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Rigidité et non-rigidité d'actions de groupes sur les espaces Lp non-commutatifs

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Résumé en français

1 Introduction du sujet

1.1 Propriété (T_B) pour un espace de Banach B

La propriété (T) de Kazhdan est une propriété de rigidité des groupes introduite par D.Kazhdan dans les annés 60 dans [47] pour montrer que les réseaux en rang supérieur sont de type fini et possèdent un abélianisé fini. Depuis qu'elle a été introduite, la propriété (T) a été étudiée par de nombreux auteurs ; elle possède de nombreuses applications dans des domaines des mathématiques très variés : en théorie ergodique, dans la théorie des algèbres d'opérateurs, en informatique théorique...De ce fait, des variantes de la propriété (T) sont naturellement apparues pour l'étude des groupes topologiques et des phénomènes de rigidité liés à leurs actions. Dans ce travail de thèse, nous nous sommes intéressés à une variante de la propriété (T) introduite récemment par Bader, Furman, Gelander et Monod dans [4].

Soit G un groupe topologique. Etant donné un espace de Banach B, notons $O(B)$ son groupe d'isométries, c'est-à-dire le groupe des bijections de B dans B qui sont linéaires et isométriques. Une représentation orthogonale de G dans B est un homomorphisme $\pi : G \to O(B)$ qui est continu, au sens où $g \to \pi(g)x$ est continue pour tout x dans B. On dit que la représentation π possède presque des vecteurs invariants si

$$\forall K \subset G \text{ compact }, \forall \epsilon > 0, \exists x \in B, \ \sup_{g \in K} ||\pi(g)x - x|| < \epsilon ||x||.$$

Pour tout sous-groupe fermé H de G, on note $B^{\pi(H)} = \{x \in B | \ \forall g \in H, \pi(g)x = x \ \}$ l'espace des vecteurs invariants par $\pi(H)$. Si de plus H est normal dans G, une représentation $\pi : G \to O(B)$ induit alors naturellement une représentation $\pi' : G \to O(B/B^{\pi(H)})$ de G sur l'espace de Banach quotient $B/B^{\pi(H)}$.

Définition 1.1. [4]Soit B un espace de Banach. Soit G un groupe topologique et H un sous-groupe fermé normal de G. La paire (G, H) possède la propriété

(T_B) (on dit aussi que G possède la propriété (T_B) relativement à H), si pour toute représentation linéaire isométrique $\pi : G \to O(B)$, la représentation π' : $G \to O(B/B^{\pi(H)})$ sur $B/B^{\pi(H)}$ ne possède pas presque des vecteurs invariants. Un groupe G possède la propriété (T_B) si la paire (G, G) possède la propriété (T_B) relative.

Lorsque B est un espace de Hilbert, la définition précédente correspond à la propriété (T) de Kazhdan. Les auteurs de [4] ont posé les bases pour l'étude de la propriété (T_B) dans le cadre des espaces de Banach B uniformément convexes, dont le dual est également uniformément convexe (ces espaces de Banach sont appelés "ucus"dans [4]).

La propriété (T) se reformule également en termes d'actions par isométries affines sur un espace de Hilbert. Un groupe topologique G possède la propriété de point fixe (FH) si toute action continue de G par isométries affines sur un espace de Hilbert $\mathcal{H}$ possède un point fixe. Pour les groupes localement compacts et σ-compacts, un théorème de Delorme-Guichardet montre que la propriété (T) est équivalente à la propriété (FH) (voir [25] et [33]). Les auteurs de [4] ont aussi étudié l'analogue de cette propriété.

Définition 1.2. Soit B un espace de Banach. Un groupe topologique G possède la propriété (F_B) si toute action continue de G par isométries affines sur B possède un point fixe.

Les propriétés (T_B) et (F_B) que nous allons étudier sont distinctes (plus faibles) des propriétés banachiques introduites par V. Lafforgue dans [52] (voir aussi [51] pour un renforcement de la propriété (T)), où l'auteur considère de plus grandes classes de représentations. Nous n'étudierons pas ici ces dernières propriétés et renvoyons aux articles [51] et [52] pour le lecteur intéressé.

1.2 Les propriétés (T_{L_p}) et (F_{L_p}) pour les espaces L_p classiques

Rappelons qu'un espace borélien standard est un espace mesurable associé à la tribu borélienne d'un espace métrique séparable complet. Les principaux résultats de [4] concernent la classe des espaces $L_p(X, \mu)$ associés à un espace mesuré (X, μ), où X est un espace borélien standard. Plus précisément, les auteurs de [4] étudient les liens entre la propriété (T) de Kazhdan et leur variante sur ces espaces L_p que l'on appellera par la suite commutatifs ou classiques. Voici un de leurs résultats principaux.

Théorème 1.3. [4] *Soit G un groupe localement compact à base dénombrable. Si G possède la propriété (T), alors G possède la propriété (T_B) pour tout espace de Banach B appartenant à la liste suivante :*

1. *les espaces $L_p(X,\mu)$ pour toute mesure μ σ-finie sur un espace borélien standard X et tout $1 \leq p < \infty$;*

2. *les sous-espaces fermés des espaces $L_p(X,\mu)$ pour tout $1 < p < \infty$ avec $p \neq 4, 6, 8...$;*

3. *les espaces quotients des espaces $L_p(X,\mu)$ pour tout $1 < p < \infty$ avec $p \neq \frac{4}{3}, \frac{6}{5}, \frac{8}{7}...$.*

Si, de plus, μ est sans atomes et G possède la propriété $(T_{L_p(X,\mu)})$, alors G possède la propriété (T) de Kazhdan.

Le résultat de [4] concernant les liens entre la propriété (T) et la propriété (F_B) s'énonce comme suit.

Théorème 1.4. [4] *Soit G un groupe localement compact à base dénombrable. Alors :*

1. *(F_B) implique (T_B) pour tout espace de Banach ;*

2. *(T) implique (F_B) pour tout sous-espace fermé B de $L_p(X,\mu)$, où (X,μ) est un espace mesuré avec une mesure μ σ-finie et pour tout $1 < p \leq 2$;*

3. *(T) implique (F_B) pour tout sous-espace fermé B de $L_p(X,\mu)$, avec $2 \leq p \leq 2 + \epsilon(G)$, pour une certaine constante $\epsilon(G) > 0$ dépendant du groupe G.*

Il est à remarquer que la restriction $p \leq 2$ dans le point 2 du théorème précédent est nécessaire. En effet, il existe des groupes avec la propriété (T) de Kazhdan qui n'ont pas la propriété de point fixe $(F_{L_p(X,\mu)})$ pour $p > 2$ suffisamment grand (voir pour cela, par exemple, [23] ou [85]).

D'autre part, les groupes de Lie simples de rang supérieur ou égal à 2, ainsi que leurs réseaux (tels que $SL_3(\mathbb{Z})$) possèdent la propriété $(F_{L_p(X,\mu)})$ pour tout $1 < p < \infty$ (voir le théorème B dans [4]). Un résultat beaucoup plus fort est conjecturé dans [4] : les groupes précédents possèdent la propriété (T_B) pour tout espace de Banach B "ucus".

Une autre classe de groupes pour lesquels la propriété $(F_{L_p(X,\mu)})$ a été démontrée, pour tout $1 < p < \infty$, est la famille des "réseaux universels", c'est-à-dire les groupes $SL_n(\mathbb{Z}[x_1, ...x_k])$ pour $k \geq 0$ et $n \geq 4$. Ceci est un résultat M.Mimura dans [59].

Un exemple d'application de la propriété (F_{L_p}) a été donné par A.Navas, qui a utilisé cette propriété pour améliorer un de ses résultats antérieurs concernant la rigidité des actions de groupes de Kazhdan sur le cercle à travers le théorème suivant (voir par exemple [61]).

Théorème 1.5. [61] *Soit* $\alpha > 0$. *Soit* $\mathrm{Diffeo}_+^{1+\alpha}(\mathbb{S}^1)$ *le groupe des difféomorphismes du cercle* $\mathbb{S}^1$ *de classe* $1+\alpha$. *Soit* Γ *un groupe avec la propriété* (F_{L_p}), *pour* $p > \frac{1}{\alpha}$. *Si* $\Phi : \Gamma \to \mathrm{Diffeo}_+^{1+\alpha}(\mathbb{S}^1)$ *est un homomorphisme, alors* $\Phi(\Gamma)$ *est un groupe cyclique fini.*

Il est naturel de se demander dans quelle mesure ces résultats peuvent se généraliser à la classe considérablement plus riche des espaces L_p non-commutatifs. C'est le premier des deux objets principaux de cette thèse : étudier les propriétés $(T_{L_p(\mathcal{M})})$ et $(F_{L_p(\mathcal{M})})$ pour les espaces $L_p(\mathcal{M})$ associés à une algèbre de von Neumann $\mathcal{M}$, appelés espaces L_p non-commutatifs par la suite.

Rappelons que tout espace mesuré (X, μ) tel que X soit un espace borélien standard est isomorphe comme espace mesuré à $[0, 1]$, muni de la mesure $\mu_1 \oplus \mu_2$, où $\mu_1 = \lambda$ est la mesure de Lebesgue et μ_2 est une mesure atomique. Les algèbres de von Neumann abéliennes sont alors toutes isomorphes à une algèbre $L^\infty([0, 1], \mu)$, et les espaces L_p commutatifs sont tous des espaces $L_p([0, 1], \mu)$. En comparaison, la variété des algèbres de von Neumann $\mathcal{M}$, objets qui peuvent être vues comme des espaces mesurés non-commutatifs, est infiniment plus vaste (voir exemples plus loin), et ceci est également valable pour les classes d'isométries des espaces $L_p(\mathcal{M})$ associés (voir la section 5 du chapitre 1).

Pour étudier la propriété $(T_{L_p(\mathcal{M})})$, nous avons dû étendre les méthodes de [4] au cadre non-commutatif. Le maniement des algèbres de von Neumann non-commutatives pose de nombreuses difficultés techniques telles que, par exemple, l'extension dans le cadre des opérateurs de certaines inégalités connues dans le cas commutatif (voir l'extension de l'inégalité de Ando dans le théorème 1.1.4). D'autre part, le groupe des isométries d'un espace L_p non-commutatif est souvent d'une complexité beaucoup plus grande que celui d'un espace L_p classique.

Les outils que nous avons développés nous permettent également d'étudier la propriété (H) de Haagerup dans le cadre des espaces L_p non-commutatifs. C'est le deuxième objet de ce mémoire. Rappelons qu'un groupe G possède la propriété (H), ou est appelé groupe a-T-menable, s'il existe une représentation unitaire de G sur un espace de Hilbert, qui est C_0 (c'est-à-dire dont les coefficients matriciels tendent vers 0 à l'infini) et possède des vecteurs presque invariants. Cette propriété peut être vue comme une propriété de non-rigidité forte des groupes G, en opposition extrême avec la propriété (T). De manière équivalente, G possède (H) s'il admet une action propre par isométries affines sur un espace de Hilbert. Nous menons une étude de ces deux versions de la propriété (H) dans le cadre des espaces L_p non-commutatifs. Les analogues de ces deux formulations de la propriété (H) sur les espaces L_p ne sont plus équivalentes, et ceci déjà dans le cas des espaces L_p classiques pour $p > 2$. Mentionnons cependant le résultat suivant de Nowak, annoncé dans [63] et prouvé dans [64]. Ce résultat a été aussi prouvé par Chatterji, Drutu, Haglund dans [12].

Théorème 1.6. [64] *Soit G un groupe localement compact à base dénombrable.*

1. *Si $1 \leq p < \infty$, et G possède la propriété (H), alors G possède une action propre par isométries affines sur $L_p([0,1])$.*

2. *Si $1 < p < 2$, et G possède une action propre par isométries affines sur $L_p([0,1])$, alors G possède la propriété (H).*

2 Plan détaillé de la thèse et résultats

Dans cette partie, nous présentons nos principaux résultats et les motivations qui nous y ont amené à travers un plan détaillé de cette thèse. Pour chaque résultat énoncé dans cette introduction, nous donnons la numérotation correspondante intervenant dans le corps du texte.

2.1 Les espaces L_p non-commutatifs

Une algèbre de von Neumann $\mathcal{M}$ joue le même rôle pour l'espace non-commutatif $L_p(\mathcal{M})$ associé que l'algèbre $L^\infty(X,\mu)$ pour l'espace $L_p(X,\mu)$ classique. Les exemples d'algèbres de von Neumann sont nombreux : les algèbres de von Neumann commutatives $L^\infty(X,\mu)$, l'algèbre $\mathcal{M}_n$ des matrices de taille $n\times n$, l'algèbre $\mathcal{B}(\mathcal{H})$ des opérateurs bornés sur un espace de Hilbert $\mathcal{H}$, le facteur hyperfini R de type II_1, les algèbres de von Neumann de groupes, les algèbres de von Neumann associés à des actions de groupes,...

Soit $\mathcal{M}$ une algèbre de von Neumann. On peut définir $L_p(\mathcal{M})$ pour tout $1 \leq p < \infty$. Nous nous contentons de rappeler cette construction dans le cadre des algèbres de von Neumann semi-finies. Une telle algèbre de von Neumann $\mathcal{M}$ possède une trace τ fidèle et semi-finie (τ joue un rôle analogue à celui de l'intégrale dans le cadre commutatif). L'espace L_p non-commutatif $L_p(\mathcal{M},\tau)$, noté $L_p(\mathcal{M})$ lorsqu'il n'y a pas de confusion possible, associé à l'espace mesuré non-commutatif $(\mathcal{M},\tau)$, est obtenu comme le complété de l'ensemble

$$\{x \in \mathcal{M} \mid ||x||_p < \infty \}$$

pour la norme $||x||_p = \tau(|x|^p)^{\frac{1}{p}}$. Quelques exemples de tels espaces sont :
- les espaces $L_p(X,\mu)$ classiques obtenus avec les algèbres de von Neumann commutatives $L^\infty(X,\mu)$;
- les idéaux de Schatten $C_p = \{ x \in \mathcal{B}(\mathcal{H}) \mid \mathrm{Tr}(|x|^p) < \infty \}$ pour l'algèbre de von Neumann $\mathcal{M} = \mathcal{B}(\mathcal{H})$;
- l'espace $S_p = \{x = \oplus_n x_n \mid x_n \in \mathcal{M}_n,\ \sum_n \mathrm{Tr}_n(|x_n|^p) < \infty \}$ pour l'algèbre de

von Neumann $(\oplus\mathcal{M}_n)_\infty = \{x = \oplus_n x_n \mid \sup_n ||x_n|| < \infty \}$.

L'étude des représentations orthogonales sur les espaces L_p passe par l'étude de leurs groupes d'isométries $O(L_p)$. Nous rappelons les nombreux résultats concernant la description de ces isométries : pour les isométries sur l_p, $L^p(X,\mu)$ et les idéaux de Schatten $C_p = \{ x \in \mathcal{B}(\mathcal{H}) \mid \mathrm{Tr}(|x|^p) < \infty \}$, jusqu'au théorème de Yeadon [84] donnant une description simple et générale des isométries surjectives sur $L_p(\mathcal{M})$, dans le cas où $\mathcal{M}$ est une algèbre semi-finie. Plus précisément, une isométrie U sur un tel espace admet une unique décomposition $U = uBJ$ avec u un certain opérateur unitaire, B un certain opérateur positif, et J un isomorphisme de Jordan. En utilisant la description des isométries donnée par Sherman dans [74] sur les éléments positifs, et une technique de prolongement d'un isomorphisme de Jordan sur $\mathcal{M}$ à un isomorphisme de Jordan sur son produit croisé, utilisée par Watanabe dans [81], nous donnons une description analogue des isométries sur $L_p(\mathcal{M})$ pour $\mathcal{M}$ une algèbre de von Neumann quelconque, non nécessairement semi-finie.

Soient $1 \leq p, q < \infty$. Un outil crucial pour le passage d'une représentation sur $L_p(\mathcal{M})$ à une représentation sur $L_q(\mathcal{M})$, déjà utilisé dans [4] dans le cas commutatif, est l'application suivante, appelée application de Mazur :

$$\begin{aligned} M_{p,q} :\ & L_p(\mathcal{M}) \to L_q(\mathcal{M}) \\ & x = \alpha|x| \mapsto \alpha|x|^{\frac{p}{q}} \end{aligned}$$

avec $x = \alpha|x|$ la décomposition polaire de x. L'inégalité suivante, qui généralise l'inégalité de Ando pour des matrices (voir [1]), montre que $M_{p,q}$ est localement uniformément continue en restriction aux éléments positifs.

Proposition 2.1. (Proposition 1.1.4) *Pour $a, b \in L_p(\mathcal{M})_+$, et $1 \leq p < q < \infty$, on a l'inégalité suivante :*

$$||a^{\frac{p}{q}} - b^{\frac{p}{q}}||_q \leq ||a - b||_p^{\frac{p}{q}}.$$

On montre aussi que, si J est un isomorphisme de Jordan de $\mathcal{M}$, alors l'application $M_{p,q}$ satisfait à la relation suivante :

$$M_{p,q} \circ J \circ M_{q,p} = J.$$

En particulier, cet outil crucial permet le passage de représentations orthogonales sur $L_p(\mathcal{M})$ à des représentations unitaires sur un espace de Hilbert.

Certains résultats valables dans le cas commutatif ne se généralisent pas au cas non-commutatif. Nous remarquons en particulier que la structure d'une isométrie sur un sous-espace fermé d'un espace $L_p(\mathcal{M})$ non-commutatif n'est connue que dans certains cas très particuliers (voir le théorème 1.6.4). D'autre part, lorsque $1 \leq p \leq 2$, l'espace $L_p(X, \mu)$ classique se plonge dans un espace de Hilbert $\mathcal{H}$, ce qui permet un passage entre représentations sur $L_p(X, \mu)$ et représentations sur $\mathcal{H}$. Ce plongement n'est plus possible pour de nombreux espaces $L_p(\mathcal{M})$ non-commutatifs, par exemple pour les idéaux de Schatten C_p (voir la section 4.2 dans le chapitre 1).

2.2 La propriété (T) pour les représentations sur $L_p(\mathcal{M})$

La propriété (T) implique la propriété $(T_{L_p(\mathcal{M})})$

Trouver les liens entre la propriété (T) de Kazhdan et sa variante $(T_{L_p(\mathcal{M})})$ nécessite d'avoir des transports entre les représentations sur un espace de Hilbert $\mathcal{H}$ et les représentations sur ces espaces L_p. Comme nous l'avons indiqué plus haut, la conjugaison par l'application de Mazur permet ce transport. Nous obtenons ainsi le théorème suivant, ayant fait l'objet d'une publication acceptée dans les Proceedings de l'AMS.

Théorème 2.2. (Theorem 2.5.3) *Soit G un groupe topologique et H un sous-groupe fermé normal de G. Si la paire (G, H) possède la propriété (T) relative, alors (G, H) possède la propriété $(T_{L_p(\mathcal{M})})$ relative pour toute algèbre de von Neumann $\mathcal{M}$, et tout $1 < p < \infty$. En particulier, si G possède la propriété (T), alors il possède la propriété $(T_{L_p(\mathcal{M})})$ pour toute algèbre de von Neumann $\mathcal{M}$, et tout $1 < p < \infty$.*

Dans [59], M.Mimura a démontré indépendamment et simultanément ce théorème pour les espaces de Schatten C_p, avec des méthodes pouvant se généraliser au cas semi-fini. Nos méthodes couvrent également le cas des algèbres de type III, qui présente de considérables difficultés techniques.

Une étape-clé de la démonstration du Théorème 2.2 est la proposition suivante, qui est basée sur une analyse approfondie des représentations orthogonales de groupes sur les espaces $L_p(\mathcal{M})$.

Proposition 2.3. (Proposition 2.5.1) *Soient G un groupe topologique, $\mathcal{M}$ une algèbre de von Neumann, $1 \leq p < \infty$, $p \neq 2$. Soit π^p une représentation orthogonale de G sur $L_p(\mathcal{M})$. Si π^p possède une suite de vecteurs presque invariants dans le complément $L_p(\mathcal{M})'$ des vecteurs $\pi^p(G)$-invariants, alors π^2 possède une suite de vecteurs presque invariants dans le complément $L_2(\mathcal{M})'$.*

Etude de $(T_{L_p(\mathcal{M})})$ pour des algèbres $\mathcal{M}$ "diffuses"

Concernant la réciproque du Théorème 2.2, on ne peut espérer obtenir un résultat général pour toute algèbre de von Neumann $\mathcal{M}$. En effet, si le groupe d'isométries $O(L_p(\mathcal{M}))$ est "petit", comme par exemple le groupe $O(l_p)$, un groupe G pourra avoir la propriété $(T_{L_p(\mathcal{M})})$ sans posséder la propriété (T). A l'opposé, si on considère certaines algèbres $\mathcal{M}$ plus "diffuses", on peut montrer que $(T_{L_p(\mathcal{M})})$ implique (T) ; cela a déjà été démontré dans [4] pour le cas $\mathcal{M} = L^\infty(X, \mu)$. Nous prouvons le résultat suivant.

Théorème 2.4. (Theorem 2.5.6) *Soit $\mathcal{M} = \mathcal{B}(\mathcal{H})$, ou $\mathcal{M}$ le facteur* II_1 *hyperfini R. Soit G un groupe topologique localement compact à base dénombrable. Si G a la propriété $(T_{L_p(\mathcal{M})})$, alors G a la propriété (T) de Kazhdan.*

La propriété $(T_{L_p(\mathcal{M})})$ pour des algèbres $\mathcal{M}$ discrètes telle que $\mathcal{M} = l^\infty$ est souvent strictement plus faible que la propriété (T) et mérite une étude indépendante. Nous avons mené une telle étude pour la propriété (T_{l_p}). Elle fait l'objet d'une publication en cours de rédaction avec B.Bekka.

Etude de (T_{l_p})

Nous caractérisons pour un groupe G la propriété (T_{l_p}) par la propriété d'isolation de la représentation triviale 1_G de G dans l'ensemble des représentations monomiales de G. On rappelle qu'une représentation unitaire σ de G est monomiale si σ est unitairement équivalente à une représentation induite $\mathrm{Ind}_H^G \chi$, où H est un sous-groupe fermé de H et $\chi : H \to \mathbb{S}^1$ est un caractère untitaire sur H.

Théorème 2.5. (Theorems 2.6.5 and 2.6.9) *Soit G un groupe localement compact à base dénombrable.*
(i) G possède (T_{l_p}) pour $1 < p < \infty$ et $p \neq 2$ si et seulement si 1_G est isolée dans l'ensemble des représentations monomiales $\mathrm{Ind}_H^G \chi$, associées aux sous-groupes ouverts H de G.
Si de plus, G est un groupe totalement discontinu, on a :
(ii) G possède (T_{l_p}) pour $1 < p < \infty$ et $p \neq 2$ si et seulement si 1_G est isolée dans l'ensemble des représentations quasi-régulières $(\lambda_{G/H}, l_2(G/H))$, associées aux sous-groupes ouverts H de G.

Pour G totalement discontinu, le Théorème 2.5 montre bien la différence entre (T) et (T_{l_p}) : (T) fait intervenir toutes les représentations unitaires de G alors que (T_{l_p}) ne concerne que les représentations quasi-régulières associées à des sous-groupes ouverts. On déduit de ces caractérisations que certains groupes sans la propriété (T) possèdent la propriété (T_{l_p}) pour $p \neq 2$. C'est, par exemple, le cas de $SL_2(\mathbb{Q}_l)$, où $\mathbb{Q}_l$ est l'ensemble des nombres l-adiques, avec l un nombre premier.

(T_F) pour F un sous-espace fermé de $L_p(\mathcal{M})$

Nous avons également cherché à étendre le Théorème 1.3 aux représentations sur des sous-espaces fermés de $L_p(\mathcal{M})$. Cependant, comme mentionné précédemment, la structure des isométries d'un tel sous-espace n'est pas très bien connue. Un cadre naturel plus adapté dans le cas non-commutatif est celui des isométries complètes. Ceci nous conduit à introduire une variante plus faible de (T_F).

Définition 2.6. Soit $\mathcal{M}$ une algèbre de von Neumann finie et $1 \leq p < \infty$. Soit F un sous-espace fermé de $L_p(\mathcal{M})$ tel que $1 \in F$. Un groupe topologique G possède la propriété $(T_F^{c.i.})$ si, pour toute représentation orthogonale $G \to O^{c.i.}(F)$ de G par isométries complètes de F préservant l'unité 1, la restriction $\pi_{|F'(\pi)}$ de π sur $F'(\pi)$ n'a pas presque des vecteurs invariants.

Nous avons obtenu l'analogue suivant du point 2. dans le Théorème 1.3.

Théorème 2.7. (Theorem 2.7.4) *Soit $1 \leq p < \infty$, $p \notin 2\mathbb{N}$. Soit $\mathcal{M}$ une algèbre de von Neumann finie, et F un sous-espace fermé de $L_p(\mathcal{M})$ tel que $F \subset \mathcal{M}$ et $1 \in F$. Supposons que G soit un groupe topologique avec la propriété (T). Alors G a la propriété $(T_F^{c.i.})$.*

2.3 Propriétés de point fixe pour les actions sur $L_p(\mathcal{M})$

Nous nous sommes aussi intéressés aux propriétés de point fixe $(F_{L_p(\mathcal{M})})$, et plus généralement aux actions par isométries affines sur les espaces L_p non-commutatifs. Les groupes de rang supérieur et leurs réseaux fournissent des exemples de groupes avec $(F_{L_p(\mathcal{M})})$. Nous rappelons leur définition.

Définition 2.8. Pour $1 \leq i \leq m$, soient k_i des corps locaux et $\mathbb{G}_i(k_i)$ les k_i-points de groupes $\mathbb{G}_i$ algébriques sur k_i, connexes et simples. Si chacun des facteurs simples $\mathbb{G}_i$ est de rang supérieur ou égal à 2 sur k_i, le groupe $G = \prod_{i=1}^{m} \mathbb{G}_i(k_i)$ est appelé groupe de rang supérieur.

En utilisant le Théorème 2.2, ainsi que les techniques développées dans [4], on obtient le résultat suivant.

Théorème 2.9. (Theorem3.2.3) *Soit G un groupe de rang supérieur et $\mathcal{M}$ une algèbre de von Neumann. Alors G, ainsi que les réseaux dans G, possèdent la propriété $(F_{L_p(\mathcal{M})})$ pour $1 < p < \infty$.*

Ces mêmes techniques ont été utilisées dans [59] pour montrer que les groupes $SL_n(\mathbb{Z}[x_1, ...x_k])$ possédaient la propriété $(F_{L_p(\mathcal{M})})$ pour $n \geq 4$. Le résultat du théorème précédent a été démontré par Puschnigg dans [68] dans le cas particulier où $\mathcal{M} = \mathcal{B}(\mathcal{H})$, et donc $L_p(\mathcal{M}) = C_p$. Il en a donné une application à l'existence de modules de Fredholm au sens de Connes.

Nous avons déjà mentionné que la propriété (F_B) implique la propriété (T_B) dans le cadre des groupes localement compacts σ-compacts, et que la réciproque est fausse pour $B = L_p([0,1])$ et $p > 2$. Nous avons cherché à généraliser au cadre non-commutatif le point 2. dans le Théorème 1.4. La preuve des auteurs de [4] dans le cas commutatif est basée sur le fait que L_p se plonge dans L_2 pour $1 < p < 2$. Ceci n'est plus le cas pour les espaces L_p non-commutatifs : un espace $L_p(\mathcal{M})$ contenant $\mathcal{M}_2(\mathbb{R})$ ne se plonge pas dans un espace de Hilbert. Nous ignorons si le résultat reste quand même valable dans le cadre non-commutatif.

Par contre, le point 3. du théorème 1.4 se généralise parfaitement au cas non-commutatif, en adaptant la preuve donnée dans [4].

Théorème 2.10. (Theorem 3.3.1) *Soit $\mathcal{M}$ une algèbre de von Neumann. Soit G un groupe topologique avec la propriété (T), alors il existe une constante $\epsilon > 0$ telle que G a la propriété de point fixe (F_B) , pour tout $p \in]2-\epsilon, 2+\epsilon[$, et pour tout sous-espace fermé B de $L_p(\mathcal{M})$.*

2.4 La propriété de Haagerup pour des actions sur les espaces L_p

Une autre propriété de groupes, qui a été beaucoup étudiée, est la propriété (H) de Haagerup. Elle est partagée par de nombreux groupes : groupes moyennables, groupes libres, groupes de Coxeter...Une obstruction bien connue à la propriété (H) est l'existence d'une paire de groupes (G, H) avec la propriété (T), où H est un sous-groupe non-compact de G [1]. En ce sens, la propriété (H) peut être considérée comme une négation forte de la propriété (T).

Rappelons qu'un groupe G localement compact à base dénombrable possède la propriété (H) de Haagerup (ou est a-T-menable) s'il existe une représentation unitaire de G, sur un espace de Hilbert $\mathcal{H}$, qui est C_0 (voir la définition plus bas) et qui possède presque des vecteurs invariants. Il est connu que ceci est équivalent à l'existence d'une action propre de G par isométries affines sur un espace de Hilbert $\mathcal{H}$.

Nous nous sommes donc intéressés à la traduction de ces deux variantes de la propriété (H) de Haagerup dans le cadre des espaces $L_p(\mathcal{M})$ non-commutatifs associés à des algèbres de von Neumann $\mathcal{M}$ semi-finies.

On rappelle qu'une fonction $f : X \to \mathbb{C}$ sur un espace topologique X est C_0 si :

$$\forall \epsilon > 0, \exists K \subset X \text{ compact tel que } |f(x)| < \epsilon \text{ pour tout } x \in X \backslash K.$$

Les coefficients matriciels d'une représentation π d'un groupe G sur un espace vectoriel V sont les fonctions $g \mapsto <\pi(g)v, w>$ de G dans $\mathbb{C}$, pour $v \in V$ et

[1] ce n'est pas la seule obstruction, voir par exemple le résultat de Cornulier [21]

$w \in V^*$. On dit qu'une représentation π est C_0 si tous ses coefficients matriciels sont C_0.

Définition 2.11. Soit $\mathcal{M}$ une algèbre de von Neumann semi-finie.
On dit que G a la propriété $(H_{L_p(\mathcal{M})})$ s'il existe une représentation $\pi : G \to O(L_p(\mathcal{M}))$ qui est C_0 et avec presque des vecteurs invariants.

A notre connaissance, la propriété $(H_{L_p(\mathcal{M})})$ n'a pas été étudiée jusqu'à présent, même dans le cas des espaces L_p commutatifs. L'existence d'une paire de groupes (G, H) avec la propriété $(T_{L_p(\mathcal{M})})$, où H est un sous-groupe normal fermé non-compact de G, est là aussi une obstruction à la propriété $(H_{L_p(\mathcal{M})})$. Il est bien connu que les sous-groupes fermés d'un groupe du type $\prod_{i\in I} S_i$, où I est fini et chaque S_i est soit le groupe $SO(n_i, 1)$, soit le groupe $SU(m_i, 1)$, possède la propriété (H). Nous montrons un résultat plus fort.

Théorème 2.12. (Theorem 4.4.1) *Soit G un sous-groupe fermé d'un groupe du type $\prod_{i\in I} S_i$, où I est fini et chaque S_i est soit le groupe $SO(n_i, 1)$, soit le groupe $SU(m_i, 1)$ pour $n_i \geq 2$, $m_i \geq 1$. Alors G possède la propriété $(H_{L_p([0,1])})$ pour tout $1 < p < \infty$.*

Les liens entre la propriété (H) et celles pour des espaces L_p non-commutatifs dépendent de l'algèbre de von Neumann considérée, comme l'indiquent les résultats que nous allons énoncer. En particulier, concernant la question de savoir si (H) implique $(H_{L_p(\mathcal{M})})$, nos résultats montrent que la réponse est positive pour certaines algèbres de von Neumann, et négatives pour d'autres.

Considérons d'abord le cas de l'algèbre $\mathcal{M} = l^\infty$. Nous montrons que seule une classe restreinte de groupes possède la propriété (H_{l_p}).

Théorème 2.13. (Theorems 4.3.1 and 4.3.2) *Soit G un groupe localement compact à base dénombrable.*
(i) Si G est connexe, alors G possède (H_{l_p}) si et seulement si G est compact.
(ii) Si G est totalement discontinu, alors G possède (H_{l_p}) si et seulement si G est moyennable.

Lorsque le groupe d'isométries de l'espace $L_p(\mu)$ considéré est plus "gros", la propriété $(H_{L_p(\mu)})$ est plus fortement liée à la propriété (H). C'est ce que montre le théorème qui suit, où nous caractérisons les groupes de Lie connexes linéaires ayant la propriété $(H_{L_p([0,1])})$.

Théorème 2.14. (Theorem 4.4.1) *Soit G un groupe de Lie connexe linéaire. Soit $1 < p < \infty$. Alors les assertions suivantes sont équivalentes :*
(i) G possède la propriété $(H_{L_p([0,1])})$;
(ii) G possède la propriété (H) de Haagerup ;
(iii) G est localement isomorphe à un produit $\prod_{i\in I} S_i \times M$, où I est fini, M est un groupe moyennable, et pour tout $i \in I$, S_i est soit le groupe $SO(n_i, 1)$ soit $SU(m_i, 1)$ avec $n_i \geq 2$, $m_i \geq 1$.

L'exemple suivant traite le cas des algèbres discrètes $\mathcal{B}(\mathcal{H})$ et $(\oplus\mathcal{M}_n)_\infty = \{x = \oplus_n x_n \mid \sup_n ||x_n|| < \infty\}$ pour lesquelles les espaces L_p correspondants sont C_p et S_p.

Théorème 2.15. (Theorems 4.5.1 and 4.5.2) *Soit G un groupe localement compact à base dénombrable. Soit $1 < p < \infty$, $p \neq 2$. Alors on a*
(i) G possède (H_{C_p}) si et seulement si G possède (H).
(ii) G possède (H_{S_p}) si et seulement si G est compact.

Les groupes apparaissant dans le Théorème 2.12 possèdent en fait des versions plus fortes de $(H_{L_p([0,1])})$, que nous définissons pour des classes plus restreintes de représentations sur les espaces $L_p(\mathcal{M})$.

Définition 2.16. Soit $\mathcal{M}$ une algèbre de von Neumann semi-finie, munie d'une trace fidèle, normale et semi-finie τ.
On dit que G a la propriété $(H_{L_p(\mathcal{M}),+})$ (resp. $(H_{L_p(\mathcal{M}),\tau})$) s'il existe une représentation positive (resp. une représentation préservant la trace) $\pi : G \to O(L_p(\mathcal{M}))$ qui est C_0 et avec presque des vecteurs invariants.

Notre prochain résultat montre que les propriétés $(H_{L_p(\mathcal{M}),+})$ et $(H_{L_p(\mathcal{M}),\tau})$ impliquent la propriété (H).

Théorème 2.17. (Theorem 4.6.4) *Soient $1 \leq p < \infty$, et G un groupe localement compact à base dénombrable.*
(i) Soit $\mathcal{M}$ une algèbre de von Neumann semi-finie. Si G possède la propriété $(H_{L_p(\mathcal{M}),\tau})$, alors G possède la propriété (H).
(ii) Soit $\mathcal{M}$ une algèbre de von Neumann finie. Si G possède la propriété $(H_{L_p(\mathcal{M}),+})$, alors G possède la propriété (H).

2.5 Actions fortement mélangeantes sur $L_p(\mathcal{M})$

Nous nous sommes aussi intéressés aux représentations fortement mélangeantes sur les espaces $L_p(\mathcal{M})$ associés à des algèbres de von Neumann finies. On peut introduire la variante suivante de la propriété (H), déjà considérée par Jolissaint dans [13].

Définition 2.18. Soit $\mathcal{M}$ une algèbre de von Neumann finie. On dit que G a la propriété $(H^{mix}_{L_p(\mathcal{M})})$ s'il existe une représentation $\pi : G \to O(L_p(\mathcal{M}))$ qui est fortement mélangeante et qui possède presque des vecteurs invariants dans le complément $L_p(\mathcal{M})'$ des vecteurs $\pi(G)$-invariants.

Les résultats suivants semblent indiquer que la propriété $(H^{mix}_{L_p(\mathcal{M})})$ est plus étroitement liée à la propriété (H) que la propriété $(H_{L_p(\mathcal{M})})$.

Théorème 2.19. (Theorems 5.2.3 and 5.2.4) *Soit $1 \leq p < \infty$ et G un groupe localement compact à base dénombrable.*

1. *Si G a la propriété de Haagerup* $(H^{mix}_{L_p(\mathcal{M})})$*, alors il a la propriété de Haagerup* (H).

2. *Si G a la propriété de Haagerup* (H)*, alors il a la propriété* $(H^{mix}_{L_p(R)})$ *pour le facteur hyperfini R de type* II_1.

3. *Si G a la propriété de Haagerup* (H)*, alors il a la propriété* $(H^{mix}_{L^p([0,1])})$.

2.6 Actions propres par isométries affines sur $L_p(\mathcal{M})$

Nous nous intéressons maintenant aux liens entre la propriété (H) et les actions propres par isométries affines sur les espaces L_p. Nous reprenons dans cette définition la terminologie utilisée dans [12].

Définition 2.20. Soit $\mathcal{M}$ une algèbre de von Neumann et $1 \le p < \infty$. Un groupe G localement compact à base dénombrable est dit a-$FL_p(\mathcal{M})$-menable s'il existe une action propre de G par isométries affines sur $L_p(\mathcal{M})$.

Pour $p = 2$, c'est la propriété de Haagerup.

Nous montrons le résultat suivant qui donne un lien entre la propriété $(H_{L_p(\mathcal{M})})$ et l'a-$FL_p(\mathcal{M})$-menabilité pour certains facteurs $\mathcal{M}$.

Théorème 2.21. (Proposition 6.2.2) *Soit G un groupe localement compact à base dénombrable. Soit* $\mathcal{M}$ *un facteur de type* II_∞*, et* $1 \le p < \infty$*. Si G possède la propriété* $(H_{L_p(\mathcal{M})})$*, alors G est a-*$(FL_p(\mathcal{M}))$*-menable.*

Rappelons que le Théorème 1.6 de Nowak montre l'équivalence entre a-$FL_p([0,1])$-menabilité et la propriété (H) pour $1 \le p < 2$. Nous avons obtenu une extension de la première partie de ce résultat pour le cas $\mathcal{M} = l^\infty \otimes R$, où R est le facteur hyperfini de type II_1.

Théorème 2.22. (Theorem 6.2.5 and Corollary 6.2.6) *Soit G un groupe localement compact à base dénombrable avec la propriété de Haagerup* (H)*, et soit* $1 \le p < \infty$*. Alors il existe une action propre de G par isométries affines sur* $L_p(\mathcal{M})$*, où* $\mathcal{M} = l^\infty \otimes R$*. De même, il existe une action propre de G par isométries affines sur* $L_p(\mathcal{M})$*, où* $\mathcal{M} = \mathcal{B}(l_2) \otimes R$ *est le facteur hyperfini de type* II_∞.

Nous ignorons si l'analogue du résultat 2. dans le Théorème 1.6 est vrai dans ce cadre, la difficulté étant, comme mentionné plus haut, que les distances associées aux normes $||.||_p$ n'induisent pas de noyau conditionnellement de type négatif.

L'a-$FL_p(\mathcal{M})$-menabilité et (H) sont des propriétés distinctes : en effet, il existe des groupes de Kazhdan avec des actions propres par isométries affines sur $L_p([0,1])$ (voir [11], [66], [85], [23], [62]). Soit Γ est un groupe hyperbolique. Utilisant des techniques développées par Mineyev [60], Yu [85] a montré que Γ possède une action propre par isométries affines sur $l_p(\Gamma\times\Gamma)$ pour p suffisamment grand. Nous adaptons sa construction pour montrer que Γ possède une action propre par isométries affines sur C_p.

3 Quelques questions ouvertes

Nous donnons ici une liste non-exhaustive de questions qui sont intervenues au cours de ce travail, et qui restent actuellement sans réponse.

- Pour $p \neq 2$, existe-t-il des groupes discrets possédant la propriété (T_{l_p}) et la propriété (H) (pour des exemples de groupes discrets avec (T_{l_p}) et sans la propriété (T), voir le preprint [9]) ?

- Pour $1 < p < 2$ et $\mathcal{M}$ une algèbre non-commutative, les groupes G avec la propriété (T) possèdent-t-il la propriété $(F_{L_p(\mathcal{M})})$?

- Pour $p \neq 2$, le revêtement universel de $SU(n,1)$ possède-t-il la propriété $(H_{L_p([0,1])})$ (voir le Theorem 4.4.1 et la remarque qui suit) ?

- Pour $p \neq 2$ et $q \neq 2$, les propriétés $(H_{L_p(\mathcal{M})})$ et $(H_{L_q(\mathcal{M})})$ sont-elles équivalentes? Plus précisément, la conjugaison par l'application de Mazur préserve-t-elle le caractère C_0 d'une représentation orthogonale sur $L_p(\mathcal{M})$ (voir Remark 4.6.6) ?

- Les propriétés (H) et $(H^{mix}_{L_p(\mathcal{M})})$ sont-elles équivalentes pour tout $1 < p < \infty$ et toute algèbre de von Neumann finie $\mathcal{M}$ (voir Chapter 5 section 2) ?

