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Investissement optimal et évaluation d'actifs sous certaines imperfections de marché

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Résumé. Dans cette thèse, nous nous intéressons à des sujets différents en mathématiques financières, tous liés aux imperfections de marché et à la technique fondamentale de la maximisation d'utilité. Elle comporte trois parties. Dans la première, qui se base sur deux papiers, nous considérons le problème d'investissement optimal sur un marché financier avec coûts de transaction proportionnels. On commence par étudier le problème d'investissement dans le cas où la fonction d'utilité est multivariée (ce qui s'adapte particulièrement bien aux marchés des devises) et l'agent a une dotation initiale aléatoire, qui peut s'interpréter comme une option ou un autre contrat dérivé. Après avoir analysé les propriétés du problème et de son dual, nous utilisons ces résultats pour examiner, dans ce contexte, certains aspects d'une technique de pricing devenue populaire dans le cadre des marchés incomplets, l'évaluation par indifférence d'utilité. Dans le deuxième chapitre, nous étudions le problème d'existence d'un ensemble de prix (appelés "prix fictifs" ou "shadow prices") qui offrirait la même utilité maximale à l'agent si le marché n'avait pas de frictions. Ces résultats sont utiles pour clarifier le lien entre la théorie classique des marchés sans frictions et la littérature en croissance rapide sur les coûts de transaction. Dans la deuxième partie de cette thèse, nous considérons le problème d'évaluation de produits dérivés par indifférence d'utilité dans des marchés incomplets, où la source d'incomplétude provient du fait que certains actifs ne peuvent pas être échangés sur le marché, ce qui est le cas par exemple dans le cadre des modèles structurels pour le prix de l'électricité. Sous certaines hypothèses, nous dérivons une caractérisation en terme d'équations différentielles stochastiques rétrogrades (EDSR) pour le prix, et nous nous concentrons ensuite sur les options européennes en établissant en particulier l'existence d'une stratégie de couverture optimale, même lorsque le payoff présente des discontinuités et est éventuellement non borné. Dans la dernière partie, nous analysons un simple problème de principal-agent à horizon fini, où le principal est essentiellement interprété comme un régulateur et l'agent comme une entreprise qui produit certaines émissions polluantes. Nous traitons séparément les problèmes du principal et de l'agent et nous utilisons la théorie des EDSR pour fournir des conditions nécessaires et suffisantes d'optimalité. Nous effectuons également des analyses de sensibilité et nous montrons des résultats numériques dans le but de fournir une meilleure compréhension du comportement des agents.

Mots clés. Théorie de l'utilité; Investissement optimal; Coûts de transaction proportionnels; Prix fictifs; Evaluation par indifférence d'utilité; Problèmes de principal-agent; Dualité; EDSR; Marchés d'électricité.

Abstract. In this thesis we deal with different topics in financial mathematics, that are all related to market imperfections and to the fundamental technique of utility maximization. The work consists of three parts. In the first one, which is based on two papers, we consider the problem of optimal investment on a financial market with proportional transaction costs. We initially study the investment problem in the case where the utility function is multivariate (which is particularly suitable on currency markets) and the agent is endowed with a random claim, which can be interpreted as an option or another derivative contract. After analyzing the properties of the primal and dual problems, we apply those results to investigate, in this context, some aspects of a popular pricing technique in incomplete markets, i.e. utility indifference evaluation. In the second contribution to the transaction costs literature, we investigate the existence problem for a set of prices (called *shadow prices*) that would provide the same maximal utility to the agent if the market did not have frictions. These results shed some light on the link between the classical theory of frictionless markets and the quickly growing literature on transaction costs. In the second part of this thesis we consider the utility indifference pricing problem in incomplete markets, where the source of incompleteness comes from the fact that some assets in the market cannot be actively traded, which is the case for example in the framework of structural models for electricity prices. We provide a BSDE characterization for the price under mild assumptions, and then focus on the case of European claims by establishing in particular the existence of an optimal hedging strategy even when the claim presents discontinuities and is possibly unbounded. In the last contribution we analyze a simple principal-agent problem in finite time horizon, where the principal is mainly interpreted as a regulator and the agent as a firm producing some kind of polluting emissions. We separately treat both the agent's and the principal's problems and use the BSDE theory for providing necessary and sufficient conditions for optimality. We also perform some sensitivity analyses and give numerical results in order to provide a better understanding of the agents' behavior.

Keywords. Utility theory; Optimal investment; Proportional transaction costs; Shadow prices; Utility indifference pricing; Principal-Agent problems; Duality; BSDEs; Electricity markets.

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Chapter 1

Introduction (en Français)

L'étude des modèles économiques et financiers commence généralement en supposant le cadre le plus simple possible permettant de capturer les idées principales en demeurant analytiquement (et/ou numériquement) traitable. Les détails qui ne sont pas pris en compte à ce stade sont parfois considérés comme négligeables, vu que leur contribution semble limitée à des "effets de second ordre", tout en laissant le message général pratiquement inchangé. D'autres fois, ces caractéristiques supplémentaires sont tout simplement trop difficiles à traiter d'un point de vue technique. Il arrive souvent, cependant, qu'une étude ultérieure de ces "imperfections" révèle qu'elles ont une influence beaucoup plus importante que prévu initialement, parfois au point de changer radicalement certains résultats bien établis, en ouvrant la voie à des nouveaux développements des théories mathématiques sous-jacentes. Un exemple frappant sort de l'introduction des coûts de transaction proportionnels (arbitrairement petits) dans la modélisation classique des marchés financiers et des processus de richesse associés (voir [LS97], [CPT99], [GRS08]): il s'avère, en effet, que ce nouveau détail (apparemment inoffensif) rend le plus petit prix de sur-réplication d'une option call exactement égal au prix du sous-jacent!
Le traitement classique de la théorie de l'arbitrage a aussi subi un changement radical avec l'introduction des coûts de transaction (voir [KS09], [S04], [JK95], [GRS10]), et la même chose est vraie pour la maximisation d'utilité ([DN90], [Bo02], [CO10], [CK96], [Gu02], [DPT01], [LPS98]). De manière similaire, le développement de la théorie de l'utilité dans des marchés incomplets s'est avéré être beaucoup plus compliquée que dans le cas classique (voir [KS99], [OZ09], [CSW01], [HG04]) en raison de la présence de plusieurs mesures de martingale qui nécessitent généralement d'analyser plus finement les propriétés des espaces de mesures. Il commence à être clair que même la plus petite imperfection de marché peut profondément modifier l'ensemble des stratégies de trading admissibles et donc la façon dont les agents couvrent leurs positions, le résultat étant généralement la perte de la possibilité d'une réplication parfaite. Coûts de transaction ou incomplétude de marché donnent alors lieu à une question pratique assez naturelle: comment faut-il évaluer les produit dérivés sur ces marchés? Comme les arguments classiques d'arbitrage ne sont généralement plus suffisants pour déterminer un prix unique, d'autres méthodes ont été proposées dans dans les dernières années, dont plusieurs profitent des résultats de la littérature sur la maximisation de l'utilité (voir [HH09]).
Un autre type d'imperfection dans les modèles économiques analysant les contrats opti-

maux est l'aléa moral, un problème qui est tellement omniprésent qu'il a suscité l'intérêt des chercheurs depuis très longtemps (voir [Ho79] pour l'une des premières contributions). Dans les dernières années, le sujet a de nouveau commencé à attirer une attention particulière, en raison du développement de la théorie de l'optimisation avec des processus en temps continu, qui s'est avérée être très utile pour apporter un nouveau regard sur ce vieux problème (voir [Sa08] pour une contribution originale et [CZ12] pour une analyse plus approfondie des problèmes mathématiques liés).
Les sujets que nous traitons dans cette thèse, qui pourraient sembler assez différents à première vue, sont tous liés à l'idée qu'un (ou plusieurs) agent(s) maximise(nt) leurs utilités espérées en présence de quelque sorte d'imperfection du marché.
La première partie de ce manuscrit traite des marchés avec coûts de transaction, selon la formalisation très générale qui a été mise en avant par Kabanov (et développée après par d'autres auteurs), qui est bien résumée dans l'ouvrage [KS09]. Cette partie est divisée en deux chapitres qui se concentrent sur des problèmes différents. Dans le premier, nous examinons le problème de maximisation d'utilité dans le cas d'une fonction d'utilité multivariée et d'une dotation initiale qui peut éventuellement être aléatoire (par exemple, le cas d'un contrat dérivé). Dans le second, nous étudions le problème d'existence de processus de prix fictifs, un concept qui fait le lien entre le la littérature sur les marchés sans frictions et celle sur les marchés avec coûts de transaction. Ces contributions sont introduites dans les Sections 1.1 et 1.2 ci-dessous.
Dans la deuxième partie, qui est résumée dans la Section 1.3, nous examinons un marché financier incomplet, où la source d'incomplétude découle du fait que certains actifs ne peuvent pas être échangés activement par l'investisseur. Nous étudions en particulier le problème d'évaluation par indifférence d'utilité (exponentielle) qui a été initialement introduit par [ER00] dans un modèle légèrement différent, en assouplissant certaines des hypothèses pour les adapter à notre cadre spécifique. La motivation de cette analyse provient principalement de la littérature sur les modèles structurels pour les prix de l'électricité, qui est bien resumée dans [CC12].
La dernière partie de ce manuscrit traite un problème d'aléa moral sous la forme d'un modèle économique du type "principal-agent" en temps continu, qui a été principalement motivé par la question des incitations optimales pour la réduction des émissions polluantes. Ce chapitre est sans doute le moins ambitieux mathématiquement, vu que la plupart des techniques utilisées ont déjà été employées par d'autres auteurs dans des contextes similaires. Le problème, cependant, ainsi que la motivation sous-jacente, est nouveau et il est traité en essayant de garder un oeil constant sur les interprétations qualitatives des résultats obtenus à travers des expériences numériques et des analyses de comparaison. Cette contribution est introduite dans la Section 1.4.

1.1 Marchés avec coûts de transaction I: Investissement optimal avec une fonction d'utilité multivariée et une dotation initiale aléatoire

Dans cette première partie, nous cherchons à généraliser les résultats du papier [CO10], traitant un problème d'investissement optimal avec coûts de transaction (dynamiques) et

une fonction d'utilité multivariée, dans le cas où la dotation initiale de l'agent n'est pas déterministe, mais est donnée par une variable aléatoire (habituellement interprétable comme un actif contingent) généralement non bornée. D'autres contributions importantes dans la littérature sur l'investissement optimal avec dotation aléatoire comprennent [Bo02] (dans le cas d'une fonction d'utilité univariée et dotation bornée) et [DPT01] (avec une fonction d'utilité multivariée, mais des coûts de transaction constants). Nous supposons qu'il y a D actifs dans notre marché financier. Le problème de maximisation de l'utilité que nous traitons dans cette section peut s'écrire

$$u(\mathcal{E}) := \sup \left\{ \mathbb{E}\left[U(X+\mathcal{E})\right] : X \in \mathcal{A}_T^0 \right\}, \tag{1.1.1}$$

où

- La fonction valeur u est définie sur un ensemble de variables aléatoires $\mathcal{F}_T$-mesurables $\mathcal{E}$, qui peut être naturellement interprété comme un ensemble de contrats dérivés ou d'actifs contingents en général. Plus précisément, nous considérons principalement des claims qui sont bornés inférieurement par une constante et supérieurement par une stratégie admissible, et nous appelons cet ensemble $\mathcal{O}$. Cette situation comprend clairement le cas d'une dotation déterministe, déjà traité dans [CO10]. Ici $T > 0$ est une échéance fixée: on n'est donc intéressé que par les positions finales à cette date dans ce contexte.
- La fonction d'utilité U est supposé être multivariée *, c'est à dire qu'elle est définie sur $\mathbb{R}_+^D$. On suppose en outre qu'elle est strictement concave et croissante par rapport à l'ordre généré par l'orthant positif (dit plus simplement, elle augmente si toutes les composantes de son argument augmentent). D'autres propriétés plus techniques et un peu moins intuitives sur U seront introduites dans le texte.
- La maximisation a lieu sur l'ensemble $\mathcal{A}_T^0$, qui représente toutes les positions (aléatoires) qui peuvent être atteintes à la maturité T en investissant sur le marché de façon admissible à partir d'une dotation initiale égale à zéro (ce qui a un sens grâce à la structure linéaire de $\mathcal{A}_T^0$ et au fait que s'il y a une partie de dotation déterministe elle peut être inclue dans $\mathcal{E}$).

Les fonctions d'utilité multivariées ne sont que rarement utilisées dans la littérature, elles sont cependant intéressantes en tant que généralisations naturelles de leurs homologues univariées et parce qu'elles sont particulièrement adaptées aux marchés dans lesquels il semble quelque peu arbitraire de fixer un numéraire dès le début (par exemple, les marchés des devises).

Dans cette formulation plutôt abstraite les coûts de transaction sont cachés à l'intérieur de $\mathcal{A}_T^0$, car les positions qui peuvent être atteintes à l'échéance en investissant sur le marché sont clairement influencées par le coût qu'on paye à chaque fois que l'on échange un actif contre l'autre. Plus précisément, et en laissant de côté pour l'instant les subtilités techniques, l'ensemble $\mathcal{A}_T^0$ est constitué de toutes les valeurs finales de portefeuilles V_T telles que la dynamique du processus V vérifie la contrainte

$$dV_t \in -K_t,$$

∗. Dans l'article, nous travaillons avec la fonction $\tilde{U}$, définie à partir de U avec le but de permettre à l'investisseur de ne s'intéresser qu'à un sous-ensemble des actifs disponibles à la négociation. Cette extension n'est cependant pas essentielle ici afin d'introduire les idées principales.

où K_t est un cône (aléatoire) qui décrit l'ensemble des positions qui sont solvables à la date t et qui peut s'écrire

$$K_t = \text{cone}\{e^k, \Pi_t^{ij} e^i - e^j : 1 \leq k, i, j \leq D\}.$$

Ici, les e^i forment la base canonique de $\mathbb{R}^D$ et Π_t^{ij} modélise les conditions d'échange d'un actif contre un autre. Le processus Π satisfait les conditions suivantes:

- $\Pi^{ij} > 0$, $1 \leq i, j \leq D$.
- $\Pi^{ii} = 1$, $1 \leq i \leq D$.
- $\Pi^{ij} \leq \Pi^{ik}\Pi^{kj}$, $1 \leq i, j, k \leq D$.

En particulier, les coûts de transaction sont cachés dans la dernière condition qui dit, vulgairement, qu'il est toujours plus rentable d'échanger un actif i pour un autre actif j d'une manière directe au lieu de passer par l'achat d'un autre actif intermédiaire k (ce qui amène à payer des coûts de transaction inutiles). Le processus Π (et donc les coûts de transaction) étant éventuellement discontinu, on voit aisément que ce type de modélisation, mise avant par Kabanov ([KS09]), comprend et généralise toutes les autres descriptions plus explicites qui sont apparues dans la littérature.

Au début, nous nous intéressons aux propriétés de la fonction valeur, et plus précisément à la mesure dans laquelle elle hérite des caractéristiques de la fonction d'utilité d'origine U. Nous traitons cette question dans la Proposition 3.3.1 où nous montrons que u est bien définie, concave, croissante sur $\mathcal{O}$, et d'autres propriétés plus techniques telles que la caractérisation de la fermeture de son domaine effectif.

Il se révèle souvent utile, lorsque l'on étudie un problème d'optimisation tel que (1.1.1), d'écrire son *problème dual* correspondant, qui dans notre cas transforme une maximisation sur des stratégies en une minimisation sur un ensemble approprié de mesures. Au delà de son intérêt théorique, le problème dual peut être pratique pour étudier certaines des propriétés du problème initial (comme l'existence d'une solution). Dans ce contexte particulier, il jouera également un rôle clé dans la construction des prix fictifs (introduits dans la section suivante). Le problème dual peut s'écrire

$$v(\mathcal{E}) := \inf_{m \in \mathcal{D}} \left\{ \mathbb{E}\left[U^*\left(\frac{dm^c}{dP} \right) \right] + m(\mathcal{E}) \right\},$$

où U^* est la conjuguée de U définie comme

$$U^*(x^*) := \sup_{x \in \mathbb{R}^D} \{U(x) - \langle x, x^* \rangle\}$$

et l'ensemble de variables duales

$$\mathcal{D} := \left\{ m \in \text{ba}(\mathbb{R}^D) : m(X) \leq 0 \text{ for all } X \in L^\infty(\mathbb{R}^D) \cap \mathcal{A}_T^0 \right\}$$

est composé de mesures finiment additives sur $\mathbb{R}^D$. La notation m^c désigne ici la partie σ-additive dans la décomposition de Yosida-Hewitt de m (un résume sur les mesures finiment additives est contenu dans Section 3.2.3). Dans le Théorème 3.3.1 on montre que pour la classe de dotations initiales que l'on considère il n'y a pas de *gap de dualité*, c'est à dire que

$$u(\mathcal{E}) = v(\mathcal{E}).$$

En outre, la borne inférieure dans la définition de v est toujours atteinte. En utilisant des arguments de dualité, nous sommes également en mesure de prouver le Théorème 3.3.2 qui montre, sous l'hypothèse supplémentaire d'une dotation bornée, que le problème d'optimisation primal 1.1.1 admet une solution qui est égale à

$$\hat{X} := I\left(\frac{d\hat{m}^c}{dP}\right) - \mathcal{E},$$

où $\hat{m}$ est le minimiseur dans le problème dual et I atteint le maximum dans la définition de la conjuguée de U.
Comme déjà mentionné, en présence de coûts de transaction, on perd généralement la possibilité de réplication parfaite des actifs contingents, ce qui crée naturellement un problème d'évaluation. Avec cette motivation, dans la dernière partie du Chapitre 3, nous utilisons certains des résultats précédemment obtenus et certaines techniques développées dans [OZ09] pour étudier les prix par indifférence d'utilité (UIP) dans ce cadre specifique (les UIPs seront également l'objet du Chapitre 5, quoique dans un contexte différent). Plus précisément, l'UIP (d'achat) $p_j(B) = p_j(B; U, \mathcal{E}) \in \mathbb{R}$ pour l'actif contingent B (exprimé en unités de l'actif j) est implicitement défini comme la solution de l'équation

$$u(\mathcal{E} + B - e_j p_j) = u(\mathcal{E}). \tag{1.1.2}$$

En d'autres termes, nous cherchons un prix qui rend l'agent indifférent entre les deux possibilités suivantes: 1) acheter le claim B en payant son prix et 2) ne rien acheter, en prenant en compte que dans les deux cas il peut aussi bien investir dans le marché pour couvrir (partiellement) sa position. Il convient de noter que le prix dépendra en général de la dotation initiale $\mathcal{E}$ (contrairement au cas de l'utilité exponentielle que nous allons examiner dans le Chapitre 5). Dans la Proposition 3.4.1 nous montrons que la définition de l'UIP est bien posée, c.-à-d. $p_j(B)$ existe et est unique, et que ce prix satisfait en particulier les propriétés d'invariance, monotonie et convexité qui caractérisent une mesure de risque convexe définie sur un ensemble de variables aléatoires vectorielles (comparer, par exemple, avec [JMT04], [HHR10]). Nous fournissons également une représentation duale du prix, qui peut être exprimé comme le résultat d'un problème de minimisation sur un ensemble approprié de mesures.
Nous introduisons enfin le UIP moyen pour l'achat de β unités de l'actif B (en termes de l'actifs j), défini comme

$$p_j^\beta(B) := \frac{p_j(\beta B)}{\beta},$$

et nous étudions les propriétés de la fonction $\beta \mapsto p_j^\beta(B)$ dans la Proposition 3.4.2. À titre d'exemple, nous constatons qu'elle est décroissante en β et qu'on a les bornes suivantes:

$$\underline{m}_j(B) \leq p_j^\beta(B) \leq \hat{m}_j(B),$$

où $\underline{m}$ et $\hat{m}$ sont deux mesures convenablement définies.

1.2 Marchés avec coûts de transaction II: Existence des prix fictifs

Le deuxième chapitre de cette thèse se situe encore dans le cadre de marchés avec coûts de transaction, en utilisant la même modélisation qui a été introduite dans la section précedente. Plus précisément, nous nous intéressons ici au problème de prix fictifs (*shadow prices* en anglais), une question qui a attiré l'attention de beaucoup de chercheurs dans les dernières années pour plusieurs raisons. Tout d'abord, à cause de son intérêt purement théorique lié au fait que les prix fictifs font en quelque sorte le pont entre la littérature sur les marchés avec coûts de transaction et celle sur les marchés sans frictions, en permettant de trouver (dans un certain sens) un marché classique équivalent à un marché avec coûts de transaction donné au départ. Deuxièmement, les prix fictifs ont été utilisés dans certains cas pour résoudre des problèmes de maximisation d'utilité qui semblaient particulièrement difficiles à aborder en utilisant des approches plus standards (voir [GMS10],[GGMS11], [KM11], [KS10],[CSZ12], [GM11],[LPS98], [CK96]).
Nous allons maintenant décrire intuitivement le problème ainsi que les principales difficultés liées à son traitement.
Considérons un marché A avec coûts de transaction proportionnels. Comme nous l'avons déjà vu dans la section précédente, le problème de maximisation d'utilité dans ce marché peut s'écrire

$$J^A(x) := \sup_{f \in \mathcal{A}_T^x(A)} E[U(f)]. \tag{1.2.1}$$

où $\mathcal{A}_T^x(A)$ représente l'ensemble des richesses atteignables au temps T en investissant sur le marché A avec une dotation initiale x. Nous supposons également ici que à maturité toute la richesse est liquidée dans le premier actif (numéraire), de sorte que la fonction d'utilité dans cette section sera toujours univariée (et définie sur la droite réelle positive).
Prenons maintenant un marché classique sans frictions B, et supposons que les prix sur ce marché évoluent à l'intérieur de la fourchette bid-ask du marché d'origine. Dans le cas unidimensionnel avec un bond et un stock S cela peut être formalisé en écrivant simplement

$$\underline{S} \leq S \leq \bar{S},$$

où $\underline{S}$ et $\bar{S}$ sont les prix bid et ask pour le stock. Dans un cadre multidimensionnel, le même concept est décrit d'une manière un peu plus complexe à l'aide de l'ensemble polaire (dans le sens de l'analyse convexe) du cône de solvabilité K_t défini dans la section précédente. Par définition, dans le marché B, l'agent échange les actifs à des conditions toujours plus favorables par rapport au marché initial A, et par conséquent on a :

$$J^A(x) \leq J^B(x).$$

Le problème du prix fictif peut se résumer par la question suivante: existe-t-il un marché (particulièrement défavorable) B tel que $J^A(x) = J^B(x)$? Si c'est le cas, nous appelons un tel marché *marché fictif* et ses prix $\hat{S}$ *prix fictifs*.
Très intuitivement, les stratégies de trading optimales dans les deux marchés devront également coïncider, ce qui permet de résoudre entièrement le problème d'origine lorsque un marché fictif est effectivement disponible.

Si un prix fictif existe, cela signifie que le comportement de l'agent face à des coûts de transaction peut être en quelque sorte réduit à son comportement sur un marché sans frictions bien choisi. L'avantage immédiat réside dans le fait que la théorie des marchés classiques est beaucoup plus développée et, en général, bien plus simple que la théorie des marchés avec coûts de transaction. De plus, travailler sur un marché sans frictions permet de réduire la dimension du problème en évitant de considérer les prix bid et ask séparément. D'un autre côté, si la réponse s'avère négative, cela veut dire que le choix du portefeuille dans le marché avec coûts de transaction présente vraiment des aspects qualitativement différents par rapport à la théorie classique.
Comment peut-on trouver un prix fictif? Basiquement, nous aimerions trouver le marché sans frictions avec la plus petite fonction valeur, en espérant que ce marché se révèle être un marché fictif. Une idée est donc de calculer

$$\inf_{B\in\mathcal{B}_A}\sup_{f\in\mathcal{A}_T^x(A)} E[U(f)],$$

où $\mathcal{B}_A$ est l'ensemble de marchés sans frictions avec les prix qui évoluent dans le spread de A. De façon heuristique, on peut éliminer l'"inf sup" en utilisant la dualité dans les marchés classiques pour obtenir

$$\inf_{B\in\mathcal{B}_A, y>0}\left\{\mathbb{E}\left[U^*\left(y\frac{dQ(B)}{dP}\right)\right]+yx\right\},$$

où $Q(B)$ est la mesure martingale équivalente du marché B (supposée unique). S'il existe un optimum $(\hat{y},\hat{Q})$, et un prix correspondant $\hat{S}$, il reste à vérifier que la stratégie optimale qui en résulte est également admissible dans le marché avec coûts de transaction, ce qui nécessite généralement de vérifier que cette stratégie ne change pas en dehors des ensembles $\{\hat{S}=\underline{S}\}$ et $\{\hat{S}=\bar{S}\}$. Si c'est le cas, l'écart sera comblé et nous aurons alors

$$J^A(x)=E\left[U^*\left(\hat{y}\frac{d\hat{Q}}{dP}\right)\right]+\hat{y}x.$$

Cette approche est particulièrement utile pour comprendre comment la théorie des marchés classiques peut donner un coup de main dans la résolution des problèmes où des coûts de transaction sont impliqués. La puissance de cette méthode a été montrée en particulier dans l'article récent [CSZ12], où les auteurs l'utilisent pour résoudre le problème de Merton (avec la consommation) en présence de coûts de transaction en toute généralité, en supprimant les hypothèses sur les valeurs des paramètres qui avaient été imposées précédemment.
Nous pouvons également remarquer que, formellement, la procédure précédente consiste en une minimisation sur l'ensemble de martingales Z (en dimension D) telles que Z/Z^1 est dans le spread bid-ask, avec Z^1 jouant le rôle de mesure martingale équivalente. Les processus Z ayants ces propriétés sont appelés *systèmes de prix consistants* (voir [JK95]): ils jouent un rôle crucial dans la théorie de l'arbitrage et aussi, dans ce contexte, pour la construction des prix fictifs.
Nous avons mentionné que des prix fictifs ont été construits dans de nombreux cas particuliers. D'autres résultats partiels d'existence ont été obtenus dans [KM10] ou [LPS98]. La question naturelle que nous abordons dans ce travail est: peut-on s'attendre à ce qu'ils

existent en général? Le premier résultat frappant que nous présentons dans le Chapitre 4 est que les prix fictifs peuvent très bien ne pas exister, et cela peut se produire même dans des modèles discrets à deux périodes très simples avec un seul actif risqué. En effet, un contre-exemple que nous présentons au début du chapitre montre que même dans un marché si basique (avec des paramètres bien choisis), l'hypothèse d'existence d'un prix fictif conduirait automatiquement à des arbitrages et, par conséquent, à une fonction valeur divergente, atteignant donc la contradiction souhaitée.

Qu'est-ce qui ne marche pas dans le contre-exemple et comment cela peut-il être modifié afin d'obtenir un résultat positif? Afin d'avoir une intuition, il est utile de faire appel à nouveau à la théorie de la dualité. Cependant, dans ce contexte, il est plus fructueux de considérer le problème dual directement sur le marché avec coûts de transaction (plutôt que sur une famille de marchés sans frictions, comme nous l'avons fait ci-dessus, dans le but de trouver un prix fictif de façon heuristique). Ici, les variables duales d'intérêt (voir [CMS11]) sont données par l'ensemble des surmartingales Z à valeurs dans le spread bid-ask (ce qui, dans notre formalisme, signifie dans K^*) telles que VZ est une surmartingale pour toute stratégie admissible V. Suivant la même idée que précédemment, en écrivant $VZ = Z^1\left(V\frac{Z}{Z^1}\right)$, nous aimerions interpréter Z/Z^1 comme étant le prix sur un marché sans frictions, et Z^1 comme sa densité de martingale. La question cruciale est maintenant la suivante: si $\hat{Z}$ est un minimiseur dual sur le marché avec coûts de transaction, est-ce que $\hat{Z}^1$ sera le minimiseur dual dans le marché sans frictions avec prix $\hat{Z}/\hat{Z}^1$? Si la réponse est positive, on peut facilement voir que $\hat{Z}/\hat{Z}^1$ est un prix fictif. Néanmoins, c'est précisément ici que la situation est délicate, et ce même dans des contextes très simples. En fait, à moins que $\hat{Z}^1$ ne soit une martingale (ce qui serait une hypothèse plutôt forte, voir [CK96] pour une hypothèse semblable), $\hat{Z}^1$ pourrait très bien ne pas être une variable duale dans le marché sans frictions. Pour le voir, il suffit d'appliquer la formule d'intégration par parties pour obtenir

$$d(\hat{Z}^1_t X_t) = V_t\left(d\hat{Z}_t - \frac{\hat{Z}_t}{\hat{Z}^1_t}d\hat{Z}^1_t\right) + X_t d\hat{Z}^1_t$$

où $dX_t = V_t d(\hat{Z}/\hat{Z}^1)_t$ représente un processus de richesse dans le marché candidat (sans frictions). Nous voyons que si $\hat{Z}^1$ est seulement une surmartingale alors $\hat{Z}^1_t X_t$ n'est pas forcement une surmartingale, la raison étant que les intégrales par rapport à des surmartingales ne sont pas nécessairement surmartingales. Une façon naturelle pour en faire des surmartingales, cependant, pourrait être de forcer l'intégrand V_t à être positif, ce qui correspond à l'interdiction de ventes à découvert sur le marché. Par conséquent, cet argument formel ainsi que le fait que notre contre-exemple repose en grande partie sur la possibilité de vendre à découvert, suggère que l'interdiction de ce type de comportement pourrait conduire à un résultat positif d'existence pour les prix fictifs. Comme nous allons le voir, ce sera effectivement le cas.

L'une des principales difficultés pour obtenir le résultat souhaité reside dans le fait que nous n'avons pas un théorème de dualité pour des marchés avec coûts de transaction et des contraintes de positivité sur les stratégies (et non sur les processus de richesse!). Pour cette raison, il semble plus facile de travailler directement avec le problème primal, en suivant une approche similaire à celle présentée dans [Lo00] dans le cas d'un modèle brownien à deux actifs. Cependant, l'absence de dualité introduit des difficultés pour

prouver l'existence d'une stratégie optimale dans le problème primal (sous contraintes), ce qui est en fait présenté en tant qu'hypothèse initiale dans [Lo00]. Au contraire, nous avons pu prouver l'existence dans la Proposition 4.3.1 sans utiliser la dualité, mais plutôt en profitant de certains résultats de compacité pour les stratégies établis dans [CS06] et d'une adaptation d'un argument employé dans [Gu02].
Afin de trouver un marché fictif, nous définissons d'abord une famille convenable de marchés classiques à partir de leurs prix Z/Z^1, pour tous les *systèmes de prix consistants surmartingales* Z (même si nous sommes dans un cadre multidimensionnel, nous prenons le premier actif comme numéraire). Ensuite dans la Proposition 4.4.1, nous établissons des conditions suffisantes pour qu'un tel prix soit réellement un prix fictif. Nous construisons finalement un prix fictif canditat à partir de son système de prix consistant surmartingale associé, défini comme

$$\hat{Z}^i_t := \lim_{\epsilon \downarrow 0} \frac{J(\hat{V} + e_i\epsilon, t) - J(\hat{V}, t)}{\epsilon},$$

(où $\hat{V}$ est la stratégie optimale) et dans le Théorème 4.4.1, nous montrons qu'il satisfait les conditions suffisantes, de sorte que $\hat{Z}/\hat{Z}^1$ s'avère être finalement un prix fictif dans notre marché avec contraintes. Même si la dualité n'est pas explicite dans cette construction, nous pouvons remarquer qu'elle semble tout de même être cachée en coulisses, comme cela est suggéré par l'une des conditions suffisantes

$$E[Z^1_T \hat{V}_T] = Z_0 x,$$

qui rappelle fortement les conditions typiques de dualité.
Après la publication de ce papier, d'autres études ont confirmé l'impossibilité d'obtenir des résultats généraux d'existence de prix fictifs sans imposer de contraintes supplémentaires. En particulier, l'article récent [CMS11] contient un autre contre-exemple qui semble être encore plus frappant que le nôtre, car il est basé sur un marché discret à deux périodes où les prix bid et ask sont supposés bornés (alors que dans notre cas, l'absence d'une borne supérieure pour le prix ask à la date $t = 1$ joue un rôle fondamental pour atteindre la contradiction). En outre, ce contre-exemple (qui est toujours basé sur la vente à découvert de l'actif risqué) montre que l'absence d'un prix fictif n'implique pas nécessairement la présence d'opportunités d'arbitrage sur le marché fictif candidat: au contraire, il est parfaitement possible que le "pire" marché sans frictions (ayant les prix dans le spread) ne soit tout simplement pas assez mauvais pour égaliser l'utilité espérée maximale du marché d'origine. Comme le soulignent les auteurs, il n'y a rien de trop étrange en cela, vu que chaque marché fictif candidat admet toujours un ensemble plus large de stratégies de trading par rapport au marché d'origine. Par conséquent, les contraintes de vente à découvert que nous avons introduites dans notre étude semblent être très proches d'une hypothèse minimale, si le but est de trouver un résultat général d'existence de prix fictifs.

1.3 Evaluation par indifférence d'utilité pour des payoffs non réguliers dans un marché avec des actifs non échangeables

Cette partie de la thèse traite de l'évaluation d'actifs contingents dans des marchés incomplets selon l'approche par indifférence d'utilité (exponentielle). Vu que la motivation principale de cette étude provient du pricing des instruments dérivés sur l'électricité dans un cadre structurel, nous considérons utile de donner ici un bref aperçu des principaux enjeux soulevés lors de la modélisation des prix d'électricité.
L'électricité est généralement considérée comme une "flow commodity", dans le sens où elle n'est utile que si elle est livrée en continu sur une période de temps. En outre, elle a la caractéristique importante d'être, sauf pour de rares cas, un bien non stockable. Elle est donc produite directement pour la consommation, avec une possibilité très limitée de la stocker pour une utilisation future. Elle est saisonnière, la demande d'électricité étant clairement affectée par la période de l'année (et même par le moment de la journée), et elle présente généralement des comportements de retour à la moyenne. Enfin, en raison de changements inattendus de l'offre ou de la demande (par exemple, à cause d'une augmentation de la température ou de panne d'une centrale), les prix d'électricité montrent souvent des pointes, sous la forme d'augmentations soudaines des prix suivies généralement par des reprises rapides. Nous renvoyons le lecteur intéressé à l'ouvrage [CC12] pour plus de détails.
La modélisation du prix spot de l'électricité est la première étape vers l'évaluation des dérivés d'énergie (et donc aussi des centrales électriques à travers l'utilisation des options sur spread), le déploiement de stratégies de couverture optimales et pour des fins de gestion des risques. Pour ces raisons, cette modélisation doit être aussi réaliste que possible, en satisfaisant en particulier les principaux faits stylisés mentionnés ci-dessus, tout en maintenant une certaine traitabilité afin de fournir des formules fermées (ou au moins des méthodes numériques efficaces) pour la tarification des produits dérivés. Deux façons classiques de procéder apparaissent généralement dans la littérature : les modèles à forme réduite et les modèles structurels. Alors que les premiers se concentrent directement sur la dynamique des prix spot (par exemple [BCK07]), les derniers essaient généralement d'exprimer le prix à partir de l'interaction de plusieurs facteurs économiques fondamentaux (par exemple [Ba02], [CV08], [PJ08], [CCS12]). Les facteurs typiques dans la modélisation structurelle des prix d'électricité comprennent:

- Les prix des carburants (utilisés dans la production d'électricité) S_t^i, $1 \leq i \leq n$,
- La demande d'électricité D_t (en MW),
- Les capacités maximales des carburants C_t^i (en MW), $1 \leq i \leq n$,
- Les taux de conversion h_i (où $h_i S_t^i$ est exprimé en €/MWh), $1 \leq i \leq n$.

L'idée sans doute plus intuitive d'un point de vue économique est de décrire le prix spot de l'électricité P_t en termes du coût marginal du combustible qui est utilisé pour sa production, en écrivant par exemple

$$P_t = \sum_{i=1}^{n} h_i S_t^i \mathbf{1}_{\left\{\sum_{k=1}^{i-1} C_t^k \leq D_t \leq \sum_{k=1}^{i} C_t^k\right\}}. \tag{1.3.1}$$

Cette formulation a été utilisée dans [ACNT02], alors que dans [ACL10] les auteurs améliorent le modèle en ajoutant à (1.3.1) un facteur multiplicatif qui augmente de manière non linéaire en $\sum_{k=1}^{i-1} C_t^k - D_t$, afin de tenir compte des pointes qui sont généralement observées lorsque le système électrique est sous tension. Une description similaire, mais plus complexe, est obtenue dans [CC12].
Il est immédiatement évident que l'on a affaire, en général, avec des payoffs qui dépendent aussi de certains facteurs qui ne sont pas directement négociables sur un marché financier (comme les capacités et la demande), créant ainsi de l'incomplétude et donc la nécessité de trouver des procédures d'évaluation alternatives. On remarque également que cette dépendance peut être très irrégulière dans les actifs non échangeables, ce qui rend la couverture beaucoup plus délicate, à moins que l'on soit prêt à utiliser le critère de surréplication (qui est cependant généralement considéré comme étant excessivement cher). Une solution possible est d'utiliser le paradigme de *minimisation locale du risque* (ou *local risk minimisation*, voir [Sc01]), qui a été appliqué dans ce contexte par [ACL10] et qui a l'avantage de fournir des formules fermées d'évaluation des produits dérivés, lorsque les dynamiques sous-jacentes sont relativement simples et les deux classes d'actifs (échangés et non échangés) sont supposées être non corrélées (ce qui semble être une approximation raisonnable dans ce cadre). Un autre paradigme d'évaluation, qui peut être considéré comme étant plus flexible vu qu'il permet de jouer avec un paramètre supplémentaire, est justement l'*évaluation par indifférence d'utilité* : c'est celui-ci que nous allons étudier plus en profondeur dans la suite. L'idée est assez simple: selon ce critère, le prix d'un actif contingent doit être tel que l'agent est indifférent entre acheter l'actif (en payant son prix) et ne rien faire. Plus précisement, le prix d'achat p^b d'un claim $\mathcal{F}_T$-mesurable est défini implicitement par l'équation suivante:

$$\sup_{\pi} E\left[U\left(V_T^{v-p^b}(\pi) + f\right)\right] = \sup_{\pi} E\left[U(V_T^v(\pi))\right], \tag{1.3.2}$$

où $V_T^x(\pi)$ est la valeur au temps T d'un portefeuille qui vaut x à la date zéro et qui suit la stratégie π. Le prix de vente p^s est défini de manière similaire. Nous soulignons que la stratégie π doit être contrainte à investir uniquement dans les actifs qui sont effectivement négociables (même si elle peut clairement dépendre de toute l'information du marché). En l'absence d'un marché financier, cette définition se réduit essentiellement (comme nous le montrons dans la Remarque 5.2.3) au concept économique bien connu d'*équivalent certain*, qui a été exploré par [BCK07] dans le contexte des marchés d'électricité. Nous considérons le cas d'une utilité exponentielle, c'est à dire $U(x) = -e^{-\gamma x}$ pour un certain $\gamma > 0$, ce qui a pour principal avantage de rendre le prix indépendant de la richesse initiale v.
Nous nous plaçons dans un cadre markovien, où les actifs échangés (c.-à-d. les carburants) S suivent une dynamique de Black-Scholes

$$\frac{dS_t^i}{S_t^i} = \mu_i dt + \sigma_i . dW_t^S$$

alors que les actifs non échangés (c.-à-d. capacités et demande) X suivent le processus de retour à la moyenne :

$$dX_t^i = (b^i(t) - \alpha_i(t) X_t^i)dt + \beta_i .(t) dW_t^X,$$

où W^S et W^X sont des mouvements browniens indépendants avec dimensions respectives n et d. Tous les paramètres dans les formules sont supposés être soit des constantes soit des fonctions déterministes du temps (par exemple la fonction $b(t)$ peut généralement décrire les oscillations saisonnières bien documentées dans la demande d'électricité). Le sous-marché composé seulement des actifs échangés est donc complet et admet par conséquent une mesure martingale unique, qui est aussi la *mesure martingale minimale* Q^0 du marché et qui joue un rôle fondamental dans le pricing par minimisation locale du risque (LRM). Nous supposons que le claim f est sous- et sur-répliquable, c'est à dire qu'il vérifie :

$$V_T^{v_1}(\pi^1) \leq f \leq V_T^{v_2}(\pi^2)$$

presque sûrement pour certaines stratégies de portefeuille données. Nous montrons d'abord dans le Lemme 5.2.1 la relation suivante entre les prix d'indifférence d'utilité, les prix de sous- ou sur-réplication et le prix LRM:

$$v_1 \leq p^b \leq E^0[f] \leq p^s \leq v_2.$$

Même si le résultat était essentiellement déjà connu ([Ho05]), nous offrons ici une nouvelle preuve qui repose uniquement sur des arguments de dualité. Nous montrons ensuite que, lorsque f est borné, le prix initial d'achat p^b vérifie $p^b = Y_0$, où (Y, Z) résout l'EDSR

$$Y_t = f - \int_t^T \frac{\gamma}{2}\|Z_s^X\|^2 ds - \int_t^T Z_s dW_s^0, \tag{1.3.3}$$

où W^0 est un mouvement brownien sous la mesure martingale minimale et la décomposition $y = (y^S, y^X)$ sépare les parties du vecteur y liées aux actifs échangés et non échangés. En outre, le processus Z est (à une constante près) la stratégie de couverture optimale, définie comme la différence entre les maximiseurs dans (1.3.2). Le résultat était essentiellement connu depuis le papier [ER00], mais encore une fois nous fournissons ici une preuve différente basée sur une technique développée dans [HIM05]. Dans le Lemme 6.7.2 nous étendons ce résultat en prouvant l'existence d'une solution de (1.3.3) lorsque f est uniquement sous- et sur-répliquable, et nous donnons des conditions permettant d'interpréter Y en tant que prix d'indifférence d'utilité aussi dans ce cas. Ces conditions sont satisfaites, en particulier, lorsque f est un claim européen ayant une croissance polynômiale dans les actifs négociés uniformément par rapport aux autres.

Cependant, la caractérisation précédente ne permet pas en général d'en savoir beaucoup sur le processus Z: nous ne connaissons pas, par exemple, s'il a des trajectoires continues, s'il est unique ou s'il est encore lié à la stratégie de couverture optimale (en raison de l'absence, en général, de la propriété BMO). Afin d'obtenir davantage d'informations à ce sujet, la deuxième partie du chapitre est consacrée au cas des claims européens. Les hypothèses 5.4.1 et 5.4.2 énumèrent les propriétés techniques que nous supposons sur f. Il suffit de mentionner ici que nous permettons des points de non-différentiabilité de f (ce qui s'applique généralement aux options) et même des points de discontinuité par rapport aux actifs non négociés, comme c'est le cas dans notre exemple principal (1.3.1).

Dans le Théorème 5.4.1, qui est le résultat principal du chapitre, nous montrons que nous pouvons exprimer la solution de (2.3.3) comme $Y_t = \varphi(t, S_t, X_t)$, où φ est solution de viscosité de l'EDP :

$$\begin{cases} \mathcal{L}\varphi - \frac{\gamma}{2}\sum_{j=1}^d (\beta'_{\cdot j}\varphi_x)^2 = 0 \\ \varphi(T, s, x) = f(s, x) \end{cases} \tag{1.3.4}$$

et $\mathcal{L}$ est l'opérateur de Dynkin standard correspondant à la dynamique de l'actif. Ce qui est particulièrement intéressant est que φ est aussi continûment différentiable dans toutes les variables d'espace, un fait qui nous permet notamment de représenter Z comme $\sigma(S_t)\varphi_s(t, S_t, X_t)$ (ce qui implique donc que Z ait des trajectoires continues), où $\sigma(S)$ est la matrice dont la i-ème ligne est donnée par $\sigma_i.S^i$. De cette façon, nous pouvons également identifier la stratégie de couverture optimale $\Delta_t = -\sigma^{-1}\sigma(S_t)\varphi_s(t, S_t, X_t)$.
La preuve du Théorème 5.4.1 est plutôt longue, mais l'idée principale est la suivante: nous commençons par réécrire l'EDSR (1.3.3) de telle sorte que son générateur soit uniformément Lipschitz, en réexprimant (1.3.4) de manière qu'elle puisse être représentée comme la solution d'un problème de contrôle stochastique où l'espace des contrôles est compact. La clé pour y parvenir est d'utiliser une technique employée dans [Ph02] qui consiste à modifier le terme de source dans (1.3.4) en utilisant la fonction

$$h^m(q) = \sup_{\delta \in \mathcal{B}^m(\mathbb{R}^d)} \left\{ -q\delta - \frac{1}{2\gamma}\|\delta\|^2 \right\},$$

où $\mathcal{B}^m(\mathbb{R}^d)$ est la boule dans $\mathbb{R}^d$ centrée en zéro et de rayon $m > 0$, puis en faisant tendre m vers l'infini. Le payoff doit également être approché par régularisation quand il n'est pas suffisamment lisse. L'astuce fonctionne bien lorsque f est suffisamment régulier pour permettre de bonnes estimations des dérivés de φ, alors que la preuve nécessite un peu de travail supplémentaire en présence de discontinuités dans f. L'existence et la continuité des dérivées spatiales de φ est obtenue en utilisant des représentations du type Malliavin qui ont été développées dans [MZ02] (et qui ont déjà été utilisées à des fins similaires dans [Zh05]).
L'existence d'une stratégie de couverture optimale et sa caractérisation en termes de φ_s sont également obtenues par un argument d'approximation, en utilisant le fait que pour des payoffs bornés nous savons déjà grace à [HIM05] que Z est interprétable de cette façon. Notre dernière contribution dans cette partie est liée aux développements asymptotiques du prix et de la stratégie de couverture optimale en termes du prix sous la mesure martingale minimale (qui a été étudié dans [ACL10] dans le même cadre). L'expansion du prix est obtenue en utilisant un résultat de [Mo12] qui emploie des idées du calcul de Malliavin, et l'expansion de la stratégie est dérivée en utilisant des arguments similaires conduisant finalement à

$$\varphi_{s^i}(t, a) = E^0_{t,s,x}\left[f_{s^i}(S_T, X_T)\right] - \gamma E^0_{t,s,x}\left[f_{s^i}(S_T, X_T)\int_t^T \beta\varphi^0_x dW^X_u\right] + O(\gamma^2),$$

où $\varphi^0_{x^i}(t, s, x) = E^0_{t,s,x}\left[f_{x^i}(S_T, X_T)\right]$. Nous avons finalement calculé explicitement l'expansion du prix pour des contrats à terme avec deux carburants.
En conclusion, nous avons fourni une contribution au pricing et à la couverture des produits dérivés d'énergie dans le cadre de la modélisation structurelle proposée recemment dans [ACL10], où une partie des actifs n'est pas négociable et la plupart des payoffs d'intérêt présentent des irrégularités. Les produits dérivés s'écrivent naturellement dans ce cadre comme des fonctions des actifs échangés et non échangés, ce qui n'est pas fréquent dans la littérature sur les prix d'indifférence d'utilité (voir [HH09], ou l'introduction du Chapitre 5 pour des références plus précises), sauf dans certains cas particuliers comme [SZ04]. Nous

croyons que certaines idées pourraient être aussi utiles dans d'autres contextes, généralement en présence de variables latentes qui produisent des caractéristiques de changement de régimes. Cela pourrait par exemple être le cas pour les produits d'assurance, qui présentent souvent une composante financière liée à des actifs négociables et une partie d'assurance stricte qui dépend de certaines variables qui ne peuvent pas, en général, être négociées sur un marché (et que nous pouvons raisonnablement supposer décorrélées des actifs du marché).

1.4 Un problème de principal-agent et réduction d'émissions

Dans cette dernière partie de la thèse, nous traitons un problème de principal-agent en temps continu et à horizon fini. Le modèle est similaire à d'autres déjà paru dans la littérature (voir [CZ12] pour un apperçu). Cependant, il diffère en ce qui concerne la motivation principale, la définition des mesures d'incitation (qui peuvent ici être de deux sortes, comme nous le verrons dans un instant) et la forme des stratégies admissibles, qui se caractérisent ici d'une manière plus simple. Globalement, le but de cette section est d'étudier un modèle particulier en privilégiant la simplicité sur la généralité, et d'obtenir des résultats numériques et qualitatifs qui puissent aider dans l'interprétation et la compréhension de ce type de modèles (qui souvent deviennent rapidement assez complexes). Afin d'avoir une idée générale de ce qu'est un problème de principal-agent, nous commençons par rappeler rapidement les principales idées qui sont derrière la première contribution fondamentale dans ce domaine, qui est due à [Sa08]. Le contexte est celui de la théorie des contrats optimaux avec aléa moral, dans lequel un employeur et un salarié ont des intérêts (au moins partiellement) contraires et les actions de l'employé (c.-à-d. sa quantité/qualité de travail) ne sont pas totalement observables par son patron. En particulier, l'employé (agent) maximise la fonctionnelle

$$E\left[\int_0^\infty e^{-rt}(u(s_t) - h(k_t))dt\right] \tag{1.4.1}$$

pour toutes les politiques d'action k_t. Ici le processus s_t représente les salaires de l'agent, qui doivent être décidés par l'employeur (principal) en maximisant

$$E\left[\int_0^\infty e^{-rt}(k_t - s_t)dt\right], \tag{1.4.2}$$

où intuitivement les quantités jouent ici dans des directions opposées par rapport à (1.4.1). L'hypothèse cruciale qui caractérise les problèmes de principal-agent est que le principal n'observe pas les actions de l'agent k_t, mais il est néanmoins censé connaître la fonctionnelle (1.4.1) que l'agent cherche à optimiser. Par conséquent, le principal est capable de deviner la réponse optimale de l'agent par rapport à une politique salariale (ou politique incitative) donnée s_t : ainsi, lorsque le principal optimise (1.4.2), il est naturel pour lui de remplacer le processus k par $k(s)$, l'effort optimal de l'agent compte tenu de s, pour choisir enfin le meilleur s. Il semble immédiatement évident qu'une bonne caractérisation de l'effort optimal de l'agent est une étape essentielle dans la résolution du problème du principal, ce qui explique pourquoi les deux sont habituellement traités séparément l'un après l'autre. Les principales techniques qui sont utilisées dans [Sa08] pour résoudre

le problème sont le théorème de représentation des martingales (MRT) et la théorie des EDO, qui permet notamment de caractériser toutes les quantités optimales en termes de la solution d'une équation différentielle (qui peut aussi être discrétisée afin d'obtenir des résultats numériques).
Après cette contribution fondamentale, la théorie des problèmes de principal-agent en temps continu a évolué et, en particulier, son extension en horizon fini s'est avérée être particulièrement complexe. Ici, l'équivalent du MRT simple est la théorie de équations différentielles stochastiques forward-backward (FBSDEs), tandis que les EDO sont généralement remplacées par des équations aux dérivées partielles (EDP). Une enquête exhaustive des contributions importantes dans ce domaine peut être trouvée dans [CZ12], alors qu'une bonne référence pour la théorie mathématique sous-jacente est [MY99]. Le modèle particulier que nous considérons dans cette section s'insère dans cette littérature, mais la motivation principale est légèrement différente puisque nous prenons plutôt le point de vue d'un régulateur qui vise à fournir à une entreprise des incitations optimales afin qu'elle réduise la production de ses émissions polluantes (même si cela n'est pas la seule interprétation possible du modèle). On se pose un double objectif : d'abord, nous voulons analyser un modèle qui est similaire, mais non directement inclus, dans les cas déjà examinés par d'autres auteurs (en étant aussi différemment motivé). Deuxièmement, nous voulons garder le cadre de modélisation suffisamment simple pour que être en mesure d'extraire aussi des résultats qualitatifs et d'interpréter quelques unes des quantités optimales grâce à des approches numériques (ce qui n'est généralement pas évident à faire dans le cas de modèles plus complexes).
On commence par modéliser le *processus d'émissions* (prenant des valeurs dans $\mathbb{R}_+$) avec la dynamique suivante :

$$X_t = x + \int_0^t X_r l(k_r) dr + \int_0^t X_r \sigma dW_r. \tag{1.4.3}$$

Le processus X (qui est positif) peut soit être interprété comme le niveau des émissions à une certaine date t, soit comme la perception du marché des émissions cumulées jusqu'à une date donnée t (de façon similaire à [CDET13] dans un cadre complètement différent). Ici, le processus k représente l'effort de l'agent, qui peut être interprété par exemple en termes de technologie utilisée afin de réduire les émissions. Les niveaux d'effort prennent des valeurs dans $\mathbb{R}_+$ (où des valeurs plus élevées correspondent à une meilleure technologie), alors que la fonction l reflète l'effet de la technologie sur les émissions (intuitivement, $l'(k) \leq 0$). L'hypothèse est que le principal n'observe pas les actions de l'agent k (par exemple, en raison de coûts de surveillance excessifs), mais il peut toutefois constater le niveau du processus X, qui modélise les conséquences (bruitées) de l'effort de l'agent. Le régulateur ne peut donc pas forcer directement l'agent à choisir le k souhaité, mais ce qu'il peut faire est de le *motiver* en choisissant une politique d'incitations appropriée, qui ne peut que dépendre du processus observé X. Une politique d'incitations est un couple (s, p), où s modélise un flux de paiements continus alors que p décrit un paiement à une échéance finale fixée T (on pense généralement à une pénalité proportionnelle ou forfaitaire basée sur les émissions finales, par exemple $p(X_T) = (X_T - \Lambda)^+$ pour une certaine valeur

du seuil $\Lambda > 0$). L'utilité espérée de l'agent s'écrit

$$V(k) = E\left[\int_0^T u(s_t - c(k_t))dt - p\right] \tag{1.4.4}$$

où u est une fonction d'utilité et c une fonction de coût convexe. Ici, nous passons volontairement sur certains détails techniques qui seront traités dans le texte grâce à des techniques de changement de mesure: pour en avoir une idée, il suffit d'observer par exemple que la valeur espérée du paiement final doit aussi dépendre de la politique d'effort k, alors que cela ne semble pas être le cas en regardant la fonctionnelle (1.4.4).
La caractérisation de l'effort optimal qui maximise (1.4.4) est effectuée dans les Propositions 6.2.1 et 6.2.2. L'outil principal qui est utilisé pour déterminer les conditions nécessaires d'optimalité est le principe du maximum stochastique, qui est souvent utilisé dans la littérature sous différentes formulations (voir, par exemple, [CWZ08] ou [Wi08]): ici la simplicité de notre modèle (en particulier, les hypothèses de bornitude pour l et s) nous permet en particulier de le faire fonctionner sous une version moins lourde d'admissibilité de l'effort.
Des conditions suffisantes, d'autre part, sont obtenues en profitant de la théorie standard des EDSR et de la convexité des fonctions du modèle. Plus précisément, nous construisons dans le Lemme 6.2.1 une fonction F qui inverse les conditions nécessaires d'optimalité et qui nous permet d'écrire l'EDSR suivante :

$$\begin{cases} dY_t = [-Z_t l(F(s_t, Z_t))/\sigma - u(s_t - c(F(s_t, Z_t)))]dt + Z_t dW_t \\ Y_T = -p \end{cases} \tag{1.4.5}$$

dont la solution fournit l'effort optimal de l'agent en réponse aux incitations (s, p) par la relation $k_t = F(s_t, Z_t)$.
Compte tenu de l'importance de la réponse optimale de l'agent aux politiques incitatives, il est assez naturel de se demander si il serait possible de mieux caractériser l'effort k, par exemple en trouvant sa dynamique, ou d'exploiter la théorie des EDSR pour analyser la façon dont l'effort réagit à une modification des incitations ou des valeurs des paramètres. Il est assez intuitif de réaliser que des incitations continues s_t plus grandes ou une pénalité plus petite p permettront d'augmenter l'utilité de l'agent (ce qui est prouvé dans le Lemme 6.3.1), mais il est moins évident d'analyser quelle incidence auront des changements similaires sur l'effort de l'agent. Motivés par cela, nous avons réussi à dériver, sous des hypothèses supplémentaires (principalement en supposant des incitations markoviennes), une EDSR directement résolue par l'effort optimal k_t, que dans le cas simple d'incitations constantes ($s_t = s$) s'écrit :

$$\begin{cases} -dk_t = \left[G(s, k_t)\Theta_t^2 + \frac{l(k_t)}{\sigma}\Theta_t\right] dt - \Theta_t dW_t \\ k_T = F(s, -\sigma X_T p'(X_T)). \end{cases} \tag{1.4.6}$$

Le cas où s est une fonction de t et X_t est légèrement plus complexe et il est présenté dans la Proposition 6.3.1. L'équation (1.4.6) est une EDSR quadratique dont le coefficient G est une fonction des paramètres du modèle, et qui est bornée sous des hypothèses convenables (grossièrement, p' ne doit pas être trop "grand", ce qui revient à dire que la pénalité

terminale ne doit pas être trop sensible aux petites perturbations de X). En plus, nous montrons également que la solution Z à (1.4.5) admet la représentation

$$Z_t = -E^* \left[\sigma X_T p'(X_T) \mid \mathcal{F}_t \right],$$

où l'espérance est définie à l'aide d'un changement de mesure approprié qui dépend de l'effort optimal. Avec une EDSR pour k à disposition, nous réalisons finalement des analyses de sensibilité en regardant, en particulier, la façon dont le processus réagit aux changements de l'aversion au risque de l'agent ou du taux d'impatience (où le taux d'impatience est simplement défini de manière similaire à [Sa08] en réécrivant (1.4.4) avec un facteur d'escompte δ). Nous constatons en particulier que l'effort optimal est toujours décroissant par rapport au taux d'impatience, alors qu'il est croissant par rapport à l'aversion au risque de l'agent si certaines conditions sont vérifiées (encore une fois, si p' n'est pas trop grand). Enfin, dans la Section 6.4, nous proposons un schéma numérique pour le calcul de l'effort, en adaptant des méthodes de discrétisation existantes pour les EDP non linéaires, et dans la Section 6.4.1 nous discutons quelques résultats obtenus pour différents types d'incitations. Des trajectoires simulées de X sont également fournies afin d'obtenir une meilleure intuition du comportement typique de l'agent.
La dernière partie du chapitre est consacrée au problème du principal. Comme nous venons de le voir, le but de l'agent est de choisir de façon optimale un effort k en réponse à une politique incitative (s, p) donnée. Intuitivement, le principal devra décider de manière optimale un couple (s, p), en tenant compte du comportement de l'agent selon ce choix. La fonctionnelle de profit pour le principal est définie comme

$$E\left[p_1(p) - p_2(X_T) - \int_0^T u_1(s_r) dr \right], \tag{1.4.7}$$

où p_1 modélise l'utilité correspondante au paiement (éventuel) de la penalité p, p_2 décrit les coûts sociaux liés au niveau des émissions, et u_1 prend en compte la perte d'utilité à cause des incitations payées en temps continu. Il s'avère, cependant, qu'une optimisation directe sur (s, p) n'est pas la meilleure façon de procéder: il est en effet plus pratique pour le principal d'optimiser directement sur (s, k), à condition qu'il modifie ensuite la pénalité p de façon appropriée afin que l'agent réalise effectivement l'effort souhaité k. Cette intuition (on peut remarquer la similitude avec les idées décrites au début de la section) est mise en pratique en remplaçant p dans (1.4.7) par l'expression

$$R - \int_0^T [u(s_r - c(k_r)) + g(k_r, s_r) l(k_r)/\sigma] dr + \int_0^T g(k_r, s_r) dW_r, \tag{1.4.8}$$

obtenue en utilisant les conditions d'optimalité de l'agent et en fixant son utilité initiale à un niveau prédéterminé R (pour que le problème soit bien posé). La fonction g est définie à partir des dérivés de u, l et c, et elle est interprétable comme l'élasticité de l'utilité de l'agent par rapport aux changements du taux de croissance des émissions.
Nous écrivons finalement le problème sous la forme d'un problème de maximisation sur (s, k), pour lequel nous fournissons des conditions nécessaires d'optimalité dans la Proposition 6.5.1, en utilisant encore le principe de maximum stochastique (mais ici avec deux variables d'état/adjointes). Il ne serait pas difficile de dériver aussi des conditions suffisantes, mais le problème principal est qu'elles nécessitent généralement un certain degré

de convexité de la fonction hamiltonienne qui est difficile à vérifier dans la pratique. Pour cette raison la dernière partie du chapitre est consacrée à l'analyse de la situation simplifiée où $p_1(x) = x$, quand la désutilité de l'agent liée au paiement de la pénalité correspond à une utilité du principal de même montant. Le problème devient alors beaucoup plus simple car une variable d'état peut être éliminée, ce qui nous laisse avec une formulation du problème qui est assez similaire à celle de l'agent. Dans ce cas, cependant, l'inversion des conditions nécessaires est un peu plus complexe, car il faut tenir compte conjointement des équations d'optimalité de l'agent et du principal. Même si la possibilité de le faire doit être vérifiée cas par cas, nous commençons par supposer que l'inversion est possible et que cette procédure donne lieu à deux fonctions I et L qui définissent naturellement une nouvelle EDSR candidate de la forme

$$\begin{cases} dY_t = -[Z_t l(L(Z_t))/\sigma + u(I(L(Z_t)) - c(L(Z_t)))]dt + Z_t dW_t \\ Y_T = -p_2(X_T) - \int_0^T u_1(I(L(Z_t)))dt - a \end{cases} \tag{1.4.9}$$

où le paramètre $a \in \mathbb{R}$ doit être choisi de telle sorte que l'utilité initiale de l'agent soit respectée. Dans la Proposition 6.5.2 nous montrons que la solution de (1.4.9) donne effectivement l'effort et les incitations continues optimales en posant $k_t = L(Z_t)$ et $s_t = I(k_t) = I(L(Z_t))$. La fonction de pénalité optimale est alors donnée par $p_2(X_T) + \int_0^T u_1(s_r)dr + a$. Remarquons que (1.4.9) est très similaire à (1.4.5), à l'exception de s qui est maintenant récupéré de façon endogène à partir de la solution de (1.4.9).
Nous concluons finalement la partie sur le principal avec un exemple qui montre comment les fonctions I et L peuvent être calculées dans le cas particulier d'une fonction d'utilité puissance, et nous donnons un exemple numérique qui montre en particulier le choix optimal des incitations continues dans ce cas spécifique.

