior as $|x| \to \infty$ like $Ce^{\lambda|x|^\kappa}$ or $C|x|^\lambda$, where $C > 0$, $\lambda \in \mathbb{R}$ and $\kappa \in [0, 1 - \frac{\alpha}{2})$ belongs to $\overline{\mathcal{D}}_\alpha$. The class $\overline{\mathcal{D}}_2$ is smaller, it contains all function growing at most as $e^{\lambda(\ln |x|)^\kappa}$ as $|x| \to \infty$ with $\lambda \in \mathbb{R}$ and $\kappa < 1$. One can take any function having exact asymptotic behavior as $e^{\lambda(\ln |x|)^\kappa}$ and also $\ln |x|$ for examples. The following nice properties of the functions in $\overline{\mathcal{D}}_\alpha$ will be frequently used in the proofs :

$$\text{For } \alpha \in [0, 2), \varphi \in \overline{\mathcal{D}}_\alpha, \qquad \forall \lambda \in \mathbb{R}, \lambda > 0, \qquad \lim_{|x|\to\infty} \varphi(x) e^{\lambda|x|^{1-\frac{\alpha}{2}}} = \infty \qquad \lim_{|x|\to\infty} \varphi(x) e^{-\lambda|x|^{1-\frac{\alpha}{2}}} = 0. \tag{2.30}$$

$$\text{For } \alpha = 2, \varphi \in \overline{\mathcal{D}}_2, \qquad \forall \lambda \in \mathbb{R}, \lambda > 0, \qquad \lim_{|x|\to\infty} \varphi(x)|x|^\lambda = \infty \qquad \lim_{|x|\to\infty} \varphi(x)|x|^{-\lambda} = 0. \tag{2.31}$$

Now, under the sense of λ_α, Hypothesis 1 extends (2.5) and (2.6) in two aspects. First, for general q, (H1a) and (H1b) suggest to look at the eigenvalue λ_α defined on $\mathbb{R}^N$ of the upper operator of $-\mathcal{L}$ near infinity, where the upper operator of $-\mathcal{L}$ near infinity means

$$-\overline{\mathcal{L}}[\phi] = -a_{ij}(x)\partial_{ij}\phi - Kq(x).\nabla\phi - \mu(x)\phi \qquad \text{and} \qquad \limsup_{|x|\to\infty} \{|x|^\alpha (f_s(x, 0) - \mu(x))\} \leq 0.$$

Second, if a_{ij} is the identity matrix and q is divergence-free, (H1c) suggests a more concise condition, that is looking at the sign of λ_α of $-\Delta - \mu(x)$ defined on $\mathbb{R}^N$. As a typical example for f, one can think of

$$f(x, u) = a(x)u - b(x)u^p, \qquad b(x) > 0, p > 1,$$

and $a(x)$ has exact asymptotic behavior as $O(-|x|^{-\alpha})$ near infinity. There, one can take a positive constant to be a test-function for λ_α such that (H1b) and (H1c) are satisfied.

Hypothesis 3 requires the existence of supersolution having the same growth as the functions in $\overline{\mathcal{D}}_\alpha$ but not necessarily require the growth of the gradient. This allows us to treat more general functions than the functions $f(x,u)$ satisfying the following assumption

$$\exists S \in \mathbb{R}, S > 0 \text{ such that } f(x,s) \leq 0, \forall x \in \mathbb{R}^N,\ s \geq S\ .$$

For the existence of sharp threshold, although the conclusions of Theorem (2.1.5) and (2.1.6) are similar but the intrinsic phenomena are different. Especially, in the compactness argument, for $\alpha \in [0,2)$ the eigenfunction φ_K associated with $\lambda_1(-\mathcal{L}_K, \mathbb{R}^N) < 0$ decays exponentially while for $\alpha = 2$ it decays polynomially. Here, thanks to (H1a) and (H1c), we can use the new criteria in [20] to achieve the minimal growth at infinity of the eigenfunctions (see (2.2.4)-(2.2.5)) to prove the decays of the eigenfunctions.

The interesting point of the long time behavior in Theorem (2.1.7) and Theorem (2.1.8) is that we provide upper critical values of the growth rates for initial data so that $u(t,x)$ converges as $t \to \infty$ to the unique stationary solution (maybe zero) in $L^p(\mathbb{R}^N)$, for all $p > 0$. These results are actually obtained thanks to the decays of stationary solutions, where the negativity of λ_α plays an important role. Since the stationary solutions may decay very weakly depending on α, they may not be able to keep up with the evolution of $\partial_t u$. The weight $(|x|+1)^{-\alpha}$ in front of $\partial_t u$ aims to pull down the growth of $\partial_t u$ so that $u(t,x)$ will converge if started by a sufficiently decaying initial data. Although we impose some decays on initial data, but these theorems are strong enough since the class of test-functions $\overline{\mathcal{D}}_\alpha$ for λ_α is considerable and in addition that, by hypothesis (H3), we allow Eq. (2.1) to possess quite large supersolutions, which may be unbounded with suitable growth rates. Moreover, by (2.25), (2.27) and (2.28), we obtain the exact convergences in $L^p(\mathbb{R}^N)$-weighted norms, which are stronger than $L^p(\mathbb{R}^N)$ norms. One of the main tools is the extension Phragmén-Lindelöf principle (Theorem 1, [44]) for parabolic operators with unbounded coefficients, which does not seem to be very well known.

2.2.2 Earlier results

Theorem 2.2.1. *Let Ω be smooth and coefficients of $\mathcal{L}$ be bounded. Then, one has*

i. [Theorem 1.7, (iii)[20]] $\lambda_1''(-\mathcal{L},\Omega) \leq \lambda_1'(-\mathcal{L},\Omega) \leq \lambda_1(-\mathcal{L},\Omega)$.

ii. [Theorem 1.8 [20]] $\lambda_1''(-\mathcal{L},\Omega) = \lambda_1(-\mathcal{L},\Omega)(= \lambda_1'(-\mathcal{L},\Omega))$ *if one of the followings holds true:*

a) $\mathcal{L}$ is a self-adjoint, a_{ij} are bounded and either $N = 1$ or $\Omega = \mathbb{R}^N$ and $\mathcal{L}$ is radially symmetric.

b) $\mathcal{L} = \tilde{\mathcal{L}} + \gamma(x)$, where $\tilde{\mathcal{L}}$ is such that $\lambda_1''(-\tilde{\mathcal{L}},\Omega) = \lambda_1(-\tilde{\mathcal{L}},\Omega)$ and $\gamma \in L^\infty(\Omega)$ is nonnegative and $\lim\limits_{x\in\Omega,|x|\to\infty} \gamma(x) = 0$.

c) a_{ij} are bounded, $\mathcal{L}$ is a self-adjoint operator and or in non-divergence form satisfying

$$\lim_{x\in\Omega,|x|\to\infty} |q(x)| = 0; \quad \forall r > 0, \forall \beta < \limsup_{x\in\Omega,|x|\to\infty} f_s(x,0), \quad \exists B_r(x_0) \subset \Omega \quad s.t. \quad \inf_{B_r(x_0)} f(x,0) > \beta.$$

d)

$$\lambda_1(-\mathcal{L},\Omega) \leq -\limsup_{|x|\to\infty} f(x,0).$$

iii. [Proposition 2.2, (vi),[20]] *If $\mathcal{L}$ is self-adjoint, one has*

$$\lambda_1(-\mathcal{L},\Omega)=\inf_{\phi\in H_0^1(\Omega),\phi\neq 0}\frac{\int_\Omega a_{ij}(x)\partial_i\phi\partial_j\phi-f_s(x,0)\phi^2}{\int_\Omega\phi^2}.$$

Theorem 2.2.2 (Proposition 9.2, (ii),[20]). *Assume that $a_{ij}^n,b_i^n,c^n\in L^\infty(\mathbb{R}^N)$. If a_{ij} are uniformly Hölder continuous, b_i c are uniformly continuous and $a_{ij}^n\to a$, $b_i^n\to b_i$ $c^n\to c$ in $L^\infty(\mathbb{R}^N)$, then*

$$\lambda_1(-L_n,\mathbb{R}^N)\to\lambda_1(-L,\mathbb{R}^N)\qquad as\ n\to\infty,$$

where $L_nu=a_{ij}^n(x)u_{ij}+b_i^n(x)u_i+c^n(x)u$ and $Lu=a_{ij}(x)u_{ij}+b_i(x)u_i+c(x)u$.

Theorem 2.2.3 (Proposition 4.2 [9]). *Let $\lambda_R:\mathbb{R}^+\to\mathbb{R}$ be the Dirichlet principal eigenvalue of $-L$ in (2.8) in B_R. Then λ_R is decreasing and satisfies*

$$\lim_{R\to\infty}\lambda_R=\lambda_1(-L,\mathbb{R}^N).$$

Furthermore, there exists a positive eigenfunction $\varphi\in W_{loc}^{2,p}(\mathbb{R}^N)$ for any $1\le p<\infty$ associated with $\lambda_1(-L,\mathbb{R}^N)$, namely

$$-L\varphi=\lambda_1(-L,\mathbb{R}^N)\varphi\quad a.e\ in\ \mathbb{R}^N. \tag{2.32}$$

Definition 2.2.4. Let Ω be unbounded. A positive function $u\in W_{loc}^{2,p}(\Omega)$ satisfying

$$\mathcal{L}[u]=0\ \text{a.e in}\ \Omega \tag{2.33}$$

is said to be a solution of minimal growth at infinity if for any $\rho>0$ and any positive function $v\in W_{loc}^{2,p}(\Omega\setminus B_\rho)$ satisfying $\mathcal{L}[v]\le 0$ a.e in $\Omega\setminus B_\rho$, one finds $R\ge\rho$ and $k>0$ such that $ku\le v$ in $\Omega\setminus B_R$.

Theorem 2.2.5 (Theorem 8.4 [20]). *If Ω is unbounded, smooth and $\lambda_1(-\mathcal{L},\Omega)$ satisfies*

$$\lambda_1(-\mathcal{L},\Omega)<\lim_{r\to\infty}\lambda_1(-\mathcal{L},\Omega\setminus\overline{B}_r),$$

then the eigenfunction φ associated with $\lambda_1(-\mathcal{L},\Omega)$ of (2.32) is a solution of minimal growth at infinity and, therefore, $\lambda_1(-\mathcal{L},\Omega)$ is simple in the class of positive functions.

2.3 A-priori estimates

Here, we establish a-priori estimates for solutions (subsolutions) satisfying suitable growth conditions of the linear equation

$$\mathcal{L}[u]=a_{ij}(x)\partial_{ij}u+Kq(x)\cdot\nabla u+f_s(x,0)u\ge 0 \tag{2.34}$$

Theorem 2.3.1. *Let $\alpha\in[0,2)$ and Hypothesis (H1) hold. Let u be a nonnegative subsolution of Eq. (2.34) satisfying (2.13). One of the followings holds true*

i) If (H1a)-(H1b) and $\limsup_{|x|\to\infty}|q(x)||x|^{\frac{\alpha}{2}}\le\theta$ are fulfilled, then for all $\gamma\in(0,\overline{\gamma})$, where $\overline{\gamma}$ are given by (2.14), one finds $\varphi\in\overline{\mathcal{D}}_\alpha$, $C(\gamma)>0$, $R(\gamma)>0$

$$u(x)\le C(\gamma)e^{-\gamma|x|^{1-\frac{\alpha}{2}}}\varphi(x)\qquad for\ |x|\ge R(\gamma).$$

ii) If (H1a), (H1c) and (2.15) are fulfilled, then for all $\gamma \in (0, \underline{\gamma})$, where $\underline{\gamma}$ are given by (2.16), one finds $\varphi \in \overline{\mathcal{D}}_\alpha$, $C(\gamma) > 0$, $R(\gamma) > 0$

$$e^{-\psi_K(x)}u(x) \le C(\gamma)e^{-(\gamma+\frac{2K\theta}{2-\alpha})|x|^{1-\frac{\alpha}{2}}}\varphi(x) \quad \textit{and} \quad u(x) \le C(\gamma)e^{-\gamma|x|^{1-\frac{\alpha}{2}}}\varphi(x) \qquad \textit{for } |x| \ge R(\gamma).$$

In particular, $u(x)$ decays exponentially.

Proof. Since Hypothesis (H1) is twofold, the proof is distinguished into two cases :

(i) Case 1. (H1a)-(H1b) and $\limsup_{|x|\to\infty} |q(x)||x|^{\frac{\alpha}{2}} \le \theta$ hold.

There exist $\alpha \in [0, 2)$ and $\mu \in L^\infty(\mathbb{R}^N)$ such that for any $\delta > 0$, one finds $R = R(\delta) > 0$ satisfying

$$f_s(x, 0) \le \mu(x) + \delta|x|^{-\alpha} \quad \text{and} \quad q(x)|x|^{\frac{\alpha}{2}} \le (\theta + \delta) \qquad \text{for } |x| \ge R.$$

By assumption (H1b), $\overline{\eta} > 0$, for any $\lambda \in (0, \overline{\eta})$, one finds a test function $\varphi \in \overline{\mathcal{D}}_\alpha$ and $h_\alpha \in C(\mathbb{R}^N)$ satisfying $\lim_{|x|\to\infty} h_\alpha(x)|x|^\alpha = 1$ such that

$$a_{ij}(x)\partial_{ij}\varphi + Kq(x) \cdot \nabla\varphi + \mu(x)\varphi \le -\lambda h_\alpha(x)\varphi \quad \text{for } |x| \ge R. \tag{2.35}$$

Set

$$\mathcal{L}_\mu[\phi] = a_{ij}(x)\partial_{ij}\phi + Kq(x) \cdot \nabla\phi + (\mu(x) + \delta)|x|^{-\alpha}\phi,$$

we see that, for any $p > 0$, one has $\mathcal{L}_\mu[u] \ge 0$ in $B_{R+p} \setminus B_R$. We look for a supersolution of Eq. (2.34) in $B_{R+p} \setminus B_R$ of the form

$$w_p(x) = w_p^1(x) + w_p^2(x) = e^{(R+p)^\beta(\tau-\gamma)}e^{\gamma|x|^\beta}\varphi(x) + e^{R^\beta(\tau+\gamma)}e^{-\gamma|x|^\beta}\varphi(x), \qquad \gamma, \beta > 0.$$

Let us denote for brief that $C_1 = e^{(R+p)^\beta(\tau-\gamma)}$ and $C_2 = e^{R^\beta(\tau+\gamma)}$ and I is the dirac function, we treat first w_p^1, careful computation yields that

$$\begin{aligned}
\mathcal{L}_\mu\left(\frac{1}{C_1}w_p^1\right) &= a_{ij}(x)\partial_j\left(\gamma\beta(|x|^{\beta-2}x_i e^{\gamma|x|^\beta}\varphi(x)) + e^{\gamma|x|^\beta}\partial_i\varphi(x)\right) + \gamma\beta Kq(x) \cdot x|x|^{\beta-2}e^{\gamma|x|^\beta}\varphi(x) \\
&+ Kq(x) \cdot \nabla\varphi(x)e^{\gamma|x|^\beta} + \mu(x)\varphi(x)e^{\gamma|x|^\beta}. \\
&= a_{ij}(x)\gamma\beta\left[(\beta-2)x_i x_j|x|^{\beta-4} + I(i-j)|x|^{\beta-2} + \gamma\beta x_i x_j|x|^{2\beta-4} + 2x_i|x|^{\beta-2}\frac{\partial_j\varphi}{\varphi}\right] \times \\
&\times e^{\gamma|x|^\beta}\varphi + \gamma\beta Kq(x) \cdot x|x|^{\beta-2}e^{\gamma|x|^\beta}\varphi + (a_{ij}(x)\partial_{ij}\varphi + Kq(x) \cdot \nabla\varphi + (\mu(x) + \delta)\varphi)\, e^{\gamma|x|^\beta}
\end{aligned} \tag{2.36}$$

We choose $\beta = 1 - \dfrac{\alpha}{2} > 0$. Thanks to (2.35), using the Cauchy-Schwarz inequality, we estimate the right-hand side (RHS) of (2.36) as follow

$$\begin{aligned}
\frac{\text{(RHS) of (2.36)}}{\varphi e^{\gamma|x|^\beta}} &\le a_{ij}(x)\left[\gamma\left(1 + \frac{\alpha}{2}\right)N|x|^{-\frac{\alpha}{2}-1} + \gamma I(i-j)|x|^{-\frac{\alpha}{2}-1} + \gamma^2\left(1 - \frac{\alpha}{2}\right)N^2|x|^{-\alpha}\right. \\
&+ \left. 2\gamma N|x|^{-\frac{\alpha}{2}}\frac{|\nabla\varphi|}{\varphi}\right]\left(1 - \frac{\alpha}{2}\right) + \gamma\left(1 - \frac{\alpha}{2}\right)K|q(x)||x|^{-\frac{\alpha}{2}} - \lambda h_\alpha(x) + \delta|x|^{-\alpha}.
\end{aligned} \tag{2.37}$$

Since $\varphi \in \overline{\mathcal{D}}_\alpha$, one has $|\nabla\varphi(x)| \le \mathcal{D}_\alpha |x|^{-\frac{\alpha}{2}}\varphi(x)$ when $|x| \ge R$ with R large enough, the (RHS) of (2.37) is bounded from above by

$$\begin{aligned}\frac{\text{(RHS) of (2.37)}}{\varphi e^{\gamma|x|^\beta}} &\le \overline{\Lambda}|x|^{-\alpha}\left[\gamma\left(1-\frac{\alpha^2}{4}\right)N|x|^{\frac{\alpha}{2}-1} + \gamma\left(1-\frac{\alpha}{2}\right)N|x|^{\frac{\alpha}{2}-1} + \gamma^2\left(1-\frac{\alpha}{2}\right)^2 N^2 + \right.\\ &+ \left. 2\gamma\left(1-\frac{\alpha}{2}\right)N\mathcal{D}_\alpha + \gamma\left(1-\frac{\alpha}{2}\right)\frac{K}{\overline{\Lambda}}(\theta+\delta) - \frac{\lambda-\delta}{\overline{\Lambda}}\frac{h_\alpha(x)}{|x|^{-\alpha}}\right] \quad \text{for } |x| \ge R. \quad (2.38)\end{aligned}$$

Similarly, one has the same computations for w_p^2, in which γ is replaced by $-\gamma$. Collecting the all of computations and noting that $|x|^{\frac{\alpha}{2}-1} \to 0$ as $|x| \to \infty$, for any fixed K, α, in order to have the (RHS) of (2.38) to be negative in $\mathbb{R}^N \setminus B_R$, it is sufficient to find γ such that

$$\gamma^2\left(1-\frac{\alpha}{2}\right)^2 N^2 + 2\gamma\left(1-\frac{\alpha}{2}\right)N\mathcal{D}_\alpha + \gamma\left(1-\frac{\alpha}{2}\right)\frac{K}{\overline{\Lambda}}(\theta+\delta) - \frac{\lambda-\delta}{\overline{\Lambda}} < 0. \quad (2.39)$$

Observe that as $\delta \to 0$, λ is arbitrarily close to $\overline{\eta}_\alpha$. Let $\overline{\gamma}$ be the positive solution of (2.14), one sees that for any for $\gamma \in (0, \overline{\gamma})$, there exist λ, $\delta > 0$ and R large enough such that (2.39) is satisfied. This implies that $\mathcal{L}_\mu w_p \le 0$ in $B_{R+p} \setminus B_R$, $\forall p > 0$.

Now, we want to estimate on the boundary of $B_{R+p} \setminus B_R$. Since $\varphi \in \overline{\mathcal{D}}_\alpha$, there exists a function $h(x) \to 0$ as $|x| \to \infty$ such that

$$\varphi(x) \ge e^{h(x)|x|^{1-\frac{\alpha}{2}}}, \qquad \text{for } |x| \ge R.$$

On the other hand, since u satisfies (2.13), for any $\kappa > 0$, there exists $C = C(\kappa) > 0$ such that

$$u(x) \le Ce^{\kappa|x|^{1-\frac{\alpha}{2}}}, \qquad \text{for } |x| \ge R.$$

Choosing arbitrarily $\gamma \in (0, \overline{\gamma})$, $\tau \in (0, \gamma)$ and $\kappa = \tau/2$, we derive, for R large,

$$Ce^{\tau|x|^{1-\frac{\alpha}{2}}}\varphi(x)/u(x) \ge e^{(\tau+h(x)-\kappa)|x|^{1-\frac{\alpha}{2}}} \ge 1, \qquad \text{for } |x| \ge R.$$

In particular, for any $p > 0$, one has

$$\begin{cases} Cw_p(x) \ge Cw_p^1(x) = Ce^{\tau(R+p)^{1-\frac{\alpha}{2}}}\varphi(x) \ge u(x) & \text{for } x \in \partial B_{R+p}, \\ Cw_p(x) \ge Cw_p^2(x) = Ce^{\tau R^{1-\frac{\alpha}{2}}}\varphi(x) \ge u(x) & \text{for } x \in \partial B_R. \end{cases}$$

To make use the maximum principle, we set $z = \dfrac{u - Cw_p}{\varphi}$. Evidently, $\mathcal{L}_\mu(z\varphi) \ge 0$ in $B_{R+p} \setminus B_R$ for all $p > 0$ and $z \le 0$ on $\partial(B_{R+p} \setminus B_R)$. Direct calculation yields

$$\begin{aligned} 0 \le \frac{\mathcal{L}_\mu(z\varphi)}{\varphi} &= \partial_{ij}z + 2\frac{\partial_j\varphi}{\varphi}\partial_i z + Kq(x)\cdot\nabla z + \frac{1}{\varphi}\left(\partial_{ij}\varphi + Kq(x)\cdot\nabla\varphi + \mu(x)\varphi\right)z \\ &\le \partial_{ij}z + 2\frac{\partial_j\varphi}{\varphi}\partial_i z + Kq(x)\cdot\nabla z - \lambda h_\alpha(x)z. \end{aligned}$$

The zero-order's coefficient of operator satisfied by z is nonpositive, for such R fixed, the maximum principle applies implies that $z \le 0$ in $B_{R+p} \setminus B_R$ for all $p > 0$. As a result,

$$\forall x \in B_{R+p} \setminus B_R, \quad u(x) \le Cw_p(x) = Ce^{(\tau-\gamma)(R+p)^{1-\frac{\alpha}{2}}}e^{\gamma|x|^{1-\frac{\alpha}{2}}}\varphi(x) + Ce^{(\tau+\gamma)R^{1-\frac{\alpha}{2}}}e^{-\gamma|x|^{1-\frac{\alpha}{2}}}\varphi(x).$$

Lastly, sending $p \to \infty$, we achieve

$$x \in \mathbb{R}^N \setminus B_R, \quad u(x) \le Ce^{(\tau+\gamma)R^{1-\frac{\alpha}{2}}} e^{-\gamma|x|^{1-\frac{\alpha}{2}}} \varphi(x).$$

In particular, since $\varphi \in \overline{\mathcal{D}}_\alpha$, u decays exponentially with the rate $O(e^{-\gamma|x|^{1-\frac{\alpha}{2}}})$.

(ii) Case 2. (H1a), (H1c) and (2.15) hold.
Let ψ be given in assumption (H1c), using the transformation

$$u(x) = e^{\psi_K(x)}\widetilde{u}(x),$$

where $\psi_K = K\psi$ satisfying $2I_N \cdot \nabla\psi_K(x) = -Kq(x)$, we convert Eq. (2.34) into self-adjoint form. Plugging u to Eq. (2.34), since $\mathrm{div} q = 0$, we derive after simplifying

$$(\mathcal{L}[u])e^{-\psi_K(x)} = \Delta\widetilde{u} + \left(-|\nabla\psi_K(x)|^2 + f_s(x,0)\right)\widetilde{u} \ge 0$$

We would like to use the same type of supersolution as of Case 1, namely

$$\widetilde{w}_p(x) = \widetilde{w}_p^1(x) + \widetilde{w}_p^2(x) = e^{(R+p)^{1-\frac{\alpha}{2}}(\tau-\kappa)} e^{\kappa|x|^{1-\frac{\alpha}{2}}} \varphi(x) + e^{R^{1-\frac{\alpha}{2}}(\tau+\kappa)} e^{-\kappa|x|^{1-\frac{\alpha}{2}}} \varphi(x),$$

with suitable positive constants R, τ, γ. Observe that, by assumption (H1a), there exists $\alpha \in [0,2)$ such that for any $\delta > 0$, there exists $R = R(\delta) > 0$ satisfying

$$f_s(x,0) \le \mu(x) + \delta|x|^{-\alpha} \qquad \text{for } |x| \ge R. \tag{2.40}$$

From (H1c), for any $\lambda \in (0, \underline{\eta})$, where $\underline{\eta} = \lambda_\alpha(-\Delta - \mu(x), \mathbb{R}^N) > 0$, we find a test function $\varphi \in \overline{\mathcal{D}}_\alpha$ and $h_\alpha \in C(\mathbb{R}^N)$ satisfying $\lim_{|x|\to\infty} h_\alpha(x)|x|^\alpha = 1$ such that

$$\Delta\varphi + \mu(x)\varphi \le -\lambda h_\alpha(x)\varphi \quad \text{for } |x| \ge R. \tag{2.41}$$

From (2.15), one has

$$|\nabla\psi_K(x)|^2 \ge \frac{K^2\theta^2 - \delta}{|x|^\alpha} \qquad \text{for } |x| \ge R = R(\delta) \text{ large enough.}$$

and thus

$$\Delta\widetilde{u} + \left(-\frac{K^2\theta^2 - \delta}{|x|^\alpha} + f_s(x,0)\right)\widetilde{u} \ge 0.$$

For $\delta > 0$, we set

$$\widetilde{\mathcal{L}}_\mu[\phi] = \Delta\phi + \mu(x)\phi - \frac{K^2\theta^2 - 2\delta}{|x|^\alpha}\phi \qquad x \in \mathbb{R}^N \setminus \{0\}.$$

Obviously, $\widetilde{\mathcal{L}}_\mu[\widetilde{u}] \ge 0$ for $|x| \ge R$. The constants λ and δ will be chosen later. As in Case 1, we want to find $\gamma > 0$ small enough and $R > 0$ large enough such that $\widetilde{\mathcal{L}}_\mu w_p \le 0$ in $B_{R+p} \setminus B_R$, $\forall p > 0$. Since $|x|^{\frac{\alpha}{2}-1} \to 0$ as $|x| \to \infty$, similar computations as (2.36)-(2.38) lead us to determine γ in such the way that

$$\kappa^2\left(1 - \frac{\alpha}{2}\right)^2 + 2\kappa\left(1 - \frac{\alpha}{2}\right)\mathcal{D}_\alpha - (\lambda + K^2\theta^2 - 2\delta) < 0.$$

As $\delta \to 0$, λ is arbitrarily close to $\underline{\eta}$. Let γ^+ be the positive solution of

$$\kappa^2 \left(1 - \frac{\alpha}{2}\right)^2 + 2\kappa \left(1 - \frac{\alpha}{2}\right) \mathcal{D}_\alpha - (\underline{\eta}_\alpha + K^2\theta^2) = 0,$$

that is

$$\gamma^+ = \frac{-2\mathcal{D}_\alpha + 2\sqrt{\mathcal{D}_\alpha^2 + \underline{\eta}_\alpha + K^2\theta^2}}{2-\alpha},$$

we see that for any $\kappa \in (0, \gamma^+)$, there exist λ, $\delta > 0$ and R large enough such that $\forall p > 0$, $\widetilde{\mathcal{L}}_\mu w_p \leq 0$ in $B_{R+p} \setminus B_R$. Also from (2.15)

$$\limsup_{|x|\to\infty} \frac{|\psi(x)|}{|x|^{1-\frac{\alpha}{2}}} = \frac{2\theta}{2-\alpha}, \qquad \theta < \frac{\underline{\eta}_\alpha}{2K\mathcal{D}_\alpha},$$

hence for any $\kappa \in (\frac{2K\theta}{2-\alpha}, \gamma^+)$, we readily find $\lambda \in (0, \underline{\eta}_\alpha)$ and $\delta > 0$ such that

$$\limsup_{|x|\to\infty} \frac{|\psi_K(x)|}{|x|^{1-\frac{\alpha}{2}}} \leq \frac{2K\theta}{2-\alpha} < \kappa = \frac{-2\mathcal{D}_\alpha + 2\sqrt{\mathcal{D}_\alpha^2 + \lambda + K^2\theta^2 - 2\delta}}{2-\alpha} < \gamma^+. \tag{2.42}$$

The estimates on the boundary $B_{R+p} \setminus B_R$ are similar in Case 1. Since u satisfies (2.13), for any $0 < \varsigma$, there exists $C = C(\varsigma) > 0$ such that

$$u(x) \leq Ce^{\varsigma|x|^{1-\frac{\alpha}{2}}}, \qquad \text{in } \mathbb{R}^N.$$

We choose τ and κ such that $0 < -\frac{2K\theta}{2-\alpha} + \tau < -\frac{2K\theta}{2-\alpha} + \kappa < \underline{\gamma}$ and $\varsigma = \frac{1}{2}(-\frac{2K\theta}{2-\alpha} + \tau)$. Hence, for R large, one has

$$\frac{Ce^{\tau|x|^{1-\frac{\alpha}{2}}}\varphi(x)}{\widetilde{u}(x)} = \frac{Ce^{\tau|x|^{1-\frac{\alpha}{2}}}\varphi(x)e^{\psi_K(x)}}{u(x)} = e^{(\tau - \frac{2K\theta}{2-\alpha} - \varsigma + h(x))|x|^{1-\frac{\alpha}{2}}} \geq 1 \qquad \text{for } |x| \geq R.$$

In particular, for any $p > 0$, $C\widetilde{w}_p \geq \widetilde{u}$ in $\partial(B_{R+p} \setminus B_R)$. The remainder of the proof goes exactly as of Case 1. As a consequence, by letting $p \to \infty$, one has

$$e^{-\psi_K(x)}u(x) = \widetilde{u}(x) \leq Ce^{R^{1-\frac{\alpha}{2}}(\tau+\kappa)}e^{-\kappa|x|^{1-\frac{\alpha}{2}}}\varphi(x) \qquad \text{in } \mathbb{R}^N \setminus B_R.$$

Due to (2.42), for R large,

$$u(x) \leq Ce^{R^{1-\frac{\alpha}{2}}(\tau+\kappa)}e^{-\kappa|x|^{1-\frac{\alpha}{2}}}e^{\psi_K(x)}\varphi(x) \leq Ce^{R^{1-\frac{\alpha}{2}}(\tau+\kappa)}e^{-(\kappa - \frac{2K\theta}{2-\alpha})|x|^{1-\frac{\alpha}{2}}}\varphi(x) \qquad \text{in } \mathbb{R}^N \setminus B_R. \tag{2.43}$$

Set

$$\gamma = \kappa - \frac{2K\theta}{2-\alpha} \in (0, \underline{\gamma}) \qquad \underline{\gamma} = \frac{-2\mathcal{D}_\alpha + 2\sqrt{\mathcal{D}_\alpha^2 + \underline{\eta}_\alpha + K^2\theta^2}}{2-\alpha} - \frac{2K\theta}{2-\alpha},$$

we achieve statement ii). Since $\varphi \in \overline{\mathcal{D}}_\alpha$, $u(x)$ decays exponentially and the proof is thereby complete. □

Theorem 2.3.2. *Let $\alpha = 2$ and Hypothesis (H1) hold. Let u be a nonnegative subsolution of Eq. (2.34) satisfying (2.17). One of the followings holds true*

i) If (H1a)-(H1b) and $\limsup_{|x|\to\infty} |q(x)||x| \le \theta$ are fulfilled, then for all $\beta \in (0, \overline{\sigma})$, where $\overline{\sigma}$ is given in (2.18), one finds $\varphi \in \overline{\mathcal{D}}_2$, $C(\beta) > 0$, $R(\beta) > 0$

$$u(x) \le C(\beta)|x|^{-\beta}\varphi(x) \qquad \text{in } |x| > R(\beta).$$

ii) If (H1a), (H1c), (2.19), (2.20) are fulfilled, then for all $\beta \in (0, \overline{\beta})$, where $\overline{\beta}$ is given by

$$\overline{\beta} = \frac{(N-2+2\mathcal{D}_2) + \sqrt{(N-2+2\mathcal{D}_2)^2 + 4\underline{\eta}_2 + 4K^2\theta^2}}{2} - K\theta, \tag{2.44}$$

one finds $\varphi \in \overline{\mathcal{D}}_2$, $C(\beta) > 0$, $R(\beta) > 0$

$$e^{-\psi_K(x)}u(x) \le C(\beta)|x|^{-(\beta+K\theta)}\varphi(x) \quad \text{and} \quad u(x) \le C(\beta)|x|^{-\beta}\varphi(x) \qquad \text{in } |x| > R(\beta).$$

In particular, $u(x)$ decays polynomially.

Proof. The scheme of the proof of Theorem 2.3.1 can be applied, thus we only outline the points, which are mainly different from the former.

(i) Case 1: (H1a)-(H1b) and $\limsup_{|x|\to\infty} |q(x)||x| \le \theta$

By Hypothesis (H1), there exists $\mu \in L^\infty(\mathbb{R}^N)$ such that for any $\delta > 0$, one finds $R = R(\delta) > 0$ satisfying

$$f_s(x,0) \le \mu(x) + \delta|x|^{-2} \quad \text{and} \quad q(x)|x| \le (\theta + \delta) \qquad \text{for } |x| \ge R.$$

Assumption (H1b) yields that for any $\lambda \in (0, \overline{\eta}_2)$, one finds a test function $\varphi \in \overline{\mathcal{D}}_2$ and $h_2 \in C(\mathbb{R}^N)$ satisfying $\lim_{|x|\to\infty} h_2(x)|x|^2 = 1$ such that

$$a_{ij}(x)\partial_{ij}\varphi + Kq(x)\cdot\nabla\varphi + \mu(x)\varphi \le -\lambda h_2(x)\varphi \quad \text{for } |x| \ge R.$$

Set

$$\mathcal{L}_\mu[\phi] = a_{ij}(x)\partial_{ij}\phi + Kq(x)\cdot\nabla\phi + (\mu(x)+\delta)|x|^{-2}\phi,$$

obviously, for any $p > 0$, one has $\mathcal{L}_\mu[u] \ge 0$ in $B_{R+p} \setminus B_R$. It is expected to find a supersolution of Eq. (2.34) in $B_{R+p} \setminus B_R$ of the form

$$w_p(x) = w_p^1(x) + w_p^2(x) = \frac{C}{\ln(R+p)}|x|^\beta\varphi(x) + C|x|^{-\beta}\varphi(x), \qquad \beta > 0.$$

Direct computations similarly to Case 1 of Theorem 2.3.1, we see that w_p is a supersolution of Eq. (2.34) if β satisfies the following algebraic inequality

$$N\overline{\Lambda}\beta^2 + (3N\overline{\Lambda} + 2\overline{\Lambda}\mathcal{D}_2 + K\theta)\beta - \lambda + \delta \le 0$$

where the constant $\mathcal{D}_2$ appears due to $\varphi \in \overline{\mathcal{D}}_2$. As $\delta \to 0$, λ is arbitrarily close to $\overline{\eta}_2$, thus β necessarily satisfies

$$0 < \beta < \overline{\sigma} = \frac{-(3N\overline{\Lambda} + 2\overline{\Lambda}\mathcal{D}_2 + K\theta) + \sqrt{(3N\overline{\Lambda} + 2\overline{\Lambda}\mathcal{D}_2 + K\theta)^2 + 4\overline{\Lambda}N\overline{\eta}_2}}{2N\overline{\Lambda}}.$$

Thanks to (2.17), for all $\sigma > 0$, there exists $C = C(\sigma)$ such that

$$u(x) \le C(\sigma)|x|^{\sigma} \qquad \text{for} \quad |x| \ge R$$

and the growth condition of φ yields that there exist $g_1(x), g_2(x) \to 0$ as $|x| \to \infty$ and large R

$$|x|^{g_1(x)} \le \varphi(x) \le |x|^{g_2(x)} \qquad |x| \ge R.$$

Fix such R, choosing arbitrarily $\beta \in (0, \overline{\sigma})$ and $\sigma \in (0, \beta)$, then

$$\liminf_{|x|=R+p, p\to\infty} \frac{w_p}{u} \ge \liminf_{|x|=R+p, p\to\infty} \frac{w_p^1}{u} \ge \liminf_{|x|=R+p, p\to\infty} \frac{C|x|^{\sigma}}{u(x)} \frac{|x|^{\beta-\sigma}|x|^{g_1(x)}}{\ln(R+p)} = \infty.$$

By using the same argumentation as of Case 1 of Theorem 2.3.1 and the weak comparison principle, we imply that

$$u(x) \le w_p(x) \qquad x \in B_{R+p} \setminus B_R,$$

where $C = C(\beta)$ is a large positive constant independent of u and p.

By letting $p \to \infty$, thanks to the growth assumption of φ, we obtain the polynomial decay

$$u(x) \le C|x|^{-\beta}\varphi(x) \qquad |x| > R(\beta).$$

(ii) Case 2 : (H1a), (H1c) and (2.19)

We omit the computations because we use the same technique as of Case 2 of Theorem 2.3.1. However, we will precisely provide the type of the supersolution on $B_{R+p} \setminus B_R$, which is important to derive the decay of solution.

Thanks to $\mathrm{div} q = 0$, we can convert Eq. (2.34) into self-adjoint form by the transformation

$$\widetilde{u}(x) = e^{-\psi_K(x)} u(x),$$

where $\psi_K = K\psi$ satisfying $2I_N \cdot \nabla\psi_K(x) = -Kq(x)$, namely $\widetilde{u}$ satisfies

$$\Delta\widetilde{u} + \left(-|\nabla\psi_K(x)|^2 + f_s(x,0)\right)\widetilde{u} \ge 0 \qquad x \in \mathbb{R}^N.$$

For $C\beta_1, \beta_2 > 0$, $\varphi \in \overline{\mathcal{D}}_2$, by plugging the function

$$w_p = w_p^1 + w_p^2 = \frac{C}{(\ln(R+p))^2}|x|^{\beta_1}\varphi(x) + C|x|^{-\beta_2}\varphi(x),$$

to Eq. (2.34), we see that w_p is a supersolution of (2.34) if β_1 and β_2 satisfy the following algebraic equations for some $\delta > 0$

$$\beta_1^2 + (N - 2 + 2\mathcal{D}_2)\beta_1 - (\underline{\eta}_2 + K^2\theta^2) + \delta \le 0$$
$$\beta_2^2 - (N - 2 + 2\mathcal{D}_2)\beta_2 - (\underline{\eta}_2 + K^2\theta^2) + \delta \le 0.$$

Necessarily,

$$0 < \beta_1 < \beta_1^+ = \frac{-(N - 2 + 2\mathcal{D}_2) + \sqrt{(N - 2 + 2\mathcal{D}_2)^2 + 4\underline{\eta}_2 + 4K^2\theta^2}}{2}$$

$$0 < \beta_2 < \beta_2^+ = \frac{(N - 2 + 2\mathcal{D}_2) + \sqrt{(N - 2 + 2\mathcal{D}_2)^2 + 4\underline{\eta}_2 + 4K^2\theta^2}}{2} = \overline{\beta} + K\theta.$$

By (2.19), we have $\theta + \frac{\theta}{2\mathcal{D}_2}(N-2) < \frac{\eta_2}{2K\mathcal{D}_2}$, which implies

$$\limsup_{|x|\to\infty} \frac{\ln \widetilde{u}(x)}{\ln |x|} \leq K\theta + \limsup_{|x|\to\infty} \frac{\ln u(x)}{\ln |x|} < \beta_1^+.$$

Taking arbitrary constants $\beta_1 \in (K\theta, \beta_1^+)$ and $\beta_2 \in (K\theta, \beta_2^+)$, there exist $C = C(\beta), R = R(\beta) > 0$, $\varphi \in \overline{\mathcal{D}}_2$ large enough such that for all $p > 0$

$$\widetilde{u}(x) \leq \frac{C}{(\ln(R+p))^2}|x|^{-\beta_1}\varphi(x) + C|x|^{-\beta_2}\varphi(x) \qquad x \in B_{R+p} \setminus B_R,$$

where we applied the comparison principle on $B_{R+p} \setminus B_R$. Note that $K\theta < \beta_2^+$ and $\beta_2 - K\theta \in (0, \overline{\beta})$. By sending $p \to \infty$, thanks to the growth rate of φ, we obtain the polynomial decay:

$$u(x) \leq C|x|^{-(\beta_2 - K\theta)}\varphi(x) \qquad |x| > R(\beta). \tag{2.45}$$

□

2.4 Existence/Nonexistence and Uniqueness

In this section, we investigate the existence/nonexistence and uniqueness of Eq. (2.1) for every fixed constant K. Since Eq. (2.1) is not self-adjoint in general, the variational approach, which was used in Theorem 2.3 [18], does not succeed to apply here. However, replied on the a-priori estimates in Section 2, we are able to prove Theorem 2.1.2 by exploiting the strong maximum principle. To this aim, we will prove a comparison principle, which covers both two cases of Hypothesis 1. First, we observe that, for any fixed K, from condition (H1c), u is a solution (resp. sub or supersolution) of Eq. (2.1) if and only if $\widetilde{u}(x) = e^{-\psi_K(x)}u(x)$ with $\psi_K = K\psi$ is a solution (resp. sub or supersolution) of following equation

$$\Delta\widetilde{u}(x) - |\nabla\psi_K(x)|^2\widetilde{u} + f(x, \widetilde{u}e^{\psi_K(x)})e^{-\psi_K(x)} = 0 \tag{2.46}$$

Therefore, the existence/nonexistence and uniqueness of Eq. (2.46) is equivalent to Eq. (2.1). It will be convenient to denote the following operators and functions

$$\mathcal{M}_1[u] = a_{ij}(x)\partial_{ij}u(x) + Kq(x)\cdot\nabla u \quad \text{and} \quad \mathcal{M}_2[\widetilde{u}] = \Delta\widetilde{u} - |\nabla\psi_K(x)|^2\widetilde{u};$$

$$f_1(x,s) = f(x,s) \quad \text{and} \quad f_2(x,s) = f(x, se^{\psi_K(x)})e^{-\psi_K(x)}.$$

Our comparison principle is based on the ideas of the analogous result, Theorem 3.3 in [18]. However, the interesting novelties are due to the new hypothesis (H1), which allows $f_s(x,0)$ to change sign and it possibly decays fast up to Hardy potential beside the presence of the nonconstant drift term and nonconstant behavior near infinity of $f_s(x,0)$.

Theorem 2.4.1. *Let $\alpha \in [0,2]$, u_i and v_i, $i \in \{1,2\}$ respectively satisfy the following inequalities*

$$\mathcal{M}_i[u_i] + f_i(x,u_i) \geq 0 \quad \textit{and} \quad \mathcal{M}_i[v_i] + f_i(x,v_i) \leq 0 \qquad \textit{a.e in } \mathbb{R}^N.$$

Assume that Hypothesis (H2) holds and $\inf_{B_r} v_i(x) > 0$ for all r and $i \in \{1,2\}$. Then, if (H1a) and (H1b) hold, one has $u_1(x) \leq v_1(x)$ in $\mathbb{R}^N$. Alternatively, if (H1a), (H1c) and (2.15) hold, then $u_2(x) \leq v_2(x)$ in $\mathbb{R}^N$.

Proof. Thanks to Hypothesis (H2), one has $\partial_s f_i(x,0)s \geq f_i(x,s)$ for $i \in \{1,2\}$. Therefore,

$$\mathcal{M}_i[u] + \partial_s f_i(x,0)u_i \geq 0 \qquad \text{in } \mathbb{R}^N.$$

Since $\partial_s f_i(x,0)$ satisfies Hypothesis (H1), there exists $\alpha \in [0,2]$ such that for any $\delta > 0$, there exists $R = R(\delta) > 0$ satisfying

$$\partial_s f_i(x,0) \leq \mu(x) + \delta(|x|+1)^{-\alpha} \qquad \text{for } |x| \geq R.$$

One finds $\lambda_i > 0$, $\varphi_i \in \overline{\mathcal{D}}_\alpha$ and $h_\alpha \in C(\mathbb{R}^N)$ such that $\limsup_{|x|\to\infty} h_\alpha(x)|x|^\alpha = 1$ and

$$\mathcal{M}_i[\varphi_i] + \mu(x)\varphi_i \leq -\lambda_i h_\alpha(x)\varphi_i \quad \text{for } |x| \geq R. \tag{2.47}$$

From Theorem (2.3.1)-(2.3.2), there exists a function $g_i^\alpha(x) > 0$ such that for R large enough

$$u_i(x) \leq g_i^\alpha(x)\varphi_i(x), \qquad \text{in } \mathbb{R}^N \setminus B_R,$$

where $g_i^\alpha(x) = C_i e^{-\gamma_i |x|^{1-\frac{\alpha}{2}}}$ if $\alpha \in [0,2)$ and $g_i^\alpha(x) = D_i|x|^{-\beta_i}$ if $\alpha = 2$ for some $C_i, \gamma_i, D_i, \beta_i > 0$.

