Université Nice Sophia-Antipolis - UFR Sciences

Ecole Doctorale Sciences Fondamentales et Appliquées

T H È S E

pour obtenir le titre de

Docteur en Science

de l'Université Nice Sophia-Antipolis

Spécialité : Géométrie Algébrique

présentée et soutenue par

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Cohomology of Algebraic Plane Curves
Cohomologie des Courbes Planes Algébriques

Thèse dirigée par Alexandru DIMCA

soutenue le 11 Juin 2014

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ACKNOWLEDGEMENT

The completion of this thesis would not have been possible without the help and the support of some dear people for whom I shall dedicate the following words.

First and foremost, I would like to express my sincere gratitude to my advisor Alexandru DIMCA. Thank you for the continuous support you gave me, for your patience, motivation and immense knowledge. Your guidance helped me in all the time of research and writing of this thesis. It is and it will always be an honor to work with you.

My deepest appreciation goes to my reading committee members, Lucian BADESCU and Michel GRANGER for spending time reading, reviewing and commenting my work. Michel GRANGER, thank you for the interest you had in my research, and I wish that one day I will have the honor to work with you.

I would like to thank my thesis jury members who accepted to judge my work. It is to my honor and privilege to present my work to brilliant specialist like yourselves.

I gratefully acknowledge the support of the Lebanese National Council for Scientific Research (CNRS-L), without which the present study could not have been completed. Charles, thank you for your support and for helping me to have the financial support.

Thank you for the municipality of Cheikh Taba, my beloved village, for giving me a complement financial support.

I owe my deepest gratitude again to my advisor, to Adam PARUSINSKI, Michel MERLE, Pauline BAILET and Jean-Baptiste CAMPESATO for listenning to my presentations, commenting my work, giving me advice. This work group was very important to me. It has increased myself confidence,

and took my fear away when I speak in front of an audience. Special thanks also to André for being there for me whenever I needed him.

I could not find words to express my gratitude to you. Louli, you made my dream come true, the dream of entering the world of research. I am grateful for you for the rest of my life.

The last years of Ph.D. studies weren't that easy for me, and the most difficult part was to know how to take the right decisions, especially when it comes to my future. You always give me faith whenever I feel down. Bina, I am grateful for you for helping me get through all of these difficulties.

I am indebted to all my colleagues and friends for providing a stimulating and fun environment in which to learn and grow. I am especially grateful to Ahed, Ben, Bienvenu, Brice and Giovanni for the special moments we have spent together. Pauline, my thesis becomes more fun and less difficult with you! Thank you for helping me get through the difficult times, and for all the emotional support, entertainment, and caring you have provided. Ghina, we went through this long journey together, and now we finish together, we shared the same difficult moments, and most importantly, we shared lovely moments with "3ab2a"! Maher I appreciate you being there for me, helping me in all my problems, especially the administrative issues. Thank you for the fun times we spent together.

From Beirut to Paris to Angola, I want to thank you all my dear friends. Sadi2eeee! I did it! I am a doctor like you now, a different kind of doctors but who cares! We both did it! Che, not only I would like to thank you for what you have done to me, but I have to tell you that you are one of the few who was able to draw a sincere smile on my face, even during the most difficult times I passed through. Bébé, maybe you are not aware of what you have done to me, but living away from my country, family, and friends was like impossible for me. Yet, you knew how to encourage me and make it easier on me. I am lucky to have a friend like you!

Hita and Risha, words could never express how much I appreciate you standing by me. You were more like sisters to me! I would never imagine having friends as supportive, helpful, caring and fun as you. "If a friendship lasts longer than 7 years, psychologist say that it will last a lifetime", and

now I am sure that our friendship will last more than 7 years.

Maybe we haven't spent too much time together, but what we have spent was so special. Thank you Hu for being there for me when I needed you, for bringing me up when I felt down, for making me laugh when I felt sad, for listening to me when I needed to talk, and for everything you have done to me. It is not easy to tolerate the nagging of a woman, but bravo you did it!

Lastly and most importantly, I wish to thank all my family. 'Cheikh El Chabeb', you always bring joy to my life ever since you were a child. I am truly blessed to have you in my life. Thank you dear for being the best brother I could ever have. "A sister is a gift to the heart, a friend to the spirit, a golden thread to the meaning of life." Nisso, I do not want to thank you for always being there for me, I want to thank you for being my sister. You were always special for all the family, but to me you are the most precious gift I could ever have. Mom, Dad, I am speechless when it comes to you. Dad, you always trusted me and you called me "batale". I am "batale" because of the faith and strength you gave me. Without you I would not be strong enough to get through all the difficult times and lonely moments. Mom, you believed in me, and you believed that I am able to reach my goals. If I succeeded in my life, it is because of you. I know that you are proud of me now, but you would rather be proud of yourself because you knew how to play your role, the role of being the perfect mother. I am grateful for each day you gave me, my lovely family, you bore me, raised me, supported me, taught me, and loved me. To you I dedicate this thesis.

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Abstract (Français)

Soit $S = \mathbb{C}[x_0, \cdots, c_n]$ l'anneau gradué des polynômes en $x_0, \cdots, x_n$ à coefficient complexes, $S = \bigoplus_{r \geq 0} S_r$ où S_r désigne l'espace vectoriel des polynômes homogènes de degré r. Pour un polynôme homogène, $f \in S_N$, de degré N, on définit l'algèbre de Milnor (ou du Jacobien) par $M(f) = S/J_f$, où J_f est l'idéal Jacobien de f, i.e. l'idéal engendré par les dérivées partielles $f_0 = \frac{\partial f}{\partial x_0}, \cdots, f_n = \frac{\partial f}{\partial x_n}$. $M(f)$ est une $\mathbb{C}$-algèbre graduée, dont la graduation est induite par celle de S.

L'étude de l'algèbre de Milnor est liée aux singularités de l'hypersurface H définie par $f = 0$ dans l'espace projectif $\mathbb{P}^n$, ainsi qu'à la structure de Hodge mixte sur la cohomologie de H et de son complémentaire $U = \mathbb{P}^n \setminus H$. En effet, l'algèbre de Milnor de f est égale, avec un décalage de graduation, au groupe d'homologie d'ordre 0 ou le groupe de cohomologie d'ordre $n+1$ du complexe de Koszul des dérivées partielles de f, donc c'est normal d'étudier les autres groupes de (co)homologie de ce complexe.

Dans la première partie de cette thèse, on étudie la relation entre la théorie de Hodge mixte du complémentaire U de H et les singularités de H. L'importance de la théorie de Hodge est qu'il existe une structure de Hodge mixte sur les groupes de cohomologie de toute variété algébrique X, compatible avec les morphismes induits par les applications régulières $u : X \to Y$. Cette structure consiste essentiellement de deux filtrations, une filtration décroissante F^s, la filtration de Hodge, et une filtration croissante W_m, la filtration par le poids. On s'intéresse au calcul des dimensions des groupes gradués associés à la filtration de Hodge dans le cas où H est une courbe dans $\mathbb{P}^2$, qu'on notera C dans la suite, qui admet des points doubles et triples ordinaires comme singularités. En particulier, on a obtenu le résultat suivant.

Theorem 0.1. *Soit $C \subset \mathbb{P}^2$ une courbe de degré N. Supposons que C n'admet que n noeuds et t points triples comme singularités. Notons $U = \mathbb{P}^2 \backslash C$. Soient $C = \bigcup_{j=1,r} C_j$ la décomposition de C en union de composantes irréductibles, $\nu : \tilde{C}_j \to C_j$ les normalisations, et $g_j = g(\tilde{C}_j)$, le genre de $\tilde{C}_j$. On a alors*

$$\dim Gr_F^1 H^2(U, \mathbb{C}) = \sum_{j=1}^{r} g_j$$

et

$$\dim Gr_F^2 H^2(U,\mathbb{C}) = \frac{(N-1)(N-2)}{2} - t.$$

Ce théorème nous permet de calculer tous les nombres de Hodge mixtes du groupe de cohomologie $H^2(U)$ de U, le complémentaire de C, et par conséquent, les nombres de Betti correspondants qui sont des invariants topologiques importants. Un cas spécial est celui des courbes rationnelles, où $g_i = 0$ pour tout j. Dans ce cas, $H^2(U)$ est pure de type $(2,2)$. Ceci est une propriété connue dans le cas des complémentaires des arrangements de droites.

On étudie ensuite le cas où C admet des singularités isolées quelconques, et on trouve que le même résultat pour la dimension $\dim Gr_F^1 H^2(U,\mathbb{C})$. Le cas où C admet des singularités de multiplicités atteignant l'ordre 4 montre qu'on ne peut pas s'attendre à trouver des formules simples pour $\dim Gr_F^2 H^2(U,\mathbb{C})$.

Dans la deuxième partie de cette thèse, on s'intéresse à trouver les dimension de $M(f)_r$ qui sont des invariants projectifs de l'hypersurface $H : f = 0$. Le cas lisse est déjà étudié, et la série de Poincaré définie par $HP(M(f))(t) = \sum_r \dim M(f)_r t^r$ est complètement déterminée. Explicitement on a,

$$HP(M(f))(t) = \frac{(1-t^{N-1})^{n+1}}{(1-t)^{n+1}}.$$

L'étude du cas où H admet des singularités isolées, $a_1, \cdots, a_p$ a commencé par A. Dimca et A.D.R Choudary dans [3], qui ont montré que $\dim M(f)_r$ devient stable à partir du rang $(n+1)(N-2)$. Dans ce cas, $M(f)_r = \tau(H)$, où $\tau(H)$ est la somme de tous les nombres de Tjurina $\tau(H, a_j)$ pour $j = 1, \cdots, p$. Puis l'étude continue avec A. Dimca et G. Sticlaru dans [15] qui ont montré que pour les premières valeurs de r, $\dim M(f)_r$ coïncide avec les dimensions de $M(f_s)_r$ d'un polynôme homogène f_s qui définit une hypersurface lisse de même degré que celui de H. En particulier, ils ont montré que $\dim M(f)_r = \dim M(f_s)_r$ pour tout $r \leq N-2$.

Pour $n = 2$, c.à.d. quand H est une courbe $C \subset \mathbb{P}^2$, l'étude devient plus simple et on a des résultats plus intéressants. Quand C est une courbe nodale, A. Dimca et G. Sticlaru ont trouvé que $\dim M(f)_r = \dim M(f_s)_r$ pour tout $r \leq 2N-4$, c.à.d. les dimensions de $M(f)_r$ sont connus jusqu'au

rang $2N-4$, et pour le rang suivant on a,

$$\dim M(f)_{2N-3} = n(C) + \sum_{j=1}^{r} g_j, \tag{0.1}$$

où $n(C)$ est le nombre des noeuds de C. Si C est une courbe nodale rationnelle, alors $\dim M(f)_{2N-3} = n(C) = \tau(C)$, et par conséquent la série de Poincaré est complètement déterminée en fonction du nombre des noeuds et du degré de C.

D'après cette étude, on peut se poser la question: Que se passe-t-il si C admet d'autres singularités que les noeuds? On restreint l'étude dans ce qui suit sur le cas des courbes ayant des points doubles et triples comme singularités, par exemple la courbe définie par $(x^2-y^2)(y^2-z^2)(x^2-z^2)=0$, qui est l'union de 6 droites se coupant en 4 points triples et 3 points doubles. Contrairement au cas nodale, dans ce cas l'inclusion $F^s \subset P^s$ dûe à Deligne-Dimca, où P^s est la filtration par l'ordre du pôle, peut être stricte, ce qui rend l'étude plus difficile.

On généralise dans ce cas l'équation 0.1. Plus précisément, on obtient le résultat suivant.

Theorem 0.2. *Soit $C \subset \mathbb{P}^2$ une courbe de degré N. Supposons que C admet n noeuds (A_1) et t points triples (D_4). Soit $C = \bigcup_{j=1,r} C_j$ la décomposition de C en union de composantes irréductibles, $\nu : \tilde{C}_j \to C_j$ les normalisations et notons $g_j = g(\tilde{C}_j)$ les genres de $\tilde{C}_j$. Soit $\tau = n + 4t$ le nombre de Tjurina global de C. On a,*

$$0 \leq \dim M(f)_{2N-3} - \tau \leq \sum_{j=1}^{r} g_j.$$

En particulier,

(i) Si $g_i = 0$ pour tout i, on a $\dim M(f)_{2N-3} = \tau$.

(ii) L'égalité $\dim M(f)_{2N-3} - \tau = \sum_{j=1}^{r} g_j$ est vérifiée si et seulement si $H^2(U)$ vérifie $F^2H^2(U) = P^2H^2(U)$.

On considère le sous-module gradué de S, $AR(f) \subset S^{n+1}$, de toutes les relations entre les dérivées partielles f_j du polynôme f, soit

$$a = (a_0, ..., a_n) \in AR(f)_m$$

si et seulement si $a_0 f_0 + a_1 f_1 + ... + a_n f_n = 0$.

Dans le module $AR(f)$ il y a un sous-module de S des *relations de Koszul*, nommé aussi le sous-module des *relations triviales*, engendré par les relations $t_{ij} \in AR(f)_{d-1}$ pour $0 \leq i < j \leq n$, où la $i^{ième}$ coordonné de t_{ij} est égale à f_j, sa $j^{ième}$ coordonné égale à $-f_i$ et les autres sont nulles.

On appelle le module quotient $ER(f) = AR(f)/KR(f)$, le module des *relations essentielles*, ou les relations non triviales, car c'est le module des relations qu'on doit ajouter aux relations de Koszul pour avoir toutes les relations, ou les syzygies, entre les f_j.

On décrit dans cette thèse les dimensions de l'espace de syzygies de l'idéal Jacobien de degré $N-2$. Plus précisément, on a obtenu le résultat suivant.

Theorem 0.3. *Avec les mêmes hypothèses du théorème précédent, on a*

$$max(r - 1 + t - \sum_{j=1}^{r} g_j, r - 1) \leq \dim ER(f)_{N-2} \leq r - 1 + t.$$

En particulier, $\dim ER(f)_{N-2} = r - 1 + t$ *si* $g_j = 0$ *pour tout* j.

Exemple 5.6 montre que les courbes avec points doubles et triples sont plus compliquées que les courbes nodales. En particulier, contrairement au cas nodal rationnel, dans le cas des courbes rationnelles à points doubles et triples ordinaires, la série de Poincaré $HP(M(f))$ n'est pas complètement déterminée par N, le nombre des composantes irréductibles et le nombre de points doubles et triples. Il montre aussi que c'est difficile de contrôler les composantes homogènes $M(f)_r$ pour $r \neq 2N - 3$.

On voudrait continuer au futur l'étude de ces questions intéressantes et difficiles. Les relations entre les syzygies de l'idéal Jacobian et la cohomologie de Rham de la fibre de Milnor définie par $F : f = 1$ est aussi un autre sujet d'investigation.

Abstract (English)

Let $S = \mathbb{C}[x_0, \cdots, c_n] = \bigoplus_{r \geq 0} S_r$ be the graded ring of polynomial functions in $x_0, \cdots, x_n$ with complex coefficients, where S_r denotes the vector space of homogeneous polynomials of degree r. Let $f \in S_N$ be a homogeneous polynomial of degree N, and define $M(f) = S/J_f$ to be the Milnor (or Jacobian) algebra of f, where J_f is the Jacobian ideal of f, i.e. the ideal generated by the first order partial derivatives $f_j = \frac{\partial f}{\partial x_j}$ for $j = 0, 1, ..., n$ of f.

The study of such Milnor algebras is related to the singularities of the hypersurface $H \subset \mathbb{P}^n$ defined by $f = 0$ in the complex projective space $\mathbb{P}^n$, as well as to the mixed Hodge structure on the cohomology of H, and of its complement $U = \mathbb{P}^n \setminus H$. In fact the Milnor algebra of f can be seen up to a twist of grading as the first (respectively the last) homology (respectively cohomology) group of the Koszul complex of the partial derivatives of f, so it is natural to study the other (co)homology groups of this complex as well.

In the first part of this thesis, we study the relation between the mixed Hodge theory of the complement of the hypersurface H and the singularities of H. The importance of Hodge theory is the existence of a mixed Hodge structure on the cohomology groups of each algebraic variety X, compatible with the morphisms induced by regular mappings $u : X \to Y$. This structure consists essentially of two filtrations, the decreasing Hodge filtration F^s and the increasing weight filtration W_m. We are interested in computing the dimensions of the associated graded groups of the former one in the case where H is a curve in $\mathbb{P}^2$, that we will denote in the sequel by C, having only ordinary double and triple points as singularities. In particular we have obtained the following result.

Theorem 0.4. *Let $C \subset \mathbb{P}^2$ be a curve of degree N and set $U = \mathbb{P}^2 \backslash C$. Suppose that C has only n nodes (A_1) and t triple points (D_4) as singularities. Let $C = \bigcup_{j=1,r} C_j$ be the decomposition of C as a union of irreducible components, let $\nu : \tilde{C}_j \to C_j$ be the normalization mappings and let $g_j = g(\tilde{C}_j)$ be the corresponding genera. Then one has*

$$\dim Gr_F^1 H^2(U, \mathbb{C}) = \sum_{j=1}^{r} g_j$$

and

$$\dim Gr_F^2 H^2(U,\mathbb{C}) = \frac{(N-1)(N-2)}{2} - t.$$

This theorem allows us to compute all the mixed Hodge numbers of the second cohomology group $H^2(U)$ of the complement U of C, and consequently the correspondant Betti numbers which are important topological invariants. A special case is the case of rational curves, where $g_j = 0$ for all j. In this case $H^2(U)$ is pure of type $(2,2)$, a well known property in the case of line arrangement complements.

Then we study the case where C has more general isolated singularities, and we find the same result for $\dim Gr_F^1 H^2(U,\mathbb{C})$. The case where C has singularities of multiplicities up to 4 shows that we cannot expect simple formulas for $\dim Gr_F^2 H^2(U,\mathbb{C})$.

In the second part of this thesis, we are interested in finding the dimensions of $M(f)_r$ which are projective invariants of the hypersurface $H : f = 0$. The case where H is smooth is already known, and the Hilbert-Poincaré series, defined by $HP(M(f))(t) = \sum_r \dim M(f)_r t^r$, is all determined. More explicitely,

$$HP(M(f))(t) = \frac{(1-t^{N-1})^{n+1}}{(1-t)^{n+1}}.$$

The study of the case where H has isolated singularities, say at the points $a_1, \cdots a_p$, has begun by A. Dimca and A.D.R. Choudary in [3], who proved that $\dim M(f)_r$ stabilizes for $r > (n+1)(N-2)$. In this case $\dim M(f)_r = \tau(H)$, where $\tau(H)$ is the sum of all Tjurina numbers $\tau(H, a_j)$ for $j = 1, \cdots, p$. Then A. Dimca and G. Sticlaru proved in [15] that $\dim M(f)_r$ coincides with the dimensions of $M(f_s)_r$ of a homogeneous polynomial f_s defining a smooth hypersurface of same degree of H for small r. Indeed, they have noticed that for $r \leq N-2$, $\dim M(f)_r = \dim M(f_s)_r$.

When $n = 2$, i.e. H is a curve C in $\mathbb{P}^2$, the study becomes simpler, and we have more interesting results. In particular, if C is a nodal curve , $\dim M(f)_r = \dim M(f_s)_r$ for all $r \leq 2N-4$, and the next dimension is given by

$$\dim M(f)_{2N-3} = n(C) + \sum_{j=1}^{r} g_j, \tag{0.2}$$

where $n(C)$ is the number of nodes of C. If C is a rational nodal curve, then $\dim M(f)_{2N-3} = n(C) = \tau(C)$, and therefore the Poincaré series is all determined in terms of the number of nodes and the degree of C.

This study gives rise to the open question: What will happen if C has singularities other than nodes? We restrict our studies to the curves having only ordinary double and triple points as singularities, for instance, the curve defined by $(x^2-y^2)(y^2-z^2)(x^2-z^2)=0$, which is the union of 6 lines intersecting in 4 triple points and 3 nodes. This case is more subtle since, unlike the nodal case, the well-known inclusion due to Deligne-Dimca, $F^s \subset P^s$, may be strict, where P^s is the pole order filtration.