Chapter 2

Introduction (in English)

The study of economic and financial models typically starts by assuming the simplest possible setting that allows to capture the main relevant ideas by remaining analytically (and/or computationally) tractable. The details that are left out at this stage are sometimes thought to be negligible and to contribute only to "second order effects", while leaving the general message substantially unchanged. Other times, these additional features are simply too challenging from a technical point of view. It often happens, however, that a subsequent study of these "imperfections" reveals that they have a much larger influence than initially expected, sometimes to the point of radically changing well-established results and/or even paving the way for completely new developments of the underlying mathematical theories. One striking example for all comes from the introduction of (arbitrarily small) proportional transaction costs inside the classical modeling of financial markets and wealth processes in continuous time (see [LS97], [CPT99], [GRS08]): it turns out, in fact, that this (apparently innocuous) new feature makes the smallest super-replicating price for a call option exactly equal to the price of its underlying!
The classical treatment of arbitrage theory is also quite drastically changed with the introduction of transaction costs (see [KS09], [S04], [JK95], [GRS10]), and the same is true for utility maximization ([DN90], [Bo02], [CO10], [CK96], [Gu02], [DPT01], [LPS98]). In a similar way, the treatment of utility theory on incomplete markets has proven to be substantially more involved than in the complete case (see [KS99], [OZ09], [CSW01], [HG04]) due to the presence of multiple martingale measures that generally require new and deeper insights into the fine properties of measure spaces. It starts being clear that even the smallest market imperfection may fundamentally modify the set of admissible trading strategies and therefore the way agents optimally hedge their positions, the result usually being the loss of the possibility of perfect replication. Both transaction costs or market incompleteness then give rise to a natural practical question: how should derivatives be evaluated on these markets? Since simple arbitrage arguments are generally no longer sufficient to come up with a unique price, other methods have been proposed in the last years, many of which take advantage of results from the literature on utility maximization (see [HH09]).
Another kind of imperfection in economic models dealing with optimal contracts is moral hazard, an issue that is so pervasive that it has attracted the interest of researchers for a very long time (see [Ho79] for one of the earliest contributions). However in the re-

cent years the topic has again started to draw a particular attention as a consequence of the development of the theory of continuous time processes and optimization which has turned out to be very useful to shed new light on this old problem (see [Sa08] for a seminal contribution and [CZ12] for a deeper analysis of the related mathematical problems).
The topics that we deal with in this thesis, which might may seem quite different at first sight, are all concerned with the idea of agents maximizing their expected utilities in the presence of some kind of market imperfections.
The first part of this manuscript deals with markets with transaction costs, according to the very general formalization that has been put forward by Kabanov (and further developed by a number of authors) and which is comprehensively explained in the book [KS09]. It is further split into two chapters that focus on different problems. In the first one we look at the utility maximization problem in the case of a multivariate utility function and an initial endowment that may possibly be random (i.e., a derivative contract), while in the second we study the problem of existence of shadow price processes, a concept that makes the link between the the literature on frictionless markets and that on markets with transaction costs. These contributions are introduced in Sections 2.1 and 2.2 below.
In the second part, which is summarized in Section 2.3, we deal with an incomplete financial market, where the source of incompleteness arises from the fact that certain assets cannot be actively traded by the investor. We study in particular the problem of (exponential) utility indifference evaluation that was initiated by the seminal paper [ER00] in a slightly different model, by relaxing some of the assumptions to fit our particular framework. The motivation for this comes from the literature on structural models for electricity prices, which is thoroughly reviewed in [CC12].
The last part of the work deals with a moral hazard problem under the form of a principal-agent economic model in continuous time, that was mainly motivated by the problem of finding optimal incentives for the reduction of polluting emissions. This chapter is probably the less mathematically involved, as many of the techniques used therein have been already employed in the literature in similar contexts. The problem however, along with the underlying motivation, is new and it is treated by trying to keep a constant eye on the qualitative interpretations of the results obtained through numerical experiments and comparison analyses. This contribution is introduced in Section 2.4.

2.1 Markets with transaction costs I: Optimal investment with a multivariate utility function and random endowment

In this first part we aim at generalizing the results of [CO10], dealing with optimal investment with (time varying) transaction costs and a multivariate utility function, to the case where the agent's initial endowment is not deterministic but is given instead by a random variable (usually interpretable as a contingent claim) that we would like to keep as general as possible (in particular, without assuming it to be bounded). Other important contributions in the literature on optimal investment with random endowment include [Bo02] (with a univariate utility function and bounded endowment) and [DPT01] (with a multivariate utility function but constant transaction costs).

We assume that there are D assets in our financial market. The utility maximization problem that we deal with in this section can be written

$$u(\mathcal{E}) := \sup\left\{\mathbb{E}\left[U(X+\mathcal{E})\right] : X \in \mathcal{A}_T^0\right\}, \tag{2.1.1}$$

where

- The value function u is defined over a set of $\mathcal{F}_T$-measurable random variables $\mathcal{E}$ which can be naturally interpreted as derivative contracts or contingent claims in general. More precisely, we mainly consider claims that are bounded from below by a constant and from above by an admissible strategy, and we call this set $\mathcal{O}$. This situation clearly includes the case of a deterministic endowment treated in [CO10]. Here $T > 0$ is a fixed maturity: only final positions at this date matter in this context.
- The utility function U is assumed to be multivariate*, i.e. it is defined on $\mathbb{R}_+^D$. It is supposed to be strictly concave and increasing with respect to the preorder generated by the positive orthant (in words, it increase if all the components of its argument increase). Other more technical and less intuitive properties of U will be introduced in the text.
- The maximization takes place over the set $\mathcal{A}_T^0$, which represents all the (random) positions that can be attained at maturity T by trading on the market in an admissible way starting from zero initial endowment (which makes sense by the linear structure of $\mathcal{A}_T^0$ and since any deterministic endowment can be embedded in $\mathcal{E}$).

Multivariate utility functions are only rarely used in the literature, they are however interesting as a natural generalization of their univariate counterpart and because they are particularly suited to model markets in which it is seems somehow arbitrary to fix a numéraire asset from the beginning (for example, currency markets).
In this rather abstract formulation transaction costs are hidden inside $\mathcal{A}_T^0$, since the positions that can be attained at maturity by trading on the market are clearly influenced by how much it costs to trade. More precisely, and leaving aside unnecessary (for the moment) technicalities, the set $\mathcal{A}_T^0$ consists on all final portfolio values V_T such that the dynamics of the process V satisfy the constraint

$$dV_t \in -K_t,$$

where K_t is a (random) cone that describes the set of solvable positions at time t and can be written

$$K_t = \text{cone}\{e^k, \Pi_t^{ij} e^i - e^j : 1 \leq k, i, j \leq D\}.$$

Here, the e^i's form the canonical basis of $\mathbb{R}^D$ and Π_t^{ij} models the terms of trading of an asset with the other. The process Π is naturally required to satisfy the following conditions:

- $\Pi^{ij} > 0$, $1 \leq i, j \leq D$,
- $\Pi^{ii} = 1$, $1 \leq i \leq D$,
- $\Pi^{ij} \leq \Pi^{ik}\Pi^{kj}$, $1 \leq i, j, k \leq D$.

∗. In the article we work with the function $\tilde{U}$, defined starting from U with the aim of allowing the investor to be interested only in a subset of all the assets available for trading. This extension is however not relevant here for an exposition of the main ideas and results.

In particular transaction costs are hidden behind the last condition which says, in words, that it is always more profitable to exchange one asset i for another asset j in a direct way instead of trading another asset k as an intermediate step (which involves paying unnecessary transaction costs). The process Π (and therefore transaction costs) being possibly discontinuous, we easily see that this kind of modelisation, due to Kabanov ([KS09]), includes and generalizes all the other more explicit descriptions that have appeared in the literature.

At first we are interested in the properties of the value function, and more precisely in the degree at which it inherits the properties of the utility function U. We deal with this issue in Proposition 3.3.1 where we show that u is well defined, concave, increasing on $\mathcal{O}$, and other more technical properties such as the characterization of the closure of its effective domain.

It is often useful, when studying an optimization problem such as (2.1.1), to introduce its corresponding *dual problem*, which in our case transforms a maximization over strategies into a minimization over a suitable set of measures. Apart from its theoretical interest, the dual problem can be useful to study some of the properties of the original problem (such as existence of a solution), and in this particular context it also plays a key role in the construction of shadow prices (see the next section). The dual problem is defined as

$$v(\mathcal{E}) := \inf_{m \in \mathcal{D}} \left\{ \mathbb{E}\left[U^* \left(\frac{dm^c}{dP} \right) \right] + m(\mathcal{E}) \right\}$$

where U^* is the conjugate of U (sometimes called Legendre transform) defined as

$$U^*(x^*) := \sup_{x \in \mathbb{R}^D} \left\{ U(x) - \langle x, x^* \rangle \right\},$$

and the set of dual variables

$$\mathcal{D} := \left\{ m \in \mathrm{ba}(\mathbb{R}^D) : m(X) \leq 0 \text{ for all } X \in L^\infty(\mathbb{R}^D) \cap \mathcal{A}_T^0 \right\}$$

is composed of finitely additive measures on $\mathbb{R}^D$. The notation m^c denotes the countably additive part in the Yosida-Hewitt decomposition of m (a review on key concepts about finitely additive vector measures is given in Section 3.2.3). In Theorem 3.3.1 we show that for the class of endowments that we consider there is no *duality gap*, that is

$$u(\mathcal{E}) = v(\mathcal{E}).$$

Moreover, the infimum in the definition of v is always attained. By using duality arguments, we are also able to prove in Theorem 3.3.2 that, under the additional assumption of boundedness of the endowment, the primal optimization problem 2.1.1 admits a solution which is equal to

$$\hat{X} := I\left(\frac{d\hat{m}^c}{dP} \right) - \mathcal{E},$$

where $\hat{m}$ is the minimizer in the dual problem and I is the argmax in the definition of the conjugate of U.

As already mentioned, in the presence of transaction costs one generally loses the possibility of perfectly replicating contingent claims, which naturally creates an evaluation

problem. With this motivation, in the last part of Chapter 2 we use some of the previous results and certain techniques due to [OZ09] to study the utility indifference pricing paradigm in this framework (utility indifference prices will also be the subject of Chapter 5, though in a different context). More precisely, the utility indifference (bid) price (UIP) $p_j(B) = p_j(B; U, \mathcal{E}) \in \mathbb{R}$ for the contingent claim B (expressed in units of asset j) is implicitly defined as the solution to the equation

$$u(\mathcal{E} + B - e_j p_j) = u(\mathcal{E}). \tag{2.1.2}$$

In words, we are looking for a price that makes the agent indifferent between buying the claim B by paying its price and and not buying it at all, by taking into account that he can also invest in the transaction cost market to (partially) hedge its position. It is worth noticing that the price will in general depend on the initial claim $\mathcal{E}$ (contrarily to the case of exponential utility that we will consider in Chapter 5). In Proposition 3.4.1 we show that the definition of UIP is well-posed, i.e. $p_j(B)$ exists unique, and that it satisfies in particular the properties of cash-invariance, monotonicity and convexity characterizing a convex risk measure defined on vector-valued random variables (compare, for example, with [JMT04], [HHR10]). We also provide a dual representation for the price, that can be expressed as the result of a minimization problem over a given set of measures.
We finally introduce the average utility indifference purchase price for β units of the contingent claim B (in terms of asset j), defined as

$$p_j^\beta(B) := \frac{p_j(\beta B)}{\beta},$$

and we study the properties of the function $\beta \mapsto p_j^\beta(B)$ in Proposition 3.4.2. As an example, we find that it is non-increasing in β and that the following bounds hold:

$$\underline{m}_j(B) \leq p_j^\beta(B) \leq \hat{m}_j(B),$$

where $\underline{m}$ and $\hat{m}$ are two suitably defined measures.

2.2 Markets with transaction costs II: Existence of shadow prices

The second chapter of this thesis is still concerned with markets with transaction costs, under the same modeling framework that was introduced in the previous chapter. More precisely, we deal here with the shadow price problem, an issue that has attracted the attention of quite a lot of researchers in the recent years for several reasons. First, a purely theoretical interest linked to the fact that shadow prices make somehow the bridge between the literature on markets with transaction costs and the classical literature on frictionless markets, by allowing to find (in a certain sense) a frictionless equivalent to the original transaction cost market. Secondly, shadow prices have been used to find the solution to maximization problems that looked particularly difficult to attack by using standard and more direct approaches ([GMS10],[GGMS11], [KM11], [KS10],[CSZ12], [GM11],[LPS98], [CK96]).

We will now intuitively describe the problem setting and the main difficulties concerning its treatment.
Consider a market A with proportional transaction cost. As we have seen in the previous section, the utility maximization problem on this market can be written

$$J^A(x) := \sup_{f \in \mathcal{A}_T^x(A)} E[U(f)]. \tag{2.2.1}$$

where $\mathcal{A}_T^x(A)$ represents the set of claims that are attainable at time T by trading on market A starting with initial endowment x. We also suppose that all wealth is liquidated to the first (numéraire) asset at maturity, so that the utility function in this section will be univariate (and defined on the positive real line).
Take now a frictionless market B, and assume that prices on this market evolve inside the bid-ask spread of the original market. In the simple one-dimensional case with a bond and a stock S this can be formalized by writing

$$\underline{S} \leq S \leq \bar{S},$$

where $\underline{S}$ and $\bar{S}$ are the bid and ask prices for the stock. In a multidimensional setting, the same concept is described in a slightly more complex way by using the polar (in the sense of convex analysis) of the solvency cones K_t defined in the previous section. By definition, on market B the agent always buys at lower prices and sells at higher prices with respect to the original market A, and therefore

$$J^A(x) \leq J^B(x).$$

The shadow price problem can be summarized with the following question: does there exist a (particularly unfavorable) market B such that $J^A(x) = J^B(x)$? If so, we refer to such a market as a *shadow market* and to its prices $\hat{S}$ as *shadow prices*. Quite intuitively, the optimal trading strategies in the two markets will also coincide, which allows to fully solve the original problem if a shadow market can actually be found.
When shadow prices do exist, it means that the behavior of the agent facing transaction costs can be somehow reduced to his behaviour on a suitable frictionless market. The immediate advantage lies in the fact that the theory of frictionless markets is much more developed and, usually, substantially simpler than the transaction cost theory. Moreover, working on a frictionless market allows to reduce the dimensionality of the problem by avoiding to consider bid and ask prices separately. On the other side, if the answer turns out to be negative, then it means that portfolio choice on the transaction cost market really presents some qualitatively different features with respect to the classical theory.
How can we find a shadow price? Loosely speaking, we would like to find the frictionless market with the smallest value function, hoping that this market will turn out to be a shadow market. One idea is therefore to compute

$$\inf_{B \in \mathcal{B}_A} \sup_{f \in \mathcal{A}_T^x(A)} E[U(f)],$$

where $\mathcal{B}_A$ is the set of frictionless markets with prices in the spread of A. Now we can get rid of the "inf sup" by using frictionless duality ([KS99]) and obtain

$$\inf_{B \in \mathcal{B}_A, y>0} \left\{ \mathbb{E}\left[U^*\left(y \frac{dQ(B)}{dP} \right) \right] + yx \right\},$$

where $Q(B)$ is the (supposedly unique) equivalent martingale measure of market B. If there exists an optimal $(\hat{y}, \hat{Q})$, and a corresponding price $\hat{S}$, it remains to verify that the resulting optimal strategy is also admissible in the transaction cost market, which typically requires checking that the strategy "doesn't move" outside the sets $\{\hat{S} = \underline{S}\}$ and $\{\hat{S} = \bar{S}\}$. If so, the gap is filled and we will have

$$J^A(x) = E\left[U^*\left(\hat{y}\frac{d\hat{Q}}{dP}\right)\right] + \hat{y}x.$$

This approach is particularly useful to understand how the theory of frictionless markets can give a helping hand in the solution of problems where transaction costs are involved. The power of this methodology has been shown in the recent paper [CSZ12], where the authors use it to solve the Merton's consumption problem with transaction costs in full generality, by removing all the previous strong assumptions on the parameter values. We can also notice that, formally, the previous procedure consists in a minimization over the set of (d-dimensional) martingales Z such that Z/Z^1 lies in the bid-ask spread, with Z^1 playing the role of the equivalent martingale measure. The processes Z with these properties are called *consistent price systems* (see [JK95]) and play a crucial role both in arbitrage theory and, most importantly here, in the construction of shadow prices.
We mentioned that shadow prices have been found in many particular cases. Other partial existence results have been obtained in [KM10] or [LPS98]. The natural question that we address in this work is: can we expect them to exist in general? The first striking result that we present in Chapter 4 is that shadow prices may very well fail to exist, and this can happen even in very simple two-period models with one risky asset. Indeed this is the setting of a counterexample that we present at the beginning of the chapter and that shows that assuming the existence of a shadow price would automatically lead to arbitrage and hence to an exploding value function, thus reaching the desired contradiction.
What goes wrong in the counterexample and how can that be fixed in order to get a positive result? In order to work out some intuition, it is useful to appeal again to duality theory: however in this context it is more fruitful to consider the dual problem directly on the transaction cost market (rather than on a family of frictionless markets, as we did above with the aim of finding heuristically a shadow price). Here, the dual variables of interest (see [CMS11]) can be taken to be the set of supermartingales Z with values in the bid/ask spread (which, in our formalism, means in K^*) such that VZ is a supermartingale for any admissible strategy V. Following the same idea exposed before, by writing $VZ = Z^1\left(V\frac{Z}{Z^1}\right)$ we would like to interpret Z/Z^1 as prices on a frictionless market, and Z^1 as a martingale density. The crucial question now is: if $\hat{Z}$ is a dual minimizer on the transaction cost market, is $\hat{Z}^1$ dual minimizer for the frictionless market with price $\hat{Z}/\hat{Z}^1$? If the answer is positive, one can easily see that $\hat{Z}/\hat{Z}^1$ would be a shadow price process. However, this is exactly where things can go wrong, even in very simple situations. In fact, unless $\hat{Z}^1$ is assumed to be (local) martingale (which is a rather strong assumption, see [**?**] for a similar one), $\hat{Z}^1$ may very well fail to be a dual variable in the frictionless market. To see this, it suffices to apply integration by parts to get

$$d(\hat{Z}^1_t X_t) = V_t\left(d\hat{Z}_t - \frac{\hat{Z}_t}{\hat{Z}^1_t}d\hat{Z}^1_t\right) + X_t d\hat{Z}^1_t,$$

where $dX_t = V_t d(\hat{Z}/\hat{Z}^1)_t$ represents the wealth process in the candidate frictionless market. We see that if $\hat{Z}^1$ is only a supermartingale then $\hat{Z}^1_t X_t$ may fail to be a supermartingale (which is required for frictionless duality to work), the reason being that integrals with respect to supermartingales are not necessarily supermartingales. One natural way to make them supermartingales, however, could be to force the integrand V_t to be positive, i.e. prohibiting short selling in the market. Therefore this formal argument, and the fact that our counterexample heavily relies on the possibility of selling short, suggests that banning short selling may lead to an existence result for shadow prices. As we will show, this will actually be the case.

One of the main difficulties to get to the desired result is that we do not have a duality result for markets with transaction costs and positivity constraints on the strategies (and not on the wealth processes!). For this reason, it seems easier to work directly with the primal problem, by following an approach similar to that presented in [Lo00] in the case of a two-asset Brownian model. However the lack of duality makes it harder to prove existence of an optimal strategy in the constrained primal problem, which is in fact taken as a standing hypothesis in [Lo00]. We were able to prove existence in Proposition 4.3.1 without using duality, but rather some compactness results for strategies established in [CS06] and an adaptation of an argument in [Gu02].

In order to find a shadow market, we initially define a suitable family of frictionless markets starting from their prices Z/Z^1, for any supermartingale consistent price system Z (even if we are in a multidimensional framework, we take the first asset as a numéraire). Then in Proposition 4.4.1 we establish some sufficient conditions in order for such a price to be actually a shadow price. We finally construct a shadow price starting from its associated supermartingale consistent price system defined as

$$\hat{Z}^i_t := \lim_{\epsilon \downarrow 0} \frac{J(\hat{V} + e_i\epsilon, t) - J(\hat{V}, t)}{\epsilon},$$

and in Theorem 4.4.1 we show that it satisfies our sufficient conditions, so that $\hat{Z}/\hat{Z}^1$ is finally proved to be a shadow price in our constrained market. Even if duality is not explicit, we just remark that it still seems to be hidden behind the scenes, as it is suggested by one of the sufficient conditions

$$E[Z^1_T \hat{V}_T] = Z_0 x,$$

(where $\hat{V}$ is the optimal strategy) which is heavily reminiscent of duality theory.

After the publication of this work, other studies have confirmed the impossibility to obtain general existence results for shadow prices without imposing additional constraints. In particular the recent paper [CMS11] contains another counterexample that appears to be even more striking than ours, as it is based on a two-period discrete market where bid and ask prices are also assumed to be bounded (while in our case the unboundedness of the ask price at $t = 1$ plays a fundamental role for reaching the contradiction). Moreover, that counterexample (which is still based on short selling) shows that the absence of a shadow price does not necessarily imply the presence of strong arbitrage opportunities in the candidate shadow market: on the contrary, it is perfectly possible that the "worst" frictionless market with prices in the spread is simply not bad enough to match the maximal expected utility of the original transaction cost market. As the authors point out,

there is nothing too strange in this, as any candidate shadow market still allows in general for a larger set of admissible trading strategies than the original one. As a consequence, the short selling constraints that we introduced in our study seem to be very close to a minimal assumption when the aim is to find a general existence result for shadow prices.

2.3 Utility indifference valuation for non-smooth payoffs on a market with some non tradable assets

This part of the thesis deals with the pricing of claims in incomplete markets under the (exponential) utility indifference approach. Since the main motivation for this study comes from the evaluation of power derivatives under a structural framework, we consider useful to provide here a short overview of the main issues and concerns arising when modeling electricity prices.

Electricity is generally considered to be a "flow commodity", in the sense that it is only useful if delivered continuously over a period of time. Moreover, it has the additional important feature of being a non-storable good, except for rare cases such as hydro: it is therefore produced directly for consumption, with a very limited possibility to stock it for future use. It is seasonal, as demand is clearly affected by the period of the year (and even the time during the day), and usually presents some kind of mean-reverting behavior. Finally, electricity prices often exhibit spikes, in the form of sudden price increases usually followed by rapid recoveries, due to unexpected change of demand/supply such as a sudden temperature increase or a power plant breakdown (see the survey [CC12] for additional details). The modelisation of the electricity spot price is the first step towards the evaluation of power derivatives (and therefore of power plants through the use of spread options), for the deployment of optimal (in some sense) hedging strategies and for risk management purposes. For these reasons it should be as realistic as possible, satisfying in particular the stylized facts mentioned above, while maintaining a certain tractability in order to provide closed-form formulas or at least efficient numerical methods for derivative pricing. Two classical ways of proceeding typically appear in the literature: reduced-form and structural models. While the former directly concentrate on the spot price dynamics (for example [BCK07]), the latter usually try to express the price as a result of the interaction of multiple fundamental economic factors (for example [Ba02], [CV08], [PJ08], [CCS12]). The typical factors in structural electricity price modeling include:

- Fuel prices S_t^i, $1 \leq i \leq n$,
- Electricity demand D_t (in MW),
- Fuel capacities C_t^i (in MW), $1 \leq i \leq n$,
- Heat rates h_i (where $h_i S_t^i$ is expressed in €/MWh), $1 \leq i \leq n$.

Perhaps the most intuitive idea from an economic point of view is to describe the electricity spot price P_t as the marginal cost of the fuel that is used in the production, by writing for example

$$P_t = \sum_{i=1}^{n} h_i S_t^i \mathbf{1}_{\left\{\sum_{k=1}^{i-1} C_t^k \leq D_t \leq \sum_{k=1}^{i} C_t^k\right\}}. \tag{2.3.1}$$

This is the formulation that is used in [ACNT02], while in [ACL10] the authors improve the model by adding to (2.3.1) a multiplicative factor that increases nonlinearly in $\sum_{k=1}^{i-1} C_t^k -$

D_t, in order to account for the spikes that are usually observed when the electric system is under stress. A similar, but more complex, description is obtained in [CC12].
It is immediately evident that one has to deal in this context with payoffs that typically also depend on factors that are not directly tradable on a financial market (like capacities and demand), thus creating market incompleteness and therefore the need for alternative evaluation procedures. Notice also that this dependency can be highly irregular in the nontraded assets, which makes hedging much harder, unless one is willing to use super-replication (that is however generally considered to be too expensive). One possible way out is to use the *local risk minimization* paradigm (see [Sc01]), that was applied in this context by [ACL10] and which has the advantage to provide useful closed-form formulas for derivatives pricing, when the underlying dynamics are reasonably simple and the two asset classes (traded and nontraded) are assumed to be uncorrelated (which seems like a sound approximation in this framework). Another evaluation paradigm, which can be considered more flexible as it allows to play with one additional parameter, is *utility indifference pricing* and it is the one we will focus on in the sequel. The idea is pretty simple: the price of a claim should be such that the agent will be indifferent between possessing the claim, by paying its price, and doing nothing. In formulas the buying price p^b of an $\mathcal{F}_T$-measurable claim is defined implicitly by the following equation:

$$\sup_\pi E\left[U\left(V_T^{v-p^b}(\pi)+f\right)\right] = \sup_\pi E\left[U(V_T^v(\pi))\right], \tag{2.3.2}$$

where $V_T^x(\pi)$ is the value at time T of a portfolio that starts with x at time zero and follows the strategy π. The selling price p^s is defined analogously. We stress that the strategy π must be constrained to invest only in those assets that are actually tradable (though it may clearly depend on the whole market information). In the absence of a financial market, this definition essentially boils down (as we show in Remark 5.2.3) to the well-known economic concept of *certainty equivalent*, that was explored by [BCK07] in the context of electricity markets. We will only consider the case of exponential utility, i.e. when $U(x) = -e^{-\gamma x}$ for some $\gamma > 0$, which is particularly convenient since in this way the price will not depend on the initial wealth v.
We place ourselves in a Markovian framework, where traded assets (i.e. fuels) S follow the simple Black-Scholes dynamics

$$\frac{dS_t^i}{S_t^i} = \mu_i dt + \sigma_i . dW_t^S$$

while nontradable assets (i.e. capacities and demand) X follow the mean-reverting process

$$dX_t^i = (b^i(t) - \alpha_i(t)X_t^i)dt + \beta_i.(t)dW_t^X,$$

where W^S and W^X are independent Brownian motions with respective dimensions n and d. All the parameters in the processes are either assumed to be constants or deterministic functions of time (for example the function $b(t)$ may typically account for the well-known seasonal oscillations in electricity demand). The sub-market composed of traded assets is complete and admits a unique martingale measure, which is also the *minimal martingale measure* Q^0 of the whole market and which plays a fundamental role in local risk minimization (LRM). Our standing assumption on the payoff f is that it must be sub- and

super-replicable, i.e. it must satisfy

$$V_T^{v_1}(\pi^1) \leq f \leq V_T^{v_2}(\pi^2)$$

almost surely for some given portfolio strategies. We first prove in Lemma 5.2.1 the following relation between utility indifference prices, sub- or super-replicating costs and the LRM price:

$$v_1 \leq p^b \leq E^0[f] \leq p^s \leq v_2.$$

Although the result was essentially already known ([Ho05]), we provide a new proof which is based on duality arguments. We then show that, when f is bounded, the buying price at time 0 p^b verifies $p^b = Y_0$, where (Y, Z) solves the BSDE

$$Y_t = f - \int_t^T \frac{\gamma}{2}\|Z_s^X\|^2 ds - \int_t^T Z_s dW_s^0. \tag{2.3.3}$$

where W^0 is a Brownian Motion under the minimal martingale measure and the decomposition $y = (y^S, y^X)$ separates what is related to traded and nontraded assets. Moreover, the process Z is (up to a constant) the optimal hedging strategy, defined as the difference of the maximizers in (2.3.2). The result was essentially known since [ER00], but again we provide here different proof based on a technique developed in [HIM05]. In Lemma 6.7.2 we extend this result by proving existence of (2.3.3) when f is only sub- and super-replicable, and providing conditions allowing to interpret Y as the utility indifference price also in this case. This will be the case, in particular, when f is a European claim having a polynomial growth in the traded assets which is uniform in the nontraded ones.
The previous characterization, however, does not allow in general to know much about the process Z: we do not know, for example, if it has continuous paths, if it is unique or if it is even linked with the optimal hedging strategy (due to the absence, in general, of the BMO property). In order to get some more information on this, the second part of the paper is devoted to the case of European payoffs. Assumptions 5.4.1 and 5.4.2 enumerate the technical properties that we need to impose on the final claim. We only mention here that we allow for points of nondifferentiability of f (which usually applies to options) and even for f to be discontinuous in the nontraded assets, as is the case in our leading example (2.3.1).
In Theorem 5.4.1, which is the main result of the chapter, we show that we can express the solution of (2.3.3) as $Y_t = \varphi(t, S_t, X_t)$, where φ solves in the viscosity sense the associated PDE:

$$\begin{cases} \mathcal{L}\varphi - \frac{\gamma}{2}\sum_{j=1}^d (\beta'_{\cdot j}\varphi_x)^2 = 0 \\ \varphi(T, a) = f(a) \end{cases} \tag{2.3.4}$$

where $\mathcal{L}$ is the standard Dynkin operator corresponding to the dynamics of the assets. What is particularly interesting to prove is that φ is also continuously differentiable in all the state variables thus providing a natural way of representing Z as $\sigma(S_t)\varphi_s(t, S_t, X_t)$ (implying therefore that Z has continuous paths), where $\sigma(S)$ is the matrix whose i-th row is given by $\sigma_{i\cdot}S^i$. In this way, we can also identify the optimal hedging strategy as $\Delta_t = -\sigma^{-1}\sigma(S_t)\varphi_s(t, S_t, X_t)$.
The proof of Theorem 5.4.1 is rather lengthy, but the main idea is the following: We

start by rewriting BSDE (2.3.3) in such a way that its generator is uniformly Lipschitz, or equivalently by rewriting (2.3.4) so that it can be represented as the solution of a stochastic control problem where the space of controls is forced to be compact. The key for doing this is to make use of a technique employed in [Ph02] that consists in modifying the source term in (2.3.4) by using the function

$$h^m(q) = \sup_{\delta \in \mathcal{B}^m(\mathbb{R}^d)} \left\{ -q\delta - \frac{1}{2\gamma}\|\delta\|^2 \right\},$$

where $\mathcal{B}^m(\mathbb{R}^d)$ is the ball in $\mathbb{R}^d$ centered at zero and of radius $m > 0$, and then letting m tend to infinity. The payoff needs also to be approximated by mollification when it is not sufficiently smooth. The trick works well when f is regular enough to allow for good estimates on the derivatives of φ, while it requires some additional work in the presence of discontinuities in f. The existence and continuity of the spacial derivatives of φ is obtained by using some Malliavin-type representations that have been developed in [MZ02] (and already used for similar purposes in [Zh05]).
The existence of an optimal hedging strategy and its characterization in terms of φ_s is also obtained by an approximating argument, using the fact that for bounded payoffs we already know from [HIM05] that Z is interpretable in this way.
Our last contribution in this part is related to asymptotic expansions for the price and the optimal hedging strategy in terms of the price under the minimal martingale measure (i.e. that studied in [ACL10] under the same framework). The price expansion is obtained using a result in [Mo12] which employs ideas from Malliavin calculus, while the strategy expansion is derived using similar arguments finally leading to

$$\varphi_{s^i}(t,a) = E^0_{t,s,x}\left[f_{s^i}(S_T, X_T)\right] - \gamma E^0_{t,s,x}\left[f_{s^i}(S_T, X_T)\int_t^T \beta \varphi^0_x dW^X_u\right] + O(\gamma^2),$$

where $\varphi^0_{x^i}(t,s,x) = E^0_{t,s,x}\left[f_{x^i}(S_T, X_T)\right]$. We finally computed the price expansion explicitly for the forward contracts with two fuels.
As a conclusion, we provided a contribution to the pricing and hedging of power derivatives under the recent structural modeling framework proposed in [ACL10], where some of the assets are not tradable and most claims of interest often present irregularities. Derivatives are naturally written in this context as functions of both traded and nontraded assets, which is not so common in the literature on utility indifference pricing (see [HH09], or the introduction of Chapter 5 for more precise references), except some particular cases like [SZ04]. We believe, however, that some of the ideas may come useful also in other contexts and frameworks, typically in the presence of latent variables that produce regime-changing features. This might for example be the case for insurance products, which typically present both a financial (tradable) component and a strictly insurance-type part depending on variables that cannot, in general, be traded on a market (and which we can reasonably assume to be uncorrelated with the tradable ones).

2.4 A principal-agent problem and emission reduction

In this last part of the thesis we deal with a principal-agent problem in continuous time with finite horizon. The model is similar to the ones already appeared in the literature (see

[CZ12] for a survey), however there are differences for example in the main motivation, in the definition of incentives (that here can be of two kinds, as we will see in a moment) and in the form of admissible strategies, that here are characterized in a simpler way. Overall the aim of this section is to study a particular model by privileging simplicity over generality, and trying to obtain also some numerical and qualitative results that hopefully help in the interpretation and understanding of this kind of models (that often quickly become rather complex).
In order to get a grasp of what a principal-agent problem is, we start by quickly recapping the main ideas behind the first seminal contribution in this field, which is due to [Sa08]. The context is the theory of optimal contracts with moral hazard, where an employer and an employee have (at least partially) contrasting interests and the actions of the employee (i.e. his work amount/quality) are not completely observable by his principal. In particular, the employee (agent) maximizes the functional

$$E\left[\int_0^\infty e^{-rt}(u(s_t) - h(k_t))dt\right] \tag{2.4.1}$$

over all actions policies k_t. Here s_t represent the agent's salaries, which must be decided by an employer (principal) maximizing

$$E\left[\int_0^\infty e^{-rt}(k_t - s_t)dt\right], \tag{2.4.2}$$

where intuitively the quantities play here on opposite directions with respect to (2.4.1). The crucial assumption that characterizes principal-agent problems is that the principal does not observe the agent's actions k_t, but he is nonetheless supposed to know the functional (2.4.1) that the agent tries to optimize. Therefore the principal is able to guess the optimal agent's response in relation to a given salary policy (or incentive policy) s_t: as a consequence, when the principal optimizes (2.4.2), it is natural for him to replace the process k with $k(s)$, the optimal agent's effort given s, and then finally choose the best s. It is immediately clear that a good characterization of the agent's optimal effort is an essential step in the resolution of the principal's problem, which is why the two are usually treated separately one after the other. The main techniques that are used in [Sa08] to solve the problems are the martingale representation theorem (MRT) and ODE theory, which allows to characterize all the optimal quantities in terms of the solution of a differential equation (which can be discretized to get numerical results).
After this seminal contribution, the theory of continuous time principal-agent problems has evolved and it has proven to be particularly challenging to extend the same ideas to the case of a finite time horizon: here the equivalent to the simple MRT is the theory of Forward-Backward Stochastic Differential Equations (FBSDEs), while ODEs are typically replaced by PDEs. An exhaustive survey of the relevant contributions in this area can be found in [CZ12], while a good reference for the underlying mathematical theory is [MY99]. The particular model that we consider in this section belongs to this stream of literature, however the main motivation is slightly different since we rather take the point of view of a regulator that aims at providing optimal incentives in order to reduce the production of polluting emissions by a firm (though this is not the only possible interpretation). The intent is twofold: first, we want to analyze a newly motivated model which is similar but

not included in the cases already considered by other authors; second, we want to keep the modeling framework sufficiently simple in order to be able to extract also some qualitative results and to visualize and interpret some of the optimal quantities through numerical approaches (which is usually not straightforward, due to the fact that these models can very quickly become rather involved).
We start by modeling the *emission process* (taking values in $\mathbb{R}_+$) with the following dynamics:

$$X_t = x + \int_0^t X_r l(k_r) dr + \int_0^t X_r \sigma dW_r. \tag{2.4.3}$$

The process X (which is clearly forced to be positive) can either be interpreted as the emissions level at a certain date t, or as the market perception of cumulative emissions up to a given date t (in a similar fashion as in [CDET13] though under a completely different framework). Here the process k represents the agent's effort, which can be interpreted for example as the technology that is used for emission reducing purposes. Effort levels take values in $\mathbb{R}_+$ (where a higher value stands for a better technology), and l captures the effect of technology on the emissions (intuitively, $l'(k) \leq 0$). The assumption here is that the principal does not observe the agent's actions k (for example, because of excessive monitoring costs) but he does observe the process X, which models the (noisy) consequences of the agent's effort. Therefore the regulator cannot directly force the agent to choose a desired k, but what he can do is to *motivate* him to do so by choosing an appropriate incentive policy, which can only depend on the process X that he can observe. An incentive policy is a couple (s, p), where s models a continuous payment flow while p describes a lump sum payment at a fixed final maturity T (we typically think of it as a proportional or forfeitary payment based on final emissions, such as $p(X_T) = (X_T - \Lambda)^+$ for some threshold value $\Lambda > 0$). The expected utility of the agent is written as

$$V(k) = E\left[\int_0^T u(s_t - c(k_t))dt - p\right], \tag{2.4.4}$$

where u is a utility function and c a convex cost function. Here we voluntarily skip some technicalities that will be treated in the text through measure change techniques: to get a hint of what they relate to, remark for example that the expected value of the final payment must also depend on the effort policy k, though it does not look so from the simplified functional (2.4.4). The characterization of the optimal effort that maximizes (2.4.4) is carried out in Propositions 6.2.1 and 6.2.2. The main tool that is used to derive necessary conditions for optimality is the stochastic maximum principle, which is often used in the literature under slightly different formulations (see, for example, [CWZ08] or [Wi08]): here the simplicity of our model (in particular, the boundedness assumptions for l and s) allows us in particular to make it work under a less cumbersome version of effort admissibility. Sufficient conditions, on the other hand, take advantage of standard BSDE theory and the convexity structure of the model. More precisely, we construct in Lemma 6.2.1 a function F that inverts the necessary conditions for optimality and that allows us to write the BSDE

$$\begin{cases} dY_t = [-Z_t l(F(s_t, Z_t))/\sigma - u(s_t - c(F(s_t, Z_t)))]dt + Z_t dW_t \\ Y_T = -p \end{cases} \tag{2.4.5}$$

which provides the agent's optimal effort given incentives (s, p) through the relation $k_t = F(s_t, Z_t)$.
Given the importance of the agent's optimal response to incentive policies, it is pretty natural to wonder if it is possible to characterize it a little bit better, for example by finding its dynamics, or to exploit BSDE theory to analyze how the effort is modified by a change in the incentives or in the parameter values. It is quite intuitive to realize that higher continuous-time incentives s or a lower penalty p will increase the agent's utility (this is actually proved in Lemma 6.3.1), but it is less obvious to analyze how similar changes will affect the agent's effort. Motivated by this we managed to derive, under additional assumptions (mainly markovianity of incentives), a BSDE that is directly solved by the optimal effort k_t, that under the simplest case of constant continuous-time incentives s follows

$$\begin{cases} -dk_t = \left[G(s, k_t)\Theta_t^2 + \frac{l(k_t)}{\sigma}\Theta_t\right] dt - \Theta_t dW_t \\ k_T = F(s, -\sigma X_T p'(X_T)). \end{cases} \tag{2.4.6}$$

The case where s is a function of t and X_t is slightly more involved and is presented in Proposition 6.3.1. Equation (2.4.6) is a quadratic BSDE whose quadratic coefficient G is a function of the parameters of the model that can be proved to be bounded under suitable assumptions (basically, p' must not be "too big", i.e. the final penalty must not be too sensitive to small perturbations of X). As a byproduct, we also show that the solution Z to (2.4.5) admits the representation

$$Z_t = -E^* \left[\sigma X_T p'(X_T) \mid \mathcal{F}_t\right],$$

where the expectation is defined by using a suitable change of measure that depends on the optimal effort. With a BSDE for k at hand, we finally perform some sensitivity analyses by looking, in particular, at how the process reacts to changes in the agent's risk aversion or impatience rate (where the impatience rate is simply defined similarly as in [Sa08] by rewriting (2.4.4) using a discount factor δ). We find that the optimal effort is always decreasing in the impatience rate, while it is increasing in the agent's risk aversion if some conditions are satisfied (again, if p' is not too big). Finally in Section 6.4 we propose a numerical scheme for the computation of the effort, by adapting existing discretization schemes for nonlinear PDEs, and in Section 6.4.1 we discuss some numerical results obtained under different incentive specifications. Simulated trajectories of X are also provided to get a better intuition of the typical agent's behavior.
The last part of the work is devoted to the principal's problem. As we have just seen, the agent's task it to optimally choose an effort k by taking an incentive policy (s, p) as given. Quite intuitively the principal, on his side, has to optimally decide over incentives (s, p), by taking into account the optimal agent's response to his particular choice. The principal profit functional is defined as

$$E\left[p_1(p) - p_2(X_T) - \int_0^T u_1(s_r)dr\right], \tag{2.4.7}$$

where p_1 relates the final (dis)utility of the agent to the final utility of the principal, p_2 captures the social costs related to the level of emissions, and u_1 is a utility function accounting for continuous-time payments to the agent. It turns out, however, that a direct

optimization over (s, p) is not the easiest way to proceed: it is indeed more convenient for the principal to directly optimize over (s, k), provided that he then modifies p appropriately so that the agent will actually perform the desired effort k. This intuition (remark the similarity with the ideas described at the beginning of this section) is put into practice by replacing p in (2.4.7) with

$$R - \int_0^T [u(s_r - c(k_r)) + g(k_r, s_r)l(k_r)/\sigma]dr + \int_0^T g(k_r, s_r)dW_r, \tag{2.4.8}$$

which is obtained by using the agent's optimality conditions and by fixing his initial utility to a predetermined level R (in order for the problem to be well-posed). The function g is defined starting from the derivatives of u, l and c, and it is shown to be interpretable as the elasticity of the agent's utility with respect to a change in the emissions growth rate over a little lapse of time.
We finally managed to write the problem as a maximization problem over (s, k), for which we provide necessary optimality conditions in Proposition 6.5.1 by still using the stochastic maximum principle (but with two state/adjoint variables). It would not be hard to state also some sufficient conditions for optimality, however they typically require some degree of convexity of the Hamiltonian function which is hard to verify in practice. This is the reason why the very last part of this work is devoted to analysis of the simplified situation in which $p_1(x) = x$, which corresponds to the case where the final agent's disutility linked to the payment of the fee corresponds to a principal's utility of the same amount. The problem becomes now much simpler because one state/adjoint variable can be eliminated, which leaves us with a problem formulation which is not that different from the agent's one. In this case, however, the inversion of necessary conditions is slightly more complex because the principal's and agent's optimality equations must be taken into account jointly. The possibility of doing this should be checked case by case, so we start by assuming that the inversion is possible and that this procedure gives rise to two functions I and L that naturally define a new candidate BSDE of the form

$$\begin{cases} dY_t = -[Z_t l(L(Z_t))/\sigma + u(I(L(Z_t)) - c(L(Z_t)))]dt + Z_t dW_t \\ Y_T = -p_2(X_T) - \int_0^T u_1(I(L(Z_t)))dt - a \end{cases} \tag{2.4.9}$$

where the parameter $a \in \mathbb{R}$ must be chosen in such a way that the initial agent's utility is respected. In Proposition 6.5.2 we show that the solution to (2.4.9) actually gives the optimal effort and continuous-time incentives by setting $k_t = L(Z_t)$ and $s_t = I(k_t) = I(L(Z_t))$. The optimal penalty function is then given by $p_2(X_T) + \int_0^T u_1(s_r)dr + a$. Remark that (2.4.9) is very similar to the only-agent equivalent (2.4.5), except for the fact that the variable s no longer appears in the equation as it is also found endogenously from the solution of (2.4.9).
We finally terminate the principal's part with an example that shows how the I and L can be computed in the special case of a power utility function, and showing a numerical example that highlights in particular the optimal choice of continuous-time incentives.

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Chapter 3

Markets with transaction costs I: Optimal investment with a multivariate utility function and random endowment

Abstract: In this chapter we deal with a utility maximization problem at finite horizon on a continuous-time market with conical (and time varying) constraints (particularly suited to model a currency market with proportional transaction costs). In particular, we extend some recent results of Campi and Owen (2010) to the situation where the agent is initially endowed with a random and possibly unbounded quantity of assets. We start by studying some basic properties of the value function (which is now defined on a space of random variables), then we dualize the problem following some convex analysis techniques which have proven very useful in this field of research. We finally prove the existence of a solution to the dual and (under an additional boundedness assumption on the endowment) to the primal problem. The last section of the paper is devoted to an application of our results to utility indifference pricing.

The content of this chapter is based on the paper "Multivariate Utility Maximization with Proportional Transaction Costs and Random Endowment", with Luciano Campi, published on *SIAM J. Control and Optimization* 50(3): 1283-1308 (2012).

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3.1 Introduction

We place ourselves in the framework of a continuous-time market with proportional transaction costs as described in [CO10] and in [CS06]. The agent's objective is to maximize his utility at a fixed terminal date T by trading in the available assets. The model is very general, as it allows the portfolio process to be driven by any cone-valued process, provided it satisfies some regularity assumptions. In the most common version of the model, the cones are generated by the evolution of bid-ask prices (which may possibly have jumps) and therefore they describe market frictions due to transaction costs. Also in this framework, the model preserves a great generality as the modeling of bid-ask prices does not pass through asset prices and transaction costs dynamics separately. This approach, based on the key concept of solvency cones, was first introduced in [Ka99] and it has been further developed by many authors in the last decade (for more details, see the recent book [KS09] and the references therein).
The agent's preferences are described by a multivariate utility function (see Section 3.2.2) supported on $\mathbb{R}^d_+$, reflecting the idea that the agent will not necessarily liquidate his positions to a single numeraire at the final date (which is realistic, in particular, on a currency market). We also make the following assumptions.

Assumption 3.1.1. The utility function $U : \mathbb{R}^d \to [-\infty, \infty)$ satisfies the following conditions:

1. U is measurable;
2. U is strictly concave on the interior of $\mathbb{R}^d_+$;
3. U is essentially smooth and its gradient diverges at the boundary of $\mathbb{R}^d_+$ (see Definition 3.2.5);
4. U is asymptotically satiable (see Definition 3.2.6).

As in [Ka01], the utility function is then extended to $D > d$ assets in order to model the investor's preferences towards a restricted set of assets in a larger economy. This is motivated by the fact that the agent may be ultimately interested in consuming a small set of assets at the date T, but he will trade in all available assets in order to reach his objective). Hence we define $\tilde{U} : \mathbb{R}^D \to [-\infty, +\infty)$ by

$$\tilde{U}(x) = \begin{cases} U(x_1, ..., x_d), & \text{if } x \in \mathbb{R}^D_+; \\ -\infty & \text{otherwise.} \end{cases} \tag{3.1.1}$$

In the formulation of [CO10] the investor is initially endowed with a deterministic amount $x \in \mathbb{R}^D$ of different assets, while in this paper we extend those results by assuming that the initial endowment is a random variable, that we call $\mathcal{E} := (\mathcal{E}_1, \ldots, \mathcal{E}_D)$. For example the agent may have no assets at the beginning but he may have access to some contingent claims on these assets (such as a right to buy or to sell some of them at a future date).
The earliest work on optimal investment using convex duality methods (with no transaction costs and deterministic endowments) dates back to [KLSX91]. The first introduction of a (bounded) random endowment is due to [CW01], where the authors considered univariate utility functions and used some of the ideas already developed in [KS99]. Important contributions in the same direction have later been given, among others, in [HG04] and

[OZ09], where the boundedness condition on the endowment is relaxed and replaced by weaker requirements (those in [OZ09], in particular, have inspired the ones which are employed in this paper).
In the literature of market with frictions, [Bo02] first accounted for transaction costs in the optimization problem * (with bounded random endowment) by adapting the underlying mathematical framework, using the already mentioned idea of solvency cones introduced by Kabanov in a series of papers (see [KS09] for a reference). This new modeling approach paved the way for the more general model in [CS06] (with time varying and random proportional transaction costs), which in turn provided the necessary tools for the results in [CO10], where multivariate utility functions are introduced in the optimization problem (with deterministic endowment). See also [DPT01], where the topic of multivariate utility maximization has been studied for the first time (in a constant transaction cost framework).
The subject of utility-based pricing of contingent claims, that we investigate in the last section, has been an active (and quite natural) area of research since the introduction and development of incomplete market models, in which the replication paradigm is no longer sufficient to find a unique price (hence utility comes in as an additional criterion of choice). The idea of utility indifference pricing has been first introduced in a dynamic hedging framework by [HN89] and it has been further extended by other authors in different settings, possibly under different names, see for example [Mu99] and [OZ09] (which is our main reference). In fact, the underlying concept of *certainty equivalent* is quite pervasive in the whole economics literature, because of its natural and intuitive interpretation. We refer to [HH09] for a more detailed overview on this subject.
Before proceeding, Section 3.2 will give some details on the transaction cost model we work on, as well as some preliminaries on the main mathematical tools that we are going to employ. The main results are presented in Section 3.3, while in Section 3.4 we propose an application to utility-based pricing of contingent claims.

3.2 Preliminaries

In this section we present all the preliminary concepts and notation which are required for the analysis of the optimization problem.

3.2.1 Cones and transaction costs

A general and convenient description of a large class or market constraints and/or frictions can be provided by a Kabanov-type market model, which is centered on the idea of cone-valued processes (evolving in continuous time in our framework). Let $(\Omega, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions and supporting all processes appearing in this paper. We will use the notation χ_A for the indicator function of a set A and cone(A) to denote the cone generated by any set A in $\mathbb{R}^D$.

A $\mathcal{C}$-valued process is defined as a sequence of set-valued mappings $K = (K_t)_{t\in[0,T]}$ specified by a countable sequence of adapted $\mathbb{R}^D$-valued processes $X^n = (X^n_t)$ such that,

∗. In fact, utility maximization with transaction costs had already been studied in [CK96], but on a market with only one risky asset, constant transaction costs and no random endowment.

for all t and ω, only a finite but nonzero number of $X_t(\omega)$ is different from zero and

$$K_t(\omega) := \text{cone}\{X_t^n(\omega), n \in N\}$$

which implies that $K_t(\omega)$ is a polyhedral cone (by the so-called Farkas-Minkowski-Weyl Theorem, see e.g. Section 5.1 in the Appendix in [KS09]). The cones K_{t-} are the ones generated by the left limits of the generators. As we shall see in a moment, these cones are there to describe the trading possibilities of an investor over time, i.e. to model the evolution of the portfolio processes.
Let $K_{s,t}(\omega)$ denote the closure of $\text{cone}\{K_r(\omega), s \leq r < t\}$, and let

$$K_{s,t+}(\omega) := \bigcap_{\epsilon>0} K_{s,t+\epsilon}(\omega), \;\; K_{s-,t}(\omega) := \bigcap_{\epsilon>0} K_{s-\epsilon,t}(\omega).$$

In order to derive useful results one needs some regularity assumptions that we list here. Recall that a cone K is proper if $K \cap (-K) = \{0\}$.

Assumption 3.2.1.
1. The cones K_t and K_{t-} are proper and contain $\mathbb{R}^D_+$ (Efficient friction)
2. $K_{t,t+} = K_t$, $K_{t-,t} = K_{t-}$ and $K_{t-,t+} = cone\{K_{t-}, K_t\}$ for all t.

Remark 3.2.1. It can be shown (see [KS09], p.165) that (2) is verified if (1) is true and all cones K_t and K_{t-} can be generated by a finite number of càdlàg vector processes.

Example 3.2.1. Even though all the results of this paper are true just under the above assumptions, we give here an example of how cone processes can be constructed in a particular (but still quite general) model of a market with transaction costs, which is the main situation we have in mind (and which justifies the title of the paper). In such a model, formalized in [CS06] (see also [S04]), all agents can trade in D assets according to a random and time varying bid-ask matrix. A $D \times D$ matrix $\Pi = (\pi^{ij})_{1 \leq i,j \leq D}$ is called a *bid-ask matrix* if (i) $\pi^{ij} > 0$ for every $1 \leq i, j \leq D$, (ii) $\pi^{ii} = 1$ for every $1 \leq i \leq D$, and (iii) $\pi^{ij} \leq \pi^{ik}\pi^{kj}$ for every $1 \leq i, j, k \leq D$.
Given a bid-ask matrix Π, the *solvency cone* $K(\Pi)$ is defined as the convex polyhedral cone in $\mathbb{R}^D$ spanned by the canonical basis vectors e^i, $1 \leq i \leq D$ of $\mathbb{R}^D$, and the vectors $\pi^{ij}e^i - e^j$, $1 \leq i, j \leq D$. The convex cone $-K(\Pi)$ should be interpreted as those portfolios available at price zero.
We must now introduce randomness and time in the model. An adapted, càdlàg process $(\Pi_t)_{t\in[0,T]}$ taking values in the set of bid-ask matrices will be called a *bid-ask process*. Once a bid-ask process $(\Pi_t)_{t\in[0,T]}$ has been fixed, one can drop it from the notation by writing K_τ instead of $K(\Pi_\tau)$ for a stopping time τ, coherently with the framework introduced above. Under the hypothesis of efficient friction (1), part (2) of Assumption 3.2.1 is automatically satisfied in this case by Remark 3.2.1.

In accordance with the framework developed in [CS06] we make the following technical assumption throughout the paper. The assumption is equivalent to disallowing a final trade at time T, but it can be relaxed via a slight modification of the model (see [CS06, Remark 4.2]). For this reason, we shall not explicitly mention the assumption anywhere.

Assumption 3.2.2. $\mathscr{F}_{T-} = \mathscr{F}_T$ and $\Pi_{T-} = \Pi_T$ a.s.

Given a cone K in $\mathbb{R}^D$, its (positive) polar cone is defined by

$$K^* = \left\{ w \in \mathbb{R}^D \,:\, \langle v, w \rangle \geq 0, \forall v \in K \right\}.$$

Definition 3.2.1. An adapted, $\mathbb{R}^D_+ \setminus \{0\}$-valued, càdlàg martingale $Z = (Z_t)_{t\in[0,T]}$ is called a *consistent price process* for the $\mathcal{C}$-valued process K if $Z_t \in K_t^*$ a.s. for every $t \in [0,T]$. Moreover, Z will be called a *strictly consistent price process* if $Z_t \in \operatorname{int}(K_t^*)$ and $Z_{t-} \in \operatorname{int}(K_{t-}^*)$ a.s. for every $t \in [0,T]$. The set of all (strictly) consistent price processes will be denoted by $\mathcal{Z}$ ($\mathcal{Z}^s$).

The following assumption, which is used extensively in [CS06], will also hold throughout the paper.

Assumption 3.2.3 (SCPS)**.** Existence of a strictly consistent price system: $\mathcal{Z}^s \neq \emptyset$.

This assumption is intimately related to the absence of arbitrage (see also [JK95, GRS10, GK12, DGR11]).

Definition 3.2.2. Suppose that $K = (K_t)_{t\in[0,T]}$ is a $\mathcal{C}$-valued process such that Assumption 3.2.3 holds true. An $\mathbb{R}^D$-valued process $V = (V_t)_{t\in[0,T]}$ is called a self-financing portfolio process for the process K if it satisfies the following properties:

1. It is predictable and a.e. path has finite variation (not necessarily right-continuous).
2. For every pair of stopping times $0 \leq \sigma \leq \tau \leq T$, we have

$$V_\tau - V_\sigma \in -K_{\sigma,\tau}$$

A self-financing portfolio process V is called *admissible* if it satisfies the additional property

3. There is a constant $a > 0$ such that $V_T + a\mathbf{1} \in K_T$ a.s. and $\langle V_\tau + a\mathbf{1}, Z_\tau^s \rangle \geq 0$ a.s. for all $[0,T]$-valued stopping times τ and for every strictly consistent price process $Z^s \in \mathcal{Z}^s$. Here, $\mathbf{1} \in \mathbb{R}^D$ denotes the vector whose entries are all equal to 1.

Let $\mathcal{A}^x$ denote the set of all admissible, self-financing portfolio processes with initial endowment $x \in \mathbb{R}^D$, and let

$$\mathcal{A}_T^x := \{V_T \,:\, V \in \mathcal{A}^x\}$$

be the set of all contingent claims attainable at time T with initial endowment x. Note that $\mathcal{A}_T^x = x + \mathcal{A}_T^0$ for all $x \in \mathbb{R}^D$.

For the convenience of the reader we present a reformulation of [CS06, Theorem 4.1].

Theorem 3.2.1 (Super-replication)**.** *Let $x \in \mathbb{R}^D$ and let X be an $\mathscr{F}_T$-measurable, $\mathbb{R}^D_+$-valued random variable. Under Assumption 3.2.3 we have*

$$X \in \mathcal{A}_T^x \qquad \textit{if and only if} \qquad \mathbb{E}[\langle X, Z_T^s \rangle] \leq \langle x, Z_0^s \rangle \ \textit{for all } Z^s \in \mathcal{Z}^s.$$

This result will be used in particular in the proof of Theorem 3.3.2 to show that our candidate for the optimizer in the utility maximization problem (with random endowment) is indeed an attainable contingent claim, i.e. the terminal value of an admissible portfolio.

3.2.2 Convex analysis and utility functions

The material of this section is mostly taken from Sections 2.2 and 2.3 in [CO10], where all the proofs can be found. We report here those results that we are going to use in our proofs for reader's convenience.

Let $(\mathscr{X}, \tau)$ be a locally convex topological vector space, and let $\mathscr{X}^*$ denote its dual space. Given a set $S \subseteq \mathscr{X}$ we let $\mathrm{cl}(S)$, $\mathrm{int}(S)$, $\mathrm{ri}(S)$ and $\mathrm{aff}(S)$ denote respectively the closure, interior, relative interior and affine hull of S. We shall say that a set $C \subseteq \mathscr{X}$ is a convex cone if $\lambda C + \mu C \subseteq C$ for all $\lambda, \mu \geq 0$. Given set $S \subseteq \mathscr{X}$, we denote its polar cone by

$$S^* := \{x^* \in \mathscr{X}^* \,:\, \langle x, x^* \rangle \geq 0 \; \forall x \in S\}\,.$$

Note that S^* is weak* closed. A convex cone $C \subseteq \mathscr{X}$ induces a preorder $\succeq_C$ on $\mathscr{X}$: We say that $x, x' \in \mathscr{X}$ satisfy $x' \succeq_C x$ if and only if $x' - x \in C$. When we do not specify the cone in the notation, we always mean that it is $\mathbb{R}^D_+$.

Definition 3.2.3 (Dual functionals)**.** 1. If $\mathfrak{U} : \mathscr{X} \to [-\infty, \infty)$ is proper concave then we define its dual functional $\mathfrak{U}^* : \mathscr{X}^* \to (-\infty, \infty]$ by

$$\mathfrak{U}^*(x^*) := \sup_{x \in \mathscr{X}} \{\mathfrak{U}(x) - \langle x, x^* \rangle\}\,. \tag{3.2.1}$$

The dual functional $\mathfrak{U}^*$ is a weak* lower semi-continuous, proper convex functional on $\mathscr{X}^*$. Note that $\mathfrak{U}^* = (\mathrm{cl}(\mathfrak{U}))^*$ (see e.g. [Za02, Theorem 2.3.1]).

2. If $\mathfrak{V} : \mathscr{X}^* \to (-\infty, \infty]$ is proper convex then we define the pre-dual functional ${}^*\mathfrak{V} : \mathscr{X} \to [-\infty, \infty)$ by

$${}^*\mathfrak{V}(x) := \inf_{x^* \in \mathscr{X}^*} \{\mathfrak{V}(x^*) + \langle x, x^* \rangle\}\,.$$

Similarly, ${}^*\mathfrak{V}$ is a weakly [†] upper semi-continuous, proper concave functional.

We say that $\mathfrak{U}$ is *increasing* with respect to a preorder $\succeq$ on $\mathscr{X}$, if $\mathfrak{U}(x') \geq \mathfrak{U}(x)$ for all $x, x' \in \mathscr{X}$ such that $x' \succeq x$.

Lemma 3.2.1. *[CO10, Lemma 2.8] Let $\mathfrak{U} : \mathscr{X} \to [-\infty, \infty)$ be proper concave. Then $\mathfrak{U}^*$ is decreasing with respect to the preorder induced by $(\mathrm{dom}(\mathfrak{U}))^*$. Suppose furthermore that $\mathfrak{U}$ is increasing with respect to the preorder induced by some cone C. Then $\mathrm{dom}(\mathfrak{U}^*) \subseteq C^*$.*

Definition 3.2.4 (Utility function)**.** We shall say that a proper concave function $U : \mathbb{R}^d \to [-\infty, \infty)$ is a *(multivariate) utility function* if

1. $C_U := \mathrm{cl}(\mathrm{dom}(U))$ is a convex cone which contains the non-negative orthant $\mathbb{R}^d_+$; and
2. U is increasing with respect to the preorder induced by the closed convex cone C_U.

We call C_U the *support* (or support cone) of U, and say that U is supported on C_U.

†. A concave functional is weakly upper semi-continuous if and only if it is upper semi-continuous with respect to the original topology τ

Throughout the whole paper the agent's utility function U is assumed to be supported on $\mathbb{R}^d_+$, the extended utility function $\tilde{U}$ defined by (3.1.1) is therefore supported on $\mathbb{R}^D_+$. It is shown in [CO10] (Proposition 3.1) that under Assumption 3.3.1 the value function $\bar{u}$ is a utility function which is supported on $\mathbb{R}^D \cap (-\mathcal{A}^0_T)$, a cone which is strictly larger than $\mathbb{R}^D_+$. It follows that $\bar{u}$ is finite on $\mathcal{I} := \operatorname{int}(\mathbb{R}^D \cap (-\mathcal{A}^0_T))$, a fact that we will use later.

We now review the analogues of the well known "Inada conditions" for the case of a multivariate utility function. For the proofs of the results, as well as for a more detailed discussion, we refer the reader to [CO10].
The first condition, which we recall from [Ro72], is well known within the field of convex analysis.

Definition 3.2.5. A proper concave function $U : \mathbb{R}^d \to [-\infty, \infty)$ is said to be *essentially smooth* if

1. $\operatorname{int}(\operatorname{dom}(U))$ is non-empty;
2. U is differentiable throughout $\operatorname{int}(\operatorname{dom}(U))$;
3. $\lim_{i\to\infty} |\nabla U(x_i)| = +\infty$ whenever $x_1, x_2, \ldots$ is a sequence in $\operatorname{int}(\operatorname{dom}(U))$ converging to a boundary point of $\operatorname{int}(\operatorname{dom}(U))$.

A proper *convex* function V is said to be essentially smooth if $-V$ is essentially smooth.

Lemma 3.2.2. *[CO10, Lemma 2.12] Let U be a proper concave function which is essentially smooth and strictly concave on $\operatorname{int}(\operatorname{dom}(U))$. Then U^* is strictly convex on $\operatorname{int}(\operatorname{dom}(U^*))$, and essentially smooth.*
Moreover, the maps $\nabla U : \operatorname{int}(\operatorname{dom}(U)) \to \operatorname{int}(\operatorname{dom}(U^*))$ and $\nabla U^* : \operatorname{int}(\operatorname{dom}(U^*)) \to -\operatorname{int}(\operatorname{dom}(U))$ are bijective and $(\nabla U)^{-1} = -\nabla U^*$.

The next condition was first introduced by [CO10] and it plays an important role in the paper.

Definition 3.2.6. We say that a utility function U is *asymptotically satiable* if for all $\epsilon > 0$ there exists an $x \in \mathbb{R}^d$ such that $\partial(\operatorname{cl}(U))(x) \cap [0, \epsilon)^d \neq \emptyset$.

Lemma 3.2.3. *[CO10, Lemma 2.14] A sufficient condition for asymptotic satiability of U is that for all $\epsilon > 0$ there exists an $x \in \operatorname{int}(\operatorname{dom}(U))$ such that $\partial U(x) \cap [0, \epsilon)^d \neq \emptyset$. If U is closed, or essentially smooth then the condition is both necessary and sufficient for asymptotic satiability.*

The next proposition clarifies the effects of asymptotic satiability on the dual function.

Proposition 3.2.1. *[CO10, Proposition 2.15] Let U be a utility function. The following conditions are equivalent:*

1. *U is asymptotically satiable;*
2. *$0 \in \operatorname{cl}(\operatorname{dom}(U^*))$;*
3. *$\operatorname{cl}(\operatorname{dom}(U^*)) = (C_U)^*$; and*
4. *$\operatorname{cl}(\operatorname{dom}(U^*))$ is a convex cone.*

If U is asymptotically satiable then we define the closed convex cone $C_{U^*} := \operatorname{cl}(\operatorname{dom}(U^*))$, so that condition (iii) can be written more succinctly as $C_{U^*} = (C_U)^*$.

We note that for a utility function U supported on $\mathbb{R}^D_+$, the previous proposition states that if U is asymptotically satiable then $\mathrm{cl}(\mathrm{dom}(U^*)) = \mathbb{R}^D_+$.

