THÈSE

Pour l'obtention du grade de

DOCTEUR DE L'UNIVERSITÉ DE PAU ET DES PAYS DE L'ADOUR

Faculté des Sciences Fondamentales et Appliquées

Diplôme National - Arrêté du 7 août 2006

DOMAINE DE RECHERCHE : MATHÉMATIQUES APPLIQUÉES - Option Analyse des EDP

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Équations de Stokes et d'Oseen en domaine extérieur avec diverses conditions aux limites

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Thèse préparée au sein du Laboratoire de Mathématiques et de leurs Applications de Pau (L.M.A.P.),
dans l'école doctorale des sciences exactes et leurs applications de l'U.P.P.A.

Pour mes parents, avec amour et admiration.

Pour ma femme, Hela.

Pour ma petite fille, Yasmine.

Pour ma soeur, Hajer.

Remerciements

La liste des personnes que je souhaite remercier est longue, et réussir à exprimer toute ma gratitude envers ces personnes avec les mots justes me semble aujourd'hui bien plus délicat que de rédiger une thèse.

Bien évidemment, c'est à mon directeur de thèse Chérif Amrouche que vont mes premiers remerciements. Il a été pour moi, tout au long de ma thèse, un mentor remarquable. Il m'a montré comment construire mon intuition mathématique, comment prendre du recul, comment rédiger de manière lisible... Mais au-delà de ses indéniables qualités mathématiques, c'est par ses qualités humaines qu'il a su rendre ces années agréables et enrichissantes. Il a toujours été très attentif à ce que je ressentais, et pas seulement aux maths qui nous liaient formellement. Il m'est parfois arrivé au cours de ma thèse d'être démotivé ; mais, invariablement, en sortant de son bureau, j'étais de nouveau confiant et prêt à attaquer avec le sourire des questions passionnantes. Je lui suis aussi reconnaissant d'avoir su rester très disponible jusqu'à la fin de ma thèse, malgré une vie professionnelle chargée.

Je suis très honoré que Paul Deuring, Werner Varnhorn et Mohamed Majdoub aient accepté de rédiger un rapport sur mon travail. Leur patience et leurs conseils appellent mes sincères remerciements.

Merci à toute l'équipe EDP de LMAP , qui a été ma maison mathématique pendant ces années de thèse ; en particulier, merci à Guy Vallet, à Gilles Carbou et à Jacques Giacomoni.

J'ai pu profiter d'un séjour mathématique au sein de l'Institute mathématique de Prague, grâce à une invitation de Šárka Nečasová. Je lui en suis très reconnaissant pour l'accueil chaleureux et les échanges mathématiques très instructifs ; une partie de cette thèse lui doit beaucoup.

Un grand merci à mes amis proches, Abderrahman Benosman et sa femme Nour El Houda Selloula ainsi que mes frères Ahmed Rejaiba, El Othmany Hammou et Ibrahim Bougrara.

Je voudrais aussi remercier mes amis de Toulouse où j'ai fait mon master II de recherche, Slim Kammoun et sa femme Rahma Bouaziz car ils étaient toujours à mes cotés au moments difficiles de la thèse.

Je remercie aussi mes collègues: Nelly Barrau, Hai Ha Pham, Hind Albaba, Paul Sauvy, Caroline Bauzet, Benoît Guerville, Loic Bourdin...

Je remercie ma chère épouse pour son soutien quotidien indéfectible et son enthousiasme contagieux à l'égard de mes travaux comme de la vie en général. Notre couple a grandi en même temps que mon projet scientifique.

Enfin, mes remerciements vont à tous ceux qui m'ont accompagné, de près ou de loin, ces dernières années.

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Introduction générale

En mécanique des fluides, les équations de Navier-Stokes sont des équations aux dérivées partielles non-linéaires qui décrivent le mouvement des fluides dans l'approximation des milieux continus. Elles gouvernent par exemple les mouvements de l'air, de l'atmosphère, les courants océaniques, l'écoulement de l'eau dans un tuyau, et de nombreux autres phénomènes d'écoulement de fluides. Ces équations peuvent être considérées pour toutes les dimensions d'espace mais elles n'ont un sens physique qu'en dimension trois. Etant donné un ouvert Ω de $\mathbb{R}^3$, nous considérons plus précisément, le problème stationnaire suivant :

$$(\mathcal{NS})\begin{cases} -\nu\Delta\,\boldsymbol{u} + \rho\,\boldsymbol{u}\cdot\nabla\,\boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} \quad \text{dans} \quad \Omega, \\ \operatorname{div}\boldsymbol{u} = h \quad \text{dans} \quad \Omega. \end{cases}$$

Le but de cette thèse est d'étudier l'existence de solutions généralisées et de solutions fortes de $(\mathcal{NS})$ dans un cadre général non nécessairement hilbertien puis de passer au cas des solutions dites très faibles. On considèra aussi bien des conditions aux limites classiques de type Dirichlet :

$$\boldsymbol{u} = \boldsymbol{u}_0 \quad \text{sur} \quad \partial\Omega = \Gamma,$$

que des conditions aux limites non standard portant sur certaines composantes du champ de vitesses, du tourbillon, voire du champ de pression :

$$\boldsymbol{u}\cdot\boldsymbol{n} = g \quad \text{et} \quad \mathbf{rot}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{\chi}\times\boldsymbol{n} \quad \text{sur} \quad \Gamma,$$

ou bien

$$\pi = \pi_0 \quad \text{et} \quad \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n} \quad \text{sur} \quad \Gamma.$$

Ici ν représente le coefficient de viscosité, ρ est la densité du fluide, $\boldsymbol{f}$ sont les forces extérieures, h est la condition de compressibilité, $\boldsymbol{u}_0$ est la valeur du champ de vitesses sur le bord, $\boldsymbol{\chi}$ et π_0 sont données et $\boldsymbol{n}$ est la normale au bord de Ω. Le problème consiste à trouver le champ de vitesses $\boldsymbol{u}$ du fluide et le champ de pression π qui vérifient $(\mathcal{NS})$.

Dans le cas où Ω est un domaine extérieur ($\Omega = \mathbb{R}^3 \setminus \overline{\Omega'}$ avec Ω' un ouvert borné connexe), auquel nous nous intéressons ici, ce système d'équations aux dérivées partielles permet d'écrire **l'écoulement d'un fluide visqueux autour d'un obstacle**. Il faut distinguer deux cas différents, tous les deux physiquement intéressants, concernant **le comportement à l'infini des solutions** : ou bien elles tendent vers zéro, ou bien elles tendent vers un vecteur constant. Dans ce dernier cas, l'écoulement présente un sillage dans la région située derrière l'obstacle.

Ces équations, relativement simples du point de vue physique, sont pertinentes pour décrire nombre de situations réelles. Pour le mathématicien, ce modèle pose de nombreuses questions mathématiques qui restent jusqu'à aujourd'hui sans réponse. Il est donc indispensable et naturel de commencer par analyser des problèmes linéaires les approchant. De ce fait, on s'intéresse d'abord dans ce travail aux équations stationnaires de Navier-Stokes linéarisées, il s'agit ici des équations d'Oseen et des équations de Stokes. Ces systèmes linéarisés sont utiles par exemple pour étudier l'écoulement de l'air autour du fuselage et des voilures dans les allures à faible vitesse par rapport à celle du son, des circulations de fluides dans les corps poreux ou encore pour approcher les modèles comportant des équations de transport ou de diffusion (dans le cas de la magnétohydrodynamique par exemple). Ces équations sont posées dans des domaines infinis, comme les domaines extérieurs en dimension trois et l'espace tout entier $\mathbb{R}^3$. Par commodité, l'origine du repère est placée à l'intérieur de l'obstacle.

Les espaces de Sobolev classiques ne sont pas adaptés à l'étude de ce problème pour une telle géométrie. Donc pour une bonne analyse, il est important de considérer des espaces de Sobolev avec des poids (voir chapitre 1 pour plus de détails concernant ces espaces). Ces espaces sont des extensions des espaces de Sobolev classiques, munis de poids de type $\rho(\mathbf{x})^\alpha = \Big((1 + |\mathbf{x}|^2)^{1/2}\Big)^\alpha$ qui permettent de contrôler la croissance ou la décroissance des fonctions à l'infini. En faisant varier le paramètre α, on dispose alors d'une grande liberté de choix quant au comportement à l'infini des fonctions considérées. Plus fondamental encore, les poids sont choisis de sorte que des inégalités de Hardy se substituent à l'inégalité de Poincaré défaillante dans $\mathbb{R}^3$. Nous appliquons ce cadre fonctionnel à la résolution des équations de Navier-Stokes linéarisées. Plus précisément, nous caractérisons les données qui permettent de trouver une solution dans un espace avec poids donné.

En particulier, dès que l'espace où l'on cherche les solutions contient des fonctions polynomiales, l'unicité n'est plus assurée dans cet espace. Si au contraire, on impose des contraintes fortes de décroissance à la solution, alors leur existence est subordonnée au fait que $\boldsymbol{f}$, g vérifient des conditions de compatibilité (par exemple, $\boldsymbol{f}$ est d'intégrale nulle).

Cette thèse est organisée de la façon suivante :

Le premier chapitre est naturellement dévolu aux notations, aux définitions et aux propriétés des espaces de Sobolev avec poids sur lesquels nous nous appuyons. Il s'agit principalement de présenter des résultats de densité et d'injections liés à ces espaces voir par exemple Hanouzet [42], Cantor

[21] et Kudrjavcev [45]. Nous rappelons aussi les inégalités de Hardy (voir chapitre I Proposition 2.1) qui jouent un rôle clé dans la résolution des problèmes aux limites elliptiques. Notons également que, pour certaines valeurs critiques de p, l'introduction du poids $\rho(\mathbf{x})$ est insuffisante pour établir ces inégalités. Il convient donc de rajouter un facteur logarithmique, défini par $\ln(2+|\mathbf{x}|^2)$ pour lever partiellement ces restrictions (*cf* J. Giroire [41]).

Dans le deuxième chapitre, on considère les **équations de Stokes** stationnaires dans un domaine extérieur Ω connexe avec $\partial\Omega = \Gamma$ de classe $C^{1,1}$ et avec **des conditions aux limites de type Dirichlet**. Ces équations modélisent en première approximation des écoulements stationnaires lents de fluides visqueux autour d'un obstacle :

$$(\mathcal{S})\begin{cases} -\nu\Delta\,\boldsymbol{u}+\nabla\,\pi=\boldsymbol{f} & \text{dans} \quad \Omega, \\ \operatorname{div}\boldsymbol{u}=h & \text{dans} \quad \Omega, \\ \boldsymbol{u}=\boldsymbol{g} & \text{sur} \quad \partial\Omega. \end{cases}$$

L'étude de solutions généralisées et de solutions fortes de ce problème a été faite par un grand nombre d'auteurs, de différents points de vue; Borchers et Sohr [18], Finn [31], Fujita [32] , Giga et Sohr [37], Sohr et Varnhorn [50] , Specovius-Neugebauer [51], Girault [38] et [40], Girault, Giroire et Sequeira [39] et récemment par Alliot et Amrouche [3]. Mais dans certains problèmes de la mécanique de fluides, il est possible de se trouver face à des données qui ne sont pas regulières, c'est pourquoi nous nous intéressons ici à la recherche des solutions dites **très faible de type** $(\boldsymbol{u},\pi)\in\boldsymbol{W}^{0,p}_{\alpha}(\Omega)\times W^{-1,p}_{\alpha}(\Omega)$ avec $\alpha=0$ ou $\alpha=-1$, que l'on obtient par des arguments de dualité. Une des difficultés consiste à donner un sens aux traces de fonctions très peu régulières et à obtenir, par le biais de lemmes de densité, les formules de Green adéquates. D'autre part, lorsque le domaine Ω est borné, la notion de solutions très faibles pour le problème de Stokes $(\mathcal{S})$ a été développé ces dernières années par Giga [36] (dans un domaine Ω de classe C^{∞}), Amrouche et Girault [5] (dans un domaine Ω de classe $C^{1,1}$) et plus récemment, par Galdi et al [35], Farwig et Galdi [22] (dans un domaine Ω de classe $C^{2,1}$), Schumacher [49] et en 2011 par Amrouche et Rodriguez-Bellido [13].

Pour trouver les solutions très faibles nous utilisons ici un argument de dualité avec les solutions fortes dont on connait l'existence (voir Alliot et Amrouche [3]). La géométrie du domaine impose de chercher ces dernières dans les espaces de Sobolev avec poids, introduits dans le premier chapitre. On renvoie au travail de Amrouche et al [10] qui ont traité le cas du demi-espace avec le même cadre fonctionnel, bien qu'un peu plus compliqué à cause de la nature du bord du demi-espace également non borné contrairement aux domaines extérieurs.

Récemment, une autre méthode a été adaptée par Farwig et al [27] qui consiste à écrire l'opérateur de Stokes A_q comme composé de la projection de Helmholtz P_q avec l'opérateur de Laplace.

Le domaine de $A_q = -P_q\,\Delta$ est défini comme suit :

$$\mathcal{D}(A_q) = \boldsymbol{L}^q_\sigma(\Omega) \cap \boldsymbol{W}^{1,q}_0(\Omega) \cap \boldsymbol{W}^{2,q}(\Omega) \quad \text{avec} \quad 1 < q < \infty, \tag{1}$$

où $\boldsymbol{W}^{1,q}_0(\Omega)$ et $\boldsymbol{W}^{2,q}(\Omega)$ sont les espaces de Sobolev classiques tandis que $\boldsymbol{L}^q_\sigma(\Omega)$ est le sous espace de $\boldsymbol{L}^q(\Omega)$ à divergence nulle dans Ω. La démonstration de l'existence de solutions très faibles passe par des arguments de dualité où les propriétés de l'opérateur de Stokes A_q sont utilisées pour obtenir des inégalités de type :

$$||\nabla\,\boldsymbol{u}||_{\boldsymbol{L}^q(\Omega)} \leq C||A_q^{1/2}\boldsymbol{u}||_{\boldsymbol{L}^q(\Omega)}, \quad 1 < q < 3, \quad \boldsymbol{u} \in \mathcal{D}(A_q^{1/2}),$$

et

$$||\nabla^2\,\boldsymbol{u}||_{\boldsymbol{L}^q(\Omega)} \leq C||A_q^{1/2}\boldsymbol{u}||_{\boldsymbol{L}^q(\Omega)}, \quad 1 < q < 3/2, \quad \boldsymbol{u} \in \mathcal{D}(A_q).$$

De plus, dans ce papier les auteurs ont supposé que $\partial\Omega$ est de classe $C^{2,1}$ et que les données $\boldsymbol{f} = \operatorname{div}\mathbb{F}_0$, h et $\boldsymbol{g}$ vérifient :

$$\mathbb{F}_0 \in \boldsymbol{L}^r(\Omega),\ h \in L^r(\Omega),\ \boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}(\Gamma),\ 3 < p < \infty,\ \frac{1}{3} + \frac{1}{p} = \frac{1}{r},$$

ce qui donne $\frac{3}{2} < r < 3$.

Dans ce chapitre, nous avons amélioré ce dernier résultat, dans un premier temps en supposant que $\partial\Omega$ est seulement de classe $C^{1,1}$ et dans un deuxième temps, en prouvant l'existence et l'unicité de deux types de solutions très faibles. Le premier type de solution est le même que celui montré dans [27] mais avec un choix plus large pour p et donc aussi pour r. Le second type de solutions consiste à montrer l'existence de $(\boldsymbol{u}, \pi)$ dans $\boldsymbol{W}^{0,p}_{-1}(\Omega) \times W^{-1,p}_{-1}(\Omega)$ lorsque les données vérifient :

$$\mathbb{F}_0 \in \boldsymbol{W}^{0,r}_{-1}(\Omega),\ h \in W^{0,r}_{-1}(\Omega) \quad \text{et} \quad \boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}(\Gamma),$$

avec

$$\frac{3}{2} < p < \infty, \quad p \neq 3 \quad \text{et} \quad \frac{1}{3} + \frac{1}{p} = \frac{1}{r}.$$

Notons également que pour $p = 3$ (valeur critique), l'existence et l'unicité des solutions très faibles peuvent être étudiées en introduisant des poids logarithmiques supplémentaires (voir [6]).

Les résultats de ce chapitre ont fait l'objet d'une publication dans les annales de Ferrara [8] et d'une conférence internationale avec acte [9].

Dans le troisième chapitre, on considère le problème **stationnaire d'Oseen** dans un domaine extérieur Ω (connexe avec $\partial\Omega = \Gamma$ de classe $C^{1,1}$) obtenu formellement par la linéarisation du système

$(\mathcal{NS})$:

$$\mathcal{O}(\Omega) \begin{cases} -\Delta\,\boldsymbol{u} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{u}) + \nabla\,\pi = \boldsymbol{f} \quad \text{dans} \quad \Omega \\ \operatorname{div}\boldsymbol{u} = h \quad \text{dans} \quad \Omega, \\ \boldsymbol{u} = \boldsymbol{g} \quad \text{sur} \quad \Gamma, \end{cases}$$

avec $\boldsymbol{v}$ vecteur donné dans $\boldsymbol{L}^3(\Omega)$ et à divergence nulle, ce qui permet d'écrire formellement l'égalité suivante :

$$\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{u}) = \boldsymbol{v}\cdot\nabla\,\boldsymbol{u}.$$

En fait, les équations d'Oseen sont typiques pour modéliser un écoulement qui se produit **autour d'un obstacle**. Elles décrivent des propriétés physiques d'un système constitué par un objet en mouvement avec une vitesse constante dans un liquide visqueux. Mais, dans un domaine borné, l'approximation d'Oseen perd sa signification physique, tandis que, du point de vue mathématique, celle-ci ne présente aucune difficulté et peut être étudiée comme une variante de la théorie développée pour le système de Stokes. Il convient d'observer, toutefois que lorsque $\boldsymbol{v} = k\,\mathbf{e}_1$ avec $k > 0$, le système $\mathcal{O}(\Omega)$ s'écrit :

$$-\Delta\,\boldsymbol{u} + k\frac{\partial\boldsymbol{u}}{\partial x_1} + \nabla\,\pi = \boldsymbol{f} \quad \text{et} \quad \operatorname{div}\boldsymbol{u} = h \quad \text{dans} \quad \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \quad \text{sur} \quad \Gamma. \tag{2}$$

Faisons à présent un rapide survol de quelques travaux consacrés au problème (2). À notre connaissance, les premières études complètes sont dues à Faxén [30] qui généralise les méthodes introduites par Odqvist [47] pour le problèmes de Stokes. En utilisant la méthode de Galerkin, Finn [31] établit l'existence de solutions pour le problème (2). Lorsque $\Omega = \mathbb{R}^3$, Babenko [15] utilise le théorème des multiplicateurs de Lizorkin pour établir l'existence de solutions de (2). Les résultats de Finn et de Babenko sont ensuite étendus et généralisés par Galdi dans [33] et plus détaillés dans [34] chapitre VII. Nous citons également Farwig [22], [23] et [28] qui étudie le problème (2) dans les espaces L^2 avec poids anisotrope η_β^α définis par :

$$L^2_{\alpha,\beta}(\Omega) = \left\{ u \in L^2_{\mathrm{loc}}(\Omega),\;\; \eta_\beta^\alpha u \in L^2(\Omega) \right\}.$$

où $\eta_\beta^\alpha(\mathbf{x}) = (1+|\mathbf{x}|)^\alpha (1+|\mathbf{x}|-x_1)^\beta$. Notons aussi le travail de Kračmar et al [43] sur les estimations des noyaux d'Oseen dans des espaces L^p avec divers poids.

Pour notre part, nous nous intéressons à la résolution du problème d'Oseen $\mathcal{O}(\Omega)$, notamment à l'existence et l'unicité de solutions généralisées $(\boldsymbol{u},\pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ et de solutions fortes $(\boldsymbol{u},\pi) \in \boldsymbol{W}_\alpha^{2,p}(\Omega) \times W_\alpha^{1,p}(\Omega)$ avec $1 < p < \infty$ et $\alpha = 0$ ou $\alpha = 1$ puis de passer au cas de solutions très faible en adaptant les même techniques du chapitre II. Notons également que notre travail a été fait sans passer par la théorie générale de Agmon, Douglis et Nirenberg [1] pour l'étude de systèmes elliptiques.

Le fait que la donnée $\boldsymbol{v}$ appartient seulement à $\boldsymbol{L}^3(\Omega)$ et à divergence nulle rend l'analyse plus difficile. Pour y faire face, nous étudions tout d'abord le problème d'Oseen dans $\mathbb{R}^3$:

$$\mathcal{O}(\mathbb{R}^3) \qquad -\Delta\, \boldsymbol{u} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{u}) + \nabla\,\pi = \boldsymbol{f} \quad \text{et} \quad \operatorname{div}\boldsymbol{u} = h \quad \text{dans} \quad \mathbb{R}^3.$$

Ceci nous permet de nous concentrer uniquement sur le comportement à l'infini des solutions. Le problème $\mathcal{O}(\mathbb{R}^3)$ peut être considéré comme le problème de base à étudier. Notre analyse est basée sur l'idée que l'étude de problèmes extérieurs linéaires peut se faire en combinant les propriétés connues dans $\mathbb{R}^3$ et dans un ouvert borné. Dans [4], les auteurs ont étudié l'existence de solutions généralisées et de solutions fortes en imposant une condition de petitesse très forte sur la donnée $\boldsymbol{v}$: il existe une constante positive k qui dépend seulement de p telle que

$$||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)} < k. \tag{3}$$

Notre but dans ce chapitre est d'améliorer, dans un premier temps, le résultat prouvé dans [4], en éliminant la condition (3) et dans un deuxième temps de l'étendre à un domaine extérieur Ω. Les résultats de ce chapitre sont soumis au "J. of Math. soc. of japan".

Dans le quatrième chapitre, on suppose que Ω est le complémentaire d'un ouvert borné Ω' de classe $C^{1,1}$, simplement connexe et avec un bord $\Gamma = \partial\Omega'$ connexe. Et on considère le problème **stationnaire de Stokes** avec **deux types de conditions aux limites sur le bord** :

$$(\mathcal{S}_T)\ \begin{cases} -\Delta\,\boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} \quad \text{et} \quad \operatorname{div}\boldsymbol{u} = \chi & \text{dans} \quad \Omega,\\ \boldsymbol{u}\cdot\boldsymbol{n} = g \quad \text{et} \quad \mathbf{rot}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} & \text{sur} \quad \Gamma,\end{cases}$$

et

$$(\mathcal{S}_N)\ \begin{cases} -\Delta\,\boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} \quad \text{et} \quad \operatorname{div}\boldsymbol{u} = \chi \quad \text{dans} \quad \Omega,\\ \pi = \pi_0, \quad \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n} \quad \text{sur} \quad \Gamma \quad \text{et} \quad \displaystyle\int_\Gamma \boldsymbol{u}\cdot\boldsymbol{n}\,d\sigma = 0.\end{cases}$$

À notre connaissance, ces conditions dites non standard n'ont jamais été considérées pour ce type de géométrie. Lorsque Ω est un domaine borné, les problèmes $(\mathcal{S}_N)$ et $(\mathcal{S}_T)$ ont été étudiés par Amrouche et Seloula [20] en théorie L^p. Nous nous intéressons dans ce chapitre à l'existence et l'unicité de solutions généralisées et solutions fortes du problème $(\mathcal{S}_T)$ et du problème $(\mathcal{S}_N)$ dans un cadre hilbertien. Une parmi les difficultés que nous rencontrons dans notre étude, c'est que le lemme de Lax-Milgram ne s'applique pas toujours pour établir l'existence de solutions bien que nous soyons dans un cadre hilbertien. Pour y faire face, nous montrons tout d'abord deux **conditions Inf-Sup** :

$$\inf_{\substack{\boldsymbol{\varphi}\in\boldsymbol{V}^2_{0,T}(\Omega)\\ \boldsymbol{\varphi}\neq 0}} \ \sup_{\substack{\boldsymbol{\psi}\in\boldsymbol{V}^2_{-2,T}(\Omega)\\ \boldsymbol{\psi}\neq 0}} \frac{\int_\Omega \mathbf{rot}\,\boldsymbol{\psi}\cdot\mathbf{rot}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\psi}\|_{\boldsymbol{X}^2_{-2,T}(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{X}^2_{0,T}(\Omega)}} \geq \beta \tag{4}$$

et

$$\inf_{\substack{\varphi \in \boldsymbol{V}^2_{-2,N}(\Omega) \\ \varphi \neq 0}} \sup_{\substack{\psi \in \boldsymbol{V}^2_{0,N}(\Omega) \\ \psi \neq 0}} \frac{\int_\Omega \mathbf{rot}\, \boldsymbol{\psi} \cdot \mathbf{rot}\, \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\psi}\|_{\boldsymbol{X}^2_{0,N}(\Omega)} \|\boldsymbol{\varphi}\|_{\boldsymbol{X}^2_{-2,N}(\Omega)}} \geq \beta. \tag{5}$$

Une fois ces conditions satisfaites, nous pouvons utiliser le théorème de Babuška-Brezzi [16, 19] pour montrer l'existence et l'unicité de solutions. Les preuves de ces deux conditions sont basées sur deux types d'inégalités prouvées par V. Girault dans [38]. La première concerne **les champs de vecteurs tangents à la frontière** :

$$||\boldsymbol{\varphi}||_{\boldsymbol{W}^{1,2}_k(\Omega)} \leq C \left(||\mathrm{div}\, \boldsymbol{\varphi}||_{W^{0,2}_k(\Omega)} + ||\mathbf{rot}\, \boldsymbol{\varphi}||_{\boldsymbol{W}^{0,2}_k(\Omega)} + \sum_{j=2}^{N(-k)} | \int_\Gamma \boldsymbol{\varphi} \cdot \nabla\, w(q_j)\, d\sigma| \right), \tag{6}$$

où $\{q_j\}_{j=2}^{N(-k)}$ est une base de $\left\{q \in \mathcal{P}^\Delta_{-k} :\ q(\mathbf{0}) = 0\right\}$, $\mathcal{P}^\Delta_{-k}$ est l'espace des polynômes harmoniques sur $\mathbb{R}^3$ de degré inférieur ou égale à $-k$, $N(-k)$ est la dimension de $\mathcal{P}^\Delta_{-k}$ et $w(q_j)$ vérifie le problème de Neumann suivant :

$$\Delta\, w(q_j) = 0 \quad \text{dans} \quad \Omega \quad \text{et} \quad \frac{\partial w(q_j)}{\partial \boldsymbol{n}} = 0 \quad \text{sur} \quad \Gamma. \tag{7}$$

La deuxième concerne **les champs de vecteurs normaux à la frontière** :

$$||\boldsymbol{\varphi}||_{\boldsymbol{W}^{1,2}_k(\Omega)} \leq C \left(||\mathrm{div}\, \boldsymbol{\varphi}||_{W^{0,2}_k(\Omega)} + ||\mathbf{rot}\, \boldsymbol{\varphi}||_{\boldsymbol{W}^{0,2}_k(\Omega)} + | \int_\Gamma (\boldsymbol{\varphi} \cdot \boldsymbol{n}) d\sigma| + \sum_{j=1}^{N(-k)} | \int_\Gamma (\boldsymbol{\varphi} \cdot \boldsymbol{n}) q_j\, d\sigma| \right), \tag{8}$$

où $\{q_j\}_{j=1}^{N(-k)}$ est une base de $\mathcal{P}^\Delta_{-k}$.

Ces deux inégalités jouent aussi un rôle fondamental pour montrer la coercivité de certaines applications bilinéaires pour pouvoir appliquer ensuite le lemme de Lax-Milgram. Notons que les espaces fonctionnels $\boldsymbol{V}^2_{k,T}(\Omega)$ et $\boldsymbol{V}^2_{k,N}(\Omega)$ sont choisis de sorte que les termes $| \int_\Gamma (\boldsymbol{\varphi} \cdot \boldsymbol{n}) d\sigma|$, $\sum_{j=1}^{N(-k)} | \int_\Gamma (\boldsymbol{\varphi} \cdot \boldsymbol{n}) q_j\, d\sigma|$ et $\sum_{j=2}^{N(-k)} | \int_\Gamma \boldsymbol{\varphi} \cdot \nabla\, w(q_j)\, d\sigma|$ qui se trouvent dans (6) et dans (8) soient nuls. Pour commencer notre étude, nous considérons d'abord le problème de Stokes avec des conditions aux limites portant sur la composante normale du champ de vitesses et la composante tangentielle du tourbillon. Les conditions aux limites données dans $(\mathcal{S}_T)$ permettent en fait d'obtenir la pression π directement comme solution d'un problème de Neumann. C'est la raison pour laquelle nous sommes naturellement conduits à étudier le problème elliptique suivant :

$$(E_T) \qquad \begin{cases} -\Delta\, \boldsymbol{z} = \boldsymbol{f} \quad \text{et} \quad \mathrm{div}\, \boldsymbol{z} = 0 & \text{dans} \quad \Omega, \\ \boldsymbol{z} \cdot \boldsymbol{n} = g \quad \text{et} \quad \mathbf{rot}\, \boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} & \text{sur} \quad \Gamma. \end{cases}$$

On commence donc par établir l'existence de solutions faibles pour ce dernier problème, ce que l'on fait grâce au lemme de Lax-Milgram et à l'inégalité (6). En suivant le même schéma, nous abordons ensuite l'étude du problème de Stokes avec des conditions aux limites portant sur la pression et sur la composante tangentielle du champ de vitesses. Ici encore, la pression peut être obtenue directement comme solution d'un problème de Dirichlet et ceci nous permet de nous ramener au problème suivant

:

$$(E_N)\begin{cases} -\Delta\,\boldsymbol{\xi} = \boldsymbol{f} \quad \text{et} \quad \operatorname{div}\boldsymbol{\xi} = 0 \quad \text{dans} \quad \Omega, \\ \boldsymbol{\xi}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n} \quad \text{sur} \quad \Gamma \quad \text{et} \quad \displaystyle\int_\Gamma (\boldsymbol{\xi}\cdot\boldsymbol{n})q\,d\sigma = 0,\ \forall\ q\in\mathcal{P}_k^\Delta. \end{cases}$$

Les dernières conditions dans (E_N) sont des conditions essentielles pour assurer l'unicité de la solution. Pour résoudre le problème (E_N), nous avons utilisé la condition Inf-Sup (5) avec $k=0$ et le lemme de Lax-Milgram si $k=-1$. Enfin, on passe à l'étude de la régularité de la solution et plus précisement aux solutions fortes du problème $(\mathcal{S}_N)$ et du problème $(\mathcal{S}_T)$. Pour celà, on étend les inégalités (6) et (8), dans un premier temps, au cas où la trace tangentielle ou la trace normale ne sont pas nulles mais appartiennent à l'espace de trace $H^{1/2}(\Gamma)$ ou $\boldsymbol{H}^{1/2}(\Gamma)$. Dans un deuxième temps, nous établissons les inégalités suivantes :

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}_k^{2,2}(\Omega)} \le C\big(\|\boldsymbol{\varphi}\|_{\boldsymbol{W}_{k-1}^{1,2}(\Omega)} + \|\mathbf{rot}\,\boldsymbol{\varphi}\|_{\boldsymbol{W}_k^{1,2}(\Omega)} + \|\operatorname{div}\boldsymbol{\varphi}\|_{W_k^{1,2}(\Omega)} + \|\boldsymbol{\varphi}\cdot\boldsymbol{n}\|_{H^{3/2}(\Gamma)}\big) \tag{9}$$

ou

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}_k^{2,2}(\Omega)} \le C\big(\|\boldsymbol{\varphi}\|_{\boldsymbol{W}_{k-1}^{1,2}(\Omega)} + \|\mathbf{rot}\,\boldsymbol{\varphi}\|_{\boldsymbol{W}_k^{1,2}(\Omega)} + \|\operatorname{div}\boldsymbol{\varphi}\|_{W_k^{1,2}(\Omega)} + \|\boldsymbol{\varphi}\times\boldsymbol{n}\|_{\boldsymbol{H}^{3/2}(\Gamma)}\big). \tag{10}$$

La preuve de l'inégalité (9) est basée sur l'existence de solution régulière du problème de Neumann extérieur. Concernant l'égalité (10), nous sommes amenés à l'étude du problème auxiliaire suivant :

$$\begin{cases} -\Delta\,\boldsymbol{\xi} = \mathbf{rot}\,\boldsymbol{v} \quad \text{et} \quad \operatorname{div}\boldsymbol{\xi} = 0 \quad \text{dans} \quad \Omega, \\ \boldsymbol{\xi}\cdot\boldsymbol{n} = 0 \quad \text{et} \quad (\mathbf{rot}\,\boldsymbol{\xi} - \boldsymbol{v})\times\boldsymbol{n} = \mathbf{0} \quad \text{sur} \quad \Gamma, \\ \displaystyle\int_\Gamma \boldsymbol{\xi}\cdot\nabla\,(w(q)-q)\,d\sigma = 0, \end{cases}$$

où $(w(q)-q)$ est une solution du problème (7). Les résultats de ce chapitre sont soumis à "Integral of Differential Equations".

Chapter I

Basic Concepts on Weighted Sobolev spaces

Nous présentons ici les propriétés des espaces de Sobolev avec poids sur lesquels nous nous appuyons et nous donons des résultats fondamentaux liés à ces espaces. Il s'agit principalement à présenter des résultats introduites par Hanouzet [42], Cantor [21], Kudrjavcev [45]. Celle-ci est ensuite généralisés avec l'introduction de poids logarithmique, voir par exemple J. Giroire [41].

1 Weighted Sobolev Spaces

Let $\mathbf{x} = (x_1, x_2, x_3)$ be a typical point in $\mathbb{R}^3$ and let $r = |\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ denotes its distance to the origin. In order to control the behaviour at infinity of our functions and distributions we use for basic weights the quantity $\rho(\mathbf{x}) = (1 + r^2)^{1/2}$ which is equivalent to r at infinity, and to one on any bounded subset of $\mathbb{R}^3$ and the quantity $\ln(2 + r^2)$.

Let Ω' be a bounded connected open set in $\mathbb{R}^3$ with boundary $\partial\Omega' = \Gamma$ representing an obstacle and let Ω its complement *i.e.* $\Omega = \mathbb{R}^3 \diagdown \overline{\Omega'}$. In all the sequel, Ω is supposed of class $C^{1,1}$ except in some cases where we will precise that the boundary can be only Lipschitz-continuous.

We define $\mathcal{D}(\Omega)$ to be the linear space of infinite differentiable functions with compact support on Ω. Now, let $\mathcal{D}'(\Omega)$ denote the dual space of $\mathcal{D}(\Omega)$, often called the space of distributions on Ω. We denote by $< .,. >$ the duality pairing between $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$. For each $p \in \mathbb{R}$ and $1 < p < \infty$, the conjugate exponent p' is given by the relation

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

We use the customary multi-index notation

$$|\lambda| = \sum_{i=1}^{3} \lambda_i, \qquad D^\lambda = \frac{\partial^{|\lambda|}}{\partial x_1^{\lambda_1} \partial x_2^{\lambda_2} \partial x_3^{\lambda_3}}$$

for any nonnegative integers λ_i.

Then, for any nonnegative integers m and real numbers $p > 1$ and α, setting

$$k = k(m, p, \alpha) = \begin{cases} -1, & \text{if} \quad \frac{3}{p} + \alpha \notin \{1, ..., m\}, \\ m - \frac{3}{p} - \alpha, & \text{if} \quad \frac{3}{p} + \alpha \in \{1, ..., m\}, \end{cases}$$

we define the following space:

$$\begin{aligned} W_{\alpha}^{m,p}(\Omega) = \{u \in \mathcal{D}'(\Omega); \\ & \forall\, \lambda \in \mathbb{N}^3 : \, 0 \leq |\lambda| \leq k, \, \rho^{\alpha-m+|\lambda|}(\ln(2+r^2))^{-1} D^{\lambda} u \in L^p(\Omega); \\ & \forall\, \lambda \in \mathbb{N}^3 : \, k+1 \leq |\lambda| \leq m, \rho^{\alpha-m+|\lambda|} D^{\lambda} u \in L^p(\Omega)\}. \end{aligned}$$

It is a reflexive Banach space equipped with its natural norm:

$$\begin{aligned} ||u||_{W_{\alpha}^{m,p}(\Omega)} = \Bigg(& \sum_{0 \leq |\lambda| \leq k} ||\rho^{\alpha-m+|\lambda|}(\ln(2+r^2))^{-1} D^{\lambda} u||^p_{L^p(\Omega)} \\ & + \sum_{k+1 \leq |\lambda| \leq m} ||\rho^{\alpha-m+|\lambda|} D^{\lambda} u||^p_{L^p(\Omega)} \Bigg)^{1/p}. \end{aligned}$$

We note that the logarithmic weight only appears if $\dfrac{3}{p} \in \{1, ..., m\}$ and all the local properties of $W_{\alpha}^{m,p}(\Omega)$ coincide with those of the corresponding classical Sobolev spaces $W^{m,p}(\Omega)$. We set $\mathring{W}_{\alpha}^{m,p}(\Omega)$ as the adherence of $\mathcal{D}(\Omega)$ for the norm $\| \, . \, \|_{W_{\alpha}^{m,p}(\Omega)}$. Then, the dual space of $\mathring{W}_{\alpha}^{m,p}(\Omega)$, denoting by $W_{-\alpha}^{-m,p'}(\Omega)$, is a space of distributions with the norm

$$|| \, u \, ||_{W_{-\alpha}^{-m,p'}(\Omega)} = \sup_{v \in \mathring{W}_{\alpha}^{m,p}(\Omega)} \frac{\langle u, v \rangle}{|| \, v \, ||_{W_{\alpha}^{m,p}(\Omega)}}.$$

Note that when $\Omega = \mathbb{R}^3$, we have $W_{\alpha}^{m,p}(\mathbb{R}^3) = \mathring{W}_{\alpha}^{m,p}(\mathbb{R}^3)$.

We give now some examples of weighted Sobolev spaces when $m \in \{0, 1, 2\}$:

For $m = 0$, we set

$$W_{\alpha}^{0,p}(\Omega) = \{u \in \mathcal{D}'(\Omega); \;\; \rho^{\alpha} \, u \in L^p(\Omega)\}.$$

For $m = 1, 2$ and $\alpha = 0$:

$$W_0^{1,p}(\Omega) := \left\{ v \in \mathcal{D}'(\Omega) : \frac{v}{w_0} \in L^p(\Omega), \nabla \, v \in \boldsymbol{L}^p(\Omega) \right\},$$

with

$$w_0 = \begin{cases} \rho(\mathbf{x}) & \text{if} \quad p \neq 3, \\ \rho(\mathbf{x}) \ln(1 + \rho(\mathbf{x})) & \text{if} \quad p = 3. \end{cases}$$

$$W_0^{2,p}(\Omega) := \left\{ v \in \mathcal{D}'(\Omega) : \frac{v}{w_1} \in L^p(\Omega), \frac{\nabla v}{w_0} \in \boldsymbol{L}^p(\Omega), \nabla^2 v \in L^p(\Omega) \right\},$$

with

$$w_1 = \begin{cases} \rho(\mathbf{x})^2 & \text{if} \quad p \notin \{\frac{3}{2}, 3\}, \\ \rho(\mathbf{x})^2 \ln(1+\rho(\mathbf{x})) & \text{if} \quad p \in \{\frac{3}{2}, 3\}. \end{cases}$$

If Ω is a Lipschitz exterior domain, then we have

$$\mathring{W}_\alpha^{1,p}(\Omega) = \left\{ v \in W_\alpha^{1,p}(\Omega), \; v = 0 \quad \text{on} \quad \partial\Omega \right\},$$

and if Ω is a $C^{1,1}$ exterior domain, we set

$$\mathring{W}_\alpha^{2,p}(\Omega) = \left\{ v \in W_\alpha^{2,p}(\Omega), \; v = \frac{\partial v}{\partial \boldsymbol{n}} = 0 \quad \text{on} \quad \partial\Omega \right\},$$

where $\dfrac{\partial v}{\partial \boldsymbol{n}}$ is the normal derivate of v.

For all $\lambda \in \mathbb{N}^3$ where $0 \leq |\lambda| \leq 2m$ with $m = 1$ or $m = 2$, the mapping

$$u \in W_\alpha^{m,p}(\Omega) \longrightarrow \partial^\lambda u \in W_\alpha^{m-|\lambda|,p}(\Omega)$$

is continuous.

2 Basic concepts and notations

The spaces $W_\alpha^{1,p}(\Omega)$ or $W_\alpha^{2,p}(\Omega)$ sometimes contain some polynomial functions. We have for $m = 1$ or $m = 2$:

$$\mathcal{P}_j \subset W_\alpha^{m,p}(\Omega) \qquad \text{with} \qquad \begin{cases} j = [m - (3/p + \alpha)] & \text{if} \quad 3/p + \alpha \notin \mathbb{Z}^-, \\ j = m - (3/p + \alpha) - 1 & \text{if} \quad 3/p + \alpha \in \mathbb{Z} \end{cases} \tag{I.1}$$

where $[s]$ denotes the integer part of the real number s and $\mathcal{P}_j$ is the space of polynomials of degree less then j.
We now recall a fundamental property of space $W^{1,p}$ (see [7] and [6]):

Proposition 2.1 ***(Hardy's inequalities)*** *Let $\alpha \in \mathbb{R}$ and let $1 < p < \infty$.*

i) Let Ω an exterior domain. There exists a constant $C = C(p, \alpha, \Omega) > 0$ such that

$$\forall \; u \in \mathring{W}_\alpha^{1,p}(\Omega), \quad ||u||_{W_\alpha^{1,p}(\Omega)} \leq C ||\nabla u||_{\boldsymbol{W}_\alpha^{0,p}(\Omega)}.$$

ii) There exists $C = C(p, \alpha) > 0$ such that

$$\forall \quad u \in W^{1,p}_{\alpha}(\mathbb{R}^3), \qquad \begin{cases} ||u||_{W^{1,p}_{\alpha}(\mathbb{R}^3)} \leq ||\nabla\, u||_{\boldsymbol{W}^{0,p}_{\alpha}(\mathbb{R}^3)}, & \text{if } 3/p + \alpha > 1, \\ ||u||_{W^{1,p}_{\alpha}(\mathbb{R}^3)/\mathcal{P}_0} \leq ||\nabla\, u||_{\boldsymbol{W}^{0,p}_{\alpha}(\mathbb{R}^3)}, & \text{otherwise,} \end{cases}$$

We recall the following Sobolev embeddings for any real values α and $1 < p < 3$,

$$W^{1,p}_{\alpha}(\Omega) \hookrightarrow W^{0,p*}_{\alpha}(\Omega) \quad \text{where} \quad p* = \frac{3p}{3-p}. \tag{I.2}$$

and, by duality, we have

$$W^{0,q}_{-\alpha}(\Omega) \hookrightarrow W^{-1,p'}_{-\alpha}(\Omega) \quad \text{where} \quad q = \frac{3p'}{3+p'}.$$

Moreover, if $1 < p < 3/2$ we have

$$W^{2,p}_{\alpha}(\Omega) \hookrightarrow W^{0,q}_{\alpha}(\Omega) \quad \text{where} \quad q = \frac{3p}{3-2p}. \tag{I.3}$$

and, by duality, we have

$$W^{0,q}_{-\alpha}(\Omega) \hookrightarrow W^{-2,p'}_{-\alpha}(\Omega) \quad \text{where} \quad q = \frac{3p'}{3+2p'}.$$

Note also that if $v \in W^{2,p}_0(\Omega)$ with $3/2 \leq p < 3$ and $\nabla\, v \in \boldsymbol{L}^r(\Omega)$ for some r, then $\nabla\, v \in \boldsymbol{L}^q(\Omega)$ for all $q \geq r$ if $p = 3/2$ and $\nabla\, v \in \boldsymbol{L}^r(\Omega) \cap \boldsymbol{L}^\infty(\Omega)$ if $3/2 < p < 3$.

On the other hand, if $\frac{3}{p} + \alpha \notin \{1, ..., m\}$, we have the following continuous embedding:

$$W^{m,p}_{\alpha}(\Omega) \hookrightarrow W^{m-1,p}_{\alpha-1}(\Omega) \hookrightarrow ... \hookrightarrow W^{0,p}_{\alpha-m}(\Omega). \tag{I.4}$$

Finally, let $1 \leq m \leq 2$ and $u \in \mathcal{D}'(\mathbb{R}^3)$ such that

$$\forall \lambda \in \mathbb{N}^3 : \quad |\lambda| = m, \quad \partial^\lambda u \in L^p(\mathbb{R}^3).$$

i) If $1 < p < 3$, then there exists a polynomial $K(u) \in \mathcal{P}_{m-1}$ such that $u + K(u)$ belongs to $W^{m,p}_0(\mathbb{R}^3)$ and

$$\inf_{\mu \in \mathcal{P}_{[m-3/p]}} ||\, u + K(u) + \mu\, ||_{W^{m,p}_0(\mathbb{R}^3)} \leq C ||\nabla\, u||_{W^{m-1,p}_0(\mathbb{R}^3)}.$$

ii) If $p \geq 3$, then $u \in W^{m,p}_0(\mathbb{R}^3)$ and we have

$$\inf_{\mu \in \mathcal{P}_{m-1}} ||\, u + \mu\, ||_{W^{m,p}_0(\mathbb{R}^3)} \leq C ||\nabla\, u||_{W^{m-1,p}_0(\mathbb{R}^3)}.$$

In this work, we shall also denote by B_R the open ball of radius $R > 0$ centered at the origin with boundary Σ. In particular, since Ω' is bounded, we can find some R_0 such that $\Omega' \subset B_{R_0}$ and we

introduce, for any $R \geq R_0$, the set
$$\Omega_R = \Omega \cap B_R.$$
We also introduce the following spaces for $k \in \mathbb{N}^*$:
$$A_k(\mathbb{R}^3) = \left\{x \in \mathbb{R}^3;\ k < |x| < 2k\right\}, \qquad B^k(\mathbb{R}^3) = \left\{x \in \mathbb{R}^3;\ |x| > k\right\},$$
and
$$C_k(\mathbb{R}^3) = \left\{x \in \mathbb{R}^3;\ e^{\frac{k}{2}} < |x| < e^k\right\}, \qquad D^k(\mathbb{R}^3) = \left\{x \in \mathbb{R}^3;\ |x| > e^{\frac{k}{2}}\right\}.$$
Given a Banach space B, with dual space B' and a closed subspace X of B, we denote by $B' \perp X$ the subspace of B' orthogonal to X, i.e.
$$B' \perp X = \{f \in B'; < f, v >= 0\ \forall\, v \in X\} = (B/X)'.$$
The space $B' \perp X$ is also called the polar space of X in B'.
Finally, we use bold type characters to denote vector distributions or spaces of vector distributions with 3 components and $C > 0$ usually denotes a generic constant the value of which may change from line to line.

Chapter II

Very Weak Solutions for the Stokes Problem

Nous montrons ici l'existence et l'unicité de deux types de solutions dites très faible pour le problème de Stokes dans un domaine extérieur Ω. Notre méthode consiste à utiliser un argument de dualité avec les solutions fortes du même problème.

1 Introduction

Let Ω' be a bounded connected open set in $\mathbb{R}^3$ with boundary $\partial\Omega' = \Gamma$ of class $C^{1,1}$ representing an obstacle and let Ω its complement *i.e.* $\Omega = \mathbb{R}^3 \diagdown \overline{\Omega'}$. We consider the Stokes problem in Ω: for given vectors fields $\boldsymbol{f}$ and $\boldsymbol{g}$ and a scalar function h, we look for a very weak solution which fulfill:

$$(\mathcal{S}) \quad -\Delta \boldsymbol{u} + \nabla q = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = h \quad \text{in } \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma,$$

where $\boldsymbol{u}$ denote the velocity and q the pressure and both are unknown. This exterior Stokes problem has been studied by a large number of authors, from different points of view, and it would be too long to list them all here so we give some examples: Borchers and Sohr [18], Finn [31], Fujita [32], Giga and Sohr [37], Sohr and Varnhorn [50], Specovius-Neugebauer [51], Girault [38] and [40], Girault, Giroire and Sequeira [39] and in the last years, Alliot and Amrouche [3].

On the other hand, the notion of very weak solutions $(\boldsymbol{u}, q) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega)$ for the Stokes problem when Ω is bounded, has been developed in the last years by Giga [36] (in a domain Ω of class C^∞), Amrouche and Girault [5] (in a domain Ω of class $C^{1,1}$) and more recently by Galdi et al [35], Farwig and Galdi [22] (in a domain Ω of class $C^{2,1}$, see also Schumacher [49]) and in 2011 by Amrouche and Angeles Rodriguez-Bellido [13]. It is well known that it is not possible to extend the result of very weak solutions to the case of unbounded domains, such as the whole space or the exterior domain in which we are interested, here the spaces $W^{m,p}(\Omega)$ are not adequate. Therefore, a specific functional framework is necessary which also has to take into account the behaviour of the functions at infinity. Our approach is to use the weighted Sobolev spaces $W^{m,p}_\alpha(\Omega)$ introduced by Hanouzet [42], Cantor [21], Kudrjavcev [45]. In the half-space, the notion of very weak solution for the Stokes problem is well studied in this kind of weighted Sobolev spaces, see Amrouche et al [10] for more details. In the last years, different methods have been developed to study these type of solution in exterior domain. One idea is well done in 2005 by Farwig et al [27], in which they prove the existence and the uniqueness of very weak solution $(\boldsymbol{u}, q)$ belonging to $\boldsymbol{L}^p(\Omega) \times W_0^{-1,p}(\Omega)$, with data $\boldsymbol{f} = \operatorname{div} \mathbb{F}_0$, h and $\boldsymbol{g}$ satisfy

$$\mathbb{F}_0 \in \boldsymbol{L}^r(\Omega),\ h \in L^r(\Omega),\ \boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}(\Gamma),\ 3 < p < \infty,\ \frac{1}{3} + \frac{1}{p} = \frac{1}{r}$$

yielding $\frac{3}{2} < r < 3$. In this article, we investigate two types of very weak solutions for the Stokes problem. One type is the same as Farwig et al see [27] but with larger range of p, *i.e* with the following data:

$$\boldsymbol{f} = \operatorname{div} \mathbb{F}_0 + \nabla f_1, \quad h \in L^r(\Omega) \quad \text{and} \quad \boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}(\Gamma),$$

with

$$\mathbb{F}_0 \in \boldsymbol{L}^r(\Omega),\ \ f_1 \in W_0^{-1,p}(\Omega),\ \ \frac{3}{2} < p < \infty, \qquad \text{and} \quad \frac{1}{3} + \frac{1}{p} = \frac{1}{r}.$$

The second one is to prove that $(\boldsymbol{u}, q)$ belongs to $\boldsymbol{W}^{0,p}_{-1}(\Omega) \times W^{-1,p}_{-1}(\Omega)$ when the data satisfying

$$\mathbb{F}_0 \in \boldsymbol{W}^{0,r}_{-1}(\Omega),\ \ f_1 \in W^{-1,p}_{-1}(\Omega) \quad \text{and} \quad h \in W^{0,r}_{-1}(\Omega),$$

with

$$\frac{3}{2} < p < \infty,\ \ p \neq 3 \quad \text{and} \quad \frac{1}{3} + \frac{1}{p} = \frac{1}{r}.$$

To do this, we adapt a method employed in [13] for a bounded domain and in [10] for the half-space, which consists in the use of an argument of duality via the strong solutions of the Stokes problem proved in [3].

This chapter is organized as follows: in Section 2 we recall the definition of some spaces and their respective norms, besides some density results, characterization of dual space and trace theorems. The main results of this chapter are presented at first in Theorem 3.1 which proves the existence and uniqueness of very weak solution $(\boldsymbol{u}, q)$ in $\boldsymbol{L}^p(\Omega) \times W_0^{-1,p}(\Omega)$ and secondly in Theorem 4.1 which proves the second type of very weak solution $(\boldsymbol{u}, q)$ in $\boldsymbol{W}^{0,p}_{-1}(\Omega) \times W^{-1,p}_{-1}(\Omega)$.

2 Preliminary results

In this sequel, we need to introduce the following space:

$$\boldsymbol{X}^{\ell}_{r,p}(\Omega) = \left\{ \boldsymbol{\varphi} \in \mathring{\boldsymbol{W}}^{1,r}_{\ell}(\Omega);\ \operatorname{div} \boldsymbol{\varphi} \in \mathring{W}^{1,p}_{\ell}(\Omega) \right\}.$$

Thanks to Poincaré-type inequality (see [7]), this space can be equipped with the following norm:

$$\| \boldsymbol{\varphi} \|_{\boldsymbol{X}^{\ell}_{r,p}(\Omega)} = \sum_{1 \leq i,j \leq 3} \| \frac{\partial \varphi_i}{\partial x_j} \|_{W^{0,r}_{\ell}(\Omega)} + \| \operatorname{div} \boldsymbol{\varphi} \|_{W^{1,p}_{\ell}(\Omega)}.$$

Then, we show some density results that are essential for the proofs below. We begin by the following density.

Lemma 2.1 *Suppose that Ω is only a Lipschitz open set and suppose that $0 \leqslant \frac{1}{p} - \frac{1}{r} \leqslant \frac{1}{3}$. We have the following properties:*

i) The space $\mathfrak{D}(\Omega)$ *is dense in* $\boldsymbol{X}^1_{r,p}(\Omega)$.

ii) If in addition $p \neq 3$ *and* $r \neq 3$, *then the space* $\mathfrak{D}(\Omega)$ *is dense in* $\boldsymbol{X}^0_{r,p}(\Omega)$.

Proof. The proof of point *i)* and *ii)* are very similar, so we do only the proof of the first result. The density of $\mathfrak{D}(\Omega)$ in $\boldsymbol{X}^1_{r,p}(\Omega)$ relies on an adequate truncation procedure and regularization. The truncation function that we shall use has been defined by:
$\varphi \in \mathcal{D}(\mathbb{R}^3)$ such that $0 \leqslant \varphi(t) \leqslant 1$ for any $t \in \mathbb{R}^3$, and

$$\varphi(t) = \begin{cases} 1 & \text{if} \quad 0 \leqslant \mid t \mid \leqslant 1, \\ 0 & \text{if} \quad \mid t \mid \geqslant 2. \end{cases}$$

Now, let $\boldsymbol{v} \in \boldsymbol{X}^1_{r,p}(\Omega)$ and $\widetilde{\boldsymbol{v}}$ be the extension by $\mathbf{0}$ of $\boldsymbol{v}$ to $\mathbb{R}^3$. Then we have $\widetilde{\boldsymbol{v}} \in \boldsymbol{X}^1_{r,p}(\mathbb{R}^3)$. We begin to apply the cut off functions φ_k, defined on $\mathbb{R}^3$ for any $k \in \mathbb{N}^*$, by $\varphi_k(x) = \varphi(\frac{x}{k})$. Set $\boldsymbol{v}_k = \varphi_k \widetilde{\boldsymbol{v}}$. The main idea of the proof is to prove that $\boldsymbol{v}_k \to \widetilde{\boldsymbol{v}}$ in $\boldsymbol{X}^1_{r,p}(\mathbb{R}^3)$ when $k \to \infty$.
1) Convergence of $\boldsymbol{v}_k$ in $\boldsymbol{W}^{1,r}_1(\mathbb{R}^3)$:
On one hand we have:

$$\int_{\mathbb{R}^3} |\boldsymbol{v}_k - \widetilde{\boldsymbol{v}}|^r \, d\mathbf{x} = \int_{\mathbb{R}^3} |\varphi_k - 1|^r |\widetilde{\boldsymbol{v}}|^r \, d\mathbf{x} \leqslant 2^r \int_{B^k(\mathbb{R}^3)} |\widetilde{\boldsymbol{v}}|^r \, d\mathbf{x}.$$

Using dominated convergence theorem, we deduce that this integral converges to zero when $k \to \infty$. On the other hand we have:

$$\begin{aligned} \int_{\mathbb{R}^3} |\rho\nabla(\varphi_k\widetilde{\boldsymbol{v}}) - \rho\nabla\,\widetilde{\boldsymbol{v}}|^r \quad & \leqslant \quad C\left(\int_{\mathbb{R}^3} \rho^r |\nabla\,\varphi_k|^r |\widetilde{\boldsymbol{v}}|^r \, d\mathbf{x} + \int_{\mathbb{R}^3} |\varphi_k - 1|^r |\rho\nabla\,\widetilde{\boldsymbol{v}}|^r \, d\mathbf{x}\right) \\ & \leqslant \quad C\left(\frac{(1+2k)^r}{k^r} \int_{A_k(\mathbb{R}^3)} |\widetilde{\boldsymbol{v}}|^r \, d\mathbf{x} + \int_{B^k(\mathbb{R}^3)} |\rho\nabla\,\widetilde{\boldsymbol{v}}|^r \, d\mathbf{x}\right), \end{aligned}$$

where C is independent of k. As above, the two last integrals converge to zero when $k \to \infty$.

2) Convergence of $\operatorname{div} \boldsymbol{v}_k$ **in** $W^{1,p}_1(\mathbb{R}^3)$:
We begin by proving that $\varphi_k \operatorname{div} \widetilde{\boldsymbol{v}} \to \operatorname{div} \widetilde{\boldsymbol{v}}$ in $W^{1,p}_1(\mathbb{R}^3)$. On one hand we have

$$\int_{\mathbb{R}^3} |\varphi_k \operatorname{div} \widetilde{\boldsymbol{v}} - \operatorname{div} \widetilde{\boldsymbol{v}}|^p \, d\mathbf{x} = \int_{\mathbb{R}^3} |\varphi_k - 1|^p |\operatorname{div} \widetilde{\boldsymbol{v}}|^p \, d\mathbf{x} \leq 2^p \int_{B^k(\mathbb{R}^3)} |\operatorname{div} \widetilde{\boldsymbol{v}}|^p \, d\mathbf{x},$$

where the last integral converges to zero when $k \to \infty$. On the other hand we have

$$\begin{aligned} \int_{\mathbb{R}^3} \rho^p |\nabla(\varphi_k \operatorname{div} \widetilde{\boldsymbol{v}}) - \nabla(\operatorname{div} \widetilde{\boldsymbol{v}})|^p \quad & \leq \quad C(\int_{\mathbb{R}^3} \rho^p |\nabla\,\varphi_k|^p |\operatorname{div} \widetilde{\boldsymbol{v}}|^p \, d\mathbf{x} + \int_{\mathbb{R}^3} |\varphi_k - 1|^p |\rho\nabla(\operatorname{div} \widetilde{\boldsymbol{v}})|^p \, d\mathbf{x}) \\ & \leq \quad C(\frac{(1+2k)^p}{k^p} \int_{A_k(\mathbb{R}^3)} |\operatorname{div} \widetilde{\boldsymbol{v}}|^p \, d\mathbf{x} + \int_{B^k(\mathbb{R}^3)} |\rho\nabla(\operatorname{div} \widetilde{\boldsymbol{v}})|^p \, d\mathbf{x}), \end{aligned}$$

where C is independent of k. As above, the two last integrals converge to zero. To finish the convergence of $\operatorname{div} \boldsymbol{v}_k$ we should prove that $\widetilde{\boldsymbol{v}} \cdot \nabla\varphi_k \longrightarrow 0$ in $W^{1,p}_1(\mathbb{R}^3)$. We start by proving that $\widetilde{\boldsymbol{v}} \cdot \nabla\varphi_k \longrightarrow 0$

in $L^p(\mathbb{R}^3)$. Indeed,

$$\int_\Omega |\widetilde{\boldsymbol{v}} \cdot \nabla \varphi_k|^p \, d\mathbf{x} \leqslant C \frac{1}{k^p} \int_{A_k(\mathbb{R}^3)} |\widetilde{\boldsymbol{v}}|^p \, d\mathbf{x} \tag{II.1}$$

and using Hölder inequality we have

$$\begin{aligned} \int_{A_k(\mathbb{R}^3)} |\widetilde{\boldsymbol{v}}|^p \, d\mathbf{x} &\leqslant \left(\int_{A_k(\mathbb{R}^3)} |\widetilde{\boldsymbol{v}}|^{p\frac{r}{p}} \, d\mathbf{x} \right)^{\frac{p}{r}} \left(\int_{A_k(\mathbb{R}^3)} 1^{\frac{r}{r-p}} \, d\mathbf{x} \right)^{\frac{r-p}{r}} \\ &\leqslant Ck^{\frac{3r-3p}{r}} \left(\int_{A_k(\mathbb{R}^3)} |\widetilde{\boldsymbol{v}}|^r \, d\mathbf{x} \right)^{\frac{p}{r}}. \end{aligned} \tag{II.2}$$

By combining (II.1) and (II.2) and using that $0 \leqslant \frac{1}{p} - \frac{1}{r} \leqslant \frac{1}{3}$, we deduce thanks to dominated convergence theorem that $\widetilde{\boldsymbol{v}} \cdot \nabla \varphi_k$ tends to 0 when $k \to \infty$.

Now, we prove that $\rho \nabla (\widetilde{\boldsymbol{v}} \cdot \nabla \varphi_k) \longrightarrow 0$ in $\boldsymbol{L}^p(\mathbb{R}^3)$:

$$\begin{aligned} \int_{\mathbb{R}^3} |\rho \nabla (\widetilde{\boldsymbol{v}} \cdot \nabla \varphi_k)|^p \, d\mathbf{x} &= \int_{\mathbb{R}^3} \rho^p |\nabla \widetilde{\boldsymbol{v}} \cdot \nabla \varphi_k + \widetilde{\boldsymbol{v}} \cdot \nabla^2 \varphi_k|^p \, d\mathbf{x} \\ &\leq 2^p \int_{\mathbb{R}^3} \rho^p |\nabla \widetilde{\boldsymbol{v}} \cdot \nabla \varphi_k|^p \, d\mathbf{x} + 2^p \int_{\mathbb{R}^3} \rho^p |\widetilde{\boldsymbol{v}} \nabla^2 \varphi_k|^p \, d\mathbf{x} \\ &\leq C \left(\frac{1}{k^p} \int_{A_k(\mathbb{R}^3)} |\rho \nabla \widetilde{\boldsymbol{v}}|^p \, d\mathbf{x} + \frac{1}{k^{2p}} (1+2k)^p \int_{A_k(\mathbb{R}^3)} |\widetilde{\boldsymbol{v}}|^p \, d\mathbf{x} \right), \end{aligned}$$

where C is independent of k. As above the first term in the left-hand side tends to zero because

$$\int_{A_k(\mathbb{R}^3)} |\rho \nabla \, \widetilde{\boldsymbol{v}}|^p \, d\mathbf{x} \leqslant C \, k^{\frac{3r-3p}{r}} \left(\int_{A_k(\mathbb{R}^3)} |\rho \nabla \, \widetilde{\boldsymbol{v}}|^r \, d\mathbf{x} \right)^{\frac{p}{r}}.$$

and using (II.2), we prove that the second term tends also to zero. Thus we obtain that
$\boldsymbol{v}_k \to \widetilde{\boldsymbol{v}}$ in $\boldsymbol{X}^1_{r,p}(\mathbb{R}^3)$ when $k \to \infty$.

Finally , we start the regularization of our sequence $\boldsymbol{v}_k$. In a first step we consider that Ω' is strictly star-shaped with respect to one of its points which is taken to the origin. Under this assumption, we set $\boldsymbol{v}_{k,\theta}(x) = \boldsymbol{v}_k(\theta x)$ for any real number $\theta > 1$ and $x \in \mathbb{R}^3$. Then $\boldsymbol{v}_{k,\theta} \in \boldsymbol{X}^1_{r,p}(\mathbb{R}^3)$ and $\operatorname{supp} \boldsymbol{v}_{k,\theta}$ is compact in Ω when θ is close to 1. Moreover

$$\lim_{\theta \to 1} \boldsymbol{v}_{k,\theta} = \boldsymbol{v}_k \ \text{ in } \ \boldsymbol{X}^1_{r,p}(\mathbb{R}^3).$$

Consequently, for any real number $\epsilon > 0$ small enough, the restriction of $\rho_\epsilon * \boldsymbol{v}_{k,\theta}$ to Ω belongs to $\boldsymbol{\mathcal{D}}(\Omega)$ and

$$\lim_{\epsilon \to 0} \lim_{\theta \to 1} \lim_{k \to \infty} \rho_\epsilon * \boldsymbol{v}_{k,\theta} = \widetilde{\boldsymbol{v}} \ \text{ in } \ \boldsymbol{X}^1_{r,p}(\mathbb{R}^3),$$

where ρ_ϵ is a mollifier. Consequently, $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{X}^1_{r,p}(\Omega)$. In the case where Ω' is only a Lipschitz open set in $\mathbb{R}^3$, we have to recover Ω' by a finite number of star open sets and partitions of unity. Clearly, it suffices to apply the above argument to each of these sets to derive the desired result on the entire domain. □

Now, for $p \neq 3$ and $r = 3$, we have the following result:

Lemma 2.2 *Suppose that Ω is only a Lipschitz open set. For $r = 3$ and $3/2 \leq p < 3$ the space $\mathcal{D}(\Omega)$ is dense in $\boldsymbol{X}^0_{3,p}(\Omega)$ and for all $q \in W_0^{-1,p'}(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{X}^0_{3,p}(\Omega)$, we have*

$$\langle \nabla q, \boldsymbol{\varphi} \rangle_{[\boldsymbol{X}^0_{3,p}(\Omega)]' \times \boldsymbol{X}^0_{3,p}(\Omega)} = -\langle q, \operatorname{div} \boldsymbol{\varphi} \rangle_{W_0^{-1,p'}(\Omega) \times \mathring{W}_0^{1,p}(\Omega)} . \tag{II.3}$$

Proof. The main idea of the proof is relied on adequate truncation procedure. The truncation function that we shall use has been introduced by Bolley-Camus [17]. First, let $\phi \in C^\infty([0, \infty[)$ be such that

$$\phi(t) = 0, \quad \forall t \in [0, 1], \quad 0 \leq \phi(t) \leq 1, \quad \forall t \in [1, 2], \quad \phi(t) = 1, \quad \forall t \geq 2.$$

Next, for $k \in \mathbb{N}$ we define ϕ_k by

$$\phi_k(x) = \begin{cases} \phi(\frac{k}{ln|x|}), & \forall x \in \mathbb{R}^3 : |x| > 1, \\ 1, & \text{otherwise.} \end{cases}$$

Then we see that for all $x \in \mathbb{R}^3$:

$$0 \leq \phi_k(x) \leq 1, \quad \text{if} \quad |x| \in [e^{\frac{k}{2}}, e^k] \quad \text{and} \quad \phi_k(x) = \begin{cases} 1 & \text{if} \quad |x| \leqslant e^{\frac{k}{2}} \\ 0 & \text{if} \quad |x| \geq e^k \end{cases} ,$$

so that multiplication by ϕ_k is indeed a truncation process. Note that this truncation process is adapted to the logarithmic weights. Now, let $\boldsymbol{v} \in \boldsymbol{X}^0_{3,p}(\Omega)$ and $\widetilde{\boldsymbol{v}}$ be the extension by $\mathbf{0}$ of $\boldsymbol{v}$ to $\mathbb{R}^3$, then we have $\widetilde{\boldsymbol{v}} \in \boldsymbol{X}^0_{3,p}(\mathbb{R}^3)$. Set $\boldsymbol{v}_k = \phi_k \widetilde{\boldsymbol{v}}$.
We start by proving that $\boldsymbol{v}_k \to \widetilde{\boldsymbol{v}}$ in $\boldsymbol{X}^0_{3,p}(\mathbb{R}^3)$ when $k \to \infty$.
1) Convergence of $\boldsymbol{v}_k$ in $\boldsymbol{W}^{1,3}_0(\mathbb{R}^3)$:
On one hand we have:

$$\int_{\mathbb{R}^3} |\frac{\boldsymbol{v}_k - \widetilde{\boldsymbol{v}}}{\rho \ln(2 + r^2)}|^3 \, d\mathbf{x} = \int_{\mathbb{R}^3} |\phi_k - 1|^3 |\frac{\widetilde{\boldsymbol{v}}}{\rho \ln(2 + r^2)}|^3 \, d\mathbf{x} \leqslant C \int_{D^k(\mathbb{R}^3)} |\frac{\widetilde{\boldsymbol{v}}}{\rho \ln(2 + r^2)}|^3 \, d\mathbf{x},$$

where C is independent of k. Using dominated convergence theorem, we deduce that this integral converges to zero when $k \longrightarrow \infty$. On the other hand we have:

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla(\phi_k \widetilde{\boldsymbol{v}}) - \nabla \widetilde{\boldsymbol{v}}|^3 \, d\mathbf{x} &\leqslant \int_{\mathbb{R}^3} |\nabla \phi_k|^3 |\widetilde{\boldsymbol{v}}|^3 \, d\mathbf{x} + \int_{\mathbb{R}^3} |\phi_k - 1|^3 |\nabla \widetilde{\boldsymbol{v}}|^3 \, d\mathbf{x} \\ &\leqslant C \left(\int_{C_k(\mathbb{R}^3)} \rho^3 \ln^3(2 + r^2) |\nabla \phi_k|^3 |\frac{\widetilde{\boldsymbol{v}}}{\rho \ln(2 + r^2)}|^3 + \int_{D^k(\mathbb{R}^3)} |\nabla \widetilde{\boldsymbol{v}}|^3 \right), \end{aligned}$$

where C is independent of k. It follows from Lemma 7.1 of [6] that

$$|\nabla \phi_k| \leqslant \frac{C_1}{\rho \ln(2 + r^2)}, \tag{II.4}$$

where C_1 is a constant independent of k. Using (II.4) and dominated convergence theorem, we deduce that

$$\int_{\mathbb{R}^3} |\nabla(\phi_k \widetilde{\boldsymbol{v}}) - \nabla \widetilde{\boldsymbol{v}}|^3 \, d\mathbf{x} \leqslant C\left(\int_{C_k(\mathbb{R}^3)} |\frac{\widetilde{\boldsymbol{v}}}{\rho \ln(2+r^2)}|^3 \, d\mathbf{x} + \int_{D^k(\mathbb{R}^3)} |\nabla \widetilde{\boldsymbol{v}}|^3 \, d\mathbf{x}\right) \to 0,$$

when $k \to \infty$.

