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Inclusions différentielles d'évolution associées à des ensembles sous-lisses

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Abstract

This dissertation is devoted to the study of the existence of solutions for some evolution problems. The study is concerned with perturbed sweeping process associated on the one hand with prox-regular sets and on the other hand with subsmooth sets. It is assumed that the sets move either in a Lipschitz way or in an absolutely continuous way.

Cette thèse est consacrée à l'étude d'existence de solutions pour certains problèmes d'évolution. Il s'agit de processus de rafle perturbés associés d'une part à des ensembles prox-réguliers et d'autre part à des ensembles sous-lisses. Les ensembles sont supposés évoluer de façon Lipschitzienne ou absolument continue.

Contents

Introduction générale

La thèse est constituée d'un chapitre préliminaire puis de cinq chapitres traitant des inclusions différentielles d'évolution régies par des cônes normaux et des perturbations multivoques. Le chapitre préliminaire rappelle divers concepts et résultats d'analyse variationnelle non lisse utilisés dans le développement de la thèse. Nous allons ci-dessous résumer brièvement les résultats principaux de chacun des cinq autres articles.

Chapitre I: Ensemble sous-lisse et processus de rafle.

Ce premier chapitre de la thèse est consacré à l'étude de processus de rafle régi par des ensembles sous-lisses. Ces ensembles correspondent à une propriété de sous-monotonie du cône normal; cette propriété est dans la ligne de l'hypomonotonie du cône normal des ensembles prox-réguliers.

Un sous-ensemble C d'un espace de Hilbert H est dit être p*rox-régulier* en $u_0 \in C$, quand il existe $r > 0$ et $\delta > 0$ tels que pour tous $u_1, u_2 \in B(u_0, \delta) \cap C$ et $\xi_i \in N_C(u_i) \cap \mathbb{B}$ avec $i = 1, 2$, on a

$$\langle \xi_1 - \xi_2, u_1 - u_2 \rangle \geq -\frac{1}{r} \|u_1 - u_2\|^2,$$

où $N_C(u_i)$ désigne le cône normal de Clarke à C en u_i et $\mathbb{B}$ la boule unité fermée de l'espace de Hilbert H centrée à l'origine.

Plusieurs exemples, propriétés et caractérisations d'ensembles prox-réguliers ont été donnés dans [2, 3, 4, 5, 11, 12].

Récemment, dans leur article "Subsmooth sets: functional characterizations and related concepts" publié en 2005 dans Transactions of American Mathematical Society (voir, [1]), D. Aussel, A Daniilidis et L. Thibault ont considéré comme une extension du concept de prox-régularité, les ensembles sous-lisses.

Ensemble sous-liss.

On dit qu'un sous-ensemble non vide C de H est *sous-lisse* en $u_0 \in C$, si pour chaque $\varepsilon > 0$ il existe $\delta > 0$ tel que pour tous $u_1, u_2 \in B(u_0, \delta) \cap C$ et $\xi_i \in N_C(u_i) \cap \mathbb{B}$ avec $i = 1, 2$, on a

$$\langle \xi_1 - \xi_2, u_1 - u_2 \rangle \geq -\varepsilon \|u_1 - u_2\|. \tag{1}$$

Ceci nous a amené à introduire le concept de famille équi-uniformément sous-lisse d'ensembles.

Définition 0.0.1. *Soit E un ensemble non vide et $(C(t))_{t\in E}$ une famille de sous-ensembles non vides de H. On dit que cette famille est équi-uniformément sous-lisse, si pour chaque $\varepsilon > 0$ il existe $\delta > 0$ tel que (1) ait lieu pour tout $t \in E$ et tous $u_1, u_2 \in C(t)$ avec $\|u_1 - u_2\| \leq \delta$ et tous $\xi_i \in N_C(u_i) \cap \mathbb{B}$ avec $i = 1, 2$.*

Les deux lemmes suivants ont été utilisés dans la démonstration du Théorème 0.0.1 ci-dessous. Le premier lemme est un résultat bien connu pour les ensembles sous-lisses, voir [1]. Il nous dit que les sous-différentiels (resp. les cônes normaux) de Clarke et de Fréchet $\partial d_C(u)$ et $\partial_F d_C(u)$ (resp. $N_C(u)$ et $N_C^F(u)$) coincident quand C est sous-lisse en $u \in C$ et fermé autour de u. De facon précise:

Lemme 0.0.1. *Si C est un sous-ensemble de H qui est sous-lisse en $u_0 \in C$ et fermé autour de u_0, alors les assertions suivantes ont lieu:*

(a) $\partial d_C(u_0) = \partial_F d_C(u_0)$.

(b) $N_C(u_0) = N_C^F(u_0)$.

Le second lemme apporte un résultat nouveau. Il correspond à une propriété de fermeture du sous-différentiel de la fonction distance à des ensembles sous-lisses. Dans son énoncé et par la suite, pour une multi-application M entre deux ensembles non vides X et Y, nous notons $\operatorname{gph} M$ son graphe, c'est-à-dire

$$\operatorname{gph} M := \{(x, y) \in X \times Y : y \in M(x)\}.$$

Lemme 0.0.2. *Soient E un espace métrique, $(C(t))_{t\in E}$ une famille de sous-ensembles non vides de H, $\eta > 0$ un réel strictement positif, $Q \subset E$ et $s_0 \in adhQ$, où $adhQ$ désigne l'adhérence de Q dans E. On suppose que la famille $(C(t))_{t\in E}$ est équi-uniformément sous-lisse. Alors, les assertions suivantes ont lieu:*

(a) Pour tout $(s, u) \in \operatorname{gph} C$, on a $\eta \partial d_{C(s)}(u) \subset \eta \mathbb{B}$.

(b) Pour toutes suites généralisées $(s_j)_{j\in J} \in Q$, $(u_j)_{j\in J} \in H$ *et* $(\zeta_j)_{j\in J} \in H$ *telles que* $s_j \to s_0$ *dans* E, $u_j \to u \in C(s_0)$ *dans* $(H, \|\cdot\|)$ *avec* $u_j \in C(s_j)$ *et* $d_{C(s_j)}(y) \to 0$ *pour chaque* $y \in C(s_0)$ *et* $\zeta_j \to \zeta$ *dans* $(H, w(H,H))$ *avec* $\zeta_j \in \eta\partial d_{C(s_j)}(u_j)$, *on a* $\zeta \in \eta\partial d_{C(s_0)}(u)$.