Corollary 3.2.1. *[CO10, Corollary 2.16] Let $U : \mathbb{R}^d \to [-\infty, \infty)$ be a utility function which is supported on $\mathbb{R}^d_+$, and which satisfies Assumption 3.1.1. Recall that by definition of the dual function we have*

$$U^*(x^*) \geq U(x) - \langle x, x^* \rangle \tag{3.2.2}$$

for all $x, x^* \in \mathbb{R}^d$. If $x^* \in \mathrm{int}(\mathbb{R}^d_+)$ then we have equality in (3.2.2) if and only if $x = I(x^*) := -\nabla U^*(x^*)$.

Given $D \geq d$, define $\tilde{U} : \mathbb{R}^D \to [-\infty, \infty)$ by (3.1.1). Again, by definition of the dual function we have

$$\tilde{U}^*(x^*) \geq \tilde{U}(x) - \langle x, x^* \rangle, \tag{3.2.3}$$

for all $x, x^* \in \mathbb{R}^D$. Define $P : \mathbb{R}^D \to \mathbb{R}^d$ by

$$P(x_1, \ldots, x_d, x_{d+1}, \ldots, x_D) := (x_1, \ldots, x_d), \tag{3.2.4}$$

and $\tilde{I} : \mathrm{int}(\mathbb{R}^d_+) \times \mathbb{R}^{D-d}_+ \to \mathrm{int}(\mathbb{R}^d_+) \times \mathbb{R}^{D-d}_+$ by

$$\tilde{I}(x^*) := (-\nabla U^*(P(x^*)), \underline{0}), \tag{3.2.5}$$

where $\underline{0}$ denotes the zero vector in $\mathbb{R}^{D-d}$. Then, (i) *if $x^* \in \mathrm{int}(\mathbb{R}^d_+) \times \mathbb{R}^{D-d}_+$ then we have equality in (3.2.3) whenever $x = \tilde{I}(x^*)$ and* (ii) *if $x^* \in \mathrm{int}(\mathbb{R}^D_+)$ then there is equality in (3.2.3) if and only if $x = \tilde{I}(x^*)$.*

3.2.3 Euclidean vector measures

A function m from a field $\mathscr{F}$ of subsets of a set Ω to a Banach space $\mathscr{X}$ is called a finitely additive vector measure, or simply a vector measure if $m(A_1 \cup A_2) = m(A_1) + m(A_2)$, whenever A_1 and A_2 are disjoint members of $\mathscr{F}$. In this paper, we will be concerned with the special case where $\mathscr{X} = \mathbb{R}^D$; we refer to the associated vector measure as a "Euclidean vector measure", or simply a "Euclidean measure". Let us recall a few definitions from the classical, one-dimensional setting. The *total variation* of a (finitely additive) measure $m : \mathscr{F} \to \mathbb{R}$ is the function $|m| : \mathscr{F} \to [0, \infty]$ defined by

$$|m|(A) := \sup \sum_{j=1}^{n} |m(A_j)|,$$

where the supremum is taken over all finite sequences $(A_j)_{j=1}^n$ of disjoint sets in $\mathscr{F}$ with $A_j \subseteq A$. A measure m is said to have *bounded total variation* if $|m|(\Omega) < \infty$. A measure m is said to be *bounded* if $\sup\{|m(A)| : A \in \mathscr{F}\} < \infty$. A measure m is said to be *purely finitely additive* if $0 \leq \mu \leq |m|$ and μ is countably additive imply that $\mu = 0$. A measure m is said to be *weakly absolutely continuous* with respect to $\mathbb{P}$ if $m(A) = 0$ whenever $A \in \mathscr{F}$ and $\mathbb{P}(A) = 0$.

We turn now to the D-dimensional case. A Euclidean measure m can be decomposed into its one-dimensional coordinate measures $m_i : \mathscr{F} \to \mathbb{R}$ by defining $m_i(A) :=$

$\langle e^i, m(A)\rangle$, where e^i is the i-th canonical basis vector of $\mathbb{R}^D$. In this way, $m(A) = (m_1(A), \dots, m_D(A))$ for every $A \in \mathscr{F}$. We shall say that a Euclidean measure m is *bounded*, *purely finitely additive* or *weakly absolutely continuous* with respect to $\mathbb{P}$ if each of its coordinate measures is bounded, purely finitely additive or weakly absolutely continuous with respect to $\mathbb{P}$.

We denote $\mathrm{ba}(\mathbb{R}^D) = \mathrm{ba}(\Omega, \mathscr{F}_T, \mathbb{P}; \mathbb{R}^D)$ the vector space of bounded Euclidean measures $m : \mathscr{F}_T \to \mathbb{R}^D$ which are weakly absolutely continuous with respect to $\mathbb{P}$, and $\mathrm{ca}(\mathbb{R}^D)$ the subspace of countably additive members of $\mathrm{ba}(\mathbb{R}^D)$. Equipped with the norm

$$\|m\|_{\mathrm{ba}(\mathbb{R}^D)} := \sum_{i=1}^{D} |m_i|(\Omega),$$

the spaces $\mathrm{ba}(\mathbb{R}^D)$ and $\mathrm{ca}(\mathbb{R}^D)$ are Banach spaces.

Let $\mathrm{ba}(\mathbb{R}^D_+)$ denote the convex cone of $\mathbb{R}^D_+$-valued measures within $\mathrm{ba}(\mathbb{R}^D)$. The next proposition is an immediate extension of its univariate counterpart.

Proposition 3.2.2. *Given any $m \in \mathrm{ba}(\mathbb{R}^D)$ there exists a unique Yosida-Hewitt decomposition $m = m^c + m^p$ where $m^c \in \mathrm{ca}(\mathbb{R}^D)$ and m^p is purely finitely additive. If $m \in \mathrm{ba}(\mathbb{R}^D_+)$ then $m^c, m^p \in \mathrm{ba}(\mathbb{R}^D_+)$.*

It is well known that $L^\infty(\mathbb{R}^D)^*$, the set of linear functionals on the space of (essentially) bounded $\mathbb{R}^D$-valued random variables, can be identified with $\mathrm{ba}(\mathbb{R}^D)$. Another standard result in functional analysis is that $(\mathrm{ba}(\mathbb{R}^D), \|.\|_{\mathrm{ba}(\mathbb{R}^D)})$ has a $\sigma(\mathrm{ba}(\mathbb{R}^D), L^\infty(\mathbb{R}^D))$-compact unit ball. For any $m \in \mathrm{ba}(\mathbb{R}^D)$ we will denote

$$m(X) := \int_\Omega \langle X, \underset{\cdot}{\mathrm{m}}\rangle := \sum_{i=1}^{D} \int_\Omega X_i \underset{\cdot}{\mathrm{m}}_i.$$

Given $x \in \mathbb{R}^D$ and $A \in \mathscr{F}_T$ we clearly have $m(x\chi_A) = \langle x, m(A)\rangle$. In the special case where $A = \Omega$, we have $m(x) = \langle x, m(\Omega)\rangle$.

Let $L^0(\mathbb{R}^D_+)$ and $L^\infty(\mathbb{R}^D_+)$ denote respectively the convex cones of random variables in $L^0(\mathbb{R}^D)$ and $L^\infty(\mathbb{R}^D)$ which are $\mathbb{R}^D_+$-valued a.s. Note that if $m \in \mathrm{ba}(\mathbb{R}^D_+)$ and $X \in L^\infty(\mathbb{R}^D_+)$ then $m(X) \geq 0$ (see [RR83, Theorem 4.4.13]). This observation allows us to extend the definition of $m(X)$ to cover the case where $m \in \mathrm{ba}(\mathbb{R}^D_+)$ and $X \in L^0(\mathbb{R}^D_+)$ (not necessarily bounded from above) by setting

$$m(X) := \sup_{n\in\mathbb{N}} m\left(X \wedge (n\mathbf{1})\right), \tag{3.2.6}$$

where $(x_1, \dots, x_D) \wedge (y_1, \dots, y_D) := (x_1 \wedge y_1, \dots, x_D \wedge y_D)$. It is trivial that (3.2.6) is consistent with the definition of $m(X)$ for $X \in L^\infty(\mathbb{R}^D)$. Furthermore, the supremum in (3.2.6) can be replaced by a limit, since the sequence of numbers is increasing. It follows that given $m_1, m_2 \in \mathrm{ba}(\mathbb{R}^D_+)$, $\lambda_1, \lambda_2, \mu_1, \mu_2 \geq 0$ and $X_1, X_2 \in L^0(\mathbb{R}^D_+)$, we have

$$(\lambda_1 m_1 + \lambda_2 m_2)(\mu_1 X_1 + \mu_2 X_2) = \lambda_1\mu_1 m_1(X_1) + \lambda_1\mu_2 m_1(X_2) + \lambda_2\mu_1 m_2(X_1) + \lambda_2\mu_2 m_2(X_2).$$

Given $m \in \mathrm{ca}(\mathbb{R}^D)$ and $X \in L^\infty(\mathbb{R}^D)$ we have $m(X) = \mathrm{E}\left[\left\langle X, \frac{dm}{d\mathbb{P}}\right\rangle\right]$, where $\frac{dm}{d\mathbb{P}}$ is the vector of Radon-Nikodym derivatives. It is easy to show that this property is also true under the extended definition (3.2.6).

More details on finitely additive measures (which are sometimes referred to as *charges*) can be found in [RR83].

3.3 Utility maximization problem with random endowment

In this section we will elaborate on the main optimization problem that was defined in (3.3.2), with a particular focus on the issue of existence of a solution. We start by investigating some useful properties of the value function in Proposition 3.3.1. We then proceed by dualizing the problem in Section 3.3.1, using some convex duality techniques that are commonly used in optimization (see, for example, [Bo02], [OZ09], [CO10] among others). Lemma 3.3.1 will give another convenient representation of the dual functional, while Theorem 3.3.1 will establish the absence of duality gap and the existence of a solution to the dual problem under some rather weak conditions on $\mathcal{E}$ (see condition (3.3.1) below). Finally, in Section 3.3.2, we show the existence of a solution to the primal problem in Theorem 3.3.2 under the additional assumptions of asymptotic satiability of the value function and boundedness of the endowment.

For technical reasons that will be clear later in the proofs, we will mainly consider endowments of this form: $\mathcal{E} \in L^0(\mathbb{R}^D, \mathcal{F}_T)$ and there exist x', $x'' \in \mathcal{I} := \text{int}(-\mathcal{A}_T^0 \cap \mathbb{R}^D)$ and $X'' \in \mathcal{A}_T^0$ such that

$$x' \preceq \mathcal{E} \preceq x'' + X''. \tag{3.3.1}$$

We call $\mathcal{O}$ the convex set of endowments satisfying (3.3.1) for some $x', x'' \in \mathcal{I}$ and X''. For any $\mathcal{E} \in \mathcal{O}$ we define the primal optimization problem as

$$u(\mathcal{E}) := \sup \left\{ \mathbb{E}\left[\tilde{U}(X + \mathcal{E})\right] : X \in \mathcal{A}_T^0 \right\}. \tag{3.3.2}$$

When $\mathcal{E} = x$ is deterministic, this reduces to the formulation in [CO10]:

$$\bar{u}(x) := \sup \left\{ \mathbb{E}\left[\tilde{U}(X)\right] : X \in \mathcal{A}_T^x \right\}.$$

We denote $\text{dom}(u) := \{\mathcal{E} \in L^0(\mathbb{R}^D) : u(\mathcal{E}) > -\infty\}$ and

$$\tilde{\mathcal{A}}_T^0 := \left\{ X \in \mathcal{A}_T^0 : \exists \epsilon > 0 : X + \epsilon \mathbf{1} \in \mathcal{A}_T^0 \right\}.$$

The set $\tilde{\mathcal{A}}_T^0$ is clearly not empty as it contains the constants in the (strictly) negative orthant. The following mild assumption is fairly natural in any optimization problem (compare [CO10, Assumption 1.2]).

Assumption 3.3.1. $\bar{u}(x) < +\infty$ for some $x \in \text{int}(\text{dom}(\bar{u}))$.

Under this assumption, we can rephrase condition (3.3.1) as follows: $x' \preceq \mathcal{E} \preceq x'' + X''$ for some initial portfolios x', x'' in $\text{int}(\text{dom}\, \bar{u})$ and some final portfolio $X'' \in \mathcal{A}_T^0$. Indeed, it has been established in [CO10, Proposition 3.1] that, under Assumption 3.3.1, one has $\text{cl}(\text{dom}\, \bar{u}) = -\mathcal{A}_T^0 \cap \mathbb{R}^D$.

Remark 3.3.1. Take any $\mathcal{E} \in \mathcal{O}$. Notice that $u(\mathcal{E} - \epsilon\mathbf{1}) \geq \mathbb{E}\left[\tilde{U}(X + \mathcal{E} - \epsilon\mathbf{1})\right] \geq \mathbb{E}\left[\tilde{U}(X + x' - \epsilon\mathbf{1})\right]$ for all $X \in \mathcal{A}_T^0$, so that $u(\mathcal{E} - \epsilon\mathbf{1}) \geq \bar{u}(x' - \epsilon\mathbf{1}) > -\infty$ for some $\epsilon > 0$ since $x' \in \mathcal{I}$. This simple observation will be used in the proof of the following proposition.

Proposition 3.3.1. *The value function $u : L^0(\mathbb{R}^D) \to [-\infty, \infty]$ has the following properties:*

1. *u is concave on $\mathcal{O}$ and increasing with respect to $L^0(\mathbb{R}^D_+)$.*
2. *$u(\mathcal{E}) \in \mathbb{R}$ for any $\mathcal{E} \in \mathcal{O}$, so that in particular $\mathcal{O} \subseteq \mathrm{dom}(u)$;*
3. *$u(\mathcal{E}) < \infty$ for any $\mathcal{E} \in \mathcal{O} \cap L^\infty(\mathbb{R}^D)$;*
4. *$\mathrm{cl}(\mathrm{dom}(u)) = -\mathrm{cl}(\mathcal{A}^0_T)$ in the topology of convergence in probability;*
5. *u is increasing w.r.t. the preorder generated by $\mathrm{dom}(u)$. If U is l.s.c. then u is also increasing w.r.t. the preorder generated by $\mathrm{cl}(\mathrm{dom}(u))$.*

Proof. (i) Concavity follows from the fact that $\mathcal{A}^0_T$ is convex and $\tilde{U}$ is concave. The second property follows from the same property for U.

(ii) Observe that $u(\mathcal{E}) \geq \mathbb{E}\left[\tilde{U}(X + \mathcal{E})\right] \geq \mathbb{E}\left[\tilde{U}(X + x')\right]$ for all $X \in \mathcal{A}^0_T$, so that $u(\mathcal{E}) \geq u(x') > -\infty$ since $x' \in \mathcal{I} \subseteq \mathrm{int}(\mathrm{dom}(\bar{u}))$, where we recall that $\bar{u}$ is the restriction of the value function u on $\mathbb{R}^D$. Also note that $u(\mathcal{E}) \leq \mathbb{E}\left[\tilde{U}(X + x' + X'')\right] \leq \bar{u}(x'') < \infty$ whenever $x'' \in \mathcal{I}$ (See Section 3.2.2). Hence $u(\mathcal{E}) \in \mathbb{R}$.

(iii) We show that $u(\mathcal{E}) < \infty$ for any $\mathcal{E} \in \mathcal{O} \cap L^\infty$. Suppose for a contradiction that there exists some $\tilde{\mathcal{E}} \in L^\infty$ such that $u(\tilde{\mathcal{E}}) = \infty$. Let $\mathcal{E} \in \mathcal{O}$, so that $u(\mathcal{E}) < \infty$. We can find an $a > 0$ such that $\mathcal{E}_1 := \mathcal{E} + a\mathbf{1} \succeq \tilde{\mathcal{E}}$ a.s.. We have $u(\mathcal{E}_1) \geq u(\tilde{\mathcal{E}}) = \infty$.
By Remark 3.3.1 there exists an $\epsilon > 0$ such that $\mathcal{E}_0 := \mathcal{E} - \epsilon\mathbf{1} \in \mathrm{dom}(u)$, so that $u(\mathcal{E}_0) > -\infty$. We also have $u(\mathcal{E}_0) \leq u(\mathcal{E}) < \infty$, hence $u(\mathcal{E}_0) \in \mathbb{R}$. This implies that we may find an $X_0 \in \mathcal{A}^0_T$ such that $\mathbb{E}\left[\tilde{U}(X_0 + \mathcal{E}_0)\right] =: c \in \mathbb{R}$. Since $u(\mathcal{E}_1) = \infty$, given any $R \in \mathbb{R}$ we may also find an $X_1 \in \mathcal{A}^0_T$ such that $\mathbb{E}\left[\tilde{U}(X_1 + \mathcal{E}_1)\right] \geq R$. Define $\lambda := \epsilon/(a + \epsilon)$ and $X := (1 - \lambda)X_0 + \lambda X_1$. So we have

$$\begin{aligned} u(\mathcal{E}) &\geq \mathbb{E}\left[\tilde{U}(X + \mathcal{E})\right] = \mathbb{E}\left[\tilde{U}((1 - \lambda)(X_0 + \mathcal{E}_0) + \lambda(X_1 + \mathcal{E}_1))\right] \\ &\geq (1 - \lambda)\mathbb{E}\left[\tilde{U}(X_0 + \mathcal{E}_0)\right] + \lambda\mathbb{E}\left[\tilde{U}(X_1 + \mathcal{E}_1)\right] \geq (1 - \lambda)c + \lambda R \end{aligned}$$

which is a contradiction since R can be taken arbitrarily large.

(iv) Take $X_0 \in \tilde{\mathcal{A}}^0_T$. There exists $\epsilon > 0$ such that $X_0 + \epsilon\mathbf{1} \in \mathcal{A}^0_T$, then

$$u(-X_0) \geq \mathbb{E}\left[\tilde{U}(\epsilon\mathbf{1})\right] > -\infty$$

hence $-X_0 \in \mathrm{dom}(u)$, so $-\tilde{\mathcal{A}}^0_T \subseteq \mathrm{dom}(u)$.
Suppose that $\mathcal{E} \in \mathrm{dom}(u)$. Necessarily then $\mathcal{A}^{\mathcal{E}}_T \cap L^0(\mathbb{R}^D_+) \neq \emptyset$, where $\mathcal{A}^{\mathcal{E}}_T := \mathcal{E} + \mathcal{A}^0_T = \{Y \in L^0 : Y = X + \mathcal{E}, X \in \mathcal{A}^0_T\}$. Take $X \in \mathcal{A}^{\mathcal{E}}_T \cap L^0(\mathbb{R}^D_+)$, then $0 = X - X \in \mathcal{A}^{\mathcal{E}}_T - L^0(\mathbb{R}^D_+) \subseteq \mathcal{A}^{\mathcal{E}}_T$, hence $0 \in \mathcal{A}^{\mathcal{E}}_T$, which implies $\mathcal{E} \in -\mathcal{A}^0_T$. So

$$-\tilde{\mathcal{A}}^0_T \subseteq \mathrm{dom}(u) \subseteq -\mathcal{A}^0_T$$

and the claim follows from $\mathrm{cl}(-\tilde{\mathcal{A}}^0_T) = \mathrm{cl}(-\mathcal{A}^0_T)$. To see the last equality, remark first that $\mathrm{cl}(\tilde{\mathcal{A}}^0_T) \subseteq \mathrm{cl}(\mathcal{A}^0_T)$. Now take $X \in \mathrm{cl}(\mathcal{A}^0_T)$, then (up to a subsequence) there exists $(X_n)_{n\geq 0} \in \mathcal{A}^0_T$ such that $X_n \to X$ almost surely. Let $(\epsilon_n)_{n\geq 0} > 0$ be such that $\epsilon_n \to 0$ and remark that $Y_n := X_n - \epsilon_n\mathbf{1}$ belongs to $\tilde{\mathcal{A}}^0_T$ and $Y_n \to X$ almost surely. Hence $Y_n \to X$ in probability yielding $X \in \mathrm{cl}(\tilde{\mathcal{A}}^0_T)$.

(v) We only prove the second part of the claim. Take $\mathcal{E} \in L^0(\mathbb{R}^D)$ such that $u(\mathcal{E}) < \infty$ and $\mathcal{E}_1 \in \mathrm{cl}(\mathrm{dom}(u))$, so by property iv) there exists $(Y_n)_{n\geq 0} \in \mathcal{A}_T^0$ such that $Y_n \to -\mathcal{E}_1$ almost surely (up to a subsequence). By definition, for any $\epsilon > 0$ there exists a $X \in \mathcal{A}_T^0$ such that $\mathbb{E}\left[\tilde{U}(X+\mathcal{E})\right] \geq u(\mathcal{E}) - \epsilon$. Since $X + Y_n \in \mathcal{A}_T^0$ we have that $u(\mathcal{E}+\mathcal{E}_1) \geq \mathbb{E}\left[\tilde{U}(X+Y_n+\mathcal{E}+\mathcal{E}_1)\right]$ for all $n \in \mathbb{N}$, hence

$$\begin{aligned} u(\mathcal{E}+\mathcal{E}_1) &\geq \liminf_n \mathbb{E}\left[\tilde{U}(X+Y_n+\mathcal{E}+\mathcal{E}_1)\right] \geq \mathbb{E}\left[\liminf_n \tilde{U}(X+Y_n+\mathcal{E}+\mathcal{E}_1)\right] \\ &\geq \mathbb{E}\left[\tilde{U}(X+\mathcal{E})\right] \geq u(\mathcal{E}) - \epsilon \end{aligned}$$

where we used the fact that $\tilde{U}$ is l.s.c. Since ϵ is arbitrary the claim follows.
If $u(\mathcal{E}) = \infty$, then we can find an $X \in \mathcal{A}_T^0$ such that $\mathbb{E}\left[\tilde{U}(X+\mathcal{E})\right] \geq R$ for any $R > 0$. With the same arguments as above we can say that $u(\mathcal{E}+\mathcal{E}_1) \geq R$, hence $u(\mathcal{E}+\mathcal{E}_1) = u(\mathcal{E}) = \infty$. □

3.3.1 Dual representation of the optimization problem

In this section we show that the value function of our optimization problem with random endowment can be represented as the value function of a suitably defined dual minimization problem. To do so, let us define the functional

$$\mathbb{U}_{\mathcal{E}}(X) := \mathbb{E}\left[\tilde{U}(X+\mathcal{E})\right]$$

and its dual

$$\mathbb{U}^*_{\mathcal{E}}(m) := \sup_{X \in L^\infty(\mathbb{R}^D)} [\mathbb{U}_{\mathcal{E}}(X) - m(X)]. \tag{3.3.3}$$

Lemma 3.3.1. *If $\mathcal{E} \in \mathcal{O}$ then we have the following representation:*

$$\mathbb{U}^*_{\mathcal{E}}(m) = \begin{cases} \mathbb{E}\left[\tilde{U}^*\left(\frac{dm^c}{dP}\right)\right] + m(\mathcal{E}) & \text{if } m \in \mathrm{ba}(\mathbb{R}^D_+) \\ \infty & \text{otherwise} \end{cases} \tag{3.3.4}$$

Proof. Remark first that since $\mathbb{U}_0$ is increasing with respect to the preorder induced by $L^\infty(\mathbb{R}^D_+)$ it follows from Lemma 3.2.1 that $\mathrm{dom}(\mathbb{U}^*_0) \subseteq L^\infty(\mathbb{R}^D_+)^* = \mathrm{ba}(\mathbb{R}^D_+)$.
Now take $m \in \mathrm{ba}(\mathbb{R}^D_+)$ and define

$$\mathbb{U}_{\mathcal{E},n}(X) := \mathbb{E}\left[\tilde{U}(X + \mathcal{E}\chi_{\mathcal{E} \preceq \mathbf{1}n})\right].$$

It is clear that by monotone convergence one has

$$\lim_{n\to\infty} \mathbb{U}_{\mathcal{E},n}(X) = \sup_n \mathbb{U}_{\mathcal{E},n}(X) = \mathbb{U}_{\mathcal{E}}(X)$$

Thus we get

$$\begin{aligned}
\mathbb{U}^*_{\mathcal{E}}(m) &= \sup_{X\in L^\infty(\mathbb{R}^D)} [\mathbb{U}_{\mathcal{E}}(X) - m(X)] \\
&= \sup_{X\in L^\infty(\mathbb{R}^D)} \sup_n [\mathbb{U}_{\mathcal{E},n}(X) - m(X)] \\
&= \sup_n \sup_{X\in L^\infty(\mathbb{R}^D)} [\mathbb{U}_{\mathcal{E},n}(X) - m(X)] \\
&= \lim_{n\to\infty} \sup_{X\in L^\infty(\mathbb{R}^D)} [\mathbb{U}_0(X + \mathcal{E}\chi_{(\mathcal{E}\preceq n\mathbf{1})}) - m(X + \mathcal{E}\chi_{(\mathcal{E}\preceq n\mathbf{1})})] + m(\mathcal{E}\chi_{\mathcal{E}\preceq \mathbf{1}n}) \\
&= \lim_{n\to\infty} \sup_{\tilde{X}\in L^\infty(\mathbb{R}^D)} [\mathbb{U}_0(\tilde{X}) - m(\tilde{X})] + m(\mathcal{E}\chi_{\mathcal{E}\preceq \mathbf{1}n}) \\
&= \mathbb{U}^*_0(m) + \lim_{n\to\infty} m(\mathcal{E}\chi_{\mathcal{E}\preceq 1n}) \\
&= \mathbb{U}^*_0(m) + m(\mathcal{E}).
\end{aligned}$$

Now that we have isolated the contribution of the random endowment, it suffices to study the case of zero endowment to conclude the proof. This has already been done in [CO10], where it is shown that $\mathbb{U}^*_0(m) = \mathbb{U}^*_0(m^c) = \mathbb{E}\left[\tilde{U}^*\left(\frac{dm^c}{dP}\right)\right]$, which yields the claim. □

Remark 3.3.2. A consequence of previous Lemma 3.3.1 is that if $\mathcal{E} \in \mathcal{O}$ then $\mathrm{dom}(\mathbb{U}^*_{\mathcal{E}}) = \mathrm{dom}(\mathbb{U}^*_0)$. Measures in this set are sometimes said to have *finite relative entropy* (see, for example, [OZ09]).

Consider now the abstract maximization problem

$$\sup_{X\in\mathcal{C}} \mathbb{U}_{\mathcal{E}}(X)$$

where

$$\mathcal{C} := L^\infty(\mathbb{R}^D) \cap \mathcal{A}^0_T.$$

It is immediately clear that

$$\sup_{X\in\mathcal{C}} \mathbb{U}_{\mathcal{E}}(X) \le u(\mathcal{E}).$$

Its abstract dual problem is defined as

$$\inf_{m\in\mathcal{D}} \mathbb{U}^*_{\mathcal{E}}(m)$$

where $\mathbb{U}^*_{\mathcal{E}}$ is defined as in (3.3.3) and

$$\mathcal{D} := \left\{ m \in \mathrm{ba}(\mathbb{R}^D) : m(X) \le 0 \text{ for all } X \in \mathcal{C} \right\}.$$

Since $-L^\infty(\mathbb{R}^D_+) \subseteq \mathcal{C}$, one clearly has $\mathcal{D} \subseteq \mathrm{ba}(\mathbb{R}^D_+)$. We introduce the Lagrangian $L(X,m) := \mathbb{U}_{\mathcal{E}}(X) - m(X)$ and note that

$$\sup_{X\in\mathcal{C}} \mathbb{U}_{\mathcal{E}}(X) \le \sup_{X\in L^\infty} \inf_{m\in\mathcal{D}} L(X,m) \le \inf_{m\in\mathcal{D}} \sup_{X\in L^\infty} L(X,m) = \inf_{m\in\mathcal{D}} \mathbb{U}^*_{\mathcal{E}}(m). \tag{3.3.5}$$

Remark 3.3.3. It is important to notice that, given any $Z \in \mathcal{Z}^s$ we can construct a corresponding $m \in \text{ca}(\mathbb{R}^D_+)$ by setting $m(A) := \mathbb{E}[Z_T \mathbf{1}_A]$ for each $A \in \mathcal{F}_T$. We call m^Z the measure associated to the price process Z. We have that

$$\mathcal{Z}^s \subseteq \mathcal{D} \cap \text{ca}(\mathbb{R}^D_+) \subseteq \mathcal{Z}.$$

To see that, begin with the first inclusion (that was already established in [CO10, Remark 3.10]) : Take $Z \in \mathcal{Z}^s$ and $X \in \mathcal{C}$. Then $\mathbb{E}[\langle X, Z_T \rangle] = m^Z(X) \leq 0$ by Theorem 3.2.1, where $m^Z \in \text{ca}(\mathbb{R}^D_+)$.
For the second inclusion, take $m \in \mathcal{D} \cap \text{ca}(\mathbb{R}^D_+)$, so that $m(X) \leq 0$ for any $X \in \mathcal{C}$. Take any $X \in L^\infty(-K_t, \mathcal{F}_t)$ for some t, then $X \in \mathcal{A}^0_T$ (consider the strategy that just trades at time t for an amount equal to X). So $X \in \mathcal{C}$ and then $m(X) \leq 0$, which implies $Z^m_t := \mathbb{E}\left[\frac{dm}{dP}|\mathcal{F}_t\right] \in K^*_t$ a.s. and so $Z^m \in \mathcal{Z}$. By monotone convergence, this is also true for unbounded X.

Define

$$\mathcal{P} := \{m \in \text{ba}(\mathbb{R}^D_+) : P\left(\frac{dm^c}{dP}\right) \text{ is } \text{int}(\mathbb{R}^d_+) - \text{valued a.s.}\}$$

where P is defined in Corollary 3.2.1.

Lemma 3.3.2. *Suppose that $\hat{m}$ is a minimizer for the problem $\inf_{m \in \mathcal{D}} \mathbb{U}^*_{\mathcal{E}}(m)$. Then $\hat{m} \in \mathcal{P}$. If the utility function U is strictly concave then the minimizer is unique.*

Proof. We will use the same arguments as in [CO10], Proposition 3.9, with minor modifications. However, we will give the details of the proof for reader's convenience.
By Lemma 3.3.1, $\hat{m} \in \text{ba}(\mathbb{R}^D_+)$. Suppose that $\hat{m} \notin \mathcal{P}$. By definition some of the components of $\frac{d\hat{m}^c}{dP}$ are zero on a set $A \in \mathcal{F}$ with positive probability. Take $Z \in \mathcal{Z}^s$ and let m^Z be its associated measure as in Remark 3.3.3. For $\lambda > 0$ let $m_\lambda := \lambda m^Z + \hat{m} \in \mathcal{D}$ and $\nu_\lambda := \tilde{U}^*\left(\frac{dm^c_\lambda}{dP}\right)$. By Lemma 3.2.1, $\mathbb{U}^*_0$ is decreasing with respect to the preorder induced by $\text{ba}(\mathbb{R}^D_+)$, implying that $m_\lambda \in \text{dom}(\mathbb{U}^*_0)$. Since ν_λ is convex as a function of λ, the integrable random variables $(\nu_\lambda - \nu_0)/\lambda$ are monotone increasing in λ. By the monotone convergence theorem,

$$\begin{aligned}
&\lim_{\lambda \to 0} \mathbb{E}\left[\chi_A \left(\frac{\nu_\lambda - \nu_0}{\lambda}\right)\right] = \mathbb{E}\left[\chi_A \lim_{\lambda \to 0}\left(\frac{\nu_\lambda - \nu_0}{\lambda}\right)\right] \\
&= \mathbb{E}\left[\chi_A \lim_{\lambda \to 0}\left(\frac{\tilde{U}^*\left(\lambda \frac{dm^Z}{dP} + \frac{d\hat{m}^c}{dP}\right) - \tilde{U}^*\left(\frac{d\hat{m}^c}{dP}\right)}{\lambda}\right)\right] \\
&= \mathbb{E}\left[\chi_A \lim_{\lambda \to 0}\left(\frac{U^*\left(\lambda P(\frac{dm^Z}{dP}) + P(\frac{d\hat{m}^c}{dP})\right) - U^*\left(P(\frac{d\hat{m}^c}{dP})\right)}{\lambda}\right)\right] = -\infty
\end{aligned}$$

since, being U^* essentially smooth (by Lemma 3.2.2), its gradient diverges on the boundary points of its domain. Hence $\lim_{\lambda \to 0} \frac{1}{\lambda}\mathbb{E}[\nu_\lambda - \nu_0] = -\infty$. By Lemma 3.3.1, the optimality of $\hat{m}$, the assumptions on the endowment and Theorem 3.2.1 we have that $\mathbb{E}[\nu_\lambda - \nu_0] = \mathbb{U}^*_{\mathcal{E}}(m_\lambda) - m_\lambda(\mathcal{E}) - \mathbb{U}^*_{\mathcal{E}}(\hat{m}) + \hat{m}(\mathcal{E}) \geq -\lambda m^Z(\mathcal{E}) \geq -\lambda m^Z(x'' + X'') \geq -\lambda m^Z(x'') = -\lambda \langle x'', \mathbb{E}[Z_T] \rangle = -\lambda \langle x'', Z_0 \rangle > -\infty$, therefore $\frac{1}{\lambda}\mathbb{E}[\nu_\lambda - \nu_0] \geq \langle x'', Z_0 \rangle > -\infty$ and so the limit as $\lambda \to 0$ cannot be $-\infty$, which is a contradiction.
Uniqueness follows easily from strict convexity of the dual function. □

Motivated by Lemma 3.3.1, we define

$$v(\mathcal{E}) := \inf_{m\in\mathcal{D}} \left\{ \mathbb{E}\left[\tilde{U}^*\left(\frac{dm^c}{dP}\right)\right] + m(\mathcal{E}) \right\}$$

Take $X \in \mathcal{C}$ and $m \in \mathcal{D}$. We can consider in what follows that $X + \mathcal{E} \in L^0(\mathbb{R}^D_+)$ otherwise the results are trivial. We have

$$\begin{aligned} m(X+\mathcal{E}) &= \sup_n m\left((X+\mathcal{E}) \wedge \mathbf{1}n\right) \leq \sup_n m\left(X \wedge \mathbf{1}n\right) + \sup_n m\left(\mathcal{E} \wedge \mathbf{1}n\right) \\ &= m\left(X\right) + m(\mathcal{E}) \leq m(\mathcal{E}). \end{aligned}$$

We also remark that

$$m(X+\mathcal{E}) \geq m^c(X+\mathcal{E}) = \mathbb{E}\left[\left\langle X+\mathcal{E}, \frac{dm^c}{dP}\right\rangle\right].$$

By combining these considerations and using the definition of the dual function we get

$$\begin{aligned} \mathbb{E}\left[\tilde{U}(X+\mathcal{E})\right] &\leq \mathbb{E}\left[\tilde{U}^*\left(\frac{dm^c}{dP}\right) + \left\langle \frac{dm^c}{dP}, X+\mathcal{E}\right\rangle\right] \\ &\leq \mathbb{E}\left[\tilde{U}^*\left(\frac{dm^c}{dP}\right)\right] + m(\mathcal{E}). \end{aligned} \tag{3.3.6}$$

After all these preliminaries, we can finally prove the existence result.

Theorem 3.3.1. *If $\mathcal{E} \in \mathcal{O}$ then*

$$\sup_{X\in\mathcal{C}} \mathbb{U}_{\mathcal{E}}(X) = u(\mathcal{E}) = v(\mathcal{E}) = \min_{m\in\mathcal{D}} \mathbb{U}^*_{\mathcal{E}}(m) < \infty. \tag{3.3.7}$$

If the utility function is strictly concave, then the minimizer is unique.

Proof. The proof can be split into two parts.

1. We first use the Lagrange duality theorem as reported in the Appendix of [CO10] to show that $\sup_{X\in\mathcal{C}} \mathbb{U}_{\mathcal{E}}(X) = v(\mathcal{E}) = \min_{m\in\mathcal{D}} \mathbb{U}^*_{\mathcal{E}}(m) < \infty$. Take $\mathcal{E} \in \mathcal{O}$, let $\mathcal{X} = L^\infty(\mathbb{R}^D)$ and define the concave functional $\mathfrak{U} : \mathcal{X} \to [-\infty, \infty)$ by $\mathfrak{U} = \mathbb{U}_{\mathcal{E}}$. By Remark 3.3.1 there exists $\epsilon > 0$ such that $-Y := \mathcal{E} - 3\epsilon\mathbf{1} \in \mathrm{dom}(u) = -\,\mathrm{cl}(\mathcal{A}^0_T)$. Suppose first that $Y \in \mathcal{A}^0_T$, so by using [KS09, Lemma 3.6.7] we can find a sequence $Y_n \in \mathcal{C} \subseteq -\,\mathrm{dom}(u)$ such that $Y_n \to Y$.
By definition $p := -\mathbf{1}\epsilon$ belongs to the interior of $\mathcal{C}$ (with the norm of L^∞), and we can assume that $Y_n \succeq 2\epsilon\mathbf{1} - \mathcal{E}$ for n sufficiently large. Hence $\mathfrak{U}(p + Y_n) = \mathfrak{U}(-\mathbf{1}\epsilon + Y_n) \geq \mathbb{E}\left[\tilde{U}(\mathbf{1}\epsilon)\right] > -\infty$ if n is sufficently large. By property (ii) in Proposition 3.3.1 we have

$$\sup_{X\in\mathcal{C}} \mathfrak{U}(X) \leq u(\mathcal{E}) < \infty$$

This verifies the hypotheses of part 1 of Theorem A.1 in [CO10], hence the claim follows.

If we have instead $Y \in \mathrm{cl}(\mathcal{A}^0_T)$ then there exists $\tilde{Y}^k \in \mathcal{A}^0_T$ such that $\tilde{Y}^k \to Y$ a.s. (up to a subsequence) and for each $\tilde{Y}^k$ we can find a sequence $\tilde{Y}^k_n \in \mathcal{C}$ such that $\tilde{Y}^k_n \to \tilde{Y}^k$. Then by the same arguments as above we have $\mathfrak{U}(p + Y^k_n) \geq \mathbb{E}\left[\tilde{U}(\mathbf{1}\epsilon)\right] > -\infty$ for n and k sufficently large.

2. It remains to show that

$$\sup_{X\in\mathcal{C}} \mathbb{U}_{\mathcal{E}}(X) = u(\mathcal{E})$$

Clearly $\sup_{X\in\mathcal{C}} \mathbb{U}_{\mathcal{E}}(X) \leq u(\mathcal{E})$. To show the other inequality, take a sequence $X^n \in \mathcal{A}_T^0$ such that $\mathbb{U}_{\mathcal{E}}(X_n) \to u(\mathcal{E})$. By step 1, there exists $Y \in \mathcal{C}$ such that $Y + \mathcal{E} \succeq \mathbf{1}\epsilon$, so we can assume w.l.o.g. that $X^n + \mathcal{E} \in \mathrm{int}(\mathbb{R}_+^D)$ for all n. For any $\epsilon > 0$ we can find n_0 such that $\mathbb{U}_{\mathcal{E}}(X_n) \geq u(\mathcal{E}) - \epsilon$ for all $n \geq n_0$. By [KS09, Lemma 3.6.7], the set $\mathcal{C} = \mathcal{A}_T^0 \cap L^\infty$ is Fatou-dense in $\mathcal{A}_T^0$.‡ Thus, for any $X^n \in \mathcal{A}_T^0$ there is a sequence $X_n^k \in \mathcal{C}$ such that $X_n^k \to X^n$, and since U is continuous on $\mathrm{int}(\mathbb{R}_+^D)$ by [Ro72] Theorem 10.1, we can find k_0 such that $\mathbb{U}_{\mathcal{E}}(X_n^k) \geq \mathbb{U}_{\mathcal{E}}(X_n) - \epsilon$ for any $k \geq k_0$. This implies that $\mathbb{U}_{\mathcal{E}}(X_n^k) \geq u(\mathcal{E}) - 2\epsilon$ when n and k are sufficiently large. Since ϵ is arbitrary we finally get the opposite inequality by letting n and k tend to infinity.
Uniqueness follows easily by strict concavity of the utility function. □

Take $\mathcal{E} \in \mathcal{O}$ and let $\hat{m}$ be the corresponding minimizer in the abstract dual problem above, so that

$$v(\mathcal{E}) = \mathbb{E}\left[\tilde{U}^*\left(\frac{d\hat{m}^c}{dP}\right)\right] + \hat{m}(\mathcal{E}) \in \mathbb{R}.$$

For $x \in \mathbb{R}_+^D$, define

$$\mathcal{D}(x) := \{m \in \mathcal{D}, m(\Omega) = x\}.$$

Take $x \in \mathbb{R}_+^D$ and $m \in \mathcal{D}(x)$, then

$$m(\mathcal{E}) = \lim_n m(\mathcal{E}\chi_{(\mathcal{E}\preceq n\mathbf{1})}) \leq \langle x, x''\rangle < \infty$$

hence if $m \in \mathrm{dom}(\mathbb{U}_0^*)$ then

$$\mathbb{U}_{\mathcal{E}}^*(m) = \mathbb{E}\left[\tilde{U}^*\left(\frac{dm^c}{dP}\right)\right] + m(\mathcal{E}) < \infty.$$

Define

$$v_{\mathcal{E}}(x) := \inf_{m\in\mathcal{D}(x)} \left\{\mathbb{E}\left[\tilde{U}^*\left(\frac{dm^c}{dP}\right)\right] + m(\mathcal{E})\right\}. \tag{3.3.8}$$

We define, for $x \in \mathbb{R}_+^D$, $u_{\mathcal{E}}(x) := u(\mathcal{E} + x)$. Hence we have§

$$\begin{aligned} u_{\mathcal{E}}(x) = v_{\mathcal{E}+x} &= \min_{m\in\mathcal{D}\cap\mathrm{dom}(\mathbb{U}_0^*)} \left\{\mathbb{E}\left[\tilde{U}^*\left(\frac{dm^c}{dP}\right)\right] + m(\mathcal{E}+x)\right\} \\ &= \inf_{y\in\mathbb{R}_+^D} \inf_{m\in\mathcal{D}(y)\cap\mathrm{dom}(\mathbb{U}_0^*)} \left\{\mathbb{E}\left[\tilde{U}^*\left(\frac{dm^c}{dP}\right)\right] + m(\mathcal{E}) + \langle x, y\rangle\right\} \\ &= \min_{y\in\mathbb{R}_+^D} \{v_{\mathcal{E}}(y) + \langle x, y\rangle\} \end{aligned}$$

‡. We recall that a sequence of $\mathbb{R}^D$-valued random variables X^n is Fatou-convergent to X if $X^n \to X$ a.s. and $X^n + a\mathbf{1} \in L^0(K_T, \mathcal{F}_T)$ for some a. A set $\mathcal{A}_0$ is said to be Fatou-dense in $\mathcal{A}$ if any element of $\mathcal{A}$ is a limit of a Fatou-convergent sequence of elements from $\mathcal{A}_0$.

§. Let $x \in \mathbb{R}_+^D$. Notice that, since $\mathbb{R}_+^D \subseteq \mathcal{I} := \mathrm{int}(-\mathcal{A}_T^0 \cap \mathbb{R}^D)$, one has that $x + x' \in \mathcal{I}$ whenever $x' \in \mathcal{I}$. This implies that if $\mathcal{E} \in \mathcal{O}$ and $x \in \mathbb{R}_+^D$, then $x + \mathcal{E}$ belongs to $\mathcal{O}$ and the duality representation (3.3.7) can be applied to $x + \mathcal{E}$ as well.

where the second equality is due to the fact that $\operatorname{dom}\mathbb{U}_0^* = \operatorname{dom}\mathbb{U}_{\mathcal{E}}^*$ whenever $\mathcal{E} \in \mathcal{O}$ (see Remark 3.3.2). A consequence, $v_{\mathcal{E}}(y)$ is the convex conjugate of $u_{\mathcal{E}}(x)$, which implies $u_{\mathcal{E}}^*(y) = v_{\mathcal{E}}(y)$.

Lemma 3.3.3. *The infimum in (3.3.8) is attained whenever $v_{\mathcal{E}}(x)$ is finite.*

Proof. Set $L^\infty = L^\infty(\mathbb{R}^D)$ and $\mathrm{ba} = \mathrm{ba}(\mathbb{R}^D)$ for the sake of simplicity. Take $x \in \mathbb{R}_+^D$ such that $v_{\mathcal{E}}(x)$ is finite. We first show that $\mathcal{D}(x)$ is $\sigma(\mathrm{ba}, L^\infty)$-compact. To see this, remark first that the set $\mathcal{D}$ is $\sigma(\mathrm{ba}, L^\infty)$-closed: for any sequence $(\mu_n)_{n\geq 0} \subseteq \mathcal{D}$ such that $\mu_n \to \mu$ in $\sigma(\mathrm{ba}, L^\infty)$ we also have $\mu(X) = \lim_n \mu_n(X) \leq 0$ for any $X \in \mathcal{C}$. To show closedness of $\mathcal{D}(x)$ take $(\mu_n)_{n\geq 0} \subseteq \mathcal{D}(x)$ such that $\mu_n \to \mu$ in $\sigma(\mathrm{ba}, L^\infty)$, then $\mu(\Omega) = \lim_n \mu_n(\Omega) = x$ and $\mu \in \mathcal{D}(x)$. The set $\mathcal{D}(x)$ only contains positive measures, for which $\|\mu\| \leq \mu(\Omega)$, hence it can be seen as a closed subset of the $\sigma(\mathrm{ba}, L^\infty)$-compact ball $\{\mu \in \mathrm{ba} : \|\mu\| \leq x\}$. Hence $\mathcal{D}(x)$ is $\sigma(\mathrm{ba}, L^\infty)$-compact.

It follows from basic properties of dual functions that $\mathbb{U}_{\mathcal{E}}^*(m)$ is $\sigma(\mathrm{ba}, L^\infty)$-lower semi-continuous (being the supremum of a sequence of affine functions). Then if $(\mu_n)_{n\geq 0} \subseteq \mathcal{D}(x)$ is a minimizing sequence in (3.3.8), we can extract a subsequence μ_{n_k} converging to μ in $\sigma(\mathrm{ba}, L^\infty)$ as $k \to \infty$ and we have $\mathbb{U}_{\mathcal{E}}^*(\mu) \leq \liminf_k \mathbb{U}_{\mathcal{E}}^*(\mu_{n_k}) = \inf_{m\in\mathcal{D}(x)} \mathbb{U}_{\mathcal{E}}^*(m)$. Hence $\mathbb{U}_{\mathcal{E}}^*(\mu) = \inf_{m\in\mathcal{D}(x)} \mathbb{U}_{\mathcal{E}}^*(m)$ and μ attains the infimum in (3.3.8). □

3.3.2 Existence of the optimizer

Let $\mathcal{E} \in \mathcal{O}$. We now show that $v_{\mathcal{E}} : \mathbb{R}_+^D \to \mathbb{R}$ is a proper convex function. It is clearly proper by Proposition 3.3.1 (ii) and Lemma 3.3.1. Now, we turn to convexity. Let m_1 and m_2 be the minimizers in $v_{\mathcal{E}}(x_1)$ and $v_{\mathcal{E}}(x_2)$ and let $x = (1-\lambda)x_1 + \lambda x_2$, $m = (1-\lambda)m_1 + \lambda m_2 \in \mathcal{D}(x) \cap \operatorname{dom}(\mathbb{U}_0^*)$. We have

$$\begin{aligned}
&(1-\lambda)v_{\mathcal{E}}(x_1) + \lambda v_{\mathcal{E}}(x_2)\\
&= (1-\lambda)\left\{\mathbb{E}\left[\tilde{U}^*\left(\frac{dm_1^c}{dP}\right)\right] + m_1(\mathcal{E})\right\} + \lambda\left\{\mathbb{E}\left[\tilde{U}^*\left(\frac{dm_2^c}{dP}\right)\right] + m_2(\mathcal{E})\right\}\\
&= \mathbb{E}\left[(1-\lambda)\tilde{U}^*\left(\frac{dm_1^c}{dP}\right) + \lambda\tilde{U}^*\left(\frac{dm_2^c}{dP}\right)\right] + (1-\lambda)m_1(\mathcal{E}) + \lambda m_2(\mathcal{E})\\
&\geq \mathbb{E}\left[\tilde{U}^*\left((1-\lambda)\frac{dm_1^c}{dP} + \lambda\frac{dm_2^c}{dP}\right)\right] + (1-\lambda)m_1(\mathcal{E}) + \lambda m_2(\mathcal{E})\\
&= \mathbb{E}\left[\tilde{U}^*\left(\frac{dm^c}{dP}\right)\right] + m(\mathcal{E}) \geq v_{\mathcal{E}}(x).
\end{aligned}$$

Consider any $m \in \mathcal{D} \cap \operatorname{dom}(\mathbb{U}_0^*)$ and $m_\lambda := \lambda m + (1-\lambda)\hat{m} \in \mathcal{D} \cap \operatorname{dom}(\mathbb{U}_0^*)$ for $\lambda \in [0,1]$. The function

$$h(\lambda) = \mathbb{E}\left[\tilde{U}^*\left(\frac{dm_\lambda^c}{dP}\right)\right] + m_\lambda(\mathcal{E})$$

is convex and has a minimum at zero, therefore by monotone convergence one has

$$\begin{aligned}
0 \leq h'_+(0) &= \lim_{\lambda \downarrow 0} \frac{h(\lambda) - h(0)}{\lambda} \\
&= \lim_{\lambda \downarrow 0} \left\{ \mathbb{E}\left[\frac{\tilde{U}^*\left(\frac{dm^c_\lambda}{dP}\right) - \tilde{U}^*\left(\frac{d\hat{m}^c}{dP}\right)}{\lambda} \right] + \frac{m_\lambda(\mathcal{E}) - \hat{m}(\mathcal{E})}{\lambda} \right\} \\
&= \mathbb{E}\left[\lim_{\lambda \downarrow 0} \frac{\tilde{U}^*\left(\frac{dm^c_\lambda}{dP}\right) - \tilde{U}^*\left(\frac{d\hat{m}^c}{dP}\right)}{\lambda} \right] + m(\mathcal{E}) - \hat{m}(\mathcal{E}) \\
&= \mathbb{E}\left[\left\langle -\tilde{I}\left(\frac{d\hat{m}^c}{dP}\right), \frac{dm^c}{dP} - \frac{d\hat{m}^c}{dP} \right\rangle \right] + m(\mathcal{E}) - \hat{m}(\mathcal{E})
\end{aligned}$$

so that

$$\mathbb{E}\left[\left\langle \tilde{I}\left(\frac{d\hat{m}^c}{dP}\right), \frac{dm^c}{dP} \right\rangle \right] - m(\mathcal{E}) \leq \mathbb{E}\left[\left\langle \tilde{I}\left(\frac{d\hat{m}^c}{dP}\right), \frac{d\hat{m}^c}{dP} \right\rangle \right] - \hat{m}(\mathcal{E}).$$

Since $\mathbb{U}_0^*$ is decreasing with respect to the preorder induced by $\mathrm{ba}(\mathbb{R}^D_+)$, if we take any $\tilde{m} \in \mathcal{D}$ we have that $m := \hat{m} + \tilde{m} \in \mathcal{D} \cap \mathrm{dom}(\mathbb{U}_0^*)$. It follows that

$$\mathbb{E}\left[\left\langle \tilde{I}\left(\frac{d\hat{m}^c}{dP}\right), \frac{d\tilde{m}^c}{dP} \right\rangle \right] \leq \tilde{m}(\mathcal{E}). \tag{3.3.9}$$

At this point, we would like to prove that we have equality in (3.3.9) when $\tilde{m} = \hat{m}$. To do so, we need to impose an additional property to the value $u_{\mathcal{E}}(\cdot)$ which is the asymptotic satiability.

Assumption 3.3.2. Let $\mathcal{E} \in \mathcal{O} \cap L^\infty(\mathbb{R}^D_+)$. The function $u_{\mathcal{E}} : \mathbb{R}^D_+ \to \mathbb{R}$ is asymptotically satiable.

Since $u_{\mathcal{E}}(\cdot)$ is asymptotically satiable, by Proposition 3.2.1 if $\mathcal{E} \in L^\infty$ there exists a $y \in \mathrm{dom}(u^*_{\mathcal{E}})$ such that $\|\mathcal{E}\|_\infty \|y\|_1 \leq \epsilon$ for any $\epsilon > 0$, where $\|y\|_1 = \sum_{i=1}^D |y_i|$. Also, by duality, there must exist an $m \in \mathcal{D}(y) \cap \mathrm{dom}(\mathbb{U}_0^*)$. Clearly $m(\mathcal{E}) \leq \|m\|_{\mathrm{ba}(\mathbb{R}^D)} \|\mathcal{E}\|_\infty \leq \epsilon$ so that

$$\begin{aligned}
-\epsilon \leq -m(\mathcal{E}) &\leq \mathbb{E}\left[\left\langle \tilde{I}\left(\frac{d\hat{m}^c}{dP}\right), \frac{dm^c}{dP} \right\rangle \right] - m(\mathcal{E}) \\
&\leq \mathbb{E}\left[\left\langle \tilde{I}\left(\frac{d\hat{m}^c}{dP}\right), \frac{d\hat{m}^c}{dP} \right\rangle \right] - \hat{m}(\mathcal{E}) \leq 0
\end{aligned}$$

and, being ϵ arbitrary, this implies

$$\mathbb{E}\left[\left\langle \tilde{I}\left(\frac{d\hat{m}^c}{dP}\right), \frac{d\hat{m}^c}{dP} \right\rangle \right] = \hat{m}(\mathcal{E}). \tag{3.3.10}$$

Inequalities (3.3.9) and equality (3.3.10) allow us to prove the existence of the optimizer for the original maximization problem with random endowment $\mathcal{E} \in \mathcal{O}$, under the additional assumption that $\mathcal{E}$ is bounded.

Theorem 3.3.2. *Let $U : \mathbb{R}^d \to [-\infty, \infty)$ be a utility function supported on $\mathbb{R}^d_+$. Given any $\mathcal{E} \in \mathcal{O} \cap L^\infty$, if the value function verifies Assumption (3.3.2) then the optimal investment problem (3.3.2) has a unique solution $\hat{X} := \tilde{I}\left(\frac{d\hat{m}^c}{dP}\right) - \mathcal{E}$, where $\hat{m}$ is any optimizer in the dual problem.*

Proof. Take any $Z \in \mathcal{Z}^s$ and let $m^Z \in \mathcal{D}$ be its corresponding measure as in Remark 3.3.3. It follows from (3.3.9) that $\mathbb{E}\left[\left\langle \hat{X} + \mathcal{E}, Z_T \right\rangle\right] = \mathbb{E}\left[\left\langle \tilde{I}\left(\frac{d\hat{m}^c}{dP}\right), \frac{dm^Z}{dP} \right\rangle\right] \leq m^Z(\mathcal{E})$, hence $\mathbb{E}\left[\left\langle \hat{X}, \frac{dm^Z}{dP} \right\rangle\right] \leq 0$. It now follows from Theorem 3.2.1 that $\hat{X} \in \mathcal{A}^0_T$. Hence by using (3.3.10) we can write

$$\begin{aligned}\mathbb{E}\left[\tilde{U}(\hat{X} + \mathcal{E})\right] &= \mathbb{E}\left[\tilde{U}\left(\tilde{I}\left(\frac{d\hat{m}^c}{dP}\right)\right)\right] = \mathbb{E}\left[\tilde{U}^*\left(\frac{d\hat{m}^c}{dP}\right)\right] + \mathbb{E}\left[\left\langle \tilde{I}\left(\frac{d\hat{m}^c}{dP}\right), \frac{d\hat{m}^c}{dP} \right\rangle\right] \\ &= \mathbb{E}\left[\tilde{U}^*\left(\frac{d\hat{m}^c}{dP}\right)\right] + \hat{m}(\mathcal{E}) = \mathbb{U}^*_{\mathcal{E}}(\hat{m}).\end{aligned}$$

It is now easy to conclude by using Theorem 3.3.1. Uniqueness follows by the same arguments used in [CO10, Theorem 3.12]. □

Remark 3.3.4. It is important to stress that the boundedness assumption on the random endowment $\mathcal{E}$ is needed only to prove the existence of the optimal portfolio, while to obtain the duality characterization and the existence of the minimizer in the dual problem it suffices to require the weaker property $\mathcal{E} \in \mathcal{O}$, i.e. the random endowment can be unbounded from above.

Remark 3.3.5. Consider the case of a deterministic endowment $\mathcal{E} = x$, and make the (quite strong) assumption that the infimum in the dual problem is reached for some $\hat{Z} \in \mathcal{Z}$, as in Kabanov (99), Proposition 4.2. Then (3.3.10) becomes

$$\mathbb{E}\left[\left\langle \hat{V}_T, \hat{Z}_T \right\rangle\right] = \mathbb{E}\left[\left\langle x + \hat{X}_T, \hat{Z}_T \right\rangle\right] = \left\langle x, \hat{Z}_0 \right\rangle = \left\langle \hat{V}_0, \hat{Z}_0 \right\rangle,$$

Integration by parts gives (we suppress the brackets for notational simplicity)

$$\hat{Z}_t\hat{V}_t = \hat{Z}_0\hat{V}_0 + \int_0^t \hat{V}_u d\hat{Z}_u + \int_0^t \hat{Z}_u\hat{V}^c_u dVar_u(\hat{V}^c) + \sum_{u\leq t} \hat{Z}_{u-}\Delta\hat{V}_u + \sum_{u<t} \hat{Z}_u\Delta_+\hat{V}_u, \quad (3.3.11)$$

hence the first integral is a true martingale and we must have

$$\mathbb{E}\left[\int_0^t \hat{Z}_u\hat{V}^c_u dVar_u(\hat{V}^c) + \sum_{u\leq t} \hat{Z}_{u-}\Delta\hat{V}_u + \sum_{u<t} \hat{Z}_u\Delta_+\hat{V}_u\right] = 0.$$

Since all the integrals are lower or equal to zero then they must be equal to zero. In the classical transaction cost model we have, formally, $d\hat{V}^i = (1/S^i)dB^i$ where

$$B^i = \sum_{j=1}^D L^{ji} - \sum_{j=1}^D (1 + \lambda^{ij})L^{ij},$$

hence we can write

$$\hat{Z}_t \hat{V}_t^c = \sum_{i,j} \frac{dL_t^{ij}}{dVar_u(\hat{V}^c)} \left[\frac{\hat{Z}_t^j}{S_t^j} - \frac{(1+\lambda_t^{ij})\hat{Z}_t^i}{S_t^i} \right].$$

$$\hat{Z}_{t-} \Delta \hat{V}_t = \sum_{i,j} \Delta L_t^{ij} \left[\frac{\hat{Z}_{t-}^j}{S_t^j} - \frac{(1+\lambda_t^{ij})\hat{Z}_{t-}^i}{S_t^i} \right].$$

$$\hat{Z}_t \Delta_+ \hat{V}_t = \sum_{i,j} \Delta_+ L_t^{ij} \left[\frac{\hat{Z}_t^j}{S_t^j} - \frac{(1+\lambda_t^{ij})\hat{Z}_t^i}{S_t^i} \right].$$

We conclude that L^{ij} must be flat outside the set $\left\{ \frac{Y^j}{S^j} - \frac{(1+\lambda^{ij})Y^i}{S^i} = 0 \right\}$.

3.3.3 Sufficient conditions for existence and liquidation

We can now give some conditions which ensure asymptotic satiability of $u_{\mathcal{E}}(x)$. In order to check them easily, it is useful, in general, to look for conditions that only concern the utility function U (or possibly its dual). We start by defining a growth condition in the version of [CO10] (even if similar conditions have appeared in different papers, for example in [DPT01]).

Definition 3.3.1. Let $U : \mathbb{R}^d \to [-\infty, \infty)$ be a utility function. We shall say that the dual function U^* satisfies the *growth condition* if there exists a function $\zeta : (0,1] \to [0,\infty)$ such that for all $\epsilon \in (0,1]$ and all $x^* \in \text{int}(\mathbb{R}^d_+)$

$$U^*(\epsilon x^*) \leq \zeta(\epsilon)(U^*(x^*)^+ + 1). \tag{3.3.12}$$

The following result is the analogue of [CO10, Corollary 3.7]. The proof is essentially the same with some minor modifications. Nonetheless, we decided to give the details for reader's convenience.

Lemma 3.3.4. *Take $\mathcal{E} \in \mathcal{O} \cap L^\infty(\mathbb{R}^D)$. If U^* satisfies the growth condition* (3.3.12) *then both U and $u_{\mathcal{E}}(\cdot)$ are asymptotically satiable.*

Proof. Take $m \in \mathcal{D} \cap \text{dom}(\mathbb{U}_0^*)$ (for example the minimizer in the dual problem (3.3.7)), define $x^* := m(\Omega)$. Then, since $\mathcal{E} \preceq x'' + X''$ with $x'' \in \mathcal{I}$ and $X'' \in \mathcal{C}$, one has

$$\begin{aligned} u_{\mathcal{E}}^*(\epsilon x^*) = v_{\mathcal{E}}(\epsilon x^*) &\leq \mathbb{E}\left[\tilde{U}^*\left(\epsilon \frac{dm^c}{dP}\right)\right] + \epsilon m(\mathcal{E}) \leq \mathbb{E}\left[U^*\left(\epsilon P\left(\frac{dm^c}{dP}\right)\right)\right] + \epsilon \langle x^*, x'' \rangle \\ &\leq \zeta(\epsilon)\mathbb{E}\left[U^*\left(P\left(\frac{dm^c}{dP}\right)\right)^+ + 1\right] + \epsilon \langle x^*, x'' \rangle \\ &= \zeta(\epsilon)\mathbb{E}\left[\tilde{U}^*\left(\frac{dm^c}{dP}\right)^+ + 1\right] + \epsilon \langle x^*, x'' \rangle < \infty \end{aligned}$$

for any $\epsilon \in (0,1]$. Hence $\epsilon x^* \in \text{dom}(u_{\mathcal{E}}^*)$. Taking the limit as $\epsilon \to 0$ shows that $0 \in \text{cl}(\text{dom}(u_{\mathcal{E}}^*))$ and hence $u_{\mathcal{E}}$ is asymptotically satiable by Proposition 3.2.1. The proof for U follows the same lines but in an easier way, by directly using the growth condition and the characterization of Proposition 3.2.1 as in [CO10, Corollary 3.7]. □

One might look for sufficient conditions to check that the growth condition (3.3.12) actually holds. In [CO10] the notion of *reasonable asymptotic elasticity* of U is introduced in order to ensure the growth condition in the case of multivariate utility functions which are multivariate risk-averse and bounded from below. If U is bounded from above then (3.3.12) trivially holds with $\zeta(\epsilon) := \sup_{x^* \in \mathbb{R}^d_+} U^*(x^*) = U^*(0) = \sup_{x \in \mathbb{R}^d} U(x) < \infty$. It is also satisfied if the quantity $-\langle \nabla U^*(\epsilon x^*), x^* \rangle$ is bounded from above in x^* (as in the case of the sum of logarithms, a utility function which is neither bounded from above nor from below).

Remark 3.3.6. Some papers dealing with optimal investment assume that the agent liquidates his assets at the terminal date to one (ore more) reference assets. As in [CO10], it is possible to show that the problem treated here is essentially equivalent to the investment problem with final liquidation, provided that U is upper semi-continuous. In particular we have that

$$u(\mathcal{E}) = \sup_{W \in \mathcal{A}^0_{T-}} \mathbb{E}\left[\bar{U}(W + \mathcal{E})\right] \tag{3.3.13}$$

where

$$\bar{U}(W) := \sup\left\{U(\xi) \,:\, \xi \in \mathbb{R}^d_+,\ (\xi, \underline{0}) - W \in -K_T\right\}, \quad W \in L^0(K_T, \mathscr{F}_{T-}) \tag{3.3.14}$$

and $\underline{0}$ denotes the zero vector in $\mathbb{R}^{D-d}$. The proof follows the same lines as [CO10], Proposition 4.3 with minor modifications.
If $\mathcal{E} \in \mathcal{O} \cap L^\infty$ (which ensures the existence of a solution in the primal problem) then we can argue as in [CO10, Proposition 4.4] to conclude that the supremum in (3.3.13) is attained at some $\hat{W} \in \mathcal{A}^0_{T-}$ and that $(\hat{\xi}(\hat{W} + \mathcal{E}), \underline{0}) = \hat{X} + \mathcal{E}$ a.s., where $\hat{\xi}(\hat{W} + \mathcal{E})$ is the maximizer in (3.3.14).