2) Convergence of $\operatorname{div} \boldsymbol{v}_k$ **in** $W_0^{1,p}(\mathbb{R}^3)$:

We begin by proving that $\phi_k \operatorname{div} \widetilde{\boldsymbol{v}} \to \operatorname{div} \widetilde{\boldsymbol{v}}$ in $W_0^{1,p}(\mathbb{R}^3)$.

$$\int_{\mathbb{R}^3} |\frac{\phi_k \operatorname{div} \widetilde{\boldsymbol{v}} - \operatorname{div} \widetilde{\boldsymbol{v}}}{\rho}|^p \, d\mathbf{x} = \int_{\mathbb{R}^3} |\phi_k - 1|^p |\frac{\operatorname{div} \widetilde{\boldsymbol{v}}}{\rho}|^p \, d\mathbf{x} \leq C \int_{D^k(\mathbb{R}^3)} |\frac{\operatorname{div} \widetilde{\boldsymbol{v}}}{\rho}|^p \, d\mathbf{x},$$

where C is independent of k. Using dominated convergence theorem, we deduce that this integral converges to zero when $k \to \infty$. Moreover,

$$\begin{aligned}
\int_{\mathbb{R}^3} |\nabla(\phi_k \operatorname{div} \widetilde{\boldsymbol{v}}) - \nabla(\operatorname{div} \widetilde{\boldsymbol{v}})|^p \, d\mathbf{x} &\leqslant \int_{\mathbb{R}^3} |(\nabla \phi_k) \operatorname{div} \widetilde{\boldsymbol{v}}|^p \, d\mathbf{x} + \int_{\mathbb{R}^3} |\phi_k - 1|^3 |\nabla(\operatorname{div} \widetilde{\boldsymbol{v}})|^p \, d\mathbf{x} \\
&\leqslant C\left(\int_{C_k(\mathbb{R}^3)} \rho^p |\frac{(\nabla \phi_k) \operatorname{div} \widetilde{\boldsymbol{v}}}{\rho}|^p \, d\mathbf{x} + \int_{D^k(\mathbb{R}^3)} |\nabla(\operatorname{div} \widetilde{\boldsymbol{v}})|^p \, d\mathbf{x}\right),
\end{aligned}$$

where C is independent of k. Using (II.4), we prove

$$\int_{\mathbb{R}^3} |\nabla(\phi_k \operatorname{div} \widetilde{\boldsymbol{v}}) - \nabla(\operatorname{div} \widetilde{\boldsymbol{v}})|^p \, d\mathbf{x} \leqslant C\left(\int_{C_k(\mathbb{R}^3)} \frac{1}{k^p} |\frac{\operatorname{div} \widetilde{\boldsymbol{v}}}{\rho}|^p \, d\mathbf{x} + \int_{D^k(\mathbb{R}^3)} |\nabla(\operatorname{div} \widetilde{\boldsymbol{v}})|^p \, d\mathbf{x}\right),$$

and then we use the dominated convergence theorem to prove that the right hand side converge to zero when $k \to \infty$.

Now, we shall prove that $\widetilde{\boldsymbol{v}} \cdot \nabla \phi_k \longrightarrow 0$ in $W_0^{1,p}(\mathbb{R}^3)$. On one hand, using Hölder inequality, (since $p < 3$), we have

$$\begin{aligned}
\int_{\mathbb{R}^3} |\frac{\widetilde{\boldsymbol{v}} \cdot \nabla \phi_k}{\rho}|^p \, d\mathbf{x} &= \int_{\mathbb{R}^3} |\frac{\widetilde{\boldsymbol{v}}}{\rho \ln(2+r^2)}|^p (\ln(2+r^2))^p |\nabla \phi_k|^p \, d\mathbf{x} \\
&\leqslant C\left(\int_{\mathbb{R}^3} |\frac{\widetilde{\boldsymbol{v}}}{\rho \ln(2+r^2)}|^{p\frac{3}{p}} \, d\mathbf{x}\right)^{\frac{p}{3}} \left(\int_{\mathbb{R}^3} (\ln(2+r^2)|\nabla \phi_k|)^{\frac{3p}{3-p}} \, d\mathbf{x}\right)^{\frac{3-p}{3}}. \quad \text{(II.5)}
\end{aligned}$$

Using (II.4) and polar coordinates, we have

$$\begin{aligned}
\left(\int_{\mathbb{R}^3} (\ln(2+r^2)|\nabla \phi_k|)^{\frac{3p}{3-p}} \, d\mathbf{x}\right)^{\frac{3-p}{3}} &\leq C\left(\int_{C_k(\mathbb{R}^3)} \left(\frac{1}{(1+|\mathbf{x}|^2)^{1/2}}\right)^{\frac{3p}{3-p}} \, d\mathbf{x}\right)^{\frac{3-p}{3}} \\
&\leq C\left(\int_{e^{\frac{k}{2}}}^{e^k} \left(\frac{1}{r}\right)^{\frac{3p}{3-p}} r^2 \, dr\right)^{\frac{3-p}{3}} \\
&\leq C\left(e^{\frac{k}{2}(3-\frac{3p}{3-p})} - e^{k(3-\frac{3p}{3-p})}\right)^{\frac{3-p}{3}}. \quad \text{(II.6)}
\end{aligned}$$

Since $p \geq 3/2$, this last term is equivalent to $e^{\frac{k}{2}(3-2p)}$ when $k \to \infty$. It follows from (II.5) and (II.6) that $\widetilde{\boldsymbol{v}} \cdot \nabla \phi_k \longrightarrow 0$ in $W^{0,p}_{-1}(\mathbb{R}^3)$. On the other hand we have

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla(\widetilde{\boldsymbol{v}} \cdot \nabla \phi_k)|^p \, d\mathbf{x} &= \int_{\mathbb{R}^3} |\nabla \widetilde{\boldsymbol{v}} \cdot \nabla \phi_k + \widetilde{\boldsymbol{v}} \cdot \nabla^2 \phi_k|^p \, d\mathbf{x} \\ &\leq C \left(\int_{\mathbb{R}^3} |\nabla \widetilde{\boldsymbol{v}}|^p |\nabla \phi_k|^p \, d\mathbf{x} + \int_{\mathbb{R}^3} \rho^p \mathrm{ln}^p (2+r^2) |\nabla^2 \, \phi_k|^p | \frac{\widetilde{\boldsymbol{v}}}{\rho \mathrm{ln}(2+r^2)}|^p \, d\mathbf{x} \right). \end{aligned}$$

It follows from Lemma 7.1 of [6], that

$$|\nabla^2 \, \phi_k| \leqslant \frac{C_2}{\rho^2 \mathrm{ln}(2+r^2)}, \tag{II.7}$$

where C_2 is a constant independent of k. As above, using (II.4), (II.7) and Hölder inequality, we deduce that

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla(\widetilde{\boldsymbol{v}} \cdot \nabla \phi_k)|^p \, d\mathbf{x} &\leqslant C \frac{1}{k^p} \left(e^{\frac{k}{2}(3 - \frac{3p}{3-p})} - e^{k(3-\frac{3p}{3-p})} \right)^{\frac{3-p}{3}} \left(\int_{\mathbb{R}^3} |\nabla \widetilde{\boldsymbol{v}}|^3 \, d\mathbf{x} \right)^{\frac{p}{3}} \\ &+ C \left(e^{\frac{k}{2}(3 - \frac{3p}{3-p})} - e^{k(3-\frac{3p}{3-p})} \right)^{\frac{3-p}{3}} \left(\int_{\mathbb{R}^3} | \frac{\widetilde{\boldsymbol{v}}}{\rho \mathrm{ln}(2+r^2)}|^3 \, d\mathbf{x} \right)^{\frac{p}{3}}. \end{aligned}$$

Since $p \geq 3/2$, we deduce that these two terms converge to zero when $k \to \infty$. Then we obtain that $\boldsymbol{v}_k \to \widetilde{\boldsymbol{v}}$ in $\boldsymbol{X}^0_{3,p}(\mathbb{R}^3)$ when $k \to \infty$. For the rest of the proof, we adapt the same method employed in the proof of Lemma 2.1. Finally (II.3) holds. □

Remark 2.1

In this chapter, we are not interested in the density of $\mathfrak{D}(\Omega)$ in $\boldsymbol{X}^0_{r',p'}(\Omega)$ when $p' = r' = 3$ or $p' = 3$ and $r' \neq 3$, but with the same method employed in the proof of Lemma 2.2 and using Hölder inequality, we can prove this result. In the case when $p' = 3$ and $r' \neq 3$, we shall suppose in addition that $r' \geq 3/2$.

As a consequence of Lemma 2.1 and Lemma 2.2 , we have the following Green's formulas:

Corollary 2.1 *Suppose that* $0 \leq \frac{1}{r} - \frac{1}{p} \leq \frac{1}{3}$, *then*

i) For all $q \in W^{-1,p}_{-1}(\Omega)$ *and* $\boldsymbol{\varphi} \in \boldsymbol{X}^1_{r',p'}(\Omega)$, *we have*

$$\langle \nabla q, \boldsymbol{\varphi} \rangle_{[\boldsymbol{X}^1_{r',p'}(\Omega)]' \times \boldsymbol{X}^1_{r',p'}(\Omega)} = - \langle q, \mathrm{div}\, \boldsymbol{\varphi} \rangle_{W^{-1,p}_{-1}(\Omega) \times \mathring{W}^{1,p'}_1(\Omega)}. \tag{II.8}$$

ii) If in addition $p' \neq 3$, *then for all* $q \in W^{-1,p}_0(\Omega)$ *and* $\boldsymbol{\varphi} \in \boldsymbol{X}^0_{r',p'}(\Omega)$, *we have*

$$\langle \nabla q, \boldsymbol{\varphi} \rangle_{[\boldsymbol{X}^0_{r',p'}(\Omega)]' \times \boldsymbol{X}^0_{r',p'}(\Omega)} = - \langle q, \mathrm{div}\, \boldsymbol{\varphi} \rangle_{W^{-1,p}_0(\Omega) \times \mathring{W}^{1,p'}_0(\Omega)}. \tag{II.9}$$

In the following lemma, we will characterize the dual space of $\boldsymbol{X}^\ell_{r,p}(\Omega)$. This result and its proof are classical.

Lemma 2.3 *Let* $\boldsymbol{f} \in (\boldsymbol{X}^{\ell}_{r,p}(\Omega))'$ *with* $\ell = 1$ *or* $\ell = 0$. *Then there exist* $\mathbb{F}_0 = (f_{ij})_{1\leq i,j\leq 3} \in \boldsymbol{W}^{0,r'}_{-\ell}(\Omega)$ *and* $f_1 \in W^{-1,p'}_{-\ell}(\Omega)$ *such that:*

$$\boldsymbol{f} = \operatorname{div} \mathbb{F}_0 + \nabla f_1. \tag{II.10}$$

Moreover,

$$\| \boldsymbol{f} \|_{[\boldsymbol{X}^{\ell}_{r,p}(\Omega)]'} = \max\left\{ \| f_{ij} \|_{W^{0,r'}_{-\ell}(\Omega)}, 1 \leq i,j \leq 3, \| f_1 \|_{W^{-1,p'}_{-\ell}(\Omega)} \right\}.$$

Conversely, if $\boldsymbol{f}$ *satisfies* (II.10), *then* $\boldsymbol{f} \in (\boldsymbol{X}^{\ell}_{r,p}(\Omega))'$.

Proof.

Let $\mathbf{E} = \boldsymbol{W}^{0,r}_{\ell}(\Omega) \times \mathring{W}^{1,p}_{\ell}(\Omega)$ which is equipped by the following norm:

$$\forall \boldsymbol{h} = (\mathbb{H}_0, h_1) \in \mathbf{E}, \ \| \boldsymbol{h} \|_{\mathbf{E}} = \sum_{1\leq i,j\leq 3} \| h_{ij} \|_{W^{0,r}_{\ell}(\Omega)} + \| h_1 \|_{W^{1,p}_{\ell}(\Omega)},$$

being $\mathbb{H}_0 = (h_{ij})_{1\leq i,j\leq 3}$. The mapping $T : \boldsymbol{\varphi} \in \boldsymbol{X}^{\ell}_{r,p}(\Omega) \mapsto (\nabla \boldsymbol{\varphi}, \operatorname{div} \boldsymbol{\varphi}) \in \mathbf{E}$ is an isometry from $\boldsymbol{X}^{\ell}_{r,p}(\Omega)$ into $\mathbf{E}$. Suppose $\mathbf{G} = T(\boldsymbol{X}^{\ell}_{r,p}(\Omega))$ with the $\mathbf{E}$-topology. Let $S = T^{-1} : \mathbf{G} \mapsto \boldsymbol{X}^{\ell}_{r,p}(\Omega)$. Thus, we can define the following mapping:

$$\boldsymbol{h} \in \mathbf{G} \mapsto \langle \boldsymbol{f}, S\boldsymbol{h} \rangle_{[\boldsymbol{X}^{\ell}_{r,p}(\Omega)]' \times \boldsymbol{X}^{\ell}_{r,p}(\Omega)} \quad \text{for } \boldsymbol{f} \in [\boldsymbol{X}^{\ell}_{r,p}(\Omega)]'$$

which is a linear continuous form on $\mathbf{G}$. Thanks to Hahn-Banach's Theorem, such form can be extended to a linear continuous form on $\mathbf{E}$, denoted by $\boldsymbol{\Pi}$ such that $||\boldsymbol{\Pi}||_{\mathbf{E}'} = ||\boldsymbol{f}||_{[\boldsymbol{X}^{\ell}_{r,p}(\Omega)]'}$. From the Riesz's Representation Lemma, there exist $\mathbb{F}_0 = (f_{ij})_{1\leq i,j\leq 3}$ such that $\mathbb{F}_0 \in \boldsymbol{W}^{0,r'}_{-\ell}(\Omega)$ and $f_1 \in W^{-1,p'}_{-\ell}(\Omega)$ such that for any $\boldsymbol{h} = (\mathbb{H}_0, h_1) \in \mathbf{E}$,

$$\begin{aligned} \langle \boldsymbol{\Pi}, \boldsymbol{h} \rangle_{\mathbf{E}' \times \mathbf{E}} &= \langle \mathbb{F}_0, \mathbb{H}_0 \rangle_{\boldsymbol{W}^{0,r'}_{-\ell}(\Omega) \times \boldsymbol{W}^{0,r}_{\ell}(\Omega)} + \langle f_1, h_1 \rangle_{W^{-1,p'}_{-\ell}(\Omega) \times \mathring{W}^{1,p}_{\ell}(\Omega)} \\ &= \sum_{i,j=1}^{3} \langle f_{ij}, h_{ij} \rangle_{W^{0,r'}_{-\ell}(\Omega) \times W^{0,r}_{\ell}(\Omega)} + \langle f_1, h_1 \rangle_{W^{-1,p'}_{-\ell}(\Omega) \times \mathring{W}^{1,p}_{\ell}(\Omega)}, \end{aligned}$$

with $\| \ \boldsymbol{\Pi} \ \|_{\mathbf{E}'} = \max\left\{ \| f_{ij} \|_{W^{0,r'}_{-\ell}(\Omega)}, 1 \leq i,j \leq 3, \| f_1 \|_{W^{-1,p'}_{-\ell}(\Omega)} \right\}$. In particular, if $\boldsymbol{h} = T\boldsymbol{\varphi} \in \mathbf{G}$, where $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega)$, we have:

$$\langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{[\boldsymbol{X}^{\ell}_{r,p}(\Omega)]' \times \boldsymbol{X}^{\ell}_{r,p}(\Omega)} = \langle -\operatorname{div} \mathbb{F}_0 - \nabla f_1, \boldsymbol{\varphi} \rangle.$$

To finish it is easy to verify that the reciprocal holds. □

Giving a meaning to the trace of a very weak solution of the Stokes problem is not trivial: remember that we are not in the classical variational framework. In this way, we need to introduce some spaces. First, we consider the space:

$$\boldsymbol{Y}_{p',\ell}(\Omega) = \left\{ \boldsymbol{\psi} \in \boldsymbol{W}^{2,p'}_{\ell}(\Omega), \ \boldsymbol{\psi}|_{\Gamma} = 0, \ \operatorname{div} \boldsymbol{\psi}|_{\Gamma} = 0 \right\}.$$

The following lemma gives another characterization to the space $\boldsymbol{Y}_{p',\ell}(\Omega)$ very useful in the Green's formula defined in Corolllary 2.2 (see below).

Lemma 2.4 *We have the identity*

$$\boldsymbol{Y}_{p',\ell}(\Omega) = \left\{\boldsymbol{\psi} \in \boldsymbol{W}^{2,p'}_{\ell}(\Omega), \boldsymbol{\psi}|_\Gamma = \boldsymbol{0}, \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n}|_\Gamma = 0\right\} \tag{II.11}$$

and the range space of the normal derivative $\gamma_1 : \boldsymbol{Y}_{p',\ell}(\Omega) \longrightarrow \boldsymbol{W}^{1/p,p'}(\Gamma)$ *is*

$$\boldsymbol{Z}_{p'}(\Gamma) = \left\{\boldsymbol{z} \in \boldsymbol{W}^{1/p,p'}(\Gamma);\;\; \boldsymbol{z}\cdot\boldsymbol{n} = 0\right\}.$$

Proof. Let $\boldsymbol{u} \in \boldsymbol{W}^{2,p'}_{\ell}(\Omega)$ such that $\boldsymbol{u} = \boldsymbol{0}$ on Γ. Then $\operatorname{div} \boldsymbol{u} = \dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n}$ on Γ and the identity (II.11) holds. Moreover, it is clear that $\operatorname{Im}(\gamma_1) \subset \boldsymbol{Z}_{p'}(\Gamma)$. Conversely, let $\boldsymbol{\mu} \in \boldsymbol{Z}_{p'}(\Gamma)$. As Ω' is bounded, we can fix once for all a ball B_{R_o}, centered at the origin and with radius R_0, such that $\overline{\Omega'} \subset B_{R_o}$. Thus we have the existence of $\boldsymbol{u} \in \boldsymbol{W}^{2,p'}(\Omega_{R_0})$ such that $\boldsymbol{u} = \boldsymbol{0}$ on $\Gamma \cup \partial B_{R_o}$, $\dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} = \boldsymbol{\mu}$ on Γ and $\dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} = 0$ on ∂B_{R_o}, note that $\Omega_{R_0} = \Omega \cap B_{R_0}$. The function $\boldsymbol{u}$ can be extended by zero outside B_{R_o} and owing to its boundary conditions on ∂B_{R_o}, the extended function, still denoted by $\boldsymbol{u}$, belongs to $\boldsymbol{W}^{2,p'}_{\ell}(\Omega)$, for any ℓ since its support is bounded. Since $\boldsymbol{\mu}\cdot\boldsymbol{n} = 0$ on Γ, we have $\boldsymbol{u} \in \boldsymbol{Y}_{p',\ell}(\Omega)$ and $\boldsymbol{\mu} \in \operatorname{Im}(\gamma_1)$. $\square$

Secondly, we shall use the space:

$$\boldsymbol{T}^{\ell}_{r,p}(\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{W}^{0,p}_{-\ell}(\Omega);\;\Delta \boldsymbol{v} \in [\boldsymbol{X}^{\ell}_{r',p'}(\Omega)]'\right\},$$

equipped with the norm:

$$\| \boldsymbol{v} \|_{\boldsymbol{T}^{\ell}_{r,p}(\Omega)} = \| \boldsymbol{v} \|_{\boldsymbol{W}^{0,p}_{-\ell}(\Omega)} + \| \Delta \boldsymbol{v} \|_{[\boldsymbol{X}^{\ell}_{r',p'}(\Omega)]'}\,.$$

We also introduce the following space:

$$\boldsymbol{H}^{r}_{p,\ell}(\operatorname{div},\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{W}^{0,p}_{\ell-1}(\Omega); \operatorname{div} \boldsymbol{v} \in W^{0,r}_{\ell-1}(\Omega)\right\}.$$

This space is equipped with the graph norm. The following lemma is essential to ensure the proof of Lemma 2.6 (see below).

Before starting, let define:

$$M(\mathbb{R}^3) = \left\{u \in W^{0,p}_{-1}(\mathbb{R}^3);\;\; \nabla u \in \boldsymbol{W}^{0,r}_{-1}(\mathbb{R}^3)\right\}.$$

Lemma 2.5 *Suppose that* $0 \leqslant \frac{1}{r} - \frac{1}{p} \leqslant \frac{1}{3}$, *then* $\mathcal{D}(\mathbb{R}^3)$ *is dense in* $M(\mathbb{R}^3)$.

Proof. Let $u \in M(\mathbb{R}^3)$, as in Lemma 2.1, since $0 \leqslant \frac{1}{r} - \frac{1}{p} \leqslant \frac{1}{3}$, we prove thanks to dominated convergence theorem that $\varphi_k u$ converges to u in $M(\mathbb{R}^3)$ when $k \to \infty$, where φ_k is a cut off functions.

In other words, the set of functions of $M(\mathbb{R}^3)$ with bounded support is dense in $M(\mathbb{R}^3)$ and we may assume that u has a bounded support. The result is then proved by regularization. Indeed, for any real number $\epsilon > 0$ small enough, we have that $\rho_\epsilon * u$ belongs to $\mathcal{D}(\mathbb{R}^3)$. Since $u \in W^{0,p}_{-1}(\mathbb{R}^3)$ and has a compact support we deduce that $u \in L^p(\mathbb{R}^3)$. Using that $\frac{1}{\rho} \in L^\infty(\mathbb{R}^3)$, we prove that $\rho_\epsilon * u \longrightarrow u$ in $W^{0,p}_{-1}(\mathbb{R}^3)$ when $\epsilon \longrightarrow 0$ and by the same way that $\rho_\epsilon * \nabla u \longrightarrow \nabla u$ in $\boldsymbol{W}^{0,r}_{-1}(\mathbb{R}^3)$ when $\epsilon \longrightarrow 0$. Then we have $\rho_\epsilon * u \longrightarrow u$ in $M(\mathbb{R}^3)$. Hence u is the limit in $M(\mathbb{R}^3)$. $\square$

Now, we prove that the tangential trace of functions $\boldsymbol{v} \in \boldsymbol{T}^\ell_{r,p}(\Omega)$ belongs to the dual space of $\boldsymbol{Z}_{p'}(\Gamma)$, which is

$$(\boldsymbol{Z}_{p'}(\Gamma))' = \left\{\boldsymbol{\mu} \in \boldsymbol{W}^{-1/p,p}(\Gamma); \boldsymbol{\mu} \cdot \boldsymbol{n} = 0\right\}. \tag{II.12}$$

Observe that we can decompose $\boldsymbol{v}$ into its tangential and normal parts, that is:

$$\boldsymbol{v} = \boldsymbol{v}_\tau + (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}.$$

The proof of the following lemma is similar to the case of bounded domain (see [13]).

Lemma 2.6 *Suppose that $\frac{3}{2} < p < \infty$ and $\frac{1}{p} + \frac{1}{3} = \frac{1}{r}$. Then*
i) For $\ell = 0$, the space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{T}^0_{r,p}(\Omega)$.

ii) For $\ell = 1$ and $p \neq 3$, the space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{T}^1_{r,p}(\Omega)$.

Proof. Let $\boldsymbol{\chi} \in (\boldsymbol{T}^\ell_{p,r}(\Omega))'$, with $\ell = 0$ or $\ell = 1$, such that for any $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$, we have $\langle \boldsymbol{\chi}, \boldsymbol{v}\rangle = 0$. We want to prove that $\boldsymbol{\chi} = \boldsymbol{0}$. There exists $(\boldsymbol{f}, \boldsymbol{g}) \in \boldsymbol{W}^{0,p'}_\ell(\Omega) \times \boldsymbol{X}^\ell_{r',p'}(\Omega)$ such that: for any $\boldsymbol{v} \in \boldsymbol{T}^\ell_{p,r}(\Omega)$,

$$\langle \boldsymbol{\chi}, \boldsymbol{v}\rangle = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} d\mathbf{x} + \langle \Delta \boldsymbol{v}, \boldsymbol{g}\rangle_{[\boldsymbol{X}^\ell_{r',p'}(\Omega)]' \times \boldsymbol{X}^\ell_{r',p'}(\Omega)}\,.$$

Since we have $\langle \boldsymbol{\chi}, \boldsymbol{v}\rangle = 0$ for any $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$ then we have also $\langle \boldsymbol{\chi}, \boldsymbol{v}\rangle = 0$ for any $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega)$, thus $\boldsymbol{f} + \Delta \boldsymbol{g} = \boldsymbol{0}$ in $\boldsymbol{\mathcal{D}}'(\Omega)$. Observe that we can easily extend by zero the functions $\boldsymbol{f}$ and $\boldsymbol{g}$, in such a way that

$$\widetilde{\boldsymbol{f}} \in \boldsymbol{W}^{0,p'}_\ell(\mathbb{R}^3) \ \text{ and } \ \widetilde{\boldsymbol{g}} \in \boldsymbol{X}^\ell_{r',p'}(\mathbb{R}^3).$$

Now we take $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3)$. Then we have by assumption that:

$$\int_{\mathbb{R}^3} \widetilde{\boldsymbol{f}} \cdot \boldsymbol{\varphi} d\mathbf{x} + \int_{\mathbb{R}^3} \widetilde{\boldsymbol{g}} \cdot \Delta\boldsymbol{\varphi} d\mathbf{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} d\mathbf{x} + \int_\Omega \boldsymbol{g} \cdot \Delta\boldsymbol{\varphi} = \langle \boldsymbol{\chi}, \boldsymbol{\varphi}\rangle = 0,$$

and we deduce that $\widetilde{\boldsymbol{f}} + \Delta\widetilde{\boldsymbol{g}} = \boldsymbol{0}$ in $\boldsymbol{\mathcal{D}}'(\mathbb{R}^3)$ and $\Delta\widetilde{\boldsymbol{g}} \in \boldsymbol{W}^{0,p'}_\ell(\mathbb{R}^3)$.
i) At first, observe that since $\frac{3}{2} < p < \infty$ and $\frac{1}{p} + \frac{1}{3} = \frac{1}{r}$ and using (I.2), we deduce that $\boldsymbol{W}^{2,p'}_0(\mathbb{R}^3) \hookrightarrow \boldsymbol{W}^{1,r'}_0(\mathbb{R}^3)$. It follows from [6] Theorem 6.6 that there exists $\boldsymbol{\lambda} \in \boldsymbol{W}^{2,p'}_0(\mathbb{R}^3)$ such that $\Delta\boldsymbol{\lambda} = \Delta\widetilde{\boldsymbol{g}}$. Thus the harmonic function $\boldsymbol{\lambda} - \widetilde{\boldsymbol{g}}$ belongs to $\boldsymbol{W}^{1,r'}_0(\mathbb{R}^3)$. Note that if $p' < \frac{3}{2}$, we would have $r' < 3$ and thus $\boldsymbol{\lambda} = \widetilde{\boldsymbol{g}} \in \boldsymbol{W}^{2,p'}_0(\mathbb{R}^3)$ and if $p' \geq \frac{3}{2}$, we would have $r' \geq 3$ and thus there exists a polynomial $\boldsymbol{K} \in \boldsymbol{W}^{1,r'}_0(\mathbb{R}^3)$ satisfying $\widetilde{\boldsymbol{g}} = \boldsymbol{\lambda} + \boldsymbol{K}$. Using (I.1) and the fact that $\frac{1}{p} + \frac{1}{3} = \frac{1}{r}$ we deduce that $\boldsymbol{K} \in \mathcal{P}_{[2-\frac{3}{p'}]} \subset \boldsymbol{W}^{2,p'}_0(\mathbb{R}^3)$. Hence since $\boldsymbol{g} \in \boldsymbol{W}^{2,p'}_0(\Omega)$, we deduce that $\boldsymbol{g} \in \mathring{\boldsymbol{W}}^{2,p'}_0(\Omega)$. As $\boldsymbol{\mathcal{D}}(\Omega)$ is

dense in $\mathring{\boldsymbol{W}}_0^{2,p'}(\Omega)$, there exists a sequence $\boldsymbol{g}_k \in \boldsymbol{\mathcal{D}}(\Omega)$ such that $\boldsymbol{g}_k \to \boldsymbol{g}$ in $\boldsymbol{W}_0^{2,p'}(\Omega)$, when $k \to \infty$. Then $\operatorname{div} \boldsymbol{g}_k \to \operatorname{div} \boldsymbol{g}$ in $W_0^{1,p'}(\Omega)$. In particular, $\boldsymbol{g}_k \to \boldsymbol{g}$ in $\boldsymbol{X}_{r',p'}(\Omega)$. Now, we consider $\boldsymbol{v} \in \boldsymbol{T}^0_{p,r}(\Omega)$ and we want to prove that $\langle \boldsymbol{\chi}, \boldsymbol{v}\rangle = 0$. Observe that:

$$\langle \boldsymbol{\chi}, \boldsymbol{v}\rangle = -\int_\Omega \Delta \boldsymbol{g} \cdot \boldsymbol{v} d\mathbf{x} + \langle \Delta \boldsymbol{v}, \boldsymbol{g}\rangle_{[\boldsymbol{X}_{r',p'}(\Omega)]' \times \boldsymbol{X}_{r',p'}(\Omega)}$$

$$= \lim_{k\to\infty} (-\int_\Omega \Delta \boldsymbol{g}_k \cdot \boldsymbol{v} d\mathbf{x} + \langle \Delta \boldsymbol{v}, \boldsymbol{g}_k\rangle_{[\boldsymbol{X}_{r',p'}(\Omega)]' \times \boldsymbol{X}_{r',p'}(\Omega)})$$

$$= \lim_{k\to\infty} (-\int_\Omega \Delta \boldsymbol{g}_k \cdot \boldsymbol{v} d\mathbf{x} + \int_\Omega \boldsymbol{v} \cdot \Delta \boldsymbol{g}_k d\mathbf{x}) = 0.$$

ii) At first, observe that since $\frac{3}{2} < p < \infty$ and $\frac{1}{p} + \frac{1}{3} = \frac{1}{r}$ and using (I.2), we deduce that $\boldsymbol{W}_1^{2,p'}(\mathbb{R}^3) \hookrightarrow \boldsymbol{W}_1^{1,r'}(\mathbb{R}^3)$. In addition, we have $\widetilde{\boldsymbol{g}} \in \boldsymbol{W}_1^{1,r'}(\mathbb{R}^3)$ and $\Delta\widetilde{\boldsymbol{g}} = -\widetilde{\boldsymbol{f}} \in \boldsymbol{W}_1^{0,p'}(\mathbb{R}^3)$. Then for any $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3)$ we have:

$$\langle \Delta\widetilde{\boldsymbol{g}}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}_1^{0,p'}(\mathbb{R}^3)\times \boldsymbol{W}_{-1}^{0,p}(\mathbb{R}^3)} = -\langle \nabla\widetilde{\boldsymbol{g}}, \nabla\boldsymbol{\varphi}\rangle_{\boldsymbol{W}_1^{0,r'}(\mathbb{R}^3)\times \boldsymbol{W}_{-1}^{0,r}(\mathbb{R}^3)}. \tag{II.13}$$

Using Lemma 2.5, the equality (II.13) is still valid for $\boldsymbol{\varphi} \in M(\mathbb{R}^3)^3$. We can deduce by (II.13) that $\Delta\widetilde{\boldsymbol{g}} \in \boldsymbol{W}_1^{0,p'}(\mathbb{R}^3) \perp \mathcal{P}_{[1-\frac{3}{p}]}$. Since $p \neq 3$, we conclude that there exists $\boldsymbol{\theta} \in \boldsymbol{W}_1^{2,p'}(\mathbb{R}^3)$ such that $\Delta\boldsymbol{\theta} = \Delta\widetilde{\boldsymbol{g}}$ (see [2] Proposition 2.2). Thus the harmonic function $\boldsymbol{\theta} - \widetilde{\boldsymbol{g}}$ belongs to $\boldsymbol{W}_1^{1,r'}(\mathbb{R}^3)$ and thus we deduce that $\boldsymbol{\theta} = \widetilde{\boldsymbol{g}} \in \boldsymbol{W}_1^{2,p'}(\mathbb{R}^3)$. Since $\boldsymbol{g} \in \boldsymbol{W}_1^{2,p'}(\Omega)$, we deduce that $\boldsymbol{g} \in \mathring{\boldsymbol{W}}_1^{2,p'}(\Omega)$. The rest of the proof is unchanged.

□

As a consequence of this lemma, we have the following result:

Corollary 2.2 *Let $\frac{3}{2} < p < \infty$ and $\frac{1}{p} + \frac{1}{3} = \frac{1}{r}$. Then the mapping $\gamma_\tau : \boldsymbol{v} \longrightarrow \boldsymbol{v}_\tau|_\Gamma$ on the space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping, still denoted by γ_τ, from $\boldsymbol{T}^\ell_{r,p}(\Omega)$ into $\boldsymbol{W}^{-1/p,p}(\Gamma)$ for $\ell = 0$ and if $p \neq 3$ for $\ell = 1$ and we have the Green formula: for any $\boldsymbol{v} \in \boldsymbol{T}^\ell_{r,p}(\Omega)$ and $\boldsymbol{\psi} \in \boldsymbol{Y}_{p',\ell}(\Omega)$,*

$$\langle \Delta \boldsymbol{v}, \boldsymbol{\psi}\rangle_{[\boldsymbol{X}^\ell_{r',p'}(\Omega)]' \times \boldsymbol{X}^\ell_{r',p'}(\Omega)} = \int_\Omega \boldsymbol{v} \cdot \Delta\boldsymbol{\psi} d\boldsymbol{x} - \left\langle \boldsymbol{v}_\tau, \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{n}}\right\rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma)\times \boldsymbol{W}^{1/p,p'}(\Gamma)}. \tag{II.14}$$

The following lemma gives a precise sense to the normal trace of functions $\boldsymbol{v} \in \boldsymbol{H}^r_{p,\ell}(\operatorname{div}, \Omega)$.

Lemma 2.7 *Let Ω be a Lipschitz open set in $\mathbb{R}^3$. Suppose that $0 \leqslant \frac{1}{r} - \frac{1}{p} \leqslant \frac{1}{3}$ and $\ell = 0$ or $\ell = 1$. Then*
i) The space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{H}^r_{p,\ell}(\operatorname{div}, \Omega)$.

ii) The mapping $\gamma_n : \boldsymbol{v} \longrightarrow \boldsymbol{v}\cdot\boldsymbol{n}|_\Gamma$ on the space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping, still denoted by γ_n, from $\boldsymbol{H}^r_{p,\ell}(\operatorname{div}, \Omega)$ into $\boldsymbol{W}^{-1/p,p}(\Gamma)$. If in addition $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$ and $\frac{3}{2} < p < \infty$, we have the following Green formula: for any $\boldsymbol{v} \in \boldsymbol{H}^r_{p,\ell}(\operatorname{div}, \Omega)$ and $\varphi \in W^{1,p'}_{1-\ell}(\Omega)$,

$$\int_\Omega \boldsymbol{v}\cdot\nabla\varphi\, d\boldsymbol{x} + \int_\Omega \varphi \operatorname{div} \boldsymbol{v}\, d\boldsymbol{x} = \langle \boldsymbol{v}\cdot\boldsymbol{n}, \varphi\rangle_{W^{-1/p,p}(\Gamma)\times W^{1/p,p'}(\Gamma)}. \tag{II.15}$$

Proof. **i)** The proof is very similar to that of Lemma 2.5. Let $\boldsymbol{v} \in \boldsymbol{H}^r_{p,\ell}(\mathrm{div}\,,\Omega)$, we have to prove that $\boldsymbol{v}$ is a limit in $\boldsymbol{H}^r_{p,\ell}(\mathrm{div}\,,\Omega)$ of vector functions of $\boldsymbol{\mathcal{D}}(\overline{\Omega})$. We first approximate $\boldsymbol{v}$ by functions of $\boldsymbol{H}^r_{p,\ell}(\mathrm{div}\,,\Omega)$ with compact support in $\overline{\Omega}$. Let φ_k be a cut off functions introduced in Lemma 2.1. We prove that $\varphi_k\boldsymbol{v} \in \boldsymbol{H}^r_{p,\ell}(\mathrm{div}\,,\Omega)$ and since $0 \leqslant \frac{1}{r} - \frac{1}{p} \leqslant \frac{1}{3}$, we prove thanks to dominated convergence theorem that $\varphi_k\boldsymbol{v}$ converges to $\boldsymbol{v}$ in this space as $k \longrightarrow \infty$. In other words, the set of functions of $\boldsymbol{H}^r_{p,\ell}(\mathrm{div}\,,\Omega)$ with bounded support is dense in $\boldsymbol{H}^r_{p,\ell}(\mathrm{div}\,,\Omega)$ and we may assume that $\boldsymbol{v}$ has a bounded support. Then the lemma follows from the fact that in a bounded Lipschitz-continuous domain $\mathcal{O}$, the space $\boldsymbol{\mathcal{D}}(\overline{\mathcal{O}})$ is dense in $\boldsymbol{H}_{p,r}(\mathrm{div}\,,\mathcal{O})$ see Lemma 13 of [13]. In this case, $\boldsymbol{H}_{p,r}(\mathrm{div}\,,\mathcal{O})$ is defined by

$$\boldsymbol{H}_{p,r}(\mathrm{div},\mathcal{O}) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\mathcal{O}); \mathrm{div}\,\boldsymbol{v} \in L^r(\mathcal{O})\}\,.$$

ii) Let $\varphi \in \mathcal{D}(\overline{\Omega})$ and $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$. Then (II.15) holds. As $\mathcal{D}(\overline{\Omega})$ is dense in $W^{1,p'}_{1-\ell}(\Omega)$, (II.15) is still valid for $\varphi \in W^{1,p'}_{1-\ell}(\Omega)$ and $\boldsymbol{v} \in \mathcal{D}(\overline{\Omega})$. Let $\mu \in W^{1/p,p'}(\Gamma)$. As Ω' is bounded, we can fix once for all a ball B_{R_o}, centered at the origin and with radius R_0, such that $\overline{\Omega'} \subset B_{R_o}$. Thus we have the existence of $\varphi \in W^{1,p'}(\Omega_{R_0})$ such that $\varphi = \mu$ on Γ and $\varphi = 0$ on ∂B_{R_o}. The function φ can be extended by zero outside B_{R_o} and owing to its boundary conditions on ∂B_{R_o}, the extended function, still denoted by φ belongs to $W^{1,p'}_k(\Omega)$, for any weight k since its support is bounded, then in particular belongs to $W^{1,p'}_{1-\ell}(\Omega)$. And thus, we have

$$\| \varphi \|_{W^{1,p'}_{1-\ell}(\Omega)} \leq C \| \mu \|_{W^{1/p,p'}(\Gamma)}\,. \tag{II.16}$$

Therefore, for any $\boldsymbol{v} \in \mathcal{D}(\overline{\Omega})$ we have:

$$|\,\langle \boldsymbol{v}\cdot\boldsymbol{n}, \mu\rangle_\Gamma\,| = |\,\langle \varphi, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma\,| \leqslant C \| \varphi \|_{W^{1,p'}_{1-\ell}(\Omega)} \| \boldsymbol{v} \|_{\boldsymbol{H}^r_{p,\ell}(\mathrm{div}\,,\Omega)}$$

$$\leqslant C \| \mu \|_{W^{1/p,p'}(\Gamma)} \| \boldsymbol{v} \|_{\boldsymbol{H}^r_{p,\ell}(\mathrm{div}\,,\Omega)}\,.$$

Thus, using (II.16), we obtain

$$\| \boldsymbol{v}\cdot\boldsymbol{n} \|_{W^{-1/p,p}(\Gamma)} \leqslant C \| \boldsymbol{v} \|_{\boldsymbol{H}^r_{p,\ell}(\mathrm{div}\,,\Omega)}\,.$$

Therefore, the linear mapping $\gamma_n : \boldsymbol{v} \to \boldsymbol{v}\cdot\boldsymbol{n}|_\Gamma$ defined on $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is continuous for the norm of $\boldsymbol{H}^r_{p,\ell}(\mathrm{div}\,,\Omega)$. Since $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{H}^r_{p,\ell}(\mathrm{div}\,,\Omega)$,$\gamma_n$ can be extended by continuity to a mapping still called $\gamma_n \in \mathcal{L}(\boldsymbol{H}^r_{p,\ell}(\mathrm{div}\,,\Omega); W^{-1/p,p}(\Gamma))$. As $\frac{1}{r} - \frac{1}{p} = \frac{1}{3}$ and $\frac{3}{2} < p < \infty$, we note that $W^{1,p'}_{1-\ell}(\Omega)$ is embedded in $W^{0,r'}_{1-\ell}(\Omega)$ for $\ell = 1$ or $\ell = 0$ and thus (II.15) holds. $\square$

Before stating the theorem of the existense and the uniqueness of the very weak solution for Stokes problem, we need to introduce the following null spaces for $\alpha \in \{-1,0,1\}$ and $k \in \{0,1,2\}$:

$$\mathcal{N}^{k,p}_\alpha(\Omega) = \Big\{(\boldsymbol{u},\pi) \in \boldsymbol{W}^{k,p}_\alpha(\Omega) \times W^{k-1,p}_\alpha(\Omega);\; T(\boldsymbol{u},\pi) = (\boldsymbol{0},0) \text{ in } \Omega \quad\text{and}\quad \boldsymbol{u}|_\Gamma = \boldsymbol{0}\Big\}\,,$$

with

$$T(\boldsymbol{u},\pi) = (-\Delta\,\boldsymbol{u} + \nabla\,\pi,\ \operatorname{div} u).$$

The following lemma proves the identity between some null spaces.

Lemma 2.8 *Assume that Ω is of class $C^{1,1}$ and let $p \neq 3$, then*

$$\mathcal{N}_0^{1,p}(\Omega) = \mathcal{N}_{-1}^{0,p}(\Omega).$$

Proof. Let $(\boldsymbol{u},\pi) \in \boldsymbol{W}_{-1}^{0,p}(\Omega) \times W_{-1}^{-1,p}(\Omega)$ such that

$$-\Delta\boldsymbol{u} + \nabla\,\pi = \boldsymbol{0} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \quad \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma.$$

Note that if $\boldsymbol{u} \in \boldsymbol{W}_{-1}^{0,p}(\Omega)$ and $-\Delta\,\boldsymbol{u} + \nabla\,\pi = \boldsymbol{0}$ in Ω with $\pi \in W_{-1}^{-1,p}(\Omega)$, then the tangential component $\boldsymbol{u}_\tau$ of $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and if $\operatorname{div}\boldsymbol{u} = 0$ in Ω, then $\boldsymbol{u}\cdot\boldsymbol{n} \in W^{-1/p,p}(\Gamma)$. That means that $\boldsymbol{u} = \boldsymbol{0}$ on Γ makes sense.

Now, let λ and μ be two nonnegative functions in $C^\infty(\mathbb{R}^3)$ that satisfy

$$\forall x \in B_{R_0}, \quad \lambda(x) = 1, \quad \operatorname{supp}\lambda \subset B_{R_0+1}, \quad \forall\ x \in \mathbb{R}^3, \quad \lambda(x) + \mu(x) = 1.$$

Let Ω_{R_0+1} denote the intersection $\Omega \cap B_{R_0+1}$ and let C_{R_0} denote the exterior (*i.e.* the complement) of B_{R_0}. Then, we can write

$$\boldsymbol{u} = \lambda\,\boldsymbol{u} + \mu\,\boldsymbol{u}, \qquad \pi = \lambda\,\pi + \mu\,\pi.$$

It follows from Corollary 1.3 of [3] that there exists $\boldsymbol{z} \in \boldsymbol{W}_{-1}^{0,p}(\Omega)$ such that $\operatorname{div}\boldsymbol{z} = \pi$. Let us extend $(\boldsymbol{u},\boldsymbol{z})$ by zero in Ω'. Owing to the boundary conditions imposed, the extended function, that we denote by $(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{z}})$ belongs to $\boldsymbol{W}_{-1}^{0,p}(\mathbb{R}^3) \times \boldsymbol{W}_{-1}^{0,p}(\mathbb{R}^3)$. Set $\tilde{\pi} = \operatorname{div}\tilde{\boldsymbol{z}}$. It suffices to prove that $(\lambda\,\boldsymbol{u},\lambda\,\pi)$ belongs to $\boldsymbol{W}^{1,p}(\Omega_{R_0+1}) \times L^p(\Omega_{R_0+1})$ and that $(\mu\,\tilde{\boldsymbol{u}},\mu\,\tilde{\pi})$ belongs to $\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$.
After an easy calculation, we obtain that the pair $(\mu\,\tilde{\boldsymbol{u}},\mu\,\tilde{\pi})$ satisfies the following equations in $\mathbb{R}^3$:

$$-\Delta(\mu\,\tilde{\boldsymbol{u}}) + \nabla\,(\mu\,\tilde{\pi}) = \boldsymbol{f}_1 \quad \text{and} \quad \operatorname{div}(\mu\,\tilde{\boldsymbol{u}}) = e_1 \quad \text{in} \quad \mathbb{R}^3,$$

with

$$\boldsymbol{f}_1 = -(\Delta\,\mu)\tilde{\boldsymbol{u}} + (\nabla\,\mu)\tilde{\pi} - 2\nabla\,\mu\cdot\nabla\,\tilde{\boldsymbol{u}} \quad \text{and} \quad e_1 = \nabla\,\mu\cdot\tilde{\boldsymbol{u}} \quad \text{in} \quad \mathbb{R}^3.$$

Thus, we are led to study the regularity of a Stokes problem in $\mathbb{R}^3$. Since the right-hand sides $\boldsymbol{f}_1$ and e_1 have indeed a bounded supports, it is easy to check that $(\boldsymbol{f}_1, e_1) \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$. We would like to show that:

$$\forall\boldsymbol{\chi} \in \mathcal{P}_{[1-\frac{3}{p'}]}, \quad \langle\boldsymbol{f}_1,\boldsymbol{\chi}\rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)\times\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} = 0. \tag{II.17}$$

Let $\varphi \in \mathcal{D}(\mathbb{R}^3)$. Then we have:

$$\langle -\Delta(\mu\,\tilde{\boldsymbol{u}}), \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-2,p}_{-1}(\mathbb{R}^3)\times \boldsymbol{W}^{2,p'}_{1}(\mathbb{R}^3)} = \langle \nabla(\mu\,\tilde{\boldsymbol{u}}), \nabla\boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1,p}_{-1}(\mathbb{R}^3)\times \boldsymbol{W}^{1,p'}_{1}(\mathbb{R}^3)}, \tag{II.18}$$

and

$$\langle \nabla\,(\mu\,\tilde{\pi}), \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-2,p}_{-1}(\mathbb{R}^3)\times \boldsymbol{W}^{2,p'}_{1}(\mathbb{R}^3)} = -\,\langle \mu\,\tilde{\pi}, \operatorname{div}\boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1,p}_{-1}(\mathbb{R}^3)\times \boldsymbol{W}^{1,p'}_{1}(\mathbb{R}^3)}. \tag{II.19}$$

Since $\mathcal{D}(\mathbb{R}^3)$ is dense in $\boldsymbol{W}^{2,p'}_{1}(\mathbb{R}^3)$, then (II.18) and (II.19) are still valid for $\boldsymbol{\varphi}$ belongs to $\boldsymbol{W}^{2,p'}_{1}(\mathbb{R}^3)$. $\mathcal{P}_{[1-\frac{3}{p'}]} \subset \boldsymbol{W}^{2,p'}_{1}(\mathbb{R}^3)$, we deduce (II.17). Thus it follows from Theorem 3.3 of [2] that there exists $(\boldsymbol{w}, q) \in \boldsymbol{W}^{1,p}_{0}(\mathbb{R}^3)/\mathcal{P}_{[1-3/p]} \times L^p(\mathbb{R}^3)$ such that

$$-\Delta\,\boldsymbol{w} + \nabla\,q = \boldsymbol{f}_1 \quad \text{and} \quad \operatorname{div}\boldsymbol{w} = e_1 \quad \text{in} \quad \mathbb{R}^3.$$

Let $\boldsymbol{v} = \mu\,\tilde{\boldsymbol{u}} - \boldsymbol{w}$ and $\theta = \mu\,\tilde{\pi} - q$, then

$$-\Delta\,\boldsymbol{v} + \nabla\,\theta = 0 \quad \text{and} \quad \operatorname{div}\boldsymbol{v} = 0 \quad \text{in} \quad \mathbb{R}^3,$$

with $\theta \in W^{-1,p}_{-1}(\mathbb{R}^3)$ and $\boldsymbol{v} \in \boldsymbol{W}^{0,p}_{-1}(\mathbb{R}^3)$. As $\Delta\,\theta = 0$ in $\mathbb{R}^3$, then $\theta = 0$ and thus $\boldsymbol{v}$ is harmonic and belongs to $\mathcal{P}_{[1-\frac{3}{p}]}$. Note that $\mu\,\tilde{\boldsymbol{u}} = \boldsymbol{w}$ if $p < 3$ and $\mu\,\tilde{\boldsymbol{u}} = \boldsymbol{w} + \boldsymbol{k} \in \boldsymbol{W}^{1,p}_{0}(\Omega)$ with $\boldsymbol{k} \in \mathbb{R}^3$, if $p \geqslant 3$. Consequently, $(\mu\,\tilde{\boldsymbol{u}}, \mu\,\tilde{\pi})$ belongs to $\boldsymbol{W}^{1,p}_{0}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$. As above, we obtain that $(\lambda\,\boldsymbol{u}, \lambda\,\pi)$ satisfies the following equations in Ω_{R_0+1}:

$$-\Delta(\lambda\,\boldsymbol{u}) + \nabla\,(\lambda\,\pi) = \boldsymbol{f}_2, \quad \operatorname{div}(\lambda\,\boldsymbol{u}) = e_2 \quad \text{and} \quad (\lambda\,\boldsymbol{u})|_{\Gamma} = \boldsymbol{0}, \quad (\lambda\,\boldsymbol{u})|_{\partial B_{R_0+1}} = \boldsymbol{0},$$

with

$$\boldsymbol{f}_2 = -(\Delta\,\lambda)\boldsymbol{u} + (\nabla\,\lambda)\pi - 2\nabla\,\lambda\cdot\nabla\,\boldsymbol{u} \quad \text{and} \quad e_2 = \nabla\,\lambda\cdot\boldsymbol{u}.$$

Note that the right-hand sides $\boldsymbol{f}_2$ and e_2 have indeed their supports in Ω_{R_0+1}, then $\boldsymbol{f}_2$ belongs to $\boldsymbol{W}^{-1,p}(\Omega_{R_0+1})$ and e_2 belongs to $L^p(\Omega_{R_0+1})$. Then the regularity results for the Stokes problem in a bounded domain of class $C^{1,1}$ (cf. [13] Theorem 10) and using the same argument as above, we show that $(\lambda\,\boldsymbol{u}, \lambda\,\pi)$ belongs to $\boldsymbol{W}^{1,p}(\Omega_{R_0+1}) \times L^p(\Omega_{R_0+1})$. Let us extend $(\lambda\,\boldsymbol{u}, \lambda\,\pi)$ by zero outside Ω_{R_0+1}. Owing to the boundary conditions imposed and the compact support, the extended function, that we still denote by $(\lambda\,\boldsymbol{u}, \lambda\,\pi)$ belongs to $\boldsymbol{W}^{1,p}_{0}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$. Then $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}^{1,p}_{0}(\Omega) \times L^p(\Omega)$ and thus $\mathcal{N}^{0,p}_{-1}(\Omega) \subset \mathcal{N}^{1,p}_{0}(\Omega)$. For the other inclusion, because $p \neq 3$, it suffices to use (I.4). $\square$

Remark 2.2

With the same method employed in Lemma 2.8, It is easy to prove that if $p \neq 3/2$,

$$\mathcal{N}^{2,p}_{1}(\Omega) = \mathcal{N}^{1,p}_{0}(\Omega).$$

3 Very weak solutions in $\boldsymbol{L}^p(\Omega) \times \boldsymbol{W}_0^{-1,p}(\Omega)$

In this section, we prove the existence and the uniqueness of very weak solutions to the Stokes problem via an argument of duality. We begin by precising the meaning of very weak variational formulation.

Let:

$$\boldsymbol{f} \in [\boldsymbol{X}^0_{r',p'}(\Omega)]', \; h \in L^r(\Omega) \; \text{ and } \; \boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}(\Gamma), \tag{II.20}$$

with

$$\frac{3}{2} < p < \infty, \qquad \text{and} \qquad \frac{1}{p} + \frac{1}{3} = \frac{1}{r}, \tag{A$_1$}$$

yielding $1 < r < 3$.

Definition 3.1 ***(Very weak solution for the Stokes problem)*** *Suppose that* (A_1) *is satisfied and let* $\boldsymbol{f}$*,* h *and* $\boldsymbol{g}$ *verifying* (II.20)*. We say that* $(\boldsymbol{u}, q) \in \boldsymbol{L}^p(\Omega) \times W_0^{-1,p}(\Omega)$ *is a very weak solution of* (S) *if the following equalities hold: For any* $\boldsymbol{\varphi} \in \mathbf{Y}_{p',0}(\Omega)$ *and* $\pi \in W_0^{1,p'}(\Omega)$*,*

$$-\int_\Omega \boldsymbol{u} \cdot \Delta \boldsymbol{\varphi} d\boldsymbol{x} - \langle q, \operatorname{div} \boldsymbol{\varphi} \rangle_{W_0^{-1,p}(\Omega) \times \mathring{W}_0^{1,p'}(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \right\rangle_\Gamma \tag{II.21}$$

$$\int_\Omega \boldsymbol{u} \cdot \nabla \pi d\boldsymbol{x} = -\int_\Omega h \pi d\boldsymbol{x} + \langle \boldsymbol{g} \cdot \boldsymbol{n}, \pi \rangle_{W^{-1/p,p}(\Gamma) \times W^{1/p,p'}(\Gamma)} \tag{II.22}$$

where the dualities on Ω *and* Γ *are defined by:*

$$\langle ., . \rangle_\Omega = \langle ., . \rangle_{[\boldsymbol{X}^0_{r',p'}(\Omega)]' \times \boldsymbol{X}^0_{r',p'}(\Omega)}, \; \langle ., . \rangle_\Gamma = \langle ., . \rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma) \times \boldsymbol{W}^{1/p,p'}(\Gamma)}.$$

Under (A_1), we have:

$$W_0^{1,p'}(\Omega) \hookrightarrow L^{r'}(\Omega) \quad \text{and} \quad \boldsymbol{Y}_{p',0}(\Omega) \hookrightarrow \boldsymbol{X}^0_{r',p'}(\Omega),$$

which means that all the brackets and integrals have a sense.

Proposition 3.1 *Under the assumptions of Definition 3.1, the following two statements are equivalent:*

i) $(\boldsymbol{u}, q) \in \boldsymbol{L}^p(\Omega) \times W_0^{-1,p}(\Omega)$ *is a very weak solution of* (S)*.*

ii) $(\boldsymbol{u}, q) \in \boldsymbol{L}^p(\Omega) \times W_0^{-1,p}(\Omega)$ *satisfies the system* (S) *in the sense of distributions.*

Proof. i)⇒ii) Let $(\boldsymbol{u}, q) \in \boldsymbol{L}^p(\Omega) \times W_0^{-1,p}(\Omega)$ a very weak solution of (S), then if we take $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega)$ and $\pi \in \mathcal{D}(\Omega)$ we can deduce by (II.21)-(II.22) that

$$-\Delta \boldsymbol{u} + \nabla \boldsymbol{q} = \boldsymbol{f} \qquad \text{and} \quad \operatorname{div} \boldsymbol{u} = h \; \text{ in } \; \Omega,$$

and that $\boldsymbol{u} \in \boldsymbol{T}^0_{r,p}(\Omega)$. Now let $\boldsymbol{\varphi} \in \boldsymbol{Y}_{p',0}(\Omega) \subset \boldsymbol{X}^0_{r',p'}(\Omega)$, then we have

$$\langle -\Delta \boldsymbol{u}, \boldsymbol{\varphi} \rangle_\Omega = \langle -\nabla \boldsymbol{q} + \boldsymbol{f}, \boldsymbol{\varphi} \rangle_\Omega .$$

As (A_1) is satisfied, it follows from Corollary 2.2 that

$$\langle -\Delta \boldsymbol{u}, \boldsymbol{\varphi}\rangle_\Omega = -\int_\Omega \boldsymbol{u}\cdot\Delta\boldsymbol{\varphi} d\mathbf{x} + \left\langle \boldsymbol{u}_\tau, \frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\right\rangle_\Gamma$$

and since $\frac{1}{r}-\frac{1}{p}=\frac{1}{3}$, it follows from (II.9) that

$$\langle \nabla q, \boldsymbol{\varphi}\rangle_\Omega = -\langle q, \operatorname{div}\boldsymbol{\varphi}\rangle_{W_0^{-1,p}(\Omega)\times \mathring{W}_0^{1,p'}(\Omega)}.$$

Thus we have

$$-\int_\Omega \boldsymbol{u}\Delta\boldsymbol{\varphi} d\mathbf{x} + \left\langle \boldsymbol{u}_\tau, \frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\right\rangle_\Gamma = \langle q, \operatorname{div}\boldsymbol{\varphi}\rangle_{W_0^{-1,p}(\Omega)\times \mathring{W}_0^{1,p'}(\Omega)} + \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_\Omega,$$

and we can deduce that for any $\boldsymbol{\varphi}\in \boldsymbol{Y}_{p',0}(\Omega)$

$$\left\langle \boldsymbol{u}_\tau, \frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\right\rangle_\Gamma = \left\langle \boldsymbol{g}_\tau, \frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\right\rangle_\Gamma.$$

Now let $\boldsymbol{\mu}\in \boldsymbol{W}^{1/p,p'}(\Gamma)$, then we have $\langle \boldsymbol{u}_\tau - \boldsymbol{g}_\tau, \boldsymbol{\mu}\rangle_\Gamma = \langle \boldsymbol{u}_\tau - \boldsymbol{g}_\tau, \boldsymbol{\mu}_\tau\rangle_\Gamma$. It is clear that $\boldsymbol{\mu}_\tau \in \boldsymbol{Z}_{p'}(\Gamma)$, thus it can follows from Lemma 2.4 that there exists $\boldsymbol{\varphi}\in\boldsymbol{Y}_{p',0}(\Omega)$ such that $\dfrac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}} = \boldsymbol{\mu}_\tau$ on Γ, then from this we can deduce that $\boldsymbol{u}_\tau = \boldsymbol{g}_\tau$ in $\boldsymbol{W}^{-1/p,p}(\Gamma)$. From the equation $\operatorname{div}\boldsymbol{u} = h$, we deduce that $\boldsymbol{u}\in\boldsymbol{H}^r_{p,1}(\operatorname{div},\Omega)$, then it follows from Lemma 2.7 ii), that for any $\pi\in W_0^{1,p'}(\Omega)$,

$$\langle \boldsymbol{u}\cdot\boldsymbol{n}, \pi\rangle_\Gamma = \langle \boldsymbol{g}\cdot\boldsymbol{n}, \pi\rangle_\Gamma.$$

Consequently $\boldsymbol{u}\cdot\boldsymbol{n} = \boldsymbol{g}\cdot\boldsymbol{n}$ in $W^{-1/p,p}(\Gamma)$ and finally $\boldsymbol{u}=\boldsymbol{g}$ on Γ.

ii)⇒i) We suppose that $(\boldsymbol{u}, q)\in \boldsymbol{L}^p(\Omega)\times W_0^{-1,p}(\Omega)$ satisfies the system (S) in the sense of distributions. Then for any $\boldsymbol{\varphi}\in\boldsymbol{Y}_{p',0}(\Omega)\hookrightarrow \boldsymbol{X}^0_{r',p'}(\Omega)$ we have

$$\langle -\Delta\boldsymbol{u}, \boldsymbol{\varphi}\rangle_\Omega = \langle \boldsymbol{f} - \nabla q, \boldsymbol{\varphi}\rangle_\Omega,$$

Using corollary 2.2 and (II.9) we prove (II.21). Now from the equation $\operatorname{div}\boldsymbol{u} = h$, we can deduce that for any $\pi\in W_0^{1,p'}(\Omega)$

$$\int_\Omega \pi \operatorname{div}\boldsymbol{u}\, d\mathbf{x} = \int_\Omega h\pi d\mathbf{x},$$

this integral has a sense because we have $W_0^{1,p'}(\Omega)\hookrightarrow L^{r'}(\Omega)$. Using Lemma 2.7 ii) we deduce (II.22). □

Theorem 3.1 *Let Ω be an exterior domain with $C^{1,1}$ boundary and let p and r satisfy* (A_1) *and let $\boldsymbol{f}$, h and $\boldsymbol{g}$ satisfying* (II.20)*. Then the Stokes problem (S) has exactly one solution $\boldsymbol{u}\in\boldsymbol{L}^p(\Omega)$ and $q\in W_0^{-1,p}(\Omega)$ if and only if for any $(\boldsymbol{v},\eta)\in\mathcal{N}_0^{2,p'}(\Omega)$:*

$$\langle \boldsymbol{f}, \boldsymbol{v}\rangle - \langle h, \eta\rangle + \langle \boldsymbol{g}, (\eta I - \nabla\boldsymbol{v})\cdot\boldsymbol{n}\rangle_\Gamma = 0.$$

Moreover, there exists a constant $C > 0$ depending only on p, r and Ω such that:

$$\| \boldsymbol{u} \|_{\boldsymbol{L}^p(\Omega)} + \| q \|_{W_0^{-1,p}(\Omega)} \leq C(\| \boldsymbol{f} \|_{[\boldsymbol{X}^0_{r',p'}(\Omega)]'} + \| h \|_{L^r(\Omega)} + \| \boldsymbol{g} \|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}). \tag{II.23}$$

Proof. It remains to consider the equivalent problem: Find $(\boldsymbol{u}, q) \in \boldsymbol{L}^p(\Omega) \times W_0^{-1,p}(\Omega)$ such that for any $\boldsymbol{w} \in \boldsymbol{Y}_{p',0}(\Omega)$ and $\pi \in W_0^{1,p'}(\Omega)$ it holds:

$$\int_\Omega \boldsymbol{u} \cdot (-\Delta \boldsymbol{w} + \nabla \pi) d\mathbf{x} - \langle q, \operatorname{div} \boldsymbol{w} \rangle_{W_0^{-1,p}(\Omega) \times \mathring{W}_0^{1,p'}(\Omega)} =$$

$$\langle \boldsymbol{f}, \boldsymbol{w} \rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \right\rangle_\Gamma + \langle \boldsymbol{g} \cdot \boldsymbol{n}, \pi \rangle_{W^{-1/p,p}(\Gamma) \times W^{1/p,p'}(\Gamma)} - \int_\Omega h\, \pi d\mathbf{x}.$$

Let T be a linear form defined by:

$$\begin{aligned} T: \quad & \mathbf{L}^{p'}(\Omega) \times \mathring{W}_0^{1,p'}(\Omega) \longrightarrow \mathbb{R} \\ & (\mathbf{F}, \varphi) \longmapsto \langle \boldsymbol{f}, \boldsymbol{w} \rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \right\rangle_\Gamma + \langle \boldsymbol{g} \cdot \boldsymbol{n}, \pi \rangle_{W^{-1/p,p}(\Gamma) \times W^{1/p,p'}(\Gamma)} - \int_\Omega h\, \pi d\mathbf{x}, \end{aligned}$$

with $(\boldsymbol{w}, \pi) \in \boldsymbol{W}_0^{2,p'}(\Omega) \times W_0^{1,p'}(\Omega)$ is a solution of the following Stokes problem:

$$-\Delta \boldsymbol{w} + \nabla \pi = \mathbf{F} \quad \text{and} \quad \operatorname{div} \boldsymbol{w} = \varphi \quad \text{in } \Omega, \quad \boldsymbol{w} = 0 \quad \text{on } \Gamma,$$

and satisfying the following estimate: (see [3] Theorem 3.1)

$$\inf_{(\mathbf{v},\eta) \in \mathcal{N}_0^{2,p'}(\Omega)} (||\boldsymbol{w} + \boldsymbol{v}||_{\boldsymbol{W}_0^{2,p'}(\Omega)} + ||\pi + \eta||_{W_0^{1,p'}(\Omega)}) \leqslant C(||\mathbf{F}||_{\boldsymbol{L}^{p'}(\Omega)} + ||\varphi||_{W_0^{1,p'}(\Omega)}). \tag{II.24}$$

Then we have for any pair $(\mathbf{F}, \varphi) \in \mathbf{L}^{p'}(\Omega) \times \mathring{W}_0^{1,p'}(\Omega)$ and for any $(\boldsymbol{v}, \eta) \in \mathcal{N}_0^{2,p'}(\Omega)$

$$| \langle \boldsymbol{f}, \boldsymbol{w} \rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \right\rangle_\Gamma + \langle \boldsymbol{g} \cdot \boldsymbol{n}, \pi \rangle_\Gamma - \int_\Omega h\, \pi d\mathbf{x} \,| =$$

$$| \langle \boldsymbol{f}, \boldsymbol{w} + \boldsymbol{v} \rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial (\boldsymbol{w} + \boldsymbol{v})}{\partial \boldsymbol{n}} \right\rangle_\Gamma + \langle \boldsymbol{g} \cdot \boldsymbol{n}, \pi + \eta \rangle_\Gamma - \int_\Omega h\, (\pi + \eta) d\mathbf{x} \,| \leq$$

$$C \left(\| \boldsymbol{f} \|_{[\mathbf{X}^0_{r',p'}(\Omega)]'} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/p,p}(\Omega)} + \| h \|_{L^r(\Omega)} \right) \left(\| \boldsymbol{w} + \boldsymbol{v} \|_{\mathbf{W}_0^{2,p'}(\Omega)} + \| \pi + \eta \|_{W_0^{1,p'}(\Omega)} \right).$$

Using (II.24) , we prove that

$$| \langle \boldsymbol{f}, \boldsymbol{w} \rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \right\rangle_\Gamma + \langle \boldsymbol{g} \cdot \boldsymbol{n}, \pi \rangle_\Gamma - \int_\Omega h\, \pi d\mathbf{x} \,| \leq$$

$$C \left(||\boldsymbol{f}||_{[\mathbf{X}^0_{r',p'}(\Omega)]'} + ||\boldsymbol{g}||_{\mathbf{W}^{-1/p,p}(\Omega)} + ||h||_{L^r(\Omega)} \right) \left(||\mathbf{F}||_{\mathbf{L}^{p'}(\Omega)} + ||\varphi||_{W_0^{1,p'}(\Omega)} \right),$$

from this we can deduce that the linear form T is continuous on $\mathbf{L}^{p'}(\Omega) \times \mathring{W}_0^{1,p'}(\Omega)$ and we deduce that there exists a unique $(\boldsymbol{u}, q) \in \mathbf{L}^p(\Omega) \times W_0^{-1,p}(\Omega)$ solution of (S) satisfying the appropriate estimate. $\square$

4 Very weak solutions in $\boldsymbol{W}^{0,p}_{-1}(\Omega) \times \boldsymbol{W}^{-1,p}_{-1}(\Omega)$

Here, we are interested in the case of the following assumptions:

$$\boldsymbol{f} \in [\boldsymbol{X}^1_{r',p'}(\Omega)]', \; h \in W^{0,r}_{-1}(\Omega) \;\text{ and }\; \boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}(\Gamma), \tag{II.25}$$

with

$$\frac{3}{2} < p < \infty, \quad p \neq 3 \quad \text{and} \qquad \frac{1}{p} + \frac{1}{3} = \frac{1}{r}, \tag{A$_2$}$$

yielding $1 < r < 3$.