Chapter 1

Non-commutative L_p-spaces

Non-commutative L_p-spaces were introduced by Dixmier [26] and studied by various authors, among them Yeadon [83] and Haagerup [35]. We recall in this chapter some basic facts on these L_p-spaces, which share some common properties with their commutative brothers, but have sometimes strong differences with them. The survey here is far from exhaustive; for a more complete survey on these spaces, see Pisier and Xu [67]. Since the variant of property (T) on L_p-spaces strongly depends on the structure of isometries on such spaces, we are going to study in this thesis the structure of such isometries. Moreover, we will prove a few results (among them a generalization of Ando's inequality) and present some tools (Mazur map, structure of the group of isometries of L_p-spaces, non-embeddability of some non-commutative L_p-spaces in Hilbert spaces) which will be needed in later chapters.

1.1 $L_p(\mathcal{M})$-spaces associated with semi-finite von Neumann algebras $\mathcal{M}$

We first review some basic properties of $L_p(\mathcal{M})$ in the case of semi-finite von Neumann algebras $\mathcal{M}$.

1.1.1 Definition and examples

Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$.

$\mathcal{M}$ is said to be *semi-finite* if it admits a normal semi-finite trace τ, that is, a linear map $\tau : \mathcal{M}^+ \to [0, +\infty]$ with the following properties :
- for all $u \in \mathcal{M}$, $\tau(u^*u) = \tau(uu^*)$,
- for any bounded increasing net (x_α) in $\mathcal{M}^+$, $\sup_\alpha \tau(x_\alpha) = \tau(\sup_\alpha x_\alpha)$,
- for any non-zero $x \in \mathcal{M}^+$, there is a non-zero $y \in \mathcal{M}^+$ such that $y \leq x$ and $\tau(y) < +\infty$,

- if $\tau(x) = 0$, then $x = 0$.

If $\tau(1) < +\infty$, the von Neumann algebra $\mathcal{M}$ is said to be *finite*.

$\mathcal{M}$ is said to be *hyperfinite* if there exists an increasing sequence of finite dimensional von Neumann algebras with dense (in the strong operator topology) union in $\mathcal{M}$.

Let us denote by $\mathcal{M}'$ the commutant of $\mathcal{M}$. The von Neumann algebra $\mathcal{M}$ is called a *factor* if $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}1$.

Let $\mathcal{M}$ be a semi-finite von Neumann algebra and $1 \leq p < +\infty$. We define the $L_p(\mathcal{M}, \tau)$-space as the completion of the set

$$\{x \in \mathcal{M} \mid ||x||_p < \infty \}$$

with respect to the norm $||x||_p = \tau(|x|^p)^{\frac{1}{p}}$.

We now give some examples of such spaces, and set some notations. If $1 \leq p \leq +\infty$, then we will always denote by p' the conjugate exponent of p.

Examples

1. Let (X, μ) be a measured space, and let $\mathcal{M} = L^\infty(X, \mu)$ be the commutative von Neumann algebra, equipped with the trace $\tau : f \longrightarrow \int_X f \; d\mu$. Then the associated L_p-space is the classical L_p-space $L_p(\mathcal{M}, \tau) = L_p(X, \mu)$.

2. The p-Schatten ideals C_p are the $L_p(\mathcal{M}, \tau)$-spaces associated to $\mathcal{M} = \mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a separable Hilbert space, and $\tau = \mathrm{Tr}$ the usual trace on $\mathcal{H}$; thus,
$$C_p = \{x \in \mathcal{B}(\mathcal{H}) \mid \mathrm{Tr}(|x|^p) < \infty \}.$$

3. Denote by $\mathcal{M}_n$ the algebra of complex $n \times n$ matrices. Consider the von Neumann algebra
$$\mathcal{M} = (\oplus_n \mathcal{M}_n)_\infty = \{\oplus_n x_n \mid x_n \in \mathcal{M}_n, \; \sup_n ||x_n|| < \infty\},$$
equipped with the trace $\tau = \sum_n \mathrm{Tr}_n$, where Tr_n is the usual trace on $\mathcal{M}_n$. The associated L_p-space $L_p(\mathcal{M}, \tau)$ will be denoted by S_p.

4. The space $L_p(R)$ associated with the hyperfinite II_1 factor R. Recall that the hyperfinite II_1 factor R can be described as the von Neumann algebra $R = \otimes_n \mathcal{M}_2$, the von Neumann infinite tensor product of copies of $\mathcal{M}_2$. R is equipped with the trace $\tau = \otimes_n \mathrm{Tr}_n$.

A few general properties of non-commutative L_p-spaces

As in the commutative case, a basic property of non-commutative L_p-spaces are the following Clarkson type inequalities (for a proof in the more general case of Haagerup L_p-spaces, see [50]; see also [70] for the proof of the equality case when $1 \leq p < 2$).

Proposition 1.1.1. *Let $\mathcal{M}$ be a von Neumann algebra. For all $x, y \in L_p(\mathcal{M})$ we have*

$$(\frac{1}{2}(||x+y||_p^{p'} + ||x-y||_p^{p'}))^{\frac{1}{p'}} \leq (||x||_p^p + ||y||_p^p)^{\frac{1}{p}} \text{ for } 1 \leq p \leq 2$$

and

$$(\frac{1}{2}(||x+y||_p^{p} + ||x-y||_p^{p}))^{\frac{1}{p}} \leq (||x||_p^{p'} + ||y||_p^{p'})^{\frac{1}{p'}} \text{ for } 2 \leq p \leq +\infty.$$

The equality case occurs in the previous inequalities if and only if $xy^* = y^*x = 0$

The equality case in these inequalities is a crucial tool in the study of the structure of $O(L_p(\mathcal{M}))$, the group of bijective linear isometries of $L_p(\mathcal{M})$. Recall the usual following formula, for $x \in L_p(\mathcal{M}, \tau)$,

$$||x||_p = \sup_{y \in L_{p'}(\mathcal{M},\tau), ||y||_{p'}=1} \tau(xy).$$

Now let $1 < p < \infty$. The dual of $L_p(\mathcal{M})$ can be identified to $L_{p'}(\mathcal{M})$ by means of the duality map $(x, y) \mapsto \tau(xy)$. A straightforward consequence of the Clarkson's inequalities is that $L_p(\mathcal{M})$ is uniformly convex, and uniformly smooth. We now recall the notions of uniformly convexity, uniformly smoothness for Banach spaces, since this properties of convexity about L_p-spaces are a crucial fact for our study of property (T_{L_p}).

Some definitions about uniformly convex Banach spaces

Let B be a Banach space. The convexity modulus of B is the function $\epsilon \mapsto \delta(\epsilon)$ defined by

$$\delta(\epsilon) = \inf\{1 - ||\frac{u+v}{2}|| \mid ||u||, ||v|| \leq 1 \text{ and } ||u-v|| \geq \epsilon \}.$$

B is said to be *uniformly convex* if $\delta(\epsilon) > 0$, whenever $\epsilon > 0$. B is said to be *uniformly smooth* if its dual space B^* is uniformly convex. We will say that B is *ucus* if it is uniformly convex and uniformly smooth.

Let B be a stricly convex Banach space, that is a Banach space satisfying

$$||\frac{x+y}{2}|| < 1 \text{ for all } x, y \text{ in the unit sphere } S(B) \text{ of } B.$$

The *duality map* $* : S(B) \to S(B^*)$ is the unique map that associates to each unit vector $x \in B$, the unit vector x^* in B^* such that $< x, x^* >= 1$. Moreover, if B is a ucus Banach space, then the map $*$ is uniformly continuous with a uniformly continuous inverse (see [10] for more details). We will describe this map in Section 1.3 of this chapter in the special case of the L_p-spaces, and we will use it in our proofs later.

1.1.2 τ-measurable operators and s-generalized numbers

Let $\mathcal{M}$ be a semi-finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and equipped with a trace τ. Let us denote by $\mathcal{P}(\mathcal{M})$ the set of projections in $\mathcal{M}$. The elements of $L_p(\mathcal{M}, \tau)$ can be seen as closed densily defined operators on $\mathcal{H}$. Recall that a densily defined closed operator x on $\mathcal{H}$ is affiliated with $\mathcal{M}$ if $xu = ux$ for every unitary u in the commutant $\mathcal{M}'$ of $\mathcal{M}$. A densily defined closed operator x with domain $D(x)$ affiliated with $\mathcal{M}$ is called τ-measurable if for every $t > 0$, there exists $P \in \mathcal{P}(\mathcal{M})$ such that $\tau(P) \leq t$, $(1 - P)(\mathcal{H}) \subset D(x)$, and $x(1-P) \in \mathcal{M}$. We will denote by $L_0(\mathcal{M}, \tau)$ the set of τ-measurable operators and $L_0(\mathcal{M}, \tau)_+$ the set of positive operators in $L_0(\mathcal{M}, \tau)$. In particular, elements of $L_p(\mathcal{M}, \tau)$ can be seen as elements of $L_0(\mathcal{M}, \tau)$.

Recall that the measure topology on $L_0(\mathcal{M}, \tau)$ is by definition the topology whose fundamental system of neighborhoods of 0 is given by

$$V(\epsilon, \delta) = \{x \in L_0(\mathcal{M}, \tau) \mid \exists P \in \mathcal{P}(\mathcal{M}),\ ||xP|| \leq \epsilon \text{ and } \tau(1 - P) \leq \delta\ \}.$$

For details on the τ-measurable operators and the measure topology, we refer to the preliminaries in [29], and Chapter I in [79].

In order to generalize Ando's inequality, we need to introduce the notion of s-generalized numbers (see the article [29] by Fack and Kosaki for more details on s-numbers). These numbers are a generalization of the singular values of matrices or the singular values of operators in $\mathcal{B}(\mathcal{H})$. The s-generalized numbers are defined for $x \in L_0(\mathcal{M}, \tau)$, and $s > 0$, by

$$\mu_s(x) = \inf_{P \in \mathcal{P}(\mathcal{M}), \tau(1-P) \leq s} (||xP||).$$

For $x \in L_0(\mathcal{M}, \tau)_+$, we have

$$\mu_s(x) = \inf_{P \in \mathcal{P}(\mathcal{M}), \tau(1-P) \leq s} \Big(\sup_{\xi \in P(\mathcal{H}), ||\xi||=1} < x\xi, \xi > \Big).$$

In the next proposition, we recall some useful properties of the s-numbers.

Proposition 1.1.2. [29] *Let $x, y \in L_0(\mathcal{M}, \tau)$. Let $1 \leq p < \infty$. Then :*
(i) $\mu_s(xy) \leq ||x|| \mu_s(y)$.

(ii) $\mu_s(x) \leq \mu_s(y)$ *if* $0 \leq x \leq y$.
(iii) $\mu_s(f(|x|)) = f(\mu_s(|x|))$ *for every continuous increasing function* f *on* $\mathbb{R}^+$ *such that* $f(0) \geq 0$.
(iv) For $x \in L_p(\mathcal{M}, \tau)$, *we have* $||x||_p = (\int_0^\infty \mu_s(x)^p ds)^{\frac{1}{p}}$.
(v) If $xy \in L_1(\mathcal{M}, \tau)$, *we have* $|\tau(xy)| \leq \int_0^\infty \mu_s(x)\mu_s(y)ds$.
(vi) $\mu_s(xy) = \mu_s(yx)$ *for* $x, y \geq 0$ *such that* $xy \geq 0$.

Properties $(i) - (v)$ are established in [29]. Notice that if x, y are positive operators such that $xy \geq 0$, then

$$< yx\xi, \xi > = < \xi, xy\xi > \geq 0$$

hence $yx \geq 0$ and (vi) follows from the definition of μ_s.

We need to recall a few basic properties of elements in the $L_p(\mathcal{M}, \tau)$-spaces.

Lemma 1.1.3. *Let* x, y *be self-adjoint elements in* $L_p(\mathcal{M}, \tau)$. *Let* $1 \leq p < \infty$.
(i) If $x \geq y$ *then* $||x^+||_p \geq ||y^+||_p($ *with* $x^+ = \max\{x, 0\}$ $)$.
(ii) If $xy = yx = 0$ *then* $||x + y||_p^p = ||x||_p^p + ||y||_p^p$.

Proof. (i) If $x \geq y$ and $r = (p-1)/2$, then

$$\begin{aligned} ||y^+||_p^p &= \tau(y^{+r} y y^{+r}) \\ &\leq \tau(y^{+r} x y^{+r}) \\ &\leq \tau(y^{+r} x^+ y^{+r}) \\ &\leq ||y^+||_p^{p-1} ||x^+||_p \text{ (using Hölder's inequality) }. \end{aligned}$$

(ii) Observe that $xy = yx = 0$ implies that the C^*-algebra generated by $\{x, y\}$ is abelian. Hence, we can assume that x and y are functions with disjoint supports $C^0(X)$, X being a topological space. It is then obvious that $|x + y|^p = |x|^p + |y|^p$. ☐

1.1.3 Generalization of Ando's inequality

In [1], Ando proved the following inequality : let a, b be positive $n \times n$-complex matrices, f a non-negative operator monotone function, and $|| \cdot ||$ a unitarily invariant norm on $\mathcal{M}_n(\mathbb{C})$; then

$$||f(a) - f(b)|| \leq ||f(|a - b|)||.$$

We extend this inequality to measurable operators in the special case of the operator monotone function $f : \lambda \mapsto \lambda^{\frac{p}{q}}$ and the norm $|| \cdot ||_q$ when $p < q$. This will be used later in order to show the local uniformly continuity of the Mazur map in the case of a semi-finite von Neumann algebra.

Proposition 1.1.4. *Let $\mathcal{M}$ be a semi-finite von Neumann algebra. For $a, b \in L_0(\mathcal{M}, \tau)_+$, and $1 \leq p < q < \infty$, we have*

$$||a^{\frac{p}{q}} - b^{\frac{p}{q}}||_q \leq ||a - b||_p^{\frac{p}{q}}.$$

Recall the following integral representation for operator monotone functions, called the Löwner integral representation.

Proposition 1.1.5. [71] *Let f be a real-valued continuous function on $(0, \infty)$. Then f is operator monotone if and only if*

$$f(\lambda) = \alpha\lambda + \beta + \int_0^\infty \frac{\lambda t}{\lambda + t} d\nu(t)$$

for some $\alpha \in \mathbb{R}$, $\beta \geq 0$ and a positive measure ν on $(0, \infty)$ such that $\int_0^\infty \frac{t}{1+t} d\nu(t) < \infty$.

Remark 1.1.6. If $p < q$, the mapping $\lambda \mapsto \lambda^{\frac{p}{q}}$ is operator monotone, and by the previous Proposition 1.1.5, it admits a Löwner decomposition. In fact, the decomposition in this case is well-known and we have

$$\lambda^{\frac{p}{q}} = \frac{\sin(\frac{p}{q}\pi)}{\pi} \int_0^\infty \frac{\lambda t}{\lambda + t} \, t^{\frac{p}{q}-2} dt.$$

Proof of Proposition 1.1.4. We first prove the inequality in the case $a \geq b \geq 0$.

We have, by Remark 1.1.6, an integral representation of the following form for the operator monotone mapping $\lambda \mapsto \lambda^{\frac{p}{q}}$:

$$\lambda^{\frac{p}{q}} = \int_0^\infty \frac{\lambda t}{\lambda + t} d\nu(t) \ ,$$

where ν is a positive measure on $(0, \infty)$ such that $\int_0^\infty \frac{t}{1+t} d\nu(t) < \infty$.

Let $c = a - b \geq 0$. We have to prove that

$$||(b + c)^{\frac{p}{q}} - b^{\frac{p}{q}}||_q \leq ||c^{\frac{p}{q}}||_q. \tag{1.1}$$

For simplicity, let us denote by $a(t)$ the operator $t((b+c)(b+c+t1)^{-1} - b(b+t1)^{-1})$, coming from the Löwner integral representation of $(b + c)^{\frac{p}{q}} - b^{\frac{p}{q}}$.

Now we check that

$$\mu_s(a(t)) \leq \mu_s(tc(c + t1)^{-1}) \text{ for } t > 0 \text{ and } s > 0. \tag{1.2}$$

To prove this, observe that we can assume that $t = 1$ by replacing b, c by $\frac{1}{t}b, \frac{1}{t}c$. As $x(x+1)^{-1} = 1 - (x+1)^{-1}$, inequality (1.2) is equivalent to the following inequality :

$$\mu_s((b+1)^{-1} - (b+c+1)^{-1}) \leq \mu_s(1-(c+1)^{-1}). \tag{1.3}$$

Since

$$(b+1)^{-1} - (b+c+1)^{-1} = (b+1)^{-\frac{1}{2}}(1-(1+(b+1)^{-\frac{1}{2}}c(b+1)^{-\frac{1}{2}})^{-1})(b+1)^{-\frac{1}{2}}$$

and $||(b+1)^{-1}|| \leq 1$, using successively Properties (vi) and (i) of Proposition 1.1.2, we get

$$\mu_s((b+1)^{-1} - (b+c+1)^{-1}) \leq \mu_s(1-(1+(b+1)^{-\frac{1}{2}}c(b+1)^{-\frac{1}{2}})^{-1}).$$

Since $(b+1)^{-1} \leq 1$, we have $c^{\frac{1}{2}}(b+1)^{-1}c^{\frac{1}{2}} \leq c$ and therefore

$$1-(1+c^{\frac{1}{2}}(b+1)^{-1}c^{\frac{1}{2}})^{-1} \leq 1-(1+c)^{-1}.$$

Now with Properties (vi) (applied to $(b+1)^{-\frac{1}{2}}$ and c) and (ii) of Proposition 1.1.2, we obtain

$$\mu_s(1-(1+(b+1)^{-\frac{1}{2}}c(b+1)^{-\frac{1}{2}})^{-1}) = \mu_s(1-(1+c^{\frac{1}{2}}(b+1)^{-1}c^{\frac{1}{2}})^{-1}) \leq \mu_s(1-(c+1)^{-1}).$$

This proves inequality 1.3 and hence inequality 1.2.

Let $y \in L_{q'}(\mathcal{M})$; using the inequality 1.2 and inequality (v) in 1.1.2, we have

$$\begin{aligned}
\tau(((b+c)^{\frac{p}{q}} - b^{\frac{p}{q}})y) &= \int_0^\infty \tau(a(t)y)d\nu(t) \\
&\leq \int_0^\infty \int_0^\infty \mu_s(a(t))\mu_s(y)d\nu(t)ds \\
&\leq \int_0^\infty \int_0^\infty \mu_s(tc(c+t1)^{-1})\mu_s(y)d\nu(t)ds \\
&= \int_0^\infty \mu_s(c^{\frac{p}{q}})\mu_s(y)ds \\
&\leq ||c^{\frac{p}{q}}||_q||y||_{q'}
\end{aligned}$$

where we used Hölder inequality (iv) in Proposition 1.1.2 in the last inequality. The inequality (1.1) follows by taking the supremum over y in the unit sphere of $L_{q'}(\mathcal{M})$.

Next, we consider the general case of arbitrary $a, b \in L_0(\mathcal{M}, \tau)_+$. For this, we proceed as in the proof of Theorem 1 in [1].

Let $a, b \geq 0$. Since $a + (b-a)^+ \geq b$, we have

$$b^{\frac{p}{q}} - a^{\frac{p}{q}} \leq (a + (b-a)^+)^{\frac{p}{q}} - a^{\frac{p}{q}}.$$

The operator $(a+(b-a)^+)^{\frac{p}{q}} - a^{\frac{p}{q}}$ is positive since $x \mapsto x^{\frac{p}{q}}$ is operator monotone. Hence

$$((a + (b-a)^+)^{\frac{p}{q}} - a^{\frac{p}{q}})^+ = (a + (b-a)^+)^{\frac{p}{q}} - a^{\frac{p}{q}}.$$

Therefore, by (i) in Lemma 1.1.3, we obtain that

$$||(b^{\frac{p}{q}} - a^{\frac{p}{q}})^+||_q \leq ||(a + (b-a)^+)^{\frac{p}{q}} - a^{\frac{p}{q}}||_q.$$

Applying the first case to a and $a + (b-a)^+$, it follows that

$$||(a + (b-a)^+)^{\frac{p}{q}} - a^{\frac{p}{q}}||_q \leq ||(b-a)^{+\frac{p}{q}}||_q$$

and hence

$$||(b^{\frac{p}{q}} - a^{\frac{p}{q}})^+||_q \leq ||(b-a)^{+\frac{p}{q}}||_q.$$

Exchanging the role of a and b, we also get

$$||(a^{\frac{p}{q}} - b^{\frac{p}{q}})^+||_q \leq ||(a-b)^{+\frac{p}{q}}||_q.$$

Using part (ii) of Lemma 1.1.3 and the inequalities above, we have

$$\begin{aligned} ||a^{\frac{p}{q}} - b^{\frac{p}{q}}||_q^q &= ||(a^{\frac{p}{q}} - b^{\frac{p}{q}})^+||_q^q + ||(b^{\frac{p}{q}} - a^{\frac{p}{q}})^+||_q^q \\ &\leq ||(a-b)^{+\frac{p}{q}}||_q^q + ||(b-a)^{+\frac{p}{q}}||_q^q \\ &= ||(a-b)^{+\frac{p}{q}} + (b-a)^{+\frac{p}{q}}||_q^q \\ &= |||a-b|^{\frac{p}{q}}||_q^q \end{aligned}$$

since $(a-b)^{+\frac{p}{q}}(b-a)^{+\frac{p}{q}} = 0$ and $(a-b)^{+\frac{p}{q}} + (b-a)^{+\frac{p}{q}} = |a-b|^{\frac{p}{q}}$. The result follows. □

1.2 General Haagerup L_p-spaces

The previous construction of non-commutative L_p-spaces associated with semi-finite von Neumann algebras does not apply to von Neumann algebras of type III, which do not admit a normal faithful semi-finite trace. However, it is known that any von Neumann algebra admits a faithful semi-finite weight (an additive homogeneous functional on the positive cone with values in $[0, +\infty]$). In [35], Haagerup gave a construction of L_p-spaces using a crossed product to reduce von Neumann of type III to semi-finite von Neumann algebras with a trace. Another construction, using complex interpolation, is due to Kosaki (see[49] and [79]). We recall here the construction given by Haagerup.

Let $\mathcal{M}$ be a von Neumann algebra, acting on a Hilbert space $\mathcal{H}$, and equipped with a normal faithful semi-finite weight φ_0. Let $t \mapsto \sigma_t^{\varphi_0}$ be the one-parameter group of modular automorphisms of $\mathcal{M}$ associated with φ_0 (see chapter VIII and theorem 1.2 in [78] for more details). We denote by

$$\mathcal{N}_{\varphi_0} = \mathcal{M} \rtimes_{\varphi_0} \mathbb{R}$$

the crossed product von Neumann algebra, which is a von Neumann algebra acting on $L^2(\mathbb{R}, \mathcal{H})$, and generated by the operators $\pi_{\varphi_0}(x)$ and λ_s, defined for $x \in \mathcal{M}$ and $s \in \mathbb{R}$ by

$$\begin{aligned} &\pi_{\varphi_0}(x)(\xi)(t) = \sigma_{-t}^{\varphi_0}(x)\xi(t) \\ &\lambda_s(\xi)(t) = \xi(t-s) \qquad \text{for any } \xi \in L^2(\mathbb{R}, \mathcal{H}) \text{ and } t \in \mathbb{R}. \end{aligned}$$

Denote by $s \mapsto \theta_s$ the dual action of $\mathbb{R}$ on $\mathcal{N}_{\varphi_0}$, which is defined by

$$\theta_s(\pi_{\varphi_0}(x)) = \pi_{\varphi_0}(x),\ \theta_s(\lambda_t) = e^{-its}\lambda_t \text{ for all } x \in \mathcal{M} \text{ and } t, s \in \mathbb{R}.$$

By Lemma 5.2 in [37], there exists a semi-finite normal trace τ_{φ_0} on $\mathcal{N}_{\varphi_0}$ satisfying

$$\tau_{\varphi_0} \circ \theta_s = \mathrm{e}^{-s}\tau_{\varphi_0} \text{for all } s \in \mathbb{R}.$$

We denote by $L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$ the *-algebra of τ_{φ_0}-measurable operators affiliated with $\mathcal{N}_{\varphi_0}$. For $1 \leq p \leq \infty$, the Haagerup non-commutative L_p-space associated with $\mathcal{M}$ is defined by

$$L_p(\mathcal{M}) = \{\ x \in L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0}) \mid \theta_s(x) = \mathrm{e}^{-s/p}x \text{ for all } s \in \mathbb{R}\}.$$

It is known that this space is independent of a weight φ_0 up to isomorphism.

The space $L_1(\mathcal{M})$ is isomorphic to $\mathcal{M}_*$ (see Chapter 2 in [79] for more details). The identification is as follows : there exists a normal faithful semi-finite operator valued weight from $\mathcal{N}_{\varphi_0}$ to $\mathcal{M}$ defined by

$$\Phi_{\varphi_0}(x) = \pi_{\varphi_0}^{-1}(\int_{\mathbb{R}} \theta_s(x)\mathrm{d}s)\ ,\ \text{for } x \in \mathcal{N}_{\varphi_0}.$$

Now, if $\varphi \in \mathcal{M}_*^+$, and $\hat{\varphi}$ denotes the extension of φ to a normal weight on $\hat{\mathcal{M}}^+$, the extended positive part of $\mathcal{M}$ (see Definition 1.1 in [36]), we then put

$$\tilde{\varphi}^{\varphi_0} = \hat{\varphi} \circ \Phi_{\varphi_0}.$$

We associate to φ the Radon-Nikodym derivative $h_\varphi = \frac{d\tilde{\varphi}^{\varphi_0}}{d\tau_{\varphi_0}}$ of $\tilde{\varphi}^{\varphi_0}$ with respect to the trace τ_{φ_0}, which is the unique operator in $L_1(\mathcal{M})^+$ satisfying

$$\tilde{\varphi}^{\varphi_0}(y) = \tau_{\varphi_0}(h_\varphi y) \text{ for all } y \in \mathcal{N}_{\varphi_0}.$$

The map $\varphi \mapsto h_{\varphi_0} = \frac{d\tilde{\varphi}^{\varphi_0}}{d\tau_{\varphi_0}}$ gives an isomorphism between $\mathcal{M}_*^+$ and $L_1(\mathcal{M})^+$, which extends to the whole spaces by linearity.

If $x \in L_1(\mathcal{M})$, and φ_x is the element of $\mathcal{M}_*^+$ associated to x, we define a linear functional Tr by

$$\mathrm{Tr}(x) = \varphi_x(1)$$

and we have, p' being the conjugate exponent of p,

$$\mathrm{Tr}(xy) = \mathrm{Tr}(yx) \text{ for } x \in L_p(\mathcal{M}),\ y \in L_{p'}(\mathcal{M})$$

For $1 \leq p < \infty$, if $x = u|x|$ is the polar decomposition of $x \in L_p(\mathcal{M})$, we define

$$||x||_p = \mathrm{Tr}(|x|^p)^{1/p}.$$

Equipped with $||.||_p$, $L_p(\mathcal{M})$ is a Banach space. The dual space of $L_p(\mathcal{M})$ is isometrically isomorphic to $L_{p'}(\mathcal{M})$. For $1 < p < \infty$, the space $L_p(\mathcal{M})$ is ucus.

If $\mathcal{M}$ is a von Neumann algebra with a semi-finite trace τ, $L_p(\mathcal{M}, \tau)$ is isometrically isomorphic to the Haagerup L_p-space constructed with the weight τ.

1.3 The Mazur map

Let $\mathcal{M}$ be a von Neumann algebra, and φ_0 a normal faithful semi-finite weight on $\mathcal{M}$. Let $(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$ be the crossed-product von Neumann algebra associated to φ_0, and equipped with the corresponding trace τ_{φ_0}, as described in the previous section 1.2.

A useful tool which relates isometries of $L_p(\mathcal{M})$ to isometries of $L_q(\mathcal{M})$ is the Mazur map.

Definition 1.3.1. Let $1 \leq p, q < \infty$. For an operator x, let $\alpha|x|$ be its polar decomposition. The map

$$\begin{aligned} M_{p,q} :& L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0}) \to L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0}) \\ & x = \alpha|x| \mapsto \alpha|x|^{\frac{p}{q}} \end{aligned}$$

is called the *Mazur map*.

We now give a few useful properties of the Mazur map.

Lemma 1.3.2. *Let $1 \leq p, q, r < \infty$. Then $M_{r,q} \circ M_{p,r} = M_{p,q}$.*

Proof. Let $\alpha|x|$ be the polar decomposition of $x \in L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$. Let $\beta > 0$, and set $y = \alpha|x|^\beta$. We claim that the polar decomposition of y is given by α and $|x|^\beta$. To show this, it suffices to prove that $\overline{\mathrm{Im}(|x|^\beta)} = \overline{\mathrm{Im}(|x|)}$.
By taking orthogonals, we have to show that $\mathrm{Ker}(|x|) = \mathrm{Ker}(|x|^\beta)$ for all $\beta > 0$. Let $\beta > 0$. Recall that the domain $D(|x|^\beta)$ of $|x|^\beta$ is

$$D(|x|^\beta) = \{\xi \mid \int_0^\infty \lambda^{2\beta} d\mu_\xi(\lambda) < \infty\}$$

where μ_ξ denotes the spectral measure associated to ξ.
If $\xi \in \mathrm{Ker}(|x|)$, we have

$$< |x|\xi, \xi >= \int_0^\infty \lambda d\mu_\xi(\lambda) = 0.$$

In particular, $\mu_\xi(]0, \infty[) = 0$. So $\xi \in D(|x|^\beta)$ and $\xi \in \mathrm{Ker}(|x|^\beta)$ thanks to

$$||\ |x|^\beta \xi\ ||^2 =< |x|^\beta \xi, |x|^\beta \xi >= \int_0^\infty \lambda^{2\beta} d\mu_\xi(\lambda) = 0.$$

By exchanging the role of $|x|$ and $|x|^\beta$, we get the equality.
Let $1 \leq p, q, r < \infty$, and $\beta = p/r$; then $M_{p,r}(x) = \alpha|x|^\beta$. It follows from what we have just seen that $M_{r,q}(M_{p,r}(x)) = \alpha|x|^{\frac{p}{q}} = M_{p,q}(x)$. □

Proposition 1.3.3. *Let $1 \leq p, q < \infty$, and $x \in L_p(\mathcal{M})$. Then*

$$||M_{p,q}(x)||_q^q = ||x||_p^p.$$

Moreover, $M_{p,q}(L_p(\mathcal{M})) \subset L_q(\mathcal{M})$.

Proof. Let $x = \alpha|x|$ be the polar decomposition of $x \in L_p(\mathcal{M})$ and $s \in \mathbb{R}$. We have already seen that $|M_{p,q}(a)| = |a|^{\frac{p}{q}}$. So we have

$$\mathrm{Tr}(|M_{p,q}(a)|^q) = \mathrm{Tr}(|a|^p).$$

By uniqueness in the polar decomposition, we have $\theta_s(\alpha) = \alpha$ and $\theta_s(|x|) = \mathrm{e}^{-s/p}|x|$, and then

$$\begin{aligned} \theta_s(M_{p,q}(x)) &= \theta_s(\alpha)\theta_s(|x|^{\frac{p}{q}}) \\ &= \alpha(\theta_s(|x|)^{\frac{p}{q}}) \\ &= \mathrm{e}^{-s/q} M_{p,q}(x). \end{aligned}$$

□

In the case of L_p-spaces, an explicit formula gives the duality map, by means of the Mazur map.

Proposition 1.3.4. *Let $p \in]1,\infty[$ and p' its conjugate exponent. The map*

$$\begin{aligned} S(L_p(\mathcal{M})) &\to S(L_{p'}(\mathcal{M})) \\ x &\mapsto M_{p,p'}(x)^* \end{aligned}$$

is the duality map from $S(L_p(\mathcal{M}))$ to $S(L_{p'}(\mathcal{M}))$.

Proof. Thanks to the defining property of the duality map, we just have to check that $\mathrm{Tr}(M_{p,p'}(x)^*x) = 1$ for x in the unit sphere $S(L_p(\mathcal{M}))$ of $L_p(\mathcal{M})$.
Let $x = \alpha|x| \in S(L_p(\mathcal{M}))$; then $M_{p,p'}(x) = \alpha|x|^{\frac{p}{p'}}$. Since $\alpha^*\alpha|x| = |x|$, it follows that

$$\mathrm{Tr}(|x|^{\frac{p}{p'}}\alpha^*\alpha|x|) = \mathrm{Tr}(|x|^{\frac{p}{p'}}|x|) = \mathrm{Tr}(|x|^p) = 1.$$

□

Proposition 1.3.5. *Let $a, b \in L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$ and let e, f be two central projections in $\mathcal{N}_{\varphi_0}$ such that $ef = 0$. Then $M_{p,q}(ae + bf) = M_{p,q}(ae) + M_{p,q}(bf)$.*

Proof. As is easily checked, we have

$$|ae + bf| = |a|e + |b|f.$$

Let γ be the partial isometry occuring in the polar decomposition of $ae+bf$, and let $a = \alpha|a|$, $b = \beta|b|$ be the polar decompositions of a and b. We claim that $\gamma = \alpha e + \beta f$. Indeed, we have

$$\begin{aligned} &ae + bf = \gamma|ae + bf| \text{ and} \\ &ae + bf = (\alpha e)(|a|e) + (\beta f)(|b|f) = (\alpha e + \beta f)|ae + bf|. \end{aligned}$$

Since αe is zero on $\mathrm{Ker}(|a|e)$ and βf is zero on $\mathrm{Ker}(|b|f)$, $\alpha e + \beta f$ is zero on $\mathrm{Im}(|ae+bf|)^\perp = \mathrm{Ker}(|ae+by|) = \mathrm{Ker}(|a|e) \cap \mathrm{Ker}(|b|f)$, since $ef = 0$. This shows that

$$ae + bf = (\alpha e + \beta f)|ae + bf|$$

is the polar decomposition of $ae + bf$.
Using again the fact that $ef = 0$ and that e, f are central elements, we deduce that

$$\begin{aligned} M_{p,q}(ae + bf) &= (\alpha e + \beta f)|ae + bf|^{\frac{p}{q}} \\ &= (\alpha e + \beta f)(e|a|^{\frac{p}{q}} + f|b|^{\frac{p}{q}}) \\ &= M_{p,q}(ae) + M_{p,q}(bf). \end{aligned}$$

□

Recall that a Jordan-homomorphism from a C^*-algebra $\mathcal{A}$ to a C^*-algebra $\mathcal{B}$ is a $*$-preserving linear map $J : \mathcal{A} \to \mathcal{B}$ such that $J(a^2) = (J(a))^2$ for every $a \in \mathcal{A}$. The structure of a Jordan isomorphism between von Neumann algebra is given in the following theorem (see Theorem 10 in [46] and Lemma 3.2 in [75].

Theorem 1.3.6. *([46]) Let $\mathcal{M}$, $\mathcal{N}$ von Neumann algebras, and $J : \mathcal{M} \to \mathcal{N}$ be a Jordan isomorphism from $\mathcal{M}$ to $\mathcal{N}$. Then we have a decomposition $J = J_1 + J_2$ with the following properties : J_1 is a *-homomorphism of algebra on $\mathcal{M}$, J_2 is a *-anti-homomorphism of algebra, and $J_1(x) = J(x)e$, $J_2(x) = J(x)f$ for all $x \in \mathcal{M}$, with e, f two orthogonal and central projections in $\mathcal{N}$ such that $e+f = I$.*

The previous decomposition and elementary properties of the Mazur map allow us to show that a Jordan-homomorphism on a von Neumann algebra remains unchanged after conjugation by the Mazur map.

Proposition 1.3.7. *Let J be a Jordan-isomorphism of $\mathcal{N}_{\varphi_0}$, and let $1 \leq p, q < \infty$. Then we have*

$$J(x) = M_{p,q} \circ J \circ M_{q,p}(x) \text{ for all } x \in \mathcal{N}_{\varphi_0}.$$

Proof. Take the decomposition $J = J_1 + J_2$ as in the previous Theorem 1.3.6. Observe first that, for $a \in \mathcal{N}_{\varphi_0}$ with $a \geq 0$ and a positive real number r, we have

$$J_1(a^r) = J_1(a)^r$$

and the same is true for J_2.
If α is a partial isometry, then $J_1(\alpha)$ and $J_2(\alpha)$ are partial isometries with initial supports $J_1(\alpha^*\alpha)$ and $J_2(\alpha\alpha^*)$, and final supports $J_1(\alpha\alpha^*))$ and $J_2(\alpha^*\alpha))$ respectively.
Let $x = \alpha|x| \in \mathcal{N}_{\varphi_0}$. Since the supports of J_1 and J_2 are orthogonal, it follows from Proposition 1.3.5 that

$$\begin{aligned} M_{p,q} \circ J \circ M_{q,p}(x) &= M_{p,q}(J_1(M_{q,p}(x)) + J_2(M_{q,p}(x))) \\ &= M_{p,q}(J_1(M_{q,p}(x))) + M_{p,q}(J_2(M_{q,p}(x))). \end{aligned}$$

Moreover, we have

$$\begin{aligned} M_{p,q}(J_1(M_{q,p}(x))) &= M_{p,q}(J_1(\alpha|x|^{\frac{q}{p}})) \\ &= M_{p,q}(J_1(\alpha)J_1(|x|)^{\frac{q}{p}}) \\ &= J_1(x) \end{aligned}$$

and

$$\begin{aligned} M_{p,q}(J_2(M_{q,p}(x))) &= M_{p,q}(J_2(\alpha|x|^{\frac{q}{p}})) \\ &= M_{p,q}(J_2(\alpha|x|^{\frac{q}{p}}\alpha^*\alpha)) \\ &= M_{p,q}(J_2(\alpha)J_2(\alpha|x|^{\frac{q}{p}}\alpha^*)) \\ &= M_{p,q}(J_2(\alpha)J_2((\alpha|x|\alpha^*)^{\frac{q}{p}})) \\ &= M_{p,q}(J_2(\alpha)J_2(\alpha|x|\alpha^*)^{\frac{q}{p}}) \\ &= J_2(x). \end{aligned}$$

$\square$

The following result about the local uniform continuity of $M_{p,q}$ is proved for general Haagerup L_p-spaces in Lemma 3.2 of [69] (for an independent proof in the case $L_p(\mathcal{M},\tau)=S_p$, see [68]). We give a proof in the semi-finite case, using Ando's inequality .

Theorem 1.3.8. [69] *For $1 \leq p,q < \infty$, the Mazur map $M_{p,q}$ is uniformly continuous on the unit sphere $S(L_p(\mathcal{M}))$.*

Until the end of the section, $\mathcal{M}$ will always denote a semi-finite von Neumann algebra. We will need the following elementary lemma.

Lemma 1.3.9. *For $r \geq 2$, and $a,b \in L_r(\mathcal{M})$, the following inequality holds :*

$$|||a|^2-|b|^2||_{\frac{r}{2}} \leq (||a||_r+||b||_r)||a-b||_r$$

Proof. Notice that

$$a^*a-b^*b=a^*(a-b)+(a^*-b^*)b$$

and use successively the triangle inequality and Hölder 's inequality. □

Proof of Theorem 1.3.8 in the semi-finite case. Let us first establish the uniform continuity on the subset of positive elements of $S(L_p(\mathcal{M}))$. Denote by $p^{'},q^{'}$ the conjugate exponents of p and q. To simplify notation, let denote until the end of the proof by $M_{p,q}$ the restriction of the Mazur map to the unit sphere.

- *First case* : $p<q$.

In this case, the uniform continuity of $M_{p,q}$ is a consequence of the generalization of Ando's inequality (Proposition 1.1.4).

- *Second case* : $p>q$ and $q \geq 2$.

Since p and p' are conjugate, $M_{p,p'}$ is uniformly continuous by Proposition 1.3.4. By the first case, $M_{p',q}$ is uniformly continuous ($p'<q$). Since, by Lemma 1.3.2, we have

$$M_{p,q}=M_{p^{'},q}\circ M_{p,p^{'}},$$

it follows that $M_{p,q}$ is uniformly continuous.

- *Third case* : $p>q$ and $q<2$.

Assume that $p<q^{'}$. Then $M_{p',q}$ and $M_{p,p'}$ are uniformly continuous, as before. Hence $M_{p,q}=M_{p^{'},q}\circ M_{p,p^{'}}$ is also uniformly continuous. Assume now that $p>q^{'}$. Since $p^{'}<q^{'}$, $M_{p',q'}$ is uniformly continuous and so is $M_{p,q}=M_{q^{'},q}\circ M_{p^{'},q^{'}}\circ M_{p,p^{'}}$.