Let ε be an arbitrary positive constant, we define the set

$$K_\varepsilon^i := \{k > 0 : kv_i \geq u_i - \varepsilon\varphi_i \text{ in } \mathbb{R}^N\}.$$

Evidently, K_ε^i is nonempty because $g_i^\alpha(x) \to 0$ as $|x| \to \infty$, $u_i > 0$ in B_r, $\forall r > 0$ and

$$u_i(x) - \varepsilon\varphi_i(x) \leq (g_i^\alpha(x) - \varepsilon)\varphi_i \leq 0, \qquad \text{for } |x| \geq R(\varepsilon) \text{ large enough.} \tag{2.48}$$

Hence there exists the infimum $k_i(\varepsilon) := \inf K_\varepsilon^i$. The function $k_i : \mathbb{R}^+ \to \mathbb{R}$ is nonincreasing. Assume by contradiction that

$$k_i^* = \lim_{\varepsilon\to 0^+} k_i(\varepsilon) > 1.$$

Taking $0 < \varepsilon < \sup_{\mathbb{R}^N}\{u_i/\varphi\}$, we get $k_i(\varepsilon) > 0$ and $w_i^\varepsilon = k_i(\varepsilon)v_i - u_i + \varepsilon\varphi_i \geq 0$. By the definition of $k_i(\varepsilon)$ one finds a sequence $(x_n^\varepsilon) \in \mathbb{R}^N$ such that

$$\left(k_i(\varepsilon) - \frac{1}{n}\right) v_i(x_n^\varepsilon) < u_i(x_n^\varepsilon) - \varepsilon\varphi_i(x_n^\varepsilon).$$

Fix $\varepsilon > 0$, (2.48) implies that $(x_n^\varepsilon) \in B_{R(\varepsilon)}$ for n large enough. Thus, it converges up to a subsequence to some $x(\varepsilon) \in B_{R(\varepsilon)}$. Then, one finds

$$w_i^\varepsilon(x(\varepsilon)) = 0. \tag{2.49}$$

The case that $\liminf_{\varepsilon\to 0^+} |x(\varepsilon)| < \infty$ is ruled out. In fact, if there is $x_0 \in \mathbb{R}$ such that $|x_0| = \liminf_{\varepsilon\to 0^+} |x(\varepsilon)|$, by (2.49), $k^* < \infty$, the function $w_i^\infty = k^* v_i - u_i$ is nonnegative, vanishes at x_0 and satisfies

$$-\mathcal{M}_i[w_i^\infty] \geq k_i^* f_i(x,v_i) - f_i(x,u_i) \geq [f_i(x,k_i^* v_i) - f_i(x,u_i)] \geq \sigma_i(x) w_i^\infty,$$

where $\sigma_i(x) \in L^\infty_{\text{loc}}(\mathbb{R}^N)$. But, by Hypothesis 2, the inequality is strictly in D with $|D| > 0$. This contradicts to the strong maximum principle, which asserts that $w_i^\infty \equiv 0$ in $\mathbb{R}^N$. It remains to consider the case $\lim_{\varepsilon\to 0^+} |x(\varepsilon)| = \infty$. As above, we know that $w_i^\varepsilon \geq 0$ and it vanishes at $x(\varepsilon) \in B_{R(\varepsilon)}$. Hence, there

exists a neighborhood $\mathcal{O}(x(\varepsilon))$ of $(x(\varepsilon))$ such that $k_i(\varepsilon)v_i < u_i$ in $\mathcal{O}(x(\varepsilon))$. For ε small enough, $k_i(\varepsilon) > 1$, we derive from Hypothesis 2, for $x \in \mathcal{O}(x(\varepsilon))$:

$$\begin{aligned} - \quad & \mathcal{M}_i[w_i^\infty] \geq k_i(\varepsilon) f_i(x, v_i) - f_i(x, u_i) + \mu(x)\varepsilon\varphi_i + \lambda_i h_\alpha(x)\varepsilon\varphi_i \\ \geq \quad & f_i(x, k_i(\varepsilon)v_i) - f_i(x, u_i) + \mu(x)\varepsilon\varphi_i + \lambda_i h_\alpha(x)\varepsilon\varphi_i \\ \geq \quad & \frac{f_i(x, k(\varepsilon)v_i)}{k_i(\varepsilon)v_i}(k(\varepsilon)v_i - u_i + \varepsilon\varphi_i) + \left(\frac{\lambda_i}{2}h_\alpha(x) + \mu(x) - \frac{f_i(x, k(\varepsilon)v_i)}{k(\varepsilon)v_i}\right)\varepsilon\varphi_i + \frac{\lambda_i}{2}h_\alpha(x)\varepsilon\varphi_i. \quad (2.50) \end{aligned}$$

Taking $0 < \varepsilon \ll 1$ then $|x(\varepsilon)|\gamma 1$, we derive from (H1a) and Hypothesis (H2) that

$$\frac{f_i(x, k(\varepsilon)v_i)}{k(\varepsilon)v_i} \leq \partial_s f_i(x, 0) < \mu(x) + \frac{\lambda_i h_\alpha(x)}{2}, \quad \forall x \in \mathcal{O}(x(\varepsilon)),$$

shrinking $\mathcal{O}$ if necessary. We get from (2.50) that

$$-\mathcal{M}_i[w_i^\varepsilon] - \varrho_i(x)w_i^\varepsilon > 0 \quad \text{in } \mathcal{O}(x(\varepsilon)),$$

where $\varrho_i(x) = \frac{f_i(x,k(\varepsilon)v_i)}{k(\varepsilon)v_i} \in L^\infty(\mathcal{O}(x(\varepsilon)))$. This contradicts again to the strong maximum principle, which asserts that $w_i^\varepsilon \equiv 0$ in $\mathcal{O}(x(\varepsilon))$. Therefore, $k_i^* > 1$ is ruled out, we actually have $k_i^* \leq 1$.

Letting $\varepsilon \to 0^+$,

$$u_i \leq \lim_{\varepsilon \to 0^+} (k_i(\varepsilon)v_i + \varepsilon\varphi_i) \leq v_i \quad \text{in } \mathbb{R}^N.$$

This completes the proof of theorem. □

Proof of Theorem 2.1.2. Theorem (2.4.1) implies in particular that Eq. (2.1) cannot possess more than one positive solution. It remains to prove the existence and nonexistence of Eq. (2.1).

• $\lambda_1(-\mathcal{L}, \mathbb{R}^N) \geq 0$: nonexistence.

Observe that, if (H1c) holds, the transformation $\widetilde{u}(x) = e^{-\psi_K(x)}u(x)$ converts Eq. (2.1) to be self-adjoint, namely

$$\Delta\widetilde{u} - |\psi_K(x)|^2\widetilde{u} + f(x, \widetilde{u}e^{\psi_K(x)})e^{-\psi_K(x)} = 0. \quad (2.51)$$

Using the same notations $\mathcal{M}_i$ and f_i as before, we see that

$$\lambda_1(-\mathcal{M}_i - \partial_s f_i(x, 0), \mathbb{R}^N) = \lambda_1(-\mathcal{L}, \mathbb{R}^N) \geq 0.$$

We argue by contradiction that there exists at least a positive solution u of Eq. (2.1) or Eq. (2.51). Let us call ϕ_i, $i \in \{1, 2\}$, with normalization $\phi_i(0) < u(0)$, the eigenfunctions associated with $\lambda_1(-\mathcal{M}_i - \partial_s f_i(x, 0), \mathbb{R}^N)$, which are obtained by Theorem (2.2.3). Hypothesis 2 yields

$$\mathcal{M}_i[\phi_i] + f_i(x, \phi_i) \leq \mathcal{M}_i[\phi_i] + \partial_s f_i(x, 0)\phi_i = -\lambda_1(-\mathcal{M}_i - \partial_s f_i(x, 0), \mathbb{R}^N)\phi_i \leq 0.$$

Therefore, ϕ_i is a supersolution of the equation

$$\mathcal{M}_i[u] + f_i(x, u) = 0.$$

Theorem (2.4.1) applies, we get $u(x) \leq \phi_i(x)$ in $\mathbb{R}^N$. This contradiction confirms the nonexistence.

• $\lambda_1(-\mathcal{L}, \mathbb{R}^N) < 0$: existence.

Thanks to the property (i) of Theorem (2.2.1), we have $\lambda_1(-\mathcal{L}, \mathbb{R}^N) \geq \lambda_1'(-\mathcal{L}, \mathbb{R}^N)$. Whence, $\lambda_1'(-\mathcal{L}, \mathbb{R}^N) < 0$. This implies that there exists a bounded test function $\phi > 0$ and a negative constant λ close enough to $\lambda_1'(-\mathcal{L}, \mathbb{R}^N)$ such that

$$a_{ij}(x)\partial_{ij}\phi(x) + Kq_i\phi_i(x) + f_s(x,0)\phi(x) + \lambda\phi(x) \geq 0.$$

By assumption $f(x,s)$ is $C^{1,\gamma}(\mathbb{R}^N)$ uniformly with respect to $s \in [0,\delta]$ for some $\delta > 0$, one sees that the function $\widetilde{\phi} = \kappa\phi$ with κ small enough is a subsolution of Eq. (2.1), namely

$$a_{ij}(x)\partial_{ij}\widetilde{\phi}(x) + Kq_i\widetilde{\phi}_i(x) + f(x,\widetilde{\phi}_i) \geq 0 \qquad \text{in } \mathbb{R}^N.$$

On the other hand, by Theorem (2.3.1), we know that there exists positive constants C, γ and $\varphi \in \overline{\mathcal{D}}_\alpha$ such that for R large enough

$$\phi(x) \leq Ce^{-\gamma|x|^{1-\frac{\alpha}{2}}}\varphi(x) \qquad \text{for } |x| \geq R.$$

Thanks to Hypothesis 3, there exists a positive supersolution, $S_K(x)$ of Eq. (2.1) satisfying

$$\lim_{|x|\to\infty} \ln|S_K(x)||x|^{-1+\frac{\alpha}{2}} = 0.$$

Since $\lim_{|x|\to\infty} \ln\varphi(x)|x|^{-1+\frac{\alpha}{2}} = 0$, one has

$$\liminf_{|x|\to\infty} \frac{S_K(x)}{\phi(x)} \geq \liminf_{|x|\to\infty} \frac{S_K(x)e^{\frac{\gamma}{2}|x|^{1-\frac{\alpha}{2}}}}{Ce^{-\frac{\gamma}{2}|x|^{1-\frac{\alpha}{2}}}\varphi(x)} = \infty.$$

Therefore, there exists $R > 0$ large enough such that $S_K(x) \geq \phi(x)$ in $\mathbb{R}^N \setminus B_R$. Obviously, for κ sufficiently small, $S_K(x) \geq \widetilde{\phi}(x)$ in $\mathbb{R}^N$. The classical iteration method allows one to find a positive solution u for Eq. (2.1) satisfying $\widetilde{\phi} \leq u \leq S_K$. Theorem (2.3.1) implies that $u(x)$ decays exponentially as $|x| \to \infty$ and this completes the proof. □

Remark 11. The existence/nonexistence of positive bounded solution are still true when the supersolution $S_K(x)$ is allowed to satisfy

$$\limsup_{|x|\to\infty} \ln(S_K(x))|x|^{1-\frac{\alpha}{2}} < \gamma^\star,$$

where $\gamma^\star$ is given in (i) of Theorem (2.1.2). In fact, due to Theorem (2.3.1), this growth condition is sufficient for proving exponential decay once a solution exists. Here, we use the lower bound of $S_K(x)$ to construct a positive solution between $\widetilde{\phi}$ and S_K, but there is another way to construct a compactly supported subsolution and then using the strong maximum principle to derive the positivity on $\mathbb{R}^N$ for solutions. We refer to [18] for this construction.

As previously mentioned, Theorem (2.1.3) only differs from Theorem 2.1.2 at the growth condition of supersolution $S_K(x)$ given by Hypothesis (H3). For $\alpha = 2$, Hypothesis (H3) provides the existence of supersolution, which grows slower than any polynomials and it is enough due to Theorem (2.3.2) to derive the polynomial decay for solutions of Eq. (2.1). This result presents the fact that if $f_s(x,0)$ decays fast enough, for instance, like a Hardy potential, existence of positive bounded solutions for Eq. (2.1) may not be derived from the existence of fast growing supersolution. Theorem (2.1.3) is actually a direct consequence of Theorem (2.3.2) and Theorem (2.1.2). Its proof is omitted.

2.5 Sharp threshold of drift

In this section, we prove the existence of sharp threshold of drift term $K^\star$ for which Eq. (2.1) (when (a_{ij}) is the identity matrix) admits positive solutions. From Theorem (2.1.2), we know that the existence of positive solutions of Eq. (2.1) is straightly conditioned by the sign of the generalized principal eigenvalue of the linearized operator of Eq. (2.1) about 0. For the simplicity of presentation, we denote by

$$\mathcal{L}_K[\phi] = \Delta\phi + Kq_i\phi_i + f_s(x,0)\phi,$$

$\lambda_K = \lambda_1(-\mathcal{L}_K, \mathbb{R}^N)$ and φ_K is an associated eigenfunction of λ_K given by Theorem (2.2.3), namely

$$\mathcal{L}_K[\varphi_K] = -\lambda_K\varphi_K \qquad \text{in } \mathbb{R}^N. \tag{2.52}$$

In order to prove the existence of $K^\star$, we aim at showing that

- There exist K_1 and K_2 such that $\lambda_{K_1} < 0$ and $\lambda_{K_2} \geq 0$.
- λ_K is continuous and nondecreasing with respect to K.

These facts are indeed the conclusions of the following lemmas :

Lemma 2.5.1. *λ_K is continuous and nondecreasing with respect to K. Moreover,*

$$\lim_{K\to 0}\lambda_K = \inf_{\phi\in H^1(\mathbb{R}^N),\|\phi\|_{L^2(\mathbb{R}^N)}=1}\int_{\mathbb{R}^N}|\nabla\phi|^2 - f_s(x,0)\phi^2.$$

Lemma 2.5.2. *Let $\alpha\in[0,2)$, $K>0$ be such that $\lambda_K<0$. If (H1a), (H1c), (2.15) with $\theta=0$ hold, then one can normalize φ_K in such the way $\sup_{\mathbb{R}^N}\varphi_K = 1$ and there exist positive constants γ, $C(\gamma)$ independent of K such that*

$$\varphi_K(x) \leq C(\gamma)e^{-\gamma|x|^{1-\frac{\alpha}{2}}}, \qquad \text{in } \mathbb{R}^N.$$

Thus, for such K, φ_K decays exponentially.

Lemma 2.5.3. *Let $\alpha\in[0,2)$, $\mathcal{I}_q$ be the set of all first integrals defined in (2.21). If (H1a), (H1c), (2.15) with $\theta=0$ hold and*

$$\inf_{\phi\in\mathcal{I}_q}\frac{\int_{\mathbb{R}^N}|\nabla\phi|^2 - f_s(x,0)\phi^2}{\int_{\mathbb{R}^N}\phi^2} > 0,$$

with the convention that the infimum is $+\infty$ if $\mathcal{I}_q$ being empty. Then there exists $K^\star>0$ such that $\lambda_K\geq 0$ for $K\geq K^\star$.

Lemma 2.5.4. *Let $\alpha=2$, $K>0$ be such that $\lambda_K<0$. If (H1a), (H1c), (2.19) with $\theta=0$ hold. For any $\beta\in(0,\overline{\beta})$, where $\overline{\beta}$ given in (2.44), one can normalize φ_K in such the way $\sup_{\mathbb{R}^N}\varphi_K=1$ and there exists $C(\beta)>0$ independent of K such that*

$$\varphi_K(x) \leq C(\beta)|x|^{-\beta}\phi(x), \qquad \text{in } \mathbb{R}^N.$$

Thus, for such K, φ_K decays polynomially .

Lemma 2.5.5. *Let $\alpha = 2$, $\mathcal{I}_q$ be the set of all first integrals defined in (2.21). If (H1a), (H1c), (2.19) with $\theta = 0$ hold and*

$$\inf_{\phi \in \mathcal{I}_q} \frac{\int_{\mathbb{R}^N} |\nabla \phi|^2 - f_s(x,0)\phi^2}{\int_{\mathbb{R}^N} \phi^2} > 0,$$

with the convention that the infimum is $+\infty$ if $\mathcal{I}_q$ being empty. Then the conclusion of Lemma (2.5.3) remains true.

Assume for the moment that these lemmas hold true, we go through to the

Proof of Theorem 2.1.5 and 2.1.6. The proof is a transparent consequence of above lemmas. By Theorem (2.1.2) and (2.1.3), Eq. (2.1) possesses positive solutions if and only if $\lambda_K < 0$. Hence, if λ_K is continuous, monotone in K and there exists K_1, K_2 such that $\lambda_{K_1} < 0$ and $\lambda_{K_2} \geq 0$, one finds immediately $K^\star$ such that $\lambda_{K^\star} = 0$. Therefore, we imply that Eq. (2.1) possesses positive solutions if and only if $K < K^\star$. The lemmas (2.5.1)-(2.5.5) confirm our statement. □

Now, we respectively prove above-mentioned lemmas

Proof of Lemma 2.5.1. Thanks to (H1c), let $\psi_K = K\psi$, the transformation $\widetilde{\varphi}_K(x) = e^{-\psi_K(x)}\varphi_K(x)$ converts Eq. (2.52) into the self-adjoint form

$$\mathcal{L}[\widetilde{\varphi}_K] = \Delta\widetilde{\varphi}_K + \left(-|\nabla\psi_K|^2 + f_s(x,0)\right)\widetilde{\varphi}_K = \Delta\widetilde{\varphi}_K + \left(-\frac{K^2}{4}|q(x)|^2 + f_s(x,0)\right)\widetilde{\varphi}_K = -\lambda_K\widetilde{\varphi}_K \qquad (2.53)$$

Clearly, λ_K is increasing with respect to K. Since $|q(x)|$ and $f_s(x,0)$ are bounded, one has the uniform convergence

$$-\frac{K^2}{4}|q(x)|^2 + f_s(x,0) \to f_s(x,0) \qquad \text{as } K \to 0.$$

Theorem (2.2.2) implies that λ_K is continuous with respect to K and $\lambda_K \to \lambda_1(-\Delta - f_s(x,0), \mathbb{R}^N)$ as $K \to \infty$. Finally, the variational characterization of $\lambda_1(-\Delta - f_s(x,0), \mathbb{R}^N)$ is derived from the part (iii) of Theorem (2.2.1). □

Proof of Lemma 2.5.2. From (H1a), there exists $\alpha \in [0,2)$ such that for any $\delta > 0$, there exist $R(\delta)$ such that

$$f_s(x,0) \leq \mu(x) + \delta|x|^{-\alpha} \qquad \text{for } |x| \geq R(\delta).$$

By (H1c), there exist $\phi \in \overline{\mathcal{D}}_\alpha$, $\xi \in (0, \underline{\eta}_\alpha)$ and $R_1 > 0$ such that

$$\Delta\phi + \mu(x)\phi + \xi h_\alpha(x)\phi \leq 0 \qquad \text{in } \mathbb{R}^N \setminus B_{R_1}.$$

Set $z_K = \widetilde{\varphi}_K/\phi$, one obtains, by direct computation from (2.53) that

$$(\mathcal{L}'_K + \lambda_K)[z_K] = \Delta z_K + 2\nabla z_K \cdot \frac{\nabla\phi}{\phi} + z_K\frac{\Delta\phi}{\phi} + \left(-|\nabla\psi_K|^2 + f_s(x,0)\right)z_K + \lambda_K z_K = 0. \qquad (2.54)$$

Therefore, z_K is an eigenfunction of $\mathcal{L}'_K$ associated with the principal eigenvalue λ_K. Since $\lambda_K < 0$, choosing $\delta < \xi$, the following inequalities hold

$$\lim_{r\to\infty} \lambda_1(-\mathcal{L}'_K, \mathbb{R}^N \setminus \overline{B}_r) \geq \lim_{r\to\infty} \lambda''_1(-\mathcal{L}'_K, \mathbb{R}^N \setminus \overline{B}_r) \geq \lim_{r\to\infty}(-\sup_{\mathbb{R}^N\setminus\overline{B}_r} c(x)) \geq 0 > \lambda_K,$$

where

$$c(x) = \frac{\Delta\phi + f_s(x,0)\phi(x) - |\nabla\psi_K|^2\phi}{\phi} \leq (-\xi+\delta)h_\alpha(x) \leq 0 \quad \text{as } |x| \to \infty.$$

Theorem 2.2.5 implies that z_K has minimal growth at infinity. Moreover, for $|x| \geq R(\delta)$

$$(\mathcal{L}'_K + \lambda_K)[1] \leq \frac{\Delta\phi + (\mu(x) + \delta(|x|+1)^{-\alpha})\phi}{\phi} + \lambda_K \leq (-\xi+\delta)(|x|+1)^{-\alpha} + \lambda_K < 0.$$

Definition (2.2.4) yields that z_K is bounded. Since $\theta = 0$ and $\phi \in \overline{\mathcal{D}}_\alpha$, one has

$$\limsup_{|x|\to\infty} \frac{\ln \varphi_K(x)}{|x|^{1-\frac{\alpha}{2}}} = \limsup_{|x|\to\infty} \frac{\ln z_K(x)\phi(x)e^{\psi_K(x)}}{|x|^{1-\frac{\alpha}{2}}} = 0.$$

Since $\mathcal{L}\varphi_K = -\lambda_K\varphi_K \geq 0$, thanks to (H1a), (H1c), (2.15), Theorem 2.3.1 part ii) implies that for any $\kappa \in \left(0, \frac{-2\mathcal{D}_\alpha + 2\sqrt{\mathcal{D}_\alpha^2 + \underline{\eta}_\alpha}}{2-\alpha}\right)$, there exist $C(\kappa), R(\kappa) > 0$ and $\tau \in (0,\kappa)$ independent of K such that

$$\varphi_K(x) \leq C(\kappa)e^{R^{1-\frac{\alpha}{2}}(\tau+\kappa)}e^{-\kappa|x|^{1-\frac{\alpha}{2}}} \quad \text{for } |x| > R(\kappa).$$

Indeed, this is a direct observation from (2.43) by substituting $\theta = 0$. Therefore, φ_K are bounded solutions of the linear equation $\mathcal{L}\varphi_K = -\lambda_K\varphi_K \geq 0$. We can normalize φ_K in such the way $\sup_{\mathbb{R}^N}\varphi_K = 1$ and finally obtain the uniform bound for some $C_1(\kappa) \geq C(\kappa)$

$$\varphi_K(x) \leq C_1(\kappa)e^{R^{1-\frac{\alpha}{2}}(\tau+\kappa)}e^{-\kappa|x|^{1-\frac{\alpha}{2}}} \quad \text{in } \mathbb{R}^N.$$

The proof is thereby complete. □

Proof of Lemma 2.5.3. Observe that if there exists $K^\star > 0$ such that $\lambda_{K^\star} \geq 0$, we complete the proof since λ_K is nondecreasing with respect to K. Recall that the eigenpair (λ_K, φ_K) satisfies

$$\Delta\varphi_K(x) + Kq_i(x)\partial_i\varphi_K(x) + f_s(x,0)\varphi_K + \lambda_K\varphi_K = 0 \qquad \text{in } \mathbb{R}^N. \tag{2.55}$$

Assume by contradiction that $\lambda_K < 0$ for all $K > 0$, then

$$\limsup_{K\to\infty} \lambda_K \leq 0. \tag{2.56}$$

By Lemma (2.5.2), we can normalize φ_K in such the way $\|\varphi_K\|_{L^\infty(\mathbb{R}^N)} = 1$. First, we consider the case $\mathcal{I}_q$ is not empty. Let $z \in \mathcal{I}_q$ and $\varepsilon > 0$, multiplying Eq. (2.55) by $\frac{z^2}{\varphi_K+\varepsilon} \in W^{1,1}(\mathbb{R}^N)$ and integrating over $\mathbb{R}^N$, one gets

$$\begin{aligned} -\int_{\mathbb{R}^N} \Delta\varphi_K \frac{z^2}{\varphi_K+\varepsilon} - K\int_{\mathbb{R}^N} q\cdot\nabla(z^2\ln(\varphi_K+\varepsilon)) + 2K\int_{\mathbb{R}^N} z\ln(\varphi_K+\varepsilon)q\cdot\nabla z + \\ -\int_{\mathbb{R}^N} f_s(x,0)\frac{z^2\varphi_K}{\varphi_K+\varepsilon} = \lambda_K\int_{\mathbb{R}^N}\frac{z^2\varphi_K}{\varphi_K+\varepsilon}. \end{aligned} \tag{2.57}$$

By Lemma 2.5.2, the interior elliptic estimates and Harnack inequality yields that for $1 \leq p < \infty$

$$\|\varphi_K\|_{W^{2,p}(B_1(x))} \leq C_2(K,\gamma)\|\varphi_K\|_{L^\infty(B_2(x))} \leq C_3(K,\gamma)\varphi_K(x), \qquad \text{in } \mathbb{R}^N.$$

Hence the Sobolev injection of $W^{2,p}$ in C^1 for $p > N$ implies that $\|\nabla\varphi_K(x)\|_{L^\infty(B_1(x))} \leq C_3(K,\gamma)\varphi_K(x)$ in $\mathbb{R}^N$. We have $|\nabla\varphi_K(x)| \to 0$ as $|x| \to \infty$, then

$$\lim_{R\to\infty}\left|\int_{\partial B_R}\frac{z^2}{\varphi_K+\varepsilon}\nabla\varphi_K\cdot\mathbf{n}dS\right| \leq \lim_{R\to\infty}\left\|\frac{z^2}{\varphi_K+\varepsilon}\right\|_{L^1(\partial B_R)}\|\nabla\varphi_K\cdot\mathbf{n}\|_{L^\infty(\partial B_R)} = 0. \tag{2.58}$$

Therefore, by Stokes theorem, the first term of (2.57) can be estimated by

$$\begin{aligned}-\int_{\mathbb{R}^N}\Delta\varphi_K\frac{z^2}{\varphi_K+\varepsilon} = \int_{\mathbb{R}^N}\nabla\varphi_K\cdot\nabla\frac{z^2}{\varphi_K+\varepsilon} &= \int_{\mathbb{R}^N}\frac{2z(\varphi_K+\varepsilon)\nabla\varphi_K\cdot\nabla z - z^2\nabla\varphi_K\cdot\nabla\varphi_K}{(\varphi_K+\varepsilon)^2}\\ &\leq \int_{\mathbb{R}^N}|\nabla z|^2.\end{aligned} \tag{2.59}$$

Since $\varphi_K > 0$ is bounded, $z^2\ln(\varphi_K+\varepsilon) \in W^{1,1}(\mathbb{R}^N)$, it can be approximated in $W^{1,1}$ norm by a sequence in $(u_n) \in C_c^\infty(\mathbb{R}^N)$. Moreover, q is divergence-free, the integration by part yields

$$\int_{\mathbb{R}^N} q\cdot\nabla u_n = \int_{\mathrm{spt}\,u_n} q(x)\cdot\nabla u_n = 0,$$

and thus the second term of (2.57) is zero. The third term of (2.57) is obviously equal to zero since $z \in \mathcal{I}_q$. As a consequence, we obtain

$$\int_{\mathbb{R}^N} f_s(x,0)\frac{z^2\varphi_K}{\varphi_K+\varepsilon} + \lambda_K\int_{\mathbb{R}^N}\frac{z^2\varphi_K}{\varphi_K+\varepsilon} \leq \int_{\mathbb{R}^N}|\nabla z|^2.$$

Since $f_s(x,0)$ is bounded and $z \in H^1(\mathbb{R}^N)$, by the Lebesgue dominated convergence theorem, we derive the following limit in $L^1(\mathbb{R}^N)$

$$g_\varepsilon(x) = f_s(x,0)\frac{z^2\varphi_K}{\varphi_K+\varepsilon} + \lambda_K\frac{z^2\varphi_K}{\varphi_K+\varepsilon} \to f_s(x,0)z^2 + \lambda_K z^2 \qquad \text{as } \varepsilon\to 0$$

It follows that

$$\lambda_K \leq M = \frac{\int_{\mathbb{R}^N}|\nabla z|^2 - f_s(x,0)z^2}{\int_{\mathbb{R}^N} z^2}. \tag{2.60}$$

Multiplying φ_K to (2.55) and integrating over $\mathbb{R}^N$, we derive by Stokes theorem that

$$-\int_{\mathbb{R}^N}|\nabla\varphi_K|^2 + \frac{K}{2}\int_{\mathbb{R}^N} q(x)\cdot\nabla(\varphi_K)^2 + \int_{\mathbb{R}^N} f_s(x,0)\varphi_K^2 = -\lambda_K\int_{\mathbb{R}^N}\varphi_K^2. \tag{2.61}$$

Thanks to the exponential decay of φ_K^2 and Stokes theorem, the integration by part again yields

$$\int_{\mathbb{R}^N} q(x).\nabla(\varphi_K)^2 = -\int_{\mathbb{R}^N}\varphi_K^2\mathrm{div}q = 0. \tag{2.62}$$

Consequently,

$$\lambda_K\int_{\mathbb{R}^N}\varphi_K^2 = \int_{\mathbb{R}^N}|\nabla\varphi_K|^2 - \int_{\mathbb{R}^N} f_s(x,0)\varphi_K^2. \tag{2.63}$$

From Lemma 2.5.2, there exists an exponential decay $\overline{\varphi}(x) = C(\gamma)e^{-\gamma|x|^{1-\frac{\alpha}{2}}}$ independent of K such that $\varphi_K \leq \overline{\varphi}$ in $\mathbb{R}^N$. Since $\lambda_K < 0$

$$\int_{\mathbb{R}^N} |\nabla\varphi_K|^2 dx \leq \int_{\mathbb{R}^N} f_s(x,0)\varphi_K^2 \leq \sup_{\mathbb{R}^N} f_s(x,0)\|\overline{\varphi}\|_{L^2(\mathbb{R}^N)}.$$

By the normalization of φ_K, we deduce that there exists a function $\varphi_\infty \in H^1(\mathbb{R}^N)$, which is not identically equal to zero, such that φ_{K_n} converges (up to subsequence) to φ_∞ weakly in $H^1(\mathbb{R}^N)$ and strongly in $L^2_{loc}(\mathbb{R}^N)$ due to the compact injection $H^1_{loc}(\mathbb{R}^N) \hookrightarrow L^2_{loc}(\mathbb{R}^N)$. The Lebesgue dominated convergence theorem and the lower semicontinuity property imply that

$$\lim_{n\to\infty} \|\varphi_{K_n}\|_{L^2(\mathbb{R}^N)} = \|\varphi_\infty\|_{L^2(\mathbb{R}^N)} \qquad \text{and} \qquad \int_{\mathbb{R}^N} |\nabla\varphi_\infty|^2 \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla\varphi_{K_n}|^2.$$

Moreover, we see that $\Delta\varphi_K \rightharpoonup \Delta\varphi_\infty$ in the sense of distributions. Dividing (2.55) by K_n and passing to the limit $n \to \infty$, one gets $q \cdot \nabla\varphi_\infty = 0$ a.e in $\mathbb{R}^N$. Therefore, $\varphi_\infty \in \mathcal{I}_q$. Since $f_s(x,0) \in L^\infty(\mathbb{R}^N)$, $\varphi_K \to \varphi_\infty$ strongly in $L^2(\mathbb{R}^N)$, moreover λ_{K_n} is nondecreasing and $\lambda_{K_n} \leq M$ by (2.60), we can pass to the limit from (2.63)

$$\liminf_{n\to\infty} \lambda_{K_n} \int_{\mathbb{R}^N} \varphi_\infty^2 \quad \geq \quad \int_{\mathbb{R}^N} |\nabla\varphi_\infty|^2 - \int_{\mathbb{R}^N} f_s(x,0)\varphi_\infty^2 \tag{2.64}$$

By (2.22), we achieve the contradiction of (2.56)

$$\liminf_{K\to\infty} \lambda_K \geq \inf_{\phi\in\mathcal{I}_q} \frac{\int_{\mathbb{R}^N} |\nabla\phi|^2 - f_s(x,0)\phi^2}{\int_{\mathbb{R}^N} \phi^2} > 0.$$

This implies that there exists $K^\star$ such that $\lambda_{K^\star} \geq 0$ and the monotonicity of λ_K Lemma (2.5.1) implies that $\lambda_K \geq 0$ for $K \geq K^\star$. Actually (2.60) and (2.64) also implies that $\lambda_{K_n} \to +\infty$ as $n \to \infty$ if $\mathcal{I}_q$ is empty. The proof is therefore complete. □

The arguments of the proofs of lemmas (2.5.2) and (2.5.3) remain valid for lemmas (2.5.4) and (2.5.5), therefore we shall just point out the main differences without necessarily providing details.

Proof of Lemma 2.5.4. To prove Lemma (2.5.4), we need to apply Theorem (2.3.2) part ii) instead of Theorem (2.3.1) part ii) to derive the polynomial decay for the principal eigenfunctions. □

Proof of Lemma 2.5.5. The proof of Lemma 2.5.5 can be followed by the scheme of Lemma (2.5.3). However, we need to verify by line that all the arguments remain valid. We see that (2.58),(2.59) are true for any $|\nabla\varphi_K(x)| \to 0$ as $|x| \to \infty$, without concerning the rate of decay. But in order to apply the Stokes theorem to derive the formulations at (2.61), (2.62), φ_K and $|\nabla\varphi_K|$ have to decay fast enough. More precisely, $\varphi_K \in L^2(\mathbb{R}^N)$ and $\lim_{R\to\infty} \int_{\partial B_R} |\varphi_K||\nabla\varphi_K| ds = 0$. By Lemma 2.5.4, to guarantee that there exists $\beta \in (0, \overline{\beta})$ such that both the requirements are true, it suffices to have

$$2\overline{\beta} > N \qquad \Longleftrightarrow \qquad N - 2 + 2\mathcal{D}_2 + \sqrt{(N-2+2\mathcal{D}_2)^2 + 4\underline{\eta}_2} > N,$$

where we use assumption (2.19) with $\theta = 0$. By a simple calculation, we see that the last inequality is equivalent to condition (2.23). The proof is therefore complete. □

Remark 12. If q is a constant vector field, the results of theorems 2.1.5 and 2.1.6 are the consequence of theorems 2.1.2 and 2.1.3. In fact, by the Liouville transformation, we get

$$\lambda_1(-\Delta - q\cdot\nabla - f_s(x,0),\mathbb{R}^N) = \frac{q^2}{4} + \lambda_1(-\Delta - f_s(x,0),\mathbb{R}^N). \tag{2.65}$$

Under the hypotheses (H1), (H2), and (H3), theorems 2.1.2 and 2.1.3 assert that the existence and nonexistence of (2.1) are directly conditioned by the sign of $\lambda_1(-\Delta - q\cdot\nabla - f_s(x,0),\mathbb{R}^N)$. More precisely, by (2.65), we see that (2.1) admits a positive solution iff $|q| < \sqrt{-2\lambda_1(-\Delta - f_s(x,0),\mathbb{R}^N)}$ and this is the threshold value.

2.6 Long time behavior in L^∞ and L^p norms

Proof of Theorem (2.1.7). Thanks to Hypothesis 3, for fixed K and $\alpha \in [0,2)$, there exists a supersolution of Eq. (2.1), $S_K(x)$ satisfying $\lim_{|x|\to\infty} \ln S_K(x)|x|^{\frac{\alpha}{2}-1} = 0$. Since $u_0(x)/S_K(x) \in L^\infty(\mathbb{R}^N)$, there exists a constant $M > 1$ such that $|u_0(x)| \le MS_K(x)$. Obviously $MS_K(x)$ is also a supersolution of stationary equation of Eq. (2.2). Let w be the solution of Eq. (2.2) starting by initial condition $w(0,x) = MS_K(x)$, w satisfies

$$\begin{cases} \partial_t w \le (|x|+1)^\alpha [a_{ij}(x)\partial_{ij}w(x) + Kq(x)\cdot\nabla w + f_s(x,0)w] & t\in\mathbb{R}^+, x\in\mathbb{R}^N \\ w(0,x) = MS_K(x). \end{cases} \tag{2.66}$$

Since $\alpha \in [0,2]$, thanks to the growth conditions of $a_{ij}(x), q(x), f_s(x,0)$ and (2.24), we can apply the extension Phragmén-Lindelöf principle (Theorem 1, [44]) for parabolic operators with unbounded coefficients in the following particular way $\widetilde{a}_{ij}(x) = (|x|+1)^\alpha a_{ij}(x) \le C_a|x|^\alpha$, $|\widetilde{q}(x)| = |(|x|+1)^\alpha q(x)| \le C_q|x|^{\frac{\alpha}{2}}$ and $(|x|+1)^\alpha f_s(x,0) \in L^\infty(\mathbb{R}^N)$ and for $k > \gamma_0$

$$\liminf_{R\to\infty} \sup_{|x|=R, t>0} \frac{w(t,x)}{e^{k(1+R^2)^{1/2}}} \le 0.$$

We infer that $t \mapsto w(t,x)$ is nonincreasing and $w(t,x)$ converges as $t\to\infty$ to a function $W(x)$ pointwise in $\mathbb{R}^N$ such that

$$\limsup_{t\to\infty} u(t,x) \le \limsup_{t\to\infty} w(t,x) = W(x) \le MS_K(x).$$

Since the coefficients are smooth and locally bounded in $\mathbb{R}^N$, the L^p regularity for second order parabolic equation in [72] yields that for any $R > 0$, there exists $C_R > 0$ such that

$$\|w\|_{(N+1),(t_0,t_0+R)\times B_R(x_0)} \le C_R.$$

Therefore, as $t\to\infty$, $w(t,x)$ converges to $W(x)$ locally in $\mathbb{R}^N$, $\partial_i w(t,x)$ and $\partial_{ij}w(t,x)$ also converge respectively weakly locally in $L^{N+1}(\mathbb{R}^+,\mathbb{R}^N)$ to $\partial_i W(x)$ and $\partial_{ij}W(x)$ satisfying

$$\begin{cases} a_{ij}(x)W_{ij}(x) + Kq_i(x)W_i(x) + f(x,W) = 0 & \text{in } \mathbb{R}^N \\ 0 \le W(x) \le MS_K(x). \end{cases} \tag{2.67}$$

Furthermore, we have the following lemma

Lemma 2.6.1. *Let $(t_n) \in \mathbb{R}^+$ and $(x_n) \in \mathbb{R}^N$ satisfying $\lim_{n\to\infty} t_n = \lim_{n\to\infty} |x_n| = +\infty$. There exists $\gamma' \in (\gamma_0, \gamma^\star)$ such that*

$$\lim_{n\to\infty} u(t_n, x_n) e^{\gamma' |x_n|^{1-\frac{\alpha}{2}}} = 0 \tag{2.68}$$

Let us postpone the proof of this lemma for a moment to first show that how (2.68) leads to (2.25) for $p = \infty$, i.e the uniform convergence.

• $\lambda_1(-\mathcal{L}, \mathbb{R}^N) \geq 0$.

Owing to Theorem 2.1.2, we must have $W \equiv 0$. This shows that $u(t, x)$ converges locally in $\mathbb{R}^N$ to zero as $t \to \infty$. To prove the uniform convergence, namely (2.25) with $p = \infty$, we assume by contradiction that it is not true, then there exists $\varepsilon > 0$, (t_n) and (x_n), such that $t_n, |x_n| \to +\infty$ and $u(t_n, x_n) e^{\kappa_0 |x_n|^{1-\frac{\alpha}{2}}} \geq \varepsilon$ for every $n \in \mathbb{N}$. This contradicts (2.68) and therefore the statement (i) is confirmed.

• $\lambda_1(-\mathcal{L}, \mathbb{R}^N) < 0$.

Let λ_R and φ_R be respectively the Dirichlet principal eigenvalue and eigenfunction of $\mathcal{L}$ on B_R, which are given by Theorem (2.2.3). We have shown that for $\kappa > 0$ small enough, there exists $R > 0$ such that $\lambda_R < 0$ and the function

$$\underline{U}(x) := \begin{cases} \kappa \varphi_R(x) & x \in B_R(x) \\ 0 & \text{otherwise} \end{cases}$$

is a weak subsolution of Eq. (2.67). Without loss of generality, we may assume $U(x) \leq u(1, x)$ and $u(1, x) > 0$. The Phragmén-Lindelöf principle (Theorem 1, [44]) implies that the solution v of Eq. (2.2) starting by $v(0, x) = \underline{U}(x)$ is nondecreasing in t and satisfies

$$\underline{U}(x) \leq v(t, x) \leq u(t+1, x) \leq w(t+1, x), \qquad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N.$$

The routine argument implies that $v(t, x)$ converges locally uniformly to a function V satisfying $\underline{U} \leq V \leq W \leq MS_K(x)$ and $a_{ij}(x) W_{ij}(x) + Kq_i(x) W_i(x) + f(x, W) = 0$ a.e in $\mathbb{R}^N$. The strong maximum principle implies that $V > 0$ in $\mathbb{R}^N$ and Theorem 2.1.2 yields $U \equiv V \equiv W$, where U is the unique solution of (2.67). The local convergence in $\mathbb{R}^N$, $u(t, x) \to U(x)$ as $t \to \infty$ is confirmed. If this convergence is not uniform in $\mathbb{R}^N$, there exist the sequences (t_n), (x_n) and $\varepsilon > 0$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} |x_n| = +\infty, \qquad |u(t_n, x_n) - U(x_n)| e^{\gamma_0 |x_n|^{1-\frac{\alpha}{2}}} \geq \varepsilon \qquad \text{for any } n \in \mathbb{N}.$$

By Theorem (2.3.1), for all $\gamma_1 \in (\gamma_0, \gamma^\star)$, one has $\lim_{|x|\to\infty} U(x_n) e^{\gamma_1 |x_n|^{1-\frac{\alpha}{2}}} = 0$. It follows immediately

$$\frac{\varepsilon}{2} < u(t_n, x_n) e^{\gamma_0 |x_n|^{1-\frac{\alpha}{2}}} \qquad \text{for } n \text{ large enough.}$$

This contradicts Lemma (2.6.1).

It remains to prove Lemma (2.6.1). Assume by contradiction that the claim is not true, one finds $\varepsilon > 0$ and the sequences $(t_n) \to \infty$ and $|x_n| \to \infty$ as $n \to \infty$ such that

$$u(t_n, x_n) e^{\gamma_0 |x_n|^{1-\frac{\alpha}{2}}} \geq \varepsilon \qquad \text{for any } n \in \mathbb{N}. \tag{2.69}$$

Observe that, by Hypothesis 2, we have

$$\partial_t u - (|x| + 1)^\alpha (a_{ij}(x) \partial_{ij} u + Kq(x) \cdot \nabla u + f_s(x, 0) u) \leq 0 \qquad t \in \mathbb{R}, x \in \mathbb{R}^N.$$

To achieve a contradiction, we shall look for a supersolution of Eq. (2.2), which decays as both t and $|x|$ tend to infinity. The proof is distinguished by two cases :

Case 1. (H1a) and (H1b) hold.
Let

$$P_\delta = \partial_t - (|x|+1)^\alpha a_{ij}(x)\partial_{ij} - K(|x|+1)^\alpha q(x)\cdot\nabla - (|x|+1)^\alpha \mu(x) - \delta,$$

we see that for all $\delta > 0$, there exists $R(\delta)$ such that $u(t,x)$ is a subsolution of $P_\delta[\phi] = 0$ in $\mathbb{R}^N \setminus B_R$.

Let $\overline{\gamma}$ be the positive solution of Eq. (2.14). According to Case 1 of Theorem (2.3.1), for any $\gamma \in (0,\overline{\gamma})$, there exist $\tau, \delta \in (0,\gamma)$, $R = R(\delta)$ large enough and a function $\varphi \in \overline{\mathcal{D}}_\alpha$ associated with $\lambda \in (0,\overline{\eta})$ such that for all $p > 0$ the function

$$w_p^\gamma(x) = [e^{(R+p)^{1-\frac{\alpha}{2}}(\tau-\gamma)}e^{\gamma|x|^{1-\frac{\alpha}{2}}} + e^{R^{1-\frac{\alpha}{2}}(\tau+\gamma)}e^{-\gamma|x|^{1-\frac{\alpha}{2}}}]\varphi(x)$$

satisfies

$$(|x|+1)^\alpha(a_{ij}(x)\partial_{ij}w_p^\gamma + Kq(x)\cdot\nabla w_p^\gamma + f_s(x,0)w_p^\gamma) - \delta w_\rho^\gamma \le 0 \qquad \text{in } B_{R+p}\setminus B_R,\ \forall p > 0. \tag{2.70}$$

By (2.24), there exist a sequence $(r_n) \to \infty$ as $n \to \infty$ and a large constant M_1 such that

$$u(t,x) \le M_1 e^{\gamma_0|x|^{1-\frac{\alpha}{2}}} \qquad \text{for } |x| = r_n,\ \forall n \in \mathbb{N},\ t \in (0,r_n).$$

There also exist $\gamma_1 \in (\gamma_0,\overline{\gamma})$, $\varphi \in \overline{\mathcal{D}}_\alpha$ and a constant $M_2 \ge M_1$ such that for all $p > 0$

$$u_0(x) \le M_2 w_p^{\gamma_1}(x) \qquad \text{in } B_{R+p}\setminus B_R.$$

Without loss of generality, we can extract a subsequence of (r_n), which is increasing and still denoted by (r_n). Let n_0 be such that $r_n \ge R(\delta)$ for $n > n_0$ and $p_n = r_n - r_{n_0}$. We choose $\tau \in (\gamma_0,\gamma_1)$. The functions $M_2 w_{p_n}^{\gamma_1}(x)$ satisfy

$$\begin{cases} M_2 w_{p_n}^{\gamma_1}(x) \ge M_2 e^{\tau(r_{n_0})^{1-\frac{\alpha}{2}}}\varphi(x) \ge M_2 e^{\gamma_0|x|^{1-\frac{\alpha}{2}}} \ge u(t,x) & \text{as } |x| = r_{n_0} \\ M_2 w_{p_n}^{\gamma_1}(x) \ge M_2 e^{\tau(r_{n_0}+p_n)^{1-\frac{\alpha}{2}}}\varphi(x) \ge M_2 e^{\gamma_0|x|^{1-\frac{\alpha}{2}}} \ge u(t,x) & \text{as } |x| = r_n \end{cases} \qquad \forall t \in (0,r_n).$$

Since $w_{p_n}^{\gamma_1}(x)$ satisfies stationary equation (2.72), the parabolic comparison principle implies that $u(t,x) \le M_2 w_{p_n}^{\gamma_1}(x)$ for $(t,x) \in (0,r_n)\times B_{r_n}\setminus B_{r_{n_0}}$, $\forall n > n_0$. Letting $n \to \infty$, we obtain

$$u(t,x) \le M_2 e^{(r_{n_0})^{1-\frac{\alpha}{2}}(\tau+\gamma_1)}e^{-\gamma_1|x|^{1-\frac{\alpha}{2}}}\varphi(x) \qquad \text{in } [0,\infty)\times\mathbb{R}^N\setminus B_{r_{n_0}}$$

Consequently, since $\gamma_0 < \gamma_1$ and $\varphi \in \overline{\mathcal{D}}_\alpha$, taking $\gamma' \in (\gamma_0,\gamma_1)$ one has

$$u(t,x)e^{\gamma'|x|^{1-\frac{\alpha}{2}}} \le M_2 e^{(r_{n_0})^{1-\frac{\alpha}{2}}(\tau+\gamma_1)}e^{-(\gamma_1-\gamma')|x|^{1-\frac{\alpha}{2}}}\varphi(x) \to 0 \qquad \text{as } |x| \to \infty. \tag{2.71}$$

This contradicts to (2.69) and therefore confirms the first case when $p = \infty$.

Case 2. (H1a), (H1c) and (2.15).
Observe that the function $\widetilde{u}(t,x) = e^{-\psi_K(x)}u(t,x)$, where $\psi_K = K\psi$ satisfying $2I_N\cdot\nabla\psi_K(x) = -Kq(x)$ satisfies

$$\begin{cases} \partial_t\widetilde{u} - (|x|+1)^\alpha\left[\Delta\widetilde{u} - |\nabla\psi_K(x)|^2\widetilde{u} + f_s(x,0)\widetilde{w}(x)\right] \le 0 & \text{in } \mathbb{R}^N \\ \widetilde{u}(0,x) = e^{-\psi_K(x)}u_0(x). \end{cases}$$

Let

$$Q_\delta = \partial_t - (|x|+1)^\alpha(\Delta + \mu(x) - |\nabla\psi_K(x)|^2) - \delta,$$

we see that for any $\delta \in (0, \underline{\eta})$, there exists $R = R(\delta)$ such that $\widetilde{u}(t,x)$ is a subsolution of $Q_\delta[\phi] = 0$ in $\mathbb{R}^N \setminus B_{R(\delta)}$. According to Case 2 of Theorem (2.3.1), we know that for any $\gamma \in (\frac{2K\theta}{2-\alpha}, \gamma^+)$ where γ^+ is given in (2.16), $\tau \in (\frac{2K\theta}{2-\alpha}, \gamma)$, there exists $\varphi \in \overline{\mathcal{D}}_\alpha$ associated with $\lambda \in (0, \underline{\eta})$ such that the function

$$v_p^\gamma(x) = [e^{(R+p)^{1-\frac{\alpha}{2}}(\tau-\gamma)}e^{\gamma|x|^{1-\frac{\alpha}{2}}} + e^{R^{1-\frac{\alpha}{2}}(\tau+\gamma)}e^{-\gamma|x|^{1-\frac{\alpha}{2}}}]\varphi(x)$$

satisfies

$$(|x|+1)^\alpha(\Delta + \mu(x) - |\nabla\psi_K(x)|^2)v_p + \delta v_p \le 0 \qquad \text{in } B_{R+p} \setminus B_R,\ \forall p > 0. \tag{2.72}$$

By (2.24), there exist large constant $M_3 > 0$ and an increasing sequence $(r_n) \in \mathbb{R}$ such that

$$\lim_{n\to\infty} r_n = +\infty, \quad \widetilde{u}(t,x) \le M_3 e^{(\frac{2K\theta}{2-\alpha}+\delta)|x|^{1-\frac{\alpha}{2}}} e^{\gamma_0|x|^{1-\frac{\alpha}{2}}} = M_3 e^{\left(\gamma_0+\frac{2K\theta}{2-\alpha}+\delta\right)|x|^{1-\frac{\alpha}{2}}} \quad \text{for } |x| = r_n,\ t \in (0, r_n),$$

moreover for δ small enough, we find $M_4 \ge M_3$ such that

$$u_0(x) \le M_4 e^{(-\gamma_0 - \frac{4K\theta}{2-\alpha} - 3\delta)|x|^{1-\frac{\alpha}{2}}}\varphi(x).$$

It follows, for all $p > 0$ and $x \in B_{R(\delta)+p} \setminus B_{R(\delta)}$, that

$$\widetilde{u}(0,x) \le M_4 e^{(\frac{2K\theta}{2-\alpha}+\delta)|x|^{1-\frac{\alpha}{2}}} u_0(x) \le M_4 e^{-\left(\gamma_0+\frac{2K\theta}{2-\alpha}+2\delta\right)|x|^{1-\frac{\alpha}{2}}}\varphi(x) \le v_p^{\gamma_1}(x).$$

where $\gamma_1 = \gamma_0 + \frac{2K\theta}{2-\alpha} + 2\delta$.