De ce qui précède on a déduit la proposition suivante:

Proposition 0.0.1. *Soient* I *un intervalle de* $\mathbb{R}$ *et* $(C(t))_{t\in I}$ *une famille de sous-ensembles non vides de* H *et* $\eta > 0$ *un réel strictement positif. On suppose que* $(C(t))_{t\in I}$ *est équi-uniformément sous-lisse et qu'il existe une fonction continue* $\vartheta : I \to \mathbb{R}_+$ *tel que, pour tous* $y \in H$ *et* $s, t \in I$ *avec* $s \leq t$,

$$d\big(y, C(t)\big) \leq d\big(y, C(s)\big) + \vartheta(t) - \vartheta(s).$$

Alors, les assertions suivantes ont lieu:

(a) Pour tout $(s, u) \in \operatorname{gph} C$, *on a* $\eta\partial d_{C(s)}(u) \subset \eta\mathbb{B}$.

(b) Pour toutes suites $(s_n)_n \in I$, $(u_n)_n \in H$ *telles que* $s_n \to s$ *avec* $s_n \geq s$, $u_n \to u \in C(s)$ *avec* $u_n \in C(s_n)$ *et pour chaque* $\xi \in H$, *on a*

$$\limsup_{n\to\infty} \sigma\big(\xi, \eta\partial d_{C(s_n)}(u_n)\big) \leq \sigma\big(\xi, \eta\partial d_{C(s)}(u)\big),$$

où $\sigma\big(\cdot, \eta\partial d_{C(s)}(u)\big)$ *désigne la fonction d'appuie de l'ensemble* $\eta\partial d_{C(s)}(u)$.

Résultats principaux

Soient T_0, T deux nombres réels positifs avec $0 \leq T_0 < T$. Soient $C : [T_0, T] \rightrightarrows H$ et $\Gamma : [T_0, T] \times H \rightrightarrows H$ deux multi-applications, la première étant à valeurs fermées non vides et la seconde prenant des valeurs convexes fermées non vides et vérifiant l'hypothèse de croissance suivante:

$$d\big(0, \Gamma(t, x)\big) \leq \alpha(t)(1 + \|x\|)$$

pour tout $t \in [T_0, T]$ et tout $x \in C([T_0, t]) := \bigcup_{T_0 \leq s \leq t} C(s)$, où α est une fonction Lebesgue intégrable sur $[T_0, T]$ à valeurs réelles positives. On a établi notre théorème d'existence sous les hypothèses suivantes:

$(\mathcal{H}_{1,1})$ Pour chaque $t \in [T_0, T]$, $C(t)$ est un sous-ensemble boule-compact de H; il existe une fonction absolument continue croissante $\vartheta : [T_0, T] \to \mathbb{R}_+$ telle que, pour tous $y \in H$ et $s, t \in [T_0, T]$ avec $s \leq t$

$$d\big(y, C(t)\big) \leq d\big(y, C(s)\big) + \vartheta(t) - \vartheta(s);$$

($\mathcal{H}_{2,1}$) La famille $(C(t))_{t\in[T_0,T]}$ est équi-uniformément sous-lisse;

($\mathcal{H}_{3,1}$) La multi-application Γ est $\mathcal{L}([T_0,T])\otimes\mathcal{B}(H)$-mesurable et semi-continue supérieurement par rapport à la seconde variable, où $\mathcal{L}([T_0,T])$ désigne la tribu de Lebesgue de $[T_0,T]$ et $\mathcal{B}(H)$ la tribu de Borel de H.

Théorème 0.0.1. *On suppose que les hypothèses $\mathcal{H}_{1,1}, \mathcal{H}_{2,1}, \mathcal{H}_{3,1}$ ci-dessus sont satisfaites pour l'intervalle $I=[T_0,T]$. Alors, il existe une application absolument continue $x : I \to H$ solution de l'inclusion différentielle*

$$(\mathcal{E}) \quad \begin{cases} \dot{x}(t) \in -N_{C(t)}\big(x(t)\big) + \Gamma\big(t, x(t)\big) \quad p.p.\ t \in I \\ x(t) \in C(t) \quad \forall t \in I \\ x(T_0) = x_0 \in C(T_0). \end{cases}$$

Pour faciliter la lecture, rappelons qu'un sous-ensemble S de H est boule-compact quand l'ensemble $S \cap r\mathbb{B}$ est compact dans $(H, \|\cdot\|)$ pour chaque $r>0$, où $\mathbb{B}$ désigne comme ci-dessus la boule unité fermée de H. Rappelons aussi, étant donné un espace mesurable $(\Omega, \mathcal{T})$, qu'une multi-application $M : \Omega \rightrightarrows H$ est $\mathcal{T}-$mesurable quand pour tout ouvert U de H nous avons

$$M^{-1}(U) \in \mathcal{T}, \text{ où } M^{-1}(U) := \{\omega \in \Omega : M(\omega) \cap U \neq \emptyset\}.$$

Le résultat suivant est une conséquence directe du Théorème 0.0.1.

Corollaire 0.0.1. *Soit $\Gamma : [T_0, +\infty[\times H \rightrightarrows H$ une multi-application $\mathcal{L}([T_0,T]) \otimes \mathcal{B}(H)$-mesurable et semi-continue supérieurement par rapport à la seconde variable. On suppose que les hypothèses suivantes sont satisfaites:*

- *Il existe une fonction positive $\beta(\cdot) \in \mathrm{L}^{\infty}_{\mathrm{loc}}(\mathbb{R}_+)$ telle que*

$$d\big(0, \Gamma(t,x)\big) \leq \beta(t)(1 + \|x\|)$$

pour tous $t \in [T_0, +\infty[$ et $x \in C([T_0,t])$;

- *La famille $(C(t))_{t\in[T_0,+\infty[}$ est équi-uniformément sous-lisse;*

- *Pour chaque $t \in [T_0,+\infty[$, $C(t)$ est un sous-ensemble boule-compact de H; il existe une fonction localement absolument continue croissante $\vartheta : [T_0,+\infty[\to \mathbb{R}_+$ telle que, pour tous $y \in H$ et $s,t \in [T_0,+\infty[$ avec $s \leq t$*

$$d\big(y, C(t)\big) \leq d\big(y, C(s)\big) + \vartheta(t) - \vartheta(s).$$

Alors, pour chaque x_0 donné dans H avec $x_0 \in C(T_0)$, il existe une application localement absolument continue $x : [T_0, +\infty[\to H$ solution du problème suivant:

$$(\mathcal{E}_\infty) \quad \begin{cases} \dot{x}(t) \in -N_{C(t)}\big(x(t)\big) + \Gamma\big(t, x(t)\big) & p.p.\ t \in [T_0, +\infty[\\ x(t) \in C(t) \quad \forall t \in [T_0, +\infty[\\ x(T_0) = x_0 \in C(T_0). \end{cases}$$