Let us prove now the uniform continuity on the whole sphere $S(L_p(\mathcal{M},\tau))$. Let us first consider the case $p>q$.
Assume also that $p>2$. We have

$$\begin{aligned} M_{p,q}(a)-M_{p,q}(b) &= a|a|^{\frac{p}{q}-1}-b|b|^{\frac{p}{q}-1} \\ &= (a-b)|a|^{\frac{p}{q}-1}+b(|a|^{\frac{p}{q}-1}-|b|^{\frac{p}{q}-1}) \end{aligned}$$

It follows from the triangle inequality and Hölder's inequality that

$$||M_{p,q}(a) - M_{p,q}(b)||_q \leq ||a-b||_p + |||a|^{\frac{p}{q}-1} - |b|^{\frac{p}{q}-1}||_{\frac{pq}{p-q}}.$$

The uniform continuity of $M_{p,q}$ follows from the uniform continuity of the map $a \mapsto |a|^{\frac{p}{q}-1}$ on the sphere (see Lemma 1.3.9) and from the uniform continuity of $M_{\frac{p}{2},\frac{pq}{p-q}}$ on positive elements (here, we require $p > 2$).
Now assume that $2 > p > q$. Then we have $p' > 2 > q$, and by the previous case $M_{p',q}$ is uniformly continuous. Hence $M_{p,q} = M_{p',q} \circ M_{p,p'}$ is also uniformly continuous.

The case $p < q$ is proved by similar arguments as the second and third cases above (by exchanging the roles of p and q). □

1.4 On classification and embeddings of non-commutative L_p-spaces

1.4.1 A result on classification of non-commutative L_p-spaces

Since we study groups actions by isometries on L_p-spaces, we are interested in classifying L_p-spaces up to isometric isomorphism. In the commutative case for $L_p(X,\mu)$, with (X,μ) a Borel standard space, the situation is simple : if the measure μ is atomic, then $L_p(X,\mu)$ is isometrically isomorphic to a discrete l_p ; if the measure is non-atomic, $L_p(X,\mu)$ is isometrically isomorphic to $L_p([0,1],\lambda)$, denoted once and for all L_p. Thus, a general $L_p(X,\mu)$ is isometrically isomorphic to a sum $l_p \oplus^p L_p$.

The situation in the non-commutative case is much more complicated, and despite new important results in the recent years, we don't have a complete classification of these spaces up to isomorphism. The classification up to isomorphism of the $L_p(\mathcal{M})$-spaces is not complete but a remarkable classification was obtained by Haagerup, Rosenthal, and Sukochev in [39] when $\mathcal{M}$ is hyperfinite (see also [77] for a survey on such classifications for semi-finite von Neumann algebras, and [76] for proofs in the type I case).

Theorem 1.4.1. *[39] Let $\mathcal{M}$ be a hyperfinite semi-finite Von Neumann algebra. Let $1 \leq p < \infty$, $p \neq 2$. Then $L_p(\mathcal{M})$ is isomorphic to precisely one of the following spaces :*

$$l_p,\ L_p,\ S_p,\ C_p,\ L_p \oplus S_p,\ L_p \oplus C_p,\ L_p(S_p),\ C_p \oplus L_p(S_p),$$

$$L_p(C_p),\ L_p(R),\ C_p \oplus L_p(R),\ L_p(C_p) \oplus L_p(R),\ L_p(R \otimes \mathcal{B}(l_2)).$$

For $1 \leq p < 2$, we have the following refinement of this theorem (see [67] for remarks on these theorems and accurate references).

Theorem 1.4.2. *Let $1 \leq p < 2$ and $\mathcal{M}$ as in Theorem 1.4.1. If $X \neq Y$ are spaces listed in the following figure, then Y contains an isomorphic copy of X if and only if X can be joined to Y by descending arrows :*

$$
\begin{tikzcd}
 & & l_p \arrow[dl] \arrow[dr] & & \\
 & S_p \arrow[dl] \arrow[dr] & & L_p \arrow[dl] & \\
C_p \arrow[dr] & & S_p \oplus_p L_p \arrow[dl] \arrow[dr] & & \\
 & C_p \oplus_p L_p \arrow[dr] & & L_p(S_p) \arrow[dl] \arrow[dr] & \\
 & & C_p \oplus_p L_p(S_p) \arrow[dl] \arrow[dr] & & L_p(R) \arrow[dl] \\
 & L_p(C_p) \arrow[dr] & & C_p \oplus_p L_p(R) \arrow[dl] & \\
 & & L_p(C_p) \oplus_p L_p(R) \arrow[d] & & \\
 & & L_p(R \otimes \mathcal{B}(l_2)) & &
\end{tikzcd}
$$

The following theorem, due to Sherman in [74], exactly gives the condition on the von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ in order to have isometrically isomorphic L_p-spaces for $p \neq 2$: the von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ have to be Jordan-*-isomorphic.

Theorem 1.4.3. ([74]) *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. Let $1 \leq p < \infty$. The following propositions are equivalent.*
(i) $\mathcal{M}$ and $\mathcal{N}$ are Jordan--isomorphic.*
(ii) $L_p(\mathcal{M})$ and $L_p(\mathcal{N})$ are isometrically isomorphic.

Remark 1.4.4. 1. Let $\mathcal{M}$ and $\mathcal{N}$ be factors. Then a Jordan-*-isomorphism $J : \mathcal{M} \to \mathcal{N}$ is a *-algebra morphism or a *-algebra antimorphism (see Theorem 1.3.6). Hence, in this case, $L_p(\mathcal{M})$ and $L_p(\mathcal{N})$ can be isometrically isomorphic only if $\mathcal{M}$ and $\mathcal{N}$ are isomorphic as *-algebras, or anti-isomorphic.

2. There exists esamples of von Neumann algebras which are Jordan-*-isomorphic, but not isomorphic as *-algebras. Indeed, in [16], Connes constructed a factor $\mathcal{M}$ of type III such that $\mathcal{M}$ is not anti-isomorphic to itself. Then, $\mathcal{M}$ and $\mathcal{N} = \mathcal{M}^{op}$ (the algebra $\mathcal{M}$ with the *opposite* algebra law $a.b := ba$) are not isomorphic as *-algebras, but they are Jordan-*-isomorphic.

1.4.2 Embeddings of L_p-spaces into Hilbert spaces

It is well-known that $L_p([0,1])$ equipped with the metric $(f,g) \mapsto ||f-g||_p^{p/2}$ embeds isometrically in $L_2([0,1])$ for $1 \leq p \leq 2$ (see [82]). We will see here that it is no longer true for non-commutative L_p-spaces.

Embeddings of metric spaces in Hilbert spaces are intimatly linked to the notion of kernels conditionally of negative type.

Definition 1.4.5. A kernel conditionally of negative type on a set X is a function $\rho : X \times X \to \mathbb{R}$ with the following properties :
(i) $\rho(x,x) = 0$ for all $x \in X$.
(ii) $\rho(x,y) = \rho(y,x)$ for all $x, y \in X$.
(iii) for any $n \in \mathbb{N}$, any $x_1, ..., x_n$ in X, and any real numbers $\lambda_1, ...\lambda_n$ such that $\sum_{k=1}^n \lambda_k = 0$, the following equality holds :

$$\sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j \rho(x_i, x_j) \leq 0.$$

If $f : X \to \mathcal{H}$ is a mapping with values in a Hilbert space, then $(x,y) \mapsto ||f(x) - f(y)||^2$ defines a kernel conditionally of negative type on X. Conversely, if ρ is conditionally of negative type on X, then there exists a Hilbert space $\mathcal{H}$ and an embedding $f : X \to \mathcal{H}$ such that $\rho(x,y) = ||f(x) - f(y)||^2$ (see [8]). The isometric embedding of a commutative L_p-space into L_2 for $1 \leq p \leq 2$, mentioned above, can be rephrased as follows.

Theorem 1.4.6. *Let $1 \leq p \leq 2$, let (X,μ) be a measure space and $L_p(X,\mu)$ the associated commutative L_p-space. The kernel $(x,y) \mapsto ||x-y||_p^p$ is conditionally of negative type on $L_p(X,\mu)$.*

The previous theorem does not longer hold for non-commutative L_p-spaces. The following fact is well-known (see [57] where similar computations occur).

Theorem 1.4.7. *Let $p \neq 2$, and let $\mathcal{M}$ be a von Neumann algebra such that $L_p(\mathcal{M})$ contains an isometric copy of $(\mathcal{M}_2(\mathbb{R}), ||.||_p)$. Then the kernel*

$$(x,y) \mapsto ||x-y||_p^p$$

is not conditionally of negative type on $L_p(\mathcal{M})$. Therefore there is no embedding j of $L_p(\mathcal{M})$ into a separable Hilbert space $\mathcal{H}$ such that

$$||j(x) - j(y)||^2 = ||x - y||_p^p \text{ for all } x, y \in L_p(\mathcal{M}).$$

Proof. We can assume that $\mathcal{H} = l_2$. Suppose that such an embedding $j : L_p(\mathcal{M}) \to l_2$ exists. Then we consider $\mathcal{M}_2(\mathbb{R})$ as a 4-dimensional subspace of $L_p(\mathcal{M})$. We can suppose that $j(0) = 0$. Then, by the Mazur-Ulam Theorem (see Theorem 2 in chapter XI of [6]), j is $\mathbb{R}$-linear on $\mathcal{M}_2(\mathbb{R})$. The subspace $\mathcal{M}_2(\mathbb{R})$ is embedded as a subspace of l_2 generated by four sequences $(a_n)_n, (b_n)_n, (c_n)_n, (d_n)_n$ as follows :

$$(a_n)_n = j\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\ (b_n)_n = j\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\ (c_n)_n = j\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},\ (d_n)_n = j\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},$$

$$||\begin{pmatrix}a & b\\ c & d\end{pmatrix}||_p^p = ||a(a_n) + b(b_n) + c(c_n) + d(d_n)||^2 \text{ for all } a, b, c, d \in \mathbb{R}. \ (*)$$

By simple computations, we have $||(a_n)_n|| = ||(b_n)_n|| = ||(c_n)_n|| = ||(d_n)_n|| = 1$.

On the one hand, we have

$$||\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}||_p^p = \operatorname{Tr}\begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}^{\frac{p}{2}} = 2^{\frac{p}{2}}$$

and

$$||\begin{pmatrix}1 & 0\\ -1 & 0\end{pmatrix}||_p^p = \operatorname{Tr}\begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}^{\frac{p}{2}} = 2^{\frac{p}{2}}.$$

But

$$||(a_n)_n + (c_n)_n||^2 + ||(a_n)_n - (c_n)_n||^2 = 2\ (||(a_n)_n||^2 + ||(c_n)_n||^2) = 4.$$

Since $j\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} = (a_n)_n + (c_n)_n$ and $j\begin{pmatrix}1 & 0\\ -1 & 0\end{pmatrix} = (a_n)_n - (c_n)_n$, it follows from $(*)$ that $4 = 2.2^{\frac{p}{2}}$. This is a contradiction since $p \neq 2$. □

1.5 Isometries on non-commutative L_p-spaces

Let $O(L_p(\mathcal{M}))$ be the group of linear bijective isometries of $L_p(\mathcal{M})$. We need to know the structure of $O(L_p(\mathcal{M}))$. We will state in this section some general results on $O(L_p(\mathcal{M}))$, and will give a more precise description of this group in a few special cases.

Let us start with the description of the group $O(l_p)$ for the usual space l_p of p-summable sequences. The following result appears in [6] (see chapter XI).

Theorem 1.5.1. *Let $1 \leq p < +\infty$ and $p \neq 2$. Let $U \in O(l_p)$. There exist a sequence $(c(n))_{n\in\mathbb{N}}$ with values in $\mathbb{S}^1$ and a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that*

$$U(f(n)) = c(n)f(\sigma(n)) \text{ for all } f \in l_p(\mathbb{N}).$$

More generally, one has the following description of $O(L_p(X,\mu))$ due to Banach and Lamperti.

Theorem 1.5.2. ([7] and [53]) *Let $1 < p < \infty$ and $p \neq 2$. Let $U \in L_p(X,\mu)$. There exist a measurable, measure-class preserving map T of (X,μ), and a measurable function h with $|h(x)| = 1$ almost everywhere, such that*

$$(U(f))(x) = h(x)\Big(\frac{dT_*\mu}{d\mu}(x)\Big)^{\frac{1}{p}} f(T(x)) \text{ for all } f \in L_p(X,\mu).$$

On the non-commutative side, the first result is due to Arazy, who describes $O(C_p)$ for the Schatten ideals C_p.

Theorem 1.5.3. [3] *Let $1 \leq p < +\infty$ and $p \neq 2$, and let $U \in O(C_p)$. Then there exist two unitaries u and v in $\mathcal{B}(\mathcal{H})$ such that :*

$$U(x) = uxv \text{ or } U(x) = u^t xv \text{ for all } x \in C_p.$$

The next description was given by Yeadon, who described $O(L_p(\mathcal{M}))$ for $L_p(\mathcal{M})$ the non-commutative L_p-space associated to a semi-finite von Neumann algebra $\mathcal{M}$. Here is the result.

Theorem 1.5.4. [84] *Let $1 \leq p < \infty$ and $p \neq 2$. Let $\mathcal{M}$ be a Von Neumann algebra equipped with a semi-finite trace τ. A linear map*

$$U : L_p(\mathcal{M},\tau) \to L_p(\mathcal{M},\tau)$$

is a surjective isometry if and only if there exist

1. *a normal Jordan *-isomorphism $J : \mathcal{M} \to \mathcal{M}$,*
2. *a unitary $u \in \mathcal{M}$,*
3. *a positive self-adjoint operator B affiliated with $\mathcal{M}$ such that the spectral projections of B commute with $\mathcal{M}$, the support of B is $s(B) = 1$, and $\tau(x) = \tau(B^p J(x))$ for all $x \in \mathcal{M}^+$,*

satisfying

$$U(x) = uBJ(x) \text{ for all } x \in \mathcal{M} \cap L_p(\mathcal{M}).$$

Moreover, such a decomposition is unique.

Remark 1.5.5. Notice that if $\mathcal{M}$ is a factor, the operator B in the previous decomposition is a scalar multiple of 1. Indeed, since B is affiliated to $\mathcal{M}$, its spectral projections are elements in $\mathcal{M}$. Moreover, they commute with $\mathcal{M}$ since B commutes with $\mathcal{M}$. Therefore every spectral projection of B is a scalar multiple of 1, and hence B itself is a scalar multiple of 1.

Explicit description of $O(L_p(\mathcal{M}))$ in a few special cases

We now give an explicit description of the isometries of S_p. Recall that $S_p = \{\oplus_{n\in\mathbb{N}^*} x_n \mid x_n \in \mathcal{M}_n, \sum_n \mathrm{Tr}(|x_n|^p) < \infty\}$.

Proposition 1.5.6. *Let $U \in O(S_p)$. There exist bijective isometries U_n of $\mathcal{M}_n$ such that $U = \oplus_n U_n$. More precisely, there exist sequences (u_n), (v_n) of unitaries in $\mathcal{M}_n$ such that*

$$U_n(x) = u_n x v_n \text{ or } U_n(x) = u_n({}^t x)v_n \text{ for all } x \in S_p.$$

We will need the two following lemmas for the proof of the previous proposition.

Lemma 1.5.7. *The two-sided ideals of $(\oplus_n \mathcal{M}_n)_\infty$ are the subspaces $\oplus_{i\in I}\mathcal{M}_i$ for $I \subset \mathbb{N}^*$.*

Proof. For every subset $I \subset \mathbb{N}^*$, it is clear that $\oplus_{i\in I}\mathcal{M}_i$ is a two-sided ideal of $\mathcal{M}$.

Conversely, let $\mathcal{A}$ is a two-sided ideal of $\mathcal{M}$. Let $I \subset \mathbb{N}^*$ be a minimal subset of $\mathbb{N}^*$ such that $\mathcal{A} \subset \oplus_{i\in I}\mathcal{M}_i$. Let $i \in I$. Then $\mathcal{M}_i \cap \mathcal{A}$ is a two-sided ideal of $\mathcal{M}_i$. By minimality of I, $\mathcal{M}_i \cap \mathcal{A}$ is non-zero. So $\mathcal{M}_i \cap \mathcal{A} = \mathcal{M}_i$. Thus $\oplus_i \mathcal{M}_i \subset \mathcal{A}$. ☐

Lemma 1.5.8. *If $\mathcal{N}$ is a von Neumann algebra, J a Jordan isomorphism of $\mathcal{N}$, and $\mathcal{A}$ an ideal of $\mathcal{N}$, then $J(\mathcal{A})$ is an ideal in $J(\mathcal{N})$.*

Proof. Recall from Theorem 1.3.6 that $J = J^1 + J^2$ with J^1 an algebra isomorphism and J^2 an algebra anti-isomorphism. More precisely, there exist two central projections $P_1, P_2 \in \mathcal{N}$ such that $J^1(x) = J(P_1 x)$ and $J^2(x) = J(P_2 x)$ for all $x \in \mathcal{N}$. We also have $P_1 P_2 = 0$ which implies that $J^1(x)J^2(y) = 0$ for all $x, y \in \mathcal{N}$. Let $a \in \mathcal{A}$, and $b, c \in \mathcal{N}$. Then $J(a) \in J(\mathcal{A})$, $J(b), J(c) \in J(\mathcal{N})$, and we have

$$\begin{aligned} J(b)J(a)J(c) &= J(b)J^1(a)J(c) + J(b)J^2(a)J(c) \\ &= J^1(b)J^1(a)J^1(c) + J^2(b)J^2(a)J^2(c) \\ &= J^1(bac) + J^2(cab) \\ &= J(P_1 bac P_1 + P_2 cab P_2) \in J(\mathcal{A}). \end{aligned}$$

☐

Proof of Proposition 1.5.6. Set $\mathcal{M} = (\oplus_n \mathcal{M}_n)_\infty = \{\oplus_n x_n \mid x_n \in \mathcal{M}_n,\ \sup_n ||x_n|| < \infty\}$. Let $U \in O(S_p)$. By Theorem 1.5.4, we know that U is given by the formula $U(x) = uBJ(x)$ for all $x \in \mathcal{M} \cap L_p(\mathcal{M})$, with u a unitary in $\mathcal{M}$ and J a Jordan isomorphism of $\mathcal{M}$.

Write $u = \oplus_n u_n$ with $u_n \in \mathcal{M}_n$ for all n. Let $v = u^{-1} = u^*$. Then $v \in \mathcal{M}$ and $v = \oplus_n v_n$. It follows that $v_n = u_n^*$, and that $u_n v_n = 1$ for all n. Hence all the u_n's are unitaries.

Since B belongs to the commutant $\mathcal{M}'$ of $\mathcal{M}$ and since the support of B is $s(B) = 1$, there exist non-zero scalars λ_n such that $B = \sum_n \lambda_n 1$.

Let $n \geq 1$. By Lemma 1.5.8, $J(\mathcal{M}_n)$ is a two-sided ideal of $\mathcal{M}$. Hence

$$J(\mathcal{M}_n) = \oplus_{i \in I_n} \mathcal{M}_i$$

for $I_n \subset \mathbb{N}^*$ by Lemma 1.5.7. For every $i \in I_n$, $J^{-1}(\mathcal{M}_i)$ is an ideal of $\mathcal{M}_n$ (since $\mathcal{M}_i \neq 0$ and J is bijective) and therefore $J^{-1}(\mathcal{M}_i) = \mathcal{M}_n$. For dimension reasons, it follows that $i = n$, that is $I_n = \{n\}$ and $J(\mathcal{M}_n) = \mathcal{M}_n$.

In summary, we have

$$U(x) = \oplus_n \lambda_n u_n J(x_n) \text{ for all } (x_n) \in \mathcal{M}.$$

Therefore $U = \oplus_n U_n$ for a sequence $(U_n)_n$ with $U_n \in O(\mathcal{M}_n)$. Since isometries of $\mathcal{M}_n$ are of the form given in Proposition 1.5.6, this completes the proof. ☐

Now we give a description of the group $O(L_p \oplus^p S_p)$.

Proposition 1.5.9. *We have the following decomposition* $O(L_p \oplus^p S_p) = O(L_p) \oplus O(S_p)$.

Proof. By Theorem 1.5.4, it suffices to prove such a decomposition on a Jordan isomorphism J of the von Neumann algebra $\mathcal{N} = L^\infty \oplus (\oplus_n \mathcal{M}_n)_\infty$.

Recall that a projection P in a von Neumann algebra $\mathcal{N}$ is said to be minimal if there is no projection Q in $\mathcal{N}$ such that $0 < Q < P$. Since a Jordan morphism preserves the projections and the order on the set of projections, J preserves the minimal projections.

Clearly, the minimal projections of $L^\infty \oplus (\oplus_n \mathcal{M}_n)_\infty$ are the rank one projections in $(\oplus_n \mathcal{M}_n)_\infty$ and they generate the algebra $(\oplus_n \mathcal{M}_n)_\infty$. Then we have

$$J((\oplus_n \mathcal{M}_n)_\infty) \subset (\oplus_n \mathcal{M}_n)_\infty$$

and the same argument for J^{-1} gives the equality $J((\oplus_n \mathcal{M}_n)_\infty) = (\oplus_n \mathcal{M}_n)_\infty$. Since J is an isomorphism of $\mathcal{N}$, we have also $J(L^\infty) = L^\infty$. ☐

Description of $O(L_p(\mathcal{M}))$ for a von Neumann algebra $\mathcal{M}$ of type III

We are now interested in the general case, that is the case of general Haagerup L_p-spaces. The following result, which we proved in [65], is a consequence of the description given in [74] and [81] of the action of isometries of Haagerup L_p-spaces on positive elements.

Theorem 1.5.10. *Let $1 \leq p < \infty$ and $p \neq 2$. Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal faithful semi-finite weight φ_0, and U a linear bijective isometry of the Haagerup L_p-space $L_p(\mathcal{M})$. Then there exist a unitary $w \in \mathcal{M}$ and a Jordan-isomorphism J of $L_0(\mathcal{M} \rtimes_{\varphi_0} \mathbb{R})$ such that U extends to the whole $L_0(\mathcal{M} \rtimes_{\varphi_0} \mathbb{R})$ with the form*

$$U(x) = wJ(x) \text{ for all } x \in L_0(\mathcal{M} \rtimes_{\varphi_0} \mathbb{R}).$$

Proof. We recall that for $\varphi \in \mathcal{M}_*^+$, h_φ is the unique operator in $L_1(\mathcal{M})^+$ satisfying

$$\tilde{\varphi}^{\varphi_0}(y) = \tau_{\varphi_0}(h_\varphi y) \text{ for all } y \in \mathcal{N}_{\varphi_0} = \mathcal{M} \rtimes_{\varphi_0} \mathbb{R}.$$

By Theorem 1.2 in [74], there exist a Jordan-isomorphism J of $\mathcal{M}$ and a unitary $w \in \mathcal{M}$ such that

$$U(h_\varphi^{1/p}) = w(h_{\varphi \circ J^{-1}})^{1/p} \text{ for all } \varphi \in \mathcal{M}_*^+.$$

It was shown in [81] that J extends to a Jordan-*-isomorphism $\widetilde{J}$ between $L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$ and $L_0(\mathcal{N}_{\varphi_0 \circ J^{-1}}, \tau_{\varphi_0 \circ J^{-1}})$; moreover, $\widetilde{J}$ is an extension of an isomorphism between $\mathcal{N}_{\varphi_0}$ and $\mathcal{N}_{\varphi_0 \circ J^{-1}}$ as well as a homeomorphism for the measure topology on $L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$ and $L_0(\mathcal{N}_{\varphi_0 \circ J^{-1}}, \tau_{\varphi_0 \circ J^{-1}})$. The isomorphism $\widetilde{J}$ satisfies the relations

$$\tau_{\varphi_0} \circ \widetilde{J}^{-1} = \tau_{\varphi_0 \circ J^{-1}}$$
$$J^{-1} \circ \Phi_{\varphi_0 \circ J^{-1}} = \Phi_{\varphi_0} \circ \widetilde{J}^{-1}$$

We claim that for $\varphi \in \mathcal{M}_*^+$, we have

$$\frac{d\tilde{\varphi}^{\varphi_0}}{d\tau_{\varphi_0}} = \widetilde{J}^{-1}\left(\frac{d\varphi \circ \tilde{J}^{-1\,\varphi_0 \circ J^{-1}}}{d\tau_{\varphi_0 \circ J^{-1}}}\right). \quad (1)$$

Indeed, let $\varphi \in \mathcal{M}_*^+$. We have

$$\begin{aligned}
\tau_{\varphi_0}(\frac{d\tilde{\varphi}^{\varphi_0}}{d\tau_{\varphi_0}}\ .\ ) &= \varphi \circ \Phi_{\varphi_0} \\
&= \varphi \circ J^{-1} \circ \Phi_{\varphi_0 \circ J^{-1}} \circ \widetilde{J} \\
&= \tau_{\varphi_0 \circ J^{-1}}(\frac{d\varphi \circ \tilde{\ } J^{-1}{}^{\varphi_0 \circ J^{-1}}}{d\tau_{\varphi_0 \circ J^{-1}}} \widetilde{J}(\ .\ )) \\
&= \tau_{\varphi_0} \circ \widetilde{J}^{-1}(\frac{d\varphi \circ \tilde{\ } J^{-1}{}^{\varphi_0 \circ J^{-1}}}{d\tau_{\varphi_0 \circ J^{-1}}} \widetilde{J}(\ .\ )) \\
&= \tau_{\varphi_0}(\widetilde{J}^{-1}(\frac{d\varphi \circ \tilde{\ } J^{-1}{}^{\varphi_0 \circ J^{-1}}}{d\tau_{\varphi_0 \circ J^{-1}}})\ .\ )\ ,
\end{aligned}$$

where in the last equality we used the fact that $\widetilde{J}$ is Jordan homomorphism.

In Lemma 2.1 in [80], it is shown that there exists a $*$-isomorphism $\widetilde{\mathcal{K}}$ between $L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$ and $L_0(\mathcal{N}_{\varphi_0 \circ J^{-1}}, \tau_{\varphi_0 \circ J^{-1}})$, which is continuous with respect to the measure topology, and which satisfies the following relation for the Radon-Nikodym derivatives :

$$\widetilde{\mathcal{K}}(\frac{d\tilde{\varphi}^{\varphi_0}}{d\tau_{\varphi_0}}) = \frac{d\tilde{\varphi}^{\varphi_0 \circ J^{-1}}}{d\tau_{\varphi_0 \circ J^{-1}}} \text{ for all } \varphi \in \mathcal{M}_*^+.$$

From the equality (1), we obtain

$$\frac{d\varphi \circ \tilde{\ } J^{-1}{}^{\varphi_0}}{d\tau_{\varphi_0}} = \widetilde{\mathcal{K}}^{-1} \circ \widetilde{J}(\frac{d\tilde{\varphi}^{\varphi_0}}{d\tau_{\varphi_0}}) \text{ for all } \varphi \in \mathcal{M}_*^+.$$

This last equality shows that

$$h_{\varphi \circ J^{-1}} = \widetilde{\mathcal{K}}^{-1} \circ \widetilde{J}(h_\varphi) \text{ for all } \varphi \in \mathcal{M}_*^+$$

and, since $\widetilde{\mathcal{K}}^{-1} \circ \widetilde{J}$ is a Jordan isomorphism, we have also

$$(h_{\varphi \circ J^{-1}})^{\frac{1}{p}} = \widetilde{\mathcal{K}}^{-1} \circ \widetilde{J}(h_\varphi^{\frac{1}{p}}) \text{ for all } \varphi \in \mathcal{M}_*^+.$$

As a consequence, the linear and bijective isometry U of $L_p(\mathcal{M})$ is given by the following relation on positive elements :

$$U(x) = w\ (\widetilde{\mathcal{K}}^{-1} \circ \widetilde{J}(x)) \text{ for all } x \in L_p(\mathcal{M})^+.$$

This relation extends by linearity to the whole space $L_p(\mathcal{M})$. $\square$

1.6 Isometries of closed subspaces of L_p-spaces

In this section, we are interested in $O(F)$ for F a closed subspace of $L_p(\mathcal{M})$. We first recall the results in the commutative case.

1.6.1 The commutative case

The following Theorem due to Hardin shows that every isometry of a closed subspace of $L_p(X, \mu)$ extends uniquely to $L_p(X', \mu')$ for some measure space (X', μ').

Theorem 1.6.1. [40] *Let $(X, \mathcal{B}, \mu)$ be a measure space. For every closed subspace $F \subset L^p(X, \mu)$, there is a canonical extension $F \subset \tilde{F} \subset L^p(\mu)$ which is isometric to $L^p(X', \mu')$ for som other measure space (X', μ'). Furthermore, if $1 < p \notin 2\mathbb{N}$, then every linear isometry $U : F \to L^p(Y, \nu)$ extends to a surjective linear isometry*

$$\tilde{U} : \tilde{F} \to \tilde{UF} \subset L^p(Y, \nu).$$

As a consequence of the previous Theorem 1.6.1, we have that a linear bijective isometry of a closed subspace of L^p has the form given in Theorem 1.5.2.

1.6.2 The non-commutative case

There is no general result about the description of isometries of closed subspaces in non-commutative L_p-spaces. In the non-commutative context, it is natural to consider complete isometries instead of isometries. Recall the definition of n-isometries and complete isometries.

Definition 1.6.2. Let $1 \leq p < \infty$. Let $\mathcal{M}$ and $\mathcal{N}$ be finite von Neumann algebras. A linear map $U : L_p(\mathcal{M}) \to L_p(\mathcal{N})$ is said to be a n-isometry from $L_p(\mathcal{M})$ to $L_p(\mathcal{N})$ if the map

$$id \otimes U : L_p(\mathcal{M}_n \otimes \mathcal{M}) \to L_p(\mathcal{M}_n \otimes \mathcal{N})$$

is an isometry.
U is said to be a *complete isometry* if U is a n-isometry for all $n \geq 1$. U is said to be *unital* if $U(1) = 1$.

Example 1.6.3. (i) Let $\mathcal{M} = L^\infty(X, \mu)$ be a commutative von Neumann algebra. Let $1 \leq p < \infty$, $p \neq 2$. Let $U \in O(L_p(X, \mu))$ be a linear bijective isometry of $L_p(X, \mu)$. By Banach-Lamperti Theorem 1.5.2, there exist a measurable, measure-class preserving bijection T of (X, μ), and a measurable function h with $|h(x)|$ almost everywhere, such that

$$(U(f))(x) = h(x)(\frac{dT_*\mu}{d\mu}(x))^{\frac{1}{p}} f(T(x)) \text{ for all } f \in L_p(X, \mu).$$

Let $n \geq 1$. We can identify $L_p(\mathcal{M}_n \otimes \mathcal{M})$ with the space $L_p(X, \mathcal{M}_n)$ of measurable matrix functions $F : X \to \mathcal{M}_n$ with finite norm

$$||F||_p = (\int_X \mathrm{Tr}(|F(x)|^p) d\mu(x))^{\frac{1}{p}}.$$

Let $F \in L_p(X, \mathcal{M}_n)$. Then, for all $x \in X$,

$$(id \otimes U)F(x) = \lambda(x)F(Tx), \text{ where } \lambda(x) = h(x)(\frac{dT_*\mu}{d\mu}(x))^{\frac{1}{p}}.$$

Hence,

$$\begin{aligned} ||(id \otimes U)F||_p &= (\int_X \mathrm{Tr}(|\lambda(x)|^p |F(Tx)|^p) d\mu(x))^{\frac{1}{p}} \\ &= (\int_X \mathrm{Tr}(|F(x)|^p) d\mu(x))^{\frac{1}{p}} \\ &= ||F||_p. \end{aligned}$$

This shows that U is completely isometric.

(ii) Let $\mathcal{M} = \mathcal{M}_2(\mathcal{C})$. Let $p = n$ for $n \in \mathbb{N}$, $n \geq 3$. We claim that the isometry of $L_p(\mathcal{M})$ defined by the anti-isomorphism

$$\begin{aligned} T: \ &\mathcal{M} \to \mathcal{M} \\ &x \mapsto^t x \end{aligned}$$

is not 2-isometric. Indeed, let

$$A = \left(\begin{array}{|cc|cc|} \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline \end{array} \right) \in \mathcal{M}_2(\mathcal{M}_2(\mathbb{C})).$$

Then $(id \otimes T)(A) = B$ for

$$B = \left(\begin{array}{|cc|cc|} \hline 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ \hline \end{array} \right).$$

We have $A^* = A$ and $A^2 = I$. Hence

$$||A||_p^p = \mathrm{Tr}(|A|^n) = \mathrm{Tr}(I) = 4.$$

On the other hand, $B^* = B$ and $B^2 = 2B$. Hence

$$||B||_p^p = \mathrm{Tr}(|B|^n) = 2^{n-1}\mathrm{Tr}(B) = 2^n.$$

So, $||(id \otimes T)(A)||_p \neq ||A||_p$. This example can obviously be generalized to the case $\mathcal{M} = \mathcal{M}_n(\mathbb{C})$ or $\mathcal{M} = \mathcal{B}(l_2)$. In view of Arazy's result (Theorem 1.5.3), we see that $U : C_p \to C_p$ is a complete surjective isometry of C_p if and only if there exist two unitaries u and v in $\mathcal{B}(l_2)$ such that $U(x) = uxv$ for all $x \in C_p$.

When $\mathcal{M}$ is a finite von Neumann algebra, we have the following remarkable result due to De La Salle about the extension of a unital complete isometry on a closed subspace F of $L_p(\mathcal{M})$.

Theorem 1.6.4. [24] *Let $\mathcal{M}$ be a finite von Neumann algebra with normalized trace τ. Let F be a subspace of $L_p(\mathcal{M}, \tau)$ and let $U : F \to L_p(\mathcal{M}, \tau)$ be a linear map. Let $1 \leq p < \infty$ such that $p \notin 2\mathbb{N}$.*
Assume that : $F \subset \mathcal{M}$.
Assume also that for all $n \in \mathbb{N}$ and all $X \in \mathcal{M}_n(F)$, the following equality holds

$$||1_{\mathcal{M}_n} \otimes 1_{\mathcal{M}} + X||_p = ||1_{\mathcal{M}_n} \otimes 1_{\mathcal{M}} + (id \otimes U)(X)||_p.$$

Let $\mathrm{VN}(F)$ denote the von Neumann subalgebra generated by F in $\mathcal{M}$. Then $U(F) \subset \mathcal{M}$ and U extends to a von Neumann algebra isomorphism $U : \mathrm{VN}(F) \to \mathrm{VN}(U(F))$ that preserves the trace, and this extension is unique.

Remark 1.6.5. With the assumptions of the previous Theorem 1.6.4, if $U : F \to F$ is a linear bijective complete isometry of F, then U extends uniquely to $\mathcal{M}' = \mathrm{VN}(F)$ and therefore extends uniquely to a linear bijective complete isometry on $L_p(\mathcal{M}')$.

Chapter 2

Property $(T_{L_p(\mathcal{M})})$

Property (T) for locally compact was introduced by D.Kazhdan in the end of the 60's in [47] to show that some lattices were finitely generated and have finite abelianization. It is a rigidity property of the unitary representation theory of the groups considered. Property (T) found many applications in diverse areas : ergodic theory, random walks, operator algebras, combinatorics, theoretical computer science...Variants of property (T) have been considered by several authors. We will be interested in property (T_B) defined by the authors of [4] for orthogonal representations on a Banach space B. We will study the case where B is a non-commutative L_p-space. In section 1, we recall some general facts about unitary representations and property (T). Section 2 is devoted to orthogonal representations on a Banach space B and the definition of property (T_B). In section 3, we recall the main result of the authors of [4] relative to property T_{L_p} for commutative L_p-spaces. The conjugation of an orthogonal representation on $L_p(\mathcal{M})$ by the Mazur map, which is a crucial tool for our proofs, is explained in section 4. In section 5, we give the proof of the main theorem of this chapter, which is that property (T) implies property $(T_{L_p(\mathcal{M})})$ for any von Neumann algebra $\mathcal{M}$. We study the special case of property (T_{l_p}) in section 6. In section 7, we introduce a weaker version of property $(T_{L_p(\mathcal{M})})$ for representations by complete isometries on $L_p(\mathcal{M})$.

2.1 Introduction

2.1.1 Unitary representations of groups

We recall here basic facts about unitary representations that we will need later. For the general facts concerning unitary representations, see Part II in [8]. Let G be a topological group, and let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{U}(\mathcal{H})$ be the group of all unitary operators on $\mathcal{H}$.

Definition 2.1.1. A *unitary representation* $(\pi, \mathcal{H})$ of G on $\mathcal{H}$ is a group homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$ such that the mapping

$$\begin{aligned} G &\to \mathcal{H} \\ g &\mapsto \pi(g)\xi \end{aligned}$$

is continuous for every $\xi \in \mathcal{H}$.

We recall the notion of containment and weak containment of unitary representations :

Definition 2.1.2. Let π and σ be unitary representations of G.

(i) We say that σ is *contained* in π (in symbols $\sigma \subset \pi$) if σ is unitarily equivalent to a subrepresentation of π.

(ii) We say that σ is *weakly contained* in π (in symbols $\sigma \prec \pi$) if every matrix coefficient $g \mapsto < \pi(g)\xi, \xi >$ can be approximated, uniformly on compact subsets of G, by finite sums of matrix coefficients $g \mapsto < \sigma(g)\eta, \eta >$.

Let $(\pi, \mathcal{H})$ be a unitary representation of G. We will denote by

$$\mathcal{H}^{\pi(G)} = \{\xi \in \mathcal{H} \mid \pi(g)\xi = \xi \text{ for all } g \in G\}$$

the closed subspace of $\pi(G)$-invariant vectors in $\mathcal{H}$.

Definition 2.1.3. For a subset Q of G and a real number $\epsilon > 0$, a vector $\xi \in \mathcal{H}$ is called (Q, ϵ)-*invariant* if

$$\sup_{g \in Q} ||\pi(g)\xi - \xi|| < \epsilon ||\xi||.$$

We say that the representation $(\pi, \mathcal{H})$ *almost has invariant vectors* if it has (Q, ϵ)-invariant vectors for every compact subset Q of G and every $\epsilon > 0$.

We will sometimes use sequences of almost invariant vectors for π :

Definition 2.1.4. A sequence $(\xi_n)_n$ of vectors of $\mathcal{H}$ is called a *sequence of almost invariant vectors* for the representation π if :
- $||\xi_n|| = 1$ for all $n \in \mathbb{N}$.
- $\lim_n \sup_{g \in K} ||\pi(g)\xi_n - \xi_n|| = 0$ for every compact subset $K \subset G$.

If π has a sequence of almost invariant vectors, then π almost has invariant vectors. The converse is true when the group G is σ-compact.

In the special case where $\sigma = 1_G$ is the trivial representation of G, the notions of containment and weak containment can be rephrased as follows :

Proposition 2.1.5. *Let $(\pi, \mathcal{H})$ be a unitary representation of G.*

(i) $1_G \subset \pi \Leftrightarrow \mathcal{H}^{\pi(G)} \neq 0$ *;*

(ii) $1_G \prec \pi \Leftrightarrow \pi$ *almost has invariant vectors.*

Examples

1. The *regular* representation λ_G of a locally compact group G on $L_2(G, dg)$, where dg stands for the left Haar measure on G, is given by
$$\lambda_G(g)f(h) = f(g^{-1}h) \text{ for all } f \in L_2(G, dg), g, h \in G.$$

2. Let H be a closed subgroup of a locally compact group G. Then G/H carries a quasi-invariant measure μ. The unitary representation $\lambda_{G/H}$ of G defined on $L_2(G/H, \mu)$ by
$$\lambda_{G/H}(g)\xi(xH) = (\frac{\mathrm{d}g^{-1}\mu}{\mathrm{d}\mu}(xH))^{1/2}\xi(g^{-1}xH) \text{ for all } g, x \in G, \xi \in L_2(G/H)$$
is called the *quasi-regular* representation of G associated to H.

3. Let G be a locally compact group and H a closed subgroup of G. Let X be a Borel subset of G which is a fondamental domain for the action of H by right translations on G; thus, $G = \bigcup_{x \in X} xH$ and $xH \cap yH$ has Haar measure 0 if $x \neq y$.
For $g \in G$ and $x \in X$, let $\alpha(g, x) \in H$ and $g.x \in X$ be defined by
$$gx = (g.x)\alpha(g, x).$$
Then $G \times X \to X$, $(g, x) \mapsto g.x$ is an action of G on X for which the Haar measure on X is quasi-invariant. Moreover, $\alpha : G \times X \to H$ is a cocycle.
Let now $(\pi, \mathcal{H})$ be a unitary representation of H. The *induced representation* $\mathrm{Ind}_H^G \pi$ is the unitary representation of G on $L_2(X, \mathcal{H}, \mu)$ defined by
$$\mathrm{Ind}_H^G(g)\xi(x) = (\frac{\mathrm{d}g^{-1}\mu}{\mathrm{d}\mu}(x))^{1/2}\pi(\alpha(g^{-1}, x))\xi(g^{-1}.x)$$
for all $g \in G, x \in X, \xi \in L_2(X, \mathcal{H})$.
The induced representation can also be realized on the Hilbert space $\tilde{\mathcal{H}}$ of measurable mappings $f : G \to \mathcal{H}$ such that :
(i) $f(gh) = \pi(h^{-1})f(g)$ for all $h \in H$ and almost every $g \in G$,
(ii) $||f||^2 = \int_{G/H} ||f(g)||^2 d\mu(gH) < \infty$.
This realization is given by the formula
$$\mathrm{Ind}_H^G(g)f(xH) = (\frac{\mathrm{d}g^{-1}\mu}{\mathrm{d}\mu}(x))^{1/2} f(g^{-1}xH)$$
for all $g \in G, x \in X$ and $f \in \tilde{\mathcal{H}}$.

Remark 2.1.6. In the particular case where $\pi = 1_H$, $\mathrm{Ind}_H^G 1_H$ is equivalent to the quasi-regular representation $\lambda_{G/H}$.