Choosing δ such that $\gamma_1 \in (\frac{2K\theta}{2-\alpha}, \gamma^+)$ and then taking arbitrary constant $\tau \in \left(\gamma_0 + \frac{2K\theta}{2-\alpha} + \delta, \gamma_1\right)$, we see that the functions $M_4 v_{p_n}^{\gamma_1}(0,x)$, where $p_n = r_n - r_{n_0}$ satisfies

$$\begin{cases} M_4 v_{p_n}^{\gamma_1}(x) \ge M_4 e^{\tau(r_{n_0})^{1-\frac{\alpha}{2}}}\varphi(x) \ge M_4 e^{(\gamma_0+\frac{2K\theta}{2-\alpha}+\delta)|x|^{1-\frac{\alpha}{2}}} \ge \widetilde{u}(t,x) & \text{as } |x| = r_{n_0} \\ M_4 v_{p_n}^{\gamma_1}(x) \ge M_4 e^{\tau(r_{n_0}+p_n)^{1-\frac{\alpha}{2}}}\varphi(x) \ge M_4 e^{(\gamma_0+\frac{2K\theta}{2-\alpha}+\delta)|x|^{1-\frac{\alpha}{2}}} \ge \widetilde{u}(t,x) & \text{as } |x| = r_n, \end{cases} \qquad \forall t \in (0, r_n)$$

where n_0 is fixed in such the way $r_{n_0} \ge R(\delta)$.

Since $v_{p_n}^{\gamma_1}$ satisfies (2.72), the parabolic comparison principle implies that for all $n > n_0$

$$\widetilde{u}(t,x) \le M_4 v_{p_n}^{\gamma_1}(x) \qquad \text{for } (0, r_n) \times B_{r_{n_0}+p_n} \setminus B_{r_{n_0}}.$$

Letting $n \to \infty$, we obtain

$$\widetilde{u}(t,x) \le M_4 e^{R^{1-\frac{\alpha}{2}}(\tau+\gamma_1)}e^{-\gamma_1|x|^{1-\frac{\alpha}{2}}}\varphi(x) \qquad \text{in } (0,\infty) \times \mathbb{R}^N \setminus B_{r_{n_0}}.$$

Converting back $\widetilde{u}$ to u, since $\varphi \in \overline{\mathcal{D}}_\alpha$ and $\gamma_1 - \frac{2K\theta}{2-\alpha} - \gamma_0 > 0$, taking $\gamma' \in (\gamma_0, \gamma_1 - \frac{2K\theta}{2-\alpha})$, we get

$$e^{\gamma'|x|^{1-\frac{\alpha}{2}}}u(t,x) \le M_4 e^{R^{1-\frac{\alpha}{2}}(\tau+\gamma_1)}e^{-\left(\gamma_1-\frac{2K\theta}{2-\alpha}-\gamma'\right)|x|^{1-\frac{\alpha}{2}}}\varphi(x) \to 0 \qquad \text{as } |x| \to \infty. \tag{2.73}$$

This contradiction confirms Lemma (2.6.1). Therefore the uniform convergence is achieved.

Finally, the convergences in $L^p(\mathbb{R}^N)$-norms for $p \in (0,\infty)$ can be also resulted by Lemma (2.6.1). Indeed, since $u(t,x) \geq 0$ and $\varphi \in \overline{\mathcal{D}}_\alpha$, we see that the functions in the right-hand side of inequalities (2.71) and (2.73) are independent of t and decay exponentially as $|x| \to \infty$. Those functions belong to $L^p(\mathbb{R}^N)$ for all $p > 0$. Either one of the followings holds for all $p > 0$:

• $\lambda_1(-\mathcal{L}, \mathbb{R}^N) \geq 0$. As shown above, $u(t,x) \to 0$ locally uniformly (pointwise) in $\mathbb{R}^N$. The Lebesgue dominated convergence theorem rapidly implies $\|e^{\gamma_0|x|^{1-\frac{\alpha}{2}}} u(t,x)\|_{L^p(\mathbb{R}^N)} \to 0$ as $t \to \infty$.

• $\lambda_1(-\mathcal{L}, \mathbb{R}^N) < 0$. Let us consider the functions $\overline{u}(t,x)$ and $\underline{u}(t,x)$ with the initial condition respectively $\overline{u}(0,x) = \max\{u_0(x), U(x)\}$ and $\underline{u}(0,x) = \min\{u_0(x), U(x)\}$, where $U(x)$ is the unique positive solution of Eq. (2.67). As above, $\overline{u}(t,x)$ (resp.) $\underline{u}(t,x)$ converge nonincreasingly (resp. nondecreasingly) to $U(x)$ as $t \to \infty$, locally in $x \in \mathbb{R}^N$. Hence, $\underline{u}(t,x) \leq U(x) \leq \overline{u}(t,x)$, $\forall t \geq 0$. Observe that the function $\overline{w}(t,x) = \overline{u}(t,x) - U(x) \geq 0$ satisfies Eq. (2.2) with the nonlinearity

$$f(x,\overline{u}) - f(x,U) \leq f(x,U)\overline{u}/U - f(x,U) \leq f_s(x,0)\overline{w}$$

and grows at most as $S_K(x)$. The local convergence implies that $e^{\gamma_0|x|^{1-\frac{\alpha}{2}}}\overline{w}(t,x) \to 0$ as $t \to \infty$. Theorem (2.3.1), (2.71) and (2.73) assert that $e^{\gamma_0|x|^{1-\frac{\alpha}{2}}} w(t,x)$ is bounded by the functions of $L^p(\mathbb{R}^N)$, the Lebesgue dominated convergence theorem again implies that $\|e^{\gamma_0|x|^{1-\frac{\alpha}{2}}}\overline{w}(t,x)\|_{L^p(\mathbb{R}^N)} \to 0$ as $t \to \infty$. Similarly, the function $\underline{w}(t,x) = U(x) - \underline{u}(t,x) \geq 0$ satisfies $\|e^{\gamma_0|x|^{1-\frac{\alpha}{2}}}\underline{w}(t,x)\|_{L^p(\mathbb{R}^N)} \to 0$ as $t \to \infty$.

The proof is complete.

□

The proof of Theorem (2.1.8) can be obtained by the same technique of Theorem (2.1.7). Therefore, we shall only specify and prove the main points, which yield significant differences with the former.

Proof of Theorem 2.1.8. First, we emphasize that the extension Phragmén-Lindelöf principle (Theorem 1, [44]) can still be applied in the case $\alpha = 2$. Arguing as in Theorem 2.1.8, we see that the locally large time behavior is thus followed from the L^p regularity for second order parabolic equation in [72]. For the $L^p(\mathbb{R}^N)$, $0 < p \leq \infty$ large time behavior, it is in fact a consequence of the following lemma.

Lemma 2.6.2. *Let $(t_n) \in \mathbb{R}^+$ and $(x_n) \in \mathbb{R}^N$ satisfying $\lim_{n\to\infty} t_n = \lim_{n\to\infty} |x_n| = +\infty$. There exists $\beta \in (\sigma_0, \sigma^\star)$ such that*

$$\lim_{n\to\infty} u(t_n, x_n)|x_n|^\beta = 0 \tag{2.74}$$

The proof of this Lemma is transparently analogous to Lemma (2.6.1). Just note that, the decaying condition of initial datum and (2.26) allow us to invoke the parabolic comparison principle under Hypothesis 1. We omit the details.

For proving uniform convergence in (2.27), this is the direct consequence of Lemma (2.6.2) and the fact that for all $\beta \in (\sigma_0, \sigma^\star)$, $\limsup_{|x|\to\infty} U(x)|x|^\beta = 0$, which derived from Theorem (2.3.2).

The L^p convergences are immediately obtained thanks to Lebesgue dominated theorem and the observation that for some $\beta \in (\sigma_0, \sigma^\star)$, $\varphi \in \overline{\mathcal{D}}_2$, $C > 0$

$$u(t,x) \leq C(|x|+1)^{-\sigma_0}(|x|+1)^{-(\beta-\sigma_0)}\varphi(x) \qquad U(x) \leq C(|x|+1)^{-\sigma_0}(|x|+1)^{-(\beta-\sigma_0)}\varphi(x),$$

where $\varphi(x)|x|^{-\kappa} \to 0$ as $|x| \to \infty$ for all $\kappa > 0$. Obviously, it is sufficient to achieve (2.28) is $p\sigma_0 > N$. The proof is thereby complete.

□

Chapitre 3

Liouville-type result for a quasilinear equations with possibly decaying potentials in $\mathbb{R}^N$

The contents of this chapter are based on the joint work :

Liouville-type result for quasilinear equations with possibly decaying potentials in $\mathbb{R}^N$

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3.1 Introduction

3.1.1 Background and the main results

We consider quasilinear elliptic equations of the form

$$\mathcal{K}_a[u] + b(x)g(u) = 0 \qquad \text{in } \mathbb{R}^N, \tag{3.1}$$

where $\mathcal{K}_a[u] = -\text{div}(|\nabla u|^{p-2}\nabla u) - a(x)u^{p-1}$, $p \geq 2$, $a, b \in L^\infty(\mathbb{R}^N)$, $b \not\equiv 0$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function. A typical model of (3.1) is

$$-\Delta_p u - a(x)u^{p-1} + u^r = 0 \quad \text{in } \mathbb{R}^N,$$

with $r > p - 1$.

By a solution of (3.1) we mean a function $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N)$ such that $g(u) \in L^1_{\text{loc}}(\mathbb{R}^N)$ and u satisfies (3.1) in the sense of distribution, namely

$$\int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u.\nabla\phi - a(x)|u|^{p-2}u\phi + b(x)g(u)\phi)dx = 0 \qquad \forall \phi \in C^\infty_c(\mathbb{R}^N). \tag{3.2}$$

A function $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ is called a supersolution (respectively subsolution) of (3.1) if $g(u) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and (3.2) holds with "=" replaced by "$\geq$" (resp. "$\leq$") and with nonnegative test function $\phi \in C_c^\infty(\mathbb{R}^N)$.

It is well-known that, the type of equation (3.1) has wide applications in various aspects of mathematical biology, physics and especially in population dynamics. In a pioneering work, Berestycki [3] has studied the existence and uniqueness of the semilinear equation

$$\begin{cases} -\Delta u &= f(u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

where Ω is bounded, smooth and f is of KPP type ($f(0) = 0$ and $f(u)/u$ is stricly descreasing). Afterward, this is a subject of extensive research and much more general results have been obtained for the equations in bounded or periodic domains. Amongst those, we cite [26] and [10], where interesting biological interpretation of (3.1) is carefully discussed. However, works involving unbounded domains (for instance $\mathbb{R}^N$) are much less. In their groundbreaking papers [5, 10, 18], Berestycki et al. studied more general elliptic equations of the form

$$-\sum_{i,j} a_{ij}(x)\partial_{ij}u - \sum_i q_i(x)\partial_i u = f(x,u) \qquad \text{in } \mathbb{R}^N \tag{3.4}$$

where the condition imposed is either (see [10])

$$\liminf_{|x|\to\infty}(4\underline{\alpha}(x)f_s(x,0) - |q(x)|^2) > 0 \quad \text{with } \underline{\alpha}(x) = \inf_{\xi\in\mathbb{R}^N,|\xi|=1} a_{ij}(x)\xi_i\xi_j > \alpha^* > 0 \tag{3.5}$$

or (see [5, 18])

$$\limsup_{|x|\to\infty} f_s(x,0) < 0. \tag{3.6}$$

They established a necessary and sufficient condition for existence of positive solutions of (3.4) in terms of the sign of the *generalized principle eigenvalue* of the operator

$$L[\phi] = -\sum_{i,j} a_{ij}(x)\partial_{ij}\phi - \sum_i q_i(x)\partial_i\phi - f_s(x,0)\phi$$

in $\mathbb{R}^N$. Furthermore, they also dealt with the question of uniqueness of positive bounded solutions. Such type of result is called *nonlinear Liouville-type result.*

Motivated by the above works, the present paper aims at providing a sharp criterion that characterizes existence/nonexistence and uniqueness of positive solutions of (3.1). It is important to understand that which kinds of result in [5, 10, 18, 26] remain true under framework of p-Laplacian. Before stating the main results, let us describe the hypotheses on a, b and g. The novelty of our study is that the potential a is allowed to be sign-changing and satisfies the following property

$$\limsup_{|x|\to\infty} a(x)|x|^\alpha < -m \quad \text{for some } \alpha \in [0,p] \text{ and } m > 0. \tag{C1}$$

Condition (**C1**) involves the case that (3.5) is not covered and may be regarded as an extension of (3.6) in the sense that a possibly decays at infinity. As will be shown in the sequel, the decay rate of the potential a has a significant effect on that of solutions of (3.1): solutions decay exponentially when a is a slow decay potential (i.e. $\alpha \in [0,p)$), while they decay polynomially when a is a Hardy potential (i.e. $\alpha = p$).

The nonlinearity g is required to satisfy

$$\lim_{s\to 0}\frac{g(s)}{s^{p-1}}=0 \quad \text{and} \quad \frac{g(s)}{s^{p-1}} \text{ is increasing on } (0,\infty). \tag{C2}$$

Next, we assume that b verifies the following relation for some $s_0>0$,

$$-a(x)s_0^{p-1}+b(x)g(s_0)\geq 0 \quad \text{ a.e. in } \mathbb{R}^N. \tag{C3}$$

Owing to (**C2**), it is easy to see that (**C3**) also holds for every $s\geq s_0$. This reveals that any constant s greater than s_0 is a supersolution of (3.1). Clearly, (**C3**) is fulfilled if $\inf_{\mathbb{R}^N} b>0$ and $\lim_{s\to\infty}\frac{g(s)}{s^{p-1}}=\infty$.

The above assumptions have biological meanings. Condition (**C1**) refers to the environment being unfavorable, unfavorably neutral or nearly neutral near infinity, which are corresponding to the cases of $\alpha=0$, $\alpha\in(0,p)$ or $\alpha=p$. This kind of assumption is recently used to describe the effect of global warming (see [5],[95]). Condition (**C2**) means that the intrinsic growth rate decreases when the population density is increasing. This is due to the intraspecific competition for resources and lastly, condition (**C3**) expresses a saturation effect: when the population density is very large, the death rate is higher than the birth rate and the population decreases.

Under theses assumptions, we will prove that the existence and nonexistence result for (3.1) is directly conditioned by the sign of "an eigenvalue" of $\mathcal{K}_a$ in the whole domain $\mathbb{R}^N$. For the sake of convenience, let us recall briefly some basic facts concerning the first eigenvalue of $\mathcal{K}_a$ in bounded domains. It is well-known that for every $C^{1,\beta}$ bounded domain $\Omega\subset\mathbb{R}^N$, the operator $\mathcal{K}_a$, under Dirichlet boundary conditions, admits a unique first eigenvalue $\lambda_\Omega(a)$ associated with a unique (up to multiplication) positive eigenfunction $\varphi_\Omega\in W_0^{1,p}(\Omega)$ (see, e.g., [60, Lemma 3]). More precisely, φ_Ω is a positve solution to

$$\begin{cases} \mathcal{K}_a[\varphi_\Omega] &= \lambda_\Omega(a)\varphi_\Omega^{p-1} & \text{in } \Omega \\ \varphi_\Omega &= 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore, $\varphi_\Omega\in C^{1,\beta}(\overline{\Omega})$ for some $\beta\in(0,1)$ and $\lambda_\Omega(a)$ is variationally characterized by

$$\lambda_\Omega(a)=\inf_{\phi\in W_0^{1,p}(\Omega)}\frac{\displaystyle\int_\Omega |\nabla\phi|^p-a(x)|\phi|^p dx}{\displaystyle\int_\Omega |\phi|^p dx}. \tag{3.7}$$

Henceforth, for every $R>0$, denote $\lambda_R(a):=\lambda_{B_R}(a)$. If $a\in L^\infty(\Omega)$, it is easy to check that the sequence $\{\lambda_R(a)\}$ is nonincreasing and bounded from below by a $-\sup_\Omega a(x)$, which leads to

Definition 3.1.1. We set

$$\lambda_\infty(a):=\lim_{R\to\infty}\lambda_R(a). \tag{3.8}$$

It is easily seen that $\lambda_\infty(a)>-\sup_{\mathbb{R}^N}a(x)$ and thus defines a real number. As we will see in Section 4, $\lambda_\infty(a)$ can be regarded as an "eigenvalue" of $\mathcal{K}_a$ in the whole domain $\mathbb{R}^N$. In fact, when $p=2$, it is known [10, 18] that $\lambda_\infty(a)$ coincides with the generalized principle eigenvalue of $\mathcal{K}_a$ in $\mathbb{R}^N$ defined by

$$\lambda_1(a)=\sup\{\lambda\in\mathbb{R},\exists\phi\in W^{2,N}_{loc}(\mathbb{R}^N),\phi>0,(\mathcal{K}_a+\lambda)[\phi]\leq 0 \text{ a.e in } \mathbb{R}^N\}.$$

When $p \neq 2$, such a result still remains open. We will show that $\lambda_\infty(a)$ plays a key role in proving existence/nonexistence result.

For $\alpha \in [0,p)$ and $m > 0$, we define a class of positive solutions of (3.1) as follows

$$\mathcal{S}_\alpha = \{u > 0 \mid u \text{ is a weak solution of (3.1) such that } \limsup_{|x|\to\infty} \frac{u(x)}{e^{\overline{\omega}|x|^{1-\frac{\alpha}{p}}}} < \infty\}$$

where

$$\overline{\omega} = \frac{p}{p-\alpha}\left(\frac{2m}{Np-N}\right)^{1/p}.$$

Theorem 3.1.1. *Let $p \geq 2$. Assume $\alpha \in [0,p)$ and conditions **(C1)-(C3)** hold. Then (3.1) possesses a positive solution in $\mathcal{S}_\alpha$ if and only if $\lambda_\infty(a) < 0$. Moreover, the positive solution is unique in $\mathcal{S}_\alpha$ and decays exponentially when exists.*

When $\alpha = p$, denote by γ_0 the unique positive solution of algebraic equation

$$(Np+1)\gamma^p + (3p+1)N\gamma^{p-1} - m = 0. \tag{3.9}$$

Define

$$\mathcal{S}_p = \{u > 0 \mid u \text{ is a weak solution of (3.1) such that } \limsup_{|x|\to\infty} |x|^{-\gamma_0} u(x) < \infty\}.$$

Theorem 3.1.2. *Let $p \geq 2$. Assume $\alpha = p$, $m > N^p(Np+3p+2)$ and conditions **(C1)-(C3)** hold. Then (3.1) possesses a positive solution in $\mathcal{S}_p$ if and only if $\lambda_\infty(a) < 0$. Moreover, the positive solution is unique in $\mathcal{S}_p$ and decays polynomially when exists.*

3.1.2 The strategy of proofs and organization of the paper

The rest of the paper is organized as follows.

In Section 2, due to a delicate construction of supersolutions, we show that any nonnegative solution of (3.1) in $\mathcal{S}_\alpha$ decays at the rate depending on α, with $\alpha \in [0,p]$. Interestingly, the rates of decay are remarkable between two cases $\alpha \in [0,p)$ and $\alpha = p$. To be more precise, the solutions of (3.1) decay exponentially in the first case, while they decay polynomially in the second case. The construction relies highly on the weak comparison principle of [45] and a-prior growth conditions of solutions. It is noteworthy that throughout the construction, boundedness of solutions is not required.

In Section 3, by using a variational approach, we establish a comparison principle for (3.1) in $\mathbb{R}^N$ under an additional assumption of integrability on subsolution of (3.1). Although the technique is inspired from [50], we need to overcome the new difficulties arising from the lack of compactness of the domain. It is worth emphasizing that the sliding arguments and variational arguments that are used in the case of linear operators [5, 18] fail to apply in this framework. Our comparison principle handles to compare a sub and a supersolution of (3.1) without further assumption on the gradient of supersolution as in [18, Theorem 2.3], which only enables us to compare a solution and a supersolution of (3.1).

To conclude the paper, in Section 4, we prove Theorem 3.1.1 and Theorem 3.1.2. To this aim, we construct an eigenfunction associated to $\lambda_\infty(a)$ defined in (3.8). This task can be done due to the help of Harnack inequality (see [92],[98]). The existence of positive solution of (3.1) is derived from the negativity of $\lambda_\infty(a)$ and (**C3**), while the nonexistence and uniqueness will be consequences of the comparison principle.

3.2 A-priori estimates

In this section, we establish a priori estimates on positive solutions of (3.1). We start with the case $\alpha \in [0, p)$.

Proposition 3.2.1. *Let $\alpha \in [0, p)$, $m > 0$ and u be a positive function satisfying*

$$\liminf_{|x|\to\infty} |x|^\alpha \left(\frac{\Delta_p u}{u^{p-1}} - \frac{m}{|x|^\alpha} \right) > 0 \qquad \text{and} \qquad \limsup_{|x|\to\infty} \frac{u(x)}{e^{\overline{\omega}|x|^{1-\frac{\alpha}{p}}}} < \infty, \tag{3.10}$$

where $\overline{\omega} = \frac{p}{p-\alpha} \left(\frac{2m}{Np-N} \right)^{1/p}$. Then,

$$\lim_{|x|\to\infty} u(x) e^{\overline{\omega}|x|^{1-\frac{\alpha}{p}}} = 0. \tag{3.11}$$

Proof. By assumption, for any $\varepsilon > 0$ there exists $R = R(\varepsilon)$ such that

$$\Delta_p u \geq \frac{m+\varepsilon}{|x|^\alpha} u^{p-1} \qquad \text{for } |x| \geq R.$$

Set $\mathcal{L}_\varepsilon[\phi] = -\Delta_p \phi + (m+\varepsilon)|x|^{-\alpha}\phi^{p-1}$, we see that $\mathcal{L}_\varepsilon u \leq 0$ as $|x| \geq R$. For any $\rho > 0$, consider the function

$$w_\rho(x) = e^{(R+\rho)^{1-\frac{\alpha}{p}}(\tau-\omega)} e^{\omega|x|^{1-\frac{\alpha}{p}}} + e^{R^{1-\frac{\alpha}{p}}(\tau+\omega)} e^{-\omega|x|^{1-\frac{\alpha}{p}}} := w_\rho^1(x) + w_\rho^2(x),$$

where ω, τ and R will be chosen later. Observe that, for any $\phi \in C^2_{loc}(\Omega)$, we have the expansion formula of p-Laplacian

$$\Delta_p \phi = (p-2)|\nabla\phi|^{p-4} \left\langle D^2\phi \nabla\phi, \nabla\phi \right\rangle + |\nabla\phi|^{p-2}\Delta\phi \qquad x \in \Omega. \tag{3.12}$$

Set $\overline{w}_\rho(x) = \max_{ij} |\partial_{ij} w_\rho(x)|$, by Cauchy-Schwarz inequality, one has

$$\Delta_p w_\rho \leq (|p-2|N\overline{w}_\rho + |\Delta w_\rho|)|\nabla w_\rho|^{p-2}. \tag{3.13}$$

Direct computation yields

$$\nabla w_\rho(x) = \omega|x|^{-\frac{\alpha}{p}} \left(1 - \frac{\alpha}{p}\right) \frac{x}{|x|} w_\rho^1 - \omega|x|^{-\frac{\alpha}{p}} \left(1 - \frac{\alpha}{p}\right) \frac{x}{|x|} w_\rho^2,$$

$$|\nabla w_\rho(x)| \leq \omega|x|^{-\frac{\alpha}{p}} \left(1 - \frac{\alpha}{p}\right) w_\rho,$$

$$\begin{aligned}
\partial_{ij} w_\rho(x) &= \left[\omega^2 \left(1 - \frac{\alpha}{p}\right)^2 x_i x_j |x|^{-2-\frac{2\alpha}{p}} + \omega \left(1 - \frac{\alpha^2}{p^2}\right) x_i x_j |x|^{-3-\frac{\alpha}{p}} \right] w_\rho^1 \\
&+ \left[\omega^2 \left(1 - \frac{\alpha}{p}\right)^2 x_i x_j |x|^{-2-\frac{2\alpha}{p}} + \omega \left(1 - \frac{\alpha^2}{p^2}\right) x_i x_j |x|^{-3-\frac{\alpha}{p}} \right] w_\rho^2 \\
&+ \delta_{i,j}\omega \left(1 - \frac{\alpha}{p}\right) |x|^{-1-\frac{\alpha}{p}} w_\rho^1 - \delta_{i,j}\omega \left(1 - \frac{\alpha}{p}\right) |x|^{-1-\frac{\alpha}{p}} w_\rho^2 \\
&\leq \left[\frac{\omega^2}{2} \left(1 - \frac{\alpha}{p}\right)^2 |x|^{1-\frac{\alpha}{p}} + \frac{\omega}{2} \left(1 - \frac{\alpha^2}{p^2}\right) + \omega \left(1 - \frac{\alpha}{p}\right) \right] |x|^{-1-\frac{\alpha}{p}} w_\rho \\
\overline{w}_\rho(x) &\leq \left[\frac{\omega^2}{2} \left(1 - \frac{\alpha}{p}\right)^2 |x|^{1-\frac{\alpha}{p}} + \frac{\omega}{2} \left(1 - \frac{\alpha^2}{p^2}\right) + \omega \left(1 - \frac{\alpha}{p}\right) \right] |x|^{-1-\frac{\alpha}{p}} w_\rho(x) \\
\Delta w_\rho(x) &\leq N \left[\frac{\omega^2}{2} \left(1 - \frac{\alpha}{p}\right)^2 |x|^{1-\frac{\alpha}{p}} + \frac{\omega}{2} \left(1 - \frac{\alpha^2}{p^2}\right) + \omega \left(1 - \frac{\alpha}{p}\right) \right] |x|^{-1-\frac{\alpha}{p}} w_\rho(x).
\end{aligned}$$

Applying these computations to (3.13), we derive

$$\begin{aligned}&\Delta_p w_\rho(x)\\ &\leq N(p-1)\omega^{p-1}\left(1-\frac{\alpha}{p}\right)^{p-1}\left[\frac{\omega}{2}\left(1-\frac{\alpha}{p}\right)|x|^{1-\frac{\alpha}{p}}+\frac{3p+\alpha}{2p}\right]|x|^{-\alpha-1+\frac{\alpha}{p}}w_\rho^{p-1}(x).\end{aligned}$$

Therefore, set $A=N(p-1)\omega^{p-1}\left(1-\frac{\alpha}{p}\right)^{p-1}\frac{3p+\alpha}{2p}$, one has

$$\mathcal{L}_\varepsilon w_\rho \;=\; |x|^{-\alpha}w_\rho^{p-1}\left[-\frac{N(p-1)}{2}\omega^p\left(1-\frac{\alpha}{p}\right)^p-A|x|^{\frac{\alpha}{p}-1}+m+\varepsilon\right].$$

Since $\alpha<p$, $|x|^{\frac{\alpha}{p}-1}\to 0$ as $|x|\to\infty$. For any $\rho>0$, to assign w_ρ to be a supersolution of $\mathcal{L}_\varepsilon\phi=0$ in $B_{R+\rho}\setminus B_R$, we choose $R=R(\varepsilon)$ and ω such that

$$A|x|^{\frac{\alpha}{p}-1}\leq\frac{\varepsilon}{2}\quad\text{as } |x|\geq R\qquad\text{and}\qquad\omega=\left(\frac{2m+\varepsilon}{Np-N}\right)^{1/p}\frac{p}{p-\alpha}>\overline{\omega}.$$

On the other hand, by the growth condition of u, one finds $C>0$ such that $u(x)\leq Ce^{\overline{\omega}|x|^{1-\frac{\alpha}{p}}}$ in $\mathbb{R}^N$. Therefore, we can take τ arbitrarily in $(\overline{\omega},\omega)$ and R sufficiently large such that for any $\rho>0$, one has

$$\begin{cases} w_\rho(x)\geq e^{R^{1-\frac{\alpha}{p}}\tau}\geq Ce^{R^{1-\frac{\alpha}{p}}\overline{\omega}}\geq u(x), & \text{as } |x|=R\\ w_\rho(x)\geq e^{(R+\rho)^{1-\frac{\alpha}{p}}\tau}\geq Ce^{(R+\rho)^{1-\frac{\alpha}{p}}\overline{\omega}}\geq u(x), & \text{as } |x|=R+\rho.\end{cases}$$

Fix such ω,τ and R. Applying the weak comparison principle on the bounded domain [60] $B_{R+\rho}\setminus B_R$, we obtain for all $\rho>0$ that

$$u(x)\leq w_\rho(x)=e^{(R+\rho)^{1-\frac{\alpha}{p}}(\tau-\omega)}e^{\omega|x|^{1-\frac{\alpha}{p}}}+e^{R^{1-\frac{\alpha}{p}}(\tau+\omega)}e^{-\omega|x|^{1-\frac{\alpha}{p}}}\qquad\text{in } B_{R+\rho}\setminus B_R.$$

Sending $\rho\to\infty$, we obtain

$$u(x)\leq e^{R^{1-\frac{\alpha}{p}}(\tau+\omega)}e^{-\omega|x|^{1-\frac{\alpha}{p}}}\qquad\text{in } \mathbb{R}^N\setminus B_R.$$

The fact $\omega>\overline{\omega}$ confirms the proof. □

When $\alpha=p$, solutions under consideration decay polynomially as stated in the following proposition.

Proposition 3.2.2. *Let $\alpha=p$, $m>0$ and u be a positive function satisfying*

$$\liminf_{|x|\to\infty}|x|^p\left(\frac{\Delta_p u}{u^{p-1}}-\frac{m}{|x|^p}\right)>0\qquad\textit{and}\qquad\limsup_{|x|\to\infty}\frac{u(x)}{|x|^{\gamma_0}}<\infty,\tag{3.14}$$

where γ_0 is the unique positive solution of (3.9). Then,

$$\lim_{|x|\to\infty}u(x)|x|^{\gamma_0}=0.\tag{3.15}$$

Proof. By assumption, for any $\varepsilon > 0$, there exists $R = R(\varepsilon)$ such that

$$\Delta_p u \geq \frac{m+\varepsilon}{|x|^p} u^p \qquad \text{for } |x| \geq R$$

We set $\mathcal{L}_\varepsilon[\phi] = -\Delta_p \phi + (m+\varepsilon)|x|^{-p}\phi^{p-1}$, obviously $\mathcal{L}_\varepsilon[u] \leq 0$ in $\mathbb{R}^N \setminus B_R$. Similar to Proposition 3.2.1, we will look for a suitable decaying supersolution of $\mathcal{L}_\varepsilon[\phi] = 0$ on $B_{R+\rho} \setminus B_R$ for all $\rho > 0$. However, the supersolution is no longer of exponential decay. Let $C_1, C_2, \gamma > 0$, we consider

$$w_\rho(x) = \frac{C_1}{\ln(R+\rho)}|x|^\gamma + C_2|x|^{-\gamma}$$

From (3.13), for any $\phi \in C^2_{\rm loc}(\Omega)$, we have

$$\Delta_p \phi \leq (|p-2|N\overline{\phi} + |\Delta\phi|)|\nabla\phi|^{p-2} \leq (pN\overline{\phi} + |\Delta\phi|)|\nabla\phi|^{p-2} \qquad x \in \Omega,$$

where $\overline{\phi}(x) = \max_{ij} |\partial_{ij}\phi(x)|$. Direct computations yield

$$|\nabla w_\rho(x)|^{p-2} \leq \gamma^{p-2}|x|^{-(p-2)} w_\rho^{p-2}(x),$$

$$|\Delta w_\rho(x)| \leq (\gamma^2 + |N-2|\gamma)|x|^{-2} w_\rho(x) \leq (\gamma^2 + N\gamma)|x|^{-2} w_\rho(x)$$

$$\overline{w}_\rho(x) \leq (\gamma^2 + 3\gamma)|x|^{-2} w_\rho(x).$$

Therefore,

$$\mathcal{L}_\varepsilon[w_\rho](x) \geq [-(Np+1)\gamma^p - (3p+1)N\gamma^{p-1} + m + \varepsilon]|x|^{-p} w_\rho(x)^{p-1}.$$

Let γ_ε be the unique positive solution of

$$-(Np+1)\gamma^p - (3p+1)N\gamma^{p-1} + m + \varepsilon = 0.$$

Obviously $\gamma_\varepsilon > \gamma_0$, which is the unique positive solution of (3.9).

By taking $\gamma = \gamma_\varepsilon$, we see that $\mathcal{L}_\varepsilon[w_\rho] \geq 0$ in $B_{R+\rho} \setminus B_R$ for all $\rho > 0$. To assign w_ρ to be a supersolution of $\mathcal{L}_\varepsilon[\phi] = 0$ on $B_{R+\rho} \setminus B_R$ for all $\rho > 0$, we verify the boundary condition. By the growth assumption of u, we have

$$\limsup_{|x|\to\infty} \frac{u(x)}{|x|^{\gamma_0}} < \infty.$$

This yields that there exist $C > 0$ such that $u(x) \leq C|x|^{\gamma_0}$ for $|x| \geq R$. Choosing $C_1 = C$, for ρ large, one has

$$u(x) \leq C|x|^{\gamma_0} \leq C\frac{|x|^{\gamma_\varepsilon - \gamma_0}}{\ln(R+\rho)}|x|^{\gamma_0} \leq w_\rho(x) \qquad x \in \partial B_{R+\rho}.$$

It is easily chosen C_2 large enough such that $u(x) \leq w(x)$ on ∂B_R. The weak comparison principle implies, for $\rho > 0$, that

$$u(x) \leq w_\rho(x) = \frac{C_1}{\ln(R+\rho)}|x|^{\gamma_\varepsilon} + C_2|x|^{-\gamma_\varepsilon} \qquad x \in B_{R+\rho} \setminus B_R.$$

Since C_1, C_2 are independent of ρ, by letting $\rho \to \infty$, we finally derive

$$u(x) \leq C_2|x|^{-\gamma_\varepsilon} \qquad x \in \mathbb{R}^N \setminus B_R.$$

The proof is complete due to $\gamma_\varepsilon > \gamma_0$. □

Now, we are able to prove the comparison principle.

3.3 Comparison principle

In this section, we demonstarte the comparison principle for (3.1) in $\mathbb{R}^N$ which serve to obtain nonexistence and uniqueness result in the next section.

Theorem 3.3.1. *Let $u_1, u_2 \in C^1_{\text{loc}}(\mathbb{R}^N)$ be respectively positive supersolution and subsolution of (3.1) such that*

$$u_2 \in L^1(\mathbb{R}^N) \quad \text{and} \quad \lim_{|x|\to\infty} u_2(x) = 0. \tag{3.16}$$

Then $u_1 \geq u_2$ in $\mathbb{R}^N$.

Proof. For any $\varepsilon \geq 0$, $R \in (0, \infty]$, set

$$\mathcal{D}_R(\varepsilon) = \{x \in B_R : u_2(x) + \varepsilon > u_1(x) + 2\varepsilon\}.$$

We will show that $D_\infty(0) = \emptyset$. It is sufficient to prove that $D_{R_0}(0) = \emptyset$ for any $R_0 > 0$ such that $b \not\equiv 0$ in B_{R_0}.

Fix a such R_0 and let $\varepsilon \in (0, 1)$. By (3.16), there exists $R = R(\varepsilon) > R_0$ such that $u_2(x) < \varepsilon/2$ for every $x \in (B_R)^c$. Denote $\varepsilon_1 = 2\varepsilon$, $\varepsilon_2 = \varepsilon$ and

$$v_i = \frac{[(u_2 + \varepsilon_2)^p - (u_1 + \varepsilon_1)^p]_+}{(u_i + \varepsilon_i)^{p-1}} \quad i = 1, 2$$

then $v_i \in W^{1,p}_0(K_R)$ with some $K_R \subset\subset B_R$ and $v_i = 0$ out side K_R. Therefore, v_i can be approximated by a sequence of functions in $C^\infty_0(K_R)$. We can use v_i, $i = 1, 2$, as test functions to get

$$\int_{B_R} (|\nabla u_1|^{p-2}\nabla u_1.\nabla v_1 - a(x)u_1^{p-1}v_1 + b(x)g(u_1)v_1)dx \geq 0 \tag{3.17}$$

$$\int_{B_R} (|\nabla u_2|^{p-2}\nabla u_2.\nabla v_2 - a(x)u_2^{p-1}v_2 + b(x)g(u_2)v_2)dx \leq 0 \tag{3.18}$$

Subtracting (3.18) from (3.17) yields

$$\begin{aligned} &\int_{B_R} (|\nabla u_1|^{p-2}\nabla u_1.\nabla v_1 - |\nabla u_2|^{p-2}\nabla u_2.\nabla v_2)dx \\ &\qquad \geq \int_{B_R} a(x)(u_1^{p-1}v_1 - u_2^{p-1}v_2)dx - \int_{B_R} b(x)(g(u_1)v_1 - g(u_2)v_2)dx. \end{aligned} \tag{3.19}$$

Set

$$w_i = u_i + \varepsilon_i, \quad V_i = \nabla(\ln(u_i + \varepsilon_i)) = \frac{\nabla u_i}{u_i + \varepsilon_i} \quad i = 1, 2 \tag{3.20}$$

and

$$I := |\nabla u_2|^{p-2}\nabla u_2 \cdot \nabla v_2 - |\nabla u_1|^{p-2}\nabla u_1 \cdot \nabla v_1.$$

By a computation, we obtain

$$I = w_2^p(|V_2|^p - |V_1|^p - p|V_1|^{p-2}V_1 \cdot (V_1 - V_2)) + w_1^p(|V_1|^p - |V_2|^p - p|V_2|^{p-2}V_2 \cdot (V_1 - V_2)). \tag{3.21}$$

By the Clarkson's inequality [74], for all vectors $X, Y \in \mathbb{R}^N$, we have

$$|X|^p - |Y|^p - p|Y|^{p-2}Y(X-Y) \geq c_p \frac{|X-Y|^{p+(2-p)_+}}{(|X|+|Y|)^{(2-p)_+}} \tag{3.22}$$

where

$$c_p = \begin{cases} \dfrac{1}{2^p - 1} & \text{if } 1 < p < 2 \\ \dfrac{3p(p-1)}{16} & \text{if } p \geq 2. \end{cases}$$

It follows from (3.21) and (3.22) that

$$I \geq c_p (w_1^p + w_2^p) \frac{|V_1 - V_2|^{p+(2-p)_+}}{(|V_1|+|V_2|)^{(2-p)_+}} \quad \forall p > 1. \tag{3.23}$$

Combining (3.19) and (3.23) and using (3.20), we obtain

$$\begin{aligned} -c_p \int_{\mathcal{D}_R(\varepsilon)} (w_1^p + w_2^p) \frac{|V_1 - V_2|^{p+(2-p)_+}}{(|V_1|+|V_2|)^{(2-p)_+}} dx \\ \geq -\int_{\mathcal{D}_R(\varepsilon)} a(x) \left[\left(\frac{u_2}{w_2}\right)^{p-1} - \left(\frac{u_1}{w_1}\right)^{p-1} \right] (w_2^p - w_1^p) dx \\ + \int_{\mathcal{D}_R(\varepsilon)} b(x) \left[\frac{g(u_2)}{w_2^{p-1}} - \frac{g(u_1)}{w_1^{p-1}} \right] (w_2^p - w_1^p) dx. \end{aligned} \tag{3.24}$$

Consequently,

$$\begin{aligned} c_p \int_{\mathcal{D}_R(\varepsilon)} \frac{|V_2 - V_1|^{p+(2-p)_+}}{(|V_1|+|V_2|)^{(2-p)_+}} [w_1^p + w_2^p] dx + \int_{\mathcal{D}_R(\varepsilon)} b \left[\frac{g(u_2)}{w_2^{p-1}} - \frac{g(u_1)}{w_1^{p-1}} \right] (w_2^p - w_1^p) dx \\ \leq \|a\|_{L^\infty(\mathbb{R}^N)} \int_{\mathcal{D}_R(\varepsilon)} \left| \left(\frac{u_2}{w_2}\right)^{p-1} - \left(\frac{u_1}{w_1}\right)^{p-1} \right| (w_2^p - w_1^p) dx. \end{aligned} \tag{3.25}$$

Suppose by contradiction that $\mathcal{D}_{R_0}(0) \neq \emptyset$. Thanks to the continuity of u_i, $i = 1, 2$, $\mathcal{D}_{R_0}(0)$ is an open set. Let B be a small ball such that $\overline{B} \subset \mathcal{D}_{R_0}(0)$. On the one hand, as $B \subset \mathcal{D}_R(\varepsilon)$ for all ε small and g is nonnegative, it follows that

$$c_p \int_B \frac{|\nabla \ln u_2 - \nabla \ln u_1|^{p+(2-p)_+}}{(|\nabla \ln u_1| + |\nabla \ln u_2|)^{(2-p)_+}} [u_1^p + u_2^p] dx + \int_B b \left[\frac{g(u_2)}{u_2^{p-1}} - \frac{g(u_1)}{u_1^{p-1}} \right] (u_2^p - u_1^p) dx \leq J_\varepsilon \tag{3.26}$$

where J_ε is the third term in (3.25). Observe that, for $\varepsilon < 1$ and $x \in \mathcal{D}(\varepsilon)$, one has

$$\begin{aligned} \left| \left(\frac{u_2}{w_2}\right)^{p-1} - \left(\frac{u_1}{w_1}\right)^{p-1} \right| (w_2^p - w_1^p) &\leq 2p(w_2 - w_1) w_2^{p-1} \\ &\leq 2p(\|u_2\|_{L^\infty(\mathbb{R}^N)} + 1)^{p-1} u_2 \end{aligned}$$

and

$$\lim_{\varepsilon \to 0} \left| \left(\frac{u_2}{w_2}\right)^{p-1} - \left(\frac{u_1}{w_1}\right)^{p-1} \right| = 0 \quad \text{pointwise in } \mathbb{R}^N,$$

Thanks to (3.16), $u_2 \in L^1(\mathbb{R}^N)$, we deduce from Lebesgue dominated convergence theorem that

$$\lim_{\varepsilon \to 0} J_\varepsilon = 0. \tag{3.27}$$

Hence by letting $\varepsilon \to 0$ in (3.26) we obtain

$$\int_B \frac{|\nabla \ln u_2 - \nabla \ln u_1|^{p+(2-p)_+}}{(|\nabla \ln u_1| + |\nabla \ln u_2|)^{(2-p)_+}} [u_1^p + u_2^p] dx = \int_B b \left[\frac{g(u_2)}{u_2^{p-1}} - \frac{g(u_1)}{u_1^{p-1}} \right] (u_2^p - u_1^p) dx = 0,$$

which, together with condition (**C2**) on g, implies

$$|\nabla(\ln u_2 - \ln u_1)| \equiv b \equiv 0 \quad \text{in } \mathcal{D}_{R_0}(0). \tag{3.28}$$

Since $b \not\equiv 0$ in B_{R_0}, it follows that $D_{R_0}(0) \subsetneq B_{R_0}$ and $\partial D_{R_0}(0) \cap B_{R_0} \neq \emptyset$. Hence there is a connected component $\mathcal{O}$ of $D_{R_0}(0)$ such that $\partial D_{R_0}(0) \cap \partial \mathcal{O} \cap B_{R_0} \neq \emptyset$. From (3.28), we deduce that $\ln u_1 - \ln u_2 \equiv$ constant in $\mathcal{O}$, which in turn implies $u_1 = \ell u_2$ in $\mathcal{O}$ for some $\ell > 0$. As $u_1 = u_2$ on $\partial D_{R_0}(0) \cap \partial \mathcal{O} \cap B_{R_0}$, it follows that $\ell = 1$, which contradicts $\mathcal{O} \subset D_{R_0}(0)$. Therefore, we must have $D_{R_0}(0) = \emptyset$ and thus $u_1 \geq u_2$ in B_{R_0}. Since $R_0 > 0$ is arbitrarily large, we conclude that $u_1 \geq u_2$ in $\mathbb{R}^N$. □

3.4 Existence/Nonexistence and Uniqueness

We start with an auxilary result which is obtained thanks to Harnack inequality.

Proposition 3.4.1. *There exists a positive function $\varphi_\infty \in C^1_{loc}(\mathbb{R}^N)$ satisfying*

$$\mathcal{K}_a[\varphi_\infty] = \lambda_\infty(a) \varphi_\infty^{p-1} \quad \textit{weakly in } \mathbb{R}^N. \tag{3.29}$$

Furthermore, $\lambda_\infty(a)$ can be variationally characterized by

$$\lambda_\infty(a) = \inf_{\phi \in W^{1,p}(\mathbb{R}^N)} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla \phi|^p - a(x)|\phi|^p dx}{\displaystyle\int_{\mathbb{R}^N} |\phi|^p dx}. \tag{3.30}$$

Proof. For any $k > 0$, we see that the sequence $\{\lambda_R(a)\}_{R>k}$ is bounded. Let φ_R be the associated eigenfunction of $\lambda_R(a)$, normalized by $\varphi_R(0) = 1$. By Harnack's inequality (see, e.g., [92, Theorem 5], [98, Theorem 1]), there exists a constant $C(k) > 0$ independent of R such that

$$\sup_{B_k} \varphi_R(a) < C(k) \inf_{B_k} \varphi_R(a) \leq C(k).$$

Hence, in light of local regularity for elliptic equations and standard argument, up to a subsequence, $\{\varphi_R\}$ converges in $C^1_{\text{loc}}(\mathbb{R}^N)$ to a function φ_∞, which solves (3.29). Moreover, $\varphi_\infty(0) = 1$ and therefore $\varphi_\infty > 0$ in $\mathbb{R}^N$ by Harnack's inequality.