We give in this case a generalization of equation (0.2). More precisely, we have obtained the following result.

Theorem 0.5. *Let $C \in \mathbb{P}^2$ be a curve of degree N. Suppose C has n nodes (A_1) and t triple points (D_4). Let $C = \bigcup_{j=1,r} C_j$ be the decomposition of C as a union of irreducible components, let $\nu : \tilde{C}_j \to C_j$ be the normalization mappings and set $g_j = g(\tilde{C}_j)$. Then we have the following.*

$0 \leq \dim M(f)_{2N-3} - \tau \leq \sum_{j=1}^{r} g_j$. *In particular,*

(i) If all $g_i = 0$, one has $\dim M(f)_{2N-3} = \tau$.

(ii) One has equality, i.e. $\dim M(f)_{2N-3} - \tau = \sum_{j=1}^{r} g_j$ if and only if $H^2(U)$ satisfies $F^2H^2(U) = P^2H^2(U)$.

One can consider the graded $S-$submodule $AR(f) \subset S^{n+1}$ of *all relations* involving the partial derivatives f_j's of the polynomial f, namely

$$a = (a_0, ..., a_n) \in AR(f)_m$$

if and only if $a_0f_0 + a_1f_1 + ... + a_nf_n = 0$.

Inside the module $AR(f)$ there is the $S-$submodule of *Koszul relations* $KR(f)$, called also the submodule of *trivial relations*, spanned by the relations $t_{ij} \in AR(f)_{d-1}$ for $0 \leq i < j \leq n$, where t_{ij} has the i-th coordinate equal to f_j, the j-th coordinate equal to $-f_i$ and the other coordinates zero.

The quotient module $ER(f) = AR(f)/KR(f)$ may be called the module of *essential relations*, or non trivial relations, since it tells us which are the

relations which we should add to the Koszul relations in order to get all the relations, or syzygies, involving the f_j's.

We describe in this thesis the dimension of the space of syzigies of the Jacobian ideal of degree $N-2$. More precisely, we have obtained the following result.

Theorem 0.6. *Under the same hypothesis of the previous theorem, we have*

$$max(r-1+t-\sum_{j=1}^{r} g_j, r-1) \leq \dim ER(f)_{N-2} \leq r-1+t.$$

In particular, $\dim ER(f)_{N-2} = r-1+t$ *if* $g_j = 0$ *for all* j.

Example 5.6 shows that the curves with ordinary nodes and triple points are much more subtle than the nodal curves. In particular, for rational curves with ordinary nodes and triple points the Poincaré series $HP(M(f))$ is not determined by N, the number of irreducible components, the number of double and triple points (as was the case for rational nodal curves). It also shows that it is rather difficult to control the dimensions of the homogeneous components $M(f)_r$ for $r \neq 2N-3$.

We plan to continue the study of these difficult and interesting questions in the future. The relations between syzygies of the Jacobian ideal and the de Rham cohomology of Milnor fibers given by $F : f = 1$ is also a subject of further investigations.

Chapter 1

Introduction

1.1 Introduction (version Française)

Les variétés algébriques sont un objet central de la géométrie algébrique. Une des plus importantes problématiques dans ce domaine est l'étude des singularités des hypersurfaces, qui s'étaient remarquées dans le cas des courbes, comme le point double ordinaire de la courbe définie par $y^2 = x^2 - x^3$, le point de rebroussement de la courbe $y^2 = x^3$ et le point triple ordinaire de la courbe $y^3 = x^3 - x^4$.

Soit $S = \mathbb{C}[x_0, \cdots, x_n]$ l'anneau de polynômes à $n+1$ variables à coefficients dans $\mathbb{C}$. S est un anneau gradué dont les éléments homogènes de degré r sont les polynômes homogènes de degré r. On note $S = \bigoplus_{r \geq 0} S_r$. Pour un polynôme $f \in S_N$, on définit l'algèbre de Milnor par $M(f) = S/J_f$, où J_f est l'idéal Jacobian de f, c.à.d. l'idéal engendré par les dérivées partielles $f_0 = \frac{\partial f}{\partial x_0}, \cdots, f_n = \frac{\partial f}{\partial x_n}$. $M(f)$ est une $\mathbb{C}$-algèbre graduée, dont la graduation est induite par celle de S.

Dans cette thèse, on étudie l'algèbre de Milnor $M(f)$ d'un polynôme homogène f et sa relation avec l'hypersurface projective correspondante, $V(f) : f = 0$. En particulier, on s'intéresse à trouver les dimensions des composantes homogènes $M(f)_r$ de cette algèbre de Milnor qui sont des invariants projectifs des hypersurfaces. Le cas lisse étant complètement étudié, on s'interesse au cas où $V(f)$ admet des singularités isolées, plus précisément aux courbes qui ont des points doubles et triples ordinaires. De telles ques-

tions ont récemment attiré beaucoup d'intérêt, voir [3], [11], [12], [13], [14], [15], [17], [24], [25], et [26].

D'autre part, la théorie de Hodge joue un rôle important dans la théorie de singularités. D'après P. Deligne, il existe une structure de Hodge mixte sur les groupes de cohomologie de toute variété algébrique, voir [6]. Cette structure définie une filtration (la filtration de Hodge) dont les dimensions des groupes gradués associés seront calculées pour certains cas. Plus précisement, on étudie dans cette thèse la structure de Hodge mixte de la cohomologie du complémentaire des courbes singulières dans $\mathbb{P}^2$, dont les singularités sont des points doubles et triples, en particulier la relation entre la filtration de Hodge et la filtration par l'ordre du pôle.

On commence par le chapitre 2 où on rappelle les notions de base des germes et des singularités isolées des hypersurfaces, les suites régulières et les intersections complètes. Notons que les dérivées partielles $f_0, \cdots, f_n$ forment une suite régulière si et seulement si $V(f)$ est une hypersurface lisse, ce qui explique le fait que l'étude de l'algèbre de Milnor $M(f)$ est simple dans ce cas. Puis on rappelle le théorème de Bézout, le théorème de Cayley-Bacharach, et les défauts des systèmes linéaires par rapport aux sous-ensembles finis (ou sous-schémas de dimension 0) dans $\mathbb{P}^n$. Ceux-ci jouent un rôle important dans la compréhension de l'algèbre de Milnor $M(f)$ quand $V(f)$ admet des singularités isolées, voir [11].

Dans le chapitre 3, on introduit le complexe de Koszul $K^*(f)$ des dérivées partielles d'un polynôme homogène f, dont le groupe de cohomologie d'ordre $n+1$ est égale à l'algèbre de Milnor avec un décalage de graduation. Quand $V(f)$ admet des singularités isolées, le seul groupe de cohomologie non nul de $K^*(f)$ distinct de H^{n+1} décrit les syzygies de l'idéal Jacobian J_f. Cette relation nous donne des résultats importants sur les composantes homogènes de $M(f)$. Le cas des courbes nodales a été étudié par A. Dimca et G. Sticlaru dans [15]. En particulier, dans le cas des courbes nodales rationnelles, où chaque composante irréductible C_i de $C = V(f)$ est rationnelle, la série de Hilbert-Poincaré de $M(f)$ est donnée explicitement en fonction du degré de f et le nombre de noeuds, voir Corollaire 3.3. On généralise partiellement ce résultat au cas des courbes dans $\mathbb{P}^2$ à points doubles et triples ordinaires, voir Théorème 5.5, et on montre dans l'exemple 5.6 qu'on ne peut pas espérer à le généraliser complètement.

Dans le chapitre 4, on introduit les structures de Hodge mixtes et leurs relations avec les singularités des hypersurfaces. Dans la première partie de ce chapitre, on rappelle quelques définitions et propriétés introduites par P. Deligne [6]. Le premier résultat de la thèse, énoncé dans la deuxième partie de ce chapitre, relie la théorie de Hodge du complémentaire de la courbe plane $C = V(f)$, admettant des points doubles et triples ordinaires comme singularités, à la topologie des composantes irréductibles C_i de C ainsi qu'au nombre de points triples. Avec ce résultat on peut calculer les nombres de Hodge mixtes du groupe de cohomologie d'ordre 2 du complémentaire des courbes à points doubles et triples, et par conséquent les nombres de Betti correspondants. Puis on considère le cas où C est une courbe plane à singularités isolées quelconques, où on calcul le polynôme de Hodge-Deligne de C et de son complémentaire U. On généralise ainsi le théorème 4.2 aux arrangements des courbes ayant des singularités ordinaires et se coupant transversalement. Dans la dernière section, on montre que le cas des courbes planes à singularités ordinaires dont les multiplicités atteignent l'ordre 4 (sans l'hypthèse que les courbes se coupent transversalement) est beaucoup plus compliqué.

On commence le chapitre 5 par donner des exemples de syzygies de l'idéal Jacobian J_f et les relier au calcul de la série de Hilbert-Poincaré de l'algèbre de Milnor $M(f)$ en utilisant le logiciel Singular. Puis on donne une brève présentation des suites spectrales associées à un complexe filtré, et on l'applique au complexe de Rham du complémentaire d'une hypersurface afin de définir la filtration par l'ordre du pôle. Cette filtration, qu'on note P^s, et la filtration de Hodge F^s vérifient la relation $F^s \subset P^s$ démontrée par Deligne-Dimca dans [7]. Dans le cas des courbes nodales, cette relation devient une égalité, ce qui explique pourquoi la théorie est plus simple dans ce cas. Pour les courbes à points doubles et triples ordinaires, l'inclusion peut être stricte, voir Exemple 5.5.

L'idée principale est la description de la suite spectrale associée à la filtration par l'ordre du pôle en fonction des composantes homogènes de la cohomologie du complexe de Koszul $K^*(f)$, voir la section (5.4) et le fait que cette suite spectrale dégénère au terme E_2 dans le cas des courbes planes aux singularités quasi-homogènes, voir [15]. Une approche différente de ces résultats est donnée dans le nouvel article de A. Dimca et M. Saito, voir [12].

Dans les Propositions 5.4 et 5.5, on montre comment on peut trouver des représentants pour des classes de cohomologie de Rham dans la cohomologie de la fibre de Milnor pour deux arrangements classiques de droites en utilisant les syzygies de l'idéal Jacobian. Notons qu'au contraire de la cohomologie du complémentaire d'un arrangement d'hyperplans, où la description en fonction des formes différentielles est due à Arnold, Brieskorn, Orlik et Solomon, voir [22], nos propositions 5.4 et 5.5 sont les premiers résultats de ce type concernant la cohomologie de la fibre de Milnor.

Le deuxième résultat de cette thèse, voir Théorème 5.5 décrit la dimension de $M(f)_{2N-3}$ et celle de l'espace de syzygies

$$EF(f)_{N-2} = \{(a,b,c) \in S^3_{N-2} : af_x + bf_y + cf_z = 0\}$$

de degré $N-2$ d'une courbe dans $\mathbb{P}^2$ de degré N à points doubles et triples. Contrairement au cas nodal, on trouve dans notre cas un encadrement pour ces invariants, et le fait que ces inégalités deviennent des égalités dépend de l'égalité entre la filtration par l'ordre du pôle P^s et la filtration de Hodge F^s.

On montre que ces égalités sont vraies quand les composantes irréductibles C_i de $C = V(f)$ sont rationnelles, et cela entraîne une nouvelle situation où on a une égalité $F^s = P^s$. Cependant, même dans ce cas, Exemple 5.6 montre qu'on ne peut pas avoir des formules simples pour $\dim M(f)_s$ comme dans le cas des courbes nodales décrites dans le Corollaire 3.3.

1.2 Introduction (English version)

Algebraic varieties are a fundamental object of study in Algebraic Geometry. One of the most important problems in this field is the study of the singularities of hypersurfaces, which were first noticed in the case of curves, for instance, the ordinary double point of the curve defined by $y^2 = x^2 - x^3$, the cusp of the curve given by $y^2 = x^3$ and the ordinary triple point $y^3 = x^3 - x^4$.

Denote by $S = \mathbb{C}[x_0, \cdots, x_n]$ the polynomial ring over $\mathbb{C}$ in $n+1$ variables, with its natural grading, $S = \oplus_{r \geq 0} S_r$, where S_r is the vector space of homogeneous polynomials in S of degree r. For a polynomial $f \in S_N$,

we define the Milnor algebra by $M(f) = S/J_f$, where J_f is the ideal of S spanned by the partial derivatives $f_0 = \frac{\partial f}{\partial x_0}, \cdots, f_n = \frac{\partial f}{\partial x_n}$. Note that $M(f)$ is a graded $\mathbb{C}$-algebra, whose grading is the one induced from the grading of S.

In this thesis, we study the Milnor algebra $M(f)$ of a homogeneous polynomial f and its relation to the corresponding hypersurface $V(f) : f = 0$ in $\mathbb{P}^n$. In particular, we are interested in finding the dimensions of the graded pieces of this Milnor algebra, which are projective invariants of hypersurfaces. The smooth case being completely studied, we are only interested in the case when $V(f)$ has isolated singularities, specifically the case of curves in $\mathbb{P}^2$ with ordinary double and triple points. Such questions have attracted a lot of interest recently, see [3], [11], [12], [13], [14], [15], [17], [24], [25], [26].

On the other hand, Hodge theory plays an important role in Singularity Theory. Following P. Deligne, there exists a mixed Hodge structure on the cohomology groups of any algebraic variety, see [6]. As part of such a structure, there are Hodge filtrations on these cohomology groups whose dimensions are computed thereafter in some cases. More precisely, we study in this thesis the mixed Hodge structures on the cohomology of the complements of singular curves in $\mathbb{P}^2$ with ordinary double and triple points, in particular the relation between the Hodge filtration and the pole order filtration.

In chapter 2 we recall the basic facts on germs and isolated hypersurface singularities, regular sequences and complete intersections. Note that $f_0, ..., f_n$ is a regular sequence if and only if $V(f)$ is a smooth hypersurface, which explains why the study on the Milnor algebra $M(f)$ is easy in this case. Then we discuss the Bézout Theorem, the Cayley-Bacharach Theorem and the defects of linear systems with respect to finite subsets (or 0-dimensional subschemes) in $\mathbb{P}^n$. Such defects play a key role in understanding the Milnor algebra $M(f)$ when $V(f)$ has isolated singularities, see [11].

In chapter 3, we introduce the Koszul complex $K^*(f)$ of the partial derivative of a homogeneous polynomial f, whose top cohomology group is up to a shift in grading the Milnor algebra $M(f)$ of f. When $V(f)$ has isolated singularities, the only other nonzero cohomology group of $K^*(f)$ describes the syzygies of the Jacobian ideal J_f, i.e. the (homogeneous) syzygies among the derivatives $f_0, ..., f_n$. This relation gives interested results about the dimensions of the homogeneous components of $M(f)$. An important case is

the one of nodal curves studied by A. Dimca and G. Sticlaru in [15]. In particular, in the case of rational nodal curves, i.e. each irreducible component C_i of $C = V(f)$ is rational, the Hilbert-Poincaré series is given explicitely in terms of the degree of f and the number of nodes, see Corollary 3.3. We partially generalize this result in the case of curves in $\mathbb{P}^2$ with ordinary double and triple points, see Theorem 5.5 and show in Example 5.6 that a complete generalization cannot be hoped for.

In chapter 4 we give a brief introduction about Hodge structures and their relation to hypersurfaces singularities. In the first part of this chapter, we recall some definitions and properties introduced by P. Deligne [6]. In the second part, we state and prove our first result, see Theorem 4.2, which relates the mixed Hodge theory of the complement of a plane curve $C = V(f)$ with double and triple points and the topology of the irreducible components C_i of the curve C and the number of triple points. Using this result, we can compute all mixed Hodge numbers of the second cohomology group of the complement of curves with double and triple points, and therefore we can find the corresponding Betti numbers.We consider then the case where C has more general isolated singularities, where we compute the Hodge-Deligne polynomial of C and of its complement U. We generalize Theorem 4.2 to the case of arrangements of curves having ordinary singularities and intersecting transversely at smooth points. In the last section we show that the case of plane curves with ordinary singularities of multiplicity up to 4 (without assuming transverse intersection) is definitely more complicated.

In chapter 5 we start by giving some examples of syzygies of the Jacobian ideal J_f, and relate them to the computation of the Poincaré series of the Milnor algebra $M(f)$ via the SINGULAR software. Then we give a quick presentation of the spectral sequence associated to a filtered complex, and apply this to the global algebraic de Rham complex of a hypersurface complement to define the pole order filtration. The pole order filtration P^s and the Hodge filtration F^s satisfies the inclusion $F^s \subset P^s$, as shown by Deligne-Dimca in [7]. In the case of nodal curves, this inclusion becomes an equality and this explains why the theory is much simpler in this case. For curves with double and triple points, this inequality can be strict, and the first such examples are given in Example 5.5.

The main technical ingredient is the description of the pole order spec-

tral sequence refered to above in terms of the homogeneous components of the cohomology of the Koszul complex $K^*(f)$, see section (5.4) and the fact that this spectral sequence degenerates at the E_2-term in the case of plane curves having only weighted homogeneous singularities, see [15]. A different approach for these results can be found in the recent paper by A. Dimca and M. Saito, see [12].

In Propositions 5.4 and 5.5 we show for two classical line arrangements how to get de Rham representatives in the cohomology of the Milnor fiber using the syzygies of the Jacobian ideal. Note that unlike the cohomology of a hyperplane arrangement complement, where the description in terms of differential forms is due to Arnold, Brieskorn, Orlik and Solomon see [22], Propositions 5.4 and 5.5 are the first such results for the Milnor fiber cohomology.

The second main result of this thesis is Theorem 5.5 in which we describe the dimension of $M(f)_{2N-3}$ and the dimension of the space of syzygies of degree $N-2$ for a degree N curve in $\mathbb{P}^2$ with double and triple points. Unlike the nodal case, here we obtain just some bounds for these invariants, and the equalities between the invariants and the bounds depend on the equality between the pole order filtration P^s and the Hodge filtration F^s.

We prove that such equalities hold when all the irreducible components C_i of $C = V(f)$ are rational, and this yields a new situation when one has an equality $F^s = P^s$. However, even in this case, Example 5.6 show that simple formulas for all the dimensions $\dim M(f)_s$ as in the case of nodal curves described in Corollary 3.3 cannot be expected.

Chapter 2

Preliminaries

2.1 Germs, equivalence relations for germs and singularities

Germs of Complex Analytic Mappings

Let $M = \{(U, f)$ where U is an open neighborhood of the origin in $\mathbb{C}^n$ and $f : U \to \mathbb{C}^p$ an analytic map$\}$. Define on M an equivalence relation given by, $(U_1, f_1) \sim (U_2, f_2)$ if and only if $f_1|_{U_0} = f_2|_{U_0}$ for some neighborhood U_0 of the origin with $U_0 \subset U_1 \cap U_2$.
An equivalence class of this relation is called a germ of analytic map from $\mathbb{C}^n$ to $\mathbb{C}^p$ at the origin and is denoted by (U, f), or simply f.
The set of all these germs is denoted by $E_{n,p}$. When $p = 1$, we simply write E_n.

Proposition 2.1. *E_n is a local $\mathbb{C}$-algebra with unique maximal ideal $m_n = \{f \in E_n : f(0) = 0\}$.*

Let $E^0_{n,p}$ be the set of germs $f \in E_{n,p}$ with $f(0) = 0$. An element of $E^0_{n,p}$ will also be denoted by $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$.

Contact Equivalence

Denote by D_n the group of complex map germs $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ which are analytic isomorphisms, the group operation being the composition

of map germs, and by $M_{n,p}$ the group $Gl(p, E_n)$, i.e. the group of invertible matrices of order p with entries in E_n.