2.1.2 Kazhdan's property (T)

For details concerning this section, see [8]. It is useful to introduce the definition of property (T) for pairs of groups.

Definition 2.1.7. Let G be a topological group and H a closed subgroup of G. The pair (G, H) is said to have property (T) (or G is said to have the relative property (T) with respect to H) if there exist a compact subset Q of G and $\epsilon > 0$ such that : whenever a unitary representation π of G on a Hilbert space $\mathcal{H}$ has a (Q, ϵ)-invariant vector, then π has a non-zero $\pi(H)$-invariant vector. The pair (Q, ϵ) is called a Kazhdan pair.
A topological group G is said to have property (T) if the pair (G, G) has property (T).

Property (T) can be rephrased as follows :

Remark 2.1.8. Let G be a topological group.

1. G has property (T) $\Leftrightarrow$ For every unitary representation π of G, if $1_G \prec \pi$, then $1_G \subset \pi$.

2. G has property (T) $\Leftrightarrow$ For every unitary representation π of G on a Hilbert space $\mathcal{H}$, the restriction π' of π to the orthogonal complement of $\mathcal{H}^{\pi(G)}$ does not almost have invariant vectors.

Consequences of property (T)

Here are some important consequences properties of property (T).

Proposition 2.1.9. *Let G be a locally compact topological group with property (T). Then G has the following properties :*
(i) G is compactly generated ;
(ii) the abelianised group $G/\overline{[G, G]}$ is compact ;
(iii) G is unimodular ;
(iv) if G is amenable, then G is compact.

An important fact about property (T) is that it is inherited by lattices or more generally by subgroups with finite covolume.

Proposition 2.1.10. *Let G be a locally compact group and let H be a closed subgroup of G such that G/H has a finite invariant regular Borel measure. The following properties are equivalent :*
- G has property (T).

- H has property (T).
In particular, if Γ is a lattice in G then Γ has property (T) if and only if G has property (T).

Examples

1. Trivial examples of groups with property (T) are compact groups.

2. Important examples of pairs of groups with property (T) are the pairs $(\mathbb{R}^2 \rtimes SL_2(\mathbb{R}), \mathbb{R}^2)$ and $(\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z}), \mathbb{Z}^2)$.

3. Let $\mathbb{K}$ a local field. Then $SL_n(\mathbb{K})$ for $n \geq 3$ and $Sp_{2n}(\mathbb{K})$ for $n \geq 2$ have property (T). More generally, *higher rang groups* (see Chapter III) have property (T).

4. The groups $Sp(n, 1)$ for $n \geq 2$ have property (T).

2.2 Property (T) for representations on Banach spaces

We give in this section general facts about a variant of property (T) relative to a certain class of Banach spaces B, namely the *ucus* Banach spaces. It is called property (T_B), and was introduced in [4], where more details and proofs concerning general facts about property (T_B)) can be found.

Let B be a Banach space. Denote by $O(B)$ the group of linear bijective isometries of B. Let $\pi : G \to O(B)$ be a homomorphism from a topological group G to $O(B)$ such that the maps $g \mapsto \pi(g)x$ from G to B are continuous for every $x \in B$. Such a continuous homomorphism is called an *orthogonal representation* of the group G on the space B.

Let G be a topological group and H a closed subgroup of G. Let $\pi : G \to O(B)$ be an orthogonal representation of G on B. We denote by $B^{\pi(H)}$ the subspace of $\pi(H)$-invariant vectors in B. We can define almost invariant vectors for π and sequences of almost invariant vectors for π as in the previous section. Observe that if the subgroup H is normal in G, then the subspace $B^{\pi(H)}$ is $\pi(G)$-invariant.

Definition 2.2.1. Let G be a topological group and H be a closed *normal* subgroup of G. The pair (G, H) has relative property (T_B) for a Banach space B if, for any orthogonal representation $\pi : G \to O(B)$, the quotient representation $\pi' : G \to O(B/B^{\pi(H)})$ does not almost have $\pi'(G)$-invariant vectors.
A topological group G has property (T_B) if the pair (G, G) has relative property (T_B).

If $B = \mathcal{H}$ is a Hilbert space, the representation π' is equivalent to the restriction of π to the orthogonal complement of the space of invariant vectors $\mathcal{H}^{\pi(G)}$, and the definition of (T_B) agrees with that of Kazhdan's property (T) (see Remark 2.1.8).

We will deal with the case where B is a ucus Banach space, that is a uniformly convex Banach space with a uniformly convex dual space. In this situation, we will have a canonical complement of $B^{\pi(G)}$.

From now on, let B be a ucus Banach space. We recall that the duality map $* : S(B) \to S(B^*)$ associates to each unit vector $x \in B$, the unique unit vector $*x$ in B^* such that $< x, *x >= 1$. Recall that since B is ucus, the duality map $*$ is uniformly continuous on the unit sphere $S(B)$ with inverse uniformly continuous on $S(B^*)$.

Let $\pi : G \to O(B)$ be an orthogonal representation of a topological group G on B. The *contragredient* representation $\pi^* : G \to O(B^*)$ of the representation π is defined by

$$< x, \pi^*(g)y >=< \pi(g^{-1})x, y > \text{ for all } x \in B, y \in B^*, g \in G.$$

Let $B^{\pi(G)}$ be the space of $\pi(G)$-invariant vectors. Notice that we have the equality

$$*(B^{\pi(G)}) = (B^*)^{\pi^*(G)}.$$

The following theorem asserts that the space $B^{\pi(G)}$ admits a canonical $\pi(G)$-invariant complement B'.

Theorem 2.2.2. (Proposition 2.6 in [4]) *Let G be a topological group, and let B be a ucus Banach space. Let $\pi : G \to O(B)$ be an orthogonal representation of G on B. Denote by $B' = B'(\pi)$ the annihilator of $(B^*)^{\pi^*(G)}$ in B, that is*

$$B' = \{x \in B \mid \forall y \in (B^*)^{\pi^*(G)}, < x, y >= 0 \}.$$

Then

$$B = B^{\pi(G)} \oplus B'.$$

Notice that when $B = \mathcal{H}$ is a Hilbert space, the complement B' is the orthogonal complement of $\mathcal{H}^{\pi(G)}$. As for property (T) (see Remark 2.1.8), property (T_B) can be rephrased as follows :

Corollary 2.2.3. *Let G be a topological group, and let B be a ucus Banach space. Then the following assertions are equivalent :*
(i) G has property (T_B) ;
(ii) For every orthogonal representation $\pi : G \to O(B)$ of G on B, the restriction π' of π to $B' = B'(\pi)$, does not almost have invariant vectors.

Remark 2.2.4. 1. Let $\pi_1 : G \to O(B_1)$, $\pi_2 : G \to O(B_2)$ be orthogonal representations of G on B_1 and B_2. Let $\phi : B_1 \to B_2$ be a linear bijective isometry intertwining the representations π_1 and π_2. Denote by $p(\pi_i)$ and $p'(\pi_i)$ the projections on $B_i^{\pi_i(G)}$ and $B_i'(\pi_i)$. Then the following diagrams are commutative :

$$\begin{tikzcd} B_1 \arrow[r, "\phi"'] \arrow[d, "p(\pi_1)"'] & B_2 \arrow[d, "p(\pi_2)"'] \\ B_1 \arrow[r, "\phi"'] & B_2 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B_1 \arrow[r, "\phi"'] \arrow[d, "p'(\pi_1)"'] & B_2 \arrow[d, "p'(\pi_2)"'] \\ B_1 \arrow[r, "\phi"'] & B_2 \end{tikzcd}$$

2. The authors of [4] studied representations on “superreflexive”Banach spaces, which is a class of Banach spaces that contains the class of ucus Banach spaces, and give an analog of Theorem 2.2.2 in this context.

We have the following basic facts concerning property (T_B) (see Proposition 2.15 and Corollary 2.12 in [4]).

Proposition 2.2.5. *Let B be a ucus Banach space and G a topological group. Then :*
(i) any compact group has property (T_B);
(ii) if G has property (T_B), then any quotient group of G has property (T_B);
(iii) if $G = G_1 \times ... \times G_n$ is a finite product of topological groups, then G has property (T_B) if and only if every G_i have property (T_B);
(iv) G has property (T_B) if and only if G has property (T_{B^*}).

2.3 Property $(T_{L_p(X,\mu)})$

The authors of [4] studied the particular case of the Banach spaces $L_p(X,\mu)$, for a non-atomic σ-finite measure μ on a Borel space $(X, \mathcal{B})$, and $1 < p < \infty$. Notice that $L_1(X,\mu)$ and $L_\infty(X,\mu)$ are not ucus (they are not even stricly convex).

Recall that the Mazur map is the map defined by

$$\begin{aligned} M_{p,q} : \quad & L_p(X,\mu) \to L_q(X,\mu) \\ & f = (f/|f|)|f| \mapsto (f/|f|)|f|^{p/q}. \end{aligned}$$

It is locally uniformly continuous and, if p' is the conjugate exponent of p, $M_{p,p'}$ is the duality map between $L_p(X,\mu)$ and $L_{p'}(X,\mu) \simeq (L_p(X,\mu))^*$. The following theorem is one of the main results in [4].

Theorem 2.3.1. (Theorem A in [4]) *Let G be a locally compact second countable group. If G has property (T), then G has property (T_B) for Banach spaces B of the following types.*

1. *$L_p(X,\mu)$ for any σ-finite measure μ on a Borel space X, any $1 \leq p < \infty$;*
2. *closed subspaces of $L_p(X,\mu)$ for $1 < p < \infty$ with $p \neq 4, 6, 8...$;*
3. *quotient spaces of $L_p(X,\mu)$ for $1 < p < \infty$ with $p \neq \frac{4}{3}, \frac{6}{5}, \frac{8}{7}....$*

If μ is moreover non-atomic and G has property $(T_{L_p}(X,\mu))$, then G has Kazhdan's property (T).

Remark 2.3.2. We give here some remarks concerning the proof of the theorem above.

1. The proof of item 2 follows from the proof for L_p-spaces, once we notice that by Theorem 1.6.1, a representation on such a closed subspace extends to a representation on a space $L_p(X',\mu')$ containing F, and hence has the form given in Banach-Lamperti's Theorem 1.5.2. A generalization of Hardin's Theorem 1.6.1 for extensions of isometries of closed subspaces of non-commutative L_p-spaces, is not known in general context. However, a result of De La Salle (see Theorem 1.6.4) shows that complete isometries admit sometimes such extensions. This will allow us to extend item 2 of Theorem 2.3.1 to the non-commutative context (see Theorem 2.7.4).

2. The result for quotient spaces is deduced from the one for closed subspaces, using (iv) in Proposition 2.2.5.

2.4 The conjugate of a representation by the Mazur map

In order to study property $(T_{L_p(\mathcal{M})})$, a crucial tool will be the possibility to transfer a representation on $L_p(\mathcal{M})$ to a representation on $L_2(\mathcal{M})$ (and from $L_2(\mathcal{M})$ to $L_p(\mathcal{M})$ if possible). In this section, we establish some general facts about group representations on non-commutative L_p-spaces. More precisely, we show how to construct a representation π^q on $L_q(\mathcal{M})$ from a representation π^p on $L_p(\mathcal{M})$. The proofs of this section rely essentially on the properties of the Mazur map, and on the structure of the group of isometries $O(L_p(\mathcal{M})$. For general properties on the Mazur map and Jordan morphisms on von Neumann algebras, we refer to Chapter 1 of this thesis.

Let $\mathcal{M}$ be a von Neumann algebra, and $1 \leq p, q < \infty$. Let

$$M_{p,q} : L_p(\mathcal{M}) \to L_q(\mathcal{M})$$

be the Mazur map, as defined in Section 3 of Chapter 1. Recall that $O(L_p(\mathcal{M}))$ denotes the group of linear bijective isometries of the space $L_p(\mathcal{M})$.

Proposition 2.4.1. *Let $p \neq 2$. For $U \in O(L_p(\mathcal{M}))$, we have*

$$V = M_{p,q} \circ U \circ M_{q,p} \in O(L_q(\mathcal{M})).$$

Moreover, if $\mathcal{M}$ is semi-finite and $U = WBJ$ is the Yeadon decomposition of U, then the isometry V is given by the formula

$$V(x) = WB^{\frac{p}{q}}J(x) \text{ for all } x \in \mathcal{M} \cap L_q(\mathcal{M}, \tau).$$

Proof. The fact that the map V is norm-preserving on $L_q(\mathcal{M})$ is a consequence of the norm-preserving property of the Mazur map (see Proposition 1.3.3). It remains to show that V is linear. We first give the proof in the semi-finite case, and then the more involved proof for general Haagerup L_p-spaces.

• *Semi-finite case* : Assume that $\mathcal{M}$ admits a faithful semi-finite normal trace τ. By Yeadon's Theorem 1.5.4, there exist a Jordan-isomorphism J of $\mathcal{M}$, a positive operator B commuting with $\mathcal{M}$, and a partial isometry W in $\mathcal{M}$ with the property that W^*W is the support of B, such that

$$U(x) = WBJ(x) \text{ for all } x \in \mathcal{M} \cap L_p(\mathcal{M}, \tau).$$

From Proposition 1.3.7, we have the equality

$$M_{p,q} \circ J \circ M_{q,p} = J.$$

Since B commutes with every $x \in \mathcal{M} \cap L_p(\mathcal{M})$, the polar decomposition of By is $By = \alpha B|y|$ if $y = \alpha|y|$ is the polar decomposition of y. Hence

$$M_{p,q}(BJ(M_{q,p}(x)) = B^{\frac{p}{q}} M_{p,q}(J(M_{q,p}(x)) \text{for all } x \in \mathcal{M} \cap L_q(\mathcal{M}, \tau).$$

Therefore, we have

$$\begin{aligned} V(x) &= WM_{p,q}(BJ(M_{q,p}(x)) \\ &= WB^{\frac{p}{q}} M_{p,q}(J(M_{q,p}(x)) \\ &= WB^{\frac{p}{q}} J(x) \end{aligned}$$

for all $x \in \mathcal{M} \cap L_q(\mathcal{M}, \tau)$. This shows that V is linear on $\mathcal{M} \cap L_q(\mathcal{M}, \tau)$. The linearity on the whole space $L_q(\mathcal{M}, \tau)$ follows from the density of $\mathcal{M} \cap L_q(\mathcal{M}, \tau)$ in $L_q(\mathcal{M}, \tau)$ and the continuity of V.

• *General case* : Let φ_0 be a normal semi-finite faithful weight on $\mathcal{M}$. Recall from Theorem 1.5.10 that there exist a unitary w in $\mathcal{M}$ and a Jordan isomorphism J of $L_0(\mathcal{M} \rtimes_{\varphi_0} \mathbb{R}, \tau_{\varphi_0}) = L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$ such that the linear isometry U has the form

$$U(x) = wJ(x) \text{ for all } x \in L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0}).$$

Recall that the restriction $J : \mathcal{N}_{\varphi_0} \to \mathcal{N}_{\varphi_0}$ is an isomorphism. Hence, by Proposition 1.3.7, we have

$$\begin{aligned} V(x) &= M_{p,q} \circ U \circ M_{q,p}(x) \\ &= w(M_{p,q} \circ J \circ M_{q,p}(x)) \\ &= wJ(x) \end{aligned}$$

for all $x \in \mathcal{N}_{\varphi_0}$. Recall from [69] that the Mazur map is continuous for the measure topology on $L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$. So by density of $\mathcal{N}_{\varphi_0}$ in $L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$ for the measure topology, we have

$$V(x) = wJ(x) \text{ for all } x \in L_q(\mathcal{M})$$

which gives the linearity of V on $L_q(\mathcal{M})$. □

Corollary 2.4.2. *Let G be a topological group, and let $\mathcal{M}$ be a von Neumann algebra. Let $p \neq 2$, and $1 \leq q < \infty$. Let $\pi^p : G \to O(L_p(\mathcal{M}))$ be a representation of G on $L_p(\mathcal{M})$. For $g \in G$, define $\pi^q(g) : L_q(\mathcal{M}) \to L_q(\mathcal{M})$ by*

$$\pi^q(g) = M_{p,q} \circ \pi^p(g) \circ M_{q,p}.$$

Then $\pi^q : g \mapsto \pi^q(g)$ is an orthogonal representation of G on $L_q(\mathcal{M})$.

Proof. By the previous proposition, $\pi^q(g) \in O(L_2(\mathcal{M}))$ for every g in G. Moreover, for every $x \in L_q(\mathcal{M})$, the map $g \mapsto \pi^q(g)x$ is continuous, since $g \mapsto \pi^p(g)M_{q,p}(x)$ and $M_{p,q} : L_p(\mathcal{M}) \to L_q(\mathcal{M})$ are continuous (see Proposition 1.3.8). It remains to check that π^q is a homomorphism. Let $g_1, g_2 \in G$. Using Lemma 1.3.2,

$$\begin{aligned} \pi^q(g_1)\pi^q(g_2) &= M_{p,q} \circ \pi^p(g_1) \circ M_{q,p} \circ M_{p,q} \circ \pi^p(g_2) \circ M_{q,p} \\ &= M_{p,q} \circ \pi^p(g_1) \circ \pi^p(g_2) \circ M_{q,p} \\ &= M_{p,q} \circ \pi^p(g_1 g_2) \circ M_{q,p} \\ &= \pi^q(g_1 g_2). \end{aligned}$$

□

Let $\mathcal{M}$ be a semi-finite von Neumann algebra and $\pi^p : G \to O(L_p(\mathcal{M}))$ an orthogonal representation of G on $L_p(\mathcal{M})$. Then, by Yeadon's Theorem 1.5.4, every $\pi(g)$ has a decomposition

$$\pi(g) = u_g B_g J_g(x) \ , \ x \in \mathcal{M} \cap L_p(\mathcal{M}),$$

where u_g is a unitary in $\mathcal{M}$, B_g a positive operator commuting with $\mathcal{M}$ and J_g a Jordan isomorphism of $\mathcal{M}$. We will need a description of the decomposition for the conjugate isometries

$$\pi^q(g) = M_{p,q} \circ \pi^p \circ M_{q,p}$$

as well as the decomposition of the product

$$\pi^p(g_1)\pi^p(g_2) \text{ for } g_1, g_2 \in G.$$

Lemma 2.4.3. *Let $\mathcal{M}$ be a semi-finite von Neumann algebra. Let u be a unitary in $\mathcal{M}$, B a positive operator affiliated with $\mathcal{M}$ and commuting with $\mathcal{M}$, and J a Jordan isomorphism of $\mathcal{M}$. Then, for all $y \in L_p(\mathcal{M})_+$, we have*

$$J(uBy) = J(u)J(B)(J^1(y) + J^2(uyu^*))$$

where J^1 is a Jordan $*$-algebra isomorphism, and J^2 is a $*$-algebra anti-isomorphism.

Proof. Recall from Theorem 1.3.6 that $J = J^1 + J^2$ where J^1 is a Jordan $*$-algebra isomorphism, and J^2 is a $*$-algebra anti-isomorphism. Let $y \in L_p(\mathcal{M})_+$. Then

$$\begin{aligned} J(uBy) &= J^1(uBy) + J^2(uyu^*Bu) \\ &= J^1(u)J^1(B)J^1(y) + J^2(u)J^2(B)J^2(uyu^*) \\ &= J(u)J(B)(J^1(y) + J^2(uyu^*)). \end{aligned}$$

□

Theorem 2.4.4. *Let G be a topological group. Let $\mathcal{M}$ be a semi-finite von Neumann algebra. Let $1 \leq p, q < \infty$. Let π^p be an orthogonal representation of a topological group G on $L_p(\mathcal{M})$ such that, for every $g \in G$, $\pi^p(g)$ has the decomposition*

$$\pi^p(g)(x) = u_g B_g J_g(x) \text{ for all } x \in L_p(\mathcal{M}),$$

where u_g is a unitary in $\mathcal{M}$, B_g a positive operator commuting with $\mathcal{M}$ and J_g a Jordan isomorphism of $\mathcal{M}$. Then

$$\pi^q(g)(x) = u_g B_g^{\frac{p}{q}} J_g(x) \text{ for all } x \in L_q(\mathcal{M}) \text{ and all } g \in G.$$

Moreover, the following relations hold for all $g_1, g_2 \in G$ and all $x \in L_p(\mathcal{M})$,

$$\begin{aligned} u_{g_1 g_2} &= u_{g_1} J_{g_1}(u_{g_2}), \\ B_{g_1 g_2} &= B_{g_1} J_{g_1}(B_{g_2}), \\ J_{g_1 g_2}(x) &= J^1_{g_1}(J_{g_2}(x)) + J^2_{g_1}(u_{g_2} J_{g_2}(x) u^*_{g_2}). \end{aligned}$$

Proof of Theorem 2.4.4. The fact that π^q has the claimed form follows from Proposition 2.4.1.

Let $g_1, g_2 \in G$. By Lemma 2.4.3, and using that the B_g's commute with $\mathcal{M}$, we have

$$\begin{aligned} \pi^p(g_1 g_2)(x) &= \pi(g_1)(\pi(g_2)(x)) \\ &= u_{g_1} B_{g_1} J_{g_1}(u_{g_2} B_{g_2} J_{g_2}(x)) \\ &= u_{g_1} J_{g_1}(u_{g_2}) B_{g_1} J_{g_1}(B_{g_2})(J^1_{g_2}(x) + J^2_{g_2}(u_{g_2} x u^*_{g_2})) \end{aligned}$$

for $g_1, g_2 \in G$ and $x \in L_p(\mathcal{M})$. Hence, by the uniqueness of the decomposition in Yeadon's theorem 1.5.4, we obtain the relations in 2.4.4. □

Remark 2.4.5. (i) Let $\mathcal{M}$ be a von Neumann algebra, and let $1 \leq p, q < \infty$. Let $\pi^p : G \to O(L_p(\mathcal{M}))$ be an orthogonal representation of the group G. Assume that, for every $g \in G$, the isometry $\pi^p(g)$ has the form

$$\pi^p(g)(x) = u_g J_g(x) \text{ for all } x \in L_p(\mathcal{M}) \quad (*)$$

for a unitary u_g in $\mathcal{M}$, and a Jordan isomorphism J_g of $\mathcal{M}$. Then, for every $g \in G$, the conjugate isometry $\pi^q(g) = M_{p,q} \circ \pi^p(g) \circ M_{q,p} \in O(L_q(\mathcal{M}))$ has the form

$$\pi^q(g)(x) = u_g J_g(x) \text{ for all } x \in L_q(\mathcal{M}).$$

(ii) Let $\mathcal{M}$ be a semi-finite factor or $\mathcal{M} = l^\infty$. Let $1 \leq p < \infty$, $p \neq 2$. Let $\pi^p : G \to O(L_p(\mathcal{M}))$ be an orthogonal representation of the group G. Then, for every $g \in G$, the isometry $\pi^p(g)$ has the form $(*)$ as above.

Now we give a useful description of the contragradient representation $(\pi^p)^*$ of an orthogonal representation $\pi^p : G \to O(L_p(\mathcal{M}))$ of a topological group G. Recall that Tr is the linear functional on $L_1(\mathcal{M})$ defined (for any von Neumann algebra $\mathcal{M}$) in Chapter 1 Section 1.2 and the duality bracket between $L_p(\mathcal{M})$ and $L_{p'}(\mathcal{M})$ is given by $(x, y) \mapsto \mathrm{Tr}(xy)$. The contragradient representation of $\pi^p : G \to O(L_p(\mathcal{M}))$ is therefore equivalent to the representation on $L_{p'}(\mathcal{M})$, also denoted by $(\pi^p)^*$ for simplicity, defined by the formula

$$\mathrm{Tr}((\pi^p)^*(g)(x)y) = \mathrm{Tr}(x\pi^p(g^{-1})(y)) \text{ for all } x \in L_{p'}(\mathcal{M}), y \in L_p(\mathcal{M}), g \in G.$$

We recall that we denote by $* : L_p(\mathcal{M}) \to L_{p'}(\mathcal{M})$ the duality map.

Proposition 2.4.6. *Let G be a topological group, $\mathcal{M}$ a von Neumann algebra, and $1 \leq p < \infty$, $p \neq 2$. Let $\pi : G \to O(L_p(\mathcal{M}))$ be an orthogonal representation. Let $g \in G$. Then*

$$(\pi^p)^*(g)x = * \circ \pi^p(g) \circ *^{-1}(x) \text{ for all } x \in S(L_{p'}(\mathcal{M})). \quad (1)$$

In particular, if $\mathcal{M}$ is semi-finite and if $\pi^p(g) = u_g B_g J_g$ is the Yeadon decomposition of $\pi^p(g)$, then we have

$$(\pi^p)^*(g)x = u_g^* B_g^{\frac{p}{p'}} u_g J_g(x) u_g^* \text{ for all } x \in L_{p'}(\mathcal{M}). \quad (2)$$

Proof. Let $x \in S(L_{p'}(\mathcal{M}))$. We have

$$\begin{aligned} \mathrm{Tr}((\pi^p)^*(g)(x)\ \pi^p(g)(*^{-1}x)) &= \mathrm{Tr}(x\pi^p(g^{-1})\pi^p(g)(*^{-1}x)) \\ &= \mathrm{Tr}(x *^{-1} x) \\ &= 1. \end{aligned}$$

This shows the equality (1) for x, by the defining property of the duality map. By linearity of the maps $x \mapsto (\pi^p)^*(g)x$ and $x \mapsto u_g^* B_g^{\frac{p}{p'}} u_g J_g(x) u_g^*$, it suffices to show the equality (2) on positive elements $x \in S(L_{p'}(\mathcal{M}))$. By Proposition 1.3.4, the duality map is given by the formula $*y = (M_{p,p'}y)^* = |y|^{\frac{p}{p'}}\alpha^*$ for all $y = \alpha|y| \in S(L_p(\mathcal{M}))$. Let now $x \in S(L_{p'}(\mathcal{M}))$, $x \geq 0$. Then, by the equality (1), we have

$$\begin{aligned}
(\pi^p)^*(x) &= *(\pi^p(g)(M_{p',p}(x)^*)\\
&= *(\pi^p(g)x^{\frac{p'}{p}})\\
&= *(u_g B_g J_g(x^{\frac{p'}{p}}))\\
&= M_{p,p'}(u_g B_g J_g(x^{\frac{p'}{p}}))^*\\
&= B_g^{\frac{p}{p'}} J_g(x)u_g^*\\
&= u_g^* B_g^{\frac{p}{p'}} u_g J_g(x)u_g^*.
\end{aligned}$$

□

2.5 Property $(T_{L_p(\mathcal{M})})$ for non-commutative $L_p(\mathcal{M})$-spaces

In this section, we generalize item 1 in Theorem 2.3.1 to all non-commutative L_p-spaces, that is, we show that property (T) implies property $(T_{L_p(\mathcal{M})})$ for $1 < p < \infty$ and for any von Neumann algebra $\mathcal{M}$. We then show that the converse is true for some von Neumann algebras $\mathcal{M}$ whose group of isometries $O(L_p(\mathcal{M}))$ is sufficiently large.

2.5.1 Property (T) implies property $(T_{L_p(\mathcal{M})})$

Let $\mathcal{M}$ be a von Neumann algebra. Let $1 < p < \infty$ and $p \neq 2$. Let G be a topological group. Let π^p be an orthogonal representation of G on a non-commutative $L_p(\mathcal{M})$. The space of $\pi^p(G)$-invariant vectors in $L_p(\mathcal{M})$ is

$$L_p(\mathcal{M})^{\pi^p(G)} = \{x \in L_p(\mathcal{M}) \mid \pi^p(g)x = x \text{ for all } g \in G\ \}$$

Let p' be the conjugate exponent of p, and let $(\pi^p)^*$ be the contragradient representation of π^p on $L_p(\mathcal{M})^*$. Recall that $(\pi^p)^*$ is equivalent to a representation (also denoted by $(\pi^p)^*$) on $L_{p'}(\mathcal{M})$, and since $L_p(\mathcal{M})$ is a ucus space, the space

$$L_p(\mathcal{M})'(\pi^p) = \{v \in L_p(\mathcal{M}) \mid \mathrm{Tr}(vc) = 0 \text{ for all } c \in L_{p'}(\mathcal{M})^{(\pi^p)^*(G)}\}$$

is a topological complement for $L_p(\mathcal{M})^{\pi^p(G)}$.

Let now H be a closed *normal* subgroup of G. Then $L_p(\mathcal{M})^{\pi^p(H)}$ as well as its complement $L_p(\mathcal{M})'(\pi^p_{/H})$ are $\pi^p(G)$-invariant.

The following proposition will play a crucial role in our study of property $(T_{L_p(\mathcal{M})})$. It will be also used in Chapter 4 for our study of property $(H_{L_p(\mathcal{M})})$.

Proposition 2.5.1. *Let G be a topological group, and H a closed normal subgroup of G. Let $\mathcal{M}$ be a von Neumann algebra. Let $p \neq 2$, and $1 \leq q < \infty$. Let π^p be a representation of G on $L_p(\mathcal{M})$. Suppose that π^p almost has invariant vectors in $L_p(\mathcal{M})$ (resp. in $L_p(\mathcal{M})'(\pi^p_{/H})$) for G. Then its conjugate by the Mazur map π^q defined by*

$$\pi^q(g) = M_{p,q} \circ \pi^p(g) \circ M_{q,p} \text{ for all } g \in G,$$

almost has invariant vectors in $L_q(\mathcal{M})$ (resp. in $L_q(\mathcal{M})'(\pi^q_{/H})$) for G.

The proof of Proposition 2.5.1 depends on an essential way on the following Lemma 2.5.2.

Lemma 2.5.2. *With the notations as in Proposition 2.5.1, let $v \in S(L_p(\mathcal{M})'(\pi^p_{/H}))$. Then*

$$d(v, L_p(\mathcal{M})^{\pi^p(H)}) \geq \frac{1}{2}.$$

Proof. Assume, by contradiction, that there exists $b \in L_p(\mathcal{M})^{\pi^p(H)}$ such that

$$||v - b||_p < \frac{1}{2}.$$

Then $\frac{1}{2} \leq ||b||_p \leq \frac{3}{2}$.
Set $c = \dfrac{b}{||b||_p}$. Then $||c||_p = 1$ and $||b - c||_p \leq \frac{1}{2}$.
We claim that $M_{p,p'}(c)^* \in L_{p'}(\mathcal{M})^{(\pi^p)^*(H)}$. Indeed, $M_{p,p'}(c)^* = *c$ by Proposition 1.3.4. Moreover, $||c||_p = 1$, and by Proposition 2.4.6, for all $g \in G$, we have

$$\begin{aligned}(\pi^p)^*(g)(*c) &= (* \circ \pi^p(g) \circ *^{-1}) * c \\ &= *\pi^p(g)c \\ &= *c\end{aligned}$$

since $c \in L_p(\mathcal{M})^{\pi^p(G)}$. Hence

$$\mathrm{Tr}((c - v)M_{p,p'}(c)^*) = \mathrm{Tr}(cM_{p,p'}(c)^*) = ||c||_p^p = 1.$$

On the other hand, using Hölder's inequality, we have

$$\begin{aligned}1 &= \mathrm{Tr}((c - v)M_{p,p'}(c)^*) \\ &\leq ||c - v||_p ||M_{p,p'}(c)^*||_{p'} \\ &= ||c - v||_p ||c||_p^{\frac{p}{p'}} \\ &= ||c - v||_p.\end{aligned}$$

This implies that $||c - v||_p \geq 1$ and we have

$$||v - b||_p \geq ||v - c||_p - ||c - b||_p \geq \frac{1}{2}.$$

This is a contradiction. $\square$

We are now in position to give the proof of Proposition 2.5.1.

Proof of Proposition 2.5.1. Assume that π^p almost has invariant vectors in $L_p(\mathcal{M})'(\pi^p_{/H})$ (the proof is identical in the case of almost invariant vectors in $L_p(\mathcal{M})$). Let Q be a compact subset in G, and take $\epsilon > 0$. We have to show that π^q has a (Q, ϵ)-invariant vector in $L_q(\mathcal{M})'(\pi^q_{/H})$.

We can find, for every n, a unit vector $v_n \in L_p(\mathcal{M})'(\pi^p_{/H})$ such that

$$\sup_{g \in Q} ||\pi^p(g)v_n - v_n||_p < \frac{1}{n}.$$

Let w_n be the projection of $M_{p,q}(v_n)$ on the canonical complement $L_q(\mathcal{M})'(\pi^q_{/H})$ of $L_q(\mathcal{M})^{\pi^q(H)}$. We claim that w_n is (Q, ϵ)-invariant for π^q for n sufficiently large.

We first show that there exists $\delta' > 0$ such that

$$d(M_{p,q}(v_n), L_q(\mathcal{M})^{\pi^q(H)}) \geq \delta' \text{ for every } n.$$

Indeed, otherwise for some n_0, there exists a sequence $(a_k)_k$ in $\in L_q(\mathcal{M})^{\pi^q(H)}$ such that

$$||M_{p,q}(v_{n_0}) - a_k||_q \xrightarrow[k\to\infty]{} 0.$$

By Proposition 1.3.3, we have

$$||M_{p,q}(v_{n_0})||_q = ||v_{n_0}||_p^{\frac{p}{q}} = 1.$$

Since $||a_k||_q \xrightarrow[k\to\infty]{} ||M_{p,q}(v_{n_0})||_q = 1$, we can assume that $||a_k||_q = 1$. Recall that

$$M_{q,p}(L_q(\mathcal{M})^{\pi^q(H)}) = L_p(\mathcal{M})^{\pi^p(H)}.$$

Hence, $M_{q,p}(a_k)$ belongs to $L_p(\mathcal{M})^{\pi^p(H)}$ for every k. Moreover

$$||v_{n_0} - M_{q,p}(a_k)||_p \xrightarrow[k\to\infty]{} 0$$

by the uniform continuity of $M_{q,p}$ on the unit sphere (see Proposition 1.3.8). This is a contradiction to lemma 2.5.2.
In particular, we have

$$||w_n||_q = d(M_{p,q}(v_n), L_q(\mathcal{M})^{\pi^q(H)}) \geq \delta', \text{ for all } n.$$

Since H is normal in G, the projection on the complement $L_q(\mathcal{M})'(\pi^q_{/H})$ commutes with $\pi^q(g)$ for every $g \in G$. Hence, for $g \in Q$, we have

$$\begin{aligned}||\pi^q(g)w_n - w_n||_q &\leq ||\pi^q(g)M_{p,q}(v_n) - M_{p,q}(v_n)||_q \\ &= ||M_{p,q}(\pi^p(g)v_n) - M_{p,q}(v_n)||_q.\end{aligned}$$

Recall that $||v_n||_p^{\frac{p}{2}} = 1$ and that

$$\sup_{g\in Q} ||\pi^p(g)v_n - v_n||_p < \frac{1}{n} \text{ for all } n.$$

Hence, by the uniform continuity of $M_{p,q}$ on $S(L_q(\mathcal{M}))$, there exists an integer N (depending only on (Q, ϵ)) such that

$$\sup_{g\in Q} ||\pi^q(g)w_n - w_n||_q < \epsilon\delta' \text{ for } n \geq N.$$

Since $||w_n||_q \geq \delta'$, it follows that

$$\sup_{g\in Q} ||\pi^q(g)w_n - w_n||_q < \epsilon||w_n||_q \text{ for } n \geq N.$$

This shows that w_n is (Q, ϵ)-invariant for $\pi^q_{/H}$ when $n \geq N$. This finishes the proof of Proposition 2.5.1. □

Here is one of the main results of this thesis.

Theorem 2.5.3. *Let G be a topological group and H a closed normal subgroup of G. Assume that the pair (G, H) has property (T). Let $\mathcal{M}$ be a von Neumann algebra, and $1 < p < \infty$. Then the pair (G, H) has property $(T_{L_p(\mathcal{M})})$.*

Proof. We follow the strategy of the proof of Theorem A in [4]. Let $p \in]1, \infty[$. Let H be a closed normal subgroup of G such that the pair (G, H) has property (T). Assume by contradiction that the pair (G, H) does not have property $(T_{L_p(\mathcal{M})})$. Then there exists an orthogonal representation $\pi^p : G \to O(L_p(\mathcal{M}))$ almost having invariant vectors in $L_p(\mathcal{M})'(\pi^p, H)$, the complement of $L_p(\mathcal{M})^{\pi^p(H)}$.

Now define $\pi^2 = M_{p,2} \circ \pi^p \circ M_{2,p}$. By Corollary 2.4.2, π^2 is an orthogonal representation of G on $L_2(\mathcal{M})$. Then by Proposition 2.5.1, π^2 almost has invariant vectors in $L_2(\mathcal{M})'(\pi^2_{/H})$, which is the orthogonal subspace of $L_2(\mathcal{M})^{\pi^p(H)}$. This contradicts the fact that the pair (G, H) has property (T).

□

The following stronger version of property (T_B) for some pairs of groups was used in [4] in order to establish some rigidity results for higher rank groups. Let H be a closed normal subgroup of G and let L be a closed group of G. Assume that $G = L \ltimes H$. The following strong relative property (T_B) was considered in [4] :

Definition 2.5.4. The pair $(L \ltimes H, H)$ has the strong property (T_B) if, for every orthogonal representation $\rho : L \ltimes H \to O(B)$, the quotient representation $\rho' : L \to O(B/B^{\rho(H)})$ does not almost have $\rho'(L)$-invariant vectors.

A straightforward modification of our proof of Theorem 2.5.3 shows that we also have the following result :

Theorem 2.5.5. *Let $\mathcal{M}$ be a von Neumann algebra. Let $(L \ltimes H, H)$ be a pair with the strong relative property (T). Then $(L \ltimes H, H)$ has the strong relative property $(T_{L_p(\mathcal{M})})$ for $1 < p < \infty$.*

2.5.2 Property $(T_{L_p(\mathcal{M})})$ implies property (T) for some algebras $\mathcal{M}$

The authors of [4] showed the following theorem for the classical L_p-spaces $L_p(X, \mu)$ associated to a standard Borel space (X, μ) equipped with a *non-atomic* measure μ. We will see in the next section that the result is no longer true for the space l_p, that is, there exist groups with property (T_{l_p}) for $p \neq 2$ and without Kazhdan's property (T).

Theorem 2.5.6. *Let $1 \leq p < \infty$. Let G be a second countable locally compact group. Assume that G has property $(T_{L_p(\mathcal{M})})$ for one of the following von Neumann algebras :*
- $\mathcal{M} = L^\infty(X, \mu)$ (with μ non-atomic),
- $\mathcal{M} = R$ the hyperfinite II_1 factor,
- $\mathcal{M} = \mathcal{B}(\mathcal{H})$.
Then G has property (T).

The result for $\mathcal{M} = L^\infty(X, \mu)$ was given in [4].

Proof. Assume that G does not have property (T). We are going to show that G does not have property $(T_{L_p(\mathcal{M})})$.

- *Case $\mathcal{M} = R$:* Assume that G does not have property (T). Let τ be the normalized trace on $L_2(R)$. The Hilbert space $L_2(R)$ is defined as the completion of R for the norm
$$||x||_2 = \tau(x^*x)^{\frac{1}{2}},\ x \in R.$$
As G does not have property (T), a construction of Araki and Choda [2] gives an action α of G by automorphisms of R, which has a non-trivial asymptotically invariant sequence $(e_n)_n$ of projections (see Definition 5.1.3). The action α induces a unitary representation $\pi^2 : G \to \mathcal{U}(L_2(R))$ given on the dense subspace R of $L_2(R)$ by
$$\pi^2(g)x = \alpha(g)(x) \text{ for all } g \in G, x \in R.$$

Set, for $n \geq 1$,

$$e'_n = \frac{e_n - \tau(e_n)1}{||e_n - \tau(e_n)1||_2}.$$

Then, for $n \geq 1$, e'_n belongs to the orthogonal complement $L_2(R)'$ of the space of $\pi^2(G)$-invariant vectors in $L_2(R)$, and $(e_n)_n$ is a sequence of almost invariant vectors for π^2.
The representation π^p of G on $L_p(R)$ associated to π^2, which is given on the dense subspace R of $L_p(R)$ by the same formula as π^2, almost has invariant vectors in $L_p(R)'$ by Proposition 2.5.1.

- *Case* $\mathcal{M} = \mathcal{B}(\mathcal{H})$: Since G does not have property (T), there exists a unitary representation $\rho : G \to \mathcal{U}(\mathcal{H})$ with almost invariant vectors and without non-zero finite-dimensional subrepresentation (see Remark 2.12.11 in [8]).
Define $\pi^p : G \to O(C_p)$ by

$$\pi^p(g)x = \rho(g)x\rho(g^{-1}) \text{ for all } g \in G, x \in C_p.$$

The corresponding conjugate representation π^2 on C_2 is given by the same formula as π^p. Moreover, π^p does not have non-zero invariant vector in C_p since ρ has no non-zero finite-dimensional subrepresentations.
Now we show that π^p almost has invariant vectors. In view of Proposition 2.5.1, it suffices to prove that π^2 almost has invariant vectors. For $\xi \in \mathcal{H}$ with $||\xi|| = 1$, denote by $P_\xi \in C_2$ the orthogonal projection on the subspace $\mathbb{C}\xi$. Observe that $||P_\xi||_2 = 1$ and for ξ, η two unit vectors in $\mathcal{H}$, we have

$$||P_\xi - P_\eta||_2 \leq 2||\xi - \eta||_2.$$

Let $(\xi_n)_n$ be a sequence of almost invariant vectors for π. Set $v_n = P_{\xi_n}$ for all n. Then, for every $g \in G$, $\pi^2_g(v_n) = P_{\rho(g)\xi_n}$. The previous inequality therefore shows that $(v_n)_n$ is a sequence of almost invariant vectors for π^2.
Hence π^p almost has invariant vectors but has no non-zero invariant vector, and G does not have property (T_{C_p}). □

2.6 Property (T_{l_p})

In this section, we will study the property (T_{l_p}), that is the property $(T_{L_p(\mathcal{M})})$ for $\mathcal{M} = l^\infty$. As we will show, there exist examples of groups with property (T_{l_p}) and without property (T). Our study of property (T_{l_p}) is made prossible by the simple structure of the isometry group $O(l_p)$ of l_p. We start by establishing some general facts about group representations on l_p. Recall that l_p is the space of p-summable *complex* valued sequences.