From above we have $\lambda_R(a) \geq \lambda_{R+1}(a) \geq \lambda_\infty(a)$ and $\lim_{R\to\infty} \lambda_R(a) = \lambda_\infty(a)$. Obviously, $\lambda_\infty(a) > -\sup_{\mathbb{R}^N} a(x)$. Let us call λ_v the right-hand side of (3.30), we shall prove that

$$\lim_{R \to \infty} \lambda_R(a) = \lambda_v.$$

Indeed, by the definition of λ_v, for all $\delta > 0$, there exists $\phi_\delta \in W^{1,p}(\mathbb{R}^N)$ such that

$$\lambda_v + \delta \geq \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla \phi_\delta|^p - a(x)|\phi_\delta|^p dx}{\displaystyle\int_{\mathbb{R}^N} |\phi_\delta|^p dx}.$$

Let $\chi_R \in C_c^1(\mathbb{R}^N)$ be compactly supported in B_R and $\chi_R = 1$ in B_{R-1}, we set $\phi_{\delta,R} = \chi_R \phi_\delta$. The variational characterization $\lambda_R(a)$ (3.7) yields

$$\frac{\displaystyle\int_{\mathbb{R}^N} |\nabla \phi_{\delta,R}|^p - a(x)|\phi_{\delta,R}|^p dx}{\displaystyle\int_{\mathbb{R}^N} |\phi_{\delta,R}|^p dx} \geq \lambda_R(a).$$

We find a constant $k > 1$ depending only on $\sup_{R>1} \|\chi_R\|_{W^{1,\infty}(\mathbb{R}^N)}$ such that

$$|\nabla \phi_{\delta,R}| = |\nabla \phi_\delta \chi_R + \phi_\delta \nabla \chi_R| \leq |\nabla \phi_\delta| + k|\phi_\delta|.$$

Since $\phi_\delta \in W^{1,p}(\mathbb{R}^N)$, the Lebesgue dominated convergence theorem implies that

$$\lambda_v + \delta \geq \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla \phi_\delta|^p - a(x)|\phi_\delta|^p dx}{\displaystyle\int_{\mathbb{R}^N} |\phi_\delta|^p dx} \geq \lim_{R\to\infty} \lambda_R(a) = \lambda_\infty(a).$$

The arbitrariness of δ confirms (3.30).

□

We are ready to present the

Proof of Theorem 3.1.1. i. Existence.

By the definition of $\lambda_\infty(a)$, there exists $R_0 > 0$ such that $\lambda_R(a) < 0$. Let φ_{R_0} be the associated eigenfunction of $\lambda_{R_0}(a)$ with normalization $\varphi_{R_0}(0) = 1$. Set $\underline{u}_{R_0}(x) = \delta \varphi_{R_0}$ where $\delta > 0$ to be chosen later and $M_{R_0} := \max_{B_{R_0}} \varphi_{R_0}$. In B_{R_0}, we get

$$\begin{aligned} -\Delta_p \underline{u}_{R_0} - a(x)\underline{u}_{R_0}^{p-1} + b(x)g(\underline{u}_{R_0}) &= \delta^{p-1}\varphi_{R_0}^{p-1}\left(\lambda_{R_0}(a) + \|b\|_{L^\infty(\mathbb{R}^N)} \frac{g(\delta\varphi_{R_0})}{(\delta\varphi_{R_0})^{p-1}}\right) \\ &\leq \delta^{p-1}\varphi_{R_0}^{p-1}\left(\lambda_{R_0}(a) + \|b\|_{L^\infty(\mathbb{R}^N)} \frac{g(\delta M_{R_0})}{(\delta M_{R_0})^{p-1}}\right) \end{aligned} \tag{3.31}$$

By (**C2**), one can choose δ small, depending only on R_0, in such a way that $\delta M_{R_0} < s_0$, where s_0 is the constant in (**C3**), and the RHS of (3.31) is negative. Put $\underline{u} = \chi_{B_{R_0}} \underline{u}_{R_0}$. For any $R > R_0$, we see that $\underline{u}$ and s_0 are respectively sub and supersolution of (3.1) in B_R and $\underline{u} < s_0$ on ∂B_R. By applying the sub-supersolution theorem (see for instance [73, Theorem 3.1]), we derive that there exists a weak solution u_R of (3.1) in B_R such that $\underline{u} \leq u_R \leq s_0$ in B_R. By regularity result for quasilinear elliptic equations (see [49]) and standard argument, up to a subsequence, $\{u_R\}$ converges in $C^1_{\text{loc}}(\mathbb{R}^N)$ to a function u^*, which is a weak solution of (3.1) in $\mathbb{R}^N$ and satisfies $\underline{u} \leq u^* \leq s_0$. Since $u^*(0) \geq \underline{u}(0) = \delta > 0$, by Harnack inequality

[92], [98], we obtain $u^* > 0$ in $\mathbb{R}^N$. Finally, rom Proposition 3.2.1, we deduce that u^* decays exponentially and thus $u^* \in S_\alpha$.

ii. Nonexistence and uniqueness.

Suppose by contradiction that there exists a function $u \in \mathcal{S}_\alpha$. Thanks to (**C1**), Proposition (3.2.1) implies that u satisfies (3.11). In particular, u satisfies (3.16). Let φ_∞ be a positive solution of (3.29) normalized by $\varphi_\infty(0) < u(0)$. Since $\lambda_\infty(a) \geq 0$, it follows that φ_∞ is a positive supersolution of (3.1). By Theorem 3.3.1, we have $u \leq \varphi_\infty$ in $\mathbb{R}^N$. This contradiction completes the proof.

The uniqueness is a direct consequence of the exponential decay property and comparison principle Theorem 3.3.1. □

Proof of Theorem 3.1.2. Existence part can be obtained by the same argument as in the proof of Theorem 3.1.1. In order to prove the nonexistence and uniqueness we also follow the above scheme, employing Proposition (3.2.2) instead of Proposition (3.2.1). The proof would be completed if we could show that if $u \in S_p$ then u satisfies (3.16) and $u \in L^1(\mathbb{R}^N)$. Indeed, by Proposition (3.2.2), we know that if $u \in S_p$ then u satisfies (3.15). Therefore $u \in L^1(\mathbb{R}^N)$ if $\gamma_0 > N$. Since the map $\gamma \mapsto (Np+1)\gamma^p + (3p+1)N\gamma^{p-1}$ is strictly increasing, it follows that $m > N^p(Np+3p+2)$ if and only if $\gamma_0 > N$. Thus $u \in L^1(\mathbb{R}^N)$. This completes the proof. □

Chapitre 4

Generalized principal eigenvalues for nonlocal operator

The contents of this chapter are based on the joint work :

Nonlocal heterogeneous KPP equations in $\mathbb{R}^N$

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In this chapter, we collect some new results about the relation of λ_p, λ_p' and λ_v in bounded and unbounded domains (eventually $\mathbb{R}^N$). For the sake of coherence, we will present here a common introduction for the two next chapters and do not repeat again in chapter 5. In section 4.1, we will present the derivation of the model, the common hypotheses that are used also for chapter 5 and explain why the investigation of the generalized principal eigenvalues plays the central role.

4.1 Introduction

In this article, we are interested in finding survival criteria for a species that has a long range dispersal strategy. As a model species, we can think as trees whose seed and pollens are disseminate on a long range. In ecology a commonly used model that integrate such long range dispersal [55, 62, 66, 70, 75, 97] is the following:

$$\frac{\partial u}{\partial t}(t,x) = J \star u(t,x) - u + f(x,u(t,x)) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N, \tag{4.1}$$

where $u(t,x)$ is the density of the considered population, J is a dispersal kernel, $f(x,s)$ is a KPP type non-linearity describing the growth rate of the population. The possibility of a long range dispersal is well known in ecology, where numerous data are now available to support this assumptions [32, 33, 34, 99, 48]. In this setting the tail of the kernel is a measure of the frequency at which long dispersal events occur. A biological explanation that support the used of (4.2) to describe the evolution of the population comes from the observation that the intrinsic variability in the capacity of the individuals to disperse generates, at

the scale of a population, a long range dispersal of the population. The effect of such variability has been investigate in [64, 82] by means of the study of correlated random walks. In such framework, all individuals follow a simple random walks where the diffusion coefficient follows a probability law, it can be checked that the probability of the density of population will then follow an integrodifferential equation [64, 82, 97] where a dispersal kernel J describe the probability to jump from one location to another.

Along this paper we will always make the following assumptions on the dispersal kernel J:

(H1) $J \in C(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ is nonnegative, radially symmetric .

(H2) $J(0) > 0$ and has unit mass (i.e. $\int_{\mathbb{R}^N} J(z)dz = 1$).

In the present paper, we focus our analysis on species that have a bounded ecological niche. A simple way to model such spatial repartition consists by considering that the environment is hostile to the species outside a bounded set. This fact translates in our modelling by assuming that f satisfies:

(H3) $f \in C^{1,\alpha}(\mathbb{R}^{N+1})$ is of KPP type, that is :

$$\begin{cases} f(\cdot, 0) \equiv 0, \\ \text{For all } x \in \mathbb{R}^N, f(x,s)/s \text{ is decreasing with respect to } s \text{ on } (0, +\infty). \\ \text{There exists } S(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \text{ such that } f(x, S(x)) \leq 0 \text{ for } \forall x \in \mathbb{R}^N. \end{cases}$$

(H4) $\limsup_{|x|\to\infty} \frac{f(x,s)}{s} < 0, \quad$ uniformly in $\quad s \geq 0.$

A typical example of such nonlinearity is given by $f(x,s) := s(a(x) - b(x)s)$ with $b(x) > 0$ and $a(x)$ so that $\limsup_{|x|\to\infty} a(x) < 0$.

Our main purpose is to seek conditions on J and f that characterize the persistence of the species modelled by (4.1). In this task, we focus our analysis on the description of the set of positive stationary solution of (4.1). That is the set of positive solution of the equation below

$$J \star u(x) - u(x) + f(x, u(x)) = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{4.2}$$

This description is expected to provide useful persistence criteria.

In the literature, persistence criteria have been well studied for the reaction diffusion version of (4.1)

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + f(x,t,u(t,x)) \quad \text{in} \quad \mathbb{R}^+ \times \Omega, \tag{4.3}$$

where Ω is the domain of $\mathbb{R}^N$ possibly $\mathbb{R}^N$ itself. Survival criteria have been obtained for various media, ranging from periodic media to ergodic media [10, 9, 18, 28, 27, 30, 56, 77, 78, 83, 93]. In the context of global warming, survival criteria have been investigate in [18, 5, 19]. For such reaction diffusion equations the survival criteria are often obtained by looking at the sign of the first eigenvalue of the linear problem obtained by linearising (4.3) around the 0 solution. That is the sign of the first eigenvalue $\lambda_1(\Delta + \partial_s f(x,0), \Omega)$ of the spectral problem

$$\Delta\varphi(x) + \partial_s f(x,0)\phi(x) + \lambda_1\varphi(x) = 0 \quad \Omega, \tag{4.4}$$

associated with the proper boundary conditions (if $\Omega \neq \mathbb{R}^N$). In most situations, for KPP like nonlinearities, the existence of a positive stationary solution to (4.3) is uniquely conditioned by the sign of λ_1. More precisely, there exists a unique positive stationary solution if and only if $\lambda_1 < 0$. If such type of criteria seems reasonable in problems defined on bounded set, it is less obvious for problems in unbounded domains. In particular, in unbounded domains, one of the main difficulty concerns the definition of λ_1.

As shown in [17, 9], the notion of first eigenvalue in unbounded domain can be ambiguous and several definition of λ_1 exists rendering the establishment of a sharp survival criteria quite involved.

For the non-local equation (4.2) less is known and to our knowledge survival criteria have been essentially investigated in some specific situations, periodic media [41, 42, 94] or for a version of the problem (4.2) defined in a bounded domain Ω,

$$\frac{\partial u}{\partial t}(t,x) = \int_\Omega J(x-y)u(t,y)\,dy - u(t,x) + f(x,u(t,x)) \quad \text{in} \quad \mathbb{R}^+ \times \Omega, \tag{4.5}$$

[2, 39, 43, 57, 68, 94]. We also quote [21] for an analysis of a persistence criteria in periodic media for a non-local version of (4.3) involving a fractional diffusion and [88] for survival criteria in time periodic versions of (4.5). Similarly to the local diffusion case, for KPP like non-linearities, the existence of a positive solution to the non-local equation (4.5) can be characterised by the sign of a spectral quantity λ_p, called the generalised principal eigenvalue of

$$\int_\Omega J(x-y)\phi(y)\,dy - \phi(x) + \partial_s f(x,0)\phi(x) + \lambda_p\phi(x) = 0 \quad \text{in} \quad \Omega. \tag{4.6}$$

In fact, λ_p is defined by

$$\lambda_p := \sup\left\{\lambda \in \mathbb{R} \,|\, \exists \varphi \in C(\Omega), \varphi > 0,\ \int_\Omega J(x-y)\varphi(y)\,dy + (\partial_s f(x,0) - 1 + \lambda)\varphi(x) \le 0 \text{ in } \Omega\right\}.$$

Unlike the elliptic PDE case, due to the lack of a regularising effect of the diffusion operator, the above spectral problem may not have a solution in any reasonable space of functions i.e $(L^p(\Omega), C(\Omega))$[41, 37, 68]. As a consequence, even in bounded domain, simple sharp survival criteria are already quite involved to obtain. Another difficulty inherent to the study of nonlocal equations (4.6) in unbounded domain concerns the lack of "reasonable" *a priori* estimates for the solution thus making standard approximations difficult to use in most cases.

4.1.1 Notations

To simplify the presentation, we introduce some notations and various linear operator that we use along this paper:

- $B_R(x_0)$ will denote the standard ball of radius R centred at the point x_0
- χ_R will always refer to the characteristic function of $B_R(0)$.
- $\mathcal{S}(\mathbb{R}^N)$ denotes the Schwartz space.
- For a positive integrable function $J \in \mathcal{S}(\mathbb{R}^N)$, the constant $\int_{\mathbb{R}^N} J(z)|z|^2\,dz$ will refer to

$$\int_{\mathbb{R}^N} J(z)|z|^2\,dz := \int_{\mathbb{R}^N} J(z)\left(\sum_{i=0}^N z_i^2\right)dz$$

- We denote $\mathcal{L}_\Omega$ the continuous linear operator

$$\begin{array}{rccl} \mathcal{L}_\Omega: & C(\bar\Omega) & \to & C(\bar\Omega) \\ & u & \mapsto & \int_\Omega K(x,y)u(y)\,dy, \end{array} \tag{4.7}$$

  where $\Omega \subset \mathbb{R}^N$ and $K \in C(\bar\Omega \times \bar\Omega)$ is a non negative function satisfying $K(x,\cdot) \in L^1(\Omega)$ for all $x \in \Omega$.

- $\mathcal{L}_R$ correspond to the continuous operator $\mathcal{L}_\Omega - Id$ where $K(x,y) = J(x-y)$ and $\Omega = B_R(0)$
- We will use $\mathcal{M}$ to denote the operators $\mathcal{L}_\Omega - Id$ where $K(x,y) = J(x-y)$ and $\Omega = \mathbb{R}^N$.
- Finally, $\mathcal{M}_\varepsilon$ will denote the operator $\mathcal{M}$ with a rescaled kernel $\frac{1}{\varepsilon^N} J\left(\frac{z}{\varepsilon}\right)$ and $\mathcal{M}_{\varepsilon,m} := \frac{1}{\varepsilon^m}\mathcal{M}_\varepsilon$

Sometimes, we also use the notation $\beta(x) := \partial_s f(x,0)$ to avoid the repeatability.

4.2 Preliminaries

In this section, we recall some results on the principal eigenvalue of a linear non-local operator $\mathcal{L}_\Omega + a(x)$ where $a(x) \in C(\bar{\Omega}) \cap L^\infty(\Omega)$ and some known results about the KPP equation below

$$\frac{\partial u}{\partial t}(t,x) = \mathcal{L}_\Omega[u] + f(x,u(t,x)) \quad \text{in} \quad \mathbb{R}^+ \times \Omega, \tag{4.8}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. For simplicity, we divide this section into two subsections, one devoted to the principal eigenvalue and the other dedicated to known survival criteria of (4.8).

4.2.1 The principal eigenvalue problem in bounded domains

In this subsection, we focus on the properties of the solution of the spectral problem

$$\mathcal{L}_\Omega[\varphi] + a(x)\varphi + \lambda\varphi = 0 \quad \text{in} \quad \Omega. \tag{4.9}$$

In contrast with elliptic operators, when $a(x) \not\equiv Cste$, neither $\mathcal{L}_\Omega + a(x) + \lambda$ nor its inverse are compact operators and the description of the spectrum of $\mathcal{L}_\Omega + a$ using the Krein-Rutman Theory fails. However as shown in [39], some variational formula introduced in [12] to characterise the first eigenvalue of elliptic operators $\mathcal{E} := a_{ij}\partial_{ij} + b_i(x)\partial_i + c(x)$,

$$\lambda_1(\mathcal{E}) := \sup\{\lambda \in \mathbb{R} \,|\, \exists\, \varphi \in W^{2,n}(\Omega), \varphi > 0 \ \text{ so that } \ \mathcal{E}[\varphi] + \lambda\varphi \leq 0\}, \tag{4.10}$$

can be transposed to the operator $\mathcal{L}_\Omega + a(x)$. Namely, the following quantity is well defined

$$\lambda_p(\mathcal{L}_\Omega + a(x)) := \sup\{\lambda \in \mathbb{R} \,|\, \exists\, \varphi \in C(\Omega), \varphi > 0 \ \text{ so that } \ \mathcal{L}_\Omega[\varphi] + a(x)\varphi + \lambda\varphi \leq 0\}. \tag{4.11}$$

Also noted in [39], the generalised principal eigenvalue given by the definition (4.11) is not always achieved. This means that there is not always a positive continuous eigenfunction associated with λ_p. However, as proved in [39, 68, 94], when Ω is an open bounded set we can find some conditions on the coefficients that guarantee the existence of a positive continuous eigenfunction. For example there exists a positive continuous principal eigenfunction, if we assume that the function $a(x)$ satisfies

$$\frac{1}{\sup_\Omega a - a(x)} \notin L^1(\Omega') \quad \text{for some open bounded domain} \quad \Omega' \subset \bar{\Omega}.$$

Another useful criteria that guarantee the existence of a continuous principal eigenfunction was remark in [42, 43]:

Proposition 4.2.1. *Let Ω be a bounded open set and let $\mathcal{L}_\Omega$ be as in (4.7) then there exists a positive continuous eigenfunction associated to λ_p* ***if and only if*** *$\lambda_p(\mathcal{L}_\Omega + a) < -\sup_\Omega a$.*

We point the interested reader to [37] for a more complete description of the positive solution associated to λ_p when the domain Ω is bounded.

Next we recall some properties of the principal eigenvalue λ_p that we will constantly use along this paper:

Proposition 4.2.2. *(i) Assume $\Omega_1 \subset \Omega_2$, then for the two operators*

$$\mathcal{L}_{\Omega_1}[u] + a(x)u := \int_{\Omega_1} K(x,y)u(y)\,dy + a(x)u$$
$$\mathcal{L}_{\Omega_2}[u] + a(x)u := \int_{\Omega_2} K(x,y)u(y)\,dy + a(x)u$$

respectively defined on $C(\Omega_1)$ and $C(\Omega_2)$ we have

$$\lambda_p(\mathcal{L}_{\Omega_1} + a(x)) \geq \lambda_p(\mathcal{L}_{\Omega_2} + a(x)).$$

(ii) Fix Ω and assume that $a_1(x) \geq a_2(x)$, then

$$\lambda_p(\mathcal{L}_{\Omega} + a_2(x)) \geq \lambda_p(\mathcal{L}_{\Omega} + a_1(x)).$$

(iii) $\lambda_p(\mathcal{L}_\Omega + a(x))$ is Lipschitz continuous in $a(x)$. More precisely,

$$|\lambda_p(\mathcal{L}_\Omega + a(x)) - \lambda_p(\mathcal{L}_\Omega + b(x))| \leq \|a(x) - b(x)\|_\infty$$

(iv) We always have the following estimate

$$-\sup_\Omega \left(a(x) + \int_\Omega K(x,y)\,dy \right) \leq \lambda_p(\mathcal{L}_\Omega + a) \leq -\sup_\Omega a.$$

We refer to [39, 43] for the proofs of $(i) - (iv)$.

4.2.2 Existence criteria for the kpp-equation in bounded domains

Equipped with the above notion of principal eigenvalue, it has been shown [2, 39] that on bounded domains the existence of a positive stationary solution of (4.8) is conditioned by the sign of the principal eigenvalue λ_p of the linear operator $\mathcal{L}_\Omega + \partial_s f(x,0)$. That is to say

Theorem 4.2.3 ([2, 39])**.** *Let Ω be a bounded open set and let $\mathcal{L}_\Omega$ be as in (4.7). Assume that f satisfy $(H3)$. Then there exists a unique positive continuous stationary solution $\bar{u}$ to (4.8)* ***if and only if*** *$\lambda_p(\mathcal{L}_\Omega + \partial_s f(x,0)) < 0$. Moreover, if $\lambda_p \geq 0$ then any non negative uniformly bounded stationary solution of (4.8) is identically zero. In addition, we have the following dynamics for any positive bounded continuous solution of (4.8):*

(i) When $\lambda_p \geq 0$,

$$\lim_{t\to\infty} u(t,x) \to 0 \quad \text{uniformly in } \Omega,$$

(ii) When $\lambda_p < 0$,

$$\lim_{t\to\infty} u(t,x) \to \bar{u} \quad \text{uniformly in } \Omega.$$

Remark 13. This existence criteria is similar to those known for the reaction diffusion version of (4.8) [10, 28, 27, 30, 53, 56].

4.3 Generalized principal eigenvalues in unbounded domain

For the rest of the article it is important to understand the eigenvalue problem

$$\mathcal{L}_\Omega[\varphi] + a(x)\varphi + \lambda\varphi = 0 \quad \text{in} \quad \Omega \tag{4.12}$$

when the domain $\Omega \subset \mathbb{R}^N$ is unbounded and in particular the eigenvalue problem

$$\mathcal{M}[\varphi] + a(x)\varphi = -\lambda\varphi \quad \text{in} \quad \mathbb{R}^N. \tag{4.13}$$

The purpose of this section is to study their properties. To this end, motivated by the works [17, 9, 18] for elliptic operators, we start by introducing two definitions of the principal eigenvalue.

Definition 4.3.1. Let $\mathcal{L}_\Omega + a(x)$ be as in (4.7). We set

$$\lambda_p(\mathcal{L}_\Omega + a(x)) := \sup\{\lambda \in \mathbb{R} \,|\, \exists \varphi > 0, \varphi \in C(\Omega), \text{ s.t } \mathcal{L}_\Omega[\varphi] + a(x)\varphi + \lambda\varphi \leq 0 \text{ in } \Omega\} \tag{4.14}$$

$$\lambda_p'(\mathcal{L}_\Omega + a(x)) := \inf\{\lambda \in \mathbb{R} \,|\, \exists \varphi > 0, \varphi \in C(\Omega) \cap L^\infty(\Omega), \text{ s.t } \mathcal{L}_\Omega[\varphi] + a(x)\varphi + \lambda\varphi \geq 0 \text{ in } \Omega\}. \tag{4.15}$$

Another useful definition that holds only for self-adjoint operators is

Definition 4.3.2. Let $\mathcal{L}_\Omega + a(x)$ be as in (4.7). Assume further that $K(\cdot,\cdot)$ is symmetric (i.e. $K(x,y) = K(y,x)$ for all $x, y \in \Omega$). We set

$$\lambda_v(\mathcal{L}_\Omega + a(x)) := \inf_{\varphi \in L^2(\Omega), \varphi \not\equiv 0} \frac{\int_\Omega \int_\Omega K(x,y)(\varphi(x) - \varphi(y))^2 \, dxdy - \int_\Omega (a(x) - k(x))\varphi^2(x) \, dx}{\|\varphi\|^2_{L^2(\Omega)}}. \tag{4.16}$$

where $k(x) := \int_\Omega K(y,x) \, dy$.

In the context of the study of nonlocal operators, these definitions are natural extension to the nonlocal case of the definitions known for an elliptic operator. It is worth to mention that those definitions have already been used in the context of the study of (4.2) in several papers [41, 42, 43, 57, 67], but the relation between all the formulations has never been clarified.

For elliptic operators, it is known that the analogous of the three formulations above are equivalent on bounded domain [12]. This is not necessarily the case for unbounded domains, where examples can be constructed [17, 9, 20] showing that $\lambda_1 > \lambda_1'$. Since $\mathcal{L}_\Omega + a(x)$ shares many properties with elliptic operators, it is then expected that the three definitions are not necessarily equivalent. It is then important do understand the relation between the different quantities λ_p, λ_p' and λ_v.

From the definition of λ_p' and λ_v, we can check that the monotone behaviours and the Lipschitz continuity properties satisfied by λ_p ($(i) - (iii)$ of Proposition (4.2.2)) are still true for λ_p' and λ_v.

Also, as already been identified for $\lambda_p(\mathcal{L}_\Omega + a(x))$ in a bounded domain, an intrinsic difficulty related to this type of nonlocal operator comes from the possible non-existence of a positive continuous eigenfunction associated to the definition of λ_p, λ_p' or to λ_v. In particular in unbounded domain, the existence of an associated positive principal eigenfunction is not known in general. To our knowledge, the only general result known concerns the case of dispersal kernel K that satisfy the following condition:

(H5) There exist positive constants $r_0 \geq r_1; C_0 \geq c_0$ so that $\forall x, y \in \Omega$,

$$C_0 \chi_{B_{r_0}(x)} \geq K(x,y) \geq c_0 \chi_{B_{r_1}(x)}$$

For such dispersal kernel, we can show that

Theorem 4.3.3 (Sufficient condition [39]). *Let $\Omega \subset \mathbb{R}^N$, $a \in C(\Omega) \cap L^\infty(\Omega)$ and $K \in C(\bar{\Omega} \times \bar{\Omega})$ non negative, satisfying condition (H7). Let us denote $\sigma := \sup_{\bar{\Omega}} a(x)$ and assume further that the function $a(x)$ satisfies $\frac{1}{\sigma - a(x)} \notin L^1(\Omega_0)$ for some bounded domain $\Omega_0 \subset \bar{\Omega}$. Then there exists a principal eigen-pair (λ_p, φ_p) solution of*

$$\mathcal{L}_\Omega[\varphi] + (a(x) + \lambda)\varphi = 0.$$

Moreover, $\varphi_p \in C(\Omega)$, $\varphi_p > 0$ and we have the following estimate

$$-\sigma' < \lambda_p < -\sigma,$$

where $\sigma' := \sup_{x \in \Omega} \left[a(x) + \int_\Omega K(x,y)\, dy\right]$.

4.3.1 Main results

We show that three notions of eigenvalue λ_p, λ_p', and λ_v are equivalent when J is a symmetric kernel satisfying $(H1 - H2)$ with compact support. Namely, we show

Theorem 4.3.4. *Let J be compactly supported and satisfy $(H1 - H2)$. Assume $a(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ then we have*

$$\lambda_p(\mathcal{M} + a(x)) = \lambda_p'(\mathcal{M} + a(x)) = \lambda_v(\mathcal{M} + a(x)).$$

It is worth noticing that the condition for such equivalence remains an open problem for general kernels. As a side result, we show that on bounded domain and for any symmetric kernel J satisfying $(H1 - H2)$ we have $\lambda_p(\mathcal{L}_R + a(x)) = \lambda_p'(\mathcal{L}_R + a(x)) = \lambda_v(\mathcal{L}_R + a(x))$, where

$$\lambda_v(\mathcal{L}_R + a(x)) := \inf_{\varphi \in L^2(B_R), \varphi \not\equiv 0} \frac{\frac{1}{2}\iint_{B_R \times B_R} J(x-y)[\varphi(x) - \varphi(y)]^2\, dxdy + \int_{B_R}\left[\int_{\mathbb{R}^N \setminus B_R} J(x-y)\, dy\right]\varphi^2(x)\, dx}{\|\varphi\|^2_{L^2(B_R)}} - \frac{\int_{B_R} a(x)\varphi^2(x) dx}{\|\varphi\|^2_{L^2(B_R)}}.$$

We also obtain a full description of the asymptotic behaviour with respect to ε of the rescaled dispersal operator $\mathcal{M}_{\varepsilon,m} := \frac{1}{\varepsilon^m}\mathcal{M}_\varepsilon$. More precisely, we obtain

Theorem 4.3.5. *Let J be compactly supported and satisfy $(H1 - H2)$. Assume $a(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ then we have*

- *When $0 < m \leq 2$* $\quad \lim_{\varepsilon \to +\infty} \lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)) = -\sup_{\mathbb{R}^N} a(x)$
- *When $0 \leq m < 2$,* $\quad \lim_{\varepsilon \to 0} \lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)) = -\sup_{\mathbb{R}^N} a(x)$
- *When $m = 0$,* $\quad \lim_{\varepsilon \to +\infty} \lambda_p(\mathcal{M}_\varepsilon + a(x)) = 1 - \sup_{\mathbb{R}^N} a(x)$

  In this case, if $a(x)$ is radially symmetric non increasing and $\sup_{\mathbb{R}^N} a(x) < 1$ then there is a sharp threshold ε_0, in the sense that $\lambda_p(\mathcal{M}_{\varepsilon,m} + a(x))$ is nonincreasing with respect to ε and it is negative if and only if $\varepsilon < \varepsilon_0$.

- *When $m = 2$ and $a(x)$ is globally Lipschitz, then*

$$\lim_{\varepsilon\to 0} \lambda_p(\mathcal{M}_{\varepsilon,2} + a(x)) = \lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right)$$

where

$$D_2(J) := \int_{\mathbb{R}^N} J(z)z^2\,dz, \quad K_{2,N} := \frac{1}{|S^{N-1}|}\int_{S^{N-1}} (\sigma.e_1)^2\,d\sigma = \frac{1}{N}$$

and

$$\lambda_1\left(K_{2,N}D_2(J)\Delta + a(x)\right) := \inf_{\varphi\in H_0^1(\mathbb{R}^N),\varphi\not\equiv 0} \frac{K_{2,N}D_2(J)}{2}\frac{\int_{\mathbb{R}^N}|\nabla\varphi|^2(x)\,dx}{\|\varphi\|_2^2} - \frac{\int_{\mathbb{R}^N} a(x)\varphi^2(x)\,dx}{\|\varphi\|_2^2}.$$

4.3.2 First properties of these three definitions

Equipped with this criteria, we start by exploring the behaviour of $\lambda_p(\mathcal{L}_\Omega + a(x))$ with respect to the domain Ω. Namely, we show

Lemma 4.3.6. *Assume that a achieves its maximum in Ω and let $\mathcal{L}_\Omega + a(x)$ be defined as in (4.7) with K satisfying (H7). Let $(\Omega_n)_{n\in\mathbb{R}}$ be a sequence of subset of Ω so that $\lim_{n\to\infty}\Omega_n = \Omega$, $\Omega_n \subset \Omega_{n+1}$. Then we have*

$$\lim_{n\to\infty} \lambda_p(\mathcal{L}_{\Omega_n} + a(x)) = \lambda_p(\mathcal{L}_\Omega + a(x))$$

Proof. By a straightforward application of the monotone properties of λ_p with respect to the domain ((i) of Proposition 4.2.2) we get the inequality

$$\lambda_p(\mathcal{L}_\Omega + a(x)) \le \lim_{n\to\infty} \lambda_p(\mathcal{L}_{\Omega_n} + a(x)). \tag{4.17}$$

To prove the equality, we argue by contradiction. So let us assume

$$\lambda_p(\mathcal{L}_\Omega + a(x)) < \lim_{n\to\infty} \lambda_p(\mathcal{L}_{\Omega_n} + a(x)), \tag{4.18}$$

and let $\lambda \in \mathbb{R}$ so that

$$\lambda_p(\mathcal{L}_\Omega + a(x)) < \lambda < \lim_{n\to\infty} \lambda_p(\mathcal{L}_{\Omega_n} + a(x)). \tag{4.19}$$

We claim

Claim 4.3.7. There exists $\varphi > 0$, $\varphi \in C(\Omega)$ so that (λ, φ) is an adequate test function. That is, φ satisfies

$$\mathcal{L}_\Omega[\varphi] + (a(x) + \lambda)\varphi \le 0 \quad \text{in} \quad \Omega.$$

Assume for the moment that the above claim holds then by definition of $\lambda_p(\mathcal{L}_\Omega + a(x))$ we get a straightforward contraction

$$\lambda_p(\mathcal{L}_\Omega + a(x)) < \lambda \le \lambda_p(\mathcal{L}_\Omega + a(x)).$$

Hence,

$$\lim_{n\to\infty} \lambda_p(\mathcal{L}_{\Omega_n} + a(x)) = \lambda_p(\mathcal{L}_\Omega + a(x))$$

□

Let us now prove the Claim

Proof of the Claim. First let us denote $\sigma := \sup_\Omega a(x)$. Since by assumption the function $a(\cdot)$ achieves its maximum in Ω, there exists $x_0 \in \Omega$ so that $\sigma = a(x_0)$. Moreover there exists n_0 so that for all $n \geq n_0$, $\sigma = \sup_\Omega a(x) = \sup_{\Omega_n} a(x)$.

Now let χ_ε be the following cut-off" functions : $\chi_\varepsilon(x) := \chi\left(\frac{\|x_0 - x\|}{\varepsilon}\right)$ where $\varepsilon > 0$ is to be chosen later on and χ is a smooth function such that $0 \leq \chi \leq 1$, $\chi(z) = 0$ for $|z| \geq 2$ and $\chi(z) = 1$ for $|z| \leq 1$. Finally, let us consider the continuous functions $a_\varepsilon(\cdot)$, defined by $a_\varepsilon(x) := \sup\{a(x), \sigma\chi_\varepsilon(x)\}$. Observe that by construction we have

$$\|a - a_\varepsilon\|_\infty \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Since $\Omega_n \to \Omega$, by the Lipschitz continuity of $\lambda_p(\mathcal{L}_\Omega + a(x))$ with respect to $a(x)$ ((iii) Proposition 4.2.2), for ε small enough the inequality (4.19) yields to

$$\lambda_p(\mathcal{L}_\Omega + a_\varepsilon(x)) < \lambda < \lim_{n\to\infty} \lambda_p(\mathcal{L}_{\Omega_n} + a_\varepsilon(x)). \tag{4.20}$$

By construction, by taking n_0 larger if necessary we also have $B_\varepsilon(x_0) \subset \Omega_n$ for all $n \geq n_0$. Moreover, $\sup_{\Omega_n} a_\varepsilon(x) = \sup_\Omega a_\varepsilon(x) = \sigma$.

Observe now that since $a_\varepsilon \equiv \sigma$ on $B_{\frac{\varepsilon}{2}}(x_0)$, for all $n \geq n_0$ the function $\frac{1}{\sigma - a_\varepsilon(x)} \notin L^1_{loc}(B_\varepsilon(x_0))$ and therefore by Theorem 4.3.3 for all $n \geq n_0$ there exists $\varphi_n \in C(\bar{\Omega}_n)$, $\varphi_n > 0$ associated with $\lambda_p(\mathcal{L}_{\Omega_n} + a_\varepsilon(x))$. Since $x_0 \in \bigcap_{n\geq n_0} \Omega_n$, without loss of generality we can assume that for all $n \geq n_0$ the φ_n can be normalised by $\varphi_n(x_0) = 1$. Recall that for all n, φ_n satisfies

$$\mathcal{L}_{\Omega_n}[\varphi_n] + (a_\varepsilon(x) + \lambda_p(\mathcal{L}_{\Omega_n} + a_\varepsilon(x)))\varphi_n = 0 \quad \text{in} \quad \Omega_n,$$

so from (4.20), it follows that (φ_n, λ) satisfies

$$\mathcal{L}_{\Omega_n}[\varphi_n] + (a_\varepsilon(x) + \lambda)\varphi_n(x) < \mathcal{L}_{\Omega_n}[\varphi_n] + (a_\varepsilon(x) + \lambda_p(\mathcal{L}_{\Omega_n} + a_\varepsilon(x)))\varphi_n = 0 \quad \text{in} \quad \Omega_n. \tag{4.21}$$

Let us now define $b_n(x) := -\lambda_p(\mathcal{L}_{\Omega_n} + a_\varepsilon(x)) - a_\varepsilon(x)$. Then φ_n satisfies

$$\mathcal{L}_{\Omega_n}[\varphi_n] = b_n(x)\varphi_n \quad \text{in} \quad \Omega_n. \tag{4.22}$$

By construction for all $n \geq n_0$ we have $b_n(x) \leq -\lambda_p(\mathcal{L}_{\Omega_{n_0}} + a_\varepsilon(x)) - \sigma > 0$, therefore the Harnack inequality (Theorem 1.4 in [38]) applies to φ_n. Thus for $n \geq n_0$ fixed and for all compact set $\omega \subset\subset \Omega_n$ there exists a constant $C_n(\omega)$ such that

$$\varphi_n(x) \leq C_n(\omega)\varphi_n(y) \quad \forall \quad x, y \in \omega.$$

Moreover the constant $C_n(\omega)$ only depends on $\bigcup_{x\in\omega} B_{r_0}(x)$ and is monotone decreasing with respect to $\inf_{x\in\Omega_n} b_n(x)$. For all $n \geq n_0$, the function $b_n(x)$ being uniformly bounded from below by a constant independent of n, the constant C_n is bounded from above independently of n by a constant $C(\omega)$. Thus we have

$$\varphi_n(x) \leq C(\omega)\varphi_n(y) \quad \forall \quad x, y \in \omega.$$

From a standard argumentation, using the normalization $\varphi_n(x_0) = 1$, we deduce that the sequence $(\varphi_n)_{n\geq n_0}$ is bounded in $C_{loc}(\Omega)$ topology. Moreover, from a standard diagonal extraction argument, there exists a subsequence still denoted $(\varphi_n)_{n\geq n_0}$ such that $(\varphi_n)_{n\geq n_0}$ converges locally uniformly to a continuous function φ. Furthermore, φ is a nonnegative non trivial function and $\varphi(x_0) = 1$.

Since K satisfies condition (H7), we can pass to the limit in equation (4.21) using the Lebesgue monotone convergence theorem and get

$$\mathcal{L}_\Omega[\varphi] + (a_\varepsilon(x) + \lambda)\varphi(x) \leq 0 \quad \text{in} \quad \Omega.$$

Hence, we have

$$\mathcal{L}_\Omega[\varphi] + (a(x) + \lambda)\varphi(x) \leq 0 \quad \text{in} \quad \Omega,$$

since $a(\cdot) \leq a_\varepsilon(\cdot)$.

□

Next we investigate the relation between λ'_p and λ_p. We first show that

Lemma 4.3.8. *Let $\mathcal{L}_\Omega + a$ be as in Lemma 4.3.6, then we always have*

$$\lambda'_p(\mathcal{L}_\Omega + a(x)) \leq \lambda_p(\mathcal{L}_\Omega + a(x)).$$

Proof. First observe that to get the inequality $\lambda'_p(\mathcal{L}_\Omega + a(x)) \leq \lambda_p(\mathcal{L}_\Omega + a(x))$, it is enough to show that for any $\delta > 0$:

$$\lambda'_p(\mathcal{L}_\Omega + a(x)) \leq \lambda_p(\mathcal{L}_\Omega + a(x)) + \delta.$$

Let $\delta > 0$ be fixed and let us consider the operator $\mathcal{L}_\Omega + b(x)$ where $b(x)$ denotes the function $b(x) := a(x) + \lambda_p(\mathcal{L}_\Omega + a(x)) + \delta$.

We claim that

Claim 4.3.9. There exists $\varphi \geq 0$ so that $\varphi \in C_c(\Omega) \cap L^\infty(\Omega)$ that satisfies

$$\mathcal{L}_\Omega[\varphi] + b(x)\varphi \geq 0 \quad \text{in} \quad \Omega.$$

Note that by proving the claim, we prove the Lemma. Indeed, assume for the moment that the claim holds. Then by construction $(\varphi, \lambda_p(\mathcal{L}_\Omega + a(x)) + \delta)$ satisfies

$$\mathcal{L}_\Omega[\varphi] + (a(x) + \lambda_p(\mathcal{L}_\Omega + a(x) + \delta)\varphi \geq 0 \quad \text{in} \quad \Omega.$$

Thus by definition of $\lambda'_p(\mathcal{L}_\Omega + a(x))$, we have $\lambda'_p(\mathcal{L}_\Omega + a(x)) \leq \lambda_p(\mathcal{L}_\Omega + a(x)) + \delta$.

The constant δ being arbitrary, we get for all $\delta > 0$:

$$\lambda'_p(\mathcal{L}_\Omega + a(x)) \leq \lambda_p(\mathcal{L}_\Omega + a(x)) + \delta.$$

□

Proof of the Claim. By construction $\lambda_p(\mathcal{L}_\Omega + b(x)) < 0$, so by Lemma 4.3.6 there exists a bounded open set ω so that $\lambda_p(\mathcal{L}_\omega + b(x)) < 0$. By taking ω larger if necessary, we can assume that $\sup_\Omega b(x) = \sup_\omega b(x)$ and $b(x)$ achieves its maximum in ω. By arguing as in the proof of Claim 4.3.7, we can find b_ε so that

$$\begin{aligned} &\|b - b_\varepsilon\|_{\infty,\omega} = \|b - b_\varepsilon\|_{\infty,\Omega} \leq \varepsilon, \\ &\lambda_p(\mathcal{L}_\omega + b_\varepsilon(x)) + \varepsilon < 0 \end{aligned}$$

and there is $\varphi_p \in C(\bar{\omega})$, $\varphi_p > 0$ associated to $\lambda_p(\mathcal{L}_\omega + b_\varepsilon(x))$. That is φ_p satisfies

$$\mathcal{L}_\omega[\varphi_p] + b_\varepsilon(x)\varphi_p = -\lambda_p(\mathcal{L}_\omega + b_\varepsilon(x))\varphi_p \quad \text{in} \quad \omega. \tag{4.23}$$

Without loss of generality we can assume that $\varphi_p \leq 1$.

Let σ denotes the maximum of b_ε in $\bar{\omega}$, then by Proposition 4.2.2 there exists $\tau > 0$ so that $-\lambda_p(\mathcal{L}_\omega + b_\varepsilon(x)) - \varepsilon - \sigma \geq \tau > 0$. Moreover, since φ_p satisfies (4.23), there exists $c_0 > 0$ so that $\inf_\omega \varphi_p \geq c_0$.

Let us choose $\omega' \subset\subset \omega$ such that

$$|\omega \setminus \omega'| \leq \frac{c_0 \inf\{\tau, -\lambda_p(\mathcal{L}_\omega + b_\varepsilon) - \varepsilon\}}{2\|K\|_\infty},$$

where for a set A, $|A|$ denotes the Lebesgue measure of A.

Since $\bar{\omega}' \subset\subset \omega$ and $\partial\omega$ are two disjoint closed sets, by the Urysohn's Lemma there exists a continuous function η such that $0 \leq \eta \leq 1$, $\eta(x) = 1$ in ω', $\eta(x) = 0$ in $\partial\omega$. Consider now $\varphi_p\eta$ and let us compute $\mathcal{L}_\omega[\varphi_p\eta] + b_\varepsilon(x)\varphi_p\eta$. Then we have

$$\begin{aligned}
\mathcal{L}_\omega[\varphi_p\eta] + b(x)\varphi_p\eta &\geq -\lambda_p(\mathcal{L}_\omega + b_\varepsilon(x))\varphi_p - \|K\||\omega \setminus \omega'| - b_\varepsilon(x)\varphi_p(1-\eta) - (b_\varepsilon(x) - b(x))\varphi_p(x) \\
&\geq -(\lambda_p(\mathcal{L}_\omega + b_\varepsilon(x)) + \|b - b_\varepsilon\|_{\infty,\omega})\varphi_p - \frac{c_0 \inf\{\tau, -\lambda_p(\mathcal{L}_\omega + b_\varepsilon) - \varepsilon\}}{2} \\
&\quad - b_\varepsilon(x)\varphi_p(1-\eta) \\
&\geq -(\lambda_p(\mathcal{L}_\omega + b_\varepsilon(x)) + \varepsilon)\varphi_p - \frac{c_0 \inf\{\tau, -\lambda_p(\mathcal{L}_\omega + b_\varepsilon) - \varepsilon\}}{2} - \max\{\sigma, 0\}\varphi_p \\
&\geq -(\lambda_p(\mathcal{L}_\omega + b_\varepsilon(x)) + \varepsilon + \max\{\sigma, 0\})\varphi_p - \frac{c_0 \inf\{\tau, -\lambda_p(\mathcal{L}_\omega + b_\varepsilon) - \varepsilon\}}{2}.
\end{aligned}$$

Since by assumption $-\lambda_p(\mathcal{L}_\omega + b_\varepsilon(x)) - \varepsilon > 0$ and $-\lambda_p(\mathcal{L}_\omega + b_\varepsilon(x)) - \varepsilon - \sigma \geq \tau > 0$ it follows from the above inequality that

$$\begin{aligned}
\mathcal{L}_\omega[\varphi_p\eta] + b_\varepsilon(x)\varphi_p\eta &\geq -(\lambda_p(\mathcal{L}_\omega + b_\varepsilon(x)) + \varepsilon + \max\{\sigma, 0\})c_0 - \frac{c_0 \inf\{\tau, -\lambda_p(\mathcal{L}_\omega + b_\varepsilon) - \varepsilon\}}{2} \\
&\geq \frac{c_0 \inf\{\tau, -\lambda_p(\mathcal{L}_\omega + b_\varepsilon(x)) - \varepsilon\}}{2} \geq 0.
\end{aligned}$$

By construction we have $\varphi_p\eta \in C(\omega)$ that satisfies

$$\begin{aligned}
&\mathcal{L}_\omega[\varphi_p\eta] + b(x)\varphi_p\eta \geq 0 \quad \text{in} \quad \omega, \\
&\varphi_p\eta = 0 \quad \text{on} \quad \partial\omega.
\end{aligned}$$

Finally by extending $\varphi_p\eta$ by 0 outside ω and denoting φ this extension, we get our desired test function. Indeed, by construction $\varphi \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is a non negative function which satisfies

$$\begin{aligned}
&\mathcal{L}_\Omega[\varphi] + b(x)\varphi = \mathcal{L}_\omega[\varphi] + b(x)\varphi \geq 0 \quad \text{in} \quad \omega, \\
&\mathcal{L}_\Omega[\varphi] + b(x)\varphi = \mathcal{L}_\omega[\varphi] \geq 0 \quad \text{in} \quad \Omega \setminus \omega.
\end{aligned}$$

Hence, $\varphi \geq 0, \varphi \in C(\Omega) \cap L^\infty$ is our desired test function. □

Remark 14. Note that assumption $(H7)$ on the kernel K is only needed to reduce the problem defined on a unbounded domain to a problem defined on a bounded problem. Therefore the proof holds as well for

Ω bounded and Lemma 4.3.8 holds true without condition $(H7)$. Namely, let Ω be a bounded domain and let $\mathcal{L}_\Omega + a(x)$ be as in (4.7) then

$$\lambda_p'(\mathcal{L}_\Omega + a(x)) \leq \lambda_p(\mathcal{L}_\Omega + a(x)).$$

In particular, for any $\delta > 0$, there exists $\varphi \in C_c(\Omega)$ so that

$$\mathcal{L}_\Omega[\varphi](x) + (a(x) + \delta + \lambda_p'(\mathcal{L}_\Omega + a(x)))\varphi(x) \geq 0 \quad \text{in} \quad \Omega.$$

Let us prove that in bounded domains, then the three definitions λ_p, λ_p' and λ_v are equivalent.