Definition 2.1. Let $G = M_{n,p} \rtimes D_n$ be the semi-direct product given by the multiplication rule:

$$(A_1, h_1).(A_2, h_2) = (A_1(A_2 \circ h_1^{-1}), h_1 h_2)$$

where composition with h_1^{-1} refers to all the entries of the matrix A_2.
Define an action on G by

$$\begin{aligned} G \times E_{n,p}^0 &\to E_{n,p}^0 \\ ((A, h), f) &\mapsto (A, h).f = A(f \circ h^{-1}) \end{aligned}$$

where the germs f and $f \circ h^{-1}$ are considered as column vectors with entries in $m_n \subset E_n$.
An equivalence relation associated to this action is given by $f_1 \sim f_2$ if and only if there exists $(A, h) \in G$ such that $f_2 = (A, h).f_1$ and is called contact equivalence or K-equivalence. When $p = 1$, $M_{n,p} = E_n$, two germs f_1 and f_2 are K-equivalent if and only if there exist a map germ $u \in E_n$ and $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ an analytic isomorphism, such that $f_2 = u.(f_1 \circ h^{-1})$.

Example 2.1. *Let $f \in m_n^2$, and assume that $V(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}) = \{x \in \mathbb{C}^n; \frac{\partial f}{\partial x_i} = 0 \ i = 1, \cdots, n\} \subseteq \{0\}$, i.e. f has at most isolated singularities.*

(i) If in addition $corank f = n - rank(\frac{\partial^2 f}{\partial x_i \partial x_j})_{i,j=1,\cdots,n} \leq 1$,then f is K-equivalent to the normal form $A_k : f_k = x_1^{k+1} + x_2^2 + \cdots x_n^2$. In particular, if $corank f = 0$, i.e. $rank(\frac{\partial^2 f}{\partial x_i \partial x_j})_{i,j=1,\cdots,n} = n$, we have a nondegenerate singularity, and by Morse lemma f is K-equivalent to the normal form $A_1 : f_1 = x_1^2 + x_2^2 + \cdots + x_n^2$ which corresponds to a nodal hypersurface.

(ii) The simplest singularities of corank 2 are the polynomials f K-equivalent to the normal form $D_4 : x_1^3 + x_1 x_2^2 + x_3^2 + \cdots + x_n^2$. When $n = 2$, this corresponds to a triple point of a plane curve.

One can define in a similar way germs of analytic sets and one has the following, see [8, p.23],

Theorem 2.1. *$f_1 \sim f_2$ if and only if $(f_1^{-1}(0), 0) \simeq (f_2^{-1}(0), 0)$.*

2.2 Regular Sequences and Complete Intersections

Definition 2.2. Let $X = f^{-1}(0)$ be the analytic set germ defined by f, where $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is an analytic map germ. X is called a complete intersection if $\dim X = n - p$.

Example 2.2. *If $p = 1$, then X is a hypersurface of dimension $n - 1$. Therefore, X is a complete intersection called a hypersurface singularity.*

Definition 2.3. Let E be a ring, $a_1, \cdots, a_p$ non-invertible elements in E. The sequence $a_1, \cdots, a_p$ is called a regular sequence in E if a_j is not a zero divisor in the quotient ring $E/(a_1, \cdots, a_{j-1})$ for $j = 1, \cdots, p$. A maximal regular sequence $(a_1, \cdots, a_p)$ is a regular sequence that cannot be extended to a regular sequence $(a_1, \cdots, a_{p+1})$, i.e. every element of the maximal ideal of $S/(a_1, \cdots, a_p)$ is a zero divisor.

One has the following, see [8, p.109].

Theorem 2.2. *Denote by $f_1, \cdots, f_n$ n homogeneous polynomials in $S = \mathbb{C}[x_1, \cdots, x_n]$. Then $\{f_1, \cdots, f_p\}$ forms a regular sequence in S if and only if the algebraic variety $V = V(f_1, \cdots, f_n) = \{x \in \mathbb{C}^n; f_1(x) = \cdots = f_n(x) = 0\}$ is a complete intersection, i.e. $\dim(V, x) = n - p$ for every $x \in V$.*

2.3 Milnor Number and Tjurina Number

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a function germ. Denote by J_f the Jacobian ideal of f, i.e. the ideal generated by $\frac{\partial f}{\partial x_i}$, $i = 1, \cdots, n$ in the local ring E_n.

Definition 2.4. (i) The analytic Milnor algebra of the germ f is defined by $M(f) = E_n/J_f$. Its dimension $\mu(f) = \dim_{\mathbb{C}} M(f)$ is called the *Milnor number* of the germ f.

(ii) The Tjurina algebra of the germ f is defined by $T(f) = E_n/(f, J_f)$. Its dimension $\tau(f) = \dim_{\mathbb{C}} T(f)$ is called the *Tjurina number* of the germ f.

Example 2.3. *(i)* $\mu(A_k) = \tau(A_k) = k$.

(ii) $\mu(D_4) = \tau(D_4) = 4$.

Let $S = \mathbb{C}[x_0, \cdots, x_n]$ the polynomial ring over $\mathbb{C}$ in $n+1$ variables, with its natural grading, $S = \oplus_{d \geq 0} S_d$, where S_d is the vector space of homogeneous polynomials in S of degree d. For a polynomial $f \in S_d$, we define the algebraic Milnor algebra by $M(f)^a = S/J_f^a$, where J_f^a is the ideal of S spanned by the partial derivatives $\frac{\partial f}{\partial x_0}, \cdots, \frac{\partial f}{\partial x_n}$.

One has the following result, see [8, p.111]

Theorem 2.3. *If f has an isolated singularity at the origin, then the natural morphism $S \longrightarrow E_{n+1}$ induces an isomorphism of local $\mathbb{C}-$algebras $M(f)^a \longrightarrow M(f)$.*

We will use in the sequel the same notation for the algebraic and the analytic setting, namely $M(f) = S/J_f$. Note that $M(f)$ is a graded $\mathbb{C}$-algebra, whose grading is the one induced from the grading of S.

Definition 2.5. For any graded module $M = \oplus_{s \geq s_0} M_s$ over a graded $\mathbb{C}$-algebra of finite type, define the *Poincaré Series* by

$$P_M(t) = \sum_{s \geq s_0} (\dim_{\mathbb{C}} M_s) t^s.$$

Theorem 2.4. [8, p.108] Let $f_1, \cdots, f_p \in S$ be homogeneous polynomials of degree $d_1, \cdots, d_p$ respectively, and assume that $\{f_1, \cdots, f_p\}$ form a regular sequence in S. If $I = (f_1, \cdots, f_p)$ then $M = S/I$ is a graded S-module and

$$P_M(t) = \frac{(1-t^{d_1}) \cdots (1-t^{d_p})}{(1-t)^{n+1}}.$$

2.4 Bézout's Theorem, Cayley-Bacharach Theorem and defects of linear systems

Let C and D be distinct curves in $\mathbb{P}^2$ defined by $F(x,y,z) = 0$ and $G(x,y,z) = 0$ respectively. Suppose that C and D have no common components, and let $P \in C \cap D$ such that $z_P \neq 0$. Regard P as a point in the affine plane with coordinates $(x, y, 1)$. Let f and g be the dehomogenization of F and G respectively with respect to z. We define the *intersection multiplicity* of C and D at the point P by

$$i(C, D; P) = \dim \frac{\mathcal{O}_P}{(f, g)_P},$$

where $\mathcal{O}_P$ is the local ring of germs of regular functions at P, and $(f,g)_P$ is the ideal generated by the germs of f and g in $\mathcal{O}_P$.

Remark 2.1. *If $z_P = 0$, then either x_P or y_P must be nonzero. In this case we dehomogenize F and G with respect to the nonzero coordinate, and we define the intersection multiplicity in an analogous way as above.*

Example 2.4. *Suppose we want to find the intersection multiplicity of each point of intersection of the curves $C : F(x,y,z) = xz - y^2 = 0$ and $D : G(x,y,z) = y - z = 0$. C and D have two points of intersection, namely, $P_1(1:1:1)$ and $P_2(1:0:0)$. Since $z_{P_1} \neq 0$, let $f_1(x,y) = x - y^2$ and $g_1(x,y) = y - 1$ be the dehomogenizations of F and G respectively. Then,*

$$\begin{aligned}\frac{\mathcal{O}_{P_1}}{(f_1,g_1)_{P_1}} &\simeq \frac{\mathbb{C}[x,y]_{(x-1,y-1)}}{(x-y^2,y-1)_{(x-1,y-1)}} \\ &\simeq \left(\frac{\mathbb{C}[x,y]}{(x-y^2,y-1)}\right)_{(x-1,y-1)} \\ &\simeq \left(\frac{\mathbb{C}[x]}{(x-1)}\right)_{(x-1)} \simeq \mathbb{C}.\end{aligned}$$

Therefore, $i(C,D;P_1) = 1$.
Now $z_{P_2} = 0$, then P_2 can be considered as the point at infinity. In this case, one dehomogenizes with respect to $x \neq 0$. Let $f_2(y,z) = z - y^2$ and $g_2(x,z) = y - z$ be the corresponding dehomogenizations. Then,

$$\begin{aligned}\frac{\mathcal{O}_{P_2}}{(f_2,g_2)_{P_2}} &\simeq \frac{\mathbb{C}[y,z]_{(y;z)}}{(z-y^2,y-z)_{(y,z)}} \\ &\simeq \left(\frac{\mathbb{C}[y,z]}{(z-y^2,y-z)}\right)_{(y,z)} \\ &\simeq \left(\frac{\mathbb{C}[y]}{(y(1-y))}\right)_{(y)} \simeq \mathbb{C}.\end{aligned}$$

Hence, $i(C,D;P_2) = 1$

Theorem 2.5. *(Bézout's Theorem,* [20, p.54])
Let C and D be distinct curves in $\mathbb{P}^2$, having degrees d and e respectively. Let $C \cap D = \{P_1, \cdots, P_s\}$. Then

$$\sum_{j=1}^{s} i(C,D;P_j) = de.$$

Example 2.5. *If C and D are the curves in example 2.4. The degree of C is* 2 *and that of D is* 1. *Then by Bézout's theorem, $i(C, D; P_1)+i(C, D; P_2) = 2$, which is the result we got in the previous example.*

Suppose that Γ is a set of γ distinct points in $\mathbb{C}^n$. One may ask about "the failure of Γ to impose independent conditions on polynomials of degree $\leq k$" for some positive integer k. The classical Cayley-Bacharach theorem gives an answer in the case of homogeneous polynomials of degree 3 in $\mathbb{P}^2$. The extended and thought-provoking history of the result starts with a prominent result by Papus of Alexandria proved in the fourth century A.D., and then it developed to have nine versions of the Cayley-Bacharach Theorem from which we will state only one, see [16, CB7].

Defects of Ideals and Defects of Linear Systems

Let I be a homogeneous ideal of S, we define the saturation $\widehat{I}$ of I to be the set of elements $s \in S$ such that for each i there exists a positive integer m_i such that $x_i^{m_i} s \in I$. $\widehat{I}$ is a homogeneous ideal of S.

Definition 2.6. Let Y be a 0-dimensional subscheme in $\mathbb{P}^n$ defined by a homogeneous ideal I. We introduce the corresponding sequence of defects

$$def_k Y = \dim H^0(Y, \mathcal{O}_Y) - \dim \frac{S_k}{\widehat{I}_k}.$$

When Y is a subscheme defined by the Jacobian ideal of a homogeneous polynomial f, then

$$def_k Y = \tau(V) - \dim \frac{S_k}{\widehat{J}_k},$$

where V is the hypersurface defined by $f = 0$, $J = J_f$, and $\tau(V)$ is the sum of all the Tjurina numbers of the singularities of V.

This positive integer is called the failure of Y to impose independent conditions on homogeneous polynomials of degree k.

Remark 2.2. *For a nodal hypersurface D with $\mathcal{N}$ nodes, $h \in \widehat{J}_k$ if and only if h vanishes on the set $\mathcal{N}$ and we get*

$$def_k(\mathcal{N}) = |\mathcal{N}| - codim\{h \in S_k | h(a) = 0 \ for \ any \ a \in \mathcal{N}\}.$$

Cayley-Bacharach Theorem

Before we state the Cayley-Bacharach theorem, we shall define the notion of *residual subscheme* Γ'' to a subscheme Γ' of a zero-dimensional scheme Γ. We want this definition to have the following two properties: (α) the sum of the degrees of Γ' and Γ'' should be equal to that of Γ, (β) the residual subscheme to Γ'' should be again Γ'. We begin first by defining the *Gorenstein* local ring.

Definition 2.7. Let A be a local Artinian ring, and $\mathfrak{m} \subset A$ its maximal ideal. We say that A is *Gorenstein* if the annihilator of $\mathfrak{m}$ has dimension one as a vector space over $K = A/\mathfrak{m}$.

Definition 2.8. A local ring R is Gorenstein if for every maximal regular sequence $(F_0, \cdots, F_k)$ of elements of R the quotient $A = R/(F_0, \cdots, F_k)$ is a Gorenstein Artinian ring.

Proposition 2.2. [16] The local rings of a zero-dimensional complete intersection scheme are Gorenstein.

Definition 2.9. Let Γ be a zero-dimensional scheme with coordinate ring $A(\Gamma) = S/I(\Gamma)$. Let $\Gamma' \subset \Gamma$ be a closed subscheme and $I_{\Gamma'} \subset A(\Gamma)$ its ideal. By the subscheme of Γ *residual* to Γ' we mean the subscheme defined by the ideal

$$I_{\Gamma''} = Ann(I_{\Gamma'}) \subset A(\Gamma).$$

Remark 2.3. *If Γ is a zero-dimensional complete intersection scheme, then Γ' and Γ'' verify the properties (α) and (β) cited above. More generally, if Γ is a zero-dimensional Gorenstein scheme, then Γ' and Γ'' are residual to each other, since $I_{\Gamma''} = (I_{\Gamma'})^{\perp}$, where the orthogonal complement $\perp$ is taken with respect to the natural paring $Q : A(\Gamma) \times A(\Gamma) \longrightarrow \mathbb{C}$, $(u, v) \mapsto p(u, v)$, with $p : A(\Gamma) \longrightarrow \mathbb{C}$ is a linear map inducing isomorphisms $p_x : \mathfrak{m}_x^{\perp} \longleftrightarrow \mathbb{C}$ for any point $x \in \Gamma$.*

Theorem 2.6. *(Cayley-Bacharach) Let $X_1, \cdots, X_n$ be hypersurfaces in $\mathbb{P}^n$ of degrees $d_1, \cdots, d_n$, and suppose that the intersection $\Gamma = X_1 \cap \cdots \cap X_n$ is zero-dimensional. Let Γ' and Γ'' be subschemes of Γ residual to one another in Γ, and set $s = \sum d_i - n - 1$. If $k \leq s$ is a nonnegative integer, then the dimension of the family of hypersurfaces in $\mathbb{P}^n$ of degree k containing Γ', which coincides with $(\widehat{I_{\Gamma'}})_k$, (modulo those containing all of Γ, which*

are exactly the elements of $(\widehat{I_\Gamma})_k$*) is equal to the failure of* Γ'' *to impose independant conditions on hypersurfaces of complementary degree* $s-k$*.*

Example 2.6. *Let* C *be the degree* 3 *cuspidal curve defined by* $f = x^3 - y^2z = 0$*. The Jacobian ideal of* f *is* $J_f = (x^2, y^2, yz)$*, therefore* $\widehat{J_f} = (x^2, y)$*. Using Definition 2.6, we get* $def_0C = 1$ *and* $def_kC = 0$ *for* $k \geq 1$*. Therefore, by Cayley-Bacharach Theorem, the dimension of the family of hypersurfaces of degree* 1 *(respectively* 0*) which contains* $\widehat{J_f}$ *is* 1 *(respectively* 0*).*

Chapter 3

Koszul Complexes

In this chapter, we recall the definitions of the Koszul complexes of the partial derivatives of a homogeneous polynomial f in S. The importance of these complexes is that they are related to the Milnor algebra of f, $M(f)$, and they are useful tools in the computation of the dimensions of the graded pieces of $M(f)$. Note that a homology type complex can be seen as a cohomology type one. Indeed, if we consider the complex $K_* : 0 \to K_n \to K_{n-1} \to \cdots \to K_0 \to 0$, and if we set $L^s = K_{-s}$, we get the equivalent cohomology type complex $L^* : 0 \to L^{-n} \to L^{-n+1} \to \cdots \to L^0 \to 0$. Therefore, all definitions given below about the homology type complexes remain true for cohomology type complexes, and vice versa.

3.1 Homogeneous Koszul Complexes

Let f be a homogeneous polynomial in S of degree e. Define a complex

$$K_*(f) : 0 \to Se_f \xrightarrow{d} S \to 0$$

where $d(e_f) = f$ and d is S-linear. Here we regard e_f as a homogeneous vector of degree e. Then d is a homogeneous morphism of degree 0, and can be seen as the multiplication by f. $K_*(f)$ is called the Koszul complex of f. Note that $(Se_f)_k = S_{k-e}$, in other words, $Se_f = S[-e]$ as graded S-modules. Let $f_0, \cdots, f_n$ be homogeneous polynomials of the same degree e, then the tensor product

$$K_*(\mathbf{f}) = K_*(f_0, \cdots, f_n) = K_*(f_0) \otimes \cdots \otimes K_*(f_n)$$

is the Koszul complex of $\mathbf{f}$, where $\mathbf{f} = (f_0, \cdots, f_n)$. Using the shift introduced above, we have $K_m(\mathbf{f}) = S^{k_m}[-me]$, with $k_m = \binom{n+1}{m}$, or explicitly

$$0 \to S[-(n+1)e] \to S^{n+1}[-ne] \to \cdots \to S^{n+1}[-e] \to S \to 0.$$

Let $K : 0 \to K_{n+1} \to \cdots \to K_0 \to 0$ be a (homology type) Koszul complex of graded finite type S-modules with a degree zero (i.e. homogeneous) differential. And let $K_s : 0 \to K_{s,n+1} \to \cdots \to K_{s,0} \to 0$ for every $s \in \mathbb{N}$ be the complex obtained by taking the homogeneous components of degree s. Define the Euler characteristic of K_s by:

$$\chi(K_s) = \sum_{j=0}^{n+1} (-1)^j \dim(K_{s,j}).$$

Then Poincaré series associated to K is defined by:

$$P(K)(t) = \sum_{s \geq 0} \chi(K_s) t^s.$$

Example 3.1. *Let $K : 0 \to \cdots \to 0 \to K_0 \to 0$, then $P(K)(t) = \sum_{s \geq 0} \chi(K_s) t^s = P(K_0)(t)$ as in Definition 2.5.*

Lemma 3.1. *Let $K : 0 \to K_{n+1} \xrightarrow{d} \ldots \xrightarrow{d} K_0 \to 0$ be a complex of graded finite type S-modules, then $\chi(K_s)(t) = \chi(H_*(K_s))(t)$, and hence, $P(K)(t) = P(H_*(K))(t)$, where H_* can be viewed as a complex $H_* : 0 \to H_{n+1} \xrightarrow{d} \ldots \xrightarrow{d} H_0 \to 0$, with $d = 0$.*

Proof. It is known that for any homomorphism $d_j : K_j \to K_{j-1}$, $K_j = imd_j + \ker d_j$, therefore the Euler characteristic $\chi(K_s)(t)$ can be given by

$$\begin{aligned} \chi(K_s)(t) &= \dim K_{s,0} - \dim imd_1 - \dim \ker d_1 + \dim imd_2 - \dim \ker d_2 \\ &\quad + \cdots + (-1)^{n+1} \dim imd_{n+1} + (-1)^{n+1} \dim \ker d_{n+1} \\ &= \dim coker d_1 - \dim \frac{\ker d_1}{imd_2} + \cdots + (-1)^{n+1} \dim \ker d_{n+1} \end{aligned}$$

For $j = 1, \cdots, n$, $H_j(K_s) = \frac{\ker d_j}{imd_{j+1}}$, $H_0 = \frac{\ker d_0}{imd_1} = coker d_1$, and $H_{n+1} = \ker d_{n+1}$. This proves the first equality. Taking the sum with respect to s, we get the second equality.