Remark 3.3.7. If $d = 1$ our optimization problem is similar to that treated in [Bo02]. In that paper, however, the utility function is defined on the whole real line, which permits to avoid recurring to singular measures. In a sense, we generalize their results in that we do not require the underlying asset processes to be continuous, nor the transaction costs to be constant (we work in the framework set out in [CS06] which is much more general). Moreover, we allow for a liquidation to many assets, which forces us to introduce multivariate utility functions. Finally, many of our results (e.g. the duality characterization) do not require the boundedness of the endowment which is instead assumed in [Bo02].

3.4 Utility indifference pricing

In this section we will examine some applications of the above results to the pricing of contingent claims in an incomplete market. The analysis that follows is motivated by the fast growing interest in new pricing paradigms (alternative to replication) in the context of incomplete financial markets. We adopt some of the techniques used in [OZ09], where the authors studied a similar investment problem but in a framework of frictionless financial markets and with univariate utility functions (defined on the whole real line).
We start by proving some continuity properties of the value function.

Lemma 3.4.1. *1. If $(\mathcal{E}_n)_{n\in\mathbb{N}}$ is a sequence of endowments in $\mathcal{O}$ and such that*

$$\sup_{m\in\mathcal{D}\cap\mathrm{dom}(\mathbb{U}_0^*)} m(\mathcal{E}_n-\mathcal{E})\to 0 \text{ and } \inf_{m\in\mathcal{D}\cap\mathrm{dom}(\mathbb{U}_0^*)} m(\mathcal{E}_n-\mathcal{E})\to 0$$

as $n\to\infty$ with $\mathcal{E}\in\mathcal{O}$, then $u(\mathcal{E}_n)\to u(\mathcal{E})$.

2. *If U is lower semi-continuous then u is as well on $\mathcal{O}$ equipped with the topology of convergence in probability.*
3. *If $(x_n+\mathcal{E})_{n\in\mathbb{N}}\in\mathcal{O}$ and $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}^D$ such that $x_n\to x$ and $a\preceq x_n\preceq b$ for some $a,b\in\mathbb{R}^D$, then $x+\mathcal{E}\in\mathcal{O}$ and*

$$u(\mathcal{E}+x)=\lim_n u(\mathcal{E}+x_n).$$

4. *If $(\mathcal{E}_n)_{n\in\mathbb{N}}$ is a sequence of endowments in $\mathcal{O}\cap L^\infty(\mathbb{R}^D)$ which uniformly satisfy equation (3.3.1) (in the sense that the upper and lower bounds do not depend on n) and such that $\mathcal{E}_n\to\mathcal{E}$ in $L^\infty(\mathbb{R}^D)$ then we have*

$$u(\mathcal{E})=\lim_n u(\mathcal{E}_n).$$

Proof. (i) We have

$$\begin{aligned}u(\mathcal{E}_n)-u(\mathcal{E}) &= \inf_{m\in\mathcal{D}\cap\mathrm{dom}(\mathbb{U}_0^*)}\left\{\mathbb{E}\left[\tilde{U}^*\left(\frac{dm^c}{dP}\right)\right]+m(\mathcal{E}_n)\right\}\\ &\quad-\inf_{m\in\mathcal{D}\cap\mathrm{dom}(\mathbb{U}_0^*)}\left\{\mathbb{E}\left[\tilde{U}^*\left(\frac{dm^c}{dP}\right)\right]+m(\mathcal{E})\right\}\\ &\le \sup_{m\in\mathcal{D}\cap\mathrm{dom}(\mathbb{U}_0^*)} m(\mathcal{E}_n-\mathcal{E})\end{aligned}$$

but also

$$u(\mathcal{E}_n)-u(\mathcal{E})\ge \inf_{m\in\mathcal{D}\cap\mathrm{dom}(\mathbb{U}_0^*)} m(\mathcal{E}_n-\mathcal{E})$$

hence $|u(\mathcal{E}_n)-u(\mathcal{E})|\to 0$ as $n\to\infty$.
(ii) Let $(\mathcal{E}_n)_{n\in\mathbb{N}}$ be a sequence of endowments in $\mathcal{O}$ such that $\mathcal{E}_n\to\mathcal{E}$ in probability ($\mathcal{E}\in\mathcal{O}$). Then a subsequence (that we still call in the same way) converges a.s. and we have, by semi-continuity of U and Fatou's lemma

$$\begin{aligned}u(\liminf_n\mathcal{E}_n) &= \sup_{X\in\mathcal{A}_T^0}\mathbb{E}\left[\tilde{U}(X+\liminf_n\mathcal{E}_n)\right]\le\sup_{X\in\mathcal{A}_T^0}\mathbb{E}\left[\liminf_n\tilde{U}(X+\mathcal{E}_n)\right]\\ &\le\sup_{X\in\mathcal{A}_T^0}\liminf_n\mathbb{E}\left[\tilde{U}(X+\mathcal{E}_n)\right]\le\liminf_n\sup_{X\in\mathcal{A}_T^0}\mathbb{E}\left[\tilde{U}(X+\mathcal{E}_n)\right]=\liminf_n u(\mathcal{E}_n)\end{aligned}$$

which implies the claim.

(iii) Let $(x_n + \mathcal{E})_{n\in\mathbb{N}} \in \mathcal{O}$ with $(x_n)_{n\in\mathbb{N}}$ a sequence in $\mathbb{R}^D$ such that $x_n \to x$ and $a \preceq x_n \preceq b$ for some $a, b \in \mathbb{R}^D$, then $x + \mathcal{E} \in \mathcal{O}$ (since $-\mathcal{A}_T^0 \cap \mathbb{R}^D$ is closed) and we have

$$\begin{aligned} u(\mathcal{E}+x) &= \inf_{m\in\mathcal{D}\cap\mathrm{dom}(\mathbb{U}_0^*)} \left\{ \mathbb{E}\left[\tilde{U}^*\left(\frac{dm^c}{dP}\right)\right] + m(\mathcal{E}) + m(\limsup_n x_n) \right\} \\ &= \inf_{m\in\mathcal{D}\cap\mathrm{dom}(\mathbb{U}_0^*)} \left\{ \mathbb{E}\left[\tilde{U}^*\left(\frac{dm^c}{dP}\right)\right] + m(\mathcal{E}) + \limsup_n m(x_n) \right\} \\ &\geq \limsup_n \inf_{m\in\mathcal{D}\cap\mathrm{dom}(\mathbb{U}_0^*)} \left\{ \mathbb{E}\left[\tilde{U}^*\left(\frac{dm^c}{dP}\right)\right] + m(\mathcal{E}+x_n) \right\} \\ &= \limsup_n u(\mathcal{E}+x_n) \end{aligned}$$

hence u is continuous along such sequences.

(iv) Let $(\mathcal{E}_n)_{n\in\mathbb{N}}$ be a sequence of endowments in $\mathcal{O}\cap L^\infty(\mathbb{R}^D)$ which uniformly satisfy equation (3.3.1) and such that $\mathcal{E}_n \to \mathcal{E}$ in $L^\infty(\mathbb{R}^D)$. Thus, we have

$$\begin{aligned} u(\mathcal{E}) &= \inf_{m\in\mathcal{D}\cap\mathrm{dom}(\mathbb{U}_0^*)} \left\{ \mathbb{E}\left[\tilde{U}^*\left(\frac{dm^c}{dP}\right)\right] + m(\lim_n \mathcal{E}_n) \right\} \\ &= \inf_{m\in\mathcal{D}\cap\mathrm{dom}(\mathbb{U}_0^*)} \left\{ \mathbb{E}\left[\tilde{U}^*\left(\frac{dm^c}{dP}\right)\right] + \lim_n m(\mathcal{E}_n) \right\} \\ &\geq \limsup_n \inf_{m\in\mathcal{D}\cap\mathrm{dom}(\mathbb{U}_0^*)} \left\{ \mathbb{E}\left[\tilde{U}^*\left(\frac{dm^c}{dP}\right)\right] + m(\mathcal{E}_n) \right\} \\ &= \limsup_n u(\mathcal{E}_n) \end{aligned}$$

since $\mathcal{E}_n \to \mathcal{E}$ in $\sigma(L^\infty(\mathbb{R}^D), \mathrm{ba}(\mathbb{R}^D))$. Hence u is continuous also along these sequences as well. □

For $j = 1, ..., d$ define

$$m_j(X) := \sum_{i=1}^D \int_\Omega X_i \frac{dm_i}{m_j(\Omega)}$$

and

$$\underline{m}_j(X) := \inf_{m\in\mathcal{D}\cap\mathrm{dom}(\mathbb{U}_0^*)} \left\{ \sum_{i=1}^D \int_\Omega X_i \frac{dm_i}{m_j(\Omega)} \right\}$$

For $B \in L^0(\mathbb{R}_+^D)$ denote $u_\mathcal{E}(B) := u(\mathcal{E} + B)$ (sometimes we will write $u_\mathcal{E}$ instead of $u(\mathcal{E})$). The following lemma will be useful for the characterization of utility indifference prices, which will be introduced immediately after.

Lemma 3.4.2. *If $\mathcal{E} \in \mathcal{O}$ and $\mathcal{E} + B - e_j\hat{m}_j(B) \in \mathcal{O}$ then*

$$u_\mathcal{E}\left(B - e_j\hat{m}_j(B)\right) \leq u_\mathcal{E} \leq u_\mathcal{E}\left(B - e_j\underline{m}_j(B)\right)$$

for all $j = 1, ..., d$.

Proof. Remark first that the conditions above imply also that $\mathcal{E} + B - e_j \underline{m}_j(B) \in \mathcal{O}$. Using the duality characterization in Theorem 3.3.1 together with the definitions of m_j and $\underline{m}_j$ yields

$$\begin{aligned}
&u_{\mathcal{E}}\left(B - e_j \underline{m}_j(B)\right) = \\
&= \inf_{m \in \mathcal{D} \cap \mathrm{dom}(\mathbb{U}_0^*)} \left\{ \mathbb{E}\left[\tilde{U}^*\left(\frac{dm^c}{dP}\right)\right] + m(\mathcal{E}) + m\left(B - e_j \underline{m}_j(B)\right) \right\} \\
&\geq \inf_{m \in \mathcal{D} \cap \mathrm{dom}(\mathbb{U}_0^*)} \left\{ \mathbb{E}\left[\tilde{U}^*\left(\frac{dm^c}{dP}\right)\right] + m(\mathcal{E}) \right\} + \inf_{m \in \mathcal{D} \cap \mathrm{dom}(\mathbb{U}_0^*)} \left\{ m\left(B - e_j \underline{m}_j(B)\right) \right\} \\
&= \inf_{m \in \mathcal{D} \cap \mathrm{dom}(\mathbb{U}_0^*)} \left\{ \mathbb{E}\left[\tilde{U}^*\left(\frac{dm^c}{dP}\right)\right] + m(\mathcal{E}) \right\} \\
&\quad + \inf_{m \in \mathcal{D} \cap \mathrm{dom}(\mathbb{U}_0^*)} \left\{ \left(\sum_{i=1}^{D} \int_\Omega B_i \frac{dm_i}{m_j(\Omega)} - \underline{m}_j(B) \right) m_j(\Omega) \right\} \\
&= v_{\mathcal{E}} + 0 = u_{\mathcal{E}}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
&u_{\mathcal{E}}\left(B - e_j \hat{m}_j(B)\right) = \\
&= \inf_{m \in \mathcal{D} \cap \mathrm{dom}(\mathbb{U}_0^*)} \left\{ \mathbb{E}\left[\tilde{U}^*\left(\frac{dm^c}{dP}\right)\right] + m(\mathcal{E}) + m\left(B - e_j \hat{m}_j(B)\right) \right\} \\
&\leq \mathbb{E}\left[\tilde{U}^*\left(\frac{d\hat{m}^c}{dP}\right)\right] + \hat{m}(\mathcal{E}) + \hat{m}\left(B - e_j \hat{m}_j(B)\right) = \mathbb{E}\left[\tilde{U}^*\left(\frac{d\hat{m}^c}{dP}\right)\right] + \hat{m}(\mathcal{E}) + 0 = u_{\mathcal{E}}.
\end{aligned}$$

which yields the other inequality. □

Definition 3.4.1. For $j = 1, \ldots, d$ the *utility indifference (bid) price* (UIP) $p_j(B) = p_j(B; U, \mathcal{E}) \in \mathbb{R}$ for the contingent claim B (expressed in units of asset j) is implicitly defined as the solution to the equation

$$u(\mathcal{E} + B - e_j p_j) = u(\mathcal{E}) \tag{3.4.1}$$

In the next proposition we show that the definition of UIP is well-posed, i.e. $p_j(B)$ exists unique, and that it satisfies in particular the properties of cash-invariance, monotonicity and convexity characterizing a convex risk measure defined on vector-valued random variables (compare [BR06, JMT04, HHR10]).

Proposition 3.4.1. *Let $j = 1, ..., d$. Under the assumptions of Lemma 3.4.2 there exists a unique solution to (3.4.1). The UIP $p_j(B)$ is therefore well defined and it verifies the following properties:*

1. $\underline{m}_j(B) \leq p_j(B) \leq \hat{m}_j(B)$;
2. *if $B \in \mathcal{A}_T^0$ then $p_j(B) \leq 0$ for any $j = 1, \ldots, d$;*
3. *for $c \in \mathbb{R}$ we have $p_j(B + e_j c) = p_j(B) + c$;*
4. *if $B \preceq C$ then $p_j(B) \leq p_j(C)$ for any $j = 1, \ldots, d$;*

5. *given contingent claims* B_1, B_2 *and* $\lambda \in [0,1]$

$$p_j(\lambda B_1 + (1-\lambda)B_2) \geq \lambda p_j(B_1) + (1-\lambda)p_j(B_2)$$

for any $j = 1, \dots, d$;

6. *the utility indifference price can be expressed as*

$$p_j(B) = \inf_{m \in \mathcal{D}^j(1) \cap \mathrm{dom}(\mathbb{U}_0^*)} \{m_j(B) + \alpha_j(m)\}$$

where

$$\mathcal{D}^j(k) := \{m \in \mathcal{D} : m_j(\Omega) = k\}$$

$$\alpha_j(m) := \inf_{k>0} \frac{1}{k} \left\{ \mathbb{E}\left[\tilde{U}^*\left(\frac{dm^{k,c}}{dP}\right)\right] + m^k(\mathcal{E}) - v_{\mathcal{E}} \right\}$$

and m^k *is such that* $m_i^k = m_i$ *if* $i \neq j$ *and* $m_j^k = k m_j$;

7. *if* $(B_n)_{n\in\mathbb{N}}$ *is a sequence of contingent claims such that*

$$\inf_{m \in \mathcal{D}^j(1) \cap \mathrm{dom}(\mathbb{U}_0^*)} m_j(B_n - B) \to 0 \text{ and } \sup_{m \in \mathcal{D}^j(1) \cap \mathrm{dom}(\mathbb{U}_0^*)} m_j(B_n - B) \to 0$$

then $p_j(B_n) \to p_j(B)$.

Proof. Remark first that if $\mathcal{E}_1$ belongs to $\mathcal{O}$ and $\mathcal{E}_2 := \mathcal{E}_1 + x$ with $x \in \mathbb{R}_+^D$ and $x_j > 0$ for some $j \leq d$, then $u(\mathcal{E}_2) > u(\mathcal{E}_1)$. Indeed we have

$$\begin{aligned} u(\mathcal{E}_1) &= \mathbb{E}\left[\tilde{U}^*\left(\frac{d\hat{m}_1^c}{dP}\right)\right] + \hat{m}_1(\mathcal{E}_1) \leq \mathbb{E}\left[\tilde{U}^*\left(\frac{d\hat{m}_2^c}{dP}\right)\right] + \hat{m}_2(\mathcal{E}_2 - x) \\ &< \mathbb{E}\left[\tilde{U}^*\left(\frac{d\hat{m}_2^c}{dP}\right)\right] + \hat{m}_2(\mathcal{E}_2) = u(\mathcal{E}_2) \end{aligned} \tag{3.4.2}$$

where $\hat{m}_1$ (resp. $\hat{m}_2$) is the minimizer in the dual problem with endowment $\mathcal{E}_1$ (resp. $\mathcal{E}_2$). Existence and uniqueness follow from Lemma 3.4.1(iii) and the above considerations. Property (i) is clear from Lemma 3.4.2. Property (ii) follows from the definition of the primal problem by noting that $X + B \in \mathcal{A}_T^0$ if $X, B \in \mathcal{A}_T^0$. In particular we have $u(\mathcal{E} + B) \leq u(\mathcal{E}) = u(\mathcal{E} + B - e_j p_j)$ which implies the claim. Property (iii) is straightforward from the definition of UIP and (iv) follows by monotonicity of $u_{\mathcal{E}}(\cdot)$.
(v) By concavity of $u_{\mathcal{E}}(\cdot)$

$$\begin{aligned} u_{\mathcal{E}}(\lambda B_1 &+ (1-\lambda)B_2 - e_j \lambda p_j(B_1) - e_j(1-\lambda)p_j(B_2)) \\ &\geq \lambda u_{\mathcal{E}}(B_1 - e_j p_j(B_1)) + (1-\lambda) u_{\mathcal{E}}(B_2 - e_j p_j(B_2)) = u_{\mathcal{E}} \\ &= u_{\mathcal{E}}(\lambda B_1 + (1-\lambda)B_2 - e_j p_j(\lambda B_1 + (1-\lambda)B_2)) \end{aligned}$$

by definition of UIP. The claim follows by monotonicity of $u_{\mathcal{E}}(\cdot)$.

(vi) By monotonicity of $u_{\mathcal{E}}(\cdot)$ and Lemma 3.4.1(iii), we have

$$\begin{aligned}
p_j(B) &= \inf\{p : u_{\mathcal{E}}(B - e_j p) < v_{\mathcal{E}}\} \\
&= \inf\left\{p : \inf_{m\in\mathcal{D}\cap\mathrm{dom}(\mathbb{U}_0^*)}\left\{\mathbb{E}\left[\tilde{U}^*\left(\frac{dm^c}{dP}\right)\right] + m(\mathcal{E}) + m(B) - pm_j(\Omega)\right\} < v_{\mathcal{E}}\right\} \\
&= \inf\left\{p : \inf_{m\in\mathcal{D}^j(1)\cap\mathrm{dom}(\mathbb{U}_0^*)}\inf_{k>0}\left\{\mathbb{E}\left[\tilde{U}^*\left(\frac{dm^{k,c}}{dP}\right)\right] + m^k(\mathcal{E}) + km_j(B) - v_{\mathcal{E}} - kp\right\} < 0\right\} \\
&= \inf\left\{p : \inf_{m\in\mathcal{D}^j(1)\cap\mathrm{dom}(\mathbb{U}_0^*)} m_j(B) + \inf_{k>0}\frac{1}{k}\left\{\mathbb{E}\left[\tilde{U}^*\left(\frac{dm^{k,c}}{dP}\right)\right] + m^k(\mathcal{E}) - v_{\mathcal{E}}\right\} < p\right\} \\
&= \inf_{m\in\mathcal{D}^j(1)\cap\mathrm{dom}(\mathbb{U}_0^*)}\left\{m_j(B) + \inf_{k>0}\frac{1}{k}\left\{\mathbb{E}\left[\tilde{U}^*\left(\frac{dm^{k,c}}{dP}\right)\right] + m^k(\mathcal{E}) - v_{\mathcal{E}}\right\}\right\} \\
&= \inf_{m\in\mathcal{D}^j(1)\cap\mathrm{dom}(\mathbb{U}_0^*)}\{m_j(B) + \alpha_j(m)\}
\end{aligned}$$

where we recall that

$$\mathcal{D}^j(k) := \{m \in \mathcal{D} : m_j(\Omega) = k\}$$

$$\alpha_j(m) := \inf_{k>0}\frac{1}{k}\left\{\mathbb{E}\left[\tilde{U}^*\left(\frac{dm^{k,c}}{dP}\right)\right] + m^k(\mathcal{E}) - v_{\mathcal{E}}\right\}$$

and m^k is such that $m_i^k = m_i$ if $i \neq j$ and $m_j^k = km_j$.
(vii) Remark that

$$\begin{aligned}
\inf_{m\in\mathcal{D}^j(1)\cap\mathrm{dom}(\mathbb{U}_0^*)} m_j(B_n - B) &= \inf_{m\in\mathcal{D}^j(1)\cap\mathrm{dom}(\mathbb{U}_0^*)}[(m_j(B_n) + \alpha_j(m)) - (m_j(B) + \alpha_j(m))] \\
&\leq \inf_{m\in\mathcal{D}^j(1)\cap\mathrm{dom}(\mathbb{U}_0^*)}[m_j(B_n) + \alpha_j(m)] - \inf_{m\in\mathcal{D}^j(1)\cap\mathrm{dom}(\mathbb{U}_0^*)}[m_j(B) + \alpha_j(m)] \\
&= p_j(B_n) - p_j(B) \leq \sup_{m\in\mathcal{D}^j(1)\cap\mathrm{dom}(\mathbb{U}_0^*)} m_j(B_n - B)
\end{aligned}$$

which implies the claim. □

Example 3.4.1. Point (ii) of the previous proposition implies that if a contingent claim is super-replicable starting from h units of asset j then $p_j(B) \leq h$. The inequality is usually strict, as it is shown in the following example. Let us first briefly recall the setting of the one period market of Example 3.13 in [CO10]. The stock price S satisfies $S_0 = 1$ and $S_1 = s_n$ with probability p_n ($n \in \mathbb{N}$), where $s_0 = 2$, $s_n = 1/n$ for $n \geq 1$, $p_0 = 1 - \alpha$ and $p_n = \alpha 2^{-n}$ with α sufficiently small. Frictions are modeled by the bid-ask matrices

$$\Pi_0 = \begin{pmatrix} 1 & S_0 \\ 2/S_0 & 1 \end{pmatrix} \text{ and } \Pi_1 = \begin{pmatrix} 1 & 2S_1 \\ 1/S_1 & 1 \end{pmatrix}.$$

Consider now a call option C on the stock with strike $K = 1$. By using the general form $X_x^\lambda = x + \lambda(S_1 - S_0)$ of the payoff of a strategy starting from x units of bonds we deduce that the minimal super-replicating price of the call is equal to $1/2$. Indeed, considering the case $S_1 = 2$ we must have $x + \lambda \geq 1$, while in the case $S_1 = 1/n$ we have $x + \lambda(1/n - 1) \geq 0$ for all $n \geq 1$, that is $x - \lambda \geq 0$. Hence in this case the super-replication price is the minimal

x such that there exists λ satisfying $x \geq \lambda$ and $x + \lambda \geq 1$, that is $x = 1/2$. On the other hand, the UIP p (with a log utility) solves the equation $u(x) = u(x - p + C)$, that is

$$\log x + E[\log(S_1)] = E[\log((x-p)S_1 + \mathbf{1}_{S_1=2})],$$

which simplifies to

$$(x - p + 1/2)^{1-\alpha}(x-p)^{\alpha} = x.$$

It follows that $p < 1/2$ (recall that $\alpha > 0$).
This pricing paradigm can be considered a good solution in situations where the super-replicating price is unreasonably high (which is often the case with transaction costs).

Definition 3.4.2. The average utility indifferent purchase price for β units of the contingent claim B (in terms of asset j) is defined by

$$p_j^{\beta}(B) := \frac{p_j(\beta B)}{\beta}.$$

In the next proposition we present some properties of the function $\beta \mapsto p_j^{\beta}(B)$. Differently from [OZ09], we were not able to prove that $\lim_{\beta \to 0} p_j^{\beta}(B) = \hat{m}_j(B)$. To establish this result we would need a stronger version of point (i) of Lemma 3.4.1, in which the supremum is only computed over measures with fixed total variation. In [OZ09] they are able to obtain this result because the dual $U^*(y)$ of a utility function U defined over the whole real line goes to infinity as $y \to \infty$, a fact that does not have an analogue in our context.

Proposition 3.4.2. *If $\mathcal{E} \in \mathcal{O}$ and $\mathcal{E} + \beta(B - e_j \hat{m}_j(B)) \in \mathcal{O}$ for all $\beta > 0$ then the function $\beta \mapsto p_j^{\beta}(B)$ verifies the following properties:*

1. *It is non-increasing in β;*
2. *$\underline{m}_j(B) \leq p_j^{\beta}(B) \leq \hat{m}_j(B)$ for all $\beta > 0$;*
3. *$\lim_{\beta \to \infty} p_j^{\beta}(B) = \underline{m}_j(B)$;*

Proof. Remark first that the conditions of Lemma 3.4.2 are automatically satisfied by βB for all $\beta > 0$.
(i) Take $0 < \beta_1 \leq \beta_2$. Then by concavity (Proposition 3.4.1 (v)) we have

$$p_j^{\beta_1}(B) := \frac{1}{\beta_1} p_j\left(\frac{\beta_1}{\beta_2}\beta_2 B\right) \geq \frac{1}{\beta_2} p_j(\beta_2 B) + \left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right) p_j(0) = \frac{1}{\beta_2} p_j(\beta_2 B) = p_j^{\beta_2}(B).$$

(ii) It is clear from Proposition 3.4.1.
(iii) Suppose for a contradiction that there exists $\tilde{m} \in \mathcal{D} \cap \mathrm{dom}(\mathbb{U}_0^*)$ such that $\tilde{m}_j(B) < \lim_{\beta \to \infty} p_j^{\beta}(B)$. Then for any $\beta > 0$

$$\begin{aligned} u_{\mathcal{E}} = u_{\mathcal{E}}(\beta B - e_j p_j(\beta B)) &= \inf_{m \in \mathcal{D} \cap \mathrm{dom}(\mathbb{U}_0^*)} \left\{ \mathbb{E}\left[\tilde{U}^*\left(\frac{dm^c}{dP}\right)\right] + m(\mathcal{E}) + \beta m\left(B - e_j p_j^{\beta}(B)\right)\right\} \\ &\leq \mathbb{E}\left[\tilde{U}^*\left(\frac{d\tilde{m}^c}{dP}\right)\right] + \tilde{m}(\mathcal{E}) + \beta \tilde{m}_j(\Omega)\left[\tilde{m}_j(B) - p_j^{\beta}(B)\right] \end{aligned}$$

and we get the desired contradiction by sending β to infinity. □

Let us look for an interpretation of the previous result. Assume the agent has purchased the claim B paying p units of asset j and now wants to eliminate all the risk arising from this position by super-hedging the claim $-B$. By Theorem 3.2.1 he will be able to reach his objective if and only if

$$\inf_{Z\in\mathcal{Z}^s} \mathbb{E}\left[\left\langle B, \frac{Z_T}{Z_0^j}\right\rangle\right] \geq p,$$

hence the highest price he will accept to pay for the claim B (in units of asset j) by remaining sure that he will run no risks at maturity is

$$\bar{p}_j(B) := \inf_{Z\in\mathcal{Z}^s} \mathbb{E}\left[\left\langle B, \frac{Z_T}{Z_0^j}\right\rangle\right].$$

Now suppose that B is bounded, then (by definition of $\bar{p}_j(B)$ and Theorem 3.2.1) there exists $X \in \mathcal{C}$ such that $-e^j\bar{p}_j(B)+X \succeq -B$. Thus for any $m \in \mathcal{D}$ we have $\bar{p}_j(B) \leq \frac{m(B)}{m_j(\Omega)}$, implying $\bar{p}_j(B) \leq \underline{m}_j(B)$.
It is natural to ask under which condition we also have $\bar{p}_j(B) \geq \underline{m}_j(B)$, that would imply $\bar{p}_j(B) = \underline{m}_j(B)$ for bounded B. By Remark 3.3.3 we know that $\mathcal{Z}^s \subseteq \mathcal{D}$, hence it would be easy to get the desired inequality if

$$\underline{m}_j(B) = \inf_{m\in\mathcal{D}\cap\mathrm{dom}(\mathbb{U}_0^*)} \frac{m(B)}{m_j(\Omega)} = \inf_{m\in\mathcal{D}} \frac{m(B)}{m_j(\Omega)}.$$

This condition on B (which looks hard to verify in practice) is, for example, automatically satisfied if the utility function U is bounded from above (which implies that $0 \in \mathrm{dom}(\mathbb{U}_0^*)$, hence $\mathrm{dom}(\mathbb{U}_0^*) = \mathrm{ba}(\mathbb{R}_+^D)$ by Lemma 3.2.1 and Proposition 3.2.1).
Therefore if B is in $L^\infty(\mathbb{R}_+^D)$ and U is bounded from above, point (iii) of Proposition 3.4.2 tells us that the average price (in terms of any of the first d assets) a risk averse agent is ready to pay to buy more and more units of a contingent claim and get always the same utility approaches a price that allows him to trade as to bear zero risk at maturity. If we only have boundedness of B, which is the case for most common claims like call and put options (recall that we are working with units and not with prices) then, in general, the agent will keep some risk also in the limiting case.

Remark 3.4.1. The definition of UIP can be further generalized to account for the case where we seek a "price" in terms of more than one asset. Let $n \leq d$ and denote $\bar{p} := (p, 0) \in \mathbb{R}^D$ where $p \in \mathbb{R}^n$ and 0 is now the zero vector in $\mathbb{R}^{D-n}$. One can define $p(B) \in \mathbb{R}^n$, the UIP for B expressed in terms of the first n assets, as a solution to $u_{\mathcal{E}+B-\bar{p}} = u_{\mathcal{E}}$, with $\mathcal{E} \in \mathcal{O}$ and $\mathcal{E}+B-\bar{p} \in \mathcal{O}$. The subspace of $\mathbb{R}^n$ of the solutions to the previous inequality is closed if we only consider endowments in $L^\infty(\mathbb{R}^D)$ (by Lemma 3.4.1(iv)). A more thorough treatment of such vector UIP's is postponed to future research.

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Chapter 4

Markets with transaction costs II: Existence of shadow prices

Abstract: For utility maximization problems under proportional transaction costs, it has been observed that the original market with transaction costs can sometimes be replaced by a frictionless *shadow market* that yields the same optimal strategy and utility. However, the question of whether or not this indeed holds in generality has remained elusive so far. In this paper we present a counterexample which shows that shadow prices may fail to exist. On the other hand, we prove that short selling constraints are a sufficient condition to warrant their existence, even in very general multi-currency market models with possibly discontinuous bid-ask-spreads.

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4.1 Introduction

Transaction costs have a severe impact on portfolio choice: If securities have to be bought for an ask price which is higher than the bid price one receives for selling them, then investors are forced to trade off the gains and costs of rebalancing. * Consequently, utility maximization under transaction costs has been intensely studied in the literature. We refer the reader to [CO11] for general existence and duality results, as well as a survey of the related literature.

It has been observed that the original market with transaction costs can sometimes be replaced by a fictitious frictionless "shadow market", that yields the same optimal strategy and utility. However – unlike in the contexts of local risk minimization ([LPS98]), no-arbitrage '[GRS08], [JK95], [Sc04]), and superhedging [CPT99] (also cf. [KS09] for an overview) — the question of whether or not such a least favorable frictionless market extension indeed exists has only been resolved under rather restrictive assumptions so far.

More specifically, Cvitanić and Karatzas [CK96] answer it in the affirmative in an Itô process setting, however, only under the assumption that the minimizer in a suitable dual problem exists and is a martingale. † Yet, subsequent work by Cvitanić and Wang [CW01] only guarantees existence of the minimizer in a class of supermartingales. Hence, this result is hard to apply unless one can solve the dual problem explicitly.

A different approach leading closer to an existence result is provided by Loewenstein [Lo00]. Here, the existence of shadow prices is established for continuous bid-ask prices in the presence of short selling constraints. In contrast to [CK96], Loewenstein [Lo00] constructs his shadow market directly from the primal rather than the dual optimizer. However, his analysis is also based on the assumption that the starting point for his analysis, in this case his constrained primal optimizer, actually does exist.

Finally, Kallsen and Muhle-Karbe [KM11] show that shadow prices always exist for utility maximization problems in finite probability spaces. But, as usual in Mathematical Finance, it is a delicate question to what degree this transfers to more general settings.

The present study contributes to this line of research in two ways. On the one hand, we present a counterexample showing that shadow prices do not exist in general without further assumptions. On the other hand, we establish that Loewenstein's approach can be used to come up with a positive result, even in Kabanov's [Ka99] general multi-currency market models with possibly discontinuous bid-ask-spreads. The crucial assumption – which is violated in our counterexample – is the prohibition of short sales for all assets under consideration. Like Loewenstein [Lo00], we construct our shadow price from the primal optimizer. Existence of the latter is established by extending the argument of Guasoni [Gu02] to the general setting considered here. This is done by making use of a compactness result for predictable finite variation processes established in Campi and Schachermayer [CS06].

The paper is organized as follows. In Section 4.2, we present our counterexample, which is formulated in simple setting with one riskless and one risky asset for better readability.

∗. For example, Liu and Loewenstein [LL02] reckon that "even small transaction costs lead to dramatic changes in the optimal behavior for an investor: from continuous trading to virtually buy-and-hold strategies."

†. That is, a *consistent price system* in the terminology of [Sc04].

Afterwards, the general multi-currency framework with transaction costs and short selling constraints is introduced. In this setting, we then show that shadow prices always exist. Finally, Section 4.5 concludes.

4.2 A counterexample

In this section, we show that even a simple discrete-time market can fail to exhibit a shadow price,[‡] if the unconstrained optimal strategy involves short selling. Consider a market with one riskless and one risky asset traded at the discrete times $t = 0, 1, 2$: The bid and ask prices of the riskless asset are equal to 1 and the bid-ask spread[§] $[\underline{S}, \overline{S}]$ of the risky asset is defined as follows. The bid prices are deterministically given by

$$\underline{S}_0 = 3, \quad \underline{S}_1 = 2, \quad \underline{S}_2 = 1,$$

while the ask prices satisfy $\overline{S}_0 = 3$ and

$$\begin{aligned} \mathbb{P}(\overline{S}_1 = 2) &= 1 - 2^{-n}, \\ \mathbb{P}(\overline{S}_1 = 2 + k) &= 2^{-n-k}, \quad k = 1, 2, 3, \ldots, \end{aligned}$$

where $n \in \mathbb{N}$ is chosen big enough for the subsequent argument to work. Moreover, we set

$$\begin{aligned} \mathbb{P}(\overline{S}_2 = 3 + k | \overline{S}_1 = 2 + k) &= 2^{-n-k}, \\ \mathbb{P}(\overline{S}_2 = 1 | \overline{S}_1 = 2 + k) &= 1 - 2^{-n-k} \end{aligned}$$

for $k = 0, 1, 2, \ldots$ The corresponding bid-ask process is illustrated in Figure 4.1. One readily verifies that a strictly consistent price system exists in this market. Now, consider the maximization of expected logarithmic utility from terminal wealth, where the maximization takes place over all self-financing portfolios that liquidate at $t = 2$.

It is not hard to determine the optimal trading strategy in this setup. Buying a positive amount of stock at time 0 is suboptimal because expected gains would be negative. Consequently, zero holdings are preferable to positive ones. A negative position in the first period, on the other hand, is impossible as well because it may lead to negative terminal wealth. Hence it is optimal to do nothing at time 0, i.e., the optimal strategy $\hat{V}$ satisfies $\hat{V}_0^2 = 0$. In the second period, a positive stock holding would be again suboptimal because prices are still falling on average. By contrast, building up a negative position is worthwhile. The stock can be sold short at time 1 for $\underline{S}_1 = 2$ and it can be bought back at time 2 for $\overline{S}_2 = 1$ with overwhelming probability and for $\overline{S}_2 = 3$ resp. $3 + k$ with very small probability. Consequently, the optimal strategy satisfies $\hat{V}_1^2 < 0$ in any state.

If a shadow price process $(1, \widetilde{S})$ for this market exists, then $\widetilde{S}$ must coincide with the bid resp. ask price if a transaction takes place in the optimal strategy $\hat{V}$. Otherwise, one could achieve strictly higher utility trading $\tilde{S}$, by performing the same purchases and sales at sometimes strictly more favorable prices. Consequently, we must have $\widetilde{S}_0 = \overline{S}_0 = 3$, $\widetilde{S}_1 = \underline{S}_1 = 2$, $\widetilde{S}_2 = \overline{S}_2$. However, $(1, \widetilde{S})$ cannot be a shadow price process. Indeed, $\widetilde{S}$ is decreasing deterministically by 1 in the first period and would allow for unbounded expected utility and in fact even for arbitrage.

‡. Cf. Definition 4.3.4 below for the formal definition

§. In the general multicurrency notation introduced below, this corresponds to $[1/\pi^{21}, \pi^{12}]$, where π^{ij} denotes the number of units of asset i for which the agent can buy one unit of asset j.

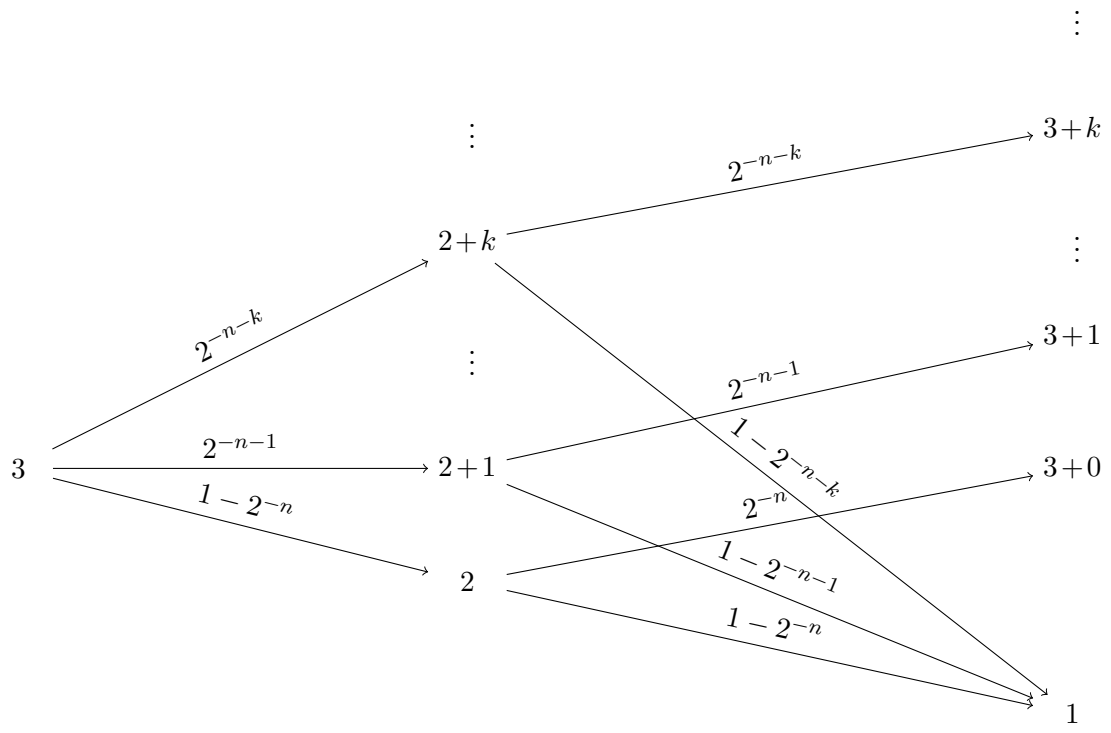

Figure 4.1: Illustration of the ask price in the counterexample. The corresponding bid price decreases deterministically from 3 to 2 to 1.

Remark 4.2.1. It is important to note that the market discussed in this section *does* admit a shadow price if one imposes short selling constraints. In this case, it is evidently optimal not to trade at all in the original market with transaction costs: Positive positions are not worthwhile because prices are always falling on average, whereas negative positions are ruled out by the constraints. In this market *any* supermartingale $(1, \widetilde{S})$ with values in the bid-ask spread, i.e., $\widetilde{S}_0 = 3$, $2 \leq \widetilde{S}_1 \leq \overline{S}_1$ and $\widetilde{S}_2 = 1$, is a shadow price (showing that even if a shadow price exists, it need not be unique). Indeed, Jensen's inequality yields that positive positions are suboptimal, and negative holdings are again prohibited by the constraints. Hence it is optimal not to trade at all, as in the original market with transaction costs. This confirms that $(1, \widetilde{S})$ is indeed a shadow price if short selling is ruled out. However, it is clearly not a shadow price in the unconstrained market, as it would allow for obvious arbitrage.

4.3 The General Multi-Currency Model

Henceforth, we work in the general transaction cost framework of Campi and Schachermayer [CS06], with slight modifications to incorporate short selling constraints. We describe here the main features of the model, but refer to the original paper for further details. For any vectors x, y in $\mathbb{R}^d$, we write $x \succeq y$ if $x - y \in \mathbb{R}^d_+$ and xy for the Euclidean scalar product.

Let $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions and supporting all processes appearing in this paper; the initial σ-field is assumed to be trivial.

We consider an agent who can trade in d assets according to some *bid-ask matrix* $\Pi = (\pi^{ij})_{1\leq i,j\leq d}$, where π^{ij} denotes the number of units of asset i for which the agent can buy one unit of asset j. To recapture the notion of currency exchanges, one naturally imposes as in [Sc04] that:

(i) $\pi^{ij} \in (0, \infty)$ for every $1 \leq i, j \leq d$;

(ii) $\pi^{ii} = 1$ for every $1 \leq i \leq d$;

(iii) $\pi^{ij} \leq \pi^{ik}\pi^{kj}$ for every $1 \leq i, j, k \leq d$.

The first condition means that the bid-ask prices of each asset in terms of the others are positive and finite, while the interpretation of the second is evident. The third implies that direct exchanges are not dominated by several successive trades. In the spirit of [Ka99], the entries of the bid-ask matrix can also be interpreted in terms of the prices $S^1, \ldots, S^d$ of the assets and proportional transaction costs λ^{ij} for exchanging asset i into asset j, by setting

$$\pi^{ij} = (1 + \lambda^{ij})\frac{S^j}{S^i}.$$

We will use both notations in the sequel, one being shorter and the other providing a better financial intuition. Given a bid-ask matrix Π, the *solvency cone* $K(\Pi)$ is defined as the convex polyhedral cone in $\mathbb{R}^d$ spanned by the canonical basis vectors e_i, $1 \leq i \leq d$, of $\mathbb{R}^d$, and the vectors $\pi^{ij}e_i - e_j$, $1 \leq i, j \leq d$.¶ The convex cone $-K(\Pi)$ should be

¶. $K(\Pi)$ contains precisely the *solvent* portfolios that can be liquidated to zero by trading according to the bid-ask matrix Π and possibly throwing away positive asset holdings.

interpreted as those portfolios available at price zero in a market without short selling constraints. Given a cone K, its (positive) polar cone is defined by

$$K^* = \left\{ w \in \mathbb{R}^d \,:\, vw \geq 0, \forall v \in K \right\}.$$

We now introduce randomness and time in the model. An adapted, càdlàg process $(\Pi_t)_{t\in[0,T]}$ taking values in the set of bid-ask matrices will be called a *bid-ask process*. Once a bid-ask process $(\Pi_t)_{t\in[0,T]}$ has been fixed, one can drop it from the notation by writing K_τ instead of $K(\Pi_\tau)$ for any stopping time τ, coherently with the framework introduced above. In accordance with the framework developed in [CS06] we make the following technical assumption throughout the paper. It means basically that no price changes take place at time T, which serves only as a date for liquidating the portfolio. This assumption can be relaxed via a slight modification of the model (see [CS06, Remark 4.2]). For this reason, we shall not explicitly mention it in the following.

Assumption 4.3.1. $\mathcal{F}_{T-} = \mathcal{F}_T$ and $\Pi_{T-} = \Pi_T$ a.s.

In markets with transaction costs, consistent price systems play a role similar to martingale measures for the frictionless case (compare, e.g., [Sc04, GRS08, KS09]). With shortselling constraints, this notion has to be extended, just as it is necessary to pass from martingale measures to supermartingale densities in the frictionless case:

Definition 4.3.1. An adapted, $\mathbb{R}^d_+ \setminus \{0\}$-valued, càdlàg supermartingale $Z = (Z_t)_{t\in[0,T]}$ is called a *supermartingale consistent price system* (supermartingale-CPS) if $Z_t \in K_t^*$ a.s. for every $t \in [0,T]$. Moreover, Z is called a *supermartingale strictly consistent price system* (supermartingale-SCPS) if it satisfies the following additional condition: for every $[0,T]\cup\{\infty\}$-valued stopping time τ, we have $Z_\tau \in \operatorname{int}(K^*_\tau)$ a.s. on $\{\tau < \infty\}$, and for every predictable $[0,T]\cup\{\infty\}$-valued stopping time σ, we have $Z_{\sigma-} \in \operatorname{int}(K^*_{\sigma-})$ a.s. on $\{\sigma < \infty\}$. The set of all supermartingale-(S)CPS is denoted by $\mathcal{Z}_{\text{sup}}$ (resp. $\mathcal{Z}^s_{\text{sup}}$).

As in [CS06], trading strategies are described by the numbers of physical units of each asset held at time t:

Definition 4.3.2. An $\mathbb{R}^d$-valued process $V = (V_t)_{t\in[0,T]}$ is called a *self-financing portfolio process* for the process K of solvency cones if it satisfies the following properties:

1. It is predictable and a.e. (not necessarily right-continuous) path has finite variation.
2. For every pair of stopping times $0 \leq \sigma \leq \tau \leq T$, we have

$$V_\tau - V_\sigma \in -K_{\sigma,\tau},$$

where $K_{s,t}(\omega)$ denotes the closure of $\operatorname{cone}\{K_r(\omega), s \leq r < t\}$.

A self-financing portfolio process V is called *admissible* if it satisfies the *no short selling constraint* $V \succeq 0$.

We need some more notation related to such processes. For any predictable process of finite variation V, we can define its continuous part V^c and its left (resp. right) jump process $\Delta V_t := V_t - V_{t-}$ (resp. $\Delta_+ V_t := V_{t+} - V_t$), so that $V = V^c + \Delta V + \Delta_+ V$. The continuous part, V^c, is itself of finite variation, so we can define its Radon-Nykodim

derivative $\dot{V}^c_t$ with respect to its total variation process $\mathrm{Var}_t(V^c)$, for all $t \in [0,T]$ (see [CS06, Section 2] for details).

We will work under the following assumption, which is the analogue of the existence of a supermartingale density in frictionless markets.

Assumption 4.3.2. $\mathcal{Z}^s_{\sup} \neq \emptyset$.

Remark 4.3.1. It is interesting to investigate the effects of Assumption 4.3.2 on the arbitrage opportunities in the market. We immediately notice that $\mathcal{A}^{0,ss}$ only contains processes identically equal to zero, since any trade would violate the short selling constraints. It follows that no positive wealth can be attained starting from zero initial endowment, thus ruling out arbitrage in the classical sense. It might however still be possible to start out with some wealth x and trade to get to $V_T \succeq x$ a.s. with strict inequality in some components. For example, if ask prices in the counterexample were identical to bid prices (i.e. strictly decreasing) it would easily be possible to start out with $(1,1)$ and get to a final wealth of $(2,2)$. The impossibility of selling short would in any case prevent to make arbitrary high profits from such opportunities: in fact, by the supermartingale property that will be proved in Lemma 4.4.1 we know that $E[Z_T V_T] \leq Z_0 x$ for any supermartingale CPS, which implies that V_T must be finite a.s. Had we assumed instead the existence of a true martingale CPS, those opportunities would also disappear: in fact $V_T \succeq x$ a.s. with strict inequality on some set of positive probability would imply $E[(V_T - x)Z_T] > 0$, which would contradict the supermartingale property (remark that this also holds more generally for the preorder $\succeq_{K_T}$ generated by K_T)

We now turn to the utility maximization problem. Here, we restrict our attention without loss of generality to admissible and self-financing portfolio processes that start out with some initial endowment $x \in \mathbb{R}^d_+ \setminus \{0\}$, and such that V_T is nonzero only in the first component (that is, the agent liquidates his wealth to the first asset at the final date). The set of those processes is denoted by $\mathcal{A}^{x,ss}$, and the set

$$\mathcal{A}^{x,ss}_T := \left\{ V^1_T \,:\, V \in \mathcal{A}^{x,ss} \right\}$$

consists of all terminal payoffs (in the first asset) attainable at time T from initial endowment x. Moreover, the set

$$\mathcal{A}^{x,ss}_{T-} := \{ V_{T-} \,:\, V \in \mathcal{A}^{x,ss} \}$$

contains the pre-liquidation values of admissible strategies.

The utility maximization problem considered in this paper is the following:

$$J(x) := \sup_{f \in \mathcal{A}^{x,ss}_T} \mathbb{E}[U(f)]. \tag{4.3.1}$$

Here, $U : (0,\infty) \to \mathbb{R}$ is a utility function in the usual sense, i.e., a strictly concave, increasing, differentiable function satisfying

1. the *Inada conditions* $\lim_{x \downarrow 0} U'(x) = \infty$ and $\lim_{x \uparrow \infty} U'(x) = 0$, and
2. the condition of *reasonable asymptotic elasticity* (RAE): $\limsup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1$.

We write $U^*(y) = \sup_{x>0}[U(x) - xy]$, $y > 0$ for the conjugate function of U, and $I := (U')^{-1}$ for the inverse function of its derivative. To rule out degeneracies, we assume throughout that the maximal expected utility is finite:

Assumption 4.3.3. $J(x) = \sup_{f \in \mathcal{A}_T^{x,ss}} \mathbb{E}[U(f)] < \infty$.

A unique maximizer for the utility maximization problem (4.3.1) indeed exists: ‖

Proposition 4.3.1. *Fix an initial endowment $x \in \mathbb{R}^d_+ \setminus \{0\}$. Under Assumptions 4.3.2 and 4.3.3, the utility maximization problem* (4.3.1) *admits a unique solution $\hat{f} \in \mathcal{A}_T^{x,ss}$.*

Proof. Step 1: The compactness result for predictable finite variation processes established in [CS06, Proposition 3.4] also holds in our setting where, in particular, the existence of a strictly consistent price system is replaced by the weaker Assumption 4.3.2. To see this, it suffices to show that the estimate in [CS06, Lemma 3.2] is satisfied under Assumption 4.3.2; then, the proof of [CS06, Proposition 3.4] can be carried through unchanged. To this end, we can use the arguments in [CS06, Section 3] with the following minor changes:

1. First, notice that in the proof of [CS06, Lemma 3.2] the martingale property of Z is only used to infer that this strictly positive process satisfies $\inf_{t\in[0,T]} \|Z_t\|_d > 0$ a.s., which is also true for supermartingales by [DM82, VI, Theorem 17].
2. [CS06, Lemma 3.2] is formulated for strategies starting from a zero initial position, but can evidently be generalized to strategies starting from any initial value x. Moreover, in our case the admissible strategies are all bounded from below by zero due to the short selling constraints. Consequently, [CS06, Lemma 3.2] holds under the weaker Assumption 4.3.2 for $V \in \mathcal{A}^{x,ss}$ and, in our case, [CS06, Equation (3.5)] reads as $\mathbb{E}_Q[\mathrm{Var}_T(V)] \leq C\|x\|$ for a constant $C \geq 0$ and an equivalent probability Q.

Step 2: Pick a maximizing sequence $(V^n)_{n\geq 1} \in \mathcal{A}^{x,ss}$ for (4.3.1) such that $\mathbb{E}[U(V_T^n)] \to J(x)$ as $n \to \infty$. By Step 1, we can assume (up to a sequence of convex combinations) that $V_T^n \to V_T^0$ a.s. for some $V^0 \in \mathcal{A}^{x,ss}$. The rest of the proof now proceeds as in [Gu02, Theorem 5.2]: by means of RAE assumption, we can prove that $\lim_{n\to\infty} \mathbb{E}[U(V_T^n)] \leq \mathbb{E}[U(V_T^0)]$, implying that V^0 is the optimal solution to (4.3.1). Uniqueness follows from the strict concavity of U. □

Consider now a supermartingale-CPS $Z \in \mathcal{Z}_{\sup}$. By definition, Z lies in the polar K^* of the solvency cone (we omit dependence on time for clarity); since $Z \neq 0$ this implies in particular that all components of Z are strictly positive. Moreover, taking any asset, say the first one, as a numeraire, it means that

$$\frac{1}{1+\lambda^{i1}} \frac{S^i}{S^1} \leq \frac{Z^i}{Z^1} \leq (1+\lambda^{1i}) \frac{S^i}{S^1}, \tag{4.3.2}$$

for any $i = 1, \ldots, d$. In other words, the frictionless price process

$$S^Z := Z/Z^1$$

‖. In the absence of constraints, similar existence results have been established for increasingly general models of the bid-ask spread by [CK96, DPT01, Bo02, Gu02, CO11].

evolves within the corresponding bid-ask spread. This implies that the terms of trade in this frictionless economy are at least as favorable for the investor as in the original market with transaction costs. For S^Z, we use the standard notion of a self-financing strategy: **

Definition 4.3.3. Let $S^Z := Z/Z^1$ for some $Z \in \mathcal{Z}_{\text{sup}}$. Then, a predictable, $\mathbb{R}^d$-valued, S^Z-integrable process $V = (V_t)_{t\in[0,T]}$ is called a *self-financing portfolio process* in the frictionless market with price process S^Z, if it satisfies

$$V_t S_t^Z = xS_0^Z + \int_0^t V_u dS_u^Z, \quad t \in [0,T]. \tag{4.3.3}$$

It is called *admissible* if it additionally satisfies the no short selling constraint $V \succeq 0$. We sometimes write Z-admissible to stress the dependence on a specific $Z \in \mathcal{Z}_{\text{sup}}$.

The set $\mathcal{A}_T^{x,ss}(Z)$ consists of all payoffs $V_T S_T^Z$ that are attained by a Z-admissible strategy V starting from initial endowment $x \in \mathbb{R}^d_+ \setminus \{0\}$.

This notion is indeed compatible with Definition 4.3.2, in the sense that any payoff in the original market with transaction costs can be dominated in the potentially more favorable frictionless markets evolving within the bid-ask spread:

Lemma 4.3.1. *Fix $Z \in \mathcal{Z}_{\text{sup}}$. For any admissible strategy V in the sense of Definition 4.3.2 there is a strategy $\tilde{V} \succeq V$ which is Z-admissible in the sense of Definition 4.3.3.*

Proof. Let V be self-financing in the original market with transaction costs in the sense of Definition 4.3.2. Then, since V is of finite variation, applying the integration by parts formula as in the proof of [CS06, Lemma 2.8] yields

$$\begin{aligned} V_t S_t^Z &= xS_0^Z + \int_0^t V_u dS_u^Z + \int_0^t S_u^Z \dot{V}_u^c d\text{Var}_u(V^c) + \sum_{u\le t} S_{u-}^Z \Delta V_u + \sum_{u<t} S_u^Z \Delta_+ V_u \\ &\le xS_0^Z + \int_0^t V_u dS_u^Z \end{aligned} \tag{4.3.4}$$

because, since $Z \in K^*$ and $Z^1 > 0$ imply $S^Z \in K^*$, we can use [CS06, Lemma 2.8] to get

$$\int_0^t S_u^Z \dot{V}_u^c d\text{Var}_u(V^c) + \sum_{u\le t} S_{u-}^Z \Delta V_u + \sum_{u<t} S_u^Z \Delta_+ V_u \le 0.$$

Now, define the portfolio process $\tilde{V}$ by

$$\tilde{V}_t^1 := xS_0^Z + \int_0^t V_u dS_u^Z - \sum_{i=2}^d V_t^i S_t^{Z,i} = xS_0^Z + \int_0^{t-} V_u dS_u^Z - \sum_{i=2}^d V_t^i S_{t-}^{Z,i},$$

and $\tilde{V}_t^i := V_t^i$ for $i = 2, \dots, d$. Then, $\tilde{V}$ is self-financing in the sense of Definition 4.3.3 by construction. Moreover, again by definition and due to (4.3.4), we have

$$\tilde{V}_t S_t^Z = xS_0^Z + \int_0^t V_u dS_u^Z \ge V_t S_t^Z$$

and in turn $\tilde{V} \succeq V$. □

**. In particular, we do *not* restrict ourselves to finite variation strategies here.

In view of Lemma 4.3.1, the maximal expected utility in the frictionless market S^Z associated to *any* supermartingale-CPS $Z \in \mathcal{Z}_{\sup}$ is greater than or equal to its counterpart in the original market with transaction costs:

$$\sup_{f \in \mathcal{A}_T^{x,ss}} \mathbb{E}[U(f)] \leq \sup_{f \in \mathcal{A}_T^{x,ss}(Z)} \mathbb{E}[U(f)].$$

The natural question that arises here, and which we address in the sequel, is whether we can find some particularly unfavorable $Z \in \mathcal{Z}_{\sup}$ such that this inequality becomes an equality.

Definition 4.3.4. Fix an initial endowment $x \in \mathbb{R}^d_+ \setminus \{0\}$. The process $S^Z = Z/Z^1$ corresponding to some $Z \in \mathcal{Z}_{\sup}$ is called a *shadow price process*, if

$$\sup_{f \in \mathcal{A}_T^{x,ss}} \mathbb{E}[U(f)] = \sup_{f \in \mathcal{A}_T^{x,ss}(Z)} \mathbb{E}[U(f)].$$

Some remarks are in order here.

1. Even if a shadow price exists it need not be unique, cf. Remark 4.2.1.
2. By Lemma 4.3.1, any payoff that can be attained in the original market with transaction costs can be dominated in frictionless markets with prices evolving within the bid-ask spread. Hence, strict concavity implies that the optimal payoff $\hat{f}$ must be the same for a shadow price as in the transaction cost market. In order not to yield a strictly higher utility in the shadow market, the optimal strategy $\hat{V}$ that attains $\hat{f}$ with transaction costs must therefore also do so in the shadow market (cf. Remark 4.4.1). Put differently, a shadow price must match the bid resp. ask prices whenever the optimal strategy $\hat{V}$ entails purchases resp. sales.

4.4 Existence of Shadow prices under short selling constraints

In this section, we prove that a shadow price always exists if short selling is prohibited (cf. Corollary 4.4.1). To this end, we first derive some sufficient conditions, and then verify that these indeed hold. Throughout, we assume that Assumptions 4.3.2 and 4.3.3 are satisfied.

The following result crucially hinges on the presence of short selling constraints.

Lemma 4.4.1. *For any supermartingale-CPS $Z \in \mathcal{Z}_{\sup}$ the following holds:*

1. *The process ZV is a supermartingale for any portfolio process V admissible in the sense of Definition 4.3.2.*
2. *The process $ZV = Z^1 V S^Z$ is a supermartingale for any portfolio process V which is Z-admissible in the sense of Definition 4.3.3.*

Proof. (i) Integration by parts gives

$$Z_t V_t - Z_0^1 x = \int_0^t V_u dZ_u + \int_0^t Z_u \dot{V}_u^c d\mathrm{Var}_u(V^c) + \sum_{u \leq t} Z_{u-} \Delta V_u + \sum_{u < t} Z_u \Delta_+ V_u. \qquad (4.4.1)$$

The first integral is a local supermartingale as V is positive, while the other terms are decreasing processes by [CS06, Lemma 2.8]. This implies that ZV is a positive local supermartingale and thus a true supermartingale.

(ii) Let V be any Z-admissible portfolio process. By [GK00, Proposition 2.1], the frictionless self-financing condition (4.3.3) is equivalent to the same condition in undiscounted terms, i.e, $d(Z_tV_t) = V_t dZ_t$. Since ZV is positive, it is therefore a positive local supermartingale and hence a supermartingale. □

The next result presents sufficient conditions for the existence of a shadow price:

Proposition 4.4.1. *Let $x \in \mathbb{R}^d_+ \setminus \{0\}$. Suppose there are a supermartingale-CPS Z and an a.s. strictly positive $\mathcal{F}_T$-measurable random variable $\hat{f} \in L^0$ satisfying:*

1. $\hat{f} \in \mathcal{A}_T^{x,ss}$;
2. $Z_T^1 = U'(\hat{f})$;
3. $S_T^{Z,i} = Z_T^i/Z_T^1 = 1/\pi_T^{i1}$ *for* $i = 1, \dots, d$;
4. $\mathbb{E}[Z_T^1 \hat{f}] = Z_0 x$.

Then, $\hat{f}$ is the optimal payoff both for the frictionless price process S^Z and in the original market with transaction costs. Consequently, S^Z is a shadow price.

Proof. We first prove that $\hat{f}$ is the optimal solution to the utility maximization problem (4.3.1) under transaction costs and short selling constraints. By (i), the payoff $\hat{f}$ is attained by some $\hat{V} \in \mathcal{A}^{x,ss}$ after liquidation of $\hat{V}_{T-}$ at time T. Now, take any $X \in \mathcal{A}_{T-}^{x,ss}$, whose liquidation value to the first asset is $f = X/\pi_{T-}^{\cdot 1} = X/\pi_T^{\cdot 1} = XZ_T/Z_T^1$ by (iii) and Assumption 4.3.1. Here $1/\pi^{\cdot 1}$ is the vector whose components are given by $1/\pi^{i1}$ for $i = 1, \dots, d$. Then, in view of (iv), we have

$$\mathbb{E}[Z_T^1 \hat{f}] = Z_0 x \geq \mathbb{E}[Z_T X] = \mathbb{E}[Z_T^1 f],$$

where the inequality is just the supermartingale property of ZV for an admissible V leading to X at time T. Concavity of U together with Property (ii) in turn gives

$$\mathbb{E}[U(\hat{f}) - U(f)] \geq \mathbb{E}[Z_T^1(\hat{f} - f)] \geq 0,$$

implying that $\hat{f}$ is the optimal solution to the transaction cost problem (4.3.1).