Theorem 4.3.10. *Let Ω be a bounded domain of $\mathbb{R}^N$ and let $\mathcal{L}_\Omega + a(x)$ be defined as in (4.7). Then we have*

$$\lambda_p(\mathcal{L}_\Omega + a(x)) = \lambda_p'(\mathcal{L}_\Omega + a(x))$$

In addition, when $\mathcal{L}_\Omega + a(x)$ is self adjoined, we have

$$\lambda_p'(\mathcal{L}_\Omega + a(x)) = \lambda_p(\mathcal{L}_\Omega + a(x)) = \lambda_v(\mathcal{L}_\Omega + a(x)).$$

Proof. By Lemma 4.3.8 we already have

$$\lambda_p'(\mathcal{L}_\Omega + a(x)) \leq \lambda_p(\mathcal{L}_\Omega + a(x)).$$

It remains to prove the converse inequality. We argue by contradiction and assume that

$$\lambda_p'(\mathcal{L}_\Omega + a(x)) < \lambda_p(\mathcal{L}_\Omega + a(x)).$$

Choose $\lambda \in (\lambda_p'(\mathcal{L}_\Omega + a(x)), \lambda_p(\mathcal{L}_\Omega + a(x)))$. Then by definition of λ_p and λ_p' there exists φ and ψ non negative continuous function so that

$$\mathcal{L}_\Omega[\varphi](x) + (a(x) + \lambda)\varphi(x) \leq 0 \quad \text{in} \quad \Omega, \tag{4.24}$$
$$\mathcal{L}_\Omega[\psi](x) + (a(x) + \lambda)\psi(x) \geq 0 \quad \text{in} \quad \Omega. \tag{4.25}$$

Moreover, $\varphi > 0$ in $\bar{\Omega}$. By taking λ smaller if necessary, we can assume that φ satisfies

$$\mathcal{L}_\Omega[\varphi](x) + (a(x) + \lambda)\varphi(x) < 0 \quad \text{in} \quad \Omega.$$

By a direct computation, we have

$$\int_\Omega K(x,y)\varphi(y)\left(\frac{\psi(y)}{\varphi(y)} - \frac{\psi(x)}{\varphi(x)}\right) dy > 0.$$

Since $\frac{\psi}{\varphi} \in C(\bar{\Omega})$, the function $\frac{\psi}{\varphi}$ achieve a maximum at some point $x_0 \in \bar{\Omega}$. At this point we get the contradiction,

$$0 < \int_\Omega K(x_0,y)\varphi(y)\left(\frac{\psi(y)}{\varphi(y)} - \frac{\psi(x_0)}{\varphi(x_0)}\right) dy \leq 0.$$

Thus,

$$\lambda_p'(\mathcal{L}_\Omega + a(x)) = \lambda_p(\mathcal{L}_\Omega + a(x)).$$

In the self-adjoined case to prove the equality of all the three definition, it is enough to prove that $\lambda_p'(\mathcal{L}_\Omega + a(x)) = \lambda_v(\mathcal{L}_\Omega + a(x))$. Let $\lambda > \lambda_p'(\mathcal{L}_\Omega + a(x))$, then by definition there exists $\psi \geq 0$ so that $\psi \in C(\Omega) \cap L^\infty(\Omega)$ and

$$\mathcal{L}_\Omega[\psi](x) + (a(x) + \lambda)\psi(x) \geq 0 \quad \text{in} \quad \Omega. \tag{4.26}$$

By construction $\psi \in L^2(\Omega)$ and let us multiply (4.26) by $-\psi$ and integrate over Ω. Then we get

$$-\int_\Omega \int_\Omega K(x,y)\psi(x)\psi(y)\,dxdy - \int_\Omega a(x)\psi(x)^2\,dx \leq \lambda \int_\Omega \psi^2(x)\,dx \tag{4.27}$$

$$\frac{1}{2}\int_\Omega \int_\Omega K(x,y)\left(\psi(x) - \psi(y)\right)^2\,dxdy - \int_\Omega (a(x) + k(x))\psi(x)^2\,dx \leq \lambda \int_\Omega \psi^2(x)\,dx \tag{4.28}$$

$$\lambda_v(\mathcal{L}_\Omega + a(x)) \int_\Omega \psi^2(x)\,dx \leq \lambda \int_\Omega \psi^2(x)\,dx. \tag{4.29}$$

Therefore, $\lambda_v(\mathcal{L}_\Omega + a(x)) \leq \lambda_p'(\mathcal{L}_\Omega + a(x))$. Let us prove now the converse inequality. We argue by contradiction and assume that

$$\lambda_v(\mathcal{L}_\Omega + a(x)) < \lambda_p'(\mathcal{L}_\Omega + a(x)). \tag{4.30}$$

Let us first observe that by density of $C(\bar{\Omega})$ in $L^2(\Omega)$, we can easily check that

$$\begin{aligned}
-\lambda_v(\mathcal{L}_\Omega + a(x)) &= -\inf_{\varphi \in L^2(\Omega), \varphi \not\equiv 0} \frac{\frac{1}{2}\int_\Omega \int_\Omega K(x,y)(\varphi(x) - \varphi(y))^2\,dydx - \int_\Omega (a(x) + k(x))\varphi(x)^2\,dx}{\|\varphi\|^2_{L^2(\Omega)}}, \\
&= -\inf_{\varphi \in L^2(\Omega), \varphi \not\equiv 0} \frac{-\int_\Omega \int_\Omega K(x,y)\varphi(x)\varphi(y)\,dydx - \int_\Omega a(x)\varphi(x)^2\,dx}{\|\varphi\|^2_{L^2(\Omega)}}, \\
&= \sup_{\varphi \in L^2(\Omega), \varphi \not\equiv 0} \frac{\int_\Omega \left[\mathcal{L}_\Omega[\varphi](x) + a(x)\varphi(x)\right]\varphi(x)\,dx}{\|\varphi\|^2_{L^2(\Omega)}}, \\
&= \sup_{\varphi \in C(\bar{\Omega}), \varphi \not\equiv 0} \frac{\int_\Omega \left[\mathcal{L}_\Omega[\varphi](x) + a(x)\varphi(x)\right]\varphi(x)\,dx}{\|\varphi\|^2_{L^2(\Omega)}}.
\end{aligned}$$

By (iv) of Proposition 4.2.2 since $\lambda_p(\mathcal{L}_\Omega + a(x)) = \lambda_p(\mathcal{L}_\Omega + a(x))$ we deduce from (4.30) that σ_+ defined by

$$\sigma_+ = \sup_{\varphi \in C(\overline{\Omega})} \frac{\int_\Omega (\mathcal{L}_\Omega[\varphi] + a(x)\varphi)\varphi}{\int_\Omega \varphi^2} \tag{4.31}$$

satisfies

$$\sigma_+ > \max_{\overline{\Omega}} a. \tag{4.32}$$

Now, using the same argument as in [41, 66] we deduce that the supremum in (4.31) is achieved. Indeed, it is standard [23] that the spectrum of $\mathcal{L}_\Omega + a(x)$ is to the left of σ_+ and that there exists a sequence $\varphi_n \in C(\overline{\Omega})$ such that $\|\varphi_n\|_{L^2(\Omega)} = 1$ and $\|(\mathcal{L}_\Omega + a(x) - \sigma_+)\varphi_n\|_{L^2(\Omega)} \to 0$ as $n \to +\infty$. By compactness of $\mathcal{L}_\Omega : L^2(\Omega) \to C(\overline{\Omega})$ for a subsequence, $\lim_{n\to+\infty} \mathcal{L}_\Omega[\varphi_n]$ exists in $C(\overline{\Omega})$. Then, using (4.32), we see that $\varphi_n \to \varphi$ in $L^2(\Omega)$ for some φ and $(\mathcal{L}_\Omega + a(x))\varphi = \sigma_+\varphi$. This equation implies $\varphi \in C(\overline{\Omega})$, and σ_+ is an

eigenvalue for the operator $\mathcal{L}_\Omega + a(x)$. Moreover, $\varphi \geq 0$, since φ^+ is also minimizer. Indeed, we have

$$\begin{aligned}\sigma_+ &= \frac{\int_\Omega [\mathcal{L}_\Omega[\varphi] + a(x)\varphi(x)]\varphi^+(x)\,dx}{\|\varphi^+\|^2_{L^2(\Omega)}},\\ &= \frac{\int_\Omega [\mathcal{L}_\Omega[\varphi^+] + a(x)\varphi^+(x)]\varphi^+(x)\,dx}{\|\varphi^+\|^2_{L^2(\Omega)}} + \frac{\int_\Omega \int_\Omega K(x,y)\varphi^-(x)\varphi^+(y)\,dydx}{\|\varphi^+\|^2_{L^2(\Omega)}},\\ &\leq \frac{\int_\Omega [\mathcal{L}_\Omega[\varphi^+] + a(x)\varphi^+(x)]\varphi^+(x)\,dx}{\|\varphi^+\|^2_{L^2(\Omega)}} \leq \sigma_+.\end{aligned}$$

Thus, there exists a non-negative continuous φ so that

$$\mathcal{L}_\Omega[\varphi] + (a(x) + \lambda_v)\varphi = 0 \quad \text{in} \quad \Omega.$$

Since $\lambda_v < \lambda_p$, we can argue as above and get the desired contradiction. Hence, $\lambda_v = \sigma_+ = \lambda_p = \lambda'_p$.

□

Finally, we investigate the relation between λ_p, λ'_p and λ_v when $\mathcal{M}$ is a self-adjoined operator. precisely we establish

Lemma 4.3.11. *Assume that $a(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, J is symmetric and J is compactly supported. Then one has*

$$\lambda_p(\mathcal{M} + a(x)) \leq \liminf_{n\to+\infty} \lambda_v(\mathcal{L}_{B_n} + a(x)) \leq \lambda'_p(\mathcal{M} + a(x)),$$

where $(B_n)_{n\in\mathbb{N}}$ be the increasing sequence of ball of radius n centred at 0

Before proving this Lemma, we start by showing some scaling invariance of $\lambda_p(\mathcal{M} + a(x))$. Namely, we have

Lemma 4.3.12. *Assume that $a(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then for all $\varepsilon > 0$ one has*

$$\lambda_p(\mathcal{M} + a(x)) = \lambda_p(\mathcal{M}_\varepsilon + a_\varepsilon(x)).$$

where $a_\varepsilon(x) := a\left(\frac{x}{\varepsilon}\right)$ and $\mathcal{M}_\varepsilon[\varphi] := \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N} J\left(\frac{x-y}{\varepsilon}\right)\varphi(y)\,dy - \varphi(x)$.

Proof. The proof of this invariance is a consequence of the following observation. By definition of $\lambda_p(\mathcal{M} + a(x))$, we have for all $\lambda < \lambda_p(\mathcal{M} + a(x))$,

$$\mathcal{M}[\varphi](x) + (a(x) + \lambda)\varphi(x) \leq 0 \quad \text{in} \quad \mathbb{R}^N,$$

for some positive $\varphi \in C(\mathbb{R}^N)$. Let $X = \frac{x}{\varepsilon}$ and $\psi(x) := \varphi(\frac{x}{\varepsilon})$then we can rewrite the above inequality as follows

$$\begin{aligned}&\int_{\mathbb{R}^N} J(X-y)\,\varphi(y)\,dy - \varphi(X) + (a(X) + \lambda)\varphi(X) \leq 0 \quad \text{in} \quad \mathbb{R}^N,\\ &\int_{\mathbb{R}^N} J\left(\frac{x}{\varepsilon} - y\right)\psi(\varepsilon y)\,dy - \psi(x) + (a_\varepsilon(x) + \lambda)\psi(x) \leq 0 \quad \text{in} \quad \mathbb{R}^N\\ &\int_{\mathbb{R}^N} J_\varepsilon(x-z)\,\psi(z)\,dy - \psi(x) + (a_\varepsilon(x) + \lambda)\psi(x) \leq 0 \quad \text{in} \quad \mathbb{R}^N,\end{aligned}$$

with $J_\varepsilon(z) := \frac{1}{\varepsilon^N} J\left(\frac{z}{\varepsilon}\right)$ and $a_\varepsilon(x) := a\left(\frac{x}{\varepsilon}\right)$. Thus ψ is a positive continuous function that satisfies

$$\mathcal{M}_\varepsilon \psi(x) + (a_\varepsilon(x) + \lambda)\psi(x) \leq 0 \quad \text{in} \quad \mathbb{R}^N.$$

Therefore, $\lambda \leq \lambda_p(\mathcal{M}_\varepsilon + a_\varepsilon(x))$ and as a consequence $\lambda_p(\mathcal{M} + a(x)) \leq \lambda_p(\mathcal{M}_\varepsilon + a_\varepsilon(x))$. The above argument holds true by replacing the role of $\lambda_p(\mathcal{M} + a(x))$ and $\lambda_p(\mathcal{M}_\varepsilon + a_\varepsilon(x))$. Hence, we get

$$\lambda_p(\mathcal{M} + a(x)) = \lambda_p(\mathcal{M}_\varepsilon + a_\varepsilon(x)).$$

□

Let us now turn our attention to the proof of Lemma 4.3.11.

Proof of Lemma 4.3.11. The proof follows some ideas developed in [18, 21, 41, 39]. Let us call $\lambda_p = \lambda_p(\mathcal{L}_\Omega + a(x))$ and $\lambda_p' = \lambda_p'(\mathcal{L}_\Omega + a(x))$. By Lemma 4.3.12, without any loss of generality we can assume that the support of J is contained in the unit ball.

First recall that that for a domain Ω bounded, we have

$$\lambda_p = \lambda_p' = \lambda_v.$$

Let $(B_n)_{n\in\mathbb{N}}$ be the increasing sequence of ball of radius n centred at 0. By monotonicity of λ_p with respect to the domain, we have

$$\lambda_p(\mathcal{M} + a(x)) \leq \lambda_p(\mathcal{L}_{B_n} + a(x)) = \lambda_v(\mathcal{L}_{B_n} + a(x))$$

Therefore

$$\lambda_p(\mathcal{M} + a(x)) \leq \liminf_{n\to\infty} \lambda_v(\mathcal{L}_{B_n} + a(x)).$$

Thanks to the last inequality, to show $\lambda_p(\mathcal{M} + a(x)) \leq \lambda_p'(\mathcal{M} + a(x))$ it is enough to prove that

$$\liminf_{n\to\infty} \lambda_v(\mathcal{L}_{\Omega_n} + a(x)) \leq \lambda_p'(\mathcal{M} + a(x)). \tag{4.33}$$

Observe that to prove (4.33), it is enough to show that for any $\delta > 0$

$$\liminf_{n\to\infty} \lambda_v(\mathcal{L}_{B_n} + a(x)) \leq \lambda_p'(\mathcal{M} + a(x)) + \delta. \tag{4.34}$$

Let us now fix $\delta > 0$ and let us denote $\mu := \lambda_p'(\mathcal{M} + a(x)) + \delta$. By definition of $\lambda_p'(\mathcal{M} + a(x))$ there exist a function $\varphi \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\varphi \geq 0$ satisfying

$$\mathcal{M}[\varphi] + a(x)\varphi(x) + \mu\varphi(x) \geq 0 \quad \text{in} \quad \mathbb{R}^N. \tag{4.35}$$

Without loss of generality we can also assume that $\|\varphi\|_{L^\infty(\mathbb{R}^N)} = 1$.

Let χ_n be the characteristic function of B_n and let $w_n = \varphi\chi_n$. By definition of $\lambda_v(\mathcal{L}_{B_n} + a(x))$, since $w_n \in L^2(B_n)$ we have

$$\lambda_v(\mathcal{L}_{B_n} + a(x))\|w_n\|^2_{L^2(B_n)} \leq \int_{B_n} \left(-\mathcal{L}_{B_n}[w_n](x) - a(x)w_n(x)\right) w_n(x)\, dx. \tag{4.36}$$

Since $\mathcal{M}[\varphi]w_n \in L^1(B_n)$, from (4.36) and by using (4.35) we achieve

$$\lambda_v(\mathcal{L}_{B_n} + a(x))\|w_n\|^2_{L^2(B_n)} \leq \int_{B_n} \left(-\mathcal{L}_{B_n}[w_n](x) - a(x)w_n(x) - \mu w_n + \mu w_n\right) w_n(x)\,dx \tag{4.37}$$

$$\leq \mu\|w_n\|^2_{L^2(B_n)} + \int_{B_n} \left(-\mathcal{L}_{B_n}[w_n](x) + \mathcal{M}[\varphi](x)\right) w_n(x)\,dx \tag{4.38}$$

$$\leq \mu\|w_n\|^2_{L^2(B_n)} + \int_{B_n} \left(\int_{\mathbb{R}^N\setminus B_n} J(x-y)\varphi(y)\,dy\right) w_n(x)\,dx \tag{4.39}$$

$$\leq \mu\|w_n\|^2_{L^2(B_n)} + I_n, \tag{4.40}$$

where I_n denotes

$$I_n := \int_{B_n} \left(\int_{\mathbb{R}^N\setminus B_n} J(x-y)\varphi(y)\,dy\right) \varphi(x)\,dx \tag{4.41}$$

Observe that we achieve (4.34) by proving

$$\liminf_{n\to\infty} \frac{I_n}{\|\varphi\|^2_{L^2(B_n)}} = 0. \tag{4.42}$$

Recall that by assumption there exists $C > 0$ such that $J(x-y) \leq C\chi_1(|x-y|))$. So we get

$$I_n \leq \int_{B_n} \left(\int_{B_{n+1}\setminus B_n} J(x-y)\varphi(y)\,dy\right) w_n(x)\,dx \tag{4.43}$$

By using the Fubini Theorem, Jensen inequality and the Cauchy-Schwarz inequality in (4.43), we obtain

$$I_n \leq \left(\int_{B_{n+1}\setminus B_n} \varphi^2(y)\,dy\right)^{1/2} \left(\int_{B_{n+1}\setminus B_n} \left(\int_{B_n} J(x-y)\varphi(x)\,dx\right)^2 dy\right)^{1/2}, \tag{4.44}$$

$$\leq \|\varphi\|_{L^2(B_{n+1}\setminus B_n)} \left(\int_{B_{n+1}\setminus B_n} \left(\int_{B_n} J^2(x-y)\varphi^2(x)\,dx\right)\,dy\right)^{1/2}, \tag{4.45}$$

$$\leq \|\varphi\|_{L^2(B_{n+1}\setminus B_n)} \left(\int_{B_n} \left(\int_{B_{n+1}\setminus B_n} J^2(x-y)\,dy\right)\varphi^2(x)\,dx\right)^{1/2}, \tag{4.46}$$

$$\leq \|J\|_\infty \|\varphi\|_{L^2(B_{n+1}\setminus B_n)}\|\varphi\|_{L^2(B_n)}. \tag{4.47}$$

Dividing (4.47) by $\|\varphi\|^2_{L^2(B_n)}$ we achieve

$$\frac{I_n}{\|\varphi\|^2_{L^2(B_n)}} \leq C\frac{\|\varphi\|_{L^2(B_{n+1}\setminus B_n)}}{\|\varphi\|_{L^2(B_n)}}.$$

Thanks to Lemma 2.6 in [21] the right hand side of the above inequality tends to 0. Hence, we get

$$\liminf_{n\to\infty} \lambda_v(\mathcal{L}_{B_n} + a(x)) \leq \mu + \liminf_{n\to\infty} \frac{I_n}{\|\varphi_n\|^2_{L^2(B_n)}} = \lambda'_p(\mathcal{M} + a(x)) + \delta \tag{4.48}$$

Since in the above argumentation δ can be chosen arbitrary the inequality (4.48) holds true for any $\delta > 0$ and the Lemma is proved.

□

When $N = 1$, the restriction on the kernel can be weakened and we can prove that

Lemma 4.3.13. *Assume that $a(x) \in L^\infty(\mathbb{R})$, K is symmetric and K satisfies $0 \le K(x,y) \le C(1+|x-y|)^\alpha$ for some $\alpha > \frac{3}{2}$. Then one has*

$$\lambda_p(\mathcal{M} + a(x)) \le \liminf_{n\to+\infty} \lambda_v(\mathcal{L}_{(-n,n)} + a(x)) \le \lambda_p'(\mathcal{M} + a(x)).$$

Proof. By arguing as in the above proof, for any $\delta > 0$ there exists $\varphi \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}^N)$ so that

$$\mathcal{M}[\varphi] + (a(x) + \lambda_p(\mathcal{M} + a(x)) + \delta)\varphi(x) \ge 0 \quad \text{in} \quad \mathbb{R}.$$

and

$$\lambda_v(\mathcal{L}_{(-n,n)} + a(x))\|w_n\|^2_{L^2((-n,n))} \le \mu\|w_n\|^2_{L^2((-n,n))} + I_n,$$

where $\mu := \lambda_p(\mathcal{M} + a(x)) + \delta)$, $w_n := \varphi\chi_{(-n,n)}$ and I_n denotes

$$I_n := \int_{(-n,n)} \left(\int_{\mathbb{R}\setminus(-n,n)} K(x,y)\varphi(y)\,dy \right) \varphi(x)\,dx \tag{4.49}$$

As above we achieve our proof by showing

$$\liminf_{n\to\infty} \frac{I_n}{\|\varphi\|^2_{L^2((-n,n))}} = 0. \tag{4.50}$$

Let us now treat two cases independently:

Case 1: $\varphi \in L^2(\mathbb{R})$

Assume that $\varphi \in L^2(\mathbb{R})$, then by the Cauchy-Schwarz inequality in (4.49), we obtain

$$I_n \le \|\varphi\|_{L^2(\mathbb{R}\setminus(-n,n))} \left[\int_{\mathbb{R}\setminus(-n,n)} \left(\int_{-n}^{n} K(x,y)\varphi(x)\,dx \right)^2 dy \right]^{\frac{1}{2}} \tag{4.51}$$

$$I_n \le C\|\varphi\|_{L^2(\mathbb{R}\setminus(-n,n))}\|\varphi\|_{L^2((-n,n))}. \tag{4.52}$$

Thus,

$$\liminf_{n\to\infty} \frac{I_n}{\|\varphi\|^2_{L^2((-n,n))}} \le \liminf_{n\to\infty} \frac{\|\varphi\|_{L^2(\mathbb{R}\setminus(-n,n))}}{\|\varphi\|_{L^2((-n,n))}} = 0.$$

Case 2: $\varphi \notin L^2(\mathbb{R})$

Assume now that $\varphi \notin L^2$, then we argue as follows. By using the Fubini Theorem and the Cauchy-Schwarz inequality in (4.49), we obtain

$$I_n \le \|\varphi\|_{L^2((-n,n))} \left[\int_{-\infty}^{-n} \left(\left(\int_{-n}^{n} K(x,y)^2\,dx \right)^{\frac{1}{2}} \right) \varphi(y)\,dy + \int_{n}^{+\infty} \left(\left(\int_{-n}^{n} K(x,y)^2\,dx \right)^{\frac{1}{2}} \right) \varphi(y)\,dy \right] \tag{4.53}$$

$$\le \|\varphi\|_{L^2((-n,n))} \left[\tilde{I}_n^- + \tilde{I}_n^+ \right] \tag{4.54}$$

Recall that by assumption there exists $C > 0$ such that $K(x,y) \leq C(1+|x-y|)^{-\alpha}$ with $\alpha > \frac{3}{2}$. So, we have

$$\tilde{I}_n^- \leq C \int_{-\infty}^{-n} \left(\left(\int_{-n}^{n} (1+|x-y|)^{-2\alpha}\, dx \right)^{\frac{1}{2}} \right) \varphi(y)\, dy$$
$$\tilde{I}_n^+ \leq C \int_{n}^{+\infty} \left(\left(\int_{-n}^{n} (1+|x-y|)^{-2\alpha}\, dx \right)^{\frac{1}{2}} \right) \varphi(y)\, dy$$

To complete our proof, we have to show that $\frac{\tilde{I}_n^{\pm}}{\|\varphi\|_{L^2((-n,n))}} \to 0$. The proof being similar in both cases, we only prove it for $\tilde{I}_n^+$. We claim that

Claim 4.3.14. There exists $C > 0$ so that for all $n \in \mathbb{N}$,

$$\int_{n}^{+\infty} \left(\left(\int_{-n}^{n} (1+|x-y|)^{-2\alpha}\, dx \right)^{\frac{1}{2}} \right) \varphi(y)\, dy \leq C.$$

Assume for the moment that the claim hold true, then (4.50) holds true.
Indeed, thanks to (4.54) we have

$$\frac{I_n}{\|\varphi\|^2_{L^2((-n,n))}} \leq \frac{C}{\|\varphi\|_{L^2((-n,n))}} \to 0 \qquad \text{when} \quad n \to \infty.$$

Hence, in both situation we get

$$\liminf_{n\to\infty} \lambda_v(\mathcal{L}_{(-n,n)} + a(x)) \leq \mu + \liminf_{n\to\infty} \frac{I_n}{\|\varphi_n\|^2_{L^2((-n,n))}} = \lambda_p'(\mathcal{M} + a(x)) + \delta$$

Since in the above argumentation δ can be chosen arbitrary the above inequality holds true for any $\delta > 0$ and the Lemma is proved

□

Proof of the Claim. Since $\varphi \in L^\infty$ and $y \geq n$ then $x \leq y$ and we have

$$\begin{aligned}
\tilde{I}_n^+ &\leq \|\varphi\|_\infty \int_{n}^{+\infty} \left(\left(\int_{-n}^{n} (1+y-x)^{-2\alpha}\, dx \right)^{\frac{1}{2}} \right) dy \\
&\leq \frac{\|\varphi\|_\infty}{\sqrt{2\alpha-1}} \int_{n}^{+\infty} (1+y-n)^{\alpha-\frac{1}{2}}\, dy, \\
&\leq C \int_{0}^{+\infty} (1+z)^{\alpha-\frac{1}{2}}\, dz.
\end{aligned}$$

□

Open problem 1. Is the result in $1D$ still valid in for $N > 1$?

We can now prove Theorem 4.3.4.

Proof of Theorem 4.3.4. As in the proof of Lemma 4.3.11, without loss of generality we can assume that the support of J is contained in the unit ball. Now from Lemma 4.3.8 and 4.3.11, we deduce that

$$\lim_{R\to\infty} \lambda_v(\mathcal{L}_R + a(x)) \le \lambda'_p(\mathcal{M} + a(x)) \le \lambda_p(\mathcal{M} + a(x)) \le \lim_{R\to\infty} \lambda_v(\mathcal{L}_R + a(x)) \le \lambda'_p(\mathcal{M} + a(x)).$$

Therefore,

$$\lim_{R\to\infty} \lambda_v(\mathcal{L}_R + a(x)) = \lambda'_p(\mathcal{M} + a(x)) = \lambda_p(\mathcal{M} + a(x)).$$

It remains to prove that $\lambda_v(\mathcal{M} + a(x)) = \lambda'_p(\mathcal{M} + a(x)) = \lambda_p(\mathcal{M} + a(x))$. By definition of $\lambda'_p(\mathcal{M} + a(x))$ and using the Claim 4.3.9, we can check that

$$\lambda_v(\mathcal{M} + a(x)) \le \lambda'_p(\mathcal{M} + a(x)).$$

On another hand by definition of $\lambda_v(\mathcal{M} + a(x))$ for any $\delta > 0$ there exists $\varphi_\delta \in L^2(\mathbb{R}^N)$ so that

$$\frac{\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} J(x-y)(\varphi_\delta(x)-\varphi_\delta(y))^2\,dydx - \int_{\mathbb{R}^N} a(x)\varphi_\delta^2(x)\,dx}{\|\varphi_\delta\|^2_{L^2(\mathbb{R}^N)}} \le \lambda_v(\mathcal{M} + a(x)) + \delta.$$

Let us compute

$$\mathcal{I}_R(\varphi_\delta) = \frac{\frac{1}{2}\iint_{B_R\times B_R} J(x-y)(\varphi_\delta(x)-\varphi_\delta(y))^2\,dydx + \int_{B_R} k_R(x)\varphi_\delta^2(x)\,dx - \int_{B_R} a(x)\varphi_\delta^2(x)\,dx}{\|\varphi_\delta\|^2_{L^2(B_R)}},$$

with $k(x) := \int_{\mathbb{R}^N\setminus B_R} J(x-y)\,dy$. Since for all $x \in \mathbb{R}^N, \lim_{R\to\infty} k_R(x) = 0$, $a(x) \in L^\infty$ and $\varphi_\delta \in L^2(\mathbb{R}^N)$ by Lebesgue's monotone convergence Theorem, we get

$$\int_{B_R} k_R(x)\varphi_\delta^2(x)\,dx - \int_{B_R} a(x)\varphi_\delta^2(x)\,dx \le \delta\|\varphi_\delta\|^2_{L^2(B_R)} - \int_{\mathbb{R}^N} a(x)\varphi_\delta^2(x)\,dx \quad \text{for } R \text{ large enough.}$$

Thus we have for R large enough

$$\begin{aligned}
\mathcal{I}_R(\varphi_\delta) &\le \frac{\frac{1}{2}\iint_{R^N\times R^N} J(x-y)(\varphi_\delta(x)-\varphi_\delta(y))^2\,dydx - \int_{\mathbb{R}^N} a(x)\varphi_\delta^2(x)\,dx + \delta\|\varphi_\delta\|^2_{L^2(B_R)}}{\|\varphi_\delta\|^2_{L^2(B_R)}},\\
&\le \frac{\|\varphi_\delta\|^2_{L^2(\mathbb{R}^N)}}{\|\varphi_\delta\|^2_{L^2(B_R)}}(\lambda_v(\mathcal{M} + a(x)) + \delta) + \delta,\\
&\le \lambda_v(\mathcal{M} + a(x)) + C\delta,
\end{aligned}$$

for some universal constant $C > 0$.

By using the definition of $\lambda_v(\mathcal{L}_R + a(x))$ we then achieve

$$\lambda_v(\mathcal{L}_R + a(x)) \le \mathcal{I}_R(\varphi_\delta) \le \lambda_v(\mathcal{M} + a(x)) + C\delta \quad \text{for } R \text{ large enough.}$$

Therefore

$$\lim_{R\to\infty} \lambda_v(\mathcal{L}_R + a(x)) \le \lambda_v(\mathcal{M} + a(x)) + C\delta. \tag{4.55}$$

Since (4.55) holds true for any δ, we get

$$\lim_{R\to\infty} \lambda_v(\mathcal{L}_R + a(x)) \leq \lambda_v(\mathcal{M} + a(x)).$$

As a consequence, we obtain

$$\lambda_p'(\mathcal{M} + a(x)) \leq \lambda_p(\mathcal{M} + a(x)) \leq \lim_{R\to\infty} \lambda_v(\mathcal{L}_R + a(x)) \leq \lambda_v(\mathcal{M} + a(x)) \leq \lambda_p'(\mathcal{M} + a(x)),$$

which enforces

$$\lambda_v(\mathcal{M} + a(x)) = \lambda_p'(\mathcal{M} + a(x)).$$

□

4.4 Further properties of the principal eigenvalue

In this section we investigate further the properties of the principal eigenvalue $\lambda_p(\mathcal{M}+a(x))$ and in particular its behaviour with respect to some scaling limit. We focus on the behaviour of the principal eigenvalue of the spectral problem

$$\mathcal{M}_{\varepsilon,m}[\varphi] + (a(x) + \lambda)\varphi = 0 \quad \text{in} \quad \mathbb{R}^N,$$

where $\mathcal{M}_{\varepsilon,m}[\varphi] := \frac{1}{\varepsilon^m}\left(\int_{\mathbb{R}^N} J(z)[\varphi(x+\varepsilon z) - \varphi(x)]\,dz\right)$. Depending on m, we establish some limits for $\lambda_p(\mathcal{M}_{\varepsilon,m} + a(x))$ when $\varepsilon \to 0$ and $\varepsilon \to \infty$. We analyse three situations: $m = 0, 0 < m < 2$ and $m = 2$. $m = 0$ and $m = 2$ are two critical situations, so we postpone for the moment their study and start by the simplest case to analyse $0 < m < 2$. In this situation, we have

Theorem 4.4.1. *Let $J \in C_c(\mathbb{R}^N)$ be a radially symmetric non negative function of unit mass and assume that $0 \leq m < 2$. Then we have the following:*

- *When $0 \leq m < 2$,* $\lim_{\varepsilon\to 0} \lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)) = -\sup_{\mathbb{R}^N} a(x)$
- *When $0 < m < 2$* $\lim_{\varepsilon\to+\infty} \lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)) = -\sup_{\mathbb{R}^N} a(x)$
- *When $m = 0$,* $\lim_{\varepsilon\to+\infty} \lambda_p(\mathcal{M}_{\varepsilon} + a(x)) = 1 - \sup_{\mathbb{R}^N} a(x)$.

In this case, if $a(x)$ is radially symmetric non increasing and $\sup_{\mathbb{R}^N} a(x) < 1$ then there is a sharp threshold ε_0, in the sense that $\lambda_p(\mathcal{M}_{\varepsilon,m} + a(x))$ is nonincreasing with respect to ε and it is negative if and only if $\varepsilon < \varepsilon_0$.

Remark 15. The proof of the sharp threshold is linked to the study of the limits of solutions of the rescaling nonlinear equations. Therefore, we postpone it to Theorem 5.1.2, Chapter 5.

When $m = 2$, the limit as $\varepsilon \to 0$ is more delicate and is related to some characterisation of Sobolev Spaces [22, 25, 86]. In this situation, we have

Theorem 4.4.2. *Let $J \in C_c(\mathbb{R}^N)$ be a radially symmetric non negative function of unit mass and assume $a(x)$ is globally Lipschitz . Then*

$$\lim_{\varepsilon\to 0} \lambda_p(\mathcal{M}_{\varepsilon,2} + a(x)) = \lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right)$$

where

$$D_2(J) := \int_{\mathbb{R}^N} J(z) z^2 \, dz, \quad K_{2,N} := \frac{1}{|S^{N-1}|} \int_{S^{N-1}} (\sigma.e_1)^2 \, d\sigma = \frac{1}{N}$$

and

$$\lambda_1 \left(K_{2,N} D_2(J) \Delta + a(x) \right) := \inf_{\varphi \in H_0^1(\mathbb{R}^N), \varphi \not\equiv 0} \frac{K_{2,N} D_2(J)}{2} \frac{\int_{\mathbb{R}^N} |\nabla \varphi|^2(x) \, dx}{\|\varphi\|_2^2} - \frac{\int_{\mathbb{R}^N} a(x) \varphi^2(x) \, dx}{\|\varphi\|_2^2}.$$

Moreover we have

$$\lim_{\varepsilon \to +\infty} \lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)) = -\sup_{\mathbb{R}^N} a(x).$$

To obtain these limits, we start by recalling a known inequality.

Lemma 4.4.3. *Let $J \in \mathcal{S}(\mathbb{R}^N)$, $J \geq 0$, J symmetric. Then for all $\varphi \in H^1(\mathbb{R}^N)$ we have*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y)(\varphi(x) - \varphi(y))^2 \, dxdy \leq \int_{\mathbb{R}^N} J(z)|z|^2 \, dz \|\nabla \varphi\|_2^2.$$

Proof. Let $\varphi \in C_c^\infty$, then by applying the standard Taylor expansion we have

$$\varphi(x+z) - \varphi(x) = \int_0^1 z_i \partial_i \varphi(x+tz) \, dt \tag{4.56}$$

$$= z_i \partial_i \varphi(x) + \int_0^1 t \left(\int_0^1 z_i z_j \partial_{ij} \varphi(x+tsz) \, ds \right) dt \tag{4.57}$$

where use the Einstein summation convention $a_i b_i = \sum_{i=1}^N a_i b_i$. Let us denote

$$\mathcal{I}(\varphi) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y)(\varphi(x) - \varphi(y))^2 \, dxdy.$$

By plugging the Taylor expansion of φ (4.56) in the above equality we see that

$$\begin{aligned}
\mathcal{I}(\varphi) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(z)(\varphi(x+z) - \varphi(x))^2 \, dzdx &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(z) \left(\int_0^1 z_i \partial_i \varphi(x+tz) dt \right)^2 dzdx \\
&\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(z) \left(|z_i| \left[\int_0^1 |\partial_i \varphi(x+tz)|^2 dt \right]^{\frac{1}{2}} \right)^2 dzdx \\
&\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(z) \left(\sum_{i=1}^N z_i^2 \right) \left[\sum_{i=1}^N \int_0^1 |\partial_i \varphi(x+tz)|^2 dt \right] dzdx
\end{aligned}$$

where we use in the last inequality the standard inequality $(\sum_i a_i b_i)^2 \leq (\sum_{i=1}^N a_i^2)(\sum_{i=1}^N b_i^2)$.

So by using Fubini's Theorem and rearranging the terms in the above inequality, it follows that

$$\mathcal{I}(\varphi) \leq \left(\int_{\mathbb{R}^N} J(z)|z|^2 \, dz \right) \|\nabla \varphi\|_2^2$$

By density of $C_c^\infty(\mathbb{R}^N)$ in H^1, the above inequality holds true for $\varphi \in H^1$, since obviously the functional $\mathcal{I}(\varphi)$ is continuous in $L^2(\mathbb{R}^N)$. Thus (i) holds true.

□

We will now use the notation

$$\mathcal{I}_{\varepsilon,m}(\varphi) := \frac{\frac{1}{\varepsilon^m}\left(\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J_\varepsilon(x-y)(\varphi(x)-\varphi(y))^2\,dxdy\right) - \int_{\mathbb{R}^N} a(x)\varphi^2(x)\,dx}{\|\varphi\|_2^2},$$
$$\mathcal{J}(\varphi) := \frac{D_2(J)}{2}\frac{\int_{\mathbb{R}^N}|\nabla\varphi|^2(x)\,dx}{\|\varphi\|_2^2},$$
$$\mathcal{A}(\varphi) := \frac{\int_{\mathbb{R}^N} a(x)\varphi^2(x)\,dx}{\|\varphi\|_2^2},$$

where $D_2(J) := \int_{\mathbb{R}^N} J(z)|z|^2\,dz$.

By Lemma 4.4.3, for any $\varphi \in H^1$ we get

$$\mathcal{I}_{\varepsilon,m}(\varphi) \le \varepsilon^{2-m}\mathcal{J}(\varphi) - \mathcal{A}(\varphi). \tag{4.58}$$

As a straightforward consequence of this inequality, we get the limits in Theorem 4.4.1

Remark 16. When $m = 2$, another consequence of (4.58) is $\lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)) \le \lambda_1(D_2(J)\Delta + a(x))$ for all $\varepsilon > 0$.

Proof of Theorem 4.4.1. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ so that $supp(\varphi) \subset supp(a^+)$. Then by Lemma 4.3.11 and (4.58) we get for ε small enough

$$\lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)) \le \mathcal{I}_{\varepsilon,m}(\varphi) \le \varepsilon^{2-m}\mathcal{J}(\varphi) - \mathcal{A}(\varphi).$$

To obtain the limit, we just have to remark that by Theorem 4.3.3 we have

$$-\sup_{\mathbb{R}^N}\left(\frac{1}{\varepsilon^m}\left[\int_{\mathbb{R}^N} J_\varepsilon(x-y)\,dy - 1\right] + a(x)\right) = -\sup_{\mathbb{R}^N} a(x) \le \lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)).$$

Therefore

$$-\sup_{\mathbb{R}^N} a(x) \le \liminf_{\varepsilon\to 0}\lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)).$$

On another hand, since a achieves its maximum, says at x_0, for any $\rho > 0$ there exists $\varphi_\rho \in H^1$ so that $supp(\varphi_\rho) \subset B_\rho(x_0)$ and from the above inequality, we deduce that

$$\limsup_{\varepsilon\to 0}\lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)) \le \mathcal{A}(\varphi_\rho) = -\frac{\int_{B_\rho(x_0)} a(x)\varphi_\rho^2(x)\,dx}{\|\varphi_\rho\|_2^2} \le -\min_{B_\rho(x_0)} a^+(x).$$

Hence,

$$-\sup_{\mathbb{R}^N} a(x) = -a(x_0) \le \liminf_{\varepsilon\to 0}\lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)) \le \limsup_{\varepsilon\to 0}\lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)) \le \lim_{\rho\to 0} -\min_{B_\rho(x_0)} a^+(x) = -a(x_0).$$

Let us now look at the limits of $\lambda_p(\mathcal{M}_{\varepsilon,m}+a(x))$ when $\varepsilon \to +\infty$. The limits when $m > 0$ is straightforward consequence of the definition of $\lambda_p(\mathcal{M}_{\varepsilon,m}+a(x))$. From the definition and by using constant as test function, we always have this estimate

$$-\sup_{\mathbb{R}^N}\left(\frac{1}{\varepsilon^m}\left[\int_{\mathbb{R}^N} J_\varepsilon(x-y)\,dy - 1\right] + a(x)\right) \le \lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)) \le -\sup_{\mathbb{R}^N}\left(-\frac{1}{\varepsilon^m} + a(x)\right).$$

Therefore,

$$-\sup_{\mathbb{R}^N} a(x) \leq \lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)) \leq -\sup_{\mathbb{R}^N}\left(-\frac{1}{\varepsilon^m} + a(x)\right)$$

and for $m > 0$ we have

$$-\sup_{\mathbb{R}^N} a(x) \leq \lim_{\varepsilon\to+\infty} \lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)) \leq -\sup_{\mathbb{R}^N} a(x).$$

Finally, we obtain the last limit as $\varepsilon \to \infty$ in the case $m = 0$. Recall that by Theorem 4.3.3 we have for all $\varepsilon > 0$

$$-\sup_{\mathbb{R}^N}(a(x)) \leq \lambda_p(\mathcal{M}_\varepsilon + a(x)) \leq 1 - \sup_{\mathbb{R}^N}(a(x)).$$

So we have

$$\limsup_{\varepsilon\to\infty} \lambda_p(\mathcal{M}_\varepsilon + a(x)) \leq 1 - \sup_{\mathbb{R}^N}(a(x)).$$

On another hand, for any $\varphi \in C_c(\mathbb{R}^N)$ we have for all ε,

$$\begin{aligned}
\mathcal{I}_\varepsilon(\varphi) &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J_\varepsilon(x-y)(\varphi(x)-\varphi(y))^2 dxdy - \int_{\mathbb{R}^N} a(x)\varphi^2(x)\,dx,\\
&= -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J_\varepsilon(x-y)\varphi(x)\varphi(y)dxdy + \int_{\mathbb{R}^N}\varphi^2(x)\,dx - \int_{\mathbb{R}^N} a(x)\varphi^2(x)\,dx,\\
&\geq -\|\varphi\|_{L^2(\mathbb{R}^N)}\left(\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} J_\varepsilon(x-y)\varphi(x)\,dx\right)^2 dy\right)^{1/2} + \int_{\mathbb{R}^N}\varphi^2(x)\,dx - \sup_{\mathbb{R}^N} a(x)\int_{\mathbb{R}^N}\varphi^2(x)\,dx,\\
&\geq -\|J_\varepsilon\|_\infty\int_{\mathbb{R}^N}\varphi^2(x)\,dx + \int_{\mathbb{R}^N}\varphi^2(x)\,dx - \sup_{\mathbb{R}^N} a(x)\int_{\mathbb{R}^N}\varphi^2(x)\,dx,\\
&\geq -\frac{\|J\|_\infty}{\varepsilon^N}\int_{\mathbb{R}^N}\varphi^2(x)\,dx + \int_{\mathbb{R}^N}\varphi^2(x)\,dx - \sup_{\mathbb{R}^N} a(x)\int_{\mathbb{R}^N}\varphi^2(x)\,dx.
\end{aligned}$$

Therefore by density of $C_c(\mathbb{R}^N)$ in $L^2(\mathbb{R}^N)$, we have for any $\varphi \in L^2(\mathbb{R}^N)$

$$\frac{\mathcal{I}_\varepsilon(\varphi)}{\|\varphi\|^2_{L^2(\mathbb{R}^N)}} \geq -\frac{\|J\|_\infty}{\varepsilon^N} + 1 - \sup_{\mathbb{R}^N} a(x).$$

Thus

$$\lambda_p(\mathcal{M}_\varepsilon + a(x)) \geq -\frac{\|J\|_\infty}{\varepsilon^N} + 1 - \sup_{\mathbb{R}^N} a(x),$$

and

$$\liminf_{\varepsilon\to+\infty} \lambda_p(\mathcal{M}_\varepsilon + a(x)) \geq 1 - \sup_{\mathbb{R}^N} a(x).$$

Hence,

$$1 - \sup_{\mathbb{R}^N} a(x) \leq \liminf_{\varepsilon\to+\infty} \lambda_p(\mathcal{M}_\varepsilon + a(x)) \leq \limsup_{\varepsilon\to+\infty} \lambda_p(\mathcal{M}_\varepsilon + a(x)) \leq 1 - \sup_{\mathbb{R}^N} a(x).$$

□

Let us now deal with the case $m = 2$ and prove Theorem 4.4.2.