□

A regular differential p-form on $\mathbb{C}^{n+1}$ is a differential form

$$\omega = \sum_I c_I dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

where $I = (i_1, \cdots, i_p)$ and $c_I \in S = \mathbb{C}[x_0, \cdots, x_n]$.
Denote by Ω^p be the $\mathbb{C}$-vector space of all the p-forms on $\mathbb{C}^{n+1}$. Ω^p can be made into a graded S-module. Indeed, if f is a homogeneous polynomial of degree N, we set

$$\deg(f dx_{i_1} \wedge \cdots \wedge dx_{i_p}) = N + p.$$

Let $f_0, \cdots, f_n$ be homogeneous polynomials in S_e. Let $K^*(f_0, \cdots, f_n)$ be the (cohomology type) Koszul complex of f_j which can be represented as follows:

$$K^*(\mathbf{f}) = K^*(f_0, \cdots, f_n) : 0 \xrightarrow{\omega\wedge} \Omega^0 \xrightarrow{\omega\wedge} \Omega^1 \to \cdots \to \Omega^{n+1} \to 0$$

where $\omega = f_0 dx_0 + f_1 dx_1 + \cdots + f_n dx_n$. Note that $|\omega| = e + 1$, so in order to have homogeneous differentials we have to consider the complex $\tilde{K}^*(\mathbf{f})$:

$$\tilde{K}^*(\mathbf{f}) : 0 \to \Omega^0[-(n+1)(e+1)] \to \cdots \to \Omega^n[-e-1] \xrightarrow{\omega\wedge} \Omega^{n+1} \to 0$$

Theorem 3.1. *Let $f_0, \cdots, f_n$ be homogeneous polynomials of degree e, and let $K_*(f_0, \ldots, f_n)$ and $K^*(f_0, \ldots, f_n)$ be the Koszul complexes defined as above, then*

$$H_p(K_*(\boldsymbol{f}))_s \simeq H^{n+1-p}(K^*(\boldsymbol{f}))_{s+n+1-p(e+1)} = H^{n+1-p}(\tilde{K}^*(\boldsymbol{f}))_{s+n+1}.$$

Proof. Let e_i be a basis for the free S-module Se_i, where e_i is a homogeneous vector of degree e, for $i = 0, \cdots, n$. Then $\{e_I\}_{|I|=p}$ where $I = \{i_1, \cdots i_p\}$ with $i_1 < i_2 < \cdots < i_p$, and $e_I = e_{i_1} \otimes \cdots \otimes e_{i_p}$, form a basis for K_p. On the other hand, a basis for Ω^{n+1-p} is given by $\{dx_J\}_{|J|=n+1-p}$, where $J = \{j_1, \cdots j_{n+1-p}\}$ with $j_1 < j_2 < \cdots < j_{n+1-p}$, and $dx_J = dx_{j_1} \wedge \cdots \wedge dx_{j_{n+1-p}}$. Define the isomorphisms $\alpha_p : K_p \longrightarrow \Omega^{n+1-p}$ such that $\alpha_p(e_I) = sign(\sigma) dx_J$ where $J = \{0, \cdots, n\} \setminus I$ and $sign(\sigma)$ is the sign of the permutation that transforms (I, J) into $(0, \cdots, n)$. These isomorphisms α_p induce isomorphisms between the complexes $K_*(\mathbf{f})$ and $K^*(\mathbf{f})$ (or $\tilde{K}^*(\mathbf{f})$ up to a sign, and they give the isomorphisms between $H_p(K_*)_s$ and $H^{n+1-p}(K^*)_{s+n+1-p(e+1)}$.
The grading on $K^*(\mathbf{f})$ is simpler to define and it will be used in the sequel of this thesis. The grading on $\tilde{K}^*(\mathbf{f})$ is useful in some questions since this makes the differential homogeneous of degree 0. □

Remark 3.1. *For $p = 0$, $H^{n+1}(K^*) \simeq H_0(K_*) = M(\boldsymbol{f})$. In particular, $H^{n+1}(K^*)_{s+n+1} \simeq H_0(K_*)_s \simeq M(\boldsymbol{f})_s$.*

3.2 Poincaré Series associated to the Jacobian ideal

In this section, we consider the hypersurface $H \subset \mathbb{P}^n$ defined by $f = 0$ and we study the relation between the homology (repectively cohomology) groups of the Koszul complex of the partial derivatives $f_0, \cdots, f_n$ of f, denoted by $K_*(f)$ (respectivly $K^*(f)$), and the singularities of H.

Remark 3.2. *Note that $H_0(K_*(f))$ is the Milnor algebra of f given by $M(f) = S/J_f$.*

For any hypersurface H, we have the following due to Saito [23].

Proposition 3.1. *Let Σ be the singular locus of the hypersurface H. Then*

$$H_k(K) = 0 \ for \ k > \dim(\Sigma) + 1.$$

The case when H is smooth is already known and we have the following, see [8, p.109], as well as Proposition 3.1, Theorem 2.2 and Theorem 2.4.

Proposition 3.2. *The following are equivalent:*

(i) $H_k(K_*(f)) = 0$ *for* $k > 0$ *and* $P(M(f))(t) = (1 - t^{N-1})^{n+1}/(1 - t)^{n+1}$.

(ii) The hypersurface H is smooth.

Proposition 3.3. *For a hypersurface H with isolated singularities, we have:*

$$P(H_0(K_*))(t) - P(H_1(K_*))(t) = (1 - t^{N-1})^{n+1}/(1 - t)^{n+1}.$$

Proof. Since the Poincaré series of a complex does not depend on differentials, then consider a homogeneous polynomial f_s of degree N, such that $D_s : f_s = 0$ is a smooth hypersurface in $\mathbb{P}^n$, and let ${}_sK_*$ be the Koszul complex associated to its partial derivatives, therefore we have by Lemma 3.1

$$P(H_\bullet(K_*))(t) = P(K_*)(t) = P({}_sK_*)(t) = P(H_\bullet({}_sK_*))(t)$$

Since f_s is smooth, then by $(ii) \Rightarrow (i)$ of Proposition 3.2,

$$P(H_\bullet({}_sK_*))(t) = P(H_0({}_sK_*))(t) = (1 - t^{N-1})^{n+1}/(1 - t)^{n+1}.$$

On the other hand, we have by definition

$$P(H_\bullet(K_*))(t) = P(H_0(K_*))(t) - P(H_1(K_*))(t).$$

And the result follows. □

If H is a hypersurface with only isolated singularities, $a_1, \cdots, a_p$, then its singular locus Σ has dimension 0, therefore by Proposition 3.1 and Proposition 3.3, it is enough to study $H_0(K_*)$, so we are only left to compute $P(M(f))(t)$. A. D. R. Choudary and A. Dimca proved in [3] that:

Theorem 3.2. *(A) The Euler characteristic $\chi(H)$ and $\dim M(f)_k$ for $k > (n+1)(N-2)$ depend only on local invariants of the singularities (H, a_j), for example their Milnor numbers $\mu(H, a_j)$ and their Tjurina numbers $\tau(H, a_j)$.*

(B) The Betti numbers $b_m(H)$ and the Poincaré series $P(M(f))$ depend in general not only on local invariants but also on the position of the singularities of the hypersurface H.

(C) Let $H : f = 0$ and $\bar{H} : \bar{f} = 0$ be two hypersurfaces as above. Consider the statements

(α) $P(M(f)) = P(M(\bar{f}))$.

(β) The pairs $(\mathbb{P}^n, H)$ and $(\mathbb{P}^n, \bar{H})$ are homeomorphic.

Then neither $(\alpha) \Rightarrow (\beta)$ nor $(\beta) \Rightarrow (\alpha)$ holds in general.

And we have the following results, making the claim (A) more precise:

Corollary 3.1. [3] The morphism $M(f)_{k-1} \to M(f)_k$ given by the multiplication by a generic linear form $l \in S_1$, is an epimorphism for $k > n(N-2)$ and an isomorphism for $k > (n+1)(N-2)+1$.

Corollary 3.2. [3] $\dim M(f)_k = \tau(H)$ for all $k > (n+1)(N-2)$, where $\tau(H)$ is the sum of all Tjurina numbers $\tau(H, a_j)$.

Proposition 3.4. [9, p.162] Let $\tilde{H}$ be a smooth hypersurface in $\mathbb{P}^n$ having the same degree as H. Then

$$\chi(H) = \chi(\tilde{H}) + (-1)^n \mu(H),$$

where $\mu(H)$ is the sum of all Milnor numbers $\mu(H, a_i)$.

Proposition 3.5. *Let H be a hypersurface in $\mathbb{P}^n$ of degree N with isolated singularities and $\tilde{K}^*$ the homogeneous Koszul complex associated to its partial derivatives defined in section 3.1, then*

$$P(H^n(\tilde{K}^*(f)))(t) = P(H^{n+1}(\tilde{K}^*(f))) - P(H^{n+1}(\tilde{K}^*(f_s))),$$

where f_s is a homogeneous polynomial of degree N defining a smooth hypersurface in $\mathbb{P}^n$.

Proof. The equation follows from the isomorphism between the homology and the cohomology of the Koszul complexes given in Theorem 3.1, as well as Proposition 3.3. □

Proposition 3.6. *[15] Under the same hypothesis of the previous proposition, we have:*

$$\dim H^n(K^*(f))_j = \dim M(f)_{j+N-n-1} - \dim M(f_s)_{j+N-n-1}.$$

For a hypersurface $H : f = 0$ in $\mathbb{P}^n$ with isolated singularities, A. Dimca and G. Sticlaru introduced three integers, as follows, see [15]:

Definition 3.1. (i) the *coincidence threshold* $ct(H)$ defined as

$$ct(H) = max\{q : \dim M(f)_k = \dim M(f_s)_k \ for\ all\ k \leq q\},$$

with f_s a homogeneous polynomial in S of degree N such that H_s: $f_s = 0$ is a smooth hypersurface in $\mathbb{P}^n$.

(ii) the *stability threshold st(H)* defined as

$$st(H) = min\{q : \dim M(f)_k = \tau(H)\ for\ all\ k \geq q\}$$

where $\tau(H)$ is the total number of H, i.e. the sum of all the Tjurina numbers of the singularities of H.

(iii) the *minimal degree of nontrivial syzygy* $mdr(H)$ defined as

$$mdr(H) = min\{q : H^n(K^*(f))_{q+n} \neq 0\}.$$

Using Proposition 3.6, we can prove that $ct(H) = mdr(H) + N - 2$, and since by Proposition 3.2 $\dim M(f_s)_k = 0$ for $k > (n+1)(N-2)$, then $N - 2 \leq ct(H) \leq (n+1)(N-2)$. Also by Corollary 3.2, we have $st(H) \leq (n+1)(N-2)+1$. We have the following results, see [15].

Proposition 3.7. *Let $C : f = 0$ be nodal curve of degree N in $\mathbb{P}^2$. Then one has $ct(C) \geq 2N - 4$.*

Therefore, the dimensions of $M(f)_k$ are determined for all $k < 2N-3$, and the next dimension is given by

$$\dim M(f)_{2N-3} = n(C) + \sum_{j=1}^{r} g_j = g + r - 1 \tag{3.1}$$

where $n(C) = \tau(C)$ is the total number of nodes of C and g_j are the genera of the normalizations of the irreducible components C_j of C whose number is r, and

$$g = \frac{(N-1)(N-2)}{2}.$$

Corollary 3.3. *If C is a rational nodal curve, then the Hilbert-Poincaré series $P(M(f))$ is completely determined by the degree N and the number of nodes $n(C)$. In particular, $st(C) = 2N-3$ unless C is a generic line arrangement and then $st(C) = 2N-4$.*

Proof. For a rational curve, $g_j = 0$ for $j = 1, \cdots, r$, then by Equation 3.1 $\dim M(f)_{2N-3} = n(C) = \tau(C)$. On the other hand, by Corollary 3.1 $\dim M(f)_k$ is decreasing for $k \geq 2N-3$ and constant for $k > 3N-5$, this proves that $st(C) = 2N-3$. If C is a generic line arrangement, then $\dim M(f)_{2N-3} = n(C) = \binom{N}{2}$, $\dim M(f)_{2N-4} = \dim M(fs)_{2N-4} = \dim M(f_s)_{N-2} = \binom{N}{2}$, and $\dim M(f)_{2N-5} = \dim M(f_s)_{2N-5} > \binom{N}{2}$. □

Example 3.2. *Let C be the degree 4 curve defined by $f = x(x^3+y^3+z^3)$. Then C has 3 collinear nodes. $st(C) \leq 3N-5 = 7$ and $ct(C) \geq 2N-4 = 4$. Indeed a computation using Singular [5] yields the following Hilbert-Poincaré series*

$$HP(M(f))(t) = 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 4t^5 + 3(t^6 + t^7 + \cdots)$$

and hence $ct(C) = 4$ and $st(C) = 6$.

Example 3.3. *Let C be a generic line arrangement defined by $f = xyz(x+y+z) = 0$. Then C has 6 nodes, by Corollory 3.3, $HP(M(f))$ is all determined and we have $st(C) = 2N-4 = 4$ and $ct(C) \geq 4$. Therefore*

$$HP(M(f))(t) = 1 + 3t + 6t^2 + 7t^3 + 6(t^4 + t^5 + \cdots),$$

which implies $ct(C) = 4$.

Theorem 3.3. *(nodal case) Let $H : f = 0$ be a degree N nodal hypersurface in $\mathbb{P}^n$ and $\mathcal{N}$ denotes the set of its nodes. Then*

$$\dim H^n(K^*(f))_{nN-n-1-k} = def_k(\mathcal{N})$$

for $0 \leq k \leq nN-2n-1$ and $\dim H^n(K^*(f))_j = \tau(H) = |\mathcal{N}|$ for $j \geq n(N-1)$. In other words,

$$\dim M(f)_{T-k} = \dim M(f_s)_k + def_k(\mathcal{N})$$

for $0 \leq k \leq nN - 2n - 1$, where $T = (n, N) = (n + 1)(N - 2)$. In particular $\dim M(f)_T = \tau(H)$, i.e. $st(H) \leq T$.

For the general case of H with isolated singularities, see Theorem 1 in [11].

Example 3.4. *Let $C : f = x(x^3 + y^3 + z^3)$. C has 3 collinear nodes. We have by Corollary 3.2, $st(C) \leq 7$, and by Proposition 3.7, $ct(C) \geq 4$. The remaining dimensions, $\dim M(f)_5$ and $\dim M(f)_6$, can be computed by Theorem 3.3. $\dim M(f)_6 = \dim M(f_s)_0 + def_0(\mathcal{N}) = 1 + 2 = 3$ using Definition 2.6. Similarly, one gets $\dim M(f)_5 = 4$. Hence, the corresponding Poincaré series is*

$$P(M(f))(t) = 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 4t^5 + 3(t^6 + t^7 + \cdots).$$

Example 3.5. *Every cubic cuspidal curve is K-equivalent to the normal form $f = x^3 - zy^2$ whose Tjurina number is $\tau(C) = 2$. We have $st(C) \leq 4$, and $ctC) \geq 3 - 2 = 1$. Using Theorem 1 in [11] and Example 2.6, we get $M(f)_2 = 3$ and $M(f)_3 = 2$. Hence, the corresponding Poincaré series is*

$$P(M(f))(t) = 1 + 3t + 3t^2 + 2(t^4 + t^5 + \cdots).$$

Example 3.6. *Let $C : f = x^p y^q + z^N = 0$ where $p > 0$, $q > 0$ and $p + q = N > 2$. $st(C) \leq 3(N - 2) + 1 = 3N - 5$ and since $qxf_x - pyf_y = 0$, then $mdr(C) = 1$. Therefore $ct(C) = N - 1$. By Proposition 1 in [11], one can compute $\dim M(f)_{3N-6-k}$ for $0 \leq k \leq 2N - 5$, or $\dim M(f)_j$ for $d - 1 \leq j \leq 3d - 6$. Hence, the corresponding Poincaré series is known for every $N > 2$.*

In the case of plane curves with ordinary double and triple points we have the following results, see [13], Example 2.2 (iii) and Example 2.8 (iii).

Proposition 3.8. *Let $C : f = 0$ be a curve of degree N in $\mathbb{P}^2$ with ordinary double and triple points. Then one has $mdr(C) \geq 2N/3 - 2$ and $ct(C) \geq 5N/3 - 4$.*

Chapter 4

Hodge Structures

4.1 Basic Facts about Mixed Hodge Theory

A *(pure) Hodge structure* is an algebraic structure introduced by W. V. D. Hodge that applies to smooth and compact Kähler manifolds. In 1970, P. Deligne defined the *mixed Hodge structure* which is a generalization of the pure Hodge structure, that applies to all complex algebraic varieties. In this section, we give some basic definitions and properties. For more details, we refer to Deligne [6], Voisin [27] and Appendix C [9].

Definition 4.1. A (pure) *Hodge Structure of weight m* is a pair (H, F), where H is a finite dimensional $\mathbb{Q}$-vector space and F is a decreasing filtration on $H_{\mathbb{C}} = H \otimes \mathbb{C}$ (called the *Hodge filtration*) such that:

(i) F is a finite filtration, i.e., there exist $s, t \in \mathbb{Z}$ with $F^s H_{\mathbb{C}} = H_{\mathbb{C}}$ and $F^t H_{\mathbb{C}} = 0$;

(ii) $H_{\mathbb{C}} = F^p H_{\mathbb{C}} \oplus \overline{F^{m-p+1}}$ for all $p \in \mathbb{Z}$, where the conjugation on $H_{\mathbb{C}}$ is induced from the complex conjugation on $\mathbb{C}$.

Or, equivalently, if we set $H^{p,q} = F^p H_{\mathbb{C}} \cap \overline{(F^q H_{\mathbb{C}})}$ for any pair (p, q) with $p + q = m$, then we have the following relations:

(α) $H_{\mathbb{C}} = \bigoplus_{p+q=m} H^{p,q}$.

(β) $\overline{H^{p,q}} = H^{q,p}$.

Conversly, given (α) and (β), H has a *Hodge Structure* where the *Hodge filtration* is defined by the formula

$$F^p H_{\mathbb{C}} = \bigoplus_{s \geq p} H^{s,m-s}.$$

Definition 4.2. The *Hodge numbers* of $H_{\mathbb{C}}$ are defined by the formula

$$h^{p,q}(H) = \dim_{\mathbb{C}} H^{p,q}$$

Remark 4.1. *Since* $\overline{H^{p,q}} = H^{q,p}$*, then* $h^{p,q} = h^{q,p}$*.*

Example 4.1. *If X is a smooth projective variety, then the cohomology group $H^m(X, \mathbb{Q})$ has a pure Hodge structure of weight m, for any $m \geq 0$. In particular the Hodge numbers $h^{p,q}(H) = h^{p,q}(H^{p+q}(X, \mathbb{C}))$ are well-defined.*

We recall now the definition and some properties about mixed Hodge structure.

Definition 4.3. A *mixed Hodge structure* (MHS) is a triple (H, W, F) where:

(i) H is a finite dimensional $\mathbb{Q}$-vector space;

(ii) W is a finite increasing filtration on H called the *weight filtration*;

(iii) F is a finite decreasing filtration on $H_{\mathbb{C}}$ called the *Hodge filtration*, such that $(Gr_k^W H, F)$ is a Hodge structure of weight k for all k.

where

$$Gr_k^W H = W_k H / W_{k-1} H,$$

and the induced filtration is given by

$$F^p (Gr_k^W H)_{\mathbb{C}} = (F^p H_{\mathbb{C}} \cap W_k H_{\mathbb{C}} + W_{k-1} H_{\mathbb{C}}) / W_{k-1} H_{\mathbb{C}}.$$

When (H, W, F) is a MHS we can define the *mixed Hodge numbers* by

$$h^{p,q}(H) = \dim \ Gr_F^p Gr_{p+q}^W H_{\mathbb{C}}.$$

The importance of the MHS is that the cohomology of any algebraic variety has such a structure. More precisely, we have the following theorem.