Next, we prove that $\hat{f}$ is also optimal in the frictionless market with price process S^Z. To this end, take a strategy V which is Z-admissible in the sense of Definition 4.3.3 with $V_0 = x$, so that the portfolio value at time t is given by $W_t := V_t S_t^Z$. The process $Z_t^1 W_t = Z_t V_t$ is a supermartingale by Lemma 4.4.1(ii), hence we have $\mathbb{E}[Z_T^1 W] \leq Z_0 x$ for any $W \in \mathcal{A}_T^{x,ss}(Z)$. Now note that the optimization problem in this frictionless market,

$$\sup_{W \in \mathcal{A}_T^{x,ss}(Z)} \mathbb{E}[U(W)],$$

is dominated by the static problem

$$\sup\{\mathbb{E}[U(W)] : W \in L^0(\mathbb{R}_+), \mathbb{E}[Z_T^1 W] \leq Z_0 x\},$$

which by monotonicity of U can be written as

$$\sup\{\mathbb{E}[U(W)] : W \in L^0(\mathbb{R}_+), \mathbb{E}[Z_T^1 W] = Z_0 x\}.$$

This problem admits a solution which is – recalling that $I = (U')^{-1}$ – given by

$$\hat{W} := I(Z_T^1) = \hat{f} = \hat{V}_{T-} Z_T / Z_1 = \hat{V}_{T-} S_T^Z.$$

Indeed, the definition of the conjugate function gives

$$\mathbb{E}[U(W)] \leq \mathbb{E}[U^*(Z_T^1)] + Z_0 x \tag{4.4.2}$$

for all random variables W satisfying $\mathbb{E}[Z_T^1 W] = Z_0 x$. The random variable $\hat{W} = I(Z_T^1)$ attains the supremum (pointwise) in the definition of $U^*(Z_T^1)$ and moreover, by assumption, $\mathbb{E}[Z_T^1 \hat{W}] = Z_0 x$. Hence, (4.4.2) becomes an equality and it follows that $\hat{W} = \hat{f}$ is also an optimal payoff in the frictionless market with price process S^Z. The latter therefore is a shadow price as claimed. □

Remark 4.4.1. By the previous discussion and Lemma 4.3.1 it follows that the optimal strategy $\hat{V}$ that attains $\hat{f}$ with transaction costs must also do so in the shadow market. Hence $\hat{V}$ is Z-admissible, i.e.

$$\hat{V}_t S_t^Z = x S_0^Z + \int_0^t \hat{V}_u dS_u^Z, \quad t \in [0, T].$$

However $\hat{V}$ is of finite variation, hence integration by parts gives

$$\hat{V}_t S_t^Z = x S_0^Z + \int_0^t \hat{V}_u dS_u^Z + \int_0^t S_u^Z \dot{\hat{V}}_u^c d\mathrm{Var}_u(\hat{V}^c) + \sum_{u \leq t} S_{u-}^Z \Delta \hat{V}_u + \sum_{u<t} S_u^Z \Delta_+ \hat{V}_u.$$

Combining the two yields

$$\int_0^t S_u^Z \dot{\hat{V}}_u^c d\mathrm{Var}_u(\hat{V}^c) + \sum_{u \leq t} S_{u-}^Z \Delta \hat{V}_u + \sum_{u<t} S_u^Z \Delta_+ \hat{V}_u = 0$$

a.s. for all $t \in [0, T]$. Since all the three factors are naturally negative and decreasing we must have

$$\begin{cases} S_t^Z \dot{\hat{V}}_t^c = 0, \quad d\mathbb{P}d\mathrm{Var}(\hat{V}^c) - a.e. \quad t \in [0, T]; \\ S_{t-}^Z \Delta \hat{V}_t = 0, \quad \text{for all } t \in [0, T] \ a.s.; \\ S_t^Z \Delta_+ \hat{V}_t = 0, \quad \text{for all } t \in [0, T) \ a.s. \end{cases} \tag{4.4.3}$$

By first focusing on the continuous part, the optimality of $\hat{V}$ implies that

$$\dot{\hat{V}}_t^c = \sum_{i \neq j} \alpha_t^{ij}(-\pi_t^{ij} e_i + e_j), \quad \alpha_t^{ij} \geq 0, \quad \alpha_t^{ij} \alpha_t^{ji} = 0.$$

Now since $S_t^Z \in K_t^*$ and $-\pi_t^{ij} e_i + e_j \in -K_t$ for all $i, j = 1, ..., d$ we have that $S_t^Z(-\pi_t^{ij} e_i + e_j) \leq 0$ and therefore by (4.4.3) $S_t^Z \alpha^{ij}(-\pi_t^{ij} e_i + e_j) = 0$ for all $i, j = 1, ..., d$.
As a consequence, when a transaction takes place at time t in the optimal strategy, say

from asset i to asset j (so that $\alpha_t^{ij} > 0$) we must have $\frac{S_t^{Z,j}}{S_t^{Z,i}} = \pi_t^{ij}$. In words, the shadow price must match the bid resp. ask price whenever the optimal strategy entails purchases resp. sales, as was already pointed out in the previous discussion.
The same argument can be repeated with minor modifications for the jump parts of the optimal strategy. In particular, when the optimal strategy trades at $t-$ we will have $\frac{S_{t-}^{Z,j}}{S_{t-}^{Z,i}} = \pi_{t-}^{ij}$.

With Theorem 4.4.1 at hand, we now show that shadow prices can indeed correspond to *strict* supermartingale consistent price systems.

Example 4.4.1. We first briefly recall the one period market of [CO11, Example 3.13] with two assets, i.e. $d = 2$. The price of the riskless asset is taken identically one and the price S of the risky asset satisfies $S_0 = 1$ and $S_1 = s_n$ with probability p_n ($n \in \mathbb{N}$), where $s_0 = 2$, $s_n = 1/n$ for $n \geq 1$, $p_0 = 1 - \alpha$ and $p_n = \alpha 2^{-n}$ with α sufficiently small. Frictions are modeled by the bid-ask matrices

$$\Pi_0 = \begin{pmatrix} 1 & S_0 \\ 2/S_0 & 1 \end{pmatrix} \text{ and } \Pi_1 = \begin{pmatrix} 1 & 2S_1 \\ 1/S_1 & 1 \end{pmatrix}.$$

The initial endowment is set at $(1, 0)$. The authors prove that a shadow price exists in this market and that it is equal to $(1, S)$. We will show how it is possible to construct it following the sufficient conditions given in Theorem 4.4.1. We already know that $\hat{f} = S_1$ (see the computation in [CO11, Example 3.13]), so that we must have $Z_1^1 = U'(\hat{f}) = \frac{1}{S_1}$, $Z_0^1 = 1$ and $Z_1^2 = Z_1^1/\pi_1^{21} = 1$. Finally Z_0^2 must be selected in such a way that $Z_0^2 \geq 1$ (supermartingale property) and $1/2 \leq Z_0^2 \leq 1$ ($Z \in K^*$), thus $Z_0^2 = 1$. The (unique) shadow price process is then $S^Z := (1, Z^2/Z^1) = (1, S)$, confirming the original result. It is important to notice that in this case the process Z^1 is a strict supermartingale, since $\mathbb{E}[Z_1^1] < 1 = Z_0^1$, which justifies the introduction of supermartingale CPSs.

Example 4.4.2. We now show that the market presented in Section 4.2 does admit a shadow price if we impose short selling constraints. Starting with $x = (\beta, 0)$ with $\beta > 0$, the optimal strategy in this case is to do nothing until the end (buying the risky asset is suboptimal since prices are falling on average, while short selling is forbidden), thus reaching a final utility of $\log \beta$. An application of Theorem 4.4.1 shows that any process $(1, S)$ such that S is a supermartingale and $S_0 = 3$, $S_1 \geq 2$ and $S_2 = 1$ is a shadow price process (one can take $Z^1 \equiv 1/\beta$ and $S = Z^2$ in the Theorem). To confirm that this is true, consider a frictionless market with price process given by $(1, S)$. At $t = 1$ it is clearly optimal to keep zero units of the risky asset, as prices are falling deterministically in the next period. At $t = 0$ one might be tempted to sell a fraction α of the first asset and buy some stocks, in order to take advantage of an unbounded potential profit by selling at $t = 1$. However by Jensen's inequality and the supermartingale property of S the expected utility of this strategy is

$$\begin{aligned} \mathbb{E}\left[\log\left(\frac{\alpha\beta}{3}S_1 + (1-\alpha)\beta\right)\right] &\leq \log\left(\frac{\alpha\beta}{3}\mathbb{E}[S_1] + (1-\alpha)\beta\right) \\ &\leq \log\left(\frac{\alpha\beta}{3}S_0 + (1-\alpha)\beta\right) = \log\beta, \end{aligned}$$

which implies that doing nothing is still optimal even on this frictionless market, confirming that $(1, S)$ is indeed a shadow price. Clearly, it is not a shadow price on the unconstrained market as it would allow for obvious arbitrage.

Using the sufficient conditions from Proposition 4.4.1, we now establish the existence of a shadow price in our multi-currency market model with transaction costs. We proceed similarly as in [Lo00], adapting the arguments to our more general setting, and using that we have shown existence of an optimal solution $\hat{f}$ to the utility maximization problem in Proposition 4.3.1 above. First, notice that the value function $J(x)$ is concave, increasing and finite for all x in the set $\mathbb{R}^d_+ \setminus \{0\}$. By [Ro70, Theorem 23.4] it is also superdifferentiable for all $x \in \mathbb{R}^d_+ \setminus \{0\}$.†† We recall that the superdifferential $\partial\varphi(x)$ of any concave function φ at some point x is defined as the set of all $y \in \mathbb{R}^d$ such that

$$\varphi(z) \leq \varphi(x) + y(z - x), \quad \text{for all } z \in \mathbb{R}^d.$$

Proposition 4.4.2. *Fix $x \in \mathbb{R}^d_+ \setminus \{0\}$, the associated optimal solution $\hat{f}$, and take $h = (h_1, \ldots, h_d)$ in the superdifferential $\partial J(x)$. Then, the following properties hold:*

1. $h_1 \geq \mathbb{E}[U'(\hat{f})]$;
2. $h_i \geq \mathbb{E}\left[U'(\hat{f})/\pi_T^{1i}\right]$ *for* $i = 2, \ldots, d$;
3. $h \in K_0^*$;
4. $hx = \mathbb{E}[U'(\hat{f})\hat{f}]$.

In particular, the optimal payoff $\hat{f}$ is a.s. strictly positive.

Proof. For $\epsilon > 0$ we have $J(x + e_1\epsilon) \geq \mathbb{E}[U(\hat{f} + \epsilon)]$ because one can just hold the extra endowment in asset 1. Hence, the definition of the superdifferential gives

$$h_1 \geq \frac{J(x + e_1\epsilon) - J(x)}{\epsilon} \geq \mathbb{E}\left[\frac{U(\hat{f} + \epsilon) - U(\hat{f})}{\epsilon}\right].$$

Since U is concave, the monotone convergence theorem yields $h_1 \geq \mathbb{E}[U'(\hat{f})]$. In view of the Inada condition $\lim_{x \downarrow 0} U'(x) = \infty$, this also shows that $\hat{f}$ is a.s. strictly positive.

For $i = 2, \ldots, d$, we have $J(x + e_i\epsilon) \geq \mathbb{E}[U(\hat{f} + \epsilon/\pi_T^{1i})]$ because one can hold the extra endowment in asset i and then liquidate it to asset 1 at time T. Hence, as before, we find $h_i \geq \mathbb{E}[U'(\hat{f})/\pi_T^{1i}]$.

Now notice that, for any $i, j = 1, \ldots, d$, one can exchange π_0^{ij} units of asset i for 1 unit of asset j at time zero. Hence, $J(x + e_i\epsilon) \geq J(x + e_j\epsilon/\pi_0^{ij})$, and the definition of the superdifferential yields

$$0 \leq J(x + e_i\epsilon) - J\left(x + e_j\epsilon/\pi_0^{ij}\right) \leq \epsilon\left(h_i - h_j/\pi_0^{ij}\right).$$

Together with $h_i \geq 0$, $i = 1, \ldots, d$, we obtain $h \in K_0^*$.

Finally,

$$hx(\lambda - 1) \geq J(\lambda x) - J(x) \geq \mathbb{E}[U(\lambda\hat{f}) - U(\hat{f})]$$

††. In fact, J can be seen as the restriction to $\mathbb{R}^d_+ \setminus \{0\}$ of some other concave function defined on K_0 that allows for negative initial endowment (but forces the agent to make an instantaneous trade at time 0 in that case).

because $\mathcal{A}_T^{\lambda x,ss} = \lambda \mathcal{A}_T^{x,ss}$. Hence if $\lambda > 1$, then

$$hx \geq \mathbb{E}\left[\frac{U(\lambda \hat{f}) - U(\hat{f})}{\lambda - 1}\right],$$

and the argument of the expectation increases as $\lambda \downarrow 1$ by concavity of U. Analogously, for $\lambda < 1$, the inequality is reversed and the argument of the expectation decreases as $\lambda \uparrow 1$. Hence, monotone and dominated convergence yield $hx = \mathbb{E}[U'(\hat{f})\hat{f}]$ as claimed. □

For any admissible portfolio process V, now define the conditional value process

$$J(V,t) := \operatorname{ess\,sup}_{f \in \mathcal{A}_{t,T}^{V,ss}} \mathbb{E}[U(f) \mid \mathcal{F}_t], \tag{4.4.4}$$

where $\mathcal{A}_{t,T}^{V,ss}$ denotes the terminal values of admissible portfolio processes which agree with V on $[0,t]$. Let $\hat{V}$ be the portfolio process in $\mathcal{A}^{x,ss}$ leading to the optimal solution $\hat{f}$ to (4.3.1). We can apply [EK79, Théorème 1.17] to get the following martingale property for the optimal value process $J(\hat{V},t)$ over the whole time interval $[0,T]$:

Lemma 4.4.2 (Dynamic Programming Principle)**.** *The following equality holds a.s.:*

$$J(\hat{V},s) = \mathbb{E}[J(\hat{V},t) \mid \mathcal{F}_s], \quad 0 \leq s \leq t \leq T.$$

For $i = 1, \ldots, d$, now define a process $\tilde{Z}$ as follows:

$$\tilde{Z}_t^i := \lim_{\epsilon \downarrow 0} \frac{J(\hat{V} + e_i\epsilon, t) - J(\hat{V},t)}{\epsilon}, \quad t \in [0,T), \quad \tilde{Z}_T^i := \frac{U'(\hat{V}_T^1)}{\pi_T^{i1}}.$$

Proposition 4.4.3. *$\tilde{Z}$ is a (not necessarily càdlàg) supermartingale satisfying $\tilde{Z}_t \in K_t^*$ a.s. for all $t \in [0,T]$.*

Proof. We adapt the argument of [Lo00, Lemma 4]. Consider $\epsilon_1, \epsilon_2 > 0$ with $\epsilon_2 < \epsilon_1$. Using the concavity of the utility function U and the properties of the essential supremum yields

$$\begin{aligned} J(\hat{V} + e_i\epsilon_2, t) &= J\left(\frac{\epsilon_2}{\epsilon_1}(\hat{V} + e_i\epsilon_1) + \left(1 - \frac{\epsilon_2}{\epsilon_1}\right)\hat{V}, t\right) \\ &\geq \frac{\epsilon_2}{\epsilon_1} J\left(\hat{V} + e_i\epsilon_1, t\right) + \left(1 - \frac{\epsilon_2}{\epsilon_1}\right) J\left(\hat{V}, t\right). \end{aligned}$$

As a consequence, $\tilde{Z}_t^i$ is well-defined as the limit of an increasing sequence. For the remainder of the proof, we drop the superscript "ss" to ease notation. Since the family $\{\mathbb{E}[U(f) \mid \mathcal{F}_t] : f \in \mathcal{A}_{t,T}^{\hat{V}+\epsilon e_i}\}$ is directed upwards, the properties of the essential supremum (see, e.g., [**?**, Proposition VI-1-1]) allow to write it as a limit which is monotone increasing in n:

$$J(\hat{V} + e_i\epsilon, t) = \operatorname{ess\,sup}_{f \in \mathcal{A}_{t,T}^{\hat{V}+\epsilon e_i}} \mathbb{E}[U(f) \mid \mathcal{F}_t] = \lim_{n \to \infty} \mathbb{E}[U(f^n) \mid \mathcal{F}_t],$$

where $(f^n)_{n\geq 0}$ is a sequence of elements of $\mathcal{A}_{t,T}^{\hat{V}+e_i\epsilon}$. As $\mathcal{A}_{t,T}^{\hat{V}+e_i\epsilon} \subseteq \mathcal{A}_{s,T}^{\hat{V}+e_i\epsilon}$ for $0 \leq s \leq t < T$,

$$\begin{aligned} J(\hat{V}+e_i\epsilon,s) &= \operatorname{ess\,sup}_{f\in\mathcal{A}_{s,T}^{\hat{V}+e_i\epsilon}} \mathbb{E}[U(f) \mid \mathcal{F}_s] \\ &\geq \operatorname{ess\,sup}_{f\in\mathcal{A}_{t,T}^{\hat{V}+e_i\epsilon}} \mathbb{E}[U(f) \mid \mathcal{F}_s] \\ &\geq \mathbb{E}[U(f^n) \mid \mathcal{F}_s] = \mathbb{E}[\mathbb{E}[U(f^n) \mid \mathcal{F}_t] \mid \mathcal{F}_s] \end{aligned}$$

for all $n \geq 0$. But then monotone convergence gives

$$\begin{aligned} J(\hat{V}+e_i\epsilon,s) \geq \lim_{n\to\infty} \mathbb{E}[\mathbb{E}[U(f^n) \mid \mathcal{F}_t] \mid \mathcal{F}_s] &= \mathbb{E}[\lim_{n\to\infty} \mathbb{E}[U(f^n) \mid \mathcal{F}_t] \mid \mathcal{F}_s] \\ &= \mathbb{E}[J(\hat{V}+e_i\epsilon,t) \mid \mathcal{F}_s]. \end{aligned}$$

Now, Lemma 4.4.2 implies

$$\frac{J(\hat{V}+e_i\epsilon,s)-J(\hat{V},s)}{\epsilon} \geq \mathbb{E}\left[\frac{J(\hat{V}+e_i\epsilon,t)-J(\hat{V},t)}{\epsilon}\Big|\mathcal{F}_s\right]$$

and the supermartingale property of $\tilde{Z}$ on $[0,T)$ follows by monotone convergence for $\epsilon \downarrow 0$. In order to verify it for the terminal time T as well notice that, for $0 \leq s < T$,

$$J(\hat{V}+e_i\epsilon,t)-J(\hat{V},t) \geq \mathbb{E}\left[U(\hat{V}_T^1+\epsilon/\pi_T^{i1})-U(\hat{V}_T^1)|\mathcal{F}_t\right]$$

because it is admissible to hold the ϵ extra units of asset i before liquidating them into $\epsilon/\pi_{T-}^{i1} = \epsilon/\pi_T^{i1}$ units of asset 1, and $J(\hat{V},t) = \mathbb{E}[U(\hat{V}_T^1)|\mathcal{F}_t]$ by Lemma 4.4.2. Then, monotone convergence yields

$$\tilde{Z}_t^i \geq \mathbb{E}[U'(\hat{V}_T^1)/\pi_T^{i1}|\mathcal{F}_t] = \mathbb{E}[\tilde{Z}_T^i|\mathcal{F}_t], \quad i=1,\dots,d,$$

such that $\tilde{Z}$ is indeed a supermartingale on $[0,T]$. In particular, it is finite-valued.

It remains to show that $\tilde{Z}_t \in K_t^*$ for all $t \in [0,T]$. To this end first fix $t \in [0,T)$ and let $(k_l^n)_{l\geq 0}$ be a partition of $[0,\infty)$ with mesh size decreasing to zero as n increases. Note that, for all $\epsilon > 0$, on the set $\{k_l^n < \pi_t^{ij} \leq k_{l+1}^n\}$ we have

$$J(\hat{V}+e_i\epsilon,t)-J(\hat{V},t) \geq J\left(\hat{V}+e_j\epsilon/k_{l+1}^n,t\right)-J(\hat{V},t)$$

because it is admissible to exchange the ϵ extra units of asset i for at least ϵ/k_{l+1}^n units of asset j immediately after time t. Again using monotone convergence, this in turn implies

$$\tilde{Z}_t^i k_{l+1}^n \mathbb{1}_{\{k_l^n<\pi_t^{ij}\leq k_{l+1}^n\}} \geq \tilde{Z}_t^j \mathbb{1}_{\{k_l^n<\pi_t^{ij}\leq k_{l+1}^n\}},$$

and thus

$$\tilde{Z}_t^i \sum_{l\geq 0} k_{l+1}^n \mathbb{1}_{\{k_l^n<\pi_t^{ij}\leq k_{l+1}^n\}} \geq \tilde{Z}_t^j.$$

Then, letting $n \to \infty$ we obtain $\tilde{Z}_t^i \pi_t^{ij} \geq \tilde{Z}_t^j$ for all $i,j = 1,\dots,d$. Hence, $\tilde{Z}_t \in K_t^*$ for $t \in [0,T)$. For the terminal time T, this follows directly from the definition and property (iii) in the definition of a bid-ask matrix. □

The process $\tilde{Z}$ constructed above is a supermartingale but not necessarily càdlàg. Therefore, we pass to the regularized càdlàg process $\hat{Z}$ defined by $\hat{Z}_T = \tilde{Z}_T$ and

$$\hat{Z}_t^i := \lim_{s \downarrow t, s \in \mathbb{Q}} \tilde{Z}_s^i$$

for all $i = 1, \dots, d$ and $t \in [0, T)$. Note that the limit exists by [KS88, Proposition 1.3.14(i)].

We can now establish our main result, the existence of shadow prices under short selling constraints subject only to the existence of a supermartingale strictly consistent price system (Assumption 4.3.2) and finiteness of the maximal expected utility (Assumption 4.3.3).

Theorem 4.4.1. *The process $\hat{Z}$ belongs to $\mathcal{Z}_{\text{sup}}$. Moreover, it satisfies the sufficient conditions of Proposition 4.4.1. Consequently, $S^{\hat{Z}} = \hat{Z}/\hat{Z}^1$ is a shadow price process.*

Proof. By [KS88, Proposition 1.3.14(iii)], the process $\hat{Z}$ is a càdlàg supermartingale. Moreover, since the bid-ask matrix is right continuous, we have $\hat{Z} \in \mathcal{Z}_{\text{sup}}$. By definition, we have $\tilde{Z}_T^1 = U'(\hat{f})$ and $\tilde{Z}_T^i / \tilde{Z}_T^1 = 1/\pi_T^{i1}$ for $i = 1, \dots, d$. Since $\tilde{Z}$ and $\hat{Z}$ are equal in T, it therefore remains to verify condition (iv) in Proposition 4.4.1. By Proposition 4.4.2,

$$hx = \mathbb{E}[U'(\hat{f})\hat{f}] = \mathbb{E}[\hat{Z}_T^1 \hat{V}_T^1] = \mathbb{E}[\hat{Z}_T \hat{V}_T], \tag{4.4.5}$$

for the portfolio process $\hat{V}$ attaining $\hat{f}$. The definition of the superdifferential then gives

$$h_i \geq \frac{J(x + e_i\epsilon) - J(x)}{\epsilon}$$

for any $\epsilon > 0$. Hence, $h_i \geq \tilde{Z}_0^i \geq \tilde{Z}_{0+}^i = \hat{Z}_0^i$ for all $i = 1, \dots, d$ by [KS88, Proposition 1.3.14(ii)]. Combined with (4.4.5) and because x has positive components, we obtain

$$\mathbb{E}[\hat{Z}_T \hat{V}_T] = hx \geq \hat{Z}_0 x$$

Conversely, since $\hat{Z} \in \mathcal{Z}_{\text{sup}}$ we can apply the supermartingale property established in Lemma 4.4.1 which gives $\mathbb{E}[\hat{Z}_T \hat{V}_T] \leq \hat{Z}_0 x$ and hence $\mathbb{E}[\hat{Z}_T \hat{V}_T] = \hat{Z}_0 x$. Thus, the sufficient conditions in Proposition 4.4.1 are satisfied and the proof is completed. □

Corollary 4.4.1. *Under short selling constraints and subject to Assumptions 4.3.2 and 4.3.3, a shadow price in the sense of Definition 4.3.4 exists.*

4.5 Conclusion

We have shown that shadow prices always exist in the presence of short selling constraints, even in general multi-currency markets with random, time-varying, and possibly discontinuous bid-ask spreads. On the other hand, we have presented a counterexample showing that existence generally does not hold beyond finite probability spaces if short selling is permitted. Yet, in simple concrete models the presence of short selling does not preclude the existence of shadow prices, compare [GGMS11]. It is therefore an intriguing question for future research to identify additional assumptions on the market structure

that warrant their existence. We also conjecture that shadow prices should always exist for utilities defined on the whole real line, where there is no solvency constraint that can become binding as in our counterexample. Making this precise first requires settling the existence of a primal optimizer. (The argument of [CS06] breaks down if wealth processes are no longer guaranteed to be bounded from below.) Secondly, the present "primal" approach only seems to be able to produce a suitable candidate shadow price process in the special case of an exponential utility function, exploiting the cash-additivity of the latter. In general, a "dual" approach in the spirit of [CMS12] seems to be required. Dealing with both issues crucially hinges on resolving the ubiquitous issue of admissibility by using a class of strategies both large enough to contain the optimizer and small enough to ensure the supermartingale property of admissible wealth processes under all dual measures. Here, the approach of [BC11] appears to be promising.

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Chapter 5

Utility indifference valuation for non-smooth payoffs on a market with some non tradable assets

Abstract: We consider the problem of exponential utility indifference valuation under the simplified framework where traded and nontraded assets are uncorrelated but where the claim to be priced possibly depends on both. Traded asset prices follow a multivariate Black and Scholes model, while nontraded asset prices evolve as generalized Ornstein-Uhlenbeck processes. We provide a BSDE characterization of the utility indifference price (UIP) for a large class of non-smooth, possibly unbounded, payoffs depending simultaneously on both classes of assets. Focusing then on European claims and using the gaussian structure of the model allows us to employ some BSDE techniques (in particular, a Malliavin-type representation theorem due to [MZ02]) to prove the regularity of Z and to characterize the UIP for possibly discontinuous European payoffs as a viscosity solution of a suitable PDE with continuous space derivatives. The optimal hedging strategy is also identified essentially as the delta hedging corresponding to the UIP. Since there are no closed-form formulas in general, we also obtain asymptotic expansions for prices and hedging strategies when the risk aversion parameter is small. Finally, our results are applied to pricing and hedging power derivatives in various structural models for energy markets.

MS Classification: 49L25, 49N15, 60H30

5.1 Introduction

This paper deals with the pricing and hedging of derivatives in incomplete markets, where the source of incompleteness comes from the fact that some of the assets are assumed not to be traded. As it is well known, such a situation generally prevents from constructing a perfect hedge and therefore to obtain a unique price as a result of classical no-arbitrage arguments (at least when contingent claims also depend on non-traded assets). In the absence of a unique equivalent martingale measure, indeed, arbitrage theory only allows to identify intervals of viable prices, which makes it necessary to develop other criteria to actually choose a unique price. The easiest and most conservative choice would be (for the seller) to pick the super-replicating price, thus eliminating all the risks by transferring to the buyer the entire cost of the incompleteness. Unfortunately this procedure often gives rise to unreasonably high prices which do not usually match with real data, as it is quite unlikely that one counterpart will completely refuse to take any risk at all. For this reason, other paradigms have been introduced in the literature: one example is Local Risk Minimization (see [Sc01]) which does not aim at canceling the hedging risk but rather at minimizing it according to some suitable criterion. Another (partial) way out is the idea of introducing in the market some new assets which are correlated to the non-tradable ones and can therefore be exchanged in the hope of improving the quality of the hedge (see [Dav97]). Of course when dealing with the optimal balancing of risks, the standard mathematical way to tackle the problem is the introduction of utility functions, which allow to describe in an easy and concise fashion the amount of uncertainty that an agent is willing to bear. This is at the basis of the well established economic principle of the *certainty equivalent*, stating that the price of a claim should be the one that makes the agent indifferent between possessing the claim or its (certain) price. Such a method has the advantage of being both economically sound and mathematically and computationally simple, requiring at most the numerical evaluation of an equation. This procedure, however, does not seem so appropriate when at least some of the assets can be traded on a financial market: in fact, if the agent is in the position of performing some kind of partial hedging, this should be incorporated in the pricing paradigm, and investors can no longer be expected to passively require an equivalent compensation for claims without engaging in any trading activity. This idea is at the heart of the pricing method that we consider in this paper, i.e. *utility indifference pricing*, a subject that has attracted quite a lot of attention in recent years (see Henderson and Hobson's survey [HH09]), in particular as a consequence of the important developments in the theory of optimal investment.

In this article we consider a model for traded and nontraded assets, that are supposed to be uncorrelated. This type of model is usually called semi-complete product market model (as in, e.g., [Be03]). The prices of traded assets follow a complete multivariate Black-Scholes model, while the prices of non traded ones evolve as generalized Ornstein-Uhlenbeck processes. This is mainly motivated by the recent literature on structural models for electricity markets, which aim at describing electricity prices as a result of the interaction of some underlying structural factors that can be either exchanged on a financial markets (like fuels) or not (like demand and fuel capacities), and which are often supposed to have simple Gaussian dynamics.
In our framework the payoff is supposed to be a function of both traded and nontraded

assets, contrarily to most of the literature where the payoff depends only on the nontraded assets which are assumed to be correlated to the traded ones, so that one usually works directly with the correlation of the traded assets with the payoff to be hedged (see, for example, [He02], [Be06], [AID07], [FS08], [IRR12]). An exception is [SZ04], where the payoff considered depends on both types of assets in a bidimensional stochastic volatility framework where the payoff is assumed to be smooth and bounded. Relying on correlation can be advantageous in some situations but not, in general, in the context of structural models, where the expressions for correlations usually become quite complex even if the model is relatively simple. In these cases it is often more convenient avoid the computation of correlation, by leaving the payoff expressed as a function of both traded and nontraded assets (by eventually exploiting their particular structure, for example their independence or gaussian properties, to simplify the problem).
The typical tool that is used to analyse utility indifference prices is the theory of (quadratic) BSDEs, that was first introduced in a similar context by the seminal paper [ER00] and which is particularly convenient as it generalizes with no additional effort to a large class of (possibly non-Markovian) settings (for example [Be06], [AID07]). Classical results require, however, boundedness or at least exponential integrability of the claim and they are only capable to identify the optimal hedging strategy when the final claim is bounded. This is a serious drawback if we notice that common payoff functions in structural models for electricity prices are linear functions of geometric brownian motions (wich are neither bounded nor exponentially integrable).
The first contribution of this work is therefore to prove the existence of (exponential) utility indifference prices without requiring boundedness or exponential integrability for the payoff, but only using sub- and super-replicability instead. Nonetheless, the question remains of whether we can actually interpret the Z-part of the BSDE in terms of the optimal hedging strategy in this case, given in particular that we lack the BMO property that is generally used to verify this (see [HIM05]). With this motivation in mind, we proceeded to study the regularity and to get some estimates on Z, by using the stochastic control representation of the problem or some Malliavin-type formulas for BSDEs in the spirit of [Zh05] or [MZ02]. This is why in the second part of the paper we focus on European payoffs, by allowing them in particular to be possibly discontinuous, which is often the case in models aiming to describe regime-changing features. Given our simple gaussian modeling framework, considering European payoffs leads naturally to a link with PDEs: our second contribution, indeed, is to describe the price as the viscosity solution of a PDE and, most importantly, to prove that the solution is sufficiently regular to possess continuous first derivatives (in space), providing a useful representation for Z which allows to write the candidate optimal hedging strategy in a similar way as the usual delta hedge. This candidate strategy is then proved to be optimal under some growth assumptions on the payoff (which does not, however, need to be bounded). Since there is in general no hope to solve the PDE explicitly, we also provide asymptotic expansions for the price (adapting a result in [Mo12]) and (under some additional regularity) on the optimal hedging strategy. As already mentioned, we finally provide an application to the pricing of power derivatives under a structural modeling framework.

The paper is organized as follows. We introduce the model in Section 5.2, along with the definition of trading strategies and utility indifference prices, by also deriving some

bounds and pointing out the connection with the related concept of certainty equivalent. In Section 5.3 we use some results of the theory of optimal investment (due to [HIM05] and [OZ09]) in order to derive a BSDE representation of the price, without the assumption of boundedness or exponential integrability of the claim that are usually encountered in the literature on quadratic BSDEs (for example [Ko00] or [BH07]). In Section 5.4 we focus on the Markovian case and we express the price and the optimal hedge in terms of the viscosity solution of a certain PDE. Particular attention is devoted to the case of discontinuous payoffs, that we are able to treat by extending some of the techniques found in [Zh05]. Asymptotic expansions are also derived following essentially the lines of [Dav97] and [Mo12]. In Section 5.5 we finally present some applications to electricity markets and provide some numerical examples.

Some useful notation: Let $T > 0$ be a finite time horizon and let $(\Omega, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ be a filtered probability space satisfying the usual conditions. For any real numbers $p > 0$, we will denote $\mathbb{H}^p(\mathbb{R}^n)$ (resp. $\mathbb{H}^p_{\text{loc}}(\mathbb{R}^n)$) the set of all $\mathbb{F}$-predictable $\mathbb{R}^n$-valued processes $Z = (Z_t)_{0\leq t\leq T}$ such that $E[\int_0^T \|Z_t\|^p dt] < +\infty$ a.s. (resp. $\int_0^T \|Z_t\|^p dt < +\infty$). For a vector x, we denote x' its transpose and $\text{diag}(x)$ the diagonal matrix such that $\text{diag}(x)_{ii} = x_i$ for all i. For a matrix α, we denote $\alpha_{i\cdot}$, $\alpha_{\cdot j}$ its i'th row or j'th column and $\alpha^{-n} := (\alpha^{-1})^n$. For any positive integer $d \geq 1$, we denote 0_d the d-dimensional zero vector.

5.2 The model

We place ourselves on a filtered probability space $(\Omega, \mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}, P)$, where $\mathbb{F}$ is the natural filtration generated by the $(n + d)$-dimensional Brownian motion $W = (W^S, W^X)$ and satisfying the usual conditions of right-continuity and P-saturatedness. Throughout the paper we will use the notation y^S and y^X to distinguish the first n and last d components of a vector $y = (y^S, y^X)$ of size $n+d$. The distinction is useful, as we will see, to separate tradable and non tradable assets. Moreover, we will denote $\mathbb{F}^S = (\mathcal{F}^S_t)$ and $\mathbb{F}^X = (\mathcal{F}^X_t)$ the natural filtrations generated, respectively, by W^S and W^X. The notation E_t will denote conditional expectations under P and with respect to the σ-field $\mathcal{F}_t$.

Tradable assets. We consider a finite horizon multivariate Black and Scholes market model with n tradable risky assets with dynamics

$$\frac{dS^i_t}{S^i_t} = \mu_i dt + \sigma_{i\cdot} dW^S_t, \quad i = 1, \dots, n \tag{5.2.1}$$

where σ is a $n \times n$ invertible matrix and $\sigma_{i\cdot}$ denotes its i-th row. We assume for the sake of simplicity that the interest rate is zero.

Remark 5.2.1. The results of this paper can be easily extended to the case where the drift and the volatilities in the dynamics of the tradable assets S are bounded functions of these assets, i.e. of the form $\mu(S_t)$ and $\sigma(S_t)$. For the sake of simplicity, we will work under the assumption that they are constant as in (5.2.1).

Nontradable assets. Apart from traded assets, we introduce non traded assets following the (generalized) Ornstein-Uhlenbeck processes

$$dX_t^i = (b_i(t) - \alpha_i X_t^i)dt + \beta_i . dW_t^X, \quad i = 1, \dots, d, \tag{5.2.2}$$

where $b_i : [0,T] \to \mathbb{R}$ is a bounded measurable function and the $\beta_i.$ is the i-th row of the $d \times d$-dimensional matrix β. It is important to remark that as they are defined, tradable and non tradable assets are independent. This is a crucial assumption in what follows. From the modeling viewpoint this is a pretty natural assumption since the application we have in mind is to energy markets where the non tradable assets typically are the electricity demand and the power plant capacities, while the tradable ones are the fuels used in the power production process such as, for instance, gas, oil and coal.

Equivalent martingale measures. If the market filtration were $\mathbb{F}^S$ (i.e. that generated by W^S only), then the market would be complete and the unique martingale measure Q^0 would be defined by the measure change

$$\frac{dQ^0}{dP} = \mathcal{E}_T(-\theta \cdot W^S),$$

where $\theta = \sigma^{-1}\mu$ and $\mathcal{E}$ denotes the stochastic exponential. When considering the whole filtration $\mathbb{F}$, the market is clearly no longer complete and the set $\mathcal{M}$ of absolutely continuous martingale measures for $S = (S^1, \dots, S^n)$ is no longer a singleton. As is well known from the literature (see Schweizer's survey [Sc01]), the measure Q^0, which is called minimal martingale measure (MMM henceforth), still plays an important role for pricing and hedging derivatives. Remark that in our case the elements of $\mathcal{M}$ are of the form $\zeta_T = \frac{dQ^0}{dP} M_T$, where the process M is nonnegative and satisfies $E[\zeta_T] = 1$. The dynamics of M can be written as

$$dM_t = \eta_t dW_t^X \quad M_0 = 1 \tag{5.2.3}$$

for some $\mathbb{F}$-predictable process η. The choice $\eta = 0$ (i.e. $M = 1$) corresponds to the MMM. We will denote $W^{S,0} = W^S + \theta t$, $W^0 = (W^{S,0}, W^X)$, and E^0 the expectation operator under Q^0. Notice that Girsanov's theorem clearly implies that W^0 is a $(n+d)$-dimensional Brownian motion under this measure.

Trading strategies. In this model, the wealth process of an agent starting from an initial capital $v \in \mathbb{R}$ and trading in the risky assets S in a self-financing way over the period $[0,T]$ can be written

$$V_t^v(\pi) = v + \int_0^t \pi_s'(\mu ds + \sigma dW_s^S) = v + \int_0^t \pi_s' \sigma(\theta ds + dW_s^S)$$

where π_s is a $n \times 1$ vector representing the investor's trading strategy (in euros) at time s and μ is a column vector containing the μ_i's. We will need to be more precise later about admissibility conditions on strategies. It is then useful to introduce the following sets:

$$\begin{aligned}
\mathcal{H} &= \{\pi \in \mathbb{H}^2_{\text{loc}}(\mathbb{R}^n) : V^0(\pi) \text{ is a } Q - \text{supermartingale for all } Q \in \mathcal{M}_E\} \\
\mathcal{H}_M &= \{\pi \in \mathbb{H}^2_{\text{loc}}(\mathbb{R}^n) : V^0(\pi) \text{ is a } Q - \text{martingale for all } Q \in \mathcal{M}_E\} \\
\mathcal{H}_b &= \{\pi \in \mathbb{H}^2_{\text{loc}}(\mathbb{R}^n) : V^0(\pi) \text{ is uniformly bounded from below by a constant}\},
\end{aligned}$$

where $\mathcal{M}_E$ denotes the subset of measures in $\mathcal{M}$ with finite relative entropy.

Utility indifference pricing. We will focus our interest in contingent claims which can depend on both tradable and non tradable assets and which satisfies the following assumption.

Assumption 5.2.1. The claim f belongs to $L^2(Q^0, \mathcal{F}_T)$, it is super/sub-replicable, i.e.

$$V_T^{v_1}(\pi_1) \leq f \leq V_T^{v_2}(\pi_2)$$

for some $v_1, v_2 \in \mathbb{R}$ and $\pi_1 \in \mathcal{H}_M$, $\pi_2 \in \mathcal{H}$. The random variables $V_T^{v_1}(\pi_1), V_T^{v_2}(\pi_2)$ lie in $L^1(Q^0, \mathcal{F}_T)$.

We focus in this paper on the case of exponential utility $U(x) = -e^{-\gamma x}$, $\gamma > 0$, and we look at the buying utility indifference price p^b of the claim f as implicitly defined as a solution to

$$\sup_\pi E\left[U\left(V_T^{v-p^b}(\pi) + f\right)\right] = \sup_\pi E\left[U(V_T^v(\pi))\right] \tag{5.2.4}$$

where $v \in \mathbb{R}$ is the initial wealth and the supremum is either taken over $\mathcal{H}$ or $\mathcal{H}_b$. It is easily seen that under exponential utility the price is independent of the initial agent's wealth. By Theorem 1.2 in [OZ09] the suprema in definition (5.2.4) are unchanged whether the optimizing set is $\mathcal{H}$ or $\mathcal{H}_b$, though the maximum will in general be attained in the larger set $\mathcal{H}$.
We will call *optimal hedging strategy* and denote it Δ the difference between the maxima $\hat{\pi}^f$ and $\hat{\pi}^0$ in, respectively, the LHS and RHS of (5.2.4), i.e. $\Delta = \hat{\pi}^f - \hat{\pi}^0$.
The selling price p^s is defined similarly as the solution to

$$\sup_\pi E\left[U\left(V_T^{v+p^s}(\pi) - f\right)\right] = \sup_\pi E\left[U(V_T^v(\pi))\right].$$

We start with a simple preliminary result showing how these prices are related to the expected payoff under the MMM (which can also be interpreted as a price under a certain risk minimizing criterion, see [Sc01]). The next result can also be found in [Ho05], Theorem 3.1 under slightly different assumptions. We provide here another proof which is perhaps a little bit more general as it is only based on duality (without requiring their Assumption 2.2, even though it would be satisfied in our particular context), and which is also useful to compare utility indifference prices with certainty equivalents (see Remark 5.2.3).

Lemma 5.2.1. *It holds that*

$$v_1 \leq p^b \leq E^0[f] \leq p^s \leq v_2,$$

where v_1, v_2 are the same as in Assumption 5.2.1.

Proof. We start from the well-known duality result (see [OZ09], Theorem 1.1):

$$\sup_\pi E[U(V_T^{v-p^b}(\pi) + f)] = \inf_{\delta>0} \inf_{\zeta_T \in \mathcal{M}} \left\{\delta(v - p^b) + \delta E[\zeta_T f] + E[U^*(\delta\zeta_T)]\right\} \tag{5.2.5}$$

where $\zeta_T = \frac{dQ^0}{dP} M_T$ as in (5.2.3) and U^* is the conjugate of U. By taking $M = 1$ (equivalently, $\eta = 0$) we get

$$\sup_{\pi} E[U(V_T^{v-p^b}(\pi) + f)] \leq \inf_{\delta>0} \left\{ \delta \left(v - p^b + E^0[f] \right) + E\left[U^* \left(\delta \frac{dQ^0}{dP} \right) \right] \right\}.$$

Now by using (5.2.4) and (5.2.5) for $f = 0$, we get that

$$\inf_{\delta>0} \inf_{\zeta_T \in \mathcal{M}} \{\delta v + E[U^*(\delta \zeta_T)]\} \leq \inf_{\delta>0} \left\{ \delta \left(v - p^b + E^0[f] \right) + E\left[U^* \left(\delta \frac{dQ^0}{dP} \right) \right] \right\}. \quad (5.2.6)$$

We want to show that the minimizer in the LHS corresponds to the MMM. Remark now that for each $\delta > 0$ and $\zeta_T = \frac{dQ^0}{dP} M_T$ by using convexity of U^* and conditional Jensen's inequality we get

$$\begin{aligned} E[U^*(\delta\zeta_T)] &= E\left[U^* \left(\delta \frac{dQ^0}{dP} M_T \right) \right] = E\left[E\left[U^* \left(\delta \frac{dQ^0}{dP} M_T \right) | \mathcal{F}_T^S \right] \right] \\ &\geq E\left[U^* \left(\delta E\left[\frac{dQ^0}{dP} M_T | \mathcal{F}_T^S \right] \right) \right] = E\left[U^* \left(\delta \frac{dQ^0}{dP} E\left[M_T | \mathcal{F}_T^S \right] \right) \right] \quad (5.2.7) \\ &= E\left[U^* \left(\delta \frac{dQ^0}{dP} \right) \right] \end{aligned}$$

where we used the fact that $E[M_T|\mathcal{F}_T^S] = 1$ a.s., which can be shown as follows. By defining $N_t = E_t\left[\frac{dQ^0}{dP}\right] = E\left[\frac{dQ^0}{dP}|\mathcal{F}_t^S\right]$ we have that

$$E[N_T M_T] = 1 = N_0 M_0$$

since $N_T M_T$ is a martingale measure density for S. Since S and X are independent, the process M in (5.2.3) is a positive local martingale in the larger filtration $(\mathcal{F}_T^S \vee \mathcal{F}_t^X)_{0 \leq t \leq T}$, hence a supermartingale, implying in particular

$$E[M_T|\mathcal{F}_T^S] \leq E[M_0|\mathcal{F}_T^S] = 1.$$

If the previous inequality was strict on a set $F \in \mathcal{F}_T^S$ of strictly positive probability then we would get the contradiction

$$E[N_T M_T] = E\left[N_T E[M_T|\mathcal{F}_T^S] \right] < M_0 E[N_T] = 1.$$

Therefore if we had $E^0[f] - p^b < 0$ by using (5.2.6) and the previous argument we would get the contradiction

$$\inf_{\delta>0} \left\{ \delta v + E[U^*(\delta \zeta_T^0)] \right\} < \inf_{\delta>0} \left\{ \delta v + E[U^*(\delta \zeta_T^0)] \right\}.$$

This proves $p^b \leq E^0[f]$.
Now consider the super-replicating strategy π_2 for the claim f, starting from a given initial capital v_2. Since

$$\sup_{\pi} E\left[U\left(V_T^{v+v_2}(\pi) - f \right) \right] \geq E\left[U\left(V_T^v(\pi) + V_T^{v_2}(\pi_2) - f \right) \right] \geq E\left[U(V_T^v(\pi)) \right]$$

and therefore

$$\sup_\pi E\left[U\left(V_T^{v+v_2}(\pi)-f\right)\right] \geq \sup_\pi E\left[U(V_T^v(\pi)\right]$$

we deduce that the selling price p^s must verify $p^s \leq v_2$. The other inequalities are obtained by similar arguments. □

Definition (5.2.4) can be extended to the conditional case by defining the (buying) price p_t^b as the $\mathcal{F}_t$-measurable r.v. satisfying

$$\text{ess sup}_\pi\, E_t\left[U\left(V_T^{v-p_t^b}(\pi)+f\right)\right] = \text{ess sup}_\pi\, E_t\left[U(V_T^v(\pi))\right] \tag{5.2.8}$$

where the set of admissible strategies is restrained to those starting at t. We denote $p_0^b = p^b$. The previous lemma can therefore be slightly generalized to obtain that

$$V_t^{v_1}(\pi_1) \leq p_t^b \leq p_t^s \leq V_t^{v_2}(\pi_2). \tag{5.2.9}$$

Generalizing the other bounds to obtain $p_t^b \leq E_t^0[f]$ is a little bit more delicate since the duality results in [OZ09] are not generalized to the conditional primal problem. A partial result can be obtained using, e.g., BSDE-based methods (see our Remark 5.3.2).

Remark 5.2.2. The previous result confirms that utility indifference valuation gives rise to a sort of bid-ask spread and the price computed under the MMM can be interpreted as a mid price. The fact that utility indifference buying (selling) prices are always higher (lower) than sub(super)-replication prices also justifies their interest.

Remark 5.2.3. A related pricing method is given by the *certainty equivalent*, which is quite popular in the economic literature and which has been explored by Benth et al. ([BCK07]) in the context of electricity markets. In that paper, there is no financial market where the investor could possibly trade. This is the one of the main differences with respect to our approach. The certainty equivalent method provides the same prices as utility indifference evaluation when the payoff is just a bounded function of the nontraded assets. To see this, remark that when the payoff is bounded we can always make a probability change and write

$$E[U(V_T^0(\pi)+f-p^b)] = E[e^{-\gamma(f-p^b)}]E^{Q^f}[U(V_T^0(\pi))] = E^{Q^f}[cU(V_T^0(\pi))]$$

with $c>0$ and the change of measure $\frac{dQ^f}{dP} = \frac{e^{-\gamma(f-p^b)}}{E[e^{-\gamma(f-p^b)}]}$ only affecting the nontraded assets. Let U^* denote the conjugate of U. By using $(cU)^*(y) = cU^*(y/c)$, the definition (5.2.4), the duality results (5.2.5) and (5.2.7) we get

$$\inf_{\delta>0} E\left[U^*\left(\delta\frac{dQ^0}{dP}\right)\right] = \inf_{\delta>0} E^{Q^f}\left[(cU)^*\left(\delta\frac{dQ^0}{dP}\right)\right]$$

which becomes

$$\inf_{\delta>0} E\left[U^*\left(\delta\frac{dQ^0}{dP}\right)\right] = E\left[e^{-\gamma(f-p^b)}\right]\inf_{\delta>0} E\left[U^*\left(\frac{\delta}{E\left[e^{-\gamma(f-p^b)}\right]}\frac{dQ^0}{dP}\right)\right],$$

that is trivially satisfied by the certainty equivalent $p^b = -\frac{1}{\gamma}\ln E[e^{-\gamma f}]$. However, when the payoff does depend on the traded assets (as in the examples of power derivatives given in Section 5.5) the two methods can provide completely different results due to the existence of additional investment opportunities offered by some financial market as it is the case in our model. Notice for instance that the certainty equivalent applied to a payoff which is linear in s (uniformly in x) will produce an infinite buying or selling price (since geometric Brownian motion does not have exponential moments), while by the previous lemma utility indifference prices will always by finite, as the payoff is super/sub-replicable.

5.3 Utility indifference pricing via BSDEs

In this section we extend to our setting the classical characterization of the utility indifference price of a contingent claim f in terms of the solution of a suitable BSDE. This characterization has to be proved in our framework since we are not assuming boundedness of f nor that it has finite exponential moments, which are the usual conditions imposed in the existing literature. These conditions would not be satisfied in the application to power derivatives that we have in mind (see Section 5.5). From now on we will only focus on buying prices, the selling counterpart being easily obtained by symmetry (see Remark 5.3.4).
The following result shows how the utility indifference price (UIP for short) is linked to the solution of the BSDE *

$$Y_t = f - \int_t^T \left(\frac{\gamma}{2}\|Z_s^X\|^2 + \mu'\sigma^{-1}Z_s^S\right) ds - \int_t^T Z_s dW_s \tag{5.3.1}$$

which can also be written under the MMM Q^0 in the simpler form

$$Y_t = f - \int_t^T \frac{\gamma}{2}\|Z_s^X\|^2 ds - \int_t^T Z_s dW_s^0. \tag{5.3.2}$$

We start by assuming that f is bounded. The next step will consist in replacing the boundedness of f with its sub/super-replicability as in Assumption 5.2.1. The following result is essentially known since the seminal work [ER00], however we provide here a different proof.

Lemma 5.3.1. *Suppose f is bounded. Then $p_t^b = Y_t$, where (Y, Z) is the unique solution of BSDE* (5.3.1) *satisfying*

$$E\left[\sup_{0\le t\le T}|Y_t|^2 + \int_0^T \|Z_t\|^2 dt\right] < \infty.$$

Moreover, the optimal trading strategy is given by $\Delta_t = -\sigma^{-1}Z_t^S$.

Proof. We prove the lemma only in the case $t = 0$. The same arguments easily extend to any time t. We use the results in [HIM05]. By definition of UIP we are allowed

∗. It can be viewed as an uncoupled FBSDE since traded and nontraded assets entering in f have forward dynamics.

to only consider strategies in $\mathcal{H}_b$, so that the admissibility conditions in [HIM05] are satisfied (apart from square integrability, which is not necessary for what follows). By their Theorem 7, the value function in the LHS of (5.2.4) takes the form

$$-\exp(-\gamma(x-p-\tilde{Y}_0))$$

where $\tilde{Y}_0$ is defined by the unique solution $(\tilde{Y},\tilde{Z})$ to

$$\tilde{Y}_t = -f - \int_t^T \tilde{Z}_s dW_s - \int_t^T g(s,\tilde{Z}_s)ds \tag{5.3.3}$$

with

$$g(\cdot,z) = -\frac{\gamma}{2}\text{dist}^2\left(z+\frac{(\theta,0)}{\gamma},C\right) + (\theta,0)'z + \frac{1}{2\gamma}\|\theta\|^2.$$

In our case $C = \mathbb{R}^n \times \{0_d\}$, with 0_d the null vector in $\mathbb{R}^d$, and $\|\theta\|^2 = \mu'\sigma^{-2}\mu$. Thus, we have

$$g(\cdot,z) = -\frac{\gamma}{2}\|(0,z^X)\|^2 + \mu'\sigma^{-1}z^S + \frac{1}{2\gamma}\mu'\sigma^{-2}\mu.$$

When the final claim is zero then $\tilde{Z}$ in (5.3.3) vanishes so that it simply gives $\tilde{Y}_0 = -\frac{T}{2\gamma}\mu'\sigma^{-2}\mu$. Applying (5.2.4) we get

$$-\exp\left(-\gamma\left(x-p-\tilde{Y}_0\right)\right) = -\exp\left(-\gamma\left(x+\frac{T}{2\gamma}\mu'\sigma^{-2}\mu\right)\right)$$

from which we get $p = -\left(\tilde{Y}_0 + \frac{T}{2\gamma}\mu'\sigma^{-2}\mu\right) =: Y_0$ where $(Y,\tilde{Z})$ solves

$$Y_t = f + \int_t^T \left(-\frac{\gamma}{2}\|(0,\tilde{Z}_s^X)\|^2 + \mu'\sigma^{-1}\tilde{Z}_s^S\right)ds + \int_t^T \tilde{Z}_s dW_s.$$

The result then follows by defining $Z = -\tilde{Z}$. The optimal strategy in the LHS of (5.2.4) is then given by $\sigma^{-1}\tilde{Z}^S + \frac{1}{\gamma}\sigma^{-2}\mu$, and the second result follows. □

Remark 5.3.1. The result can be also easily derived by properly modifying the proof of Lemma 2.4 in [HL11]. However that approach requires a BMO property for admissible strategies which we do not assume.

Remark 5.3.2. Notice also that from the representation (5.3.2), by using the classical comparison result for quadratic BSDEs, we can also immediately generalize the result of Lemma 5.2.1 by obtaining

$$p_t^b \leq E_t^0[f] \tag{5.3.4}$$

for all $t \in [0,T]$.

We now want to show that (5.3.1) still admits a solution when f is possibly unbounded but still satisfies Assumption 5.2.1. We insist once more on the fact that the result is not immediately obvious from the standard literature since f does not necessarily possess exponential moments (for example if it depends linearly on the final value of some tradable assets as in our examples in Section 5.5 of the paper).

Lemma 5.3.2. *Under Assumption 5.2.1 BSDE* (5.3.2) *admits a solution.*

Proof. We will adapt the arguments in the proof of Proposition 3 in [BH07]. Rewrite equation (5.3.2) as

$$Y_t = f + \int_t^T g(Z_s)ds - \int_t^T Z_s dW_s^0, \tag{5.3.5}$$

with $g(z) = -\frac{\gamma}{2}\|z^X\|^2$, and denote $f_n = (-n) \vee f \wedge n$, $L_t = E_t^0[|f|] + E_t^0[|V_T^{v_1}(\pi_1)|]$, $L_t^n = E_t^0[|f_n|] + E_t^0[|V_T^{v_1}(\pi_1)|]$ (which are well defined thanks to Assumption 5.2.1). Let (Y^n, Z^n) be the minimal bounded solution to (5.3.5) where f is replaced by f_n (it exists by [Ko00], Theorem 2.3). By (5.2.9) and (5.3.4) we have that $|Y_t^n| \leq L_t^n \leq L_t$ for all n. Moreover the sequence $(Y^n)_{n\geq 1}$ is nondecreasing by the comparison theorem (see [Ko00], Theorem 2.3).
Now define

$$\tau_k = \inf\{t \in [0,T] : L_t \geq k\} \wedge T, \quad \inf \emptyset = +\infty.$$

The sequence $Y_k^n(t) = Y_{t\wedge\tau_k}^n$ remains bounded uniformly in n for each k. Setting $Z_k^n(t) = Z_t^n \mathbf{1}_{\{t\leq\tau_k\}}$ we have

$$Y_k^n(t) = Y_{\tau_k}^n + \int_t^T \mathbf{1}_{\{s\leq\tau_k\}} g(Z_k^n(s))ds - \int_t^T Z_k^n(s)dW_s^0. \tag{5.3.6}$$

Now we can apply Lemma 2 in [BH07] and obtain, for each k, a solution (Y_k, Z_k) to the BSDE

$$Y_k(t) = \xi_k + \int_t^{\tau_k} g(Z_k(s))ds - \int_t^{\tau_k} Z_k(s)dW_s^0 \tag{5.3.7}$$

where $\xi_k = \sup_n Y_{\tau_k}^n$. Defining $Y_t = Y_1(t)\mathbf{1}_{\{t\leq\tau_1\}} + \sum_{k\geq 2} Y_k(t)\mathbf{1}_{]\tau_{k-1},\tau_k]}(t)$ and similarly for Z_t we get

$$Y_t = \xi_k + \int_t^{\tau_k} g(Z_s)ds - \int_t^{\tau_k} Z_s dW_s^0 \tag{5.3.8}$$

and the result follows by sending k to infinity. □

We would like now to be able to interpret the solution Y constructed in the previous lemma as the UIP of the claim f. We borrow and adapt the next result from [OZ09], which gives some sufficient conditions ensuring this property. Those conditions are quite easy to verify in our setting for a large class of contingent claims (see Section 5.5), since the independence between tradable and non tradable assets implies a very simple product structure for the set $\mathcal{M}$ of all absolutely continuous martingale measures for S.

Lemma 5.3.3. *Let f be a contingent claim satisfying Assumptions 5.2.1 and let $f_n = (-n) \vee f \wedge n$, $n \geq 1$. If*

$$\sup_{Q\in\mathcal{M}_E} E^Q[f_n - f] \to 0, \quad \inf_{Q\in\mathcal{M}_E} E^Q[f_n - f] \to 0 \tag{5.3.9}$$

as $n \to \infty$ then $Y_0 = p^b$, where Y solves (5.3.1).

Proof. Following the previous proof, we know that $Y_0^n = p^b(f_n)$, the buying UIP of f_n, and that $Y_0^n \to Y_0$, where Y solves (5.3.1). By Proposition 5.1 (iii) in [OZ09] we know that

$$\sup_\pi E\left[-e^{-\gamma\left(V_T^{v-p(f_n)}(\pi)+f_n\right)}\right] \to \sup_\pi E\left[-e^{-\gamma\left(V_T^{v-Y_0}(\pi)+f\right)}\right]$$

which implies that $Y_0 = p^b$. □

Remark 5.3.3. Notice that the conditions in (5.3.9) are automatically satisfied whenever the super/sub-replicating portfolio strategies are $\mathbb{F}^S$-predictable and the portfolio values $V_T^{v_1}(\pi_1)$ and $V_T^{v_s}(\pi_s)$ are in $L^2(Q^0, \mathcal{F}_T)$. This follows from the fact that, for any $Q \in \mathcal{M}_E$, we have

$$\begin{aligned}
E^Q[|f_n - f|] &= E^Q[|f_n - f|\mathbf{1}_{|f|\geq n}] \leq \|f_n - f\|_{L^2(Q)} Q(|f| \geq n)^{1/2} \\
&\leq \|f_n - f\|_{L^2(Q)} Q(|V_T^{v_1}(\pi_1)| + |V_T^{v_2}(\pi_2)| \geq n)^{1/2} \\
&\leq C\|f\|_{L^2(Q)} Q(|V_T^{v_1}(\pi_1)| + |V_T^{v_2}(\pi_2)| \geq n)^{1/2} \\
&\leq C(\|V_T^{v_1}(\pi_1)\|_{L^2(Q)} + \|V_T^{v_2}(\pi_2)\|_{L^2(Q)}) Q(|V_T^{v_1}(\pi_1)| + |V_T^{v_2}(\pi_2)| \geq n)^{1/2} \\
&= C(\|V_T^{v_1}(\pi_1)\|_{L^2(Q^0)} + \|V_T^{v_2}(\pi_2)\|_{L^2(Q^0)}) Q^0(|V_T^{v_1}(\pi_1)| + |V_T^{v_2}(\pi_2)| \geq n)^{1/2} \\
&\to 0
\end{aligned}$$

as $n \to \infty$. This will be the case under the Assumptions 5.4.1 and 5.4.2 that we will introduce in the next section.

We will now focus on the Markovian case by considering European claims. This will allow, under proper assumptions, to get more information about the process Z and therefore on the hedging strategy. In particular, representation results like those found in [MZ02] or [Zh05] will reveal to be useful to study the continuity of Z and the possibility to express it starting from the spacial (classical) derivatives of the solution of a given partial differential equation. This will also allow to obtain some estimates on Z which permit to interpret it in terms of the optimal hedging strategy under some less restrictive hypotheses than the boundedness of f (which is required in the standard martingale optimality approach of [HIM05] to prove a BMO property for Z which is needed to identify it with the hedging strategy).

Remark 5.3.4. We decided to focus the discussion on buying prices, however most of the results can be adapted to selling prices. In particular the usual relation $p^s(f) = -p^b(-f)$ holds between selling and buying prices. The natural candidate for the selling price is the solution to the BSDE

$$Y_t = f + \int_t^T \frac{\gamma}{2}\|Z_s^X\|^2 ds - \int_t^T Z_s dW_s^0. \tag{5.3.10}$$

Remark immediately the sort of symmetry with (5.3.2). Existence for (5.3.10) can be obtained by following the proof of Lemma 6.7.2, but using the super (instead of sub)-replicating process in Assumption 5.2.1 as a bound. Moreover, under the same conditions as in Lemma 5.3.3 we are able to interpret this solution as the selling price. All the other results still hold for selling prices with minor modifications. In particular, Lemma 5.7.1 finds its analogue in Lemma 5.7.5. Both are relegated in the Appendix for the sake of readability.

5.4 Pricing and hedging of European payoffs

In this section we will address the problem of computing the utility indifference price and the corresponding optimal hedging strategy of a European contingent claim f, which is a function of both tradable and non tradable assets at the terminal date T, i.e. we assume (with a slight abuse of notation) that $f = f(S_T, X_T)$ for some measurable function $f : \mathbb{R}^n_+ \times \mathbb{R}^d \to \mathbb{R}$. We will denote $f_{s^i\pm}(s,x)$ and $f_{x^j\pm}(s,x)$ the right/left derivatives of the function $f(s,x)$ with respect to, respectively, s^i and x^j for $i = 1,\dots,n$ and $j = 1,\dots,d$. We will use the notation $A_t = (S_t, X_t)$ when we wish to consider asset processes with no distinction. The standard notation $E_{t,a}$ denotes expectation with respect to $\mathcal{F}_t$ given that the process A takes the value $a = (s,x)$ at time t. Our goal is to obtain a complete characterization of the optimal hedging strategy Δ as well as asymptotic expansion of the price of the contingent claim f for small risk aversion parameter γ. Combining techniques coming from both BSDEs and PDEs thanks to the Markovian framework, we are able to do so for a large class of non-smooth contingent claims. More precisely, we consider the following two kinds of assumptions for f.

Assumption 5.4.1. *(Continuous non-smooth payoffs)* The payoff function f is continuous and a.e. differentiable with left and right derivatives growing polynomially in s, uniformly in x, i.e.

$$|f_{s^i\pm}(s,x)| + |f_{x^j\pm}(s,x)| \le C(1 + \|s\|^q), \quad (s,x) \in \mathbb{R}^n_+ \times \mathbb{R}^d,$$

for all $i = 1,\dots,n$ and $j = 1,\dots d$ and for some $q \ge 1$, where the constant C does not depend on x.

Assumption 5.4.2. *(Discontinuous payoffs)* The payoff function f is bounded from below and a.e. differentiable. Moreover

1. f may have finitely many discontinuities only in the x-variables.
2. Where it exists, the derivative $f_{s^i}(s,x)$ is bounded, and in particular $f_{s^i}(s,x) = O(1/s^i)$ for s^i big enough, for all $i = 1,\dots,n$ uniformly in x.
3. Where it exists, the derivative f_{x^j} verifies $|f_{x^j}(s,x)| \le C(1+\|s\|^q)$ for all $j = 1,\dots,d$ and some $q \ge 1$, where the constant $C > 0$ does not depend on x.

We see that if we want to treat discontinuous payoffs we need stronger growth assumptions than in the continuous case. In particular the second hypothesis implies a uniform logarithmic growth of f in the traded assets. The main example we think about in this case is that of a payoff which separates the contributions of traded and nontraded assets in a multiplicative way (see Section 5.5 for some examples coming from electricity markets).

Since $f = f(A_T) = f(S_T, X_T)$ we can exploit the Markovian setting and look for a solution to (5.3.1) of the form $Y_t = \varphi(t, A_t)$ where $\varphi(t,a) = \varphi(t,s,x)$ solves the PDE

$$\begin{cases} \mathcal{L}\varphi - \frac{\gamma}{2}\sum_{j=1}^d (\beta'_{\cdot j}\varphi_x)^2 = 0 \\ \varphi(T,a) = f(a) \end{cases} \tag{5.4.1}$$

where $\beta_{\cdot j}$ denotes the j-th column of the matrix β and

$$\mathcal{L}\varphi = \varphi_t + (b - \alpha x)\varphi_x + \frac{1}{2}\sum_{i,j=1}^n \sigma_{i\cdot}\sigma_{\cdot j} s^i s^j \varphi_{s^i s^j} + \frac{1}{2}\sum_{i,j=1}^d \beta_{i\cdot}\beta_{\cdot j}\varphi_{x^i x^j}.$$

The PDE above is motivated by a formal application of Itô's lemma (to be justified later) for the function φ and recalling that with sufficient regularity we expect to have $Z^{X,i} = \beta'_{\bullet i}\varphi_x$, where φ_x is the (d-dimensional) gradient of φ with respect to x.
Now denote

$$h(q) = \frac{\gamma}{2}\|q\|^2 = \sup_{\delta\in\mathbb{R}^d}\left\{-q\delta - \frac{1}{2\gamma}\|\delta\|^2\right\}, \quad q \in \mathbb{R}^d.$$

In this way (5.4.1) can be written as

$$\begin{cases} -\mathcal{L}\varphi + h(\beta'\varphi_x) = 0 \\ \varphi(T,a) = f(a). \end{cases} \tag{5.4.2}$$

The main result of this section can be summarized in the following theorem.