Proof. Let us rewrite $\mathcal{I}_{\varepsilon,2}(\varphi)$ in a more convenient way. For $\varphi \in H^1(\mathbb{R}^N)$

$$\mathcal{I}_{\varepsilon,2}(\varphi) = \frac{1}{\|\varphi\|_2^2}\left(\frac{1}{2\varepsilon^2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J_\varepsilon(x-y)(\varphi(x)-\varphi(y))^2\,dxdy\right) - \mathcal{A}(\varphi) \tag{4.59}$$

$$= \frac{1}{\|\varphi\|_2^2}\left(\frac{1}{2\varepsilon^2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J_\varepsilon(x-y)|x-y|^2\frac{(\varphi(x)-\varphi(y))^2}{|x-y|^2}\,dxdy\right) - \mathcal{A}(\varphi) \tag{4.60}$$

$$= \frac{1}{\|\varphi\|_2^2}\left(\frac{D_2(J)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \rho_\varepsilon(x-y)\frac{(\varphi(x)-\varphi(y))^2}{|x-y|^2}\,dxdy\right) - \mathcal{A}(\varphi), \tag{4.61}$$

with $\rho_\varepsilon(z) := \frac{1}{\varepsilon^2 D_2(J)} J_\varepsilon(z)|z|^2$. By construction ρ_ε is a continuous mollifier so that

$$\begin{cases} \rho_\varepsilon \geq 0 \quad \text{in} \quad \mathbb{R}^N, \\ \int_{\mathbb{R}^N} \rho_\varepsilon(z)dz = 1 \quad \forall\, \varepsilon > 0, \\ \lim_{\varepsilon\to 0}\int_{|z|\geq\delta} \rho_\varepsilon(z)dz = 0 \quad \forall\, \delta > 0. \end{cases}$$

From the known characterisation of Sobolev spaces in [22, 25, 86], we have

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \rho_\varepsilon(x-y)\frac{(\varphi(x)-\varphi(y))^2}{|x-y|^2}\,dxdy = K_{2,N}\|\nabla\varphi\|_2^2,$$

with $K_{2,N} := \frac{1}{|S^{N-1}|}\int_{S^{N-1}}(\sigma.e_1)^2\,d\sigma$. Therefore

$$\lim_{\varepsilon\to 0}\mathcal{I}_{\varepsilon,2}(\varphi) = K_{2,N}\frac{D_2(J)}{2}\frac{\|\nabla\varphi\|_2^2}{\|\varphi\|_2^2} - \mathcal{A}(\varphi).$$

From the above inequality, we deduce that

$$\limsup_{\varepsilon\to 0}\lambda_p(\mathcal{M}_{\varepsilon,2} + a(x)) \leq \lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right).$$

Conversely, let $\delta > 0$, we claim that for all $\varepsilon > 0$ there exists $\varphi_\varepsilon \in C_c^2(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ so that

$$\mathcal{M}_{\varepsilon,2}[\varphi_\varepsilon] + (a(x) + \lambda_p(\mathcal{M}_{\varepsilon,2} + a(x)) + 2\delta)\varphi_\varepsilon \geq 0.$$

Indeed, from the Claim 4.3.9, for all ε, there exists $\psi_\varepsilon \in C_c(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ so that

$$\mathcal{M}_{\varepsilon,2}[\psi_\varepsilon] + (a(x) + \lambda_p(\mathcal{M}_{\varepsilon,2} + a(x)) + \delta)\psi_\varepsilon \geq 0.$$

Let η be a smooth mollifier of unit mass and with support in the unit ball. Consider $\eta_\tau := \frac{1}{\tau^N}\eta\left(\frac{z}{\tau}\right)$. By taking $\varphi_\varepsilon := \eta_\tau \star \psi_\varepsilon$ and observing that $\mathcal{M}_{\varepsilon,2}[\varphi_\varepsilon] = \eta^\tau \star (\mathcal{M}_{\varepsilon,2}[\psi_\varepsilon])$ we achieve

$$\begin{aligned} &\eta_\tau \star (\mathcal{M}_{\varepsilon,2}[\psi_\varepsilon] + (a(x) + \lambda_p(\mathcal{M}_{\varepsilon,2} + a(x)) + \delta)\psi_\varepsilon) \geq 0, \\ &\mathcal{M}_{\varepsilon,2}[\varphi_\varepsilon] + (\lambda_p(\mathcal{M}_{\varepsilon,2} + \delta)\varphi_\varepsilon + \eta_\tau \star (a\psi_\varepsilon)(x) \geq 0, \\ &\mathcal{M}_{\varepsilon,2}[\varphi_\varepsilon] + (a(x) + \lambda_p(\mathcal{M}_{\varepsilon,2} + \delta)\varphi_\varepsilon + \int_{\mathbb{R}^N}\eta_\tau(x-y)\psi_\varepsilon(y)(a(y)-a(x))\,dy \geq 0. \end{aligned}$$

Since $a(x)$ is uniformly Lipschitz, we can estimate the integral in the last inequality by

$$\left|\int_{\mathbb{R}^N} \eta_\tau(x-y)\psi_\varepsilon(y)(a(y)-a(x))\,dy\right| \le \int_{\mathbb{R}^N} \eta_\tau(x-y)\psi_\varepsilon(y)\left|\frac{a(y)-a(x)}{y-x}\right||x-y|\,dy$$
$$\le \kappa\tau\int_{\mathbb{R}^N}\eta_\tau(x-y)\psi_\varepsilon(y)\,dy,$$

where κ is the Lipschitz constant of a. Thus for τ small, says $\tau \le \frac{\delta}{2\kappa}$, we achieve

$$\mathcal{M}_{\varepsilon,2}[\varphi_\varepsilon] + (a(x)+\lambda_p(\mathcal{M}_{\varepsilon,2}+a(x))+2\delta)\varphi_\varepsilon \ge 0. \tag{4.62}$$

Let us normalize now φ_ε so that

$$\frac{\int_{\mathbb{R}^N}\psi_\varepsilon^2(x)\,dx}{\int_{\mathbb{R}^N}\varphi_\varepsilon^2(x)\,dx} = 1. \tag{4.63}$$

Let us now multiply $\mathcal{M}_{\varepsilon,\varepsilon}[\varphi_\varepsilon]$ by $-\varphi_\varepsilon$ and integrate it over $\mathbb{R}^N$, then we have

$$-\int_{\mathbb{R}^N}\mathcal{M}_{\varepsilon,2}[\varphi_\varepsilon]\varphi_\varepsilon(x)\,dx = -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{\varepsilon^2}J_\varepsilon(x-y)(\varphi_\varepsilon(y)-\varphi_\varepsilon(x))\varphi_\varepsilon(x)\,dydx. \tag{4.64}$$
$$= \frac{1}{2\varepsilon^2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}J_\varepsilon(x-y)(\varphi_\varepsilon(y)-\varphi_\varepsilon(x))^2\,dxdy \tag{4.65}$$
$$= \frac{D_2(J)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\rho_\varepsilon(z)\frac{(\varphi_\varepsilon(x+z)-\varphi_\varepsilon(x))^2}{|z|^2}\,dzdx \tag{4.66}$$

Combining (4.62) and (4.66) it follows that

$$\frac{D_2(J)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\rho_\varepsilon(z)\frac{(\varphi_\varepsilon(x+z)-\varphi_\varepsilon(x))^2}{|z|^2}\,dzdx - \int_{\mathbb{R}^N}a(x)\varphi_\varepsilon(x)^2\,dx \le (\lambda_p(\mathcal{M}_{\varepsilon,2}+a(x))+2\delta)\int_{\mathbb{R}^N}\varphi_\varepsilon^2(x)\,dx \tag{4.67}$$

On another hand, inspired by the proof of Theorem 2 in [25], since $\varphi_\varepsilon \in C_c^2(\mathbb{R}^N)$, for all $x, z \in \mathbb{R}^N$ we have by a Taylor expansion

$$|\varphi_\varepsilon(x+z)-\varphi_\varepsilon(x)-z\cdot\nabla\varphi_\varepsilon(x)| \le \sum_{i,j}|z_iz_j|\int_0^1 t\left(\int_0^1|\partial_{ij}\varphi_\varepsilon(x+tsz)|\,ds\right)dt.$$

Therefore

$$|z\cdot\nabla\varphi_\varepsilon(x)| \le \sum_{i,j}|z_iz_j|\int_0^1 t\left(\int_0^1|\partial_{ij}\varphi_\varepsilon(x+tsz)|\,ds\right)dt + |\varphi_\varepsilon(x+z)-\varphi_\varepsilon(x)|,$$

and for every $\theta > 0$ we have

$$|z\cdot\nabla\varphi_\varepsilon(x)|^2 \le C_\theta\left[\sum_{i,j}|z_iz_j|\int_0^1 t\left(\int_0^1|\partial_{ij}\varphi_\varepsilon(x+tsz)|\,ds\right)dt\right]^2 + (1+\theta)|\varphi_\varepsilon(x+z)-\varphi_\varepsilon(x)|^2,$$
$$\le C_\theta\sum_{i,j}|z_iz_j|^2\int_0^1 t^2\left(\int_0^1|\partial_{ij}\varphi_\varepsilon(x+tsz)|^2\,ds\right)dt + (1+\theta)|\varphi_\varepsilon(x+z)-\varphi_\varepsilon(x)|^2.$$

Thus

$$\int_{\mathbb{R}^N}\int_{|z|\le 1}\frac{\rho_\varepsilon(|z|)}{|z|^2}|z\cdot\nabla\varphi_\varepsilon(x)|^2\,dzdx \le$$
$$\le C_\theta\int_{\mathbb{R}^N}\int_{|z|\le 1}\rho_\varepsilon(|z|)\sum_{i,j}\frac{|z_iz_j|^2}{|z|^2}\int_0^1 t^2\left(\int_0^1|\partial_{ij}\varphi_\varepsilon(x+tsz)|^2\,ds\right)dt\,dzdx$$
$$+(1+\theta)\int_{\mathbb{R}^N}\int_{|z|\le 1}\rho_\varepsilon(|z|)\frac{|\varphi_\varepsilon(x+z)-\varphi_\varepsilon(x)|^2}{|z|^2}\,dzdx.$$

For ε small $supp(\rho_\varepsilon)\subset B_1(0)$, therefore

$$\int_{|z|\le 1}\frac{\rho_\varepsilon(|z|)}{|z|^2}|z\cdot\nabla\varphi_\varepsilon(x)|^2\,dz = K_{2,N}|\nabla\varphi_\varepsilon(x)|^2,$$

and we have

$$\begin{aligned}K_{2,N}\int_{\mathbb{R}^N}|\nabla\varphi_\varepsilon(x)|^2\,dx \le C_\theta\int_{\mathbb{R}^N}\int_{|z|\le 1}\rho_\varepsilon(|z|)\sum_{i,j}\frac{|z_iz_j|^2}{|z|^2}\int_0^1 t^2\left(\int_0^1|\partial_{ij}\varphi_\varepsilon(x+tsz)|^2\,ds\right)dt\,dzdx\\ +(1+\theta)\int_{\mathbb{R}^N}\int_{|z|\le 1}\rho_\varepsilon(|z|)\frac{|\varphi_\varepsilon(x+z)-\varphi_\varepsilon(x)|^2}{|z|^2}\,dzdx.\end{aligned}\tag{4.68}$$

Dividing by $\|\varphi_\varepsilon\|_2^2$ the above inequality and subtracting $\mathcal{A}(\varphi_\varepsilon)$ on both side, it follows

$$\begin{aligned}K_{2,N}\mathcal{J}(\varphi_\varepsilon)-\mathcal{A}(\varphi_\varepsilon)\le\frac{C_\theta}{\|\varphi_\varepsilon\|_2^2}\int_{\mathbb{R}^N}\int_{|z|\le 1}\rho_\varepsilon(|z|)\sum_{i,j}\frac{|z_iz_j|^2}{|z|^2}\int_0^1 t^2\left(\int_0^1|\partial_{ij}\varphi_\varepsilon(x+tsz)|^2\,ds\right)dt\,dzdx\\ +(1+\theta)\mathcal{I}_{\varepsilon,2}(\varphi_\varepsilon).\end{aligned}\tag{4.69}$$

By combining now (4.69) with (4.67) and using the definition of $\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta+a(x)\right)$ we get

$$\begin{aligned}\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta+a(x)\right)\le\frac{C_\theta}{\|\varphi_\varepsilon\|_2^2}\int_{\mathbb{R}^N}\int_{|z|\le 1}\rho_\varepsilon(|z|)\sum_{i,j}\frac{|z_iz_j|^2}{|z|^2}\int_0^1 t^2\left(\int_0^1|\partial_{ij}\varphi_\varepsilon(x+tsz)|^2\,ds\right)dt\,dzdx\\ +(1+\theta)[\lambda_p(\mathcal{M}_{\varepsilon,2}+a(x))+2\delta]\end{aligned}\tag{4.70}$$

Let us now estimate the last integral in the above inequality. By construction we have $\partial_{ij}\varphi_\varepsilon=\partial_{ij}\eta_\tau\star\psi_\varepsilon$, so by Fubini's Theorem and standard convolution estimates we get

$$\sum_{i,j}\int_{\mathbb{R}^N}\int_{|z|\le 1}\iint_{[0,1]^2}\rho_\varepsilon(|z|)\frac{|z_iz_j|^2}{|z|^2}t^2|\partial_{ij}\varphi_\varepsilon(x+tsz)|^2$$
$$\le\sum_{i,j}\int_{|z|\le 1}\iint_{[0,1]^2}\rho_\varepsilon(|z|)\frac{|z_iz_j|^2}{|z|^2}t^2\left(\int_{\mathbb{R}^N}|\partial_{ij}\eta_\tau\star\psi_\varepsilon(x+tsz)|^2\,dx\right),$$
$$\le\sum_{i,j}\int_{|z|\le 1}\iint_{[0,1]^2}\rho_\varepsilon(|z|)\frac{|z_iz_j|^2}{|z|^2}t^2\|\partial_{ij}\eta_\tau\|_{L^1(\mathbb{R}^N)}\|\psi_\varepsilon\|_2^2,$$
$$\le\frac{C_N\|\partial_{ij}\eta_\tau\|_{L^1(\mathbb{R}^N)}}{6}\|\psi_\varepsilon\|_2^2\int_{|z|\le 1}\rho_\varepsilon(|z|)|z|^2\,dz,$$

with C_N a universal constant depending only on N.

Combining this inequality with (4.70), we get

$$\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right) \leq C_{\theta,N}\|\partial_{ij}\eta_\tau\|_{L^1(\mathbb{R}^N)}\frac{\|\psi_\varepsilon\|_2^2}{\|\varphi_\varepsilon\|_2^2}\int_{|z|\leq 1}\rho_\varepsilon(|z|)|z|^2dz + (1+\theta)[\lambda_p(\mathcal{M}_{\varepsilon,2}+a(x)) + 2\delta] \tag{4.71}$$

Thanks to the normalisation (4.63), since $\int_{|z|\leq 1}\rho_\varepsilon(|z|)|z|^2dz \leq \varepsilon^2$ letting $\varepsilon \to 0$ in (4.71) yields

$$\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right) \leq (1+\theta)[2\delta + \liminf_{\varepsilon\to 0}\lambda_p(\mathcal{M}_{\varepsilon,2}+a(x))]. \tag{4.72}$$

Since (4.72) holds for every $\theta, \delta > 0$, we obtain

$$\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right) \leq \liminf_{\varepsilon\to 0}\lambda_p(\mathcal{M}_{\varepsilon,2}+a(x)).$$

Hence,

$$\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right) \leq \liminf_{\varepsilon\to 0}\lambda_p(\mathcal{M}_{\varepsilon,2}+a(x)) \leq \limsup_{\varepsilon\to 0}\lambda_p(\mathcal{M}_{\varepsilon,2}+a(x)) \leq \lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right).$$

□

Remark 17. The characterisation of the limits also holds for a operator $\frac{1}{\varepsilon^2}\mathcal{L}_{R,\varepsilon} + a(x)$. Indeed, from Remark 14 for $\delta > 0$, then for all $\varepsilon > 0$ there exists $\psi_\varepsilon \in C_c(B_R)$ so that

$$\frac{1}{\varepsilon^2}\mathcal{L}_{\varepsilon,R}[\psi_\varepsilon](x) + (a(x) + \delta + \lambda_p(\frac{1}{\varepsilon^2}\mathcal{L}_{R,\varepsilon} + a(x)))\psi_\varepsilon(x) \geq 0.$$

Thus by reproducing the above argumentation, we get

$$\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x), B_R\right) \leq \liminf_{\varepsilon\to 0}\lambda_p(\frac{1}{\varepsilon^2}\mathcal{L}_{R,\varepsilon} + a(x)).$$

To obtain the upper bound, we just observe that for any $\varphi \in C_c^2(B_R)$ then we have

$$\lambda_p(\frac{1}{\varepsilon^2}\mathcal{L}_{R,\varepsilon}+a(x)) \leq \frac{\frac{1}{2\varepsilon^2}\iint_{B_R\times B_R}J_\varepsilon(x-y)[\varphi(x)-\varphi(y)]^2\,dxdy + \frac{1}{\varepsilon^{2+N}}\int_{B_R}k_\varepsilon(x)\varphi^2(x)\,dx - \int_{B_R}a(x)\varphi^2(x)\,dx}{\|\varphi\|^2_{L^2(B_R)}},$$

where $k_\varepsilon(x) := \frac{1}{\varepsilon^2}\int_{\mathbb{R}^N\setminus B_R}J_\varepsilon(x-y)$. Since φ is compactly supported, we have $\frac{1}{\varepsilon^{2+N}}\int_{B_R}k_\varepsilon(x)\varphi^2(x)\,dx = 0$ for ε small and we deduce that

$$\limsup_{\varepsilon\to 0}\lambda_p(\frac{1}{\varepsilon^2}\mathcal{L}_{R,\varepsilon} + a(x)) \leq \lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x), B_R\right).$$

Chapitre 5

On positive solutions of nonlocal heterogeneous KPP equations

The contents of this chapter are based on the joint work :

Nonlocal heterogeneous KPP equations in $\mathbb{R}^N$

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This is a continuation of Chapter 4, where the fundamental results of generalized principal eigenvalues are applied to derive some deeper results on existence, nonexistence, uniqueness of the positive solutions of (4.3), the long time behavior of (4.1) and the asymptotic of u_ε, which are solutions of the rescaling equations

$$\mathcal{M}_{\varepsilon,m}[u_\varepsilon] + u_\varepsilon(a(x) - u_\varepsilon) = \frac{1}{\varepsilon^m}(J_\varepsilon \star u_\varepsilon - u_\varepsilon) + u_\varepsilon(a(x) - u_\varepsilon) = 0 \qquad x \in \mathbb{R}^N, \tag{5.1}$$

for $m \in [0, 2]$, as $\varepsilon \to 0$ and $\varepsilon \to \infty$.

In the last section, we further provide some extensions for non compactly supported kernels, which possess an adequate decay near infinity.

5.1 Main Results

Let us now state our main results for this chapter. We first prove a simple sharp survival criteria assuming that the dispersal kernel J satisfy an extra assumption.

Theorem 5.1.1. *Assume that J, f satisfy (H1-H4) and assume further that J is compactly supported. Then, there exists a unique positive solution to* (4.2) *iff $\lambda_p(\mathcal{M} + \partial_s f(x, 0)) < 0$, where*

$$\lambda_p(\mathcal{M} + \partial_u f(x, 0)) := \sup\{\lambda \in \mathbb{R} \,|\, \exists \varphi \in C(\mathbb{R}^N), \phi > 0 \;\; \textit{so that} \;\; \mathcal{M}[\varphi] + \partial_s f(x, 0)\varphi + \lambda\varphi \leq 0\},$$

where $\mathcal{M}$ denotes the continuous operator $\mathcal{M}[\varphi] = J \star \varphi(x) - \varphi(x)$. Moreover, for any non-negative initial data $u_0 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ we have the following asymptotic behaviour:

- *If $\lambda_p(\mathcal{M} + \partial_s f(x,0)) \geq 0$, then the solution $\|u\|_\infty(t) \to 0$ as $t \to \infty$,*
- *If $\lambda_p(\mathcal{M} + \partial_s f(x,0)) < 0$, then the solution $\|u - \tilde{u}\|_\infty(t) \to 0$ as $t \to \infty$, where $\tilde{u}$ denotes the unique positive solution to* (4.2)

In addition, if the initial data $u_0 \in C(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, then the convergence $u(t,x) \to \tilde{u}$ holds in $L^1(\mathbb{R}^N)$.

Next, we aim at understanding the effect of the dispersal kernel on the survival of the species. To this end, we analyse the behaviour of the survival criteria under some scaling of the dispersal operator. More precisely, let $J_\varepsilon := \frac{1}{\varepsilon^N} J\left(\frac{z}{\varepsilon}\right)$ and let $\mathcal{M}_\varepsilon$ denotes the operator $\mathcal{M}$ with the rescaled kernel, then we look at the behavior of the solution to (4.2) as $\varepsilon \to 0$ or $+\infty$ where the dispersal operator $\mathcal{M}$ is replace by $\alpha(\varepsilon)\mathcal{M}_\varepsilon$, with $\alpha(\varepsilon) \sim \frac{\alpha_0}{\varepsilon^m}$. These asymptotics represent two possible strategies that are observed in nature. The terms $\alpha(\varepsilon)$ refers to a dispersal budget of the species as defined in [66]. Roughly speaking, for a fixed cost, this budget is a way to measure the differences between different strategies. For a given dispersal cost function of the order of $\sim |y|^m$, the term $\alpha(\varepsilon)$ behaves like $\frac{\alpha_0}{\varepsilon^m}$ and in the analysis, the dispersal operator is then given by $\alpha(\varepsilon)\mathcal{M}_\varepsilon$. As explained in [66], the limit as $\varepsilon \to 0$ can be associated to a strategy of producing a lot of offspring but with little capacity of movement. Whereas the limit $\varepsilon \to +\infty$ corresponds to a strategy that aims at maximizing the possibility to explore the environment at the expense of the number of offspring.

Here, we analysis cases $0 \leq m \leq 2$ and $\alpha_0 = 1$, the case $m = 0$ corresponding to understand the impact of the mean distance on the survival criteria. To simplify the presentation of these asymptotics, we restrict to nonlinearities $f(x,s)$ of the form $f(x,s) = s(a(x) - s)$. However, the proofs apply more generally.

In this situation, we first obtain

Theorem 5.1.2. *Assume that J and f satisfy (H1-H4), J is compactly supported and let $m = 0$. Then there exists $\varepsilon_0 \in (0, +\infty]$ so that for all $\varepsilon \leq \varepsilon_0$ there exists a positive solution u_ε to* (4.2)*. Moreover, we have*

$$\lim_{\varepsilon \to \varepsilon_0} u_\varepsilon(x) = (a(x) - 1)^+,$$

where $a(x)^+$ denotes the positive part of a (i.e. $a^+(x) = sup\{0, a(x)\}$). Assuming further that a is smooth, at least $C^{0,1}(\mathbb{R}^N)$, we have

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = v(x).$$

where v is a non-negative bounded solution of

$$v(x)(a(x) - v(x)) = 0 \quad \text{in} \quad \mathbb{R}^N.$$

In addition, when $a(x)$ is radially symmetric non increasing and $\varepsilon_0 < +\infty$ then ε_0 is sharp, in the sense that for all $\varepsilon \geq \varepsilon_0$ there is no positive solution to (4.2)*.*

The ecological interpretation of this result is that one way of persistence for a species is to match the resource and not move too much. In some situation, $\varepsilon_0 = +\infty$ and there is no effect of the dispersal on the survival criteria of the species. A natural condition that ensures that $\varepsilon_0 = +\infty$ is

$$(a(x) - 1)^+ \neq 0.$$

In this context, the birth rates exceed all death rates and guarantee the persistence of the population no matter is the dispersal strategy. In particular, the exists a bounded positive solution to (4.2) for any positive kernel J. The uniqueness and the behaviour at infinity of the solution are still open questions for general kernels.

When $m > 0$ then the characterisation of the existence of a positive solution changes and a new picture emerges. In particular, for large ε there is always solution to (4.2) whereas for small ε it may happen that no positive solution exists. Thus, the situation is, in a sense, opposite to the case when $m = 0$. Non existence for small value of ε appears only when $m \geq 2$. More precisely we prove

Theorem 5.1.3. *Assume that J and f satisfy (H1-H4), J is compactly supported and let $0 \leq m < 2$. Then there exists $\varepsilon_0 \leq \varepsilon_1 \in (0, +\infty)$ so that for all $\varepsilon \leq \varepsilon_0$ and for all $\varepsilon \geq \varepsilon_1$ there exists a positive solution u_ε to (4.2). Moreover, we have*

$$\lim_{\varepsilon \to +\infty} \|u_\varepsilon - a^+\|_\infty = 0, \qquad \lim_{\varepsilon \to +\infty} \|u_\varepsilon - a^+\|_{L^2(\mathbb{R}^N)} = 0.$$

In addition, assuming further that a is smooth, at least $C^2(\mathbb{R}^N)$, we have

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = v(x).$$

where v is a non-negative bounded solution of

$$v(x)(a(x) - v(x)) = 0 \quad \text{in} \quad \mathbb{R}^N.$$

Remark 18. Due to the condition on $a(x)$, we must have $v = 0$ in $\{x \in \mathbb{R}^N, a(x) \leq 0\}$. The characterization of v in $\{x \in \mathbb{R}^N, a(x) < 0\}$ are still under investigating.

Theorem 5.1.4. *Assume that J and f satisfy (H1-H4), J is compactly supported and let $m = 2$. Then there exists $\varepsilon_1 \in (0, \infty)$ so that for all $\varepsilon \geq \varepsilon^*$ there exists a positive solution u_ε to (4.2). Moreover,*

$$\lim_{\varepsilon \to +\infty} \|u_\varepsilon - a^+\|_\infty = 0, \qquad \lim_{\varepsilon \to +\infty} \|u_\varepsilon - a^+\|_{L^2(\mathbb{R}^N)} = 0.$$

In addition, we have the following dichotomy

- *When $\lambda_1\left(\frac{K_{2,N} D_2(J)}{2}\Delta + a(x)\right) < 0$, there exists $\varepsilon_0 \in (0, \infty)$ so that for all $\varepsilon \leq \varepsilon_0$ there exists a positive solution to (4.2) and*

  $$u_\varepsilon \to v, \quad \text{in} \quad L^2_{loc}(\mathbb{R}^N),$$

  where v is the unique bounded non-trivial solution to

  $$\frac{K_{2,N} D_2(J)}{2}\Delta v + v(a(x) - v) = 0 \quad \text{in} \quad \mathbb{R}^N.$$

- *When $\lambda_1\left(\frac{K_{2,N} D_2(J)}{2}\Delta + a(x)\right) > 0$, then there $\varepsilon_0 \in (0, \infty)$ so that for all $\varepsilon \leq \varepsilon_0$ there exists no positive solution to (4.2).*

The last result clearly accentuates the dependence of the strategy on the cost functions and the structure of ecological niche. Especially when $m = 2$ the slower dispersal strategy may not be an Ecological Stable Strategy (ESS) in the sense of [96]. This stands in contrast with the results in [66] obtained in bounded domain neglecting the structure of the ecological niche. We also point the interested reader to [31] for recent development on this subject.

Finally, we obtained existence/ non-existence criteria when we relax the compactly supported constrain on the dispersal kernel J. In this direction, we investigate a class of kernel J that can have a fat tail but still have some decay at infinity. More precisely, assume that

(H6) $\int_{\mathbb{R}^N} J(z)|z|^{N+1} < +\infty$.

Then for such kernel we prove

Theorem 5.1.5. *Assume that J, f satisfy (H1-H4) and assume further that J satisfies $(H5)$ then we have*

(i) ***if*** $\lambda_p(\mathcal{M} + \partial_s f(x,0)) > 0$ *there is no bounded positive solution to* (4.2).

(ii) ***if*** $\lim_{R\to\infty} \lambda_p(\mathcal{L}_R + \partial_s f(x,0)) < 0$ *where*

$$\mathcal{L}_R[\varphi] := \int_{B_R(0)} J(x-y)\varphi(y)\,dy - \varphi(x),$$

then there exists a unique positive bounded solution to (4.2).

Although we do not have a sharp criteria for fat tailed kernels, the above existence and uniqueness criteria is almost sharp, in the sense that we can show that

$$\lambda_p(\mathcal{M} + \partial_s f(x,0)) \le \lim_{R\to\infty} \lambda_p(\mathcal{L}_{B(0,R)} + \partial_s f(x,0)).$$

We strongly believe that there exists a unique bounded positive solutions to (4.2) when $\lambda_p(\mathcal{M}+\partial_s f(x,0)) < 0$ and no positive solution when $\lambda_p(\mathcal{M} + \partial_s f(x,0)) \ge 0$. Possibly, more delicate analyses are required. For the moment, we leave it as an open question.

Open problem 2. Can one obtain a sharp characterization for existence and uniqueness of (4.2) when J only satisfies (H6) ?

Finally, we also want to stress that although we have a clear description of the existence/non-existence of a positive solution for small ε, the study of the convergence of u_ε as $\varepsilon \to 0$ is quite delicate. Indeed, in $L^\infty(\mathbb{R}^N)$, the problem

$$v(x)(a(x) - v(x)) = 0 \quad \text{in} \quad \mathbb{R}^N,$$

has infinitely many bounded non negative solution (e.g. for any set $Q \subset \mathbb{R}^N$, the function $a^+(x)\chi_Q$ is a solution) and owing to the lack of regularising effect of the dispersal operator, we cannot rely on standard compactness result in the usual manner to obtain a smooth limit. If for the case $m = 2$ we could rely on the elliptic regularity and the new description of Sobolev Spaces developed in [22, 25, 87, 86] to get some compactness, this characterisation does not allow us to treat the case $m < 2$. We believe that a new characterisation of Fractional Sobolev space in the spirit of the work of Bourgain, Brezis and Mironescu [22, 25] may be helpful to resolve this issue.

5.2 Existence/non existence and uniqueness of a non-trivial solution

In this section we construct a non-trivial solution to (4.2) and prove necessary and sufficient condition stated in Theorem 5.1.1. For convenience the section is split up into three subsections, each of them respectively devoted to the proofs of existence of a solution, the uniqueness and non-existence.

5.2.1 Existence of a non-trivial positive solution

The construction follows a basic approximation scheme previously used for example in [5]. To this aim, we introduce the following approximated problem :

$$\mathcal{L}_{_R}[u] + f(x,u) = 0 \quad \text{in} \quad \bar{B}(0,R) \tag{5.2}$$

where $B(0,R)$ denotes the ball of radius R centred at the origin. By Theorem 4.2.3, for any $R > 0$ the existence of a unique positive solution to (5.2) is conditioned to the sign of $\lambda_p(\mathcal{L}_{_R} + \beta(x))$ where $\beta(x) := \partial_u f(x,0)$. By Lemma 4.3.6 since

$$\lim_{R\to+\infty} \lambda_p(\mathcal{L}_{_R} + \beta(x)) = \lambda_p(\mathcal{M} + \beta(x)) < 0$$

there exists $R_0 > 0$ so that

$$\forall R \geq R_0, \lambda_p(\mathcal{L}_{_R} + \beta(x)) < 0.$$

As a consequence by Theorem 4.2.3, for all $R > R_0$ there exists a unique positive solution to (5.2) that we denote u_R. Moreover, since for all $R > 0, \sup_{B_R(0)} S(x)$ is a super-solution to (5.2), by a standard sweeping argument since the solution to (5.2) is unique, we get

$$\forall R > 0,\ u_R \leq \sup_{B_R(0)} S(x) \quad \text{in} \quad B(0,R).$$

On another hand, for any $R_1 > R_2$, the solution u_{R_1} is a super-solution to the problem

$$\mathcal{L}_{_{R_2}}[u] + f(x,u) = 0 \quad \text{in} \quad \bar{B}(0,R_2) \tag{5.3}$$

So as above by a standard sweeping argument we get

$$u_{R_2} \leq u_{R_1}(x) \quad \text{in} \quad B(0,R_2).$$

Thus the map $R \mapsto u_R$ is monotone increasing.

The idea is to obtain a positive solution to (4.2) as a limit of positive solution of (5.2). To this end we construct a uniform super-solution to (4.2).

Lemma 5.2.1. *There exists $\bar{u} \in C_0(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, $\bar{u} > 0$ so that $\bar{u}$ is a super-solution to* (4.2).

Proof. Fix $\nu > 0$ so that $\nu < -\limsup_{|x|\to\infty} \beta(x)$ and let $R_0 > 1$ be such that $\beta(x) \leq -\frac{\nu}{2}$ for all $|x| \geq R_0$. Let w be the following function

$$w(x) = Ce^{-\alpha|x|},$$

where C, α are to be chosen. By direct computations we get for all $x \in \mathbb{R}^N \setminus B_{R_0}(0)$:

$$\mathcal{M}[w](x)+\beta(x)w(x) = Ce^{-\alpha|x|}\left(\int_{\mathbb{R}^N} J(x-y)e^{-\alpha(|y|-|x|)}\,dy - 1 + \beta(x)\right)$$
$$\leq w(x)\left(\int_{\mathbb{R}^N} J(z)e^{\alpha(|z|)}\,dz - 1 - \frac{\nu}{2}\right)$$

Therefore w satisfies

$$\mathcal{M}[w](x)+\beta(x)w(x) \leq h(\alpha)w(x) \quad \text{in} \quad \mathbb{R}^N \setminus B_{R_0}(0), \tag{5.4}$$

where $h(\alpha)$ is defined by

$$h(\alpha) = -1 - \frac{\nu}{2}.$$

Since J is compactly supported, thanks to the Lebesgue's Theorems, we can check that $h(\cdot)$ is a smooth (C^2) convex increasing function of α. Moreover, we have

$$\lim_{\alpha\to 0} h(\alpha) = h(0) = -\frac{\nu}{2}.$$

Therefore, by continuity of h, we can choose α small says $\alpha \leq \alpha_0$ for some α_0 so that $h(\alpha) < 0$. For such α, we get

$$\mathcal{M}[w](x)+\beta(x)w(x) \leq h(\alpha)w(x) < 0 \quad \text{in} \quad \mathbb{R}^N \setminus B_{R_0}(0). \tag{5.5}$$

Let $M := \sup_{B_{2R_0}(0)} S(x)$ and choose now C so that $C = 2Me^{2\alpha R_0}$. Let us consider now the continuous function

$$\bar{u}(x) := \begin{cases} Ce^{-\alpha|x|} & \text{in} \quad \mathbb{R}^N \setminus B_{2R_0}(0), \\ 2M & \text{in} \quad B_{2R_0}(0). \end{cases}$$

By direct computation we can check that $\bar{u}$ is a super-solution to (4.2). Indeed, for any $x \in B_{2R_0}(0)$, we have $\bar{u} = 2M > \sup_{B_{2R_0}(0)} S(x)$ which implies that $f(x,\bar{u}) = f(x,2M) \leq 0$ and

$$\mathcal{M}[\bar{u}](x) + f(x,\bar{u}(x)) \leq 2M\int_{\mathbb{R}^N} J(x-y)\,dy - 2M + f(x,2M) \leq f(x,2M) \leq 0.$$

Whereas, for $x \in \mathbb{R}^N \setminus B_{2R_0}(0) \subset \mathbb{R}^N \setminus B_{R_0}(0)$ we have by (5.5)

$$\mathcal{M}[\bar{u}](x) + f(x,\bar{u}(x)) \leq \mathcal{M}[\bar{u}](x) + \beta(x)w(x) \leq \mathcal{M}[w](x) + \beta(x)w(x)$$
$$\leq h(\alpha)w(x) \leq 0.$$

□

We are now in position to construct a solution of (4.2). By Lemma 5.2.1, there exists $\bar{u}$ a positive super-solution to (4.2). Therefore for any $R > 0$, $\bar{u}$ is also a positive super-solution to (5.2). Therefore by using a standard sweeping argument, we can check that for all $R \geq R_0$ then the unique solution to (5.2) satisfies $u_R \leq \bar{u}$ in $B_R(0)$. By sending now $R \to \infty$ and observing that $u_R \in C(\mathbb{R}^N)$ is monotone locally uniformly bounded, by a diagonal extraction $u_R \to \tilde{u} := \lim_{R\to\infty} u_R$ a non-negative solution to (4.2), moreover $\tilde{u} \in C_0(\mathbb{R}^N)$ and is non trivial since $0 \leq \tilde{u} \leq \bar{u}$ and

$$u_R \leq \tilde{u} \quad \text{in } B(0,R), \quad \text{for all } R \geq R_0.$$

5.2.2 Uniqueness

Having construction of a $L^1(\mathbb{R}^N)$ positive solution to (4.2), the uniqueness of the solutions of (4.2) is then obtained by the following argument. Assume by contradiction that v is another positive solution and assume that $v \in L^\infty$. Then v is a supersolution to (5.2) for any $R > 0$. Therefore $v \geq u_R$ for all $R \geq R_0$. Since u_R is monotone with respect to R, it follows that $v \geq \tilde{u} := \lim_R u_R(x)$. Assume by contradiction that $v \not\equiv \tilde{u}$ and recall that v and $\tilde{u}$ satisfies:

$$\mathcal{M}[\tilde{u}] + f(x, \tilde{u}) = 0 \quad \text{in} \quad \mathbb{R}^N \tag{5.6}$$

$$\mathcal{M}[v] + f(x, v) = 0 \quad \text{in} \quad \mathbb{R}^N \tag{5.7}$$

By multiplying (5.6) by v and (5.7) by u we get after integration over $\mathbb{R}^N$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\tilde{u}(y)v(x)\,dydx - \int_{\mathbb{R}^N} \tilde{u}(x)v(x)\,dx + \int_{\mathbb{R}^N} v(x)f(x,\tilde{u}(x))\,dx = 0, \tag{5.8}$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\tilde{u}(x)v(y)\,dydx - \int_{\mathbb{R}^N} \tilde{u}(x)v(x)\,dx + \int_{\mathbb{R}^N} \tilde{u}(x)f(x,v(x))\,dx = 0. \tag{5.9}$$

Therefore by subtracting the two above equality, we get the contradiction

$$0 < \int_{\mathbb{R}^N} v(x)\tilde{u}\left[\frac{f(x,\tilde{u}(x))}{\tilde{u}(x)} - \frac{f(x,v(x))}{v(x)}\right] dx = 0,$$

since $\tilde{u} \leq v$ and $f(x,s)/s$ is decreasing.

5.2.3 Non-existence of a solution

In this section, we deal with the non-existence of positive solution when $\lambda_p(\mathcal{M} + \beta(x)) \geq 0$. To simplify the presentation of the proofs we treat the two cases: $\lambda_p(\mathcal{M}+\beta(x)) > 0$ and $\lambda_p(\mathcal{M}+\beta(x)) = 0$ separately. The proof in the second case being more involved, we start by showing the non existence results when $\lambda_p(\mathcal{M} + \beta(x)) > 0$.

Case $\lambda_p(\mathcal{M} + \beta(x)) > 0$:

In this situation we argue as follows. Assume by contradiction that a positive bounded solution u exists. By assumption, u satisfies

$$\mathcal{M}[u](x) + \beta(x)u(x) \geq 0. \tag{5.10}$$

Therefore u is a test function for $\lambda'_p(\mathcal{M} + \beta(x))$ and we get $\lambda'_p(\mathcal{M} + \beta(x)) \leq 0$. Since by Lemma 4.3.11 $\lambda_p(\mathcal{M} + \beta(x)) \leq \lambda'_p(\mathcal{M} + \beta(x))$ we get a straightforward contradiction.

Case $\lambda_p(\mathcal{M} + \beta(x)) = 0$:

In this situation, as above we argue by contradiction. Assume by contradiction that a non-negative, non identically zero, bounded solution u exists. By a straightforward application of the maximum principle, since $u \not\equiv 0$ we have $u > 0$ in $\mathbb{R}^N$. Now let us observe that in this situation, by the above argumentation we have $\lambda_p(\mathcal{M} + \beta(x)) = 0 = \lambda'_p(\mathcal{M} + \beta(x))$ and by (iv) of Proposition (4.2.2) we get the following estimate

$$\sup_{\mathbb{R}^N}(\beta(x) - 1) \leq 0. \tag{5.11}$$

Let us denote $\gamma(x) := \frac{f(x,u(x))}{u(x)}$, then we obviously have

$$J \star u(x) - u(x) + \gamma(x)u(x) = 0 \quad \text{in} \quad \mathbb{R}^N \tag{5.12}$$

Therefore by definition of λ_p' we have $\lambda_p'(\mathcal{M} + \gamma(x)) \leq 0$. By construction $\gamma(x) \leq \beta(x)$, so by combining (5.12) with Proposition 4.2.2, Lemma 4.3.11 and the definition of $\lambda_p(\mathcal{M} + \gamma(x))$ we can infer that

$$\lambda_p(\mathcal{M} + \gamma(x)) \leq \lambda_p'(\mathcal{M} + \gamma(x)) \leq 0 \leq \lambda_p(\mathcal{M} + \beta(x)) \leq \lambda_p(\mathcal{M} + \gamma(x)).$$

Therefore $\lambda_p(\mathcal{M} + \gamma(x)) = 0$. Let us denote $\eta \in C(\mathbb{R}^N)$ be a smooth regularisation of $\chi_{B_1(0)}$ the characteristic function of the unit ball. Since $\gamma(x) < \beta(x)$ in $\mathbb{R}^N$, we can find $\varepsilon_0 > 0$ small so that for all $\varepsilon \leq \varepsilon_0$

$$\gamma(x) < \gamma(x) + \varepsilon\eta(x) < \beta(x) \quad \text{in} \quad \mathbb{R}^N.$$

By (i) of Proposition 4.2.2, we then have

$$0 = \lambda_p(\mathcal{M} + \beta) \leq \lambda_p(\mathcal{M} + \gamma + \varepsilon\eta) \leq \lambda_p(\mathcal{M} + \gamma) = 0.$$

Now we claim that

Claim 5.2.2. There exists $R_1 > 0$ and $\psi > 0$, $\psi \in C(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ so that

$$J \star \psi(x) - \psi(x) + (\gamma(x) + \varepsilon\eta(x))\psi(x) = 0 \quad \text{in} \quad \mathbb{R}^N$$

Assume for a moment that the claim holds then by arguing as in the uniqueness subsection, since $\psi \in L^1$ we get the following contradiction

$$0 = -\varepsilon \int_{\mathbb{R}^N} u(x)\psi(x)\eta(x)\, dx < 0.$$

Proof of the Claim. For convenience we denote $\tilde{\gamma}(x) := \gamma + \varepsilon\eta$. By (5.11), since $\tilde{\gamma} < \beta$ we also have

$$0 < -\sup_{\mathbb{R}^N}(\tilde{\gamma} - 1). \tag{5.13}$$

From the later inequality, by using Proposition 4.2.1 and Lemma 4.3.6 we see that there exists R_0 so that for all $R \geq R_0$ there exists a positive $\varphi_R \in C(\bar{B}(0,R))$ associated to the principal eigenvalue $\lambda_p(\mathcal{L}_R + \tilde{\gamma}(x))$ of the approximated problem

$$\mathcal{L}_R[\varphi] + (\tilde{\gamma}(x) + \lambda)\varphi = 0 \quad \text{in} \quad B(0,R) \tag{5.14}$$

Take now the increasing sequence $(R_n)_{n\in\mathbb{N}} := (R_0 + n)_{n\in\mathbb{N}}$ and let $(\varphi_n)_{n\in\mathbb{N}}$ be the sequence of function where φ_n is the positive principal eigenfunction associated to $\lambda_p(\mathcal{L}_{R_n} + \tilde{\gamma}(x))$. Without loss of generality, we can assume that for all n, $\varphi_n(0) = 1$. Since $J(0) > 0$ and is compactly supported we can reproduce the arguments developed in Claim 4.3.7 and extract from the sequence $(\varphi_n)_{n\in N}$ a subsequence still denoted $(\varphi_n)_{n\in N}$ which converges locally uniformly to a non negative continuous function φ. Moreover, φ satisfies

$$\mathcal{M}[\varphi] + \tilde{\gamma}(x)\varphi = 0 \quad \text{in} \quad \mathbb{R}^N \tag{5.15}$$

To conclude the proof of this claim, we characterise the behaviour of $\varphi(x)$ for $|x| >> 1$.
Let us denote $0 < \nu < -\limsup_{|x|\to\infty} \beta(x)$ and fix R_1 so that $\beta(x) \leq -\frac{\nu}{2}$ for $|x| \geq R_1$.

Since by Lemma 4.3.6 $\lambda_p(\mathcal{L}_R+\tilde{\gamma}(x)) \to \lambda_p(\mathcal{M}+\tilde{\gamma}(x))=0$, we can take R_1 larger if necessary to achieve

$$\tilde{\gamma}(x)+\lambda_p(\mathcal{L}_R+\tilde{\gamma}(x)) \leq -\frac{\nu}{4} \quad \text{for} \quad |x|\geq R_1.$$

Now let us consider $\psi(x) := Ce^{-\alpha(|x|-R_1)}$ where C and α will be chosen later on. By a straightforward computation, we can see that for all $R > R_1$

$$\begin{aligned}\mathcal{L}_R[\psi](x)+(\tilde{\gamma}(x)+\lambda_p(\mathcal{L}_R+\tilde{\gamma}(x)))\psi(x) &\leq \psi(x)\left(\int_{\mathbb{R}^N} J(z)e^{\alpha|z|}dz-1-\frac{\nu}{4}\right) \quad \text{for} \quad |x|\geq R_1\\ &\leq h(\alpha)\psi(x) \quad \text{for} \quad |x|\geq R_1\end{aligned}$$

with

$$h(\alpha) := \left(\int_{\mathbb{R}^N} J(z)e^{\alpha|z|}dz-1-\frac{\nu}{4}\right).$$

Since J is compactly supported, by the Lebesgue Theorem, the function h is continuous and $h(0)=-\frac{\nu}{4}$. By assumption $\nu>0$, so by continuity of h there exists $\alpha_0>0$ so that $h(\alpha_0)<0$. Thus we achieve for $\alpha=\alpha_0$

$$\mathcal{L}_R[\psi](x)+(\tilde{\gamma}(x)+\lambda_p(\mathcal{L}_R+\tilde{\gamma}(x)))\psi(x)\leq 0 \quad \text{for} \quad |x|\geq R_1. \tag{5.16}$$

Recall that by construction the function φ_n satisfies

$$\mathcal{L}_{R_n}[\varphi_n](x)+(\tilde{\gamma}(x)+\lambda_p(\mathcal{L}_{R_n}+\tilde{\gamma}(x)))\varphi_n(x)=0 \quad \text{in} \quad B_{R_n}(0). \tag{5.17}$$

Since J is compactly supported and $J(0)>0$ there exists positive constants $r_0\geq r_1; M\geq m$ so that

$$M\chi_{B_{r_0}(x)} \geq J(x-y)\geq m\chi_{B_{r_1}(x)} \quad \text{for all } x,y\in\mathbb{R}^N.$$

Therefore for n large enough say $n\geq n_0$, we have $R_n > R_1+r_0$ and by the Harnack inequality, for all $n\geq n_0$ we have

$$\varphi_n(x)\leq C(B_{R_1},\lambda_p(\mathcal{L}_{R_n}+\tilde{\gamma}(x)))\varphi_n(y) \quad \text{for all} \quad x,y\in B_{R_1}(0).$$

The constant $C(B_{R_1},\lambda_p(\mathcal{L}_{R_n}+\tilde{\gamma}(x)))$ only depends on $\bigcup_{x\in B_{R_1}} B_{r_0}(x)$ and is monotone decreasing with respect to $\inf_{x\in B_{R_n}}(\tilde{\gamma}(x)+\lambda_p(\mathcal{L}_{R_n}+\tilde{\gamma}(x)))$. For all $n\geq n_0$, the function $\tilde{\gamma}(x)+\lambda_p(\mathcal{L}_{R_n}+\tilde{\gamma}(x))$ being uniformly bounded from below by a constant independent of n, the constant $C(B_{R_1},\lambda_p(\mathcal{L}_{R_n}+\tilde{\gamma}(x)))$ is bounded from above independently of n by a constant $C(B_{R_1})$. Thus we have for all $n\geq n_0$

$$\varphi_n(x)\leq C(B_{R_1})\varphi_n(y) \quad \forall \quad x,y\in B_{R_1}.$$

In particular, we have for all $n\geq n_0$,

$$\varphi_n(x)\leq C(B_{R_1})\varphi_n(0)=C(B_{R_1}) \quad \forall \quad x\in B_{R_1}.$$

By choosing $C>C(B_{R_1})$, we achieve

$$\psi(x)\geq C>C(B_{R_1})\geq\varphi_n(x) \quad \forall \quad x\in B_{R_1}.$$

Set now $w_n:=\psi-\varphi_n$, from (5.16) and (5.17) we get

$$\mathcal{L}_{R_n}[w_n](x)+(\tilde{\gamma}(x)+\lambda_p(\mathcal{L}_{R_n}+\tilde{\gamma}(x)))w_n(x)\leq 0 \quad \text{for} \quad R_1\leq|x|<R_n, \tag{5.18}$$

$$w_n>0 \quad \text{for} \quad |x|<R_1. \tag{5.19}$$

By a straightforward application of the Maximum principle, it follows that for all $n \geq n_0$ we have $\varphi_n(x) \leq \psi$. Indeed, since w_n is continuous, then it achieves a minimum at some point $x_0 \in B_{R_n}$. Assume by contradiction that $w_n(x_0) < 0$, then thanks to (5.19) $x_0 \in\in B_{R_n} \setminus B_{R_1}$ and at this point by (5.18) we have the following contradiction

$$0 \geq \mathcal{L}_{R_n}[w_n](x_0) + (\tilde{\gamma}(x_0) + \lambda_p(\mathcal{L}_{R_n} + \tilde{\gamma}(x)))w_n(x_0) \geq \int_{B_{R_n}} J(x_0 - y)w_n(y)\,dy - w_n(x_0) + \frac{\nu}{4}|w_n(x_0)|,$$
$$\geq \int_{B_{R_n}} J(x_0 - y)[w_n(y) - w_n(x_0)]\,dy + \frac{\nu}{4}|w_n(x_0)| > 0.$$

Hence, for all $n \geq n_0$ $\varphi_n \leq \psi$ in B_{R_n} which by sending $n \to \infty$ leads to $\varphi \leq \psi$ in $\mathbb{R}^N$ which concludes the proof of the Claim. □

5.3 Long time Behaviour

In this section, we investigate the long-time behaviour of the positive solution $u(t,x)$ of

$$\frac{\partial u}{\partial t}(t,x) = J \star u(t,x) - u(t,x) + f(x, u(t,x)) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N, \tag{5.20}$$
$$u(0,x) = u_0(x). \tag{5.21}$$

For any $u_0 \in C^k(\mathbb{R}^N) \cap L^\infty$ or in $C^k(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, we know that the existence of a solution $u(t,x) \in C^1((0,+\infty), C^{\min\{1,k\}}(\mathbb{R}^N))$ respectively $u(t,x) \in C^1((0,+\infty), C^{\min\{1,k\}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))$ is a straightforward consequence of the Cauchy-Lipschitz Theorem and of the KPP structure of the nonlinearity f. Before going to the proof of the asymptotic behaviour, let us recall some useful results

Lemma 5.3.1. *Assume that $u_0(x)$ is a sub-solution to (5.20), then the solution $u(t,x)$ is increasing in time. Conversely, if $u_0(x)$ is a super-solution to (5.20) then $u(t,x)$ is decreasing in time.*

The proof of this Lemma follows from a straightforward used of the parabolic maximum principle and is let to reader. Let us now prove the asymptotic behaviour of the solution of (5.20) and end the proof of Theorem 5.1.1.