Theorem 4.1. *(Deligne) There is a functorial MHS on $H^*(X;\mathbb{Q})$ for any quasi-projective variety X such that the following properties hold for all $m \geq 0$:*

(i) the weight filtration W on $H^m(X;\mathbb{Q})$ satisfies

$$0 = W_{-1} \subset W_0 \subset \cdots \subset W_{2m} = H^m(X;\mathbb{Q});$$

for $m \geq n = \dim X$, we also have $W_{2n} = \cdots = W_{2m}$;

(ii) the Hodge filtration F on $H^m(X;\mathbb{C})$ satisfies $H^m(X;\mathbb{C}) = F^0 \supset \cdots \supset F^{m+1} = 0$. For $n = \dim X$, we also have $F^{n+1} = 0$;

(iii) if X is a smooth variety, then $W_{m-1}H^m(X,\mathbb{Q}) = 0$ (i.e., all weights on $H^m(X;\mathbb{Q})$ are $\geq m$) and $W_mH^m(X,\mathbb{Q}) = j^*H^m(\overline{X},\mathbb{Q})$ for any compactification $j : X \hookrightarrow \overline{X}$;

(iv) if X is a projective variety, then $W_mH^m(X,\mathbb{Q}) = H^m(X,\mathbb{Q})$ (i.e., all weights on $H^m(X;\mathbb{Q})$ are $\leq m$) and $W_{m-1} = kerp^*$ for any proper map $p : \tilde{X} \to X$ with $\tilde{X}$ smooth.

Remark 4.2. *By (iii) and (iv) in Theorem 4.1, if X is a smooth projective variety, then $H^m(X,\mathbb{Q})$ has a pure Hodge structure of weight m, as in Example 4.1.*

For a projective variety $X \subset \mathbb{P}^n$, let $j^m : H^m(\mathbb{P}^n) \to H^m(X)$ be the morphism induced by the inclusion $j : X \to \mathbb{P}^n$. Define the primitive m-th cohomology group of X to be $H_0^m(X) = coker j^m$. The mixed Hodge numbers of the complement U of X and those of X are related by

$$h^{p,q}(H^m(U)) = h^{n-p,n-q}(H_0^{2n-m-1}(X)), \tag{4.1}$$

see [9, p.246].

Let X be a variety, and let $H_c^*(X;\mathbb{C})$ be the cohomology groups of X with compact support. These groups also admit MHS by the work of P. Deligne. If X is smooth then $H_c^m(X;\mathbb{C})$ is dual to $H^{2n-m}(X;\mathbb{C})$, where $n = \dim_{\mathbb{C}} X$, (see Theorem 6.23 in [19, p.155]), and we have the following.

Proposition 4.1. *With the above notations, we have*

$$h^{p,q}(H^m(X,\mathbb{C})) = h^{n-q,n-p}(H_c^{2n-m}(X,\mathbb{C})),$$

where $h^{p,q}$ are the mixed Hodge numbers for every $p, q \leq n$.

Definition 4.4. Let X be a quasi-projective complex variety. The *Hodge-Deligne* polynomial of X is defined by

$$P(X)(u,v) = \sum_{p,q} E^{p,q}(X) u^p v^q$$

where $E^{p,q} = \sum_s (-1)^s \dim Gr_F^p Gr_{p+q}^W H_c^s(X,\mathbb{C}) = \sum_s (-1)^s h^{p,q}(H_c^s(X,\mathbb{C}))$.

Proposition 4.2. *Let Y be a closed subvariety of X, then $P(X) = P(X\backslash Y) + P(Y)$.*

Proof. This follows immediatley from the long exact sequence of the compactly supported cohomology, namely,

$$\cdots \to H_c^{m-1}(Y) \to H_c^m(X\backslash Y) \to H_c^m(X) \to H_c^m(Y) \to H_c^{m+1}(X\backslash Y) \to \cdots$$

which is strictly compatible with the filtrations F^* and W_*, i.e. the sequence remains exact after application of $Gr_F^p Gr_{p+q}^W$. □

4.2 Hodge theory for plane curves with double and triple points

In this section, we take $X = C$ to be a curve in $\mathbb{P}^2$, and $U = \mathbb{P}^2 \setminus C$. By Theorem 4.1, $H^*(U,\mathbb{C})$ has a mixed Hodge structure. In particular, for the Hodge filtration on $H^m(U)$, one has

$$H^m(U) = F^0 = F^1 \supset F^2 \supset \cdots \supset F^m \supset F^{m+1} = 0,$$

where the equality $F^0 = F^1$ follows from the fact that $\dim Gr_F^0 H^m(U) = \dim \frac{F^0}{F^1} = h^{0,m} + h^{0,m+1} + h^{0,2m} = 0$ since

$$h^{0,q}(H^m(U)) = h^{2,2-q}(H_0^{3-m}(C)) = 0,$$

for all q since $2 > \dim C = 1$ (see Equation 4.1).

The case $m = 1$ is not interesting to be studied because the Hodge filtration on $H^1(U)$ is trivial, i.e. is given by

$$H^1(U) = F^0 = F^1 \supset F^2 = 0.$$

We are only interested in the case $m = 2$, where the Hodge filtration on $H^2(U)$ is given by

$$H^2(U) = F^0 = F^1 \supset F^2 \supset F^3 = 0, \tag{4.2}$$

and the weight filration by

$$0 = W_1 \subset W_2 \subset W_3 \subset W_4 = H^2(U), \tag{4.3}$$

Note that for $k > 2 = \dim U$ $H^k(U) = 0$ since U is an affine variety.

A. Dimca and G. Sticlaru described the dimensions of $Gr_F^s H^2(U)$ for nodal curves in the following proposition.

Proposition 4.3. *Let $C \subset \mathbb{P}^2$ be a nodal curve of degree N and set $U = \mathbb{P}^2 \backslash C$. Let $C = \bigcup_{j=1,r} C_j$ be the decomposition of C as a union of irreducible components, let $\nu : \tilde{C}_j \to C_j$ be the normalization mappings and set $g_j = g(\tilde{C}_j)$. Then one has*

$$\dim Gr_F^1 H^2(U, \mathbb{C}) = \sum_{j=1}^{r} g_j$$

and

$$\dim Gr_F^2 H^2(U, \mathbb{C}) = \frac{(N-1)(N-2)}{2}.$$

Example 4.2. *Let $C \subset \mathbb{P}^2$ be the nodal curve defined by $f = x(x^3+y^3+z^3) = 0$. C is the union of two irreducible components, a line $C_1 : f_1 = x = 0$ and a smooth curve $C_2 : f_2 = x^3+y^3+z^3 = 0$, see the figure below. Since C_1 and C_2 are smooth, the genus of each one is equal to the arithmetic genus given by the formula $p_a(C_j) = \frac{(N_j-1)(N_j-2)}{2}$ where N_j is the degree of f_j for $j = 1, 2$. Then we have $g_1 = 0$ and $g_2 = 1$. By Theorem 4.1, the Hodge filtration is given by $0 \subset F^2 \subset F^1 = H^2(U, \mathbb{C})$, and hence $\dim Gr_F^1 H^2(U, \mathbb{C}) = \dim \frac{F^1}{F^2} = 1$, and $\dim Gr_F^2 H^2(U, \mathbb{C}) = \dim F^2 = 3$.*



One of the main results of this thesis is a generalization of Proposition 4.3 to curves with ordinary double and triple points as singularities.

Theorem 4.2. *Let $C \subset \mathbb{P}^2$ be a curve of degree N. Suppose that C has only n nodes and t triple points. Set $U = \mathbb{P}^2 \backslash C$. Let $C = \bigcup_{j=1,r} C_j$ be the decomposition of C as a union of irreducible components,let $\nu : \tilde{C}_j \to C_j$ be the normalization mappings and set $g_j = g(\tilde{C}_j)$. Then one has*

$$\dim Gr_F^1 H^2(U, \mathbb{C}) = \sum_{j=1}^{r} g_j$$

and

$$\dim Gr_F^2 H^2(U, \mathbb{C}) = \frac{(N-1)(N-2)}{2} - t.$$

Proof. Suppose that the curve $C_j : f_j = 0$ has degree N_j, and has n_j nodes and t_j triple points.
Using the normalization maps ν_j and the additivity of the Hodge-Deligne polynomial, it follows that,

$$\begin{aligned} P(C_j) &= P(C_j \backslash (C_j)_{sing}) + P((C_j)_{sing}) = P(\tilde{C}_j \backslash \{(2n_j + 3t_j)\ points\}) + n_j + t_j \\ &= P(\tilde{C}_j) - P(\{2n_j + 3t_j\ points\}) + n_j + t_j \\ &= uv - g_j u - g_j v + 1 - n_j - 2t_j. \end{aligned}$$

Indeed, it is known that for a smooth curve C, the genus $g(C)$ is exactly the Hodge number $h^{1,0}(C) = h^{0,1}(C)$. Now,

$$\begin{aligned} P(C) &= P(C_1 \cup \cdots \cup C_r) \\ &= \sum_{j=1}^{r} P(C_j) - \sum_{1 \le i < j \le r} P(C_i \cap C_j) + \sum_{1 \le i < j < k \le r} P(C_i \cap C_j \cap C_k) \\ &= ruv - \left(\sum_{j=1}^{r} g_j \right) u - \left(\sum_{j=1}^{r} g_j \right) v + r - \sum_{j=1}^{r} (n_j + 2t_j) \\ &\quad -(\sum_{1 \le i < j \le r} N_i N_j - s) + t', \end{aligned}$$

where s (respectively t') denotes the total number of triple points which are intersection of only two (respectively three) curves. The term $\sum_{1 \le i < j \le r} N_i N_j -$

s is a result of Bézout's Theorem. Indeed, the total number of intersection points counted with multiplicities is $\sum_{1\leq i<j\leq r} N_iN_j$, and it is easy to see that the intersection multiplicity of each of the s triple points is 2, i.e., each one is counted twice, whereas the intersection multiplicity of each of the t' triple points is 1, in other words $3t'$ contribution to the sum $\sum_{1\leq i<j\leq r} N_iN_j - s$, each such being the intersection of two locally smooth curves in three ways. Next we have by the additivity $P(U) = P(\mathbb{P}^2) - P(C)$ where $P(\mathbb{P}^2) = u^2v^2+uv+1$. Let's look at the cohomology of the smooth surface U. The group $H^4_c(U,\mathbb{C})$ is dual to the group $H^0(U,\mathbb{C})$, which is 1-dimensional of type (0,0). It follows that $H^4_c(U,\mathbb{C})$ is 1-dimensional of type (2,2) and its contribution to $P(U)$ is exactly the term u^2v^2.
The group $H^3_c(U,\mathbb{C})$ is dual to the group $H^1(U,\mathbb{C})$, which is $(r-1)$-dimensional of type (1,1). Indeed, by Theorem 4.1, the only nonzero weights on $H^1(U,\mathbb{C})$ are $m = 1$ and $m = 2$. But W_1 is zero since $W_1 = j^*(H^1(\mathbb{P}^2)) = 0$ where $j : U \longrightarrow \mathbb{P}^2$ is the compactification of U and we apply again Theorem 4.1 (iii). It follows that the contribution of $H^3_c(U,\mathbb{C})$ to $P(U)$ is exactly the term $-(r-1)uv$.
The remaining terms come from group $H^2_c(U,\mathbb{C})$, which is dual to the group $H^2(U,\mathbb{C})$. By theorem 4.1, the only nonzero weights on $H^2(U,\mathbb{Q})$ are $m = 2$, 3 and 4. On the other hand, $W_2 = 0$, since $W_2 = j^*H^2(\mathbb{P}^2)$. But $H^2(\mathbb{P}^2) = \alpha\mathbb{Q}$, where $\alpha = c_1(L)$, c_1 denotes the Chern class of the line bundle $L = O(1)$. Therefore, by the naturality property of Chern classes (see [2]), $N\alpha = j^*(c_1(L^{\otimes N})) = c_1(j^*(L^{\otimes N})) = c_1(L^{\otimes N}|U)$ which is equal to zero since the line bundle $L^{\otimes N}|U$ has a nowhere vanishing section given by f, the defining equation of C. This implies that $H^2(U,\mathbb{C})$ has only classes of type $(2,1)$, $(1,2)$, and $(2,2)$. Therefore the dimension $\dim Gr^1_F H^2(U,\mathbb{C})$ is the number of independent classes of type (1,2), which correspond to classes of type $(1,0)$ in $H^2_c(U)$ according to Proposition 4.1, and hence to the terms in u in $P(U)$. This gives the first equality.
Now the dimension $\dim Gr^2_F H^2(U,\mathbb{C})$ is the number of independent classes of type (2,1) or (2,2), which correspond respectively to the terms in v or the constant terms in the polynomial $P(U)$. This yields

$$\dim Gr^2_F H^2(U,\mathbb{C}) = \sum_{j=1}^{r}(g_j + n_j + 3t_j - 1) + \sum_{1\leq i<j\leq r} d_id_j + 1 - (\sum_{j=1}^{r} t_j + s + t').$$

Recall the formula

$$g_j + n_j + 3t_j = p_a(C_j) = \frac{(N_j-1)(N_j-2)}{2},$$

where p_a denotes the arithmetic genus, see [20, p.298 and p.54]. Recall also that the δ-invariant of a node A_1 is 1 and the δ-invariant of a triple point D_4 is 3. Let $t := \sum_{j=1}^{r} t_j + s + t'$, using the fact that $N = \sum_{j=1}^{r} N_j$, we get:

$$\begin{aligned}
\dim Gr_F^2 H^2(U,\mathbb{C}) &= \sum_{j=1}^{r}\left(\frac{(N_j-1)(N_j-2)}{2}-1\right)+\sum_{1\le i<j\le r} N_iN_j+1-t \\
&= \sum_{j=1}^{r}\frac{N_j^2-3N_j}{2}+\sum_{1\le i<j\le r} N_iN_j+1-t \\
&= (\sum_{j=1}^{r}\frac{N_j^2}{2}+\sum_{1\le i<j\le r} N_iN_j)-\sum_{j=1}^{r}\frac{3N_j}{2}+1-t \\
&= \frac{1}{2}(\sum_{j=1}^{r} N_j)^2-\frac{3N}{2}+1-t=\frac{N^2-3N+2}{2}-t \\
&= \frac{(N-1)(N-2)}{2}-t.
\end{aligned}$$

□

Remark 4.3. *As shown in the proof above, we have $W_2H^2(U,\mathbb{C})=0$, and this implies*

$$h^{2,0}(H^2(U)) = h^{1,1}(H^2(U)) = h^{0,2}(H^2(U)) = 0.$$

On the other hand, since $F_0 = F_1$, then $h^{0,j} = h^{j,0} = 0$, for all j. We have also $h^{1,3} = h^{3,1} = 0$, since $F^3 = 0$.

This last remark and Theorem 4.2 allow us to compute all mixed Hodge numbers $h^{p,q}(H^2(U))$. Indeed, $\dim Gr_F^1 H^2(U) = h^{1,2}$, and $\dim Gr_F^2 H^2(U) = h^{2,1} + h^{2,2}$. We conclude,

Corollary 4.1. *With the above notation and assumptions, we have the following.*

(i) $h^{2,1}(H^2(U)) = h^{1,2}(H^2(U)) = \sum_{j=1}^{r} g_j$.

(ii) $h^{2,2}(H^2(U)) = \frac{(N-1)(N-2)}{2} - \sum_{j=1}^{r} g_j - t$.

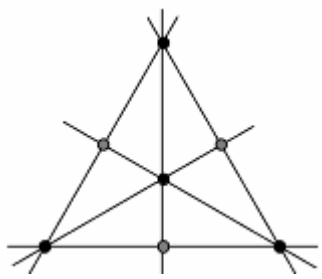
(iii) $b_2(U) = \frac{(N-1)(N-2)}{2} + \sum_{j=1}^{r} g_j - t$, *where $b_2(U)$ denotes the second Betti number of the complement U.*

In particular, it follows that $H^2(U)$ is pure of type $(2,2)$ when $g_j = 0$ for all j, a well known property in the case of line arrangements. In addition, in the case of rational curves with double and triple points, the Hodge filtration on $H^2(U)$ is given by

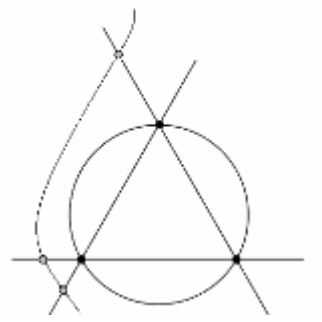
$$H^2(U) = F^0 = F^1 = F^2 \supset F^3 = 0,$$

since $Gr_F^1(H^2(U)) = \frac{F^1}{F^2} = 0$.

Example 4.3. *Let* $C: (x^2-y^2)(y^2-z^2)(x^2-z^2) = 0$. *$C$ is the union of* 6 *lines in* $\mathbb{P}^2$. *It has* 4 *triple points* $((1,1,1), (1,-1,1), (-1,1,1), (-1,-1,1))$ *and* 3 *nodes* $((0,1,0),(1,0,0),(0,0,1))$. *We have* $g_i = 0$ *for every* $i = 1, \cdots 6$, $N = 6$, *and* $t = 4$. *Then,* $\dim H^1 = 5$, *and according to the aforementioned theorem (4.2) we get* $\dim Gr_F^1 H^2(U, \mathbb{C}) = \dim \frac{F^1}{F^2} = 0$ *and* $\dim Gr_F^2 H^2(U, \mathbb{C}) = \dim F^2 = 6$. *Hence* $b_2(U) = 6$.



Example 4.4. *Let* $C: xyz(x^2y + x^2z + y^2x + y^2z + z^2x + z^2y) = 0$. *$C$ is the union of* 3 *lines giving rise to a triangle and a smooth cubic curve. It has* 3 *triple points (the vertices of the triangle) and* 3 *nodes. We have* $g_1 = g_2 = g_3 = 0$, $g_4 = 1$, $d = 6$, *and* $t = 3$. *Then* $\dim H^1(U) = 3$, $\dim Gr_F^1 H^2(U, \mathbb{C}) = 1$ *and* $\dim Gr_F^2 H^2(U, \mathbb{C}) = 7$, *and* $b_2(U) = 1 + 7 = 8$.



Example 4.5. *Let* $C: (x^3-y^3)(y^3-z^3)(x^3-z^3)=0$. *$C$ is the union of 9 lines. It has 12 triple points and no nodes. We have $g_i=0$ for every $i=1,\cdots 6$, $N=9$, and $t=12$. Then* $\dim H^1=8$, $\dim Gr_F^1H^2(U,\mathbb{C})=0$ *and* $\dim Gr_F^2H^2(U,\mathbb{C})=16=b_2(U)$.

4.3 On Hodge Theory of Singular Plane Curves

In this section, we generalize the results of the previous section to curves with more general singularities.

Suppose first C is an irreducible curve of degree N. Denote by a_k, $k=1,...,p$ the singular points of C, and let $r(C,a_k)$ be the number of irreducible branches of the germ (C,a_k). Let $\nu:\tilde{C}\to C$ be the normalization mapping. Using the normalization map ν and the additivity of the Hodge-Deligne polynomial, it follows that,

$$\begin{aligned} P(C) &= P(C\backslash(C)_{sing})+P((C)_{sing})=P(\tilde{C}\backslash(\cup_k\nu^{-1}(a_k))+p \\ &= P(\tilde{C})-\sum_k P(\nu^{-1}(a_k))+p=uv-gu-gv+1-\sum_k(r(C,a_k)-1). \end{aligned}$$

It is known that one has the formula

$$g=\frac{(N-1)(N-2)}{2}-\sum_k\delta(C,a_k), \tag{4.4}$$

relating the genus, the degree and the local singularities of C, and the δ-invariants can be computed using the formula

$$2\delta(C,a_k)=\mu(C,a_k)+r(C,a_k)-1, \tag{4.5}$$

where $\mu(C,a_k)$ is the Milnor number of the singularity (C,a_k). For both formulas above, see [21, p.85]. This proves the following result.

Proposition 4.4. *With the above notation and assumptions, we have the following for an irreducible plane curve $C\subset\mathbb{P}^2$.*

(i) The Hodge-Deligne polynomial of C is given by

$$P(C)(u,v)=uv-gu-gv+1-\sum_k(r(C,a_k)-1),$$

with g given by the formula (4.4).

(ii) $H^0(C) = \mathbb{C}$ *is pure of type* $(0,0)$.

(iii) $H^2(C) = \mathbb{C}$ *is pure of type* $(1,1)$.