2.6.1 Group representations on l_p

We begin with some preliminary remarks on permutation representations of topological groups twisted by a cocycle with values in $\mathbb{S}^1$. Let G be a topological group. Let X be a discrete space equipped with a G-action. We assume that this action is continuous, or equivalently, that the stabilizers of points in X are open subgroups of G. Let $c : G \times X \to \mathbb{S}^1$ be a continuous cocycle with values in $\mathbb{S}^1$; thus, c satisfies the cocycle relation

$$c(g_1g_2, x) = c(g_1, g_2x)c(g_2, x) \text{ for all } g_1, g_2 \in G,\ x \in X. \quad (*)$$

We associate to the G-action and the cocycle c the permutation representation twisted by c, which is the continuous representation of G on $l_2(X)$, denoted by λ_X^c and defined by the formula

$$\lambda_X^c(g)f(x) = c(g^{-1}, x)f(g^{-1}x) \text{ for all } g \in G,\ f \in l_2(X),\ x \in X.$$

The following lemma is a very special case of Mackey's imprimitivity (see theorem 3.10 in [55]).

Lemma 2.6.1. *Assume that G acts transitively on X. Let $x_0 \in X$ and denote by H the stabilizer of x_0 in G. Let $\chi : H \to \mathbb{S}^1$ be defined by $\chi(h) = c(h, x_0)$ for all $h \in H$. Then χ is a unitary character of H and λ_X^c is unitarily equivalent to the* monomial *representation* $\mathrm{Ind}_H^G\chi$.

Proof. The fact that χ is a homomorphism follows immediately from the cocycle relation $(*)$.
Fix a set $T \subset G$ of representatives for the left cosets of H. The space $l_2(X)$ is the direct sum $\oplus_{x\in X}V_x$, where V_x is the one-dimensional space $\mathbb{C}\delta_x$. The restriction of λ_X^c to H leaves V_{x_0} invariant, with the corresponding H-action given by the character χ. Moreover, we have $\lambda_X^c(t)V_{x_0} = V_{tx_0}$ for all $t \in T$. This shows that λ_X^c is equivalent to $\mathrm{Ind}_H^G\chi$, by the defining property of induced representations. ☐

Remark 2.6.2. Conversely, every monomial representation of G associated to an open subgroup H and a character χ on H can be realized as a representation of the form λ_X^c for the action of G on $X = G/H$ and a continuous cocycle $c : G \times G/H \to \mathbb{S}^1$. We recall the construction of c. Choose a section $s : G/H \to G$ for the canonical projection $G \to G/H$, with $s(H) = e$. Define a cocycle $\alpha : G \times X \to H$ with values in H, given by

$$\alpha(g, x) = s(gx)^{-1}gs(x) \text{ for all } g \in G,\ x \in X.$$

Then $c : G \times G/H \to \mathbb{S}^1$ is defined by $c(g, x) = \chi(\alpha(g, x))$.

The following corollary is an immediate consequence of Lemma 2.6.1 and the previous remark.

Corollary 2.6.3. *Let G be a topological group.*
(i) Let X be a discrete space equipped with a continuous G-action and let $c : G \times X \to \mathbb{S}^1$ be a continuous cocycle. The associated representation λ_X^c of G on $l_2(X)$ is equivalent to a direct sum of monomial representations associated to open subgroups of G.
(ii) Let $\pi = \oplus_{i\in I}\mathrm{Ind}_{H_i}^G \chi_i$ be a direct sum of monomial representations associated to open subgroups H_i of G. Set $X = \bigsqcup_{i\in I} G/H_i$, the disjoint sum of the G/H_i's, with the obvious G-action. Then π is unitarily equivalent to the representation λ_X^c of G on $l_2(X)$ for a cocycle $c : G \times X \to \mathbb{S}^1$.

Now let $1 \leq p < \infty$, $p \neq 2$. Let G be a topological group and $\pi : G \to O(l_p)$ an orthogonal representation of G on $l_p = l_p(X)$, where X is an infinite countable set. Recall from Banach's result (see Theorem 1.5.1) that there exists mappings

$$\varphi : G \to \mathrm{Sym}(X) \text{ and } c : G \times X \to \mathbb{S}^1$$

such that

$$\pi(g)f(x) = c(g^{-1}, x)f(\varphi(g^{-1})(x)) \text{ for all } g \in g,\ f \in l_p(X),\ x \in X.$$

Since π is a group homomorphism, one checks that φ is also a group homomorphism; so φ defines an action of G on X. Moreover, $c : G \times X \to \mathbb{S}^1$ satisfies the cocycle relation $(*)$.
Observe that $\{\delta_x \mid x \in X\}$ is a discrete subset of $l_p(X)$, equipped with the norm topology. Since π is continuous, it follows that the action of G on the discrete space X is continuous. Similarly, one checks that $c : G \times X \to \mathbb{S}^1$ is continuous.

In summary, to a continuous orthogonal representation π^p of G on $l_p(X)$ (for $p \neq 2$ and $1 \leq p < \infty$), is associated an action of G on X with open point stabilizers and a continuous cocycle $c : G \times X \to \mathbb{S}^1$. (It is clear that, conversely, such an action of G on X and a continuous cocycle $c : G \times X \to \mathbb{S}^1$ define an orthogonal representation of G on $l_p(X)$). Moreover, the conjugate representation by the Mazur map $\pi^2 = M_{p,2} \circ \pi^p \circ M_{2,p}$ is the permutation representation λ_X^c on $l_2(X)$ twisted by c.

2.6.2 A characterization of property (T_{l_p})

In the sequel, the sets of representations which are considered are sets of classes of unitary representations for the unitarily equivalence. We recall the notation π' for the restriction of a unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ to the orthogonal complement of the space $\mathcal{H}^{\pi(G)}$ of $\pi(G)$-invariant vectors.

Definition 2.6.4. Let G be a topological group and $\mathcal{R}$ a set of unitary representations. We say that 1_G is isolated in $\mathcal{R}$ if 1_G is not weakly contained in $\oplus_{\pi\in\mathcal{R}}\pi'$.

Notice that from Remark 2.1.8, property (T) can be rephrased as follows : G has property (T) if and only if 1_G is isolated in the set of (equivalence classes of) unitary representations of G.

Recall that a unitary representation σ of a topological group G is *monomial* if σ is unitarily equivalent to the induced representation $\mathrm{Ind}_H^G \chi$, where H is a closed subgroup of G, and $\chi : H \to \mathbb{S}^1$ a unitary character of H.

Examples of monomial representations are the quasi-regular representations $\lambda_{G/H}$ of G on $L_2(G/H)$ since $\lambda_{G/H} = \mathrm{Ind}_H^G 1_H$. Let $\mathcal{R}_{mon}$ be the set of monomial representations $\pi = \mathrm{Ind}_H^G \chi$, associated to an open subgroup H of G.

Theorem 2.6.5. *Let G be a second countable locally compact group. The following properties are equivalent.*
(i) G has property (T_{l_p}) for some $1 < p < \infty$ and $p \neq 2$.
(ii) The trivial representation of G is isolated in the set $\mathcal{R}_{mon}$.

Proof. $(ii) \Rightarrow (i)$:
Assume that G does not have property (T_{l_p}). Then there exists an orthogonal representation $\pi^p : G \to O(l_p)$ such that $(\pi^p)'$ almost has invariant vectors. By Proposition 2.5.1, the representation π^2 is such that $(\pi^2)'$ almost has invariant vectors. On the other hand, by Corollary 2.6.3, π^2 is unitarily equivalent to a direct sum of monomial representations associated to open subgroups. Then 1_G is not isolated in $\mathcal{R}_{mon}$.

$(i) \Rightarrow (ii)$:
Assume that 1_G is not isolated in $\mathcal{R}_{mon}$. Thus there exist open subgroups $(H_i)_{i \in I}$ and unitary characters $\chi_i : H_i \to \mathbb{S}^1$ with the following property : the representation $\oplus_{i \in I} \mathrm{Ind}_{H_i}^G$ of G on $\mathcal{H} = \oplus_{i \in I} l^2(G/H_i)$ almost has invariant vectors in $\mathcal{H}(\pi^2)'$. For $f \in \mathcal{H}$, the projection of f on $l^2(G/H_i)$ is non-zero for at most countably many i. It follows that we can assume that I is infinite countable (if I happens to be finite, we replace I by $I \times \mathbb{N}$ and set $H_{(i,n)} = H_i$).
Let $X = \bigsqcup_{i \in I} G/H_i$. By Corollary 2.6.3, $\oplus_{i \in I} \mathrm{Ind}_{H_i}^G$ is unitarily equivalent to the permutation representation $\pi^2 = \lambda_X^c$ of G on $l_2(X)$ associated with a cocycle $c : G \times X \to \mathbb{S}^1$.
By Proposition 2.5.1, the conjugate representation π^p of G on $l_p(X)$ almost has invariant vectors in l_p'. Therefore, G does not have property (T_{l_p}). ∎

When G is connected, the only open subgroup of G is G itself and $\mathcal{R}_{mon}$ therefore coincides with the group of unitary characters of G, that is, with the Pontrjagin dual of the abelianization $G/\overline{[G,G]}$. The following corollary is an immediate consequence of Theorem 2.6.5.

Corollary 2.6.6. *Let $1 \leq p < \infty$, $p \neq 2$. A connected second countable group G has property (T_{l_p}) if and only if its abelianization $G/\overline{[G,G]}$ is compact.*

2.6.3 Consequences of property (T_{l_p})

Groups with property (T_{l_p}) share some important properties with Kazhdan groups.

Theorem 2.6.7. *Let G be a second countable locally compact group. Assume that G has property (T_{l_p}) for some $1 < p < \infty$. The following statements hold :*
(i) G is compactly generated.
(ii) The abelianized group $G/\overline{[G,G]}$ is compact.
(iii) Every subgroup of finite index in G and every topological group containing G as a finite index subgroup has property (T_{l_p}). (In other words, property (T_{l_p}) only depends on the commensurability class of G.)
(iv) If G is amenable and totally disconnected, then G is compact.

Proof. (i) The proof is similar to Kazhdan's one in [47]. Let $\mathcal{C}$ be the family of open and compactly generated subgroups of G. Since G is locally compact, 1_G is weakly contained in the family of quasi-regular representations $(\lambda_{G/H})_{H\in\mathcal{C}}$. Hence, by Theorem 2.6.5, there exists $H \in \mathcal{C}$ such that G has a non-zero invariant vector in $l_2(G/H)$. This implies that H has finite index and therefore that G is compacly generated.

(ii) Assume, by contradiction, that $G/\overline{[G,G]}$ is not compact. Then there exists a sequence $(\chi_n)_n$ of unitary characters of G such that $\chi_n \neq 1$ and $\chi_n \to 1$ uniformly on compact subsets of G. This contradicts Theorem 2.6.5.

(iii) • Let L be a finite index subgroup of G. We want to show that L has property (T_{l_p}).
Let $\mathcal{L}$ be the set of pairs (H,χ) consisting of an open subgroup H of L and a unitary character χ of H. For $(H,\chi)\in\mathcal{L}$, denote by $\lambda_{(H,\chi)}$ the induced representation $\mathrm{Ind}_H^L\chi$. Set

$$\rho = \oplus_{(H,\chi)\in\mathcal{L}}\lambda_{(H,\chi)}.$$

Assume, by contradiction, that L does not have property (T_{l_p}) for $p \neq 2$. Then, by Theorem 2.6.5, the trivial representation 1_L of L is weakly contained in ρ'. It follows, by continuity of induction, that $\lambda_{G/L}$ is weakly contained in $\mathrm{Ind}_L^G\rho'$, which is a subrepresentation of

$$\oplus_{(H,\chi)\in\mathcal{L}}\mathrm{Ind}_L^G(\lambda_{(H,\chi)}) \simeq \oplus_{(H,\chi)\in\mathcal{L}}\mathrm{Ind}_H^G\chi.$$

On the other hand, 1_G is contained in $\lambda_{G/L}$, since G/L is finite. Therefore, 1_G is weakly contained in $\mathrm{Ind}_L^G\rho'$. However, $\mathrm{Ind}_L^G\rho'$ has no non-zero invariant vector, since ρ' has no non-zero L-invariant vector (see Theorem E.3.1 in [8]). This is a contradiction to Theorem 2.6.5. We conclude that L has property (T_{l_p}) for $p \neq 2$.

• Let $\tilde{G}$ be a group containing G as a subgroup of finite index. We want to show that $\tilde{G}$ has property (T_{l_p}).

Since G contains a normal subgroup in $\tilde{G}$ of finite index and since this subgroup has property (T_{l_p}) for $p \neq 2$, by the previous proof, we can assume that G is a normal subgroup of $\tilde{G}$.
Assume, by contradiction, that $\tilde{G}$ does not have property (T_{l_p}) for $p \neq 2$. Then there exists an orthogonal representation $\pi^p : \tilde{G} \to O(l_p)$ which has a sequence of almost invariant vectors in the complement $l'_p(\pi^p)$ of $\pi^p(\tilde{G})$-invariant vectors in l_p. Denote by $\pi^2 : G \to O(l_2)$ the conjugate of π^p by the Mazur map. By Proposition 2.5.1, π^2 has also a sequence $(\xi_n)_n$ of almost invariant vectors in $l'_2 = l'_2(\pi^2)$.

Let $P : l'_2 \to (l'_2)^{\pi^2(G)}$ be the orthogonal projection on the subspace of $\pi^2(G)$-invariant vectors in l'_2. Observe that $(l'_2)^{\pi^2(G)}$ is invariant under $\pi^2(\tilde{G})$, since G is normal in $\tilde{G}$. For every $n \in \mathbb{N}$, the vector $\xi_n - P\xi_n$ belongs to the orthogonal complement of $(l_2{}')^{\pi^2(G)}$ in l'_2. Hence, $\xi_n - P\xi_n$ belongs to the orthogonal complement in l_2 of the space $l_2^{\pi^2(G)}$, since $l_2^{\pi^2(G)} = (l_2{}')^{\pi^2(G)} \oplus l_2^{\pi^2(\tilde{G})}$.
Moreover, we have

$$\lim_n ||\pi^2(g)(\xi_n - P\xi_n) - (\xi_n - P\xi_n)|| = 0 \text{ for all } g \in G.$$

It follows that $\inf_n ||\xi_n - P\xi_n|| = 0$; indeed, otherwise, $\frac{1}{||\xi_n - P\xi_n||}(\xi_n - P\xi_n)$ would be a sequence of almost invariant vectors in the orthogonal complement of $l_2^{\pi^2(G)}$ in l_2 and, by Proposition 2.5.1, this would contradict the fact that G has property (T_{l_p}). Hence, upon passing to a subsequence, we can assume that $\lim_n ||\xi_n - P\xi_n|| = 0$.

Since $P\xi_n$ is $\pi^2(G)$-invariant, we can define the following sequence $(\eta_n)_n$ of vectors in l_2 :

$$\eta_n = \frac{1}{|\tilde{G}/G|} \sum_{t \in \tilde{G}/G} \pi^2(t) P\xi_n.$$

It is clear that η_n is $\pi^2(\tilde{G})$-invariant. Moreover, we have

$$\begin{aligned}
||\eta_n - \xi_n|| &\leq \frac{1}{|\tilde{G}/G|} \sum_{t \in \tilde{G}/G} ||\pi^2(t) P\xi_n - \xi_n|| \\
&\leq \frac{1}{|\tilde{G}/G|} \sum_{t \in \tilde{G}/G} ||\pi^2(t) P\xi_n - \pi^2(t)\xi_n|| + \frac{1}{|\tilde{G}/G|} \sum_{t \in \tilde{G}/G} ||\pi^2(t)\xi_n - \xi_n|| \\
&= \frac{1}{|\tilde{G}/G|} \sum_{t \in \tilde{G}/G} ||P\xi_n - \xi_n|| + \frac{1}{|\tilde{G}/G|} \sum_{t \in \tilde{G}/G} ||\pi^2(t)\xi_n - \xi_n||.
\end{aligned}$$

It follows that $\lim_n ||\eta_n - \xi_n|| = 0$. Hence, $\eta_n \neq 0$ for sufficiently large n, since $||\xi_n||$=1.
For every $t \in \tilde{G}$, the vector $\pi^2(t)(P\xi_n)$ belongs to $(l'_2)^{\pi^2(G)}$, since $(l'_2)^{\pi^2(G)}$ is invariant under $\pi^2(\tilde{G})$. It follows that $\eta_n \in (l'_2)^{\pi^2(G)}$ and, in particular, $\eta_n \in l'_2$.

This is a contradiction, as there are no non-zero $\pi^2(\tilde{G})$-invariant vector in l_2'.

(iv) Since G is totally disconnected, we can find a compact open subgroup K of G, by van Dantzig's theorem (see Theorem 7.7 in [41]). The amenability of G implies the amenability of its action on G/K : 1_G is weakly contained in $\lambda_{G/K}$ (see Theorem p.28 in [28]). As G has property (T_{l_p}), it follows from Theorem 2.6.5 that G has a non-zero invariant vector in $l_2(G/K)$. Hence, K has finite index in G and G is compact.

□

Remark 2.6.8. 1. It follows from the previous theorem that, for instance, (abelian or non-abelian) free groups as well as the groups $SL_n(\mathbb{Q})$ do not have property (T_{l_p}).

2. Property (T_{l_p}) for $p \neq 2$ is not inherited by lattices, even in the totally disconnected case. Indeed, $SL_2(\mathbb{Q}_l)$ has property (T_{l_p}) for $p \neq 2$ (see example 2.6.12 below), whereas torsion-free discrete subgroups in $SL_2(\mathbb{Q}_l)$ are free groups (see Chap. II, théorème 5 in [73]).

2.6.4 Property (T_{l_p}) for totally disconnected groups

The next result shows that, when G is totally disconnected, isolation of 1_G in the set of quasi-regular representations associated to open subgroups suffices to characterize property (T_{l_p}). Let $\mathcal{R}_{quasi-reg}$ be the set of quasi-regular representations $(\lambda_{G/H}, l_2(G/H))$, associated to an open subgroup H of G.

Theorem 2.6.9. *Let G be a totally disconnected, second countable locally compact group. The following properties are equivalent.*
(i) G has property (T_{l_p}) for some $1 < p < \infty$ and $p \neq 2$.
(ii) The trivial representation of G is isolated in $\mathcal{R}_{quasi-reg}$.

The proof of Theorem 2.6.9 will be an easy consequence of the following lemma.

Lemma 2.6.10. *Let G be a locally compact totally disconnected group, H an open subgroup of G, and χ a continuous unitary character of H. There exists an open subgroup L of G contained in H such that the monomial representation $\mathrm{Ind}_H^G\chi$ is weakly contained in the quasi-regular representation $\lambda_{G/H}$.*

Proof. Since G, and hence H, is totally disconnected, every neighborhood of the group unit in H contains a compact open subgroup, by van Dantzig's theorem. By continuity of χ, there exists a compact open subgroup K of H such that

$$|\chi(k) - 1| < 1 \text{ for all } k \in K.$$

For every $k \in K$, we then have $|\chi(k)^n - 1| < 1$ for all $n \in \mathbb{N}$ and hence $\chi(k) = 1$. Therefore χ is trivial on K.
Let L be the subgroup of G generated by $K \cup [H, H]$. Then L is a normal and open subgroup of H and χ is trivial on L. So, χ factorizes to a unitary character $\overline{\chi}$ of the abelian quotient group $\overline{H} = H/L$.
Since $\overline{H}$ is amenable, $\overline{\chi}$ is weakly contained in the regular representation $\lambda_{\overline{H}}$ of $\overline{H}$, by the Hulanicki-Reiter theorem (see Theorem G.3.2 in [8]). Hence, χ is weakly contained in the quasi-regular representation $\lambda_{H/L}$, since $\lambda_{H/L} = \lambda_{\overline{H}} \circ p$, where $p : H \to \overline{H}$ is the quotient homomorphism. By continuity of induction (see Theorem F.3.5 in [8]), it follows that $\mathrm{Ind}_H^G \chi$ is weakly contained in

$$\mathrm{Ind}_H^G \lambda_{G/H} \simeq \lambda_{G/L}.$$

□

Proof of Theorem 2.6.9. By Theorem 2.6.5, it suffices to show that if 1_G is isolated in $\mathcal{R}_{quasi-reg}$, then 1_G is isolated in $\mathcal{R}_{mon}$ for G a second countable locally compact and totally disonnected group.
Assume that for such a group G, 1_G is not isolated in $\mathcal{R}_{mon}$. Then there exists a family $(H_i, \chi_i)_{i \in I}$ of open subgroups H_i with unitary characters χ_i such that 1_G is weakly contained in the restriction π' of

$$\pi = \oplus_{i \in I} \mathrm{Ind}_{H_i}^G \chi_i$$

to the orthogonal complement of the $\pi(G)$-invariant vectors. On the other hand, by Lemma 2.6.10, there exists a family $(L_i)_{i \in I}$ of open subgroups L_i of H_i such that π is weakly contained in

$$\rho := \oplus_{i \in I} \lambda_{G/L_i}.$$

This implies that π' is weakly contained in ρ'. Hence, 1_G is not isolated in $\mathcal{R}_{quasi-reg}$. □

The next result will provide us with a class of examples of totally disconnected non discrete groups with property (T_{l_p}) for $p \neq 2$ and without property (T).
A locally compact group has the Howe-Moore property if, for every unitary representation π of G without non-zero invariant vectors, the matrix coefficients of π are in $C_0(G)$. For an extensive study of groups with this property, see [15].

Corollary 2.6.11. *Let G be a totally disconnected, second countable locally compact group with the Howe-Moore property. Assume that G is non-amenable. Then G has property (T_{l_p}) for every $1 < p < \infty$ and $p \neq 2$.*

Proof. Since G has the Howe-Moore property, every proper open subgroup of G is compact (see Proposition 3.2 in [15]). It follows that, for every proper open subgroup H, the space $l_2(G/H)$ can be identified with the G-invariant subspace

of $L_2(G)$ of functions on G which are right H-invariant. As a consequence, we see that $\lambda_{G/H}$ is a subrepresentation of the regular representation λ_G. Denoting by $\mathcal{L}$ the set of proper open subgroups of G, this implies that $\oplus_{H\in\mathcal{L}}\lambda_{G/H}$ is weakly contained in the regular representation λ_G.
On the other hand, since G is not amenable, 1_G is not weakly contained in λ_G, by the Hulanicki-Reiter theorem. It follows that 1_G is not weakly contained in $\oplus_{H\in\mathcal{L}}\lambda_{G/H}$ and Theorem 2.6.9 shows that G has property (T_{l_p}). $\square$

Examples 2.6.12.

1. Let k be a non-archimedean local field, $\mathbb{G}$ a simple linear algebraic group over k and $G = \mathbb{G}(k)$ the group of k-points in $\mathbb{G}$ (an example is $G = SL_n(\mathbb{Q}_l)$ for $n \geq 2$, where $\mathbb{Q}_l$ is the field of l-adic numbers for a prime number l). Then G has the Howe-Moore property (see Theorem 5.1 in [43]). Moreover, G is amenable if and only if G is compact. So, G has property (T_{l_p}) for $p \neq 2$. Observe that if $k - \mathrm{rank}(\mathbb{G}) = 1$, then G does not have property (T) (see Remark 1.6.3 in [8]). This is, for instance, the case for $G = SL_2(\mathbb{Q}_l)$.

2. Let $G = Aut(T)$ be the group of color preserving automorphisms of a k-regular tree of type (m, n) for $m, n \geq 3$. Then G is a totally disonnected locally compact group and, as shown in [54], G has the Howe-Moore property. Since G is non-amenable, it has property (T_{l_p}) for $p \neq 2$. Observe that G does not have property (T), since G acts without fixed point on a tree (see chapter 2 section 3 in [8]).

2.7 Property (T) and complete isometries

In a purely non-commutative context, it is more natural to use complete isometries instead of isometries. We have already noticed (see Remark 2.3.2) that, for example, results on isometries of closed subspaces of L_p-spaces require to deal with complete isometries. Hence we introduce a weak version of property (T_{L_p}) using representations by unital complete isometries on non-commutative L_p-spaces.

2.7.1 Weak property (T_F) for closed subspaces of $L_p(\mathcal{M})$

Let $\mathcal{M}$ be a semi-finite von Neumann algebra, $1 \leq p < \infty$. Recall that a complete isometry on $L_p(\mathcal{M})$ is a linear map $U : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ such that

$$id \otimes U : L_p(\mathcal{M}_n \otimes \mathcal{M}) \to L_p(\mathcal{M}_n \otimes \mathcal{M})$$

is an isometry for all $n \geq 1$. We say that U is unital if $U(1) = 1$. In particular, if $p \neq 2$, a unital complete isometry U on $L_p(\mathcal{M})$ is the extension of a Jordan isomorphism of $\mathcal{M}$, in view of the Yeadon decomposition (Theorem 2.7.1 shows that U is in fact an automorphism of $\mathcal{M}$).

If $\mathcal{M}$ is finite, we define the notion of complete isometries for a closed subspace $F \subset L_p(\mathcal{M})$ containing 1. A complete isometry on F is a linear map $U : F \to F$ such that

$$||id \otimes U(X)||_p = ||X||_p \text{ for all } X \in \mathcal{M}_n(F) \text{ and all } n \in \mathbb{N}.$$

We say that U is unital if $U(1) = 1$.

Denote by $O^{c.i.}(L_p(\mathcal{M}))$ (resp. $O^{c.i.}(F)$) the group of unital complete isometries of $L_p(\mathcal{M})$ (resp. $F \subset L_p(\mathcal{M})$). Recall from Example 1.6.3 that in general $O^{c.i.}(L_p(\mathcal{M})) \neq O(L_p(\mathcal{M}))$. Junge, Ruan and Sherman obtained a result on the structure of complete isometries of general non-commutative $L_p(\mathcal{M})$-spaces. We recall it in the case where the von Neumann algebra $\mathcal{M}$ is semi-finite.

Theorem 2.7.1. [45] *Let $\mathcal{M}$ be a semi-finite von Neumann algebra, and $1 \leq p < \infty$, $p \neq 2$. For an isometry $U = uBJ : L_p(\mathcal{M}) \to L_p(\mathcal{M})$, the following statements are equivalent :*
(i) U is a complete isometry,
(ii) U is a 2-isometry,
(iii) the Jordan map $J : \mathcal{M} \to \mathcal{M}$ is multiplicative.

Let G be a topological group, and let $\pi : G \to O^{c.i.}(L_p(\mathcal{M}))$ be a representation of G by unital complete isometries. It follows from the previous result that every $\pi(g)$ has a form

$$\pi(g)(x) = J_g(x) \text{ for all } x \in L_p(\mathcal{M})$$

where $J : G \to Aut(\mathcal{M})$ is a morphism of the group G in the group $Aut(\mathcal{M})$ of automorphisms of $\mathcal{M}$.

Definition 2.7.2. Let $\mathcal{M}$ be a finite von Neumann algebra. Let F be a closed subspace of a non-commutative space $L_p(\mathcal{M})$ such that $1 \in F$. A topological group G is said to have property $(T_F^{c.i.})$ if, for every representation $G \to O^{c.i.}(F)$, the restriction $\pi_{/F'(\pi)}$ of π on $F'(\pi)$ does not almost have invariant vectors.

Let $\pi : G \to O(B)$ be an orthogonal representation of a topological group on a ucus Banach space B. Let F be a $\pi(G)$-invariant closed subspace of B. The following lemma shows that the decomposition $F = F^{\pi(G)} \oplus F'(\pi)$ from Theorem 2.2.2 is coherent with the decomposition $B = B^{\pi(G)} \oplus B'(\pi)$.

Lemma 2.7.3. *Let B be a ucus Banach space, and $\pi : G \to O(B)$ be an orthogonal representation of a topological group G on B. Let F be a closed $\pi(G)$-invariant subspace of B. Then $F'(\pi) \subset B'(\pi)$.*

Proof. Notice that the contragradient representation $\pi^* : G \to O(B^*)$ on B^* defines also a representation $\pi^* : G \to O(F^*)$ on F^*, since F is $\pi(G)$-invariant. Now let $x \in F'(\pi)$ and $\varphi \in (B^*)^{\pi^*(G)}$. It is straightforward that the restriction $\varphi_{/F}$ of φ to F belongs to the subspace $(F^*)^{\pi^*(G)}$. Recall that

$$F'(\pi) = \{y \in F \mid \forall \psi \in (F^*)^{\pi^*(G)}, < \psi, y >= 0 \ \}.$$

Hence, by definition of $F'(\pi)$, we have

$$< \varphi, x >_{B^*,B} = < \varphi_{/F}, x >_{F^*,F} = 0.$$

Notice that Lemma 2.7.3 is also a consequence of Remark 2.2.4 (with $B_1 = F$, $B_2 = B$ and φ the canonical inclusion). □

Our next result generalizes item 2 in Theorem 2.3.1 to the case of non-commutative L_p-spaces. The proof shows that property $(T^{c.i.}_{L_p(\mathrm{VN}(F))})$ implies property $(T^{c.i.}_F)$ for some subspaces F of $L_p(\mathcal{M})$.

Theorem 2.7.4. *Let $1 \leq p < \infty$, $p \notin 2\mathbb{N}$. Let $\mathcal{M}$ be a finite von Neumann algebra, and F a closed subspace of $L_p(\mathcal{M})$ such that $F \subset \mathcal{M}$, and $1 \in F$. Assume that G is a topological group with property (T). Then G has property $(T^{c.i.}_F)$.*

Proof. By contradiction, assume that there exists a representation

$$\pi : G \to O^{c.i.}(F)$$

of G on F by unital complete isometries with almost invariant vectors in $F'(\pi)$. By Theorem 1.6.4 and Remark 1.6.5, every $\pi(g)$ extends uniquely to a complete isometry on $L_p(\mathrm{VN}(F))$, also denoted by $\pi(g)$. Then $\pi : G \to O^{c.i.}(L_p(\mathrm{VN}(F)))$ defines a representation of G by unital complete isometries on $L_p = L_p(\mathrm{VN}(F))$, denoted again by π. By the previous Lemma 2.7.3, π has almost invariant vectors in $L'_p(\pi)$. By Theorem 2.5.3, this contradicts the fact that G has Kazhdan's property (T). □

2.7.2 Relationship between property (T_{L_p}) and property $(T^{c.i.}_{L_p})$

We give an analog of Definition 2.7.2 in the case of $L_p(\mathcal{M})$-spaces associated to semi-finite von Neumann algebras $\mathcal{M}$.

Definition 2.7.5. Let $\mathcal{M}$ be a semi-finite von Neumann algebra. A topological group G is said to have property $(T^{c.i.}_{L_p(\mathcal{M})})$ if, for every representation $G \to O^{c.i.}(L_p(\mathcal{M}))$, the restriction $\pi_{/L_p(\mathcal{M})'}$ of π on $L_p(\mathcal{M}')$ does not almost have invariant vectors.

It is obvious that property (T_{L_p}) implies property $(T^{c.i.}_{L_p})$. We can show that the converse implication holds for some specific von Neumann algebras.

Theorem 2.7.6. *Let $1 \leq p < \infty$, $p \neq 2$.*
(i) Let $\mathcal{M}$ be a von Neumann algebra, and let G be a topological group with Kazhdan's property (T). Then G has property $(T^{c.i.}_{L_p(\mathcal{M})})$.
(ii) Let $\mathcal{M}$ be one of the following von Neumann algebras :
- $\mathcal{M} = L^\infty(X,\mu)$ (μ being non-atomic);
- $\mathcal{M} = R$ the hyperfinite II_1 factor;
- $\mathcal{M} = \mathcal{B}(\mathcal{H})$.
Let G be a locally compact second countable group such that G has property $(T^{c.i.}_{L_p(\mathcal{M})})$. Then G has property (T).

Proof. (i) This follows from Theorem 2.5.3 and the obvious fact that $(T_{L_p(\mathcal{M})})$ implies $(T^{c.i.}_{L_p(\mathcal{M})})$.

(ii) The three cases are consequences of the construction of a representation $\pi^p : G \to O(L_p(\mathcal{M}))$ given in the proof of Theorem 2.5.6, which almost has invariant vectors in $L_p(\mathcal{M})'$ when the group G does not have property (T). In every case, π^p is a representation by unital complete isometries.

- For the case $\mathcal{M} = L^\infty(X,\mu)$, π^p is the representation induced by a measure-preserving action α on (X,μ) (see the proof of Theorem A in [4]). By Example 1.6.3, π^p is a complete isometry. It is unital since the action α is measure-preserving.

- For the case $\mathcal{M} = R$, π^p is the representation induced by an action α of G on R by automorphisms. For $n \geq 1$, and $g \in G$, $id \otimes \pi^p(g)$ coincides with the automorphism $id \otimes \alpha_g$ on $\mathcal{M}_n \otimes R$, which is dense in $L_p(\mathcal{M}_n \otimes R)$. Hence $id \otimes \pi^p(g)$ is a unital isometry in $O(L_p(\mathcal{M}_n \otimes R))$. This shows that $\pi^g \in O^{c.i.}(L_p(R))$.

- For the case $\mathcal{M} = \mathcal{B}(\mathcal{H})$, every $\pi^p(g)$ is given by the following formula

$$\pi^p(g)(x) = \rho(g)x\rho(g)^{-1} \text{ for } x \in C_p,$$

where every $\rho(g)$ is a unitary in $\mathcal{U}(\mathcal{H})$. Therefore it is clear that $\pi^p(g) \in O^{c.i.}(C_p)$. □

We cannot expect that property (T) and property $(T^{c.i.}_{L_p(\mathcal{M})})$ are equivalent for every von Neumann algebra $\mathcal{M}$. In the next proposition, we give an example of a group G without property (T), such that G does not have property $(T^{c.i.}_{L_p(\mathcal{M})})$ for $p \neq 2$ and for some von Neumann algebras $\mathcal{M}$.

Let $\mathcal{M}$ be a II_1 factor, with trace τ. We view $\mathcal{M}$ as a subalgebra of $L_2(\mathcal{M})$, where $L_2(\mathcal{M})$ is the completion of $\mathcal{M}$ for the norm

$$||x||_2 = \tau(x^* x)^{\frac{1}{2}}.$$

We define a topology on $Aut(\mathcal{M})$, as the group topology given by the following fundamental system of neighborhoods of $id_{\mathcal{M}}$:

$$V(x_1, ..., x_n, \epsilon) = \{\alpha \in Aut(\mathcal{M}) \mid ||\alpha(x_i) - x_i||_2 < \epsilon \text{ for all } i = 1, ..., n \}.$$

Then $Aut(\mathcal{M})$ is a polish group.
Let Γ be a group such that every non-trivial conjugacy class is infinite. Then the group von Neumann algebra $\mathcal{M} = L(\Gamma)$ is a II_1 factor. Assume that Γ has property (T). Connes proved in [17] that the subgroup $Inn(\mathcal{M})$ of inner automorphisms is open, and hence $Out(\mathcal{M}) = Aut(\mathcal{M})/Int(\mathcal{M})$ is countable discrete.

Proposition 2.7.7. *Let $\mathcal{M}$ be a II_1 factor such that $Out(\mathcal{M}) = Aut(\mathcal{M})/Int(\mathcal{M})$ is discrete. Then $SL_2(\mathbb{R})$ has property $(T^{c.i.}_{L_p(\mathcal{M})})$.*

Proof. Set $SL_2(\mathbb{R}) = G$, and let $\pi : G \to O^{c.i.}(L_p(\mathcal{M}))$ be a representation by unital complete isometries on $L_p(\mathcal{M})$. We have already noticed that the representation has the form

$$\pi(g)(x) = J_g(x) \text{ for all } g \in G, x \in L_p(\mathcal{M})$$

where the map $J : G \to Aut(\mathcal{M})$ is a continuous group homomorphism (see Theorem 2.4.4). Let $\tilde{J} : G \to Out(\mathcal{M})$ be the composition map of J with the quotient map $Aut(\mathcal{M}) \to Out(\mathcal{M})$.

Since G is connected, the continuous map $\tilde{J}$ is constant, as $Out(\mathcal{M})$ is discrete. So, $J_g \in Int(\mathcal{M})$ for every $g \in G$ since $J_e \in Int(\mathcal{M})$.

There exists a continuous homomorphism $\varphi : G \to \mathcal{U}(\mathcal{M})$, satisfying $J_g(x) = \varphi_g x \varphi_g^{-1}$ for all $g \in G$ and $x \in \mathcal{M}$. Let τ be the normalized trace on $\mathcal{M}$; let $L_2(\mathcal{M})$ be the completion of $\mathcal{M}$ for the inner product

$$< x, y >= \tau(y^* x)$$

and let $\rho : \mathcal{M} \to \mathcal{B}(L_2(\mathcal{M}))$ be the $*$-homomorphism defined by $\rho(x)y = xy$ for all $x, y \in \mathcal{M}$. Then

$$\tau(x) =< \rho(x)1, 1 > \text{ for all } x \in \mathcal{M}.$$

Set $\psi = \tau \circ \varphi : G \to \mathbb{R}$. Then ψ is a continuous positive definite function on G, which is constant on every conjugacy class. We claim that ψ is constant on

G, that is $\psi_g = 1$ for all $g \in G$.

Indeed, let

$$T^+ = \{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{R}\} \text{ and } T^- = \{\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \mid a \in \mathbb{R}\}.$$

For $a > 0$ and $b \in \mathbb{R}$, we have

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & a^2 b \\ 0 & 1 \end{pmatrix}.$$

Hence, by continuity of ψ at e, we have $\psi(g) = 1$ for all $g \in T^+$. Similarly, we have $\psi(g) = 1$ for all $g \in T^-$. As

$$\psi(g) =< \rho(\varphi_g)1, 1 >,$$

the equality case of Cauchy-Schwarz inequality shows that

$$\rho(\varphi_g) = 1 \text{ for all } g \in T^+ \sqcup T^-.$$

Since $T^+ \sqcup T^-$ generates G as a group, this implies that

$$\rho(\varphi_g) = 1 \text{ for all } g \in G.$$

Therefore we have $\varphi_g = 1$ for all $g \in G$, that is, $J_g = id$ for all $g \in G$.

Then J has to be trivial. Hence the representation π is trivial and the group has property $(T_{L_p}^{c.i.})$. $\square$

Remark 2.7.8. Ioana, Peterson and Popa constructed in [44] II_1 factors $\mathcal{M}$ such that $Out(\mathcal{M})$ is trivial.

Chapter 3

Fixed-point property $(F_{L_p(\mathcal{M})})$

Property (T), which was originally defined in terms of unitary group representations, can be rephrased, at least for σ-compact locally compact groups, in terms of property (FH), a fixed-point property for actions by affine isometries on a real Hilbert space. A similar fixed-point property (F_B) can be defined for every Banach space B. It is known that, for σ-compact locally compact groups, property (F_B) always implies property (T_B). However, in general, property (T_B) is weaker than property (F_B); this happens already for the classical L_p-spaces $L_p([0,1])$, and p sufficiently large ([66]).

3.1 Introduction

3.1.1 Property (F_B)

We recall the basic facts concerning property (F_B). We refer to Chapter 2 in [8] for more details when $B = \mathcal{H}$ is a Hilbert space, and to Section 2.d in [4] for details about group actions by affine isometries on Banach spaces.

Let B be an affine real Banach space. Denote by $\mathrm{Isom}(B)$ the group of bijective isometries of B. Observe that, by the Mazur-Ulam theorem [56], every bijective isometry is an affine map.

An affine isometric action of a topological group G on B is a group homomorphism $\alpha : G \to \mathrm{Isom}(B)$ such that the map $G \to B,\ g \mapsto \alpha(g)x$ is continuous for every $x \in \mathcal{H}$.

Let $\alpha : G \to \mathrm{Isom}(B)$ be such an action. For every $g \in G$, let $\pi(g)$ be the linear part of $\alpha(g)$, and $b(g)$ the translation part; thus

$$\alpha(g)x = \pi(g)x + b(g) \text{ for all } g \in G, x \in B.$$

Then $\pi : G \to O(B)$ is an orthogonal representation and $b : G \to B$ is a continuous map satisfying the following cocycle relation:

$$b(gh) = b(g) + \pi(g)(b(h))$$

for all $g, h \in G$ and $x \in \mathcal{H}$. So, b is a continuous 1-cocycle of G with values in B, associated to the representation π. The set of all such 1-cocycles is denoted by $Z^1(G, \pi)$; this is a real vector space under the pointwise operations.
The action α has a fixed point $x \in B$ if and only if

$$b(g) = \pi(g)(-x) - (-x) \text{ for all } g \in G.$$

Cocycles of this form are called 1-coboundaries with respect to π. The set of all 1-coboudaries is a subspace of $Z^1(G, \pi)$, denoted by $B^1(G, \pi)$. The quotient vector space

$$H^1(G, \pi) = Z^1(G, \pi)/B^1(G, \pi)$$

is called the first cohomology group with coefficients in π.