Proof. Let $z(t,x)$ be the solution to

$$\frac{\partial z}{\partial t} = J \star z - z + f(x, z(t,x)) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \tag{5.22}$$
$$z(0,x) = C\|u_0\|_\infty \tag{5.23}$$

Since $S(x) \in L^\infty$ by choosing C large enough, the constant $C\|u_0\|_\infty$ is a super-solution to (5.20) therefore $z(t,x)$ is a decreasing function and by the parabolic maximum principle we have $u(t,x) \leq z(t,x)$ for all $(t,x) \in [0,+\infty) \times \mathbb{R}^N$, leading to

$$\limsup_{t\to\infty} u(t,x) \leq \limsup_{t\to\infty} z(t,x) \quad \text{for all} \quad x \in \mathbb{R}^N. \tag{5.24}$$

Now let us consider the approximated parabolic problem

$$\frac{\partial v_R}{\partial t}(t,x) = \int_{B_R(0)} J(x-y)v_R(t,y)\,dy - v_R(t,x) + f(x,v_R(t,x)) \quad \text{in} \quad \mathbb{R}^+ \times B_R(0), \tag{5.25}$$

$$v_R(0,x) = \eta_R u_0(x), \tag{5.26}$$

where $\eta_R := \eta\left(\frac{|x|}{R}\right)$ with $\eta \in C(\mathbb{R}^+)$ a smooth cut-off function so that $\eta \geq 0$, $\eta \equiv 1$ in $[0,1]$ and $\eta \equiv 0$ in $\mathbb{R}^+ \setminus [0,2]$. By Theorem 4.2.3, for R large enough the solution $v_R(t,x)$ converges to $u_R(x)$ the unique positive stationary solution of (5.25). By construction since $u(t,x)$ is a super-solution of the problem (5.25), by the parabolic comparison principle we have for all R large enough $v_R(t,x) \leq u(t,x)$ for all $(t,x) \in [0,+\infty) \times B_R(0)$. Therefore we have for all R large enough

$$\liminf_{t\to\infty} u(t,x) \geq u_R(x) \quad \text{for all} \quad x \in B_R(0) \tag{5.27}$$

By taking the limit as $R \to \infty$, in the above inequality we get

$$\liminf_{t\to\infty} u(t,x) \geq \lim_{R\to\infty} u_R(x) = \tilde{u}(x) \quad \text{for all} \quad x \in \mathbb{R}^N \tag{5.28}$$

Note that we can reproduce the above arguments with $z(t,x)$, thus we also get

$$v_R(t,x) \leq z(t,x) \quad \text{for all} \quad (t,x) \in [0,+\infty) \times B_R(0) \tag{5.29}$$

$$\liminf_{t\to\infty} z(t,x) \geq \lim_{R\to\infty} u_R(x) = \tilde{u}(x) \quad \text{for all} \quad x \in \mathbb{R}^N \tag{5.30}$$

By (5.29) $z(t,x)$ is locally uniformly bounded from below and since $z(t,x)$ is a decreasing function of t we get $\lim_{t\to\infty} z(t,x) = \bar{z}(x) > 0$ for all $x \in \mathbb{R}^N$. Moreover $\bar{z}$ is a bounded stationary solution to (5.20). By uniqueness of the positive stationary solution, we conclude that $\bar{z} = \tilde{u}$. Thus we have

$$\lim_{t\to\infty} z(t,x) = \tilde{u}(x) \quad \text{for all} \quad x \in \mathbb{R}^N \tag{5.31}$$

Hence by collecting (5.24),(5.28), (5.31) we get for all $x \in \mathbb{R}^N$

$$\tilde{u}(x) \leq \liminf_{t\to\infty} u(t,x) \leq \limsup_{t\to\infty} u(t,x) \leq \limsup_{t\to\infty} z(t,x) = \lim_{t\to\infty} z(t,x) = \tilde{u}(x).$$

Now, to complete the proof we are left to show that $\|u - \tilde{u}\|_\infty \to 0$ as $t \to \infty$. To this end we follow the argument in [18]. We argue by contradiction and assume there exists $\varepsilon > 0$ and the sequences $(t_n) \in \mathbb{R}^+$, $(x_n) \in \mathbb{R}^N$ such that

$$\lim_{n\to\infty} t_n = \infty, \qquad |u(t_n,x_n) - \tilde{u}(x_n)| > \varepsilon, \qquad \forall n \in \mathbb{N}. \tag{5.32}$$

By (5.31), we already know that $u \to \tilde{u}$ locally uniformly in $\mathbb{R}^N$, so without loss of generality, we can assume that $|x_n| \to \infty$. From the construction of $\tilde{u}$, Subsection 5.2.1, we have $\lim_{|x|\to\infty} \tilde{u}(x) = 0$. Therefore for some $R_0 > 0$, we have $\tilde{u}(x) \leq \frac{\varepsilon}{2}$ for all $|x| \geq R_0$. The latter combined with (5.31) and (5.32) enforces

$$z(t_n,x_n) - \tilde{u}(x_n) \geq u(t_n,x_n) - \tilde{u}(x_n) > \varepsilon, \qquad \forall n \in \mathbb{N}. \tag{5.33}$$

We claim that

Claim 5.3.2. For all sequences $(t_n)_{n\in\mathbb{N}}, (x_n)_{n\in\mathbb{N}}$ so that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} |x_n| = +\infty$, then $z(t_n, x_n) \to 0$.

Assume for the moment the claim holds true. Then we obtain a straightforward contradiction

$$0 = \lim_{n\to\infty} z(t_n, x_n) - \tilde{u}(x_n) \geq \lim_{n\to\infty} u(t_n, x_n) - \tilde{u}(x_n) > \varepsilon.$$

Let us prove the Claim. Again we argue by contradiction and assume there exists $\varepsilon > 0$ and sequences $(t_n)_{n\in\mathbb{N}}, (x_n)_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty} t_n = \lim_{n\to\infty} |x_n| = \infty$ so that $z(t_n, x_n) > \varepsilon$ for all $n \in \mathbb{N}$. Let us define $z_n(t, x) := z(t, x + x_n)$ then by definition z_n satisfies

$$\begin{aligned}
&\frac{\partial z_n}{\partial t}(t,x) = \int_{\mathbb{R}} J(x-y) z_n(y)\, dy - z_n(t,x) + f(x+x_n, z_n(t,x)) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N,\\
&z_n(0,x) = C\|u_0\|_\infty,
\end{aligned}$$

and $0 < z_n(t, x) < C\|u_0\|_\infty$. Since for all n, $z_n(0, x) \in C^\infty$ by the Cauchy Lipschitz Theorem we see that $z_n \in C^1(\mathbb{R}^+, C^1(\mathbb{R}^N))$. Thus, there exists $C_0 > 0$ independent of n so that $\|z_n\|_{C^{1,1}(\mathbb{R}^+, C(\mathbb{R}^N))} < C_0$. From these estimates, the sequence $(z_n)_{n\in\mathbb{N}}$ is uniformly bounded in $C^{1,1}((0,T), C^{0,1}(\mathbb{R}^N))$ for any $T > 0$. By a diagonal extraction, there exists a subsequence of $(z_n)_{n\in\mathbb{N}}$ that converges locally uniformly to $\tilde{z}(t, x)$. Moreover, thanks to $\lim_{|x|\to\infty} \frac{f(x,s)}{s} < 0$, there exists $\kappa > 0$ so that $\tilde{z}(x, t)$ satisfies

$$\frac{\partial \tilde{z}}{\partial t}(t,x) \leq \int_{\mathbb{R}} J(x-y)\tilde{z}(t,y)\, dy - \tilde{z}(t,x) - \kappa \tilde{z}(t,x)) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N, \tag{5.34}$$

$$\tilde{z}(0,x) = C\|u_0\|_\infty. \tag{5.35}$$

In addition, for all $t > 0$, $\tilde{z}(t, 0) = \lim_{n\to\infty} z_n(t, 0) \geq \varepsilon$. Since $\tilde{z}(0, x)$ is a super-solution of (5.34), by Lemma 5.3.1 the function $\tilde{z}(t, x)$ is monotone decreasing in time. By sending $t \to \infty$, since $\tilde{z} \geq 0$, $\tilde{z}$ converges locally uniformly to a non-negative function $\bar{z}$ that satisfies

$$\begin{aligned}
&\int_{\mathbb{R}} J(x-y)\bar{z}(y)\, dy - \bar{z}(x) - \kappa\bar{z}(x)) \geq 0 \quad \text{in} \quad \mathbb{R}^N,\\
&0 \leq \bar{z} \leq C\|u_0\|_\infty,\\
&\bar{z}(0) \geq \varepsilon.
\end{aligned}$$

Now let consider the function $w(x) := \frac{\varepsilon}{2} e^{\alpha|x|} - \bar{z}$ with α to be chosen, then w satisfies

$$\int_{\mathbb{R}} J(x-y)w(y)\, dy - w(x) - \kappa w(x) \leq \rho e^{\alpha|x|} \left(\int_{\mathbb{R}^N} J(z) e^{\alpha|z|}\, dy - 1 - \kappa \right) \quad \text{in} \quad \mathbb{R}^N.$$

The left hand side of the inequality is well defined and continuous with respect to α since J is compactly supported. Thanks to $\int_{\mathbb{R}^N} J(z)dz = 1$, by choosing α small enough, we achieve

$$\int_{\mathbb{R}} J(x-y)w(y)\, dy - w(x) - \kappa w(x) < 0 \quad \text{in} \quad \mathbb{R}^N.$$

By construction, since $\bar{z}$ is bounded $\lim_{|x|\to\infty} w(x) = +\infty$ and w achieves a minimum in $\mathbb{R}^N$ says at x_0. Since $w(0) = \frac{\varepsilon}{2} - \bar{z}(0) \leq -\frac{\varepsilon}{2}$, we have $w(x_0) < 0$. Now at this point, we get the following contradiction

$$0 < \int_{\mathbb{R}} J(x_0 - y)[w(y) - w(x_0)]\, dy - \kappa w(x_0) < 0 \quad \text{in} \quad \mathbb{R}^N.$$

□

Finally we establish the long time behaviour of the solution $u(t,x)$ starting from an integrable initial datum u_0,i.e $u_0 \in L^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$. To do so, we define two auxiliary functions $h(t,x)$ and $v(t,x)$ that are respectively solution to

$$\begin{cases} \frac{\partial h}{\partial t}(t,x) = J \star h(t,x) - h(t,x) + f(x,h(t,x)) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N, \\ h(0,x) = \sup\{\tilde{u}(x), u_0(x)\}, \end{cases} \tag{5.36}$$

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) = J \star v(t,x) - v(t,x) + f(x,v(t,x)) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N, \\ v(0,x) = \inf\{\tilde{u}(x), u_0(x)\}. \end{cases} \tag{5.37}$$

By construction, from the comparison principle we deduce that $v(t,x) \leq u(t,x) \leq h(t,x)$ for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N$. Therefore

$$\|u - \tilde{u}\|_{L^1(\mathbb{R}^N)} \leq \sup\{\|h - \tilde{u}\|_{L^1(\mathbb{R}^N)}, \|v - \tilde{u}\|_{L^1(\mathbb{R}^N)}\}.$$

Thus to prove that $\|u - \tilde{u}\|_{L^1(\mathbb{R}^N)} \to 0$ it is enough to show that h and v converge to $\tilde{u}$ in $L^1(\mathbb{R}^N)$.

Let us show that v converges to $\tilde{u}$ in $L^1(\mathbb{R}^N)$. Since $\tilde{u}(x)$ is a super solution to (5.37) we deduce $v(t,x) \leq \tilde{u}(x)$ for all $x \in \mathbb{R}^N$. Let $\varepsilon > 0$ be fixed and choose R so that $\int_{\mathbb{R}^N \setminus B(0,R)} \tilde{u}(x)\,dx \leq \frac{\varepsilon}{4}$ then we have

$$\begin{aligned} \|\tilde{u} - v\|_{L^1(\mathbb{R}^N)} &= \int_{\mathbb{R}^N \setminus B_R(0)} (\tilde{u}(x) - v(t,x))\,dx + \int_{B_R(0)} (\tilde{u}(x) - v(t,x))\,dx, \\ &\leq 2\int_{\mathbb{R}^N \setminus B_R(0)} \tilde{u}(x)\,dx + \int_{B_R(0)} (\tilde{u}(x) - v(t,x))\,dx, \\ &\leq \frac{\varepsilon}{2} + \int_{B_R(0)} (\tilde{u}(x) - v(t,x))\,dx. \end{aligned}$$

Recall that v converges pointwise to $\tilde{u}$ as t tends to infinity. Therefore, by Lebesgue Theorem for some $t(\varepsilon)$ we get for all $t \geq t(\varepsilon)$, $\int_{B_R(0)} (\tilde{u}(x) - v(t,x))\,dx \leq \frac{\varepsilon}{2}$ which enforces

$$\|\tilde{u} - v\|_{L^1(\mathbb{R}^N)} \leq \varepsilon.$$

ε being chosen arbitrary, the latter inequality shows that $\lim_{t\to\infty} \|\tilde{u} - v\|_{L^1(\mathbb{R}^N)} = 0$ which proves that v converges to $\tilde{u}$ in $L^1(\mathbb{R}^N)$.

To obtain that $\|h - \tilde{u}\|_{L^1(\mathbb{R}^N)} \to 0$ we argue as follow. By construction $\tilde{u}$ is a sub solution to (5.36), thus $\tilde{u}(x) \leq h(t,x)$ for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N$. Let us denote $w(t,x) := h(t,x) - \tilde{u}(x)$. Then w satisfies for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N$:

$$\begin{aligned} \frac{\partial w}{\partial t}(t,x) &= J \star w(t,x) - w(t,x) + \left(\frac{f(x,h(t,x))}{h(t,x)} - \frac{f(x,\tilde{u}(x))}{\tilde{u}}\right) h(t,x) + \frac{f(x,\tilde{u}(x))}{\tilde{u}} w(t,x), \\ &\leq J \star w(t,x) - w(t,x) + \frac{f(x,\tilde{u}(x))}{\tilde{u}} w(t,x). \end{aligned}$$

Now thanks to $\lim_{|x|\to\infty} \frac{f(x,s)}{s} < 0$, there exists $\kappa > 0$ and R_0 so that w satisfies

$$\frac{\partial w}{\partial t}(t,x) \leq J \star w(t,x) - w(t,x) - \kappa w(t,x) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N \setminus B_{R_0}(0). \tag{5.38}$$

Fix now $\varepsilon > 0$. Recall that $h(t,x)$ converges pointwise to $\tilde{u}$, then by Lebesgue Theorem there exists t_0 so that for all $t \geq t_0$,

$$\int_{B_{R_0}(0)} w(t,x)\,dx \leq \kappa\varepsilon.$$

Now let us estimate $\int_{\mathbb{R}^N\setminus B_{R_0}(0)} w(x)\,dx$ for $t \geq t_0$. By integrating (5.38) over $\mathbb{R}^N \setminus B_{R_0}(0)$ it yields

$$\frac{\partial \int_{\mathbb{R}^N\setminus B_{R_0}(0)} w(t,x)\,dx}{\partial t} \leq \int_{\mathbb{R}^N\setminus B_{R_0}(0)} J\star w(t,x)\,dx - \int_{\mathbb{R}^N\setminus B_{R_0}(0)} w(t,x)\,dx - \kappa\int_{\mathbb{R}^N\setminus B_{R_0}(0)} w(t,x)\,dx.$$

By using Fubini's Theorem, the uniform estimate on $\|w\|_\infty$ and the unit mass of the kernel, we can check that for $t \geq t_0$

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_{R_0}(0)} J\star w(t,x)\,dx &= \int_{\mathbb{R}^N\setminus B_{R_0}(0)} w(t,y)\left(\int_{\mathbb{R}^N\setminus B_{R_0}(0)} J(x-y)\,dx\right)dy + \int_{B_{R_0}(0)} w(t,y)\left(\int_{\mathbb{R}^N\setminus B_{R_0}(0)} J(x-y)\,d\right.\\ &\leq \int_{\mathbb{R}^N\setminus B_{R_0}(0)} w(t,y)\,dy + \int_{B_{R_0}(0)} w(t,y)\,dy,\\ &\leq \int_{\mathbb{R}^N\setminus B_{R_0}(0)} w(t,y)\,dy + \kappa\varepsilon.\end{aligned}$$

Therefore for $t \geq t_0$, w satisfies

$$\frac{\partial \int_{\mathbb{R}^N\setminus B_{R_0}(0)} w(t,x)\,dx}{\partial t} \leq \kappa\varepsilon - \kappa\int_{\mathbb{R}^N\setminus B_{R_0}(0)} w(t,x)\,dx.$$

From the later differential inequality, there exists $t(\varepsilon) \geq t_0$ so that for all $t \geq t(\varepsilon)$ we have

$$\int_{\mathbb{R}^N\setminus B_{R_0}(0)} w(t,x)\,dx \leq 2\varepsilon.$$

Hence, we have for all $t \geq t(\varepsilon)$

$$\|w\|_{L^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N\setminus B_{R_0}(0)} w(t,x)\,dx + \int_{B_{R_0}(0)} w(t,x)\,dx \leq \left(2 + \frac{\kappa}{|B_{R_0}(0)|}\right)\varepsilon,$$

As above, ε being chosen arbitrary, the latter inequality shows that $\lim_{t\to\infty}\|w\|_{L^1(\mathbb{R}^N)} = 0$ which proves that h converges to $\tilde{u}$ in $L^1(\mathbb{R}^N)$.

5.4 Some asymptotics

In this section we analyse the qualitative behaviour of the solution of (4.2) with respect to the size of the support of J. For convenience we investigate the particular situation

$$\frac{1}{\varepsilon^m}(J_\varepsilon \star u - u) + u(a(x) - u) = 0 \quad \text{in} \quad \mathbb{R}^N \tag{P_ε}$$

where $J_\varepsilon(z) = \frac{1}{\varepsilon^N}J\left(\frac{z}{\varepsilon}\right)$ with $supp(J) = B(0,1)$ and $a \in C^1(\mathbb{R}^N)$ so that $a^+ \not\equiv 0$.

The later condition on a is necessary to observe the possible existence of a solution. Indeed, if $a^+ \equiv 0$ then for any positive constant c_0 we have

$$\mathcal{M}[c_0] + a(x)c_0 \leq 0$$

therefore $\lambda_p(\mathcal{M}[c_0] + a(x)c_0) \geq 0$ and for all ε there is no solution to (P_ε) besides 0.

We look for the behaviour of u_ε when $\varepsilon \to 0$ and $\to +\infty$ and try to understand the influence of m on the resulting limits.

We start by showing some *a priori* estimate for the solution u_ε.

Lemma 5.4.1. *There exist positive constants C_1, C_2, C_3 so that we have for any positive bounded solution u_ε of (P_ε)*

(i) $\|u_\varepsilon\|_{L^2(\mathbb{R}^N)} \leq \|a^+(x)\|_{L^2(\mathbb{R}^N)} \leq C_1, \quad \|u_\varepsilon\|_\infty < C_3,$

(ii) $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J_\varepsilon(x-y)(u_\varepsilon(x) - u_\varepsilon(y))^2\,dxdy \leq C_2\varepsilon^m$

(iii) $\sup_{supp(a^+)} u_\varepsilon \geq -\frac{\lambda_p(\mathcal{M}_{\varepsilon,m}+a(x))}{2}.$

(iv) $u_\varepsilon \geq (a(x) - \frac{1}{\varepsilon^m})^+,$

Proof. Since by construction the solution is unique and $u_\varepsilon \in L^1(\mathbb{R}^N) \cap L^\infty$. Moreover by (P_ε) $u_\varepsilon \leq M = \|a\|_\infty$. We obtain (i) by integrating (P_ε) over $\mathbb{R}^N$. We get, by Cauchy-Schwart inequalities that

$$\int_{\mathbb{R}^N} u_\varepsilon^2(x)\,dx = \int_{\mathbb{R}^N} a(x)u_\varepsilon(x)\,dx \leq \int_{\mathbb{R}^N} a^+(x)u_\varepsilon(x)\,dx \leq \sqrt{\int_{\mathbb{R}^N}[a^+(x)]^2dx}\sqrt{\int_{\mathbb{R}^N}[u_\varepsilon(x)]^2dx}.$$

Therefore,

$$\|u_\varepsilon\|_{L^2(\mathbb{R}^N)} \leq \|a^+(x)\|_{L^2(\mathbb{R}^N)} \leq C_1.$$

To obtain (ii), let us multiply (P_ε) by u_ε and integrate over $\mathbb{R}^N$, then we get

$$\frac{1}{2\varepsilon^m}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u_\varepsilon(x) - u_\varepsilon(y))^2\,dxdy = \int_{\mathbb{R}^N} u_\varepsilon^2(x)(a(x) - u_\varepsilon(x))\,dx$$

Since u_ε and $a(x)$ are uniformly bounded independently, (ii) holds true with $C_2 := 4C_1M$. Observe that $(a(x) - \frac{1}{\varepsilon^m})^+$ is always a sub-solution to (P_ε), so by a standard sweeping principle $u_\varepsilon \geq (a(x) - \frac{1}{\varepsilon^m})^+$ and (iv) holds true. Finally to obtain (iii) we argue as follows. Since u_ε is a positive bounded solution of (P_ε) by Theorem 5.1.1 we have $\lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)) < 0$. Now since $\lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)) < 0$ and J is compactly supported, by regularising a if necessary, we can find (see the proof of Lemma 4.3.8) $\varphi_\varepsilon \in C_c(\mathbb{R}^N)$ so that

$$\mathcal{M}_{\varepsilon,m}[\varphi_\varepsilon] + a(x) + \frac{\lambda_p}{2}\varphi_\varepsilon \geq 0 \quad \text{in} \quad \mathbb{R}^N.$$

Moreover, we can normalised φ_ε so that $\|\varphi_\varepsilon\|_\infty = 1$. Plugging $\theta\varphi_\varepsilon$ with in (P_ε) it follows that

$$\mathcal{M}_{\varepsilon,m}[\theta\varphi_\varepsilon] + \theta\varphi_\varepsilon(a(x) - \theta\varphi_\varepsilon) \geq \theta\varphi_\varepsilon(-\frac{\lambda_p}{2} - \theta\varphi_\varepsilon).$$

Therefore for $0 < \theta \leq -\frac{\lambda_p}{2}$, the function $\theta\varphi_\varepsilon$ is a sub-solution to (P_ε). By a standard sweeping argument, we get

$$-\frac{\lambda_p}{2}\varphi_\varepsilon \leq u_\varepsilon \quad \text{and} \quad \sup_{\mathbb{R}^N} u_\varepsilon \geq -\frac{\lambda_p}{2}.$$

Since $u_\varepsilon \in L^1(\mathbb{R}^N)$, u_ε achieves its maximum at some point, says x_0. At this point from (P_ε) we have

$$0 \geq \mathcal{M}_{\varepsilon,m}[u_\varepsilon](x_0) = -u_\varepsilon(x_0)(a(x_0) - u_\varepsilon(x_0)).$$

Thus $x_0 \in supp(a^+)$ and $\|u_\varepsilon\|_\infty = \sup_{supp(a^+)} u_\varepsilon$ which proves (iii). □

Next we obtain derive some useful super-solution for large ε.

Lemma 5.4.2. *There exists $\varepsilon_0 > 0$ so that for all $m \geq 0$ and $\varepsilon \geq \varepsilon_0$ any positive bounded solution u_ε of (P_ε) satisfies*

$$u_\varepsilon \leq a^+(x) + \frac{1}{\varepsilon^{\frac{N}{4}}}$$

Proof. Let $\delta \in (0, \frac{N}{2})$ and consider the function $\zeta_\varepsilon(x) := \frac{1}{\varepsilon^{\frac{N}{2}-\delta}} + a^+(x)$. We will show that ζ_ε is a super-solution to (P_ε) when $\varepsilon >> 1$.

Indeed, we have

$$\begin{aligned}\mathcal{M}_{\varepsilon,m}[\zeta_\varepsilon](x) + \zeta_\varepsilon(x)(a(x) - \zeta_\varepsilon(x)) &\leq \frac{\|J\|_\infty}{\varepsilon^{N+m}} \int_{\mathbb{R}^N} a^+(y)\,dy + \left(\frac{1}{\varepsilon^{\frac{N}{2}-\delta}} + a^+(x)\right)\left[a(x) - \frac{1}{\varepsilon^{\frac{N}{2}-\delta}} - a^+(x)\right] \\ &\leq \frac{\|J\|_\infty}{\varepsilon^{N+m}} \int_{\mathbb{R}^N} a^+(y)\,dy - \frac{1}{\varepsilon^{N-2\delta}}.\end{aligned}$$

where we use in the last inequality that

$$\left(\frac{1}{\varepsilon^{\frac{N}{2}-\delta}} + a^+(x)\right)\left[a(x) - \frac{1}{\varepsilon^{\frac{N}{2}-\delta}} - a^+(x)\right] \leq -\frac{1}{\varepsilon^{N-2\delta}} \quad \text{for all} \quad x \in \mathbb{R}^N.$$

Thus for $\varepsilon >> 1$, we achieve

$$\mathcal{M}[\zeta_\varepsilon](x) + \zeta_\varepsilon(x)(a(x) - \zeta_\varepsilon(x)) \leq \frac{\|J\|_\infty}{\varepsilon^{N+m}} \int_{\mathbb{R}^N} a^+(y)\,dy - \frac{1}{\varepsilon^{N-2\delta}} < 0.$$

Therefore for $\varepsilon >> 1$, by a sweeping argument we get $u_\varepsilon \leq \zeta_\varepsilon$. We end the proof by taking $\delta = \frac{N}{4}$.

□

Remark 19. When $m = 0$ and $(a(x) - 1)^+ \not\equiv 0$, the above computation holds as well with $\zeta_\varepsilon(x) := \frac{1}{\varepsilon^{\frac{N}{2}-\delta}} + (a(x) - 1)^+$. Thus in this case we have for large ε

$$u_\varepsilon(x) \leq \frac{1}{\varepsilon^{\frac{N}{4}}} + (a(x) - 1)^+.$$

Next, we prove some continuity of $\lambda_p(\mathcal{L}_{R,\varepsilon} + a(x))$ with respect to ε.

Lemma 5.4.3. *Let R, ε be fixed and positive then for all $\eta > 0$ there exists $\delta > 0$ so that*

$$|\lambda_p(\mathcal{L}_R + a_\varepsilon(x)) - \lambda_p(\mathcal{L}_R + a_{\varepsilon+\delta}(x))| \leq \eta,$$

where $a_\varepsilon(x) := a(\varepsilon x)$.

Let us prove now the continuity of $\lambda_p(\mathcal{L}_R + a_\varepsilon(x))$.

Proof of the Claim. Let $\varepsilon > 0$ and $R > 0$ be fixed. We observe that for all $|\delta| < \varepsilon$ we have for all $x \in \mathbb{R}^N, a_{\varepsilon+\delta}(x) = a_\varepsilon\left(\frac{\varepsilon+\delta}{\varepsilon}x\right)$ therefore

$$\|a_\varepsilon - a_{\varepsilon+\delta}\|_{\infty,R} = \sup_{B(0,R)} \left\| a_\varepsilon(x) - a_\varepsilon\left(\frac{\varepsilon+\delta}{\varepsilon}x\right)\right\|.$$

Since a_ε is a Lipschitz in $\mathbb{R}^N$, we have

$$\left\| a_\varepsilon(x) - a_\varepsilon\left(\frac{\varepsilon+\delta}{\varepsilon}x\right)\right\| \leq K(\varepsilon)\varepsilon\delta\|x\|,$$

where $K(\varepsilon)$ is the Lipschitz constant of a_ε. Thus

$$\|a_\varepsilon - a_{\varepsilon+\delta}\|_{\infty,R} \leq K(\varepsilon)R\varepsilon\delta.$$

Hence, by (ii) of Proposition 4.2.2 we get

$$|\lambda_p(\mathcal{L}_R + a_\varepsilon(x)) - \lambda_p(\mathcal{L}_R + a_{\varepsilon+\delta}(x))| \leq K(\varepsilon)R\varepsilon\delta.$$

□

Finally, we establish a useful identity, which can be seen as a Green-formula for nonlocal operators.

Proposition 5.4.4. *Let $\rho \in C_c^\infty(\mathbb{R}^N)$ be a radial function, then for all $u \in L^2(\mathbb{R}^N), \varphi \in C_c^\infty(\mathbb{R}^N)$ we have*

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)[u(x+z) - u(x)]\varphi(x)\,dzdx = \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)u(x)[\varphi(x+z) - 2\varphi(x) + \varphi(x-z)]\,dzdx.$$

Proof. Thanks to the symmetry of ρ, using standard changes variables we have

$$\begin{aligned}
&\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)[u(x+z) - u(x)]\varphi(x) = \\
&\qquad = \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)[u(x+z) - u(x)]\varphi(x) + \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(-z)[u(x-z) - u(x)]\varphi(x),\\
&\qquad = \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)[u(x+z) - u(x)]\varphi(x) + \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)[u(x) - u(x+z)]\varphi(x+z),\\
&\qquad = -\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)[u(x+z) - u(x)][\varphi(x+z) - \varphi(x)],\\
&\qquad = -\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)u(x)[\varphi(x) - \varphi(x-z)] + \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)u(x)[\varphi(x+z) - \varphi(x)],\\
&\qquad = \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(z)u(x)[\varphi(x+z) - 2\varphi(x) + \varphi(x-z)].
\end{aligned}$$

□

From the Proposition, for all $u \in L^2(\mathbb{R}^N), \varphi \in C_c^\infty(\mathbb{R}^N)$ we straightforwardly get the following identity

$$\int_{\mathbb{R}^N} \mathcal{M}_{\varepsilon,m}[u](x)\varphi(x)\,dx = \frac{\varepsilon^{2-m}D_2(J)}{2} \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\rho_\varepsilon(z)}{|z|^2} u_\varepsilon(x)[\varphi(x+z) - 2\varphi(x) + \varphi(x-z)]\,dxdz \tag{5.39}$$

where $\rho_\varepsilon(z) = \frac{1}{\varepsilon^N D_2(J)} J\left(\frac{z}{\varepsilon}\right)\frac{|z|^2}{\varepsilon^2}$.

Equipped with all these *apriori* estimates, we can now analyse the asymptotic behaviour of u_ε.

5.4.1 The case $m = 0$

In this situation, from Theorem 4.4.1 we know that

$$\lim_{\varepsilon\to 0} \lambda_p(\mathcal{M}_\varepsilon + a(x)) = -\sup_{\mathbb{R}^N} a(x) \tag{5.40}$$

$$\lim_{\varepsilon\to +\infty} \lambda_p(\mathcal{M}_\varepsilon + a(x)) = 1 - \sup_{\mathbb{R}^N} a(x) \tag{5.41}$$

As a consequence for ε small enough we have $\lambda_p(\mathcal{M}_\varepsilon + a(x)) \le -\frac{\sup_{\mathbb{R}^N} a(x)}{2} < 0$ and by Theorem (5.1.1) there exists a solution to (P_ε). Moreover the following quantity is well defined

$$\varepsilon^* := \sup\{\varepsilon > 0 \,|\, \text{ for all } \varepsilon' < \varepsilon, \text{ there exists a positive solution to } (P_{\varepsilon'})\}.$$

In view of (5.41) $\varepsilon^* \in (0, +\infty]$ and $\varepsilon^* < +\infty$ **if and only if** $(a(x) - 1)^+ \not\equiv 0$.

Let us now deal with the limit of u_ε as $\varepsilon \to 0$ and $\varepsilon \to +\infty$ and let us start by proving that

$$\lim_{\varepsilon\to 0} u_\varepsilon(x) = v(x) \quad \text{a.e.} \tag{5.42}$$

where v is a non negative bounded solution of

$$v(x)(a(x) - v(x)) = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{5.43}$$

Let $w_\varepsilon := a(x) - u_\varepsilon$, then from (P_ε), w_ε satisfy

$$-J_\varepsilon \star w_\varepsilon + w_\varepsilon + u_\varepsilon(x)w_\varepsilon(x) = a(x) - J_\varepsilon \star a(x). \tag{5.44}$$

Multiplying the above equation by w_ε^+ and integrating over $\mathbb{R}^N$, it follows that

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)((w_\varepsilon^+)^2(x) - w_\varepsilon(y)w_\varepsilon^+(x))\,dxdy + \int_{\mathbb{R}^N} u_\varepsilon(x)(w_\varepsilon^+)^2(x) = \int_{\mathbb{R}^N} w_\varepsilon^+(x)g_\varepsilon(x)\,dx$$

with $g_\varepsilon(x) := a(x) - J_\varepsilon \star a(x)$.

Let us now estimate the above integrals. First we observe that the double integral is positive. Indeed, since $w(y) = w^+(y) - w^-(x)$ we get

$$\begin{aligned}\iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)((w_\varepsilon^+)^2(x) - w_\varepsilon(y)w_\varepsilon^+(x))\,dxdy &= \iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)((w_\varepsilon^+)^2(x) - w_\varepsilon^+(y)w_\varepsilon^+(x))\,dxdy \\ &\quad + \iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)w_\varepsilon^-(y)w_\varepsilon^+(x)\,dxdy\end{aligned}$$

Thus

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)((w_\varepsilon^+)^2(x)-w_\varepsilon(y)w_\varepsilon^+(x))\,dxdy=\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)((w_\varepsilon^+)(x)-w_\varepsilon^+(y))^2\,dxdy + \iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)w_\varepsilon^-(y)w_\varepsilon^+(x)\,dxdy \tag{5.45}$$

Let us denote $Q:=supp(a^+)$. Since u_ε is positive and uniformly bounded, we have $supp(w^+)\subset Q$ and

$$\left|\int_{\mathbb{R}^N} w_\varepsilon^+(x)g_\varepsilon(x)\,dx\right|\le C\int_Q |g_\varepsilon|(x)\,dx.$$

Since a is Lipschitz, by using a Taylor expansion, we can see that $|g_\varepsilon(x)|\le\varepsilon D_2(J)\|\nabla a\|_\infty$. Therefore

$$\left|\int_{\mathbb{R}^N} w_\varepsilon^+(x)g_\varepsilon(x)\,dx\right|\le C|Q|\varepsilon. \tag{5.46}$$

Collecting (5.45),(5.46), we get

$$\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)((w_\varepsilon^+)(x)-w_\varepsilon^+(y))^2\,dxdy+\iint_{\mathbb{R}^N\times\mathbb{R}^N} J_\varepsilon(x-y)w_\varepsilon^-(y)w_\varepsilon^+(x)\,dxdy+\int_{\mathbb{R}^N} u_\varepsilon(w_\varepsilon^+)^2(x)\,dx\le C\varepsilon.$$

Thus

$$\int_{\mathbb{R}^N} u_\varepsilon(w_\varepsilon^+)^2(x)\,dx\le C\varepsilon$$

and $u_\varepsilon w_\varepsilon^+(x)\to 0$ almost everywhere in Q.

Recall that

$$\int_{\mathbb{R}^N} u_\varepsilon w_\varepsilon(x)\,dx=0$$

then from above estimates we conclude that

$$\int_{\mathbb{R}^N\setminus Q} u_\varepsilon(a(x)-u_\varepsilon)=\int_Q u_\varepsilon(a(x)-u_\varepsilon)\to 0\quad\text{when}\quad\varepsilon\to 0.$$

Since $u_\varepsilon(a(x)-u_\varepsilon)\le 0$ in $\mathbb{R}^N\setminus Q$, from the above inequality it follows that $u_\varepsilon(x)w_\varepsilon(x)\to 0$ almost everywhere in $\mathbb{R}^N\setminus Q$. Since $u_\varepsilon>0$ and $w_\varepsilon=(a(x)-u_\varepsilon(x))\le 0$, it follows that $u_\varepsilon(x)\to 0$ almost everywhere in $\mathbb{R}^N\setminus Q$. Thus u_ε converges pointwise almost everywhere to a bounded non-neqative solution of (5.43).

Remark 20. Note that the above proof can be easily adapted to $\mathcal{M}_{\varepsilon,m}$ for $m<2$ as soon as the function a is smooth enough. Indeed, for $a\in C^2(\mathbb{R}^N)$, following the above arguments, we get by using the Taylor expansion up to order 2 of a

$$\int_{\mathbb{R}^N} u_\varepsilon(w_\varepsilon^+)^2(x)\,dx\le C\varepsilon^{2-m},$$

with C a constant depending on $\|\nabla^2 u\|_\infty$. When a is only Lipschitz, the above argument holds only for $\mathcal{M}_{\varepsilon,m}$ with $m<1$.

Finally, to complete our analysis, we need to check that

$$\lim_{\varepsilon\to\varepsilon^*} u_\varepsilon = (a(x)-1)^+. \tag{5.47}$$

We treat separately the following two cases : (i) $\varepsilon^* < +\infty$, (ii) $\varepsilon^* = \infty$. The latter arise when $\sup_{\mathbb{R}^N}(a(x)-1) > 0$. In this situation, there exists $R_0 > 0$ so that the continuous function $\varphi = (a(x)-1)^+ \not\equiv 0$ in $B_R(0)$ for $R \geq R_0$ and we can check that φ is a sub-solution to the approximated problem:

$$\int_{B_R(0)} J_\varepsilon(x-y)u(y)\,dy - u(x) + u(x)(a(x)-u) = 0 \quad \text{in} \quad B_R(0). \tag{5.48}$$

Since large constants are super-solutions of (5.48) for any $\varepsilon \geq 0, R > R_0$ there exists a unique solution $\varphi \leq u_{\varepsilon,R} \leq M$. By sending $R \to \infty$ and by the uniqueness of the solution to (P_ε) we have $\varphi \leq u_\varepsilon \leq M$ in $\mathbb{R}^N$.

Case $\varepsilon^* = +\infty$:

Thanks to Lemma 5.4.1 and Remark 19, for all $x \in \mathbb{R}^N$ for large ε we have

$$(a(x)-1)^+ \leq u_\varepsilon(x) \leq (a(x)-1)^+ + \frac{1}{\varepsilon^{\frac{N}{4}}}.$$

Hence, u_ε converge uniformly to $(a(x)-1)^+$.

Case $\varepsilon^* < +\infty$:

In this situation, the function $(a(x)-1)^+ \equiv 0$ in $\mathbb{R}^N$ and we are reduce to prove that

$$\lim_{\varepsilon\to\varepsilon^*} u_\varepsilon(x) = 0 \quad \text{for all} \quad x \in \mathbb{R}^N.$$

Note that by definition of ε^* we must have $\lambda_p(\mathcal{M}_{\varepsilon^*} + a(x)) \geq 0$. Indeed, if not then $\lambda_p(\mathcal{M}_{\varepsilon^*} + a(x)) < 0$ and by Lemma 4.3.12 $\lambda_p(\mathcal{M} + a_{\varepsilon^*}(x)) < 0$. Therefore for some R we have $\lambda_p(\mathcal{L}_R + a_{\varepsilon^*}(x)) < 0$. By continuity of $\lambda_p(\mathcal{L}_R + a_{\varepsilon^*}(x))$ with respect to ε, (Lemma 5.4.3) we get for some $\delta_0 > 0$, $\lambda_p(\mathcal{L}_R + a_{\varepsilon^*+\delta}(x)) < 0$ for any $\delta \leq \delta_0$. Hence, $\lambda_p(\mathcal{M}_{\varepsilon^*+\delta} + a(x)) = \lambda_p(\mathcal{M} + a_{\varepsilon^*+\delta}(x)) < 0$ for any $\delta \leq \delta_0$ and by Theorem 5.1.1 there exists a positive solution to (P_ε) for all $\varepsilon \leq \varepsilon^* + \delta_0$ contradicting the definition of ε^*.

Note also that since $\varepsilon^* < +\infty$, the construction of the supersolution in Section 5.2 holds for any $\varepsilon \in [\frac{\varepsilon^*}{2}, \varepsilon^*]$, thus u_ε is uniformly bounded in $L^1(\mathbb{R}^N)$.

Let $g(x,s) := s(a(x)-1-s)$ then for all ε we have

$$J_\varepsilon \star u_\varepsilon = -g(x, u_\varepsilon(x)) \quad \text{in} \quad \mathbb{R}^N.$$

Now since J is C^1 and ε^* then for $\varepsilon \in [\frac{1}{2}\varepsilon^*, \varepsilon^*)$, we have

$$\begin{aligned}
|g(x,u_\varepsilon(x)) - g(z,u_\varepsilon(z))| &= \left| \int_{\mathbb{R}^N} [J_\varepsilon(x-y) - J(z-y)]u_\varepsilon(y)\,dy \right|, \\
&\leq |x-z| \int_{\mathbb{R}^N} \frac{|J_\varepsilon(x-y) - J(z-y)|}{|x-z|} u_\varepsilon(y)\,dy, \\
&\leq C(\varepsilon^*)|x-z|.
\end{aligned}$$

Therefore, we get

$$\begin{aligned} C(\varepsilon^*)|x-z| &\geq |[1-a(x)+u_\varepsilon(x)+u_\varepsilon(z)][u_\varepsilon(x)-u_\varepsilon(z)]+[a(z)-a(x)]u_\varepsilon(x)|, \\ &\geq |[1-a(x)+u_\varepsilon(x)+u_\varepsilon(z)]||u_\varepsilon(x)-u_\varepsilon(z)|-|x-y|\frac{|a(z)-a(x)|}{|x-z|}M, \end{aligned}$$

and for any $x \in Q := \{y \in \mathbb{R}^N \,|\, a(y) < 1\}$ u_ε is uniformly Lipschitz in x with a constant independent of ε. Thus $(u_\varepsilon)_{\varepsilon\in[\frac{1}{2}\varepsilon^*,\varepsilon^*)}$ is uniformly bounded in $C^{0,\frac{1}{2}}_{loc}(Q)$. If $Q^c = \emptyset$, then $(u_\varepsilon)_{\varepsilon\in[\frac{1}{2}\varepsilon^*,\varepsilon^*)}$ is uniformly bounded in $C^{0,\frac{1}{2}}_{loc}(\mathbb{R}^N)$. Otherwise, $Q^c \neq \emptyset$ and on Q^c $a(x) \equiv 1$. Therefore $u_\varepsilon^2(x) = J_\varepsilon \star u_\varepsilon$ and the $C^{0,\frac{1}{2}}(Q^c)$ norm of u_ε is bounded independently of ε. Hence,

$$(u_\varepsilon)_{\varepsilon\in[\frac{1}{2}\varepsilon^*,\varepsilon^*)} \quad \text{is uniformly bounded in} \quad C^{0,\frac{1}{2}}_{loc}(Q)\cap C^{0,\frac{1}{2}}(Q^c). \tag{5.49}$$

In both case, since $a(x) < 0$ for $|x| >> 1$, Q^c is a compact set and $|\bar{Q}\cap Q^c| = 0$. From (5.49), for all sequence $\varepsilon_n \to \varepsilon^*$ by a diagonal extraction procedure there exists a subsequence still denoted $(u_{\varepsilon_n})_{n\in\mathbb{N}}$ that converges locally uniformly in $R^N \setminus (\bar{Q}\cap Q^c)$ to some non-negative function v. By passing to the limit in (P_ε) we can see that v is a bounded non negative solution of

$$J_{\varepsilon^*} \star v(x) - v(x) + v(x)(a(x)-v(x)) = 0 \quad \text{in} \quad \mathbb{R}^N \setminus (\bar{Q}\cap Q^c).$$

Since $\bar{Q}\cap Q^c$ is of zero measure v is a solution to

$$\mathcal{L}_{\varepsilon^*,\mathbb{R}^N\setminus(\bar{Q}\cap Q^c)}[v] + v(x)(a(x)-v(x)) = 0 \quad \text{in} \quad \mathbb{R}^N \setminus (\bar{Q}\cap Q^c).$$

Since $0 \leq \lambda_p(\mathcal{M}_{\varepsilon^*} + a(x)) \leq \lambda_p(\mathcal{L}_{\varepsilon^*,\mathbb{R}^N\setminus(\bar{Q}\cap Q^c)} + a(x))$, we deduce that $v \equiv 0$ which proves the limit.

Remark 21. When $a(x)$ is a radially symmetric non-increasing function we remark that ε^* is a sharp threshold. That is for all $\varepsilon \geq \varepsilon^*$ then (P_ε) does not have any positive solutions. Indeed in this situation the function $a_\varepsilon(x)$ is monotone non increasing with respect to ε. Thus by (i) of Proposition 4.2.2, for all $\varepsilon \geq \varepsilon^*$ we have

$$0 = \lambda_p(\mathcal{M} + a_{\varepsilon^*}(x)) \leq \lambda_p(\mathcal{M} + a_\varepsilon(x)).$$

Hence, by Theorem 5.1.1, 0 is the unique non negative solution to (P_ε) for $\varepsilon \geq \varepsilon^*$.

5.4.2 The case $0 < m < 2$

In this situation, from Theorem 4.4.1 we know that

$$\lim_{\varepsilon\to 0} \lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)) = -\sup_{\mathbb{R}^N} a(x) \tag{5.50}$$

$$\lim_{\varepsilon\to +\infty} \lambda_p(\mathcal{M}_{\varepsilon} + a(x)) = -\sup_{\mathbb{R}^N} a(x) \tag{5.51}$$

As a consequence for ε small enough and for large ε we have $\lambda_p(\mathcal{M}_\varepsilon + a(x)) \leq -\frac{\sup_{\mathbb{R}^N} a(x)}{2} < 0$. Therefore by Theorem (5.1.1) there exists a solution to (P_ε) for small and large ε. The limit of u_ε when $\varepsilon \to \infty$ is easy to obtain. Indeed it is a straightforward consequence of (iv) of Lemma 5.4.1 and Lemma 5.4.2 since we have for ε large

$$(a(x) - \frac{1}{\varepsilon^m})^+ \leq u_\varepsilon \leq a^+(x) + \frac{1}{\varepsilon^{\frac{N}{4}}}.$$

To obtain the limits in L^2, we just observe that since by Lemma u_ε is uniformly bounded in L^2 and converges pointwise to a^+, we get $u_\varepsilon \rightharpoonup a^+$ in L^2. Moreover by Fatou's Lemma, we have

$$\int_{\mathbb{R}^N} (a^+)^2(x)\,dx \leq \liminf_{\varepsilon\to\infty} \int_{\mathbb{R}^N} u_\varepsilon^2(x)\,dx.$$

On another hand, by (i) Lemma (5.4.1), we have

$$\|u_\varepsilon\|_{L^2(\mathbb{R}^N)} \leq \|a^+(x)\|_{L^2(\mathbb{R}^N)}.$$

It follows that

$$\int_{\mathbb{R}^N} (a^+)^2(x)\,dx \leq \liminf_{\varepsilon\to\infty} \int_{\mathbb{R}^N} u_\varepsilon^2(x)\,dx \leq \limsup_{\varepsilon\to+\infty} \int_{\mathbb{R}^N} u_\varepsilon^2(x)\,dx \leq \int_{\mathbb{R}^N} (a^+)^2(x)\,dx$$

Hence, $\|u_\varepsilon\|_2 \to \|a^+\|_2$ and by the parallelogram identity $u_\varepsilon \to a^+$ in $L^2(\mathbb{R}^N)$ since u_ε converges weakly to a^+ in L^2.