(iv) The mixed Hodge numbers of the MHS on $H^1(C)$ *are given by*

$$h^{0,0}(H^1(C)) = \sum_k (r(C,a_k)-1), \quad h^{1,0}(H^1(C)) = h^{0,1}(H^1(C)) = g.$$

In particular, one has the following formulas for the first Betti number of C.

$$b_1(C) = \sum_k (r(C,a_k)-1) + 2g = (N-1)(N-2) - \sum_k \mu(C,a_k).$$

Now we consider the case of a curve C having several irreducible components. More precisely, let $C = \bigcup_{j=1,r} C_j$ be the decomposition of C as a union of irreducible components C_j, let $\nu_j : \tilde{C}_j \to C_j$ be the normalization mappings and set $g_j = g(\tilde{C}_j)$. Suppose that the curve C_j has degree N_j, denote by a_k^j for $k = 1, ..., p_j$ be the singular points of C_j and let $r(C_j, a_k^j)$ be the number of branches of the germ (C_j, a_k^j). Then the formulas (4.4) and (4.5) can be applied to each irreducible curve C_j, as well as Proposition 4.4.

Let A be the union of the singular sets of the curves C_j. Let B be the set of points in C sitting on at least two distinct components C_i and C_j. For $b \in B$, let $n(b)$ be the number of irreducible components C_j passing through b. By definition, $n(b) \geq 2$. Moreover, note that the sets A and B are not disjoint in general, and their union is precisely the singular set of C.

Using the additivity of Hodge-Deligne polynomials we get

$$P(C) = P(C_1 \cup \cdots \cup C_r) = \sum_{j=1}^{r} P(C_j) + (-1)^{l-1} \sum_{0 \leq i_1 < \cdots < i_l \leq r} P(C_{i_1} \cap \cdots \cap C_{i_l}).$$

The first sum is easy to determine using Proposition 4.4.

$$\sum_{j=1}^{r} P(C_j) = ruv - \left(\sum_{j=1}^{r} g_j\right) u - \left(\sum_{j=1}^{r} g_j\right) v + r - \sum_{j,k} ((r(C_j, a_k^j) - 1).$$

Consider now the alternated sum, where $l \geq 2$. The only points of C that give a contribution to this sum are the points in B. Now, for a point $b \in B$, its contribution to the alternated sum is clearly given by

$$c(b) = -\binom{n(b)}{2} + \binom{n(b)}{3} - ... + (-1)^{n(b)-1}\binom{n(b)}{n(b)} = -n(b) + 1.$$

Proposition 4.5. *With the above notation and assumptions, we have the following for a reducible plane curve $C = \bigcup_{j=1,r} C_j$.*

(i) The Hodge-Deligne polynomial of C is given by

$$\begin{aligned} P(C)(u,v) &= ruv - \left(\sum_{j=1}^{r} g_j\right) u - \left(\sum_{j=1}^{r} g_j\right) v + r - \sum_{j,k}((r(C_j, a_k^j) - 1) \\ &\quad - \sum_{b \in B}(n(b) - 1). \end{aligned}$$

with g_j given by the formula (4.4).

(ii) $H^0(C) = \mathbb{C}$ is pure of type $(0,0)$.

(iii) $H^2(C) = \mathbb{C}^r$ is pure of type $(1,1)$.

(iv) The mixed Hodge numbers of the MHS on $H^1(C)$ are given by

$$h^{0,0}(H^1(C)) = \sum_{j,k}((r(C_j, a_k^j) - 1) + \sum_{b \in B}(n(b) - 1) - r + 1,$$

$$h^{1,0}(H^1(C)) = h^{0,1}(H^1(C)) = \sum_j g_j.$$

In particular, one has the following formula for the first Betti number of C.

$$b_1(C) = \sum_{j,k}((r(C_j, a_k^j) - 1) + \sum_{b \in B}(n(b) - 1) - r + 1 + 2\sum_j g_j.$$

Note that a point in the intersection $A \cap B$ will give a contribution to the last two sums in the above formula for $P(C)$.

Example 4.6. *Suppose C is a nodal curve. Then for each singularity $a_k^j \in A$ one has $a_k^j \notin B$ (otherwise we get worse singularities than nodes) and $r(a_k^j) = 2$. Moreover, each point $b \in B$ satisfies $n(b) = 2$. It follows that in this case we get*

$$P(C)(u,v) = ruv - \left(\sum_{j=1}^{r} g_j\right) u - \left(\sum_{j=1}^{r} g_j\right) v + r - n_2,$$

with n_2 the number of nodes of C. More precisely, in this case we have $n_2 = n_2' + n_2''$, where n_2' (resp. n_2'') is the number of nodes of C in A (resp. in B) and one clearly has

$$n_2' = S_1 := \sum_{j,k}((r(C_j, a_k^j) - 1), \quad n_2'' = S_2 := \sum_{b \in B}(n(b) - 1).$$

Example 4.7. *Suppose C has only nodes and ordinary triple points as singularities. Then let n_3 be the number of triple points and note that we can write as above $n_3 = n_3' + n_3''$, where n_3' (resp. n_3'') is the number of triple points of C in $A_0 = A \setminus B$ (resp. in B). For a point $a \in A_0$, the contribution to the sum S_1 is 2, while the contribution to the sum S_2 is 0.*

A point $b \in B$ can be of two types. The first type, corresponding to the partition $3 = 1 + 1 + 1$, is when b is the intersection of three components C_j, all smooth at b. The contribution of such a point b is 0 to the sum S_1 and 2 to the sum S_2. The second type, corresponding to the partition $3 = 2 + 1$, is when b is the intersection of two components, say C_i and C_j, such that C_i has a node at b, and C_j is smooth at b. The contribution of such a point b is 1 to the sum S_1 and 1 to the sum S_2.

It follows that the contribution of any triple point to the sum $S_1 + S_2$ is equal to 2. Since the double points in C can be treated exactly as in Example 4.6, this yields the following.

$$P(C)(u,v) = ruv - \left(\sum_{j=1}^{r} g_j\right) u - \left(\sum_{j=1}^{r} g_j\right) v + r - n_2 - 2n_3.$$

When there are only triple points in B of the first type, then we obviously have the following additional relations

$$S_1 = n_2' + 2n_3', \quad S_2 = n_2'' + 2n_3''.$$

Example 4.8. *Suppose C has only ordinary points of multiplicity 2, 3 and 4 as singularities. Then let n_4 be the number of points of multiplicity 4 and note that we can write as above $n_4 = n_4' + n_4''$, where n_4' (resp. n_4'') is the number of points of multiplicity 4 of C in $A_0 = A \setminus B$ (resp. in B). For a point $a \in A_0$ of multiplicity 4, the contribution to the sum S_1 is 3, while the contribution to the sum S_2 is 0.*

A point $b \in B$ can be of 4 types. The first type, corresponding to the partition $4 = 1+1+1+1$, is when b is the intersection of 4 components C_j, all smooth at b. The contribution of such a point b is 0 to the sum S_1 and 3 to the sum S_2.

The second type, corresponding to the partition $4 = 2+1+1$, is when b is the intersection of 3 components, say C_i, C_j and C_k, such that C_i has a node at b, and C_j and C_k are smooth at b. The contribution of such a point b is 1 to the sum S_1 and 2 to the sum S_2.

The third type, corresponding to the partition $4 = 2+2$, is when b is the intersection of 2 components, say C_i and C_k, such that C_i and C_k have a node at b. The contribution of such a point b is 2 to the sum S_1 and 1 to the sum S_2.

The fourth type, corresponding to the partition $4 = 3+1$, is when b is the intersection of 2 components, say C_i and C_k, such that C_i has a triple point at b, and C_k is smooth at b. The contribution of such a point b is 2 to the sum S_1 and 1 to the sum S_2.

It follows that the contribution of any point of multiplicity 4 to the sum $S_1 + S_2$ is equal to 3. Since the double and triple points in C can be treated exactly as in Example 4.7, this yields the following.

$$P(C)(u,v) = ruv - \left(\sum_{j=1}^{r} g_j\right) u - \left(\sum_{j=1}^{r} g_j\right) v + r - n_2 - 2n_3 - 3n_4.$$

When there are only points of multiplicity 4 in B of the first type, then we obviously have the following additional relations

$$S_1 = n_2' + 2n_3' + 3n_4'', \quad S_2 = n_2'' + 2n_3'' + 3n_4''.$$

Let's look now at the cohomology of the smooth surface $U = \mathbb{P}^2 \setminus C$. By the additivity we get $P(U) = P(\mathbb{P}^2) - P(C)$ where $P(\mathbb{P}^2) = u^2v^2 + uv + 1$. This yields the following consequence.

Corollary 4.2.

$$P(U)(u,v) = u^2v^2 - (r-1)uv + \left(\sum_{j=1}^{r} g_j\right) u + \left(\sum_{j=1}^{r} g_j\right) v - (r-1) +$$

$$+ \sum_{j,k}((r(C_j, a_k^j) - 1) + \sum_{b \in B}(n(b) - 1).$$

As in the proof of Theorem 4.2, the contribution of $H_c^4(U, \mathbb{C})$ to $P(U)$ is the term u^2v^2, and that of $H_c^3(U, \mathbb{C})$ is the term $-(r-1)uv$. Moreover, the dimension $\dim Gr_F^1 H^2(U, \mathbb{C})$ is the number of independent classes of type (1,2), which correspond to classes of type $(1, 0)$ in $H_c^2(U)$, and hence to the terms in u in $P(U)$. This proves the following result.

Theorem 4.3.

$$\dim Gr_F^1 H^2(U, \mathbb{C}) = \sum_{j=1}^{r} g_j$$

and

$$\dim Gr_F^2 H^2(U, \mathbb{C}) = \sum_{j=1}^{r} g_j + \sum_{j,k}((r(C_j, a_k^j) - 1) + \sum_{b \in B}(n(b) - 1) - r + 1.$$

In particular, all the components C_j of the curve C are rational if and only if $H^2(U)$ is pure of type $(2, 2)$.

Example 4.9. *Suppose C has only ordinary points of multiplicity 2, 3 and 4 as singularities. Then let n_k be the number of points of multiplicity k, for $k = 2, 3, 4$ Then using Example 4.8, we get the formula*

$$\dim Gr_F^2 H^2(U, \mathbb{C}) = \sum_{j=1}^{r} g_j - r + 1 + n_2 + 2n_3 + 3n_4.$$

4.4 Arrangements of transversely intersecting curves

Recall that $C = \bigcup_{j=1,r} C_j$ is the decomposition of C as a union of irreducible components C_j, and the curve C_j has degree N_j. In this section we assume that any curve C_j has only ordinary multiple points as singularities and let $n_k(C_j)$ denote the number of ordinary points on C_j of multiplicity k. We also assume that the intersection of any two distinct components C_i and C_j is transverse, i.e. the points in $C_i \cap C_j$ are nodes of the curve $C_i \cup C_j$. This implies in particular that $A \cap B = \emptyset$. The formulas (4.4) and (4.5) yield the equality.

$$g_j = \frac{(N_j-1)(N_j-2)}{2} - \frac{1}{2}\sum_k \left(\mu(C_j, a_k^j) + r(C, a_k^j) - 1\right), \tag{4.6}$$

Using this, Theorem 4.3 gives the formula

$$\dim Gr_F^2 H^2(U,\mathbb{C}) = \sum_{j=1}^{r} \frac{(N_j-1)(N_j-2)}{2} - \frac{1}{2}\sum_{j,k}\left(\mu(C_j, a_k^j) - r(C, a_k^j) + 1\right) +$$

$$+ \sum_{b\in B}(n(b)-1) - r + 1.$$

If a_k^j is an ordinary m-multiple point on the curve C_j, one has $\mu(C_j, a_k^j) = (m-1)^2$ and hence

$$\mu(C_j, a_k^j) - r(C, a_k^j) + 1 = (m-1)(m-2).$$

If we denote by n'_m (resp. n''_m) the number of m-multiple points of C coming from just one component C_j (resp. from the intersection of several components C_j), we see that we have

$$\sum_{j,k}\left(\mu(C_j, a_k^j) - r(C, a_k^j) + 1\right) = \sum_m (m-1)(m-2)n'_m.$$

This equality explains the contribution of the points in A. Now let $b \in B$ such that $n(b) = m$. The number of such points is precisely n''_m. It follows that

$$\sum_{b\in B}(n(b)-1) = \sum_m (m-1)n''_m.$$

Let $1 \leq i < j \leq r$ and consider the intersection $C_i \cap C_j$. It contains exactly $N_i N_j$ points, since C_i and C_j intersects transversely. The sum $S = \sum_{1 \leq i < j \leq r} N_i N_j$ represents the number of all such intersection points. Note that a point $b \in B$ is counted in this sum exactly $\binom{n(b)}{2}$ times. This yields the following formula

$$2S = \sum_m m(m-1)n''_m.$$

These formulas give the following result.

Theorem 4.4. *With the above assumptions and notation, one has*

$$\dim Gr_F^2 H^2(U, \mathbb{C}) = \frac{(N-1)(N-2)}{2} - \sum_m \binom{m-1}{2} n_m,$$

with $n_m = n'_m + n''_m$ the number of ordinary m-tuple points of C.

4.5 Curves with ordinary singularities of multiplicity ≤ 4

Let $C \subset \mathbb{P}^2$ be a curve of degree N having only ordinary singular points of multiplicity at most 4. Set $U = \mathbb{P}^2 \setminus C$, and let $C = \cup_{j=1}^r C_j$ be the decomposition of C in irreducible components. Then,

$$\begin{aligned} P(C) &= \sum_{j=1}^r P(C_j) - \sum_{0 \leq i < j \leq r} P(C_i \cap C_j) + \sum_{0 \leq i < j < k \leq r} P(C_i \cap C_j \cap C_k) \\ &- \sum_{0 \leq i < j < k < l \leq r} P(C_i \cap C_j \cap C_k \cap C_l). \end{aligned}$$

Let a_m^j denote the number of singular points of multiplicity m that belong to the component C_j (note that a point can be singular on two components, being a node on each of them).

Denote by b_3^k (respectively b_4^k) the number of triple points (respectively points of multiplicity 4) of C that are intersection of exactly k components, for $k = 2, 3$ (respectively $k = 3, 4$). Let b_4^2 (respectively $\tilde{b_4^2}$) be the number of singular points p of multiplicity 4 in C representing the intersection of exactly

2 components, such that one of which has a triple point at p (respectively each one has a node at p). Then one has

$$\sum_{0\leq i<j\leq r} P(C_i\cap C_j) = \sum_{0\leq i<j\leq r} N_iN_j - b_3^2 - 3\tilde{b_4^2} - 2b_4^2 - 2b_4^3.$$

Indeed, a point of type b_3^2 (resp. b_4^2, resp. $\tilde{b_4^2}$) occurs only in one intersection $C_i\cap C_j$, and has the multiplicy 2 (resp.3, resp. 4) in this intersection. A point of type b_4^3 occurs in 3 intersections $C_i\cap C_j$ with multiplicitities $1,2,2$, and this accounts for the correction term $-2b_4^3$. Then one has

$$\sum_{0\leq i<j<k\leq r} P(C_i\cap C_j\cap C_k) = b_3^3 + b_4^3 + \binom{4}{3} b_4^4,$$

and

$$\sum_{0\leq i<j<k<l\leq r} P(C_i\cap C_j\cap C_k\cap C_l) = b_4^4.$$

Hence, by Proposition 4.4, we get the following.

$$\begin{aligned} P(C) &= ruv - (\sum_{j=1}^r g_j)u - (\sum_{j=1}^r g_j)v - \sum_{j=1}^r (a_2^j + 2a_3^j + 3a_4^j) - \sum N_iN_j \\ &\quad + b_3^2 + 3\tilde{b_4^2} + 2b_4^2 + 3b_4^3 + b_3^3 + 3b_4^4. \end{aligned}$$

Therefore, as above, we obtain

$$\begin{aligned} P(U) &= u^2v^2 - (r-1)uv + 1 - r + (\sum_{j=1}^r g_j)u + (\sum_{j=1}^r g_j)v + \sum_{j=1}^r (a_2^j + 3a_3^j + 6a_4^j) \\ &\quad - \sum_{j=1}^r (a_3^j + 3a_4^j) + \sum N_iN_j - b_3^2 - 3\tilde{b_4^2} - 2b_4^2 - 3b_4^3 - b_3^3 - 3b_4^4. \end{aligned}$$

Finally we get

$$\begin{aligned} \dim Gr_F^2 H^2(U) &= \sum_{j=1}^r (g_j + a_2^j + 3a_3^j + 6a_4^j - 1) + \sum N_iN_j + 1 \\ &\quad - (\sum_{j=1}^r a_3^j + b_3^2 + b_3^3) - 3(\sum_{j=1}^r a_4^j + \tilde{b_4^2} + b_4^2 + b_4^3 + b_4^4) + b_4^2 \\ &= \frac{(N-1)(N-2)}{2} - n_3 - 3n_4 + b_4^2, \end{aligned}$$

with n_m the number of ordinary m-tuple points of C.

Theorem 4.5. *Let $C \subset \mathbb{P}^2$ be a curve of degree N having only ordinary singular points of multiplicity at most 4. If $U = \mathbb{P}^2 \setminus C$, then one has*

$$\dim Gr_F^2 H^2(U, \mathbb{C}) = \frac{(N-1)(N-2)}{2} - \sum_{m=3,4} \binom{m-1}{2} n_m + b_4^2,$$

with n_m the number of ordinary m-tuple points of C and b_4^2 the number of singular points p of C which are smooth on one component C_i of C and have multiplicity 3 on the other component C_j of C passing through p.

Chapter 5

Syzygies of Jacobian Ideals for curves with double and triple points

Let $f \in S$ be a homogeneous polynomial of degree N, $f_0, \cdots, f_n$ the corresponding partial derivatives, and H the hypersurface defined by $f = 0$. Denote by $AR(f)$ the graded S-submodule in S^{n+1} of all the relations involving the $f_j's$, namely $a = (a_0, \cdots, a_n) \in AR(f)_q$ if and only if $R(q) : a_0f_0 + \cdots + a_nf_n = 0$. Let $KR(f) \subset AR(f)$ be the S-submodule of Koszul relations or the trivial relations. Note that a trivial relation is an element of the S-submodule in $AR(f)$ spanned by the relations R_{N-1} with a_i equal to f_j and a_j equal to $-f_i$, and the other coordinates are zero, i.e. $(f_i)f_j + (-f_j)f_i = 0$. The quotient module $ER(f) = AR(f)/KR(f)$, called the module of essential relations, is the graded S-submodule of nontrivial syzygies. By Theorem 3.1 in [11],

$$\dim ER(f)_{nN-2n-1-k} = \dim H^n(K^*(f))_{nN-n-1-k} = def_k\Sigma_f \tag{5.1}$$

and

$$def_k\Sigma_f = \dim M(f)_{T-k} - \dim M(f_s)_k \tag{5.2}$$

for $0 \leq k \leq nN-2n-1$, where $T = (n+1)(N-2)$, and $\dim ER(f)_{nN-2n+k} = \tau(H)$ for $k \geq 0$. These formulas allow us to compute $\dim ER(f)_s$ for all s if we know the corresponding Poincaré series $HP(M(f))$, e.g. by computing it using the Singular software. We begin this chapter by some examples that

will be used later to give us an idea about the geometric interpretation of these syzygies.