Théorème 0.0.2. *Supposons que $\mathcal{H}_{2,1}$ et les hypothèses suivantes soient satisfaites:*

(a) Pour chaque $t \in I$, $C(t)$ est un sous-ensemble non vide compact de H et il existe une fonction absolument continue $\vartheta : I \to \mathbb{R}_+$ telle que, pour tous $y \in H$ et $s, t \in I$ avec $s \leq t$,

$$|d\big(y, C(t)\big) - d\big(y, C(s)\big)| \leq |\vartheta(t) - \vartheta(s)|;$$

(b) Pour chaque sous-ensemble borné S de H, il existe deux fonctions α_S et β_S dans $\mathrm{L}^1_{\mathbb{R}_+}(I)$ telles que

$$d\big(0, \Gamma(t, x)\big) \leq \alpha_S(t) + \beta_S(t)\|x\| \quad \textit{pour tout } (t, x) \in I \times S.$$

Alors, il existe une application absolument continue $x : I \to H$ solution de l'inclusion différentielle $(\mathcal{E})$

Le prochain résultat est un corollaire du Théorème 0.0.2.

Corollaire 0.0.2. *On suppose que les hypothèses suivantes sont vérifiées:*

(a) Pour chaque $t \in [T_0, +\infty[$, $C(t)$ est un sous-ensemble non vide boule-compact de H; Il existe une fonction localement absolument continue $\vartheta : [T_0, +\infty[\to \mathbb{R}_+$ telle que, pour tous $y \in H$ et $s, t \in [T_0, +\infty[$

$$|d\big(y, C(t)\big) - d\big(y, C(s)\big)| \leq |\vartheta(t) - \vartheta(s)|;$$

(b) La famille $(C(t))_{t \in [T_0, +\infty[}$ est équi-uniformément sous-lisse;

(c) La multi-application Γ est $\mathcal{L}([T_0, +\infty[) \otimes \mathcal{B}(H)$-mesurable et semi-continue supérieurement par rapport à la seconde variable et Γ satisfait la condition de croissance: pour chaque sous-ensemble borné S de H, il existe deux fonctions α_S et β_S dans $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+)$ telles que

$$d\big(0, \Gamma(t, x)\big) \leq \alpha_S(t) + \beta_S(t)\|x\| \quad \textit{pour tout } (t, x) \in [T_0, +\infty[\times S.$$

Alors, il existe une application localement absolument continue $x : [T_0, +\infty[\to H$ solution de l'inclusion différentielle $(\mathcal{E}_\infty)$.

Chapitre II: Perturbation avec mémoire de processus de rafle gouverné par des ensembles sous-lisses.

Le chapitre II étudie l'inclusion différentielle $(\mathcal{E})$ du chapitre I dans le cas où intervient un retard dans la multi-application Γ.

Soient H un espace de Hilbert et $r > 0$ un réel strictement positif. Soient aussi $C : [0,T] \rightrightarrows H$ et $\Gamma : [0,T] \times \mathcal{C}_H(-r,0) \rightrightarrows H$ deux multi-applications données. L'inclusion différentielle du chapitre I en présence de retard se présente alors sous la forme suivante:

$$(\mathcal{E}_r) \quad \begin{cases} \dot{x}(t) \in -N_{C(t)}(x(t)) + \Gamma(t, \Lambda(t)x) \quad \text{p.p. } t \in [0,T]; \\ \\ x(t) \in C(t) \quad \forall t \in \ [0,T]; \\ \\ x(\cdot) = \varphi(\cdot) \quad \text{in } [-r,0], \end{cases}$$

où $\Lambda(t)$ est l'application de $\mathcal{C}_H(-r,T)$ dans $\mathcal{C}_H(-r,0)$ définie, pour tout $x \in \mathcal{C}_H(-r,T)$, par $\Lambda(t)x(s) := x(t+s)$ pour tout $s \in [-r,0]$, et φ est un élément de $\mathcal{C}_H(-r,0)$ tel que $\varphi(0) \in C(0)$. Ici nous notons $\mathcal{C}_H(-r,T)$ l'espace des applications continues de $[-r,T]$ dans H.

Nous avons étudié l'existence de solutions pour l'inclusion différentielle $(\mathcal{E}_r)$. Nous appelons solution de $(\mathcal{E}_r)$ toute application $x : [-r,T] \to H$ telle que

1. pour chaque $s \in [-r,0]$, nous avons $x(s) = \varphi(s)$;
2. $x(t) \in C(t)$ pour tout $t \in [0,T]$;
3. la restriction $x|_{[0,T]}$ de x à $[0,T]$ est absolument continue et sa dérivée satisfait presque partout l'inclusion

$$\dot{x}(t) \in -N_{C(t)}\big(x(t)\big) + \Gamma\big(t, \Lambda(t)x\big) \quad \text{p.p. } t \in [0,T].$$

Les hypothèses concernant l'ensemble $C(t)$ et la multi-application Γ avec lesquels nous avons travaillé sont les suivantes:

$(\mathcal{H}_{1,2})$ Pour tout $t \in [0,T]$, $C(t)$ est un sous-ensemble boule-compact de H; l'ensemble $C(t)$ bouge de facon absolument continue, c'est-à-dire, il existe une fonction absolument continue $\vartheta : [0,T] \to \mathbb{R}$ telle que, pour chaque $y \in H$ et tous $s,t \in [0,T]$

$$|d\big(y, C(t)\big) - d\big(y, C(s)\big)| \leq |\vartheta(t) - \vartheta(s)|;$$

($\mathcal{H}_{2,2}$) La famille $(C(t))_{t\in[0,T]}$ est équi-uniformément sous-lisse;

($\mathcal{H}_{3,2}$) La multi-application Γ à valeurs non-vides convexes fermées, est $\mathcal{L}([0,T])\otimes\mathcal{B}(\mathcal{C}_H(-r,0))$-mesurable et semi-continue supérieurement par rapport à $\phi \in \mathcal{C}_H(-r,0)$ pour presque tout $t \in [0,T]$. De plus

$$d\big(0, \Gamma(t,\phi)\big) \leq \alpha(t)(1 + \|\phi\|_\infty)$$

pour tout $t \in [0,T]$ et tout $\phi \in \mathcal{C}_H(-r,0)$, où α est une fonction intégrable sur $[0,T]$ à valeurs réelles positives.