Definition 3.1.1. Let B be a Banach space. A topological group G has property (F_B) if every affine isometric action of G on a real affine Banach space B has a fixed-point.

Remark 3.1.2. Let G be a topological group. Then G has property (F_B) if and only if $H^1(G, \pi) = 0$ for every orthogonal representation $\pi : G \to O(B)$.

Recall the following “lemma of the center”(see Remark (5) in [4]):

Lemma 3.1.3. *Let B a ucus Banach space, and A a non-empty bounded subset $A \subset B$. Then there exists a unique $x \in B$ minimizing* $\inf\{r > 0 \mid A \subset \overline{B}(x, r)\ \}$. *The point $x = x(A)$ is called the Chebyshev center of A.*

A consequence of the previous lemma is that, for affine actions by isometries on ucus Banach spaces, it is equivalent to have bounded orbits or to have a fixed point.

Proposition 3.1.4. *Let B be a ucus real Banach space. Let π be an orthogonal representation of G on B, and b a 1-cocycle with respect to π. Let α the associated affine isometric action associated to π and b. Then the following assertions are equivalent :*
(i) α has a fixed point in B ;
(ii) b is bounded ;
(iii) all the orbits of α are bounded ;
(iv) some orbit of α is bounded.

The following proposition shows that, for σ-compact locally compact groups, property (F_B) is always stronger than property (T_B).

Proposition 3.1.5. *Let G be a σ-compact locally compact group, and let B be a Banach space. If G has property (F_B), then G has property (T_B).*

For the proof of the previous proposition, see Theorem 1.3 in [4]. We will see later (Remark 3.1.8) that the converse of this proposition is not true in general.

3.1.2 Property (FH)

A topological group G is said to have property (FH) if every affine action of G by isometries on a real Hilbert space has a fixed-point. The following theorem shows that property (T) and property (FH) are equivalent for σ-compact locally compact groups. It is due to Delorme (Theorem V.1 in [25]) and Guichardet (Theorem 1 in [33]).

Theorem 3.1.6. *Let G be a topological group.*

1. *If G has property (T), then G has property (FH).*
2. *If G is a σ-compact locally compact group and if G has property (FH), then G has property (T).*

Property (FH) does not imply property (T) when G is not σ-compact: it is shown by de Cornulier in [22] that the group of all permutations of an infinite set has property (FH), but not property (T).

3.1.3 Property $(F_{L_p(X,\mu)})$

The authors of [4] proved the following theorem which relates property (T) and property $(F_{L_p(X,\mu)})$ for $L_p(X,\mu)$ a commutative L_p-space.

Theorem 3.1.7. ([4] and see [5] for the point 2.)
Let G be a locally compact second countable group.

1. *Let $1 < p \leq 2$, and let B be a closed subspace of L_p. If G has property (T), then G has property (F_B) ;*
2. *If G has property (T), then G has property (F_{L_1}) ;*
3. *If G has property (T), then there exists $\epsilon(G) > 0$ such that, for $2 \leq p < \epsilon(G)$ and for every closed subspace B of L_p, G has property (F_B).*

Remark 3.1.8. The analog of the Delorme-Guichardet theorem 3.1.6 is no longer true in general for actions on Banach spaces. It was shown by several authors that property (T) does not imply property (F_{L_p}) for some commutative L_p-spaces, and for $p > 2$ sufficiently large :

(i) Pansu in [66] proved that $Sp(n, 1)$ and cocompact lattices in these groups admit fixed-point-free affine isometric actions on $L^p(G)$ for $p > 4n + 2$.

(ii) Bourdon and Pajot ([11]) more generally proved that a non-elementary hyperbolic group does not have property (F_{L^p}) for p large enough.

(iii) Cornulier, Tessera and Valette in [23] proved that, for G a rank-one Lie (or algebraic) group, if p is sufficiently large, there is a proper affine isometric action of G on $L^p(G)$ whose linear part is the regular representation.

(iv) Yu also gave a short proof that any hyperbolic group Γ admits a proper action by affine isometries on $l^p(\Gamma \times \Gamma)$ if p is large enough (see [85]); in Chapter 4, we will use his construction to define a proper action on a non-commutative L_p-space.

(v) Nica ([62]) uses the latter result to prove that, if Γ is a non-elementary hyperbolic group with boundary $\partial\Gamma$, then for p large enough Γ admits an affine isometric action on $L^p(\partial\Gamma \times \partial\Gamma)$ that is proper.

3.2 Property $(F_{L_p(\mathcal{M})})$ for higher rank groups

Property $(F_{L_p(X,\mu)})$ was established in [4] for higher rank groups and their lattices. We first give the definition of higher rank groups.

Definition 3.2.1. For $1 \leq i \leq m$, let k_i be local fields and $\mathbb{G}_i(k_i)$ be the k_i-points of connected simple k_i-algebraic groups $\mathbb{G}_i$. Assume that each simple factor $\mathbb{G}_i$ has k_i-rank ≥ 2. The group $G = \Pi_{i=1}^m \mathbb{G}_i(k_i)$ is called a higher rank group.

Example 3.2.2. The groups $G = SL_n(\mathbb{R})$ for $n \geq 3$ and $G = Sp_{2n}(\mathbb{R})$ for $n \geq 2$ are higher rank groups. This is also true for their corresponding groups $G = SL_n(\mathbb{Q}_l)$ and $G = Sp_{2n}(\mathbb{Q}_l)$ over the field of l-adic numbers.

Our next result shows that Theorem B in [4] remains true for non-commutative L_p-spaces. In fact, it was conjectured in [4] that higher rank groups have property F_B for all superreflexive space B.

Theorem 3.2.3. *Let G be a higher rank group and $\mathcal{M}$ a von Neumann algebra. Then G, as well as every lattice in G, has property $F_{L_p(\mathcal{M})}$ for $1 < p < \infty$.*

Given our Theorem 2.5.5, the proof of Theorem 3.2.3 is a straightforward adaptation of the proof given in [4] for classical L_p-spaces. For this reason, we just give an indication of the main steps involved in this proof.

Strategy of proof of theorem 3.2.3. • We first show the result when G is a higher rank group.
Using an analogue of Howe-Moore's theorem on vanishing of matrix coefficients, it was shown in [4] that G has property (F_B) for every ucus Banach space B, whenever a certain pair $(L \ltimes H, H)$ of subgroups, which has strong property (T), has also strong property (T_B). In view of Theorem 2.5.5, this shows that G has property $(F_{L_p(\mathcal{M})})$.

• Let G be a lattice in a higher rank group. The result for G is obtained by an induction procedure exactly as in the Proposition 8.8 of [4]. □

Example 3.2.4. $\Gamma = SL_3(\mathbb{Z}[\sqrt{2}])$ is a lattice in the higher rank group $SL_3(\mathbb{R}) \times SL_3(\mathbb{R})$.

Remark 3.2.5. (i) Theorem 3.2.3 was proved by Puschnigg in [68] in the case $L_p(\mathcal{M}) = S_p$.

(ii) Others examples of groups with fixed-point property on non-commutative L_p-spaces were given by Mimura in [59], using our Theorem 3.2.3: for $n \geq 4$, $k \geq 1$, $1 < p < \infty$, and $\mathcal{M}$ a von Neumann algebra, the universal lattice $G = SL_n(\mathbb{Z}[x_1, ..., x_k])$ has property $(F_{L_p(\mathcal{M})})$.

3.3 Property $(F_{L_p(\mathcal{M})})$

3.3.1 Property $(F_{L_p(\mathcal{M})})$ for p close to 2

We recall that in general property (T) does not imply property (F_{L_p}) for commutative L_p-spaces and $p > 2$. However, Fisher and Margulis proved in [30] that (T_{L_p}) and (F_{L_p}) are equivalent for p close to 2. We show that this result is true for non-commutative L_p-spaces. Our method of proof is an adaptation of Fisher and Margulis' proof, as given in [4].

Theorem 3.3.1. *Let G be a locally compact group with property (T). There exists ϵ with the following property : for every von Neumann algebra $\mathcal{M}$, for every $p \in]2-\epsilon, 2+\epsilon[$, and every closed subspace B of $L_p(\mathcal{M})$, G has property (F_B).*

If α is a G-action by affine isometries on a space $(B, ||.||)$, K a subset of G and $x \in B$, we will use the notation,

$$\delta(\alpha(K)x) = \sup_{g,h \in K} ||\alpha(g)x - \alpha(h)x||.$$

Let $1 < p < \infty$ and let $\mathcal{M}$ be a von Neumann algebra. In the following, we simply denote by $L_p = L_p(\mathcal{M})$ the non-commutative L_p-space associated to $\mathcal{M}$. The main step of the proof of Theorem 3.3.1 is the following lemma.

Lemma 3.3.2. *Let G be a locally compact group with property (T) and K is a generating compact set of G. There exist constants $C < \infty$ and $\epsilon > 0$ with the following property : for every $p \in]2-\epsilon, 2+\epsilon[$, for every closed subspace $B \subset L_p$, for every action α of G by affine isometries on B, and for every $x \in B$, there exists a point $y \in B$ with*

$$||x-y||_p \leq C\delta(\alpha(K)x) \quad , \quad \delta(\alpha(K)y) \leq \frac{\delta(\alpha(K)x)}{2}.$$

Proof. Assume, by contradiction, that the lemma does not hold. Then we can find a sequence (p_n) tending to 2, affine isometric G-actions α_n on closed subspaces B_n of L_{p_n} and $x_n \in B_n$ such that

$$\delta(\alpha_n(K)y) > \frac{\delta(\alpha_n(K)x_n)}{2} \text{ for all } y \in B(x_n, n\delta(\alpha_n(K)x_n)).$$

The strict inequality above implies that $\delta(\alpha_n(K)x_n) > 0$. Set

$$\overline{x}_n = \frac{x_n}{\delta(\alpha_n(K)x_n)}$$

in order to have $\delta(\alpha_n(K)\overline{x}_n) = 1$. For every $y \in B(\overline{x}_n, n)$, we have that

$$\delta(\alpha_n(K)x_n)y \ \in \ B(x_n, n\delta(\alpha_n(K)x_n)).$$

So we have $\delta(\alpha_n(K)y) > \frac{1}{2}$.

To sum up, we get sequences $(p_n)_n$ with $\lim_n p_n = 2$ and closed subspaces $B_n \subset L_{p_n}$, affine isometric G-actions α_n on B_n and points $x_n \in B_n$ (which were the $\overline{x}_n$ above) satisfying

$$\delta(\alpha_n(K)x_n) = 1 \quad , \quad \delta(\alpha_n(K)y) > \frac{1}{2} \text{ for all } y \in B(x_n, n).$$

We now construct a Hilbert space $\mathcal{H}$, together with a G-action α by affine isometries without fixed point, which will contradict property (FH) and then property (T). Let

$$\mathcal{H}_0 = \{y = (y_n) \in \prod_n B_n \mid \sup_n ||y_n||_{p_n} < \infty \}.$$

We define a semi-norm on $\mathcal{H}_0$ by $||y|| = \overline{\lim}_n ||y_n||_{p_n}$. Then, the quotient of $\mathcal{H}_0$ by the subspace $\mathcal{N} = \{ y \in \mathcal{H}_0 \mid ||y|| = 0 \}$, denoted by $\mathcal{H}$, is a Banach space. For simplicity, we will identify the elements of $\mathcal{H}$ with one of their representative in $\mathcal{H}_0$.

We next show that $\mathcal{H}$ is a Hilbert space. In fact, by considering $[2, 2+\epsilon[$ before instead of $]2-\epsilon, 2+\epsilon[$, the p_n can be taken in $[2, \infty[$. By the Clarkson type inequalities (see Proposition 1.1.1), we have for all n and all $a_n, b_n \in B_n$,

$$||a_n + b_n||_{p_n}^{p_n} + ||a_n - b_n||_{p_n}^{p_n} \leq 2^{p_n - 1}(||a_n||_{p_n}^{p_n} + ||b_n||_{p_n}^{p_n}).$$

By passing to the limit superior, we obtain, for all $a, b \in \mathcal{H}$,

$$||a+b||^2 + ||a-b||^2 \leq 2(||a||^2 + ||b||^2).$$

And if we apply this inequality to a replaced by $a+b$ and b replaced by $a-b$, we obtain the parallelogram identity :

$$||a+b||^2 + ||a-b||^2 = 2(||a||^2 + ||b||^2) \text{ for all } a, b \in \mathcal{H}.$$

This proves that $\mathcal{H}$ is a Hilbert space.

Now we define a G-action by affine isometries on the affine Hilbert space $(x_n) + \mathcal{H}$ by

$$\alpha(g)((x_n)_n + y)_n = \alpha_n(g)(x_n + y_n)_n \text{ for all } y = (y_n)_n \in \mathcal{H} \text{ and all } g \in G.$$

This action is isometric (and so affine) since the $\alpha_n(g)$ are isometries. The homomorphism property follows as well.

Passing to the limit, the conditions on the L_{p_n}

$$\delta(\alpha_n(K)x_n) = 1 \quad \text{and} \quad \delta(\alpha_n(K)y) > \frac{1}{2} \text{ for all } y \in B(x_n, n) \subset L_{p_n}$$

imply the following conditions in $\mathcal{H}$ with $x = (x_n)$:

$$\delta(\alpha(K)x) = 1 \quad \text{and} \quad \delta(\alpha(K)y) \geq \frac{1}{2} \text{ for all } y \in \mathcal{H}.$$

In particular, the action is not degenerate and $\delta(\alpha(K)y) < \infty$ for all $y \in \mathcal{H}$.

As we deal with topological groups, we have to restrict α to an invariant subspace $\mathcal{H}_1$ on which the action is continuous, that is, such that the maps $g \mapsto \alpha(g)y$ are continuous for all $y \in \mathcal{H}_1$.

For this, we adapt the method from Section 4 in [14], which deals with the case of an ultraproduct of unitary group representations. We can assume without loss of generality that K is a neighborhood of the group unit e.

Let $n \in \mathbb{N}$. For $f \in C_c(G)$ and $v \in B_n$, we define

$$\alpha_n(f)v = \int_G f(g)\alpha_n(g)vdv \in B_n.$$

Then $\alpha_n(f)$ is an affine map from B_n to B_n and

$$||\alpha_n(f)v - \alpha_n(f)w||_{p_n} \leq ||f||_1||v - w||_{p_n} \text{ for all } v, w \in B_n. \quad (1)$$

Moreover, if f, g are positive functions on G with $\text{supp}(f) \subset K$, $\text{supp}(g) \subset K$ and $||f||_1 = ||g||_1 = 1$, we have

$$||\alpha_n(f)v - \alpha_n(g)v||_{p_n} \leq ||f - g||_1\delta(\alpha_n(K)v) \text{ for all } v \in B_n. \quad (2)$$

Indeed, we have $v = \int_G f(h)v\ dh = \int_G g(h)v\ dh$, and hence

$$\begin{aligned} ||\alpha_n(f)v - \alpha_n(g)v||_{p_n} &\leq ||\int_G f(h)(\alpha(h)v - v)\ dh - \int_G g(h)(\alpha(h)v - v)\ dh||_{p_n} \\ &\leq ||f - g||_1\delta(\alpha_n(K)v). \end{aligned}$$

For $y = (y_n)_n \in \mathcal{H}$, and $f \in C_c(G)$ with $\text{supp}(f) \subset K$, we define

$$\alpha(f)y = (\alpha_n(f)y_n)_n.$$

Notice that $\alpha(f)y \in \mathcal{H}$ by inequality (1).

Let $f, f_1, f_2 \in C_c(G)$ be such that $L = \text{supp}(f)$ satisfies $L^2 \subset K$, $L.L^{-1} \subset K$, and $(\text{supp}(f_1))(\text{supp}(f_2)) \subset K$. For $g \in L$, we have

$$\alpha({}_gf)y = \alpha(g)\alpha(f)y,\ \alpha(f_g)y = \Delta(g)\alpha(f)\alpha(g)y$$

and

$$\alpha(f_1 * f_2)y = \alpha(f_1)\alpha(f_2)y$$

where Δ is the modular function of G, and ${}_gf(h) = f(gh)$, $f_g(h) = f(gh)$ for all $g, h \in G$.

Let $L \subset K$ be a neighborhood of e such that $L^2 \subset K$ and $L.L^{-1} \subset K$. Fix an approximate identity $(f_n)_n$ of functions with supports in L, that is a sequence $(f_n)_n$ in $C_c(G)$ such that $\text{supp} f_n \subset L$, $f_n \geq 0$, $||f_n||_1 = 1$ for all n, and for every neighborhood V of e there exists N such that $\text{supp}(f_n) \subset V$ for all $n \geq N$.

Define the following subspace $\mathcal{H}_1$ of $\mathcal{H}$

$$\mathcal{H}_1 = \{y \in \mathcal{H} \mid \lim_n ||\alpha(f_n)y - y|| = 0\ \}.$$

We claim that $\mathcal{H}_1$ is independant of the choice of the approximate identity $(f_n)_n$. Indeed, let $(f_n')_n$ be another approximate identity. Let $y \in \mathcal{H}_1$. Then for all m, n, we have

$$\begin{aligned} ||\alpha(f_m{}')y - y|| &\leq ||\alpha(f_m{}')y - \alpha(f_m{}')\alpha(f_n)y|| + ||\alpha(f_m{}')\alpha(f_n)y - \alpha(f_n)y|| \\ &\quad + ||\alpha(f_n)y - y|| \\ &\leq 2||\alpha(f_n)y - y|| + ||f_m{}' * f_n - f_n||_1 \delta(\alpha(K)y) \end{aligned}$$

where we used inequality (2) to obtain the last inequality.
Since $\lim_n ||\alpha(f_n)y - y|| = 0$ and $\lim_m ||f_m{}' * f_n - f_n||_1 = 0$ for every fixed n, it follows that $\lim_m ||\alpha(f_m{}')y - y|| = 0$.

We claim that $\mathcal{H}_1$ is closed in $\mathcal{H}$. Indeed, let $y^{(l)} \in \mathcal{H}_1$ and $y \in \mathcal{H}_1$ such that $\lim_l y^{(l)} = y$. Then, for all n, l, we have

$$\begin{aligned} ||\alpha(f_n)y - y|| &\leq ||\alpha(f_n)y - \alpha(f_n)y^{(l)}|| + ||\alpha(f_n)y^{(l)} - y^{(l)}|| + ||y^{(l)} - y|| \\ &= 2||y^{(l)} - y|| + ||\alpha(f_n)y^{(l)} - y^{(l)}|| \end{aligned}$$

and the claim follows.

We claim that $\mathcal{H}_1$ is $\alpha(G)$-invariant. Indeed, let $y \in \mathcal{H}_1$ and fix $g \in G$. Let $f_n{}' \in C_c(G)$ be defined by

$$f_n{}' = \Delta(h^{-1}) f_n(ghg^{-1}) \text{ for all } h \in G.$$

Then $(f_n{}')_n$ is an approximate of the identity for n sufficiently large. Hence, $\lim_n ||\alpha(f_n{}')y - y|| = 0$ by what we have seen above. Since we have

$$\begin{aligned} ||\alpha(f_n)\alpha(g)y - \alpha(g)y|| &= ||\alpha(g^{-1})\alpha(f_n)\alpha(g)y - y|| \\ &= ||\alpha(f_n{}')y - y||, \end{aligned}$$

it follows that $\alpha(g)y \in \mathcal{H}_1$.

Let $y \in \mathcal{H}_1$. We claim that $G \to \mathcal{H}_1$, $g \mapsto \alpha(g)y$ is continuous. Indeed, let $\epsilon > 0$. Choose $n \in \mathbb{N}$ such that $||\alpha(f_n)y - y|| \leq \epsilon$ and choose a neighborhood V of e with $V\text{supp}(f_n) \subset K$ such that $||{}_g f_n - f_n||_1 \leq \epsilon$ for all $g \in V$. Then, for all $g \in V$, we have

$$\begin{aligned} ||\alpha(g)y - y|| &\leq ||\alpha(g)y - \alpha(g)\alpha(f_n)y|| + ||\alpha(g)\alpha(f_n)y - \alpha(f_n)y|| + ||\alpha(f_n)y - y|| \\ &\leq 2||\alpha(f_n)y - y|| + ||\alpha({}_g f_n)y - \alpha(f_n)y|| \\ &\leq \epsilon + ||{}_g f_n - f_n||_1 \delta(\alpha(K)y) \\ &\leq 2\epsilon + \epsilon\delta(\alpha(K)y). \end{aligned}$$

This shows the continuity of $g \mapsto \alpha(g)y$.

The action α of G by affine isometries on the Hilbert space $\mathcal{H}_1$ has clearly no $\alpha(G)$-fixed point. This contradicts property (FH) and hence property (T).

□

We are now able to prove Theorem 3.3.1.

Proof of Theorem 3.3.1. Take $\epsilon > 0$ and $C > 0$ such as in the lemma. Now let $p \in]2-\epsilon, 2+\epsilon[$, B a closed subspace of $L_p(\mathcal{M})$ and α an action of the Kazhdan group by affine isometries on B. Starting from an arbitrary point $x_0 \in B$, we can find by induction a sequence (x_n) of elements in B satisfying :

$$x_{n+1} \in B(x_n, C\delta_n) \text{ and } \delta_{n+1} \leq \frac{\delta_n}{2} \leq \frac{\delta_0}{2^{n+1}}.$$

by setting $\delta_n = \delta(\alpha(K)x_n)$.

Then, $x_n = x_0 + \sum_{k=0}^{n-1} x_{k+1} - x_k$ defines a convergent sequence in B, and its limit point is a fixed point for the action α. □

3.3.2 Consequences of embeddings between L_p-spaces

We will give a simple procedure to construct an affine isometric action on a non-commutative L_p-space from an action on a commutative L_p-space.

The next statement is obviously true in a more general context. We say that a cocycle $b : G \to L_p$ is *proper* if $\lim_{g\to\infty} ||b(g)||_p = +\infty$.

Lemma 3.3.3. *Let $1 \leq p < \infty$. Let $\mathcal{M}$ and $\mathcal{M}'$ be von Neumann algebras such that there exists an isometric linear embedding*

$$\begin{array}{rcl} L_p(\mathcal{M}) & \to & L_p(\mathcal{M}') \\ x & \mapsto & \tilde{x}. \end{array}$$

Let G be a topological group and let $\pi : G \to O(L_p(\mathcal{M}))$ be an orthogonal representation. Assume that π extends to a representation $\tilde{\pi}$ of G on $L_p(\mathcal{M}')$, that is, there exists an orthogonal representation $\tilde{\pi} : G \to O(L_p(\mathcal{M}'))$ such that $\tilde{\pi}(g)(\tilde{x}) = \widetilde{\pi(g)x}$ for all $g \in G$, $x \in L_p(\mathcal{M})$. Let $b : G \to L_p(\mathcal{M})$ be a 1-cocycle associated to π.
Then the map $\tilde{b} : G \to L_p(\mathcal{M}')$, defined by $\tilde{b}(g) = \widetilde{b(g)}$ for every $g \in G$, is a 1-cocycle associated to the representation $\tilde{\pi}$. Moreover, b is proper if and only if $\tilde{b}$ is proper.

Proof. Let $g, h \in G$. We have

$$\begin{aligned}\tilde{b}(gh) &= b(\tilde{gh}) \\ &= b(\tilde{g}) + \pi(g)\tilde{b}(h) \\ &= \tilde{b}(g) + \tilde{\pi}(g)b(\tilde{h}) \\ &= \tilde{b}(g) + \tilde{\pi}(g)\tilde{b}(h).\end{aligned}$$

So $\tilde{b}$ is a 1-cocycle. The last claim is obvious. □

Proposition 3.3.4. *Let G be a topological group. Let $\pi : G \to O(L_p([0,1]))$ be an orthogonal representation and $b : G \to L_p([0,1])$ a 1-cocycle for π. Let $\mathcal{M}$ be a finite von Neumann algebra. Then $\tilde{b} : G \to L_p(L^\infty([0,1]) \otimes \mathcal{M})$, defined by $\tilde{b}(g) = b(\tilde{g})$ for all $g \in G$, is a cocycle satisfying $||b(g)||_p = ||\tilde{b}(g)||_p$ for all $g \in G$. In particular :*
- b is proper $\Leftrightarrow$ $\tilde{b}$ is proper ;
- b is bounded $\Leftrightarrow$ $\tilde{b}$ is bounded.

Proof. Let τ be the normalized trace on $\mathcal{M}$. Recall that $L_p(L^\infty([0,1]) \otimes \mathcal{M})$ is isometrically isomorphic to the Bochner space $L_p(L_p(\mathcal{M}))$ equipped with the norm

$$||(x_t)||_p^p = \int_0^1 ||x_t||_p^p dt = \int_0^1 \tau(|x_t|^p)dt.$$

We have the natural isometric embedding of $L_p([0,1])$ into $L_p(L_p(\mathcal{M}))$ by

$$f \mapsto \tilde{f} = f \otimes 1 = f1.$$

Let $\pi : G \to O(L_p([0,1]))$ be an orthogonal representation of a topological group G. By Theorem 1.5.2, π has the form

$$\pi(g)(f)(x) = u_g(x)(\frac{d\varphi_g * \mu}{d\mu}(x))^{\frac{1}{p}} f(\varphi_g(x)) \text{ for all } g \in G \text{ and } f \in L_p([0,1]).$$

We then extend π to an orthogonal representation $\tilde{\pi}$ on $L_p(L_p(\mathcal{M}))$ by

$$\tilde{\pi}(g)((x_t)_t) = (u_g(t)(\frac{d\varphi_g * \mu}{d\mu}(t))^{\frac{1}{p}} x_{\varphi_g(t)})_t \text{ for all } g \in G \text{ and } (x_t) \in L_p(L_p(\mathcal{M})).$$

Since we have $\tilde{\pi}(g)(\tilde{f}) = \tilde{\pi}(g)(f1) = \pi(g)(f)1$ for all $g \in G$ and $f \in L_p([0,1])$, the claim is proved. □

Using the description of $O(l_p)$ given in Theorem 1.5.1, we can prove Proposition 3.3.4 when we replace $L_p([0,1])$ by l_p.

We can also prove the same result with the embedding of l_p in C_p. The construction is as follows : let $\pi : G \to O(l_p)$ be an orthogonal representation of the topological group G on l_p. By Theorem 1.5.1, π is given by

$$\pi(g)((x_n)) = (c(n,g)x_{\sigma_g(n)}) \text{ for all } (x_n) \in l_p.$$

Now take a Hilbert space $\mathcal{H}$ and an orthonormal basis (e_i) of $\mathcal{H}$. We have the following isometric linear embedding of l^p in the subspaces of diagonal operators of $C_p(\mathcal{H})$

$$(x_n) \mapsto (\tilde{x_n}) = diag(x_n).$$

Now for $g \in G$, let ρ_g be the unitary operator defined by $\rho_g(e_i) = e_{\sigma_g(i)}$. Then we define the following representation $\tilde{\pi} : G \to O(C_p)$, extending the representation π on C_p, by

$$\tilde{\pi}(g)(x) = (c(\tilde{n,} g))_n \rho_g x \rho_g^{-1} \text{ for all } g \in G \text{ and } x \in C_p.$$

We have then

$$\tilde{\pi}(g)((\tilde{x_n})) = \pi(g)\tilde{(}(x_n)) \text{ for all } g \in G \text{ and } (x_n) \in l_p.$$

Using this latter property and the fact that $x \mapsto \tilde{x}$ is a linear isometric embedding, we see, as in Proposition 3.3.4, that cocycles for π in l_p isometrically extend to cocycles for $\tilde{\pi}$ on C_p. We obtain as corollary the following result.

Corollary 3.3.5. *Let $1 \leq p < \infty$ and $\mathcal{M}$ be a finite von Neumann algebra. If a topological group G has property $(F_{L_p(L^\infty \otimes \mathcal{M})})$, then G has property (F_{L_p}).*
If a topological group G has property (F_{C_p}), then G has also property (F_{l_p}).

Chapter 4

The Haagerup property for actions on $L_p(\mathcal{M})$-spaces

The Haagerup property (H), or a-T-menability, is a strong negation of property (T). As the latter property, the Haagerup property may be defined either in terms of unitary representations or in terms of actions by affine isometries on Hilbert spaces. Several authors (Nowak, Chatterji, Drutu, Haglund) considered the Haagerup property, called a-FL_p-menability in [12], for actions by affine isometries on commutative L_p-spaces.

We will define and study in the setting of non-commutative L_p-spaces, both versions of the Haagerup property : one through orthogonal representations, and the other through affine isometric actions. We will call these properties (H_{L_p}) and a-FL_p-menability. As we will see, these properties are different from each other, and different from the classical property (H), even for classical L_p-spaces. In order to study more closely the relationships between these properties and the original Haagerup property, it will be useful to introduce further versions of property (H_{L_p}), which we call properties $(H_{L_p(\mathcal{M}),+})$ and $(H_{L_p(\mathcal{M}),\tau})$.

To our knowledge, property (H_{L_p}) has not been yet considered in the literature, even in the commutative case. We will characterize totally disconnected groups with property (H_{l_p}) as the amenable ones. We see therefore that properties (H_{l_p}) and (H) do not coincide. Property $(H_{L_p([0,1])})$ seems to be more closely related to property (H) : we prove that both properties coincide for linear Lie groups.

4.1 Introduction

4.1.1 The Haagerup property

For an accurate survey on the Haagerup property (H), see [13].

Theorem-Definition 4.1.1. *Let G be a second countable locally compact group. The following properties are equivalent :*

(i) There exists a unitary representation of G on a Hilbert space $\mathcal{H}$ which almost has invariant vectors, and has vanishing coefficients.
(ii) There exists a proper affine isometric action of the group G on some Hilbert space $\mathcal{H}$.
(iii) There exists a continuous function $\psi : G \to \mathbb{R}^+$ which is conditionally of negative type and proper, that is, $\lim_{g\to\infty} \psi(g) = +\infty$.
A group G has the Haagerup property (H), or is said to be a-T-menable, if it has one of the equivalent properties above.

In contrast with property (T), the Haagerup property was first established for some specific groups by constructing proper actions (or proper functions conditionally of negative type). For instance, Haagerup ([34]) proved in 1979 that the word length on a free group is a conditionally negative definite function on this group.

4.1.2 Some examples of groups with property (H)

Groups with the Haagerup property (H) form a large class of groups containing for example amenable groups, free groups, and more generally groups acting properly on trees, Coxeter groups (the list is non-exhaustive, more can be found in [13]). For connected Lie groups, we have the following classification result :

Theorem 4.1.2. (Theorem 4.0.1 in [13]) *Let G be a connected Lie group. Then the following assertions are equivalent :*
- G has the Haagerup property (H) ;
- if, for some closed subgroup H, the pair (G, H) has relative property (T), then H is compact ;
- G is locally isomorphic to a direct product

$$M \times SO(n_1, 1) \times ... \times SO(n_k, 1) \times SU(m_1, 1) \times ... \times SU(m_l, 1)$$

where M is an amenable Lie group.

Therefore, a non-compact simple Lie group has the Haagerup property (H) if it is locally isomorphic to $SO(n, 1)$ or $SU(n, 1)$; otherwise, it has property (T). Only compact groups have property (T) and property (H) at the same time, and a well-known general obstruction to have property (H) is the following : if a group G contains a non-compact subgroup H such that the pair (G, H) has relative property (T), then G does not have property (H). It is not the only known obstruction, as shown by de Cornulier in [21].

4.1.3 Some hereditary properties of property (H)

We list here some group constructions under which the Haagerup property is stable :

- if G is a group with the Haagerup property, and H is a closed subgroup of G, then H has also the Haagerup property ;
- if (G_n) is an increasing sequence of open subgroups of a locally compact group G, and if all G_n have the Haagerup property, then so does G (see Proposition 6.1.1 in [13]);
- if H is a closed subgroup of a locally compact group which is co-Følner in G, that is such that there exists a G-invariant state on $L^\infty(G/H)$ (see Proposition 6.1.5 in [13]), and H has property (H), then G has property (H).
- if G and H are two discrete groups with the Haagerup property and containing a finite subgroup A, then the amalgamated product $G *_A H$ has also the Haagerup property (see Proposition 6.2.3 in [13]).

4.2 Property $(H_{L_p(\mathcal{M})})$

In this section, we introduce a variant of the version (i) (see Theorem-Definition 4.1.1) of property (H) involving orthogonal representations on non-commutative L_p-spaces.

We recall that if $\mathcal{M}$ is a von Neumann algebra, and $L_p(\mathcal{M})$ is its associated Haagerup L_p-space, we denote by Tr the *Haagerup trace* defined on $L_1(\mathcal{M}) \simeq \mathcal{M}_*$.

Definition 4.2.1. Let G be a topological group, $1 \leq p < \infty$, and let $\mathcal{M}$ be a von Neumann algebra. We say that a representation $\pi : G \to O(L_p(\mathcal{M}))$ has vanishing coefficients (or π is said to be C_0) if

$$\lim_{g\to\infty} \mathrm{Tr}(\pi(g)(x)y) = 0 \text{ for all } x \in L_p(\mathcal{M}) \text{ and } y \in L_{p'}(\mathcal{M}).$$

Remark 4.2.2. Notice that if $\mathcal{M}$ is a semi-finite von Neumann algebra equipped with a trace τ, by density of the subspace $\mathcal{M} \cap L_1(\mathcal{M})$ in every $L_p(\mathcal{M})$, a representation has vanishing coefficients if and only if

$$\lim_{g\to\infty} \tau(\pi(g)(x)y) = 0 \text{ for all } x, y \in \mathcal{M} \cap L_1(\mathcal{M}).$$

The previous definition is motivated by the following theorem, called the Howe-Moore theorem. It was first proved for unitary representations (see [43]); the authors of [4] gave a proof of the theorem for representations on ucus Banach spaces (and more generally for representations on superreflexive Banach spaces).

Theorem 4.2.3. *Let I be a finite set, k_i $(i \in I)$ be local fields, $\mathbb{G}_i$ be connected semisimple simply-connected k_i-groups, $G_i = \mathbb{G}_i(k_i)$ be the locally compact group of k_i-points, and*

$$G = \prod_{i\in I} G_i.$$

Let $1 \leq p < \infty$, and let $\mathcal{M}$ be a von Neumann algebra. Let $\pi : G \to O(L_p(\mathcal{M}))$ be an orthogonal representation such that $L_p(\mathcal{M})^{\pi(G_i)} = \{0\}$ for every i. Then π has vanishing coefficients.

Now we give the definition of property $(H_{L_p(\mathcal{M})})$.

Definition 4.2.4. Let G be a topological group. Let $\mathcal{M}$ be a von Neumann algebra, and $1 \leq p < \infty$.
We say that a group G has property $(H_{L_p(\mathcal{M})})$ if there exists a representation $\pi : G \to O(L_p(\mathcal{M}))$ with vanishing coefficients , which almost has invariant vectors.

Remark 4.2.5. By analogy with property (T) and property (H), property $(H_{L_p(\mathcal{M})})$ is a strong negation of property $(T_{L_p(\mathcal{M})})$ in the following sense : if a topological group G admits a closed normal non-compact subgroup H such that the pair (G, H) has property $(T_{L_p(\mathcal{M})})$, then G does not have property $(H_{L_p(\mathcal{M})})$.

The following proposition is obvious.

Proposition 4.2.6. *Let $1 \leq p < \infty$ and let $\mathcal{M}$ be a von Neumann algebra. Property $(H_{L_p(\mathcal{M})})$ is inherited by closed subgroups.*

We will only study property $(H_{L_p(\mathcal{M})})$ for semi-finite von Neumann algebras. If $\mathcal{M}$ is a semi-finite von Neumann algebra, denote by $[\mathcal{M}]_p$ the class of von Neumann algebras $\mathcal{M}'$ such that $L_p(\mathcal{M}')$ is isometrically isomorphic to $L_p(\mathcal{M})$ (notice that such a von Neumann algebra $\mathcal{M}'$ is also semi-finite by Remark 1.4.4 since it has the same type as $\mathcal{M}$).

Proposition 4.2.7. *Let $1 \leq p < \infty$, and let $\mathcal{M}$ be a semi-finite von Neumann algebra. Let $\mathcal{M}' \in [\mathcal{M}]_p$.* Properties $(H_{L_p(\mathcal{M})})$ and $(H_{L_p(\mathcal{M}')})$ are equivalent.

Proof. Let $U : L_p(\mathcal{M}, \tau) \to L_p(\mathcal{M}', \tau')$ be an isometric isomorphism. Denote by $U^* : L_{p'}(\mathcal{M}', \tau') \to L_{p'}(\mathcal{M}, \tau)$ its dual map. Let π be a representation of group G on $L_p(\mathcal{M}, \tau)$ with almost invariant vectors and vanishing coefficients. Then $\overline{\pi} = U \circ \pi \circ U^{-1}$ defines a representation on $L_p(\mathcal{M}', \tau')$, which is easily seen to have almost invariant vectors. The representation $\overline{\pi}$ has vanishing coefficients since we have for all $x' \in L_p(\mathcal{M}', \tau')$, $y' \in L_{p'}(\mathcal{M}', \tau')$, and all $g \in G$

$$\begin{aligned}\tau'(\overline{\pi}(g)(x')y') &= \tau'(U(\pi(g)(U^{-1}(x)))\ y') \\ &= \tau(\pi(g)(U^{-1}(x))\ U^*(y')).\end{aligned}$$

□

4.3 Property (H_{l_p})

Let G be a second countable locally compact group, and let $1 \leq p < \infty$, $p \neq 2$. Let $\pi^p : G \to O(l_p)$ be a representation G on l_p.

We recall from Chapter 2 that the conjugate representation π^2 of G on l_2 is unitarily equivalent to a sum of monomial representations associated to open subgroups of G, that is, there exist open subgroups $(H_i)_{i\in I}$ of G and unitary characters $(\chi_i)_{i\in I}$ on the H_i's such that

$$\pi^2 = M_{p,2} \circ \pi^p \circ M_{2,p} \simeq \oplus_{i\in I}\mathrm{Ind}_{H_i}^G \chi_i.$$

If G is connected, the only open subgroup of G is G itself and a character on G is C_0 if and only if G is compact. Therefore we have the following result.

Theorem 4.3.1. *Let G be a connected second countable locally compact group. Let $1 \leq p < \infty$, $p \neq 2$. Then G has property (H_{l_p}) if and only if G is compact.*

Now we turn to property (H_{l_p}) for totally disconnected groups.

Theorem 4.3.2. *Let G be a totally disconnected locally compact second countable group. The following properties are equivalent :*
(i) G has property (H_{l_p}).
(ii) G is amenable.

Proof. (i) $\Rightarrow$ (ii) : Let $\pi^p : G \to O(l_p)$ be an orthogonal representation with almost invariant vectors, and with vanishing coefficients. Then we have $\pi^2 \simeq \oplus_{i\in I}\mathrm{Ind}_{H_i}^G\chi_i$ for some open subgroups H_i and unitary characters χ_i on H_i.
Since π^2 has the same form as π^p, π^2 has vanishing coefficients, and so does $\pi_i = \mathrm{Ind}_{H_i}^G\chi_i$ for every $i \in I$. Let $i \in I$ be fixed. Then $\pi^i_{/H_i}$ contains χ_i; indeed, we have

$$\pi_i(h)\delta_{H_i} = \chi_i(h)\delta_{H_i} \text{ for all } h \in H_i.$$

It follows that $\chi_i \in C_0(H_i)$ and hence H_i is compact.
Since H_i is compact, $\chi_i \subset \lambda_{H_i}$ and hence $\pi_i = \mathrm{Ind}_{H_i}^G\chi_i \subset \mathrm{Ind}_{H_i}^G\lambda_{H_i} = \lambda_G$. So, π^2 is weakly contained in λ_G. Since we have also $1_G \prec \pi^2$, it follows that $1_G \prec \lambda_G$. By Hulanicki-Reiter theorem, G is amenable.

(ii) $\Rightarrow$ (i) : Since G is totally disconnected, by van Dantzig's theorem, there exists a compact open subgroup K of G. Let $(\pi^2, l_2(G/K))$ be the quasi-regular representation of G on $l_2(G/K)$, and let $\pi^p = M_{2,p} \circ \pi^2 \circ M_{2,p}$. Notice that, for $g \in G$, every $\pi^p(g)$ is given by the same formula as $\pi^2(g)$ on the common dense subspace $l_1(G/K)$, so $\pi^p : G \to O(l_p(G/K))$ defines an orthogonal representation. Since K is compact, $\lambda_{G/K}$ is contained in the regular representation λ_G (by identifying $l_2(G/K)$ with the K-invariant functions in $L_2(G)$). Hence $\lambda_{G/K}$ has

vanishing coefficients, since all matrix coefficients of λ_G are C_0. It follows that π^p has vanishing coefficients.
Since G is amenable, the action of G on G/K is amenable (see p.28 in [28]). Hence, $\lambda_{G/K}$ almost has invariant vectors. Therefore, G has (H_{l_p}).