Definition 4.1 ***(Very weak solution for the Stokes problem)*** *Suppose that* (A_2) *is satisfied and let* $\boldsymbol{f}$, h *and* $\boldsymbol{g}$ *satisfying* (II.25). *We say that* $(\boldsymbol{u}, q) \in \boldsymbol{W}^{0,p}_{-1}(\Omega) \times W^{-1,p}_{-1}(\Omega)$ *is a very weak solution of* (S) *if the following equalities hold: For any* $\boldsymbol{\varphi} \in \boldsymbol{Y}_{p',1}(\Omega)$ *and* $\pi \in W^{1,p'}_1(\Omega)$,

$$-\int_\Omega \boldsymbol{u} \cdot \Delta \boldsymbol{\varphi} d\boldsymbol{x} - \langle q, \operatorname{div} \boldsymbol{\varphi} \rangle_{W^{-1,p}_{-1}(\Omega) \times \mathring{W}^{1,p'}_1(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{n}} \right\rangle_\Gamma \tag{II.26}$$

$$\int_\Omega \boldsymbol{u} \cdot \nabla \pi d\boldsymbol{x} = -\int_\Omega h \pi d\boldsymbol{x} + \langle \boldsymbol{g} \cdot \boldsymbol{n}, \pi \rangle_{W^{-1/p,p}(\Gamma) \times W^{1/p,p'}(\Gamma)} \tag{II.27}$$

where the dualities on Ω *and* Γ *are defined by:*

$$\langle \cdot, \cdot \rangle_\Omega = \langle \cdot, \cdot \rangle_{[\boldsymbol{X}^1_{r',p'}(\Omega)]' \times \boldsymbol{X}^1_{r',p'}(\Omega)}, \langle \cdot, \cdot \rangle_\Gamma = \langle \cdot, \cdot \rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma) \times \boldsymbol{W}^{1/p,p'}(\Gamma)}.$$

Note that if $\frac{3}{2} < p < \infty$ and $\frac{1}{p} + \frac{1}{3} = \frac{1}{r}$, we have:

$$W^{1,p'}_1(\Omega) \hookrightarrow W^{0,r'}_1(\Omega), \quad \text{and} \quad \boldsymbol{Y}_{p',1}(\Omega) \hookrightarrow \boldsymbol{X}^1_{r',p'}(\Omega),$$

which means that all the brackets and integrals have a sense. As previously we prove under the assumption (A_2), that if $\boldsymbol{f}$, h and $\boldsymbol{g}$ satisfying (II.25), then $(\boldsymbol{u}, q) \in \boldsymbol{W}^{0,p}_{-1}(\Omega) \times W^{-1,p}_{-1}(\Omega)$ is a very weak solution of (S) if and only if $(\boldsymbol{u}, q)$ satisfy the system (S) in the sense of distributions.

Theorem 4.1 *Let* Ω *be an exterior domain with* $C^{1,1}$ *boundary and let* p *and* r *satisfy* (A_2) *and let* $\boldsymbol{f}$, h *and* $\boldsymbol{g}$ *satisfying* (II.25). *Then the Stokes problem* (S) *has a solution* $\boldsymbol{u} \in \boldsymbol{W}^{0,p}_{-1}(\Omega)$ *and* $q \in W^{-1,p}_{-1}(\Omega)$ *if and only if for any* $(\boldsymbol{v}, \eta) \in \mathcal{N}^{2,p'}_1(\Omega)$:

$$\langle \boldsymbol{f}, \boldsymbol{v} \rangle - \langle h, \eta \rangle + \langle \boldsymbol{g}, (\eta I - \nabla \boldsymbol{v}) \cdot \boldsymbol{n} \rangle_\Gamma = 0.$$

In $\boldsymbol{W}^{0,p}_{-1}(\Omega) \times W^{-1,p}_{-1}(\Omega)$, *each solution is unique up to an element of* $\mathcal{N}^{0,p}_{-1}(\Omega)$ *and there exists a constant*

$C > 0$ *depending only on* p, r *and* Ω *such that:*

$$\inf_{(\boldsymbol{v},\eta)\in \mathcal{N}^{1,p}_0(\Omega)} (||\boldsymbol{u}+\boldsymbol{v}||_{\boldsymbol{W}^{0,p}_{-1}(\Omega)} + ||q+\eta||_{W^{-1,p}_{-1}(\Omega)}) \leq C(||\boldsymbol{f}||_{[\boldsymbol{X}^1_{r',p'}(\Omega)]'} + ||h||_{W^{0,r}_{-1}(\Omega)} + ||\boldsymbol{g}||_{\boldsymbol{W}^{-1/p,p}(\Gamma)}).$$

Proof. It remains to consider the equivalent problem: Find $(\boldsymbol{u}, q) \in \boldsymbol{W}^{0,p}_{-1}(\Omega) \times W^{-1,p}_{-1}(\Omega)$ such that for any $\boldsymbol{w} \in \boldsymbol{Y}_{p',0}(\Omega)$ and $\pi \in W^{1,p'}_1(\Omega)$ the following equality holds:

$$\int_\Omega \boldsymbol{u}\cdot(-\Delta \boldsymbol{w} + \nabla\pi)d\mathbf{x} - \langle q, \operatorname{div}\boldsymbol{w}\rangle_{W^{-1,p}_{-1}(\Omega)\times \mathring{W}^{1,p'}_1(\Omega)} =$$

$$\langle \boldsymbol{f}, \boldsymbol{w}\rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}}\right\rangle_\Gamma + \langle \boldsymbol{g}\cdot\boldsymbol{n}, \pi\rangle_\Gamma - \int_\Omega h\,\pi d\mathbf{x}.$$

Let T be a linear form defined from $\left(\mathbf{W}^{0,p'}_1(\Omega) \times \mathring{W}^{1,p'}_1(\Omega)) \perp \mathcal{N}^{1,p}_0(\Omega)\right)$ onto $\mathbb{R}$ by:

$$T(\mathbf{F},\varphi) = \langle \boldsymbol{f}, \boldsymbol{w}\rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}}\right\rangle_\Gamma + \langle \boldsymbol{g}\cdot\boldsymbol{n}, \pi\rangle_\Gamma - \int_\Omega h\,\pi d\mathbf{x},$$

with $(\boldsymbol{w},\pi) \in \boldsymbol{W}^{2,p'}_1(\Omega) \times W^{1,p'}_1(\Omega)$ is a solution of the following Stokes problem:

$$-\Delta \boldsymbol{w} + \nabla\,\pi = \mathbf{F} \quad \text{and} \quad \operatorname{div}\boldsymbol{w} = \varphi \quad \text{in } \Omega, \quad \boldsymbol{w} = 0 \quad \text{on } \Gamma,$$

and satisfying the following estimate: (see [3] Theorem 3.1)

$$\inf_{(\boldsymbol{v},\eta)\in \mathcal{N}^{2,p'}_1(\Omega)} (||\boldsymbol{w}+\boldsymbol{v}||_{\boldsymbol{W}^{2,p'}_1(\Omega)} + ||\pi+\eta||_{W^{1,p'}_1(\Omega)}) \leqslant C(||\mathbf{F}||_{\boldsymbol{W}^{0,p'}_1(\Omega)} + ||\varphi||_{W^{1,p'}_1(\Omega)}). \tag{II.28}$$

Then for any pair $(\mathbf{F},\varphi) \in (\mathbf{W}^{0,p'}_1(\Omega) \times \mathring{W}^{1,p'}_1(\Omega))\perp \mathcal{N}^{1,p}_0(\Omega)$ and for any $(\boldsymbol{v},\eta) \in \mathcal{N}^{2,p'}_1(\Omega)$

$$|\,\langle \boldsymbol{f}, \boldsymbol{w}\rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}}\right\rangle_\Gamma + \langle \boldsymbol{g}\cdot\boldsymbol{n}, \pi\rangle_\Gamma - \int_\Omega h\,\pi d\mathbf{x}\,| =$$

$$|\,\langle \boldsymbol{f}, \boldsymbol{w}+\boldsymbol{v}\rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial(\boldsymbol{w}+\boldsymbol{v})}{\partial \boldsymbol{n}}\right\rangle_\Gamma + \langle \boldsymbol{g}\cdot\boldsymbol{n}, \pi+\eta\rangle_\Gamma - \int_\Omega h\,(\pi+\eta) d\mathbf{x}\,|$$

$$\leq C\left(\|\,\boldsymbol{f}\,\|_{[\mathbf{X}^1_{r',p'}(\Omega]'} + \|\,\boldsymbol{g}\,\|_{\mathbf{W}^{-1/p,p}(\Omega)} + \|\,h\,\|_{W^{0,r}_{-1}(\Omega)}\right)\left(\|\,\boldsymbol{w}+\boldsymbol{v}\,\|_{\mathbf{W}^{2,p'}_1(\Omega)} + \|\,\pi+\eta\,\|_{W^{1,p'}_1(\Omega)}\right).$$

Using (II.28), we prove that

$$|\,\langle \boldsymbol{f}, \boldsymbol{w}\rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}}\right\rangle_\Gamma + \langle \boldsymbol{g}\cdot\boldsymbol{n}, \pi\rangle_\Gamma - \int_\Omega h\,\pi d\mathbf{x}|$$

$$\leq C\left(\|\,\boldsymbol{f}\,\|_{[\mathbf{X}_{p'}(\Omega]'} + \|\,\boldsymbol{g}\,\|_{\mathbf{W}^{-1/p,p}(\Omega)} + \|\,h\,\|_{W^{0,r}_{-1}(\Omega)}\right)\left(\|\,\mathbf{F}\,\|_{\boldsymbol{W}^{0,p'}_1(\Omega)} + \|\,\varphi\,\|_{W^{1,p'}_1(\Omega)}\right).$$

From this we can deduce that the linear form T is continuous on the following space $\mathbf{W}^{0,p'}_1(\Omega)\times \mathring{W}^{1,p'}_1(\Omega) \perp \mathcal{N}^{1,p}_0(\Omega)$ and we deduce that there exists $(\boldsymbol{u}, q) \in (\mathbf{W}^{0,p}_{-1}(\Omega)\times W^{-1,p}_{-1}(\Omega))$ solution of (S), unique up to an element of $\mathcal{N}_0^{1,p}(\Omega)$, satisfying the appropriate estimate. $\square$

Remark 4.1

*i)*Theorem 4.1 can be extended to other behaviors at infinity of the very weak solutions *i.e*, $(\boldsymbol{u}, q) \in \boldsymbol{W}_\ell^{0,p}(\Omega) \times W_\ell^{-1,p}(\Omega)$ for $\ell \notin \{-1, 0\}$. Indeed, on the first hand, all the preliminary results established in section 3 except Lemma 3.8 and Lemma 3.11 are still valid for any integer values ℓ. On the other hand, we use Theorem 3.1 of [3] where the existence of the velocity and the pressure is proved in the space $\boldsymbol{W}_{\ell+1}^{2,p}(\Omega) \times W_{\ell+1}^{1,p}(\Omega)$ for any integer ℓ satisfying

$$3/p + \ell \notin \mathbb{Z}^- \quad \text{and} \quad 3/p' - \ell \notin \mathbb{Z}^-.$$

But, we should take into account some conditions for the integer values ℓ. In Lemma 3.8, when we have $\Delta \widetilde{\boldsymbol{g}} \in \boldsymbol{W}_\ell^{0,p'}(\mathbb{R}^3)$, it is not always true that there exists $\boldsymbol{\lambda} \in \boldsymbol{W}_\ell^{2,p'}(\mathbb{R}^3)$ such that $\Delta\, \boldsymbol{\lambda} = \Delta \widetilde{\boldsymbol{g}}$. It depends on whether or not the isomorphisms for the laplace operator in weighted Sobolev spaces holds, for more details see [6]. In Lemma 3.11, we should respect the condition that $3/p + \ell \notin \{1, 2\}$ (see (I.4)). Finally, it is possible to study the case of real values ℓ, but it is more difficult.
ii) The uniqueness of very weak solution $(\boldsymbol{u}, q) \in \boldsymbol{W}_\ell^{0,p}(\Omega) \times W_\ell^{-1,p}(\Omega)$ of the Stokes problem depends on the characterization of the kernel $\mathcal{N}_\ell^{0,p}(\Omega)$ (see section 3 for the definition). It follows from Theorem 2.7 of [3] that:

$$\mathcal{N}_\ell^{0,p}(\Omega) = \Big\{(\boldsymbol{v}(\boldsymbol{\lambda}) - \boldsymbol{\lambda}, \eta(\boldsymbol{\lambda}) - \mu), \;\; (\boldsymbol{\lambda}, \mu) \in N_{[-\ell-3/p]}\Big\},$$

where $(\boldsymbol{v}(\boldsymbol{\lambda}), \eta(\boldsymbol{\lambda})$ denotes the unique solution in $\boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ of the equations

$$-\Delta\, \boldsymbol{v} + \nabla\, \eta = \boldsymbol{0} \quad \text{and} \quad \operatorname{div} \boldsymbol{v} = 0 \quad \text{in} \quad \Omega, \qquad \boldsymbol{v} = \boldsymbol{\lambda} \quad \text{on} \quad \Gamma$$

and for any integer k

$$N_k = \{(\boldsymbol{\lambda}, \mu) \in \boldsymbol{\mathcal{P}}_k \times \mathcal{P}_{k-1};\; \operatorname{div} \boldsymbol{\lambda} = 0,\; -\Delta\, \boldsymbol{\lambda} + \nabla\, \eta = \boldsymbol{0}\}\,.$$

In particular, recall that $N_k = \{(\boldsymbol{0}, 0)\}$ whenever $k < 0$ and that $N_0 = \boldsymbol{\mathcal{P}}_0 \times \{0\}$. Finally, the existence of very weak solution is linked to compatibility condition which we can obtained directly via Green formula.

Chapter III

Uniqueness and Regularity for the Exterior Oseen Problem

Nous montrons ici l'existence et l'unicité de solutions généralisées et de solutions fortes du problème d'Oseen, dans un premier temps, lorsque le problème est posé dans $\mathbb{R}^3$ et dans un deuxième temps lorsqu'il est posé dans un domaine extérieur puis passer au cas de solutions très faible.

1 Introduction

Let Ω' be a bounded connected open set in $\mathbb{R}^3$ with boundary $\partial\Omega' = \Gamma$ of class $C^{1,1}$ representing an obstacle and let Ω be the exterior region occupied by the fluid, *i.e.* $\Omega = \mathbb{R}^3 \diagdown \overline{\Omega'}$. We consider here the Oseen equations in Ω obtained formally by linearising of the Navier-Stokes equations: For a given vector field $\boldsymbol{f}$, a function h and a boundary value $\boldsymbol{g}$ we are looking for a velocity field $\boldsymbol{u}$ of the fluid and a pressure π which fulfil:

$$-\Delta\, \boldsymbol{u} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u}) + \nabla\, \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = h \quad \text{in} \quad \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \quad \text{on} \quad \Gamma, \tag{III.1}$$

where, $\boldsymbol{v}$ is a given velocity field belonging to $\boldsymbol{L}^3(\Omega)$ with divergence free. In fact, the Oseen approximation is typical for a flow occurring in an exterior region because it describs the physical properties of a system constituted by an object moving with a small, constant velocity in a viscous liquid, at least at large distances from the object where the viscous effects become less important. But, in bounded region, the Oseen approximation loses its physical meaning, while, from the mathematical point of view, it presents no difficulties and can be handled as a corollary to the theory developed for the Stokes system. It should be observed, however, that the Oseen problem has different structures, one of them is given by the following equations:

$$-\Delta\, \boldsymbol{u} + k\frac{\partial \boldsymbol{u}}{\partial x_1} + \nabla\, \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = h \quad \text{in} \quad \Omega, \quad \boldsymbol{u} = \boldsymbol{g} \quad \text{on} \quad \Gamma, \tag{III.2}$$

with $k > 0$. Problem (III.2) has been studied by many authors, from different points of view and it would be too long to list them all here so we give some examples. One of the first complete work on (III.2) is due to Faxén [30] who generalized the method introduced by Odqvist [47] for the Stokes problem. More recently, Finn [31] used Galerkin's method to establish existence of solutions of (III.2) including weighted estimates. For the $\Omega = \mathbb{R}^3$, Babenko [15] used the Lizorkin's Multiplier Theorem in the investigation of (III.2). The results proved by Finn and Babenko were generalized and improved by Galdi in [33] and very expanded and detail version see Chapter VII of his book [34]. Galdi approach is based in the functional framework, homogeneous Sobolev spaces,which is one of possible tools, how to describe the behaviour of solution in the large distance. In [22], Farwig used anisotropic weighted L^2 spaces for the investigation of the exterior problem. Spaces with the weight function η_0^α are also used in [23] and [28], but the weighted estimates are only obtained for the derivatives of second order of functions. We also mention [43] or [48] where the convolution with the fundamental solution of the Oseen problem is studied in L^p space with the anisotropic weight function η_β^α. Also see work of Kračmar, Penel for generalized Oseen problem [44]. Recently, problem (III.2), has been studied by Amrouche and Razafison [12] and more recently by Amrouche and Nguyen [11]. Note that, in [12] and in [11], problem (III.2) was setted in weighted Sobolev spaces in order to provide an explicit description of the behavior of the functions and all its derivatives at infinity.

When Ω is a bounded domain, the existence, uniqueness and regularity properties of the solutions for the Oseen problem (III.1) and (III.2) are well known in the classical Sobolev spaces $\boldsymbol{W}^{m,p}(\Omega)$, see [13] for example for the problem (III.1). It is well known that it is not possible to extend this result to the case of unbounded domains, for example the whole space $\mathbb{R}^3$ or the exterior domain, here the classical Sobolev spaces $\boldsymbol{W}^{m,p}(\Omega)$ are not adequate. Therefore, a specific functional framework is necessary which also has to take into account the behaviour of the functions at infinity. Our approach is similar to that [12] and [11], which is the use of the weighted Sobolev spaces $\boldsymbol{W}^{m,p}_\alpha(\Omega)$ introduced by Hanouzet [42], Cantor [21], Kudrjavcev [45] (see Section 2 for the notations and details).

In the last years, problem (III.1) has been studied when $\Omega = \mathbb{R}^3$ *i.e* without boundary condition, the idea is to suppose in addition that the norm of $\boldsymbol{v}$ in $\boldsymbol{L}^3(\mathbb{R}^3)$ is controled by a positive constant:

$$||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)} < k, \tag{III.3}$$

for more details, we can see [4]. Observe that this condition of smallness is very strong. The basic idea of our method consists, on the first hand, to improve the work done by Amrouche and Consiglieri '[4] by dropping the condition (III.3) and on the second hand to extend this work to the exterior domain Ω. Moreover we are interested also in the very weak solutions. The concept of very weak solutions for Stokes or Navier–Stokes equations was introduced by Giga in 1981, see [37], by Amrouche and Girault in 1994 in a domain class $C^{1,1}$, see [5]. More recently this concept was extended by Amrouche, Rodríguez - Bellido, see [13], Galdi, Simader, Farwig, Kozono and Sohr, see [24], [25], [26], [27], [29] to a setting in classical L^q-spaces.

This chapter is organized as follows. In Section 2, we recall the definition of some spaces and some density results. In section 3, we start our study of the Oseen problem in the whole space $\mathbb{R}^3$ by the existence of generalized solutions which is the pivot of this work and in the same section we prove the uniqueness of this solutions and finally we prove the existence and the uniqueness of strong solutions and some regularity results. The main results of this chapter are presented in section 4, in which we study the existence and the uniqueness of weak, strong and very weak solutions of the Oseen problem in exterior domain Ω.

2 Preliminary results

We begin this sequel with the introducing of the following spaces:

$$\boldsymbol{\mathcal{D}}_\sigma(\Omega) = \{\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega);\ \operatorname{div} \boldsymbol{v} = 0\}, \qquad \boldsymbol{L}^p_\sigma(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \operatorname{div} \boldsymbol{v} = 0\},$$

and

$$\boldsymbol{V}_p(\Omega) = \{\boldsymbol{v} \in \mathring{\boldsymbol{W}}^{1,p}_0(\Omega);\ \nabla \cdot \boldsymbol{v} = 0\}.$$

Note that all these definitions will also be used with Ω replaced by $\mathbb{R}^3$. We define also the following space

$$\boldsymbol{H}^3_\sigma(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\Omega);\ \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1\rangle_{W^{-1/3,3}(\Gamma)\times W^{1/3,3/2}(\Gamma)} = 0\}.$$

We prove a density result which is proved in [2] when $\Omega = \mathbb{R}^3$ and $p \neq 3/2$.

Lemma 2.1 *Let Ω' be a Lipschitz-continuous set of $\mathbb{R}^3$. Then the space $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ is dense in $\mathbf{V}_p(\Omega)$.*

Proof.

First case: If $p \notin \{3/2, 3\}$. The idea follows from Theorem 2.6 of [38]. Let $\boldsymbol{v} \in \boldsymbol{V}_p(\Omega)$ and let us extend $\boldsymbol{v}$ by $\mathbf{0}$ in Ω' such that the extended function, still denoting it by $\boldsymbol{v}$ belongs to $\boldsymbol{V}_p(\mathbb{R}^3)$. It follows from Proposition 2.9 of [2] that there exists $\boldsymbol{\psi} \in \boldsymbol{W}^{2,p}_0(\mathbb{R}^3)$ such that $\boldsymbol{v} = \mathbf{curl}\,\boldsymbol{\psi}$. The proof consists in approximating $\boldsymbol{\psi}$ by a sequence of functions with bounded support. To this end, we introduce the following truncation function: $\varphi \in \mathcal{D}(\mathbb{R}^3)$ such that $0 \leqslant \varphi(t) \leqslant 1$ for any $t \in \mathbb{R}^3$ and

$$\varphi(t) = \begin{cases} 1 & \text{if} \quad |\, t \,| \leqslant 1, \\ 0 & \text{if} \quad |\, t \,| \geqslant 2. \end{cases}$$

We begin by applying the cut off functions φ_k, defined on $\mathbb{R}^3$ for any $k \in \mathbb{N}^*$, by $\varphi_k(\mathbf{x}) = \varphi(\frac{x}{k})$. Set $\boldsymbol{\psi}_k = \varphi_k \boldsymbol{\psi}$. We will prove that:

$$\lim_{k\to\infty} \boldsymbol{\psi}_k = \boldsymbol{\psi} \quad \text{in} \quad \boldsymbol{W}^{2,p}_0(\mathbb{R}^3). \tag{III.4}$$

On one hand we have

$$\int_{\mathbb{R}^3} |\frac{\boldsymbol{\psi}_k - \boldsymbol{\psi}}{\rho^2}|^p \, d\mathbf{x} \leqslant \int_{\mathbb{R}^3} |\varphi_k - 1|^p |\frac{\boldsymbol{\psi}}{\rho^2}|^p \, d\mathbf{x} \leqslant 2^p \int_{B^k(\mathbb{R}^3)} |\frac{\boldsymbol{\psi}}{\rho^2}|^p \, d\mathbf{x}.$$

Using the dominated convergence theorem, we deduce that this integral converges to zero when $k \to \infty$. On the other hand, we have

$$\begin{aligned} \int_{\mathbb{R}^3} |\frac{\nabla(\boldsymbol{\psi}_k) - \nabla \boldsymbol{\psi}}{\rho}|^p \, d\mathbf{x} \quad &\leqslant \quad \left(\int_{\mathbb{R}^3} |\nabla \varphi_k|^p |\frac{\boldsymbol{\psi}}{\rho}|^p \, d\mathbf{x} + \int_{\mathbb{R}^3} |\varphi_k - 1|^p |\frac{\nabla \boldsymbol{\psi}}{\rho}|^p \, d\mathbf{x} \right) \\ &\leqslant \quad C \left(\frac{(1+4k^2)^{p/2}}{k^p} \int_{A_k(\mathbb{R}^3)} |\frac{\boldsymbol{\psi}}{\rho^2}|^p \, d\mathbf{x} + \int_{B^k(\mathbb{R}^3)} |\frac{\nabla \boldsymbol{\psi}}{\rho}|^p \, d\mathbf{x} \right), \end{aligned}$$

where C is independent of k. As above, the two last integrals converge to zero when $k \to \infty$. Finally, we have

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla^2 \boldsymbol{\psi}_k - \nabla^2 \boldsymbol{\psi}|^p &\leqslant \int_{\mathbb{R}^3} |\nabla^2 \varphi_k|^p |\boldsymbol{\psi}|^p + \int_{\mathbb{R}^3} 2^p |\nabla \varphi_k|^p \cdot |\nabla \boldsymbol{\psi}|^p \, d\mathbf{x} + \int_{\mathbb{R}^3} |\varphi_k - 1|^p |\nabla^2 \boldsymbol{\psi}|^p \\ &\leqslant C(\frac{(1+4k^2)^p}{k^{2p}} \int_{A_k(\mathbb{R}^3)} |\frac{\boldsymbol{\psi}}{\rho^2}|^p \, d\mathbf{x} + \frac{(1+4k^2)^{p/2}}{k^p} \int_{A_k(\mathbb{R}^3)} |\frac{\nabla \boldsymbol{\psi}}{\rho}|^p \, d\mathbf{x} \\ &+ \int_{B^k(\mathbb{R}^3)} |\nabla^2 \boldsymbol{\psi}|^p \, d\mathbf{x}),\end{aligned}$$

where C is independent of k. As above, this last integrals converge to zero when $k \to \infty$. We consider the divergence free function $\boldsymbol{v}_k = \mathbf{curl}(\boldsymbol{\psi}_k)$, which coincides with $\boldsymbol{v}$ in a neighbourhood of Γ for k large enough. Therefore, $\boldsymbol{v}_k$ belongs to $\boldsymbol{V}_p(\Omega)$, has a bounded support and

$$\lim_{k\to\infty} \boldsymbol{v}_k = \lim_{k\to\infty} \mathbf{curl}(\boldsymbol{\psi}_k) = \mathbf{curl}\,\boldsymbol{\psi} = \boldsymbol{v} \quad \text{in} \ \ \boldsymbol{W}_0^{1,p}(\Omega).$$

In other words, the set of functions of $\boldsymbol{V}_p(\Omega)$ with bounded support is dense in $\boldsymbol{V}_p(\Omega)$. The assertion of the theorem follows from the fact that in a bounded Lipschitz-continuous domain $\mathcal{O}$, the space $\mathcal{D}_\sigma(\mathcal{O})$ is dense in $\boldsymbol{V}_p(\mathcal{O})$ (for any weight, as the weight is not significant in a bounded domain). Indeed, let $\boldsymbol{u} \in \boldsymbol{V}_p(\Omega)$ with a compact support, then it follows from the density result in bounded domain that there exists $\boldsymbol{u}_k \in \mathcal{D}_\sigma(\Omega)$ such that $\operatorname{supp} \boldsymbol{u}$ and $\operatorname{supp} \boldsymbol{u}_k$ are included in a fixed compact and we have

$$\lim_{k\to\infty} ||\boldsymbol{u}_k - \boldsymbol{u}||_{\boldsymbol{W}^{1,p}(\Omega)} = 0.$$

Using the support's property, we obtain that

$$\lim_{k\to\infty} ||\boldsymbol{u}_k - \boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\Omega)} = 0.$$

Second case: When $p = 3$ or $p = 3/2$. The proof is very similar to the case $i)$. Here, we shall use the truncation function that has been introduced by Bolley-Camus [17], that after we shall use Lemma 7.1 of [6]. First, let $\phi \in C^\infty([0, \infty[)$ be such that

$$\phi(t) = 0, \quad \forall t \in [0,1], \quad 0 \leq \phi(t) \leq 1, \quad \forall t \in [1,2], \quad \phi(t) = 1, \quad \forall t \geq 2.$$

Next, for $k \in \mathbb{N}$ we define ϕ_k by

$$\phi_k(\mathbf{x}) = \begin{cases} \phi(\frac{k}{ln|x|}), & \forall x \in \mathbb{R}^3 : |x| > 1, \\ 1, & \text{otherwise.} \end{cases}$$

The rest of the proof is same.

□

The second density result is given by the following lemma:

Lemma 2.2 *The space* $\mathcal{D}_\sigma(\overline{\Omega})$ *is dense in* $\mathbf{L}^p_\sigma(\Omega)$.

Proof. It follows from the work of Miyakawa see [46], that when $\Omega = \mathbb{R}^3$ we have $\boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3)$ is dense in $\boldsymbol{L}^p_\sigma(\mathbb{R}^3)$. In the case when Ω is an exterior domain, we adapt the same proof done in [13] when Ω is a bounded set. Let $\boldsymbol{\ell}$ be a linear and continuous form in $\boldsymbol{L}^p_\sigma(\Omega)$ such that $\langle \boldsymbol{\ell}, \boldsymbol{v}\rangle = 0$ for any $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$. We want to prove that $\boldsymbol{\ell} = \mathbf{0}$. Since $\boldsymbol{L}^p_\sigma(\Omega)$ is subspace of $\boldsymbol{L}^p(\Omega)$, we can extend $\boldsymbol{\ell}$ to $\boldsymbol{\chi} \in \boldsymbol{L}^{p'}(\Omega)$, then we have $\boldsymbol{\chi}$ vanishes on $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$, thus on $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$. By De Rham's Lemma, there exists $\pi \in \mathcal{D}'(\Omega)$ unique up to an additive constant such that $\boldsymbol{\chi} = \nabla\pi$. We extend $\boldsymbol{\chi}$ by zero out of Ω and we denote the extension by $\widetilde{\boldsymbol{\chi}}$, it is clear that $\widetilde{\boldsymbol{\chi}} \in \boldsymbol{L}^{p'}(\mathbb{R}^3)$. Then for any $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3)$ such that $\operatorname{div}\boldsymbol{\varphi} = 0$ in $\mathbb{R}^3$ we have:

$$\int_{\mathbb{R}^3} \widetilde{\boldsymbol{\chi}} \cdot \boldsymbol{\varphi}\, d\mathbf{x} = \int_\Omega \nabla\pi \cdot \boldsymbol{\varphi}\, d\mathbf{x} = 0.$$

From that, we deduce again, thanks to De Rham's Lemma, that there exists $h \in \mathcal{D}'(\mathbb{R}^3)$ such that $\widetilde{\boldsymbol{\chi}} = \nabla h$. In fact we have $h \in \boldsymbol{W}^{1,p'}_{loc}(\mathbb{R}^3)$ (see [5] proposition 2.10).

First case : if $p' < 3$, we deduce from Proposition 4.3 of [6] that there exists a unique constant k (depending on h) such that $h + k \in W^{1,p'}_0(\mathbb{R}^3)$, in particular it belongs to $W^{1,p'}_0(\Omega)$. Moreover, since $\widetilde{\boldsymbol{\chi}} = \nabla h$ then $\nabla h = 0$ in Ω'. It implies that there exists a constant C_1 such that $h = C_1$ in Ω' (Ω' is connected). As h is unique to an additive constant, we can choose this constant in such a way that $h = -k$ in Ω'. Furthermore, $\nabla h = \nabla\pi$ in Ω so that $\nabla(h + k) = \nabla\pi$ in Ω and then there exists a constant C_2 such that $h+k = \pi+C_2$ in Ω (Ω is connected), from that we deduce that $\pi+C_2 \in \mathring{W}^{1,p'}_0(\Omega)$.

We know that for every $\varphi \in \mathcal{D}(\Omega)$ and $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ we have

$$\int_\Omega \nabla\varphi \cdot \boldsymbol{u}\, d\mathbf{x} = -\langle \varphi, \operatorname{div}\boldsymbol{u}\rangle_{\mathring{\boldsymbol{W}}^{1,p'}_0(\Omega)\times \boldsymbol{W}^{-1,p}_0(\Omega)}. \tag{III.5}$$

As $\mathcal{D}(\Omega)$ is dense in $\mathring{W}^{1,p'}_0(\Omega)$, this equality is still valid for $\varphi \in \mathring{W}^{1,p'}_0(\Omega)$. In consequence, for every $\boldsymbol{v} \in \boldsymbol{L}^p_\sigma(\Omega)$, we have:

$$\begin{aligned}\langle \boldsymbol{\ell}, \boldsymbol{v}\rangle = \int_\Omega \nabla\pi \cdot \boldsymbol{v}\, d\mathbf{x} &= \int_\Omega \nabla(\pi + C_2)\cdot \boldsymbol{v}\, d\mathbf{x} \\ &= -\langle (\pi + C_2), \operatorname{div}\boldsymbol{v}\rangle_{\mathring{\boldsymbol{W}}^{1,p'}_0(\Omega)\times \boldsymbol{W}^{-1,p}_0(\Omega)} = 0.\end{aligned}$$

Second case : if $p' \geq 3$ there are a slight change, we keep that $h \in \boldsymbol{W}^{1,p'}_{loc}(\Omega)$ and we have from Proposition 4.3 of [6] that $h \in W^{1,p'}_0(\Omega)$ and then we can deduce from the first case that there exists a constant C_3 such that $\pi + C_3 \in \mathring{W}^{1,p'}_0(\Omega)$.

Thus we conclude from the two cases that $\boldsymbol{\ell} = \mathbf{0}$ in $[\boldsymbol{L}^p_\sigma(\Omega)]'$. $\square$

3 Oseen problem in $\mathbb{R}^3$

We consider here the Oseen problem in the whole space:

$$-\Delta\,\boldsymbol{u} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{u}) + \nabla\,\pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = h \quad \text{in} \quad \mathbb{R}^3. \tag{III.6}$$

3.1 Generalized solutions in $\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)$

We are interested in the existence and the uniqueness of generalized solutions $(\boldsymbol{u},\pi)\in\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)\times L^p(\mathbb{R}^3)$, with $1<p<\infty$, to the Problem (III.6). We will consider the following data:

$$\boldsymbol{f}\in\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3), \quad \boldsymbol{v}\in\boldsymbol{L}^3_\sigma(\mathbb{R}^3) \quad \text{and} \quad h\in L^p(\mathbb{R}^3).$$

On the one hand if $\boldsymbol{u}\in\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)$, then we have $\boldsymbol{u}\in\boldsymbol{L}^{3/2}_{\text{loc}}(\mathbb{R}^3)$ and thus $\boldsymbol{v}\otimes\boldsymbol{u}$ belongs to $\boldsymbol{L}^1_{\text{loc}}(\mathbb{R}^3)$. It means that $\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{u})$ is well defined as a distribution in $\mathbb{R}^3$. On the other hand, if $p\geq 3/2$, we deduce that the term $\boldsymbol{v}\cdot\nabla\,\boldsymbol{u}$ is well defined and we can write $\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{u}) = \boldsymbol{v}\cdot\nabla\,\boldsymbol{u}$. Moreover, if $(\boldsymbol{u},\pi)\in\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)\times L^p(\mathbb{R}^3)$ with $p<3$ is a solution to (III.6), we have for any $\boldsymbol{\varphi}\in\boldsymbol{\mathcal{D}}(\mathbb{R}^3)$:

$$\int_{\mathbb{R}^3}((\nabla\boldsymbol{u}+\boldsymbol{v}\otimes\boldsymbol{u})\,:\,\nabla\boldsymbol{\varphi} - \pi\operatorname{div}\boldsymbol{\varphi}) = \langle\boldsymbol{f},\,\boldsymbol{\varphi}\rangle_{W_0^{-1,p}(\mathbb{R}^3)\times W_0^{1,p'}(\mathbb{R}^3)}\,. \tag{III.7}$$

Observe that in this case, $\boldsymbol{u}\in\boldsymbol{L}^{p*}(\mathbb{R}^3)$ with $\frac{1}{p*}=\frac{1}{p}-\frac{1}{3}$, so $\boldsymbol{v}\otimes\boldsymbol{u}\in\boldsymbol{L}^p(\mathbb{R}^3)$. Because $\boldsymbol{\mathcal{D}}(\mathbb{R}^3)$ is dense in $\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)$, this last relation holds for any $\boldsymbol{\varphi}\in\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)$. As this last space contains the constant vectors when $p'\geq 3$, the force $\boldsymbol{f}$ must satisfies the following compatibility condition:

$$\langle f_i, 1\rangle_{W_0^{-1,p}(\mathbb{R}^3)\times W_0^{1,p'}(\mathbb{R}^3)} = 0 \quad \text{for any } i=1,2,3 \quad \text{if} \quad p\leq 3/2. \tag{III.8}$$

If $p\geq 3$, (III.6) is equivalent to the following variational problem:

$$\int_{\mathbb{R}^3}(\nabla\boldsymbol{u}:\,\nabla\boldsymbol{\varphi} - \pi\operatorname{div}\boldsymbol{\varphi} + \boldsymbol{v}\cdot\nabla\,\boldsymbol{u}\cdot\boldsymbol{\varphi}) = \langle\boldsymbol{f},\,\boldsymbol{\varphi}\rangle_{W_0^{-1,p}(\mathbb{R}^3)\times W_0^{1,p'}(\mathbb{R}^3)}\,. \tag{III.9}$$

Remark 3.1

To simplify the study of problem (III.6), we can suppose at first that $h=0$. Indeed, if h in $L^p(\mathbb{R}^3)$, there exists $\chi\in W_0^{2,p}(\mathbb{R}^3)$ such that $\Delta\,\chi = h$ (see [2]) and satisfying

$$||\nabla\,\chi||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} \leqslant C||h||_{L^p(\mathbb{R}^3)}. \tag{III.10}$$

Set $\boldsymbol{w}_h = \nabla\,\chi\in\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)$ and $\boldsymbol{z}=\boldsymbol{u}-\boldsymbol{w}_h$. Then problem (III.6) becomes:

$$-\Delta\,\boldsymbol{z} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{z}) + \nabla\,\pi = \boldsymbol{f} + \Delta\,\boldsymbol{w}_h - \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{w}_h) \quad \text{and} \quad \operatorname{div}\boldsymbol{z} = 0 \quad \text{in} \quad \mathbb{R}^3.$$

If $1<p<3$, we have $\boldsymbol{w}_h\in\boldsymbol{L}^{p*}(\mathbb{R}^3)$ and $\boldsymbol{v}\otimes\boldsymbol{w}_h$ belongs to $\boldsymbol{L}^p(\mathbb{R}^3)$. Consequently $\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{w}_h)$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$. However when $p\geq 3$, $\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{w}_h) = \boldsymbol{v}\cdot\nabla\,\boldsymbol{w}_h$ belongs to $\boldsymbol{L}^r(\mathbb{R}^3)$, with $\frac{1}{r}=\frac{1}{3}+\frac{1}{p}$ and $\boldsymbol{L}^r(\mathbb{R}^3)\hookrightarrow\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$. This means that $\mathbf{F} := \boldsymbol{f}+\Delta\,\boldsymbol{w}_h - \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{w}_h)$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$. In addition, we have for any $i = 1, 2, 3$ and $p \leqslant \frac{3}{2}$ the equivalence

$$\langle f_i, 1\rangle_{W_0^{-1,p}(\mathbb{R}^3)\times W_0^{1,p'}(\mathbb{R}^3)} = 0 \Longleftrightarrow \langle F_i, 1\rangle_{W_0^{-1,p}(\mathbb{R}^3)\times W_0^{1,p'}(\mathbb{R}^3)} = 0. \tag{III.11}$$

This means that to solve (III.6), it is sufficient to solve the following problem:

$$-\Delta\, \boldsymbol{u} + \operatorname{div}(\boldsymbol{v}\otimes \boldsymbol{u}) + \nabla\, \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{III.12}$$

In the following theorem, we establishes the existence of generalized solutions to Problem (III.6) in the case $1 < p \le 2$. The uniqueness of the solutions will be studied later.

Theorem 3.1 *Let $1 < p \leqslant 2$. Assume that $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$ satisfies the compatibility condition* (III.8) *and let $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Then the Oseen problem* (III.12) *has a solution*
$(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ such that

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\pi||_{L^p(\mathbb{R}^3)} \leqslant C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}. \tag{III.13}$$

Proof. First, the case $p = 2$ is an immediate consequence of the following property

$$\forall \boldsymbol{w} \in \boldsymbol{W}_0^{1,2}(\mathbb{R}^3), \qquad \int_{\mathbb{R}^3} (\boldsymbol{v}\cdot\nabla)\, \boldsymbol{w}\cdot \boldsymbol{w} = 0$$

and Lax-Milgram Lemma. So we can suppose that $1 < p < 2$.

The main idea of the proof is to observe that $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$ can be approximated by a smooth function $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3)$. Given ε, there is $\boldsymbol{\psi}_\epsilon \in \boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3)$ such that

$$||\boldsymbol{v} - \boldsymbol{\psi}_\varepsilon||_{\boldsymbol{L}^3(\mathbb{R}^3)} < \varepsilon, \tag{III.14}$$

where $\varepsilon > 0$ is a constant which will be fixed as below. By (III.8) and [6], we have $\boldsymbol{f} = \operatorname{div} \mathbf{F}$ with $\mathbf{F} \in \boldsymbol{L}^p(\mathbb{R}^3)$. Let $\rho \in \mathcal{D}(\mathbb{R}^3)$, be a smooth $\mathcal{C}^\infty$ function with compact support in $B(0,1)$, such that $\rho \ge 0$, $\int_{\mathbb{R}^3} \rho(\mathbf{x})\, d\mathbf{x} = 1$. For $t \in (0,1)$, let ρ_t denote the function $x \longmapsto (\frac{1}{t^3})\rho(\frac{x}{t})$. Let $\varphi \in \mathcal{D}(\mathbb{R}^3)$ such that $0 \leqslant \varphi(\mathbf{x}) \leqslant 1$ for any $x \in \mathbb{R}^3$, and

$$\varphi(\mathbf{x}) = \begin{cases} 1 & \text{if} \quad 0 \leqslant \mid x \mid \leqslant 1, \\ 0 & \text{if} \quad \mid x \mid \geqslant 2. \end{cases}$$

We begin by applying the cut off functions φ_k, defined on $\mathbb{R}^3$ for any $k \in \mathbb{N}^*$, as $\varphi_k(\mathbf{x}) = \varphi(\frac{x}{k})$. Set $\mathbf{F}_k = \varphi_k \mathbf{F}$. Thus we obtain

$$\mathbf{G}_{t,k} = \rho_t * \mathbf{F}_k \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3) \quad \text{and} \quad \lim_{t\to 0}\lim_{k\to\infty} \mathbf{G}_{t,k} = \mathbf{F} \quad \text{in} \quad \boldsymbol{L}^p(\mathbb{R}^3). \tag{III.15}$$

Now, observe that using Young inequality, we have

$$||\rho_t * \mathbf{F}_k||_{\boldsymbol{L}^2(\mathbb{R}^3)} \leqslant ||\rho_t||_{\boldsymbol{L}^q(\mathbb{R}^3)} ||\mathbf{F}_k||_{\boldsymbol{L}^p(\mathbb{R}^3)}, \tag{III.16}$$

with $q = \dfrac{2p}{3p-2}$. Observe that $q > 1$ is equivalent to $p < 2$. After an easy calculation, we obtain that

$$||\rho_t * \mathbf{F}_k||_{\boldsymbol{L}^2(\mathbb{R}^3)} \leqslant \frac{4}{3}\pi t^{\frac{-3}{q'}} ||\mathbf{F}_k||_{\boldsymbol{L}^p(\mathbb{R}^3)}. \tag{III.17}$$

We choose $t = k^{-\alpha}$ with $\alpha > 0$ which will be precise later. Set now $\boldsymbol{f}_k = \operatorname{div} \mathbf{G}_{t,k}$ for any $k \in \mathbb{N}^*$. Then we have

$$\boldsymbol{f}_k \to \boldsymbol{f} \quad \text{in} \quad \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3).$$

It is clear that $\boldsymbol{f}_k$ satisfies the condition (III.8).

Step 1. We suppose that $\boldsymbol{v} \in \mathfrak{D}_\sigma(\mathbb{R}^3)$. Thanks to Lemma 4.1 see [4], there exists a unique solution

$$\boldsymbol{u}_k \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3), \quad \pi_k \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$$

satisfying

$$-\Delta\, \boldsymbol{u}_k + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u}_k) + \nabla\, \pi_k = \boldsymbol{f}_k, \quad \operatorname{div} \boldsymbol{u}_k = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{III.18}$$

Set $B_\varepsilon = \operatorname{supp} \boldsymbol{\psi}_\varepsilon$, then from the Stokes theory (see [2] Theorem 3.3), we obtain

$$||\boldsymbol{u}_k||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\pi_k||_{L^p(\mathbb{R}^3)} \leqslant C_1 \left(||\boldsymbol{f}_k||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + ||\boldsymbol{v} \otimes \boldsymbol{u}_k||_{\boldsymbol{L}^p(\mathbb{R}^3)}\right), \tag{III.19}$$

where C_1 doesn't depend on $k, \boldsymbol{f}_k$ and $\boldsymbol{v}$. Using Hölder inequality, we have

$$\begin{aligned} ||\boldsymbol{v} \otimes \boldsymbol{u}_k||_{\boldsymbol{L}^p(\mathbb{R}^3)} &\leqslant ||(\boldsymbol{v} - \boldsymbol{\psi}_\epsilon) \otimes \boldsymbol{u}_k||_{\boldsymbol{L}^p(\mathbb{R}^3)} + ||\boldsymbol{\psi}_\varepsilon \otimes \boldsymbol{u}_k||_{\boldsymbol{L}^p(\mathbb{R}^3)} \\ &\leqslant ||\boldsymbol{v} - \boldsymbol{\psi}_\varepsilon||_{\boldsymbol{L}^3(\mathbb{R}^3)} ||\boldsymbol{u}_k||_{\boldsymbol{L}^{p*}(\mathbb{R}^3)} + ||\boldsymbol{\psi}_\varepsilon||_{\boldsymbol{L}^3(B_\varepsilon)} ||\boldsymbol{u}_k||_{\boldsymbol{L}^{p*}(B_\varepsilon)}. \end{aligned} \tag{III.20}$$

Using the Sobolev inequality, we obtain

$$||\boldsymbol{u}_k||_{\boldsymbol{L}^{p*}(\mathbb{R}^3)} \leqslant C_2 ||\boldsymbol{u}_k||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)}. \tag{III.21}$$

By the assumption (III.14), and from (III.19), (III.20) and (III.21) it follows that

$$(1 - C_1 C_2 \varepsilon)||\boldsymbol{u}_k||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\pi_k||_{L^p(\mathbb{R}^3)} \leqslant C_1(||\boldsymbol{f}_k||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + ||\boldsymbol{\psi}_\varepsilon||_{\boldsymbol{L}^3(B_\varepsilon)} ||\boldsymbol{u}_k||_{\boldsymbol{L}^{p*}(B_\varepsilon)}). \tag{III.22}$$

Taking $0 < \varepsilon < 1/2C_1C_2$, we obtain

$$||\boldsymbol{u}_k||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\pi_k||_{L^p(\mathbb{R}^3)} \leqslant 2C_1(||\boldsymbol{f}_k||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + ||\boldsymbol{\psi}_\varepsilon||_{\boldsymbol{L}^3(B_\varepsilon)} ||\boldsymbol{u}_k||_{\boldsymbol{L}^{p*}(B_\varepsilon)}). \tag{III.23}$$

From (III.23), we prove that there exists $C > 0$ not depending of k and $\boldsymbol{v}$ such that for any $k \in \mathbb{N}^*$ we have

$$||\boldsymbol{u}_k||_{\boldsymbol{L}^{p*}(B_\varepsilon)} \leqslant C ||\boldsymbol{f}_k||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}. \tag{III.24}$$

Indeed, assuming, per absurdum, the invalidity of (III.24). Then for any $m \in \mathbb{N}^*$ there exists $\ell_m \in \mathbb{N}$, $\boldsymbol{f}_{\ell_m} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,2}(\mathbb{R}^3)$ and $\boldsymbol{v}_m \in \boldsymbol{\mathfrak{D}}_\sigma(\mathbb{R}^3)$ such that, if $(\boldsymbol{u}_{\ell_m}, \pi_{\ell_m}) \in (\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))$ denotes the corresponding solution to the following problem:

$$-\Delta\, \boldsymbol{u}_{\ell_m} + \operatorname{div}(\boldsymbol{v}_m \otimes \boldsymbol{u}_{\ell_m}) + \nabla\, \pi_{\ell_m} = \boldsymbol{f}_{\ell_m}, \quad \operatorname{div} \boldsymbol{u}_{\ell_m} = 0 \quad \text{in} \quad \mathbb{R}^3, \tag{III.25}$$

the inequality

$$||\boldsymbol{u}_{\ell_m}||_{\boldsymbol{L}^{p*}(B_\epsilon)} > m||\boldsymbol{f}_{\ell_m}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}, \tag{III.26}$$

would hold. Note that

$$\boldsymbol{f}_{\ell_m} = \operatorname{div}(\rho_t * \mathbf{F}_{\ell_m}) \quad \text{with} \quad \mathbf{F}_{\ell_m} = \varphi_{\ell_m}\mathbf{F}.$$

Setting

$$\boldsymbol{w}_m = \frac{\boldsymbol{u}_{\ell_m}}{||\boldsymbol{u}_{\ell_m}||_{\boldsymbol{L}^{p*}(B_\varepsilon)}}, \quad \theta_m = \frac{\pi_{\ell_m}}{||\boldsymbol{u}_{\ell_m}||_{\boldsymbol{L}^{p*}(B_\varepsilon)}} \quad \text{and} \quad \mathbf{R}_m = \frac{\boldsymbol{f}_{\ell_m}}{||\boldsymbol{u}_{\ell_m}||_{\boldsymbol{L}^{p*}(B_\varepsilon)}}.$$

Then for any $m \in \mathbb{N}^*$ we have

$$-\Delta\, \boldsymbol{w}_m + \operatorname{div}(\boldsymbol{v}_m \otimes \boldsymbol{w}_m) + \nabla\, \theta_m = \mathbf{R}_m \quad \text{and} \quad \operatorname{div} \boldsymbol{w}_m = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{III.27}$$

Now, using (III.27) and the fact that $\operatorname{div}(\boldsymbol{v}_m \otimes \boldsymbol{w}_m) = \boldsymbol{v}_m \cdot \nabla\, \boldsymbol{w}_m$, we obtain for any $m \in \mathbb{N}^*$ and $t > 0$

$$\int_{\mathbb{R}^3} |\nabla\, \boldsymbol{w}_m|^2\, d\mathbf{x} = -\frac{1}{||\boldsymbol{u}_{\ell_m}||_{\boldsymbol{L}^{p*}(B_\varepsilon)}} \int_{\mathbb{R}^3} \rho_t * \mathbf{F}_{\ell_m} : \nabla\, \boldsymbol{w}_m\, d\mathbf{x}.$$

Using (III.26) and Cauchy Schwartz inequality, we have

$$||\nabla\, \boldsymbol{w}_m||_{\boldsymbol{L}^2(\mathbb{R}^3)} < \frac{1}{m||\boldsymbol{f}_{\ell_m}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}} ||\rho_t * \mathbf{F}_{\ell_m}||_{\boldsymbol{L}^2(\mathbb{R}^3)}. \tag{III.28}$$

Using (III.17) and choosing $t = \dfrac{1}{m^\alpha}$ with $0 < \alpha < \frac{q'}{3}$, we deduce that

$$||\nabla\, \boldsymbol{w}_m||_{\boldsymbol{L}^2(\mathbb{R}^3)} \leqslant \frac{4\pi}{3m^{1-\frac{3\alpha}{q'}}||\boldsymbol{f}_{\ell_m}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}} ||\mathbf{F}_{\ell_m}||_{\boldsymbol{L}^p(\mathbb{R}^3)}. \tag{III.29}$$

Because the semi-norm $||\nabla \cdot||_{\boldsymbol{L}^2(\mathbb{R}^3)}$ is equivalent to the full norm $||\cdot||_{\boldsymbol{W}_0^{1,2}(\mathbb{R}^3)}$ and the right hand side of the last inequality tends to zero when m goes to ∞, we deduce that

$$\boldsymbol{w}_m \to \mathbf{0} \text{ in } \quad \boldsymbol{W}_0^{1,2}(\mathbb{R}^3). \tag{III.30}$$

Then, $\boldsymbol{w}_m \to \mathbf{0}$ in $\boldsymbol{L}^6(\mathbb{R}^3)$ and in particular in $\boldsymbol{L}^{p*}(B_\varepsilon)$. On the other hand, we have $||\boldsymbol{w}_m||_{\boldsymbol{L}^{p*}(B_\varepsilon)} = 1$, leading to a contradiction. Inequality (III.24) is therefore established. From (III.23) , (III.24) and (III.14) we obtain for any $k \in \mathbb{N}^*$

$$||\boldsymbol{u}_k||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\pi_k||_{L^p(\mathbb{R}^3)} \leqslant 2C_1(1 + C||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})||\boldsymbol{f}_k||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}. \tag{III.31}$$

Thus we can extract a subsequences of $\boldsymbol{u}_k$ and π_k, still denoted by $\boldsymbol{u}_k$ and π_k, such that

$$\boldsymbol{u}_k \rightharpoonup \boldsymbol{u} \quad \text{in} \quad \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \quad \text{and} \quad \pi_k \rightharpoonup \pi \quad \text{in} \quad L^p(\mathbb{R}^3),$$

where $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ verifies (III.12) and the following estimate

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\pi||_{L^p(\mathbb{R}^3)} \leqslant 2C_1(1 + C||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}. \tag{III.32}$$

Step 2. We suppose that $\boldsymbol{v}$ belongs only to $\boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Let $\boldsymbol{v}_\lambda \in \mathfrak{D}_\sigma(\mathbb{R}^3)$ such that

$$\boldsymbol{v}_\lambda \longrightarrow \boldsymbol{v} \qquad \text{in } \boldsymbol{L}^3(\mathbb{R}^3). \tag{III.33}$$

Using the first step, there exists $(\boldsymbol{u}_\lambda, \pi_\lambda) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ satisfying

$$-\Delta\, \boldsymbol{u}_\lambda + \operatorname{div}(\boldsymbol{v}_\lambda \otimes \boldsymbol{u}_\lambda) + \nabla\, \pi_\lambda = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u}_\lambda = 0 \quad \text{in} \quad \mathbb{R}^3, \tag{III.34}$$

and satisfying the estimate

$$||\boldsymbol{u}_\lambda||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\pi_\lambda||_{L^p(\mathbb{R}^3)} \leqslant 2C_1(1 + C||\boldsymbol{v}_\lambda||_{\boldsymbol{L}^3(\mathbb{R}^3)})||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}. \tag{III.35}$$

We can finally extract a subsequence which converges to $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ which is a solution of the Oseen problem (III.12) and verifying the estimate (III.13) when $1 < p < 2$. For $p = 2$, estimate (III.13) was proved in Theorem 3.4 of [4]. □

Remark 3.2

1) If h belongs to $L^p(\mathbb{R}^3)$ with $1 < p \leq 2$ *i.e* we are in the case of problem (III.6), the estimate (III.13) becomes:

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\pi||_{L^p(\mathbb{R}^3)} \leqslant C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + (1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})||h||_{L^p(\mathbb{R}^3)}). \tag{III.36}$$

   The proof of (III.36) when $1 < p < 2$ is a simple consequence of Remark 3.1. Note that the proof of (III.36) when $p = 2$ is done in Theorem 3.3 of [4].

2) For $p = 2$ and $h = 0$, the velocity $\boldsymbol{u}$ of the Oseen problem (III.12) satisfies the estimate

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,2}(\mathbb{R}^3)} \leqslant C||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,2}(\mathbb{R}^3)},$$

   and the energy equality

$$\int_{\mathbb{R}^3} |\nabla\, \boldsymbol{u}|^2\, d\mathbf{x} = \langle \boldsymbol{f}, \boldsymbol{u} \rangle_{\boldsymbol{W}_0^{-1,2}(\mathbb{R}^3) \times \boldsymbol{W}_0^{1,2}(\mathbb{R}^3)}.$$

   In addition, the pressure π of the Oseen problem (III.12) satisfies the following estimate:

$$||\pi||_{L^2(\mathbb{R}^3)} \leqslant C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,2}(\mathbb{R}^3)}.$$

See Theorem 3.3 of [4] for more details.

We will prove now some regularity results, when the external forces belong to the intersection of negative weighted Sobolev spaces. The first result is given by the following theorem.

Theorem 3.2 *Let $1 < p < 2$. Let $\boldsymbol{f}$ belonging to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,2}(\mathbb{R}^3)$ satisfying the compatibility condition* (III.8) *and let $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Then the Oseen problem* (III.12) *has a unique solution $(\boldsymbol{u}, \pi) \in (\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))$ such that*

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,2}(\mathbb{R}^3)} + ||\pi||_{L^p(\mathbb{R}^3)} + ||\pi||_{L^2(\mathbb{R}^3)} \leqslant C\big(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)}\big)\big(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,2}(\mathbb{R}^3)} + ||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}\big). \tag{III.37}$$

Proof. Step 1. We suppose that $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3)$. Let $\boldsymbol{f}$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,2}(\mathbb{R}^3)$ satisfying the compatibility condition (III.8). Then $\boldsymbol{f}$ can be written as $\boldsymbol{f} = \mathrm{div}\boldsymbol{F}$ with $\boldsymbol{F} \in \boldsymbol{L}^p(\mathbb{R}^3) \cap \boldsymbol{L}^2(\mathbb{R}^3)$.

Take the same sequence $\boldsymbol{f}_k$, as in the previous theorem, which converges to $\boldsymbol{f}$ in $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,2}(\mathbb{R}^3)$. Proceeding as in the first step of the previous theorem, we deduce that there exists a unique solution

$$\boldsymbol{u}_k \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3), \quad \pi_k \in L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$$

satisfying

$$-\Delta\, \boldsymbol{u}_k + \mathrm{div}(\boldsymbol{v} \otimes \boldsymbol{u}_k) + \nabla\, \pi_k = \boldsymbol{f}_k, \quad \mathrm{div}\, \boldsymbol{u}_k = 0 \quad \text{in} \quad \mathbb{R}^3 \tag{III.38}$$

and with the following estimate

$$||\boldsymbol{u}_k||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\pi_k||_{L^p(\mathbb{R}^3)} \leqslant C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})||\boldsymbol{f}_k||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}, \tag{III.39}$$

where C doesn't depend on k. On the other hand, multiplying by $\boldsymbol{u}_k$, we have also the following estimate

$$||\boldsymbol{u}_k||_{\boldsymbol{W}_0^{1,2}(\mathbb{R}^3)} + ||\pi_k||_{L^2(\mathbb{R}^3)} \leqslant C||\boldsymbol{f}_k||_{\boldsymbol{W}_0^{-1,2}(\mathbb{R}^3)}. \tag{III.40}$$

Finally, $(\boldsymbol{u}_k, \pi_k)$ is bounded in $(\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))$ and we can extract a subsequence denoted again by $(\boldsymbol{u}_k, \pi_k)$ and satisfying

$$\boldsymbol{u}_k \rightharpoonup \boldsymbol{u} \quad \text{in} \quad \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3) \quad \text{and} \quad \pi_k \rightharpoonup \pi \quad \text{in} \quad L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3). \tag{III.41}$$

We then verify that $(\boldsymbol{u}, \pi)$ is solution of (III.12) and we have the estimate (III.37).

Step 2. We suppose that $\boldsymbol{v}$ belongs only to $\boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. The proof is the same as in the previous theorem. Let $\boldsymbol{v}_\lambda \in \boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3)$ such that

$$\boldsymbol{v}_\lambda \longrightarrow \boldsymbol{v} \qquad \text{in } \boldsymbol{L}^3(\mathbb{R}^3). \tag{III.42}$$

Using the first step, there exists $(\boldsymbol{u}_\lambda, \pi_\lambda) \in (\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))$ solution of

$$-\Delta\, \boldsymbol{u}_\lambda + \operatorname{div}(\boldsymbol{v}_\lambda \otimes \boldsymbol{u}_\lambda) + \nabla\, \pi_\lambda = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u}_\lambda = 0 \quad \text{in} \quad \mathbb{R}^3, \tag{III.43}$$

and satisfying the estimate

$$||\boldsymbol{u}_\lambda||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\boldsymbol{u}_\lambda||_{\boldsymbol{W}_0^{1,2}(\mathbb{R}^3)} + ||\pi_\lambda||_{L^p(\mathbb{R}^3)} + ||\pi_\lambda||_{L^2(\mathbb{R}^3)} \leqslant$$

$$C(1 + ||\boldsymbol{v}_\lambda||_{\boldsymbol{L}^3(\mathbb{R}^3)})(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,2}(\mathbb{R}^3)} + ||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}). \tag{III.44}$$

The sequence $(\boldsymbol{u}_\lambda, \pi_\lambda)$ is bounded in $(\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))$, and we can finally extract a subsequence which converges to $(\boldsymbol{u}, \pi) \in (\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))$, which is a solution of the Oseen problem (III.12) and verifies the estimate (III.37). To finish, observe that the uniqueness is immediate because $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. □

In Theorem 3.1, we have studied the existence of weak solution of the Oseen problem when $1 < p \leq 2$. Now the question that will be discussed: if, the solution given by is Theorem 3.1 unique? If it is unique, is it for all $1 < p \leq 2$? The first answer is given in the following proposition:

Proposition 3.1 *Let $6/5 < p < 2$. Let $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$ satisfying the compatibility condition* (III.8) *and $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Then the solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ given by Theorem 3.1 is unique.*

Proof. Suppose that there exist two solutions $(\boldsymbol{u}_1, \pi_1)$ and $(\boldsymbol{u}_2, \pi_2)$ belong to $\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ and verifying Problem (III.12). Set $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and $\pi = \pi_1 - \pi_2$ then we have

$$-\Delta\, \boldsymbol{u} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u}) + \nabla\, \pi = \boldsymbol{0} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{III.45}$$

Our aim is to prove that $(\boldsymbol{u}, \pi) = (\boldsymbol{0}, 0)$. Observe that for any $\varepsilon > 0$, $\boldsymbol{v}$ can be decomposed as: $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$ with

$$\boldsymbol{v}_1 \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3), \quad ||\boldsymbol{v}_1||_{\boldsymbol{L}^3(\mathbb{R}^3)} < \varepsilon \qquad \text{and} \quad \boldsymbol{v}_2 \in \boldsymbol{\mathfrak{D}}_\sigma(\mathbb{R}^3). \tag{III.46}$$

The parameter ε will be fixed at the end of the proof.

Note that $\boldsymbol{v}_2 \in \boldsymbol{L}^1(\mathbb{R}^3) \cap \boldsymbol{L}^\infty(\mathbb{R}^3)$. Now, since $\boldsymbol{u} \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \hookrightarrow \boldsymbol{L}^{p*}(\mathbb{R}^3)$ we prove that $\boldsymbol{v}_2 \otimes \boldsymbol{u}$ belongs to $\boldsymbol{L}^{p*}(\mathbb{R}^3) \cap \boldsymbol{L}^1(\mathbb{R}^3)$. As $6/5 \leq p$, then $p* \geq 2$ and thus $\operatorname{div}(\boldsymbol{v}_2 \otimes \boldsymbol{u}) = \boldsymbol{v}_2 \cdot \nabla\, \boldsymbol{u}$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,2}(\mathbb{R}^3)$ and satisfies the compatibilty condition (III.8). Then it follows from Theorem 3.2 that there exists a unique $\boldsymbol{z} \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3)$ and $\theta \in \boldsymbol{L}^p(\mathbb{R}^3) \cap \boldsymbol{L}^2(\mathbb{R}^3)$ such that

$$-\Delta\, \boldsymbol{z} + \operatorname{div}(\boldsymbol{v}_1 \otimes \boldsymbol{z}) + \nabla\, \theta = -\boldsymbol{v}_2 \cdot \nabla\, \boldsymbol{u} \quad \text{and} \quad \operatorname{div} \boldsymbol{z} = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{III.47}$$

Because of (III.45) and (III.47), the functions $\boldsymbol{w} = \boldsymbol{z} - \boldsymbol{u}$ and $q = \theta - \pi$ satisfy:

$$-\Delta\, \boldsymbol{w} + \operatorname{div}(\boldsymbol{v}_1 \otimes \boldsymbol{w}) + \nabla\, q = \boldsymbol{0} \quad \text{and} \quad \operatorname{div} \boldsymbol{w} = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{III.48}$$

From the Stokes theory see ([2]) and Sobolev imbeddings, we obtain

$$\begin{aligned}||\boldsymbol{w}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} \leq C||\boldsymbol{v}_1 \otimes \boldsymbol{w}||_{\boldsymbol{L}^p(\mathbb{R}^3)} &\leq C||\boldsymbol{v}_1||_{\boldsymbol{L}^3(\mathbb{R}^3)}||\boldsymbol{w}||_{\boldsymbol{L}^{p*}(\mathbb{R}^3)} \\ &\leq CC^*||\boldsymbol{v}_1||_{\boldsymbol{L}^3(\mathbb{R}^3)}||\boldsymbol{w}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} \\ &\leq CC^*\varepsilon||\boldsymbol{w}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)}.\end{aligned}$$

Taking $0 < \varepsilon < 1/(CC^*)$, we conclude that $\boldsymbol{w} = \mathbf{0}$ and so $q = 0$. Thus $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}_0^{1,2}(\mathbb{R}^3) \times \boldsymbol{L}^2(\mathbb{R}^3)$ and we can write that $\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u}) = \boldsymbol{v} \cdot \nabla \boldsymbol{u}$. Using (III.45), we deduce that

$$\langle -\Delta \boldsymbol{u} + \boldsymbol{v} \cdot \nabla \boldsymbol{u} + \nabla \pi, \boldsymbol{u}\rangle_{\boldsymbol{W}_0^{-1,2}(\mathbb{R}^3) \times \boldsymbol{W}_0^{1,2}(\mathbb{R}^3)} = \mathbf{0},$$

and so

$$||\nabla \boldsymbol{u}||_{\boldsymbol{L}^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} \boldsymbol{v} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{u} \, d\mathbf{x} = 0.$$

Since $\int_{\mathbb{R}^3} \boldsymbol{v} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{u} \, d\mathbf{x} = \dfrac{1}{2} \int_{\mathbb{R}^3} \boldsymbol{v} \cdot \nabla |\boldsymbol{u}^2| \, d\mathbf{x} = 0$, we prove that $||\nabla \boldsymbol{u}||_{\boldsymbol{L}^2(\mathbb{R}^3)} = 0$ and thus $\boldsymbol{u} = \mathbf{0}$ and so $\pi = 0$. Finally, we have proved that $(\boldsymbol{u}, \pi) = (\mathbf{0}, 0)$ for any $6/5 \leq p \leq 2$. □

The second regularity result is announced in the following theorem.