5.4.3 The case $m = 2$

In this situation, from Theorem 4.4.1 we have

$$\lim_{\varepsilon\to 0} \lambda_p(\mathcal{M}_{\varepsilon,m} + a(x)) = \lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right) \tag{5.52}$$

$$\lim_{\varepsilon\to+\infty} \lambda_p(\mathcal{M}_\varepsilon + a(x)) = -\sup_{\mathbb{R}^N} a(x) \tag{5.53}$$

As a consequence for large ε we have $\lambda_p(\mathcal{M}_\varepsilon + a(x)) \leq -\frac{\sup_{\mathbb{R}^N} a(x)}{2} < 0$ and by Theorem (5.1.1) there exists a solution to (P_ε) for large ε. Whereas the existence of a positive solution for ε small is conditioned to the sign of $\lambda_1(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x))$. When $\lambda_1(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)) > 0$ then for ε small there exists no positive solution to (P_ε). The limit of u_ε when $\varepsilon \to +\infty$ can be obtain as in the case $2 > m > 0$ so we focus only on the limit when $\varepsilon \to 0$.

Assume for the moment that $\lambda_1(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)) < 0$, we will show that $u_\varepsilon \to v$ where v is the positive solution to

$$\frac{K_{2,N}D_2(J)}{2}\Delta v + v(a(x) - v) = 0 \quad \text{in} \quad \mathbb{R}^N.$$

Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive number converging to 0 and let u_n denote u_{ε_n}. By Lemma 5.4.1, $\|u_n\|_2$ is bounded uniformly and after simple algebraic computation

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho_n(z)\frac{(u_n(x+z) - u_n(x))^2}{|z|^2}\,dxdz < C,$$

with C independent of ε. Therefore for any $R > 0$, we have

$$\iint_{B_R\times B_R} \rho_\varepsilon(z)\frac{(u_n(x+z) - u_n(x))^2}{|z|^2}\,dxdz < C.$$

For $R > 0$ fixed, since $\|u_n\|_2$ is uniformly there exists a subsequence $u_n \rightharpoonup v$ in $L^2(B_R)$ and from the characterisation of Sobolev Space [86, 87], we have $u_n \to v$ in $L^2(B_R)$.

By a standard diagonal extraction argument, from the sequence $(u_n)_{n\in\mathbb{N}}$ we can then extract a subsequence still denoted $(u_n)_{n\in\mathbb{N}}$ which converges to some v in $L^2_{loc}(\mathbb{R}^N)$. Moreover by Lemma 5.4.1 u_n is uniformly bounded and there exists $\delta(\lambda_1)>0$ independent of ε so that $\max_{supp(a^+)}(u_n)>\delta$.

Multiplying (P_ε) by $\varphi\in C_c^\infty(\mathbb{R}^N)$ and integrating we get

$$\frac{D_2(J)}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{\rho_n(z)}{|z|^2}u_n(x)[\varphi(x+z)-2\varphi(x)+\varphi(x-z)]\,dxdz+\int_{\mathbb{R}^N}\varphi(x)u_n(x)(a(x)-u_n(x))\,dx=0,$$

where we use (5.39) to compute $\int_{\mathbb{R}^N}\mathcal{M}_{\varepsilon,2}[u_n](x)\varphi(x)\,dx$. Thus we get

$$\begin{aligned}\frac{D_2(J)}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{\rho_n(z)}{|z|^2}u_n(x)\,{}^tz\nabla^2\varphi(x)z\,dxdz+\int_{\mathbb{R}^N}\varphi(x)u_n(x)(a(x)-u_n(x))\,dx\\ =-\frac{D_2(J)}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{\rho_n(z)}{|z|^2}u_n(x)[\varphi(x+z)-2\varphi(x)+\varphi(x-z)-{}^tz\nabla^2\varphi(x)z]\,dxdz,\end{aligned}$$

where $\nabla^2\varphi(x):=(\partial_{ij}\varphi(x))_{i,j}$. Since $\rho_n(z)$ is radial, we can see that

$$\frac{D_2(J)}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{\rho_n(z)}{|z|^2}u_n(x)\,{}^tz\nabla^2\varphi(x)z\,dxdz=\frac{D_2(J)K_{2,N}}{2}\int_{\mathbb{R}^N}u_n(x)\Delta\varphi(x)\,dx$$

and we get

$$\begin{aligned}\frac{D_2(J)K_{2,N}}{2}\int_{\mathbb{R}^N}u_n(x)\Delta\varphi(x)\,dx+\int_{\mathbb{R}^N}\varphi(x)u_n(x)(a(x)-u_n(x))\,dx\\ =-\frac{D_2(J)}{2}\iint_{\mathbb{R}^N}\frac{\rho_n(z)}{|z|^2}u_n(x)[\varphi(x+z)-2\varphi(x)+\varphi(x-z)-{}^tz\nabla^2\varphi(x)z]\,dxdz,\end{aligned}\tag{5.54}$$

Note that since u_n converges to v in $L^2_{loc}(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N}\varphi(x)u_n(x)(a(x)-u_n(x))\,dx\to\int_{\mathbb{R}^N}\varphi(x)v(x)(a(x)-v(x))\,dx\tag{5.55}$$

Recall now that $\varphi\in C_c^\infty(\mathbb{R}^N)$, so there exists $C(\varphi)$ and $R(\varphi)$ so that

$$|\varphi(x+z)-2\varphi(x)+\varphi(x-z)-{}^tz\nabla^2\varphi(x)z|<C(\varphi)|z|^3\chi_{B_{R(\varphi)}}(x).$$

Therefore since u_n is bounded uniformly,

$$\frac{D_2(J)}{2}\iint_{\mathbb{R}^N}\frac{\rho_\varepsilon(z)}{|z|^2}u_n(x)[\varphi(x+z)-2\varphi(x)+\varphi(x-z)-{}^tz\nabla^2\varphi(x)z]\,dxdz\le CC(\varphi)\int_{\mathbb{R}^N}\rho_n(z)|z|\to 0.\tag{5.56}$$

Passing to the limit $\varepsilon\to 0$ in (5.54), thanks to (5.55) and (5.56) we get

$$\frac{D_2(J)K_{2,N}}{2}\int_{\mathbb{R}^N}v(x)\Delta\varphi(x)\,dx+\int_{\mathbb{R}^N}\varphi(x)v(x)(a(x)-v(x))\,dx=0.\tag{5.57}$$

(5.57) being true for any $\varphi\in C_c^\infty$ this implies that v satisfies

$$\frac{K_{2,N}D_2(J)}{2}\Delta v + v(a(x)-v) = 0 \quad \text{a.e. in} \quad \mathbb{R}^N.$$

Since v is bounded, by elliptic regularity v is smooth. To conclude we need to prove that v is non trivial. To do so we claim that

Claim 5.4.5. There exists R_0, τ and ε_0 positive constants so that for all $\varepsilon \leq \varepsilon_0$ we have $u_\varepsilon \geq \tau$ almost everywhere in $B_{R_0}(0)$.

From the above claim, we deduce that $v \geq \tau > 0$ *a.e.* and therefore $v \equiv u$, the unique smooth non-trivial solution of

$$\frac{K_{2,N}D_2(J)}{2}\Delta u + u(a(x)-u) = 0 \quad \text{in} \quad \mathbb{R}^N.$$

The sequence $(\varepsilon_n)_n$ being arbitrary, it follows that $u_\varepsilon \to u$ in $L^2_{loc}(\mathbb{R}^N)$.

Similarly, if we assume now that $\lambda_1(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)) = 0$ and there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}, \varepsilon_n \to 0$ of non trivial solution of (P_ε). The above argumentation then holds true and we get $u_n \to v$ in $L^2_{loc}(\mathbb{R}^N)$ with v a smooth solution to

$$\frac{K_{2,N}D_2(J)}{2}\Delta v + v(a(x)-v) = 0 \quad \text{in} \quad \mathbb{R}^N.$$

Since $\lambda_1(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)) = 0$, $v \equiv 0$ is the only solution and we get $u_n \to 0$ in $L^2_{loc}(\mathbb{R}^N)$.

Let us complete our proof and establish the claim.

Proof. Let us denote $\mathcal{L}_{R,\varepsilon}$ the operator

$$\mathcal{L}_{R,\varepsilon}[\varphi] := \frac{1}{\varepsilon^2}\left[\int_{B_R(0)} J_\varepsilon(x-y)\varphi(y)\,dy - \varphi(x)\right].$$

Since $sup_{\mathbb{R}^N} a(x)$ is achieve in $\mathbb{R}^N$ we regularise a by a_σ independently of ε, so that for all ε and $R \geq R_1$ the principal eigenvalue $\lambda_p(\mathcal{L}_{R,\varepsilon} + a_\tau(x))$ is associated to a continuous principal eigenfunction $\varphi_{p,\varepsilon}$ and

$$|\lambda_p(\mathcal{L}_{R,\varepsilon} + a_\tau(x)) - \lambda_p(\mathcal{L}_{R,\varepsilon} + a(x))| \leq \|a_\sigma(x) - a(x)\|_\infty \leq \kappa\sigma,$$

with κ the Lipschitz constant of a.

By the Lipschitz continuity of $\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right)$ with respect to a, we can choose σ small enough so that

$$\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a_\sigma(x)\right) \leq \frac{1}{2}\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right) < 0.$$

Recall that

$$\lim_{R\to\infty} \lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a_\sigma(x), B_R\right) = \lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right),$$

So we can choose R_0 large so that

$$\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a_\sigma(x), B_{R_0}\right) \leq \frac{1}{4}\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta + a(x)\right).$$

Thanks to Remark 17, we have $\lim_{\varepsilon\to 0}\lambda_p(\mathcal{L}_{R,\varepsilon}+a_\sigma(x))=\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta+a_\sigma(x),B_R\right)$ so for ε small,say $\varepsilon\le\varepsilon_0$ by choosing σ smaller if necessary, we achieve

$$\lambda_p(\mathcal{L}_{R,\varepsilon}+a_\sigma(x))\le\frac{1}{8}\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta+a(x)\right)\quad\text{for all}\quad\varepsilon\le\varepsilon_0.$$

Let $\varphi_{p,\varepsilon}$ be the principal eigenfunction associated with $\mathcal{L}_{R_0,\varepsilon}+a_\sigma(x)$, then we have

$$\mathcal{L}_{R_0,\varepsilon}[\varphi_{p,\varepsilon}](x)+a(x)\varphi_p(x)\ge\left[-\frac{1}{8}\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta+a(x)\right)-\kappa\sigma\right]\varphi_{p,\varepsilon}(x)\quad\text{for all}\quad\varepsilon\le\varepsilon_0.$$

By choosing σ smaller if necessary,

$$\left[-\frac{1}{8}\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta+a(x)\right)-\kappa\sigma\right]\ge-\frac{1}{16}\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta+a(x)\right)$$

and we achieve

$$\mathcal{L}_{R_0,\varepsilon}[\varphi_{p,\varepsilon}](x)+a(x)\varphi_p(x)\ge-\frac{1}{16}\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta+a(x)\right)\varphi_{p,\varepsilon}(x)\quad\text{for all}\quad\varepsilon\le\varepsilon_0.\tag{5.58}$$

To conclude our proof, it is then enough to show that for some well chosen normalisation of $\varphi_{p,\varepsilon}$ we have

$$\varphi_{p,\varepsilon}(x)\to\varphi_1(x),\quad\text{a.e. in}\quad B_{R_0}\tag{5.59}$$

φ_1 is a positive principal eigenfunction associated with $\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta+a_\sigma(x),B_{R_0}\right)$. Indeed, assume for the moment that (5.59) holds true. Then there exists $\alpha>0$ so that

$$\alpha\varphi_{p,\varepsilon}(x)\to\alpha\varphi_1(x)<\frac{1}{2}\quad\text{a.e. in}\quad B_{R_0}.$$

Now thanks to (5.58), we can now adapt the proof the proof of (iii) of Lemma 5.4.1 to get for ε small, says $\varepsilon\le\varepsilon_1$,

$$u_\varepsilon(x)\ge-\frac{\alpha}{32}\lambda_1\left(\frac{K_{2,N}D_2(J)}{2}\Delta+a(x)\right)\varphi_{p,\varepsilon}(x)\quad\text{a.e. in}\quad B_{R_0},\tag{5.60}$$

which combined with (5.59) enforces

$$u_\varepsilon(x)\ge\gamma\varphi_1(x)\quad\text{a.e. in}\quad B_{R_0},\quad\text{for all}\quad\varepsilon\le\varepsilon_2,$$

for some $\gamma,\varepsilon_2>0$.

Since $\varphi_1>0$ in B_{R_0}, the claim holds true in any smaller ball B_R.

To prove (5.59), let us normalise $\varphi_{p,\varepsilon}$ by $\|\varphi_{p,\varepsilon}\|_{L^2(B_{R_0})}=1$. Let k_ε be the function

$$k_\varepsilon(x):=\frac{1}{\varepsilon^2}\int_{\mathbb{R}^N\setminus B_{R_0}}J_\varepsilon(x-y)\,dy,$$

then by multiplying by $\varphi_{p,\varepsilon}$ the equation satisfied by $\varphi_{p,\varepsilon}$ and integrating it over B_{R_0}, we get

$$\frac{D_2(J)}{2}\iint_{B_{R_0}\times B_{R_0}}\rho_\varepsilon(x-y)\frac{|\varphi_{p,\varepsilon}(y)-\varphi_{p,\varepsilon}(x)|^2}{|x-y|^2}\,dxdy = \int_{B_{R_0}}(a_\sigma(x)+\lambda_{p,\varepsilon})\varphi_{p,\varepsilon}^2(x)\,dx - \int_{B_{R_0}}k_\varepsilon(x)\varphi_{p,\varepsilon}^2(x)\,dx,$$
$$\leq C.$$

Therefore by the characterisation of Sobolev space [86, 87], along a sequence we have $\varphi_{p,\varepsilon}\to\psi$ in $L^2(B_{R_0})$ with $\|\psi\|_{L^2(B_{R_0})}=1$. Moreover by extending $\varphi_{p,\varepsilon}$ and φ by 0 outside B_{R_0} and by arguing as above for any $\varphi\in C_c^2(B_{R_0})$ we have

$$\frac{D_2(J)}{2}\iint_{B_{R_0}\times\mathbb{R}^N}\frac{\rho_\varepsilon(z)}{|z|^2}\varphi_{p,\varepsilon}(x)[\varphi(x+z)-2\varphi(x)+\varphi(x-z)]\,dxdz = -\int_{B_{R_0}}\varphi(x)\varphi_{p,\varepsilon}(a(x)+\lambda_{p,\varepsilon})\,dx$$
$$+\int_{B_{R_0}}k_\varepsilon(x)\varphi_{p,\varepsilon}(x)\varphi(x)\,dx$$

Since $\varphi\in C_c^2(B_{R_0})$ we get for ε small enough $supp(k_\varepsilon)\cap supp(\varphi)=\emptyset$. Thus passing to the limit along a sequence in the above equation yields

$$\frac{D_2(J)K_{2,N}}{2}\int_{B_{R_0}}\psi(x)\Delta\varphi(x)\,dx+\int_{B_{R_0}}\varphi(x)\psi(x)(a(x)+\lambda_1)\,dx=0. \tag{5.61}$$

(5.61) being true for any φ, it follows that ψ is the smooth positive eigenfunction associated to λ_1 normalised by $\|\psi\|_{L^2(B_{R_0})}=1$. ψ being uniquely defined, we get $\varphi_{p,\varepsilon}\to\psi$ in $L^2(B_{R_0})$ when $\varepsilon\to 0$. Thus along any sequence $\varphi_{p,\varepsilon}(x)\to\varphi_1(x)$ almost everywhere in B_{R_0}.

□

5.5 Extension to non-compactly supported kernels

In this section, we discuss the extension of our survival criteria to more general dispersal kernel J and prove Theorem 5.1.5. Observe that the construction of positive solution only required that $\lambda_p(\mathcal{L}_R+\beta(x))<0$ for some R, no matter the dispersal kernel J is. Therefore as soon as $\lim_{R\to\infty}\lambda_p(\mathcal{L}_R+\beta(x))<0$ there exists a positive solution to (4.2) with no restriction on the decay of the kernel. Similarly, when $\lambda_p(\mathcal{M}+\beta(x))>0$ the proof of the non-existence of positive bounded solution essentially relies on the inequality between $\lambda_p(\mathcal{M}+\beta(x))$ and $\lambda_p'(\mathcal{M}+\beta(x))$ obtained in Lemma 4.3.11 which holds for quite general kernels including those satisfying assumption $(H5)$. Concerning the proof of the uniqueness of the positive solution, it relies on the construction of a integrable uniform super-solution of (4.2) which guarantes the existence of a positive L^1 solution to (4.2). Such super-solution still exists for kernels J that satisfies the decay assumption $H5$. Indeed, we can show

Lemma 5.5.1. *Assume that J satisfies $H5$ and there exists a periodic function $\mu(x):\mathbb{R}^N\to\mathbb{R}$ such that*

$$\limsup_{|x|\to\infty}(\beta(x)-\mu(x))\leq 0 \qquad \textit{and} \qquad \lambda_p(\mathcal{M}+\mu(x))>0.$$

Then there exists $\bar{u}\in C_0(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$, $\bar{u}>0$ so that $\bar{u}$ is a super-solution to (4.2).

Observe that the construction of the super-solution covers a larger class of nonlinearity $f(x,u)$ than those that satisfy $H4$. As a immediate consequence, the survival criteria obtained in Theorem 5.1.1 is still true for nonlinearity that satisfies:

(H7) There exists $\mu(x) \in C_p(\mathbb{R}^N)$ so that : $\begin{cases} \lambda_p(\mathcal{M} + \mu(x)) > 0, \\ \limsup_{|x|\to\infty} \left(\frac{f(x,s)}{s} - \mu(x) \right) \leq 0 \quad \text{uniformly in } s. \end{cases}$

From the ecological point of view, such nonlinearity allows to consider a more complex niche structure for the species. In particular, we can consider ecological niches that are the superposition of a compact niche structure with a periodic structure. The condition being that on the periodic structure alone, the species could not survive. The perspective offers by this approach are quite promising and we believe that it may be applied to investigate a climate change version of (4.2).

Proof. The construction of the super-solution in this situation follows the same scheme as for a compactly supported kernel. By assumption since $\limsup_{|x|\to\infty}(\beta(x) - \mu(x)) \leq 0$, for any $\delta > 0$ there exists $R_\delta > 1$ such that

$$\beta(x) \leq \mu(x) + \delta \qquad |x| \geq R_\delta.$$

Fix $\delta < \lambda_p(\mathcal{M} + \mu(x))$ and observe that by definition of $\lambda_p(\mathcal{M} + \mu(x))$ there exists a constant $\delta < \lambda < \lambda_p(\mathcal{M} + \mu(x))$ and a positive periodic function φ so that

$$\mathcal{M}[\varphi](x) + (\mu(x) + \lambda)\varphi(x) \leq 0 \quad \text{for all} \quad x \in \mathbb{R}^N. \tag{5.62}$$

Let $w = C\frac{\varphi(x)}{1+\tau|x|^{N+1}}$ with C, τ to be chosen.

$$\begin{aligned}
\mathcal{M}[w] + (\mu(x) + \delta)w(x) &= C(1+\tau|x|^{N+1})^{-1} \left(\int_{\mathbb{R}^N} J(x-y) \frac{(1+\tau|x|^{N+1})}{(1+\tau|y|^{N+1})} \varphi(y)\, dy - \varphi(x) + (\mu(x)+\delta)\varphi(x) \right), \\
&\leq C(1+\tau|x|^{N+1})^{-1} \left(\int_{\mathbb{R}^N} J(z) \left[\frac{(1+\tau|x|^{N+1})}{(1+\tau|x+z|^{N+1})} - 1 \right] \varphi(x+z)\, dz + (\delta - \lambda)\varphi(x) \right), \\
&\leq C(1+\tau|x|^{N+1})^{-1} \left(\tau \int_{\mathbb{R}^N} J(z) \left[\frac{|x|^{N+1} - |x+z|^{N+1}}{(1+\tau|x+z|^{N+1})} \right] \varphi(x+z)\, dz + (\delta - \lambda)\varphi(x) \right), \\
&\leq w(x) \left(\tau \int_{\mathbb{R}^N} J(z) \left[\frac{|x|^{N+1} - |x+z|^{N+1}}{(1+\tau|x+z|^{N+1})} \right] \frac{\varphi(x+z)}{\varphi(x)}\, dz + \delta - \lambda \right),
\end{aligned}$$

where we use (5.62) and $inf_{\mathbb{R}^N}\varphi > 0$.

Set

$$h(\tau, x) = \tau \int_{\mathbb{R}^N} J(z) \left[\frac{|x|^{N+1} - |x+z|^{N+1}}{(1+\tau|x+z|^{N+1})} \right] \frac{\varphi(x+z)}{\varphi(x)}\, dz + \delta - \lambda.$$

Thanks to $\varphi \in L^\infty(\mathbb{R}^N), \inf_{\mathbb{R}^N} \varphi > 0$ there exists a positive constant C_0 so that

$$\frac{\varphi(x+z)}{\varphi(x)} \leq C_0 \quad \text{for all} \quad x, z \in \mathbb{R}^N.$$

Thus for all $x \in \mathbb{R}^N$, we have

$$h(\tau, x) \leq C_0 \tau \int_{\mathbb{R}^N} J(z) \left[\frac{|x|^{N+1} - |x+z|^{N+1}}{(1+\tau|x+z|^{N+1})} \right] dz + \delta - \lambda. \tag{5.63}$$

Let

$$I := C_0\tau \int_{\mathbb{R}^N} J(z) \left[\frac{|x|^{N+1} - |x+z|^{N+1}}{(1+\tau|x+z|^{N+1})} \right] dz,$$

then we have

$$I = C_0\tau \int_{\{|x|\leq 2|z|\}} J(z) \left[\frac{|x|^{N+1} - |x+z|^{N+1}}{(1+\tau|x+z|^{N+1})} \right] dz + C_0\tau \int_{\{|x|>2|z|\}} J(z) \left[\frac{|x|^{N+1} - |x+z|^{N+1}}{(1+\tau|x+z|^{N+1})} \right] dz.$$

Let us estimate the first integral. Since $|x| \leq 2|z|$ we have

$$C_0\tau \int_{\{|x|\leq 2|z|\}} J(z) \left[\frac{|x|^{N+1} - |x+z|^{N+1}}{(1+\tau|x+z|^{N+1})} \right] dz \leq C_0\tau 2^{N+1} \int_{\mathbb{R}^N} J(z)|z|^{N+1}\, dz. \tag{5.64}$$

Let us now estimate second term. Since $|x+z|^{N+1} \geq (|x| - |z|)^{N+1}$, we have

$$\begin{aligned} C_0\tau \int_{\{|x|>2|z|\}} J(z) \left[\frac{|x|^{N+1} - |x+z|^{N+1}}{(1+\tau|x+z|^{N+1})} \right] dz \leq \\ \leq C_0 \sum_{i=1}^{N+1} \binom{N+1}{i} \int_{\{|x|>2|z|\}} J(z)(-1)^i |z|^i \left[\frac{\tau|x|^{N+1-i}}{(1+\tau|x+z|^{N+1})} \right] dz, \\ \leq C_0 \sum_{i=1}^{N+1} \binom{N+1}{i} \int_{\{|x|>2|z|\}} J(z)|z|^i \left[\frac{\tau|x|^{N+1-i}}{(1+\tau|x+z|^{N+1})} \right] dz. \end{aligned}$$

Since $|x| > 2|z|$, we have

$$\frac{1}{1+\tau|x+z|^{N+1}} \leq \frac{2^{N+1}}{2^{N+1}+\tau|x|^{N+1}}$$

and for $|x| \geq R_0 > 1$

$$\begin{aligned} C_0\tau \int_{\{|x|>2|z|\}} J(z) \left[\frac{|x|^{N+1} - |x+z|^{N+1}}{(1+\tau|x+z|^{N+1})} \right] dz \leq \\ \leq C_0 2^{N+1} \sum_{i=1}^{N+1} \binom{N+1}{i} \int_{\{|x|>2|z|\}} J(z) \frac{|z|^i}{|x|^i} \left[\frac{\tau|x|^{N+1}}{(2^{N+1}+\tau|x|^{N+1})} \right] dz \\ \leq \frac{C_0 2^{N+1}}{R_0} \sum_{i=1}^{N+1} \binom{N+1}{i} \int_{\mathbb{R}^N} J(z)|z|^i \left[\frac{\tau|x|^{N+1}}{2^{N+1}+\tau|x|^{N+1}} \right] dz. \end{aligned}$$

Since for all $|x|$,

$$\left[\frac{\tau|x|^{N+1}}{2^{N+1}+\tau|x|^{N+1}} \right] < 1,$$

we achieve for $|x| \geq R_0$

$$C_0\tau \int_{\{|x|>2|z|\}} J(z) \left[\frac{|x|^{N+1} - |x+z|^{N+1}}{(1+\tau|x+z|^{N+1})} \right] dz \leq \frac{C_0 2^{N+1}}{R_0} \sum_{i=1}^{N+1} \binom{N+1}{i} \int_{\mathbb{R}^N} J(z)|z|^i\, dz. \tag{5.65}$$

Combining (5.64), (5.65) and (5.63), we get for $|x| > R_0$

$$h(x,\tau) \leq \frac{C_0 2^{N+1}}{R_0} \sum_{i=1}^{N+1} \binom{N+1}{i} \int_{\mathbb{R}^N} J(z)|z|^i \, dz + C_0 \tau 2^{N+1} \int_{\mathbb{R}^N} J(z)|z|^{N+1} \, dz + \delta - \lambda.$$

Thanks to (H5), for τ small enough, says $\tau \leq \tau_1$ and R_0 large enough we achieve $h(x,\tau) \leq \frac{\delta-\lambda}{2} < 0$, Hence, for all $\tau \leq \tau_1$, we have

$$\mathcal{M}[w] + (\mu(x) + \delta)w(x) \leq w(x)h(x,\tau) \leq w(x)\frac{\delta-\lambda}{2} < 0 \quad \text{for all} \quad x \in \mathbb{R}^N \setminus B_{R_0}. \tag{5.66}$$

Fix now $\tau \leq \tau_1$ and fix $R_0 > R_\delta$ so that $h(x,\tau) < 0$ in $\mathbb{R}^N \setminus B_{R_0}(0)$.

Let $\kappa_0 := \sup_{\mathbb{R}^N \setminus B_{R_0}(0)} \frac{\varphi(x)}{1+\tau|x|^{N+1}}$. Let $0 < \kappa < \kappa_0$ and consider the set

$$\Omega_\kappa := \left\{ x \in \mathbb{R}^N \mid \frac{\varphi(x)}{1+\tau|x|^{N+1}} \leq \kappa \right\}.$$

By construction since $\varphi > 0$ in $\mathbb{R}^N$ we can choose κ small so that

$$\Omega_\kappa \subset \mathbb{R}^N \setminus B_{R_0}(0).$$

Moreover, $\mathbb{R}^N \setminus \Omega_\kappa$ is a bounded domain and $M := \sup_{\mathbb{R}^N \setminus \Omega_\kappa} S(x)$ is well defined. Choose now C so that $C = \frac{2M}{\kappa}$ and consider the continuous function

$$\bar{u}(x) := \begin{cases} C\frac{\varphi(x)}{1+\tau|x|^{N+1}} & \text{in} \quad \Omega_\kappa, \\ C\kappa \quad \text{in} \quad \mathbb{R}^N \setminus \Omega_\kappa. \end{cases}$$

By direct computation we can check that $\bar{u}$ is a super-solution to (4.2). Indeed, for any $x \in \mathbb{R}^N \setminus \Omega_\kappa$, we have $\bar{u} = C\kappa = 2M > \sup_{\mathbb{R}^N \setminus \Omega_\kappa} S(x)$ which implies that $f(x, C\kappa) \leq 0$ and

$$\mathcal{M}[\bar{u}](x) + f(x,\bar{u}(x)) \leq \int_{\mathbb{R}^N} J(x-y)\bar{u}(y) \, dy - C\kappa + f(x,C\kappa) \leq f(x,C\kappa) \leq 0.$$

Whereas, for $x \in \Omega_\kappa$ we have

$$\begin{aligned} \mathcal{M}[\bar{u}](x) + f(x,\bar{u}(x)) \leq \mathcal{M}[\bar{u}](x) + \beta(x)w(x) &\leq \mathcal{M}[w] + (\mu(x)+\delta)w(x), \\ &\leq h(x,\tau)w(x) \leq 0. \end{aligned}$$

□

Bibliographie

[1] P. Bates ; PC Fife ; X. Ren ; X. Wang Traveling waves in a convolution model for phase transitions. *Arch. Rational Mech. Anal* 138 (1997), no. 2, 105–136.

[2] P. Bates ; G. Zhao Existence, uniqueness and stability of the stationary solution to a nonlocal evolution equation arising in population dispersal. *J. Math. Anal. Appl* 332 (2007), no. 1, 428–440.

[3] H. Berestycki. Le nombre de solutions de certains problèmes semi-linéaires elliptiques. *J. Funct. Anal.* 40, 1–29 (1981).

[4] H. Berestycki ; L. A Caffarelli ; L. Nirenberg. Inequalities for second-order elliptic equations with applications to unbounded domains. I. A celebration of John F. Nash, Jr. *Duke Math. J.* 81 (1996), no. 2, 467–494.

[5] H. Berestycki, O. Diekmann, K. Nagelkerke, P. Zegeling. Can a Species Keep Pace with a Shifting Climate ? *Bull. Mathematical Biology* 71, 2008, p. 399.

[6] H. Berestycki, M.J. Esteban. Existence and bifurcation of solutions for an elliptic degenerate problem, *J. Differential Equations* 134 (1) (1997) 1–25.

[7] H. Berestycki ; F. Hamel ; N. Nadirashvili. Elliptic eigenvalue problems with large drift and applications to nonlinear propagation phenomena. *Comm. Math. Phys.* 253 (2005), no. 2, 451–480.

[8] Berestycki, H ; Hamel, F ; Nadirashvili, N. The speed of propagation for KPP type problems. I. Periodic framework. *J. Eur. Math. Soc.* 7 (2005), no. 2, 173–213.

[9] H. Berestycki, F. Hamel, L. Rossi, Liouville type results for semilinear elliptic equations in unbounded domains. *Annali Mat. Pura Appl.*, (4), 186 (2007), pp 469-507

[10] H. Berestycki, F. Hamel, and L. Roques. Analysis of the periodically fragmented environment model. I. Species persistence. *J. Math. Biol.*, 51(1) :75–113, 2005.

[11] H. Berestycki, G. Nadin. Spreading speeds for one-dimensional monostable reaction-diffusion equations. *J. Math. Phys*, 53(11)(2012).

[12] H. Berestycki, L. Nirenberg, and S. R. S. Varadhan. The principal eigenvalue and maximum principle for second-order elliptic operators in general domains. *Comm. Pure Appl. Math.*, 47(1) :47–92, 1994.

[13] H. Berestycki, L. Nirenberg. traveling fronts in cylinders. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 9 (1992), no. 5, 497 ?572

[14] H. Berestycki, L. Nirenberg. Some qualitative properties of solutions of semilinear elliptic equations in cylindrical domains, *Analysis, Et Cetera ed. P. Rabinowitz, et al., Academic Press, Boston*, 1990, pp. 115–164.

[15] H. Berestycki, P.-L. Lions. Nonlinear scalar field equations, I - Existence of a ground state. *Arch.Rational Mech. Anal.* 82, (1983), p.313–345.

[16] H. Berestycki and P.-L. Lions. Some applications of the method of super and subsolutions. *Bifurcation and nonlinear eigenvalue problems.* (Proc., Session, Univ. Paris XIII, Villetaneuse, 1978), Lecture Notes in Math., 782, Springer, Berlin, 1980, 16 ?41.

[17] Berestycki, H. and Rossi, L. On the principal eigenvalue of elliptic operators in $\mathbb{R}^N$ and applications, *J. Eur. Math. Soc. (JEMS)*, 8, 2006, 2, 195–215.

[18] H. Berestycki, L. Rossi. Reaction-diffusion equations for population dynamics with forced speed, I - The case of the whole space. *Discrete and Continuous Dynamical System Series B*, 21, 2008, p. 41-67.

[19] H. Berestycki, L. Rossi. Reaction-diffusion equations for population dynamics with forced speed, II - Cylindrical type domains. *Discrete and Continuous Dynamical Systems Series A*, 25, 2009, p. 19-61.

[20] H. Berestycki, L. Rossi. Generalizations and properties of the principal eigenvalue of elliptic operators in unbounded domains. *Comm. Pure Appl. Math*, to appear.

[21] Berestycki, H and Roquejoffre, J M and Rossi, L. The periodic patch model for population dynamics with fractional diffusion, *Discrete Contin. Dyn. Syst. S*, 4, 2011, 1–13.

[22] J. Bourgain, H. Brezis, P. Mironescu, Another look at Sobolev spaces. *Optimal Control and Partial Differential Equations*, 2001.

[23] H. Brezis. Functional analysis, Sobolev spaces and partial differential equations. *Universitext. Springer, New York*, 2011. xiv+599 pp. ISBN : 978–0–387–70913–0.

[24] H. Brezis, M. Chipot, Y. Xie Some remarks on Liouville type theorems. Recent advances in nonlinear analysis, 43–65, *World Sci. Publ., Hackensack*, NJ, 2008.

[25] Brézis, Haım How to recognize constant functions. Connections with Sobolev spaces, *Russian Mathematical Surveys*, 57, 2002, 693.

[26] A. Cañada ; P. Drábek, J. L. Gámez, *Existence of positive solutions for some problems with nonlinear diffusion*, Trans. Amer. Math. Soc. 349 (1997), 4231-4249.

[27] R.S Cantrell ; C.Cosner Diffusive logistic equations with indefinite weights : population models in disrupted environments. II. *SIAM J. Math. Anal* 22 (1991), no. 4, 1043–1064.

[28] Cantrell,R. S. and Cosner,C. Diffusive logistic equations with indefinite weights : population models in disrupted environments *Proc. Roy. Soc. Edinburgh, Section : A Mathematics*, 112, 1989, 293–318.

[29] RS Cantrell ; C. Cosner ; Y.Lou, Evolutionary stability of ideal free dispersal strategies in patchy environments. *J. Math. Biol* 65 (2012), no. 5, 943–965.

[30] RS. Cantrell ; C. Cosner On the effects of spatial heterogeneity on the persistence of interacting species. *J. Math. Biol* 37 (1998), no. 2, 103–145.

[31] RS. Cantrell, C. Cosner and Y. Lou, Evolutionary stability of ideal free dispersal strategies in patchy environments, *Journal of Mathematical Biology*, 65, 2012, 943-965.

[32] M. L. Cain, B. G. Milligan, A. E. Strand, Long-distance seed dispersal in plant populations *Am. J. Bot.*, 87, 2000, 9, 1217–1227.

[33] J. S. Clark, Why Trees Migrate So Fast : Confronting Theory with Dispersal Biology and the Paleorecord *The American Naturalist* 152, 1998, 204–224.

[34] J S Clark and C Fastie and G Hurtt and S T Jackson and C Johnson and G A King and M Lewis and J Lynch and S Pacala and C Prentice and E W Schupp and T Webb III and P Wyckoff, Reid's Paradox of rapid plant migration, *BioScience* 48, 1998, 13–24.

[35] G. Chapuisat. Existence and nonexistence of curved front solution of a biological equation. *J. Differential Equations* 236 (2007), no. 1, 237–279.

[36] C. Cortázar ; J. Coville ; M. Elgueta ; S. Martínez. A nonlocal inhomogeneous dispersal process. *J. Differential Equations* 241 (2007), no. 2, 332–358.

[37] J. Coville, Singular measure as principal eigenfunctions of some nonlocal operators. *Appl. Math. Lett* 26 (2013), no. 8, 831–835.

[38] J. Coville, Harnack type inequality for positive solution of some integral equation. *Ann. Mat. Pura Appl.* (4) 191 (2012), no. 3, 503–528.

[39] J. Coville, On a simple criterion for the existence of a principal eigenfunction of some nonlocal operators. *J. Differential Equations* 249 (2010), no. 11, 2921–2953.

[40] J. Coville, On uniqueness and monotonicity of solutions of non-local reaction diffusion equation. *Ann. Mat. Pura Appl* (4) 185 (2006), no. 3, 461–485.

[41] J. Coville ; J. Dávila ; S Martínez. Existence and uniqueness of solutions to a nonlocal equation with monostable nonlinearity. *SIAM J. Math. Anal* 39 (2008), no. 5, 1693–1709.

[42] J. Coville ; J. Dávila ; S Martínez. Pulsating fronts for nonlocal dispersion and KPP nonlinearity *Ann. Inst. H. Poincaré Anal. Non Linéaire* 30 (2013), no. 2, 179–223.

[43] Coville, J. Nonlocal refuge model with a partial control, *2013arXiv1305.7122C*, may, 2013.

[44] C. Cosner. Asymptotic behavior of solutions of second order parabolic partial differential equations with unbounded coefficients. *J. Differential Equations*. 35 (1980)

[45] L. Damascelli. Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 15 (1998), no. 4, 493–516.

[46] L. Damascelli ; A. Farina ; B. Sciunzi ; E. Valdinoci. Liouville results for m-Laplace equations of Lane-Emden-Fowler type. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 26 (2009), no. 4, 1099–1119.

[47] C. De Lellis ; L.Jr. Székelyhidi, On admissibility criteria for weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.* 195 (2010), no. 1, 225–260.

[48] Deveaux, C. and Klein, E., Estimation de la dispersion de pollen à longue distance à l'echelle d'un paysage agicole : une approche expérimentale., *Publication du Laboratoire Ecologie, Systèmatique et Evolution*, 2004.

[49] E. Dibenedetto, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, *Nonlinear Anal.* **7** (1983), 827-850.

[50] Y. Du and Z. M. Guo, Boundary blow-up solutions and their applications in quasilinear elliptic equations, *J. Anal. Math.* 89 (2003), 277-302.

[51] Y. Du, R.Peng. The periodic logistic equation with spatial and temporal degeneracies. *Trans. Amer. Math. Soc.* 364 (2012), no. 11, 6039–6070.

[52] Y. Du, R.Peng. Sharp spatiotemporal patterns in the diffusive time-periodic logistic equation. *J. Differential Equations* 254 (2013), no. 9, 3794–3816

[53] J. Englander ; R.G Pinsky, On the construction and support properties of measure-valued diffusions on $D \subset \mathbb{R}^d$ with spatially dependent branching. *Ann. Probab* 27 (1999), no. 2, 684 ?730.

[54] PC. Fife. Long-time behavior of solutions of bistable nonlinear diffusion equations. *Arch Ration Mech Anal* 70 (1979), 31–46.

[55] , Fife, P. C. An integrodifferential analog of semilinear parabolic PDEs, Partial differential equations and applications, *Lecture Notes in Pure and Appl. Math,* 177, 1996, 137–145

[56] JM. Fraile ;P. Koch Medina ; J. López-Gómez ; S. Merino Elliptic eigenvalue problems and unbounded continua of positive solutions of a semilinear elliptic equation. *J. Differential Equations* 127 (1996), no. 1, 295–319.

[57] García-Melián, J. and Rossi, J. D., On the principal eigenvalue of some nonlocal diffusion problems, *Journal of Differential Equations*, 246, 2009, 21 - 38.

[58] J. Garnier. Accelerating solutions in integro-differential equations. *SIAM J. Math. Anal* 43 (2011), no. 4, 1955–1974.

[59] J. Gartner and M. Freidlin. (1979). On the propagation of concentration waves in periodic and random media. Dokl. Acad. Nauk SSSR 249 521-525 (in Russian).

[60] J. García-Melián and J. Sabina de Lis, Maximum and Comparison principles for operators involving the $p-$Laplacian, *J. Math. Anal. Appl. 218*, 49-65 (1998).

[61] D. Gilbarg and N. S. Trudinger. "Elliptic partial differential equations of second order *2nd edition, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences.* Vol. 224, Springer-Verlag, Berlin, 1983. 48(3) :497–521, 1981.

[62] M. Grinfeld ; G. Hines ; V. Hutson ; K. Mischaikow ; G.T Vickers, Non-local dispersal. *Differential Integral Equations* 18 (2005), no. 11, 1299–1320.

[63] JS. Guo and F. Hamel. Propagation and blocking in periodically hostile environments. *Arch. Ration. Mech. Anal.* 204 (2012), no. 3, 945–975.

[64] Hapca, S. and Crawford, J. W. and Young, I. M. Anomalous diffusion of heterogeneous populations characterized by normal diffusion at the individual level *Journal of The Royal Society Interface* 6, 2009, 111–122.

[65] P. Hess. Periodic-parabolic boundary value problems and positivity *vol. 247. Longman Scientific and Technical*, London (1991).

[66] V Hutson ; S. Martinez ; K. Mischaikow ; GT. Vickers, The evolution of dispersal. *J. Math. Biol.* 47 (2003), no. 6, 483–517.

[67] IL. Ignat ; J.D Rossi ;A. San Antolin Lower and upper bounds for the first eigenvalue of nonlocal diffusion problems in the whole space. *J. Differential Equations* 252, no. 12, 6429–6447.

[68] Kao, C-Y. and Lou, Y. and Shen, W., Random Dispersal VS. Nonlocal Dispersal *Discrete and Continuous Dynamical Systems*, 26, 2010, 551–596.

[69] A. N. Kolmogorov, I. G. Petrovsky and N. S. Piskunov. *Etude de l'equation de la diffusion avec croissance de la quantite de matiere et son application a un probleme biologique*, Bull. Univ. Moskov. Ser. Internat. Sect. 1 (1937), 1–25.

[70] F. Lutscher ; E. Pachepsky ; M.A Lewis. The effect of dispersal patterns on stream populations. *SIAM Rev.* 47 (2005), no. 4, 749–772.

[71] J. López-Gómez The maximum principle and the existence of principal eigenvalues for some linear weighted boundary value problems. *J. Differential Equations* 127 (1996), no. 1, 263–294.

[72] G. M. Lieberman, Second Order Parabolic Differential Equations, *World Scientific Publishing Co. Inc.*, River Edge, NJ, 1996.

[73] V. K. Le and K. Schmitt, *Sub-Supersolution theorems for quasilinear elliptic problems : A variational approach*, Elec. J. Diff. Equ. 118 (2004), 1-7.

[74] P.Lindqvist On the equation $\mathrm{div}(\nabla u|^{p?2}\nabla u) + \lambda|u|^{p-2}u = 0$. *Proc. Amer. Math. Soc* 109 (1990), no. 1, 157-164.

[75] D. Mollison. Long-distance dispersal of wind-borne organisms, *manuscript*, 1977.

[76] G. Nadin The principal eigenvalue of a space-time periodic parabolic operator, *Ann. Mat. Pura Appl.* 188(4) (2009).

[77] G. Nadin Existence and uniqueness of the solutions of a space-time periodic reaction-diffusion equation, *J. Differential Equations.* 249(6) (2010), pp 1288-1304.

[78] G Nadin and L Rossi Propagation phenomena for time heterogeneous KPP reaction-diffusion equations, *J. Math. Pures Appl*, 2012.

[79] R.Peng, D.Wei The periodic-parabolic logistic equation on $\mathbb{R}^N$. *Discrete Contin. Dyn. Syst.* 32 (2012), no. 2, 619-641.

[80] A. B. Potapov, M. A. Lewis, Climate and competition : the effect of moving range boundaries on habitat invasibility. *Bull. Math. Biol*, 66 (2004), no. 5, 975-1008.

[81] M. H. Protter and H. F. Weinberger. Maximum principles in differential equations. *Book.* Prentice-Hall Inc., Englewood Cliffs, N.J., 1967.

[82] S. Petrovskii, A. Mashanova, V. A. A Jansen, Variation in individual walking behavior creates the impression of a Lévy flight, *Proceedings of the National Academy of Sciences* , 2011, 18704-8707.

[83] Y. Pinchover On positive solutions of second-order elliptic equations, stability results, and classification. *Duke Math. J. 57* (1988), no. 3, 955-980.

[84] Y. Pinchover Criticality and ground states for second-order elliptic equations. *J. Differential Equations* 80 (1989), no. 2, 237-250.

[85] Y. Pinchover On criticality and ground states of second order elliptic equations. II. *J. Differential Equations* 87 (1990), no. 2, 353-364.

[86] AC. Ponce, A new approach to Sobolev spaces and connections to Γ-convergence. *Calc. Var. Partial Differential Equations* 19 (2004), no. 3, 229-255.

[87] AC. Ponce, An estimate in the spirit of Poincaré ?s inequality *J. Eur. Math. Soc.(JEMS)* (2004), no. 6, 1-15.

[88] Rawal, Nar and Shen, Wenxian Criteria for the Existence and Lower Bounds of Principal Eigenvalues of Time Periodic Nonlocal Dispersal Operators and Applications, *Journal of Dynamics and Differential Equations*, 24, 2012, 927-954.

[89] L. Roques, A. Roques, H. Berestycki, and A. Kretzschmar. A population facing climate change : joint influences of Allee effects and environmental boundary geometry. *Population Ecology*, 2008, 50(2), 215–225.

[90] L. Rossi Liouville type results for periodic and almost periodic linear operators. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 26 (2009), no. 6, 2481–2502.

[91] D.H. Sattinger. Topics in stability and bifurcation theory *Book*, Springer, 1973.

[92] J. Serrin, *Local behavior of solutions of quais-linear equations*, *Acta Math. 111*, 247-301 (1964).

[93] W. Shen Traveling waves in diffusive random media, *J. Dynamics and Diff. Eqns.*, 16, 2004, 1011–1060.

[94] , Shen, W. and Zhang, A. Stationary solutions and spreading speeds of nonlocal monostable equations in space periodic habitats., *Proc. Am. Math. Soc.* , 140, 2012, 1681-1696.

[95] N. Shigesada and K. Kawasaki Biological Invasions ; Theory and Practice. Oxford University Press, 1997.

[96] Smith, J. M. The Logic of Animal Conflict, Nature, Provided by the SAO/NASA Astrophysics Data System, 246, 1973, 15-18.

[97] Quantitative Analysis of Movement : Measuring and Modeling Population Redistribution in Animals and Plants, book, 1998.

[98] N. S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations*, *Comm. Pure Appl. Math. XX* 721-747 (1967).

[99] Xhaard, C. and Fabre, F. and Coville, J. and Frey, P. and Soubeyrand, S. and Halkett, F., Inferring invasion determinants with mechanistic models and multitype samples, *Preprint.*

[100] Y. Pinchover Criticality and ground states for second-order elliptic equations. *J. Differential Equations* 80 (1989), no. 2, 237–250.

[101] Y. Pinchover On criticality and ground states of second order elliptic equations. II. *J. Differential Equations* 87 (1990), no. 2, 353–364.