Theorem 5.4.1. *Let $f = f(A_T) = f(S_T, X_T)$ be a given European type contingent claim for some measurable payoff function $f : \mathbb{R}^n_+ \times \mathbb{R}^d \to \mathbb{R}$. We have the following properties.*

1. *Under Assumption 5.4.1 or 5.4.2 the buying UIP φ of the claim f is a viscosity solution of (5.4.1) on $[0,T) \times \mathbb{R}^n_+ \times \mathbb{R}^d$, which is also differentiable in all the space variables.*
2. *The optimal hedging strategy is given by*

$$\Delta_t = -\sigma^{-1}Z^S_t = -\sigma^{-1}\sigma(S_t)\varphi_s(t, A_t),$$

where (Y,Z) is solution to (5.3.2) and $\sigma(S)$ the $n \times n$ matrix whose i-th row is given by $\sigma_i.S^i$.

The rest of this section is devoted to proving this theorem and deducing some asymptotic expansions of the price and the optimal hedging strategy for a small risk aversion parameter γ.

5.4.1 Proof of the main theorem

Before giving the technical details, we briefly sketch the main ideas underlying our proofs. Equation (5.4.2) suggests that we can look at our pricing problem as a stochastic control problem with a quadratic cost function: following this intuition, the idea of the proof is to start with a slightly modified reformulation (using some ideas developed in [Ph02]) in which the control space is forced to be compact. When the payoff is regular enough, this trick allows us to prove the existence of a smooth solution to the modified problem, which immediately extends to the original one by using some estimates on the derivatives which do not depend on the size of the control space. When the payoff is continuous but not smooth enough, we will approximate it with a sequence of smooth ones (to which our previous results apply) and study the behavior of prices in the limit: in particular, by using a Malliavin-type representation of the derivatives which does not rely on the regularity of the payoff (which is due to [MZ02]), we are able to prove that the limiting price function remains differentiable in the state variables (though it possibly fails to be $C^{1,2}$). The case of discontinuous payoffs is a little bit more delicate: again the aim

is to obtain some estimates on the derivatives which do not depend on the approximating sequence for the payoff, but here we can exploit neither the derivatives of the approximating sequence (which may explode due to the discontinuities) nor the Malliavin-type estimates in [MZ02] and [Zh05] which do not apply to quadratic BSDEs. We will tackle the problem by performing a suitable change of measure, which however requires stronger assumptions with respect to the case of continuous payoffs.

An auxiliary problem with compact control space

We start analyzing (5.4.2) by forcing the space of controls to be compact, in particular by replacing the function $h(q)$ in (5.4.2) by $h^m(q)$ defined as

$$h^m(q) = \sup_{\delta \in \mathcal{B}^m(\mathbb{R}^d)} \left\{ -q\delta - \frac{1}{2\gamma}\|\delta\|^2 \right\}$$

where $\mathcal{B}^m(\mathbb{R}^d)$ is the ball in $\mathbb{R}^d$ centered at zero and of radius $m > 0$. Thus, the PDE we consider in this section is

$$\begin{cases} -\mathcal{L}\varphi^m + h^m(\beta'\varphi^m_x) = 0 \\ \varphi^m(T,a) = f(a). \end{cases} \tag{5.4.3}$$

We also write its associated BSDE

$$Y^m_t = f - \int_t^T h^m(Z^{X,m}_r)dr - \int_t^T Z^m_r dW^0_r \tag{5.4.4}$$

that we will refer to in the sequel. Existence and uniqueness of the solution for this BSDE are guaranteed by classical results in [PP90], since the generator h^m is a Lipschitz function.
We use the standard notation $\mathrm{diag}(x)$ for the diagonal matrix whose i-th entry in its diagonal is given by x^i, for any vector x.

Lemma 5.4.1. *Let $m > 0$. If $f \in C^3$ and f and all its first derivatives have polynomial growth, then there exists a classical solution φ^m to (5.4.3). If f is only of polynomial growth (and possibly discontinuous), then φ^m is characterized as a continuous viscosity solution to (5.4.3) with continuous first derivatives in all the space variables, which have the representation*

$$\varphi^m_a(t,a) = E^0_{t,a}\left[f(A_T)N_T - \int_t^T h^m(Z^{X,m}_r)N_r dr \right] \tag{5.4.5}$$

(where φ^m_a is to be interpreted as a column vector in $\mathbb{R}^{n+d}$ containing the derivatives with respect to the traded and nontraded assets) with

$$N_r = \begin{pmatrix} \frac{1}{r-t}\sigma^{-1}(S_t)'(W^S_t - W^S_r) \\ \frac{1}{r-t}\int_t^r \mathrm{diag}(e^{-\alpha(u-t)})'\beta^{-1}dW^X_u \end{pmatrix}.$$

Moreover the following stochastic control representation holds:

$$\varphi^m(t,a) = \inf_{\delta \in \mathcal{A}^m_t} E^Q_{t,a}\left[\frac{1}{2\gamma}\int_t^T \|\delta_r\|^2 dr + f(A_T) \right] \tag{5.4.6}$$

for some auxiliary probability measure Q, under which

$$\begin{cases} \frac{dS_t^i}{S_t^i} = \sigma_i.dW_t^{S,Q}, \quad i = 1,\dots,n \\ dX_t^i = (b_i(t) - \alpha_i X_t^i + \beta_i.\delta_t)dt + \beta_i.dW_t^{X,Q} \quad i = 1,\dots,d \end{cases} \tag{5.4.7}$$

where $(W^{S,Q}, W^{X,Q})$ is a n-dimensional BM under the measure Q and $\mathcal{A}_t^m$ stands for the class of adapted $\mathbb{R}^d$-valued controls δ_s starting from time t and such that $\|\delta_s\| \leq m$.

Remark 5.4.1. Recall that only the dynamics of nontraded assets is touched under the new measure Q, while traded assets still evolve as under the MMM Q^0.

Proof. We split the proof into two main steps.

Step 1: The case where f is smooth follows by Theorem 6.2 in [FR75] (or Theorem IV.4.3 in [FS06]). The reason for introducing the index m comes from the fact that those theorems require that controls must take values in a compact space. The lack of uniform parabolicity here can be handled by a standard logarithmic transformation in the tradable assets. Under the new logarithmic variable, however, the payoff will not preserve polynomial growth in general. Therefore the result should first be applied to PDE (5.4.3) (under the new variable) where the payoff is replaced by $f(s \wedge C, x)$ for some constant $C > 0$, then undoing the logarithmic change of variable and letting $C \to \infty$ will get the final result.
The regularity of φ^m implies (by an application of Itô's lemma) that $\varphi^m(t, A_t) = Y_t^m$, where Y^m solves (5.4.4) and $Z_t^{X,m} = \beta' \varphi_x^m(t, A_t)$, $Z_t^{S,m} = \sigma(S_t)' \varphi_s^m(t, A_t)$.
We need to introduce the tangent process of A, ∇A (see, e.g., equation (2.9) in [MZ02] for a definition), which has the following characterization in our particular case: $(\nabla A_t)_{ii} = S_t^i / S_0^i$ if $i \leq n$, $(\nabla A_t)_{ii} = e^{-\alpha_i t}$ if $n+1 \leq i \leq n+d$, $(\nabla A_t)_{ij} = 0$ if $i \neq j$. Now, define $\Sigma(S)$ as the $(n+d) \times (n+d)$ matrix composed by $\sigma(S)$ on the upper left side and β on the lower right side, being zero everywhere else. The $n \times n$ matrix $\sigma^{-1}(S)$ coincides with the matrix where the i-th column is equal to the i-th column of σ^{-1} divided by S^i. Then $\Sigma^{-1}(S_t)\nabla A_t$ is equal to $\sigma^{-1}(S_0)$ on the upper-left corner and $B^{-1}(t)$ on the lower-right corner, being zero everywhere else. Define the $(n+d)$-dimensional processes

$$M_r = \int_t^r (\Sigma^{-1}(S_u)\nabla A_u)' dW_u^0 = \begin{pmatrix} \sigma^{-1}(S_0)'(W_t^{S,0} - W_r^{S,0}) \\ \int_t^r \operatorname{diag}(e^{-\alpha u})' \beta^{-1} dW_u^X \end{pmatrix}$$

and

$$N_r = \frac{1}{r-t} M_r' (\nabla A_t)^{-1} = \begin{pmatrix} \frac{1}{r-t}\sigma^{-1}(S_t)'(W_t^S - W_r^S) \\ \frac{1}{r-t}\int_t^r \operatorname{diag}(e^{-\alpha(u-t)})' \beta^{-1} dW_u^X \end{pmatrix}. \tag{5.4.8}$$

Since h^m is a Lipschitz function for all fixed $m \geq 0$, we can apply the results in [MZ02] (in particular Theorem 4.2) to the processes M and N just defined to show that (5.4.9) is true. Theorem 4.2 in [MZ02] requires uniform parabolicity which is not respected in our case, however again this is not a problem for geometric Brownian motions since only the process M defined above enters in its proof.

Step 2: In order to prove the result for a general (possibly discontinuous) f we can adapt the proof of Theorem 3.2 in [Zh05] to our framework. In particular, we can take a sequence

f^l of smooth functions with bounded first derivatives such that $f^l \to f$ a.e. as $l \to \infty$. Then we have $f^l(A_T) \to f(A_T)$ Q^0-a.s. since all the processes have absolutely continuous densities. Then one defines

$$\varphi^{m,l}(t,a) = Y_t^{m,l} = f^l - \int_t^T h^m(Z_r^{X,m,l})dr - \int_t^T Z_r^{m,l} dW_r^0$$

We have

$$\varphi_a^{m,l}(t,a) = E_{t,a}^0 \left[f^l(A_T) N_T - \int_t^T h^m(Z_r^{X,m,l}) N_r dr \right] \tag{5.4.9}$$

and with the same arguments as in [Zh05], Theorem 3.2 (slightly modified to our multivariate setting) we can also obtain the estimate

$$\|\varphi_a^{m,l}(t,a)\| \leq C \frac{\|a\|^q}{\sqrt{T-t}}. \tag{5.4.10}$$

for some $q \geq 0$. Here the constant C does not depend on l but it depends on m through the Lipschitz constant of h^m. Applying classical stability results for BSDE (see for example [MY99]), we have the convergence

$$E^0 \left[\sup_{0 \leq t \leq T} |Y_t^{m,l} - Y_t^m|^2 + \int_0^T \|Z_t^{m,l} - Z_t^m\|^2 dt \right] \to 0$$

as $l \to \infty$, where (Y^m, Z^m) solve (5.4.4) (but with a nonsmooth f as terminal condition). We deduce from Lemma 6.2 in [FS06] and the estimate (5.4.10) (which gives uniform convergence on compact subsets of $[0,T) \times \mathbb{R}^{n+d}$) that $\varphi^{m,l} \to \varphi^m$, where φ^m the a viscosity solution of (5.4.3), which is continuous except possibly at T. Following the last part of Zhang's proof of Theorem 4.2 we also obtain that φ^m is differentiable and we have

$$\varphi_a^m(t,a) = E_{t,a}^0 \left[f(A_T) N_T - \int_t^T h^m(Z_r^{X,m}) N_r dr \right].$$

It remains to prove that the stochastic representation (5.4.6) holds for φ^m. Clearly it holds for $\varphi^{m,l}$ as the approximating functions are continuous, so

$$\varphi^{m,l}(t,a) \leq E_{t,a}^Q \left[\frac{1}{2\gamma} \int_t^T \|\delta_r\|^2 dr + f^l(A_T) \right]$$

for any $\delta \in \mathcal{A}_t^m$ and therefore

$$\varphi^m(t,a) \leq E_{t,a}^Q \left[\frac{1}{2\gamma} \int_t^T \|\delta_r\|^2 dr + f(A_T) \right]$$

by dominated convergence (since f has polynomial growth), and

$$\varphi^m(t,a) \leq \inf_{\delta \in \mathcal{A}_t^m} E_{t,a}^Q \left[\frac{1}{2\gamma} \int_t^T \|\delta_r\|^2 dr + f(A_T) \right]$$

To obtain the reverse inequality it suffices to note that we can choose $f^l \geq f$. □

Continuous non-smooth payoffs

The next step is now to remove the dependence on the parameter m and to characterize the price φ. We work in this section under Assumption 5.4.1. We start with a useful probabilistic characterization of the derivatives of φ^m under this assumption (such derivatives exist even if φ^m is only a viscosity solution by Lemma 5.4.1).

Lemma 5.4.2. *Let $m > 0$. Under Assumption 5.4.1 we have the following representations:*

$$\varphi^m_{s^i}(t,a) = E^Q_{t,a}\left[f_{s^i}(A_T)\frac{S^i_T}{S^i_t}\right], \quad \varphi^m_{x^j}(t,a) = e^{-\alpha_j(T-t)}E^Q_{t,a}\left[f_{x^j}(A_T)\right] \tag{5.4.11}$$

for $i = 1,\dots,n$, $j = 1,\dots,d$, where the processes evolve as in (5.4.7) with $\delta = \hat{\delta}$, the maximizer in $h^m(\beta'\varphi^m_x)$.

Proof. We adapt the arguments in [FS06], Lemma 11.4, to our slightly different framework. First assume that f is smooth (in the sense of Lemma 5.4.1), then there exists an optimal Markov feedback $\hat{\delta} \in \mathcal{A}^m_0$ (the one achieving the max in $h^m(\beta'\varphi^m_x)$) such that

$$\varphi^m(t,a) = E^Q_{t,a}\left[\frac{1}{2\gamma}\int_t^T \|\hat{\delta}_r\|^2 dr + f(A_T)\right]$$

By using the same control but with different initial condition we clearly obtain

$$\varphi^m(t,a+\varepsilon e_i) \le E^Q_{t,a+\varepsilon e_i}\left[\frac{1}{2\gamma}\int_t^T \|\hat{\delta}_r\|^2 dr + f(A_T)\right], \quad i = 1,\dots,n+d.$$

Taking the difference and dividing by $\varepsilon > 0$ we get

$$\frac{\varphi^m(t,a+\varepsilon e_i) - \varphi^m(t,a)}{\varepsilon} \le E^Q_t\left[\frac{f(A^{t,a+\varepsilon e_i}_T) - f(A^{t,a}_T)}{\varepsilon}\right], \quad i = 1,\dots,n+d,$$

where for clarity we wrote here $A^{t,a}_T$ to stress that the process starts at time t with value a. The polynomial growth property in the traded assets of the derivatives of f allows us to apply dominated convergence (since traded assets have the same dynamics under Q and Q^0, see (5.4.7)) to get

$$\varphi^m_{a^i}(t,a) \le E^Q_t\left[f_{a^i}(A^{t,a}_T)\frac{\partial}{\partial a^i}A^{t,a,i}_T\right], \quad i = 1,\dots,n+d.$$

By repeating the argument with $-\varepsilon$ we finally obtain

$$\varphi^m_{a^i}(t,a) = E^Q_{t,a}\left[f_{a^i}(A_T)\frac{\partial}{\partial a^i}A^i_T\right]$$

for $i = 1,\dots,n+d$, which gives the result by considering traded and non traded assets separately.
The general result follows by considering an approximating sequence f^l as in the proof of Lemma 5.4.1 and using dominated convergence. □

If the payoff f is sufficiently regular we can immediately remove the dependence on m, as is shown in the next result.

Lemma 5.4.3. *If f satisfies Assumption 5.4.1 and is C^3 then* (5.4.1) *admits a classical solution φ.*

Proof. By the representation (5.4.11) we have

$$\varphi^m_{x^i}(t,a) = e^{-\alpha_i(T-t)} E^Q_{t,a}\left[f_{x^i}(A_T)\right] \leq C E^0_{t,a}\left[\|S_T\|^q\right] \leq C\|s\|^q$$

where the constant is independent of m, since this parameter only modifies through δ the dynamics of X, and by the growth assumptions on f.
For $M > 0$ arbitrarily large we can find $D > 0$ such that $\gamma\|\beta'\varphi^m_x\| \leq D$ if $\|s\| \leq M$, uniformly in m. Therefore if $m \geq D$ then

$$\sup_{\delta\in\mathcal{B}^m(\mathbb{R}^d)}\left\{-(\beta'\varphi^m_x)\delta - \frac{1}{2\gamma}\|\delta\|^2\right\} = \sup_{\delta\in\mathbb{R}^d}\left\{-(\beta'\varphi^m_x)\delta - \frac{1}{2\gamma}\|\delta\|^2\right\}, \tag{5.4.12}$$

for $\|s\| \leq M$. Since M is arbitrary, this implies that (5.4.1) admits a classical solution on the whole domain $[0,T]\times\mathbb{R}^n_+\times\mathbb{R}^d$. □

We can finally prove the part (i) in Theorem 5.4.1 for a continuous payoff f satisfying Assumption 5.4.1.

Proof of Theorem 5.4.1 (i) under Assumption 5.4.1. We approximate the payoff by a sequence of C^3 functions f^l satisfying Assumption 5.4.1 and converging pointwise to f. When a smooth f^l is used as terminal condition by Lemma 5.4.3 we can define the classical solution φ^l to PDE (5.4.1) as a limit of a sequence $\varphi^{m,l}$ when $m\to\infty$. By Lemma 5.4.2 for each m we have

$$|\varphi^{m,l}_{s^i}(t,a)| + |\varphi^{m,l}_{x^j}(t,a)| \leq C\|s\|^q \tag{5.4.13}$$

and

$$dX^i_t = (b^i(t) - \alpha_i X^i_t + \beta_i.\hat{\delta}^m_t)dt + \beta_i.dW^{X,Q}_t \quad i = 1,\dots,d,$$

where $\hat{\delta}^m$ is the maximizer in in LHS of (5.4.12). Here C is independent of m (because of the uniformity property in the nontraded assets as in Assumption 5.4.1) and of l (because of continuity). Remark therefore that, being $\gamma\beta'\varphi^l_x(t,a)$ the maximizer in the RHS of (5.4.12), one necessarily has $\|\hat{\delta}^m_t\| \leq \|\gamma\beta'\varphi^l_x(t,A_t)\|$. This yields $\hat{\delta}^m_t \to -\gamma\beta'\varphi^l_x(t,A_t)$ in $\mathbb{H}^{q'}(\mathbb{R}^d)$ for all $q' > 0$ as $m\to\infty$ (since geometric Brownian motion has moments of all orders), therefore for each l

$$\varphi^l_{x^j}(t,a) = e^{-\alpha_j(T-t)} E^Q_{t,a}\left[f^l_{x^j}(A_T)\right] \leq C\|s\|^q$$

and similarly for $\varphi^l_{s^i}$, where

$$dX^j_t = (b^j(t) - \alpha_j X^j_t - \gamma\beta_j\beta'\varphi^l_x(t,A_t))dt + \beta_j dW^{X,Q}_t \quad j = 1,\dots,d. \tag{5.4.14}$$

For fixed m we recall the Zhang representation in Theorem 3.2 (as in (5.4.9))

$$\varphi_a^{m,l}(t,a) = E_{t,a}^0\left[f^l(A_T)N_T - \int_t^T h^m(Z_r^{X,m,l})N_r dr\right]$$

where

$$Y_t^{m,l} = f^l - \int_t^T h^m(Z_r^{X,m,l})dr - \int_t^T Z_r^{m,l}dW_r^0.$$

Hence

$$\varphi_a^l(t,a) = E_{t,a}^0\left[f^l(A_T)N_T - \frac{\gamma}{2}\int_t^T \|Z_r^{X,l}\|^2 N_r dr\right]$$

by dominated convergence and the previous estimates (5.4.13) applied to

$$Z_t^{X,m,l} = \sigma(S_t)'\varphi_s^{m,l}(t,A_t),$$

and the fact that $Z^{X,m,l} \to Z^{X,l}$ in $\mathbb{H}^{q'}(\mathbb{R}^d)$ for all $q' > 0$ as $m \to \infty$ using classical results on quadratic BSDEs in [Ko00] (since we can assume without loss of generality that f^l is bounded for fixed l), where

$$Y_t^l = f^l - \int_t^T \frac{\gamma}{2}\|Z_r^{X,l}\|^2 dr - \int_t^T Z_r^l dW_r^0. \tag{5.4.15}$$

Now by using an argument like in Lemma 6.7.2 we get that $Y^l \to Y$ as $l \to \infty$ where

$$Y_t = f - \int_t^T \frac{\gamma}{2}\|Z_r^X\|^2 dr - \int_t^T Z_r dW_r^0 \tag{5.4.16}$$

and also $Z^l \to Z$ in $\mathbb{H}^{q'}(\mathbb{R}^d)$ for all $q' > 0$. By the definition of the process N in (5.4.8), we obtain that $E_t^0[\|N_T\|^p] \leq C(T-t)^{-p/2}$ for any $p \geq 1$ and some constant $C > 0$. Therefore again by dominated convergence

$$\varphi_a^l(t,a) \to g(t,a) := E_{t,a}^0\left[f(A_T)N_T - \frac{\gamma}{2}\int_t^T \|Z_r^X\|^2 N_r dr\right].$$

Similarly as in the last part of our Lemma 5.4.3, using Lemma 6.2 in [FS06] we deduce that φ^l converges, uniformly on compact sets of $[0,T] \times \mathbb{R}_+^n \times \mathbb{R}^d$, to φ, viscosity solution to (5.4.1), which is also continuous.

We will now show that g is continuous and that $g = \varphi_a$. To do so we can adapt the last part of Zhang's proof of Theorem 3.2, we give all the details for reader's convenience. For all $\varepsilon > 0$ we can choose an open set O_ε with Lebesgue measure smaller than ε and a continuous function f^ε such that $f^\varepsilon = f$ outside O_ε. Denote

$$g_\varepsilon(t,a) := E_{t,a}^0\left[f^\varepsilon(A_T)N_T - \frac{\gamma}{2}\int_t^T \|Z_r^X\|^2 N_r dr\right]$$

(where Z is solution to the limit BSDE (5.4.16), with f and not f^ε as terminal condition). Denoting g^i and g_ε^i the i-th component of, respectively, g and g_ε we get

$$\begin{aligned}|g_\varepsilon^i - g^i|(t,a) &= |E_{t,a}^0\left[(f^\varepsilon(A_T) - f(A_T))N_T^i\right]| \leq E_{t,a}^0\left[|f^\varepsilon(A_T) - f(A_T)||N_T^i|; X_T \in O_\varepsilon\right] \\ &\leq E_{t,a}^0\left[|f^\varepsilon(A_T) + f(A_T)||N_T^i|; X_T \in O_\varepsilon\right] \leq C(t,a)\sqrt{\varepsilon}.\end{aligned}$$

for some constant $C(t,a)$. Now taking a sequence (t_κ, A_κ) tending to (t,a) we have

$$\begin{aligned}
&|g^i(t_\kappa, A_\kappa) - g^i(t,a)| \\
&\quad \leq |g^i(t_\kappa, A_\kappa) - g^i_\varepsilon(t_\kappa, A_\kappa)| + |g^i_\varepsilon(t_\kappa, A_\kappa) - g^i_\varepsilon(t,a)| + |g^i_\varepsilon(t_\kappa, A_\kappa) - g^i(t,a)| \\
&\quad \leq [C(t,a) + C(t_\kappa, A_\kappa)]\sqrt{\varepsilon} + |g^i_\varepsilon(t_\kappa, A_\kappa) - g^i_\varepsilon(t,a)|.
\end{aligned}$$

Since g^i_ε is continuous and ε is arbitrary we deduce that g^i is continuous as well. Now for any $(t,\tilde{a}) \in [0,T] \times \mathbb{R}^n_+ \times \mathbb{R}^d$ we have

$$\varphi^l(t,\tilde{a}) = \varphi^l(t, I^i\tilde{a}) + \int_0^{\tilde{a}^i} \varphi^l_{a^i}(t, I^i\tilde{a} + e_i y)dy$$

where we denoted I^i the $\mathbb{R}^{n+d}$-identity matrix whose i-th diagonal entry is zero, and e_i is the canonical basis vector in $\mathbb{R}^{n+d}$. By dominated convergence (using (5.4.13)) we deduce

$$\varphi(t,\tilde{a}) = \varphi(t, I^i\tilde{a}) + \int_0^{\tilde{a}^i} g^i(t, I^i\tilde{a} + e_i y)dy,$$

implying that $g = \varphi_a$. □

Remark 5.4.2. In the representation (5.4.14) it would be tempting to pass from measure Q (coming from the stochastic control representation) to the MMM Q^0 by identifying

$$dW^{X,0}_t = dW^{X,Q}_t - \gamma\beta'\varphi^l_x(t, A_t)dt.$$

This may however not be possible in general due to the growth properties of φ^l_x and the fact that geometric Brownian motion does not have exponential moments.
We will perform a similar change of measure in the next section under more restrictive assumptions on the derivatives of the payoff function f.

Discontinuous payoffs

In this part of the paper, we show that the continuity of the payoff f can be partially removed. The price to pay for that is imposing stronger conditions on its derivatives as in Assumption 5.4.2.
The idea hat lies at the heart of the proof that follows is showing that, when we approximate our discontinuous payoff f with a smooth sequence f^l, the derivatives of the price φ^l will not explode for $t < T$. This is easily seen if we take, for example, the digital payoff $f(x) = \mathbf{1}_{[0,\infty)}(x)$ which does not depend on the traded assets. Setting $\alpha = 0$ in the dynamics (5.2.2) we have

$$\varphi^l_x(t,x) = E^Q[f^l(X_T)]$$

with

$$dX_t = -\gamma\varphi^l_x(t, X_t)dt + \beta dW^{X,Q}_t,$$

and $\varphi^l_x(T-t,x) \to g(t,x)$, where g solves the Burgers' equation

$$g_t + \gamma g_x g = \frac{1}{2}\beta^2 g_{xx}$$

which has the solution

$$g(t,x)=\frac{\beta e^{-\frac{x^2}{2\beta^2 t}}(1-e^{-\frac{\gamma}{\beta^2}})}{\gamma\sqrt{2\pi t}\left[(e^{-\frac{\gamma}{\beta^2}}-1)\Phi\left(\frac{x}{\beta\sqrt{t}}\right)+1\right]}$$

In particular we clearly have $g(t,x)\leq\frac{C}{\sqrt{t}}$, where $C=\frac{\beta}{\gamma\sqrt{2\pi}}(e^{\frac{\gamma}{\beta^2}}-1)$. Unfortunately the Burgers-type equation that results by adding traded assets does not seem to have an explicit solution, therefore we will need to employ a different method to get a similar estimate. Here is the proof of our main result concerning discontinuous payoffs.

Proof of Theorem 5.4.1 (i) under Assumption 5.4.2. Take again a sequence f^l of approximating smooth functions as in the proof of Lemma 5.4.1. Each function f^l of the sequence satisfies the assumptions of Lemma 5.4.2, so that the representation formula therein applies and we have that

$$|\varphi^l_{x^i}(t,a)|\leq C^l(1+\|s\|^q), \tag{5.4.17}$$

with the constant C^l depending on l. Remark that this is not the same constant appearing in the characterization of uniform growth with respect to x: since we are dealing with discontinuous payoffs, the derivatives of the approximating functions f^l may well explode close to the discontinuities for large l. We will have

$$|f^l_{x^i}(a)|\leq C^l(x)(1+\|s\|^q)\quad i=1,\dots,d, \tag{5.4.18}$$

where $C^l(x)$ is a function which stays bounded on compact sets which do not include discontinuity points, but that may explode at these points for large l. In order to see this, we can explicitly write the mollified sequence f^l as

$$f^l(s,x)=\int_{\mathbb{R}^d}f(s,x+y)\psi^l(y)dy=\int_{\mathbb{R}^d}f(s,z)\psi^l(z-x)dz$$

where

$$\tilde{\psi}^l(x)=K\exp\left(\frac{-1}{1-\|x\|^2}\right)\mathbf{1}_{\{\|x\|\leq 1\}},\quad \psi^l(x)=l\tilde{\psi}^l\left(lx\right)$$

Recall that ψ^l is a mollifier with support on $\mathcal{B}_d(1/l)$. If $\|x-I\|>1/l$, where I is the discontinuity point closest to x, then

$$f^l_{x^i}(s,x)=\int_{\mathbb{R}^d}f_{x^i}(s,x+y)\psi^l(y)dy$$

and so $|f^l_{x^i}(s,x)|\leq C(1+\|s\|^q)$. For $\|x-I\|\leq 1/l$ we use the representation (recall that $f(s,\cdot)$ is bounded for fixed s)

$$f^l_{x^i}(s,x)=-\int_{\mathbb{R}^d}f(s,z)\psi^l_{x^i}(z-x)dz$$

which yields

$$|f^l_{x^i}(s,x)|\leq C(1+\|s\|^q)\int_{\mathbb{R}^d}|\psi^l_{x^i}(z-x)|dz\leq Cl(1+\|s\|^q)$$

since f has uniform polynomial growth in s. Therefore

$$|f^l_{x^i}(s,x)| \leq C^l(x)(1+\|s\|^q)$$

where

$$C^l(x) = Cl\mathbf{1}_{\{\|x-I\|\leq 1/l\}}.$$

Also by Lemma 5.4.2 and Assumption 5.4.2 (iii) we have

$$|\varphi^l_{s^i}(t,a)| \leq C\frac{1}{s^i} \tag{5.4.19}$$

for s^i big enough (since discontinuities can only occur in the x-variables) and for some constant $C>0$ independent of l and x. If we consider the pricing BSDE (5.4.15) associated with f^l we can identify $Z^{X,l}_t = \beta'\varphi^l_x(t,A_t)$ and $Z^{S,l}_t = \sigma(S_t)'\varphi^l_s(t,A_t)$. By estimate (5.4.19) we deduce that $Z^{S,l}$ is bounded for each l (with a bound independent on l), and estimate (5.4.17) allows us to perform a probability measure change to get

$$\begin{aligned}
|\varphi^l_{x^i}(t,a)| &=^{(i)} \left|E^0_{t,a}\left[\frac{\mathcal{E}_T}{\mathcal{E}_t}(-\gamma Z^{X,l}\cdot W^X)e^{-\alpha_i(T-t)}f^l_{x^i}(A_T)\right]\right| \\
&\leq^{(ii)} CE^0_{t,a}\left[e^{\gamma(Y^l_t - f^l + \int_t^T Z^{S,l}_r dW^{S,0}_r)}|f^l_{x^i}(A_T)|\right] \\
&\leq^{(iii)} Ce^{\gamma Y^l_t}E^0_{t,a}\left[e^{\gamma\int_t^T Z^{S,l}_r W^{S,0}_r}|f^l_{x^i}(A_T)|\right] \\
&\leq^{(iv)} Ce^{\gamma Y^l_t}E^0_{t,a}\left[\frac{\mathcal{E}_T}{\mathcal{E}_t}(\gamma Z^{S,l}\cdot W^{S,0})|f^l_{x^i}(A_T)|\right] \\
&=^{(v)} Ce^{\gamma Y^l_t}E^{\bar{Q}}_{t,a}\left[|f^l_{x^i}(A_T)|\right] \leq^{(vi)} C\|s\|^q e^{\gamma Y^l_t}E^0_{t,x}\left[C^l(X_T)\right] \\
&\leq^{(vii)} \frac{C\|s\|^q\|x\|^{q'}}{\sqrt{T-t}}e^{\gamma Y^l_t} \leq^{(viii)} \frac{C\|s\|^q\|x\|^{q'}}{\sqrt{T-t}}e^{\gamma C(1+\|s\|^q)}
\end{aligned} \tag{5.4.20}$$

where the constant C changes from line to line and the inequalities above can be justified as follows:

1. is due to the second equality in (5.4.11) applied to the sequence $\varphi^l(t,a)$, which has bounded derivatives.
2. comes from the pricing BSDE (5.4.15) under the MMM Q^0, which implies

$$\frac{\mathcal{E}_T}{\mathcal{E}_t}(-\gamma Z^{X,l}\cdot W^X) = e^{-\gamma\left(\int_t^T Z^{X,l}_r dW^X_r + \frac{\gamma}{2}\int_t^T \|Z^{X,l}_r\|^2 dr\right)} = e^{\gamma(Y^l_t - f^l + \int_t^T Z^{S,l}_r dW^{S,0}_r)}.$$

3. is a consequence of boundedness from below of f.
4. is derived from boundedness of $Z^{S,l}$, uniformly in l (so that C does not depend on l).
5. is obtained by applying the measure change $\frac{d\bar{Q}}{dQ^0} = \mathcal{E}_T(\gamma Z^{S,l}\cdot W^{S,0})$.

6. the inequality comes from Assumption 5.4.2 (ii) and the fact that the drift changes induced by the measure change $\frac{d\bar{Q}}{dQ^0}$ are bounded and only pertain the tradable assets. In particular the dynamics of S^i under $\bar{Q}$ can be controlled by noticing

$$S^i_T = S^i_t e^{\gamma\sigma_i \cdot \int_t^T Z^{S,l}_u du - \frac{\|\sigma_i\cdot\|^2}{2}(T-t)+\sigma_i\cdot(W^{S,\bar{Q}}_T - W^{S,\bar{Q}}_t)} \leq C S^i_t e^{-\frac{\|\sigma_i\cdot\|^2}{2}(T-t)+\sigma_i\cdot(W^{S,\bar{Q}}_T - W^{S,\bar{Q}}_t)}.$$

   The inequality above is due to the fact that, by Assumption 5.4.2 (ii), there exist a threshold $M > 0$ such that $|\varphi^l_{s^i}(t, A_t)| \leq C/S^i_t$ when $|S^i_t| \geq M$, otherwise it is bounded. Thus one obtains

$$|\gamma Z^{S,l}_t| = |\gamma\sigma(S_t)'\varphi^l_s(t, A_t)| \leq C\gamma \left\|\sigma(S_t)\frac{1}{S_t}\right\|,$$

   where one can easily check that the last term on the RHS is constant.
7. is the derived from the definition of C^l and using the density of X_T (i.e. the multivariate Gaussian). In fact, taking for simplicity just one discontinuity point at zero we immediately see that

$$E^0_{t,x}\left[C^l(X_T)\right] = ClP_{t,x}(\|X_T\| \leq 1/l) \leq Cl\frac{1}{l}\frac{1}{\det(\mathrm{Var}_{t,x}(X_T))^{1/2}} \leq \frac{C}{\sqrt{T-t}}$$

   with the obvious notations for conditional variance and probability.
8. Since f has uniform polynomial growth in s, the same holds for f^l (uniformly in l). Therefore $Y^l_t \leq E^0_{t,a}[f^l(A_T)] \leq C + CE^0_{t,s}[\|S_T\|^q] \leq C(1 + \|s\|^q)$.

Using the previous estimate (5.4.20), we can apply the usual stability properties (Lemma 6.2 in [FS06]) to get $Y_t = \lim_l \varphi^l(t, A_t) = \varphi(t, A_t)$, where φ is viscosity solution of (5.4.1). We now would like to prove that φ has continuous first derivatives in all space variables. Since $Z^{X,l}$ is locally bounded uniformly in l by (5.4.20) we can use Lemma 5.7.3 componentwise (together with Lemma 5.7.1) to get the uniform integrability property allowing us to use dominated convergence and obtain

$$\varphi^l_a(t, a) \to g(t, a) := E^0_{t,a}\left[f(A_T)N_T - \frac{\gamma}{2}\int_t^T \|Z^X_r\|^2 N_r dr\right].$$

To conclude it suffices to show that g is continuous and that $g = \varphi_a$. This can be done by exactly the same arguments that we used at the end of the proof of Theorem 5.4.1 (i) under Assumption 5.4.1. For this reason, we omit this part of the proof. □

Remark 5.4.3. Had we supposed directly the multiplicative form $f(s, x) = g(x)h(s)$ with a bounded g then we could have allowed for a countable (and not simply finite) number of discontinuities in g. This is true by remarking that in (5.4.20) we could have used Theorem 3.2 in [Zh05], by considering the function $u^l(t, x) = E^0_{t,x}[g^l(X_T)]$ (corresponding to the trivial linear BSDE arising from the martingale representation theorem) and the estimates on its derivative $u^l_x(t, x) = E^0_{t,x}[g^l_x(X_T)]$.

Remark 5.4.4. Here we focused on the case of discontinuities only taking place in the x-variables, as it turns out to be the most useful case in the applications (See Section 5.5). The arguments in the previous proof (in particular estimate (5.4.20)) can, however, be easily adapted to the case where discontinuities take place only in the s variables, provided the payoff has polynomial growth in x, uniformly in s.

The optimal hedging strategy

The previous results (stating the differentiability of UIP) allows us to represent Z^S in terms of the derivatives of the solution of a PDE. Indeed, when f is bounded, the optimal strategy can be immediately recovered by $\Delta_t = -\sigma^{-1} Z_t^S$, using Lemma 5.3.1. The next result gives a slight generalization to the case where f has polynomial growth in the traded assets.

Proof of Theorem 5.4.1 (ii). Approximate f as in Lemma 5.4.1 with a sequence f^l, where each of its element can always be taken to be bounded. By Lemma 5.3.1, the corresponding optimal strategies with the claims f^l are given by $\hat{\pi}_t^l = -\sigma^{-1}\sigma(S_t)\varphi_s^l(t, A_t) + \frac{1}{\gamma}\sigma^{-2}\mu$ and the value functions are

$$u^l(t, v, a) = \sup_\pi E_{t,a}\left[-e^{-\gamma\left(V_T^v(\pi)+f^l\right)}\right] = E_{t,a}\left[-e^{-\gamma\left(V_T^v(\hat{\pi}^l)+f^l\right)}\right].$$

By the growth assumptions in s (uniform in x) we deduce that the assumptions of Lemma 5.3.3 are satisfied and therefore

$$u^l \to u \tag{5.4.21}$$

for all $(t, v, a) \in [0, T] \times \mathbb{R} \times \mathbb{R}_+^n \times \mathbb{R}^d$, where

$$u(t, v, a) = E_{t,a}\left[-e^{-\gamma\left(V_T^v(\hat{\pi})+f\right)}\right]$$

for some optimal $\hat{\pi}$. We would like to identify $\hat{\pi}$ with $\tilde{\pi}_t := -\sigma^{-1}\sigma(S_t)\varphi_s(t, A_t) + \frac{1}{\gamma}\sigma^{-2}\mu$. An application of the reverse Fatou's Lemma gives

$$\limsup_l E_{t,a}\left[-e^{-\gamma\left(V_T^v(\hat{\pi}^l)+f^l\right)}\right] \le E_{t,a}\left[\lim_l -e^{-\gamma\left(V_T^v(\hat{\pi}^l)+f^l\right)}\right], \tag{5.4.22}$$

where the limit on the left is meant to be in probability. To show that this limit exists, remark first that $\hat{\pi}^l \to \tilde{\pi}$ in $\mathbb{H}^2(\mathbb{R}^n)$, which implies that $V_T^v(\hat{\pi}^l)$ converges to $V_T^v(\tilde{\pi})$ in $L^2(\Omega, P)$, hence in probability. In the same way, $f^l \to f$ in probability. By using (5.4.21) and the continuity of the exponential function, (5.4.22) becomes

$$E_{t,a}\left[-e^{-\gamma\left(V_T^v(\hat{\pi})+f\right)}\right] \le E_{t,a}\left[-e^{-\gamma\left(V_T^v(\tilde{\pi})+f\right)}\right],$$

which implies that $\tilde{\pi}$ is indeed optimal (remark that it is in $\mathbb{H}^2(\mathbb{R}^n)$ for any measure $Q \in \mathcal{M}_V$, therefore it lies in $\mathcal{H}_M$). □

5.4.2 Asymptotic expansions

In this subsection we turn to the problem of computing effectively the UIP and the corresponding optimal hedging strategy for a given contingent claim. It is well-known that solving PDE (5.4.1) numerically can be impractical for time reasons when the number of assets is large. It is therefore useful to derive some asymptotic expansions which allow to approximate the price and the hedging strategy when the risk aversion parameter γ is small. The formulas are given in terms of the no-arbitrage price and strategy, which can usually be computed in a much simpler way either explicitly or by numerical integration

or by Monte Carlo methods.
Consider a contingent claim with payoff $f(A_T)$ integrable under the MMM Q^0, whose no-arbitrage price under Q^0 is denoted by $p^0(t,a) = E^0[f(A_T)]$. Now define

$$\zeta(t,a) := E^0_{t,a}\left[\int_t^T \|\beta' p^0_x\|^2(s, A_s)ds\right].$$

The next result is due to a recent preprint by Monoyios ([Mo12]).

Lemma 5.4.4. *Under Assumption 5.4.1 or 5.4.2 for the contingent claim $f(A_T)$, the following asymptotic expansion holds:*

$$\varphi(t,a) = p^0(t,a) - \frac{\gamma}{2}\zeta(t,a) + O(\gamma^2). \tag{5.4.23}$$

Proof. This is a reformulation of [Mo12], Theorem 5.3. It is enough to remark that our growth assumptions on f ensure that it is in $L^2(Q)$ for any $Q \in \mathcal{M}_E$. □

The next result provides asymptotic expansions for the derivatives of the price, and therefore of the optimal hedging strategy.

Lemma 5.4.5. *Suppose Assumption 5.4.1 holds, and moreover that f_x is bounded. Then the following asymptotic expansions hold*

$$\varphi_{x^i}(t,a) = E^0_{t,a}\left[f_{x^i}(A_T)\right] - \gamma E^0_{t,a}\left[f_{x^i}(A_T)\int_t^T \beta'\varphi^0_x(u,A_u)dW^X_u\right] + O(\gamma^2) \tag{5.4.24}$$

$$\varphi_{s^i}(t,a) = E^0_{t,a}\left[f_{s^i}(A_T)\right] - \gamma E^0_{t,a}\left[f_{s^i}(A_T)\int_t^T \beta'\varphi^0_x(u,A_u)dW^X_u\right] + O(\gamma^2) \tag{5.4.25}$$

where $\varphi^0_{x^i}(t,a) = E^0_{t,a}\left[f_{x^i}(A_T)\right]$.

Proof. By considering as usual a sequence of approximating functions we get that

$$\varphi^l_{x^i}(0,a) = E^0_{t,a}\left[\mathcal{E}_T(-\gamma\beta'\varphi^l_x \cdot W^X) f^l_{x^i}(A_T)\right]$$

is bounded, uniformly in l. By taking $l \to \infty$ we get

$$\varphi_{x^i}(t,a) = E^0_{t,a}\left[\mathcal{E}_T(-\gamma\beta'\varphi_x \cdot W^X) f_{x^i}(A_T)\right],$$

which is bounded. Now we write φ^γ to emphasize dependence on γ. So we have

$$\frac{\varphi^\gamma_{x^i} - \varphi^0_{x^i}}{\gamma}(t,a) = E^0_{t,a}\left[\frac{\mathcal{E}_T(-\gamma\beta'\varphi^\gamma_x \cdot W^X) - 1}{\gamma} f_{x^i}(A_T)\right].$$

In the rest of the proof for simplifying the notation, we will prove the expansions only at $t = 0$, otherwise the same arguments (conditionally to $\mathcal{F}_t$) apply and get the result for any t. Moreover, we will denote the process $\varphi^\gamma_x(t, A_t)$ by φ^γ_x with a slight abuse of notation.

Remark that, defining M^γ as the unique solution to $dM_t^\gamma = -\gamma M_t^\gamma \beta' \varphi_x^\gamma(t, A_t) dW_t^X$ with initial condition $M_0^\gamma = 1$, we have

$$\begin{aligned}
E^0\left[\left(\frac{\mathcal{E}_T(-\gamma\beta'\varphi_x^\gamma \cdot W^X) - 1}{\gamma} + \int_0^T \beta'\varphi_x^0 dW_s^X \right)^2\right] &= E^0\left[\left(\int_0^T (\beta'\varphi_x^0 - M_s^\gamma \beta'\varphi_x^\gamma) dW_s^X\right)^2\right] \\
&= E^0\left[\int_0^T \|\beta'\varphi_x^0 - M_s^\gamma \beta'\varphi_x^\gamma\|^2 ds\right] \\
&\leq 2E^0\left[\int_0^T \|\beta'\varphi_x^0 - \beta'\varphi_x^\gamma\|^2 ds\right] + 2E^0\left[\int_0^T \|\beta'\varphi_x^\gamma\|^2 (1 - M_s^\gamma)^2 ds\right] \\
&\leq CE^0\left[\int_0^T (1 - M_s^\gamma)^2 ds\right],
\end{aligned}$$

where the second equality is due to Itô's isometry, since the integrand therein belongs to $\mathbb{H}^2(\mathbb{R}^d)$. Since f_{x^i} is bounded by assumption, φ_x^γ is also bounded and this implies that $E^0[\int_0^T (1 - M_s^\gamma)^2 ds]$ tends to zero as $\gamma \to 0$ by dominated convergence. Thus

$$\frac{\mathcal{E}_T(-\gamma\beta'\varphi_x^\gamma \cdot W^X) - 1}{\gamma} \to -\int_0^T \beta'\varphi_x^0 dW_t^X$$

in L^2 as $\gamma \to 0$, and therefore

$$\frac{\partial}{\partial\gamma}\varphi_{x^i}^\gamma\Big|_{\gamma=0} = \lim_{\gamma\to 0} \frac{\varphi_{x^i}^\gamma - \varphi_{x^i}^0}{\gamma} = -E^0\left[f_{x^i}(A_T)\int_0^T \beta'\varphi_x^0 dW_s^X\right].$$

The proof for φ_{s^i} is analogous. □

We conclude this section with a lower bound on the utility indifference price of f.

Lemma 5.4.6. *Under Assumptions 5.4.1 or 5.4.2 the following bound on the price holds:*

$$\varphi(t, a) \geq -\frac{1}{\gamma} \log E_{t,a}^0\left[e^{-\gamma f(A_T)}\right].$$

Proof. Define

$$h(t, a) = E_{t,a}^0\left[e^{-\gamma f(A_T)}\right]$$

which solves

$$\begin{cases} \mathcal{L}h = 0 \\ h(T, a) = e^{-\gamma f(a)} \end{cases}$$

in the classical sense (assuming f to be smooth). Now set $g = -\frac{1}{\gamma}\log h$, so that g solves

$$\begin{cases} \mathcal{L}g - \frac{\gamma}{2}\|\sigma(S)'g_s\|^2 - \frac{\gamma}{2}\|\beta' g_x\|^2 = 0 \\ g(T, a) = f(a). \end{cases}$$

By the comparison theorem for PDEs we have that $g(t, a) \leq \varphi(t, a)$. By our approximation arguments the same bound holds true when f is not smooth. We left the details to the reader. □

5.5 Application to electricity markets

Our framework can be particularly useful to evaluate derivatives in situations where the underlying asset prices are determined by the interplay between several factors, but only some of these can be actually traded on a financial market (while the others may be of a totally different nature, for example macroeconomic or even behavioral factors). This is the case in particular for structural models of electricity prices, where the relevant components that influence the price are typically both tradable (like fuels) and non tradable (like market demand or production capacities) †.
The seminal contribution in the direction of structural electricity models has been the Barlow's model ([Ba02]), which describes the electricity spot price as a function of a one-dimensional diffusion representing the evolution of market demand. Since there is only a non tradable asset in his framework, utility indifference valuation here reduces to the computation of the certainty equivalent (see Remark 5.2.3), at least when prices are bounded (an assumption which is suggested by Barlow himself and which reflects the reality of electricity markets, where prices are usually capped). Similar considerations hold for the models in [SGI00] or [CV08], where an exponential function is used and an additional non tradable factor is added describing maximal capacity.
Building on this literature, several authors have proposed more developed structural models with the aim of capturing the contribution of other assets, notably the (marginal) fuels employed in electricity generation along with their production capacities. Since fuels are commodities which are typically traded on financial markets, their introduction fully justifies the employment of pricing techniques that allow for some kind of partial hedging (such as local risk minimization or, in our case, utility indifference pricing). For example, in [PJ08] the authors describe the spot price as the product of two components accounting for a traded and a non traded asset (following, respectively, a geometric Brownian motion and an Ornstein-Uhlenbeck process as in our framework). Multi-asset models have then followed, with the aim of considering the whole stack of available fuels, which typically present different levels of correlation with the spot price depending on their available capacities and market demand. They enter in our framework, possibly with some minor adaptations.

In this paper we will focus especially on the model introduced in [ACL10], where the authors directly model the spreads between fuels as geometric Brownian motions, hence the tradable assets of our model S_t^i can be interpreted in this case as those fuel spreads by using the relation

$$S_t^i = h_i K_t^i - h_{i-1} K_t^{i-1},$$

where K_t^i is the price at time t of i-th fuel and the h_i's are heat rates associated to each fuel. The model also includes fuel capacities C_t^i and a process D_t describing the demand for electricity, which make for $d = n + 1$ nontradable assets. In [ACL10], the dynamics postulated for tradable and nontradable assets perfectly fit into our setting, since the spread between two fuels follow a multidimensional Black-Scholes model while the non tradable ones follow Ornstein-Uhlenbeck processes with non zero mean-reversion and a seasonality component that can be embedded in the function $b(t)$ as in (5.2.2). More

†. We refer the reader to [CC12] for a comprehensive survey of structural models.

precisely, we have

$$\frac{dS_t^i}{S_t^i} = \mu_i dt + \sigma_i dW_t^{S,i}, \quad i = 1, \dots, n \tag{5.5.1}$$

$$dC_t^j = (b_{C^j}(t) - \alpha_{C^j} C_t^j)dt + \beta_{C^j} dW_t^{C^j} \quad j = 1, \dots, n \tag{5.5.2}$$

$$dD_t = (b_D(t) - \alpha_D D_t)dt + \beta_D dW_t^D, \tag{5.5.3}$$

where we also supposed that the stochastic components of the assets are independent (compare with equation (4.2) in [ACL10]), i.e. the Brownian motions W^{C^j} and W^D are assumed to be independent. The coefficients $\mu_i, \alpha_{C^j}, \alpha_D$ are arbitrary constants while $\sigma_i, \beta_{C^j}, \beta_D$ are strictly positive real numbers. Moreover, $b_{C^j}(t)$ and $b_D(t)$ are deterministic bounded functions that possibly include the seasonality component of nontraded asset dynamics.

One of the main goals of structural models for energy markets (included the one in [ACL10]) is to have a realistic and tractable setting where pricing and hedging power derivatives. One of the most important derivatives to price and hedge is the forward contract on electricity, with payoff given by the value at maturity of the electricity spot price, which in [ACL10] can be written as

$$f(a) = f(s, c, y) = g\left(\sum_{i=1}^n c^i - y\right) \sum_{i=1}^n h_i k^i \mathbf{1}_{\{y \in I^i\}} = g\left(\sum_{i=1}^n c^i - y\right) \sum_{j \le i \le n} s^j \mathbf{1}_{\{y \in I^i\}} \tag{5.5.4}$$

where g is a bounded function with bounded first derivatives, c^i and y stand for fuel capacities and market demand, and we used the fact that $h_i K_t^i = \sum_{j \le i} S_t^j$. The function g is called *scarcity function*, it has a crucial role for producing spikes in electricity spot prices (see the paper [ACL10] for further details).

The payoff (5.5.4), as it is, does not satisfy neither Assumption 5.4.1 or 5.4.2, however it can be made to satisfy

- Assumption 5.4.1 by replacing indicators with continuous functions, or
- Assumption 5.4.2 by bounding the payoff by some constant M (which makes sense since in reality, as already remarked, electricity prices are capped).

The same observations hold for the utility indifference pricing of the quite popular spread options, which present a payoff which is either bounded or linearly growing in the electricity price.

Remark 5.5.1. Substantially equivalent considerations hold for the electricity spot price model proposed in [CCS12] (equation (6)), which still uses a multiplicative form separating the contributions of traded and non traded assets (in a more involved way than in [ACL10], with the drawback of becoming rather messy when more than two assets are considered): bounding the payoff of the forward contract makes it satisfy Assumption 5.4.2 (remark that it is usually discontinuous in the non traded assets). More generally, as reported in [CC12] (Chapter 5), most of the structural models found in the literature assume lognormal fuel prices, OU-driven demand and an electricity price which is multiplicative in the marginal fuel, which justifies our standing assumptions. Markov switching models like the one described in [CC12], equation (10), can also be treated in our framework as the structure of the payoff is standard, and additional indicator functions can be added to describe the different regimes (which create discontinuities in the non traded assets).

Remark 5.5.2. Notice that the BSDE approach developed in Section 5.3 can be applied without modifying the payoffs as suggested above. Indeed, in many important examples such as forward contract and call options on spread the sufficient conditions established in Lemma 5.3.3 are easily checked, due to the simple multiplicative structure of the set of equivalent martingale densities implied by the independence between tradable and non tradable assets. On the other hand, to get asymptotic expansions of prices and strategies one should still use results in Section 5.4.

When the payoff f is linear or concave in the traded assets (as in the case of the forward contract in [ACL10]) we have the following result.

Lemma 5.5.1. *If $f(s,x)$ is concave in s, the same holds for its UIP $\varphi(t,s,x)$.*

Proof. By Lemma 5.4.1 and using an approximating sequence f^l, the price is represented as

$$\varphi^l(t,s,x) = E^Q_{t,a}\left[\frac{1}{2\gamma}\int_t^T \|\hat{\delta}_r\|^2 dr + f^l(S_T, X_T)\right]$$

and therefore, setting $\tilde{a} = (\tilde{s}, x)$, we have

$$\begin{aligned}
&\varphi^l(t, \lambda s + (1-\lambda)\tilde{s}, x) \\
&\quad \geq \lambda E^Q_{t,a}\left[\frac{1}{2\gamma}\int_t^T \|\hat{\delta}_r\|^2 dr + f^l(S_T, X_T)\right] + (1-\lambda) E^Q_{t,\tilde{a}}\left[\frac{1}{2\gamma}\int_t^T \|\hat{\delta}_r\|^2 dr + f^l(S_T, X_T)\right] \\
&\quad \geq \lambda \inf_\delta E^Q_{t,a}\left[\frac{1}{2\gamma}\int_t^T \|\delta_r\|^2 dr + f^l(S_T, X_T)\right] \\
&\quad + (1-\lambda)\inf_\delta E^Q_{t,\tilde{a}}\left[\frac{1}{2\gamma}\int_t^T \|\delta_r\|^2 dr + f^l(S_T, X_T)\right] \\
&\quad = \lambda\varphi^l(t,s,x) + (1-\lambda)\varphi^l(t,\tilde{s},x), \quad \lambda \in [0,1].
\end{aligned}$$

Now it is enough to take limits to get the result. □

Example 5.5.1 (Forward contract for $n = 2$ fuels)**.** We derive here a more explicit expression for the first term $\zeta(0,a)$ of the asymptotic expansion (5.4.23) of the price at time zero for a forward contract with two fuels as described in [ACL10], with payoff

$$f(a) = f(s,c,y) = g\left(c^1 + c^2 - y\right)(s^1 + s^2 \mathbf{1}_{\{y - c^1 > 0\}}).$$

The assets dynamics are given in (5.5.1), where we also assume the seasonality components to be zero for clearness (they would only appear as a mean component in the expressions for the derivatives of ψ below). The no-arbitrage price under the MMM Q^0 is

$$p^0(t,a) = E^0_{t,a}[f(A_T)] = \psi^1(t,x)s^1 + \psi^2(t,x)s^2$$

where $a = (s,x)$, $s = (s^1, s^2)$, $x = (c^1, c^2, y)$, and

$$\psi^i(t,x) = \int_{\mathbb{R}^2} \Psi_{C^1_T - D_T}(t,z)\Psi_{C^2_T}(t,c)g(c+z)\chi^i(z)dcdz$$

for $i = 1, 2$, where we set

$$\chi^i(z) := \mathbf{1}_{\{z<0\}} + \mathbf{1}_{\{z\geq 0, i=1\}}$$

and $\Psi_{C_T^1 - D_T}(t, \cdot)$ stands for the conditional density of $C_T^1 - D_T$ given $C_t^1 = c^1, D_t = y$ (and similarly for $\Psi_{C_T^2}(t, \cdot)$). Notice that an explicit expression for the price $p^0(t, a)$ has been obtained in [ACL10] together with an efficient numerical method to compute it. Based on the previous expression, we can obtain an explicit formula for the derivatives of $p^0(t, a)$ as an intermediate step towards the optimal hedging strategy. We have

$$p_x^0(t, a) = \begin{pmatrix} \psi_{C^1}^1(t, x)s^1 + \psi_{C^1}^2(t, x)s^2 \\ \psi_{C^2}^1(t, x)s^1 + \psi_{C^2}^2(t, x)s^2 \\ \psi_D^1(t, x)s^1 + \psi_D^2(t, x)s^2 \end{pmatrix}$$

where

$$\psi_{C^1}^i(t, x) = \frac{e^{-\alpha_{C^1}(T-t)}}{\mathrm{Var}_t(C_T^1 - D_T)} \cdot \int_{\mathbb{R}^2} (z - c^1 e^{-\alpha_{C^1}(T-t)} + y e^{-\alpha_D(T-t)}) \Psi_{C_T^1 - D_T}(t, z) \Psi_{C_T^2}(t, c) g(c + z) \chi^i(z) dc dz$$

$$\psi_{C^2}^i(t, x) = \frac{e^{-\alpha_{C^2}(T-t)}}{\mathrm{Var}_t(C_T^2)} \int_{\mathbb{R}^2} (c - c^2 e^{-\alpha_{C^2}(T-t)}) \Psi_{C_T^1 - D_T}(t, z) \Psi_{C_T^2}(t, c) g(c + z) \chi^i(z) dc dz$$

$$\psi_D^i(t, x) = -\frac{e^{-\alpha_D(T-t)}}{\mathrm{Var}_t(C_T^1 - D_T)} \cdot \int_{\mathbb{R}^2} (z - c^1 e^{-\alpha_{C^1}(T-t)} + y e^{-\alpha_D(T-t)}) \Psi_{C_T^1 - D_T}(t, z) \Psi_{C_T^2}(t, c) g(c + z) \chi^i(z) dc dz$$

for $i = 1, 2$ with Var_t denoting the conditional variance at time t, which in our case can be explicitly computed since $C^1 - D$ and C^2 are generalized Ornstein-Uhlenbeck processes with time-dependent deterministic coefficients (the details are left to the reader). By defining

$$\phi^i(j, x) = \int_0^T e^{\sigma_i^2 (T-t)} E_{0,x}[\beta_j^2 \psi_j^i(t, X_t)^2] dt, \quad \phi^{12}(j, x) = \int_0^T E_{0,x}[\beta_j^2 \psi_j^1(t, X_t) \psi_j^2(t, X_t)] dt,$$

for $i = 1, 2$ and $j \in \{C^1, C^2, D\}$, we finally obtain

$$\zeta(0, a) = \left(\sum_{j \in \{C^1, C^2, D\}} \phi^1(j, x) \right) (s^1)^2 + \left(\sum_{j \in \{C^1, C^2, D\}} \phi^2(j, x) \right) (s^2)^2 + \left(\sum_{j \in \{C^1, C^2, D\}} \phi^{12}(j, x) \right) s^1 s^2.$$

Remark 5.5.3. By direct computation as above, one can also obtain similar expressions for spread call options. Pricing spread call options is particularly important in energy markets since such derivatives constitute the building blocks for evaluating the central plants in the real option approach as in, e.g., [CCS12] (see the next section for a comparison between UIP and the non-arbitrage MMM price of spread call options).

5.5.1 Numerical examples

Consider the payoff of a forward contract with one fuel $f(s,c) = sg(c)$, where c models here the difference between demand and capacity as a unique Ornstein-Uhlenbeck process, and

$$g(c) = \min\left(M, \frac{1}{c}\right)\mathbf{1}_{\{c>0\}} + M\mathbf{1}_{\{c>0\}}.$$

Figure 5.1 shows the no-arbitrage forward price at a given time to maturity, while in Figure 5.2 we plotted the difference in the price and the hedging strategy when switching from utility indifference evaluation to no-arbitrage pricing. As expected, the no-arbitrage price is always higher than the utility indifference (buying) price, and the difference is larger in the region where the payoff is more sensitive to the nontraded assets (that is around $c = 0.8$ in our examples). The hedging strategy is also lower (in absolute value) for utility indifference valuation. As reported in [ACL10] (Section 4.3.1), the hedging performance with the no-arbitrage paradigm is particularly poor close to maturity in the regions where nontraded assets have a larger influence on the price: intuitively, the introduction of risk aversion reduces the amount of hedging precisely where it is most ineffective. In the case of a selling price, we would have obtained the opposite effect: the agent in this case would require a higher price to sell the asset, in order to finance a higher hedging activity in the riskier region.
For a similar comparison, we now take an option on spread with payoff

$$(P_T - h_1 S_T^1 - K)^+.$$

By taking a market with two fuels, and assuming an inelastic demand D and a constant spike function the payoff can be simplified to

$$(\tilde{S}_T \mathbf{1}_{\{C_T<D\}} - K)^+, \tag{5.5.5}$$

where $\tilde{S} = h_2 S^2 - h_1 S^1$ is the fuel spread. As expected, Figure 5.3 shows that the pricing differences generated by utility indifference evaluation lie mostly around the discontinuity in the capacity, and they grow with the moneyness of the option (being almost negligible when the option is at the money). All the figures were obtained using a standard explicit finite difference scheme. Parameter values are arbitrary and only for illustrative purposes.

5.6 Conclusions

In this paper we considered the utility indifference pricing problem in a particular market model that includes tradable and nontradable assets, and where the derivatives' payoffs

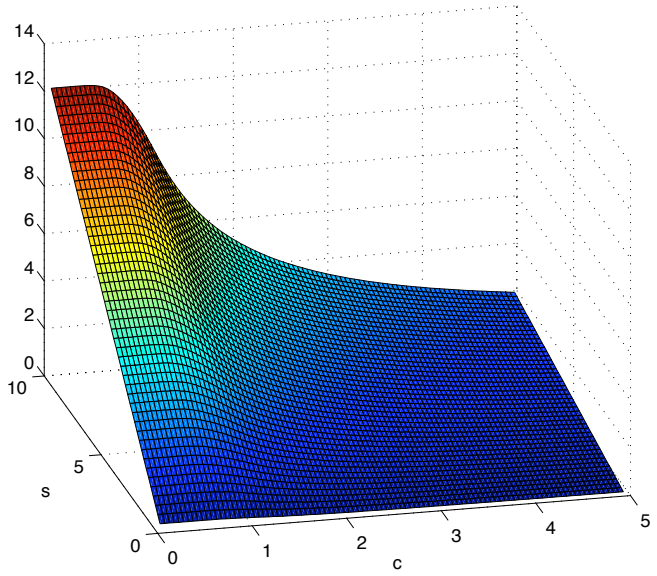

Figure 5.1: No-arbitrage price of a forward contract at a given time to maturity $T-t = 0.5$. Parameter values: $\sigma = \beta = 0.3$, $\alpha = 0.2$, $\frac{1}{M} = 0.8$.