Theorem 3.3 *Let $1 < p < r < 2$. Suppose that $\boldsymbol{f}$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)$ satisfying the compatibility condition (III.8) with respect to p and r and let $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Then the Oseen problem* (III.12) *has a solution $(\boldsymbol{u}, \pi) \in (\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,r}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3))$ such that*

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,r}(\mathbb{R}^3)} + ||\pi||_{L^p(\mathbb{R}^3)} + ||\pi||_{L^r(\mathbb{R}^3)} \leqslant C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + ||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)}). \tag{III.49}$$

Proof.

Let $\boldsymbol{f}$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)$ and satisfying the compatibility condition (III.8) with respect to p and with r. Then $\boldsymbol{f}$ can be written as $\boldsymbol{f} = \operatorname{div}\boldsymbol{F}$ with $\boldsymbol{F} \in \boldsymbol{L}^p(\mathbb{R}^3) \cap \boldsymbol{L}^r(\mathbb{R}^3)$. Take the same sequence $\boldsymbol{f}_k$, as in the proof of Theorem 3.1, which now converges to $\boldsymbol{f}$ in $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)$.

Step 1. We suppose that $\boldsymbol{v} \in \mathcal{D}_\sigma(\mathbb{R}^3)$. Proceeding as in the first step of Theorem 3.1, there exists a unique solution

$$\boldsymbol{u}_k \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3), \quad \pi_k \in L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$$

such that

$$-\Delta \boldsymbol{u}_k + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u}_k) + \nabla \pi_k = \boldsymbol{f}_k, \quad \operatorname{div} \boldsymbol{u}_k = 0 \quad \text{in} \quad \mathbb{R}^3 \tag{III.50}$$

and satisfying the estimate

$$||\boldsymbol{u}_k||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\pi_k||_{L^p(\mathbb{R}^3)} \leqslant C_p(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})||\boldsymbol{f}_k||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}, \tag{III.51}$$

where C_p doesn't depend on k. On the other hand, using an interpolation argument, we have also $\boldsymbol{u}_k \in \boldsymbol{W}_0^{1,r}(\mathbb{R}^3)$, because $p < r < 2$. Now proceeding as in Theorem 3.1, we prove that

$$||\boldsymbol{u}_k||_{\boldsymbol{W}_0^{1,r}(\mathbb{R}^3)} + ||\pi_k||_{L^r(\mathbb{R}^3)} \leqslant C_r(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})||\boldsymbol{f}_k||_{\boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)}, \tag{III.52}$$

where C_r doesn't depend on k.

Finally, $(\boldsymbol{u}_k, \pi_k)$ is bounded in $(\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,r}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3))$ and we can extract a subsequence denoted again by $(\boldsymbol{u}_k, \pi_k)$ and satisfying

$$\boldsymbol{u}_k \rightharpoonup \boldsymbol{u} \quad \text{in} \quad \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,r}(\mathbb{R}^3) \quad \text{and} \quad \pi_k \rightharpoonup \pi \quad \text{in} \quad L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3). \tag{III.53}$$

We then verify that $(\boldsymbol{u}, \pi)$ is a solution of (III.12) and we have the estimate (III.59).

Step 2. We suppose that $\boldsymbol{v}$ belongs only to $\boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. The proof is exactly the same as in Theorem 3.2 where we take the exponent r instead of the exponent 2. □

Now, we study the uniqueness of generalized solution when $1 < p \leqslant 6/5$:

Proposition 3.2 *Let $1 < p \leqslant 6/5$. Let $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$ satisfying the compatibility condition (III.8) and $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Then the solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ given by Theorem 3.1 is unique.*

Proof. We proceed as in Proposition 3.1. Let $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ and satisfying (III.45). We know that $\boldsymbol{v_2} \otimes \boldsymbol{u}$ belongs to $\boldsymbol{L}^{p*}(\mathbb{R}^3) \cap \boldsymbol{L}^p(\mathbb{R}^3)$, with $3/2 < p* \leq 2$ and thus $\mathrm{div}(\boldsymbol{v_2} \otimes \boldsymbol{u})$ belongs to $\boldsymbol{W}_0^{-1,p*}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$. Moreover $\mathrm{div}(\boldsymbol{v_2} \otimes \boldsymbol{u})$ satisfies the compatibility condition (III.8). Using Theorem 3.3, we deduce that there exists $(\boldsymbol{\xi}, \varphi) \in (\boldsymbol{W}_0^{1,p*}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,p}(\mathbb{R}^3)) \times (L^{p*}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3))$ such that

$$-\Delta\, \boldsymbol{\xi} + \mathrm{div}(\boldsymbol{v}_1 \otimes \boldsymbol{\xi}) + \nabla\, \varphi = -\mathrm{div}(\boldsymbol{v_2} \otimes \boldsymbol{u}) \quad \text{and} \quad \mathrm{div}\, \boldsymbol{\xi} = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{III.54}$$

Set $\boldsymbol{\lambda} = \boldsymbol{\xi} - \boldsymbol{u}$ and $\psi = \varphi - \pi$, we have

$$-\Delta\, \boldsymbol{\lambda} + \mathrm{div}(\boldsymbol{v}_1 \otimes \boldsymbol{\lambda}) + \nabla\, \psi = \boldsymbol{0} \quad \text{and} \quad \mathrm{div}\, \boldsymbol{\lambda} = 0 \quad \text{in} \quad \mathbb{R}^3.$$

As in Proposition 3.1, we prove that $(\boldsymbol{\lambda}, \psi) = (\boldsymbol{0}, 0)$. Then we deduce that $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}_0^{1,p*}(\mathbb{R}^3) \times L^{p*}(\mathbb{R}^3)$. Using again Proposition 3.1, we prove that $(\boldsymbol{u}, \pi) = (\boldsymbol{0}, 0)$. □

We can now summarize our existence, uniqueness and regularity results as below.

Theorem 3.4 *Assume that $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$.*

i) Let $1 < p \leq 2$, $h \in L^p(\mathbb{R}^3)$ and $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$ satisfying the compatibility condition (III.8). Then the Oseen problem (III.6) *has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ such that*

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\pi||_{L^p(\mathbb{R}^3)} \leqslant C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)}) \left(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + (1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})||h||_{L^p(\mathbb{R}^3)} \right). \tag{III.55}$$

ii) Let $1 < p < r \leqslant 2$. Suppose that $\boldsymbol{f}$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)$ and satisfying the compatibility condition (III.8) with respect to p and r. Then the Oseen problem (III.12) *has a unique solution* $(\boldsymbol{u}, \pi) \in (\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,r}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3))$ *such that*

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,r}(\mathbb{R}^3)} + ||\pi||_{L^p(\mathbb{R}^3)} + ||\pi||_{L^r(\mathbb{R}^3)} \leqslant C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + ||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)}). \tag{III.56}$$

Finally the following existence result can be stated via a dual argument.

Theorem 3.5 *For $p > 2$, let $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$, $h \in L^p(\mathbb{R}^3)$ and $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Then, the Oseen problem* (III.6) *has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ if $p < 3$ and if $p \geq 3$, $\boldsymbol{u}$ is unique up to an additive constant vector. In addition, we have*

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)/\mathcal{P}_{[1-3/p]}} + ||\pi||_{L^p(\mathbb{R}^3)} \leqslant C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^2 \left(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + ||h||_{L^p(\mathbb{R}^3)}\right). \tag{III.57}$$

Proof. On one hand, Green formula yields, for all $\boldsymbol{w} \in \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)$ and $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$

$$\langle -\Delta\, \boldsymbol{u} + \boldsymbol{v} \cdot \nabla\, \boldsymbol{u} + \nabla\, \pi, \boldsymbol{w} \rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \times \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} =$$

$$\langle \boldsymbol{u}, -\Delta\, \boldsymbol{w} - \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{w}) \rangle_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times \boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)} - \langle \pi, \operatorname{div} \boldsymbol{w} \rangle_{L^p(\mathbb{R}^3) \times L^{p'}(\mathbb{R}^3)}.$$

Taking into account that if $p > 2$, we have $\boldsymbol{w} \in \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3) \hookrightarrow \boldsymbol{L}^{3p/(2p-3)}(\mathbb{R}^3)$ and since $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$ we can conclude that $\boldsymbol{v} \otimes \boldsymbol{w} \in \boldsymbol{L}^{p'}(\mathbb{R}^3)$ and consequently $\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{w}) \in \boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)$. On the other hand, for all $\eta \in L^{p'}(\mathbb{R}^3)$,

$$\langle \boldsymbol{u}, \nabla\, \eta \rangle_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times \boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)} = -\langle \operatorname{div} \boldsymbol{u}, \eta \rangle_{L^p(\mathbb{R}^3) \times L^{p'}(\mathbb{R}^3)}.$$

Then problem (III.6) has the following equivalent variational formulation:
Find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ such that for all $(\boldsymbol{w}, \eta) \in \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3) \times L^{p'}(\mathbb{R}^3)$,

$$\langle \boldsymbol{u}, -\Delta\, \boldsymbol{w} - \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{w}) + \nabla\, \eta \rangle_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times \boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)} - \langle \pi, \operatorname{div} \boldsymbol{w} \rangle_{L^p(\mathbb{R}^3) \times L^{p'}(\mathbb{R}^3)} =$$

$$\langle \boldsymbol{f}, \boldsymbol{w} \rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \times \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} - \langle h, \eta \rangle_{L^p(\mathbb{R}^3) \times L^{p'}(\mathbb{R}^3)}. \tag{III.58}$$

According to Theorem 3.4, for each $(\boldsymbol{f}', h') \in \boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3) \times L^{p'}(\mathbb{R}^3)$ satisfying

$$\langle f_i', 1 \rangle_{W_0^{-1,p'}(\mathbb{R}^3) \times W_0^{1,p}(\mathbb{R}^3)} = 0 \quad \text{if} \quad p' \leqslant \frac{3}{2},$$

there exists a unique solution $(\boldsymbol{w}, \eta) \in \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3) \times L^{p'}(\mathbb{R}^3)$ such that

$$-\Delta\, \boldsymbol{w} - \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{w}) + \nabla\, \eta = \boldsymbol{f}', \quad \operatorname{div} \boldsymbol{w} = h' \quad \text{in} \quad \mathbb{R}^3,$$

with the estimate

$$||\boldsymbol{w}||_{\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} + ||\eta||_{L^{p'}(\mathbb{R}^3)} \leq C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})\big(||\boldsymbol{f}'||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + (1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})||h'||_{L^p(\mathbb{R}^3)}\big).$$

Observe that the mapping

$$T:\ (\boldsymbol{f}', h') \mapsto \langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)\times \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} - \langle h, \eta\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)},$$

is linear and continuous with

$$\begin{aligned} |T(\boldsymbol{f}', h')| &\leq ||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}||\boldsymbol{w}||_{\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} + ||h||_{L^p(\mathbb{R}^3)}||\eta||_{L^{p'}(\mathbb{R}^3)} \\ &\leq C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^2\left(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + ||h||_{L^p(\mathbb{R}^3)}\right)\left(||\boldsymbol{f}'||_{\boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)} + ||h'||_{L^{p'}(\mathbb{R}^3)}\right). \end{aligned}$$

Note that $\boldsymbol{f}'$ belongs to $\boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)$ and $\boldsymbol{f}' \perp \mathbb{R}^3$ if $p \geq 3$. Thus there exists of unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ if $2 < p < 3$, and a unique $(\boldsymbol{u}, \pi) \in \left(\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)/\mathcal{P}_{[1-3/p]}) \times L^p(\mathbb{R}^3)\right)$ if $p \geq 3$, such that

$$T(\boldsymbol{f}', h') = \langle \boldsymbol{u}, \boldsymbol{f}'\rangle_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)\times \boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)} - \langle \pi, h'\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)},$$

with

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)/\mathcal{P}_{[1-3/p]}} + ||\pi||_{L^p(\mathbb{R}^3)} \leq C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^2\left(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + ||h||_{L^p(\mathbb{R}^3)}\right).$$

By definition of T, it follows that

$$\langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)\times \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} - \langle h, \eta\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)} = \langle \boldsymbol{u}, \boldsymbol{f}'\rangle_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)\times \boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)} - \langle \pi, h'\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)},$$

which is the variational formulation (III.58). □

Remark 3.3

Suppose in the assumption of Theorem 3.5 that $h = 0$ and proceeding as in the proof of Theorem 3.5. Then problem (III.6) has the following equivalent variational formulation: Find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ such that for all $\boldsymbol{w} \in \boldsymbol{V}_{p'}(\mathbb{R}^3)$ and $\eta \in L^{p'}(\mathbb{R}^3)$,

$$\langle \boldsymbol{u}, -\Delta\, \boldsymbol{w} - \operatorname{div}(\boldsymbol{v}\otimes \boldsymbol{w}) + \nabla\, \eta\rangle_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)\times \boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)} = \langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)\times \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)}$$

According to Theorem 3.4, for each $\boldsymbol{f}' \in \boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)$ satisfying

$$\langle f_i', 1\rangle_{W_0^{-1,p'}(\mathbb{R}^3)\times W_0^{1,p}(\mathbb{R}^3)} = 0 \quad \text{if} \quad p' \leqslant \frac{3}{2},$$

there exists a unique solution $(\boldsymbol{w}, \eta) \in \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3) \times L^{p'}(\mathbb{R}^3)$ satisfies the problem (III.6) with the estimate

$$||\boldsymbol{w}||_{\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} + ||\eta||_{L^{p'}(\mathbb{R}^3)} \leq C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})||\boldsymbol{f}'||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}.$$

Observe that the mapping

$$T: \boldsymbol{f}' \mapsto \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \times \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)}$$

is linear and continuous with

$$|T(\boldsymbol{f}')| \quad \leq \quad ||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} ||\boldsymbol{w}||_{\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} \leq C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)}) ||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} ||\boldsymbol{f}'||_{\boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)}.$$

Thus there exists a unique velocity $\boldsymbol{u}$ in $\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)/\mathcal{P}_{[1-3/p]}$ of problem (III.6) satisfies the estimate:

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)/\mathcal{P}_{[1-3/p]}} \leqslant C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)}) ||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}.$$

In addition, we have $-\Delta\, \boldsymbol{u} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u}) - \boldsymbol{f}$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$ and satisfies

$$\langle -\Delta\, \boldsymbol{u} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u}) - \boldsymbol{f}, \boldsymbol{w} \rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \times \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} = 0$$

for all $\boldsymbol{w}$ in $\boldsymbol{V}_{p'}(\mathbb{R}^3)$. Thus we use Theorem 1 of [3] to deduce the existence of a unique pressure π in $L^p(\mathbb{R}^3)$ of problem (III.6).

Now, we prove an other regularity result when $2 < p, r < \infty$:

Lemma 3.1 *Supposing that* $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)$, $h \in L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3)$ *and* $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$, *with* $2 < r < p < \infty$. *The Oseen problem* (III.6) *has a unique solution* $(\boldsymbol{u}, \pi) \in (\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,r}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3))$ *such that*

$$\begin{aligned} ||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,r}(\mathbb{R}^3)} + ||\pi||_{L^p(\mathbb{R}^3)} + ||\pi||_{L^r(\mathbb{R}^3)} \leqslant C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^2 \times \\ \Big(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + ||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)} + ||h||_{L^p(\mathbb{R}^3)} + ||h||_{L^r(\mathbb{R}^3)} \Big). \end{aligned} \tag{III.59}$$

Proof. Step 1. We suppose that $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3)$. Let $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$ and $h \in L^p(\mathbb{R}^3)$, from Theorem 3.5 there exists a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3)/\mathcal{P}_{[1-3/p]} \times L^p(\mathbb{R}^3)$ to the Oseen problem (III.6) such that

$$||\nabla\, \boldsymbol{u}||_{\boldsymbol{L}^p(\mathbb{R}^3)} + ||\pi||_{L^p(\mathbb{R}^3)} \leqslant C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^2 (||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + ||h||_{L^p(\mathbb{R}^3)}). \tag{III.60}$$

Note that $\mathcal{P}_{[1-3/p]}$ is equal to zero if $p < 3$. Since $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3)$, we prove that $\boldsymbol{v} \cdot \nabla\, \boldsymbol{u}$ belongs to $\boldsymbol{L}^1(\mathbb{R}^3) \cap \boldsymbol{L}^p(\mathbb{R}^3)$ and using the fact that $r < p$, we prove that $\boldsymbol{v} \cdot \nabla\, \boldsymbol{u}$ belongs to $\boldsymbol{L}^r(\mathbb{R}^3)$ and has a compact support. Then $\boldsymbol{v} \cdot \nabla\, \boldsymbol{u} \in \boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)$ and according to Theorem 3.3 of [2], there exists a unique solution $(\boldsymbol{u}', \pi') \in \boldsymbol{W}_0^{1,r}(\mathbb{R}^3)/\mathcal{P}_{[1-3/r]} \times L^r(\mathbb{R}^3)$ such that

$$-\Delta\, \boldsymbol{u}' + \nabla\, \pi' = \boldsymbol{f} - \boldsymbol{v} \cdot \nabla\, \boldsymbol{u} \quad \text{and} \quad \operatorname{div} \boldsymbol{u}' = h \quad \text{in} \quad \mathbb{R}^3, \tag{III.61}$$

taking into account that $\boldsymbol{f}$ belongs also to $\boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)$ and h belongs to $L^r(\mathbb{R}^3)$.
Set $\boldsymbol{z} = \boldsymbol{u} - \boldsymbol{u}'$ and $\theta = \pi - \pi'$, we obtain

$$-\Delta\,\boldsymbol{z} + \nabla\,\theta = \boldsymbol{0} \quad \text{and} \quad \operatorname{div}\boldsymbol{z} = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{III.62}$$

The uniqueness argument implies first that the harmonic function θ belonging to $L^p(\mathbb{R}^3) + L^r(\mathbb{R}^3)$ is necessarily equal to zero and with similar argument, we obtain also $\nabla\,\boldsymbol{u} = \nabla\,\boldsymbol{u}' \in \boldsymbol{L}^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3)$. Note that
$\boldsymbol{u}' = \boldsymbol{u}$ if $2 < r < p < 3$ and $\boldsymbol{u} = \boldsymbol{u}' + \boldsymbol{k} \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3)$ with $\boldsymbol{k} \in \mathbb{R}^3$, if $2 < r < 3 < p$. Then problem (III.61) becomes

$$-\Delta\,\boldsymbol{u}' + \nabla\,\pi' = \boldsymbol{f} - \boldsymbol{v}\cdot\nabla\,\boldsymbol{u}' \quad \text{and} \quad \operatorname{div}\boldsymbol{u}' = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{III.63}$$

According to Theorem 3.5, we have

$$\|\nabla\,\boldsymbol{u}'\|_{\boldsymbol{L}^r(\mathbb{R}^3)} + \|\pi'\|_{L^r(\mathbb{R}^3)} \leqslant C(1 + \|\boldsymbol{v}\|_{\boldsymbol{L}^3(\mathbb{R}^3)})^2(\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)} + \|h\|_{L^r(\mathbb{R}^3)}). \tag{III.64}$$

Replacing $\nabla\,\boldsymbol{u}'$ with $\nabla\,\boldsymbol{u}$ and π' with π in (III.64) and using (III.60), we deduce (III.59).
Step 2. We suppose $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Let $\boldsymbol{v}_\lambda \in \boldsymbol{\mathfrak{D}}_\sigma(\mathbb{R}^3)$ such that

$$\boldsymbol{v}_\lambda \longrightarrow \boldsymbol{v} \qquad \text{in } \boldsymbol{L}^3(\mathbb{R}^3). \tag{III.65}$$

Using the first step, there exists $(\boldsymbol{u}_\lambda, \pi_\lambda) \in (\,\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,r}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3))$ solution of the Oseen problem (III.6) and satisfying the estimate

$$\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + \|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}_0^{1,r}(\mathbb{R}^3)} + \|\pi_\lambda\|_{L^p(\mathbb{R}^3)} + \|\pi_\lambda\|_{L^r(\mathbb{R}^3)}$$

$$\leqslant C(1 + \|\boldsymbol{v}_\lambda\|_{\boldsymbol{L}^3(\mathbb{R}^3)})^2 \left(\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)} + \|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + \|h\|_{L^p(\mathbb{R}^3)} + \|h\|_{L^r(\mathbb{R}^3)}\right). \tag{III.66}$$

The sequence $(\boldsymbol{u}_\lambda, \pi_\lambda)$ being bounded in $(\,\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,r}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3))$, we can finally extract a subsequence which converges to
$(\boldsymbol{u}, \pi) \in (\,\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,r}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3))$ solution of the Oseen problem (III.6) and verifying the estimate (III.59). □

Remark 3.4

Reasoning as in Lemma 3.1, we prove that if $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)$ satisfies the compatibility condition (III.8) if $r \leq 3/2$ and $h \in L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3)$ with $1 < r \leq 2 < p$ and $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$, then there exists a unique solution $(\boldsymbol{u}, \pi) \in (\,\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,r}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3))$ to the Oseen problem (III.6).

3.2 Strong solutions in $\boldsymbol{W}_0^{2,p}(\mathbb{R}^3)$ and in $\boldsymbol{W}_1^{2,p}(\mathbb{R}^3)$

We begin by proving the existence of a unique strong solution in $\boldsymbol{W}_0^{2,p}(\Omega) \times \boldsymbol{W}_0^{1,p}(\Omega)$ for $1 < p < 3$:

Theorem 3.6 *For $1 < p < 3$, let $\boldsymbol{f} \in \boldsymbol{L}^p(\mathbb{R}^3)$, $h \in W_0^{1,p}(\mathbb{R}^3)$ and $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Then problem* (III.6) *has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{2,p}(\mathbb{R}^3)/\mathcal{P}_{[2-3/p]} \times W_0^{1,p}(\mathbb{R}^3)$ such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{2,p}(\mathbb{R}^3)/\mathcal{P}_{[2-3/p]}} + \|\pi\|_{W_0^{1,p}(\mathbb{R}^3)} \leq C(1 + \|\boldsymbol{v}\|_{\boldsymbol{L}^3(\mathbb{R}^3)})^3 \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\mathbb{R}^3)} + \|h\|_{W_0^{1,p}(\mathbb{R}^3)}\right). \tag{III.67}$$

Proof. The proof is similar to that of Theorem 5.1 of [4]. Note that in Theorem 3.6 we don't need to suppose that $\boldsymbol{v}$ satisfies (III.3). Observe first that if $1 < p < 3$ we have

$$L^p(\mathbb{R}^3) \hookrightarrow W_0^{-1,3p/(3-p)}(\mathbb{R}^3),$$

because $W_0^{1,t'}(\mathbb{R}^3) \hookrightarrow L^{p'}(\mathbb{R}^3)$ with $t = \dfrac{3p}{3-p}$ and $\dfrac{1}{p'} = \dfrac{1}{t'} - \dfrac{1}{3}$.
Since $h \in L^{3p/(3-p)}(\mathbb{R}^3)$ and $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,3p/(3-p)}(\mathbb{R}^3)$, Theorem 3.4 and Theorem 3.5 guarantees the existence of a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,3p/(3-p)}(\mathbb{R}^3) \times L^{3p/(3-p)}(\mathbb{R}^3)$ to the Oseen problem (III.6). Moreover, we have

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,3p/(3-p)}(\mathbb{R}^3)/\mathcal{P}_{[2-3/p]}} + \|\pi\|_{L^{3p/(3-p)}(\mathbb{R}^3)} \leq C\left(1 + \|\boldsymbol{v}\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\right)^2 \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\mathbb{R}^3)} + \|h\|_{W_0^{1,p}(\mathbb{R}^3)}\right). \tag{III.68}$$

Note that the compatibility condition (III.8) is not required because we have $3p/(3-p) > 3/2$. Using the fact that $\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u}) = \boldsymbol{v} \cdot \nabla \boldsymbol{u}$ belongs to $\boldsymbol{L}^p(\mathbb{R}^3)$, we can apply the Stokes regularity theory, see Theorem 3.8 of [2], to deduce the existence of $(\boldsymbol{z}, \eta) \in \boldsymbol{W}_0^{2,p}(\mathbb{R}^3) \times W_0^{1,p}(\mathbb{R}^3)$ unique up to an element of $\mathcal{P}_{[2-3/p]} \times \{0\}$ verifying:

$$-\Delta \boldsymbol{z} + \nabla \eta = \boldsymbol{f} - \boldsymbol{v} \cdot \nabla \boldsymbol{u} \quad \text{and} \quad \operatorname{div} \boldsymbol{z} = h \quad \text{in } \mathbb{R}^3.$$

Moreover, we have

$$\begin{aligned} &\inf_{\boldsymbol{\lambda} \in \mathcal{P}_{[2-3/p]}} \|\boldsymbol{z} + \boldsymbol{\lambda}\|_{\boldsymbol{W}_0^{2,p}(\mathbb{R}^3)} + \|\eta\|_{W_0^{1,p}(\mathbb{R}^3)} \\ &\leq C\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\mathbb{R}^3)} + \|\boldsymbol{v}\|_{\boldsymbol{L}^3(\mathbb{R}^3)} \|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^{3p/(3-p)}(\mathbb{R}^3)} + \|h\|_{W_0^{1,p}(\mathbb{R}^3)}\right), \end{aligned}$$

$$\leq C\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\mathbb{R}^3)} + C_1 \|\boldsymbol{v}\|_{\boldsymbol{L}^3(\mathbb{R}^3)} \left(1 + \|\boldsymbol{v}\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\right)^2 \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\mathbb{R}^3)} + \|h\|_{W_0^{1,p}(\mathbb{R}^3)}\right) + \|h\|_{W_0^{1,p}(\mathbb{R}^3)}\right), \tag{III.69}$$

with C denoting a constant only dependent on p. Set $\boldsymbol{w} = \boldsymbol{z} - \boldsymbol{u}$ and $\theta = \eta - \pi$, then we have

$$-\Delta \boldsymbol{w} + \nabla \theta = 0 \quad \text{and} \quad \operatorname{div} \boldsymbol{w} = 0 \quad \text{in} \quad \mathbb{R}^3.$$

Since $\nabla \boldsymbol{z} \in \boldsymbol{L}^{3p/(3-p)}(\mathbb{R}^3)$, there exists a constant $\mathbf{k} \in \mathbb{R}^3$, depending on $\boldsymbol{z}$ ($\mathbf{k} = \mathbf{0}$ if $p \geq 3/2$), such that $\boldsymbol{z} + \mathbf{k} \in \boldsymbol{W}_0^{1,3p/(3-p)}(\mathbb{R}^3)$ and thus $\boldsymbol{w} + \mathbf{k} \in \boldsymbol{W}_0^{1,3p/(3-p)}(\mathbb{R}^3)$. As $\Delta\theta = 0$ in $\mathbb{R}^3$ and $\theta \in L^{3p/(3-p)}(\mathbb{R}^3)$, then $\theta = 0$ and so $\boldsymbol{w}$ is a harmonic function belonging to
$\boldsymbol{W}_0^{2,p}(\mathbb{R}^3) + \boldsymbol{W}_0^{1,3p/(3-p)}(\mathbb{R}^3)$. Then if $p < 3/2$, we would have $3p/(3-p) < 3$ and thus $\boldsymbol{u} = \boldsymbol{z} \in \boldsymbol{W}_0^{2,p}(\mathbb{R}^3)$. If $p \geq 3/2$, there exists a polynomial $\boldsymbol{\lambda} \in \mathcal{P}_{[2-3/p]} \subset \boldsymbol{W}_0^{2,p}(\mathbb{R}^3)$ such that $\boldsymbol{u} = \boldsymbol{z} + \boldsymbol{\lambda}$. Consequently, $\boldsymbol{u} \in \boldsymbol{W}_0^{2,p}(\mathbb{R}^3)$ and $\pi \in W_0^{1,p}(\mathbb{R}^3)$ and we obtain (III.67). □

Remark 3.5

1) Under the assumptions of Theorem 3.6 and supposing that $1 < p \leq 2$, the solution $(\boldsymbol{u}, \pi)$ satisfies the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{2,p}(\mathbb{R}^3)/\mathcal{P}_{[2-3/p]}} + \|\pi\|_{W_0^{1,p}(\mathbb{R}^3)} \leq$$
$$C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^2 \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\mathbb{R}^3)} + (1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})\|h\|_{W_0^{1,p}(\mathbb{R}^3)}\right).$$

2) If we suppose in the assumption of Theorem 3.6 that $h = 0$, we prove that the solution $(\boldsymbol{u}, \pi)$ satisfies the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{2,p}(\mathbb{R}^3)/\mathcal{P}_{[2-3/p]}} + \|\pi\|_{W_0^{1,p}(\mathbb{R}^3)} \leq C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^2 \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\mathbb{R}^3)}. \tag{III.70}$$

Estimate (III.70) is an easy consequence of estimate (III.37) and Remark (3.3).

3) For $p \geq 3$, the hypothesis of $\boldsymbol{f} \in \boldsymbol{L}^p(\mathbb{R}^3)$, $h \in W_0^{1,p}(\mathbb{R}^3)$ and $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$ is not sufficient to ensure the existence of strong solutions for problem (III.6) in $\boldsymbol{W}_0^{2,p}(\mathbb{R}^3) \times W_0^{1,p}(\mathbb{R}^3)$. Indeed, suppose that under this assumptions it would be possible to find $\boldsymbol{u} \in \boldsymbol{W}_0^{2,p}(\mathbb{R}^3)$ and $\pi \in W_0^{1,p}(\mathbb{R}^3)$ such that

$$\boldsymbol{v} \cdot \nabla \boldsymbol{u} = \Delta \boldsymbol{u} - \nabla \pi + \boldsymbol{f} \in \boldsymbol{L}^p(\mathbb{R}^3).$$

This is a contradiction, since $\boldsymbol{v} \in \boldsymbol{L}^3(\mathbb{R}^3)$ and $\nabla \boldsymbol{u} \notin \boldsymbol{L}^{3p/(3-p)}(\mathbb{R}^3)$. Thus, it is necessary to suppose in addition that $\boldsymbol{f} \in \boldsymbol{L}^q(\mathbb{R}^3)$, $h \in W_0^{1,q}(\mathbb{R}^3)$ and $\boldsymbol{v} \in \boldsymbol{L}^{3pq/q(3+p)-3p}(\mathbb{R}^3)$ for some $3p/(3+p) \leq q < 3$. Under this assumptions, we deduce that the solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{2,q}(\mathbb{R}^3) \times W_0^{1,q}(\mathbb{R}^3)$ given by Theorem 3.6 belongs also to $\boldsymbol{W}_0^{2,p}(\mathbb{R}^3) \times W_0^{1,p}(\mathbb{R}^3)$ and it satisfies

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{2,p}(\mathbb{R}^3)} + \|\pi\|_{W_0^{1,p}(\mathbb{R}^3)} \leq C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^3 (\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\mathbb{R}^3)} + \|h\|_{W_0^{1,p}(\mathbb{R}^3)}).$$

Finally, we take $\boldsymbol{f}$ in weighted $\boldsymbol{L}^p(\mathbb{R}^3)$, more precisely $\boldsymbol{f} \in \boldsymbol{W}_1^{0,p}(\mathbb{R}^3)$, and the data h in the corresponding weighted Sobolev space $W_1^{1,p}(\mathbb{R}^3)$.

Theorem 3.7 *Suppose that $1 < p < 3$ and $p \neq 3/2$. Let $h \in W_1^{1,p}(\mathbb{R}^3)$ and $\boldsymbol{f} \in \boldsymbol{W}_1^{0,p}(\mathbb{R}^3)$ such that*

$$\int_{\mathbb{R}^3} \boldsymbol{f}(\boldsymbol{x})\, d\boldsymbol{x} = 0 \quad \text{if} \quad p < 3/2, \tag{III.71}$$

and let $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Then the Oseen problem (III.6) *has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,p}(\mathbb{R}^3) \times W_1^{1,p}(\mathbb{R}^3)$ satisfying the following estimate:*

$$||\boldsymbol{u}||_{\boldsymbol{W}_1^{2,p}(\mathbb{R}^3)} + ||\pi||_{W_1^{1,p}(\mathbb{R}^3)} \leqslant C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^6 (||\boldsymbol{f}||_{\boldsymbol{W}_1^{0,p}(\mathbb{R}^3)} + ||h||_{W_1^{1,p}(\mathbb{R}^3)}). \tag{III.72}$$

Proof. First, note that we have $\boldsymbol{W}_1^{0,p}(\mathbb{R}^3) \hookrightarrow \boldsymbol{L}^1(\mathbb{R}^3)$ if $p < 3/2$ and thus $\int_{\mathbb{R}^3} \boldsymbol{f}(\mathbf{x})\, d\mathbf{x}$ is well defined. On the other hand, observe that $h \in W_1^{1,p}(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ and for $p \neq 3/2$, we have $\boldsymbol{f} \in \boldsymbol{W}_1^{0,p}(\mathbb{R}^3) \hookrightarrow \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$. Then thanks to Theorem 3.4 and Theorem 3.5, there exists a unique solution

$$\boldsymbol{u} \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3), \quad \pi \in L^p(\mathbb{R}^3)$$

satisfying

$$-\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} - \boldsymbol{v} \cdot \nabla\, \boldsymbol{u} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = h \quad \text{in} \quad \mathbb{R}^3,$$

and we have

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + \|\pi\|_{L^p(\mathbb{R}^3)} \leq C\left(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)}\right)^2 \left(\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + \|h\|_{L^p(\mathbb{R}^3)}\right). \tag{III.73}$$

Step 1. We suppose that $\boldsymbol{v} \in \mathfrak{D}_\sigma(\mathbb{R}^3)$. Observe that $\boldsymbol{v} \cdot \nabla\, \boldsymbol{u}$ belongs to $\boldsymbol{W}_1^{0,p}(\mathbb{R}^3)$ and reasoning as in Theorem 3.6 we deduce that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,p}(\mathbb{R}^3) \times W_1^{1,p}(\mathbb{R}^3)$. After an easy calculation, we obtain that the pair $(\rho\, \boldsymbol{u}, \rho\, \pi) \in \boldsymbol{W}_0^{2,p}(\mathbb{R}^3) \times W_0^{1,p}(\mathbb{R}^3)$ satisfies the following equations in $\mathbb{R}^3$:

$$-\Delta(\rho\, \boldsymbol{u}) + \boldsymbol{v} \cdot \nabla(\rho\, \boldsymbol{u}) + \nabla(\rho\, \pi) := \boldsymbol{\chi}_\rho \quad \text{and} \quad \operatorname{div}(\rho\, \boldsymbol{u}) := \xi_\rho \quad \text{in} \quad \mathbb{R}^3,$$

with

$$\boldsymbol{\chi}_\rho = \rho\, \boldsymbol{f} - 2\nabla\, \rho \cdot \nabla\, \boldsymbol{u} - (\Delta\, \rho)\boldsymbol{u} + (\nabla\, \rho)\pi + (\boldsymbol{v} \cdot \nabla\, \rho)\boldsymbol{u} \quad \text{and} \quad \xi_\rho = \rho\, h + \nabla\, \rho \cdot \boldsymbol{u}. \tag{III.74}$$

Remember that here ρ is the weight defined by $\rho(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{1/2}$. It is clear that $(\boldsymbol{\chi}_\rho, \xi_\rho)$ belongs to $\boldsymbol{L}^p(\mathbb{R}^3) \times W_0^{1,p}(\mathbb{R}^3)$, so using Theorem 3.6 we obtain

$$\begin{aligned}
\|\boldsymbol{u}\|_{\boldsymbol{W}_1^{2,p}(\mathbb{R}^3)} + \|\pi\|_{W_1^{1,p}(\mathbb{R}^3)} &\leq C\|\rho\, \boldsymbol{u}\|_{\boldsymbol{W}_0^{2,p}(\mathbb{R}^3)} + \|\rho\, \pi\|_{W_0^{1,p}(\mathbb{R}^3)} \\
&\leq C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^3 \left(\|\boldsymbol{\chi}_\rho\|_{\boldsymbol{L}^p(\mathbb{R}^3)} + \|\xi_\rho\|_{W_0^{1,p}(\mathbb{R}^3)}\right),
\end{aligned} \tag{III.75}$$

Using (III.73), and that $\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \hookrightarrow \boldsymbol{L}^{p*}(\mathbb{R}^3)$ we deduce that

$$\|\boldsymbol{\chi}_\rho\|_{\boldsymbol{L}^p(\mathbb{R}^3)} + \|\xi_\rho\|_{W_0^{1,p}(\mathbb{R}^3)} \leq$$

$$\begin{aligned}
&\leq C\left(||\boldsymbol{f}||_{\boldsymbol{W}_1^{0,p}(\mathbb{R}^3)} + \|h\|_{W_1^{1,p}(\mathbb{R}^3)} + \|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + \|\pi\|_{L^p(\mathbb{R}^3)} + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)}\|\boldsymbol{u}\|_{\boldsymbol{L}^{p*}(\mathbb{R}^3)}\right) \\
&\leq C\left(||\boldsymbol{f}||_{\boldsymbol{W}_1^{0,p}(\mathbb{R}^3)} + \|h\|_{W_1^{1,p}(\mathbb{R}^3)} + \|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + \|\pi\|_{L^p(\mathbb{R}^3)} + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)}\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)}\right) \\
&\leq C\left(||\boldsymbol{f}||_{\boldsymbol{W}_1^{0,p}(\mathbb{R}^3)} + \|h\|_{W_1^{1,p}(\mathbb{R}^3)} + (1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})(\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + \|\pi\|_{L^p(\mathbb{R}^3)})\right) \\
&\leq C\left(||\boldsymbol{f}||_{\boldsymbol{W}_1^{0,p}(\mathbb{R}^3)} + \|h\|_{W_1^{1,p}(\mathbb{R}^3)} + (1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^3(\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + \|h\|_{L^p(\mathbb{R}^3)})\right).
\end{aligned} \tag{III.76}$$

From (III.75) and (III.76) and using the fact that $W_1^{1,p}(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ and $\boldsymbol{W}_1^{0,p}(\mathbb{R}^3) \hookrightarrow \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$ for $p \neq 3/2$, we deduce that

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_1^{2,p}(\mathbb{R}^3)} + \|\pi\|_{W_1^{1,p}(\mathbb{R}^3)} \leq$$

$$C(||\boldsymbol{f}||_{\boldsymbol{W}_1^{0,p}(\mathbb{R}^3)} + \|h\|_{W_1^{1,p}(\mathbb{R}^3)})(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^3\left(1+(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^3\right).$$

Then $(\boldsymbol{u},\pi)\in \boldsymbol{W}_1^{2,p}(\mathbb{R}^3)\times W_1^{1,p}(\mathbb{R}^3)$ satisfies the estimate (III.72).

Step 2. We suppose that $\boldsymbol{v}$ belongs only to $\boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Let $\boldsymbol{v}_\lambda\in\boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3)$ such that

$$\boldsymbol{v}_\lambda \longrightarrow \boldsymbol{v} \qquad \text{in } \boldsymbol{L}^3(\mathbb{R}^3).$$

Using the first step, there exists $(\boldsymbol{u}_\lambda,\pi_\lambda)\in \boldsymbol{W}_1^{2,p}(\mathbb{R}^3)\times W_1^{1,p}(\mathbb{R}^3)$ satisfying

$$-\Delta\,\boldsymbol{u}_\lambda+\boldsymbol{v}_\lambda\cdot\nabla\,\boldsymbol{u}_\lambda+\nabla\,\pi_\lambda=\boldsymbol{f} \quad\text{and}\quad \operatorname{div}\boldsymbol{u}_\lambda=h \quad\text{in}\quad \mathbb{R}^3,$$

and satisfying the estimate

$$||\boldsymbol{u}_\lambda||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)}+||\pi_\lambda||_{L^p(\mathbb{R}^3)}\leqslant C(1+||\boldsymbol{v}_\lambda||_{\boldsymbol{L}^3(\mathbb{R}^3)})^6(||\boldsymbol{f}||_{\boldsymbol{W}_1^{0,p}(\mathbb{R}^3)}+\|h\|_{W_1^{1,p}(\mathbb{R}^3)}).$$

Thus we can extract subsequences of $\boldsymbol{u}_\lambda$ and π_λ, still denoted by $\boldsymbol{u}_\lambda$ and π_λ, such that

$$\boldsymbol{u}_\lambda\rightharpoonup\boldsymbol{u} \quad\text{in}\quad \boldsymbol{W}_1^{2,p}(\mathbb{R}^3)\quad\text{and}\quad \pi_\lambda\rightharpoonup\pi\quad\text{in}\quad W_1^{1,p}(\mathbb{R}^3),$$

where $(\boldsymbol{u},\pi)\in\boldsymbol{W}_1^{2,p}(\mathbb{R}^3)\times W_1^{1,p}(\mathbb{R}^3)$ verifies the Oseen problem (III.6) and the estimate (III.72). To finish, observe that the uniqueness of the solution $(\boldsymbol{u},\pi)\in\boldsymbol{W}_1^{2,p}(\mathbb{R}^3)\times W_1^{1,p}(\mathbb{R}^3)$ is immediate because $\boldsymbol{W}_1^{2,p}(\mathbb{R}^3)\times W_1^{1,p}(\mathbb{R}^3)\subset\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)\times L^p(\mathbb{R}^3)$ and that $(\boldsymbol{u},\pi)$ is unique in $\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)\times L^p(\mathbb{R}^3)$. □

Remark 3.6

1) For $p=3/2$, the existence result of Theorem 3.7 holds if we suppose in addition that $\boldsymbol{f}\in\boldsymbol{W}_1^{0,3/2}(\mathbb{R}^3)\cap\boldsymbol{W}_0^{-1,3/2}(\mathbb{R}^3)$.

2) Under the assumptions of Theorem 3.6 and supposing that $1<p\leq 2$, the solution $(\boldsymbol{u},\pi)$ satisfies the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_1^{2,p}(\mathbb{R}^3)}+\|\pi\|_{W_1^{1,p}(\mathbb{R}^3)}\leq C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^5\left(\|\boldsymbol{f}\|_{\boldsymbol{W}_1^{0,p}(\mathbb{R}^3)}+(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})\|h\|_{W_1^{1,p}(\mathbb{R}^3)}\right).$$

3) If we suppose in the assumption of Theorem 3.7 that $h=0$, we prove that the solution $(\boldsymbol{u},\pi)$ satisfies the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_1^{2,p}(\mathbb{R}^3)}+\|\pi\|_{W_1^{1,p}(\mathbb{R}^3)}\leq C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\mathbb{R}^3)})^4\|\boldsymbol{f}\|_{\boldsymbol{W}_1^{0,p}(\mathbb{R}^3)}. \tag{III.77}$$

4) For $p\geq 3$, the hypothesis of $\boldsymbol{f}\in\boldsymbol{W}_1^{0,p}(\mathbb{R}^3)$, $h\in W_1^{1,p}(\mathbb{R}^3)$ and $\boldsymbol{v}\in\boldsymbol{L}^3_\sigma(\mathbb{R}^3)$ is not sufficient to study the existence of strong solutions for problem (III.6) in $\boldsymbol{W}_1^{2,p}(\mathbb{R}^3)\times W_1^{1,p}(\mathbb{R}^3)$. Indeed, suppose that under this assumptions it would be possible to find $\boldsymbol{u}\in\boldsymbol{W}_1^{2,p}(\mathbb{R}^3)$ and $\pi\in W_1^{1,p}(\mathbb{R}^3)$

such that

$$\boldsymbol{v}\cdot\nabla\,\boldsymbol{u}=\Delta\,\boldsymbol{u}-\nabla\,\pi+\boldsymbol{f}\in\,\boldsymbol{W}^{0,p}_{1}(\mathbb{R}^3).$$

This is a contradiction, since $\boldsymbol{v}\in\boldsymbol{L}^3(\mathbb{R}^3)$ and $\nabla\,\boldsymbol{u}\notin\,\boldsymbol{W}^{0,p*}_{1}(\mathbb{R}^3)$.

4 Oseen problem in an exterior domain

We will recall in this section the existence and the uniqueness of weak, strong and very weak solutions of the Oseen problem in an exterior domain Ω. We are interested at first in the following data:

$$\boldsymbol{f}\in\,\boldsymbol{W}^{-1,p}_{0}(\Omega),\quad h\in L^p(\Omega),\quad \boldsymbol{v}\in\boldsymbol{H}^3_\sigma(\Omega)\quad\text{and}\quad \boldsymbol{g}\in\,\boldsymbol{W}^{1/p',p}(\Gamma).$$

4.1 Generalized solutions in $\boldsymbol{W}^{1,2}_{0}(\Omega)$

We start by proving the existence and the uniqueness of solution in the Hilbert space case *i.e* in $\boldsymbol{W}^{1,2}_{0}(\Omega)\times L^2(\Omega)$.

Theorem 4.1 *Let*

$$\boldsymbol{f}\in\,\boldsymbol{W}^{-1,2}_{0}(\Omega),\quad h\in L^2(\Omega),\quad \boldsymbol{v}\in\boldsymbol{L}^3_\sigma(\Omega)\quad\text{and}\quad \boldsymbol{g}\in\,\boldsymbol{W}^{1/2,2}(\Gamma).$$

Then, Problem (III.1) *has a unique solution* $(\boldsymbol{u},\pi)\in\,\boldsymbol{W}^{1,2}_{0}(\Omega)\times L^2(\Omega)$. *Moreover, there exist some constants* $C_1>0$ *and* $C_2>0$ *such that:*

$$\|\,\boldsymbol{u}\,\|_{\boldsymbol{W}^{1,2}_{0}(\Omega)}\leq C_1\big(\,\|\,\boldsymbol{f}\,\|_{\boldsymbol{W}^{-1,2}_{0}(\Omega)}+(1+\|\,\boldsymbol{v}\,\|_{\boldsymbol{L}^3(\Omega)})(\|\,h\,\|_{L^2(\Omega)}+\|\,\boldsymbol{g}\,\|_{\boldsymbol{W}^{1/2,2}(\Gamma)})\big),\tag{III.78}$$

$$\|\,\pi\,\|_{L^2(\Omega)}\leq C_2\big(\,\|\,\boldsymbol{f}\,\|_{\boldsymbol{W}^{-1,2}_{0}(\Omega)}+(1+\|\,\boldsymbol{v}\,\|_{\boldsymbol{L}^3(\Omega)})(\|\,h\,\|_{L^2(\Omega)}+\|\,\boldsymbol{g}\,\|_{\boldsymbol{W}^{1/2,2}(\Gamma)})\big)\tag{III.79}$$

where $C_1=C(\Omega)$ *and* $C_2=C_1\left(1+\|\,\boldsymbol{v}\,\|_{\boldsymbol{L}^3(\Omega)}\right)$.

Proof. In order to prove the existence of solution, first using Lemma 3.3 of [40] for instance, we lift the boundary and the divergence data. Then, there exists $\boldsymbol{u}_0\in\,\boldsymbol{W}^{1,2}_{0}(\Omega)$ such that $\operatorname{div}\boldsymbol{u}_0=h$ in Ω, $\boldsymbol{u}_0=\boldsymbol{g}$ on Γ and:

$$\|\,\boldsymbol{u}_0\,\|_{\boldsymbol{W}^{1,2}_{0}(\Omega)}\leq C(\|\,h\,\|_{L^2(\Omega)}+\|\,\boldsymbol{g}\,\|_{\boldsymbol{W}^{1/2,2}(\Gamma)}).\tag{III.80}$$

Therefore, it remains to find $(\boldsymbol{z},\pi)=(\boldsymbol{u}-\boldsymbol{u}_0,\pi)\in\,\mathring{\boldsymbol{W}}^{1,2}_{0}(\Omega)\times L^2(\Omega)$ such that:

$$-\Delta\boldsymbol{z}-\boldsymbol{v}\cdot\nabla\boldsymbol{z}+\nabla\,\pi=\widetilde{\boldsymbol{f}}\quad\text{and}\quad\operatorname{div}\boldsymbol{z}=0\quad\text{in}\quad\Omega,\quad \boldsymbol{z}=0\quad\text{on}\quad\Gamma.$$

being $\widetilde{\boldsymbol{f}}=\boldsymbol{f}+\Delta\boldsymbol{u}_0+\boldsymbol{v}\cdot\nabla\boldsymbol{u}_0$. Observe that $\boldsymbol{v}\cdot\nabla\boldsymbol{u}_0=\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{u}_0)$ and $\boldsymbol{u}_0\in\,\boldsymbol{W}^{1,2}_{0}(\Omega)\hookrightarrow\boldsymbol{L}^6(\Omega)$ then we have $\widetilde{\boldsymbol{f}}\in\,\boldsymbol{W}^{-1,2}_{0}(\Omega)$. Using the density of $\boldsymbol{\mathfrak{D}}_\sigma(\Omega)$ in $\boldsymbol{V}_2(\Omega)$, we see that the previous problem is equivalent to: Find $\boldsymbol{z}\in\,\boldsymbol{V}_2(\Omega)$ such that:

$$\int_\Omega\nabla\,\boldsymbol{z}\cdot\nabla\,\boldsymbol{\varphi}\,d\mathbf{x}-b(\boldsymbol{v},\boldsymbol{z},\boldsymbol{\varphi})=\left\langle\widetilde{\boldsymbol{f}},\boldsymbol{\varphi}\right\rangle_{\boldsymbol{W}^{-1,2}_{0}(\Omega)\times\mathring{\boldsymbol{W}}^{1,2}_{0}(\Omega)}\qquad\forall\quad\boldsymbol{\varphi}\in\,\boldsymbol{V}_2(\Omega),\tag{III.81}$$

where $b = \langle \mathrm{div}(\boldsymbol{v} \otimes \boldsymbol{z}), \boldsymbol{\varphi} \rangle_{\boldsymbol{W}_0^{-1,2}(\Omega) \times \mathring{\boldsymbol{W}}_0^{1,2}(\Omega)}$ is a trilinear antisymmetric form with respect to the last two variables, well-defined for $\boldsymbol{v} \in \boldsymbol{L}^3(\Omega)$, $\boldsymbol{z}$, $\boldsymbol{\varphi} \in \mathring{\boldsymbol{W}}_0^{1,2}(\Omega)$. By Lax-Milgram theorem we can deduce the existence of unique $\boldsymbol{z} \in \mathring{\boldsymbol{W}}_0^{1,2}(\Omega)$ verifying:

$$
\begin{aligned}
\| \boldsymbol{z} \|_{\boldsymbol{W}_0^{1,2}(\Omega)} &\leq C(\| \boldsymbol{f} \|_{\boldsymbol{W}_0^{-1,2}(\Omega)} + \| \Delta \boldsymbol{u}_0 \|_{\boldsymbol{W}_0^{-1,2}(\Omega)} + \| \mathrm{div}\,(\boldsymbol{v} \otimes \boldsymbol{u}_0) \|_{\boldsymbol{W}_0^{-1,2}(\Omega)}) \\
&\leq C(\| \boldsymbol{f} \|_{\boldsymbol{W}_0^{-1,2}(\Omega)} + \| \boldsymbol{u}_0 \|_{\boldsymbol{W}_0^{1,2}(\Omega)} + \| \boldsymbol{v} \otimes \boldsymbol{u}_0 \|_{\boldsymbol{L}^2(\Omega)}) \\
&\leq C(\| \boldsymbol{f} \|_{\boldsymbol{W}_0^{-1,2}(\Omega)} + \Big(1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)}\Big) \| \boldsymbol{u}_0 \|_{\boldsymbol{W}_0^{1,2}(\Omega)} \Big) \\
&\leq C(\| \boldsymbol{f} \|_{\boldsymbol{W}_0^{-1,2}(\Omega)} + \Big(1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)}\Big) \Big(\| h \|_{L^2(\Omega)} + \| \boldsymbol{g} \|_{\boldsymbol{W}^{1/2,2}(\Gamma)}\Big),
\end{aligned}
$$

which added to estimate (III.80) makes (III.78). Now, $-\Delta \boldsymbol{z} - \boldsymbol{v} \cdot \nabla \boldsymbol{z} - \widetilde{\boldsymbol{f}} \in \boldsymbol{W}_0^{-1,2}(\Omega)$ and:

$$
\forall \boldsymbol{\varphi} \in \boldsymbol{V}_2(\Omega), \quad \Big\langle -\Delta\, \boldsymbol{z} - \boldsymbol{v} \cdot \nabla\, \boldsymbol{z} - \widetilde{\boldsymbol{f}}, \boldsymbol{\varphi} \Big\rangle_{\boldsymbol{W}_0^{-1,2}(\Omega) \times \mathring{\boldsymbol{W}}_0^{1,2}(\Omega)} = 0.
$$

As a consequence to Corollary 3.2 in [40], there exists a unique $\pi \in L^2(\Omega)$ such that:

$$
-\Delta \boldsymbol{z} - \boldsymbol{v} \cdot \nabla \boldsymbol{z} + \nabla\, \pi = \widetilde{\boldsymbol{f}} \quad \text{in} \quad \Omega
$$

with $\| \pi \|_{L^2(\Omega)} \leq C \| \nabla\, \pi \|_{\boldsymbol{W}_0^{-1,2}(\Omega)}$. Finally, estimate (III.79) follows from the previous equation and estimate for $\boldsymbol{z}$. □

4.2 Generalized solutions in $\boldsymbol{W}_0^{1,p}(\Omega)$

Throughout the rest of this work, if we do not say otherwise, we assume that $\boldsymbol{v} \in \boldsymbol{H}_\sigma^3(\Omega)$ (see section 1 for definition). Firstly, we recall the definition of the kernel $\mathcal{S}_\alpha^p(\Omega)$ of the Stokes operator for any real value α and $1 < p < \infty$:

$$
\mathcal{S}_\alpha^p(\Omega) = \Big\{ (\boldsymbol{u}, \pi) \in \boldsymbol{W}_\alpha^{1,p}(\Omega) \times W_\alpha^{0,p}(\Omega);\ -\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{0} \text{ and } \mathrm{div}\, \boldsymbol{u} = 0 \text{ in } \Omega,\ \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma \Big\}.
$$

$\mathcal{S}_\alpha^p(\Omega)$ is characterized, see [3] for more details. Now, we want to characterize the kernel $\mathcal{N}_0^p(\Omega)$ of the Oseen operator with Dirichlet boundary conditions:

$$
\begin{aligned}
\mathcal{N}_0^p(\Omega) \;=\; \{ (\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega);\ & -\Delta\, \boldsymbol{u} + \mathrm{div}(\boldsymbol{v} \otimes \boldsymbol{u}) + \nabla\, \pi = \boldsymbol{0} \\
& \text{and } \mathrm{div}\, \boldsymbol{u} = 0 \text{ in } \Omega,\ \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma \},
\end{aligned}
$$

where $\boldsymbol{v} \in \boldsymbol{H}_\sigma^3(\Omega)$. We will start by $p > 2$ and we shall see at the end of this section the characterization of the kernel $\mathcal{N}_0^p(\Omega)$ when $p \leqslant 2$. We introduce the space of polynomials for each integer k:

$$
\mathbf{N}_k = \{ (\boldsymbol{\lambda}, \mu) \in \mathcal{P}_k \times \mathcal{P}_{k-1},\ \mathrm{div}\, \boldsymbol{\lambda} = 0,\ -\Delta\, \boldsymbol{\lambda} + \mathrm{div}(\boldsymbol{v} \otimes \boldsymbol{\lambda}) + \nabla\, \mu = \boldsymbol{0} \}.
$$

In particular, recall that $\mathbf{N}_k = \{(\mathbf{0}, 0)\}$ whenever $k < 0$ and that $\mathbf{N}_0 = \mathcal{P}_0 \times \{0\}$.

Theorem 4.2 *Suppose that $p > 2$.*

i) If $p < 3$, then $\mathcal{N}_0^p(\Omega) = \{(\mathbf{0}, 0)\}$.

ii) If $p \geq 3$, then

$$\mathcal{N}_0^p(\Omega) = \Big\{(\boldsymbol{z}(\boldsymbol{\lambda}) - \boldsymbol{\lambda}, \eta(\boldsymbol{\lambda}) - \mu), \quad (\boldsymbol{\lambda}, \mu) \in \boldsymbol{N}_{[1-3/p]}\Big\}, \tag{III.82}$$

where $(\boldsymbol{z}(\boldsymbol{\lambda}), \eta(\boldsymbol{\lambda}))$ denotes the unique solution in $\bigcap_{3/2<r\leq p} \boldsymbol{W}_0^{1,r}(\Omega) \times \boldsymbol{L}^r(\Omega)$ of the following equations

$$-\Delta\, \boldsymbol{z} + \boldsymbol{v} \cdot \nabla\, \boldsymbol{z} + \nabla\, \eta = \mathbf{0} \quad \text{and} \quad \operatorname{div} \boldsymbol{z} = 0 \text{ in } \Omega, \;\; \boldsymbol{z} = \boldsymbol{\lambda} \quad \text{on} \quad \Gamma. \tag{III.83}$$

Proof. The proof follows the idea of [7]. Let $(\boldsymbol{u}, \pi)$ an element of $\mathcal{N}_0^p(\Omega)$ and let extend $\boldsymbol{u}$ and π by zero in Ω'. The extended functions, denoted by $\widetilde{\boldsymbol{u}}$ and $\widetilde{\pi}$ respectively belongs to $\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)$ and $L^p(\mathbb{R}^3)$. Now, we extend $\boldsymbol{v}$ in $\mathbb{R}^3$ in the following way: We solve the following Neumann problem in Ω':

$$\Delta\, \theta = 0 \quad \text{in} \quad \Omega' \quad \text{and} \quad \frac{\partial \theta}{\partial \boldsymbol{n}} = \boldsymbol{v} \cdot \boldsymbol{n} \quad \text{on} \quad \Gamma.$$

Owing to the boundary condition, this problem has a solution $\theta \in W^{1,3}(\Omega')$. Let us take

$$\boldsymbol{w} = \nabla\, \theta \quad \text{in} \quad \Omega' \quad \text{and} \quad \boldsymbol{w} = \boldsymbol{v} \quad \text{in} \quad \Omega.$$

Then $\boldsymbol{w}$ belongs to $\boldsymbol{L}^3(\mathbb{R}^3)$. Let $\varphi \in \mathcal{D}(\mathbb{R}^3)$ then we have

$$\begin{aligned}
< \operatorname{div} \boldsymbol{w}, \varphi >_{\mathcal{D}'(\mathbb{R}^3)\times\mathcal{D}(\mathbb{R}^3)} &= -\int_{\mathbb{R}^3} \boldsymbol{w} \cdot \nabla\, \varphi\, d\mathbf{x} \\
&= -\int_{\Omega} \boldsymbol{v} \cdot \nabla\, \varphi\, d\mathbf{x} - \int_{\Omega'} \nabla\, \theta \cdot \nabla\, \varphi\, d\mathbf{x} \\
&= \; < \boldsymbol{v} \cdot \boldsymbol{n}, \varphi >_\Gamma - < \frac{\partial \theta}{\partial \boldsymbol{n}}, \varphi >_\Gamma = 0,
\end{aligned}$$

where $< .,. >_\Gamma = < .,. >_{W^{-1/3,3}(\Gamma)\times W^{1/3,3/2}(\Gamma)}$. Then $\operatorname{div} \boldsymbol{w} = 0$ in $\mathbb{R}^3$ and thus $\boldsymbol{w}$ belongs to $\boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Set

$$-\Delta\, \widetilde{\boldsymbol{u}} + \boldsymbol{w} \cdot \nabla\, \widetilde{\boldsymbol{u}} + \nabla\, \widetilde{\pi} := \mathbf{F} \quad \text{and} \quad \operatorname{div} \widetilde{\boldsymbol{u}} := e \quad \text{in} \quad \mathbb{R}^3. \tag{III.84}$$

Then $(\mathbf{F}, e)$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ and obviously they have a compact support. Since $p > 2$, we deduce that $(\mathbf{F}, e)$ belongs to $\boldsymbol{W}_0^{-1,2}(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. It follows from Remark 3.4 that there exists a solution $(\boldsymbol{z}, \eta)$ in $\boldsymbol{W}_0^{1,r}(\mathbb{R}^3) \times (L^r(\mathbb{R}^3)$ for any $r \in]3/2, p]$ such that

$$-\Delta(\widetilde{\boldsymbol{u}} - \boldsymbol{z}) + \boldsymbol{w} \cdot \nabla(\widetilde{\boldsymbol{u}} - \boldsymbol{z})) + \nabla(\widetilde{\pi} - \eta) = \mathbf{0} \quad \text{and} \quad \operatorname{div}(\widetilde{\boldsymbol{u}} - \boldsymbol{z}) = 0 \quad \text{in} \quad \mathbb{R}^3.$$

If $p < 3$, we deduce from the argument of uniqueness in Theorem 3.5 that $(\widetilde{\boldsymbol{u}} - \boldsymbol{z}, \widetilde{\pi} - \eta) = (\mathbf{0}, 0)$ and thus $\widetilde{\boldsymbol{u}}$ and $\widetilde{\pi}$ belongs respectively to $\boldsymbol{W}_0^{1,r}(\mathbb{R}^3)$ and $L^r(\mathbb{R}^3)$ for any $3/2 < r \leq p$, which implies that $(\boldsymbol{u}, \pi)$ belongs to $\mathcal{N}_0^2(\Omega)$ and so $(\boldsymbol{u}, \pi) = \{(\mathbf{0}, 0)\}$. If $p \geq 3$, using again Theorem 3.5, we necessarily have $(\widetilde{\boldsymbol{u}} - \boldsymbol{z}, \widetilde{\pi} - \eta) = (\boldsymbol{\lambda}, \mu) \in \mathbf{N}_{[1-3/p]}$ and since $\boldsymbol{u} = \mathbf{0}$ on Γ, the restriction of $(\boldsymbol{z}, \eta)$ to Ω is nothing else but $(\boldsymbol{z}(\boldsymbol{\lambda}), \eta(\boldsymbol{\lambda}))$ which verifies (III.83). Observe that in this case, $\boldsymbol{\lambda}$ is a vector constant of $\mathbb{R}^3$ and $\mu = 0$. □

Remark 4.1

Of course, we have seen at the beginning of this section that $\mathcal{N}_0^2(\Omega) = \{(\mathbf{0}, 0)\}$.
The next lemma solves Problem (III.1) with homogeneous boundary conditions and a right-hand side $\boldsymbol{f}$ and h with bounded support.

Lemma 4.1 *Assume that $p > 2$ and $\boldsymbol{g} = \mathbf{0}$ on Γ. Let $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$ and $h \in L^p(\Omega)$ such that $\boldsymbol{f}$ and h have a compact support. Then, the Oseen Problem* (III.1) *has a unique solution $\boldsymbol{u} \in \boldsymbol{W}_0^{1,2}(\Omega) \cap \boldsymbol{W}_0^{1,p}(\Omega)$ and $\pi \in L^2(\Omega) \cap L^p(\Omega)$.*

Proof. By virtue of Lemma 2.1 of [7], the right-hand side $\boldsymbol{f}$ belongs also to $\boldsymbol{W}_0^{-1,2}(\Omega)$. Since $p > 2$ and support of h is compact, we have $h \in L^2(\Omega)$. Due to Theorem 4.1, Problem (III.1) has exactly one solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$. The remainder of the proof is devoted to establish that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$. Take R_0 sufficiently large so that both the supports of $(\boldsymbol{f}, h)$ are contained in B_{R_0} and $\overline{\Omega'} \subset B_{R_0}$. Let λ and μ be two scalar, nonnegative functions in $C^\infty(\mathbb{R}^3)$ that satisfy

$$\forall\ x \in B_{R_0}, \quad \lambda(\mathbf{x}) = 1, \quad \operatorname{supp} \lambda \subset B_{R_0+1}, \quad \forall\ x \in \mathbb{R}^3, \quad \lambda(\mathbf{x}) + \mu(\mathbf{x}) = 1.$$

Let Ω_{R_0+1} denotes the intersection $\Omega \cap B_{R_0+1}$ and let C_{R_0} denote the exterior (*i.e.* the complement) of B_{R_0}. Then, we can write

$$\boldsymbol{u} = \lambda\, \boldsymbol{u} + \mu\, \boldsymbol{u}, \qquad \pi = \lambda\, \pi + \mu\, \pi.$$

As μ is very smooth and vanishes on B_{R_0}, then $\mu \boldsymbol{f} = \mathbf{0}$ and $\mu\, h = 0$. Let us extend $(\boldsymbol{u}, \pi)$ by zero in Ω'. Then, the extended distributions denoted by $(\widetilde{\boldsymbol{u}}, \widetilde{\pi})$ belongs to $\boldsymbol{W}_0^{1,2}(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ and let $\boldsymbol{w} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$ such as in Theorem 4.2.