Dans ce chapitre, nous avons appliqué le résultat du chapitre I concernant la fermeture du sous-différentiel de la fonction distance, afin d'obtenir la propriété de semi-continuité supérieure de la fonction d'appui $\sigma\big(\xi, \eta\partial d_{C(\cdot)}(\cdot)\big)$. Nous avons ainsi démontré le théorème suivant d'existence de solution pour l'inclusion diffèrentielle $(\mathcal{E}_r)$.

Théorème 0.0.3. *On suppose que les hypothèses $\mathcal{H}_{1,2}, \mathcal{H}_{2,2}, \mathcal{H}_{3,2}$ ci-dessus sont satisfaites. Alors, pour chaque φ dans $\mathcal{C}_H(-r,0)$ avec $\varphi(0) \in C(0)$, l'inclusion différentielle $(\mathcal{E}_r)$ admet au moins une solution.*

Chapitre III: Processus de rafle non-convexe avec un ensemble dépendant de l'état.

Dans le chapitre I l'évolution des ensembles $C(t)$ intervenant dans l'inclusion différentielle ne dépend que du temps. Le chapitre III est consacré au cas où l'ensemble dépend à la fois du temps et de l'état, c'est-à-dire se présente sous la forme $C(t, u(t))$.

Le théorème suivant démontre que la projection métrique sur des ensembles prox-réguliers est Hölderienne par rapport à la distance de Hausdorff. C'est une propriété importante qui a son propre intérêt en analyse variationnelle. Dans l'énoncé du théorème, pour $C \subset H$, $\gamma \in]0,1[$ et $r > 0$ nous utilisons la notation

$$U_r^\gamma(C) := \{v \in H : 0 < d(v,C) < \gamma r\},$$

désignant le tube ouvert autour de C.

Théorème 0.0.4. *Soient deux ensembles C et C' d'un espace de Hilbert H qui sont r-prox-réguliers pour une constante $r > 0$ et soit $\gamma \in]0,1[$. Alors pour tous $u \in U_r^\gamma(C)$ et $v \in U_r^\gamma(C')$ on a*

$$\|\mathrm{P}_C(u) - \mathrm{P}_{C'}(v)\| \leq (1-\gamma)^{-1}\|u - v\| + \sqrt{\frac{2\gamma r}{1-\gamma}}\Big(\mathrm{Haus}\,(C, C')\Big)^{1/2},$$

où Haus (C, C') *désigne la distance de Hausdorff entre les ensembles* C *et* C'.

Le corollaire suivant en est une conséquence immédiate.

Corollaire 0.0.3. *Soit* $C(t,u)$ *un ensemble* r*-prox-régulier de* H *pour une constante* $r > 0$ *et soit* $\gamma \in]0,1[$. *On suppose qu'il existe un réel constant* $L > 0$ *tel que pour tous* $t \in [0,T]$ *et* $, u, v \in H$, *on ait*

$$\operatorname{Haus}(C(t,u), C(t,v)) \leq L\|u - v\|.$$

Alors, pour tous $u, v \in H$ *et* $x \in U_r^\gamma(C(t,u)) \cap U_r^\gamma(C(t,v))$, *on a*

$$\|\mathrm{P}_{C(t,u)}(x) - \mathrm{P}_{C(t,v)}(x)\| \leq \sqrt{\frac{2\gamma r L}{1-\gamma}}\|u - v\|^{1/2}.$$

Cette propriété et l'extension du théorème de point fixe de Schauder de [7] ou [10] font partie des résultats cruciaux utilisés dans la démonstration du Théoréme 0.0.5 ci-dessous.

Résultats principaux

Soient $C : [0,T] \rightrightarrows H$ et $G : [0,T] \times H \rightrightarrows H$ deux multi-applications données, qui sont à valeurs non vides fermées et à valeurs non vides convexes fermées respectivement. Les hypothèses suivantes vont intervenir dans le prochain théorème:

$(\mathcal{G}_{1,3})$ La multi-application G est scalairement semi-continue supérieurement par rapport aux deux variables et il existe un certain $\alpha > 0$ tel que

$$d\big(0, G(t,u)\big) \leq \alpha$$

pour tous $t \in [0,T]$ et $u \in H$ avec $u \in C(t,u)$;

$(\mathcal{G}_{2,3})$ Pour chaque $t \in [0,T]$ et chaque $u \in H$, les ensembles $C(t,u)$ sont r-prox-reguliers pour une constante $r > 0$;

$(\mathcal{G}_{3,3})$ Il existe deux constantes réelles $L_1 > 0$, $L_2 \in]0,1[$ telles que, pour tous $t, s \in [0,T]$ et $x, y, u, v \in H$

$$\Big|d\big(x, C(t,u)\big) - d\big(y, C(s,v)\big)\Big| \leq \|x - y\| + L_1|t - s| + L_2\|u - v\|;$$

$(\mathcal{G}_{4,3})$ Pour chaque sous-ensemble borné $A \subset H$, l'ensemble $C([0,T] \times A)$ est relativement boule-compact, c'est-à-dire, l'intersection de $C([0,T] \times A)$ avec chaque boule fermée de H est un ensemble relativement compact dans H.

Théorème 0.0.5. *Supposons que les hypothèses $\mathcal{G}_{1,3}, \cdots, \mathcal{G}_{4,3}$ soient satisfaitess. Alors, pour chaque $u_0 \in H$ avec $u_0 \in C(0, u_0)$, il existe une application Lipschitzienne $u : [0, T] \to H$ telle que*

$$(\mathcal{D}) \quad \begin{cases} \dot{u}(t) \in -N_{C\left(t,u(t)\right)}\big(u(t)\big) + G\big(t, u(t)\big) \quad \text{p.p. } t \in [0, T], \\ \\ u(t) \in C\big(t, u(t)\big) \ \forall t \in [0, T], \\ \\ u(t) = u_0 + \int_0^t \dot{u}(s)ds \ \forall t \in [0, T], \end{cases}$$

c'est-à-dire, u est une solution Lipschitzienne de l'inclusion différentielle $(\mathcal{D})$ avec $\|\dot{u}(t)\| \leq \frac{L_1+2\alpha}{1-L_2}$ p.p. $t \in [0, T]$.