□

4.4 Property $(H_{L_p([0,1])})$

We give a classification of (almost all) Lie groups with property $(H_{L_p([0,1])})$; it turns out that this class of groups coincides with the class of Lie groups with the Haagerup property (H) (see Theorem 4.1.2).

Theorem 4.4.1. *Let G be a connected Lie group such that in its Levi decomposition $G = SR$, the semi-simple part S has finite center. Then the following are equivalent :*
(i) G has property $(H_{L_p([0,1])})$;
(ii) G has the Haagerup property (H) ;
(iii) G is locally isomorphic to a product $\prod_{i\in I} S_i \times M$, where M is amenable, I is finite, and for every $i \in I$, S_i is a group $SO(n_i, 1)$ or $SU(m_i, 1)$ with $n_i \geq 2$, $m_i \geq 1$.

Remark 4.4.2. (i) The previous theorem gives a classification of linear groups since, for any such group, the center of the semi-simple part in the Levi decomposition is finite (see Proposition 4.1 of Chapter XVIII in [42]).

(ii) We had to exclude groups $G = SR$ with S having infinte center, as we could not be able to prove $(iii) \Rightarrow (i)$ for the universal covers of $SO(n, 1)$ and $SU(n, 1)$.

Theorem 4.4.1 has the following immediate consequence.

Corollary 4.4.3. *Let G be a closed subgroup of a Lie group of the form $\prod_{i\in I} S_i \times M$, where M is amenable, I is finite, and for every $i \in I$, S_i is a group $SO(n, 1)$ or $SU(m, 1)$. Then G has property $(H_{L_p([0,1])})$ for all $1 \leq p < \infty$.*

The most difficult part of the proof of Theorem 4.4.1 is the proof of $(iii) \Rightarrow (i)$. Actually, we will show that groups as in Corollary 4.4.3 have a stronger version of property $(H_{L_p([0,1])})$, called property $(H_{L_p([0,1]),+})$ and introduced later in this chapter. We mention that the proof, even for the groups $SO(n, 1)$ or $SU(n, 1)$, is not elementary since it depends heavily on the fact that these groups have lattices Γ with non-trivial first Betti number, that is, lattices with infinite abelianization. The latter result was shown by Millson in [58] for the case $SO(n, 1)$, and by Kazhdan in [48] for the case $SU(n, 1)$.

In the proof of $(iii) \Rightarrow (i)$, we will need the following technical lemma which asserts that vanishing coefficients and almost invariant vectors are preserved for the quasi-regular representation, when passing from a group to a finite cover, and from the finite cover to the group.

Lemma 4.4.4. *Let G_1 and G_2 be locally compact topological groups and $p : G_1 \to G_2$ be a finite covering.*

1. *Let H_2 be a closed subgroup of G_2 such that G_2/H_2 carries a G_2-invariant measure, and the quasi-regular representation λ_{G_2/H_2} has almost invariant vectors and vanishing coefficients. Set $H_1 = p^{-1}(H_2)$. Then λ_{G_1/H_1} has almost invariant vectors and vanishing coefficients.*

2. *Let H_1 be a closed subgroup of G_1 such that G_1/H_1 carries a G_1-invariant measure, and the quasi-regular representation λ_{G_1/H_1} has almost invariant vectors and vanishing coefficients. Set $H_2 = p(H_1)$. Then H_2 is closed and λ_{G_2/H_2} has almost invariant vectors and vanishing coefficients.*

Proof. In the two cases, let $\overline{p} : G_1/H_1 \to G_2/H_2$ be the map induced by the covering map $p : G_1 \to G_2$. Observe that $\overline{p}$ is G_1-invariant, for the natural action of G_1 on G_1/H_1 and the action of G_1 on G_2/H_2 given by p :

$$g_1.(g_2H_2) = p(g_1)g_2H_2 \text{ for all } g_1 \in G_1, g_2 \in G_2.$$

Since $Z_1 = \mathrm{Ker}(p)$, the map $\overline{p}$ has finite fibers : indeed, the fiber over $p(g_1H_2)$ is $\{g_1zH_1 \mid z \in Z_1\} = \{zg_1H_1 \mid z \in Z_1\}$, as Z_1 is central.

1. Let μ_2 be a G_2-invariant measure on G_2/H_2. In this case, $Z_1 \subset H_1$ and so $\overline{p}$ is bijective. Then $\mu_1 = \overline{p}^{-1} * \mu_2$ is a G_1-invariant measure on G_1/H_1. The quasi-regular representation λ_{G_1/H_1} is equivalent to the representation $\lambda_{G_2/H_2} \circ p$.
   Since $1_{G_2} \prec \lambda_{G_2/H_2}$ and λ_{G_2/H_2} has vanishing coefficients, we have $1_{G_1} \prec \lambda_{G_1/H_1}$ and λ_{G_1/H_1} has vanishing coefficients.

2. Notice that $H_2 = p(H_1)$ is a closed subgroup of G_2 since the cover $p : G_1 \to G_2$ is finite. Let μ_1 be a G_1-invariant measure on G_1/H_1. Since the fibers of $\overline{p}$ are finite, we can define a G_2-invariant measure μ_2 on G_2/H_2 by

   $$\int_{G_2/H_2} f \, d\mu_2 = \int_{G_1/H_1} f \circ \overline{p} \, d\mu_1 \text{ for all } f \in C_c(G_2/H_2). \ (*)$$

   The induced mapping

   $$\begin{aligned} \psi : \ & L_2(G_2/H_2, \mu_2) \to L_2(G_1/H_1, \mu_1) \\ & f \ \mapsto \ f \circ \overline{p} \end{aligned}$$

is a linear isometry which intertwines the G_1-representations $\lambda_{G_2/H_2} \circ p$ and λ_{G_1/H_1}. So, $\lambda_{G_2/H_2} \circ p$ is equivalent to a subrepresentation of λ_{G_1/H_1}. Since λ_{G_1/H_1} has vanishing coefficients, the same is true for $\lambda_{G_2/H_2} \circ p$. As p is surjective and had finite kernel, it follows that the G_2-representation λ_{G_2/H_2} has vanishing coefficients.

It remains to prove that $1_{G_2} \prec \lambda_{G_2/H_2}$. To show this we claim that

$$\mathrm{Im}(\psi) = L_2(G_1/H_1)^{\lambda_{G_1/H_1}(Z_1)},$$

the space of Z_1-invariant vectors in $L_2(G_1/H_1)$.
Indeed, let $f_1 \in L_2(G_1/H_1)^{\lambda_{G_1/H_1}(Z_1)}$. As mentioned above, the fiber over $p(g_1H_2)$ is $\{zg_1H_1, z \in Z_1\}$ for every $g_1 \in H_1$. Hence, f_1 is constant on the fibers of $\overline{p}$ and there exists a map f_2 on G_2/H_2 such that $f_2 \circ \overline{p} = f_1$. It is clear that $f_2 \in L_2(G_2/H_2)$ by formula $(*)$.
Conversely, if $f \in L_2(G_2/H_2)$, it is clear that $f \circ \overline{p}$ is a Z_1-invariant function in $L_2(G_1/H_1)$.

We now show that λ_{G_2/H_2} almost has invariant vectors. It suffices to show that the restriction of λ_{G_1/H_1} to the subspace $L_2(G_1/H_1)^{\lambda_{G_1/H_1}(Z_1)}$ almost has invariant vectors. Take a sequence $(v_n)_n$ of almost invariant vectors for λ_{G_1/H_1}. For $n \in \mathbb{N}$, define

$$w_n = \frac{1}{|Z_1|} \sum_{z \in Z_1} \lambda_{G_1/H_1}(z) v_n.$$

For every $n \in \mathbb{N}$, $w_n \in L_2(G_1/H_1)^{\lambda_{G_1/H_1}(Z_1)}$. Moreover, for $g \in G$,

$$||\lambda_{G_1/H_1}(g)w_n - w_n||_2 \leq \frac{1}{|Z_1|} \sum_{z \in Z_1} ||\lambda_{G_1/H_1}(zg)v_n - v_n||_2$$

so that $\lim_n \sup_{g \in K} ||\lambda_{G_1/H_1}(g)w_n - w_n||_2 = 0$ for every compact $K \subset G$. We have

$$||w_n - v_n||_2 \leq \frac{1}{|Z_1|} \sum_{z \in Z_1} ||\lambda_{G_1/H_1}(z)v_n - v_n||_2$$

and the left side of the inequality tends to 0; hence, since $||v_n||_2 = 1$, $\lim_n ||w_n||_2 = 1$. The sequence $(\tilde{w_n})_n$, defined by $\tilde{w_n} = \frac{1}{||w_n||_2} w_n$, is a sequence of almost invariant vectors for the restriction of λ_{G_1/H_1} to $L_2(G_1/H_1)^{\lambda_{G_1/H_1}(Z_1)}$. The lemma is proved.

□

We are now able to give the proof of Theorem 4.4.1.

Proof of Theorem 4.4.1. (i) $\Rightarrow$ (ii) : Assume that G is a connected Lie group without property (H). By Theorem 3.3.1 of Cornulier in [19], there exists a *normal* subgroup R_T in G such that G/R_T has the Haagerup property, and the pair (G, R_T) has property (T). Since G/R_T has the Haagerup property (H), and G does not have property (H), the subgroup R_T is non-compact. By Theorem 2.5.3, the pair (G, R_T) has property $(T_{L_p([0,1])})$ for every $1 < p < \infty$. Hence, by Remark 4.2.5, G does not have property $(H_{L_p([0,1])})$.

(ii) $\Rightarrow$ (iii) : This is the result from [13], stated in Theorem 4.1.2.

(iii) $\Rightarrow$ (i) : We will show that G admits a closed subgroup H such that the quasi-regular representation $\lambda_{G/H} : G \to O(L_2(G/H))$ has almost invariant vectors and vanishing coefficients. Then we will conjugate this representation by the Mazur map to obtain the desired representation on $L_p(G/H)$.

Since the semi-simple part S of G has finite center, and since G is locally isomorphic to the direct product $\prod_i S_i \times M$, using Proposition 8.1 in [20], there exists a finite covering $p : G^\natural \to G$ such that $G^\natural$ is a direct product of closed connected subgroups $\prod_i S_i^\natural \times M^\natural$, where every $S_i^\natural$ is a simple Lie group with finite center, and $M^\natural$ is amenable.

Let $i \in I$. Let $S_i = SO(n_i, 1)$ or $S_i = SU(m_i, 1)$ be such that S_i is locally isomorphic to $S_i^\natural$. By the results in [58] and [48], there exists a lattice Γ_i in S_i such that $|\Gamma_i/[\Gamma_i, \Gamma_i]| = \infty$.

Set $G_1 = \prod_i S_i^\natural \times M^\natural$. We consider the closed subgroup of G_1 defined by

$$H_1 = \prod_{i \in I} [\Gamma_i, \Gamma_i] \times \{e\}.$$

We claim that the quasi-regular representation λ_{G_1/H_1} of G_1 on $L_2(G_1/H_1)$ has almost invariant vectors and vanishing coefficients. Indeed, we have

$$\lambda_{G_1/H_1} \simeq \otimes_{i \in I} \lambda_{S_i/[\Gamma_i,\Gamma_i]} \otimes \lambda_{M^\natural}, \ (1)$$

the right hand-side being the exterior tensor product of the representations.

We first show that λ_{G_1/H_1} has vanishing coefficients. Since the representation $\lambda_{M^\natural}$ has vanishing coefficients, it suffices to show that $\lambda_{S_i/[\Gamma_i,\Gamma_i]}$ has vanishing coefficients for every $i \in I$. By the Howe-Moore Theorem 4.2.3, this is the case if and only if

$$L_2(S_i/[\Gamma_i, \Gamma_i])^{\lambda_{S_i/[\Gamma_i,\Gamma_i]}(S_1)} = \{0\} \text{ for all } i \in I.$$

To show this, let $i \in I$ be fixed. Since $\Gamma_i/[\Gamma_i, \Gamma_i]$ is infinite, the space $l_2(\Gamma_i/[\Gamma_i, \Gamma_i])$ does not have non-zero $\lambda_{\Gamma_i/[\Gamma_i,\Gamma_i]}(\Gamma_i)$-invariant vector. Since S_i/Γ_i

carries a S_i-invariant finite measure, this implies by induction (see theorem E.3.1 in [8]) that

$$L_2(S_i/[\Gamma_i,\Gamma_i])^{\lambda_{S_i/[\Gamma_i,\Gamma_i]}(S_i)} = \{0\}$$

We have therefore proved that λ_{G_1/H_1} has vanishing coefficients.

Next we show that λ_{G_1/H_1} almost has invariant vectors. For this, it suffices to show that $\lambda_{M^\natural}$ and all $\lambda_{S_i/[\Gamma_i,\Gamma_i]}$ have almost invariant vectors, by formula (1). Indeed, by the Hulanicki-Reiter Theorem, this is clear for $\lambda_{M^\natural}$ since $M^\natural$ is amenable. Fix $i \in I$. Since $\Gamma_i/[\Gamma_i,\Gamma_i]$ is abelian and therefore amenable, we have $1_{\Gamma_i} \prec \lambda_{\Gamma_i/[\Gamma_i,\Gamma_i]}$. Since S_i/Γ_i has a finite S_i-invariant measure, we have also $1_{S_i} \prec \lambda_{S_i/[\Gamma_i,\Gamma_i]}$.

Denote by $\overline{S_i} = PSO(n_i,1)$ or $\overline{S_i} = PSU(m_i,1)$ the quotient of S_i by its (finite) center. Denote by G_2 and G_3 the groups

$$G_2 = \prod_{i\in I} \overline{S_i} \times M^\natural \text{ and } G_3 = \prod_{i\in I} S_i^\natural \times M^\natural.$$

Observe that we have three finite covering maps $p_1 : G_1 \to G_2$, $p_2 : G_3 \to G_2$ and p_3 $(= p) : G_3 \to G$. We apply now Lemma 4.4.4 successively to p_1, p_2 and p_3. We obtain the existence of a closed subgroup H in G such that G/H carries a G-invariant measure, and $\lambda_{G/H}$ almost has invariant vectors and has vanishing coefficients.

Let π^p be the orthogonal representation of G on $L_p(G/H)$ defined by the same formula as $\lambda_{G/H}$:

$$\pi^p(g)f(g'H) = f(g^{-1}g'H) \text{ for all } f \in L_p(G/H) \text{ and } g, g' \in G.$$

By proposition 2.5.1, π^p almost has invariant vectors. Since G/H carries a G-invariant measure, we have Moreover, for $x, y \in C_c(G/H)$, the matrix coefficient $g \mapsto < \pi^p(g)x, y >$ is in $C_0(G)$. By density of $C_c(G/H)$ in $L_p(G/H)$, the representation π^p has vanishing coefficients.

□

4.5 Properties (H_{C_p}) and (H_{S_p})

Let us study $(H_{L_p(\mathcal{M})})$ for the two discrete von Neumann algebras $\mathcal{B}(\mathcal{H})$ and $(\oplus\mathcal{M}_n)_\infty = \{x = \oplus_n x_n \mid \sup_n ||x_n|| < \infty\}$. The associated L_p-spaces are respectively the Schatten p-classes C_p and the space

$$S_p = \{x = \oplus_n x_n \mid x_n \in \mathcal{M}_n,\ \sum_n \mathrm{Tr}_n(|x_n|^p) < \infty\}.$$

We show that, for $p \neq 2$, property (H_{C_p}) is equivalent to property (H), and that only compact groups have property (H_{S_p}).

Theorem 4.5.1. *Let $1 \leq p < \infty$, $p \neq 2$. Let G be a locally compact second countable group. Then the following properties are equivalent.*
(i) G has property (H_{C_p}).
(ii) G has property (H).

Proof. (i) $\Rightarrow$ (ii) : Let $\pi^p : G \to O(C_p)$ be an orthogonal representation with vanishing coefficients and which almost has invariant vectors. Then by Remark 2.4.5, the conjugate representation $\pi^2 = M_{p,2} \circ \pi^p \circ M_{2,p}$ has the same form as π^p, hence π^2 has vanishing coefficients. By Proposition 2.5.1, π^2 almost has invariant vectors. Hence G has property (H).

(ii) $\Rightarrow$ (i) : Let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation of the group G on the Hilbert space $\mathcal{H}$ with almost invariant vectors and vanishing coefficients.
Define

$$\alpha_g(x) = \pi(g)x\pi(g^{-1}) \text{ for } x \in \mathcal{B}(\mathcal{H}).$$

Clearly, the previous formula defines an orthogonal representation $\alpha : G \to O(C_p)$. Let us show that α has vanishing coefficients.
By density of finite rank operators and linearity of the trace, it suffices to show that $\lim_{g\to\infty} \mathrm{Tr}(\alpha_g(x)y) = 0$ for x, y positive finite rank operators. This is straightforward to check. Indeed, we can write

$$x = \sum_{i=1}^{n} < ., \xi_i > \xi_i$$
$$y = \sum_{j=1}^{m} < ., \eta_j > \eta_j$$

Let (ζ_k) be an orthonormal basis of $\mathcal{H}$. Then, for vectors $\zeta_i, \eta_j \in \mathcal{H}$,

$$\begin{aligned}\mathrm{Tr}(\alpha_g(x)y) &= \sum_k < \alpha_g(x)y\zeta_k, \zeta_k > \\ &= \sum_k \sum_{j=1}^{m} \sum_{i=1}^{n} < \zeta_k, \eta_j >< \pi(g^{-1})\eta_j, \xi_i >< \pi(g)\xi_i, \zeta_k >\end{aligned}$$

Hence,

$$|\mathrm{Tr}(\alpha_g(x)y)| \leq \sum_{j=1}^{m} \sum_{i=1}^{n} | < \pi(g^{-1})\eta_j, \xi_i > |||\eta_j||||\xi_i||.$$

As π has vanishing coefficients, the right side of the inequality tends to 0 as g tends to infinity.

It remains to show that α has almost invariant vectors in C_p. In view of Proposition 2.5.1, it suffices to prove this for $p = 2$. This is the same proof as the one given for the third case of (ii) of Theorem 2.5.6.

$\square$

The following proposition implies that only compact groups have property (H_{S_p}) or property $(H_{L^p \oplus^p S_p})$ for $p \neq 2$.

Theorem 4.5.2. *Let G be a non-compact topological group. Let $p \neq 2$. There is no representation of G on S_p or on $L^p \oplus^p S_p$. Consequently, G does not have property (H_{S_p}) or property $(H_{L^p \oplus^p S_p})$.*

Proof. Assume, by contradiction, that we have a representation $\pi : G \to O(S_p)$ with vanishing coefficients. By Proposition 1.5.6, such a representation can be written as a sum $\pi = \oplus_n \pi_n$ of representations π_n on $\mathcal{M}_n$. For all n and $g \in G$, there exist $u_n(g)$, $v_n(g)$ unitaries in $\mathcal{M}_n$ such that $\pi_n(g)x = u_n(g)xv_n(g)$ or $\pi_n(g)x = u_n(g)(^t x)v_n(g)$ for all $x \in \mathcal{M}_n$. Since π has vanishing coefficients, each π_n has also vanishing coefficients. Hence

$$\lim_{g \to \infty} \mathrm{Tr}_n(u_n(g)v_n(g)x) = 0 \text{ for all } x \in \mathcal{M}_n.$$

This implies that every coefficient of the matrix $u_n(g)v_n(g)$ tends to 0 as g tends to ∞, which contradicts the facts that $||u_n(g)v_n(g)|| = 1$ for all $g \in G$ and that G is non-compact.

The claim about property $(H_{L^p \oplus^p S_p})$ is proved with a similar way, using the decomposition $O(L^p \oplus^p S_p) = O(L^p) \oplus O(S_p)$ from Proposition 1.5.9.

$\square$

4.6 Stronger versions of property $(H_{L_p(\mathcal{M})})$

We have seen that the Mazur map gives a mean to transfer orthogonal representations on $L_p(\mathcal{M})$ on orthogonal representations on $L_2(\mathcal{M})$. In order to study the relationship between properties $(H_{L_p(\mathcal{M})})$ and (H), the C_0-property of a representation has to be preserved during this process.

If we restrict to isometries with some further properties (positivity, measure preservation), the vanishing property of the involved representation is preserved. We do not know if the conjugation by the Mazur map preserves the vanishing property for all representations.

Definition 4.6.1. Let $\mathcal{M}$ be a semi-finite von Neumann algebra.
(i) $U \in O(L_p(\mathcal{M}))$ is said to be trace-preserving if it has the form $U = uJ$ in the Yeadon decomposition given in theorem 1.5.4.
(ii) U is said to be positive if it has the form $U = BJ$ in this decomposition. We denote by $O^\tau(L_p(\mathcal{M}))$ (resp. $O^+(L_p(\mathcal{M}))$) the subgroup of trace-preserving isometries in $O(L_p(\mathcal{M}))$ (resp. the subgroup of positive isometries in $O(L_p(\mathcal{M}))$).

Remark 4.6.2. The terminology in the previous definition is motivated by the two following facts :
- a trace-preserving isometry $U = uJ$ has a *Jordan* part which preserves the trace τ, that is $\tau(J(x)) = \tau(x)$ for all $x \in L_1(\mathcal{M}, \tau)$;
- a positive isometry BJ sends the positive cone $L_p(\mathcal{M})^+$ into itself.

Now we give the definitions of stronger versions of property $(H_{L_p(\mathcal{M})})$.

Definition 4.6.3. *Let G be a topological group. Let $\mathcal{M}$ be a semi-finite von Neumann algebra, and $1 \leq p < \infty$.*
We say that a group G has property $(H_{L_p(\mathcal{M}),+})$ (resp. $(H_{L_p(\mathcal{M}),\tau})$) if there exists a representation $\pi : G \to O^+(L_p(\mathcal{M}))$ (resp. a representation $\pi : G \to O^\tau(L_p(\mathcal{M}))$) which has vanishing coefficients and which almost has invariant vectors.

We have already seen that property $(H_{L_p(\mathcal{M})})$ implies property (H) when $\mathcal{M}$ is one of the following algebras : $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\mathcal{M} = L^\infty([0,1])$ for some Lie groups. The following theorem gives a similar result for the stronger versions of $(H_{L_p(\mathcal{M})})$.

Theorem 4.6.4. *Let G be a locally compact topological group. Let $\mathcal{M}$ be a semi-finite von Neumann algebra. Let $1 \leq p < \infty$.*
(i) Let $\mathcal{M}$ be a semi-finite von Neumann algebra and assume that G has property $(H_{L_p(\mathcal{M}),\tau})$. Then G has property (H).
(ii) Let $\mathcal{M}$ be a finite von Neumann algebra and assume that G has property $(H_{L_p(\mathcal{M}),+})$. Then G has property (H).

The proof of the previous theorem will be an easy consequence of the following proposition.

Proposition 4.6.5. *Let $1 \leq p < \infty$. (i) Let $\mathcal{M}$ be a semi-finite von Neumann algebra. Let $\pi^p : G \to O^\tau(L_p(\mathcal{M}))$ be a representation with vanishing coefficients. Then, for $1 \leq q < \infty$, its conjugate by the Mazur map $\pi^q = M_{p,q} \circ \pi^p \circ M_{q,p}$ has also vanishing coefficients.*
(ii) Let $\mathcal{M}$ be a finite von Neumann algebra. Let $\pi^p : G \to O^+(L_p(\mathcal{M}))$ be a representation with vanishing coefficients. Then, for $1 \leq q < \infty$, π^q has also vanishing coefficients.

Proof of proposition 4.6.5. (i) This is clear since $\pi^q = \pi^p$ on the common dense subset $\mathcal{M} \cap L_1(\mathcal{M})$ of $L_p(\mathcal{M})$ and $L_q(\mathcal{M})$ (see Remark 2.4.5).

(ii) Take a representation $\pi^p : G \to O^+(L_p(\mathcal{M}))$ with vanishing coefficients. We have $\pi^p(g)(x) = B_g J_g(x)$ for all $g \in G$ and $x \in L_p(\mathcal{M})$. If π^p has vanishing coefficients, its contragradient representation $(\pi^p)'$ has also vanishing coefficients. This latter representation is given by

$$(\pi^p)'(g)(x) = B_g^{\frac{p}{p'}} J_g(x) \text{ for all } g \in G \text{ and } x \in L_{p'}(\mathcal{M}).$$

Since every element in $\mathcal{M}$ is a linear combination of positive operators or a linear combination of unitaries, it suffices to show that the conjugate $\pi^q = M_{p,q} \circ \pi^p \circ M_{q,p} = B^{\frac{p}{q}} J$ has vanishing coefficients on positive elements or unitaries.

Assume $q \in [p, p']$ (or $q \in [p', p]$). Let $t \in [0,1]$ be such that $\frac{p}{q} = t + (1-t)\frac{p}{p'}$. Then, for x a positive element in $\mathcal{M} \cap L_1(\mathcal{M})$, and y a unitary in $\mathcal{M}$, we have

$$\begin{aligned}
\tau(B_g^{\frac{p}{q}} J_g(x)y) &= \tau(B_g^{t+(1-t)\frac{p}{p'}} J_g(x^{t+(1-t)})y^{t+(1-t)}) \\
&= \tau(B_g^t y^t J_g(x^t)\ (B_g^{\frac{p}{p'}})^{1-t} x^{1-t} y^{1-t}) \\
&\leq \tau(B_g|y^t J_g(x^t)|^{\frac{1}{t}})^t \tau(B_g^{\frac{p}{p'}} |J_g(x^{1-t})y^{1-t}|^{\frac{1}{1-t}})^{1-t}
\end{aligned}$$

using the tracial property of τ and Hölder's inequality with conjugate exponents $\frac{1}{t}$ and $\frac{1}{1-t}$. Since y is unitary, we have $|ya| = |a|$ and $|ay| = y^*|a|y$ for all a and therefore

$$\tau(B_g^{\frac{p}{q}} J_g(x)y) \leq \tau(\pi^p(g)(x))^{\frac{1}{t}} \tau((\pi^p)'(g)(x))^{\frac{1}{1-t}}.$$

Assume $q \in [1, p]$. Let $t \in [0,1]$ be such that $\frac{p}{q} = tp + (1-t)$. We have $\tau(B_g^p) = \tau(1) = 1$ for all $g \in G$. Then by the same computation as before, we have

$$\begin{aligned}
\tau(B_g^{\frac{p}{q}} J_g(x)y) &\leq \tau(B_g^p)^t \tau(B_g|J_g(x)y|^{\frac{1}{1-t}})^{1-t} \\
&= \tau(B_g|J_g(x)y|^{\frac{1}{1-t}})^{1-t} \\
&= \tau(\pi^p(g)(x^{\frac{1}{1-t}}))^{1-t}.
\end{aligned}$$

Assume that $q \in [2, \infty[$, then $q' \in [1, q]$ and by the previous case, $\pi^{q'}$ has vanishing coefficients. Then π^q has vanishing coefficients for all $q \in [1, \infty[$. □

Proof of theorem 4.6.4. (i) Let π^p be a *trace-preserving* on $L_p(\mathcal{M})$ which has vanishing coefficients and almost has invariant vectors. Then the conjugate representation π^2 has vanishing coefficients by (i) in Proposition 4.6.5 and almost has invariant vectors by Proposition 2.5.1.
(ii) The proof is similar as the previous one above, using (ii) in Proposition 4.6.5. □

Remark 4.6.6. Let $\pi : G \to O(L_p(\mathcal{M}))$ be an orthogonal representation of the group G with vanishing coefficients. We can define its positive part $\pi^+ : G \to O(L_p(\mathcal{M}))$ by $\pi^+(g) = B_g J_g$ for all $g \in G$ if $\pi(g) = u_g B_g J_g$. One might be tempted to think that π^+ is a representation with vanishing coefficients. This would imply that $(H_{L_p(\mathcal{M})})$ and $(H_{L_p(\mathcal{M},+)})$ are equivalent. However, the first problem is that π^+ is not necessarily a group homomorphism (compare with the formulas in Theorem 2.4.4). But even if π^+ happens to be a representation, π^+ can have non vanishing matrix coefficients, as the following example shows.

Let $G = SL_2(\mathbb{R})$. The free group $H = \mathbb{F}_2$ on two generators embeds as a finite index subgroup in the lattice $SL_2(\mathbb{Z})$, and hence as a lattice in G. Let χ be a non-trivial character on H (such a character exists since the abelianized group of H is non-trivial). Let $\pi = \mathrm{Ind}_H^G \chi$. There exists a cocycle $c : G \times G/H \to H$ such that

$$\pi(g)f(g'H) = \chi(c(g^{-1}, g'H))f(g^{-1}g'H) \text{ for all } f \in L_2(G/H),\ g, g' \in G.$$

The positive part π^+ of $\pi = \mathrm{Ind}_H^G \chi$ is the quasi-regular representation on $l_2(G/H)$, which has a non-zero invariant vector since H has finite covolume in G. Hence, π^+ has non-vanishing coefficients.

On the other hand, π does not have non-zero invariant vectors since χ is non-trivial on H (see Theorem E.3.1 in [8]). Hence, by the Howe-Moore theorem 4.2.3, π has vanishing coefficients.

Corollary 4.6.7. *Let $\mathcal{M}$ be a finite factor and G a locally compact group with property $(H_{L_p(\mathcal{M})})$. Then G has property (H).*

Chapter 5

Strongly mixing representations on $L_p(\mathcal{M})$

In this chapter, we define and study strongly mixing *representations* on non-commutative L_p-spaces associated with finite von Neumann algebras. We give a variant of the Haagerup property for strongly mixing representations on $L_p(\mathcal{M})$, which seems to be closer to (H) than property $(H_{L_p(\mathcal{M})})$. The proofs of our results rely on two important constructions of strongly mixing actions on measures spaces; we recall them in the first part of this chapter with some useful facts from ergodic theory. The second part of the chapter is devoted to our results.

5.1 Strongly mixing actions on measured spaces

The aim of this section is to recall two constructions from ergodic theory which we will use use in our proofs later.

5.1.1 The Connes-Weiss construction

We first recall some definitions from ergodic theory. Let G be a locally compact second countable group that acts on a standard probability space (X, μ) by measure-preserving Borel automorphisms.

Definition 5.1.1. - The action of G on (X, μ) is said to be strongly mixing if, for all Borel subsets A and B of X,

$$\lim_{g\to\infty} \mu(g^{-1}A \cap B) = \mu(A)\mu(B)$$

that is, for all $\epsilon > 0$, there exists a compact subset $K \subset G$ such that

$$|\mu(g^{-1}A \cap B) - \mu(A)\mu(B)| < \epsilon \text{ for all } g \in G \setminus K.$$

- A sequence of Borel subsets (A_n) of X is said to be asymptotically invariant if, for all compact subsets K of G,

$$\lim_{n\to\infty} \sup_{g\in K} \mu(g^{-1}A_n \triangle A_n) = 0.$$

It is said to be non-trivial if moreover $\inf_n \mu(A_n)(1-\mu(A_n)) > 0$.
- A sequence of nonnull Borel subsets (A_n) of X is said to be a Følner sequence if, for all compact subset $K \subset G$,

$$\lim_{n\to\infty} \mu(A_n) = 0 \textit{ and } \lim_{n\to\infty} \sup_{g\in K} \frac{\mu(g^{-1}A_n \triangle A_n)}{\mu(A_n)} = 0.$$

The following theorem of Connes and Weiss gives a construction of a strongly mixing action with a nontrivial asymptotically invariant sequence for every non-Kazhdan group.

Theorem 5.1.2. ([31] and [18]) *Let G be a second countable group which does not have property (T). There exists a measure-preserving G-action on a probability space (X,μ), which is strongly mixing and which has a nontrivial asymptotically invariant sequence.*

5.1.2 A result of Jolissaint

We recall here the analog of the previous definitions appearing in ergodic theory, but in a non-commutative setting.

Let G be a second countable locally compact group. Let $\mathcal{M}$ be a von Neumann algebra with separable predual, and φ be a faithful normal state on $\mathcal{M}$. Assume that there exists a G-action $\alpha : G \to Aut(\mathcal{M})$ by automorphisms of $\mathcal{M}$ such that φ is α-invariant, that is

$$\varphi(\alpha_g(x)) = \varphi(x) \text{ for all } g \in G,\ x \in \mathcal{M}.$$

Definition 5.1.3. - The action of G on $(\mathcal{M},\varphi)$ is said to be strongly mixing if, for all $x, y \in \mathcal{M}$,

$$\lim_{g\to\infty} \varphi(\alpha_g(x)y) = \varphi(x)\varphi(y).$$

- A sequence of projections (e_n) in $\mathcal{M}$ is said to be asymptotically invariant if, for all compact subsets K of G,

$$\lim_{n\to\infty} \sup_{g\in K} \varphi(|\alpha_g(e_n) - e_n|^2) = 0.$$

It is said to be non-trivial if moreover $\inf_n \varphi(e_n)(1-\varphi(e_n)) > 0$.
- A sequence of nonnull projections (e_n) in $\mathcal{M}$ is said to be a Følner sequence if, for all compact subset $K \subset G$,

$$\lim_{n\to\infty} \varphi(e_n) = 0 \textit{ and } \lim_{n\to\infty} \sup_{g\in K} \frac{\varphi(|\alpha_g(e_n)-e_n|^2)}{\varphi(e_n))} = 0.$$

The following theorem, due to Jolissaint, gives an analog of the Connes-Weiss construction, in this non-commutative context.

Theorem 5.1.4. ([13]) *Let G be a locally compact second countable group with the Haagerup property (H). Then, for each factor $\mathcal{M}$ listed below, there exist an action of G on $\mathcal{M}$ by automorphisms, and an α-invariant state φ on $\mathcal{M}$ for which α is strongly mixing, and such that $\mathcal{M}$ contains a Følner sequence and a non-trivial asymptotically invariant sequence for α and φ :*
(i) $\mathcal{M}$ is the hyperfinite II_1 factor R, and φ is the canonical trace τ.
(ii) $\mathcal{M}$ is the factor $R_{0,1} = R \otimes \mathcal{B}(l_2)$ of type II_∞ and $\varphi = \tau \otimes \omega$, where ω is a suitable normal state on $\mathcal{B}(l_2)$.
(iii) $\mathcal{M}$ is the Powers factor R_λ of type III_λ, and $\varphi = \varphi_\lambda$ is the associated Powers state.

5.2 Strongly mixing actions on $L_p(\mathcal{M})$

We assume in this section that $\mathcal{M}$ is a finite von Neumann algebra, equipped with a finite trace τ. We will always assume in this chapter that τ is *normalized*, that is $\tau(1) = 1$, and that τ is *faithful*, that is $\tau(x) > 0$ if $x \in \mathcal{M}^+$, $x \neq 0$.

5.2.1 Property $(H^{mix}_{L_p(\mathcal{M})})$

Let G be a locally compact group. Given a finite von Neumann algebra, we define a notion of strongly mixing orthogonal representation of G on $L_p(\mathcal{M})$.

Definition 5.2.1. Let $\mathcal{M}$ be a finite von Neumann algebra with trace τ. We say that a representation $\pi : G \to O(L_p(\mathcal{M}))$ is *strongly mixing* if

$$\lim_{g\to\infty} \tau(\pi(g)(x)y) = \tau(x)\tau(y) \text{ for all } x, y \in \mathcal{M}.$$

We will study the following variant of property (H) with strongly mixing orthogonal representations.

Definition 5.2.2. Let $\mathcal{M}$ be a finite von Neumann algebra. We say that a group G has property $(H^{\mathrm{mix}}_{L_p(\mathcal{M})})$ if there exists a representation $\pi : G \to O(L_p(\mathcal{M}))$ which is strongly mixing and which almost has invariant vectors in the complement $L_p(\mathcal{M})'$ of the $\pi(G)$-invariant vectors.

Here are our main results concerning the relationship between property $(H^{mix}_{L_p(\mathcal{M})})$ and property (H).

Theorem 5.2.3. *Let G be a locally compact group. Let $\mathcal{M}$ be a finite von Neumann algebra, and let $1 \leq p < \infty$. If G has property $(H^{mix}_{L_p(\mathcal{M})})$, then G has the Haagerup property (H).*

Theorem 5.2.4. *Let G be a locally compact group with the Haagerup property (H). Let $1 \leq p < \infty$, $p \neq 2$. Then G has property $(H^{mix}_{L_p(\mathcal{M})})$ in the two following cases :*
(i) $(\mathcal{M}, \tau) = (L^\infty([0,1]), \lambda)$ with λ the Lebesgue measure ;
(ii) $\mathcal{M} = R$ is the hyperfinite II_1 factor.

The proofs of the previous theorems will be given in the next subsection. The main technical tools for these proofs are the two following lemmas.

Lemma 5.2.5. *Let G be a locally compact group. Let $\mathcal{M}$ be a finite von Neumann algebra, and let $1 \leq p < \infty$, $p \neq 2$. Let $\pi : G \to O(L_p(\mathcal{M}))$ be a strongly mixing orthogonal representation. Then $\pi(g)$ is a Jordan *-automorphism of $\mathcal{M}$ for every $g \in G$.*

The following corollary is a straightforward consequence of the previous lemma, using Remark 2.4.5.

Corollary 5.2.6. *Let $1 \leq p < \infty$, $p \neq 2$, and $1 \leq q < \infty$. Let $\pi^p : G \to O(L_p(\mathcal{M}))$ be a strongly mixing representation. Then the conjugate representation π^q is strongly mixing.*

The following lemma asserts that the multiples of the unit $1 \in \mathcal{M}$ are the only invariant vectors for a strongly mixing representation. We set

$$L_p^0(\mathcal{M}) = \{\ x \in L_p(\mathcal{M}) \mid \tau(x) = 0\ \}.$$

Lemma 5.2.7. *Let G be a locally compact group. Let $\mathcal{M}$ be a finite von Neumann algebra and let $1 \leq p < \infty$. Let $\pi : G \to O(L_p(\mathcal{M}))$ be a strongly mixing representation. Then $L_p(\mathcal{M})'(\pi) = L_p^0(\mathcal{M})$.*

5.2.2 Proofs

Proof of Lemma 5.2.5. Let $g \in G$. By Yeadon's theorem 1.5.4, $\pi(g)$ has a decomposition

$$\pi(g) = u_g B_g J_g$$

with u_g a unitary in $\mathcal{M}$, B_g a positive operator affiliated with $\mathcal{M}$ such that its spectral projections commute with $\mathcal{M}$, and J_g a *-Jordan automorphism. Set

$v_g = u_g B_g$ for all $g \in G$.

We will show that $u_g = 1$ and $B_g = 1$ for all $g \in G$. We claim that it suffices to give the proof when $p > 2$. Indeed, if $p < 2$, let $\pi^p(g) = u_g B_g J_g$ for every $g \in G$. By Proposition 2.4.6, the contragradient representation $(\pi^p)'$ of π^p on $L_{p'}$, with $p' > 2$ the conjugate exponent of p, is given by the following formula

$$(\pi^p)'(g)x = u_g^* B_g^{\frac{p}{p'}} u_g J_g(x) u_g^* \text{ for all } g \in G, x \in \mathcal{M}.$$

Moreover, the contragradient is obviously strongly mixing. Hence, if the claim is true for $p' > 2$, then $u_g^* = 1 = u_g$, $B_g^{\frac{p'}{p}} = 1 = B_g$ and the claim is true for p.

So we can assume that $p > 2$. For $g \in G$, $J_g(1) = 1$ since J_g is a sum of a *-algebra morphism and a *-algebra antimorphism by Theorem 1.3.6. Since π is strongly mixing, for $x = 1$, we have

$$\lim_{g\to\infty} \tau(\pi(g)(y)) = \lim_{g\to\infty} \tau(\pi(g)(y)1) = \tau(y) \text{ for all } y \in \mathcal{M}.$$

Therefore, for $y = 1$, we obtain

$$\lim_{g\to\infty} \tau(v_g) = 1.$$

On the other hand, for $g_0 \in G$ be fixed, we have

$$\lim_{g\to\infty} \tau(v_{gg_0}) = \lim_{g\to\infty} \tau(\pi(g)\pi(g_0)(1)) = \tau(\pi(g_0)(1)) = \tau(v_{g_0}).$$

Hence $\tau(v_g) = 1$ for all $g \in G$.

Let $g \in G$ be fixed. Since $\pi(g) \in O(L_p(\mathcal{M}))$ and $1 \in L_p(\mathcal{M})$, we have $\tau(|\pi(g)1|^p) = ||1||_p^p = 1$, that is $\tau(B_g^p) = 1$. Using twice Hölder's inequality, we have

$$1 = \tau(u_g B_g) \le \tau(B_g) \le \tau(B_g^t)^{1/t} \le \tau(B_g^p)^{1/p} = 1 \text{ for } 1 \le t \le p,$$

and it follows that $\tau(B_g^2) = 1$. Now by the Cauchy-Schwarz inequality,

$$1 = \tau(v_g) \le \sqrt{\tau(B_g^2)} = 1.$$

The equality case gives that $v_g = u_g B_g = 1$. From the uniqueness in the polar decomposition, it follows that $u_g = 1$ and $B_g = 1$. Hence the lemma is proved. □

Proof of Lemma 5.2.7. Let $(\pi^p)' = \pi^{p'} : G \to O(L_{p'}(\mathcal{M}))$ be the contragradient representation of $\pi^p : G \to O(L_p(\mathcal{M}))$. Let $x \in L_p^0(\mathcal{M})$, and $y \in L_{p'}(\mathcal{M})^{\pi^{p'}(G)}$. Then,

$$\tau(yx) = \lim_{g\to\infty} (\pi^{p'}(g)(y)x) = \lim_{g\to\infty} (y\pi^p(g^{-1})(x)) = \tau(y)\tau(x) = 0.$$

Thus $L_p^0(\mathcal{M}) \subset L_p(\mathcal{M})'(\pi^p)$.