After an easy calculation, we obtain that the pair $(\mu \widetilde{\boldsymbol{u}}, \mu \widetilde{\pi})$ satisfies the following equations in $\mathbb{R}^3$:

$$-\Delta(\mu\, \widetilde{\boldsymbol{u}}) + \boldsymbol{w} \cdot \nabla(\mu\, \widetilde{\boldsymbol{u}}) + \nabla\,(\mu\, \widetilde{\pi}) := \boldsymbol{f}_1 \quad \text{and} \quad \operatorname{div}(\mu\, \widetilde{\boldsymbol{u}}) := e_1 \quad \text{in} \quad \mathbb{R}^3,$$

with

$$\boldsymbol{f}_1 = (\Delta\, \lambda)\widetilde{\boldsymbol{u}} - (\nabla\, \lambda)\widetilde{\pi} + 2\nabla\, \lambda \cdot \nabla\, \widetilde{\boldsymbol{u}} - (\boldsymbol{w} \cdot \nabla\, \lambda)\widetilde{\boldsymbol{u}} \quad \text{and} \quad e_1 = -\nabla\, \lambda \cdot \widetilde{\boldsymbol{u}}\ .$$

Moreover, owing to the supports of μ and λ, $(\boldsymbol{f}_1, e_1)$ belongs to $\boldsymbol{L}^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$. In addition, if $\mathcal{O}$ is a Lipschitzian bounded domain, we have $\boldsymbol{L}^2(\mathcal{O}) \hookrightarrow \boldsymbol{W}^{-1,q}(\mathcal{O})$ and $H^1(\mathcal{O}) \hookrightarrow L^q(\mathcal{O})$ for any $2 \leqslant q \leqslant 6$. Hence, we shall assume for the time being that $2 < p \leqslant 6$ and afterward, we shall use a bootstrap argument. Then $(\boldsymbol{f}_1, e_1)$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,2}(\mathbb{R}^3) \times L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. It follows from Remark 3.4 that there exists $(\boldsymbol{z}, \theta) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3) \times L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ such that

$$-\Delta\, \boldsymbol{z} + \boldsymbol{w} \cdot \nabla\, \boldsymbol{z} + \nabla\, \theta = \boldsymbol{f}_1 \quad \text{and} \quad \operatorname{div} \boldsymbol{z} = e_1 \quad \text{in} \quad \mathbb{R}^3.$$

and thus,

$$-\Delta(\mu\,\widetilde{\boldsymbol{u}}-\boldsymbol{z})+\boldsymbol{w}\cdot\nabla(\mu\,\widetilde{\boldsymbol{u}}-\boldsymbol{z})+\nabla(\mu\widetilde{\pi}-\theta)=\mathbf{0}\quad\text{and}\quad \operatorname{div}(\mu\,\widetilde{\boldsymbol{u}}-\boldsymbol{z})=0\quad\text{in}\quad\mathbb{R}^3,$$

with $(\mu\widetilde{\pi}-\theta)\in L^2(\mathbb{R}^3)$ and $(\mu\,\widetilde{\boldsymbol{u}}-\boldsymbol{z})\in \boldsymbol{W}_0^{1,2}(\mathbb{R}^3)$. Then

$$\langle-\Delta(\mu\,\widetilde{\boldsymbol{u}}-\boldsymbol{z})+\boldsymbol{w}\cdot\nabla(\mu\,\widetilde{\boldsymbol{u}}-\boldsymbol{z})+\nabla(\mu\widetilde{\pi}-\theta),(\mu\,\widetilde{\boldsymbol{u}}-\boldsymbol{z})\rangle_{\boldsymbol{W}_0^{-1,2}(\mathbb{R}^3)\times\boldsymbol{W}_0^{1,2}(\mathbb{R}^3)}=0,$$

and so

$$\|\nabla\,(\mu\,\widetilde{\boldsymbol{u}}-\boldsymbol{z})\|_{\boldsymbol{L}^2(\mathbb{R}^3)}=0.$$

Thus $\mu\,\widetilde{\boldsymbol{u}}-\boldsymbol{z}=\mathbf{0}$ and so $\mu\,\widetilde{\pi}-\theta=0$. Consequently, $(\mu\,\widetilde{\boldsymbol{u}},\mu\,\widetilde{\pi})$ belongs to $\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)\times L^p(\mathbb{R}^3)$.

In particular, we have $\mu\,\widetilde{\boldsymbol{u}}=\boldsymbol{u}$ and $\mu\,\widetilde{\pi}=\pi$ outside B_{R_0+1}, so the restriction of $\boldsymbol{u}$ to ∂B_{R_0+1} belongs to $\boldsymbol{W}^{1/p',p}(\partial\,B_{R_0+1})$. Therefore, $(\boldsymbol{u},\pi)$ satisfies:

$$-\Delta\,\boldsymbol{u}+\boldsymbol{v}\cdot\nabla\,\boldsymbol{u}+\nabla\,\pi=\boldsymbol{f}\quad\text{and}\quad\operatorname{div}\boldsymbol{u}=h\quad\text{in}\quad\Omega_{R_0+1},\quad \boldsymbol{u}|_{\partial B_{R_0+1}}=\widetilde{\boldsymbol{u}}\quad\text{and}\quad\boldsymbol{u}|_\Gamma=\mathbf{0}.\tag{III.85}$$

Observe that for any $\varphi\in W^{1,2}(\Omega_{R_0+1})$ we have

$$\int_{\Omega_{R_0+1}}\boldsymbol{u}\cdot\nabla\,\varphi\,d\mathbf{x}=-\int_{\Omega_{R_0+1}}\varphi\operatorname{div}\boldsymbol{u}\,d\mathbf{x}+\int_{\partial\,\Omega_{R_0+1}}\varphi\boldsymbol{u}\cdot\boldsymbol{n}\,d\mathbf{x}.$$

In particular, for $\varphi=1$, we have

$$\int_{\Omega_{R_0+1}}h(\mathbf{x})\,d\mathbf{x}=\int_{\partial\,\Omega_{R_0+1}}\boldsymbol{u}\cdot\boldsymbol{n}\,d\mathbf{x}=\int_{\partial\,B_{R_0+1}}\boldsymbol{u}\cdot\boldsymbol{n}\,d\mathbf{x}\tag{III.86}$$

and thus, according to Theorem 15 see [13], this problem has a unique $(\boldsymbol{u},\pi)\in\boldsymbol{W}^{1,p}(\Omega_{R_0+1})\times L^p(\Omega_{R_0+1})$. This implies that $(\boldsymbol{u},\pi)\in\boldsymbol{W}_0^{1,p}(\Omega)\times L^p(\Omega)$ if $2<p\le 6$. Now, suppose that $p>6$. The above argument shows that $(\boldsymbol{u},\pi)$ belongs to $\boldsymbol{W}_0^{1,6}(\Omega)\times L^6(\Omega)$ and we can repeat the same argument with $p=6$ instead of $p=2$ using the fact that if $\mathcal{O}$ is a Lipschitzian bounded domain, we have $\boldsymbol{L}^6(\mathcal{O})\hookrightarrow\boldsymbol{W}^{-1,t}(\mathcal{O})$ for any real number t and we have $L^2(\mathcal{O})\cap L^\infty(\mathcal{O})\hookrightarrow L^p(\mathcal{O})$. This establishes the existence of solution $(\boldsymbol{u},\pi)$ in $\boldsymbol{W}_0^{1,p}(\Omega)\times L^p(\Omega)$ of Problem (III.1) when $p>2$. Uniqueness follows from the fact that $\boldsymbol{W}_0^{1,2}(\Omega)$ does not contain the vector constant functions.

□

The next lemma solves Problem (III.1) with non homogeneous boundary conditions and a right-hand side $\boldsymbol{f}$ and h with bounded support.

Lemma 4.2 *Under the assumptions of Lemma 4.1, for each $\boldsymbol{g}\in\boldsymbol{W}^{1/p',p}(\Gamma)$, Problem* (III.1) *has a unique solution $(\boldsymbol{u},\pi)\in\boldsymbol{W}_0^{1,p}(\Omega)\cap\boldsymbol{W}_0^{1,2}(\Omega)\times L^p(\Omega)\cap L^2(\Omega)$.*

Proof. Let $\boldsymbol{g}\in\boldsymbol{W}^{1/p',p}(\Gamma)$ and take R sufficiently large so that $\overline{\Omega'}\subset B_R$. Set $\Omega_R=\Omega\cap B_R$, then there exists $\boldsymbol{z}\in\boldsymbol{W}^{1,p}(\Omega_R)$ solution of the problem $-\Delta\boldsymbol{z}=\mathbf{0}$ in Ω_R, $\boldsymbol{z}=\boldsymbol{g}$ on Γ and $\boldsymbol{z}=\mathbf{0}$ on ∂B_R. We extend $\boldsymbol{z}$ by zero out of B_R. The extended function denoted by $\widetilde{\boldsymbol{z}}$ has a compact support in $\overline{\Omega'}$ and belongs to $\boldsymbol{W}_0^{1,p}(\Omega)$ and once we set $\boldsymbol{u}' = \boldsymbol{u} - \widetilde{\boldsymbol{z}}$. Then Problem (III.1) is equivalent to the following problem: Find $(\boldsymbol{u}', \pi)$ such that

$$\begin{cases} -\Delta \boldsymbol{u}' + \boldsymbol{v}\cdot\nabla\boldsymbol{u}' + \nabla\pi = \boldsymbol{f} + \boldsymbol{v}\cdot\nabla\widetilde{\boldsymbol{z}} + \Delta\widetilde{\boldsymbol{z}}, \\ \operatorname{div}\, \boldsymbol{u}' = h + \operatorname{div}\, \widetilde{\boldsymbol{z}} \text{ in } \Omega, \boldsymbol{u}'|_{\partial\Omega} = \boldsymbol{0}, \end{cases} \tag{III.87}$$

where data belonging to the space $\boldsymbol{W}_0^{-1,p}(\Omega) \times L^p(\Omega)$ with the compact support in $\bar{\Omega}$. Then we will apply Lemma 4.1 . □

Corollary 4.1 *Assume that $p > 2$ and let $\boldsymbol{g} \in \boldsymbol{W}^{1/p',p}(\partial\Omega)$. Then there exists $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \cap \boldsymbol{W}_0^{1,2}(\Omega) \times L^p(\Omega) \cap L^2(\Omega)$ such that*

$$-\Delta\boldsymbol{u} + \boldsymbol{v}\cdot\nabla\boldsymbol{u} + \nabla\pi = \boldsymbol{0}, \quad \operatorname{div}\boldsymbol{u} = 0 \quad \textit{in} \quad \Omega, \quad \boldsymbol{u}|_\Gamma = \boldsymbol{g}.$$

Proof. Let $R_0 > 0$ such that $\overline{\Omega'} \subset B_{R_0}$. Take $\psi \in \mathcal{D}(\mathbb{R}^3)$ with support in Ω_{R_0} and such that

$$\int_{\Omega_{R_0}} \psi(\mathbf{x}) d\mathbf{x} + \int_{\partial\Omega} \boldsymbol{g}\cdot\boldsymbol{n} ds = 0.$$

According to Theorem 12 [13], there exists $(\boldsymbol{z}, \eta) \in \boldsymbol{W}^{1,p}(\Omega_{R_0}) \times L^p(\Omega_{R_0})$ such that

$$-\Delta\boldsymbol{z} + \boldsymbol{v}\cdot\nabla\boldsymbol{z} + \nabla\eta = 0, \ \operatorname{div}\ \boldsymbol{z} = \psi \text{ in } \Omega_{R_0}, \ \boldsymbol{z}|_{\partial B_{R_0}} = \boldsymbol{0}, \boldsymbol{z}|_\Gamma = \boldsymbol{g}.$$

If we denote the extension by $(\boldsymbol{0}, 0)$ of $(\boldsymbol{z}, \eta)$ outside B_{R_0} by $(\tilde{\boldsymbol{z}}, \tilde{\eta})$ then $(\tilde{\boldsymbol{z}}, \tilde{\eta}) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ and

$$\begin{aligned} -\Delta\tilde{\boldsymbol{z}} + \boldsymbol{v}\cdot\nabla\tilde{\boldsymbol{z}} + \nabla\tilde{\eta} &:= \boldsymbol{\xi} && \text{in} \quad \Omega, \\ \operatorname{div}\,\tilde{\boldsymbol{z}} &= \psi && \text{in} \quad \Omega, \\ \tilde{\boldsymbol{z}} &= \boldsymbol{g} && \text{on} \quad \Gamma. \end{aligned}$$

Observe that $\boldsymbol{\xi}$ belongs to $\boldsymbol{W}_0^{-1,p}(\Omega)$ with compact support in $\overline{\Omega'}$. From Theorem 4.1 we have a solution $(\boldsymbol{w}, \tau) \in \big(\boldsymbol{W}_0^{1,p}(\Omega) \cap \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega) \cap L^2(\Omega)\big)$ to the problem

$$-\Delta\boldsymbol{w} + \boldsymbol{v}\cdot\nabla\boldsymbol{w} + \nabla\tau = -\boldsymbol{\xi}, \ \operatorname{div}\ \boldsymbol{w} = -\psi \text{ in } \Omega, \boldsymbol{w}|_\Gamma = \boldsymbol{0}.$$

Then the pair $(\boldsymbol{u}, \pi) = (\tilde{\boldsymbol{z}} + \boldsymbol{w}, \tilde{\eta} + \tau)$ has the required properties. □

Theorem 4.3 *Assume that $p > 2$. Let $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$, $h \in L^p(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{W}^{1/p',p}(\Gamma)$. Then Problem* (III.1) *has a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ unique up to an element of $\mathcal{N}_0^p(\Omega)$.*

Proof.

i) **First case**: $\boldsymbol{g} = \boldsymbol{0}$.

We would like to extend data $(\boldsymbol{f}, h) \in \boldsymbol{W}_0^{-1,p}(\Omega) \times L^p(\Omega)$ to the whole space. According to Corollary 1.3 see [3] there exists a second-tensor $\boldsymbol{F} \in \boldsymbol{L}^p(\Omega)$ such that div $\boldsymbol{F} = \boldsymbol{f}$. Then we extend $\boldsymbol{F}$ (resp. h)

by zero into the whole space and we denote this extension by $\tilde{\boldsymbol{F}}$ (resp. $\tilde{h}$). Set $\tilde{\boldsymbol{f}} = \operatorname{div} \tilde{\boldsymbol{F}}$. It is clear that $(\tilde{\boldsymbol{f}}, \tilde{h}) \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$. Now, we consider the following equation:

$$-\Delta \tilde{\boldsymbol{z}} + \boldsymbol{w} \cdot \nabla \tilde{\boldsymbol{z}} + \nabla \tilde{\eta} = \tilde{\boldsymbol{f}} \quad \text{and} \quad \operatorname{div} \tilde{\boldsymbol{z}} = \tilde{h} \quad \text{in } \mathbb{R}^3,$$

with $\boldsymbol{w} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$ introduced in the proof of Theorem 4.2. Applying the theory of Oseen problem in $\mathbb{R}^3$, we deduce that this problem has a unique solution $(\tilde{\boldsymbol{z}}, \tilde{\eta}) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ if $p < 3$ and if $p \geq 3$, $\tilde{\boldsymbol{z}}$ is unique up to a constant vector. In addition, we have:

$$\|\tilde{\boldsymbol{z}}\|_{\boldsymbol{W}^{1,p}(\mathbb{R}^3)/\mathcal{P}_{[1-3/p]}} + \|\tilde{\eta}\|_{L^p(\mathbb{R}^3)} \leq C \left(\|\tilde{\boldsymbol{f}}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3} + \|\tilde{h}\|_{L^p(\mathbb{R}^3)} \right). \tag{III.88}$$

Denoting the restriction to Ω by $(\boldsymbol{z}, \eta)$ and by $\gamma \boldsymbol{z} \in \boldsymbol{W}^{1/p',p}(\Gamma)$ the trace of $\boldsymbol{z}$ on Γ. According to Corollary 4.1 , we have the existence of $(\boldsymbol{\xi}, \nu) \in \big(\boldsymbol{W}_0^{1,p}(\Omega) \cap \boldsymbol{W}_0^{1,2}(\Omega) \times L^p(\Omega) \cap L^2(\Omega)\big)$ such that

$$-\Delta \boldsymbol{\xi} + \boldsymbol{v} \cdot \nabla \boldsymbol{\xi} + \nabla \nu = 0, \ \operatorname{div} \boldsymbol{\xi} = 0 \text{ in } \Omega, \boldsymbol{\xi}|_\Gamma = -\gamma \boldsymbol{z}.$$

Hence, the pair $(\boldsymbol{u}, \pi) = (\boldsymbol{z} + \boldsymbol{\xi}, \eta + \nu)$ belongs to $\boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ and satisfies Problem (III.1) with $\boldsymbol{g} = \boldsymbol{0}$.

ii) **Second case**: Nonhomogeneous boundary data. Each $\boldsymbol{g} \in \boldsymbol{W}^{1/p',p}(\Gamma)$ has a lifting $\boldsymbol{\chi} \in \boldsymbol{W}_0^{1,p}(\Omega)$ such that

$$\|\boldsymbol{\chi}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} \leq C \|\boldsymbol{g}\|_{\boldsymbol{W}^{1/p',p}(\partial\Omega)}.$$

Setting $\boldsymbol{u}' = \boldsymbol{u} - \boldsymbol{\chi}$, then Problem (III.1) is equivalent to the following problem: Find $(\boldsymbol{u}', \pi)$ such that

$$\begin{aligned} -\Delta \boldsymbol{u}' + \boldsymbol{v} \cdot \nabla \boldsymbol{u}' + \nabla \pi &= \boldsymbol{f} + \Delta \boldsymbol{\chi} - \boldsymbol{v} \cdot \nabla \boldsymbol{\chi} && \text{in} \quad \Omega, \\ \operatorname{div} \boldsymbol{u}' &= h - \operatorname{div} \boldsymbol{\chi} && \text{in} \quad \Omega, \\ \boldsymbol{u}' &= 0 && \text{on} \quad \Gamma. \end{aligned}$$

Set $\boldsymbol{f}_{\boldsymbol{\chi}} = \boldsymbol{f} + \Delta \boldsymbol{\chi} - \boldsymbol{v} \cdot \nabla \boldsymbol{\chi}$ and $h_{\boldsymbol{\chi}} = h - \operatorname{div} \boldsymbol{\chi}$. As $p > 2$, $\boldsymbol{v} \cdot \nabla \boldsymbol{\chi} \in \boldsymbol{L}^r(\Omega)$, with $\frac{1}{r} = \frac{1}{3} + \frac{1}{p}$ and $\boldsymbol{L}^r(\Omega) \hookrightarrow \boldsymbol{W}_0^{-1,p}(\Omega)$. Hence, $\boldsymbol{f}_{\boldsymbol{\chi}}$ belongs to $\boldsymbol{W}_0^{-1,p}(\Omega)$. From previous step we know that this problem has a solution in $\mathring{\boldsymbol{W}}_0^{1,p}(\Omega) \times L^p(\Omega)$. Uniqueness follows from the definition of the kernel $\mathcal{N}_0^p(\Omega)$. $\square$

In particular, it follows from Theorem 4.3 that, for any $p \geq 2$, the Oseen operator

$$\mathcal{O} : \mathring{\boldsymbol{W}}_0^{1,p}(\Omega) \times L^p(\Omega)/\mathcal{N}_0^p(\Omega) \longrightarrow \boldsymbol{W}_0^{-1,p}(\Omega) \times L^p(\Omega)$$

defined by : $\mathcal{O}(\boldsymbol{u}, \pi) = \big(-\Delta \boldsymbol{u} + \boldsymbol{v} \cdot \nabla \boldsymbol{u} + \nabla \pi, \operatorname{div} \boldsymbol{u}\big)$ is obviously continuous and since both spaces are Banach spaces, it is an isomorphism. Thus there exists a constant $C(\boldsymbol{v})$ depending on $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\Omega)$, Ω and p such that

$$\inf_{(\boldsymbol{\lambda}, \mu) \in \mathcal{N}_0^p(\Omega)} ||\boldsymbol{u} + \boldsymbol{\lambda}||_{\boldsymbol{W}_0^{1,p}(\Omega)} + ||\pi + \mu||_{L^p(\Omega)} \leq C(\boldsymbol{v}) \big(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + ||h||_{L^p(\Omega)} \big). \tag{III.89}$$

The following existence result can be stated via a dual argument.

Theorem 4.4 *Suppose that* $1 < p < 2$ *and* $\boldsymbol{g} = \boldsymbol{0}$. *Let* $(\boldsymbol{f}, h) \in \boldsymbol{W}_0^{-1,p}(\Omega) \times L^p(\Omega)$ *such that for any* $(\boldsymbol{\lambda}, \mu) \in \mathcal{N}_0^{p'}(\Omega)$, *we have*

$$\langle \boldsymbol{f}, \boldsymbol{\lambda} \rangle_{\boldsymbol{W}_0^{-1,p}(\Omega) \times \mathring{\boldsymbol{W}}_0^{1,p'}(\Omega)} - \langle h, \mu \rangle_{L^p(\Omega) \times L^{p'}(\Omega)} = 0. \tag{III.90}$$

Then the Oseen problem (III.1) *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$.

Proof. On one hand, Green formula yields, for all $\boldsymbol{w} \in \mathring{\boldsymbol{W}}_0^{1,p'}(\Omega)$ and $(\boldsymbol{u}, \pi) \in \mathring{\boldsymbol{W}}_0^{1,p}(\Omega) \times L^p(\Omega)$

$$\langle -\Delta\, \boldsymbol{u} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u}) + \nabla\, \pi, \boldsymbol{w} \rangle_{\boldsymbol{W}_0^{-1,p}(\Omega) \times \mathring{\boldsymbol{W}}_0^{1,p'}(\Omega)} =$$

$$\langle \boldsymbol{u}, -\Delta\, \boldsymbol{w} - \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{w}) \rangle_{\mathring{\boldsymbol{W}}_0^{1,p}(\Omega) \times \boldsymbol{W}_0^{-1,p'}(\Omega)} - \langle \pi, \operatorname{div} \boldsymbol{w} \rangle_{L^p(\Omega) \times L^{p'}(\Omega)} .$$

Taking into account that $p' > 2$, we have $\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{w}) = \boldsymbol{v} \cdot \nabla\, \boldsymbol{w} \in \boldsymbol{L}^r(\Omega)$ with $\frac{1}{r} = \frac{1}{3} + \frac{1}{p'}$ and $\boldsymbol{L}^r(\Omega) \hookrightarrow \boldsymbol{W}_0^{-1,p'}(\Omega)$. On the other hand, for all $\eta \in L^{p'}(\Omega)$,

$$\langle \boldsymbol{u}, \nabla\, \eta \rangle_{\mathring{\boldsymbol{W}}_0^{1,p}(\Omega) \times \boldsymbol{W}_0^{-1,p'}(\Omega)} = - \langle \operatorname{div} \boldsymbol{u}, \eta \rangle_{L^p(\Omega) \times L^{p'}(\Omega)} .$$

Then problem (III.1) with $\boldsymbol{g} = \boldsymbol{0}$ has the following equivalent variational formulation: find $(\boldsymbol{u}, \pi) \in \mathring{\boldsymbol{W}}_0^{1,p}(\Omega) \times L^p(\Omega)$ such that for all $(\boldsymbol{w}, \eta) \in \mathring{\boldsymbol{W}}_0^{1,p'}(\Omega) \times L^{p'}(\Omega)$,

$$\langle \boldsymbol{u}, -\Delta\, \boldsymbol{w} - \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{w}) + \nabla\, \eta \rangle_{\mathring{\boldsymbol{W}}_0^{1,p}(\Omega) \times \boldsymbol{W}_0^{-1,p'}(\Omega)} - \langle \pi, \operatorname{div} \boldsymbol{w} \rangle_{L^p(\Omega) \times L^{p'}(\Omega)} =$$

$$= \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{\boldsymbol{W}_0^{-1,p}(\Omega) \times \mathring{\boldsymbol{W}}_0^{1,p'}(\Omega)} - \langle h, \eta \rangle_{L^p(\Omega) \times L^{p'}(\Omega)} . \tag{III.91}$$

According to Theorem 4.3, for each $(\boldsymbol{f}', h') \in \boldsymbol{W}_0^{-1,p'}(\Omega) \times L^{p'}(\Omega)$ there exists a unique solution $(\boldsymbol{w}, \eta) \in (\mathring{\boldsymbol{W}}_0^{1,p'}(\Omega) \times L^{p'}(\Omega))/\mathcal{N}_0^{p'}(\Omega)$ such that

$$-\Delta\, \boldsymbol{w} - \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{w}) + \nabla\, \eta = \boldsymbol{f}', \quad \operatorname{div} \boldsymbol{w} = h' \quad \text{in} \quad \Omega \quad \text{and} \quad \boldsymbol{w} = \boldsymbol{0} \quad \text{on} \quad \Gamma,$$

with the following estimate

$$\inf_{(\boldsymbol{\lambda},\mu) \in \mathcal{N}_0^{p'}(\Omega)} ||\boldsymbol{w} + \boldsymbol{\lambda}||_{\boldsymbol{W}_0^{1,p'}(\Omega)} + ||\eta + \mu||_{L^{p'}(\Omega)} \leq C(\boldsymbol{v}) \big(||\boldsymbol{f}'||_{\boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)} + ||h'||_{L^{p'}(\Omega)}\big). \tag{III.92}$$

Let T be a linear form defined from $\mathbf{W}_0^{-1,p'}(\Omega) \times L^{p'}(\Omega)$ onto $\mathbb{R}$ by:

$$T: \ (\boldsymbol{f}', h') \mapsto \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{\boldsymbol{W}_0^{-1,p}(\Omega) \times \mathring{\boldsymbol{W}}_0^{1,p'}(\Omega)} - \langle h, \eta \rangle_{L^p(\Omega) \times L^{p'}(\Omega)} .$$

Observe that for any pair $(\boldsymbol{f}', h') \in \boldsymbol{W}_0^{-1,p'}(\Omega) \times L^{p'}(\Omega)$ and for any $(\boldsymbol{\lambda}, \mu) \in \mathcal{N}_0^{p'}(\Omega)$ we have

$$\begin{aligned} |T(\boldsymbol{f}', h')| &= |\, \langle \boldsymbol{f}, \boldsymbol{w} + \boldsymbol{\lambda} \rangle_{\boldsymbol{W}_0^{-1,p}(\Omega) \times \mathring{\boldsymbol{W}}_0^{1,p'}(\Omega)} - \langle h, \eta + \mu \rangle_{L^p(\Omega) \times L^{p'}(\Omega)} \,| \\ &\leq ||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} ||\boldsymbol{w} + \boldsymbol{\lambda}||_{\boldsymbol{W}_0^{1,p'}(\Omega)} + ||h||_{L^p(\Omega)} ||\eta + \mu||_{L^{p'}(\Omega)} \end{aligned}$$

Using (III.92), we prove that

$$|T(\boldsymbol{f}', h')| \leq C(\boldsymbol{v})\big(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + ||h||_{L^p(\Omega)}\big)\big(||\boldsymbol{f}'||_{\boldsymbol{W}_0^{-1,p'}(\Omega)} + ||h'||_{L^{p'}(\Omega)}\big).$$

Thus the linear form T is continuous on the following space $\mathbf{W}_0^{-1,p'}(\Omega) \times L^{p'}(\Omega)$ and we deduce that there exists a unique $(\boldsymbol{u}, \pi) \in \mathring{\boldsymbol{W}}_0^{1,p}(\Omega) \times L^p(\Omega)$ such that

$$T(\boldsymbol{f}', h') = \big\langle \boldsymbol{u}, \boldsymbol{f}' \big\rangle_{\mathring{\boldsymbol{W}}_0^{1,p}(\Omega)\times \boldsymbol{W}_0^{-1,p'}(\Omega)} - \big\langle \pi, h' \big\rangle_{L^p(\Omega)\times L^{p'}(\Omega)},$$

with

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\Omega)} + ||\pi||_{L^p(\Omega)} \leq C(\boldsymbol{v})\big(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + ||h||_{L^p(\Omega)}\big). \tag{III.93}$$

By definition of T, it follows

$$\langle \boldsymbol{f}, \boldsymbol{w} \rangle_{\boldsymbol{W}_0^{-1,p}(\Omega)\times \mathring{\boldsymbol{W}}_0^{1,p'}(\Omega)} - \langle h, \eta \rangle_{L^p(\Omega)\times L^{p'}(\Omega)} = \big\langle \boldsymbol{u}, \boldsymbol{f}' \big\rangle_{\mathring{\boldsymbol{W}}_0^{1,p}(\Omega)\times \boldsymbol{W}_0^{-1,p'}(\Omega)} - \big\langle \pi, h' \big\rangle_{L^p(\Omega)\times L^{p'}(\Omega)},$$

which is the variational formulation (III.91).

□

Now, let prove the appropriate estimate for the Oseen problem (III.1) and we start by the case $1 < p < 2$:

Theorem 4.5 *Suppose that* $1 < p < 2$ *and let* $(\boldsymbol{u}, \pi) \in \mathring{\boldsymbol{W}}_0^{1,p}(\Omega) \times L^p(\Omega)$ *be the unique solution of the Oseen problem* (III.1) *given by Theorem 4.4 with the following data:*

$$\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega), \quad h \in L^p(\Omega), \quad \text{and} \quad \boldsymbol{g} = \mathbf{0}$$

and for any $(\boldsymbol{\lambda}, \mu) \in \mathcal{N}_0^{p'}(\Omega)$, *we have*

$$\langle \boldsymbol{f}, \boldsymbol{\lambda} \rangle_{\boldsymbol{W}_0^{-1,p}(\Omega)\times \mathring{\boldsymbol{W}}_0^{1,p'}(\Omega)} - \langle h, \mu \rangle_{L^p(\Omega)\times L^{p'}(\Omega)} = 0.$$

Then $(\boldsymbol{u}, \pi)$ *satisfies the following estimate :*

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\Omega)} + ||\pi||_{L^p(\Omega)} \leqslant C\big(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)}\big)^2\big(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + (1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)})||h||_{L^p(\Omega)}\big). \tag{III.94}$$

Proof. Since $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$, then it follows from Corollary 1.3 of [3] that $\boldsymbol{f} = \operatorname{div} \mathbf{F}$ with $\mathbf{F} \in \boldsymbol{L}^p(\Omega)$ and $||\,\mathbf{F}\,||_{\boldsymbol{L}^p(\Omega)} \leq C||\,\boldsymbol{f}\,||_{\boldsymbol{W}_0^{-1,p}(\Omega)}$. Let extend $\mathbf{F}$ and h by zero in Ω'. The extended functions, denoted by $\widetilde{\mathbf{F}}$ and $\widetilde{h}$ belong to $\boldsymbol{L}^p(\mathbb{R}^3)$ and set $\widetilde{\boldsymbol{f}} = \operatorname{div} \widetilde{\mathbf{F}}$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$. Let ϕ be a truncation function: $\phi \in \mathcal{D}(\mathbb{R}^3)$ such that $0 \leqslant \phi(t) \leqslant 1$ for any $t \in \mathbb{R}^3$ and

$$\phi(t) = \begin{cases} 1 & \text{if} \quad |\, t \,| \leqslant 1, \\ 0 & \text{if} \quad |\, t \,| \geqslant 2. \end{cases}$$

Let λ be a cut off function, defined on $\mathbb{R}^3$ by $\lambda(\mathbf{x}) = \phi(\dfrac{\mathbf{x}}{R})$ for any R sufficiently large so that $\overline{\Omega'} \subset B_R$. Set $\mu = 1 - \lambda$. Let Ω_R denote the intersection $\Omega \cap B_{2R}$. Now, let $(\boldsymbol{u}, \pi) \in \mathring{\boldsymbol{W}}_0^{1,p}(\Omega) \times L^p(\Omega)$ be the unique solution of the Oseen problem (III.1) given by Theorem 4.4 and let us extend $(\boldsymbol{u}, \pi)$ by zero in Ω'. Then, the extended distributions denoted by $(\widetilde{\boldsymbol{u}}, \widetilde{\pi})$ belongs to $\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ and let $\boldsymbol{w} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$ such as in Theorem 4.2. Then

$$
\begin{aligned}
||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\Omega)} + ||\pi||_{L^p(\Omega)} &= ||\mu\, \boldsymbol{u} + \lambda\, \boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\Omega)} + ||\mu\, \pi + \lambda\, \pi||_{L^p(\Omega)} \\
&\leq ||\mu\, \widetilde{\boldsymbol{u}}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\lambda\, \boldsymbol{u}||_{\boldsymbol{W}^{1,p}(\Omega_R)} + ||\mu\, \widetilde{\pi}||_{L^p(\mathbb{R}^3)} + ||\lambda\, \pi||_{L^p(\Omega_R)}
\end{aligned} \tag{III.95}
$$

After an easy calculation, we obtain that the pair $(\mu\widetilde{\boldsymbol{u}}, \mu\widetilde{\pi})$ satisfies the following equations in $\mathbb{R}^3$:

$$
-\Delta(\mu\, \widetilde{\boldsymbol{u}}) + \operatorname{div}(\boldsymbol{w} \otimes (\mu\, \widetilde{\boldsymbol{u}})) + \nabla\, (\mu\, \widetilde{\pi}) := \boldsymbol{f}_1 \quad \text{and} \quad \operatorname{div}(\mu\, \widetilde{\boldsymbol{u}}) := e_1 \quad \text{in} \quad \mathbb{R}^3,
$$

with

$$
\boldsymbol{f}_1 = \mu\widetilde{\boldsymbol{f}} + (\Delta\, \lambda)\widetilde{\boldsymbol{u}} - (\nabla\, \lambda)\widetilde{\pi} + 2\nabla\, \lambda \cdot \nabla\, \widetilde{\boldsymbol{u}} - (\boldsymbol{w} \cdot \nabla\, \lambda)\widetilde{\boldsymbol{u}} \quad \text{and} \quad e_1 = \mu\, \widetilde{h} - \nabla\, \lambda \cdot \widetilde{\boldsymbol{u}}\ .
$$

From Theorem 3.4, we have

$$
\begin{aligned}
&||\mu\, \widetilde{\boldsymbol{u}}||_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + ||\mu\, \widetilde{\pi}||_{L^p(\mathbb{R}^3)} \leq \\
&C(1 + ||\boldsymbol{w}||_{\boldsymbol{L}^3(\mathbb{R}^3)}) \Big(||\boldsymbol{f}_1||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + (1 + ||\boldsymbol{w}||_{\boldsymbol{L}^3(\mathbb{R}^3)}) ||e_1||_{L^p(\mathbb{R}^3)} \Big) \leq \\
&C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)}) \Big(||\boldsymbol{f}_1||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + (1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)}) ||e_1||_{L^p(\mathbb{R}^3)} \Big)\,.
\end{aligned} \tag{III.96}
$$

Now, let $\boldsymbol{\varphi} \in \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)$, then using Hölder we have

$$
\begin{aligned}
|\, \langle (\Delta\, \lambda)\widetilde{\boldsymbol{u}}, \boldsymbol{\varphi} \rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \times \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} | &= |\int_{\mathbb{R}^3} \boldsymbol{\varphi}(\Delta\, \lambda)\widetilde{\boldsymbol{u}}\, d\mathbf{x}| \\
&\leq C \int_{A_R(\mathbb{R}^3)} |\frac{1}{R^2} \boldsymbol{\varphi}(\Delta\, \phi)\widetilde{\boldsymbol{u}}|\, d\mathbf{x} \\
&\leq C\frac{(1 + 4R^2)}{R^2} \left(\int_{A_R(\mathbb{R}^3)} |\frac{\widetilde{\boldsymbol{u}}}{\rho}|^p\, d\mathbf{x}| \right)^{1/p} ||\boldsymbol{\varphi}||_{\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)}
\end{aligned}
$$

and

$$
\begin{aligned}
|\, \langle (\nabla\, \lambda)\widetilde{\pi}, \boldsymbol{\varphi} \rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \times \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} | &\leq \int_{A_R(\mathbb{R}^3)} |\frac{1}{R} \boldsymbol{\varphi}(\nabla\, \phi)\widetilde{\pi}|\, d\mathbf{x} \\
&\leq C\frac{(1 + 4R^2)^{1/2}}{R} \left(\int_{A_R(\mathbb{R}^3)} |\widetilde{\pi}|^p\, d\mathbf{x} \right)^{1/p} ||\boldsymbol{\varphi}||_{\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)},
\end{aligned}
$$

in addition, we prove that

$$\begin{aligned}
|\left\langle 2\nabla\lambda\cdot\nabla\,\widetilde{\boldsymbol{u}},\boldsymbol{\varphi}\right\rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)\times\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)}| &\leq 2\int_{A_R(\mathbb{R}^3)}|\frac{1}{R}\boldsymbol{\varphi}\nabla\,\phi\cdot\nabla\,\widetilde{\boldsymbol{u}}|\,d\mathbf{x}\\
&\leq C\frac{(1+4R^2)^{1/2}}{R}\left(\int_{A_R(\mathbb{R}^3)}|\nabla\,\widetilde{\boldsymbol{u}}|^p\,d\mathbf{x}\right)^{1/p}||\boldsymbol{\varphi}||_{\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)}
\end{aligned}$$

next, we have

$$\begin{aligned}
|\left\langle(\boldsymbol{w}\cdot\nabla\,\lambda)\widetilde{\boldsymbol{u}},\boldsymbol{\varphi}\right\rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)\times\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)}| &\leq \int_{A_R(\mathbb{R}^3)}|\frac{1}{R}\boldsymbol{\varphi}(\boldsymbol{w}\cdot\nabla\,\phi)\widetilde{\boldsymbol{u}}|\,d\mathbf{x}\\
&\leq C\frac{(1+4R^2)^{1/2}}{R}\left(\int_{A_R(\mathbb{R}^3)}|\boldsymbol{w}\otimes\widetilde{\boldsymbol{u}}|^p\,d\mathbf{x}\right)^{1/p}||\boldsymbol{\varphi}||_{\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)},
\end{aligned}$$

finally, we prove that

$$\begin{aligned}
|\left\langle\mu\widetilde{\boldsymbol{f}},\boldsymbol{\varphi}\right\rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)\times\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)}| &= |\left\langle\operatorname{div}\widetilde{\mathbf{F}},\mu\boldsymbol{\varphi}\right\rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)\times\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)}|\\
&= |\int_\Omega\mathbf{F}\nabla(\mu\boldsymbol{\varphi})\,d\mathbf{x}|\\
&\leq C||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)}||\mu\boldsymbol{\varphi}||_{\mathring{\boldsymbol{W}}_0^{1,p'}(\Omega)}\\
&\leq C||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)}||\nabla(\mu\boldsymbol{\varphi})||_{\boldsymbol{L}^{p'}(\Omega)}\\
&\leq C||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)}||\boldsymbol{\varphi}||_{\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)}.
\end{aligned}$$

Now, let $\varphi\in L^{p'}(\mathbb{R}^3)$, then using Hölder we have

$$|\int_{\mathbb{R}^3}(\mu\,\widetilde{h}-\nabla\,\lambda\cdot\widetilde{\boldsymbol{u}})\varphi\,d\,\mathbf{x}|\leq$$

$$C\left[||h||_{L^p(\Omega)}||\mu\,\varphi||_{L^{p'}(\Omega)}+\frac{(1+4R^2)^{1/2}}{R}\left(\int_{A_R(\mathbb{R}^3)}|\frac{\widetilde{\boldsymbol{u}}}{\rho}|^p\,d\mathbf{x}|\right)^{1/p}||\varphi||_{L^{p'}(\mathbb{R}^3)}\right]\leq$$

$$C\left[||h||_{L^p(\Omega)}+\frac{(1+4R^2)^{1/2}}{R}\left(\int_{A_R(\mathbb{R}^3)}|\frac{\widetilde{\boldsymbol{u}}}{\rho}|^p\,d\mathbf{x}|\right)^{1/p}\right]||\varphi||_{L^{p'}(\mathbb{R}^3)}.$$

Then, we deduce from the previous inequalities that

$$||\boldsymbol{f}_1||_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}+(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)})||e_1||_{L^p(\mathbb{R}^3)}\leq$$

$$
\begin{aligned}
&\leq C \Bigg[||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \frac{(1+4R^2)}{R^2} \left(\int_{A_R(\mathbb{R}^3)} |\nabla\, \widetilde{\boldsymbol{u}}|^p \, d\mathbf{x} \right)^{1/p} + \frac{(1+4R^2)}{R^2} \left(\int_{A_R(\mathbb{R}^3)} |\frac{\widetilde{\boldsymbol{u}}}{\rho}|^p \, d\mathbf{x} \right)^{1/p} \\
&+ \frac{(1+4R^2)^{1/2}}{R} \left\{ \left(\int_{A_R(\mathbb{R}^3)} |\widetilde{\pi}|^p \, d\mathbf{x} \right)^{1/p} + \left(\int_{A_R(\mathbb{R}^3)} |\boldsymbol{w} \otimes \widetilde{\boldsymbol{u}}|^p \, d\mathbf{x} \right)^{1/p} \right\} \\
&+ (1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)}) \left\{ \frac{(1+4R^2)^{1/2}}{R} \left(\int_{A_R(\mathbb{R}^3)} |\frac{\widetilde{\boldsymbol{u}}}{\rho}|^p \, d\mathbf{x} \right)^{1/p} + ||h||_{L^p(\Omega)} \right\} \Bigg] . \qquad \text{(III.97)}
\end{aligned}
$$

Similarly, the pair $(\lambda\, \boldsymbol{u}, \lambda\, \pi)$ satisfies the following equations in Ω_R:

$$
\begin{cases}
-\Delta(\lambda\, \boldsymbol{u}) + \operatorname{div}(\boldsymbol{v} \otimes (\lambda\, \boldsymbol{u})) + \nabla(\lambda\, \pi) := \boldsymbol{f}_2 \quad \text{and} \quad \operatorname{div}(\lambda\, \boldsymbol{u}) := e_2 \quad \text{in} \quad \Omega_R, \\
(\lambda\, \boldsymbol{u})|_{\partial B_{2R}} = \mathbf{0} \quad \text{and} \quad (\lambda\, \boldsymbol{u})|_\Gamma = \mathbf{0},
\end{cases}
$$

with

$$
\boldsymbol{f}_2 = \lambda\, \boldsymbol{f} + (\Delta\, \mu)\boldsymbol{u} - (\nabla\, \mu)\pi + 2\nabla\, \mu \cdot \nabla\, \boldsymbol{u} - (\boldsymbol{v} \cdot \nabla\, \mu)\boldsymbol{u} \quad \text{and} \quad e_2 = \lambda\, h - \nabla\, \mu \cdot \boldsymbol{u}\, .
$$

Using Theorem 15 of [13], we prove that

$$
||\lambda\, \boldsymbol{u}||_{\boldsymbol{W}^{1,p}(\Omega_R)} + ||\lambda\, \pi||_{L^p(\Omega_R)} \leq
$$

$$
C(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)})^2 \left(||\boldsymbol{f}_2||_{\boldsymbol{W}^{-1,p}(\Omega_R)} + (1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)})||e_2||_{L^p(\Omega_R)} \right) . \qquad \text{(III.98)}
$$

As in the beginning of the proof, we show that

$$
\begin{aligned}
||\boldsymbol{f}_2||_{\boldsymbol{W}^{-1,p}(\Omega_R)} \leq C \Bigg[&(1+\frac{1}{R})||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + \frac{(1+4R^2)^{1/2}}{R^2} \left(\int_{A_R(\mathbb{R}^3)} |\frac{\boldsymbol{u}}{\rho}|^p \, d\mathbf{x} \right)^{1/p} \\
&+ \frac{1}{R} \left\{ \left(\int_{A_R(\mathbb{R}^3)} |\pi|^p \, d\mathbf{x}| \right)^{1/p} + \left(\int_{A_R(\mathbb{R}^3)} |\nabla\, \boldsymbol{u}|^p \, d\mathbf{x} \right)^{1/p} + \left(\int_{A_R(\mathbb{R}^3)} |\boldsymbol{v} \otimes \boldsymbol{u}|^p \, d\mathbf{x} \right)^{1/p} \right\} \Bigg]
\end{aligned}
\qquad \text{(III.99)}
$$

and that

$$
||e_2||_{L^p(\Omega_R)} \leq C \left[\frac{(1+4R^2)^{1/2}}{R} \left(\int_{A_R(\mathbb{R}^3)} |\frac{\boldsymbol{u}}{\rho}|^p \, d\mathbf{x} \right)^{1/p} + ||h||_{L^p(\Omega)} \right] . \qquad \text{(III.100)}
$$

Using (III.95)-(III.100) and tending R to ∞, we prove thanks to dominated convergence theorem the estimate (III.94). □

Remark 4.2

Under the assumptions of Theorem 4.5 and supposing that $6/5 \leq p < 2$, the solution $(\boldsymbol{u}, \pi)$ satisfies the estimate :

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\Omega)} + ||\pi||_{L^p(\Omega)} \leqslant C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)})(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + (1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)})||h||_{L^p(\Omega)}).$$

Indeed, we shall use in the proof of Theorem 4.5 the following estimate

$$||\lambda\, \boldsymbol{u}||_{\boldsymbol{W}^{1,p}(\Omega_R)} + ||\lambda\, \pi||_{L^p(\Omega_R)} \leq C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)}) \left(||\boldsymbol{f}_2||_{\boldsymbol{W}^{-1,p}(\Omega_R)} + (1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)})||e_2||_{L^p(\Omega_R)}\right),$$

instead of (III.98), see Proposition 3 of [13].
Now, we study the nonhomogeneous boundary data *i.e* $\boldsymbol{g} \neq \boldsymbol{0}$ on Γ.

Corollary 4.2 *Suppose that* $1 < p < 2$. *Let* $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$, $h \in L^p(\Omega)$ *and* $\boldsymbol{g} \in \boldsymbol{W}^{1/p',p}(\partial\Omega)$ *such that for any* $(\boldsymbol{\lambda}, \mu) \in \mathcal{N}_0^{p'}(\Omega)$, *we have*

$$\langle \boldsymbol{f}, \boldsymbol{\lambda} \rangle_{\boldsymbol{W}_0^{-1,p}(\Omega) \times \mathring{\boldsymbol{W}}_0^{1,p'}(\Omega)} - \langle h, \mu \rangle_{L^p(\Omega) \times L^{p'}(\Omega)} + \langle \boldsymbol{g}, (\mu I - \nabla\, \boldsymbol{\lambda}) \cdot \boldsymbol{n} \rangle_\Gamma = 0. \tag{III.101}$$

Then the Oseen problem (III.1) *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ *such that*

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\Omega)} + ||\pi||_{L^p(\Omega)} \leqslant$$

$$C(1 + ||\, \boldsymbol{v}\, ||_{\boldsymbol{L}^3(\Omega)})^2 \left(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + (1 + ||\, \boldsymbol{v}\, ||_{\boldsymbol{L}^3(\Omega)})(||h||_{L^p(\Omega)} + ||\, \boldsymbol{g}\, ||_{\boldsymbol{W}^{1/p',p}(\partial\Omega)})\right). \tag{III.102}$$

Proof. Let $\boldsymbol{g} \in \boldsymbol{W}^{1/p',p}(\partial\Omega)$, then there exists $\boldsymbol{\chi} \in \boldsymbol{W}_0^{1,p}(\Omega)$ such that $\boldsymbol{\chi} = \boldsymbol{g}$ on Γ and

$$\|\boldsymbol{\chi}\|_{\boldsymbol{W}_0^{1,p}(\Omega)} \leq C\|\boldsymbol{g}\|_{\boldsymbol{W}^{1/p',p}(\partial\Omega)}. \tag{III.103}$$

Setting $\boldsymbol{u}' = \boldsymbol{u} - \boldsymbol{\chi}$, then Problem (III.1) is equivalent to the following problem: Find $(\boldsymbol{u}', q)$ such that

$$\begin{aligned} -\Delta \boldsymbol{u}' + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u}') + \nabla\pi &= \boldsymbol{f} + \Delta\boldsymbol{\chi} - \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{\chi}) && \text{in} \quad \Omega, \\ \operatorname{div} \boldsymbol{u}' &= h - \operatorname{div} \boldsymbol{\chi} && \text{in} \quad \Omega, \\ \boldsymbol{u}'|_{\partial\Omega} &= 0. \end{aligned}$$

Set $\boldsymbol{f}_{\boldsymbol{\chi}} = \boldsymbol{f} + \Delta\boldsymbol{\chi} - \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{\chi})$ and $h_{\boldsymbol{\chi}} = h - \operatorname{div} \boldsymbol{\chi}$. As $1 < p < 2$ then $\boldsymbol{\chi} \in \boldsymbol{L}^{p*}(\Omega)$ and $\boldsymbol{v} \otimes \boldsymbol{\chi} \in \boldsymbol{L}^p(\Omega)$. Thus $\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{\chi}) \in \boldsymbol{W}_0^{-1,p}(\Omega)$. Hence, $\boldsymbol{f}_{\boldsymbol{\chi}}$ belongs to $\boldsymbol{W}_0^{-1,p}(\Omega)$. It is clear that $(\boldsymbol{f}_{\boldsymbol{\chi}}, h_{\boldsymbol{\chi}})$ satisfies the compatibility condition (III.90). Then from Theorem 4.4 we know that this problem has a solution in $\mathring{\boldsymbol{W}}_0^{1,p}(\Omega) \times L^p(\Omega)$. In addition, using Theorem 4.5 we deduce that

$$||\boldsymbol{u}'||_{\boldsymbol{W}_0^{1,p}(\Omega)} + ||\pi||_{L^p(\Omega)} \leqslant C(1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)})^2(||\boldsymbol{f}_{\boldsymbol{\chi}}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + (1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)})||h_{\boldsymbol{\chi}}||_{L^p(\Omega)}).$$

It follows from (III.103) that

$$||\boldsymbol{f}_{\boldsymbol{\chi}}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + (1 + ||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)})||h_{\boldsymbol{\chi}}||_{L^p(\Omega)} \leq$$

$$C\left(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + (1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)})(||h||_{L^p(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1/p',p}(\partial\Omega)})\right). \tag{III.104}$$

Then (III.105) is a trivial consequence of the previous inequality. □

Remark 4.3

We suppose now that $p > 2$. As in Theorem 3.5, using a dual argument with the estimate (III.94) of Theorem 4.5, we prove that if $\boldsymbol{g} = \boldsymbol{0}$,

$$\inf_{(\boldsymbol{\lambda},\mu)\in\mathcal{N}_0^p(\Omega)} ||\boldsymbol{u}+\boldsymbol{\lambda}||_{\boldsymbol{W}_0^{1,p}(\Omega)} + ||\pi+\mu||_{L^p(\Omega)} \leqslant C\big(1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)}\big)^3\big(||\,\boldsymbol{f}\,||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + ||\,h\,||_{L^p(\Omega)}\big).$$

As in Corollary 4.2, when $\boldsymbol{g} \in \boldsymbol{W}^{1/p',p}(\partial\Omega)$, we prove that

$$\inf_{(\boldsymbol{\lambda},\mu)\in\mathcal{N}_0^p(\Omega)} ||\boldsymbol{u}+\boldsymbol{\lambda}||_{\boldsymbol{W}_0^{1,p}(\Omega)} + ||\pi+\mu||_{L^p(\Omega)} \leqslant C\big(1+||\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)}\big)^3 \times$$
$$\big(||\,\boldsymbol{f}\,||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + ||h||_{L^p(\Omega)} + (1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)})||\,\boldsymbol{g}\,||_{\boldsymbol{W}^{1/p',p}(\partial\Omega)}\big).$$

The next theorem summarizes the results of the existence and uniqueness of generalized solutions of Problem (III.1) when $1 < p < \infty$:

Theorem 4.6 *Let Ω be an exterior domain with $C^{1,1}$ boundary. If $p \geq 2$, for any $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$, $h \in L^p(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{W}^{1/p',p}(\Gamma)$, Problem* (III.1) *has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)/\mathcal{N}_0^p(\Omega)$ and there exists a constant C, independent of $\boldsymbol{u}$, q, $\boldsymbol{f}$, h, $\boldsymbol{g}$ and $\boldsymbol{v}$, such that*

$$\inf_{(\boldsymbol{\lambda},\mu)\in\mathcal{N}_0^p(\Omega)} ||\boldsymbol{u}+\boldsymbol{\lambda}||_{\boldsymbol{W}_0^{1,p}(\Omega)} + ||\pi+\mu||_{L^p(\Omega)} \leq C\big(1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)}\big)^3$$
$$\times\big(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + ||\,h\,||_{L^p(\Omega)} + (1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)})||\,\boldsymbol{g}\,||_{\boldsymbol{W}^{1/p',p}(\partial\Omega)}\big). \tag{III.105}$$

If $1 < p < 2$, for any $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\Omega)$, $h \in L^p(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{W}^{1/p',p}(\partial\Omega)$ that satisfy the necessary compatibility condition (III.101), *Problem* (III.1) *has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}_0^{1,p}(\Omega) \times L^p(\Omega)$ and there exists a constant C, independent of $\boldsymbol{u}$, q, $\boldsymbol{f}$, h, $\boldsymbol{g}$ and $\boldsymbol{v}$, such that*

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p}(\Omega)} + ||\pi||_{L^p(\Omega)} \leq$$

$$C\big(1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)}\big)^2\big(||\boldsymbol{f}||_{\boldsymbol{W}_0^{-1,p}(\Omega)} + (1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)})(||\,h\,||_{L^p(\Omega)} + ||\,\boldsymbol{g}\,||_{\boldsymbol{W}^{1/p',p}(\partial\Omega)})\big). \tag{III.106}$$

4.3 Strong solutions in $\boldsymbol{W}_0^{2,p}(\Omega)$ and in $\boldsymbol{W}_1^{2,p}(\Omega)$

We begin by proving the existence and uniqueness of strong solution in $\boldsymbol{W}_0^{2,p}(\Omega)$ for $1 < p < 3$ in the following sense.

Theorem 4.7 *For* $1 < p < 3$, *let* $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $h \in W_0^{1,p}(\Omega)$ *and* $\boldsymbol{g} \in \boldsymbol{W}^{2-1/p,p}(\Gamma)$. *Then the Oseen problem* (III.1) *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega)/\mathcal{N}_0^{p*}(\Omega)$ *such that*

$$\inf_{(\boldsymbol{\lambda},\mu)\in\mathcal{N}_0^{p*}(\Omega)} \|\boldsymbol{u} + \boldsymbol{\lambda}\|_{\boldsymbol{W}_0^{2,p}(\Omega)} + \|\pi + \mu\|_{W_0^{1,p}(\Omega)} \leq$$

$$C(1 + \|\,\boldsymbol{v}\,\|_{\boldsymbol{L}^3(\Omega)})^4 \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|h\|_{W_0^{1,p}(\Omega)} + (1 + \|\,\boldsymbol{v}\,\|_{\boldsymbol{L}^3(\Omega)})\|\boldsymbol{g}\|_{\boldsymbol{W}^{2-1/p,p}(\Gamma)}\right). \tag{III.107}$$

Proof. For all $1 < p < 3$, Sobolev embedding holds *i.e* $\boldsymbol{L}^p(\Omega) \hookrightarrow \boldsymbol{W}_0^{-1,p*}(\Omega)$. Observe that $h \in W_0^{1,p}(\Omega) \hookrightarrow L^{p*}(\Omega)$, $\boldsymbol{g} \in \boldsymbol{W}^{2-1/p,p}(\Gamma) \hookrightarrow \boldsymbol{W}^{1-1/p*,p*}(\Gamma)$ and $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p*}(\Omega)$. Using Theorem 4.6 (there is no compatibility condition since $p* > 3/2$ *i.e* $(p*)' < 3$), we prove that there exists a solution

$$(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p*}(\Omega) \times L^{p*}(\Omega)$$

for the Oseen problem (III.1) with the following estimate

$$||\boldsymbol{u}||_{\boldsymbol{W}_0^{1,p*}(\Omega)} + ||\pi||_{L^{p*}(\Omega)} \leq C(1 + ||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)})^3 \times$$
$$\Big((||\,\boldsymbol{f}\,||_{\boldsymbol{W}_0^{-1,p*}(\Omega)} + ||\,h\,||_{L^{p*}(\Omega)}) + (1 + ||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)})||\,\boldsymbol{g}\,||_{\boldsymbol{W}^{1-1/p*,p*}(\partial\Omega)}\Big). \tag{III.108}$$

Since $(\boldsymbol{v} \cdot \nabla)\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$, we can apply the Stokes regularity theory see [3] to deduce the existence of $(\boldsymbol{z}, \eta) \in \boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega)$ verifying:

$$-\Delta \boldsymbol{z} + \nabla \eta = \boldsymbol{f} - \boldsymbol{v} \cdot \nabla \boldsymbol{u} \quad \text{and} \quad \text{div}\boldsymbol{z} = h \quad \text{in} \quad \Omega, \quad \boldsymbol{z} = \boldsymbol{g} \quad \text{on } \Gamma.$$

Moreover, the estimate holds

$$\|\boldsymbol{z}\|_{\boldsymbol{W}_0^{2,p}(\Omega)} + \|\eta\|_{W_0^{1,p}(\Omega)} \leq$$

$$C\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{v}\|_{\boldsymbol{L}^3(\Omega)}\|\nabla\,\boldsymbol{u}\|_{\boldsymbol{L}^{p*}(\Omega)} + \|h\|_{W_0^{1,p}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{2-1/p,p}(\Gamma)}\right), \tag{III.109}$$

with C denoting a constant only dependent on p. Let $\boldsymbol{w} = \boldsymbol{z} - \boldsymbol{u}$ and $\theta = \eta - \pi$, then we have $(\boldsymbol{w}, \theta)$ belongs to $\mathcal{S}_0^{p*}(\Omega)$. Therefore, if $1 < p < 3/2$ *i.e* $3/2 < p* < 3$, we deduce from [3] that $\mathcal{S}_0^{p*} = (\boldsymbol{0}, 0)$ and thus $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega)$ and we deduce from (III.108) and (III.109) that:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{2,p}(\Omega)} + \|\pi\|_{W_0^{1,p}(\Omega)} \leq$$

$$C(1 + \|\,\boldsymbol{v}\,\|_{\boldsymbol{L}^3(\Omega)})^4 \left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|h\|_{W_0^{1,p}(\Omega)} + (1 + \|\,\boldsymbol{v}\,\|_{\boldsymbol{L}^3(\Omega)})\|\boldsymbol{g}\|_{\boldsymbol{W}^{2-1/p,p}(\Gamma)}\right). \tag{III.110}$$

If $p \geq 3/2$ *i.e* $p* \geq 3$, we deduce from the Stokes regularity theory see [3], that $(\boldsymbol{w}, \theta)$ belongs to $\boldsymbol{W}_1^{2,p}(\Omega) \times W_1^{1,p}(\Omega) \subset \boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega)$ and thus $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega)$. Now, using the following embeddings $\boldsymbol{W}_0^{2,p}(\Omega) \hookrightarrow \boldsymbol{W}_0^{1,p*}(\Omega)$ and $W_0^{1,p}(\Omega) \hookrightarrow L^{p*}(\Omega)$ and using (III.108), (III.109) and (III.110), we deduce that

$$\|\boldsymbol{w}\|_{\boldsymbol{W}_0^{1,p*}(\Omega)} + \|\theta\|_{L^{p*}(\Omega)} \leq$$

$$C(1+\|\,\boldsymbol{v}\,\|_{\boldsymbol{L}^3(\Omega)})^4\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|h\|_{W_0^{1,p}(\Omega)} + (1+\|\,\boldsymbol{v}\,\|_{\boldsymbol{L}^3(\Omega)})\|\boldsymbol{g}\|_{\boldsymbol{W}^{2-1/p,p}(\Gamma)}\right).$$

Observe that in the finite dimensional case, all norms are equivalent so we have

$$\|\boldsymbol{w}\|_{\boldsymbol{W}_0^{2,p}(\Omega)} + \|\theta\|_{W_0^{1,p}(\Omega)}) \leq$$

$$C(1+\|\,\boldsymbol{v}\,\|_{\boldsymbol{L}^3(\Omega)})^4\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|h\|_{W_0^{1,p}(\Omega)} + (1+\|\,\boldsymbol{v}\,\|_{\boldsymbol{L}^3(\Omega)})\|\boldsymbol{g}\|_{\boldsymbol{W}^{2-1/p,p}(\Gamma)}\right)$$

and thus we obtain (III.110). The uniqueness of the solution $(\boldsymbol{u},\pi)$ follows from $\boldsymbol{W}_0^{2,p}(\Omega)\times W_0^{1,p}(\Omega) \hookrightarrow \boldsymbol{W}_0^{1,p*}(\Omega)\times L^{p*}(\Omega)$ and also in $\boldsymbol{W}_0^{1,p*}(\Omega)\times L^{p*}(\Omega)$ the solution is unique up to an element of $\mathcal{N}_0^{p*}(\Omega)$. □

Theorem 4.8 *Suppose that* $1<p<3$ *and* $p\neq 3/2$. *Let* $\boldsymbol{f}\in \boldsymbol{W}_1^{0,p}(\Omega)$, $h\in W_1^{1,p}(\Omega)$, $\boldsymbol{g}\in \boldsymbol{W}^{2-1/p,p}(\Gamma)$ *that satisfy the compatibility condition* (III.101) *if* $p<2$. *Then the Oseen problem* (III.1) *has a unique solution* $(\boldsymbol{u},\pi)\in \boldsymbol{W}_1^{2,p}(\Omega)\times W_1^{1,p}(\Omega)/\mathcal{N}_0^p(\Omega)$ *such that*

$$\begin{aligned}\inf_{(\boldsymbol{\xi},\eta)\in\mathcal{N}_0^p(\Omega)} ||\boldsymbol{u}+\boldsymbol{\xi}||_{\boldsymbol{W}_1^{2,p}(\Omega)} + ||\pi+\eta||_{W_1^{1,p}(\Omega)} &\leq C(1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)})^6\\ \times\Big(||\boldsymbol{f}||_{\boldsymbol{W}_1^{0,p}(\Omega)} + (1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)})(||\,h\,||_{W_1^{1,p}(\Omega)} &+ ||\,\boldsymbol{g}\,||_{\boldsymbol{W}^{1/p',p}(\partial\Omega)})\Big).\end{aligned} \tag{III.111}$$

Proof. ***i) Regularity***:
Since the following embeddings hold $W_1^{1,p}(\Omega)\hookrightarrow L^p(\Omega)$, $\boldsymbol{W}^{2-1/p,p}(\Gamma)\hookrightarrow \boldsymbol{W}^{1/p',p}(\Gamma)$,resp. for $p\neq 3/2$ we have $\boldsymbol{W}_1^{0,p}(\Omega)\hookrightarrow \boldsymbol{W}_0^{-1,p}(\Omega)$, according to Theorem 4.6 it follows the existence of a unique solution $(\boldsymbol{u},\pi)\in \boldsymbol{W}_0^{1,p}(\Omega)\times L^p(\Omega)$ to the Oseen problem (III.1) if $p<2$ and if $p>2$ it is unique up to an element of $\mathcal{N}_0^p(\Omega)$. Moreover the following estimate is satisfied

$$\begin{aligned}\inf_{(\boldsymbol{\xi},\eta)\in\mathcal{N}_0^p(\Omega)} ||\boldsymbol{u}+\boldsymbol{\xi}||_{\boldsymbol{W}_0^{1,p}(\Omega)} + ||\pi+\eta||_{L^p(\Omega)} &\leq C(1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)})^3\\ \times(||\,\boldsymbol{f}\,||_{\boldsymbol{W}_1^{0,p}(\Omega)} + ||\,h\,||_{W_1^{1,p}(\Omega)} &+ (1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)})||\,\boldsymbol{g}\,||_{\boldsymbol{W}^{2-1/p,p}(\partial\Omega)}).\end{aligned} \tag{III.112}$$

The rest of the proof is similar to that of Lemma 4.1, we introduce the same partition of unity as in Lemma 4.1. With the same notation, we can write

$$\boldsymbol{u} = \lambda\,\boldsymbol{u} + \mu\,\boldsymbol{u}, \qquad \pi = \lambda\,\pi + \mu\,\pi.$$

Let us extend $(\mu\,\boldsymbol{u},\mu\,\pi)$ by zero in Ω'. Then, the extended distributions denoted by $(\widetilde{\mu\,\boldsymbol{u}},\widetilde{\mu\,\pi})$ belongs to $\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)\times L^p(\mathbb{R}^3)$ and let $\boldsymbol{w}\in \boldsymbol{L}_\sigma^3(\mathbb{R}^3)$ such as in Theorem 4.2. A quick computation in $\mathcal{D}'(\mathbb{R}^3)$, shows that the pair $(\widetilde{\mu\,\boldsymbol{u}}, \widetilde{\mu\,\pi})$ satisfies the following equations:

$$-\Delta(\widetilde{\mu\,\boldsymbol{u}}) + \boldsymbol{w}\cdot\nabla(\widetilde{\mu\,\boldsymbol{u}}) + \nabla\,(\widetilde{\mu\,\pi}) := \boldsymbol{f}_1 \quad \text{and} \quad \operatorname{div}(\widetilde{\mu\,\boldsymbol{u}}) := e_1 \quad \text{in} \quad \mathbb{R}^3,$$

with

$$\boldsymbol{f}_1 = \mu\,\widetilde{\boldsymbol{f}} + (\Delta\,\lambda)\widetilde{\boldsymbol{u}} - (\nabla\,\lambda)\widetilde{\pi} + 2\nabla\,\lambda\cdot\nabla\,\widetilde{\boldsymbol{u}} - (\boldsymbol{w}\cdot\nabla\,\lambda)\widetilde{\boldsymbol{u}} \quad \text{and} \quad e_1 = \widetilde{\mu\,h} - \nabla\,\lambda\cdot\widetilde{\boldsymbol{u}}\;.$$

Moreover, owing to the supports of μ and λ, $(\boldsymbol{f}_1, e_1)$ belongs to $\boldsymbol{W}^{0,p}_1(\mathbb{R}^3)\times W^{1,p}_1(\mathbb{R}^3)$. It is clear that $\boldsymbol{f}_1$ satisfies (III.71) and thus it follows from 3.7, that there exists a unique $(\boldsymbol{z},\theta) \in \boldsymbol{W}^{2,p}_1(\mathbb{R}^3)\times W^{1,p}_1(\mathbb{R}^3)$ such that

$$-\Delta\,\boldsymbol{z} + \boldsymbol{w}\cdot\nabla\,\boldsymbol{z} + \nabla\,\theta = \boldsymbol{f}_1 \quad \text{and} \quad \operatorname{div}\boldsymbol{z} = e_1 \quad \text{in} \quad \mathbb{R}^3.$$

and thus,

$$-\Delta(\widetilde{\mu\,\boldsymbol{u}} - \boldsymbol{z}) + \boldsymbol{w}\cdot\nabla(\widetilde{\mu\,\boldsymbol{u}} - \boldsymbol{z}) + \nabla(\widetilde{\mu\,\pi} - \theta) = \boldsymbol{0} \quad \text{and} \quad \operatorname{div}(\widetilde{\mu\,\boldsymbol{u}} - \boldsymbol{z}) = 0 \quad \text{in} \quad \mathbb{R}^3,$$

with $(\widetilde{\mu\,\pi} - \theta) \in L^p(\mathbb{R}^3)$ and $(\widetilde{\mu\,\boldsymbol{u}} - \boldsymbol{z}) \in \boldsymbol{W}^{1,p}_0(\mathbb{R}^3)$. Then, using the argument of uniqueness in section 4, we deduce that $\widetilde{\mu\,\boldsymbol{u}} - \boldsymbol{z} = \boldsymbol{0}$ and $\widetilde{\mu\,\pi} - \theta = 0$. Consequently, $(\widetilde{\mu\,\boldsymbol{u}}, \widetilde{\mu\,\pi})$ belongs to $\boldsymbol{W}^{2,p}_1(\mathbb{R}^3)\times W^{1,p}_1(\mathbb{R}^3)$. In particular, we have $\widetilde{\mu\,\boldsymbol{u}} = \boldsymbol{u}$ and $\widetilde{\mu\,\pi} = \pi$ outside B_{R_0+1}, so the restriction of $\boldsymbol{u}$ to ∂B_{R_0+1} belongs to $\boldsymbol{W}^{2-1/p,p}(\partial\,B_{R_0+1})$. Therefore, $(\boldsymbol{u},\pi)$ satisfies:

$$-\Delta\,\boldsymbol{u} + \boldsymbol{v}\cdot\nabla\,\boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = h \quad \text{in} \quad \Omega_{R_0+1}, \quad \boldsymbol{u}|_{\partial B_{R_0+1}} = \widetilde{\mu\,\boldsymbol{u}} \quad \text{and} \quad \boldsymbol{u}|_\Gamma = \boldsymbol{g}. \tag{III.113}$$

Observe that for any $\varphi \in W^{1,p'}(\Omega_{R_0+1})$ we have

$$\int_{\Omega_{R_0+1}} \boldsymbol{u}\cdot\nabla\,\varphi\,d\mathbf{x} = -\int_{\Omega_{R_0+1}} \varphi\operatorname{div}\boldsymbol{u}\,d\mathbf{x} + \int_{\partial\,\Omega_{R_0+1}} \varphi\boldsymbol{u}\cdot\boldsymbol{n}\,d\mathbf{x}.$$

In particular, for $\varphi = 1$, we have

$$\int_{\Omega_{R_0+1}} h(\mathbf{x})\,d\mathbf{x} = \int_{\partial\,\Omega_{R_0+1}} \boldsymbol{u}\cdot\boldsymbol{n}\,d\mathbf{x} = \int_{\partial\,B_{R_0+1}} \boldsymbol{u}\cdot\boldsymbol{n}\,d\mathbf{x} + \int_\Gamma \boldsymbol{g}\cdot\boldsymbol{n}\,d\mathbf{x}. \tag{III.114}$$

and thus, according to Theorem 14 and Corollary 7 of [13], this problem has a unique $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{2,p}(\Omega_{R_0+1})\times W^{1,p}(\Omega_{R_0+1})$. This implies that $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{2,p}_1(\Omega)\times W^{1,p}_1(\Omega)$. The uniqueness of the solution $(\boldsymbol{u},\pi)$ follows from this inclusion $\boldsymbol{W}^{2,p}_1(\Omega)\times W^{1,p}_1(\Omega) \subset \boldsymbol{W}^{1,p}_0(\Omega)\times L^p(\Omega)$ which holds for $p \neq 3/2$.

ii) Apriori Estimate:

First observe that each solution $(\boldsymbol{\xi},\eta) \in \boldsymbol{W}^{1,p}_0(\Omega)\times L^p(\Omega)$ to the Oseen problem (III.1) with null data obviously belongs to $\boldsymbol{W}^{2,p}_1(\Omega)\times W^{1,p}_1(\Omega)$. In fact the proof is very similar to that of Lemma 3.11 of [8]. Conversly, we have $\boldsymbol{W}^{2,p}_1(\Omega)\times W^{1,p}_1(\Omega) \subset \boldsymbol{W}^{1,p}_0(\Omega)\times L^p(\Omega)$. Now, considering the first step of regularity, it follows that the continuous operator

$$\mathcal{O}' : \boldsymbol{W}^{2,p}_1(\Omega)\times W^{1,p}_1(\Omega)/\mathcal{N}^p_0(\Omega) \longrightarrow \boldsymbol{W}^{0,p}_1(\Omega)\times W^{1,p}_1(\Omega)\times \boldsymbol{W}^{2-1/p,p}(\Gamma)$$

defined by : $\mathcal{O}'(\boldsymbol{u},\pi) = \big(-\Delta\,\boldsymbol{u} + \boldsymbol{v}\cdot\nabla\,\boldsymbol{u} + \nabla\,\pi, \operatorname{div}\boldsymbol{u}, \boldsymbol{u}|_\Gamma\big)$ is an isomorphism. Thus there exists a constant $C(\boldsymbol{v})$ depending on $\boldsymbol{v}\in\boldsymbol{L}^3_\sigma(\Omega)$, Ω and p such that

$$\inf_{(\boldsymbol{\lambda},\mu)\in\mathcal{N}_0^p(\Omega)} ||\boldsymbol{u}+\boldsymbol{\lambda}||_{\boldsymbol{W}^{2,p}_1(\Omega)} + ||\pi+\mu||_{W^{1,p}_1(\Omega)} \leq C(\boldsymbol{v})\big(||\boldsymbol{f}||_{\boldsymbol{W}^{0,p}_1(\Omega)} + ||h||_{W^{1,p}_1(\Omega)} + ||\boldsymbol{g}||_{\boldsymbol{W}^{2-1/p,p}(\Gamma)}\big). \tag{III.115}$$

Proceeding then as in Theorem 4.5 and Corollary 4.2, we can characterize the constant $C(\boldsymbol{v})$ and we obtain (III.111).