Le prochain résultat est une extension du Théorème 0.0.5.

Théorème 0.0.6. *Soit $G : \mathbb{R}_+ \times H \rightrightarrows H$ une multi-application scalairement semi-continue supérieurement par rapport aux deux variables. On suppose que les hypothèses suivantes sont satisfaites:*

- *Il existe une fonction positive $\beta(\cdot) \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+)$ tel que*

  $$d\big(0, G(t, u)\big) \leq \beta(t)$$

  pour tous $t \in \mathbb{R}_+$ et $u \in H$ avec $u \in C(t, u)$;

- *Pour chaque $t \in \mathbb{R}_+$ et chaque $u \in H$, les ensembles $C(t, u)$ sont r-prox-reguliers pour une constante $r > 0$;*

- *Il existe deux constantes réelles $L_1 > 0$, $L_2 \in]0, 1[$ telles que, pour tous $t, s \in \mathbb{R}_+$ et $x, y, u, v \in H$*

  $$\Big|d\big(x, C(t, u)\big) - d\big(y, C(s, v)\big)\Big| \leq \|x - y\| + L_1|t - s| + L_2\|u - v\|;$$

- *Pour chaque sous-ensemble borné $A \subset H$, l'ensemble $C([0, T] \times A)$ est relativement boule-compact.*

Alors, pour chaque u_0 donné dans H avec $u_0 \in C(0, u_0)$, il existe une application localement Lipschitzienne $u : \mathbb{R}_+ \to H$ solution du problème suivant:

$$(\mathcal{D}_{\mathbb{R}_+}) \quad \begin{cases} \dot{u}(t) \in -N_{C\left(t,u(t)\right)}\big(u(t)\big) + G\big(t, u(t)\big) \quad \text{p.p. } t \in \mathbb{R}_+, \\ \\ u(t) \in C\big(t, u(t)\big) \ \forall t \in \mathbb{R}_+, \\ \\ u(t) = u_0 + \int_0^t \dot{u}(s)ds \ \forall t \in \mathbb{R}_+. \end{cases}$$

Les deux corollaires suivants sont des conséquences directes du Théorème 0.0.5 et du Théorème 0.0.6 respectivement.

Corollaire 0.0.4. *Soit $G : [0,T] \times H \rightrightarrows H$ une multi-application scalairement semi-continue supérieurement par rapport aux deux variables. On suppose que H est un espace Euclidien de dimension finie et que les hypothèses suivantes sont satisfaites:*

- *Il existe un nombre réel positif α tel que*
$$d\big(0, G(t,u)\big) \leq \alpha$$
pour tous $t \in [0,T]$ et $u \in H$ avec $u \in C(t,u)$;

- *Pour chaque $t \in [0,T]$ et chaque $u \in H$, les ensembles $C(t,u)$ sont r-prox-reguliers pour une constante $r > 0$;*

- *Il existe deux constantes réelles $L_1 > 0$, $L_2 \in]0,1[$ telles que, pour tous $t, s \in [0,T]$ et $x, y, u, v \in H$*
$$\Big|d\big(x, C(t,u)\big) - d\big(y, C(s,v)\big)\Big| \leq \|x-y\| + L_1|t-s| + L_2\|u-v\|.$$

Alors, pour chaque u_0 donné dans H avec $u_0 \in C(0,u_0)$, il existe une application Lipschitzienne $u : [0,T] \to H$ solution du problème $(\mathcal{D})$. De plus, on a $\|\dot{u}(t)\| \leq \frac{L_1+2\alpha}{1-L_2}$ p.p. $t \in [0,T]$.

Corollaire 0.0.5. *Soit $G : \mathbb{R}_+ \times H \rightrightarrows H$ une multi-application scalairement semi-continue supérieurement par rapport aux deux variables. On suppose que H est un espace Euclidien de dimension finie et que les hypothèses suivantes sont satisfaites:*

- *Il existe une fonction positive $\beta(\cdot) \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+)$ telle que*
$$d\big(0, G(t,u)\big) \leq \beta(t)$$
pour tous $t \in \mathbb{R}_+$ et $u \in H$ avec $u \in C(t,u)$;

- *Pour chaque $t \in \mathbb{R}_+$ et chaque $u \in H$, les ensembles $C(t,u)$ sont r-prox-reguliers pour une constante $r > 0$;*

- *Il existe deux constantes réelles $L_1 > 0$, $L_2 \in]0,1[$ telles que, pour tous $t, s \in \mathbb{R}_+$ et $x, y, u, v \in H$*
$$\Big|d\big(x, C(t,u)\big) - d\big(y, C(s,v)\big)\Big| \leq \|x-y\| + L_1|t-s| + L_2\|u-v\|.$$

Alors, pour chaque u_0 donné dans H avec $u_0 \in C(0,u_0)$, il existe une application localement Lipschitzienne $u : \mathbb{R}_+ \to H$ solution du problème $(\mathcal{D}_{\mathbb{R}_+})$.

Chapitre IV: Perturbation avec mémoire de processus de rafle gouverné par des ensembles prox-réguliers dépendant de l'état.

Le chapitre IV traite le problème d'évolution $(\mathcal{D})$ du chapitre III dans le cas où intervient un retard dans la multi-application G.