To show that $L_p(\mathcal{M})'(\pi^p) \subset L_p^0(\mathcal{M})$, it suffices to show that $1 \in L_{p'}(\mathcal{M})^{\pi^{p'}(G)}$. Indeed, if $1 \in L_{p'}(\mathcal{M})^{\pi^{p'}(G)}$, then for every $x \in L_p(\mathcal{M})'(\pi^p)$, we have

$$\tau(x) = \tau(x1) = 0.$$

Now let $g \in G$. By Lemma 5.2.5, $\pi^{p'}(g)$ is a Jordan *-automorphism, so that $\pi^{p'}(g)(1) = 1$. □

Proof of Theorem 5.2.4. (ii) By Theorem 5.1.4, there exists an action π of G by automorphisms on the hyperfinite II_1 factor R such that
- π is strongly mixing,
- there exists a non-trivial asymptotically invariant sequence (e_n) of projections in R, that is there exists a sequence of projections (e_n) such that $\tau(e_n) = 1/2$ for all n and

$$\lim_{n\to\infty} \sup_{g\in K} ||\pi(g)(e_n) - e_n||_2 = 0 \text{ for all compact } K \text{ of } G.$$

This action defines a unitary representation $\pi^2 : G \to \mathcal{U}(L_2(R))$ by

$$\pi^2(g)x = \pi(g)(x) \text{ for all } g \in G,\ x \in R,$$

as well as an orthogonal representation $\pi^p : G \to O(L_p(R))$ for every $1 \le p < \infty$. It is clear that π^p is strongly mixing in the sense of Definition 5.2.1.

By Lemma 5.2.7, $L_2^0(R) = L_2(R)'(\pi^2)$. Define a sequence (v_n) in R by

$$v_n = e_n - \tau(e_n)1.$$

Then (v_n) is a sequence of almost invariant vectors for π in $L_2^0(R) = L_2(R)'(\pi^2)$. It is straightforward to check that $||v_n||_2^2 = 1/4$. Hence, π^2 has almost invariant vectors in $L_2(R)'(\pi^2)$. By Proposition 2.5.1, π^p has almost invariant vectors in $L_p(R)'(\pi^p)$.

(i) The proof is similar as the previous proof, using Theorem 5.1.2 based on the Connes-Weiss construction instead of Theorem 5.1.4 of Jolissaint. □

Proof of Theorem 5.2.3. Let π^p be a strongly mixing representation of G on $L_p(\mathcal{M})$ which almost has invariant vectors in $L_p(\mathcal{M})'(\pi^p)$. Then the conjugate representation π^2 defines a strongly mixing representation on $L_2(\mathcal{M})$ by Corollary 5.2.6.

By Proposition 2.5.1, π^2 almost has invariant vectors in $L_2(\mathcal{M})'(\pi^2)$. By Lemma 5.2.7, we have $L_2(\mathcal{M})'(\pi^2) = L_2^0(\mathcal{M})$. Since π^2 is strongly mixing, the restriction $\pi^2_{/L_2(\mathcal{M})'(\pi^2)}$ of π^2 to $L_2(\mathcal{M})'(\pi^2)$ almost has invariant vectors and has vanishing coefficients. Hence G has property (H). □

Chapter 6

Proper actions by affine isometries on $L_p(\mathcal{M})$

In this chapter, we consider a more geometric approach to the Haagerup property (H) by studying the existence of proper actions by affine isometries on non-commutative L_p-spaces. We will use a terminology already used in [12] : a group is said to be a-$FL_p(\mathcal{M})$-menable if it admits a proper action by affine isometries on $L_p(\mathcal{M})$.
We relate a-$FL_p(\mathcal{M})$-menability with the property $(H_{L_p(\mathcal{M})})$ that we have introduced in chapter 5 : we show that if a locally compact second countable group G has property $(H_{L_p(\mathcal{M})})$, then G is a-$FL_p(\tilde{\mathcal{M}})$-menable for $\tilde{\mathcal{M}} = l^\infty(\mathcal{M})$ or $\tilde{\mathcal{M}} = \mathcal{M} \otimes \mathcal{B}(l_2)$. We also show that every group with the Haagerup property (H) admits a proper action by affine isometries on $L_p(R \otimes \mathcal{B}(l_2))$, where $R \otimes \mathcal{B}(l_2)$ is the hyperfinite II_∞ factor associated to the hyperfinite II_1 factor R.
In [85], Yu showed that every hyperbolic group $\Gamma \times \Gamma$ admits a proper action by affine isometries on $l_p(\Gamma \times \Gamma)$ for p large enough. We show that his construction yields a proper action of Γ by affine isometries on the non-commutative L_p-space C_p for p sufficiently large.

6.1 Introduction

Let G be a topological group, and let B be a real Banach space. Let $\alpha : G \to \mathrm{Isom}(B)$ be a continuous action of G on B by affine isometries. Let $\pi : G \to O(B)$ be the linear part and $b : G \to B$ the translation part of α, so that

$$\alpha(g)x = \pi(g)x + b(g) \text{ for all } g \in G, x \in B.$$

Recall that b is a 1-cocycle on G with values in B.

Definition 6.1.1. (i) The action α is *proper* if, for every bounded subset X of B, the set $\{g \in G \mid \alpha(g)X \cap X \neq \emptyset\}$ is relatively compact in G.

This is equivalent to the fact that the cocycle is proper :

$$\lim_{g\to\infty} ||b(g)|| = +\infty.$$

(ii) The group G is said to be a-FB-menable if G admits a proper action by affine isometries on B.

Gromov defined in [32] a-T-menable groups as those groups G which admit a proper action by affine isometries on Hilbert spaces. It turned out that the class of a-T-menable groups coincide with the class of groups with the Haagerup property (see [13]).

Proper actions by affine isometries on commutative L_p-spaces were studied by Nowak in [63], and Chatterji-Drutu-Haglund in [12]. The following characterization of the Haagerup property was announced in [63] and proved in an updated version in [64] (see also Corollary 1.5 in [12]).

Theorem 6.1.2. ([64]) *Let $1 \leq p < 2$ and let G be a second countable locally compact group. Then the following conditions are equivalent :*

1. *G has the Haagerup property (H).*
2. *G admits a proper action on $L^p([0,1])$ by affine isometries.*

Let us mention also the following result about proper actions on some strictly convex spaces. It is due to Haagerup and Przybyszewska in [38].

Theorem 6.1.3. *Let G be a locally compact second countable group. Then G admits a proper action by affine isometries on the l_2-sum*

$$\bigoplus_{n=1}^{\infty} L_{2n}(G,\mu)$$

where μ is the Haar measure on G.

6.2 Proper actions on $L_p(\mathcal{M})$ and property (H)

The following theorem shows that a group which has the property $(H_{L_p(\mathcal{M})})$ introduced in chapter 5, has a proper action by affine isometries on a non-commutative L_p-space associated to an amplification of $\mathcal{M}$.

Theorem 6.2.1. *Let G be a locally compact second countable group, let $\mathcal{M}$ be a von Neumann algebra and let $1 \leq p < \infty$. Assume that G has property $(H_{L_p(\mathcal{M})})$. Then there exists a proper action of G by affine isometries on $L_p(l^\infty \otimes \mathcal{M})$.*

Proof. Let $\pi : G \to O(L_p(\mathcal{M}))$ a representation which has vanishing coefficients and almost invariant vectors. Let $(v_n)_n$ be a sequence of almost invariant vectors for π.

We claim that $b : G \to \bigoplus_n^p L_p(\mathcal{M})$ defined by

$$b(g) = \oplus_n \pi(g)(v_n) - v_n$$

is a proper cocycle with values in $\bigoplus_n^p L_p(\mathcal{M})$, the l_p-sum of infinite many copies of the space $L_p(\mathcal{M})$. This cocycle is associated to the orthogonal representation $\oplus_n \pi : G \to O(\bigoplus_n^p L_p(\mathcal{M}))$, defined for all $g \in G$ and all $\oplus_n x_n \in \bigoplus_n^p L_p(\mathcal{M})$ by

$$\oplus_n \pi(g)(\oplus_n x_n) = \oplus_n \ (\pi(g)x_n).$$

Let (K_n) be an increasing sequence of compact subsets such that $G = \cup_n K_n$. Since (v_n) is a sequence of almost invariant vectors for π, we can assume (passing to a subsequence of the v_n's if necessary) that

$$\sup_{g \in K_n} ||\pi(g)v_n - v_n||_p^p \leq 1/2^n \text{ for all } n \in \mathbb{N}.$$

Fix $N \in \mathbb{N}$ and let $g \in K_N$. Then $g \in K_n$ for all $n \geq N$ and hence

$$||\pi(g)v_n - v_n||_p^p \leq 1/2^n \text{ for all } n \geq N.$$

Thus $b(g)$ is well-defined as an element in $\oplus_n^p L_p(\mathcal{M})$. It is obvious that b is a cocycle for $\oplus_n \pi$.

Moreover, for all $g \in G$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} ||\pi(g)v_n - v_n||_p &= \sup_{||a||_{p'}=1} \mathrm{Tr}((\pi(g)v_n - v_n)a) \\ &\geq \mathrm{Tr}((v_n - \pi(g)v_n)M_{p,p'}(v_n)^*) \\ &= \mathrm{Tr}(v_n M_{p,p'}(v_n)^*) - \mathrm{Tr}(\pi(g)v_n M_{p,p'}(v_n)^*) \\ &= 1 - \mathrm{Tr}(\pi(g)v_n M_{p,p'}(v_n)^*). \end{aligned}$$

Since π has vanishing coefficients, it follows that

$$\lim_{g \to \infty} ||\pi(g)v_n - v_n||_p = 1.$$

This shows that $\lim_{g\to\infty} ||b(g)||_p = +\infty$. Hence, the affine isometric action associated to $(\oplus_n \pi, b)$ on $\bigoplus_n^p L_p(\mathcal{M})$. The latter space is isometrically isomorphic to $L_p(l^\infty \otimes \mathcal{M})$, hence there exists a proper cocycle of G with values in $L_p(l^\infty \otimes \mathcal{M})$. $\square$

As a consequence of the previous proposition, property $(H_{L_p(\mathcal{M})})$ implies a-$FL_p(\mathcal{M})$-menability for $\mathcal{M}$ a factor of type II_∞.

Proposition 6.2.2. *Let G be a locally compact second countable group. Let $\mathcal{M}$ be a factor of type II_∞. Let $1 \leq p < \infty$. Assume that G has property $(H_{L_p(\mathcal{M})})$. Then G is a-$FL_p(\mathcal{M})$-menable.*

We will need two lemmas for the proof of Proposition 6.2.2. The first lemma is a well-known result on von Neumann algebras.

Lemma 6.2.3. *Let $\mathcal{M}$ be a von Neumann algebra of type II_∞. Then $\mathcal{M}$ is isomorphic, as a $*$-algebra, to $\mathcal{B}(l_2) \otimes \mathcal{M}$. In particular, $L_p(\mathcal{M})$ is isometrically isomorphic to $L_p(\mathcal{B}(l_2) \otimes \mathcal{M})$ for $1 \leq p < \infty$, $p \neq 2$.*

Proof. We only recall how to get the desired isomorphism between $\mathcal{M}$ and $\mathcal{B}(l_2) \otimes \mathcal{M}$. Since $\mathcal{M}$ is of type II_∞, there exist a sequence of pairwise orthogonal projections $(P_i)_{i\in I}$ in $\mathcal{M}$, and a sequence of partial isometries $(v_i)_{i\in I}$ in $\mathcal{M}$, such that

$$\begin{aligned} 1 &= \sum_{i\in I} P_i \\ 1 &= v_i^* v_i \\ P_i &= v_i v_i^* . \end{aligned}$$

Recall from [27] (see chapter I, Proposition 4) that the elements of $\mathcal{B}(l_2) \otimes \mathcal{M}$ can be identified with matrices $(a_{i,j})$ with coefficients $a_{i,j}$ in $\mathcal{M}$. Then the $*$-algebra isomorphism between $\mathcal{M}$ and $\mathcal{B}(l_2) \otimes \mathcal{M}$ is given by

$$\begin{aligned} \Phi :\ & \mathcal{M} \to \mathcal{B}(l_2) \otimes \mathcal{M} \\ & x \ \mapsto (v_i x v_j^*)_{i,j} . \end{aligned}$$

□

Let $\mathcal{M}$ be a factor. Recall from Remark 1.4.4 that if $J : \mathcal{M} \to \mathcal{M}$ is a Jordan morphism, then J is a $*$-algebra morphism, or J is a $*$-algebra antimorphism. Moreover, the Yeadon's decomposition of an element $U \in O(L_p(\mathcal{M}))$ has Radon-Nikodym derivative B equal to 1, that is $U = uJ$, where u is a unitary in $\mathcal{M}$ and J a Jordan isomorphism of $\mathcal{M}$ (see Remark 1.5.5).

Lemma 6.2.4. *Let $\mathcal{M}$ be a von Neumann algebra which is a factor. Let $1 \leq p < \infty$, $p \neq 2$. Let $U \in O(L_p(\mathcal{M}))$ and let $U = uJ$ its Yeadon's decomposition. Define the map $T : \mathcal{B}(l_2) \to \mathcal{B}(l_2)$ by*

$$\begin{aligned} Tx &= x \text{ if } J \text{ is a } *\text{-algebra morphism },\\ Tx &= \overline{x^*} \text{ if } J \text{ is a } *\text{-algebra antimorphism.} \end{aligned}$$

Define also $\tilde{u} \in \mathcal{B}(l_2) \otimes \mathcal{M}$ and $\tilde{J} : \mathcal{B}(l_2) \otimes \mathcal{M} \to \mathcal{B}(l_2) \otimes \mathcal{M}$ by

$$\begin{aligned} \tilde{u} &= 1 \otimes u \ ,\\ \tilde{J}(x \otimes y) &= Tx \otimes J(y) \text{ for all } x \in \mathcal{B}(l_2) \text{ and } y \in \mathcal{M}. \end{aligned}$$

Then the formula $\tilde{U}(a) = \tilde{u}\tilde{J}(a)$, for $a \in L_p(\mathcal{B}(l_2) \otimes \mathcal{M})$, defines a linear bijective isometry $\tilde{U} : L_p(\mathcal{B}(l_2) \otimes \mathcal{M}) \to L_p(\mathcal{B}(l_2) \otimes \mathcal{M})$, whose Yeadon's decomposition is $\tilde{U} = \tilde{u}\tilde{J}$.

Proof. We first show that $\tilde{U}$ takes its values in $L_p(\mathcal{B}(l_2) \otimes \mathcal{M})$. By linearity, and since the linear subspace generated by $\{A \otimes x \mid A \in \mathcal{B}(l_2), x \in \mathcal{M}\}$ is dense (in the strong operator topology) in $\mathcal{B}(l_2) \otimes \mathcal{M}$, it suffices to prove

$$\tilde{\tau}(\tilde{J}(A \otimes x)) = \tilde{\tau}(A \otimes x) \text{ for all } A \in \mathcal{B}(l_2)_+, x \in \mathcal{M}_+ \cap L_0(\mathcal{M}).$$

Let $A \in \mathcal{B}(l_2)_+$ and $x \in \mathcal{M}_+ \cap L_0(\mathcal{M})$. Then we have

$$\begin{aligned}
\tilde{\tau}(\tilde{J}(A \otimes x)) &= \tilde{\tau}(T(A) \otimes J(x)) \\
&= \operatorname{Tr}(\overline{A^*})\ \tau(J(x)) \\
&= \operatorname{Tr}(A)\ \tau(x) \\
&= \tilde{\tau}(A \otimes x).
\end{aligned}$$

Now we check that the elements $\tilde{u}$, and $\tilde{J}$ give the Yeadon's decomposition of $\tilde{U}$.

It is clear that $\tilde{u}$ is a unitary in $\mathcal{B}(l_2) \otimes \mathcal{M}$, since u is a unitary in $\mathcal{M}$.

Now we check that $\tilde{J}$ is a Jordan-$*$-isomorphism of the algebra $\mathcal{B}(l_2) \otimes \mathcal{M}$. The fact that $\tilde{J}$ is a linear $*$-isomorphism of $\mathcal{B}(l_2) \otimes \mathcal{M}$ is clear. We only have to show that $\tilde{J}$ is Jordan, that is $\tilde{J}(a^2) = \tilde{J}(a)^2$ for all $a \in \mathcal{B}(l_2) \otimes \mathcal{M}$. By density, it suffices to prove the latter relation on finite sums of the form $\sum_{n \in I} A_n \otimes x_n$. We may assume that J is a $*$-algebra antimorphism: the computation when J is a $*$-algebra morphism is the same and simpler. Let I be a finite set, $A_n \in \mathcal{B}(l_2)$, $x_n \in \mathcal{M}$ for $n \in I$. Then we have

$$\begin{aligned}
\tilde{J}((\sum_{n \in I} A_n \otimes x_n)^2) &= T \otimes J(\sum_{i,j \in I} A_i A_j \otimes x_i x_j) \\
&= \sum_{i,j \in I} \overline{(A_i A_j)^*} \otimes J(x_i x_j) \\
&= \sum_{i,j \in I} \overline{A_j^* A_i^*} \otimes J(x_j) J(x_i) \\
&= (\sum_{n \in I} \overline{A_n^*} \otimes J(x_n))^2 \\
&= (\tilde{J}(\sum_{n \in I} A_n \otimes x_n))^2.
\end{aligned}$$

□

Now we give the proof of Proposition 6.2.2.

Proof of Proposition 6.2.2. Let π be an orthogonal representation of G on $L_p(\mathcal{M})$ such that π has almost invariant vectors and has vanishing coefficients. We first extend π to an orthogonal representation $\tilde{\pi}$ of G on the space $L_p(\mathcal{B}(l_2) \otimes \mathcal{M})$.

For every $g \in G$, define $T_g : \mathcal{B}(l_2) \to \mathcal{B}(l_2)$ and $\widetilde{\pi(g)}$ as in Lemma 6.2.4. By Lemma 6.2.4, $\widetilde{\pi(g)} \in O(L_p(\mathcal{B}(l_2) \otimes \mathcal{M}))$ for every $g \in G$. Then we have a map $\tilde{\pi} : G \to O(L_p(\mathcal{B}(l_2) \otimes \mathcal{M}))$ given by $\tilde{\pi}(g) = \widetilde{\pi(g)}$ for all $g \in G$. We claim that $\tilde{\pi}$ defines an orthogonal representation; we have to check that $\tilde{\pi}(g_1 g_2) = \tilde{\pi}(g_1)\tilde{\pi}(g_2)$ for all $g_1, g_2 \in G$.

For every $g \in G$, denote by $\pi(g) = u_g J_g$ the Yeadon's decomposition of $\pi(g)$, and by $\tilde{\pi}(g) = \tilde{u}_g \tilde{J}_g$ the corresponding decomposition of $\widetilde{\pi(g)} = \tilde{\pi}(g)$. By Theorem 2.4.4, to prove that $\tilde{\pi}$ is a homomorphism, we have to show the following relations, for all $g_1, g_2 \in G$, and all $y \in \mathcal{B}(l_2) \otimes \mathcal{M}$:

(1) $\tilde{u}_{g_1 g_2} = \tilde{u}_{g_1} \tilde{J}_{g_1}(\tilde{u}_{g_2})$,
(2) $\tilde{J}_{g_1 g_2}(y) = \tilde{J}_{g_1}(\tilde{J}_{g_2}(y))$ if $\tilde{J}_{g_1}$ is a $*$-morphism ,
(3) $\tilde{J}_{g_1 g_2}(y) = \tilde{J}_{g_1}(u_{g_2} \tilde{J}_{g_2}(y) u_{g_2}^*)$ if $\tilde{J}_{g_1}$ is a $*$-antimorphism.

Let $g_1, g_2 \in G$. By density of the linear subspace generated by $\{A \otimes x \mid A \in \mathcal{B}(l_2), x \in \mathcal{M}\ \}$ in $\mathcal{B}(l_2) \otimes \mathcal{M}$, it suffices to show relations (3) and (4) on elements of the form $y = A \otimes x$.
(1) : Notice that $T_g(1) = 1$ for every $g \in G$. Hence we have

$$\begin{aligned}
\tilde{u}_{g_1 g_2} &= 1 \otimes u_{g_1 g_2} \\
&= 1 \otimes u_{g_1} J_{g_1}(u_{g_2}) \\
&= (1 \otimes u_{g_1})(1 \otimes J_{g_1}(u_{g_2})) \\
&= (1 \otimes u_{g_1})(\tilde{J}_{g_1}(1 \otimes u_{g_2})) \\
&= \tilde{u}_{g_1} \tilde{J}_{g_1}(\tilde{u}_{g_2}).
\end{aligned}$$

(2) : In this case, J_{g_1} is a $*$-morphism, $T_{g_1} = id$ and $\tilde{J}_{g_1}$ is a $*$-morphism. Moreover, $J_{g_1 g_2} = J_{g_1} \circ J_{g_2}$ is a $*$-morphism if and only if J_{g_2} is a $*$-morphism; hence we have the relation $T_{g_1 g_2} = T_{g_2} = T_{g_1} \circ T_{g_2}$. Then, for $A \in \mathcal{B}(l_2)$ and $x \in \mathcal{M}$, we have

$$\begin{aligned}
\tilde{J}_{g_1 g_2}(A \otimes x) &= T_{g_1 g_2} A \otimes J_{g_1 g_2}(x) \\
&= T_{g_1} \circ T_{g_2}(A) \otimes J_{g_1} \circ J_{g_2}(x) \\
&= \tilde{J}_{g_1}(\tilde{J}_{g_2}(A \otimes x)).
\end{aligned}$$

(3) : In this case, J_{g_1} and $\tilde{J}_{g_1}$ are $*$-antimorphisms; moreover, $T_{g_1}(A) = \overline{A^*}$ for all $A \in \mathcal{B}(l_2)$.

We have $J_{g_1g_2}(x) = J_{g_1}(u_{g_2}J_{g_2}(x)u_{g_2}^*)$ for all $x \in \mathcal{M}$; hence $J_{g_1g_2}$ is a $*$-morphism if and only if J_{g_2} is a $*$-antimorphism. So the relation $T_{g_1g_2} = T_{g_1} \circ T_{g_2}$ is satisfied. Then, for $A \in \mathcal{B}(l_2)$ and $x \in \mathcal{M}$, we have

$$\begin{aligned}\tilde{J}_{g_1g_2}(A \otimes x) &= T_{g_1g_2}A \otimes J_{g_1g_2}(x)\\ &= T_{g_1} \circ T_{g_2}(A) \otimes J_{g_1}(u_{g_2}J_{g_2}(x)u_{g_2}^*)\\ &= T_{g_1} \otimes J_{g_1}(T_{g_2}(A) \otimes u_{g_2}J_{g_2}(x)u_{g_2}^*)\\ &= \tilde{J}_{g_1}((1 \otimes u_{g_2})T_{g_2} \otimes J_{g_2}(A \otimes x)(1 \otimes u_{g_2}^*))\\ &= \tilde{J}_{g_1}(\tilde{J}_{g_2}(A \otimes x)).\end{aligned}$$

So we have proved that $\tilde{\pi} : G \to O(L_p(\mathcal{B}(l_2) \otimes \mathcal{M}))$ is an orthogonal representation.

Now by Proposition 6.2.1, there exists a proper cocycle $b : G \to L_p(l_\infty \otimes \mathcal{M})$. Recall from the proof of Proposition 6.2.1 that b, viewed as a cocycle with values in $L_p(l^\infty(\mathcal{M}))$, is associated to the representation $\oplus_n \pi \in O(\bigoplus_n^p L_p(\mathcal{M}))$. We identify $l^\infty(\mathcal{M})$ with $l^\infty \otimes \mathcal{M}$, and then $\bigoplus_n^p L_p(\mathcal{M})$ with $L_p(l_\infty \otimes \mathcal{M})$. Now we claim that b, viewed as a cocycle with values in $L_p(l^\infty \otimes \mathcal{M})$, is associated to the representation $\overline{\pi} : G \to O(L_p(l^\infty \otimes \mathcal{M}))$ whose Yeadon's decomposition is given by

$$\overline{\pi}(g) = (1 \otimes u_g)(id \otimes J_g) \text{ for all } g \in G.$$

Indeed, an element $u_n \otimes x \in l^\infty \otimes \mathcal{M}$ is identified with an element $\oplus_n u_n x \in l^\infty(\mathcal{M})$. Such elements generate the von Neumann algebras $l^\infty \otimes \mathcal{M}$ and $l^\infty(\mathcal{M})$, and the latter von Neumann algebras contain a dense subspace (in the norm topology) of the respective L_p-spaces $L_p(l^\infty \otimes \mathcal{M})$ and $L_p(l^\infty(\mathcal{M}))$. Hence, it suffices to show the identification of the representation $\overline{\pi}$ with the representation $\oplus_n \pi$ on elements of the form $u_n \otimes x$ and $\oplus_n u_n x$. This is clear since for all $g \in G$, $(u_n) \in l^\infty$ and $x \in \mathcal{M}$, we have

$$\begin{aligned}&\overline{\pi}(g)(u_n \otimes x) = u_n \otimes \pi(g)(x)\\ &\text{and } (\oplus_n \pi(g))(u_n x) = \oplus_n(u_n \pi(g)(x)).\end{aligned}$$

Hence, there exists a proper cocycle $b : G \to L_p(l^\infty \otimes \mathcal{M})$ associated to the representation $\overline{\pi} : G \to O(L_p(l^\infty \otimes \mathcal{M}))$.

The space $L_p(l^\infty \otimes \mathcal{M})$ embeds linearly and isometrically in $L_p(\mathcal{B}(l_2) \otimes \mathcal{M})$. Indeed, the elements in $\mathcal{B}(l_2) \otimes \mathcal{M}$ can be identified with matrices $(a_{i,j})$ with values in $\mathcal{M}$, and with this identification, the von Neumann algebra $l^\infty(\mathcal{M}) \simeq l^\infty \otimes \mathcal{M}$ embeds diagonally in $\mathcal{B}(l_2) \otimes \mathcal{M}$. This induces a linear embedding $x \mapsto \tilde{x}$ from $L_p(l^\infty \otimes \mathcal{M})$ into $L_p(\mathcal{B}(l_2) \otimes \mathcal{M})$. Moreover, this embedding is isometric since the traces $\overline{\tau}$ and $\tilde{\tau}$, on $l^\infty \otimes \mathcal{M}$ and $\mathcal{B}(l_2) \otimes \mathcal{M}$ respectively, satisfy the following relation

$$\begin{aligned}\overline{\tau}(|(x_n)_n|^p) &= \overline{\tau}((|x_n|^p)_n)\\ &= \sum_n \tau(|x_n|^p)\\ &= \tilde{\tau}(|(\tilde{x_n})|^p)\end{aligned}$$

where $(x_n)_n \in l^\infty(\mathcal{M}) \cap L_p(l^\infty(\mathcal{M}))$ and $(\tilde{x_n})$ denotes the diagonal embedding of $(x_n)_n$ in $\mathcal{B}(l_2) \otimes \mathcal{M}$.

Now we claim that the restriction of the representation $\tilde{\pi} : G \to O(L_p(\mathcal{B}(l_2) \otimes \mathcal{M}))$ to the $\tilde{\pi}(G)$-invariant subspace $L_p(l^\infty \otimes \mathcal{M})$ coincides with the representation $\overline{\pi} : G \to O(L_p(l^\infty \otimes \mathcal{M}))$. Indeed, for every $g \in G$, the restriction of T_g to the diagonal (identified with l^∞) is the identity, that is $T_{g|l^\infty} = id$, hence the Yeadon's decompositions of $\overline{\pi}$ and $\tilde{\pi}$ coincide on $L_p(l^\infty \otimes \mathcal{M})$.

For every $g \in G$, define $\tilde{b}(g) = \tilde{b(g)}$. Then, by Lemma 3.3.3, $\tilde{b} : G \to O(L_p(\mathcal{B}(l_2) \otimes \mathcal{M}))$ is a proper cocycle. Moreover, by Lemma 6.2.3, $L_p(\mathcal{B}(l_2) \otimes \mathcal{M})$ is isometrically isomorphic to $L_p(\mathcal{M})$, since the von Neumann algebra $\mathcal{M}$ is of type II_∞. Hence, there exists a proper cocycle of G with values in $L_p(\mathcal{M})$, and G is a-$(FL_p(\mathcal{M}))$-menable.

□

The following theorem extends the implication $1 \Rightarrow 2$ of Theorem 6.1.2, in a non-commutative setting. We do not know whether the implication $2 \Rightarrow 1$ of Theorem 6.1.2 is true in this setting. The method used in [64] for the classical L_p-spaces breaks down in the non-commutative context, since in general the distance associated to the norm of a non-commutative L_p-space is no longer a kernel conditionally of negative type (see Theorem 1.4.7).

Theorem 6.2.5. *Let G be a second countable locally compact group with the Haagerup property. Then there exists a proper action of G by affine isometries on $L_p(l^\infty \otimes R)$, where R is the hyperfinite* II_1 *factor.*

Proof. We adapt the method of the proof of Theorem 3 given in [63] to our non-commutative setting. Denote by τ the normalized faithful trace on the hyperfinite II_1 factor R.

By Jolissaint's Theorem 5.1.4, there exists a trace-preserving action α of G on R by automorphisms, which is strongly mixing and has a nontrivial asymptotically invariant sequence (e_n). Recall that (see the proof of Theorem 5.2.4) α induces an orthogonal representation π^2 of G on $L_2(R)$, which is strongly mixing.

Set $v_n = e_n - \frac{1}{2}\,1$ for all $n \in \mathbb{N}$. Then $||v_n||_2^2 = \frac{1}{4}$. We have

$$||\pi^2(g)v_n - v_n||_2^2 = ||\pi^2(g)e_n - e_n||_2^2.$$

So,

$$\lim_{n\to\infty} \sup_{g\in K} ||\pi^2(g)v_n - v_n||_2 = 0 \text{ for every compact subset } K \text{ of } G.$$

Moreover, we have

$$||\pi^2(g)v_n - v_n||_2^2 = 2||v_n||_2^2 - \tau(\pi^2(g)v_n^* v_n) - \tau(\pi^2(g)v_n v_n^*).$$

Since the action is strongly mixing, it follows that

$$\lim_{g\to\infty} ||\pi^2(g)v_n - v_n||_2^2 = \frac{1}{2}.$$

Set $w_n = M_{2,p}(v_n)$ for all $n \in \mathbb{N}$, and let π^p be the conjugate representation of π^2 on $L_p(R)$. Notice that $||w_n||_p^p = \frac{1}{4}$. Since the Mazur map is locally uniformly continuous, there exists $C > 0$ such that

$$\lim_{g\to\infty} ||\pi^p(g)w_n - w_n||_p^p \geq C \text{ for all } n \in \mathbb{N}.$$

and we have (see Proposition 2.5.1)

$$\lim_{n\to\infty} \sup_{g\in K} ||\pi^p(g)w_n - w_n||_2 = 0 \text{ for every compact subset } K \text{ of } G.$$

Then $b : G \to \bigoplus_n^p L_p(R)$ defined by

$$b(g) = \oplus_n \pi^p(g)w_n - w_n$$

is 1-cocycle with values $\bigoplus_n^p L_p(R)$ associated to the representation $\oplus_n \pi^p$ on $\bigoplus_n^p L_p(R)$. We conclude the proof with arguments similar to those used in the proof of Theorem 6.2.1. □

Using the same construction as in the proof of Proposition 6.2.2, we deduce from the previous theorem the following corollary.

Corollary 6.2.6. *Let G be a second countable locally compact group with the Haagerup property. Then there exists a proper action by affine isometries of G on $L_p(R \otimes \mathcal{B}(l_2))$.*

6.3 Proper actions of hyperbolic groups on $L_p(\mathcal{M})$

Let Γ be a hyperbolic group. We recall Yu's construction from [85] of a proper cocycle on $l_p(\Gamma \times \Gamma)$ for p large enough, and then we show that it can be used to produce a proper action of Γ by affine isometries on C_p.

6.3.1 Mineyev construction and Yu's result on hyperbolic groups

In [85], Yu showed the following result.

Theorem 6.3.1. ([85]) *If Γ is a hyperbolic group, then there exists $2 \leq p < \infty$ such that Γ admits a proper isometric action on $l_p(\Gamma \times \Gamma)$.*

We recall some general facts and notations about hyperbolic groups and then explain Yu's construction. In the next subsection, we construct a proper action of Γ by isometries on C_p.

Let Γ be a hyperbolic group and G its Cayley graph with respect to a finite generating set. Let $\delta \geq 1$ be a positive integer such that all the geodesic triangles in G are δ-fine. Denote by d the path-metric and by

$$(b|c)_a = \frac{1}{2}(d(a,b) + d(a,c) - d(b,c))$$

the Gromov product for a, b, c vertices of G. For $a, b \in G$, denote by $q[a,b]$ the oriented geodesic edge-path from a to b. Let $q[a,b](t)$ be the point at distance t from a on the geodesic path $q[a,b]$.

Yu's construction is based on the following result of Mineyev from [60].

Theorem 6.3.2. ([60]) *There exists a function $h : \Gamma \times \Gamma \to l_p(\Gamma)$ satisfying the following conditions :*
(1) $||h(b,a)||_p = 1$ for all $a, b \in \Gamma$;
(2) if $d(a,b) \geq 10\delta$, then $supp(h(b,a)) \subset B(q[b,a](10\delta), \delta) \cap S(b, 10\delta)$;
(3) if $d(a,b) \leq 10\delta$, then $h(b,a) = a$;
(4) h is Γ-equivariant, that is $h(ga, gb) = gh(a,b)$ for all $g, a, b \in \Gamma$;
(5) there exist $C \geq 0$ and $0 \leq \rho < 1$ such that, for all $a, a', b \in \Gamma$,

$$||h(b,a) - h(b,a')||_p \leq C\rho^{(a|a')_b}.$$

Now let $k : \Gamma \times \Gamma \to \mathbb{R}$ be the function defined by

$$k(a,b) = h(a,e)(b) \text{ for all } a, b \in \Gamma.$$

Let $\rho : \Gamma \times \Gamma \to O(l_p(\Gamma \times \Gamma))$ be the following orthogonal representation :

$$\rho(g)(f)(a,b) = f(g^{-1}a, g^{-1}b) \text{ for all } g, a, b \in \Gamma$$

and define

$$k_g(a,b) = (\rho(g)(k)(a,b)) - k(a,b) = k(g^{-1}a, g^{-1}b) - k(a,b)$$

for all $g, a, b \in \Gamma$.

Theorem 6.3.3. *The formula above defines a cocycle* $k : \Gamma \to l_p(\Gamma \times \Gamma)$, $g \mapsto k_g$ *for the representation* ρ. *Moreover, this cocycle is proper.*

We only recall here why such a cocycle is proper. Indeed, Mineyev construction ensures that, given $g \in G$, there exists at least $d(g, e) - 100\delta$ vertices a on the oriented path $q[g, e]$ such that

$$\operatorname{supp}(h(a, g)) \cap \operatorname{supp}(h(a, e)) = \emptyset.$$

Hence, for such vertices a and b in Γ, we have

$$\begin{aligned} |k_g(a, b)| &= |h(g^{-1}a, e)(g^{-1}b) - h(a, e)(b)| \\ &= |h(a, g)(b) - h(a, e)(b)| \\ &= |h(a, g)(b)| + |h(a, e)(b)| \\ &\geq |h(a, e)(b)| \end{aligned}$$

and, for all $p \geq 1$, we have

$$\sum_{b\in\Gamma} |k_g(a, b)|^p \geq \sum_{b\in\Gamma} |h(a, e)(b)|^p = ||h(a, e)||_p^p = 1.$$

It follows that, for all $p \geq 1$, we have

$$||k_g||_p^p = \sum_{a\in\Gamma} \sum_{b\in\Gamma} |k_g(a, b)|^p \geq d(g, e) - 100\delta. \ (*)$$

6.3.2 A proper cocycle on C_p for large p

Let Γ be a hyperbolic group. We use Yu's construction from above to give a proper cocycle on C_p for large p. Let v such that the cardinal of every ball of radius n in G is less than v^n. Let ρ be as in Theorem 6.3.2.

Lemma 6.3.4. *Let* $p \geq 2$ *such that* $\rho^p v < \frac{1}{2}$, *and* p' *its conjugate exponent. We have that* $\sum_{a\in\Gamma} | \sum_{b\in\Gamma} |k_g(a, b)|^{p'}|^{\frac{p}{p'}} < \infty$.

Proof. By the condition (5) in Theorem 6.3.2, for all $a \in \Gamma$, we have

$$\begin{aligned} \sum_{b\in\Gamma} |k_g(a, b)|^{p'} &= \sum_{b\in\Gamma} |h(a, g)(b) - h(a, e)(b)|^{p'} \\ &\leq C^{p'} \rho^{p'(g|e)_a} \end{aligned}$$

and then

$$\begin{aligned}\sum_{a\in\Gamma}|\sum_{b\in\Gamma}|k_g(a,b)|^{p'}|^{\frac{p}{p'}} &\leq \sum_{a\in\Gamma} C^p\rho^{p(g|e)_a}\\ &\leq \sum_{a\in\Gamma} C^p\rho^{p(d(a,e)-d(g,e))}\\ &\leq \sum_{n=0}^{\infty} C^p\rho^{p(n-d(g,e))}v^n\\ &\leq 2C^p\rho^{-pd(g,e)}.\end{aligned}$$

□

Now denote by $\pi:\Gamma\to\mathcal{U}(l_2(\Gamma))$ be the regular representation of Γ on $l_2(\Gamma)$. And let K be the unbounded operator on $l_2(\Gamma)$ of kernel k, that is

$$K(f)(a)=\sum_{b\in\Gamma}k(a,b)f(b) \text{ for all } a\in\Gamma \text{ and } f\in l_2(\Gamma).$$

Let $\rho: G\to O(C_p)$ be defined by

$$\rho(g)x=\pi(g)x\pi(g^{-1}) \text{ for } x\in C_p.$$

Proposition 6.3.5. *(i) There exists $p_0>1$ such that*

$$K_g=\pi(g)K\pi(g^{-1})-K\in C_p$$

for all $g\in G$ and all $p\geq p_0$.
(ii) Let $p\geq p_0$. The map $\Gamma\to C_p$, $g\mapsto K_g$ is a 1-cocycle for the representation ρ.

Proof. Let $g\in\Gamma$. The operator K_g is a kernel operator with kernel $k_{g^{-1}}$. By Lemma 6.3.4, there exists $p_0>1$ such that the mixed norm, satisfies

$$||k_{g^{-1}}||_{p,p'}=\sum_{a\in\Gamma}|\sum_{b\in\Gamma}|k_{g^{-1}}(a,b)|^{p'}|^{\frac{p}{p'}}<\infty$$

for $p\geq p_0$. It follows from Russo's Theorem 1 in [72] that $K_g\in C_p$ for $p\geq p_0$.
(ii) The fact that $g\mapsto K_g$ is a cocycle for ρ is straightforward. □

Theorem 6.3.6. *Let Γ be a hyperbolic group. Then there exists $p\geq 2$ such that the cocycle $K:\Gamma\to C_p$ is proper.*

Proof. By Proposition 6.3.5, there exists p_0 such that K takes its values in C_p for all $p\geq p_0$. Fix $a,g\in\Gamma$. Then, for all $n\geq 1$, we have

$$\begin{aligned}<|K_g|^{2^n}\delta_a,\delta_a> &= |||K_g|^{2^{n-1}}\delta_a||^2\\ &=\sum_{b\in\Gamma}|<|K_g|^{2^{n-1}}\delta_a,\delta_b>|^2\\ &\geq |<|K_g|^{2^{n-1}}\delta_a,\delta_a>|^2.\end{aligned}$$

By induction on n, it follows that

$$< |K_g|^{2^n}\delta_a, \delta_a >\geq | < |K_g|^2\delta_a, \delta_a > |^{2^{n-1}}.$$

On the other hand,

$$\begin{aligned} < |K_g|^2\delta_a, \delta_a > &=< K_g\delta_a, K_g\delta_a > \\ &= \sum_{b\in\Gamma} |k_g(b,a)|^2. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} < |K_g|^{2^n}\delta_a, \delta_a > &\geq | \sum_{b\in\Gamma} |k_g(b,a)|^2|^{2^{n-1}} \\ &\geq \sum_{b\in\Gamma} |k_g(b,a)|^{2^n} \end{aligned}$$

and hence, by inequality $(*)$ of the previous subsection,

$$\sum_{a\in\Gamma} < |K_g|^{2^n}\delta_a, \delta_a >\geq d(g,e) - 100\delta.$$

Set $p = 2^n$ for n such that $2^n \geq p_0$. Then

$$\mathrm{Tr}(|K_g|^p) \geq d(g,e) - 100\delta$$

and hence

$$\lim_{g\to\infty} ||K_g||_p = +\infty.$$

$\square$

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