□

Remark 4.4

As in the case of the Oseen problem in $\mathbb{R}^3$, for $p\geq 3$ and $\alpha=0$ or $\alpha=1$, the hypothesis of $\boldsymbol{f}\in\boldsymbol{W}^{0,p}_\alpha(\Omega)$, $h\in W^{1,p}_\alpha(\Omega)$, $\boldsymbol{g}\in\boldsymbol{W}^{2-1/p,p}(\Gamma)$ and $\boldsymbol{v}\in\boldsymbol{H}^3_\sigma(\Omega)$ is not sufficient to ensure the existence of strong solutions for problem (III.1) in $\boldsymbol{W}^{2,p}_\alpha(\Omega)\times W^{1,p}_\alpha(\Omega)$.

4.4 Very weak solutions in $\boldsymbol{L}^p(\Omega)$ and in $\boldsymbol{W}^{0,p}_{-1}(\Omega)$

In this section, we are interested in the existence and the uniqueness of very weak solutions for the Oseen problem (III.1). We recall some density results and Green formulas proved in chapter II:
Let introduce the following space:

$$\boldsymbol{X}^\ell_{r,p}(\Omega) = \Big\{\boldsymbol{\varphi}\in\mathring{\boldsymbol{W}}^{1,r}_\ell(\Omega);\ \operatorname{div}\boldsymbol{\varphi}\in\mathring{W}^{1,p}_\ell(\Omega)\Big\}.$$

According to Poincaré-type inequality (see [7]), this space can be equipped with the following norm:

$$\|\,\boldsymbol{\varphi}\,\|_{\boldsymbol{X}^\ell_{r,p}(\Omega)} = \sum_{1\leq i,j\leq 3}\|\,\frac{\partial\varphi_i}{\partial x_j}\,\|_{W^{0,r}_\ell(\Omega)} + \|\,\operatorname{div}\boldsymbol{\varphi}\,\|_{W^{1,p}_\ell(\Omega)}\,.$$

Note that if $\boldsymbol{f}\in(\boldsymbol{X}^\ell_{r,p}(\Omega))'$ with $\ell=1$ or $\ell=0$ then there exist $\mathbb{F}_0=(f_{ij})_{1\leq i,j\leq 3}\in\boldsymbol{W}^{0,r'}_{-\ell}(\Omega)$ and $f_1\in W^{-1,p'}_{-\ell}(\Omega)$ such that:

$$\boldsymbol{f} = \operatorname{div}\mathbb{F}_0 + \nabla f_1. \tag{III.116}$$

Moreover, we can define

$$\|\,\boldsymbol{f}\,\|_{[\boldsymbol{X}^\ell_{r,p}(\Omega)]'} = \max\Big\{\|\,f_{ij}\,\|_{W^{0,r'}_{-\ell}(\Omega)}, 1\leq i,j\leq 3, \|\,f_1\,\|_{W^{-1,p'}_{-\ell}(\Omega)}\Big\}.$$

The first result is given by the following lemma:

Lemma 4.3 *Suppose that* $0\leq\frac{1}{r}-\frac{1}{p}\leq\frac{1}{3}$, *then*

i) For all $q\in W^{-1,p}_{-1}(\Omega)$ *and* $\boldsymbol{\varphi}\in\boldsymbol{X}^1_{r',p'}(\Omega)$, *we have*

$$\langle\nabla q,\boldsymbol{\varphi}\rangle_{[\boldsymbol{X}^1_{r',p'}(\Omega)]'\times\boldsymbol{X}^1_{r',p'}(\Omega)} = -\,\langle q,\operatorname{div}\boldsymbol{\varphi}\rangle_{W^{-1,p}_{-1}(\Omega)\times\mathring{W}^{1,p'}_1(\Omega)}\,. \tag{III.117}$$

ii) If in addition $p' \neq 3$, then for all $q \in W_0^{-1,p}(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{X}^0_{r',p'}(\Omega)$, we have

$$\langle \nabla q, \boldsymbol{\varphi} \rangle_{[\boldsymbol{X}^0_{r',p'}(\Omega)]' \times \boldsymbol{X}^0_{r',p'}(\Omega)} = - \langle q, \operatorname{div} \boldsymbol{\varphi} \rangle_{W_0^{-1,p}(\Omega) \times \mathring{W}_0^{1,p'}(\Omega)} . \tag{III.118}$$

Giving a meaning to the trace of a very weak solution of the Oseen problem is not trivial task. We need to introduce appropriate spaces. First, we consider the space:

$$\boldsymbol{Y}_{p',\ell}(\Omega) = \left\{ \boldsymbol{\psi} \in \boldsymbol{W}^{2,p'}_{\ell}(\Omega),\ \boldsymbol{\psi}|_{\Gamma} = 0,\ \operatorname{div} \boldsymbol{\psi}|_{\Gamma} = 0 \right\},$$

that can also be described (see [8]) as:

$$\boldsymbol{Y}_{p',\ell}(\Omega) = \left\{ \boldsymbol{\psi} \in \boldsymbol{W}^{2,p'}_{\ell}(\Omega), \boldsymbol{\psi}|_{\Gamma} = \mathbf{0}, \frac{\partial \boldsymbol{\psi}}{\partial \mathbf{n}} \cdot \boldsymbol{n}|_{\Gamma} = 0 \right\}. \tag{III.119}$$

Note that if $\boldsymbol{\psi} \in \boldsymbol{Y}_{p',\ell}(\Omega)$, then $\operatorname{div} \boldsymbol{\psi} \in \boldsymbol{W}^{1,p'}_{\ell}(\Omega)$ and the range space of the normal derivative $\gamma_1 : \boldsymbol{Y}_{p',\ell}(\Omega) \longrightarrow \boldsymbol{W}^{1/p,p'}(\Gamma)$ is

$$\boldsymbol{Z}_{p'}(\Gamma) = \left\{ \boldsymbol{z} \in \boldsymbol{W}^{1/p,p'}(\Gamma);\ \ \boldsymbol{z} \cdot \boldsymbol{n} = 0 \right\}.$$

Secondly, we shall use the space:

$$\boldsymbol{T}^{\ell}_{r,p}(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{W}^{0,p}_{-\ell}(\Omega);\ \Delta \boldsymbol{v} \in [\boldsymbol{X}^{\ell}_{r',p'}(\Omega)]' \right\},$$

equipped with the norm:

$$\| \boldsymbol{v} \|_{\boldsymbol{T}^{\ell}_{r,p}(\Omega)} = \| \boldsymbol{v} \|_{\boldsymbol{W}^{0,p}_{-\ell}(\Omega)} + \| \Delta \boldsymbol{v} \|_{[\boldsymbol{X}^{\ell}_{r',p'}(\Omega)]'} .$$

We also introduce the following space:

$$\boldsymbol{H}^{r}_{p,\ell}(\operatorname{div}, \Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{W}^{0,p}_{\ell-1}(\Omega); \operatorname{div} \boldsymbol{v} \in W^{0,r}_{\ell-1}(\Omega) \right\}.$$

This space is equipped with the graph norm. Moreover, we have the following result (see [8] for the proof):

Lemma 4.4 *Let $\frac{3}{2} < p < \infty$ and $\frac{1}{p} + \frac{1}{3} = \frac{1}{r}$. Then the mapping $\gamma_\tau : \boldsymbol{v} \longrightarrow \boldsymbol{v}_\tau|_{\Gamma}$ on the space $\mathfrak{D}(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping, still denoted by γ_τ, from $\boldsymbol{T}^{\ell}_{r,p}(\Omega)$ into $\boldsymbol{W}^{-1/p,p}(\Gamma)$ for $\ell = 0$ and if $p \neq 3$ for $\ell = 1$ and we have the Green formula: for any $\boldsymbol{v} \in \boldsymbol{T}^{\ell}_{r,p}(\Omega)$ and $\boldsymbol{\psi} \in \boldsymbol{Y}_{p',\ell}(\Omega)$,*

$$\langle \Delta \boldsymbol{v}, \boldsymbol{\psi} \rangle_{[\boldsymbol{X}^{\ell}_{r',p'}(\Omega)]' \times \boldsymbol{X}^{\ell}_{r',p'}(\Omega)} = \int_{\Omega} \boldsymbol{v} \cdot \Delta \boldsymbol{\psi} d\boldsymbol{x} - \left\langle \boldsymbol{v}_\tau, \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{n}} \right\rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma) \times \boldsymbol{W}^{1/p,p'}(\Gamma)} . \tag{III.120}$$

Finally, we have

Lemma 4.5 *Let Ω be a Lipschitz open set in $\mathbb{R}^3$. Suppose that $0 \leqslant \frac{1}{r} - \frac{1}{p} \leqslant \frac{1}{3}$ and $\ell = 0$ or $\ell = 1$. Then* *i) The space* $\mathcal{D}(\overline{\Omega})$ *is dense in* $\boldsymbol{H}^r_{p,\ell}(\mathrm{div},\Omega)$.

ii) The mapping $\gamma_n : \boldsymbol{v} \longrightarrow \boldsymbol{v}\cdot\boldsymbol{n}|_\Gamma$ *on the space* $\mathcal{D}(\overline{\Omega})$ *can be extended by continuity to a linear and continuous mapping, still denoted by* γ_n*, from* $\boldsymbol{H}^r_{p,\ell}(\mathrm{div}\,,\Omega)$ *into* $\boldsymbol{W}^{-1/p,p}(\Gamma)$*. If in addition* $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$ *and* $\frac{3}{2} < p < \infty$*, we have the following Green formula: for any* $\boldsymbol{v} \in \boldsymbol{H}^r_{p,\ell}(\mathrm{div}\,,\Omega)$ *and* $\varphi \in W^{1,p'}_{1-\ell}(\Omega)$*,*

$$\int_\Omega \boldsymbol{v}\cdot\nabla\varphi\, d\boldsymbol{x} + \int_\Omega \varphi\,\mathrm{div}\,\boldsymbol{v}\, d\boldsymbol{x} = \langle \boldsymbol{v}\cdot\boldsymbol{n}, \varphi\rangle_{W^{-1/p,p}(\Gamma)\times W^{1/p,p'}(\Gamma)}\,. \tag{III.121}$$

4.4.1 Very weak solutions in $\boldsymbol{L}^p(\Omega)$.

To begin with we introduce the definition of very weak solution.

Let

$$\boldsymbol{f} \in [\boldsymbol{X}^0_{r',p'}(\Omega)]', h \in L^r(\Omega), \text{ and } \boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}(\Gamma), \tag{III.122}$$

with

$$\frac{3}{2} < p < \infty \quad\text{and}\quad \frac{1}{p} + \frac{1}{3} = \frac{1}{r}, \tag{A$_1$}$$

yielding $1 < r < 3$.

Definition 4.1 ***(Very weak solution for the Oseen problem)*** *We suppose that* r *and* p *satisfy* (A$_1$) *and let* $\boldsymbol{f}$*,* h *and* $\boldsymbol{g}$ *satisfy* (III.122) *and let* $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\Omega)$*. We say that* $(\boldsymbol{u},\pi) \in \boldsymbol{L}^p(\Omega)\times W^{-1,p}_0(\Omega)$ *is a very weak solution of problem* (III.1) *if the following equalities hold: For any* $\boldsymbol{\varphi} \in \boldsymbol{Y}_{p',0}(\Omega)$ *and* $\theta \in W^{1,p'}_0(\Omega)$*,*

$$\int_\Omega \boldsymbol{u}\cdot(-\Delta\boldsymbol{\varphi} - \mathrm{div}(\boldsymbol{v}\otimes\boldsymbol{\varphi}))\, d\boldsymbol{x} - \langle \pi, \nabla\cdot\boldsymbol{\varphi}\rangle_{W^{-1,p}_0(\Omega)\times \mathring{W}^{1,p'}_0(\Omega)} = \langle \boldsymbol{f},\boldsymbol{\varphi}\rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\right\rangle_\Gamma \tag{III.123}$$

$$\int_\Omega \boldsymbol{u}\cdot\nabla\theta\, d\boldsymbol{x} = -\int_\Omega h\,\theta\, d\boldsymbol{x} + \langle \boldsymbol{g}\cdot\boldsymbol{n}, \theta\rangle_{W^{-1/p,p}(\Gamma)\times W^{1/p,p'}(\Gamma)} \tag{III.124}$$

where the dualities on Ω *and* Γ *are defined by:*

$$\langle .,.\rangle_\Omega = \langle .,.\rangle_{[\boldsymbol{X}^0_{r',p'}(\Omega)]'\times\boldsymbol{X}^0_{r',p'}(\Omega)}\,, \langle .,.\rangle_\Gamma = \langle .,.\rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma)\times\boldsymbol{W}^{1/p,p'}(\Gamma)}\,.$$

Note that if (A$_1$) is satisfied, we have:

$$W^{1,p'}_0(\Omega) \hookrightarrow L^{r'}(\Omega) \quad\text{and}\quad \boldsymbol{Y}_{p',0}(\Omega) \hookrightarrow \boldsymbol{X}^0_{r',p'}(\Omega),$$

and $\int_\Omega \boldsymbol{u}\cdot(\boldsymbol{v}\cdot\nabla\,\boldsymbol{\varphi})\, d\mathbf{x}$ is well defined , which means that all the brackets and integrals have a sense.

Proposition 4.1 *Let* p *and* r *satisfy* (A$_1$) *and let* $\boldsymbol{f} \in [\boldsymbol{X}^0_{r',p'}(\Omega)]'$*,* $h \in L^r(\Omega)$*,* $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\Omega)$ *and* $\boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}(\Gamma)$*. Then the following two statements are equivalent:*

i) $(\boldsymbol{u},\pi) \in \boldsymbol{L}^p(\Omega)\times W^{-1,p}_0(\Omega)$ *is a very weak solution of* (III.1)

ii) $(\boldsymbol{u},\pi)$ *satisfies* (III.1) *in the sense of distributions.*

Proof. *i)* $\Rightarrow$ *ii)* Let $(\boldsymbol{u},\pi) \in \boldsymbol{L}^p(\Omega) \times W_0^{-1,p}(\Omega)$ be a very weak solution of (III.1), then if we take $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega)$ and $\theta \in \mathcal{D}(\Omega)$ we can deduce by (III.123) and (III.124) that

$$-\Delta \boldsymbol{u} + \boldsymbol{v}\cdot\nabla\,\boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} \text{ in } \Omega \quad \text{and} \quad \nabla\cdot\boldsymbol{u} = h \text{ in } \Omega,$$

Since $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\Omega)$ and $\frac{1}{p}+\frac{1}{3}=\frac{1}{r}$, we can deduce by Hölder that $\boldsymbol{v}\otimes\boldsymbol{u} \in \boldsymbol{L}^r(\Omega)$. Moreover, we have $-\Delta\boldsymbol{u} = -\mathrm{div}(\boldsymbol{v}\otimes\boldsymbol{u}) - \nabla\,\pi + \boldsymbol{f} \in [\boldsymbol{X}^0_{r',p'}(\Omega)]'$ and $\boldsymbol{u} \in \boldsymbol{T}^0_{p,r}(\Omega)$. Now, let $\boldsymbol{\varphi} \in \boldsymbol{Y}_{p',0}(\Omega) \subset \boldsymbol{X}^0_{r',p'}(\Omega)$, it follows

$$\langle -\Delta\boldsymbol{u},\boldsymbol{\varphi}\rangle_\Omega = \langle -\nabla\,\pi - \mathrm{div}(\boldsymbol{v}\otimes\boldsymbol{u}) + \boldsymbol{f},\boldsymbol{\varphi}\rangle_\Omega\,.$$

Lemma 4.4 implies that

$$\langle -\Delta\boldsymbol{u},\boldsymbol{\varphi}\rangle_\Omega = \int_\Omega \boldsymbol{u}\cdot\Delta\boldsymbol{\varphi}\,d\mathbf{x} - \left\langle \boldsymbol{u}_\tau, \frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\right\rangle_\Gamma$$

and from (III.118) that

$$\langle \nabla\,\pi,\boldsymbol{\varphi}\rangle_\Omega = -\,\langle \pi,\nabla\cdot\boldsymbol{\varphi}\rangle_{W_0^{-1,p}(\Omega)\times \mathring{W}_0^{1,p'}(\Omega)}\,.$$

On the other hand, we have $\nabla\,\boldsymbol{\varphi} \in \boldsymbol{L}^{r'}(\Omega)$ and $\mathrm{div}(\boldsymbol{v}\otimes\boldsymbol{\varphi}) = \boldsymbol{v}\cdot\nabla\,\boldsymbol{\varphi} \in \boldsymbol{L}^{p'}(\Omega)$. Then we obtain

$$\begin{aligned}\langle \mathrm{div}(\boldsymbol{v}\otimes\boldsymbol{u}),\boldsymbol{\varphi}\rangle_\Omega &= \langle \mathrm{div}(\boldsymbol{v}\otimes\boldsymbol{u}),\boldsymbol{\varphi}\rangle_{\boldsymbol{W}_0^{-1,r}(\Omega)\times \mathring{\boldsymbol{W}}_0^{1,r'}(\Omega)}\\ &= -\,\langle \boldsymbol{v}\otimes\boldsymbol{u},\nabla\,\boldsymbol{\varphi}\rangle_{\boldsymbol{L}^r(\Omega)\times\boldsymbol{L}^{r'}(\Omega)}\\ &= -\int_\Omega \boldsymbol{u}\cdot\mathrm{div}(\boldsymbol{v}\otimes\boldsymbol{\varphi})\,d\mathbf{x}.\end{aligned}$$

Thus we have

$$\int_\Omega \boldsymbol{u}\Delta\boldsymbol{\varphi}\,d\mathbf{x} - \left\langle \boldsymbol{u}_\tau, \frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\right\rangle_\Gamma = \langle \pi,\nabla\cdot\boldsymbol{\varphi}\rangle_{\boldsymbol{W}_0^{-1,p}(\Omega)\times \mathring{\boldsymbol{W}}_0^{1,p'}(\Omega)} + \langle \boldsymbol{f},\boldsymbol{\varphi}\rangle_\Omega + \int_\Omega \boldsymbol{u}\cdot\mathrm{div}(\boldsymbol{v}\otimes\boldsymbol{\varphi})\,d\mathbf{x}.,$$

and we can deduce that for any $\boldsymbol{\varphi} \in \boldsymbol{Y}_{p',0}(\Omega)$

$$\left\langle \boldsymbol{u}_\tau, \frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\right\rangle_\Gamma = \left\langle \boldsymbol{g}_\tau, \frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\right\rangle_\Gamma\,.$$

Now let $\boldsymbol{\mu} \in \boldsymbol{W}^{1/p,p'}(\Gamma)$, then we have $\langle \boldsymbol{u}_\tau - \boldsymbol{g}_\tau,\boldsymbol{\mu}\rangle_\Gamma = \langle \boldsymbol{u}_\tau - \boldsymbol{g}_\tau,\boldsymbol{\mu}_\tau\rangle_\Gamma$.It is clear that $\boldsymbol{\mu}_\tau \in \boldsymbol{Z}_{p'}(\Omega)$ and it implies that there exists $\boldsymbol{\varphi} \in \boldsymbol{Y}_{p',0}(\Omega)$ such that $\dfrac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}} = \boldsymbol{\mu}_\tau$ on Γ. We can deduce that $\boldsymbol{u}_\tau = \boldsymbol{g}_\tau$ in $\boldsymbol{W}^{-1/p,p}(\Gamma)$. From the equation $\nabla\cdot\boldsymbol{u} = h$, we deduce that $\boldsymbol{u} \in \boldsymbol{H}^r_{p,1}(\mathrm{div},\Omega)$, then it follows from (III.121), that for any $\theta \in W_0^{1,p'}(\Omega)$,

$$\langle \boldsymbol{u}\cdot\boldsymbol{n},\theta\rangle_\Gamma = \langle \boldsymbol{g}\cdot\boldsymbol{n},\theta\rangle_\Gamma\,.$$

Consequently $\boldsymbol{u}\cdot\boldsymbol{n} = \boldsymbol{g}\cdot\boldsymbol{n}$ in $W^{-1/p,p}(\Gamma)$ and finally $\boldsymbol{u} = \boldsymbol{g}$ on Γ.
ii) $\Rightarrow$ *i)*] The converse is a simple consequence of (III.121), (III.118) and Lemma 4.4.

□

Theorem 4.9 *Let Ω be an exterior domain with $C^{1,1}$ boundary and let p and r satisfy* (A$_1$) *and* *let* $\boldsymbol{f}$, h, *and* $\boldsymbol{g}$ *satisfy* (III.122), $\boldsymbol{v} \in \boldsymbol{H}^3_\sigma(\Omega)$. *Then the Oseen problem* (III.1) *has a unique solution* $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ *and* $\pi \in W_0^{-1,p}(\Omega)$ *if and only if for any* $(\boldsymbol{\lambda}, \mu) \in \mathcal{N}_0^{(p')*}(\Omega)$:

$$\langle \boldsymbol{f}, \boldsymbol{\lambda} \rangle - \langle h, \mu \rangle + \langle \boldsymbol{g}, (\mu I - \nabla \boldsymbol{\lambda}) \cdot \boldsymbol{n} \rangle_\Gamma = 0.$$

Moreover, there exists a constant $C > 0$ *depending only on* p, r *and* Ω *such that:*

$$\| \boldsymbol{u} \|_{\boldsymbol{L}^p(\Omega)} + \| \pi \|_{W_0^{-1,p}(\Omega)} \leq C(1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)})^4 \left(\| \boldsymbol{f} \|_{[\boldsymbol{X}^0_{r',p'}(\Omega)]'} + \| h \|_{L^r(\Omega)} + \| \boldsymbol{g} \|_{\boldsymbol{W}^{-1/p,p}(\Gamma)} \right). \tag{III.125}$$

Proof. It remains to consider the equivalent problem: Find $(\boldsymbol{u}, \pi) \in \boldsymbol{L}^p(\Omega) \times W_0^{-1,p}(\Omega)$ such that for any $\boldsymbol{w} \in \boldsymbol{Y}_{p',0}(\Omega)$ and $\theta \in W_0^{1,p'}(\Omega)$ it holds:

$$\int_\Omega \boldsymbol{u} \cdot (-\Delta \boldsymbol{w} + \boldsymbol{v} \cdot \nabla \boldsymbol{w} + \nabla \theta) d\mathbf{x} - \langle \pi, \operatorname{div} \boldsymbol{w} \rangle_{W_0^{-1,p}(\Omega) \times \mathring{W}_0^{1,p'}(\Omega)} =$$

$$\langle \boldsymbol{f}, \boldsymbol{w} \rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \right\rangle_\Gamma + \langle \boldsymbol{g} \cdot \boldsymbol{n}, \theta \rangle_{W^{-1/p,p}(\Gamma) \times W^{1/p,p'}(\Gamma)} - \int_\Omega h\, \theta \, d\mathbf{x}.$$

Let T be a linear form defined by:

$$\begin{aligned} T: \quad & \mathbf{L}^{p'}(\Omega) \times \mathring{W}_0^{1,p'}(\Omega) \longrightarrow \mathbb{R} \\ & (\mathbf{F}, \varphi) \longmapsto \langle \boldsymbol{f}, \boldsymbol{w} \rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \right\rangle_\Gamma + \langle \boldsymbol{g} \cdot \boldsymbol{n}, \theta \rangle_{W^{-1/p,p}(\Gamma) \times W^{1/p,p'}(\Gamma)} - \int_\Omega h\, \theta d\mathbf{x}, \end{aligned}$$

with $(\boldsymbol{w}, \theta) \in \boldsymbol{W}_0^{2,p'}(\Omega) \times W_0^{1,p'}(\Omega)$ is a solution of the following Oseen problem:

$$-\Delta \boldsymbol{w} + \boldsymbol{v} \cdot \nabla \boldsymbol{w} + \nabla \theta = \mathbf{F} \quad \text{and} \quad \operatorname{div} \boldsymbol{w} = \varphi \quad \text{in } \Omega, \quad \boldsymbol{w} = 0 \quad \text{on } \Gamma,$$

and satisfying the following estimate: (see Theorem 4.7)

$$\inf_{(\boldsymbol{\lambda}, \mu) \in \mathcal{N}_0^{(p')*}(\Omega)} \| \boldsymbol{w} + \boldsymbol{\lambda} \|_{\boldsymbol{W}_0^{2,p'}(\Omega)} + \| \theta + \mu \|_{W_0^{1,p'}(\Omega)} \leq$$

$$C(1 + \| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)})^4 \left(\| \mathbf{F} \|_{\boldsymbol{L}^{p'}(\Omega)} + \| \varphi \|_{W_0^{1,p'}(\Omega)} \right). \tag{III.126}$$

Then we have for any pair $(\mathbf{F}, \varphi) \in \mathbf{L}^{p'}(\Omega) \times \mathring{W}_0^{1,p'}(\Omega)$ and for any $(\boldsymbol{\lambda}, \mu) \in \mathcal{N}_0^{(p')*}(\Omega)$

$$| \langle \boldsymbol{f}, \boldsymbol{w} \rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \right\rangle_\Gamma + \langle \boldsymbol{g} \cdot \boldsymbol{n}, \theta \rangle_\Gamma - \int_\Omega h\, \theta d\mathbf{x} \, | =$$

$$| \langle \boldsymbol{f}, \boldsymbol{w}+\boldsymbol{\lambda}\rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial(\boldsymbol{w}+\boldsymbol{\lambda})}{\partial \boldsymbol{n}}\right\rangle_\Gamma + \langle \boldsymbol{g}\cdot\boldsymbol{n}, \theta+\mu\rangle_\Gamma - \int_\Omega h\,(\theta+\mu) d\mathbf{x}\,| \leq$$

$$C\left(\| \boldsymbol{f} \|_{[\mathbf{X}^0_{r',p'}(\Omega)]'} + \| \boldsymbol{g} \|_{\mathbf{W}^{-1/p,p}(\Omega)} + \| h \|_{L^r(\Omega)}\right)\left(\| \boldsymbol{w}+\boldsymbol{\lambda} \|_{\mathbf{W}^{2,p'}_0(\Omega)} + \| \theta+\mu \|_{W^{1,p'}_0(\Omega)}\right).$$

Using (III.126), we prove that

$$| \langle \boldsymbol{f}, \boldsymbol{w}\rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}}\right\rangle_\Gamma + \langle \boldsymbol{g}\cdot\boldsymbol{n}, \theta\rangle_\Gamma - \int_\Omega h\,\theta d\mathbf{x}\,| \leq$$

$$C(1+\| \boldsymbol{v} \|_{\boldsymbol{L}^3(\Omega)})^4\left(||\mathbf{F}||_{\mathbf{L}^{p'}(\Omega)} + ||\varphi||_{W^{1,p'}_0(\Omega)}\right)\left(||\boldsymbol{f}||_{[\mathbf{X}^0_{r',p'}(\Omega)]'} + ||\boldsymbol{g}||_{\mathbf{W}^{-1/p,p}(\Omega)} + ||h||_{L^r(\Omega)}\right).$$

It implies that the linear form T is continuous on $\mathbf{L}^{p'}(\Omega)\times \mathring{W}^{1,p'}_0(\Omega)$ and moreover there exists a unique solution$(\boldsymbol{u},\pi)\in \mathbf{L}^p(\Omega)\times W^{-1,p}_0(\Omega)$ solution of the Oseen problem (III.1) satisfying estimate (III.125). $\square$

4.4.2 Very weak solutions in $\boldsymbol{W}^{0,p}_{-1}(\Omega)$.

Here, we are interested in the case of the following assumptions:

$$\boldsymbol{f}\in [\boldsymbol{X}^1_{r',p'}(\Omega)]',\ h\in W^{0,r}_{-1}(\Omega)\ \text{ and }\ \boldsymbol{g}\in \boldsymbol{W}^{-1/p,p}(\Gamma), \tag{III.127}$$

with

$$\frac{3}{2}<p<\infty,\quad p\neq 3\quad\text{and}\qquad \frac{1}{p}+\frac{1}{3}=\frac{1}{r}, \tag{A$_2$}$$

yielding $1<r<3$.

Definition 4.2 ***(Very weak solution for the Oseen problem)*** *Suppose that* (A$_2$) *is satisfied and let* $\boldsymbol{f}$, h *and* $\boldsymbol{g}$ *satisfying* (III.127) *and let* $\boldsymbol{v}\in \boldsymbol{L}^3_\sigma(\Omega)$. *We say that* $(\boldsymbol{u},\pi)\in \boldsymbol{W}^{0,p}_{-1}(\Omega)\times W^{-1,p}_{-1}(\Omega)$ *is a very weak solution of* (III.1) *if the following equalities hold: For any* $\boldsymbol{\varphi}\in \boldsymbol{Y}_{p',1}(\Omega)$ *and* $\theta\in W^{1,p'}_1(\Omega)$,

$$\int_\Omega \boldsymbol{u}\cdot(-\Delta\boldsymbol{\varphi} - \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{\varphi}))\,d\boldsymbol{x} - \langle \pi, \operatorname{div}\boldsymbol{\varphi}\rangle_{W^{-1,p}_{-1}(\Omega)\times \mathring{W}^{1,p'}_1(\Omega)} = \langle \boldsymbol{f},\boldsymbol{\varphi}\rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\right\rangle_\Gamma \tag{III.128}$$

$$\int_\Omega \boldsymbol{u}\cdot\nabla\theta\,d\boldsymbol{x} = -\int_\Omega h\,\theta d\boldsymbol{x} + \langle \boldsymbol{g}\cdot\boldsymbol{n},\theta\rangle_{W^{-1/p,p}(\Gamma)\times W^{1/p,p'}(\Gamma)} \tag{III.129}$$

where the dualities on Ω *and* Γ *are defined by:*

$$\langle .,.\rangle_\Omega = \langle .,.\rangle_{[\boldsymbol{X}^1_{r',p'}(\Omega)]'\times \boldsymbol{X}^1_{r',p'}(\Omega)}\,, \langle .,.\rangle_\Gamma = \langle .,.\rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma)\times \boldsymbol{W}^{1/p,p'}(\Gamma)}\,.$$

Note that if $\frac{3}{2} < p < \infty$ and $\frac{1}{p} + \frac{1}{3} = \frac{1}{r}$, we have:

$$W_1^{1,p'}(\Omega) \hookrightarrow W_1^{0,r'}(\Omega), \quad \text{and} \quad \boldsymbol{Y}_{p',1}(\Omega) \hookrightarrow \boldsymbol{X}^1_{r',p'}(\Omega),$$

and $\int_\Omega \boldsymbol{u} \cdot \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{\varphi})\, d\mathbf{x}$ is well defined which means that all the brackets and integrals have a sense. As previously we prove under the assumption (A$_2$), that if $\boldsymbol{f}$, h, $\boldsymbol{g}$ satisfy (III.127) and $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\Omega)$, then $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{-1}(\Omega) \times W^{-1,p}_{-1}(\Omega)$ is a very weak solution of (III.1) if and only if $(\boldsymbol{u}, \pi)$ satisfy (III.1) in the sense of distributions.

Theorem 4.10 *Let Ω be an exterior domain with $C^{1,1}$ boundary. Suppose that* (A$_2$) *is satisfied and let $\boldsymbol{f}$, h, $\boldsymbol{g}$ satisfy* (III.127) *and let $\boldsymbol{v} \in \boldsymbol{H}^3_\sigma(\Omega)$. Then the Oseen problem* (III.1) *has a solution $\boldsymbol{u} \in \boldsymbol{W}^{0,p}_{-1}(\Omega)$ and $\pi \in W^{-1,p}_{-1}(\Omega)$ if and only if for any $(\boldsymbol{\lambda}, \mu) \in \mathcal{N}^{p'}_0(\Omega)$:*

$$\langle \boldsymbol{f}, \boldsymbol{\lambda} \rangle - \langle h, \mu \rangle + \langle \boldsymbol{g}, (\eta I - \nabla \boldsymbol{\lambda}) \cdot \boldsymbol{n} \rangle_\Gamma = 0.$$

In $\boldsymbol{W}^{0,p}_{-1}(\Omega) \times W^{-1,p}_{-1}(\Omega)$, each solution is unique up to an element of $\mathcal{N}^p_0(\Omega)$ and there exists a constant $C > 0$ depending only on p, r and Ω such that:

$$\inf_{(\boldsymbol{\lambda},\mu) \in \mathcal{N}^p_0(\Omega)} (||\boldsymbol{u} + \boldsymbol{\lambda}||_{\boldsymbol{W}^{0,p}_{-1}(\Omega)} + ||\pi + \mu||_{W^{-1,p}_{-1}(\Omega)})$$

$$\leq C(1 + ||\,\boldsymbol{v}||_{\boldsymbol{L}^3(\Omega)})^7 (||\boldsymbol{f}||_{[\boldsymbol{X}^1_{r',p'}(\Omega)]'} + ||h||_{W^{0,r}_{-1}(\Omega)} + ||\boldsymbol{g}||_{\boldsymbol{W}^{-1/p,p}(\Gamma)}). \tag{III.130}$$

Proof. It remains to consider the equivalent problem: Find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,p}_{-1}(\Omega) \times W^{-1,p}_{-1}(\Omega)$ such that for any $\boldsymbol{w} \in \boldsymbol{Y}_{p',0}(\Omega)$ and $\theta \in W^{1,p'}_1(\Omega)$ the following equality holds:

$$\int_\Omega \boldsymbol{u} \cdot (-\Delta \boldsymbol{w} + \boldsymbol{v} \cdot \nabla\, \boldsymbol{w} + \nabla \theta) d\mathbf{x} - \langle \pi, \operatorname{div} \boldsymbol{w} \rangle_{W^{-1,p}_{-1}(\Omega) \times \mathring{W}^{1,p'}_1(\Omega)} =$$

$$\langle \boldsymbol{f}, \boldsymbol{w} \rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \right\rangle_\Gamma + \langle \boldsymbol{g} \cdot \boldsymbol{n}, \theta \rangle_\Gamma - \int_\Omega h\, \theta d\mathbf{x}.$$

Let T be a linear form defined from $\left(\mathbf{W}^{0,p'}_1(\Omega) \times \mathring{W}^{1,p'}_1(\Omega)) \perp \mathcal{N}^p_0(\Omega)\right)$ onto $\mathbb{R}$ by:

$$T(\mathbf{F}, \varphi) = \langle \boldsymbol{f}, \boldsymbol{w} \rangle_\Omega - \left\langle \boldsymbol{g}_\tau, \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{n}} \right\rangle_\Gamma + \langle \boldsymbol{g} \cdot \boldsymbol{n}, \pi \rangle_\Gamma - \int_\Omega h\, \theta d\mathbf{x},$$

with $(\boldsymbol{w}, \theta) \in \boldsymbol{W}^{2,p'}_1(\Omega) \times W^{1,p'}_1(\Omega)$ is a solution of the following Stokes problem:

$$-\Delta \boldsymbol{w} + \boldsymbol{v} \cdot \nabla\, \boldsymbol{w} + \nabla\, \theta = \mathbf{F} \quad \text{and} \quad \operatorname{div} \boldsymbol{w} = \varphi \quad \text{in } \Omega, \quad \boldsymbol{w} = 0 \quad \text{on } \Gamma,$$

and satisfying the following estimate: (see Theorem 4.8)

$$\inf_{(\boldsymbol{\lambda},\mu)\in\mathcal{N}_0^{p'}(\Omega)}(||\boldsymbol{w}+\boldsymbol{\lambda}||_{\boldsymbol{W}_1^{2,p'}(\Omega)}+||\theta+\mu||_{W_1^{1,p'}(\Omega)})\leqslant C$$

$$(1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)})^6\left(||\mathbf{F}||_{\boldsymbol{W}_1^{0,p'}(\Omega)}+(1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)})||\varphi||_{W_1^{1,p'}(\Omega)}\right). \qquad \text{(III.131)}$$

Then for any pair $(\mathbf{F},\varphi)\in(\mathbf{W}_1^{0,p'}(\Omega)\times\mathring{W}_1^{1,p'}(\Omega))\bot\mathcal{N}_0^{1,p}(\Omega)$ and for any $(\boldsymbol{\lambda},\mu)\in\mathcal{N}_0^{p'}(\Omega)$

$$|\,\langle\boldsymbol{f},\boldsymbol{w}\rangle_\Omega-\left\langle\boldsymbol{g}_\tau,\frac{\partial\boldsymbol{w}}{\partial\boldsymbol{n}}\right\rangle_\Gamma+\langle\boldsymbol{g}\cdot\boldsymbol{n},\theta\rangle_\Gamma-\int_\Omega h\,\theta d\mathbf{x}\,|=$$

$$|\,\langle\boldsymbol{f},\boldsymbol{w}+\boldsymbol{\lambda}\rangle_\Omega-\left\langle\boldsymbol{g}_\tau,\frac{\partial(\boldsymbol{w}+\boldsymbol{\lambda})}{\partial\boldsymbol{n}}\right\rangle_\Gamma+\langle\boldsymbol{g}\cdot\boldsymbol{n},\theta+\mu\rangle_\Gamma-\int_\Omega h\,(\theta+\mu)d\mathbf{x}\,|\leq$$

$$C\left(\|\,\boldsymbol{f}\,\|_{[\mathbf{X}^1_{r',p'}(\Omega)]'}+\|\,\boldsymbol{g}\,\|_{\mathbf{W}^{-1/p,p}(\Omega)}+\|\,h\,\|_{W^{0,r}_{-1}(\Omega)}\right)\left(\|\,\boldsymbol{w}+\boldsymbol{\lambda}\,\|_{\mathbf{W}_1^{2,p'}(\Omega)}+\|\,\theta+\mu\,\|_{W_1^{1,p'}(\Omega)}\right).$$

Using (III.131), we prove that

$$|\,\langle\boldsymbol{f},\boldsymbol{w}\rangle_\Omega-\left\langle\boldsymbol{g}_\tau,\frac{\partial\boldsymbol{w}}{\partial\boldsymbol{n}}\right\rangle_\Gamma+\langle\boldsymbol{g}\cdot\boldsymbol{n},\theta\rangle_\Gamma-\int_\Omega h\,\pi d\mathbf{x}|\leq$$

$$\begin{aligned}C(1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)})^6&\left(||\mathbf{F}||_{\boldsymbol{W}_1^{0,p'}(\Omega)}+(1+||\,\boldsymbol{v}\,||_{\boldsymbol{L}^3(\Omega)})||\varphi||_{W_1^{1,p'}(\Omega)}\right)\\&\times\left(\|\,\boldsymbol{f}\,\|_{[\mathbf{X}_{r',p'}(\Omega)]'}+\|\,\boldsymbol{g}\,\|_{\mathbf{W}^{-1/p,p}(\Omega)}+\|\,h\,\|_{W^{0,r}_{-1}(\Omega)}\right).\end{aligned}$$

From this we can deduce that the linear form T is continuous on the following space $\mathbf{W}_1^{0,p'}(\Omega)\times\mathring{W}_1^{1,p'}(\Omega)\perp\mathcal{N}_0^p(\Omega)$ and we deduce that there exists $(\boldsymbol{u},\pi)\in(\mathbf{W}^{0,p}_{-1}(\Omega)\times W^{-1,p}_{-1}(\Omega))$ solution of the Oseen problem (III.1), which is unique up to an element of $\mathcal{N}_0^p(\Omega)$, satisfying the estimate (III.130). □

Remark 4.5

Observe that each solution $(\boldsymbol{\lambda},\mu)\in\boldsymbol{W}^{0,p}_{-1}(\Omega)\times W^{-1,p}_{-1}(\Omega)$ to the Oseen problem (III.1) with null data obviously belongs to $\mathcal{N}_0^p(\Omega)$, in fact the proof is very similar to that of Lemma 3.11 of [8]. Moreover, if $p\neq 3$, we have $\mathcal{N}_0^p(\Omega)\subset\boldsymbol{W}^{0,p}_{-1}(\Omega)\times W^{-1,p}_{-1}(\Omega)$.

Chapter IV

Exterior Stokes Problem with Different Boundary Conditions

Nous montrons ici l'existence et l'unicité de solutions généralisées et de solutions fortes du problème de Stokes dans un domaine extérieur avec différentes conditions aux limites sur le bord. Nous nous intéressons, dans ce chapitre, par le cadre hilbertien.

1 Introduction

Let Ω' denotes a bounded open in $\mathbb{R}^3$ of class $C^{1,1}$, simply-connected and with a connected boundary $\partial\Omega' = \Gamma$, representing an obstacle and Ω is its complement i.e. $\Omega = \mathbb{R}^3 \diagdown \overline{\Omega'}$. Then a unit exterior normal vector to the boundary can be defined almost everywhere on Γ; it is denoted by $\boldsymbol{n}$.
The purpose of this paper is to solve the Stokes equation in Ω, with two types of non standard boundary conditions on Γ.

$$(\mathcal{S}_T) \quad \begin{cases} -\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = \chi & \text{in}\, \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = g \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} & \text{on}\, \Gamma, \end{cases}$$

and

$$(\mathcal{S}_N) \quad \begin{cases} -\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = \chi \quad \text{in} \quad \Omega, \\ \pi = \pi_0, \quad \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} \quad \text{on} \quad \Gamma \quad \text{and} \quad \displaystyle\int_\Gamma \boldsymbol{u} \cdot \boldsymbol{n}\, d\sigma = 0. \end{cases}$$

Since this problem is posed in an exterior domain, an approach adapted to the solution is the use of weighted Sobolev spaces. Let us begin by introducing these spaces. A point in Ω will be denoted by $\mathbf{x} = (x_1, x_2, x_3)$ and its distance to the origin by $r = |\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. We will use the following weights:

$$\rho = \rho(r) = (1 + r^2)^{1/2}.$$

For all m in $\mathbb{N}$ and all k in $\mathbb{Z}$, we define the weighted space

$$W_k^{m,2}(\Omega) = \{u \in \mathcal{D}'(\Omega);\, \forall\, \lambda \in \mathbb{N}^3 : \, 0 \leq |\lambda| \leq m,\, \rho(r)^{k-m+|\lambda|} D^\lambda u \in L^2(\Omega)\},$$

which is a Hilbert space for the norm

$$||u||_{W_k^{m,2}(\Omega)} = \left(\sum_{|\lambda|=0}^{m} ||\rho^{k-m+|\lambda|} D^\lambda u||^2_{L^2(\Omega)} \right)^{1/2},$$

where $||.||_{L^2(\Omega)}$ denotes the standard norm of $L^2(\Omega)$. We shall sometimes use the seminorm

$$|u|_{W_k^{m,2}(\Omega)} = \left(\sum_{|\lambda|=m} ||\rho^{k} D^\lambda u||^2_{L^2(\Omega)} \right)^{1/2}.$$

In addition, it is established by Hanouzet in [42], for domains with a Lipschitz-continuous boundary, that $\mathcal{D}(\overline{\Omega})$ is dense in $W_k^{m,2}(\Omega)$. We set $\mathring{W}_k^{m,2}(\Omega)$ as the adherence of $\mathcal{D}(\Omega)$ for the norm $\| \cdot \|_{W_k^{m,2}(\Omega)}$. Then, the dual space of $\mathring{W}_k^{m,2}(\Omega)$, denoting by $W_{-k}^{-m,2}(\Omega)$, is a space of distributions. Furthermore, as in bounded domain, we have for $m=1$ or $m=2$,

$$\mathring{W}_k^{1,2}(\Omega) = \left\{ v \in W_k^{1,2}(\Omega),\ v = 0 \quad \text{on} \quad \partial\Omega \right\},$$

and

$$\mathring{W}_0^{2,2}(\Omega) = \left\{ v \in W_0^{2,2}(\Omega),\ v = \frac{\partial v}{\partial \boldsymbol{n}} = 0 \quad \text{on} \quad \partial\Omega \right\},$$

where $\dfrac{\partial v}{\partial \boldsymbol{n}}$ is the normal derivate of v. As a consequence of Hardy's inequality, the following Poincaré inequality holds: for $m=0$ or $m=1$ and for all k in $\mathbb{Z}$ there exists a constant C such that

$$\forall\, v \in \mathring{W}_k^{m,2}(\Omega), \quad ||v||_{W_k^{m,2}(\Omega)} \leq C|v|_{W_k^{m,2}(\Omega)}, \tag{IV.1}$$

i.e., the seminorm $|.|_{W_k^{m,2}(\Omega)}$ is a norm on $\mathring{W}_k^{m,2}(\Omega)$ equivalent to the norm $||.||_{W_k^{m,2}(\Omega)}$.

In the sequel, we shall use the following properties. For all integers m and k in $\mathbb{Z}$, we have

$$\forall\ n\ \in \mathbb{Z} \quad \text{with} \quad n \leq m-k-2, \quad \mathcal{P}_n \subset W_k^{m,2}(\Omega), \tag{IV.2}$$

where $\mathcal{P}_n$ denotes the space of all polynomials (of three variables) of degree at most n, with the convention that the space is reduced to zero when n is negative. Thus the difference $m-k$ is an important parameter of the space $W_k^{m,2}(\Omega)$. We denote by $\mathcal{P}_n^\Delta$ the subspace of all harmonic polynomials of $\mathcal{P}_n$.

Using the derivation in the distribution sense, we can define the operators curl and div on $\boldsymbol{L}^2(\Omega)$. Indeed, let $<\ .,.\ >$ denote the duality pairing between $\mathcal{D}(\Omega)$ and its dual space $\mathcal{D}'(\Omega)$. For any function $\boldsymbol{v} = (v_1, v_2, v_3) \in \boldsymbol{L}^2(\Omega)$, we have for any $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3) \in \boldsymbol{\mathcal{D}}(\Omega)$,

$$\begin{aligned} \langle \mathbf{curl}\, \boldsymbol{v}, \boldsymbol{\varphi} \rangle &= \int_\Omega \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, d\boldsymbol{x} \\ &= \int_\Omega \left(v_1 (\frac{\partial \varphi_3}{\partial x_2} - \frac{\partial \varphi_2}{\partial x_3}) + v_2 (\frac{\partial \varphi_1}{\partial x_3} - \frac{\partial \varphi_3}{\partial x_1}) + v_3 (\frac{\partial \varphi_2}{\partial x_1} - \frac{\partial \varphi_1}{\partial x_2}) \right) d\boldsymbol{x}, \end{aligned}$$

and for any $\varphi \in \mathcal{D}(\Omega)$

$$\langle \operatorname{div} \boldsymbol{v},\, \varphi\rangle = -\int_\Omega \boldsymbol{v}\cdot \mathbf{grad}\,\varphi\, d\boldsymbol{x} = -\int_\Omega \left(v_1 \frac{\partial \varphi}{\partial x_1} + v_2 \frac{\partial \varphi}{\partial x_2} + v_3 \frac{\partial \varphi}{\partial x_3}\right) d\boldsymbol{x}.$$

We note that the vector-valued Laplace operator of a vector field $\boldsymbol{v} = (v_1,\, v_2,\, v_3)$ is equivalently defined by

$$\Delta\, \boldsymbol{v} = \mathbf{grad}\,(\operatorname{div} \boldsymbol{v}) - \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v} \tag{IV.3}$$

or by

$$\Delta \boldsymbol{v} = (\Delta v_1,\, \Delta v_2,\, \Delta v_3).$$

This leads to the following definitions

Definition 1.1 *For all integers $k \in \mathbb{Z}$, the space $\boldsymbol{H}^2_k(\mathbf{curl}, \Omega)$ is defined by*

$$\boldsymbol{H}^2_k(\mathbf{curl}, \Omega) = \left\{ \boldsymbol{v} \in\ \boldsymbol{W}^{0,2}_k(\Omega); \mathbf{curl}\ \boldsymbol{v} \in\ \boldsymbol{W}^{0,2}_{k+1}(\Omega)\right\},$$

and is provided with the norm:

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^2_k(\mathbf{curl},\Omega)} = \left(\|\boldsymbol{v}\|^2_{\ \boldsymbol{W}^{0,2}_k(\Omega)} + \|\mathbf{curl}\ \boldsymbol{v}\|^2_{\ \boldsymbol{W}^{0,2}_{k+1}(\Omega)}\right)^{\frac{1}{2}}.$$

The space $\boldsymbol{H}^2_k(\operatorname{div}, \Omega)$ is defined by

$$\boldsymbol{H}^2_k(\operatorname{div}, \Omega) = \left\{ \boldsymbol{v} \in\ \boldsymbol{W}^{0,2}_k(\Omega); \operatorname{div}\ \boldsymbol{v} \in W^{0,2}_{k+1}(\Omega)\right\},$$

and is provided with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^2_k(\operatorname{div},\Omega)} = \left(\|\boldsymbol{v}\|^2_{\boldsymbol{W}^{0,2}_k(\Omega)} + \|\operatorname{div}\ \boldsymbol{v}\|^2_{W^{0,2}_{k+1}(\Omega)}\right)^{\frac{1}{2}}.$$

Finally, we set

$$\boldsymbol{X}^2_k(\Omega) = \boldsymbol{H}^2_k(\mathbf{curl}, \Omega) \cap \boldsymbol{H}^2_k(\operatorname{div}, \Omega).$$

It is provided with the norm

$$\boldsymbol{X}^2_k(\Omega) = \left(\|\boldsymbol{v}\|^2_{\boldsymbol{W}^{0,2}_k(\Omega)} + \|\operatorname{div}\ \boldsymbol{v}\|^2_{W^{0,2}_{k+1}(\Omega)} + \|\mathbf{curl}\ \boldsymbol{v}\|^2_{\boldsymbol{W}^{0,2}_{k+1}(\Omega)}\right)^{\frac{1}{2}}.$$

These definitions will also be used with Ω replaced by $\mathbb{R}^3$.

The argument used by Hanouzet see [42] to prove the density of $\mathcal{D}(\overline{\Omega})$ in $W^{m,2}_k(\Omega)$ can be easily adapted to establish that $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{H}^2_k(\operatorname{div}, \Omega)$ and in $\boldsymbol{H}^2_k(\mathbf{curl}, \Omega)$ and so in $\boldsymbol{X}^2_k(\Omega)$. Therefore, denoting by $\boldsymbol{n}$ the exterior unit normal to the boundary Γ, the normal trace $\boldsymbol{v}\cdot\boldsymbol{n}$ and the tangential trace $\boldsymbol{v}\times\boldsymbol{n}$

can be defined respectively in $H^{-1/2}(\Gamma)$ for the functions of $\boldsymbol{H}^2_k(\mathrm{div},\Omega)$ and in $\boldsymbol{H}^{-1/2}(\Gamma)$ for functions of $\boldsymbol{H}^2_k(\mathbf{curl},\Omega)$, where $H^{-1/2}(\Gamma)$ denotes the dual space of $H^{1/2}(\Gamma)$. They satisfy the trace theorems i.e, there exists a constant C such that

$$\forall \boldsymbol{v} \in \boldsymbol{H}^2_k(\mathrm{div},\Omega),\ ||\boldsymbol{v}\cdot\boldsymbol{n}||_{H^{-1/2}(\Gamma)} \leq C||\boldsymbol{v}||_{\boldsymbol{H}^2_k(\mathrm{div},\Omega)}, \tag{IV.4}$$

$$\forall \boldsymbol{v} \in \boldsymbol{H}^2_k(\mathbf{curl},\Omega),\ ||\boldsymbol{v}\times\boldsymbol{n}||_{\boldsymbol{H}^{-1/2}(\Gamma)} \leq C||\boldsymbol{v}||_{\boldsymbol{H}^2_k(\mathbf{curl},\Omega)} \tag{IV.5}$$

and the following Green's formulas hold: For any $\boldsymbol{v} \in \boldsymbol{H}^2_k(\mathrm{div},\Omega)$ and $\varphi \in W^{1,2}_{-k}(\Omega)$

$$\langle \boldsymbol{v}\cdot\boldsymbol{n}, \varphi\rangle_\Gamma = \int_\Omega \boldsymbol{v}\cdot\nabla\varphi\,dx + \int_\Omega \varphi\,\mathrm{div}\,\boldsymbol{v}\,dx, \tag{IV.6}$$

where $\langle,\rangle_\Gamma$ denotes the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.
For any $\boldsymbol{v} \in \boldsymbol{H}^2_k(\mathbf{curl},\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,2}_{-k}(\Omega)$

$$\langle \boldsymbol{v}\times\boldsymbol{n}, \boldsymbol{\varphi}\rangle_\Gamma = \int_\Omega \boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,dx - \int_\Omega \mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{\varphi}dx, \tag{IV.7}$$

where $\langle,\rangle_\Gamma$ denotes the duality pairing between $\boldsymbol{H}^{-1/2}(\Gamma)$ and $\boldsymbol{H}^{1/2}(\Gamma)$.

Remark 1.1 *If $\boldsymbol{v}$ belongs to $\boldsymbol{H}^2_k(\mathrm{div},\Omega)$ for some integer $k \geq 1$, then $\mathrm{div}\,\boldsymbol{v}$ is in $L^1(\Omega)$ and Green's formula (IV.6) yields*

$$\langle \boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_\Gamma = \int_\Omega \mathrm{div}\,\boldsymbol{v}\,dx \tag{IV.8}$$

But when $k \leq 0$, then $\mathrm{div}\,\boldsymbol{v}$ is not necessarily in $L^1(\Omega)$ and (IV.8) is generally not valid. Note also that when $k \leq 0$, $W^{0,2}_{-k-1}(\Omega)$ does not contain the constants.

The closures of $\mathfrak{D}(\Omega)$ in $\boldsymbol{H}^2_k(\mathrm{div},\Omega)$ and in $\boldsymbol{H}^2_k(\mathbf{curl},\Omega)$ are denoted respectively by $\mathring{\boldsymbol{H}}^2_k(\mathbf{curl},\Omega)$ and $\mathring{\boldsymbol{H}}^2_k(\mathrm{div},\Omega)$ and can be characterized respectively by:

$$\mathring{\boldsymbol{H}}^2_k(\mathbf{curl},\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{H}^2_k(\mathbf{curl},\Omega);\ \boldsymbol{v}\times\boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\right\},$$

$$\mathring{\boldsymbol{H}}^2_k(\mathrm{div},\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{H}^2_k(\mathrm{div},\Omega);\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\right\}.$$

Their dual spaces are characterized by the following propositions:

Proposition 1.1 *A distribution $\boldsymbol{f}$ belongs to $[\mathring{\boldsymbol{H}}^2_k(\mathrm{div},\Omega)]'$ if and only if there exist $\boldsymbol{\psi} \in \boldsymbol{W}^{0,2}_{-k}(\Omega)$ and $\chi \in W^{0,2}_{-k-1}(\Omega)$, such that $\boldsymbol{f} = \boldsymbol{\psi} + \mathbf{grad}\,\chi$.*
Moreover

$$\|\boldsymbol{f}\|_{[\mathring{\boldsymbol{H}}^2_k(\mathrm{div},\Omega)]'} = \max\left\{\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{0,2}_{-k}(\Omega)}, \|\chi\|_{W^{0,2}_{-k-1}(\Omega)}\right\}. \tag{IV.9}$$

Proof. Let $\boldsymbol{\psi} \in \boldsymbol{W}^{0,2}_{-k}(\Omega)$ and $\chi \in W^{0,2}_{-k-1}(\Omega)$, we have

$$\forall \boldsymbol{v} \in \mathfrak{D}(\Omega), \quad \langle \boldsymbol{\psi} + \mathbf{grad}\,\chi,\, \boldsymbol{v}\rangle_{\mathfrak{D}'(\Omega)\times\mathfrak{D}(\Omega)} = \int_\Omega (\boldsymbol{\psi}\cdot\boldsymbol{v} - \chi\,\mathrm{div}\,\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}.$$

Therefore, the linear mapping $\boldsymbol{\ell}\,:\,\boldsymbol{v}\longmapsto\int_\Omega(\boldsymbol{\psi}\cdot\boldsymbol{v}-\chi\,\mathrm{div}\,\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}$ defined on $\boldsymbol{\mathfrak{D}}(\Omega)$ is continuous for the norm of $\mathring{\boldsymbol{H}}^2_k(\mathrm{div},\,\Omega)$. Since $\boldsymbol{\mathfrak{D}}(\Omega)$ is dense in $\mathring{\boldsymbol{H}}^2_k(\mathrm{div},\,\Omega)$, $\boldsymbol{\ell}$ can be extended by continuity to a mapping still called $\boldsymbol{\ell}\in[\mathring{\boldsymbol{H}}^2_k(\mathrm{div},\,\Omega)]'$. Thus $\boldsymbol{\psi}+\mathbf{grad}\,\chi$ is an element of $[\mathring{\boldsymbol{H}}^2_k(\mathrm{div},\,\Omega)]'$.

Conversely, Let $E=\boldsymbol{W}^{0,2}_k(\Omega)\times W^{0,2}_{k+1}(\Omega)$ equipped by the following norm:

$$\|\boldsymbol{v}\|_E=(\|\boldsymbol{v}\|^2_{\boldsymbol{W}^{0,2}_k(\Omega)}+\|\mathrm{div}\,\boldsymbol{v}\|^2_{W^{0,2}_{k+1}(\Omega)})^{\frac{1}{2}}.$$

The mapping $T\,:\,\boldsymbol{v}\in\mathring{\boldsymbol{H}}^2_k(\mathrm{div},\,\Omega)\to(\boldsymbol{v},\,\mathrm{div}\boldsymbol{v})\in E$ is an isometry from $\mathring{\boldsymbol{H}}^2_k(\mathrm{div},\Omega)$ in E. Suppose $G=T(\mathring{\boldsymbol{H}}^2_k(\mathrm{div},\,\Omega))$ with the E-topology. Let $S=T^{-1}\,:\,G\longrightarrow\,\mathring{\boldsymbol{H}}^2_k(\mathrm{div},\,\Omega)$. Thus, we can define the following mapping:

$$\boldsymbol{v}\in G\longmapsto\langle\boldsymbol{f},S\boldsymbol{v}\rangle_{[\mathring{\boldsymbol{H}}^2_k(\mathrm{div},\,\Omega)]'\times\mathring{\boldsymbol{H}}^2_k(\mathrm{div},\,\Omega)}\quad\text{for}\quad\boldsymbol{f}\in[\mathring{\boldsymbol{H}}^2_k(\mathrm{div},\,\Omega)]'$$

which is a linear continuous form on G. Thanks to Hahn-Banach's Theorem, such form can be extended to a linear continuous form on E, denoted by Υ such that

$$||\Upsilon||_{E'}=||\boldsymbol{f}||_{[\mathring{\boldsymbol{H}}^2_k(\mathrm{div},\,\Omega)]'}.\tag{IV.10}$$

From the Riesz's Representation Lemma, there exist $\boldsymbol{\psi}\in\,\boldsymbol{W}^{0,2}_{-k}(\Omega)$ and $\chi\in W^{0,2}_{-k-1}(\Omega)$, such that for any $\boldsymbol{v}=(\boldsymbol{v}_1,v_2)\in E$,

$$\langle\Upsilon,\boldsymbol{v}\rangle_{E'\times E}=\int_\Omega\boldsymbol{v}_1\cdot\boldsymbol{\psi}\,dx+\int_\Omega v_2\,\chi\,dx,$$

with $||\Upsilon||_{E'}=\max\Big\{||\boldsymbol{\psi}||_{\boldsymbol{W}^{0,2}_{-k}(\Omega)},||\chi||_{W^{0,2}_{-k-1}(\Omega)}\Big\}$. In particular, if $\boldsymbol{v}=T\boldsymbol{\varphi}\in G$, where $\boldsymbol{\varphi}\in\boldsymbol{\mathfrak{D}}(\Omega)$, we have:

$$\langle\boldsymbol{f},\boldsymbol{\varphi}\rangle_{[\mathring{\boldsymbol{H}}^2_k(\mathrm{div},\,\Omega)]'\times\mathring{\boldsymbol{H}}^2_k(\mathrm{div},\,\Omega)}=\langle\boldsymbol{\psi}-\nabla\,\chi,\boldsymbol{\varphi}\rangle_{[\mathring{\boldsymbol{H}}^2_k(\mathrm{div},\,\Omega)]'\times\mathring{\boldsymbol{H}}^2_k(\mathrm{div},\,\Omega)},$$

and (IV.9) follows imeddiatly from (IV.10). □

We skip the proof of the following result as it is similar to that of Proposition 1.1.

Proposition 1.2 *A distribution $\boldsymbol{f}$ belongs to $[\mathring{\boldsymbol{H}}^2_k(\mathbf{curl},\Omega)]'$ if and only if there exist functions $\boldsymbol{\psi}\in\,\boldsymbol{W}^{0,2}_{-k}(\Omega)$ and $\boldsymbol{\xi}\in\,\boldsymbol{W}^{0,2}_{-k-1}(\Omega)$, such that $\boldsymbol{f}=\boldsymbol{\psi}+\mathbf{curl}\,\boldsymbol{\xi}$.*
Moreover

$$\|\,\boldsymbol{f}\|_{[\mathring{\boldsymbol{H}}^2_k(\mathbf{curl},\,\Omega)]'}=\max\Big\{\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{0,2}_{-k}(\Omega)},\,\|\,\boldsymbol{\xi}\,\|_{\boldsymbol{W}^{0,2}_{-k-1}(\Omega)}\Big\}.$$

Definition 1.2 *Let $\boldsymbol{X}^2_{k,N}(\Omega)$, $\boldsymbol{X}^2_{k,T}(\Omega)$ and $\mathring{\boldsymbol{X}}^2_k(\Omega)$ be the following subspaces of $\boldsymbol{X}^2_k(\Omega)$:*

$$\boldsymbol{X}^2_{k,N}(\Omega)=\Big\{\boldsymbol{v}\in\boldsymbol{X}^2_k(\Omega);\;\boldsymbol{v}\times\boldsymbol{n}=\mathbf{0}\text{ on }\Gamma\Big\},$$

$$\boldsymbol{X}^2_{k,T}(\Omega)=\Big\{\boldsymbol{v}\in\boldsymbol{X}^2_k(\Omega);\boldsymbol{v}\cdot\boldsymbol{n}=0\text{ on }\Gamma\Big\},$$

and

$$\mathring{\boldsymbol{X}}^2_k(\Omega) = \boldsymbol{X}^2_{k,N}(\Omega) \cap \boldsymbol{X}^2_{k,T}(\Omega).$$

Now, we give some results related to solving the Dirichlet problem and Neumann problem which are essential to ensure the existence and the uniqueness of some vectors potentials and one usually forces either the normal component to vanish or the tangential components to vanish. We start by giving the definition of the kernel of the Laplace operator for any integer $k \in \mathbb{Z}$:

$$\mathcal{A}^{\Delta}_{k-1} = \Big\{\chi \in W^{1,2}_{-k}(\Omega); \quad \Delta\,\chi = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \chi = 0 \quad \text{on} \quad \Gamma\Big\}.$$

In contrast with a bounded domain, the Dirichlet problem for the Laplace operator with zero data can have nontrivial solutions in an exterior domain; it depends upon the exponent of the weight. The result that we state below is established by Giroire in [41].

Proposition 1.3 *For any integer $k \geq 1$, the space $\mathcal{A}^{\Delta}_{k-1}$ is a subspace of all functions in $W^{1,2}_{-k}(\Omega)$ of the form $v(p) - p$, where p runs over all polynomials of $\mathcal{P}^{\Delta}_{k-1}$ and $v(p)$ is the unique solution in $W^{1,2}_0(\Omega)$ of the Dirichlet problem*

$$\Delta\, v(p) = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad v(p) = p \quad \text{on} \quad \Gamma. \tag{IV.11}$$

$\mathcal{A}^{\Delta}_{k-1}$ is a finite-dimentional space of the same dimension as $\mathcal{P}^{\Delta}_{k-1}$ and $\mathcal{A}^{\Delta}_{k-1} = \{0\}$ when $k \leq 0$.