Soient H un espace de Hilbert et $r > 0$ un réel strictement positif. Soient aussi $C : [0,T] \rightrightarrows H$ et $G : [0,T] \times \mathcal{C}_H(-r,0) \rightrightarrows H$ deux multi-applications données. L'inclusion différentielle du chapitre III en présence de retard se présente alors sous la forme suivante:

$$(\mathcal{D}_r) \quad \begin{cases} \dot{u}(t) \in -N_{C\big(t,u(t)\big)}\big(u(t)\big) + G\big(t, \Lambda(t)u\big) \quad \text{a.e } t \in [0,T], \\ \\ u(t) \in C\big(t,u(t)\big) \quad \forall t \in [0,T], \\ \\ u = \varphi \text{ in } [-r,0], \end{cases}$$

où l'application $\Lambda(t)$ et l'espace $\mathcal{C}_H(-r,T)$ sont définis comme dans le chapitre II ci-dessus. Les hypothèses suivantes vont intervenir dans le prochain énoncé:

$(\mathcal{G}_{1,4})$ La multi-application G est à valeurs non vides convexes fermées et est scalairement semi-continue supérieurement par rapport aux deux variables et il existe un certain $\alpha > 0$ tel que
$$d\big(0, G(t,\varphi)\big) \leq \alpha$$
pour tous $t \in [0,T]$ et $\varphi \in \mathcal{C}_H(-r,0)$;

$(\mathcal{G}_{2,4})$ Pour chaque $t \in [0,T]$ et chaque $u \in H$, les ensembles $C(t,u)$ sont ρ-prox-réguliers pour une constante $\rho > 0$;

$(\mathcal{G}_{3,4})$ Il existe deux constantes réelles $L_1 > 0$, $L_2 \in]0,1[$ telles que, pour tous $t, s \in [0,T]$ et $x, y, u, v \in H$
$$\Big|d\big(x, C(t,u)\big) - d\big(y, C(s,v)\big)\Big| \leq \|x-y\| + L_1|t-s| + L_2\|u-v\|;$$

$(\mathcal{G}_{4,4})$ Pour chaque sous-ensemble borné $A \subset H$, l'ensemble $C([0,T] \times A)$ est relativement boule-compact.

Théorème 0.0.7. *On suppose que les hypothèses $\mathcal{G}_{1,4}, \cdots, \mathcal{G}_{4,4}$ sont satisfaites. Alors, pour chaque $\varphi \in \mathcal{C}_H(-r,0)$ et pour chaque $u_0 \in H$ avec $\varphi(0) = u_0 \in C(0,u_0)$, le problème d'évolution $(\mathcal{D}_r)$ admet au moins une solution $u : [-r,T] \to H$, qui est continue sur $[-r,T]$ et Lipschitzienne sur $[0,T]$ avec $\|\dot{u}(t)\| \leq \frac{L_1+2\alpha}{1-L_2}$ p.p. $t \in [0,T]$.*

Chapitre V: Processus de rafle perturbé régi par des ensembles sous-lisses dépendant de l'état.

Ce chapitre considère le problème d'évolution $(\mathcal{D})$ du chapitre III dans le cadre d'ensembles sous-lisses. Il s'agit d'une classe d'ensembles beaucoup plus large que celle des ensembles prox-réguliers.

La proposition suivante est une adaptation de la proposition 0.0.1 quand les ensembles dépendent à la fois du temps et de l'état. Elle a été utilisée dans la démonstration du théorème 0.0.8 ci-dessous.

Proposition 0.0.2. *Soit $\{C(t,v) : (t,v) \in [0,T] \times H\}$ une famille d'ensembles non vides fermés de H et $\eta > 0$ un réel strictement positif. On suppose que $(C(t,v))_{(t,v)\in[0,T]\times H}$ est équi-uniformément sous-lisse et qu'il existe deux constantes $L_1 > 0$ et $L_2 > 0$ tel que, pour tous $x, y, u, v \in H$ et $s, t \in [0,T]$*

$$|d\big(x, C(t,u)\big) - d\big(y, C(s,v)\big)| \leq \|x - y\| + L_1|t - s| + L_2\|u - v\|.$$

Alors, les assertions suivantes ont lieu:

(a) Pour tout $(s, v, y) \in \operatorname{gph} C$, on a $\eta\partial d_{C(s,v)}(y) \subset \eta\mathbb{B}$;

(b) Pour toutes suites $(s_n)_n \in [0,T]$, $(v_n)_n \in H$ et $(y_n)_n \in H$ telles que $s_n \to s$, $v_n \to v$ et $y_n \to y \in C(s,v)$ avec $y_n \in C(s_n, v_n)$ et pour chaque $\xi \in H$, on a

$$\limsup_{n\to\infty} \sigma\big(\xi, \eta\partial d_{C(s_n,v_n)}(y_n)\big) \leq \sigma\big(\xi, \eta\partial d_{C(s,v)}(y)\big).$$

En utilisant des idées de [6, 9] on a établi une démonstration du théorème suivant sous les mêmes hypothèses $(\mathcal{G}_{1,3}, \mathcal{G}_{3,3}, \mathcal{G}_{4,3})$ utilisées dans le chapitre III, sauf que l'hypothèse $(\mathcal{G}_{2,3})$ est remplacée par:

$(\mathcal{G}'_{2,3})$ pour tous $t \in [0,T]$ et $u \in H$, les ensembles $C(t,u)$ sont non-vides et équi-uniformément sous-lisses.

Théorème 0.0.8. *On suppose que les hypothèses $\mathcal{G}_{1,3}, \mathcal{G}'_{2,3}, \mathcal{G}_{3,3}, \mathcal{G}_{4,3}$ sont satisfaites. Alors, pour chaque $u_0 \in H$ avec $u_0 \in C(0,u_0)$, il existe une application Lipschitzienne $u : [0,T] \to H$ solution de $(\mathcal{D})$ avec $\|\dot{u}(t)\| \leq \frac{L_1+2\alpha}{1-L_2}$ p.p. $t \in [0,T]$.*

Avant d'énoncer les prochains résultats, il est important de mentionner que ce sont des extensions de ceux obtenus dans le chapitre III, puisque la classe d'ensembles utilisée, dans cette partie, contient celle du chapitre III.

Théorème 0.0.9. *Soit $G : \mathbb{R}_+ \times H \rightrightarrows H$ une multi-application scalairement semi-continue supérieurement par rapport aux deux variables. On suppose que les hypothèses suivantes sont satisfaites:*

- *Il existe une fonction positive $\beta(\cdot) \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+)$ telle que*

$$d\big(0, G(t,u)\big) \leq \beta(t)$$

  pour tous $t \in \mathbb{R}_+$ et $u \in H$ avec $u \in C(t,u)$;

- *Pour chaque $t \in \mathbb{R}_+$ et chaque $u \in H$, les ensembles $C(t,u)$ sont non-vides et équi-uniformément sous-lisses;*

- *Il existe deux constantes réelles $L_1 > 0$, $L_2 \in]0,1[$ telles que, pour tous $t, s \in \mathbb{R}_+$ et $x, y, u, v \in H$*

$$\Big|d\big(x, C(t,u)\big) - d\big(y, C(s,v)\big)\Big| \leq \|x-y\| + L_1|t-s| + L_2\|u-v\|;$$

- *Pour chaque sous-ensemble borné $A \subset H$, l'ensemble $C([0,T] \times A)$ est relativement boule-compact.*

Alors, pour un u_0 donné dans H avec $u_0 \in C(0,u_0)$, il existe une application localement Lipschitzienne $u : \mathbb{R}_+ \to H$ solution du problème $(\mathcal{D}_{\mathbb{R}_+})$.