5.1 Examples of syzygies

Example 5.1. *Let C be the curve as in Example 4.3(i). C has 3 nodes and 4 triple points, then $\dim ER(f)_{8+j} = \tau(C) = 4 \times 4 + 3 = 19$ for $j \geq 0$. Using the program Singular, we can find the Poincaré series $P(M(f))(t)$ and $P(M(f_s))(t)$. We have:*

$$P(M(f_s))(t) = 1+3t+6t^2+10t^3+15t^4+18t^5+19t^6+18t^7+15t^8+10t^9+6t^{10}+3t^{11}+t^{12},$$

and

$$P(M(f))(t) = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 18t^5 + 19(t^6 + t^7 + t^8 + \cdots).$$

Using these two Poincaré series and equation (5.2), one can find $\dim ER(f)_k$ for $0 \leq k \leq 7$. For instance, $\dim ER(f)_1 = \dim M(f)_{12-6} - \dim M(f_s)_6 = 19 - 19 = 0$, $\dim ER(f)_2 = 1$ and $\dim ER(f)_3 = 4$, etc. Therefore, the first nontrivial relations have degree 2.
Note that $f = Q_1Q_2Q_3 = Q_1Q_2(Q_1 + Q_2)$ where $Q_1 = x^2 - y^2$ and $Q_2 = y^2 - z^2$. So $df = Q_2Q_3dQ_1 + Q_1Q_3dQ_2 + Q_1Q_2(dQ_1 + dQ_2)$.
Let $\omega = dQ_1 \wedge dQ_2$, then $w \wedge df = 0$. On the other hand, $dQ_1 = 2(xdx - ydy)$ and $dQ_2 = 2(ydy - zdz)$. Therefore $\omega = 4(xydx \wedge dy - xzdx \wedge dz + yzdy \wedge dz)$. Hence, $\omega \wedge df = 4(yzf_x + xzf_y + xyf_z)dx \wedge dy \wedge dz = 0$, which gives the first relation $r : yzf_x + xzf_y + xyf_z = 0$, and a basis for $ER(f)_2$ as a $\mathbb{C}$-vector space can be given then by $\{r\}$.
Multiplying r by x, y, and z respectively, we can find 3 elements of $AR(f)_3$. Doing some computation we find

$$R : (x^3 - 2xy^2 - 2xz^2)f_x + (y^3 - 2x^2y - 2yz^2)f_y + (z^3 - 2x^2z - 2y^2z)f_z = 0$$

which is also an element in $AR(f)_4$.Since the vectors $xr = (xyz, x^2z, x^2y)$, $yr = (y^2z, xyz, xy^2)$, $zr = (yz^2, xz^2, xyz)$ and $R = (x^3 - xy^2 - xz^2, y^3 - x^2y - yz^2, z^3 - x^2z - y^2z)$ are clearly linearly independant, modulo the trivial relations which have degree $\geq N - 1 = 5$, they belong to $ER(f)_3$ which has dimension 4, hence, they form a basis for $ER(f)_3$.

Example 5.2. *In this example we consider the curve C of Example 4.3(ii) which has only 12 triple points. Then* $\dim ER(f)_{14+j} = \tau(C) = 12 \times 4 = 48$ *for* $j \geq 0$. *Using Singular again, we find:*

$$P(M(f_s))(t) = 1+3t+6t^2+10t^3+15t^4+21t^5+28t^6+36t^7+42t^8+46t^9+48t^{10}+48t^{11}+46t^{12}+42t^{13}+36t^{14}+28t^{15}+21t^{16}+15t^{17}+10t^{18}+6t^{19}+3t^{20}+t^{21},$$

and

$$P(M(f))(t) = 1+3t+6t^2+10t^3+15t^4+21t^5+28t^6+36t^7+42t^8+46t^9 +48(t^{10}+t^{11}+t^{12}\cdots .$$

By the same argument of the previous example, we find that the first non-trivial syzygies have degree 4 and doing a similar computation we find that a basis of $ER(f)_4$ can be given by
$\{r_1 : y^2z^2f_x + x^2z^2f_y + x^2y^2f_z = 0,\ r_2 : (x^4 - 2xy^3 - 2xz^3)f_x + (y^4 - 2x^3y - 2yz^3)f_y + (z^4 - 2x^3z - 2y^3z)f_z = 0\}$.
To find a basis of $ER(f)_5$ whose dimension is 6, we simply multiply r_1 and r_2 by x, y, and z respectively, i.e. $\{xr_1, yr_1, zr_1, xr_2, yr_2, zr_2\}$. Indeed, these vectors are clearly linearly independant modulo the trivial relations since they are homogeneous of degree $4 < N - 1 = 8$.

Remark 5.1. *The two example above give an equality in the inequality*

$$mdr(C) \geq 2N/3 - 2$$

from Proposition 3.8. This shows that this inequality is the best possible in general for curves with ordinary double and triple points.

5.2 Spectral sequences

Spectral sequences are algebraic tools introduced by Jean Leray in 1946, that are useful in the computation of homology and cohomology groups. These latters are usually not easy to be computed. The idea behind spectral sequences is to construct a sequence of bigraded abelian groups that converges to the homology or cohomology group we wish to obtain.

Definition 5.1. A (first quadrant, cohomological) *spectral sequence* is a sequence $\{E_r, d_r\}$ of bigraded groups

$$E_r = \bigoplus_{p,q\geq 0} E_r^{p,q},$$

together with differentials

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1},\ d_r^2 = 0,$$

such that

$$H^*(E_r, d_r) = E_{r+1}.$$

The $E_r^{p,q}$ term and the differential d_r are shown in the figure below.

Definition 5.2. A spectral sequence $\{E_r\}$ is said to be convergent, if $E_r = E_{r+1} = \cdots$ for $r \geq r_0$. The limit of E_r is denoted by E_∞.

A very common type of spectral sequences is the one associated to a filtered complex. Let $K^* : K^0 \stackrel{d}{\to} K^1 \stackrel{d}{\to} K^2 \to \cdots$ be a complex of abelian groups. A subcomplex K' of K^* is a subgroup such that $dK' \subset K^*$. A sequence of subcomplexes

$$K^* = F^0K^* \supset F^1K^* \supset \cdots$$

is called a decreasing filtration on K^*.
The *associated graded complex* to K^* is defined by

$$Gr^pK^* = \frac{F^pK^*}{F^{p+1}K^*}$$

with differential the one induced by d. The filtration on K^* induces a filtration on its cohomology $H^*(K^*) = \bigoplus_{p\geq 0} H^p(K^*)$, given by

$$F^pH^q = im(H^q(F^pK^*) \to H^q(K^*))$$

The associated graded cohomology is then

$$Gr^pH^* = \frac{F^pH*}{F^{p+1}H^*}.$$

Proposition 5.1. *Let K^* be a filtered complex. There exists a spectral sequence $\{E_r\}$ with*

$$E_0^{p,q} = Gr^pK^{p+q},$$
$$E_1^{p,q} = H^{p+q}(Gr^pK^*),$$

and

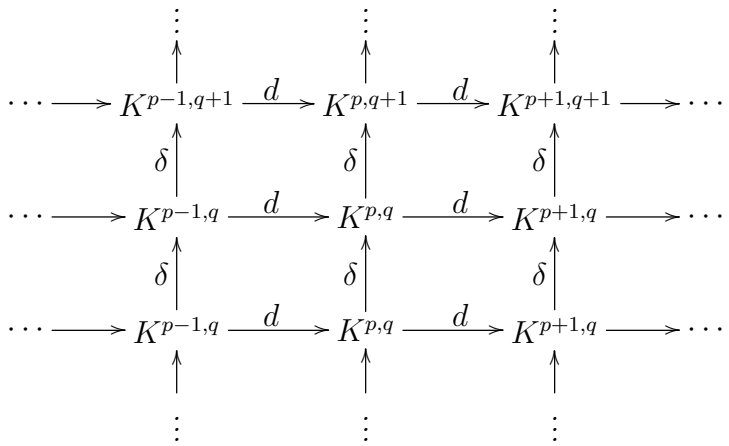
$$E_\infty^{p,q} = Gr^pH^{p+q}(K^*),$$

for any p and q. We say that the spectral sequence converges to $H^*(K*)$.

For the proof, refers to [18].

An important example of spectral sequences is the spectral sequence of a double complex.

Example 5.3. *A double complex $K^{*,*} = \bigoplus K^{p,q}$ is a complex with two differential operators, the vertical operator δ and the horizontal operator d, such that the rows and the columns of the diagram below are complexes of abelian groups, i.e. $d^2 = \delta^2 = 0$, with the condition $d\delta + \delta d = 0$.*



Out of a double complex we can form the associated single complex (K^*, D) defined by

$$K^n = \bigoplus_{p+q=n} K^{p,q}$$

and $D = d + \delta$ with $D^2 = (d+\delta)^2 = d\delta + \delta d = 0$. Define a filtration on K^* by

$$F^p K^n = \bigoplus_{\substack{p' \geq p \\ p+q=n}} K^{p',q}.$$

By Proposition 5.1, there exists a spectral sequence $\{E_r, d_r\}$ converging to $H^*(K^*)$, such that

$$E_0^{p,q} = \frac{F^p K^n}{F^{p+1} K^n} = \frac{K^{p,q} + K^{p+1,q-1} + \cdots}{K^{p+1,q-1} + \cdots} \cong K^{p,q}.$$

The differential d_0 is defined by $d_0 : E_0^{p,q} \longrightarrow E_0^{p,q+1}$. By the above isomorphism, $d_0 : K^{p,q} \longrightarrow K^{p,q+1}$. Then the differential d_0 is exactly the differential δ, and the complex $(E_0^{p,q}, d_0)$ is then the complex

$$\cdots \to K^{p,q-1} \xrightarrow{\delta} K^{p,q} \xrightarrow{\delta} K^{p,q+1} \to \cdots .$$

Consequently

$$E_1^{p,q} = H_\delta^{p+q}(K^{p,*}).$$

The differential d_1 is defined by $d_1 : E_1^{p,q} \to E_1^{p+1,q}$. The $E_2^{p,q}$ term is the cohomology of the complex $(E_1^{p,q}, d_1)$ given by

$$\cdots \to H_\delta^{p+q}(K^{p-1,*}) \xrightarrow{d_1} H_\delta^{p+q}(K^{p,*}) \xrightarrow{d_1} H_\delta^{p+q}(K^{p+1,*}) \to \cdots$$

where the differential d_1 is the one induced by d. Hence

$$E_2^{p,q} = H_d H_\delta(K^{*,*}).$$

Remark 5.2. *There exists another filtration on K^* given by*

$$F^q K^n = \bigoplus_{\substack{q' \geq q \\ p+q=n}} K^{p,q'}.$$

By symmetry of p and q, one can construct another spectral sequence $\{'E_r, 'd_r\}$ converging to $H^*(K^*)$ such that

$$E_0^{p,q} = K^{p,q}, \quad d_0 = d,$$

$$E_1^{p,q} = H_d(K^{*,q}), \quad d_1 \text{ induced by } \delta,$$

and

$$E_2^{p,q} = H_\delta H_d(K^{*,*}).$$

Definition 5.3. Given two spectral sequences $\{E_r^{*,*}, d_r\}$ and $\{'E_r^{*,*}, d_r'\}$, we define a *morphism of spectral sequences* to be a sequence of homomorphisms of bigraded groups,

$$f_r : E_r^{*,*} \longrightarrow' E_r^{*,*}$$

of bidegree $(0,0)$, such that f_r commutes with the differential d_r for every $r \geq 0$, i.e. $f_r \circ d_r = d_r' \circ f_r$, and each f_{r+1} is induced by f_r on cohomology, that is,

$$f_{r+1} : E_{r+1}^{*,*} \cong H(E_r^{*,*}, d_r) \longrightarrow H('E_r^{*,*}, d_r') \cong' E_r^{*,*}.$$

5.3 Rational differential forms and pole order filtrations

A regular differential p-form on $\mathbb{C}^{n+1}$ is a differential form

$$\omega = \sum_I c_I dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

where $I = (i_1, \cdots, i_p)$ and $c_I \in S = \mathbb{C}[x_0, \cdots, x_n]$.
Denote by Ω^p be the $\mathbb{C}$-vector space of all the p-forms on $\mathbb{C}^{n+1}$. Ω^p can be made into a graded S-module. Indeed, if f is a homogeneous polynomial of degree N, we set

$$\deg(f dx_{i_1} \wedge \cdots \wedge dx_{i_p}) = N + p.$$

Suppose f as above, and let $U(f) = \{x \in \mathbb{P}^n; f(x) \neq 0\}$. The differential p-forms on $U(f)$ can be regarded as elements of $H^0(U(f), \Omega^p_{\mathbb{P}^n})$.

Every differential p-form ω on $U(f)$ can be written as $\omega = \frac{\Delta(\gamma)}{f^s}$, where $\gamma \in \Omega^{p+1}_{sN}$ and Δ is the contraction with the Euler vector field $E = \sum_{j=0}^n x_j \frac{\partial}{\partial x_j}$, see [9, chapter 6].

Definition 5.4. Let ω be a differential p-form on $U(f)$, the minimal value of s such that one can write $\omega = \frac{\Delta(\gamma)}{f^s}$ is called the order of the pole of the rational differential form ω along the hypersurface $H : f = 0$, and is denoted by $ord_H(\omega)$.

Let $(A_0^*, d) = (H^0(U(f), \Omega^*), d)$ be the de Rham complex associated to the hypersurface $H : f = 0$, i.e. the complex given by

$$0 \to A_0^0 \xrightarrow{d} A_0^1 \xrightarrow{d} \cdots \to A_0^n \to 0.$$

The differential d is given by $d(\omega) = d(\frac{\Delta(\gamma)}{f^s}) = -\frac{d_f(\gamma))}{f^{s+1}}$, where $d_f(\gamma) = f d\gamma - \frac{|\gamma|}{N} df \wedge \gamma$.

Theorem 5.1. *[18, p.453]* (Grothendieck)

$$H^*(U(f)) = H^*(A_0^*, d).$$

Definition 5.5. The *polar filtration* on the complex (A_0^*, d) is the decreasing filtration P defined by

$$P^s A_0^m = \{\omega \in A_0^m; ord_H(\omega) \leq m - s + 1\}$$

for $m - s + 1 \geq 0$ and $P^s A_0^m = 0$ for $m - s + 1 < 0$.

The polar filtration on (A_0^*, d) induces a *polar filtration* on the cohomology groups $H^*(U)$ given by

$$P^s H^m(U) = im\{H^m(P^s A_0^*) \to H^m(A_0^*)\},$$

and we have the following,

Theorem 5.2. *[9, p.185]* (Deligne-Dimca). For any hypersurface complement $U = \mathbb{P}^n \backslash H$ we have

$$P^s H^m(U) \supset F^s H^m(U)$$

for any integers s and m.

For $a \in \{-N + 1, \cdots, -1, 0\}$ define the complexes A_a^* by

$$A_a^m = \{\frac{\Delta(\gamma)}{f^s}; \gamma \in \Omega^{m+1}_{sN+a} \ for \ some \ s \geq 0\}.$$

with differential d given by $d(\frac{\omega}{f^s}) = \frac{d\omega}{f^s} - (s + \frac{a}{N})\frac{df \wedge \omega}{f^{s+1}}$. This complex has a polar filtration given by

$$P^s A_a^m = \{\frac{\Delta(\gamma)}{f^k}; |\gamma| = kN + a \ for \ k \leq m - s + 1\}$$

for $m - s + 1 \geq 0$ and $P^s A_a^m = 0$ for $m - s + 1 < 0$. Denote by A^* the sum complex of A_a^*, i.e. $A^* = \bigoplus_a A_a^*$.

Let F be the Milnor fiber of f, i.e. the smooth hypersurface given by $f = 1$. There is a monodromy isomorphism $h : F \longrightarrow F$ given by the multiplication by $\lambda = exp(\frac{2\pi i}{N})$ and an induced monodromy operator $h^* : H^*(F) \longrightarrow H^*(F)$. The eigenvalues of h^* are exactly the Nth-roots of unity. We denote by $H^*(F)_a$ the eigenspace corresponding to λ^a, we have (see [9, p.189]),

$$H^m(F)_a = H^m(A_a^*, d). \tag{5.3}$$

Note that $H^k(F)_0 = H^k(U(f)) = H^k(A_0^*, d)$ as in Grothendieck theorem above. Define a new decreasing filtration $\bar{P}$ on A_a^* (respectively on A^*) by setting $\bar{P}^s = P^{s+1}$. By equation 5.3 and Proposition 5.1 we conclude:

Proposition 5.2. *(i) For any $a \in \mathbb{Z}/N\mathbb{Z}$ there exists a spectral sequence $(E_r(f)_a, d_r)$ on A_a^* with*

$$E_1^{s,t}(f)_a = H^{s+t}(\bar{P}^s A_a^*/\bar{P}^{s+1}A_a^*),$$

which converges to the Milnor fiber cohomology eigenspace $H^*(F)_a$.

(ii) There exists a spectral sequence $(E_r(f), d_r)$ on A^* with

$$E_1^{s,t}(f) = H^{s+t}(\bar{P}^s A^*/\bar{P}^{s+1}A^*),$$

which converges to the Milnor fiber cohomology $H^*(F)$.

5.4 Spectral Sequences Associated to the Koszul Complex

The spectral sequences associated to the de Rham cohomology defined in section 5.2 can be compared to some algebraic spectral sequences of the Milnor fiber $F : f = 1$ associated to the Koszul Complex $K^*(f)$ defined in section 3.1.

Consider the double complex (B, d', d'') defined by

$$B_a^{s,t} = \Omega^{s+t}_{(t+1)N+a} \quad s, t \in \mathbb{Z},$$

$d' = d$, and $d''(\omega) = -|\omega|N^{-1}df \wedge \omega$ for a homogeneous differential form ω. Let (B_a^*, D_f) be the associated total complex, namely, $B_a^k = \bigoplus_{s+t=k} B_a^{s,t}$, and $D_f = d' + d''$ with $d'd'' + d''d' = 0$, and $F^pB_a^k = \bigoplus_{\substack{s\geq p \\ s+t=k}} B_a^{s,t}$ the decreasing filtration on B_a^* as in Example 5.3. By Proposition 5.1 and Example 5.3, we have:

Proposition 5.3. *There exists a spectral sequence $('E_r(f), d_r)$ with*

$$'E_1^{s,t}(f)_a = H^{s+t}(B_a^{s,*})$$

and converging to the cohomology $H^*(B_a^*)$.

Remark 5.3. *The cohomology group $H^{s+t}(B_a^{s,*})$ is in fact the cohomology of the complex*

$$\cdots \to B_a^{s,t-1} \xrightarrow{d_1} B_a^{s,t} \xrightarrow{d_1} B_a^{s,t+1} \to \cdots$$

which is exactly,

$$\cdots \to \Omega^{s+t-1}_{tN+a} \xrightarrow{d_1} \Omega^{s+t}_{(t+1)N+a} \xrightarrow{d_1} \Omega^{s+t+1}_{(t+2)N+a} \to \cdots$$

for any $a \in \{-N+1, \cdots, 1, 0\}$. By Example 5.3 d_1 is induced by $d'' = -\frac{|\omega|}{N} df \wedge \omega$. Therefore, this is exactly the Koszul complex defined in section 2.2 with differential a multiple of the differential of the Koszul complex. Hence, we conclude,

Theorem 5.3. *The E_1 term of the spectral sequence of Proposition 5.3 is given by*

$$'E_1^{s,t}(f)_a = H^{s+t}(K_f^*)_{(t+1)N+a}.$$

The complexes B_a^* (respectively $B^* = \bigoplus_a B_a^*$) and A_a^* (respectively A^*) defined in Section 1.9 are related by the following morphisms of complexes

$$\bar{\delta} : B_a^* \to A_a^{*-1} \quad \bar{\delta} : B^* \to A^{*-1}$$

$$\bar{\delta}(\omega) = \Delta(\omega) f^{-(t+1)}, \quad for\ \omega \in B^{s,t}.$$

These morphisms $\bar{\delta}$ induce morphisms of spectral sequences,

$$\delta_r : ('E_r(f)_a, d_r) \longrightarrow (E_r(f)_a, d_r).$$

Denote by $\tilde{E}_r(f)_a$ the reduced spectral sequence of $E_r(f)_a$ obtained by replacing $E_r^{0,0}(f)_0$ by $E_r^{0,0}(f)_0/\mathbb{C}$ for $a = 0$, and $\tilde{E}_r(f)_a = E_r(f)_a$ for $a \neq 0$. Similarly, we define the reduced spectral sequence $'\tilde{E}_r(f)_a$ to be the spectral sequence given by replacing $'\tilde{E}_r^{0,0}(f)_0$ by $'E_r^{0,0}(f)_0/\mathbb{C}$ for $a = 0$, and $'\tilde{E}_r(f)_a = 'E_r(f)_a$ for $a \neq 0$. Let $\tilde{\delta}_r : ('\tilde{E}_r(f)_a, d_r) \longrightarrow (\tilde{E}_r(f)_a, d_r)$ be the induced morphisms.