Our second proposition is established also by Giroire in [41], it characterizes the kernel of the Laplace operator with Neumann boundary condition. For any integer $k \in \mathbb{Z}$,

$$\mathcal{N}^{\Delta}_{k-1} = \Big\{\chi \in W^{1,2}_{-k}(\Omega); \quad \Delta\,\chi = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \frac{\partial \chi}{\partial \boldsymbol{n}} = 0 \quad \text{on} \quad \Gamma\Big\}.$$

Proposition 1.4 *For any integer $k \geq 1$, $\mathcal{N}^{\Delta}_{k-1}$ the subspace of all functions in $W^{1,2}_{-k}(\Omega)$ of the form $w(p) - p$, where p runs over all polynomials of $\mathcal{P}^{\Delta}_{k-1}$ and $w(p)$ is the unique solution in $W^{1,2}_0(\Omega)$ of the Neumann problem*

$$\Delta\, w(p) = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \frac{\partial w(p)}{\partial \boldsymbol{n}} = \frac{\partial p}{\partial \boldsymbol{n}} \quad \text{on} \quad \Gamma. \tag{IV.12}$$

Here also, we set $\mathcal{N}^{\Delta}_{k-1} = \{0\}$ when $k \leq 0$; $\mathcal{N}^{\Delta}_{k-1}$ is a finite-dimentional space of the same dimension as $\mathcal{P}^{\Delta}_{k-1}$ and in particular, $\mathcal{N}^{\Delta}_0 = \mathbb{R}$.

Next, the uniqueness of the solutions of Problem $(\mathcal{S}_T)$ and Problem $(\mathcal{S}_N)$ will follow from the characterization of the kernel. For all integers k in $\mathbb{Z}$, we define

$$\boldsymbol{Y}^2_{k,N}(\Omega) = \Big\{\boldsymbol{w} \in \boldsymbol{X}^{\,2}_{-k,N}(\Omega);\; \operatorname{div} \boldsymbol{w} = 0 \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{w} = \mathbf{0} \quad \text{in} \;\; \Omega\Big\}$$

and

$$\boldsymbol{Y}^2_{k,T}(\Omega) = \Big\{\boldsymbol{w} \in \boldsymbol{X}^{\,2}_{-k,T}(\Omega);\; \operatorname{div} \boldsymbol{w} = 0 \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{w} = \mathbf{0} \quad \text{in} \;\; \Omega\Big\}.$$

The proof of the following propositions can be easily deduced from [38].

Proposition 1.5 *Let $k \in \mathbb{Z}$ and suppose that Ω' is of class $C^{1,1}$, simply-connected and with a Lipshitz-continuous and connected boundary Γ.*

If $k < 1$, then $\boldsymbol{Y}^2_{k,N}(\Omega) = \{\mathbf{0}\}$.

If $k \geq 1$, then $\boldsymbol{Y}^2_{k,N}(\Omega) = \Big\{\nabla(v(p) - p),\ \ p \in \mathcal{P}^{\Delta}_{k-1}\Big\}$, where $v(p)$ is the unique solution in $W^{1,2}_0(\Omega)$ of the Dirichlet problem (IV.11).

Proof. Let $k \in \mathbb{Z}$ and let $\boldsymbol{w} \in \boldsymbol{X}^2_{-k,N}(\Omega)$ such that $\operatorname{div} \boldsymbol{w} = 0$ and $\mathbf{curl}\, \boldsymbol{w} = \mathbf{0}$ in Ω. Then since Ω' is simply-connected, there exists $\chi \in W^{1,2}_{-k}(\Omega)$, unique up to an additive constant, such that $\boldsymbol{w} = \nabla \chi$. But $\boldsymbol{w} \times \boldsymbol{n} = \mathbf{0}$, hence, χ is constant on Γ (Γ is a connected boundary) and we choose the additive constant in χ so that $\chi = 0$ on Γ. Thus χ belongs to $\mathcal{A}^{\Delta}_{k-1}(\Omega)$.
Due to Proposition 1.3, we deduce that if $k < 1$, χ is equal to zero and if $k \geq 1$, $\chi = v(p) - p$, where p runs over all polynomials of $\mathcal{P}^{\Delta}_{k-1}$ and $v(p)$ is the unique solution in $W^{1,2}_0(\Omega)$ of problem (IV.11) and thus $\boldsymbol{w} = \nabla(v(p) - p)$. Now, to finish the proof we shall prove that $\nabla(v(p) - p)$ belongs to $\boldsymbol{Y}^2_{k,N}(\Omega)$ and this is a simple consequence of the definition of p and $v(p)$. □

We skip the proof of the following result as it is entirely similar to that of Proposition 1.5.

Proposition 1.6 *Let the assumptions of Proposition 1.6 hold.*
If $k < 1$, then $\boldsymbol{Y}^2_{k,T}(\Omega) = \{\mathbf{0}\}$.

If $k \geq 1$, then $\boldsymbol{Y}^2_{k,T}(\Omega) = \Big\{\nabla(w(p) - p),\ \ p \in \mathcal{P}^{\Delta}_{k-1}\Big\}$, where $w(p)$ is the unique solution in $W^{1,2}_0(\Omega)$ of the Neumann problem (IV.12)

The imbedding results that we state below are established by V. Girault in [38]. The first imbedding result is given by the following theorem:

Theorem 1.1 *Let $k \leqslant 2$ and assume that Ω' is of class $C^{1,1}$. Then the space $\boldsymbol{X}^2_{k-1,T}(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{1,2}_k(\Omega)$. In addition there exists a constant C such that for any $\boldsymbol{\varphi} \in \boldsymbol{X}^2_{k-1,T}(\Omega)$,*

$$||\boldsymbol{\varphi}||_{\boldsymbol{W}^{1,2}_k(\Omega)} \leq C\left(||\boldsymbol{\varphi}||_{\boldsymbol{W}^{0,2}_{k-1}(\Omega)} + ||\operatorname{div} \boldsymbol{\varphi}||_{\boldsymbol{W}^{0,2}_k(\Omega)} + ||\mathbf{curl}\, \boldsymbol{\varphi}||_{\boldsymbol{W}^{0,2}_k(\Omega)}\right). \tag{IV.13}$$

If in addition, Ω' is simply-connected, there exists a constant C such that for all $\boldsymbol{\varphi} \in \boldsymbol{X}^2_{k-1,T}(\Omega)$ we have

$$\begin{aligned} ||\boldsymbol{\varphi}||_{\boldsymbol{W}^{1,2}_k(\Omega)} &\leq C(||\operatorname{div} \boldsymbol{\varphi}||_{\boldsymbol{W}^{0,2}_k(\Omega)} + ||\mathbf{curl}\, \boldsymbol{\varphi}||_{\boldsymbol{W}^{0,2}_k(\Omega)} \\ &+ \sum_{j=2}^{N(-k)} |\int_\Gamma \boldsymbol{\varphi} \cdot \nabla\, w(q_j)\, d\sigma|), \end{aligned} \tag{IV.14}$$

where $\{q_j\}_{j=2}^{N(-k)}$ *denotes a basis of* $\left\{q \in \mathcal{P}^{\Delta}_{-k} : \ q(\mathbf{0}) = 0\right\}$, $N(-k)$ *denotes the dimension of* $\mathcal{P}^{\Delta}_{-k}$ *and* $w(q_j)$ *is the corresponding function of* $\mathcal{N}^{\Delta}_{-k}$*. Thus, the seminorm in the right-hand side of* (IV.14) *is a norm on* $\boldsymbol{X}^2_{k-1,T}(\Omega)$ *equivalent to the norm* $||\boldsymbol{\varphi}||_{\boldsymbol{W}^{1,2}_k(\Omega)}$.

The second imbedding result is given by the following theorem:

Theorem 1.2 *Let* $k \leqslant 2$ *and assume that* Ω' *is of class* $C^{1,1}$*. Then the space* $\boldsymbol{X}^2_{k-1,N}(\Omega)$ *is continuously imbedded in* $\boldsymbol{W}^{1,2}_k(\Omega)$*. In addition there exists a constant* C *such that for any* $\boldsymbol{\varphi} \in \boldsymbol{X}^2_{k-1,N}(\Omega)$,

$$||\boldsymbol{\varphi}||_{\boldsymbol{W}^{1,2}_k(\Omega)} \leq C \left(||\boldsymbol{\varphi}||_{\boldsymbol{W}^{0,2}_{k-1}(\Omega)} + ||\mathrm{div}\, \boldsymbol{\varphi}||_{\boldsymbol{W}^{0,2}_k(\Omega)} + ||\mathbf{curl}\, \boldsymbol{\varphi}||_{\boldsymbol{W}^{0,2}_k(\Omega)} \right). \tag{IV.15}$$

If in addition, Ω' *is simply-connected and its boundary* Γ *is connected, there exists a constant* C *such that for all* $\boldsymbol{\varphi} \in \boldsymbol{X}^2_{k-1,N}(\Omega)$ *we have*

$$\begin{aligned} ||\boldsymbol{\varphi}||_{\boldsymbol{W}^{1,2}_k(\Omega)} &\leq C(||\mathrm{div}\, \boldsymbol{\varphi}||_{\boldsymbol{W}^{0,2}_k(\Omega)} + ||\mathbf{curl}\, \boldsymbol{\varphi}||_{\boldsymbol{W}^{0,2}_k(\Omega)} \\ &+ |\int_\Gamma (\boldsymbol{\varphi} \cdot \boldsymbol{n}) d\sigma| + \sum_{j=1}^{N(-k)} |\int_\Gamma (\boldsymbol{\varphi} \cdot \boldsymbol{n}) q_j \, d\sigma|), \end{aligned} \tag{IV.16}$$

where the term $|\int_\Gamma (\boldsymbol{\varphi} \cdot \boldsymbol{n}) d\sigma|$ *can be dropped if* $k \neq 1$ *and where* $\{q_j\}_{j=1}^{N(-k)}$ *denotes a basis of* $\mathcal{P}^{\Delta}_{-k}$*. In other words, the seminorm in the right-hand side of* (IV.16) *is a norm on* $\boldsymbol{X}^2_{k-1,N}(\Omega)$ *equivalent to the norm* $||\boldsymbol{\varphi}||_{\boldsymbol{W}^{1,2}_k(\Omega)}$.

Finally, let us recall the abstract setting of Babuška-Brezzi's Theorem (see Babuška [16], Brezzi [19] and Amrouche-Selloula [14]).

Theorem 1.3 *Let* X *and* M *be two reflexive Banach spaces and* X' *and* M' *their dual spaces. Let* a *be the continuous bilinear form defined on* $X \times M$*, let* $A \in \mathcal{L}(X;\ M')$ *and* $A' \in \mathcal{L}(M;\ X')$ *be the operators defined by*

$$\forall v \in X,\ \forall w \in M,\ a(v, w) = < Av, w > = < v, A'w >$$

and $V = \mathrm{Ker}\, A$*. The following statements are equivalent:*

i) There exist $\beta > 0$ *such that*

$$\inf_{\substack{w \in M \\ w \neq 0}} \sup_{\substack{v \in X \\ v \neq 0}} \frac{a(v, w)}{\|v\|_X \, \|w\|_M} \geq \beta. \tag{IV.17}$$

ii) The operator $A :\ X/V \mapsto M'$ *is an isomophism and* $1/\beta$ *is the continuity constant of* A^{-1}.

iii) The operator $A' :\ M \mapsto X' \bot V$ *is an isomophism and* $1/\beta$ *is the continuity constant of* $(A')^{-1}$.

Proof. First, we note that *ii)* $\Leftrightarrow$ *iii)* because $(X/V)' = X' \bot V$ where this last space contains the elements $f \in X'$ satisfying $\langle f,\, v \rangle = 0$ for any $v \in V$. It suffices then to prove that *i)* $\Leftrightarrow$ *iii)*. We begin with the implication $i) \Rightarrow iii)$. Due to (IV.17), we deduce that there exists a constant $\beta > 0$ such that:

$$\forall w \in M, \quad \|w\|_M \leq \frac{1}{\beta} \sup_{\substack{v \in X \\ v \neq 0}} \frac{|a(v,\, w)|}{\|v\|_X}.$$

So,

$$\|w\|_M \leq \frac{1}{\beta}\|A'w\|_{X'}, \tag{IV.18}$$

and A' is injective. Moreover, $\operatorname{Im} A'$ is a closed subspace of X' where $A' : \ M \to X'$. Moreover, $\operatorname{Im} A' = (\operatorname{Ker} A)^{\perp} = X' \perp V$. It remains to prove that $iii) \Rightarrow i)$. For this, it suffices to prove that if $iii)$ holds, then (IV.18) also holds and (IV.17) follows immediately. $\square$

Remark 1.2

As consequence, if the Inf-Sup condition (IV.17) is satisfied, then we have the following properties:

i) If $V = \{0\}$, then for any $f \in X'$, there exists a unique $w \in M$ such that

$$\forall v \in X, \; a(v, w) = < f, v > \quad \text{and} \quad \|w\|_M \leq \frac{1}{\beta}\|f\|_{X'}. \tag{IV.19}$$

ii) If $V \neq \{0\}$, then for any $f \in X'$, satisfying the compatibility condition:

$\forall v \in V, \; < f, v > = 0$, there exists a unique $w \in M$ such that (IV.19).

iii) For any $g \in M'$, $\exists v \in X$, unique up an additive element of V, such that:

$$\forall w \in M, \; a(v, w) = < g, w > \quad \text{and} \quad \|v\|_{X/V} \leq \frac{1}{\beta}\|g\|_{M'}.$$

2 Preliminary results

In this sequel, we prove some imbedding results. More precisely, we show that the results of Theorem 1.1 and the result of Theorem 1.2 can be extended to the case where the boundary conditions $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ or $\boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ are replaced by inhomogeneous one. Next, we study some problems posed in an exterior domain which are essentials to prove the regularity of solution for the Problem $(\mathcal{S}_T)$ and Problem $(\mathcal{S}_N)$.

For any integers k in $\mathbb{Z}$, we introduce the following spaces:

$$\boldsymbol{Z}^2_{k,T}(\Omega) = \Big\{ \boldsymbol{v} \in \boldsymbol{X}^2_k(\Omega) \text{ and } \boldsymbol{v} \cdot \boldsymbol{n} \in H^{1/2}(\Gamma) \Big\}, \quad \boldsymbol{Z}^2_{k,N}(\Omega) = \Big\{ \boldsymbol{v} \in \boldsymbol{X}^2_k(\Omega) \text{ and } \boldsymbol{v} \times \boldsymbol{n} \in \boldsymbol{H}^{1/2}(\Gamma) \Big\}$$

and

$$\boldsymbol{M}^2_{k,T}(\Omega) = \Big\{ \boldsymbol{v} \in \boldsymbol{W}^{1,2}_{k+1}(\Omega), \quad \operatorname{div} \boldsymbol{v} \in W^{1,2}_{k+2}(\Omega), \quad \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{W}^{1,2}_{k+2}(\Omega) \quad \text{and} \quad \boldsymbol{v} \cdot \boldsymbol{n} \in H^{3/2}(\Gamma) \Big\}.$$

Proposition 2.1 *Let $k = -1$ or $k = 0$, then the space $\boldsymbol{Z}^2_{k,T}(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{1,2}_{k+1}(\Omega)$ and we have the following estimate for any $\boldsymbol{v}$ in $\boldsymbol{Z}^2_{k,T}(\Omega)$:*

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,2}_{k+1}(\Omega)} \leq C\big(\|\boldsymbol{v}\|_{\boldsymbol{W}^{0,2}_{k}(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{W}^{0,2}_{k+1}(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{W^{0,2}_{k+1}(\Omega)} + \|\boldsymbol{v}\cdot\boldsymbol{n}\|_{H^{1/2}(\Gamma)}\big). \tag{IV.20}$$

Proof. Let $k = -1$ or $k = 0$ and let $\boldsymbol{v}$ any function of $\boldsymbol{Z}^2_{k,T}(\Omega)$. Let us study the following Neumann problem:

$$\Delta\chi = \mathrm{div}\,\boldsymbol{v} \ \text{ in}\,\Omega \quad \text{ and } \quad \partial_n\chi = \boldsymbol{v}\cdot\boldsymbol{n} \ \text{ on}\,\Gamma. \tag{IV.21}$$

It is shown in Theorem 3.7 and Theorem 3.9 of [38], that Problem (IV.21) has a unique solution χ in $W^{2,2}_{k+1}(\Omega)/\mathbb{R}$ if $k = -1$ and χ is unique in $W^{2,2}_{k+1}(\Omega)$ if $k = 0$. With the estimate

$$\|\nabla\,\chi\|_{W^{1,2}_{k+1}(\Omega)} \leq C\big(\|\mathrm{div}\,\boldsymbol{v}\|_{W^{0,2}_{k+1}(\Omega)} + \|\boldsymbol{v}\cdot\boldsymbol{n}\|_{H^{1/2}(\Gamma)}\big). \tag{IV.22}$$

Let $\boldsymbol{w} = \boldsymbol{v} - \mathbf{grad}\,\chi$, then $\boldsymbol{w}$ is a divergence-free function. Since $\boldsymbol{W}^{1,2}_{k+1}(\Omega) \hookrightarrow \boldsymbol{W}^{0,2}_{k}(\Omega)$, then $\boldsymbol{w} \in \boldsymbol{X}^2_{k,T}(\Omega)$. Applying Theorem 1.1, we have $\boldsymbol{w}$ belongs to $\boldsymbol{W}^{1,2}_{k+1}(\Omega)$ and then $\boldsymbol{v}$ is in $\boldsymbol{W}^{1,2}_{k+1}(\Omega)$. According to Inequality (IV.13), we obtain

$$\|\boldsymbol{w}\|_{\boldsymbol{W}^{1,2}_{k+1}(\Omega)} \leq C\big(\|\boldsymbol{w}\|_{\boldsymbol{W}^{0,2}_{k}(\Omega)} + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{W}^{0,2}_{k+1}(\Omega)}\big).$$

Then, the inequality (IV.20) follows directly from (IV.22). $\square$

Similarly, we can prove the following imbedding result:

Proposition 2.2 *Suppose that Ω' is of class $C^{2,1}$. Then the space $\boldsymbol{M}^2_{-1,T}(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{2,2}_{1}(\Omega)$ and we have the following estimate for any $\boldsymbol{v}$ in $\boldsymbol{M}^2_{-1,T}(\Omega)$:*

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{2,2}_{1}(\Omega)} \leq C\big(\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,2}_{0}(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{W}^{1,2}_{1}(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{W^{1,2}_{1}(\Omega)} + \|\boldsymbol{v}\cdot\boldsymbol{n}\|_{H^{3/2}(\Gamma)}\big). \tag{IV.23}$$

Proof. Proceeding as in Proposition 2.1. Let $\boldsymbol{v}$ in $\boldsymbol{M}^2_{-1,T}(\Omega)$. Since Ω' is of class $C^{2,1}$, then according to Theorem 3.9 of [38], there exists a unique solution χ in $W^{3,2}_{1}(\Omega)/\mathbb{R}$ of Problem (IV.21). Setting $\boldsymbol{w} = \boldsymbol{v} - \mathbf{grad}\,\chi$. Since $W^{2,2}_{1}(\Omega)$ is imbedded in $W^{1,2}_{0}(\Omega)$, it follows from Corollary 3.16 of [38], that $\boldsymbol{w}$ belongs to $\boldsymbol{W}^{2,2}_{1}(\Omega)$ and moreover we have the following estimate

$$\|\boldsymbol{w}\|_{\boldsymbol{W}^{2,2}_{1}(\Omega)} \leq C\big(\|\boldsymbol{w}\|_{\boldsymbol{W}^{1,2}_{0}(\Omega)} + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{W}^{1,2}_{1}(\Omega)}\big).$$

Then $\boldsymbol{v} = \boldsymbol{w} + \mathbf{grad}\,\chi$ belongs to $\boldsymbol{W}^{2,2}_{1}(\Omega)$ and we have the estimate (IV.23). $\square$

Although we are under the hilbertian case but the Lax-Milgram lemma is not always valid to ensure the existence of solutions. Thus, we shall establish two "Inf-Sup" conditions in order to apply Theorem 1.3. First recall the following spaces for all integers $k \in \mathbb{Z}$:

$$\boldsymbol{V}^2_{k,T}(\Omega) = \left\{ \boldsymbol{z} \in \boldsymbol{X}^2_{k,T}(\Omega);\ \operatorname{div} \boldsymbol{z} = 0 \text{ in } \Omega \text{ and } \int_\Gamma \boldsymbol{z} \cdot \nabla\left(w(q) - q\right) d\sigma = 0,\ \forall\ (w(q) - q) \in \mathcal{N}^\Delta_{-k-1} \right\}$$

and

$$\boldsymbol{V}^2_{k,N}(\Omega) = \left\{ \boldsymbol{z} \in \boldsymbol{X}^2_{k,N}(\Omega);\ \operatorname{div} \boldsymbol{z} = 0 \text{ in } \Omega \text{ and } \int_\Gamma (\boldsymbol{z} \cdot \boldsymbol{n}) q \, d\sigma = 0,\ \forall\ q \in \mathcal{P}^\Delta_{-k-1} \right\}.$$

The first "Inf-Sup" condition is given by the following lemma:

Lemma 2.1 *The following Inf-Sup Condition holds: there exists a constant $\beta > 0$, such that*

$$\inf_{\substack{\boldsymbol{\varphi} \in \boldsymbol{V}^2_{0,T}(\Omega) \\ \boldsymbol{\varphi} \neq 0}} \sup_{\substack{\boldsymbol{\psi} \in \boldsymbol{V}^2_{-2,T}(\Omega) \\ \boldsymbol{\psi} \neq 0}} \frac{\int_\Omega \mathbf{curl}\, \boldsymbol{\psi} \cdot \mathbf{curl}\, \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\psi}\|_{\boldsymbol{X}^2_{-2,T}(\Omega)} \|\boldsymbol{\varphi}\|_{\boldsymbol{X}^2_{0,T}(\Omega)}} \geq \beta. \tag{IV.24}$$

Proof. Let $\boldsymbol{g} \in \boldsymbol{W}^{0,2}_{-1}(\Omega)$ and let us introduce the following Dirichlet problem:

$$-\Delta\, \chi = \operatorname{div} \boldsymbol{g} \quad \text{in} \quad \Omega, \quad \chi = 0 \quad \text{on} \quad \Gamma.$$

It is shown in Theorem 3.5 of [38], that this problem has a solution $\chi \in \mathring{W}^{1,2}_{-1}(\Omega)$ unique up to an element of $\mathcal{A}^\Delta_0$ and we can choose χ such that

$$\|\nabla\, \chi\|_{\boldsymbol{W}^{0,2}_{-1}(\Omega)} \leq C \|\boldsymbol{g}\|_{\boldsymbol{W}^{0,2}_{-1}(\Omega)}.$$

Set $\boldsymbol{z} = \boldsymbol{g} - \nabla\, \chi$. Then we have $\boldsymbol{z} \in \boldsymbol{W}^{0,2}_{-1}(\Omega)$, $\operatorname{div} \boldsymbol{z} = 0$ and we have

$$\|\boldsymbol{z}\|_{\boldsymbol{W}^{0,2}_{-1}(\Omega)} \leq C \|\boldsymbol{g}\|_{\boldsymbol{W}^{0,2}_{-1}(\Omega)}. \tag{IV.25}$$

Let $\boldsymbol{\varphi}$ any function of $\boldsymbol{V}^2_{0,T}(\Omega)$, by Theorem 1.1 we have $\boldsymbol{\varphi} \in \boldsymbol{X}^2_{0,T}(\Omega) \hookrightarrow \boldsymbol{W}^{1,2}_1(\Omega)$. Then due to (IV.14) we can write

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{X}^2_{0,T}(\Omega)} \leq C \|\mathbf{curl}\, \boldsymbol{\varphi}\|_{\boldsymbol{W}^{0,2}_1(\Omega)} = C \sup_{\substack{\boldsymbol{g} \in \boldsymbol{W}^{0,2}_{-1}(\Omega) \\ \boldsymbol{g} \neq 0}} \frac{|\int_\Omega \mathbf{curl}\, \boldsymbol{\varphi} \cdot \boldsymbol{g} \, \mathrm{d}\boldsymbol{x}|}{\|\boldsymbol{g}\|_{\boldsymbol{W}^{0,2}_{-1}(\Omega)}}. \tag{IV.26}$$

Using the fact that $\mathbf{curl}\, \boldsymbol{\varphi} \in \boldsymbol{H}^2_1(\operatorname{div}, \Omega)$ and applying (IV.6), we obtain

$$\int_\Omega \mathbf{curl}\, \boldsymbol{\varphi} \cdot \nabla\, \chi \, \mathrm{d}\boldsymbol{x} = 0. \tag{IV.27}$$

Now, let $\lambda \in W^{1,2}_0(\Omega)$ the unique solution of the following problem:

$$\Delta\, \lambda = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \lambda = 1 \quad \text{on} \quad \Gamma.$$

It follows from Lemma 3.11 of [38] that $\displaystyle\int_\Gamma \frac{\partial \lambda}{\partial \boldsymbol{n}}\, d\sigma = C_1 > 0$. Now, setting

$$\widetilde{\boldsymbol{z}} = \boldsymbol{z} - \frac{1}{C_1}\,\langle \boldsymbol{z}\cdot\boldsymbol{n}, 1\rangle_\Gamma \,\nabla\,\lambda.$$

It is clear that $\widetilde{\boldsymbol{z}} \in \boldsymbol{W}^{0,2}_{-1}(\Omega)$, $\operatorname{div}\widetilde{\boldsymbol{z}} = 0$ in Ω and that $\langle \widetilde{\boldsymbol{z}}\cdot\boldsymbol{n}, 1\rangle_\Gamma = 0$. Due to Theorem 3.15 of [38], there exists a potential vector $\boldsymbol{\psi} \in \boldsymbol{W}^{1,2}_{-1}(\Omega)$ such that

$$\widetilde{\boldsymbol{z}} = \mathbf{curl}\,\boldsymbol{\psi}, \quad \operatorname{div}\boldsymbol{\psi} = 0 \quad \text{in}\,\Omega \quad \text{and} \quad \boldsymbol{\psi}\cdot\boldsymbol{n} = 0 \quad \text{on}\,\Gamma. \tag{IV.28}$$

and we have

$$\forall\ \boldsymbol{v}(q) \in \mathcal{N}_1^{\Delta}, \ \int_\Gamma \boldsymbol{\psi}\cdot\nabla\,\boldsymbol{v}(q)\, d\sigma = 0. \tag{IV.29}$$

In addition, we have the estimate

$$||\boldsymbol{\psi}||_{\boldsymbol{W}^{1,2}_{-1}(\Omega)} \le C||\widetilde{\boldsymbol{z}}||_{\boldsymbol{W}^{0,2}_{-1}(\Omega)} \le C||\boldsymbol{z}||_{\boldsymbol{W}^{0,2}_{-1}(\Omega)}. \tag{IV.30}$$

Using (IV.29), we obtain that $\boldsymbol{\psi}$ belongs to $\boldsymbol{V}^{\,2}_{-2,T}(\Omega)$. Since $\boldsymbol{\varphi}$ is $\boldsymbol{H}^1$ in a neighborhood of Γ, then $\boldsymbol{\varphi}$ has an $\boldsymbol{H}^1$ extension in Ω' denoted by $\widetilde{\boldsymbol{\varphi}}$. Applying Green's formula in Ω', we obtain

$$0 = \int_{\Omega'} \operatorname{div}(\mathbf{curl}\,\widetilde{\boldsymbol{\varphi}})\,\mathrm{d}\boldsymbol{x} = \langle \mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\cdot\boldsymbol{n}, 1\rangle_\Gamma = \langle \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{n}, 1\rangle_\Gamma\,.$$

Using the fact that $\mathbf{curl}\,\boldsymbol{\varphi}$ in $\boldsymbol{H}^2_1(\operatorname{div},\Omega)$ and λ in $W^{1,2}_{-1}(\Omega)$ and applying (IV.6), we obtain

$$0 = \langle \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{n}, 1\rangle_\Gamma = \langle \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{n}, \lambda\rangle_\Gamma = \int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\nabla\,\lambda\,\mathrm{d}\boldsymbol{x}. \tag{IV.31}$$

Using (IV.27) and (IV.31), we deduce that

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{g}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{z}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\widetilde{\boldsymbol{z}}\,\mathrm{d}\boldsymbol{x}. \tag{IV.32}$$

From (IV.30), (IV.25) and (IV.32), we deduce that

$$\frac{|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{g}\,\mathrm{d}\boldsymbol{x}|}{\|\boldsymbol{g}\|_{\boldsymbol{W}^{0,2}_{-1}(\Omega)}} \le C\frac{|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\widetilde{\boldsymbol{z}}\,\mathrm{d}\boldsymbol{x}|}{\|\widetilde{\boldsymbol{z}}\|_{\boldsymbol{W}^{0,2}_{-1}(\Omega)}} = C\frac{|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{curl}\,\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}|}{\|\mathbf{curl}\,\boldsymbol{\psi}\|_{\boldsymbol{W}^{0,2}_{-1}(\Omega)}}.$$

Applying again (IV.14) and using (IV.29), we obtain

$$\frac{|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{g}\,\mathrm{d}\boldsymbol{x}|}{\|\boldsymbol{g}\|_{\boldsymbol{W}^{0,2}_{-1}(\Omega)}} \le C\frac{|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{curl}\,\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}|}{\|\boldsymbol{\psi}\|_{\boldsymbol{X}^{\,2}_{-2,T}(\Omega)}},$$

and the Inf-Sup Condition (IV.24) follows immediately from (IV.26). $\square$

The second "inf sup" condition is given by the following lemma:

Lemma 2.2 *The following Inf-Sup Condition holds: there exists a constant $\beta > 0$, such that*

$$\inf_{\substack{\boldsymbol{\varphi}\in \boldsymbol{V}^2_{-2,N}(\Omega)\\ \boldsymbol{\varphi}\neq 0}} \sup_{\substack{\boldsymbol{\psi}\in \boldsymbol{V}^2_{0,N}(\Omega)\\ \boldsymbol{\psi}\neq 0}} \frac{\int_\Omega \mathbf{curl}\,\boldsymbol{\psi}\cdot \mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\psi}\|_{\boldsymbol{X}^2_{0,N}(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{X}^2_{-2,N}(\Omega)}} \geq \beta. \tag{IV.33}$$

Proof. The proof is similar to that of Lemma 2.1. Let $\boldsymbol{g} \in \boldsymbol{W}^{0,2}_1(\Omega)$ and let us introduce the following generalized Neumann problem:

$$\mathrm{div}(\nabla\chi - \boldsymbol{g}) = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad (\nabla\chi - \boldsymbol{g})\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma. \tag{IV.34}$$

It follows from [41] that Problem (IV.34) has a solution $\chi \in W^{1,2}_1(\Omega)$ and we have

$$||\nabla\chi||_{W^{0,2}_1(\Omega)} \leq C||\boldsymbol{g}||_{\boldsymbol{W}^{0,2}_1(\Omega)}.$$

Setting $\boldsymbol{z} = \boldsymbol{g} - \nabla\chi$, then we have $\boldsymbol{z} \in \mathring{\boldsymbol{H}}^2_1(\mathrm{div},\Omega)$ and $\mathrm{div}\,\boldsymbol{z} = 0$ with the following estimate:

$$||\boldsymbol{z}||_{\boldsymbol{W}^{0,2}_1(\Omega)} \leqslant C||\boldsymbol{g}||_{\boldsymbol{W}^{0,2}_1(\Omega)}. \tag{IV.35}$$

Let $\boldsymbol{\varphi}$ be any function of $\boldsymbol{V}^2_{-2,N}(\Omega)$. Due to Theorem 1.2, we have $\boldsymbol{X}^2_{-2,N}(\Omega) \hookrightarrow \boldsymbol{W}^{1,2}_{-1}(\Omega)$ and by (IV.16) we can write

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{X}^2_{-2,N}(\Omega)} \leq C\|\mathbf{curl}\,\boldsymbol{\varphi}\|_{\boldsymbol{W}^{0,2}_{-1}(\Omega)} = C \sup_{\substack{\boldsymbol{g}\in \boldsymbol{W}^{0,2}_1(\Omega)\\ \boldsymbol{g}\neq 0}} \frac{|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{g}\,\mathrm{d}\boldsymbol{x}|}{\|\boldsymbol{g}\|_{\boldsymbol{W}^{0,2}_1(\Omega)}}. \tag{IV.36}$$

Observe that $\mathbf{curl}\,\boldsymbol{\varphi}$ belongs to $\boldsymbol{H}^2_{-1}(\mathrm{div},\Omega)$ with $\boldsymbol{\varphi}\times\boldsymbol{n} = \boldsymbol{0}$ on Γ and $\chi \in W^{1,2}_1(\Omega)$. Then using (IV.6), we obtain

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\nabla\chi\,dx = \langle \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{n}, \chi\rangle_\Gamma = 0. \tag{IV.37}$$

Due to Proposition 3.12 of [38], there exists a potential vector $\boldsymbol{\psi} \in \boldsymbol{W}^{1,2}_1(\Omega)$ such that

$$\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{\psi}, \quad \mathrm{div}\,\boldsymbol{\psi} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{\psi}\times\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma \tag{IV.38}$$

and

$$\int_\Gamma \boldsymbol{\psi}\cdot\boldsymbol{n}\,d\sigma = 0. \tag{IV.39}$$

In addition, we have

$$||\boldsymbol{\psi}||_{\boldsymbol{W}^{1,2}_1(\Omega)} \leq C||\boldsymbol{z}||_{\boldsymbol{W}^{0,2}_1(\Omega)}. \tag{IV.40}$$

Then, we deduce that $\boldsymbol{\psi}$ belongs to $\boldsymbol{V}^2_{0,N}(\Omega)$. Using (IV.35), (IV.37) and (IV.38), we deduce that

$$\frac{|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{g}\,\mathrm{d}\boldsymbol{x}|}{\|\boldsymbol{g}\|_{\boldsymbol{W}^{0,2}_1(\Omega)}} \leq C\frac{|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{z}\,\mathrm{d}\boldsymbol{x}|}{\|\boldsymbol{z}\|_{\boldsymbol{W}^{0,2}_1(\Omega)}} = C\frac{|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{curl}\,\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}|}{\|\mathbf{curl}\,\boldsymbol{\psi}\|_{\boldsymbol{W}^{0,2}_1(\Omega)}}.$$

Applying again (IV.16) and using (IV.39), we obtain

$$\frac{|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{g}\,\mathrm{d}\boldsymbol{x}|}{\|\boldsymbol{g}\|_{\boldsymbol{W}^{0,2}_1(\Omega)}} \leq C\frac{|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{curl}\,\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}|}{\|\boldsymbol{\psi}\|_{\boldsymbol{X}^2_{0,N}(\Omega)}},$$

and the Inf-Sup Condition (IV.33) follows immediately from (IV.36). □

Next, we need to study the problem:

$$(E_N)\begin{cases} -\Delta\,\boldsymbol{\xi} = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{\xi} = 0 \quad \text{in} \quad \Omega, \\ \boldsymbol{\xi}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n} \quad \text{on} \quad \Gamma \quad \text{and} \quad \displaystyle\int_\Gamma (\boldsymbol{\xi}\cdot\boldsymbol{n})q\,d\sigma = 0,\ \forall\ q\in\mathcal{P}^\Delta_k. \end{cases} \tag{IV.41}$$

Proposition 2.3 *Let $k=-1$ or $k=0$ and suppose that $\boldsymbol{g}\times\boldsymbol{n}=\boldsymbol{0}$ and let $\boldsymbol{f}\in[\mathring{\boldsymbol{H}}^2_{k-1}(\mathbf{curl},\Omega)]'$ with* $\operatorname{div}\boldsymbol{f}=0$ *in Ω and satisfying the compatibility condition:*

$$\forall \boldsymbol{v}\in \boldsymbol{Y}^2_{1-k,N}(\Omega), \qquad \langle \boldsymbol{f},\,\boldsymbol{v}\rangle_{[\mathring{\boldsymbol{H}}^2_{k-1}(\mathbf{curl},\Omega)]'\times\mathring{\boldsymbol{H}}^2_{k-1}(\mathbf{curl},\Omega)} = 0. \tag{IV.42}$$

Then, Problem (E_N) has a unique solution in $\boldsymbol{W}^{1,2}_{-k}(\Omega)$ and we have:

$$\|\,\boldsymbol{\xi}\,\|_{\boldsymbol{W}^{1,2}_{-k}(\Omega)} \leqslant C\|\,\boldsymbol{f}\,\|_{[\mathring{\boldsymbol{H}}^2_{k-1}(\mathbf{curl},\Omega)]'}. \tag{IV.43}$$

Moreover, if $\boldsymbol{f}$ in $\boldsymbol{W}^{0,2}_{-k+1}(\Omega)$ and Ω' is of class $C^{2,1}$, then the solution $\boldsymbol{\xi}$ is in $\boldsymbol{W}^{2,2}_{-k+1}(\Omega)$ and satisfies the estimate:

$$\|\,\boldsymbol{\xi}\,\|_{\boldsymbol{W}^{2,2}_{-k+1}(\Omega)} \leqslant C\|\,\boldsymbol{f}\,\|_{\boldsymbol{W}^{0,2}_{-k+1}(\Omega)}. \tag{IV.44}$$

Proof. *i)* On the first hand, observe that Problem (E_N) is reduced to the following variational problem: Find $\boldsymbol{\xi}\in\boldsymbol{V}^2_{-k-1,N}(\Omega)$ such that

$$\forall\ \boldsymbol{\varphi}\in\boldsymbol{X}^2_{k-1,N}(\Omega),\ \int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} = \langle\boldsymbol{f},\,\boldsymbol{\varphi}\rangle_\Omega, \tag{IV.45}$$

where the duality on Ω is

$$\langle\cdot,\,\cdot\rangle_\Omega = \langle\cdot,\,\cdot\rangle_{[\mathring{\boldsymbol{H}}^2_{k-1}(\mathbf{curl},\,\Omega)]'\times\mathring{\boldsymbol{H}}^2_{k-1}(\mathbf{curl},\,\Omega)}.$$

On the other hand, problem (IV.45) is equivalent to the following problem: Find $\boldsymbol{\xi}\in\boldsymbol{V}^2_{-k-1,N}(\Omega)$ such that

$$\forall\ \boldsymbol{\varphi}\in\boldsymbol{V}^2_{k-1,N}(\Omega),\ \int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} = \langle\boldsymbol{f},\,\boldsymbol{\varphi}\rangle_\Omega. \tag{IV.46}$$

Indeed, every solution of (IV.45) also solves (IV.46). Conversely, assume that (IV.46) holds, and let $\boldsymbol{\varphi}\in\boldsymbol{X}^2_{k-1,N}(\Omega)$. Let us solve the exterior Dirichlet problem:

$$-\Delta\,\chi = \operatorname{div}\boldsymbol{\varphi} \quad \text{in} \quad \Omega \quad \text{and} \quad \chi = 0 \quad \text{on} \quad \Gamma. \tag{IV.47}$$

It is shown in Theorem 3.5 of [38] that problem (IV.47) has a unique solution $\chi\in W^{2,2}_k(\Omega)/\mathcal{A}^\Delta_{-k}$.

First case: if $k=0$, we set

$$\widetilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \nabla\chi - \frac{1}{C_1}\langle \boldsymbol{\varphi} - \nabla\chi, 1\rangle_\Gamma \nabla(v(1)-1),$$

where $v(1)$ is the unique solution in $W_0^{1,2}(\Omega)$ of the Dirichlet problem (IV.11) and $C_1 = \int_\Gamma \frac{\partial v(1)}{\partial \boldsymbol{n}}\, d\sigma$. It follows from Lemma 3.11 of [38] that $C_1 > 0$ and since $\nabla(v(1)-1)$ belongs to $\boldsymbol{Y}^2_{1,N}(\Omega)$, we deduce that $\widetilde{\boldsymbol{\varphi}}$ belongs to $\boldsymbol{V}^2_{-1,N}(\Omega)$.

Second case: if $k=-1$, for each polynomial p in $\mathcal{P}_1^\Delta$, we take $\widetilde{\boldsymbol{\varphi}}$ of the form

$$\widetilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \nabla\chi - \nabla(v(p)-p)$$

where $v(p)$ is the unique solution in $W_0^{1,2}(\Omega)$ of the Dirichlet problem (IV.11). The polynomial p is chosen to satisfy the following condition:

$$\int_\Gamma (\widetilde{\boldsymbol{\varphi}}\cdot\boldsymbol{n})\, q\, d\sigma = 0 \qquad \forall q \quad \in \mathcal{P}_1^\Delta. \tag{IV.48}$$

To show that this is possible, let T be a linear form defined by:

$$\begin{aligned} T:\quad & \mathcal{P}_1^\Delta \longrightarrow \mathbb{R}^4 \\ & p \longmapsto \left(\int_\Gamma \frac{\partial(v(p)-p)}{\partial\boldsymbol{n}} d\sigma, \int_\Gamma \frac{\partial(v(p)-p)}{\partial\boldsymbol{n}}\, x_1 d\sigma, \int_\Gamma \frac{\partial(v(p)-p)}{\partial\boldsymbol{n}}\, x_2 d\sigma, \int_\Gamma \frac{\partial(v(p)-p)}{\partial\boldsymbol{n}}\, x_3 d\sigma\right), \end{aligned}$$

where $\{1, x_1, x_2, x_3\}$ denotes a basis of $\mathcal{P}_1^\Delta$. It is shown in the proof of Theorem 7 of [39], that

$$\text{if} \quad \int_\Gamma \frac{\partial(v(p)-p)}{\partial\boldsymbol{n}}\, q\, d\sigma = 0 \quad \forall q \in \mathcal{P}_1^\Delta \quad \text{then} \quad p = 0.$$

This implies that T is injective and so bijective. And so, there exists a unique p in $\mathcal{P}_1^\Delta$ so that condition (IV.48) is satisfied and since $\nabla(v(p)-p)$ belongs to $\boldsymbol{Y}^2_{2,N}(\Omega)$, we prove that $\widetilde{\boldsymbol{\varphi}} \in \boldsymbol{V}^2_{-2,N}(\Omega)$.

Finally, using (IV.42), we obtain for $k=0$ and $k=-1$ that

$$\langle \boldsymbol{f}, \nabla(v(p)-p)\rangle_\Omega = 0 \quad \text{and} \quad \langle \boldsymbol{f}, \nabla(v(1)-1)\rangle_\Omega = 0$$

and as $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\mathring{\boldsymbol{H}}^2_{k-1}(\mathbf{curl}\,,\Omega)$, we obtain that

$$\langle \boldsymbol{f}, \nabla\chi\rangle_\Omega = 0.$$

Then we have

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x} = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_\Omega.$$

Then Problem (IV.45) and Problem (IV.46) are equivalent. Now, to solve Problem (IV.46), we use Lax-Milgram lemma for $k=0$ and the Inf-Sup condition (IV.33) for $k=-1$. Let us start by $k=0$.

We consider the bilinear form $\boldsymbol{a}:\ \boldsymbol{V}^{2}_{-1,N}(\Omega)\times \boldsymbol{V}^{2}_{-1,N}(\Omega)\longrightarrow \mathbb{R}$ such that

$$\boldsymbol{a}(\boldsymbol{\xi},\boldsymbol{\varphi})=\int_{\Omega}\mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\ \mathrm{d}\boldsymbol{x}.$$

According to Theorem 1.2, $\boldsymbol{a}$ is continuous and coercive on $\boldsymbol{V}^{2}_{-1,N}(\Omega)$. Due to Lax-Milgram lemma, there exists a unique solution $\boldsymbol{\xi}\in \boldsymbol{V}^{2}_{-1,N}(\Omega)$ of Problem (IV.46). Using again Theorem 1.2, we prove that this solution $\boldsymbol{\xi}$ belongs to $\boldsymbol{W}^{1,2}_{0}(\Omega)$ and the following estimate follows immediately

$$||\,\boldsymbol{\xi}\,||_{\boldsymbol{W}^{1,2}_{0}(\Omega)}\leqslant C||\,\boldsymbol{f}\,||_{[\mathring{\boldsymbol{H}}^{2}_{-1}(\mathbf{curl}\,,\Omega)]'}. \tag{IV.49}$$

When $k=-1$, we have that Problem (IV.46) satisfies the Inf-sup condition (IV.33). Let consider the following mapping $\boldsymbol{\ell}:\ \boldsymbol{V}^{2}_{-2,N}(\Omega)\longrightarrow\mathbb{R}$ such that $\boldsymbol{\ell}(\boldsymbol{\varphi})=\langle\boldsymbol{f},\boldsymbol{\varphi}\rangle_{\Omega}$. It is clear that $\boldsymbol{\ell}$ belongs to $(\boldsymbol{V}^{2}_{-2,N}(\Omega))'$ and according to Remark 1.2, there exists a unique solution $\boldsymbol{\xi}\in \boldsymbol{V}^{2}_{0,N}(\Omega)$ of Problem (IV.46). Due to Theorem 1.2, we prove that this solution $\boldsymbol{\xi}$ belongs to $\boldsymbol{W}^{1,2}_{1}(\Omega)$. It follows from Remark 1.2 i) that

$$||\boldsymbol{\xi}||_{\boldsymbol{W}^{1,2}_{1}(\Omega)}\leqslant C||\boldsymbol{f}||_{[\mathring{\boldsymbol{H}}^{2}_{-2}(\mathbf{curl},\Omega)]'}. \tag{IV.50}$$

ii) We suppose in addition that $\boldsymbol{f}$ is in $\boldsymbol{W}^{0,2}_{-k+1}(\Omega)$ for $k=-1$ or $k=0$ and Ω' is of class $C^{2,1}$ and we set $\boldsymbol{z}=\mathbf{curl}\,\boldsymbol{\xi}$, where $\boldsymbol{\xi}\in \boldsymbol{W}^{1,2}_{-k}(\Omega)$ is the unique solution of Problem (E_N). Then we have

$$\boldsymbol{z}\in \boldsymbol{W}^{0,2}_{-k}(\Omega),\quad \mathbf{curl}\,\boldsymbol{z}=\boldsymbol{f}\in \boldsymbol{W}^{0,2}_{-k+1}(\Omega),\quad \operatorname{div}\boldsymbol{z}=0\quad\text{and}\quad \boldsymbol{z}\cdot\boldsymbol{n}=0\quad\text{on}\quad\Gamma$$

and thus $\boldsymbol{z}$ belongs to $\boldsymbol{X}^{2}_{-k,T}(\Omega)$. By Theorem 1.1, we prove that $\boldsymbol{z}$ belongs to $\boldsymbol{W}^{1,2}_{-k+1}(\Omega)$ and using (IV.14), we prove that $\boldsymbol{z}$ satisfies:

$$||\boldsymbol{z}||_{\boldsymbol{W}^{1,2}_{-k+1}(\Omega)}\leqslant C||\boldsymbol{f}||_{\boldsymbol{W}^{0,2}_{-k+1}(\Omega)}. \tag{IV.51}$$

As a consequence $\boldsymbol{\xi}$ satisfies:

$$\boldsymbol{\xi}\in \boldsymbol{W}^{1,2}_{-k}(\Omega),\quad \mathbf{curl}\,\boldsymbol{\xi}\in \boldsymbol{W}^{1,2}_{-k+1}(\Omega),\quad \operatorname{div}\boldsymbol{\xi}=0\quad\text{and}\quad \boldsymbol{\xi}\times\boldsymbol{n}=\boldsymbol{0}\quad\text{on}\quad\Gamma.$$

Applying Corollary 3.14 in [38], we deduce that $\boldsymbol{\xi}$ belongs to $\boldsymbol{W}^{2,2}_{-k+1}(\Omega)$ and using in addition the boundary condition of (IV.41) we prove that

$$||\boldsymbol{\xi}||_{\boldsymbol{W}^{2,2}_{-k+1}(\Omega)}\leqslant C||\mathbf{curl}\,\boldsymbol{\xi}||_{\boldsymbol{W}^{1,2}_{-k+1}(\Omega)}. \tag{IV.52}$$

Finally, estimate (IV.44) follows from (IV.51) and (IV.52). □

Corollary 2.1 *Let $k=-1$ or $k=0$ and let $\boldsymbol{f}\in[\mathring{\boldsymbol{H}}^{2}_{k-1}(\mathbf{curl},\Omega)]'$ with $\operatorname{div}\boldsymbol{f}=0$ in Ω and $\boldsymbol{g}\in \boldsymbol{H}^{1/2}(\Gamma)$ and satisfying the compatibility condition* (IV.42). *Then, Problem (E_N) has a unique solution $\boldsymbol{\xi}$ in $\boldsymbol{W}^{1,2}_{-k}(\Omega)$ and we have:*

$$||\,\boldsymbol{\xi}\,||_{\boldsymbol{W}^{1,2}_{-k}(\Omega)}\leqslant C\left(||\,\boldsymbol{f}\,||_{[\mathring{\boldsymbol{H}}^{2}_{k-1}(\mathbf{curl},\Omega)]'}+||\boldsymbol{g}\times\boldsymbol{n}||_{\boldsymbol{H}^{1/2}(\Gamma)}\right). \tag{IV.53}$$

Moreover, if $\boldsymbol{f}$ in $\boldsymbol{W}^{0,2}_{-k+1}(\Omega)$, $\boldsymbol{g}$ in $\boldsymbol{H}^{3/2}(\Gamma)$ and Ω' is of class $C^{2,1}$, then the solution $\boldsymbol{\xi}$ is in $\boldsymbol{W}^{2,2}_{-k+1}(\Omega)$ and satisfies

$$|| \boldsymbol{\xi} ||_{\boldsymbol{W}^{2,2}_{-k+1}(\Omega)} \leqslant C \left(|| \boldsymbol{f} ||_{\boldsymbol{W}^{0,2}_{-k+1}(\Omega)} + ||\boldsymbol{g} \times \boldsymbol{n}||_{\boldsymbol{H}^{3/2}(\Gamma)} \right). \tag{IV.54}$$

Proof. Let $k = 0$ or $k = -1$ and let $\boldsymbol{g} \in \boldsymbol{H}^{1/2}(\Gamma)$. We know that there exists $\boldsymbol{\xi}_0$ in $\boldsymbol{H}^1(\Omega)$ with compact support satisfying

$$\boldsymbol{\xi}_0 = \boldsymbol{g}_\tau \quad \text{on} \quad \Gamma \quad \text{and} \quad \operatorname{div} \boldsymbol{\xi}_0 = 0 \quad \text{in} \quad \Omega,$$

where $\boldsymbol{g}_\tau$ is the tangential component of $\boldsymbol{g}$ on Γ. Since support of $\boldsymbol{\xi}_0$ is compact, we deduce that $\boldsymbol{\xi}_0$ belongs to $\boldsymbol{W}^{1,2}_{-k}(\Omega)$ for $k = -1$ or $k = 0$ and satisfies

$$\| \boldsymbol{\xi}_0 \|_{\boldsymbol{W}^{1,2}_{-k}(\Omega)} \leq C \| \boldsymbol{g}_\tau \|_{\boldsymbol{H}^{1/2}(\Gamma)}. \tag{IV.55}$$

Setting $\boldsymbol{z} = \boldsymbol{\xi} - \boldsymbol{\xi}_0$, then Problem (E_N) is equivalent to: find $\boldsymbol{z} \in \boldsymbol{W}^{1,2}_{-k}(\Omega)$ such that

$$\begin{cases} -\Delta\, \boldsymbol{z} = \boldsymbol{f} + \Delta\, \boldsymbol{\xi}_0 \quad \text{and} \quad \operatorname{div} \boldsymbol{z} = 0 \quad \text{in} \quad \Omega, \\ \boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on} \quad \Gamma \quad \text{and} \quad \displaystyle\int_\Gamma (\boldsymbol{z} \cdot \boldsymbol{n})\, q\, d\sigma = 0, \,\forall\, q \in \mathcal{P}^{\Delta}_k. \end{cases} \tag{IV.56}$$

Observe that $\boldsymbol{F} = \boldsymbol{f} - \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\xi}_0$ belongs to $[\mathring{\boldsymbol{H}}^2_{k-1}(\mathbf{curl},\, \Omega)]'$. Since $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\mathring{\boldsymbol{H}}^2_{k-1}(\mathbf{curl},\, \Omega)$, we have for any $\boldsymbol{v} \in \boldsymbol{Y}^2_{1-k,N}(\Omega)$:

$$\langle \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\xi}_0, \boldsymbol{v} \rangle_\Omega = \int_\Omega \mathbf{curl}\,\boldsymbol{\xi}_0 \cdot \mathbf{curl}\,\boldsymbol{v}\, dx = 0.$$

Thus $\boldsymbol{F}$ satisfies the compatibility condition (IV.42). Due to Proposition 2.3, there exists a unique $\boldsymbol{z} \in \boldsymbol{W}^{1,2}_{-k}(\Omega)$ solution of problem (IV.56) such that

$$|| \boldsymbol{z} ||_{\boldsymbol{W}^{1,2}_{-k}(\Omega)} \leqslant C|| \boldsymbol{F} ||_{[\mathring{\boldsymbol{H}}^2_{k-1}(\mathbf{curl},\Omega)]'} \leq C \left(|| \boldsymbol{f} ||_{[\mathring{\boldsymbol{H}}^2_{k-1}(\mathbf{curl},\Omega)]'} + || \mathbf{curl}\,\boldsymbol{\xi}_0 ||_{\boldsymbol{W}^{0,2}_{-k}(\Omega)} \right). \tag{IV.57}$$

Then $\boldsymbol{\xi} = \boldsymbol{z} + \boldsymbol{\xi}_0$ belongs to $\boldsymbol{W}^{1,2}_{-k}(\Omega)$ is the unique solution of (E_N) and estimate (IV.53) follows immediately from (IV.55) and (IV.57).

Regularity of the solution: Suppose in addition that Ω' is of class $C^{2,1}$, $\boldsymbol{f}$ in $\boldsymbol{W}^{0,2}_{-k+1}(\Omega)$ and $\boldsymbol{g}$ in $\boldsymbol{H}^{3/2}(\Gamma)$. Then the function $\boldsymbol{\xi}_0$ defined above belongs to $\boldsymbol{H}^2(\Omega)$ with compact support and thus $\boldsymbol{\xi}_0$ belongs to $\boldsymbol{W}^{2,2}_{-k+1}(\Omega)$ and we have

$$\| \boldsymbol{\xi}_0 \|_{\boldsymbol{W}^{2,2}_{-k+1}(\Omega)} \leq C \| \boldsymbol{g}_\tau \|_{\boldsymbol{H}^{3/2}(\Gamma)}. \tag{IV.58}$$

Using again Proposition 2.3, we prove that $\boldsymbol{z}$ belongs to $\boldsymbol{W}^{2,2}_{-k+1}(\Omega)$ and satisfies

$$|| \boldsymbol{z} ||_{\boldsymbol{W}^{2,2}_{-k+1}(\Omega)} \leqslant C|| \boldsymbol{F} ||_{\boldsymbol{W}^{0,2}_{-k+1}(\Omega)}.$$

Then $\boldsymbol{\xi}$ is in $\boldsymbol{W}^{2,2}_{-k+1}(\Omega)$ and estimate (IV.54) follows from (IV.58). $\square$

The next theorem solves an other type of exterior problem:

Theorem 2.1 *Let $k=-1$ or $k=0$ and let $\boldsymbol{v}$ belongs to $\boldsymbol{W}^{0,2}_{k}(\Omega)$. Then, the following problem*

$$\begin{cases} -\Delta\,\boldsymbol{\xi} = \mathbf{curl}\,\boldsymbol{v} \quad \text{and} \quad \operatorname{div}\boldsymbol{\xi} = 0 \quad \text{in} \quad \Omega, \\ \boldsymbol{\xi}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad (\mathbf{curl}\,\boldsymbol{\xi} - \boldsymbol{v})\times\boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Gamma, \\ \displaystyle\int_{\Gamma}\boldsymbol{\xi}\cdot\nabla\,(w(q)-q)\,d\sigma = 0, \qquad \forall\,(w(q)-q)\in\mathcal{N}^{\Delta}_{-k} \end{cases} \tag{IV.59}$$

has a unique solution $\boldsymbol{\xi}$ in $\boldsymbol{W}^{1,2}_{k}(\Omega)$ and we have:

$$\|\boldsymbol{\xi}\|_{\boldsymbol{W}^{1,2}_{k}(\Omega)} \leq C\|\boldsymbol{v}\|_{\boldsymbol{W}^{0,2}_{k}(\Omega)}. \tag{IV.60}$$

Moreover, if $\boldsymbol{v}\in\boldsymbol{W}^{1,2}_{k+1}(\Omega)$ and Ω' is of class $\mathcal{C}^{2,1}$, then the solution $\boldsymbol{\xi}$ is in $\boldsymbol{W}^{2,2}_{k+1}(\Omega)$ and satisfies the estimate:

$$\|\boldsymbol{\xi}\|_{\boldsymbol{W}^{2,2}_{k+1}(\Omega)} \leq C\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,2}_{k+1}(\Omega)}. \tag{IV.61}$$

Proof. At first observe that if $\boldsymbol{\xi}\in\boldsymbol{W}^{1,2}_{k}(\Omega)$ is a solution of Problem (IV.59) for $k=-1$ or $k=0$, then $\mathbf{curl}\,\boldsymbol{\xi}-\boldsymbol{v}$ belongs to $\boldsymbol{H}^{2}_{k}(\mathbf{curl},\Omega)$ and thus $(\mathbf{curl}\,\boldsymbol{\xi}-\boldsymbol{v})\times\boldsymbol{n}$ is well defined in Γ and belongs to $\boldsymbol{H}^{-1/2}(\Gamma)$.

On the other hand, note that (IV.59) can be reduced to the following variational problem: Find $\boldsymbol{\xi}\in\boldsymbol{V}^{2}_{k-1,T}(\Omega)$ such that

$$\forall\boldsymbol{\varphi}\in\boldsymbol{X}^{2}_{-k-1,T}(\Omega) \qquad \int_{\Omega}\mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} = \int_{\Omega}\boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}. \tag{IV.62}$$

Indeed, every solution of (IV.59) also solves (IV.62). Conversely, let $\boldsymbol{\xi}\in\boldsymbol{V}^{2}_{k-1,T}(\Omega)$ a solution of the problem (IV.62). Then,

$$\forall\boldsymbol{\varphi}\in\boldsymbol{\mathcal{D}}(\Omega), \quad \langle\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\xi}-\mathbf{curl}\,\boldsymbol{v},\,\boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega)\times\boldsymbol{\mathcal{D}}(\Omega)} = 0.$$

Then

$$-\Delta\,\boldsymbol{\xi} = \mathbf{curl}\,\boldsymbol{v} \quad \text{in} \quad \Omega. \tag{IV.63}$$

Moreover, by the fact that $\boldsymbol{\xi}$ belongs to the space $\boldsymbol{V}^{2}_{k-1,T}(\Omega)$ we have $\operatorname{div}\boldsymbol{\xi}=0$ in Ω and $\boldsymbol{\xi}\cdot\boldsymbol{n}=0$ on Γ. Then, it remains to verify the boundary condition $(\mathbf{curl}\,\boldsymbol{\xi}-\boldsymbol{v})\times\boldsymbol{n}=\mathbf{0}$ on Γ. Now setting $\boldsymbol{z}=\mathbf{curl}\,\boldsymbol{\xi}-\boldsymbol{v}$, then $\boldsymbol{z}$ belongs to $\boldsymbol{H}^{2}_{k}(\mathbf{curl},\Omega)$. Therefore, (IV.63) becomes:

$$\mathbf{curl}\,\boldsymbol{z} = \mathbf{0} \quad \text{in} \quad \Omega. \tag{IV.64}$$

Let $\boldsymbol{\varphi} \in \boldsymbol{X}^2_{-k-1,T}(\Omega)$, by Theorem 1.1 we have $\boldsymbol{X}^2_{-k-1,T}(\Omega) \hookrightarrow \boldsymbol{W}^{1,2}_{-k}(\Omega)$. Thank's to (IV.7) we obtain

$$\int_\Omega \boldsymbol{z} \cdot \mathbf{curl}\, \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} = \langle \boldsymbol{z} \times \boldsymbol{n}, \boldsymbol{\varphi} \rangle_{\boldsymbol{H}^{-1/2}(\Gamma) \times \boldsymbol{H}^{1/2}(\Gamma)} + \int_\Omega \mathbf{curl}\, \boldsymbol{z} \cdot \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x}. \tag{IV.65}$$

Compare (IV.65) with (IV.62) and using (IV.64), we deduce that

$$\forall \boldsymbol{\varphi} \in \boldsymbol{X}^2_{-k-1,T}(\Omega), \quad \langle \boldsymbol{z} \times \boldsymbol{n}, \boldsymbol{\varphi} \rangle_\Gamma = 0.$$

Let now $\boldsymbol{\mu}$ any element of the space $\boldsymbol{H}^{1/2}(\Gamma)$. As Ω' is bounded, we can fix once for all a ball B_R, centered at the origin and with radius R, such that $\overline{\Omega'} \subset B_R$. Setting $\Omega_R = \Omega \cap B_R$, then we have the existence of $\boldsymbol{\varphi}$ in $\boldsymbol{H}^1(\Omega_R)$ such that $\boldsymbol{\varphi} = \mathbf{0}$ on ∂B_R and $\boldsymbol{\varphi} = \boldsymbol{\mu}_t$ on Γ, where $\boldsymbol{\mu}_t$ is the tangential component of $\boldsymbol{\mu}$ on Γ. The function $\boldsymbol{\varphi}$ can be extended by zero outside B_R and the extended function, still denoted by $\boldsymbol{\varphi}$, belongs to $\boldsymbol{W}^{1,p}_\alpha(\Omega)$, for any α since its support is bounded. Thus $\boldsymbol{\varphi}$, belongs to $\boldsymbol{W}^{1,2}_{-k}(\Omega)$. It is clear that $\boldsymbol{\varphi}$ belongs to $\boldsymbol{X}^2_{-k-1,T}(\Omega)$ and

$$\langle \boldsymbol{z} \times \boldsymbol{n}, \boldsymbol{\mu} \rangle_\Gamma = \langle \boldsymbol{z} \times \boldsymbol{n}, \boldsymbol{\mu}_t \rangle_\Gamma = \langle \boldsymbol{z} \times \boldsymbol{n}, \boldsymbol{\varphi} \rangle_\Gamma = 0. \tag{IV.66}$$

This implies that $\boldsymbol{z} \times \boldsymbol{n} = \mathbf{0}$ on Γ which is the last boundary condition in (IV.59).

On the other hand, let us introduce the following problem: Find $\boldsymbol{\xi} \in \boldsymbol{V}^2_{k-1,T}(\Omega)$ such that

$$\forall \boldsymbol{\varphi} \in \boldsymbol{V}^2_{-k-1,T}(\Omega) \quad \int_\Omega \mathbf{curl}\, \boldsymbol{\xi} \cdot \mathbf{curl}\, \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x}. \tag{IV.67}$$

Problem (IV.67) can be solved by Lax-Milgram lemma if $k = 0$ and by Lemma 2.1 if $k = -1$.

We start by the case $k = -1$. Observe that Problem (IV.67) satisfies the Inf-Sup condition (IV.24). Let consider the following mapping $\boldsymbol{\ell} \,:\, \boldsymbol{V}^2_{0,T}(\Omega) \longrightarrow \mathbb{R}$ such that $\boldsymbol{\ell}(\boldsymbol{\varphi}) = \int_\Omega \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x}$. It is clear that $\boldsymbol{\ell}$ belongs to $(\boldsymbol{V}^2_{0,T}(\Omega))'$ and according to Remark 1.2, there exists a unique solution $\boldsymbol{\xi} \in \boldsymbol{V}^2_{-2,T}(\Omega)$. Applying Theorem 1.1, we deduce that this solution $\boldsymbol{\xi}$ belongs to $\boldsymbol{W}^{1,2}_{-1}(\Omega)$. It follows from Remark 1.2 i) and Theorem 1.2 that

$$|| \boldsymbol{\xi} ||_{\boldsymbol{W}^{1,2}_{-1}(\Omega)} \leq C || \boldsymbol{\ell} ||_{(\boldsymbol{V}^2_{0,T}(\Omega))'} \leq C \|\boldsymbol{v}\|_{\boldsymbol{W}^{0,2}_{-1}(\Omega)}. \tag{IV.68}$$

For $k = 0$, let us consider the bilinear form $\boldsymbol{b} :\, \boldsymbol{V}^2_{-1,T}(\Omega) \times \boldsymbol{V}^2_{-1,T}(\Omega) \longrightarrow \mathbb{R}$ such that

$$\boldsymbol{b}(\boldsymbol{\xi}, \boldsymbol{\varphi}) = \int_\Omega \mathbf{curl}\, \boldsymbol{\xi} \cdot \mathbf{curl}\, \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x}.$$

According to Theorem 1.1, $\boldsymbol{b}$ is continuous and coercive on $\boldsymbol{V}^2_{-1,T}(\Omega)$. Due to Lax-Milgram lemma, there exists a unique solution $\boldsymbol{\xi} \in \boldsymbol{V}^2_{-1,T}(\Omega)$ of Problem (IV.67). Using again Theorem 1.1, we prove that this solution $\boldsymbol{\xi}$ belongs to $\boldsymbol{W}^{1,2}_0(\Omega)$ and estimate (IV.60) follows immediately.

Next, we want to extend (IV.67) to any test function in $\boldsymbol{X}^2_{-k-1,T}(\Omega)$. Let $\boldsymbol{\varphi} \in \boldsymbol{X}^2_{-k-1,T}(\Omega)$ and let us solve the exterior Neumann problem:

$$\Delta\,\chi = \operatorname{div}\boldsymbol{\varphi} \;\text{ in }\; \Omega \quad\text{and}\quad \frac{\partial\,\chi}{\partial\boldsymbol{n}} = 0 \;\text{ on }\; \Gamma. \tag{IV.69}$$

It is shown in Lemma 3.7 and Theorem 3.9 of [38] that this problem has a unique solution χ in $W^{1,2}_{-k-1}(\Omega)$ if $k=-1$ and unique up to a constant if $k=0$. Set

$$\widetilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \nabla\,\chi. \tag{IV.70}$$

It is clear that for $k=0$ and $k=-1$, $\displaystyle\int_\Gamma \widetilde{\boldsymbol{\varphi}}\cdot\nabla\,(w(q)-q)\,d\sigma = 0$ for any $(w(q)-q)\in\mathcal{N}^{\Delta}_k$. Then $\widetilde{\boldsymbol{\varphi}}$ belongs to $\boldsymbol{V}^2_{-k-1,T}(\Omega)$. Now, if (IV.67) holds, we have

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x} \;=\; \int_\Omega \boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}.$$

Hence, problem (IV.62) and problem (IV.67) are equivalent. This implies that problem (IV.59) has a unique solution $\boldsymbol{\xi}$ in $\boldsymbol{W}^{1,2}_k(\Omega)$ for $k=0$ or $k=-1$.