Comme conséquences directes du Théorème 0.0.8 et du Théorème 0.0.9 on obtient:

Corollaire 0.0.6. *Soit $G : [0,T] \times H \rightrightarrows H$ une multi-application scalairement semi-continue supérieurement par rapport aux deux variables. On suppose que H est un espace Euclidien de dimension finie et que les hypothèses suivantes sont satisfaites:*

- *Il existe un nombre réel positif α tel que*

$$d\big(0, G(t,u)\big) \leq \alpha$$

  pour tous $t \in [0,T]$ et $u \in H$ avec $u \in C(t,u)$;

- *Pour chaque $t \in [0,T]$ et chaque $u \in H$, les ensembles $C(t,u)$ sont non-vides et équi-uniformément sous-lisses;*

- *Il existe deux constantes réelles $L_1 > 0$, $L_2 \in]0,1[$ telles que, pour tous $t, s \in [0,T]$ et $x, y, u, v \in H$*

$$\Big|d\big(x, C(t,u)\big) - d\big(y, C(s,v)\big)\Big| \leq \|x-y\| + L_1|t-s| + L_2\|u-v\|.$$

Alors, pour chaque u_0 donné dans H avec $u_0 \in C(0, u_0)$, il existe une application Lipschitzienne $u : [0, T] \to H$ solution du problème $(\mathcal{D})$. De plus, on a $\|\dot{u}(t)\| \leq \frac{L_1+2\alpha}{1-L_2}$ p.p. $t \in [0, T]$.

Corollaire 0.0.7. *Soit $G : \mathbb{R}_+ \times H \rightrightarrows H$ une multi-application scalairement semi-continue supérieurement par rapport aux deux variables. On suppose que H est un espace Euclidien de dimension finie et que les hypothèses suivantes sont satisfaites:*

- *Il existe une fonction positive $\beta(\cdot) \in \mathrm{L}^{\infty}_{\mathrm{loc}}(\mathbb{R}_+)$ telle que*

$$d\big(0, G(t, u)\big) \leq \beta(t)$$

  pour tous $t \in \mathbb{R}_+$ et $u \in H$ avec $u \in C(t, u)$;

- *Pour chaque $t \in \mathbb{R}_+$ et chaque $u \in H$, les ensembles $C(t, u)$ sont non-vides et équi-uniformément sous-lisses;*

- *Il existe deux constantes réelles $L_1 > 0$, $L_2 \in]0, 1[$ telles que, pour tous $t, s \in \mathbb{R}_+$ et $x, y, u, v \in H$*

$$\Big| d\big(x, C(t, u)\big) - d\big(y, C(s, v)\big) \Big| \leq \|x - y\| + L_1|t - s| + L_2\|u - v\|.$$

Alors, pour chaque u_0 donné dans H avec $u_0 \in C(0, u_0)$, il existe une application localement Lipschitzienne $u : \mathbb{R}_+ \to H$ solution du problème $(\mathcal{D}_{\mathbb{R}_+})$.

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Preliminary chapter

Throughout this chapter, we give some preliminary definitions and results used in the dissertation. A great part will be focused on some useful properties both on prox-regular sets and on subsmooth sets.

Let us start with various concepts of normal cones in variational analysis.

Definition 0.0.1. *Let C be a set of the normed space $(X, \|\cdot\|)$ and $x \in C$. The Clarke normal cone $N_C^{Cl}(x)$ of C at x is the negative polar $(T_C^{Cl}(x))^0$ of the Clarke tangent cone, that is,*

$$N_C^{Cl}(x) := \big\{x^* \in X^* : \langle x^*, h\rangle \le 0 \ \forall h \in T_C^{Cl}(x)\big\},$$

where

$$T_C^{Cl}(x) := \liminf_{t \downarrow 0, u \xrightarrow[C]{} x} \frac{C - u}{t},$$

and $u \xrightarrow[C]{} x$ means $u \to x$ along with $u \in C$. Otherwise stated $h \in T_C^{Cl}(x)$ if and only if, for any sequence $(t_n)_n$ tending to 0 with $t_n > 0$ and any sequence $(x_n)_n$ in X converging to x with $x_n \in C$, there exists a sequence $(h_n)_n$ in X converging to h such that

$$x_n + t_n h_n \in C \quad \text{for all } n \in \mathbb{N}.$$

When $x \notin C$ we define both tangent and normal cones to be empty.

Through the Clarke normal cone we introduce the Clarke subdifferential as follows.

Definition 0.0.2. *Let $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be an extended real-valued function defined on the normed space X and x be a point where f is finite. The Clarke subdifferential $\partial_{Cl} f(x)$ of f at x is defined by*

$$\partial_{Cl} f(x) := \{x^* \in X^* : (x^*, -1) \in N_{\operatorname{epi} f}^{Cl}(x, f(x))\}.$$

We also put $\partial_{Cl} f(x) = \emptyset$ when $f(x)$ is not finite.

Above $\operatorname{epi} f$ denotes the epigraph of f, that is,

$$\operatorname{epi} f = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}.$$

Definition 0.0.3. *Let C be a set of the normed space $(X, \|\cdot\|)$ and $x \in C$. An element $x^* \in X^*$ is a Fréchet normal of the set C at $x \in X$ when for any real $\varepsilon > 0$ there exists some neighborhood U of x such that*

$$\langle x^*, y - x\rangle \leq \varepsilon\|y - x\| \quad \textit{for all } y \in C \cap U,$$

that is,

$$\limsup_{y \xrightarrow[C]{} x} \frac{\langle x^*, y - x\rangle}{\|y - x\|} \leq 0.$$

The set $N_C^F(x)$ of all Fréchet normals of C at x is the Fréchet normal cone of C at x.

Similarly, the Fréchet subdifferential $\partial_F f(x)$ of f at x is defined as follows:

$$\partial_F f(x) := \{x^* \in X^* : (x^*, -1) \in N_{\operatorname{epi} f}^F(x, f(x))\}.$$

We always have

$$N_C^F(x) \subset N_C^{Cl}(x) \quad \text{and} \quad \partial_F f(x) \subset \partial_{Cl} f(x).$$

We recall that a Banach space X is called *Asplund* if every separable subspace of X has a separable topological dual. In particular, every reflexive Banach space is Asplund.