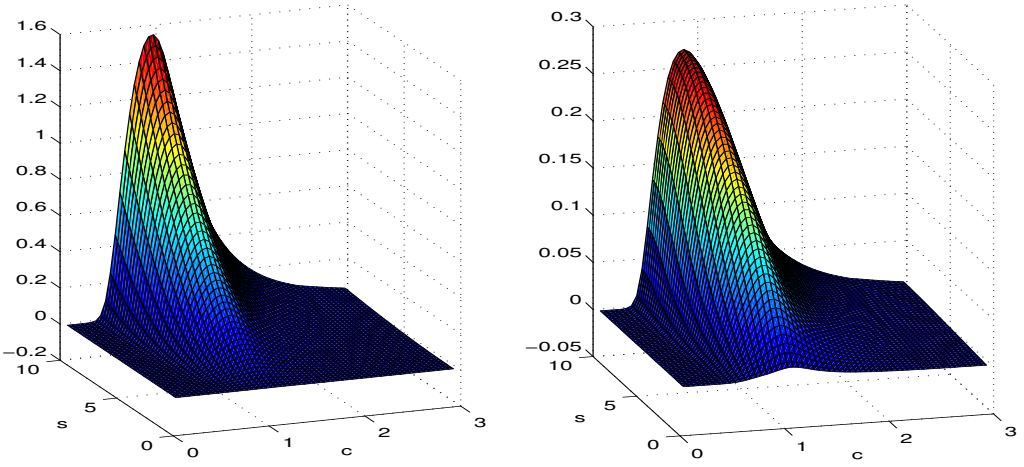

Figure 5.2: Absolute difference in the price (left) and hedging strategy (right), under no-arbitrage and utility indifference evaluation (with $\gamma = 5$) of a forward contract.

possibly depend on both classes. Using a combination of BSDE and PDE techniques, we established some existence and regularity results for the price, showing in particular how they can be applied to the pricing and hedging of power derivatives under a structural modeling framework. Although we did not aim for the greatest generality we believe that, under suitable assumptions, most of the results could be extended to a broader set of

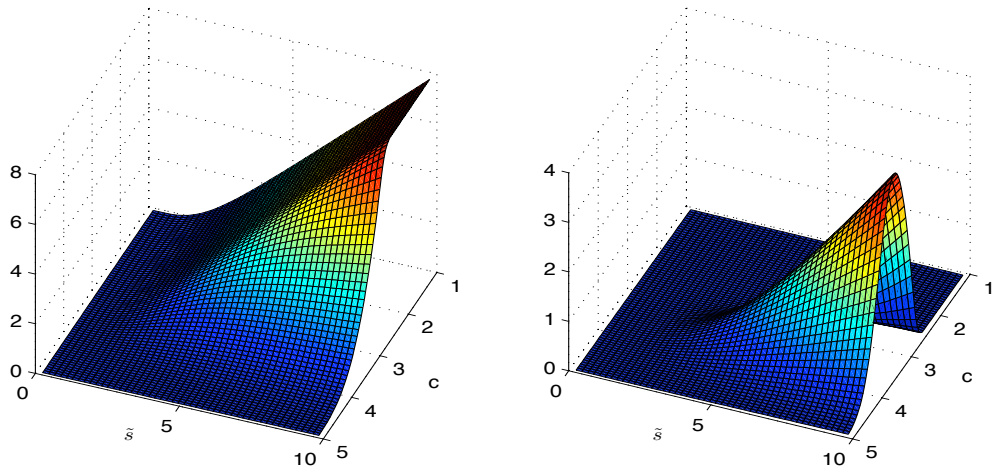

Figure 5.3: No-arbitrage price (left) and pricing difference with utility indifference evaluation (right) of an option on spread with payoff given by (5.3). Parameter values: $\sigma = \beta = 0.3$, $\alpha = 0.2$, $T - t = 2.5$, $\gamma = 2$, $D = 3$, $K = 2$.

asset dynamics. Nevertheless, we remark that our framework already allows to consider derivatives written on underlyings that possibly exhibit spikes and discontinuities (as it is the case for electricity prices).

5.7 Auxiliary results and their proofs

Lemma 5.7.1. *Let $f \in L^1(Q_0)$ be bounded from below and let (Y, Z), with $Z = (Z^S, Z^X)$, be a solution to the BSDE (5.3.2). Assume that for some $q > 0$ there exists a constant $C > 0$ such that $\|Z_t^S\| \leq C\|S_t\|^q$ for all $t \in [0, T]$. Then the solution of (5.3.2) satisfies, for all $p > 1$*

$$E^0\left[\left(\int_0^t \|Z_u^X\|^2 du\right)^p\right] \leq CE^0\left[\left(\int_0^t \|\xi_u\|^2 du\right)^{p/2} + 1\right]$$

where ξ comes from the martingale representation of f under the MMM Q^0.

Proof. Consider the BSDE (5.3.5)

$$Y_t = f + \int_t^T g(Z_r)dr - \int_t^T Z_r dW_r^0$$

write the generator as $g(z) = -\frac{\gamma}{2}\|(0, z^X)\|^2 = -\frac{\gamma}{2}\|z\|^2 + \frac{\gamma}{2}\|(z^S, 0)\|^2$. Notice that $g(Z_r)$ can also be expressed as

$$g(Z_r) = -\frac{\gamma}{2}\|Z_r\|^2 + a(t),$$

with $a(t) = \frac{\gamma}{2}\|(Z_t^S, 0_d)\|^2$, which satisfies $|a(t)| \leq C'\|S_t\|^{2q}$ for some constant $C' > 0$. We now assume that f is positive, the case where it is only bounded from below being

analogous. Consider the function

$$u(x) = \frac{1}{\gamma^2}(e^{-\gamma x} - 1 + \gamma x), \quad x \geq 0,$$

from $\mathbb{R}_+$ to itself. Remark that $u(x) \geq 0$ and $u'(x) \geq 0$ for $x \geq 0$. Moreover, $\gamma u'(x) + u''(x) = 1$ and $u(x) \leq \frac{x}{\gamma}$, $u'(x) \leq \frac{1}{\gamma}$, $u''(x) \leq 1$ for $x \geq 0$. Defining

$$\tau_\kappa = \inf\{t \geq 0 : \int_0^t \|Z_u\|^2 du \geq n\}, \quad \inf \emptyset = +\infty,$$

and applying Itô's lemma we get

$$\begin{aligned}
u(Y_0) &= u(Y_{t\wedge\tau_\kappa}) + \int_0^{t\wedge\tau_\kappa} \left(u'(Y_s)g(Z_s) - \frac{1}{2}u''(Y_s)\|Z_s\|^2\right) ds - \int_0^{t\wedge\tau_\kappa} u'(Y_s)Z_s dW_s^0 \\
&\leq u(Y_{t\wedge\tau_\kappa}) + \int_0^{t\wedge\tau_\kappa} u'(Y_s)a(s) - \int_0^{t\wedge\tau_\kappa} \frac{1}{2}\left(\gamma u'(Y_s) + u''(Y_s)\right)\|Z_s\|^2 ds \\
&\quad - \int_0^{t\wedge\tau_\kappa} u'(Y_s)Z_s dW_s^0 \\
&= u(Y_{t\wedge\tau_\kappa}) + \int_0^{t\wedge\tau_\kappa} u'(Y_s)a(s) - \int_0^{t\wedge\tau_\kappa} \frac{1}{2}\|Z_s\|^2 ds - \int_0^{t\wedge\tau_\kappa} u'(Y_s)Z_s dW_s^0
\end{aligned}$$

therefore

$$\begin{aligned}
\frac{1}{2}\int_0^{t\wedge\tau_\kappa} \|Z_s\|^2 ds &\leq u(Y_{t\wedge\tau_\kappa}) + \int_0^{t\wedge\tau_\kappa} u'(Y_s)a(s)ds - \int_0^{t\wedge\tau_\kappa} u'(Y_s)Z_s dW_s^0 \\
&\leq Y_{t\wedge\tau_\kappa} + \int_0^{t\wedge\tau_\kappa} u'(Y_s)a(s)ds + \sup_{0\leq t\leq T}\left|\int_0^{t\wedge\tau_\kappa} u'(Y_s)Z_s dW_s^0\right|
\end{aligned}$$

and using the Burholder-Davis-Gundy inequalities we obtain

$$\begin{aligned}
E^0\left[\left(\int_0^{t\wedge\tau_\kappa} \|Z_s\|^2 ds\right)^p\right] &\leq CE^0\left[Y_{t\wedge\tau_\kappa}^p + \left(\int_0^{t\wedge\tau_\kappa} u'(Y_s)a(s)ds\right)^p\right] \\
&\quad + CE^0\left[\left(\int_0^{t\wedge\tau_\kappa} u'(Y_s)^2\|Z_s\|^2 ds\right)^{p/2}\right] \\
&\leq CE^0\left[Y_{t\wedge\tau_\kappa}^p + \left(\int_0^{t\wedge\tau_\kappa} u'(Y_s)a(s)ds\right)^p + 1\right] \\
&\quad + \frac{1}{2}E^0\left[\left(\int_0^{t\wedge\tau_\kappa} \|Z_s\|^2 ds\right)^p\right]
\end{aligned}$$

where we used Young's inequality in the last line. Therefore

$$\begin{aligned}
E^0\left[\left(\int_0^{t\wedge\tau_\kappa} \|Z_s\|^2 ds\right)^p\right] &\leq CE^0\left[\sup_{r\in[0,t]}(E_r^0[f])^p + \left(\int_0^t \|S_r\|^2 dr\right)^p + 1\right] \\
&\leq CE^0\left[\left(\sup_{r\in[0,t]}\int_0^r \xi_s dW_s\right)^p + 1\right] \\
&\leq CE^0\left[\left(\int_0^t \|\xi_s\|^2 ds\right)^{p/2} + 1\right]
\end{aligned}$$

where ξ comes from the martingale representation of f under Q^0. The result follows by Fatou's lemma. □

Lemma 5.7.2. *Let W be a $\mathbb{R}^{n+d}$-valued Brownian Motion, $T > 0$, $p > 1$ and $0 < \alpha < p/2$. Define $U_t = \int_0^t u(r)dW_r$, where u is a $\mathbb{R}^{n+d}$-valued deterministic bounded process. Then*

$$E\left[\sup_{0\le t\le T}\frac{|U_t|^p}{t^\alpha}\right] < \infty$$

Proof. By Dumbis-Dubins-Schwarz representation of the martingale U_t, there exists a Brownian motion $\widetilde{W}$ such that $U_t = \widetilde{W}_{\tau_t}$ where $\tau_t = \langle U\rangle_t = \int_0^t \|u(r)\|^2 dr$ is a deterministic bounded time change. Thus, using the scaling property of Brownian motion we have

$$\begin{aligned} E\left[\sup_{0\le t\le T}\frac{|U_t|^p}{t^\alpha}\right] &= E\left[\sup_{0\le t\le T}\frac{|\widetilde{W}_{\tau_t}|^p}{t^\alpha}\right] = E\left[|\widetilde{W}_1|^p\right]\sup_{0\le t\le T}\frac{\tau_t^{p/2}}{t^\alpha} \\ &\le CE\left[|\widetilde{W}_1|^p\right]\sup_{0\le t\le T} t^{p/2-\alpha} < \infty, \end{aligned}$$

which ends the proof.

□

Lemma 5.7.3. *Let W be a $\mathbb{R}^{n+d}$-valued Brownian motion, U be defined as in Lemma 5.7.2 and let K be a process in $\mathbb{H}^{q'}(\mathbb{R})$ for some $q' \ge 1$. Suppose, moreover, that $|K_t| \le F(t, W_t)$ for all $t \in [0, T)$ for some continuous function $F : [0, T) \times \mathbb{R}^{n+d} \to \mathbb{R}$. Then there exists $p' > 1$ such that*

$$E_t\left[\left(\int_t^T \frac{U_r - U_t}{(r-t)}K_r dr\right)^{p'}\right] < \infty.$$

Proof. We have, by choosing $0 < \alpha' < 1/2$ and applying Hölder's inequality

$$\begin{aligned} E_t\left[\left(\int_t^T \frac{U_r - U_t}{(r-t)}K_r dr\right)^{p'}\right] &= E_t\left[\left(\int_t^T \frac{U_r - U_t}{(r-t)^{\alpha'}}\frac{K_r}{(r-t)^{1-\alpha'}}dr\right)^{p'}\right] \\ &\le E_t\left[\left(\sup_{t\le r\le T}\frac{|U_r - U_t|}{(r-t)^{\alpha'}}\right)^{p'}\left(\int_t^T \frac{K_r}{(r-t)^{1-\alpha'}}dr\right)^{p'}\right] \\ &\le E_t\left[\left(\sup_{t\le r\le T}\frac{|U_r - U_t|}{(r-t)^{\alpha'}}\right)^{pp'}\right]^{1/p} E_t\left[\left(\int_t^T \frac{K_r}{(r-t)^{1-\alpha'}}dr\right)^{p'q}\right]^{1/q} \\ &= E_t\left[\sup_{t\le r\le T}\frac{|U_r - U_t|^{pp'}}{(r-t)^{pp'\alpha'}}\right]^{1/p} E_t\left[\left(\int_t^T \frac{K_r}{(r-t)^{1-\alpha'}}dr\right)^{p'q}\right]^{1/q} \\ &\le CE_t\left[\left(\int_t^T \frac{K_r}{(r-t)^{1-\alpha'}}dr\right)^{p'q}\right]^{1/q} \end{aligned}$$

by Lemma 5.7.2, where the $p > 1$ used above is arbitrary. Now set $p'q = q'$ and recall that $q' > 1$ and it can be chosen arbitrarily close to 1. Now define

$$\tau = \inf\{r > t : \|W_r - W_t\| \geq M\}, \quad \inf \emptyset = +\infty,$$

and notice that, for any $0 < \varepsilon < T - t$, when $t \leq r \leq \tau \wedge (T - \varepsilon)$ we have $|K_r| \leq \tilde{M}$, where $\tilde{M}$ is a constant depending on M and on the function F. Thus we obtain

$$\begin{aligned}
&E_t\left[\left(\int_t^T \frac{K_r}{(r-t)^{1-\alpha'}}dr\right)^{q'}\right] \\
&\leq E_t\left[\left(\int_t^{\tau\wedge(T-\varepsilon)} \frac{K_r}{(r-t)^{1-\alpha'}}dr\right)^{q'} + \left(\int_{\tau\wedge(T-\varepsilon)}^T \frac{K_r}{(r-t)^{1-\alpha'}}dr\right)^{q'}\right] \\
&\leq C + E_t\left[\left(\int_{\tau\wedge(T-\varepsilon)}^T \frac{K_r}{(r-t)^{1-\alpha'}}dr\right)^{q'}\right] \\
&\leq C + E_t\left[\frac{1}{(\tau\wedge(T-\varepsilon)-t)^{q'(1-\alpha')}}\left(\int_{\tau\wedge(T-\varepsilon)}^T |K_r|dr\right)^{q'}\right] \\
&\leq C + E_t\left[\frac{1}{(\tau\wedge(T-\varepsilon)-t)^{lq'(1-\alpha')}}\right]^{1/l} E_t\left[\left(\int_{\tau\wedge(T-\varepsilon)}^T |K_r|dr\right)^{q'\frac{l}{1-l}}\right]^{\frac{1-l}{l}} \\
&\leq C + CE_t\left[\frac{1}{(\tau\wedge(T-\varepsilon)-t)^{lq'(1-\alpha')}}\right]^{1/l}.
\end{aligned}$$

To conclude the proof it suffices to show that the expectation in the RHS of the last inequality is finite. This is a straightforward consequence of Lemma 5.7.4 below since, conditionally to $\mathcal{F}_t$, $(\|W_{t+u} - W_t\|)_{u\geq 0}$ is clearly a Bessel process of dimension $n + d$ and $lq'(1 - \alpha') > 1$. □

Lemma 5.7.4. *Let R be a Bessel process of any positive integer dimension $k \geq 1$ with $R_0 = 0$. Let $\tau_b := \inf\{t \geq 0 : R_t = b\}$ (with the convention $\inf \emptyset = \infty$) its first hitting time of a level $b > 0$. Then we have that $E[\tau_b^{-p}] < \infty$ for any $p \geq 1$.*

Proof. First notice that $a^{-(n+1)} = n! \int_0^\infty x^n e^{-ax} dx$ for all $n \geq 0$. Replacing a with τ_b, taking expectations on both sides and using Fubini's theorem, we get

$$E\left[\tau_b^{-(n+1)}\right] = n! \int_0^\infty x^n E\left[e^{-x\tau_b}\right] dx.$$

The Laplace transform for the hitting time τ_b ($b > 0$) of a k-dimensional Bessel process starting from zero is given by (see, e.g., [GJY03])

$$E\left[e^{-x\tau_b}\right] = \left(\frac{x}{2}\right)^{\nu/2} \Gamma^{-1}(\nu + 1) \frac{b^\nu}{I_\nu(b\sqrt{2x})},$$

where $\nu = k/2 - 1$ is the index of the Bessel process R, Γ denotes the Gamma function and I_ν is the modified Bessel function of the first kind of order ν. Thus, to conclude the

proof it suffices to show that

$$\int_0^\infty \frac{x^{n+\frac{\nu}{2}}}{I_\nu(b\sqrt{2x})}dx = C\int_0^\infty \frac{y^{\nu+1+2n}}{I_\nu(y)}dy < \infty,$$

for a constant $C > 0$, which easily follows from the asymptotic behavior of the modified Bessel function $I_\nu(y)$ for small and large y given in [Le72] (relations 5.16.4 and 5.16.5). □

Lemma 5.7.5. *Let f be a payoff satisfying Assumption 5.2.1. Suppose that the super-replicating price $V_t := V_t^{v_1}(\pi_1)$, expressed under the MMM Q^0 as*

$$K_t = \tilde{f} - \int_t^T L_s dW_s^{S,0}, \quad \tilde{f} = V_T,$$

for some adapted process L satisfying

$$E^0\left[\left(\int_0^T \|L_s\|^2 ds\right)^p\right] < \infty$$

for some $p > 1$. Then the solution (Y, Z) of (5.3.10) also verifies

$$E^0\left[\left(\int_0^T \|Z_s\|^2 ds\right)^p\right] < \infty.$$

Proof. Define

$$U_t = V_t - Y_t = \tilde{f} - f + \frac{\gamma}{2}\int_t^T \|Z_s^X\|^2 ds - \int_t^T ((L_s, 0) - Z_s)dW_s^0.$$

Clearly $U_t \geq 0$. Now if the conditions are satisfied, then following the proof of Lemma 5.7.1 we deduce that

$$E^0\left[\left(\int_0^t \|(L_s, 0) - Z_s\|^2 ds\right)^p\right] \leq CE^0\left[\left(\int_0^t \|L_s\|^2 ds\right)^{p/2} + 1\right]$$

for some constant C, which implies the result. □

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Chapter 6

A principal-agent problem and emission reduction

Abstract: We analyze a simple principal-agent problem in continuous time with finite horizon. We mainly think of the interplay between the state (principal) and a firm (agent) that produces carbon emissions, where the aim of the state is to motivate the firm to reduce those emissions according to some social cost criterion. Incentive strategies take the form of continuous time payments and a lump sum penalty at maturity. Given an incentive policy, we solve the agent's problem using the stochastic maximum principle to derive alternative representations of the optimal effort in terms of BSDEs, and showing uniqueness of the optimal effort up a certain class of policies. This also allows to derive some sensitivity results for the optimal effort with respect to some relevant parameters such as risk aversion and impatience rates. We finally deal with the principal's problem and show that under some assumptions it becomes similar to the agent's one and can be solved using analogous techniques. We also provide numerical examples to visualize the form of the optimal quantities in some special cases.

JEL Classification: H23
AMS Classification (2010): 91B70, 91G99

6.1 Introduction

Principal-agent problems in continuous time have been an active field of research in the recent years, since the seminal work [Sa08] that introduced some of the basic ideas and methods. The usual setting is that of private contracts, where an employer (principal) needs to design a contract in such a way that the agent will i) accept it, and ii) behave afterwards according to the principal's interest. The most general approach to these problems is probably the one in [CWZ08] (see also [CZ12]), where the authors derive FBSDE systems to characterize the optimal contract under very general assumptions. Similar results in a slightly different context are shown in [Wi08]. In [Sa08], instead, the particular structure of the model and the infinite horizon of the optimization problem allow to characterize the optimal contract with an ODE, which can be solved numerically and which permits a deeper study of the properties of the optimal players' behavior.
The present paper contributes to this field by studying in detail a particular version of a principal-agent problem (slightly different to the ones already appeared in the literature), motivated in particular by the recent ongoing discussion concerning the reduction of greenhouse gases emissions via the use of appropriate incentives and/or taxation schemes. In fact, the introduction of the Kyoto Protocol in 1997 has stimulated quite an intense debate over these themes, and the academic literature has also followed this growing interest, by focusing especially on the newly created financial market for emission allowances, on its price formation mechanisms and its possible effects on the firms' production decisions (see, for example, [BFT11], [CFH09], [CDET13] and references therein). However, given that the firms' emission reducing policies are typically only partially observable by the regulator, these kinds of issues seem to be also closely related to the classical economic concept of moral hazard, whose literature has by now quite a long history (see [Ho79] for one of the first mathematical treatments) culminating with the already mentioned recent studies on principal-agent problems in continuous time.
Our aim is to apply and develop some of the ideas and techniques of the principal-agent literature to a particular toy economic model where the principal is no longer interpreted as a private company but rather as a regulator, who aims at minimizing the social cost of carbon emissions X by imposing an appropriate incentive structure, made up by a continuous incentive process s and a final penalty p at a given maturity T (which typically intervenes when emissions are too high). The state process X is assumed to follow the dynamics

$$dX_t = X_t l(k_t)dt + X_t \sigma dW_t^k, \tag{6.1.1}$$

where W^k is a Brownian Motion, $X_0 = x$ and $\sigma > 0$. The process $k_t \geq 0$ is the agent's effort, essentially interpreted here as a measure of the efficiency of emission-reducing policies put in place at time t (a higher value of k_t stands for more effort, thus more efficiency). The function $l : [0, +\infty) \mapsto \mathbb{R}$ models the impact of effort on the emissions evolution and is therefore assumed to be strictly decreasing. Other assumptions will be made in order to ensure some good properties of the optimization problem and to be able to apply the measure change techniques. The underlying assumption from the dynamics (6.1.1) is that X must stay positive and that the agent only controls the drift of the process and not its

volatility *.

The principal (state) is assumed to observe the process X, that we call "emissions process" but which may also be interpreted, in line with [BFT11], as a market perception of the cumulative emissions produced by the firm (which become completely known only at maturity T). What the principal does not observe is the agent's effort k, i.e. he observes the left-hand side of (6.1.1) but he is not able to recover the decomposition on the right-hand side. In particular, he does not observe the Brownian Motion W^k, where the superscript is common in the recent principal-agent literature and indicates that once the process k is also known then the Brownian Motion becomes observable. For technical reasons, we will not define directly the evolution in (6.1.1), but we will first introduce a reference filtration which is independent of the agent's effort and we will then get to the same representation through a suitably defined measure change. This is called *weak formulation* in the literature (see [CWZ08] and [Wi08]): it will prove to be quite powerful to treat the agent's utility maximization problem but we will also apply it to the principal's one, supposing him to know the optimal reactions of the agent to the incentive policy that he puts in place.

As mentioned above, we consider incentive policies made up by two different components: continuous-time incentives s and a final penalty p. The incentive process s is assumed to depend on X (not necessarily in a Markovian way) but not on k, which the principal does not observe. We mainly think of final penalties of the form $p(X_T)$, where p is a function from $\mathbb{R}_+$ to $\mathbb{R}_+$ that typically operates if emissions exceed a certain level $\Lambda > 0$ (see [CFH09] for a related discussion). As we will see, our model considerably simplifies the general framework considered in [CWZ08], although it still presents some basic differences (like the fact that the agent compensation/penalisation enters in the his expected utility (6.2.2) both as a final lump sum payment and inside the integral part, where continuous-time incentives compensate for the cost of effort).

We remark that production is not present in our model: the optimal agent's choices concerning his effort plan will only take into account the final regulatory fee and continuous time incentives that he might gain with his effort †. Finally, we do not model any financial market, hence we do not allow for the possibility of exchanging financial contracts on emissions before maturity (see [BFT11], [CFH09], [CDET13] for a discussion in this direction).

In the first part of the paper we introduce the agent's problem and we give an existence and uniqueness result for the optimal agent's effort. The measure change techniques used in this part are similar to the ones developed in [CWZ08] (although, due to the great generality, their hypotheses are not always easy to interpret nor to verify. Our framework also allows for simpler and more intuitive proofs). The particular model that we adopted then allows us to push the study some further and to analyse some of the properties of the optimal effort, given a fixed incentive policy ‡: for example, we show under which

∗. Here we assume for simplicity σ to be constant, but all of the discussion can be easily generalized to the case where σ is a strictly positive function of time.

†. However, another interpretation of the agent's problem might lead to interpreting $s(t,(X)_{0\leq s\leq t})$ as a continuous penalty/reward caused by the effects on production of the effort-reducing policy, and not chosen by the principal.

‡. Clearly, fixing incentives gives only a partial picture of the original problem (where the principal optimizes as well), but we believe this is still interesting and useful, both in itself as we gain a better

conditions it will be increasing in the agent's risk aversion or in the volatility of the state process. We are also able (under some regularity assumptions) to solve the problem numerically and to visualize the structure of the optimal effort in some special cases: for example, we find a bell shape (in the state variable) in the case of a forfeitary fee and a monotone behavior with a linearly increasing fee (see Section 6.4.1 for an interpretation of these results). In the last part of this work we deal with the problem of the principal, who needs to optimally choose an incentive plan (which is considered fixed in the first part). Again we are able to get optimality conditions and to propose a numerical example which shows how continuous-time incentives are optimally chosen in a special case.
The paper is structured as follows. In Section 6.2 we fix an incentive structure and we introduce the optimization problem of the firm (agent), whose solution is characterized in terms of a BSDE. A uniqueness result is also given in a particular set of effort policies. In Section 6.3 we give alternative BSDE representations of the agent's optimal effort and we find some comparison results. In particular, we look at how the agent's effort and expected utility are affected by a change in the incentive policy. Section 6.4 is devoted to deriving a (nonlinear) PDE representation for the optimal effort under certain conditions, along with a numerical scheme to solve it. Finally in Section 6.5 we deal with the principal's problem by giving some necessary and sufficient conditions for optimality. We finally show that in a particular case the problem is quite similar to the agent's one and can be solved with analogous techniques.

6.2 The agent's problem

Let W^0 be a Brownian Motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, where the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is assumed to be the one generated by W^0 augmented by all the P-null sets in $\mathcal{F}$. We define the *emissions process* X, evolving as

$$dX_t = X_t \sigma dW_t^0. \tag{6.2.1}$$

with $\sigma > 0$. The choice of this type of dynamics reflects the fact that we want the process to stay positive and we assume, for simplicity, that the agent is only able to modify its drift and not its volatility. The process X is either interpreted as the level of the agent's emissions at a given time t, or as the market perception of the cumulative emissions up to a given time t.
Notice that $(\mathcal{F}_t)_{t\in[0,T]}$ is also generated by X. In order to model the impact of the agent's action to the emissions process we introduce the function $l : [0, +\infty) \mapsto \mathbb{R}$ which verifies the following Assumption.

Assumption 6.2.1. The function $l : [0, +\infty) \mapsto \mathbb{R}$ is C^3, strictly decreasing, convex, bounded and with bounded first derivative.

We then define the change of measure $\Gamma_t^k = \mathcal{E}_t(l/\sigma \cdot W^0)$ associated to the control k (note that it is well defined, as l is bounded), with dynamics $d\Gamma_t^k = \Gamma_t^k l(k_t)/\sigma dW^0$. In this way $dW_t^k := dW_t^0 - \frac{l(k_t)}{\sigma}dt$ is a BM for the measure (induced by) Γ^k. Under this weak

understanding of the agent's behavior that follows from a given principal's decision.

formulation § we imagine the agent as regarding the process (6.2.1) through the probability change Γ^k, which he knows once he decides a technology plan k (the first introduction of this approach to optimization problems dates back to [Bi78]). We will denote E^k the expectation operator under the change of measure induced by k.
Before stating the agent's problem we introduce a utility function $u : \mathbb{R}_+ \mapsto \mathbb{R}$ and a cost function $c : \mathbb{R}_+ \mapsto \mathbb{R}$ with the following regularity properties:

Assumption 6.2.2. The following properties hold:

1. u is a C^3 utility function (i.e. strictly increasing and concave) satisfying the Inada conditions, i.e. $u'(0) = +\infty$ and $u'(\infty) = 0$. Moreover, $\frac{u''}{u'}(x) \to -\infty$ as $x \to 0$. We set $u(x) = -\infty$ for $x < 0$.
2. c is C^3, positive, strictly increasing and convex with $c'(0) = 0$ and $c''(0) > 0$.

The utility function u models the continuous-time part of the agent's total utility, the one that accounts for the effort plan and continuous-time incentives. Remark that the last condition in its characterization is satisfied in the most common cases, i.e. for power and logarithmic utilities.
The function c captures the monetary cost associated to effort at each date. It is increasing to reflect the fact that more effort is more costly, while the other technical assumptions (which are satisfied by the common quadratic cost function) will be needed to derive our BSDE representation. Remark that we only model variable costs connected to effort plans, there are no fixed costs linked to increasing or decreasing effort.

Definition 6.2.1. An admissible incentive policy $(s_t)_{0\le t\le T}$ is a positive $\mathcal{F}_t$-adapted stochastic process such that $c(0) < m \le s_t \le M$ for some $0 < m < M$.
A penalty p is an $\mathcal{F}_T$-measurable random variable, it is called admissible if $p \in L^{2+\alpha}(\Omega, P)$ for some $\alpha > 0$.

In this section we will consider an admissible incentive structure (s, p) to be fixed, and we will be concerned with the optimal agent's reaction. To this aim, we define his admissible effort strategies.

Definition 6.2.2. An admissible effort policy $(k_t)_{0\le t\le T}$ is a positive $\mathcal{F}_t$-adapted stochastic process such that

$$E\left[\int_0^T |u(s_t - c(k_t))|^{2+\alpha} dt\right] < \infty$$

for some $\alpha > 0$. It is *strongly* admissible if $k_t \le c^{-1}(s_t - \epsilon)$ a.s. $\forall t \in [0, T]$ for some $\epsilon > 0$.

§. In another formulation, called 'strong', one starts by fixing a BM, say W, and then works directly with the controlled process

$$dX_t^k = l(k_t)X_t^k dt + \sigma X_t^k dW_t,$$

which may seem more natural at first sight, and we would not need all of the assumptions on l that ensure the well posedness of the change of measure. Apart from this, we preferred the weak formulation for many reasons: for example, it allows to consider a wider class of continuous time incentive policies (i.e. those depending on the history of X) and it requires no stringent conditions on the penalty function, such as differentiability and convexity. As reported in [CWZ08], the two methods have sometimes been used together in the literature even if the connection between the two is not always clear and it may hide some subtle measurability issues. We will give some more details in the sequel.

We will also use the expression "ϵ-admissible" with the obvious meaning.

Remark 6.2.1. It is clear that a strongly admissible effort policy is admissible. Moreover, since s is bounded, an admissible k must also be bounded. If $u(0)$ is finite, then admissibility of k is equivalent to simply requiring $k_t \leq c^{-1}(s_t)$ a.s. $\forall t \in [0,T]$.

For an admissible k we can now write down the expected agent's utility as

$$\begin{aligned} V(k) = V^{(s,p)}(k) &= E^k\left[\int_0^T u(s_t - c(k_t))dt - p\right] \\ &= E\left[\int_0^T \Gamma_t^k u(s_t - c(k_t))dt - \Gamma_T^k p\right] \end{aligned} \tag{6.2.2}$$

where X evolves according to (6.2.1). The implicit assumption in (6.2.2) is that the agent's utility separates into two components: a continuous-time part which is captured by u and a lump part at maturity T which is described by p. Therefore the random variable p is to be interpreted as the (dis)utility the agent gets from the final fee payment, and not as a penalty function *tout court* (unless risk neutrality is assumed).

Remark 6.2.2. Given its link with an application of the stochastic maximum principle, our notion of admissibility is similar in spirit to the one found in [CWZ08], though considerably less technical due to our simpler framework (for example, the assumption of constant volatility, the introduction of the bounded function l influencing the drift or the different control space).

For a given admissible incentive structure (s,p), the agent needs to optimally choose his effort plan, that is he must solve the optimization problem

$$v^{(s,p)} := \sup_k V^{(s,p)}(k) \tag{6.2.3}$$

where the sup is taken over admissible effort policies k.

6.2.1 Necessary conditions

In order to characterize the solution in terms of a BSDE we will apply the stochastic maximum principle (hereafter SMP), as stated in Theorem 3.2 in [YZ99]. A crucial step in the application of this kind of results is the choice of the state variable(s), as different choices generally lead to different conditions. The peculiarity of the weak formulation lies in the fact that we can take Γ^k as the state variable in the optimization, while X_T is considered as a fixed element of the reference probability space (it does not depend on the control under this formulation). This is quite an advantage as it allows to work under no regularity assumptions on the penalty function p nor on the incentive process s. We remark in particular that no convexity requirements are imposed on p. Choosing Γ^k as state variable the Hamiltonian of the problem can be expressed as

$$H(t, k_t, s_t, Y_t, Z_t) = \Gamma_t^k[Z_t l(k_t)/\sigma + u(s_t - c(k_t))], \tag{6.2.4}$$

where (Y, Z) are the adjoint variables which follow the BSDE

$$\begin{cases} dY_t = [-Z_t l(k_t)/\sigma - u(s_t - c(k_t))]dt + Z_t dW_t^0 \\ Y_T = -p. \end{cases} \tag{6.2.5}$$

We say that a control $(k_t^*)_{0\le t\le T}$ is optimal if it reaches the supremum in the definition of $v^{(s,p)}$. The next result gives some necessary conditions for optimality.

Proposition 6.2.1. *Let k^* be an optimal strongly admissible control. Then there exist adapted processes (Y, Z) satisfying (6.2.5) with $k = k^*$ and the optimal control k^* satisfies*

$$\begin{cases} \sigma u'(s_t - c(k_t^*))c'(k_t^*) = Z_t l'(k_t^*) \text{ on } \{k_t^* > 0\} \\ \sigma u'(s_t - c(k_t^*))c'(k_t^*) \ge Z_t l'(k_t^*) \text{ on } \{k_t^* = 0\} \end{cases} \tag{6.2.6}$$

Proof. We will apply the SMP. The non-smoothness of p does not represent a problem since we use Γ^k as state variable. In the notation of [YZ99], we have $h(\Gamma_T^k) = -\Gamma_T^k p(X_T)$, where X_T is independent of the control. Hence $h'(\Gamma_T^k) = -p(X_T)$ and $h''(\Gamma_T^k) = 0$. However, in order to work as it is, the SMP requires that $(k, \Gamma) \mapsto \Gamma u(s - c(k))$ be Lipschitz in both variables. The problem is partially solved by assuming s to be bounded, but there still remains an issue when k is close to $c^{-1}(s)$, and this is where strong admissibility gets in. Let $\epsilon_n \to 0$ and define a sequence of Lipschitz functions $\tilde{u}^n(x)$ which coincide with $u(x)$ for $x \ge \epsilon_n$. Also define $V_n^{(s,p)}(k)$ by replacing u with $\tilde{u}^n$ in the definition of the problem. By definition k^* is ϵ-admissible for some $\epsilon > 0$, therefore we have that $V_n^{(s,p)}(k^*) = V^{(s,p)}(k^*)$ for $n \ge n_0$. Take $0 < \epsilon_0 < \epsilon$ and an ϵ_0-admissible k, then $V^{(s,p)}(k) \le V_n^{(s,p)}(k^*) = V^{(s,p)}(k^*)$ for $n \ge n_0$. We also have $V_n^{(s,p)}(k) = V^{(s,p)}(k)$ for $n \ge n_1$ and so $V_n^{(s,p)}(k) \le V_n^{(s,p)}(k^*)$ for $n \ge n_1$. It follows that k^* maximizes V_n over all ϵ_0-admissible k, when $n \ge n_1$. Therefore the SMP can be applied to this new problem, implying that k^* satisfies (6.2.6) replacing u with $\tilde{u}^n$ and adding a supplementary condition when $k = c^{-1}(s - \epsilon_n)$. However since k^* is ϵ-admissible the replacement is irrelevant and the supplementary condition is never satisfied, so we directly have (6.2.6). □

Remark 6.2.3. The requirement of strong admissibilty in the previous result is linked to the perturbation argument which is at the heart of the proof of the SMP. In our context it could be replaced by simple admissibility if, given the optimal admissible policy k^*, we could find a sequence k^n converging uniformly to k^* and such that k^n is optimal when we only consider controls such that $k_t \le c^{-1}(s_t - \epsilon_n)$, with $\epsilon_n \downarrow 0$. Indeed for each ϵ_n we can define a function $\tilde{u}^n$ as in the previous proof, so that $\tilde{u}^n \to u$ pointwise. Then the necessary conditions hold for (Y^n, Z^n) and k^n which are the analogous variables in the problem where u_n replaces u. By standard properties of BSDEs (see [MY99] Theorem 4.4) we have

$$\begin{aligned} &E\left[\sup_{0\le t\le T} |Y_t^m - Y_t^n|^2 + \int_0^T |Z_t^m - Z_t^n|^2 dt\right] \\ &\le CE\left[\int_0^T [(Z_t^m)^2 |l(k_t^n) - l(k_t^m)|^2 + |u_n(s_t - c(k_t^n)) - u_m(s_t - c(k_t^m))|^2]dt\right] \to 0, \end{aligned}$$

recall that l is bounded, hence (Y^n, Z^n) converge to some (Y, Z) which satisfy (6.2.5) and the Hamiltonians also converge.

Proposition 6.2.1 leads to the representation of the target volatility process Z as

$$\begin{cases} \hat{Z}_t(s,p,k) = \sigma \frac{u'(s_t - c(k_t))c'(k_t)}{l'(k_t)} \leq 0 \text{ on } \{k_t > 0\} \\ \hat{Z}_t(s,p,k) \geq 0 \text{ on } \{k_t = 0\} \end{cases} \tag{6.2.7}$$

where we emphasized the fact that Z itself depends on (s,p,k) through (6.2.5). It is called *target* (see [Wi08]) because, if the principal wants to induce a strongly admissible technology plan k, it is necessary to act on the incentives s and/or on the fee p in such a way that the volatility process in (6.2.5) satisfies (6.2.7).

Definition 6.2.3. A policy (s,p,k) is said to be
- *promise-keeping* if (s,p,k) imply a solution Y to the BSDE (6.2.5) with volatility process Z satisfying (6.2.7).
- *implementable* if, given (s,p), the agent optimally chooses the recommended actions k.

The term promise-keeping (taken from [Wi08]) expresses the idea that under this condition the volatility process Z "keeps the promise" of being equal to its target level.

6.2.2 Sufficient conditions

In Proposition 6.2.1 we proved that if (s,p,k) is implementable and k is strongly admissible, then it is promise-keeping. We now aim at proving a converse implication, for which we need a preliminary technical discussion.
For a given admissible effort process k we can rewrite (6.2.1)-(6.2.5) as

$$\begin{cases} dX_t = l(k_t)X_t dt + \sigma X_t dW_t^k \\ dY_t = -u(s_t - c(k_t))dt + Z_t dW_t^k \\ X_0 = x, Y_T = -p(X_T) \end{cases} \tag{6.2.8}$$

This is a simple kind of Forward-Backward SDE, as the link between the two equations only lies in the terminal condition. If we could prove that

$$E^k \left[\int_t^T Z_r dW_r^k \mid \mathcal{F}_t \right] = 0 \tag{6.2.9}$$

then we would obtain that

$$Y_t = E^k \left[\int_t^T u(s_r - c(k_r))dr - p(X_T) \mid \mathcal{F}_t \right]$$

that is $Y_t = V_t^{(s,p)}(k)$, where $V_t^{(s,p)}(k)$ is naturally defined as the conditional agent's expected utility. We could apply standard existence and uniqueness results directly to (6.2.8) in order to obtain $E^k \left[\int_0^T Z_r^2 dr \right] < \infty$ which would imply the result. However this can be a little bit tricky, as the system contains the process k which is $\mathcal{F}_t$-adapted, while existence results in this context would hold on the filtration $\mathcal{F}^{W^k}$, which in general differs

from $\mathcal{F} = \mathcal{F}^{W^0}$ (though we clearly have $\mathcal{F}_t^{W^k} \subseteq \mathcal{F}_t^{W^0}$). Lemma 6.7.2 in the Appendix addresses this issue by constructing, for any admissible k, an $\mathcal{F}_t$-measurable solution to (6.2.8) which exhibits the usual integrability properties of the standard BSDE theory with respect to any measure associated to an effort policy (boundedness of l will play an important role via Lemma 6.7.1).
The following result completes Proposition 6.2.1 by providing necessary and sufficient conditions for optimality. Remark that it is stated for strongly admissible policies even if the sufficient part holds with simple admissibility.

Proposition 6.2.2. *A strongly admissible policy (s, p, k) is implementable if and only if it is promise-keeping. In other words, a strongly admissible effort policy k is optimal for the agent given the incentive structure (s, p) if and only if the (unique) process Z defined as in* (6.2.8) *with (s, p, k) satisfies condition* (6.2.7).

Proof. The necessary condition is just Proposition 6.2.1. To show sufficiency, consider an admissible effort plan k^* and assume that $\hat{Z} = \hat{Z}(s, p, k^*)$ defined as in (6.2.8) with (s, p, k^*) satisfies condition (6.2.7). The agent's Hamiltonian is (we omit Γ_t^k as it is positive and does not affect the maximization)

$$H^*(t, k_t, s_t) = \hat{Z}_t(s, p, k^*)l(k_t)/\sigma + u(s_t - c(k_t)).$$

We would like to show that

$$k_t^* = \text{argmax}_{0 \le k < c^{-1}(s_t)}[\hat{Z}_t(s, p, k^*)l(k)/\sigma + u(s_t - c(k))]. \tag{6.2.10}$$

By calling $b(k)$ the function in the argument we can compute its first and second derivatives

$$b'(k) = \hat{Z}_t l'(k)/\sigma - u'(s_t - c(k))c'(k)$$

$$b''(k) = \hat{Z}_t l''(k)/\sigma + u''(s_t - c(k))(c'(k))^2 - u'(s_t - c(k))c''(k)$$

If $\hat{Z}_t < 0$ then $b''(k) \le 0$ for all $k \in [0, c^{-1}(s_t))$ and $b'(k_t^*) = 0$ since $\hat{Z}_t$ verifies (6.2.7), hence (6.2.10) is true. If $\hat{Z}_t \ge 0$ then $b'(k) \le 0$ for all $k \in [0, c^{-1}(s_t))$, meaning that $k = 0$ is optimal and again (6.2.10) is true. Therefore k^* always reaches the maximum in H^*.
It remains to verify that the agent will choose k^* when he faces incentives s and fee p. By Lemma 6.7.2, the agent's expected utility following k^* is $V^{(s,p)}(k^*) = Y_0^*$, where (Y^*, Z^*) is the solution of (6.2.5) with k^* replacing k. Then for any admissible k we have

$$\begin{aligned}
V^{(s,p)}(k) - V^{(s,p)}(k^*) =& E^k\left[\int_0^T u(s_t - c(k_t))dt - p\right] - E^k\left[Y_0^*\right] \\
=& E^k\left[\int_0^T [u(s_t - c(k_t)) - u(s_t - c(k_t^*))]dt + \int_0^T Z_t^* dW_t^{k^*}\right] \\
=& E^k\left[\int_0^T [u(s_t - c(k_t)) - u(s_t - c(k_t^*))]dt + \int_0^T Z_t^* dW_t^{k}\right] \\
&+ E^k\left[\int_0^T Z_t^*[l(k_t) - l(k_t^*)]\sigma^{-1}dt\right] \\
=& E^k\left[\int_0^T [H^*(t, k_t, s_t) - H^*(t, k_t^*, s_t)]dt\right] \le 0,
\end{aligned}$$

which implies the claim since Z^* satisfies (6.2.7) by assumption. Here we used the fact that, by definition, we have that $dW_t^{k^*} = dW_t^k + [l(k_t) - l(k_t^*)]\sigma^{-1}dt$. Moreover, Lemma 6.7.2 ensures that the expected value of the stochastic integral in the third equality is zero. □

For later use we denote g the function appearing in the first line of the optimality condition (6.2.7):

$$g(s,k) = \sigma \frac{u'(s - c(k))c'(k)}{l'(k)}. \tag{6.2.11}$$

In economic terms, this can be interpreted as the elasticity of the agent's utility with respect to a change in the emission growth rate over a little lapse of time. The following calculation may help grasping this intuition.

$$\begin{aligned}
&\frac{E^{k+\delta}\left[\int_t^{t+\epsilon} dV_r^{(s,p)}(k+\delta) \mid \mathcal{F}_t\right] - E^k\left[\int_t^{t+\epsilon} dV_r^{(s,p)}(k) \mid \mathcal{F}_t\right]}{E^{k+\delta}\left[\frac{1}{X_t}\int_t^{t+\epsilon} dX_r \mid \mathcal{F}_t\right] - E^k\left[\frac{1}{X_t}\int_t^{t+\epsilon} dX_r \mid \mathcal{F}_t\right]} \\
&= \frac{E^{k+\delta}\left[-\int_t^{t+\epsilon} u(s_r - c(k_r+\delta))dr \mid \mathcal{F}_t\right] - E^k\left[-\int_t^{t+\epsilon} u(s_r - c(k_r))dr \mid \mathcal{F}_t\right]}{E^{k+\delta}\left[e^{\int_t^{t+\epsilon} l(k_r+\delta)dr} \mid \mathcal{F}_t\right] - E^k\left[e^{\int_t^{t+\epsilon} l(k_r)dr} \mid \mathcal{F}_t\right]} \\
&\approx \frac{-u(s_t - c(k_t+\delta))\epsilon - [-u(s_t - c(k_t))]\epsilon}{e^{l(k_t+\delta)\epsilon} - e^{l(k_t)\epsilon}} \approx \frac{u'(s_t - c(k_t))c'(k_t)\delta\epsilon}{e^{l(k_t)\epsilon}l'(k_t)\delta\epsilon} \approx g(s_t,k_t)/\sigma.
\end{aligned}$$

Some of the sensitivity results in Section 6.3 will make reference to this quantity.

Remark 6.2.4. Standard procedures can be used to look for a candidate solution to (6.2.8), when the policy (s,p,k) is fixed and Markovian (i.e. s and k depend only on t and X_t, and $p = p(X_T)$). In particular, by assuming that $Y_t = \theta^{(s,p,k)}(t,X_t)$ then θ is solution (supposing it is sufficiently regular) of

$$\begin{cases} \theta_t + \frac{1}{2}\theta_{xx}x^2\sigma^2 + xl(k(t,x))\theta_x + u(s(t,x) - c(k(t,x))) = 0 \\ \theta(T,x) = -p(x) \end{cases} \tag{6.2.12}$$

and $Z_t = \theta_x^{(s,p,k)}(t,X_t)\sigma X_t$.
In this way the implementability constraints on the volatility process Z given in (6.2.7) can be re-expressed in terms of the solution of the PDE (6.2.12) ¶. We can therefore state that a Markovian policy (s,p,k) is implementable if and only if the solution $\theta^{(s,p,k)}(t,X_t)$ to (6.2.12) satisfies

$$\begin{cases} \theta_x^{(s,p,k)}(t,x) = \frac{u'(s_t - c(k_t))c'(k_t)}{xl'(k_t)} \leq 0 \text{ if } k_t > 0 \\ \theta_x^{(s,p,k)}(t,x) \geq 0 \text{ if } k_t = 0 \end{cases}$$

¶. This result also gives an idea of how one can heuristically recover the optimal effort without solving a nonlinear PDE, i.e. one can solve the PDE (6.2.12) backwards with a standard implicit finite-difference scheme by making sure that the discretized versions of the implementability conditions be satisfied at each point in space and time. This procedure gives rise, at each time step, to a nonlinear equation with a number of unknowns equal to the dimension of the spacial grid, which is usually well handled numerically by standard computing software.

This result also has a clear economic intuition. The quantity $xl'(k)$ represents the marginal average emission reduction when effort is increased, hence $xl'(k)\theta_x$ is the marginal expected utility benefit from increasing effort. On the other hand, $u'(s-c(k))c'(k)$ is the marginal cost of effort. In equilibrium, the marginal benefit should be equal to the marginal cost. When $\theta_x \geq 0$ it means that the marginal expected utility benefit from increasing effort is negative, a pathological situation that will typically only occur when p is increasing, i.e. it is no longer a penalty but a reward for polluting. In this unrealistic case the optimal effort is going to be zero.
At this stage, however, our conditions are too weak to ensure the existence of a classical solution to (6.2.12). Other PDE results will be derived in Section 6.4.

6.2.3 Existence of the optimal effort

We still consider an admissible incentive structure (s,p) to be fixed. In the following we will investigate the question of whether an optimal effort k^* exists and is unique. The main ingredient to do this is going to be the inversion of the the conditions for optimality stated in (6.2.7). This is done in the following Lemma.

Lemma 6.2.1. *Given $0 < m < s \leq M$ and $z \in \mathbb{R}$, there exists a unique $k = F(s,z)$ satisfying*

$$\begin{cases} z = \sigma \frac{u'(s-c(k))c'(k)}{l'(k)} = g(s,k) (\leq 0) \text{ if } k > 0 \\ z \geq 0 \text{ if } k = 0 \end{cases} \tag{6.2.13}$$

The function $F(s,\cdot)$ is nonincreasing, Lipschitz and continuously differentiable on $\mathbb{R}\setminus\{0\}$. If $2l''(k)^2 - l'(k)l^{(3)}(k) \geq 0$ and $c^{(3)}(k) \geq 0$ then $F(s,\cdot)$ is concave on $(-\infty, 0]$.

Proof. See Appendix. □

Inverting the conditions in (6.2.7) through the function F allows us to rewrite (6.2.5) by incorporating the implementability constraints *inside* the BSDE. In this spirit, to any admissible (s,p) we can associate the system

$$\begin{cases} dX_t = X_t \sigma dW_t^0 \\ dY_t = [-Z_t l(F(s_t, Z_t))/\sigma - u(s_t - c(F(s_t, Z_t)))]dt + Z_t dW_t^0 \\ X_0 = x, Y_T = -p \end{cases} \tag{6.2.14}$$

By Proposition 6.2.2 the existence of a unique solution to this equation is equivalent to the existence of an optimal effort policy, which is characterized by posing $k_t = F(s_t, Z_t)$. Theorem 6.2.1 gives such an existence result.

Theorem 6.2.1. *There exists an admissible optimal effort k^* for the agent's problem* (6.2.3)*. Uniqueness holds in the class of strongly admissible policies.*

Proof. We want the FBSDE (6.2.14) to have a unique solution (Y,Z). Since it is decoupled, we can treat it as a simple BSDE as far as existence is concerned. Hence it is enough to check that $f(s,z) := u(s-c(F(s,z))) + zl(F(s,z))/\sigma$ is uniformly Lipschitz continuous in z. We have

$$f_z(s,z) = l(F(s,z))/\sigma + F_z(s,z)[zl'(F(s,z))/\sigma - u'(s-c(F(s,z)))c'(F(s,z))].$$

Recall that $F(s,\cdot)$ is not differentiable in 0 and therefore the previous expression should be interpreted at first as a right/left derivative in zero. Let us focus on the second term: if $z \leq 0$ the term in brackets is zero by definition of F, while if $z > 0$ then $F_z(s,z) = 0$. Therefore f is differentiable and we have $f_z(s,z) = l(F(s,z))/\sigma$, which is bounded by assumption. By admissibility of s and p and Theorem 6.2.1 in [Ph09] equation (6.2.14) has a unique solution (Y,Z) and therefore $k_t = F(s_t, Z_t)$ is an optimal effort. We need to verify that it is admissible, i.e. that

$$E\left[\int_0^T |u(s_t - c(k_t))|^{2+\alpha} dt\right] < \infty.$$

for some $\alpha > 0$. Considering the function $a(s,z) = u(s - c(F(s,z)))$, we have for $z < 0$

$$\begin{aligned} 0 \leq a_z(s,z) &= -u'(s-c(F))c'(F)F_z \\ &= -\frac{u'(s-c(F))c'(F)l'(F)^2}{l'(F)[-u''(s-c(F))c'(F)^2 + u'(s-c(F))c''(F)] - l''(F)[u'(s-c(F))c'(F)]} \\ &= \frac{c'(F)l'(F)^2}{l'(F)[\frac{u''}{u'}(s-c(F))c'(F)^2 - c''(F)] + l''(F)c'(F)}. \end{aligned}$$

Since $\frac{u''}{u'}(x) \to \infty$ as $x \to 0$ by Assumption 6.2.2 we deduce that $a_z(s,z) \to 0$ as $z \to -\infty$. Moreover, $a_{z-}(s,0)$ is finite and $a_z(s,z) = 0$ for $z > 0$. It follows that $a(s,z)$ has sublinear growth in z, i.e. we can write $|a(s,z)| \leq K_1 + K_2|z|^{1/\beta}$ for some $\beta > 1$ and constants $K_1, K_2 > 0$ (which can be chosen independently of s taking into account $m \leq s \leq M$). Now if we take $\alpha = 2\beta - 2 > 0$ we obtain

$$E\left[\int_0^T |u(s_t - c(k_t))|^{2+\alpha} dt\right] \leq K_1 + K_2 E\left[\int_0^T |Z_t|^2 dt\right] < \infty,$$

Therefore $k_t = F(s_t, Z_t)$ is indeed admissible.
As for uniqueness, the results follows from the fact that the necessary conditions in Proposition 6.2.1 only hold in the class of strongly admissible policies. □

In the next Section we will see that under additional regularity assumptions on the penalty function we can ensure that the optimal effort coming from (6.2.14) is strongly admissible.

6.3 The optimal effort and comparison results

Having characterized the agent's value function and optimal effort in terms of the solution of a BSDE, it is now natural to look at how the solution is affected by a change in the parameters (i.e. a change in the incentive structure). Even if we still do not account here for any optimization from the principal, we believe that this kind of analysis can provide some useful insights for understanding the agent's behavior.
As a starting point, the next lemma shows that the agent's value function $v^{(s,p)}$ defined in

(6.2.3) reacts positively to higher incentives or a lower final penalty. This is quite intuitive and it could in part also be deduced directly from definition (6.2.3) (though the last claim seems to require the comparison theorem for BSDEs).

Lemma 6.3.1. *Assume that we have two admissible incentive policies (s, p) and $(\bar{s}, \bar{p})$ such that $\bar{s}_t \geq s_t$ a.s. for all t and $\bar{p} \leq p$ a.s. Then $v^{(\bar{s},\bar{p})} \geq v^{(s,p)}$. Moreover, if $\bar{p}(x) < p(x)$ on a set of strictly positive Lebesgue measure, or if $\bar{s}_t > s_t$ on a set of strictly positive measure $dt \times dP$, then $v^{(\bar{s},\bar{p})} > v^{(s,p)}$.*

Proof. The agent's conditional value function given incentives s follows the BSDE

$$\begin{cases} -dY_t = f(s_t, Z_t)dt - Z_t dW_t^0 \\ Y_T = -p \end{cases} \tag{6.3.1}$$

where $f(s, z) = zl(F(s, z)) + u(s - c(F(s, z)))$. We have that

$$\begin{aligned} f_s(s, z) &= zl'(F)F_s + u'(s - c(F))[1 - c'(F)F_s] \\ &= [zl'(F) - u'(s - c(F))c'(F)]F_s + u'(s - c(F)) \end{aligned}$$

The first term in brackets is zero when $z \leq 0$, while $F_s = 0$ when $z \geq 0$, thus

$$f_s(s, z) = u'(s - c(F)) > 0$$

since u is strictly increasing. This implies that $f(s'_t, z) \geq f(s_t, z)$ for all $z \in \mathbb{R}$ and the claim follows by standard comparison theorems for BSDEs (see [Ph09], Theorem 6.2.2). □

We now turn to the study of the optimal effort. Since it is defined as a function of the Z-part of BSDE (6.2.14), the starting point must be a better characterization of Z. To our best knowledge, comparison theorems for the Z-part of a BSDE seem to be lacking in the literature, therefore we will directly look for a new BSDE solved by Z. This procedure, however, requires stronger regularity conditions on the incentives and on the final penalty function p. For the rest of the section we will therefore work under the following additional assumption.

Assumption 6.3.1. Continuous time incentives are Markovian, i.e. $s_t = s(t, X_t)$, where (with a slight abuse of notation) $s : [0, T] \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ is $C^{1,2}$, bounded ($m \leq s \leq M$) and with bounded derivatives. Moreover $s_x(\cdot, x) \leq 0$.
The penalty p is also Markovian and the functions $x \mapsto p(x)$ and $x \mapsto xp'(x)$ defined on $\mathbb{R}_+$ are positive, bounded and C^∞.

Remark that supposing $s_x(\cdot, x) \leq 0$ and $p'(x) \geq 0$ reflects the natural assumption that higher emissions should induce lower incentives and a higher final fee ‖.
We are now able to give a BSDE characterization of the optimal effort that results from BSDE (6.2.14). Recall that it might not be the unique optimal effort, in the sense that there may exist other optimal policies which are not strongly admissible.

‖. Remark that the hypothesis $s_x(\cdot, x) \leq 0$ is indeed quite natural but it might not be always optimal for the principal: in certain cases he may be willing to offer higher incentives when emissions are high, with the aim of inducing more effort and thus reduce the final social cost at maturity. See Section 6.5.

Proposition 6.3.1. *Under Assumption 6.3.1, if the optimal effort is strongly admissible then it follows the BSDE*

$$\begin{cases} -dk_t = \left[G(t, X_t, k_t)\Theta_t^2 + D(t, X_t, k_t)\Theta_t + C(t, X_t, k_t)\right] dt - \Theta_t dW_t^0 \\ k_T = F(s(T, X_T), -\sigma X_T p'(X_T)) \end{cases} \tag{6.3.2}$$

where

$$\begin{aligned} G(t,x,k) &= \frac{1}{2}\frac{g_{kk}}{g_k} \\ D(t,x,k) &= \frac{l(k)}{\sigma} + \frac{s_x \sigma x}{g_k^2}\left(g_{kk}g_s + g_{sk}g_k\right) \\ C(t,x,k) &= \frac{g_s}{g_k}\frac{\partial}{\partial t}s + \frac{u'(s-c(k))}{g_k}s_x\sigma x + \frac{1}{2}\frac{\sigma^2 x^2}{g_k}\left(g_{ss}s_x^2 + g_s s_{xx} + \frac{g_{kk}g_s^2}{g_k^2}s_x^2\right) \end{aligned}$$

(we omit the argument (s,k) from g and its derivatives, and (t,x) from s and its derivatives for the sake of clarity).

Proof. We start from the dynamics of the optimal agent's expected utility Y

$$\begin{cases} dX_t = \sigma X_t dW_t^0 \\ dY_t = -f(s(t, X_t), Z_t)dt + Z_t dW_t^0 \\ X_0 = x \\ Y_T = -p(X_T) \end{cases} \tag{6.3.3}$$

where $f(s,z) = zl(F(s,z))/\sigma + u(s - c(F(s,z)))$.
We want to recover the dynamics of Z_t starting from (6.3.3). Recall first that f is continuously differentiable in both variables with $f_z(s,z) = l(F(s,z))/\sigma$ and $f_s(s,z) = u'(s - c(F(s,z)))$: the first is bounded by assumption while the second can also be considered bounded since we assume a strongly admissible effort policy. Therefore we can assume that $Y_t = L(t, X_t)$ where L is $C^{1,2}$ (see Chapter 4, Theorem 2.3 in [MY99]) and we can write $Z_t = L_x(t, X_t)\sigma X_t$. We also have $\nabla Y_t = L_x(t, X_t)\nabla X_t$, which implies

$$Z_t = \sigma X_t (\nabla X_t)^{-1} \nabla Y_t,$$

where ∇ denotes the derivative of the process with respect to x. The dynamics of the tangent processes are given by

$$\begin{cases} d\nabla Y_t = -[f_z(s_t, Z_t)\nabla Z_t + f_s(s_t, Z_t)s_x(t, X_t)\nabla X_t]dt + \nabla Z_t dW_t^0 \\ d\nabla X_t = \sigma \nabla X_t dW_t^0 \\ d(\nabla X_t)^{-1} = -\sigma(\nabla X_t)^{-1} dW_t^0 + \sigma^2 (\nabla X_t)^{-1} dt. \end{cases}$$

We have therefore $d[X_t(\nabla X_t)^{-1}] = 0$, so that $X_t(\nabla X_t)^{-1} = x$ and $dZ_t = \sigma x d\nabla Y_t$, hence Z follows the BSDE

$$\begin{cases} dZ_t = -[l(F(s_t, Z_t))/\sigma N_t + u'(s_t - c(F(s_t, Z_t)))s_x(t, X_t)\sigma X_t]dt + N_t dW_t^0 \\ Z_T = -\sigma X_T p'(X_T) \end{cases} \tag{6.3.4}$$

where we have identified N_t with $\sigma x \nabla Z_t$.
Now the optimal effort is given by $k_t = F(s_t, Z_t) = F(s(t, X_t), Z_t)$, therefore by Itô's rule we get

$$\begin{aligned} dk_t =& F_s(s_t, Z_t)s_x(t, X_t)dX_t + \frac{1}{2}\sigma^2 X_t^2[F_{ss}(s_t, Z_t)s_x(t, X_t)^2 + F_s(s_t, Z_t)s_{xx}(t, X_t)]dt \\ &+ F_z(s_t, Z_t)dZ_t + \frac{1}{2}F_{zz}(s_t, Z_t)N_t^2 dt + F_{sz}(s_t, Z_t)s_x(t, X_t)N_t X_t \sigma dt \\ &+ F_s(s_t, Z_t)\frac{\partial}{\partial t}s(t, X_t)dt \\ =& [-G(t, X_t, k_t)\Theta_t^2 - D(t, X_t, k_t)\Theta_t - C(t, X_t, k_t)]dt + \Theta_t dW_t^0 \end{aligned}$$

where we identified $\Theta_t = F_z(s, Z_t)N_t + F_s(s, Z_t)s_x(t, X_t)\sigma X_t$. In the previous computation we used the fact that

$$F_z(s, z) = \frac{1}{g_k(s, F(s, z))},$$

therefore

$$F_{zz}(s, z) = \frac{-g_{kk}(s, F(s, z))F_z(s, z)}{[g_k(s, F(s, z))]^2},$$

and finally

$$\frac{F_{zz}}{F_z^2}(s, z) = -\frac{g_{kk}(s, F(s, z))}{g_k(s, F(s, z))}.$$

Similarly we have that $F_s = -\frac{g_s}{g_k}$, $F_{ss} = 2\frac{g_{sk}g_s}{g_k^2} - \frac{g_{ss}}{g_k}$, $F_{sz} = -\frac{g_{sk}}{g_k^2}$.
One last thing to be remarked is that we replaced $f_s(s_t, Z_t)$ with $f_s(s_t, g(s_t, k_t))$, and this is only justified when Z in (6.3.4) is negative. To prove this, note first that the term $f_s(s, z)s_x(t, x) = u'(s - c(F(s, z)))s_x(t, x)$ in the generator of Z in (6.3.4) is negative by Assumption 6.3.1. Since we also have $Z_T \leq 0$ by Assumption 6.3.1, the comparison theorem gives that $Z_t \leq 0$.
The dynamics of the optimal effort $k_t = F(s_t, Z_t)$ is therefore given by (6.3.2) as claimed. □

Remark 6.3.1. Using a strong formulation of the problem would lead to a state/adjoint system of the type

$$\begin{cases} dX_t = l(F(s_t, \tilde{Y}_t))X_t dt + \sigma X_t dW_t \\ d\tilde{Y}_t = -\sigma u'(s_t - c(F(s_t, \tilde{Y}_t)))s_x(t, X_t)X_t dt + \tilde{Z}_t dW_t \\ \tilde{Y}_T = -\sigma p'(X_T)X_T, \end{cases} \tag{6.3.5}$$

which is very similar to (6.3.4). Hence under this formulation the adjoint variable $\tilde{Y}$ would play the role of Z in the weak formulation, which would require stronger regularity assumptions on the penalty function from the beginning. Moreover, the drift component $l(F)$ moves from the backward to the forward part of the system, thus making (6.3.5) a *coupled* FBSDE (whose solvability is in general harder to prove).