Regularity: Now, we suppose that $\boldsymbol{v}\in\boldsymbol{W}^{1,2}_{k+1}(\Omega)\hookrightarrow\boldsymbol{W}^{0,2}_k(\Omega)$ and Ω' is of class $\mathcal{C}^{2,1}$. Let $\boldsymbol{\xi}\in\boldsymbol{W}^{1,2}_k(\Omega)$ the weak solution of (IV.59) and we set $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{\xi} - \boldsymbol{v}$. It is clear that $\boldsymbol{z}$ belongs to $\boldsymbol{X}^2_{k,N}(\Omega)$. Applying Theorem 1.2, we obtain that $\boldsymbol{z}\in\boldsymbol{W}^{1,2}_{k+1}(\Omega)$ and using (IV.15) and (IV.60) we obtain that

$$\begin{aligned}
\|\,\boldsymbol{z}\,\|_{\boldsymbol{W}^{1,2}_{k+1}(\Omega)} &\le C\Big(\|\,\boldsymbol{z}\,\|_{\boldsymbol{W}^{0,2}_k(\Omega)} + \|\operatorname{div}\boldsymbol{z}\|_{W^{0,2}_{k+1}(\Omega)}\Big)\\
&\le C\Big(\|\,\mathbf{curl}\,\boldsymbol{\xi}\,\|_{\boldsymbol{W}^{0,2}_k(\Omega)} + \|\,\boldsymbol{v}\,\|_{\boldsymbol{W}^{0,2}_k(\Omega)} + \|\operatorname{div}\boldsymbol{v}\|_{W^{0,2}_{k+1}(\Omega)}\Big)\\
&\le C\|\,\boldsymbol{v}\,\|_{\boldsymbol{W}^{1,2}_{k+1}(\Omega)}.
\end{aligned} \tag{IV.71}$$

This implies that $\boldsymbol{\xi}$ satisfies

$$\boldsymbol{\xi}\in\boldsymbol{W}^{1,2}_k(\Omega),\quad \operatorname{div}\boldsymbol{\xi} = 0\in\boldsymbol{W}^{1,2}_{k+1}(\Omega),\quad \mathbf{curl}\,\boldsymbol{\xi}\in\boldsymbol{W}^{1,2}_{k+1}(\Omega)\quad\text{and}\quad \boldsymbol{\xi}\cdot\boldsymbol{n} = 0\quad\text{on}\quad\Gamma.$$

Applying Corollary 3.16 in [38], we deduce that $\boldsymbol{\xi}$ belongs to $\boldsymbol{W}^{2,2}_{k+1}(\Omega)$ and using (IV.71), we obtain

$$\begin{aligned}
\|\,\boldsymbol{\xi}\,\|_{\boldsymbol{W}^{2,2}_{k+1}(\Omega)} &\le C\Big(\|\,\boldsymbol{\xi}\,\|_{\boldsymbol{W}^{1,2}_k(\Omega)} + \|\mathbf{curl}\,\boldsymbol{\xi}\|_{\boldsymbol{W}^{1,2}_{k+1}(\Omega)}\Big)\\
&\le C\Big(\|\,\boldsymbol{v}\,\|_{\boldsymbol{W}^{0,2}_k(\Omega)} + \|\boldsymbol{z}\|_{\boldsymbol{W}^{1,2}_{k+1}(\Omega)} + \|\,\boldsymbol{v}\,\|_{\boldsymbol{W}^{1,2}_{k+1}(\Omega)}\Big)\\
&\le \|\,\boldsymbol{v}\,\|_{\boldsymbol{W}^{1,2}_{k+1}(\Omega)}.
\end{aligned}$$

This finish the proof of the theorem. □

As consequence, we can prove other imbedding results. We start by the following theorem:

Theorem 2.2 *Let $k=-1$ or $k=0$. Then the space $\boldsymbol{Z}^2_{k,N}(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{1,2}_{k+1}(\Omega)$*

and we have the following estimate for any $\boldsymbol{v}$ in $\boldsymbol{Z}^2_{k,N}(\Omega)$:

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,2}_{k+1}(\Omega)} \leq C\big(\|\boldsymbol{v}\|_{\boldsymbol{W}^{0,2}_{k}(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{W}^{0,2}_{k+1}(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{W^{0,2}_{k+1}(\Omega)} + \|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{H}^{1/2}(\Gamma)}\big). \tag{IV.72}$$

Proof. Let $k=-1$ or $k=0$ and let $\boldsymbol{v}$ be any function of $\boldsymbol{Z}^2_{k,N}(\Omega)$. We set $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{\xi} - \boldsymbol{v}$ where $\boldsymbol{\xi} \in \boldsymbol{W}^{1,2}_{k}(\Omega)$ is the solution of the problem (IV.59). Hence, $\boldsymbol{z}$ belongs to the space $\boldsymbol{X}^2_{k,N}(\Omega)$. By Theorem 1.2 and (IV.15), $\boldsymbol{z}$ even belongs to $\boldsymbol{W}^{1,2}_{k+1}(\Omega)$ with the estimate:

$$\|\boldsymbol{z}\|_{\boldsymbol{W}^{1,2}_{k+1}(\Omega)} \leq C\big(\|\boldsymbol{z}\|_{\boldsymbol{W}^{0,2}_{k}(\Omega)} + \|\mathrm{div}\,\boldsymbol{z}\|_{W^{0,2}_{k+1}(\Omega)} + \|\mathbf{curl}\,\boldsymbol{z}\|_{\boldsymbol{W}^{0,2}_{k+1}(\Omega)}\big). \tag{IV.73}$$

Then, it suffices to prove that $\mathbf{curl}\,\boldsymbol{\xi} \in \boldsymbol{W}^{1,2}_{k+1}(\Omega)$ in order to obtain $\boldsymbol{v} \in \boldsymbol{W}^{1,2}_{k+1}(\Omega)$. Setting $\boldsymbol{\omega} = \mathbf{curl}\,\boldsymbol{\xi}$. It is clear that

$$\int_\Gamma \boldsymbol{\omega}\cdot\boldsymbol{n}\,d\sigma = 0 \tag{IV.74}$$

and then $\boldsymbol{\omega}$ satisfies:

$$\begin{cases} -\Delta\,\boldsymbol{\omega} = \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v} \quad \text{and} \quad \mathrm{div}\,\boldsymbol{\omega} = 0 \quad \text{in} \quad \Omega \\ \boldsymbol{\omega}\times\boldsymbol{n} = \boldsymbol{v}\times\boldsymbol{n} \quad \text{on} \quad \Gamma \quad \text{and} \quad \displaystyle\int_\Gamma (\boldsymbol{\omega}\cdot\boldsymbol{n})q\,d\sigma = 0,\ \forall\, q \in \mathcal{P}^{\Delta}_{-k-1}. \end{cases} \tag{IV.75}$$

Note that $\mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{W}^{0,2}_{k+1}(\Omega)$ then $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v}$ is in $[\mathring{\boldsymbol{H}}^{2}_{-k-2}(\mathbf{curl},\Omega)]'$ and we have $\boldsymbol{v}\times\boldsymbol{n} \in \boldsymbol{H}^{1/2}(\Gamma)$. Since $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\mathring{\boldsymbol{H}}^{2}_{-k-2}(\mathbf{curl}\,,\Omega)$, we prove that

$$\forall \boldsymbol{\phi} \in \boldsymbol{Y}^2_{k+2,N}(\Omega), \qquad \langle\, \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v}\,,\, \boldsymbol{\phi}\,\rangle_{[\mathring{\boldsymbol{H}}^{2}_{-k-2}(\mathbf{curl}\,,\Omega)]'\times\mathring{\boldsymbol{H}}^{2}_{-k-2}(\mathbf{curl}\,,\Omega)} = 0.$$

Due to Corollary 2.1, the function $\boldsymbol{\omega}$ belongs to $\boldsymbol{W}^{1,2}_{k+1}(\Omega)$ and satisfies the estimate:

$$\begin{aligned} \|\boldsymbol{\omega}\|_{\boldsymbol{W}^{1,2}_{k+1}(\Omega)} &\leq C\big(\|\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v}\|_{[\mathring{\boldsymbol{H}}^{2}_{-k-2}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{H}^{1/2}(\Gamma)}\big) \\ &\leq C\big(\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{W}^{0,2}_{k+1}(\Omega)} + \|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{H}^{1/2}(\Gamma)}\big). \end{aligned} \tag{IV.76}$$

Finally, estimate (IV.72) can be deduced by using inequalities (IV.73) and (IV.76). $\square$

Before giving the second imbedding result, we need to introduce the following space for any integer k in $\mathbb{Z}$:

$$\boldsymbol{M}^2_{k,N}(\Omega) = \Big\{\boldsymbol{v} \in \boldsymbol{W}^{1,2}_{k+1}(\Omega), \quad \mathrm{div}\,\boldsymbol{v} \in W^{1,2}_{k+2}(\Omega), \quad \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{W}^{1,2}_{k+2}(\Omega) \quad \text{and} \quad \boldsymbol{v}\times\boldsymbol{n} \in \boldsymbol{H}^{3/2}(\Gamma)\Big\}$$

Proposition 2.4 *Suppose that Ω' is of class $C^{2,1}$. Then the space $\boldsymbol{M}^2_{-1,N}(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{2,2}_{1}(\Omega)$ and we have the following estimate for any $\boldsymbol{v}$ in $\boldsymbol{M}^2_{-1,N}(\Omega)$:*

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{2,2}_{1}(\Omega)} \leq C\big(\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,2}_{0}(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{W}^{1,2}_{1}(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{W^{1,2}_{1}(\Omega)} + \|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{H}^{3/2}(\Gamma)}\big). \tag{IV.77}$$

Proof. The proof is very similar to that of Theorem 2.2. Let $\boldsymbol{v}$ be any function of $\boldsymbol{M}^2_{-1,N}(\Omega)$ and set $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{\xi} - \boldsymbol{v}$ where $\boldsymbol{\xi} \in \boldsymbol{W}^{2,2}_0(\Omega)$ is the solution of the problem (IV.59). According to Corollary 3.14 of [38], we prove that $\boldsymbol{z}$ belongs to $\boldsymbol{W}^{2,2}_1(\Omega)$ with the estimate:

$$\|\boldsymbol{z}\|_{\boldsymbol{W}^{2,2}_1(\Omega)} \leq C\big(\|\boldsymbol{z}\|_{\boldsymbol{W}^{1,2}_0(\Omega)} + \|\mathrm{div}\,\boldsymbol{z}\|_{W^{1,2}_1(\Omega)} + \|\mathbf{curl}\,\boldsymbol{z}\|_{\boldsymbol{W}^{1,2}_1(\Omega)}\big). \qquad \text{(IV.78)}$$

Then, it suffices to prove that $\mathbf{curl}\,\boldsymbol{\xi} \in \boldsymbol{W}^{2,2}_1(\Omega)$ in order to obtain $\boldsymbol{v} \in \boldsymbol{W}^{2,2}_1(\Omega)$. We set $\boldsymbol{\omega} = \mathbf{curl}\,\boldsymbol{\xi}$. Using Theorem 3.1 of [38], we prove that $\boldsymbol{\omega}$ satisfies Problem (IV.75). Using the regularity of Corollary 2.1, we prove that $\boldsymbol{\omega}$ belongs to $\boldsymbol{W}^{2,2}_1(\Omega)$ and satisfies

$$|| \,\boldsymbol{\omega}\, ||_{\boldsymbol{W}^{1,2}_{-k}(\Omega)} \leqslant C\left(||\, \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v}\, ||_{\boldsymbol{W}^{0,2}_1(\Omega)} + ||\boldsymbol{v}\times\boldsymbol{n}||_{\boldsymbol{H}^{3/2}(\Gamma)}\right)$$

and then estimate (IV.77) follows from (IV.78). $\square$

3 Generalized solutions for $(\mathcal{S}_T)$ and $(\mathcal{S}_N)$

We start this sequel by introducing the following space:

$$\boldsymbol{E}^2(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,2}_0(\Omega);\ \Delta\,\boldsymbol{v} \in [\mathring{\boldsymbol{H}}^2_{-1}(\mathrm{div},\,\Omega)]'\}.$$

This is a Banach space for the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{E}^2(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,2}_0(\Omega)} + \|\Delta\,\boldsymbol{v}\|_{[\mathring{\boldsymbol{H}}^2_{-1}(\mathrm{div},\,\Omega)]'}.$$

We have the following preliminary result.

Lemma 3.1 *The space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{E}^2(\Omega)$.*

Proof. Let P be a continuous linear mapping from $\boldsymbol{W}^{1,2}_0(\Omega)$ to $\boldsymbol{W}^{1,2}_0(\mathbb{R}^3)$, such that $P\,\boldsymbol{v}|_\Omega = \boldsymbol{v}$ and let $\boldsymbol{\ell} \in (\boldsymbol{E}^2(\Omega))'$, such that for any $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$, we have $\langle \boldsymbol{\ell}, \boldsymbol{v}\rangle = 0$. We want to prove that $\boldsymbol{\ell} = \boldsymbol{0}$ on $\boldsymbol{E}^2(\Omega)$. Then there exists $(\boldsymbol{f}, \boldsymbol{g}) \in \boldsymbol{W}^{-1,2}_0(\mathbb{R}^3) \times \mathring{\boldsymbol{H}}^2_{-1}(\mathrm{div},\,\Omega)$ such that : for any $\boldsymbol{v} \in \boldsymbol{E}^2(\Omega)$,

$$\langle \boldsymbol{\ell}, \boldsymbol{v}\rangle = \langle \boldsymbol{f}, P\,\boldsymbol{v}\rangle_{\boldsymbol{W}^{-1,2}_0(\mathbb{R}^3)\times \boldsymbol{W}^{1,2}_0(\mathbb{R}^3)} + \langle \Delta\boldsymbol{v}, \boldsymbol{g}\rangle_{[\mathring{\boldsymbol{H}}^2_{-1}(\mathrm{div},\,\Omega)]'\times \mathring{\boldsymbol{H}}^2_{-1}(\mathrm{div},\,\Omega)}.$$

Observe that we can easily extend by zero the function $\boldsymbol{g}$ in such a way that $\widetilde{\boldsymbol{g}} \in \boldsymbol{H}^2_{-1}(\mathrm{div},\,\mathbb{R}^3)$. Now we take $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3)$. Then we have by assumption that:

$$\langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1,2}_0(\mathbb{R}^3)\times \boldsymbol{W}^{1,2}_0(\mathbb{R}^3)} + \int_{\mathbb{R}^3} \widetilde{\boldsymbol{g}}\cdot\Delta\boldsymbol{\varphi}\,dx = 0,$$

because $\langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle = \langle \boldsymbol{f}, P\,\boldsymbol{v}\rangle$ where $\boldsymbol{v} = \boldsymbol{\varphi}|_\Omega$. Thus we have $\boldsymbol{f} + \Delta\widetilde{\boldsymbol{g}} = \boldsymbol{0}$ in $\boldsymbol{\mathcal{D}}'(\mathbb{R}^3)$. Then we can deduce that $\Delta\widetilde{\boldsymbol{g}} = -\boldsymbol{f} \in \boldsymbol{W}^{-1,2}_0(\mathbb{R}^3)$ and due to Theorem 1.3 of [7], there exists a unique $\boldsymbol{\lambda} \in \boldsymbol{W}^{1,2}_0(\mathbb{R}^3)$ such that $\Delta\boldsymbol{\lambda} = \Delta\widetilde{\boldsymbol{g}}$. Thus the harmonic function $\boldsymbol{\lambda} - \widetilde{\boldsymbol{g}}$ belonging to $\boldsymbol{W}^{0,2}_{-1}(\mathbb{R}^3)$ is necessarily equal to zero. Since $\boldsymbol{g} \in \boldsymbol{W}^{1,2}_0(\Omega)$ and $\widetilde{\boldsymbol{g}} \in \boldsymbol{W}^{1,2}_0(\mathbb{R}^3)$, we deduce that $\boldsymbol{g} \in \mathring{\boldsymbol{W}}^{1,2}_0(\Omega)$. As $\mathfrak{D}(\Omega)$ is dense in $\mathring{\boldsymbol{W}}^{1,2}_0(\Omega)$, there exists a sequence $\boldsymbol{g}_k \in \mathfrak{D}(\Omega)$ such that $\boldsymbol{g}_k \to \boldsymbol{g}$ in $\boldsymbol{W}^{1,2}_0(\Omega)$, when $k \to \infty$. Then $\nabla \cdot \boldsymbol{g}_k \to \nabla \cdot \boldsymbol{g}$ in $L^2(\Omega)$. Since $\boldsymbol{W}^{1,2}_0(\Omega)$ is imbedded in $\boldsymbol{W}^{0,2}_{-1}(\Omega)$, we deduce that $\boldsymbol{g}_k \to \boldsymbol{g}$ in $\boldsymbol{H}^2_{-1}(\mathrm{div}, \Omega)$. Now, we consider $\boldsymbol{v} \in \boldsymbol{E}^2(\Omega)$ and we want to prove that $\langle \boldsymbol{\ell}, \boldsymbol{v}\rangle = 0$. Observe that:

$$\begin{aligned}\langle \boldsymbol{\ell}, \boldsymbol{v}\rangle &= -\langle \Delta\widetilde{\boldsymbol{g}}, P\boldsymbol{v}\rangle_{\boldsymbol{W}^{-1,2}_0(\mathbb{R}^3)\times \boldsymbol{W}^{1,2}_0(\mathbb{R}^3)} + \langle \Delta \boldsymbol{v}, \boldsymbol{g}\rangle_{[\mathring{\boldsymbol{H}}^2_{-1}(\mathrm{div},\Omega)]'\times\mathring{\boldsymbol{H}}^2_{-1}(\mathrm{div},\Omega)}\\
&= \lim_{k\to\infty}(-\int_\Omega \Delta\boldsymbol{g}_k\cdot\boldsymbol{v}dx + \langle\Delta\boldsymbol{v}, \boldsymbol{g}_k\rangle_{[\mathring{\boldsymbol{H}}^2_{-1}(\mathrm{div},\Omega)]'\times\mathring{\boldsymbol{H}}^2_{-1}(\mathrm{div},\Omega)}\\
&= \lim_{k\to\infty}(-\int_\Omega \Delta\boldsymbol{g}_k\cdot\boldsymbol{v}dx + \int_\Omega \boldsymbol{v}\cdot\Delta\boldsymbol{g}_k dx) = 0.\end{aligned}$$

□

As a consequence, we have the following result.

Corollary 3.1 *The linear mapping $\gamma\,:\, \boldsymbol{v} \to \mathbf{curl}\,\boldsymbol{v}|_\Gamma \times \boldsymbol{n}$ defined on $\mathfrak{D}(\overline{\Omega})$ can be extended to a linear continuous mapping*

$$\gamma\,:\, \boldsymbol{E}^2(\Omega) \longrightarrow \boldsymbol{H}^{-1/2}(\Gamma).$$

Moreover, we have the Green formula: for any $\boldsymbol{v} \in \boldsymbol{E}^2(\Omega)$ and any $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,2}_0(\Omega)$ such that $\mathrm{div}\,\boldsymbol{\varphi} = 0$ *in Ω and $\boldsymbol{\varphi}\cdot\boldsymbol{n} = 0$ on Γ,*

$$-\langle \Delta\,\boldsymbol{v},\, \boldsymbol{\varphi}\rangle_{[\mathring{\boldsymbol{H}}^2_{-1}(\mathrm{div},\Omega)]'\times\mathring{\boldsymbol{H}}^2_{-1}(\mathrm{div},\Omega)} = \int_\Omega \mathbf{curl}\,\boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} - \langle \mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n},\, \boldsymbol{\varphi}\rangle_\Gamma, \tag{IV.79}$$

where the duality on Γ is defined by $\langle\cdot,\cdot\rangle_\Gamma = \langle\cdot,\cdot\rangle_{\boldsymbol{H}^{-1/2}(\Gamma)\times\boldsymbol{H}^{1/2}(\Gamma)}$.

Proof. Let $\boldsymbol{v} \in \mathfrak{D}(\overline{\Omega})$. Observe that if $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,2}_0(\Omega)$ such that $\boldsymbol{\varphi}\cdot\boldsymbol{n} = 0$ on Γ we deduce that $\boldsymbol{\varphi} \in \boldsymbol{X}^2_{-1,T}(\Omega)$, then (IV.79) holds for such $\boldsymbol{\varphi}$. Now, let $\boldsymbol{\mu} \in \boldsymbol{H}^{1/2}(\Gamma)$, then there exists $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,2}_0(\Omega)$ such that $\boldsymbol{\varphi} = \boldsymbol{\mu}_t$ on Γ and that $\mathrm{div}\,\boldsymbol{\varphi} = 0$ with

$$||\boldsymbol{\varphi}||_{\boldsymbol{W}^{1,2}_0(\Omega)} \leqslant C||\boldsymbol{\mu}_t||_{\boldsymbol{H}^{1/2}(\Gamma)} \leqslant C||\boldsymbol{\mu}||_{\boldsymbol{H}^{1/2}(\Gamma)}. \tag{IV.80}$$

As a consequence, using (IV.79) we have

$$|\langle \mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n},\,\boldsymbol{\mu}\rangle_\Gamma| \leqslant C||\boldsymbol{v}||_{\boldsymbol{E}^2(\Omega)}||\boldsymbol{\mu}||_{\boldsymbol{H}^{1/2}(\Gamma)}.$$

Thus,

$$||\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n}||_{\boldsymbol{H}^{-1/2}(\Gamma)}| \leqslant C||\boldsymbol{v}||_{\boldsymbol{E}^2(\Omega)}.$$

We deduce that the linear mapping γ is continuous for the norm $\boldsymbol{E}^2(\Omega)$. Since $\mathfrak{D}(\overline{\Omega})$ is dense in $\boldsymbol{E}^2(\Omega)$, γ can be extended to by continuity to $\gamma \in \mathcal{L}(\boldsymbol{E}^2(\Omega), \boldsymbol{H}^{-1/2}(\Gamma))$ and formula (IV.79) holds for all $\boldsymbol{v} \in \boldsymbol{E}^2(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,2}_0(\Omega)$ such that $\mathrm{div}\,\boldsymbol{\varphi} = 0$ in Ω and $\boldsymbol{\varphi}\cdot\boldsymbol{n} = 0$ on Γ. □

Proposition 3.1 *Let* $\boldsymbol{f}$ *belongs to* $\boldsymbol{W}_1^{0,2}(\Omega)$ *with* $\operatorname{div}\boldsymbol{f}=0$ *in* Ω, $g\in H^{1/2}(\Gamma)$ *and* $\boldsymbol{h}\in\boldsymbol{H}^{-1/2}(\Gamma)$ *verify the following compatibility conditions: for any* $\boldsymbol{v}\in\boldsymbol{Y}_{1,T}^2(\Omega)$,

$$\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}+\langle\boldsymbol{h}\times\boldsymbol{n},\,\boldsymbol{v}\rangle_{\boldsymbol{H}^{-1/2}(\Gamma)\times\boldsymbol{H}^{1/2}(\Gamma)}=0, \tag{IV.81}$$

$$\boldsymbol{f}\cdot\boldsymbol{n}+\operatorname{div}_\Gamma(\boldsymbol{h}\times\boldsymbol{n})=0\quad\text{on }\Gamma, \tag{IV.82}$$

where $\operatorname{div}_\Gamma$ *is the surface divergence on* Γ. *Then, the problem*

$$(E_T)\qquad\begin{cases}-\Delta\,\boldsymbol{z}=\boldsymbol{f}\quad\text{and}\quad\operatorname{div}\boldsymbol{z}=0 & \text{in }\Omega,\\ \boldsymbol{z}\cdot\boldsymbol{n}=g\quad\text{and}\quad\mathbf{curl}\,\boldsymbol{z}\times\boldsymbol{n}=\boldsymbol{h}\times\boldsymbol{n} & \text{on }\Gamma,\end{cases}$$

has a unique solution $\boldsymbol{z}$ *in* $\boldsymbol{W}_0^{1,2}(\Omega)$ *satisfying the estimate:*

$$\|\,\boldsymbol{z}\|_{\boldsymbol{W}_0^{1,2}(\Omega)}\le C\left(\|\,\boldsymbol{f}\|_{\boldsymbol{W}_1^{0,2}(\Omega)}+\|\,g\,\|_{H^{1/2}(\Gamma)}+\|\,\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{H}^{-1/2}(\Gamma)}\right). \tag{IV.83}$$

Moreover, if $\boldsymbol{h}$ *in* $\boldsymbol{H}^{1/2}(\Gamma)$, g *in* $H^{3/2}(\Gamma)$ *and* Ω' *is of class* $C^{2,1}$, *then the solution* $\boldsymbol{z}$ *is in* $\boldsymbol{W}_1^{2,2}(\Omega)$ *and satisfies the estimate*

$$\|\,\boldsymbol{z}\|_{\boldsymbol{W}_1^{2,2}(\Omega)}\le C\left(\|\,\boldsymbol{f}\|_{\boldsymbol{W}_1^{0,2}(\Omega)}+\|\,g\,\|_{H^{3/2}(\Gamma)}+\|\,\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{H}^{1/2}(\Gamma)}\right). \tag{IV.84}$$

Proof. First, note that if $\boldsymbol{h}\in\boldsymbol{H}^{-1/2}(\Gamma)$, then $\boldsymbol{h}\times\boldsymbol{n}$ also belongs to $\boldsymbol{H}^{-1/2}(\Gamma)$.
On the other hand, let us consider the Neumann problem:

$$(\mathcal{N})\qquad \Delta\,\theta=0\ \text{ in }\Omega\quad\text{and}\quad\frac{\partial\,\theta}{\partial\,\boldsymbol{n}}=g\ \text{ on }\Gamma.$$

It is shown in Theorem 3.9 of [38], that this problem has a unique solution $\theta\in W_0^{2,2}(\Omega)/\mathbb{R}$ satisfying the estimate:

$$\|\,\theta\,\|_{W_0^{2,2}(\Omega)}\le C\|\,g\,\|_{H^{1/2}(\Gamma)}. \tag{IV.85}$$

Setting $\boldsymbol{\xi}=\boldsymbol{z}-\nabla\,\theta$, then problem (E_T) becomes: find $\boldsymbol{\xi}\in\boldsymbol{W}_0^{1,2}(\Omega)$ such that

$$\begin{cases}-\Delta\,\boldsymbol{\xi}=\boldsymbol{f}\quad\text{and}\quad\operatorname{div}\boldsymbol{\xi}=0 & \text{in }\Omega,\\ \boldsymbol{\xi}\cdot\boldsymbol{n}=0\quad\text{and}\quad\mathbf{curl}\,\boldsymbol{\xi}\times\boldsymbol{n}=\boldsymbol{h}\times\boldsymbol{n} & \text{on }\Gamma.\end{cases} \tag{IV.86}$$

Now, observe that problem (IV.86) is reduced to the following variational problem: Find $\boldsymbol{\xi}\in\boldsymbol{V}_{-1,T}^{\,2}(\Omega)$ such that

$$\forall\boldsymbol{\varphi}\in\boldsymbol{X}_{-1,T}^{\,2}(\Omega)\qquad\int_\Omega\mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}=\int_\Omega\boldsymbol{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}+\langle\,\boldsymbol{h}\times\boldsymbol{n},\,\boldsymbol{\varphi}\,\rangle_\Gamma. \tag{IV.87}$$

Indeed, every solution of (IV.86) also solves (IV.87). Conversely, let $\boldsymbol{\xi}$ a solution of the problem (IV.87). Then,

$$\forall\boldsymbol{\varphi}\in\boldsymbol{\mathcal{D}}(\Omega),\qquad\langle\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\xi}-\boldsymbol{f},\,\boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega)\times\boldsymbol{\mathcal{D}}(\Omega)}=0.$$

So $-\Delta\boldsymbol{\xi} = \boldsymbol{f}$ in Ω. Moreover, by the fact that $\boldsymbol{\xi}$ belongs to the space $\boldsymbol{V}^{\,2}_{-1,T}(\Omega)$, we have $\operatorname{div}\boldsymbol{\xi} = 0$ in Ω and $\boldsymbol{\xi}\cdot\boldsymbol{n} = 0$ on Γ. Then, it remains to verify the boundary condition $\mathbf{curl}\,\boldsymbol{\xi}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n}$ on Γ. Observe that $\boldsymbol{\xi}$ belongs to $\boldsymbol{E}^{\,2}(\Omega)$ so by (IV.79) and comparing with (IV.87) we deduce that for any $\boldsymbol{\varphi}\in\boldsymbol{X}^{\,2}_{-1,T}(\Omega)$, we have:

$$\langle\,\mathbf{curl}\,\boldsymbol{\xi}\times\boldsymbol{n},\,\boldsymbol{\varphi}\,\rangle_\Gamma = \langle\,\boldsymbol{h}\times\boldsymbol{n},\,\boldsymbol{\varphi}\,\rangle_\Gamma.$$

Proceeding as in the proof of Theorem 2.1, we prove that $\mathbf{curl}\,\boldsymbol{\xi}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n}$ on Γ.

On the other hand, let us introduce the following problem: Find $\boldsymbol{\xi}\in\boldsymbol{V}^{\,2}_{-1,T}(\Omega)$ such that

$$\forall\boldsymbol{\varphi}\in\boldsymbol{V}^{\,2}_{-1,T}(\Omega)\qquad\int_\Omega\mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} = \int_\Omega\boldsymbol{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \langle\,\boldsymbol{h}\times\boldsymbol{n},\,\boldsymbol{\varphi}\,\rangle_\Gamma. \tag{IV.88}$$

As in the proof of Theorem 2.1, we use Lax-Milgram lemma to prove the existence of a unique solution $\boldsymbol{\xi}$ in $\boldsymbol{V}^{\,2}_{-1,T}(\Omega)$ of Problem (IV.88). Using Theorem 1.1, we prove that this solution $\boldsymbol{\xi}$ belongs to $\boldsymbol{W}^{1,2}_0(\Omega)$ and the following estimate follows immediately

$$||\boldsymbol{\xi}||_{\boldsymbol{W}^{1,2}_0(\Omega)}\leqslant C\left(\|\boldsymbol{f}\|_{\boldsymbol{W}^{0,2}_1(\Omega)} + \|\,\boldsymbol{h}\times\boldsymbol{n}\,\|_{\boldsymbol{H}^{\,-1/2}(\Gamma)}\right). \tag{IV.89}$$

Next, we want to extend (IV.88) to any test function $\boldsymbol{\varphi}$ in $\boldsymbol{X}^{\,2}_{-1,T}(\Omega)$. Let $\widetilde{\boldsymbol{\varphi}}\in\boldsymbol{X}^{\,2}_{-1,T}(\Omega)$ and let us solve the exterior Neumann problem:

$$\Delta\,\chi = \operatorname{div}\widetilde{\boldsymbol{\varphi}}\ \text{ in }\Omega\quad\text{and}\quad\frac{\partial\,\chi}{\partial\boldsymbol{n}} = 0\ \text{ on }\Gamma. \tag{IV.90}$$

It is shown in Theorem 4.12 of [41] that this problem has a unique solution χ in $W^{2,2}_0(\Omega)$ up to an additive constant. Then, we set

$$\boldsymbol{\varphi} = \widetilde{\boldsymbol{\varphi}} - \nabla\,\chi. \tag{IV.91}$$

Since $W^{2,2}_0(\Omega)$ is imbedded in $W^{1,2}_{-1}(\Omega)$, then $\boldsymbol{\varphi}$ belongs to $\boldsymbol{V}^{\,2}_{-1,T}(\Omega)$. Now, if (IV.88) holds, we have

$$\begin{aligned}\int_\Omega\mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x} &= \int_\Omega\boldsymbol{f}\cdot\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x} + \langle\,\boldsymbol{h}\times\boldsymbol{n},\,\widetilde{\boldsymbol{\varphi}}\,\rangle_\Gamma\\ &- \int_\Omega\boldsymbol{f}\cdot\nabla\,\chi\mathrm{d}\boldsymbol{x} - \langle\,\boldsymbol{h}\times\boldsymbol{n},\,\nabla\,\chi\,\rangle_\Gamma.\end{aligned}$$

Using (IV.6) and (IV.82), we obtain

$$\int_\Omega\mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x} = \int_\Omega\boldsymbol{f}\cdot\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x} + \langle\,\boldsymbol{h}\times\boldsymbol{n},\,\widetilde{\boldsymbol{\varphi}}\,\rangle_\Gamma. \tag{IV.92}$$

This implies that problem (IV.87) and problem (IV.88) are equivalent and thus problem (IV.86) has a unique solution $\boldsymbol{\xi}$ in $\boldsymbol{W}^{1,2}_0(\Omega)$. Finally, we set $\boldsymbol{z} = \boldsymbol{\xi} + \nabla\,\theta\in\boldsymbol{W}^{1,2}_0(\Omega)$ the unique solution of (E_T). Finally, (IV.83) follows immediately from (IV.89) and (IV.85).
Regularity of the solution: We suppose in addition that $\boldsymbol{h}$ is in $\boldsymbol{H}^{\,1/2}(\Gamma)$, g in $H^{3/2}(\Gamma)$ and Ω' is of class $C^{\,2,1}$ and let $\boldsymbol{z}$ in $\boldsymbol{W}^{1,2}_0(\Omega)$ be the weak solution of Problem (E_T). Setting $\boldsymbol{\omega} = \mathbf{curl}\,\boldsymbol{z}$, then

$\boldsymbol{\omega}$ satisfies

$$\boldsymbol{\omega} \in \boldsymbol{L}^2(\Omega), \quad \operatorname{div} \boldsymbol{\omega} = 0 \in W_1^{0,2}(\Omega), \quad \mathbf{curl}\, \boldsymbol{\omega} = \boldsymbol{f} \in \boldsymbol{W}_1^{0,2}(\Omega) \quad \text{and} \quad \boldsymbol{\omega} \times \boldsymbol{n} = \mathbf{curl}\, \boldsymbol{z} \times \boldsymbol{n} \in \boldsymbol{H}^{1/2}(\Gamma).$$

Applying Theorem 2.2 (with $k = 0$), we prove that $\boldsymbol{\omega}$ belongs to $\boldsymbol{W}_1^{1,2}(\Omega)$. This implies that $\boldsymbol{z}$ satisfies

$$\boldsymbol{z} \in \boldsymbol{W}_0^{1,2}(\Omega), \quad \operatorname{div} \boldsymbol{z} = 0 \in W_1^{1,2}(\Omega), \quad \mathbf{curl}\, \boldsymbol{z} \in \boldsymbol{W}_1^{1,2}(\Omega) \quad \text{and} \quad \boldsymbol{z} \cdot \boldsymbol{n} \in H^{3/2}(\Gamma).$$

Applying Proposition 2.2, we prove that $\boldsymbol{z}$ belongs to $\boldsymbol{W}_1^{2,2}(\Omega)$ and we have the estimate (IV.84). □

Next, we solve the Stokes problem $(\mathcal{S}_T)$.

Theorem 3.1 (**Weak solutions for** $(\mathcal{S}_\mathrm{T})$) *Suppose that* $g = 0$ *and* $\chi = 0$. *For* $\boldsymbol{f}$ *given in* $[\mathring{\boldsymbol{H}}^2_{-1}(\mathrm{div}, \Omega)]'$ *and* $\boldsymbol{h}$ *given in* $\boldsymbol{H}^{-1/2}(\Gamma)$ *satisfying* (IV.81). *The Stokes problem* $(\mathcal{S}_T)$ *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ *and we have the following estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} + \|\pi\|_{L^2(\Omega)} \leq \big(\|\boldsymbol{f}\|_{[\mathring{\boldsymbol{H}}^2_{-1}\mathrm{div}, \Omega)]'} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{H}^{-1/2}(\Gamma)}\big). \tag{IV.93}$$

Proof. At first, observe that problem $(\mathcal{S}_T)$ is reduced to the following variational problem: Find $\boldsymbol{u} \in \boldsymbol{V}^2_{-1,T}(\Omega)$ such that

$$\begin{aligned} \forall \boldsymbol{\varphi} \in \boldsymbol{V}^2_{-1,T}(\Omega), \qquad & \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} \\ = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{[\mathring{\boldsymbol{H}}^2_{-1}(\mathrm{div}, \Omega)]' \times \mathring{\boldsymbol{H}}^2_{-1}(\mathrm{div}, \Omega)} & + \langle\, \boldsymbol{h} \times \boldsymbol{n},\, \boldsymbol{\varphi} \,\rangle_\Gamma. \end{aligned} \tag{IV.94}$$

Indeed, every solution of $(\mathcal{S}_T)$ also solves (IV.94). Conversely, let $\boldsymbol{u}$ a solution of the problem (IV.94). Then,

$$\forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega) \quad \text{such that} \quad \operatorname{div} \boldsymbol{\varphi} = 0, \quad \langle -\Delta\, \boldsymbol{u} - \boldsymbol{f},\, \boldsymbol{\varphi} \rangle_{\boldsymbol{\mathcal{D}}'(\Omega) \times \boldsymbol{\mathcal{D}}(\Omega)} = 0.$$

By De Rham theorem, there exists $q \in \mathcal{D}'(\Omega)$ such that

$$-\Delta\, \boldsymbol{u} - \boldsymbol{f} = \nabla q \quad \text{in } \Omega.$$

Note that $[\mathring{\boldsymbol{H}}^2_{-1}(\mathrm{div}, \Omega)]'$ is imbedded in $\boldsymbol{W}_0^{-1,2}(\Omega)$ and thus $-\Delta\, \boldsymbol{u} - \boldsymbol{f} \in \boldsymbol{W}_0^{-1,2}(\Omega)$. It follows from Theorem 2.7 of [38], that there exists a unique real constant C and a unique $\pi \in L^2(\Omega)$ such that π has the decompositio $q = \pi + C$.

Observe that since $\boldsymbol{f}$ and $\nabla \pi$ are two elements of $[\mathring{\boldsymbol{H}}^2_{-1}(\mathrm{div}, \Omega)]'$, it is the same for $\Delta\, \boldsymbol{u}$. Since $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\mathring{\boldsymbol{H}}^2_{-1}(\mathrm{div}, \Omega)$, we obtain for any $\boldsymbol{\varphi} \in \mathring{\boldsymbol{H}}^2_{-1}(\mathrm{div}, \Omega)$ such that $\operatorname{div} \boldsymbol{\varphi} = 0$:

$$\langle \nabla \pi, \boldsymbol{\varphi} \rangle_{[\mathring{\boldsymbol{H}}_{-1}(\mathrm{div}, \Omega)]' \times \mathring{\boldsymbol{H}}_{-1}(\mathrm{div}, \Omega)} = 0.$$

Moreover, if $\boldsymbol{\varphi} \in \boldsymbol{V}^2_{-1,T}(\Omega)$, using Corollary 3.1 we have

$$\begin{aligned}\langle -\Delta\,\boldsymbol{u},\,\boldsymbol{\varphi}\rangle_{[\mathring{\boldsymbol{H}}^2_{-1}(\mathrm{div},\Omega)]'\times \boldsymbol{H}^2_{-1}(\mathrm{div},\Omega)} &= \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} - \\ &- \langle \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n},\,\boldsymbol{\varphi}\rangle_{\boldsymbol{H}^{-1/2}(\Gamma)\times\boldsymbol{H}^{1/2}(\Gamma)}.\end{aligned}$$

We deduce that for all $\boldsymbol{\varphi} \in \boldsymbol{V}^2_{-1,T}(\Omega)$

$$\langle \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n},\,\boldsymbol{\varphi}\rangle_{\boldsymbol{H}^{-1/2}(\Gamma)\times\boldsymbol{H}^{1/2}(\Gamma)} = \langle\,\boldsymbol{h}\times\boldsymbol{n},\,\boldsymbol{\varphi}\,\rangle_{\boldsymbol{H}^{-1/2}(\Gamma)\times\boldsymbol{H}^{1/2}(\Gamma)}.$$

Let now $\boldsymbol{\mu}$ any element of the space $\boldsymbol{H}^{1/2}(\Gamma)$. So, there exists an element $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,2}_0(\Omega)$ such that $\mathrm{div}\,\boldsymbol{\varphi} = 0$ in Ω and $\boldsymbol{\varphi} = \boldsymbol{\mu}_t$ on Γ. It is clear that $\boldsymbol{\varphi} \in \boldsymbol{V}^2_{-1,T}(\Omega)$ and

$$\begin{aligned}\langle \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n},\,\boldsymbol{\mu}\rangle_\Gamma - \langle \boldsymbol{h}\times\boldsymbol{n},\,\boldsymbol{\mu}\rangle_\Gamma &= \langle \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n},\,\boldsymbol{\mu}_t\rangle_\Gamma - \langle \boldsymbol{h}\times\boldsymbol{n},\,\boldsymbol{\mu}_t\rangle_\Gamma \\ &= \langle \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n},\,\boldsymbol{\varphi}\rangle_\Gamma - \langle \boldsymbol{h}\times\boldsymbol{n},\,\boldsymbol{\varphi}\rangle_\Gamma = 0.\end{aligned}$$

This implies that $\mathbf{cur}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n}$ on Γ. As a consequence, Problem (IV.94) and $(\mathcal{S}_T)$ are equivalent. As in the proof of Theorem 2.1, we use Lax-Milgram lemma to prove the existence of a unique solution $\boldsymbol{u}$ in $\boldsymbol{V}^2_{-1,T}(\Omega)$ of Problem (IV.94). Using Theorem 1.1, we prove that this solution $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{1,2}_0(\Omega)$. Then the pair $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}^{1,2}_0(\Omega)\times \boldsymbol{L}^2(\Omega)$ is the unique solution of the problem $(\mathcal{S}_T)$. The estimate (IV.93) follows from (IV.14). □

Corollary 3.2 *Let* $\boldsymbol{f}$, χ, g, $\boldsymbol{h}$ *such that*

$$\boldsymbol{f}\in [\mathring{\boldsymbol{H}}^2_{-1}(\mathrm{div},\,\Omega)]',\ \chi\in L^2(\Omega),\ \ g\in H^{1/2}(\Gamma)\ \text{ and }\ \boldsymbol{h}\in \boldsymbol{H}^{-1/2}(\Gamma),$$

and that (IV.81) *holds. Then, the Stokes problem* $(\mathcal{S}_T)$ *has a unique solution* $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}^{1,2}_0(\Omega)\times L^2(\Omega)$ *and we have:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,2}_0(\Omega)} + \|\,\pi\,\|_{L^2(\Omega)} \leqslant C\left(\|\,\boldsymbol{f}\|_{[\mathring{\boldsymbol{H}}^2_{-1}(\mathrm{div},\Omega)]'} + \|\,\chi\,\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{H}^{-1/2}(\Gamma)}\right). \quad \text{(IV.95)}$$

Proof. First case: We suppose that $\chi = 0$. Let $\theta \in W^{2,2}_0(\Omega)$ be a solution of the exterior Neumann problem $(\mathcal{N})$. Setting $\boldsymbol{z} = \boldsymbol{u} - \nabla\,\theta$, then, problem $(\mathcal{S}_T)$ becomes: Find $(\boldsymbol{z},\pi) \in \boldsymbol{W}^{1,2}_0(\Omega)\times L^2(\Omega)$ such that

$$\begin{cases} -\Delta\,\boldsymbol{z} + \nabla\,\pi = \boldsymbol{f} \quad \text{and} \quad \mathrm{div}\,\boldsymbol{z} = 0 & \text{in}\,\Omega,\\ \boldsymbol{z}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{z}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} & \text{on}\,\Gamma,\end{cases} \quad \text{(IV.96)}$$

Due to Theorem 3.1, this problem has a unique solution $(\boldsymbol{z}\,,\,\pi) \in \boldsymbol{W}^{1,2}_0(\Omega)\times L^2(\Omega)$. Thus $\boldsymbol{u} = \boldsymbol{z} + \nabla\,\theta$ belongs to $\boldsymbol{W}^{1,2}_0(\Omega)$ and using (IV.85) and (IV.93), we deduce that

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,2}_0(\Omega)} + \|\pi\|_{L^2(\Omega)} \le C(\|\,\boldsymbol{f}\,\|_{[\mathring{\boldsymbol{H}}^2_{-1}(\mathrm{div},\,\Omega)]'} + \|g\|_{H^{1/2}(\Gamma)} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{H}^{-1/2}(\Gamma)}). \quad \text{(IV.97)}$$

Second case: We suppose that $\chi \in L^2(\Omega)$. We solve the following Neumann problem in Ω:

$$\Delta\,\theta = \chi \quad \text{in} \quad \Omega, \qquad \frac{\partial \theta}{\partial \boldsymbol{n}} = g \quad \text{on} \quad \Gamma. \tag{IV.98}$$

It follows from Theorem 3.9 in [38] that Problem (IV.98) has a unique solution θ in $W_0^{2,2}(\Omega)/\mathbb{R}$ and we have:

$$||\theta||_{W_0^{2,2}(\Omega)/\mathbb{R}} \leqslant C\left(||\chi||_{L^2(\Omega)} + ||g||_{H^{1/2}(\Gamma)}\right). \tag{IV.99}$$

Setting $\boldsymbol{z} = \boldsymbol{u} - \nabla\,\theta$, then Problem problem $(\mathcal{S}_T)$ becomes: Find $(\boldsymbol{z}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ such that

$$\begin{cases} -\Delta\,\boldsymbol{z} + \nabla\,\pi = \boldsymbol{f} + \nabla\,\chi \quad \text{and} \quad \operatorname{div} \boldsymbol{z} = 0 & \text{in}\,\Omega, \\ \boldsymbol{z}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{z}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} & \text{on}\,\Gamma, \end{cases} \tag{IV.100}$$

Observe that $\boldsymbol{f} + \nabla\,\chi$ belongs to $[\mathring{\boldsymbol{H}}^{2}_{-1}(\mathrm{div},\,\Omega)]'$ and $\langle\nabla\,\chi, \boldsymbol{v}\rangle_{[\mathring{\boldsymbol{H}}^{2}_{-1}(\mathrm{div},\,\Omega)]'\times\mathring{\boldsymbol{H}}^{2}_{-1}(\mathrm{div},\,\Omega)} = 0$ for all $\boldsymbol{v}$ in $\boldsymbol{Y}^2_{1,T}(\Omega)$. According to the first step, this problem has a unique solution $(\boldsymbol{z}\,,\,\pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$. Thus $\boldsymbol{u} = \boldsymbol{z} + \nabla\,\theta$ belongs to $\boldsymbol{W}_0^{1,2}(\Omega)$ and estimate (IV.95) follows from (IV.97) and (IV.99). □

Now, we study the problem $(\mathcal{S}_N)$:

Theorem 3.2 (**Weak solutions for** $(\mathcal{S}_\mathrm{N})$) *Assume that* $\chi = 0$. *For* $\boldsymbol{f}$ *given in* $[\mathring{\boldsymbol{H}}^{2}_{-1}(\mathbf{curl},\,\Omega)]'$, $\boldsymbol{g}$ *given in* $\boldsymbol{H}^{1/2}(\Gamma)$ *and* π_0 *in* $H^{1/2}(\Gamma)$, *satisfying the following compatibility condition:*

$$\forall \boldsymbol{\lambda} \in \boldsymbol{Y}^2_{1,N}(\Omega), \qquad \langle \boldsymbol{f},\ \boldsymbol{\lambda}\rangle_{[\mathring{\boldsymbol{H}}^{2}_{-1}(\mathbf{curl}\,,\Omega)]'\times\mathring{\boldsymbol{H}}^{2}_{-1}(\mathbf{curl}\,,\Omega)} = \langle\boldsymbol{\lambda}\cdot\boldsymbol{n}, \pi_0\rangle_\Gamma\,. \tag{IV.101}$$

Then the Stokes problem $(\mathcal{S}_N)$ *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times W_1^{1,2}(\Omega)$ *and we have:*

$$\|\,\boldsymbol{u}\,\|_{\boldsymbol{W}_0^{1,2}(\Omega)} + \|\,\pi\,\|_{W_1^{1,2}(\Omega)} \leqslant C(\|\,\boldsymbol{f}\|_{[\mathring{\boldsymbol{H}}^{2}_{-1}(\mathbf{curl},\,\Omega)]'} + \|\,\boldsymbol{g}\times\boldsymbol{n}\,\|_{\boldsymbol{H}^{1/2}(\Gamma)} + \|\,\pi_0\,\|_{H^{1/2}(\Gamma)}). \tag{IV.102}$$

Proof. First, we consider the following problem:

$$\Delta\,\pi = \operatorname{div}\boldsymbol{f} \quad \text{in} \quad \Omega, \qquad \pi = \pi_0 \quad \text{on} \quad \Gamma. \tag{IV.103}$$

Since $\boldsymbol{f} \in [\mathring{\boldsymbol{H}}^{2}_{-1}(\mathbf{curl},\,\Omega)]'$, we deduce from Proposition 1.2 that $\operatorname{div}\boldsymbol{f}$ belongs to $\boldsymbol{W}_1^{-1,2}(\Omega)$. Now, let $(v(1)-1)$ an element of $\mathcal{A}_0^\Delta$, it is clear that $\nabla(v(1)-1)$ belongs to $\boldsymbol{Y}^2_{1,N}(\Omega)$. Then using the density of $\boldsymbol{\mathcal{D}}(\Omega)$ in $\mathring{\boldsymbol{W}}^{1,2}_{-1}(\Omega)$ and (IV.101), we prove that

$$\begin{aligned} \langle\operatorname{div}\boldsymbol{f}, (v(1)-1)\rangle_{\boldsymbol{W}_1^{-1,2}(\Omega)\times\mathring{\boldsymbol{W}}^{1,2}_{-1}(\Omega)} &= -\langle\,\boldsymbol{f}\,,\,\nabla\,(v(1)-1)\rangle_{[\mathring{\boldsymbol{H}}^{2}_{-1}(\mathbf{curl}\,,\Omega)]'\times\mathring{\boldsymbol{H}}^{2}_{-1}(\mathbf{curl}\,,\Omega)} \\ &= -\,\langle\nabla\,(v(1)-1)\cdot\boldsymbol{n}, \pi_0\rangle_\Gamma\,. \end{aligned} \tag{IV.104}$$

Since (IV.104) is satisfied, we apply Theorem 3.6 of [38] to prove that Problem (IV.103) has a unique solution $\pi \in W_1^{1,2}(\Omega)$ and we have the following estimate:

$$\|\,\pi\,\|_{W_1^{1,2}(\Omega)} \leqslant C\left(\|\,\operatorname{div}\boldsymbol{f}\,\|_{\boldsymbol{W}_1^{-1,2}(\Omega)} + \|\,\pi_0\,\|_{H^{1/2}(\Gamma)}\right). \tag{IV.105}$$

Setting $\boldsymbol{F} = \boldsymbol{f} - \nabla\pi$, then $\boldsymbol{F}$ belongs to $[\mathring{\boldsymbol{H}}^2_{-1}(\mathbf{curl},\,\Omega)]'$. Thus Problem $(\mathcal{S}_N)$ becomes: Find $\boldsymbol{u} \in \boldsymbol{W}^{1,2}_0(\Omega)$ such that:

$$\begin{cases} -\Delta\,\boldsymbol{u} = \boldsymbol{F} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \\ \quad \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n} \quad \text{on } \Gamma \quad \text{and} \quad \displaystyle\int_\Gamma \boldsymbol{u}\cdot\boldsymbol{n}\,d\sigma = 0. \end{cases} \tag{IV.106}$$

Using (IV.101) and the fact that $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\mathring{\boldsymbol{H}}^2_{-1}(\mathbf{curl},\,\Omega)$, we prove that:

$$\forall\boldsymbol{\lambda}\in\,\boldsymbol{Y}^2_{1,N}(\Omega),\qquad \langle\,\boldsymbol{F}\,,\,\boldsymbol{\lambda}\,\rangle_{[\mathring{\boldsymbol{H}}^2_{-1}(\mathbf{curl}\,,\Omega)]'\times\mathring{\boldsymbol{H}}^2_{-1}(\mathbf{curl}\,,\Omega)} = 0. \tag{IV.107}$$

Therefore $\boldsymbol{F}$ satisfies the assumptions of Corollary 2.1 and thus Problem (IV.106) has a unique solution $\boldsymbol{u}\in\,\boldsymbol{W}^{1,2}_0(\Omega)$ with

$$||\boldsymbol{u}||_{\boldsymbol{W}^{1,2}_0(\Omega)} \leqslant C\left(||\boldsymbol{F}||_{[\mathring{\boldsymbol{H}}^2_{-1}(\mathbf{curl},\Omega)]'} + ||\boldsymbol{g}\times\boldsymbol{n}||_{\boldsymbol{H}^{1/2}(\Gamma)}\right). \tag{IV.108}$$

Thus estimate (IV.102) follows from (IV.117) and from (IV.105). □

Corollary 3.3 *Let $\boldsymbol{f}$, χ, $\boldsymbol{g}$, π_0 such that*

$$\boldsymbol{f}\in[\mathring{\boldsymbol{H}}^2_{-1}(\mathbf{curl},\Omega)]',\ \chi\in W^{1,2}_1(\Omega),\ \boldsymbol{g}\in\boldsymbol{H}^{1/2}(\Gamma),\ \pi_0\in H^{1/2}(\Gamma),$$

and satisfying the compatibility condition:

$$\forall\boldsymbol{\lambda}\in\,\boldsymbol{Y}^2_{1,N}(\Omega),\ \ \langle\boldsymbol{f},\,\boldsymbol{\lambda}\,\rangle_{[\mathring{\boldsymbol{H}}^2_{-1}(\mathbf{curl}\,,\Omega)]'\times\mathring{\boldsymbol{H}}^2_{-1}(\mathbf{curl}\,,\Omega)} = \langle\boldsymbol{\lambda}\cdot\boldsymbol{n},\pi_0-\chi\rangle_\Gamma\,. \tag{IV.109}$$

Then the Stokes problem $(\mathcal{S}_N)$ has a unique solution $(\boldsymbol{u},\pi)\in\,\boldsymbol{W}^{1,2}_0(\Omega)\times W^{1,2}_1(\Omega)$. Moreover, we have the following estimate:

$$\|\,\boldsymbol{u}\,\|_{\boldsymbol{W}^{1,2}_0(\Omega)} + \|\,\pi\,\|_{W^{1,2}_1(\Omega)} \leqslant C(\|\,\boldsymbol{f}\|_{[\mathring{\boldsymbol{H}}^2_{-1}(\mathbf{curl},\,\Omega)]'} + \|\,\boldsymbol{g}\times\boldsymbol{n}\,\|_{\boldsymbol{H}^{1/2}(\Gamma)} + \|\,\pi_0\,\|_{H^{1/2}(\Gamma)} + \|\,\chi\,\|_{W^{1,2}_1(\Omega)}). \tag{IV.110}$$

Proof. First, we consider the following problem:

$$\Delta\,\pi = \operatorname{div}\boldsymbol{f} + \Delta\,\chi \quad \text{in} \quad \Omega, \quad \pi = \pi_0 \quad \text{on} \quad \Gamma. \tag{IV.111}$$

Since $\boldsymbol{f}\,\in\,[\mathring{\boldsymbol{H}}^2_{-1}(\mathbf{curl},\,\Omega)]'$, we deduce from Proposition 1.2 that $\operatorname{div}\boldsymbol{f}+\Delta\,\chi$ belongs to $\boldsymbol{W}^{-1,2}_1(\Omega)$. Proceeding as in the proof of Theorem 3.2, we prove that

$$\langle\operatorname{div}\boldsymbol{f}+\Delta\,\chi,(v(1)-1)\rangle_{\boldsymbol{W}^{-1,2}_1(\Omega)\times\mathring{\boldsymbol{W}}^{1,2}_{-1}(\Omega)} = -\,\langle\nabla\,(v(1)-1)\cdot\boldsymbol{n},\pi_0\rangle_\Gamma$$

and then we apply Theorem 3.6 of [38] to prove that Problem (IV.111) has a unique solution $\pi\in W^{1,2}_1(\Omega)$ and we have the following estimate:

$$\|\,\pi\,\|_{W^{1,2}_1(\Omega)} \leqslant C\left(\|\operatorname{div}\boldsymbol{f}+\Delta\,\chi\,\|_{\boldsymbol{W}^{-1,2}_1(\Omega)} + \|\,\pi_0\,\|_{H^{1/2}(\Gamma)}\right). \tag{IV.112}$$

Thus Problem ($\mathcal{S}_N$) becomes: Find $\boldsymbol{u} \in \boldsymbol{W}_0^{1,2}(\Omega)$ such that:

$$\begin{cases} -\Delta\, \boldsymbol{u} = \boldsymbol{f} - \nabla\, \pi \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = \chi \quad \text{in } \Omega, \\ \quad \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} \quad \text{on } \Gamma \quad \text{and} \quad \displaystyle\int_\Gamma \boldsymbol{u} \cdot \boldsymbol{n}\, d\sigma = 0. \end{cases} \tag{IV.113}$$

On the other hand, let us solve the following Dirichlet problem:

$$\Delta\, \theta = \chi \quad \text{in} \quad \Omega, \quad \theta = 0 \quad \text{on} \quad \Gamma.$$

Since $W_1^{1,2}(\Omega)$ is imbedded in $L^2(\Omega)$, it follows from Theorem 3.5 of [38], that this problem has a unique solution $\theta \in W_0^{2,2}(\Omega)$ (there is no compatibility condition) and we have the following estimate:

$$\|\, \theta\, \|_{W_0^{2,2}(\Omega)} \leqslant C \|\, \chi\, \|_{L^2(\Omega)}. \tag{IV.114}$$

Setting

$$\boldsymbol{z} = \boldsymbol{u} - \left(\nabla\, \theta - \frac{1}{C_1} \langle \nabla\, \theta \cdot \boldsymbol{n}, 1 \rangle_\Gamma\, \nabla (v(1) - 1) \right),$$

where $v(1)$ is the unique solution in $W_0^{1,2}(\Omega)$ of the Dirichlet problem (IV.11) and $C_1 = \displaystyle\int_\Gamma \frac{\partial v(1)}{\partial \boldsymbol{n}}\, d\sigma$. We know from Lemma 3.11 of [38] that $C_1 > 0$ and that $\nabla(v(1) - 1)$ belongs to $\boldsymbol{Y}_{1,N}^2(\Omega)$. Then Problem (IV.113) becomes: Find $\boldsymbol{z} \in \boldsymbol{W}_0^{1,2}(\Omega)$ such that

$$\begin{cases} -\Delta\, \boldsymbol{z} = \boldsymbol{f} - \nabla\, \pi + \nabla\, \chi \quad \text{and} \quad \operatorname{div} \boldsymbol{z} = 0 \quad \text{in } \Omega, \\ \quad \boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} \quad \text{on } \Gamma \quad \text{and} \quad \displaystyle\int_\Gamma \boldsymbol{z} \cdot \boldsymbol{n}\, d\sigma = 0. \end{cases} \tag{IV.115}$$

Now, we will solve the Problem (IV.115). Setting $\boldsymbol{F} = \boldsymbol{f} - \nabla\, \pi + \nabla\, \chi$, then $\boldsymbol{F}$ belongs to $[\mathring{\boldsymbol{H}}_{-1}^2(\mathbf{curl}, \Omega)]'$. Using (IV.109) and the fact that $\boldsymbol{\mathfrak{D}}(\Omega)$ is dense in $\mathring{\boldsymbol{H}}_{-1}^2(\mathbf{curl}, \Omega)$, we prove that:

$$\forall \boldsymbol{\lambda} \in \boldsymbol{Y}_{1,N}^2(\Omega), \qquad \langle\, \boldsymbol{F}\,,\, \boldsymbol{\lambda}\, \rangle_{[\mathring{\boldsymbol{H}}_{-1}^2(\mathbf{curl}\,,\Omega)]' \times \mathring{\boldsymbol{H}}_{-1}^2(\mathbf{curl}\,,\Omega)} = 0. \tag{IV.116}$$

Therefore $\boldsymbol{F}$ satisfies the assumptions of Corollary 2.1 and thus Problem (IV.115) has a unique solution $\boldsymbol{z} \in \boldsymbol{W}_0^{1,2}(\Omega)$ with

$$||\boldsymbol{z}||_{\boldsymbol{W}_0^{1,2}(\Omega)} \leqslant C \left(||\boldsymbol{F}||_{[\mathring{\boldsymbol{H}}_{-1}^2(\mathbf{curl},\Omega)]'} + ||\boldsymbol{g} \times \boldsymbol{n}||_{\boldsymbol{H}^{1/2}(\Gamma)} \right) \tag{IV.117}$$

and estimate (IV.110) holds. □

4 Strong solutions for ($\mathcal{S}_T$) and ($\mathcal{S}_N$)

We prove in this sequel the existence and the uniqueness of strong solutions for Problem ($\mathcal{S}_T$) and ($\mathcal{S}_N$), we start by Problem ($\mathcal{S}_T$)

Theorem 4.1 *Suppose that Ω' is of class $C^{2,1}$. Let $\boldsymbol{f}$, χ, g, $\boldsymbol{h}$ such that*

$$\boldsymbol{f} \in \boldsymbol{W}_1^{0,2}(\Omega),\ \chi \in W_1^{1,2}(\Omega),\ g \in H^{3/2}(\Gamma) \text{ and } \boldsymbol{h} \in \boldsymbol{H}^{1/2}(\Gamma),$$

and that (IV.81) *holds. Then, the Stokes problem* $(\mathcal{S}_T)$ *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,2}(\Omega) \times W_1^{1,2}(\Omega)$ *and we have:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_1^{2,2}(\Omega)} + \|\pi\|_{W_1^{1,2}(\Omega)} \leqslant \|\boldsymbol{f}\|_{\boldsymbol{W}_1^{0,2}(\Omega)} + \|\chi\|_{W_1^{1,2}(\Omega)} + \|g\|_{H^{3/2}(\Gamma)} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{H}^{1/2}(\Gamma)}. \tag{IV.118}$$

Proof. **First case** $\chi = 0$. Since $\boldsymbol{W}_1^{0,2}(\Omega)$ is included in $[\mathring{\boldsymbol{H}}^2_{-1}(\text{div}, \Omega)]'$, we deduce that we are under the hypothesis of Corollary 3.2 and so Problem $(\mathcal{S}_T)$ has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$. Setting $\boldsymbol{z} = \mathbf{curl}\, \boldsymbol{u}$, then $\boldsymbol{z}$ satisfies

$$\boldsymbol{z} \in \boldsymbol{L}^2(\Omega), \quad \text{div}\, \boldsymbol{z} = 0 \in W_1^{0,2}(\Omega), \quad \mathbf{curl}\, \boldsymbol{z} = \boldsymbol{f} \in \boldsymbol{W}_1^{0,2}(\Omega) \quad \text{and} \quad \boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} \in \boldsymbol{H}^{1/2}(\Gamma).$$

Applying Theorem 2.2 (with $k = 0$), we prove that $\boldsymbol{z}$ belongs to $\boldsymbol{W}_1^{1,2}(\Omega)$. This implies that $\boldsymbol{u}$ satisfies

$$\boldsymbol{u} \in \boldsymbol{W}_0^{1,2}(\Omega), \quad \text{div}\, \boldsymbol{u} = 0 \in W_1^{1,2}(\Omega), \quad \mathbf{curl}\, \boldsymbol{u} \in \boldsymbol{W}_1^{1,2}(\Omega) \quad \text{and} \quad \boldsymbol{u} \cdot \boldsymbol{n} = g \in H^{3/2}(\Gamma).$$

Applying Proposition 2.2, we prove that $\boldsymbol{u}$ belongs to $\boldsymbol{W}_1^{2,2}(\Omega)$ and thus $\nabla \pi = \boldsymbol{f} + \Delta \boldsymbol{u} \in \boldsymbol{W}_1^{0,2}(\Omega)$. Since π is in $L^2(\Omega)$ then π is in $W_1^{1,2}(\Omega)$.
Second case χ is in $W_1^{1,2}(\Omega)$. Since Ω' is of class $C^{2,1}$, it follows from Theorem 3.9 in [38] that there exists a unique solution θ in $W_1^{3,2}(\Omega)/\mathbb{R}$ satisfies Problem (IV.98) and

$$||\theta||_{W_1^{3,2}(\Omega)/\mathbb{R}} \leqslant C \left(||\chi||_{W_1^{1,2}(\Omega)} + ||g||_{H^{3/2}(\Gamma)} \right). \tag{IV.119}$$

The rest of the proof is similar to that Corollary 3.2. □

Remark 4.1

Assume that the hypothesis of Theorem 4.1 hold and suppose in addition that $\chi = 0$. Let $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ the unique solution of Problem $(\mathcal{S}_T)$ then π satisfies the following problem

$$\text{div}(\nabla \pi - \boldsymbol{f}) = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad (\nabla \pi - \boldsymbol{f}) \cdot \boldsymbol{n} = -\text{div}_\Gamma\, (\boldsymbol{h} \times \boldsymbol{n}) \quad \text{on} \quad \Gamma. \tag{IV.120}$$

It follows from [41] that Problem (IV.120) has a solution π in $W_1^{1,2}(\Omega)$. Setting $\boldsymbol{F} = \nabla \pi - \boldsymbol{f} \in \boldsymbol{W}_1^{0,2}(\Omega)$. Then problem $(\mathcal{S}_T)$ becomes:

$$\begin{cases} -\Delta \boldsymbol{u} = \boldsymbol{F} \quad \text{and} \quad \text{div}\, \boldsymbol{u} = 0 & \text{in}\, \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = g \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} & \text{on}\, \Gamma. \end{cases}$$

Therefore, $\boldsymbol{F}$, g and $\boldsymbol{h}$ satisfy the assumptions of Proposition 3.1 and thus $\boldsymbol{u}$ belongs to $\boldsymbol{W}_1^{2,2}(\Omega)$.

Next, we study the regularity of the solution for Problem $(\mathcal{S}_N)$.

Theorem 4.2 *Suppose that Ω' is of class $C^{2,1}$. Let $\boldsymbol{f}$, χ, $\boldsymbol{g}$, π_0 such that*

$$\boldsymbol{f} \in \boldsymbol{W}_1^{0,2}(\Omega),\ \chi \in W_1^{1,2}(\Omega),\ \boldsymbol{g} \in \boldsymbol{H}^{3/2}(\Gamma),\ \pi_0 \in H^{1/2}(\Gamma),$$

and satisfying the compatibility condition (IV.109). *Then the Stokes problem* $(\mathcal{S}_N)$ *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,2}(\Omega) \times W_1^{1,2}(\Omega)$. *Moreover, we have the following estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_1^{2,2}(\Omega)} + \|\pi\|_{W_1^{1,2}(\Omega)} \leqslant C(\|\boldsymbol{f}\|_{\boldsymbol{W}_1^{0,2}(\Omega)} + \|\boldsymbol{g}\times\boldsymbol{n}\|_{\boldsymbol{H}^{3/2}(\Gamma)} + \|\pi_0\|_{H^{1/2}(\Gamma)} + \|\chi\|_{W_1^{1,2}(\Omega)}) \tag{IV.121}$$

Proof. **First case: We suppose that** $\chi = 0$. Since $\boldsymbol{W}_1^{0,2}(\Omega)$ is included in $[\mathring{\boldsymbol{H}}^2_{-1}(\mathrm{curl}, \Omega)]'$, we deduce that we are under the hypothesis of Corollary 3.2 and so Problem $(\mathcal{S}_N)$ has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,2}(\Omega) \times W_1^{1,2}(\Omega)$. Setting $\boldsymbol{z} = \mathbf{curl}\, \boldsymbol{u}$. Observe that $\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n}$ belongs to $\boldsymbol{H}^{3/2}(\Gamma)$ and thus $\mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{n}$ belongs to $H^{1/2}(\Gamma)$ and so $\boldsymbol{z}$ satisfies

$$\boldsymbol{z} \in \boldsymbol{L}^2(\Omega), \quad \mathrm{div}\,\boldsymbol{z} = 0 \in W_1^{0,2}(\Omega), \quad \mathbf{curl}\,\boldsymbol{z} = \boldsymbol{f} \in \boldsymbol{W}_1^{0,2}(\Omega) \quad \text{and} \quad \boldsymbol{z}\cdot\boldsymbol{n} = \mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{n} \in \boldsymbol{H}^{1/2}(\Gamma).$$

Applying Proposition 2.1 (with $k = 0$), we prove that $\boldsymbol{z}$ belongs to $\boldsymbol{W}_1^{1,2}(\Omega)$. This implies that $\boldsymbol{u}$ satisfies

$$\boldsymbol{u} \in \boldsymbol{W}_0^{1,2}(\Omega), \quad \mathrm{div}\,\boldsymbol{u} = 0 \in W_1^{1,2}(\Omega), \quad \mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{W}_1^{1,2}(\Omega) \quad \text{and} \quad \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n} \in H^{3/2}(\Gamma).$$

Applying Proposition 2.4, we prove that $\boldsymbol{u}$ belongs to $\boldsymbol{W}_1^{2,2}(\Omega)$.
Second case: χ **is in** $W_1^{1,2}(\Omega)$. The proof of this case is very similar to that Corollary 3.3. □

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