Theorem 5.4. *[9, p.192] $\tilde{\delta}_r$ are isomorphisms for all $r \geq 1$ and hence, by equation 5.3 they induce isomorphisms $H^k(B_a^*, D_f) = H^{k-1}(F)_a$.*

5.5 Cohomology of Milnor Fibers and of Plane Curves Complements

For a curve $C \subset \mathbb{P}^2$ with isolated singularities, the only nontrivial groups in the spectral sequence are $E_1^{0,1}(f)_a = H^2(K^*)_{N+a}$, $E_1^{1,0}(f)_a = H^2(K^*)_a$, $E_1^{0,2}(f)_a = H^3(K^*)_{2N+a}$, $E_1^{1,1}(f)_a = H^3(K^*)_{N+a}$, and $E_1^{2,0}(f)_a = H^3(K^*)_a$. But since $a \leq 0$, then $E_1^{1,0}(f)_a = E_1^{2,0}(f)_a = 0$.

A relation $af_x + bf_y + cf_z = 0$ can be written as $\omega \wedge df = 0$ where $df = f_x dx + f_y dy + f_z dz$ and $\omega = ady \wedge dz - bdx \wedge dz + cdx \wedge dy$. Therefore, it gives rise to a 2-form $\omega \in H^2(K_f^*)_{|\omega|+2}$. For the next two results, set $\lambda = exp(\frac{-2\pi i}{3})$ and let $j : F \to \mathbb{C}^3$ be the inclusion.

Proposition 5.4. *Let C be the curve defined by $f : (x^2 - y^2)(y^2 - z^2)(x^2 - z^2) = 0$, and let $F : f - 1 = 0$ be the Milnor fiber. There exists a basis for $H^{0,1}(F)_\lambda$ given by $[j^*\Delta(\omega)]$, where*

$$\omega = yzdy \wedge dz - xzdx \wedge dz + xydx \wedge dy.$$

Proof. In Example 5.1, the degree 2 relation r gives rise to $\omega \in H^2(K^*(f))_4$, therefore we should choose $a = -2$ to get $N + a = 4$. It follows that $\lambda = exp(\frac{2\pi i a}{N}) = exp(\frac{-2\pi i}{3})$. Note that $d\omega = (a_x + b_y + c_z)dx \wedge dy \wedge dz = 0$ in this case. Since d_1 is induced by d, then $d_1(\omega) = 0$ and ω survives to $E_\infty^{0,1}(f)_a = H^{0,1}(F)_\lambda$ whose dimension is 1, see Theorem 1 in [10], and the corresponding form

$$[j^*\Delta(\omega)]$$

yields a basis for $H^{0,1}(F)_\lambda$.

□

Proposition 5.5. *Let C be the curve defined by $f : (x^3 - y^3)(y^3 - z^3)(x^3 - z^3) = 0$, and let $F : f - 1 = 0$ be the Milnor fiber. There exists a basis for $H^{0,1}(F)_\lambda$ given by $\{[j^*\Delta(\omega_1)], [j^*\Delta(\omega_2]\}$, where*

$$\omega_1 = y^2z^2dy \wedge dz - x^2z^2dx \wedge dz + x^2y^2dx \wedge dy,$$

and

$$\omega_2 = (x^4 - 2xy^3 - 2xz^3)dy \wedge dz - (y^4 - 2x^3y - 2yz^3)dx \wedge dz + (z^4 - 2x^3y - 2x^3z)dx \wedge dy.$$

Proof. In Example 5.2, the relations r_1 and r_2 correspond to the forms ω_1 and $\omega_2 \in H^2(K_f^*)_6$ respectively, given by

$$\omega_1 = y^2z^2dy \wedge dz - x^2z^2dx \wedge dz + x^2y^2dx \wedge dy$$

and

$$\omega_2 = (x^4-2xy^3-2xz^3)dy\wedge dz-(y^4-2x^3y-2yz^3)dx\wedge dz+(z^4-2x^3z-2y^3z)dx\wedge dy,$$

and we have $d\omega_1 = d\omega_2 = 0$. Here we choose $a = -3$ to get $N + a = 9 + a = 6$, and therefore $\lambda = exp(\frac{2\pi ia}{N}) = exp(\frac{-2\pi i}{3})$. Theorem 1 in [10] says that $H^{0,1}(F)_\lambda = 2$ in this case. Hence, since the two 2-forms ω_1 and ω_2 survives at $E^{0,1}_{\infty,a} = H^{0,1}(F)_\lambda$, and their corresponding forms $[j^*\Delta(\omega_1)]$ and $[j^*\Delta(\omega_2)]$ form a basis for $H^{0,1}(F)_\lambda$.

□

We conclude that finding explicit Jacobian syzygies allows us to describe explicit bases for the eigenvalues of the monodromy acting on the cohomology of the Milnor fibers of homogeneous polynomials. This type of results is new even in the case of line arrangements, where the cohomology of the complement U was described in terms of differential forms since a long time, see [22].

Consider now the case when $a = 0$. We will denote in the sequel $E_r(f)_0$ and $'E_r(f)_0$ by $E_r(f)$ and $'E_r(f)$ respectively. Suppose that we have a curve $C \in \mathbb{P}^2 : f = 0$ with isolated singularities. we know that in this case the only nonzero cohomology groups of the Koszul complex $K^*(f)$ are $H^2(K^*(f))$ and $H^3(K^*(f))$. Therefore, the nontrivial terms of the E_1-spectral sequence belong to the lines $(L) :\ s + t = 3$ and $(L') :\ s + t = 2$.

Consider first the terms on the line (L). By Theorem 5.4 and Remark 3.1, we have

$$'E_1^{3-t,t}(f) = H^3(K^*(f))_{(t+1)N} = M(f)_{(t+1)N-3}.$$

By Proposition 3.6, the terms $'E_1^{2-t,t}(f) = H^2(K^*(f))_{(t+1)N}$ on the line (L') have dimensions

$$\dim' E_1^{2-t,t}(f) = \dim M(f)_{(t+2)N-3} - \dim M(f_s)_{(t+2)N-3},$$

where $f_s = 0$ defines a smooth curve in $\mathbb{P}^2$ of degree N.

A study by A. Dimca and G. Sticlaru using the spectral sequences defined above, showed that for a nodal curve $\dim M(f)_{2N-3} = n(C) + \sum_{j=1}^{r} g_j$, where $n(C)$ is the number of nodes of C. We extend now this result to the case when C has nodes and ordinary triple points as singularities.

Theorem 5.5. *Let $C \in \mathbb{P}^2$ be a curve of degree N. Suppose C has n nodes (A_1) and t triple points (D_4). Let $C = \bigcup_{j=1,r} C_j$ be the decomposition of C as a union of irreducible components, let $\nu : \tilde{C}_j \to C_j$ be the normalization mappings and set $g_j = g(\tilde{C}_j)$. Then we have the following.*

(A) $0 \leq \dim M(f)_{2N-3} - \tau \leq \sum_{j=1}^{r} g_j$. *In particular,*

- *(i) If all $g_i = 0$, one has $\dim M(f)_{2N-3} = \tau$, i.e. $st(C) \leq 2N-3$.*
- *(ii) One has equality, i.e. $\dim M(f)_{2N-3} - \tau = \sum_{j=1}^{r} g_j$ if and only if $H^2(U)$ satisfies $F^2H^2(U) = P^2H^2(U)$.*

(B) $max(r-1+t-\sum_{j=1}^{r} g_j, r-1) \leq \dim ER(f)_{N-2} \leq r-1+t$. *In particular,* $\dim ER(f)_{N-2} = r-1+t$ *if $g_j = 0$ for all j.*

Proof. (A) Consider the spectral sequence

$$E_1^{p,q}(f) = H^{p+q}(K^*(f))_{(q+1)N}$$

that converges to $H^{p+q-1}(U)$, where $U = \mathbb{P}^2 \setminus C$. By Theorem 2.4 (ii) in [15], the differential

$$d_1^t : E_1^{2-t,t} \to E_1^{3-t,t}$$

is bijective for $t \geq 2$ and injective for $t = 1$.
Consider first the case when $t = 1$. Since $d_1 : E_1^{1,1} \to E_1^{2,1}$ is injective, then $\dim E^{1,1} \leq \dim E_1^{2,1}$. $\dim E_1^{1,1} = \dim H^2(K^*(f))_{2N} = \dim ER(f)_{2N-2}$ which is equal to $\tau(C)$ by Equation 5.1. On the other hand, $\dim E_1^{2,1}(f) = \dim H^3(K^*(f))_{3N} = \dim M(f)_{2N-3}$. This proves the left hand side of the inequality in (A).
To prove the right hand side of the inequality, consider the limit term

$$E_\infty^{2,1} = \frac{P^1H^2(U)}{P^2H^2(U)}.$$

By Theorem 5.2, $P^sH^m(U) \supset F^sH^m(U)$. For $s = 1$ we have in addition $F^1H^2(U) = H^2(U)$, then $P^1H^2 = F^1H^2(U) = H^2(U)$. For $s = 2$, $P^2H^2 \supset F^2H^2(U) \supset H^2(U)$. It follows that the map

$$Gr_F^1(H^2(U)) \longrightarrow E_\infty^{2,1}$$

is an epimorphism, and hence, $\dim E_\infty^{2,1} \leq \dim Gr_F^1(H^2(U))$. By Proposition 4.2 $\dim Gr_F^1(H^2(U)) = \sum_{j=1}^r g_j$. On the other hand, by Proposition 2.4 (iii) in [15] the spectral sequence degenerate at the E_2 terms, i.e. $E_\infty^{2,1} = E_2^{2,1} = coker d_1^1$. Hence, $\dim E_\infty^{2,1} = \dim M(f)_{2N-3} - \tau$, and this proves the inequality (A).
For the proof of part (i), we refer to the proof of Corollary 3.3.
(B) To prove the second inequality, we consider the differential $d_1^0 : E_1^{2,0} \to E_1^{3,0}$. We know that $E_\infty^{2,0} = E_2^{2,0} = ker\ d_1^0 \simeq H^1(U)$ and hence $\dim(ker\ d_1^0) = r-1$. In particular, $r - 1 \leq E_1^{2,0} = \dim H^2(K^*(f))_N = \dim ER(f)_{N-2}$, and $\dim E_\infty^{3,0} = \dim(coker\ d_1^0) = g - \dim Er(f)_{N-2} + r - 1$, where $g = \frac{(N-1)(N-2)}{2} = \dim M(f)_{N-3}$. We compute now $b_2(U)$ in two different ways:

$$b_2(U) = \dim Gr_F^1 H^2(U) + \dim Gr_F^2 H^2(U) = \sum_{j=1}^r g_j + g - t,$$

by Theorem 4.2, and

$$b_2(U) = \dim E_\infty^{3,0} + \dim E_\infty^{2,1} = g - \dim Er(f)_{N-2} + r - 1 + \dim M(f)_{N-3} - \tau.$$

On the other hand, Theorem 1 in [10] implies

$$\dim Er(f)_{N-2} = \dim M(f)_{2N-3} - \dim M(f_s)_{N-3} = \dim M(f)_{2N-3} - g.$$

The last two formulas implies that $b_2(U) = 2g - \tau + r - 1$. The first formula for $b_2(U)$ now implies that $\sum_{j=1}^r g_j - t = 2g - \tau + r - 1$. If we apply part A of the theorem we get

$$\tau - g \leq \dim Er(f)_{N-2} \leq \tau + \sum g_j - g,$$

and this proves part B. □

Example 5.4. *(i) Consider curve C of degree 6 in Example 5.1. Using the equations (4.1) and (4.2),* $\dim ER(f)_4 = 19 - 10 = 9$. *On the other hand, C is the union of 6 lines, i.e $g_i = 0$ for $i = 1, \cdots, 6$. Hence, by part (B) of Theorem 5.5,* $\dim ER(f)_4 = 6 - 1 + 4 = 9$.

(ii) In the Example 5.2, $\dim ER(f)_7 = 48 - 28 = 20$. *In this example, C is the union of 9 lines, therefore by part (B) of Theorem 5.5,* $\dim ER(f)_7 = 9 - 1 + 12 = 20$.

Example 5.5. *Let C be the degree 9 curve defined by $f = (x^3+y^3+z^3)^3 + (x^3+2y^3+3z^3)^3 = 0$. C is the union of 3 smooth curves, and have 9 triple points as singularities. Using Singular we can find $\dim M(f)_{16} = \tau = 36$. Hence, one has strict inequality in (A)*

$$\dim(f)_{16} - \tau = 0 < 3 = \sum_{j=1}^{3} g_j.$$

Moreover, the inequalities in this case are

$$8 \leq 8 \leq 9+2.$$

Example 5.6. *Consider the following two distinct realizations of the Pappus configuration 9_3, see [4] and [9], Example (6.4.16), p. 213. The first one is the line arrangement*

$$\mathcal{A}_1 : f = xyz(x-y)(y-z)(x-y-z)(2x+y+z)(2x+y-z)(-2x+5y-z) = 0.$$

A Singular computation yields

$$HP(M(f))(t) = 1+3t+6t^2+10t^3+15t^4+21t^5+28t^6+36t^7+42t^8+46t^9+48t^{10}+$$
$$+48t^{11}+47t^{12}+45(t^{13}+...$$

The second one is the line arrangement $\mathcal{A}_2$ given by

$$\mathcal{A}_2 : g = xyz(x+y)(x+3z)(y+z)(x+2y+z)(x+2y+3z)(4x+6y+6z) = 0.$$

Using again the Singular software, we get

$$HP(M(g))(t) = 1+3t+6t^2+10t^3+15t^4+21t^5+28t^6+36t^7+42t^8+46t^9+48t^{10}+$$
$$+48t^{11}+46t^{12}+45(t^{13}+...$$

Both arrangements have $N = n = t = 9$ and $HP(M(f)(t) - HP(M(g)(t) = t^{12} \neq 0$. This shows once again that the curves with ordinary nodes and triple points are much more subtle than the nodal curves. In particular, for rational curves with ordinary nodes and triple points the Poincaré series $HP(M(f))$ is not determined by N, the number of irreducible components, the number of double and triple points, compare with Corollary 3.3. Not even the weaker invariant $ct(C)$ is determined by these numbers, since $ct(V(f)) = 11$ and $ct(V(g)) = 12$. It also shows that it is rather difficult to control the dimension of the homogeneous components $M(f)_r$ for $r \neq 2N-3$.

Chapter 6

Conclusion

6.1 Conclusion(version Française)

Cette thèse met en oeuvre le lien entre la théorie de Hodge et les singularités des hypersurfaces, en particulier des courbes planes.

Dans la première partie, on donne une étude détaillée des groupes de cohomologie du complémentaire $U = \mathbb{P}^2 \setminus C$ d'une courbe plane C. Le premier résultat principal montre que les dimensions des groupes gradués associés à ces groupes de cohomologie peuvent être calculées en fonction des invariants gémétriques simples dans le cas où C admet des points doubles et triples ordinaires . Une étude plus approfondie montre que le cas des singularités isolées quelconques est plus difficile à étudier. Ainsi, lorsque les singularités atteignent l'ordre 4, on ne peut pas s'attendre à trouver des formules simples.

Dans la deuxième partie, on trouve d'abord pour certains cas particuliers des représentants pour des classes de cohomologie de Rham dans la cohomologie de la fibre de Milnor d'un polynôme homogène f qui définit une courbe C à des singularités doubles et triples. Ces résultats sont les premiers résultats de ce type concernant la cohomologie de la fibre de Milnor et donnent lieu à de nouvelles questions intéressantes et difficiles.

On s'intéresse également dans cette partie, à trouver les dimensions des groupes gradués de l'algèbre de Milnor, $M(f)_r$, qui sont des invariants projectifs de l'hypersurface H définie par $f = 0$. Le deuxième résultat prin-

cipal et les exemples qui suivent montrent que contrairement au cas des courbes nodales, dans le cas des courbes à points doubles et triples ordinaires, l'inclusion $F^s \subset P^s$ dûe à Deligne-Dimca entre la filtration de Hodge F^s et la filtration par l'ordre de pôle P^s peut être stricte et rend l'étude plus difficile.

Question Ouverte

Dans la partie (B) du Théorème 5.5, on sait que $\dim ER(f)_{N-2} = r-1+t$ si $g_j = 0$ pour tout $j = 1, \cdots, r$. En d'autres termes, chaque composante irréductible C_j et chaque point triple $p \in C$ entraîne une relation et il n'y a qu'une seule relation de dépendance parmi les $r+t$ relations. Est-ce possible d'écrire les relations associées à la composante C_j et le point associé à C_j et au point p de la même manière que dans le cas nodal traité dans Théorèm 4.1 in [15]?

6.2 Conclusion(English version)

This thesis studies the relation between the Hodge theory and singularities of hypersurfaces, in particular plane curves.

In the first part, we give a detailed study of the cohomology group of the complement $U = \mathbb{P}^2 \setminus C$ of a plane curve C. The first main result shows that the dimensions of the graded quotient groups of these cohomology groups with respect to the Hodge filtration can be computed in terms of simple geometric invariants in the case where C has only ordinary double and triple points as singularities. For more complicated singularities, the study becomes more subtle. In particular, we show that for curves singularities of multiplicity 4, we can not expect simple formulas as in the previous case.

In the second part, first we find in some particular cases, de Rham representatives in the cohomology of the Milnor fiber of a homogeneous polynomial f that defines a curve with ordinary double and triple points, using the syzygies of the Jacobian ideal. These results are the first such results for the Milnor fiber cohomology, and give rise to new interesting and difficult questions.

Furthermore, we are interesting in this section, in finding the dimensions of the graded pieces of the Milnor algebra, $M(f)_r$, which are projective invariants of the hypersurfaces H given by $f = 0$. The second main result and the following example show that unlike the case of nodal curves, in the case of curves with ordinary double and triple points, the inclusion $F^s \subset P^s$ due to Deligne-Dimca between the Hodge filtration F^s and the pole order filtration P^s can be strict and and make the study more subtle.

Open Question

By Theorem 5.5, part (B), we know that $\dim ER(f)_{N-2} = r - 1 + t$ if all $g_j = 0$. In other words, each irreducible component C_j and each triple point $p \in C$ yield a relation and there is just one dependence relation among these $r + t$ relations. Is it possible to write the relations associated to the component C_j and to the point p in a similar way to the case of nodal curves treated in Theorem 4.1 in [15]?

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Cohomology of Algebraic Plane Curves

Abstract: We describe in this thesis the dimensions of the graded quotients of the cohomology of a plane complement curve with respect to the Hodge filtration in terms of simple geometric invariants. The case of curves with ordinary singularities is discussed in details. In particular, we find the Hodge-Deligne polynomial of any curve C with isolated singularities and that of its complement, from which we can compute the mixed Hodge numbers of the second cohomology group of the complement of the curve, and consequently the correspondant Betti numbers. Furthermore, in the case of curves with ordinary double and triple points, we give relations to the Milnor algebra of the homogeneous polynomial f defining C, to the syzygies of the Jacobian ideal of f and pole order filtration on the second cohomology group of the curve complement.

Key Words: Plane Curves, Milnor algebras, Syzygies, Hodge and pole ordre Filtration.

Cohomologie des Courbes Planes Algébriques

Résumé: On décrit dans cette thèse les dimensions des groupes quotients gradués associés à la cohomologie du complémentaire d'une courbe plane par rapport à la filtration de Hodge en fonction de certains invariants géométriques. Le cas des courbes à singularités ordinaires est détaillé. En particulier, on trouve le polynôme de Hodge-Deligne d'une courbe C quelconque à singularités isolées et celui de son complémentaire duquel on déduit les nombres de Hodge mixtes ainsi que les nombres de Betti correspondants. Dans le cas des courbes dont les singularités sont des noeuds et des points triples ordinaires, on donne des relations importantes avec l'algèbre de Milnor du polynôme homogène f qui définit C, les syzygies de l'idéal Jacobien de f et la filtration par l'ordre de pôle du groupe cohomologique d'ordre 2 du complémentaire de la courbe.

Mots Clés: Courbes planes, Algèbres de Milnor, Syzygies, Filtration de Hodge et par l'ordre du pôle.