6.3.1 Constant s

Let us now suppose, to simplify the analysis, that s is constant. In this case s is not interpreted as an incentive process, but either as a constant agent's revenue or simply as a cap on the monetary cost of effort that he chooses to implement. In this context incentives are only of the negative type (through the final payment of p). BSDE (6.3.2) for the agent's effort simplifies significantly to

$$\begin{cases} -dk_t = \left[G(s,k_t)\Theta_t^2 + \frac{l(k_t)}{\sigma}\Theta_t\right]dt - \Theta_t dW_t^0 \\ k_T = F(s, -\sigma X_T p'(X_T)) \end{cases} \tag{6.3.6}$$

where $G(s,k) = \frac{1}{2}\frac{g_{kk}(s,k)}{g_k(s,k)}$ **, while Z solves

$$\begin{cases} dZ_t = -\frac{l(F(s,Z_t))}{\sigma}N_t dt + N_t dW_t^0 \\ Z_T = -\sigma X_T p'(X_T) \end{cases} \tag{6.3.7}$$

Remark 6.3.2. Since we assume Z_T to be bounded, the comparison theorem gives us that Z_t is uniformly bounded, therefore the optimal effort is strongly admissible and (6.3.6) holds automatically without assuming strong admissibility as we did in Proposition 6.3.1. We can also prove that $E^k\left[\int_t^T N_r dW_r^k \mid \mathcal{F}_t\right] = 0$, where $k = F(s,Z_t)$ is the optimal effort policy. Indeed we have that $dZ_t = N_t dW_t^k$, and so Z is a uniformly bounded local martingale under the measure Γ^k associated to the optimal effort, hence a true martingale. This gives the representation

$$Z_t = -E^k\left[\sigma X_T p'(X_T) \mid \mathcal{F}_t\right].$$

We now aim at studying the effects of risk aversion on the optimal effort. In order to do so we consider the power utility function $u(x) = u^\gamma/\gamma$, parameterized by $\gamma < 1$. The next result gives some sufficient conditions for the optimal effort to be increasing with respect to risk aversion.

Proposition 6.3.2. *In the power utility case, if*

- $k_T \le c^{-1}(s-1)$, *or equivalently*

$$p'(x)x \le \frac{c'}{\sigma|l'|}\left(c^{-1}(s-1)\right) \tag{6.3.8}$$

  for all $x \in \mathbb{R}_+$,
- $c^{(3)}(k) \le 0$, $l^{(3)}(k) \le 0$,

then the optimal effort is decreasing in γ, therefore increasing in the risk aversion coefficient $1-\gamma$.

Proof. Remark that now the functions F and G also depend on γ, so we are allowed to differentiate them with respect to this variable. We want to apply the comparison theorem in [Ko00] to the quadratic BSDE (6.3.6), therefore we need to study the reaction of its generator and terminal condition to a change in γ. As for the terminal condition, remark

**. The definition of G has been slightly changed and adapted to this particular case.

that $F_\gamma(s, Z_T) = -\frac{g_\gamma(s,k_T)}{g_k(s,k_T)}$, where we recall that in this case $g(s,k) = \sigma\frac{u'(s-c(k))c'(k)}{l'(k)} = \sigma\frac{\gamma(s-c(k))^{\gamma-1}c'(k)}{l'(k)}$. Hence we have that $F_\gamma(s, Z_T) \leq 0$ if $g_\gamma(s, k_T) \leq 0$, or equivalently $k_T \leq c^{-1}(s-1)$. This gives the first condition of the Proposition (by replacing $k_T = F(s, -\sigma X_T p'(X_T))$).
We now turn to the generator of (6.3.6): we compute

$$\begin{aligned} G_\gamma(s,k) = &-\Big\{c'(k)l'(k)\Big[(\gamma-1)^2c'(k)^4l'(k) + 3(s-c(k))^2l'(k)c''(k)^2 \\ &+(2\gamma-3)(s-c(k))c'(k)^3l''(k) - (s-c(k))^2c'(k)\Big(3c''(k)l''(k) + l'(k)c^{(3)}(k)\Big) \\ &+(s-c(k))c'(k)^2\Big((3-2\gamma)l'(k)c''(k) + (s-c(k))l^{(3)}(k)\Big)\Big]\Big\} \\ &\times\left((s-c(k))\Big((\gamma-1)c'(k)^2l'(k) + (-s+c(k))l'(k)c''(k) + (s-c(k))c'(k)l''(k)\Big)^2\right)^{-1} \end{aligned}$$

and we remark that $G_\gamma(s,k) \leq 0$ if $c^{(3)}(k) \leq 0$ and $l^{(3)}(k) \leq 0$, which gives the second condition of the Proposition.
In order to conclude with the the comparison theorem stated in [Ko00], Theorem 2.6, we need to ensure that the coefficient of the quadratic term in the generator stays bounded, which is guaranteed by the following observation: since Z_t is uniformly bounded we deduce that k_t is bounded away from $c^{-1}(s) - \epsilon$ for some $\epsilon > 0$. Now take a bounded function $\tilde{G}(s,k)$ which coincides with G when $k \leq c^{-1}(s) - \epsilon$: we deduce that the optimal effort still solves (6.3.6) with G replaced by $\tilde{G}$. □

The first condition in the previous result imposes that the derivative of the final penalty function must not be too high with respect to the other parameters, i.e. the fee to be paid must not change too drastically in response to little changes in the state variable: if this was the case, the environment may become too risky (recall that X is not perfectly controlled) and a highly risk averse agent may decide to reduce his effort, thus avoiding to pay its certain cost in exchange for too uncertain consequences. Remark also that, in line with intuition, if the exogenous risk σ is small then the condition is more likely to be satisfied.
The second condition is probably less intuitive and is related to the rate of increase of costs and benefits when effort is increased. It is going to be satisfied in the model that we consider in Section 6.4, i.e. by a quadratic cost function and $l(k) = \frac{1-k}{1+k}$.
The reaction of the optimal agent's expected utility to a change in risk aversion is less clear to investigate. To get an intuition of why this is so, notice that for example under the conditions of Proposition 6.3.2 a higher value of γ reduces the optimal effort: this increases the continuous-time part of the agent's utility but will also in general increase final emissions, thus reducing the expected utility.
We now examine the effects of volatility on the effort.

Proposition 6.3.3. *Suppose the process N solution of (6.3.4) is negative for any $\sigma > 0$. Then if $l(k) \leq 0$ (resp $l(k) \geq 0$) the optimal effort is increasing (resp. decreasing) in σ.*

Proof. We use (6.3.6), and we remark that

$$\frac{\partial}{\partial\sigma}F(s, -\sigma x p'(x)) = 0$$

(since F also depends on σ through g) and that G does not depend on σ. If $N_t \leq 0$ then $\Theta_t \geq 0$ by the proof of Proposition 6.3.1, and the claim follows by the comparison theorem for quadratic BSDEs in [Ko00], using the function $\tilde{G}$ as explained in the proof of Proposition 6.3.2. □

The previous result requires the knowledge of N, which can be computed by solving the nonlinear PDE (6.4.1) that will be presented in the next section.
As for the dependence of the optimal effort on revenues s, the analysis is more complex: if we consider the terminal condition in (6.3.2), we see that $F_s(s,z) = -F_z g_s(s,F) \geq 0$, but the reaction of the generator to a change in s is harder to examine.

6.3.2 Impatience rate

Sometimes an *impatience rate* $\delta \geq 0$ is incorporated in principal-agent models (see [Sa08]) in order to account for the time preferences of the agent, in the sense that he gives a lower weight to cash flows that are far away in the future. This can be easily done in our framework by reformulating the agent's expected utility in this way:

$$V(k) = E^k \left[\int_0^T e^{-\delta t} u(s_t - c(k_t))dt - e^{-\delta T} p(X_T) \right].$$

All the results above can be readily adapted with minor modifications. In particular the agent's conditional value function given constant incentives s now follows the BSDE

$$\begin{cases} -dY_t = f^\delta(t,s,Z_t)dt - Z_t dW_t^0 \\ Y_T = -e^{-\delta T} p(X_T) \end{cases}$$

where $f^\delta(t,s,z) = zl(F^\delta(t,s,z)) + e^{-\delta t} u(s - c(F^\delta(t,s,z)))$ and $F^\delta(t,s,z)$ is now the inverse (in k) of

$$g^\delta(t,s,k) = e^{-\delta t}\sigma \frac{u'(s-c(k))c'(k)}{l'(k)}. \tag{6.3.9}$$

In the same way as before we obtain the following BSDE for Z:

$$\begin{cases} dZ_t = -\frac{l(F^\delta(t,s,Z_t))}{\sigma} N_t dt + N_t dW_t^0 \\ Z_T = -e^{-\delta T}\sigma X_T p'(X_T), \end{cases} \tag{6.3.10}$$

and the optimal effort therefore solves

$$\begin{cases} -dk_t = \left[-\delta \frac{g(t,s,k_t)}{g_k(t,s,k_t)} + G(t,s,k_t)\Theta_t^2 + \frac{l(k_t)}{\sigma}\Theta_t \right] dt - \Theta_t dW_t^0 \\ k_T = F^\delta(t,s,-e^{-\delta T}\sigma X_T p'(X_T)) = F(s,-\sigma X_T p'(X_T)) \end{cases} \tag{6.3.11}$$

where $G(t,s,k) = \frac{1}{2}\frac{g^\delta_{kk}(t,s,k)}{g^\delta_k(t,s,k)} = \frac{1}{2}\frac{g_{kk}(t,s,k)}{g_k(t,s,k)}$. Here we used the fact that $\frac{\partial}{\partial t}F^\delta(t,s,z) = \delta\frac{g^\delta(t,s,F^\delta)}{g^\delta_k(t,s,F^\delta)} = \delta\frac{g(t,s,F^\delta)}{g_k(t,s,F^\delta)}$. Now since $\frac{g^\delta}{g^\delta_k} \geq 0$ and G does not depend on δ, we deduce the following result.

Proposition 6.3.4. *The optimal effort is decreasing in the impatience rate δ.*

Since the terminal condition in (6.3.11) does not depend on δ, we see that the changes will be more relevant as the time to maturity increases. Remark that this result also holds when s is not necessarily constant but it only satisfies Assumption 6.3.1, though we presented it in this simpler case.

6.4 Numerical computation of the optimal contract

We still assume that continuous-time incentives are constant, or at least space independent (i.e. $s = s(t)$).
From (6.3.7) and Assumption 6.3.1 we can write $Z_t = \phi(t, X_t)$ where ϕ solves (in the classical sense, see [MY99], Chapter 9, Section 2.1)

$$\begin{cases} \phi_t + \frac{1}{2}\sigma^2 x^2 \phi_{xx} + l(F(s,\phi))x\phi_x = 0 \\ \phi(T,x) = -\sigma x p'(x) \end{cases} \tag{6.4.1}$$

which is usually easier to treat than (6.3.6). The idea is therefore to approximate Z first and then recover k. We set $y = \log x$ and $\theta(t,y) = \phi(T-t,x)$ from which we get

$$\begin{cases} \theta_t - \frac{1}{2}\sigma^2 \theta_{yy} - b(\theta)\theta_y = 0 \\ \theta(0,y) = -\sigma e^y p'(e^y) \end{cases} \tag{6.4.2}$$

where $b(\theta) = l(F(s,\theta)) - \frac{1}{2}\sigma^2$. The solution to (6.4.2) can be approximated numerically using a standard scheme that we briefly recall and adapt to our case (see [MY99], Chapter 9 for details).
We set the space and time steps $h > 0$, $\Delta t > 0$. We let $y_i = ih$, $i = 0, \pm 1, ..., \pm i_0$, and $t^j = j\Delta t$, $j = 0, 1, ..., N$. We denote $h_i^j = h(t^k, y_i)$ the grid value of the function h, and $h^j = h(t^j, \cdot)$. We define for each j the approximate solution w^j by the following recursive steps:

1. *Step 0*: Set $w_i^0 = -\sigma e^{y_i} p'(e^{y_i})$, $i = 0, \pm 1, ..., \pm i_0$; use linear interpolation to obtain a function $w^0(y)$ defined on $y \in \mathbb{R}$.
2. *Step j*: Suppose that $w^{j-1}(y)$ is defined for $y \in \mathbb{R}$ and set

$$\begin{cases} b_i^j = b(w_i^{j-1}) \\ \bar{y}_i^j = y_i - b_i^j \Delta t, \quad \bar{w}_i^{j-1} = w^{j-1}(\bar{y}_i^j) \\ \delta^2(w)_i^j = h^{-2}[w_{i+1}^j - 2w_i^j + w_{i-1}^j] \end{cases}$$

Obtain the grid values for the j-th step approximate solution by solving

$$\frac{w_i^j - \bar{w}_i^{j-1}}{\Delta t} = \frac{1}{2}\sigma^2 \delta^2(w)_i^j.$$

Use again linear interpolation to extend the grid values to all $y \in \mathbb{R}$.

Define the error function on the grid by $\zeta_i^j = \theta_i^j - w_i^j$, where θ_i^j represent grid values for the true solution. One can prove that

$$\sup_{j,i} |\zeta_i^j| = \mathcal{O}(h + \Delta t).$$

The approximation for the optimal effort is then recovered by setting $k_i^j = F(s, w_i^j)$. Since F has bounded derivatives, the same $\mathcal{O}(h + \Delta t)$ rate of convergence holds for the approximation of the optimal effort.
We just mention for completeness (without discussing regularity and convergence of the numerical schemes) the two other possible ways to compute the optimal effort. In the first we use (6.3.6) and supposing $k_t = \varphi(t, X_t)$ we recover φ as the solution to

$$\begin{cases} \varphi_t + \frac{1}{2}\sigma^2 x^2 \left[\varphi_{xx} + G(s,\varphi)(\varphi_x)^2\right] + l(\varphi) x \varphi_x = 0 \\ \varphi(T,x) = F(s, -\sigma x p'(x)). \end{cases} \tag{6.4.3}$$

Another idea is to write the analogue of the PDE (6.2.12) that takes into account the implementability constraints by forcing $k_t = F(s, Z_t) = F(s, \theta_x \sigma x)$:

$$\begin{cases} \theta_t + \frac{1}{2}\theta_{xx}\sigma^2 x^2 + x\theta_x l(F(s, \theta_x \sigma x)) + u(s - c(F(s, \theta_x \sigma x))) = 0, \\ \theta(T,x) = -p(x). \end{cases} \tag{6.4.4}$$

6.4.1 Interpretation of results

For our numerical experiments we take the functions $c(k) = k^2/2$, $u(x) = 2\sqrt{x}$ and $l(k) = \frac{1-k}{1+k}$ (the first two are quite standard choices, while the third is just a decreasing bounded function on $[0, \infty)$ with bounded first derivative).
Figure 6.1 shows the numerical approximation of the optimal effort by choosing (a proper regularization of) a penalty function of the type $p(x) = \lambda \mathbf{1}_{[\Lambda,\infty)}(x)$ (i.e. a fixed amount is charged when a certain level of emissions is exceeded). The economic interpretation is straightforward: since s is time and space independent, it can be more naturally considered as an income flow, and not as a real incentive policy. Therefore in this example we have in a sense isolated the effects on effort provided by the final fee to pay at maturity T. At every date the optimal effort is bell-shaped: loosely speaking, when emissions are too high the firm has little hope to reduce them and finds no reason to bear the cost of trying (the fee being fixed); on the other hand, when emissions are sufficiently small the agent can be reasonably sure that they will end up below Λ even without any positive effort. As maturity approaches, the short time left to act makes it optimal to take on some effort only when emissions are close to Λ.
The situation changes if we choose a penalty function of the type $p(x) = \lambda(x - \Lambda)^+$, corresponding to a situation where the agent is charged proportionally for each unit of emissions that exceeds a certain threshold Λ at maturity. This case is shown in Figure 6.2: we see that it is no more optimal to stop putting effort when emissions are high, since there is always an opportunity to reduce the final payment.
In Figure 6.3 we plotted some simulated paths by using (6.1.1) and the optimal effort dynamics of Figure 6.2: we observe a natural tendency for emissions to be driven just

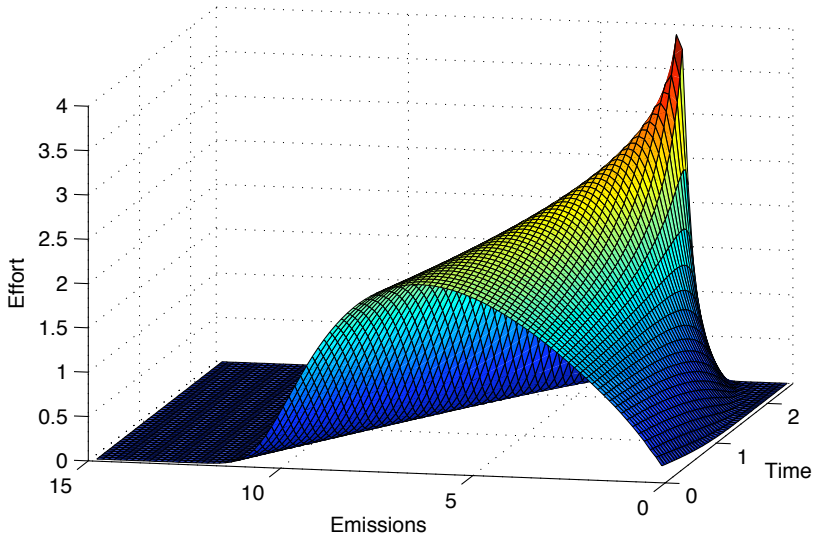

Figure 6.1: Optimal effort dynamics with a constant incentive policy $s_t = 10$ and fee $p(x) = 41_{[3,\infty)}(x)$. Parameter values: $\sigma = 0.22$, $\gamma = 0.5$, $T = 2.5$.

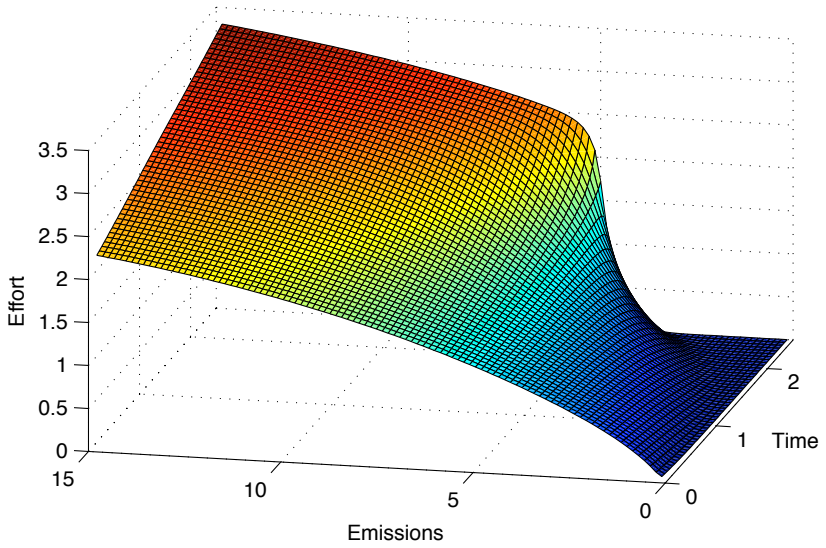

Figure 6.2: Optimal effort dynamics with a constant incentive policy $s_t = 10$ and $p(x) = (x-5)^+$

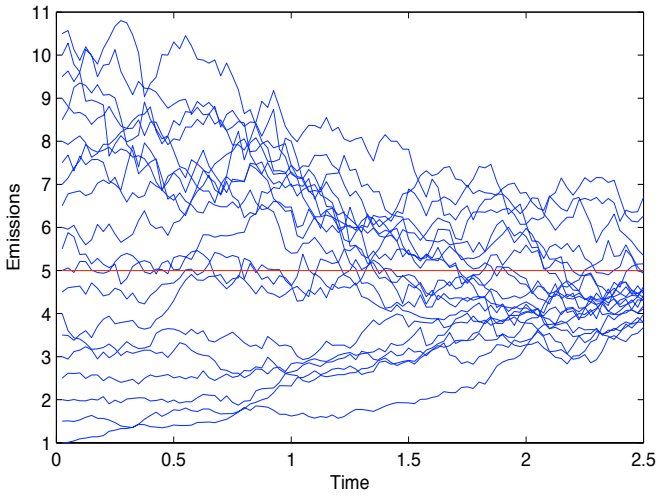

Figure 6.3: Simulation of typical emissions paths following the agent's optimal effort with a constant incentive policy $s_t = 10$ and different starting values. The red line shows $\Lambda = 5$.

below the threshold level at maturity, if the starting value x is reasonably low (otherwise the process may be left uncontrolled and become very large at maturity).

The effort dynamics of the previous examples can be considered as a benchmark situation when there are no continuous-time incentives (or they are trivially constant).

6.5 The principal's problem

We will stick to the weak formulation to treat the principal's optimization problem, in order to avoid measurability issues and inconsistencies that can arise when one switches from the two formulations (as we partially mentioned in the Introduction and as is also reported in [CWZ08]).
In the agent's case we were trying to find the best possible effort policy k given an incentive structure made up by a penalty function p and continuous-time payments s. When considering the principal, we therefore look for a criterion to find the couple (s, p) that maximizes a certain utility functional. To do so, we model the principal's expected profit given (s, p, k) as

$$\begin{aligned} I(s,p,k) &= E^k\left[p_1(p) - p_2(X_T) - \int_0^T u_1(s_r)dr\right] \\ &= E\left[\Gamma_T^k p_1(p) - \Gamma_T^k p_2(X_T) - \int_0^T \Gamma_r^k u_1(s_r)dr\right], \end{aligned} \tag{6.5.1}$$

where

1. $p_1 : \mathbb{R} \mapsto \mathbb{R}$ is a concave function, C^2 and with bounded derivatives. It relates the (dis)utility of the agent linked to the payment of the fee to the corresponding utility of the principal.

2. p_2 is a random variable in $L^{2+\alpha}$ for some $\alpha > 0$, with the role to capture the social costs related to the level of emissions.
3. $u_1 : \mathbb{R}_+ \mapsto \mathbb{R}$ is a C^2 utility function (i.e. increasing and concave) that takes into account continuous-time payments to the agent.

If we assume, however, that the principal can forecast the optimal agent's response given (s, p), it is convenient to simply consider $I(s, p)$, defined by replacing k in (6.5.1) with the optimal effort policy given (s, p) (forgetting for a moment that this might not be unique). This way of proceeding, though quite natural, is not very easy to pursue as it would involve some form of optimization over a function space to recover p.
Since the state knows the optimal agent's reaction given his choices, a more efficient way to attack the problem is to directly assume that he is able to choose the couple (s, k), provided he then adjusts the final fee structure accordingly. Quite intuitively, however, with no other constraints the problem will easily be ill-posed: in fact, if the state can arbitrarily increase the final fees then his maximal expected profit will diverge to infinity (and the agent's one to minus infinity). To avoid this issue, in the classical literature on private contracts (see [CWZ08], [Sa08], [Wi08]) the principal also has to guarantee to the agent a certain initial utility, that has to be coherent with the other opportunities available on the market. We are not in such a situation since we mainly think of a context where the agent is forced to enter the contract, however we still assume that the state chooses to provide the agent with a certain initial utility R. The assumption is not so strange if we think that the aim of the state is not to ruin the agent but rather to push it to act in some socially convenient way.
Recall that the agent's utility given (s, k, p) follows the BSDE (where we add the superscript "A" for "Agent")

$$\begin{cases} dY_t^A = [-Z_t^A l(k_t)/\sigma - u(s_t - c(k_t))]dt + Z_t^A dW_t^0 \\ Y_T^A = -p(X_T) \end{cases} \tag{6.5.2}$$

If, however, the principal chooses the initial agent's utility, the terminal condition above is replaced by the initial condition $Y_0^A = R$, thus the backward SDE (6.5.2) becomes a forward SDE for the principal and the terminal value $Y_T^A =: -C_T$ will now be an output of the initial choice of (s, k), once Z^A is fixed. We write (s, C_T, k) to refer to a policy with C_T as final penalty.
Now remark that we must also make sure that the resulting triplet (s, C_T, k) is implementable: in other words, when the agent faces the incentive structure (s, C_T) he must actually optimally choose k. By Proposition 6.2.2, this can be achieved by setting $Z_t^A = g(s_t, k_t)$ in (6.5.2), which then becomes

$$Y_t^A = R - \int_0^t [u(s_r - c(k_r)) + g(k_r, s_r)l(k_r)/\sigma]dr + \int_0^t g(k_r, s_r)dW_r^0 \tag{6.5.3}$$

Loosely speaking, the principal looks at the agent's BSDE in its forward version, by fixing its initial condition, the couple (s, k) and the process Z^A in order to be able to interpret the terminal value as a penalty payment that will actually induce the agent to choose the effort policy k. We define $C_T^{(s,k)} := -Y_T^A$, where Y^A follows (6.5.3) with (s, k). The next result is now straightforward.

Lemma 6.5.1. *The triplet $(s, C_T^{(s,k)}, k)$ is the unique implementable policy for any strongly admissible couple (s, k).*

Proof. Since (s, k) is strongly admissible, it follows that $g(s_t, k_t)$ stays bounded and consequently $C_T^{(s,k)}$ is also admissible for the agent (it is in $L^{2+\alpha}(\Omega)$ for any $\alpha > 0$). Proposition 6.2.2 and Theorem 6.2.1 then imply the result. □

Example 6.5.1. Suppose the state aims at inducing a constant level of effort $k_t = \bar{k}$ over time. Assume continuous time incentives are kept constant at some $s_t = \bar{s}$ such that $\bar{k} < c^{-1}(\bar{s})$ (strong admissibility). Denote $\bar{g} = g(\bar{s}, \bar{k})$, $\bar{u} = u(\bar{s} - c(\bar{k}))$ and $\bar{l} = l(\bar{k})$. The final penalty that has to be proposed in this case is then

$$C_T^{(s,k)} = -R + [\bar{u} + \bar{g}\bar{l}/\sigma]T + |\bar{g}|W_T^0.$$

Since $X_T = x \exp\{\sigma W_T^0 - \sigma^2 T/2\}$ we can write

$$C_T^{(s,k)} = -R + \left[\bar{u} + \bar{g}\frac{\bar{l} + \sigma^2/2}{\sigma}\right] T + \frac{|\bar{g}|}{\sigma} \log \frac{X_T}{x}.$$

We see that, even in this simple example, the final penalty is not of the form $p(X_T)$, since the initial value x of the emissions process appears in the formula. Indeed this kind of final fee penalizes the proportional increase in the emission level from the beginning of the period. Remark also that if the proportional reduction is sufficiently large then $C_T^{(s,k)}$ can become negative and therefore represents a reward more than a fee.
Conversely, once continuous time incentives are constant and fixed at $\bar{s}$, any final penalty of the form $K + BW_T^0$ for some $K \in \mathbb{R}$ and $B \in \mathbb{R}_+$ induces a constant optimal effort, which can be recovered by solving for $\bar{k}$ the equation $|g(\bar{s}, \bar{k})| = B$. The corresponding agent's initial utility is then given by $R = \left[\bar{u} + \bar{g}\frac{\bar{l}+\sigma^2/2}{\sigma}\right] T - K$.

We can now naturally redefine the expected profit of the principal as

$$J(s, k) = E\left[\Gamma_T^k p_1(C_T^{(s,k)}) - \Gamma_T^k p_2 - \int_0^T \Gamma_r^k u_1(s_r) dr\right]. \tag{6.5.4}$$

The principal's optimization problem is

$$v_P := \sup_{(s,k)} J(s, k), \tag{6.5.5}$$

where the sup is taken over all strongly admissible policies. In order to solve it, we now need one additional state equation, so that our state system becomes

$$\begin{cases} \Gamma_t^k = 1 + \int_0^t \Gamma_r^k l(k_r)/\sigma dW_r^0 \\ Y_t^A = R - \int_0^t [u(s_r - c(k_r)) + g(k_r, s_r) l(k_r)/\sigma] dr + \int_0^t g(k_r, s_r) dW_r^0. \end{cases}$$

We will apply again the SMP (Theorem 3.2 in [YZ99]). Define the two adjoint processes

$$\begin{cases} dY_t^P = [-l(k_t)/\sigma Z_t^P + u_1(s_t)]dt + Z_t^P dW_t^0 \\ dY_t^1 = Z_t^1 dW_t^0 \\ Y_T^P = p_1(-Y_T^A) - p_2 \\ Y_T^1 = -\Gamma_T^k p_1'(-Y_T^A) \end{cases} \tag{6.5.6}$$

and the Hamiltonian

$$\tilde{H}(\Gamma^k, Z^P, Y^1, Z^1, s, k) = -Y^1[u(s-c(k))+g(k,s)l(k)/\sigma]+Z^P\Gamma^k l(k)/\sigma+Z^1 g(k,s)-\Gamma^k u_1(s). \tag{6.5.7}$$

The next result gives necessary conditions for optimality.

Proposition 6.5.1. *Suppose the strongly admissible contract (s^*, k^*) is optimal for the principal's problem. Then there exist two pairs of processes given by* (6.5.6) *such that (dropping *-superscripts for clearness)*

$$\begin{cases} -Y_t^1[u'(s_t - c(k_t)) + g_s(k_t, s_t)l(k_t)/\sigma] + Z_t^1 g_s(k_t, s_t) - \Gamma_t^k u_1'(s_t) = 0 & on\ \{m < s_t < M\} \\ -Y_t^1[u'(s_t - c(k_t)) + g_s(k_t, s_t)l(k_t)/\sigma] + Z_t^1 g_s(k_t, s_t) - \Gamma^k u_1'(s_t) \geq 0 & on\ \{s_t = M\} \\ -Y_t^1[u'(s_t - c(k_t)) + g_s(k_t, s_t)l(k_t)/\sigma] + Z_t^1 g_s(k_t, s_t) - \Gamma^k u_1'(s_t) \leq 0 & on\ \{s_t = m\} \\ -Y_t^1 g_k(k_t, s_t)l(k_t)/\sigma + Z_t^P \Gamma_t^k l'(k_t)/\sigma + Z_t^1 g_k(k_t, s_t) = 0 & on\ \{k_t > 0\} \\ -Y_t^1 g_k(k_t, s_t)l(k_t)/\sigma + Z_t^P \Gamma_t^k l'(k_t)/\sigma + Z_t^1 g_k(k_t, s_t) \leq 0 & on\ \{k_t = 0\} \end{cases} \tag{6.5.8}$$

Proof. Similar to Proposition 6.2.1. Here we have the additional control s, which by admissibility (see Definition 6.2.1) takes its values in $[m, M]$. □

Sufficient conditions are much harder to derive in this case (compared to the agent's problem), and typically require some form of convexity properties that are hard to verify. For example, a long but straightforward calculation gives, for any admissible control (s, k), that

$$\begin{aligned} &J(s,k) - J(s^*,k^*) = Y_0^P - Y_0^{P^*} = E^{k^*}[Y_0^P - Y_0^{P^*}] = E[\Gamma_T^{k^*}(Y_0^P - Y_0^{P^*})] \\ &= E\left[\int_0^T [\tilde{H}(\Theta_r^*, Z_r^P, s_r, k_r) - \tilde{H}(\Theta_r^*, Z_r^{P*}, s_r^*, k_r^*) - \tilde{H}_{Z^P}(\Theta_r^*, Z_r^{P*}, s_r^*, k_r^*)(Z^P - Z^{P*})]dr\right] \end{aligned}$$

where we call $\Theta_r^* := (\Gamma_r^{k^*}, Y_r^{1*}, Z_r^{1*})$. In order to conclude that $J(s,k) - J(s^*,k^*) \leq 0$ and thus prove a sufficient condition for optimality we would need to show that $\tilde{H}$ is jointly concave in (Z^P, s, k), but this is not true because of the term $Z^P\Gamma^k l(k)/\sigma$. We will be able to give some sufficient conditions in the particular case studied in the next section.

6.5.1 The case $p_1(x) = x$

We will consider in this section the particular (and simpler) case where $p_1(x) = x$, i.e. the final agent's disutility linked to the payment of the fee corresponds to a principal's utility of the same amount. Then $Y_t^1 = -\Gamma_t^k$ and hence $Z_t^1 = -\Gamma_t^k l(k_t)/\sigma$. The necessary conditions now simply become

$$\begin{cases} u'(s_t - c(k_t)) - u_1'(s_t) = 0 & \text{on } \{m < s_t < M\} \\ u'(s_t - c(k_t)) - u_1'(s_t) \geq 0 & \text{on } \{s_t = M\} \\ u'(s_t - c(k_t)) - u_1'(s_t) \leq 0 & \text{on } \{s_t = m\} \\ l'(k_t)Z_t^P = 0 & \text{on } \{k_t > 0\} \\ l'(k_t)Z_t^P \leq 0 & \text{on } \{k_t = 0\} \end{cases} \tag{6.5.9}$$

The first condition has a clear economic meaning: the principal will choose continuous-time incentives s_t in such a way that, at any time, the marginal cost $u_1'(s_t)$ of an additional quantity be equal to the marginal benefit $u'(s_t - c(k_t))$ ††.
Similarly as in the agent's case, the key idea is to try to invert the optimality conditions and get to a BSDE whose solution will be a candidate for the optimal quantities s and k (and hence C_T). To this aim, we introduce the following assumption (which includes the definition of the two additional functions I and L), that we will later verify for a particular choice of functions.

Assumption 6.5.1. Suppose that

1. We are able to invert uniquely the first three conditions in (6.5.9) by constructing a continuous and a.e. differentiable function I such that $s = I(k)$ verifies them and $I(k)$ reaches the maximum of $s \mapsto u(s - c(k)) - u_1(s)$ in $[m, M]$ for all $k \geq 0$.
2. We can uniquely define a continuous and a.e. differentiable function $L(z)$ that solves (in k) the implicit equation $g(I(k), k) = z$ for all $z \leq 0$. We set $L(z) = 0$ if $z > 0$. This function accounts for optimality of the agent (similarly as F in the preceding sections).

To get a candidate BSDE, suppose also that $k_t > 0$ a.s. for all $t \in [0, T]$ at the optimum. We deduce from the agent's optimality conditions (6.2.6) that $Z_t^A = g(s_t, k_t)$. Then from the principal's conditions (6.5.9) we deduce $Z_t^A = g(I(k_t), k_t)$, hence $k_t = L(Z_t^A)$. Also (6.5.9) implies that at the optimum $Z_t^P = 0$, therefore

$$Y_0^P = C_T^{(s,k)} - p_2 - \int_0^T u_1(s_r)dr,$$

so that $C_T^{(s,k)} = p_2 + \int_0^T u_1(s_r)dr + c$ for some constant $c \in \mathbb{R}$ corresponding to the initial principal's expected utility. This allows to identify the optimal terminal condition to the agent's problem. Now plugging this into the agent's BSDE we get

$$\begin{cases} dY_t^A = [-Z_t^A l(L(Z_t^A))/\sigma - u(I(L(Z_t^A)) - c(L(Z_t^A)))]dt + Z_t^A dW_t^0 \\ Y_T^A = -p_2 - \int_0^T u_1(I(L(Z_t^A)))dt - c \end{cases} \tag{6.5.10}$$

where the parameter c is there to ensure that $Y_0^A = R$. We then have the following corollary to the necessary conditions.

Corollary 6.5.1. *Suppose the strongly admissible contract (s^*, k^*) is optimal for the principal's problem (6.5.5) with $p_1(x) = x$ and verifies $k_t^* > 0$ a.s. for all $t \in [0, T]$. Then, under Assumption 6.5.1, there exists a solution (Y^A, Z^A) to (6.5.10) for $c \in \mathbb{R}$ such that $Y_0^A = R$, and the optimal quantities can be recovered from this solution by setting: $k_t^* = L(Z_t^A)$, $s_t^* = I(k_t^*) = I(L(Z_t^A))$, $C_T = p_2 + \int_0^T u_1(s_r^*)dr + c$.*

††. The quantity $u'(s_t - c(k_t))$, by (6.5.3), can be seen as the average increase in the agent's final fee following an increase in s_t (recall that an increase in continuous time incentives in this context reduces the average final fee, as the initial agent's utility is fixed). Since $p_1(x) = x$, the same quantity $u'(s_t - c(k_t))$ is also interpreted as a marginal benefit to the principal.

Conversely, if a solution to (6.5.10) exists, then it will be a candidate for the optimal solution. In order to derive some sufficient conditions, we introduce the modified Hamiltonian

$$H^M(s,k,z) := -u_1(s) + u(s-c(k)) + zl(k)/\sigma.$$

Proposition 6.5.2. *Suppose Assumption 6.5.1 holds and that* (6.5.10) *admits a solution* (Y^{A*}, Z^{A*})*. Also, suppose that* $I(k)$ *reaches the maximum of* $\zeta(s) = u(s-c(k)) - u_1(s)$ *in* $[m, M]$*. Denote* $k_t^* = L(Z_t^{A*})$ *and* $s_t^* = I(k_t^*)$*. Then* (s^*, k^*) *is optimal for the principal's problem* (6.5.5) *with* $p_1(x) = x$*.*

Proof. From $C_T^{(s^*,k^*)} = -Y_T^{A*}$ we get

$$\begin{cases} dY_t^{P*} = [-l(k_t^*)/\sigma Z_t^{P*} + u_1(s_t^*)]dt + Z_t^{P*}dW_t^0 \\ Y_T^{P*} = C_T^{(s^*,k^*)} - p_2 = \int_0^T u_1(s_t^*)dt + c, \end{cases}$$

therefore $Z_t^{P*} = 0$ a.s. for all $t \in [0,T]$. Hence we have for any strongly admissible couple (s,k)

$$\begin{aligned} J(s,k) - J(s^*,k^*) &= E^k\left[p_1(C_T^{(s,k)}) - p_2 - \int_0^T u_1(s_r)dr\right] - E^k\left[Y_0^{P*}\right] \\ &= E^k\left[C_T^{(s,k)} - C_T^{(s^*,k^*)} - \int_0^T [u_1(s_r) - u_1(s_r^*)]dr\right] \\ &= E^k\left[-\int_0^T [u_1(s_r) - u_1(s_r^*)]dr\right] \\ &\quad + E^k\left[\int_0^T \{[u(s_r - c(k_r)) - u(s_r^* - c(k_r^*))]dr + g(k_r^*, s_r^*)[l(k_r) - l(k_r^*)]/\sigma\}dr\right] \\ &= E^k\left[\int_0^T [H^M(s_r, k_r, g(s_t^*, k_t^*)) - H^M(s_r^*, k_r^*, g(s_t^*, k_t^*))]dr\right] \le 0. \end{aligned}$$

The last line follows from the fact that by assumption (s_t^*, k_t^*) is the only stationary point of $H^M(\cdot,\cdot,g(s_t^*,k_t^*))$. Indeed, the FOC in s gives that $s = I(k)$, then substituting in $H^M(\cdot,\cdot,g(s_t^*,k_t^*))$ gives

$$H^M(I(k),k,g(s_t^*,k_t^*)) = -u_1(I(k)) + u(I(k)-c(k)) + g(I(k_t^*),k_t^*)l(k)/\sigma.$$

Differentiating in k we obtain

$$\begin{aligned} &H_k^M(I(k),k,g(s_t^*,k_t^*)) \\ &\quad = I'(k)[u'(I(k)-c(k)) - u_1'(I(k))] - u'(I(k)-c(k))c'(k) + g(I(k_t^*),k_t^*)l'(k)/\sigma \\ &\quad = -u'(I(k)-c(k))c'(k) + g(I(k_t^*),k_t^*)l'(k)/\sigma \end{aligned}$$

by the FOC in s (remark that $I'(k) = 0$ iff $s = I(k) = m$ or $s = I(k) = M$ since u is strictly increasing, hence the first term in the expression disappears). Equating to zero we obtain $g(I(k),k) = g(I(k_t^*),k_t^*)$, implying $k = k_t^*$ by Assumption 6.5.1 (ii). We now want to prove that this unique stationary point is a global maximum. It suffices to notice that, since $g(s_t^*,k_t^*) \le 0$, $H_k^M(I(0),0,g(s_t^*,k_t^*)) = g(s_t^*,k_t^*)l'(k)/\sigma \ge 0$ and $H_k^M(I(k),k,g(s_t^*,k_t^*)) \to -\infty$ when $k \to c^{-1}(M)$, which implies the claim. □

A special case

Here we consider for the agent a power utility function $u(x) = x^\gamma/\gamma$ (restrictions on γ will be imposed in the sequel) and we take for the principal $u_1(x) = \theta u(x)$ for some $\theta > 1$ (see Remark 6.5.1 below).
Setting $K_m = c^{-1}(m(1-\theta^{\frac{1}{\gamma-1}}))$ now the function I defined in Assumption 6.5.1 (i) takes the form

$$I(k) = \begin{cases} \frac{c(k)}{1-\theta^{\frac{1}{\gamma-1}}} & \text{if } K_m \le k \le K_M \\ m & \text{if } 0 \le k \le K_m \\ M & \text{if } k \ge K_M. \end{cases}$$

The quantity $I(k)$ reaches the maximum of $\zeta(s) = u(s - c(k)) - \theta u(s)$ in $[m, M]$ (even if ζ is not necessarily concave). To see this, compute $\zeta'(s) = u'(s - c(k)) - \theta u'(s)$ and remark that $\zeta'(c(k)) = +\infty$, $\zeta'(s)$ becomes negative for s sufficiently large (since $\theta > 1$) and $\zeta'(s) = 0$ at only one point.
In order to apply Proposition 6.5.2 we still need to verify that point (ii) in Assumption 6.5.1 holds, that is we need to show that the function $\tilde{g}(k) := \sigma \frac{u'(I(k)-c(k))c'(k)}{l'(k)}$ is strictly decreasing (as clearly $\tilde{g}(0) = 0$ and $\tilde{g}(k) \to -\infty$ as $k \to c^{-1}(M)$). Computing its derivative gives

$$\tilde{g}'(k) := \sigma \frac{[u''(I(k) - c(k))(I'(k) - c'(k))c'(k) + u'(I(k) - c(k))c''(k)]l'(k) - u'(I(k) - c(k))c'(k)l''(k)}{l'(k)^2},$$

hence a sufficient condition (remarking that the last term in the espression above is strictly positive except at $k = 0$) is

$$u''(I(k) - c(k))(I'(k) - c'(k))c'(k) + u'(I(k) - c(k))c''(k) \ge 0 \tag{6.5.11}$$

a.e. on $K_m \le k \le K_M$, which in our power utility case is equivalent to

$$(\gamma - 1)c'(k)^2 + c(k)c''(k) \ge 0.$$

With a quadratic cost function this is verified for $1/2 \le \gamma < 1$.
Defining $\tilde{Y}^A_t = Y^A_t + \int_0^t 2u(I(L(Z^A_r)))dr$ then (6.5.10) becomes

$$\begin{cases} -d\tilde{Y}^A_t = \tilde{f}(Z^A_t)dt - Z^A_t dW^0_t \\ \tilde{Y}^A_T = -p_2 - c \end{cases} \tag{6.5.12}$$

with $\tilde{f}(z) = zl(L(z))/\sigma + u(I(L(z)) - c(L(z))) - 2u(I(L(z)))$. Using the definitions of I and L we can compute

$$\begin{aligned} \tilde{f}'(z) =& l(L(z))/\sigma + zl'(L(z))L'(z)/\sigma \\ &+ u'(I(L(z)) - c(L(z)))[I'(L(z)) - c'(L(z))]L'(z) - 2u'(I(L(z)))I'(L(z))L'(z) \\ =& l(L(z))/\sigma + [zl'(L(z))/\sigma - u'(I(L(z)) - c(L(z)))c'(L(z))]L'(z) \\ &+ [u'(I(L(z)) - c(L(z))) - 2u'(I(L(z)))]I'(L(z))L'(z) \\ =& l(L(z))/\sigma. \end{aligned}$$

Remark that, even if I and L are not differentiable at two points the previous equality still holds, by considering I' and L' as a left/right derivatives in the first place.
As in the proof of Proposition 6.3.1 we can show that, under Assumption 6.3.1 for p_2 ‡‡, Z^A follows the BSDE

$$\begin{cases} dZ_t^A = -\frac{l(L(Z_t^A))}{\sigma} N_t dt + N_t dW_t^0 \\ Z_T^A = -\sigma X_T p_2'(X_T). \end{cases} \tag{6.5.13}$$

In this way the same numerical method proposed in Section 6.4 can be applied to (6.5.13) with minor modifications to recover the optimal effort $k_t = L(Z_t^A)$ and the optimal incentives $s_t = I(k_t)$. In principle, it is pretty straightforward to apply Ito's Lemma and derive a BSDE for the optimal effort as in (6.3.6) and perform similar analyses as in Section 6.3 to the principal's problem. However in this particular case of a quadratic cost the functions I and L are not twice continuously differentiable, so we did not pursue this further here.

Example 6.5.2. An example is shown in Figure 6.4, where we used again the functions $l(k) = \frac{1-k}{1+k}$, $c(k) = k^2/2$ and $u(x) = 2\sqrt{x}$. We chose (a mollified version of) the capped proportional penalty function $p_2(x) = (x-4)^+ - (x-8)^+$ and the minimal (maximal) incentive value is set to $m = 2$ ($M = 10$). Assumption 6.5.1 holds in this case by the previous discussion, since $\gamma = 1/2$.
The shape of the optimal effort is similar to the one we have already seen in Section 6.4.1. As for the optimal incentives, they are set most of the time at their minimal level $m = 2$, while they are raised towards the end of the period in order to generate a higher effort in the region where it is more effective (i.e. for emissions values between 4 and 8). Notice that continuous-time incentives are not necessarily a decreasing function of X: higher emissions (notably towards maturity) may induce the principal to increase incentives, in order to generate a higher effort which will reduce the final social cost at maturity.

Remark 6.5.1. The choice $\delta > 1$ has been done to guarantee a nontrivial solution, in the sense that in this way different levels of incentives will be chosen by the principal. By taking, for example, $u_1(x) = u(x)/2$, we would obtain $I(k) = M$ for all $k \geq 0$ (recall also that $c(0) = 0$ in our example) and the problem would be equivalent to the one where incentives are fixed (see the previous Remark).

The next section contains a discussion which uses previously obtained results to introduce an issue which is not directly related to the problem at hand, but that can still be interesting for a better understanding of the model and eventually for future research.

6.5.2 Different agents

An interesting class of related problems is suggested by the recent literature on adverse selection (see [CWY13]), and arises when considering the joint behavior of different types of agents. For example, let us introduce in our model another agent with utility function given by ϕu, with $\phi > 0$. If $\phi < 1$ ($\phi > 1$) we interpret it as a "bad" ("good") agent, with

‡‡. The corresponding assumption is that the social cost is only a function of X_T. This can be justified if we interpret X as a market perception of cumulative emissions, which only become known at a given maturity T (due to monitoring costs, for example).

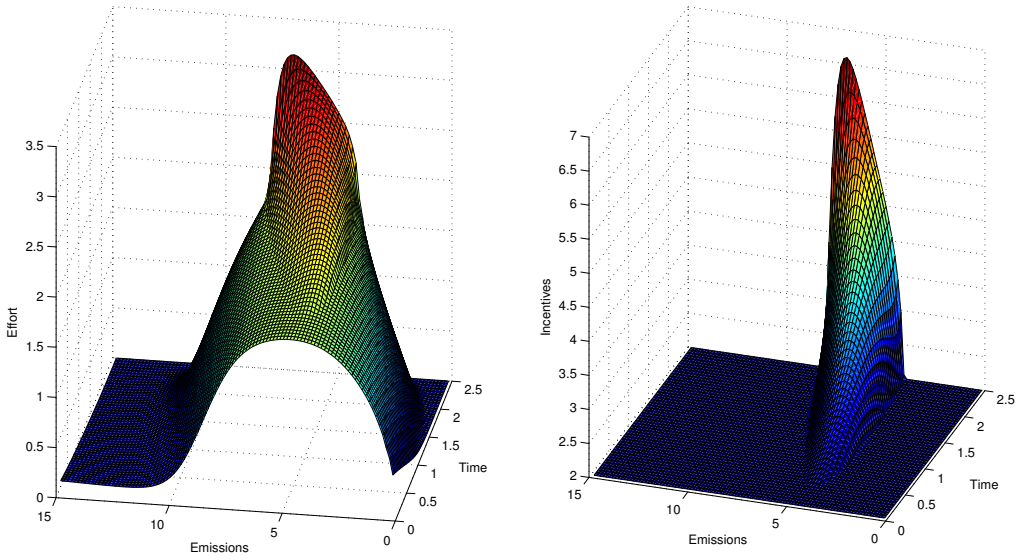

Figure 6.4: Optimal effort and optimal incentives dynamics with $p_2(x) = (x-4)^+ - (x-8)^+$, $m = 2$, $M = 10$.

respect to the first one. An interesting question to ask in this context is the following: given an initial utility R for the second agent, which is the largest initial utility that can be achieved by the first one? We call this quantity $U(R)$ and characterize it in the next proposition. This kind of information can be valuable to the principal in an adverse selection environment, supposing that the agents have the choice to enter the contract and that the principal knows their reservation values.

Proposition 6.5.3. *If $0 < \phi < 1$ then $U(R) = R + Y_0$, where Y solves the BSDE*

$$\begin{cases} dY_t = [-Z_t l(\tilde{F}(Z_t))/\sigma - (1-\phi)u(M - c(\tilde{F}(Z_t)))]dt + Z_t dW_t^0 \\ Y_T = 0. \end{cases} \tag{6.5.14}$$

and $\tilde{F}(z)$ is defined as in Lemma 6.2.1 replacing g with $(1-\phi)g$ and taking $s = M$.
If $\phi > 1$ then $U(R) = R - T(\phi - 1)u(m - c(0))$.

Proof. The second agent gets initial utility R after optimally choosing $\tilde{k}$, given incentives $(s, -L_T)$, where

$$L_t = R - \int_0^t [\phi u(s_r - c(\tilde{k}_r)) + \phi g(\tilde{k}_r, s_r) l(\tilde{k}_r)/\sigma]dr + \int_0^t \phi g(\tilde{k}_r, s_r) dW_r^0$$

The first agent gets the same incentives and optimizes over k. The quantity we look for is therefore written as

$$U(R) = \sup_{k,s,\tilde{k}} E^k \left[\int_0^T u(s_t - c(k_t))dt + L_T \right], \tag{6.5.15}$$

which after a straightforward calculation becomes

$$\begin{aligned} U(R) &= R + \sup_{k,s,\tilde{k}} E\left[\int_0^T \Gamma_t^k u(s_t - c(k_t))dt + \Gamma_T^k L_T\right] \\ &= R + \sup_{k,s,\tilde{k}} E\left[\int_0^T \Gamma_t^k \Big[u(s_t - c(k_t)) - \phi u(s_t - c(\tilde{k}_t)) \right. \\ &\quad \left. - \phi g(\tilde{k}_t, s_t)\frac{l(\tilde{k}_t)}{\sigma} + \phi g(\tilde{k}_t, s_t)\frac{l(k_t)}{\sigma}\Big] dt\right]. \end{aligned} \tag{6.5.16}$$

In order to solve the maximization on the right hand side, we define as usual the Hamiltonian

$$\begin{aligned} &H(t,k_t, s_t, Y_t, Z_t) \\ &\quad = \Gamma_t^k[Z_t l(k_t)/\sigma + u(s_t - c(k_t)) - \phi u(s_t - c(\tilde{k}_t)) - \phi g(\tilde{k}_t, s_t)\frac{l(\tilde{k}_t)}{\sigma} + \phi g(\tilde{k}_t, s_t)\frac{l(k_t)}{\sigma}], \end{aligned} \tag{6.5.17}$$

where (Y, Z) are the adjoint variables which follow the BSDE

$$\begin{aligned} dY_t = [&- Z_t l(k_t)/\sigma - u(s_t - c(k_t)) + \phi u(s_t - c(\tilde{k}_t)) + \phi g(\tilde{k}_t, s_t)\frac{l(\tilde{k}_t)}{\sigma} \\ &- \phi g(\tilde{k}_t, s_t)\frac{l(k_t)}{\sigma}]dt + Z_t dW_t^0 \end{aligned} \tag{6.5.18}$$

with $Y_T = 0$. The FOC in k and $\tilde{k}$ give (assuming interior solutions)

$$Z_t = g(k_t, s_t) - \phi g(\tilde{k}_t, s_t)$$

$$\frac{\theta}{\sigma} g_k(\tilde{k}_t, s_t)[l(k_t) - l(\tilde{k}_t)] = 0$$

Since l is stricly decreasing, the second equality gives $k_t = \tilde{k}_t$. The condition at the boundary gives

$$\frac{\theta}{\sigma} g_k(\tilde{k}_t, s_t)[l(k_t) - l(\tilde{k}_t)] \leq 0$$

when $\tilde{k}_t = 0$, that is (since $g_k \leq 0$) $l(k_t) - l(\tilde{k}_t) \geq 0$, or $l(k_t) \geq l(0)$, that is $k_t = 0$. From the first condition we get $Z_t = (1 - \phi)g(k_t, s_t)$. On the boundary we obtain $Z_t \geq 0$ when $k_t = 0$. Moreover, $s = M$ at the optimum. By substitution Y becomes

$$\begin{cases} dY_t = [-Z_t l(k_t)/\sigma - (1 - \phi)u(s_t - c(k_t))]dt + Z_t dW_t^0 \\ Y_T = 0. \end{cases}$$

and (6.5.14). With similar arguments as in the preceding proofs we find that those conditions are also sufficient.
If $\phi > 1$ then by the FOC we get $Z_t \geq 0$, implying $k_t = \tilde{k}_t = 0$. Moreover, the Hamiltonian is maximized for $s = m$. □

6.6 Conclusions

In this paper we studied a principal-agent problem with finite horizon, motivated by the discussion about optimal emissions-reducing incentive policies. We looked at the problem from the two points of view of the firm (agent) and the state (principal), by deriving optimality conditions, BSDE/PDE representations and sensitivity results. A discretization scheme along with numerical experiments in some particular cases are also provided. It would be interesting to extend our results by allowing for switching costs for changing effort regimes or by including the possibility to trade emissions contracts on a financial market. This is left for future research.

6.7 Auxiliary results

Lemma 6.7.1. *The measure changes Γ^k and $(\Gamma^k)^{-1}$ are bounded in L^q for any $q \geq 0$, uniformly on $[0,T]$.*

Proof. Denote $Q_t = (\Gamma_t^k)^q$ for some $q \in \mathbb{R}$. We have in general

$$dQ_t = Q_t \left[Kl(k_t)dW_t^0 + Cl^2(k_t)dt \right],$$

where K, C are constants depending on q. We can choose an increasing sequence T_n of stopping times such that $T_n \to \infty$ and, since l is bounded, we have

$$E[Q_{t\wedge T_n}] = 1 + CE\left[\int_0^{t\wedge T_n} Q_r l^2(k_r)dr \right] \leq 1 + CE\left[\int_0^t Q_{r\wedge T_n} dr \right],$$

Therefore by Gronwall's lemma

$$E[Q_{t\wedge T_n}] \leq e^C,$$

and hence $E[Q_t] \leq e^C$ by Fatou's lemma. Remark that C only depends on q. □

Lemma 6.7.2. *Let k be an admissible effort policy. Then system* (6.2.8) *associated to k admits a unique $\mathcal{F}_t$-measurable solution (Y,Z) which satisfies*

$$E\left[\sup_{0\leq t\leq T} |Y_t|^2 + \int_0^T |Z_t|^2 dt \right] < \infty$$

and also

$$E^{\bar{k}}\left[\sup_{0\leq t\leq T} |Y_t|^2 + \int_0^T |Z_t|^2 dt \right] < \infty.$$

for any admissible $\bar{k}$ (possibly different from k).

Proof. For any admissible k define the system

$$\begin{cases} dX_t = \sigma X_t dW_t^0 \\ d\tilde{Y}_t = -\Gamma_t^k u(s_t - c(k_t))dt + \tilde{Z}_t dW_t^0 \\ X_0 = x, \tilde{Y}_T = \Gamma_T^k \xi \end{cases} \tag{6.7.1}$$

with $\xi = -p(X_T)$. By Hölder's inequality (with $q = \frac{2+\alpha}{2}$), Lemma 6.7.1 and admissibility of k we obtain

$$E\left[\int_0^T (\Gamma_t^k)^2 u(s_t - c(k_t))^2 dt\right] \leq E\left[\int_0^T |\Gamma_t^k|^{\frac{2(2+\alpha)}{\alpha}} dt\right]^{\frac{\alpha}{2+\alpha}} E\left[\int_0^T |u(s_t - c(k_t))|^{2+\alpha} dt\right]^{\frac{2}{2+\alpha}}$$

which is finite. In a similar way we have that $\Gamma_T^k \xi \in L^2$ by Lemma 6.7.1.
By standard results on BSDEs (see [Ph09], Theorem 6.2.1), or simply by the MRT, there exists a unique solution to (6.7.1) which satisfies

$$E\left[\sup_{0\leq t\leq T} |\tilde{Y}_t|^2 + \int_0^T |\tilde{Z}_t|^2 dt\right] < \infty.$$

Now define $Y_t = \tilde{Y}_t[\Gamma_t^k]^{-1}$ and $Z_t = [\tilde{Z}_t - l(k_t)/\sigma \tilde{Y}_t][\Gamma_t^k]^{-1}$. We have

$$d\left(\frac{1}{\Gamma_t^k}\right) = -\frac{1}{\Gamma_t^k}\frac{l(k_t)}{\sigma} dW_t^k,$$

recall that $d\Gamma_t^k = \Gamma_t^k \frac{l(k_t)}{\sigma} dW_t^0$, and so

$$\begin{aligned} dY_t =& d\left(\frac{\tilde{Y}_t}{\Gamma_t^k}\right) = \left(\frac{1}{\Gamma_t^k}\right) d\tilde{Y}_t + \tilde{Y}_t d\left(\frac{1}{\Gamma_t^k}\right) - \frac{\tilde{Z}_t}{\Gamma_t^k}\frac{l(k_t)}{\sigma} dt = -u(s_t - c(k_t))dt \\ &+ \frac{\tilde{Z}_t}{\Gamma_t^k} dW_t^k + \frac{\tilde{Z}_t}{\Gamma_t^k}\frac{l(k_t)}{\sigma} dt - \frac{\tilde{Y}_t}{\Gamma_t^k}\frac{l(k_t)}{\sigma} dW_t^k - \frac{\tilde{Z}_t}{\Gamma_t^k}\frac{l(k_t)}{\sigma} dt \\ =& - u(s_t - c(k_t))dt + Z_t dW_t^k \end{aligned}$$

hence (Y, Z) solve (6.2.8) with $Y_T = \xi$. By using (6.7.1) we get

$$Y_t = E^k\left[\xi + \int_t^T u(s_r - c(k_r))dr \mid \mathcal{F}_t\right]$$

Now define the martingale

$$\hat{Y}_t = Y_t + \int_0^t u(s_r - c(k_r))dr = E^k\left[\xi + \int_0^T u(s_r - c(k_r))dr \mid \mathcal{F}_t\right].$$

We have by Doob's inequality

$$E^k\left[\left(\sup_{0\leq t\leq T} |\hat{Y}_r|\right)^{2+\alpha}\right] \leq CE^k\left[\xi^{2+\alpha}\right] + CE^k\left[\int_0^T u(s_r - c(k_r))^{2+\alpha} dr\right] < \infty$$

for $\alpha > 0$ sufficiently small, by admissibility of k. Since $d\hat{Y}_t = Z_t dW_t^k$ we can also conclude (by BDG) that

$$E^k\left[\left(\int_0^T Z_r^2 dr\right)^{\frac{2+\alpha}{2}}\right] \leq CE^k\left[\left(\sup_{0\leq t\leq T} |\hat{Y}_r|\right)^{2+\alpha}\right] < \infty$$

By taking another admissible $\bar{k}$ we get

$$E^{\bar{k}}\left[\left(\int_0^T Z_r^2 dr\right)^2\right] = E^k\left[\frac{\Gamma_T^{\bar{k}}}{\Gamma_T^k}\left(\int_0^T Z_r^2 dr\right)^2\right] < \infty$$

by using Hölder's inequality and Lemma 6.7.1 in a similar way as above. We also have by Doob's inequality

$$E^k\left[\left(\sup_{0\le t\le T}|\hat{Y}_r|^2\right)^\beta\right] \le CE^k\left[\xi^{2\beta}\right] + CE^k\left[\int_0^T u(s_r - c(k_r))^{2\beta} dr\right] < \infty$$

for $\beta > 1$ sufficiently small, hence similarly as above

$$E^{\bar{k}}\left[\sup_{0\le t\le T}|Y_r|^2\right] < \infty.$$

□

of Lemma 6.2.1. Recall that g is naturally defined for $m < s \le M$ and $0 \le k < c^{-1}(s)$. Assumptions 6.2.1 and 6.2.2 ensure that

$$g_k(s,k) = \sigma\frac{l'(k)[-u''(s-c(k))c'(k)^2 + u'(s-c(k))c''(k)] - l''(k)[u'(s-c(k))c'(k)]}{l'(k)^2} \le 0,$$

where g_k is the first derivative of g with respect to the variable k. Moreover, $g(s,0) = 0$ and $\lim_{k\to c^{-1}(s)} g(s,k) = -\infty$, which implies that for any $s \ge m$ and $z \le 0$ the equation $g(s,k) = z$ has a unique solution, i.e. $F(s,z)$. Finally we set $F(s,z) = 0$ when $z \ge 0$. Remark that $0 \le F(s,z) < c^{-1}(s)$, with $\lim_{z\to-\infty} F(s,z) = c^{-1}(s)$. We have

$$F_z(s,z) = \frac{l'(F)^2}{l'(F)[-u''(s-c(F))c'(F)^2 + u'(s-c(F))c''(F)] - l''(F)[u'(s-c(F))c'(F)]}$$

when $z < 0$, and $F_z(s,z) = 0$ when $z > 0$. When $z = 0$ then F has a right derivative $F_{z+}(s,0) = 0$ and a left derivative

$$F_{z-}(s,0) = \frac{l'(0)^2}{l'(0)u'(s-c(0))c''(0)}$$

which does not diverge since $s \le M$ and $c''(0) > 0$. Hence $F(s,\cdot)$ is Lipschitz and continuously differentiable on $\mathbb{R}\setminus\{0\}$.
Finally we can compute

$$\begin{aligned} g_{kk}(s,k) =& -2c'(k)\left(\frac{c''(k)}{l'(k)} - \frac{c'(k)l''(k)}{l'(k)^2}\right)u''(s-c(k)) \\ &+ u'(s-c(k))\left[-\frac{2c''(k)l''(k)}{l'(k)^2} + \frac{c^{(3)}(k)}{l'(k)} + c'(k)\left(\frac{2l''(k)^2}{l'(k)^3} - \frac{l^{(3)}(k)}{l'(k)^2}\right)\right] \\ &+ \frac{c'(k)\left(-c''(k)u''(s-c(k)) + c'(k)^2u^{(3)}(s-c(k))\right)}{l'(k)} \end{aligned}$$

and the last claim follows by noting that

$$F_{zz}(s,z) = \frac{-g_{kk}(s,F(s,z))F_z(s,z)}{[g_k(s,F(s,z))]^2} = -\frac{g_{kk}(s,F(s,z))}{[g_k(s,F(s,z))]^3}.$$

□

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