Definition 0.0.4. *Let $(X, \|\cdot\|)$ be an Asplund space. A continuous linear functional $x^* \in X^*$ is a Mordukhovich limiting subgradient of f at x if there exists a sequence $((x_n, f(x_n)))_n$ converging to $(x, f(x))$ and a sequence $(x_n^*)_n$ converging weakly star to x^* such that $x_n^* \in \partial^F f(x_n)$. The set $\partial_L f(x)$ of all limiting subgradients of f at x is the Mordukhovich limiting subdifferential of f at x, that is,*

$$\partial_L f(x) =^{seq} \limsup_{u \xrightarrow[f]{} x} \partial_F f(x),$$

where ${}^{seq}\limsup_{u \xrightarrow[f]{} x}$ denotes the weak star sequential limit superior when $u \to x$ and $f(u) \to f(x)$.

Thus, we define the limiting normal cone $N_C^L(x)$ as follows:

$$N_C^L(x) =^{seq} \limsup_{u \xrightarrow[C]{} x} N_C^F(x).$$

It is always true that

$$N_C^F(x) \subset N_C^L(x) \subset N_C^{Cl}(x) \quad \text{and} \quad \partial_F f(x) \subset \partial_L f(x) \subset \partial_{Cl} f(x).$$

It is known that $x^* \in \partial_F f(x)$ if and only if for any $\varepsilon > 0$ there exists some neighborhood U of x such that

$$\langle x^*, y - x\rangle \leq f(y) - f(x) + \varepsilon\|y - x\| \quad \text{for all } y \in U.$$

Definition 0.0.5. *Let H be a Hilbert space, a vector $\zeta \in H$ is a proximal normal vector to the set C at $x \in C$ when there exist a real constant $\sigma \geq 0$ and a neighborhood U of x such that*

$$\langle \zeta, y - x\rangle \leq \sigma\|y - x\|^2 \quad \text{for all } y \in U \cap C.$$

The set of such vectors is the proximal normal cone $N_C^p(x)$ of C at x.

Definition 0.0.6. *Let H be a Hilbert space and $f : H \to \mathbb{R} \cup \{-\infty, +\infty\}$ be an extended real-valued function which is finite at $x \in H$. A vector $\zeta \in H$ is a proximal subgradient of f at x provided that $(\zeta, -1) \in N^p_{\operatorname{epi} f}(x, f(x))$. The set $\partial_p f(x)$ of all vectors proximal subgradient of f at x is the proximal subdifferential of f at x and we put $\partial_p f(x) = \emptyset$ when f is not finite at x.*

It is known that $\zeta \in \partial_p f(x)$ if and only if there exist some real number $\sigma \geq 0$ and some neighborhood U of x such that

$$\langle \zeta, y - x\rangle \leq f(y) - f(x) + \sigma\|y - x\|^2 \quad \text{for all } y \in U \cap C.$$

In the Hilbert setting, we always have the following inclusions

$$N_C^p(x) \subset N_C^F(x) \subset N_C^L(x) \subset N_C^{Cl}(x),$$
$$\partial_p f(x) \subset \partial_F f(x) \subset \partial_L f(x) \subset \partial_{Cl} f(x).$$

0.1 Prox-regularity in Hilbert space

We start with the definition of prox-regular sets. Let C be a closed subset of H. The set C is known to be *prox-regular* at $u_0 \in C$, when there exist $r > 0$ and $\delta > 0$ such that for all $u \in B(u_0, \delta) \cap C$ and all $\xi \in N_C^p(u) \cap \mathbb{B}$, we have

$$u \in \operatorname{Proj}_C(u + r\xi). \tag{2}$$

In the latter inclusion, for any $v \in H$,

$$\operatorname{Proj}_C(v) := \{u \in C : d(v, C) = \|v - u\|\}$$

is the set of nearest points of v in C. When this set has a unique point, we will use the notation $\mathrm{P}_C(v)$.

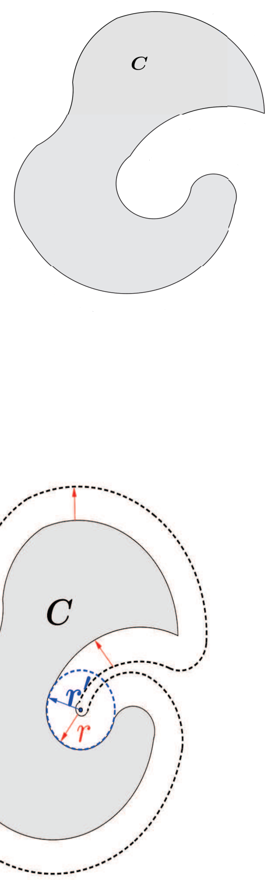


Figure 1: The set C is r-prox-regular.

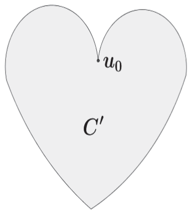


Figure 2: The set C' is not r-prox-regular at u_0.

Further, C is called *prox-regular* if it is *prox-regular* at every $u_0 \in C$. The set C is *uniformly prox-regular* or *r-prox-regular* if there exists $r > 0$ such that (2) holds for all $u \in C$ and $\xi \in N^p_C(u)$. Figure 1 provides an example of a prox-regular set C and Figure 2 an example of a non prox-regular set C' (non prox-regular at u_0).

Certain characterizations below are formulated as either the hypomonotocity property or the cocoercivity of a set-valued mapping involving a truncated normal cone. For a normed space X and an extended real $r \in]0, +\infty]$, a set-valued mapping $M : X \rightrightarrows X^*$ is said to be *r-hypomonotone* on a subset U of X provided

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\frac{1}{r}\|x_1 - x_2\|^2 \quad \text{for all } x_i \in U \subset \mathrm{Dom}\, M,\ x_i^* \in M(x_i).$$

When $U = X$ one just says that M is *r-hypomonotone*. The *r-hypomonotonicity* for $r = +\infty$ amounts to the monotonicity of the set-valued mapping M.

Th set-valued mapping M is c-coercive on U for some real $c > 0$ when

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq c\|x_1 - x_2\|^2 \quad \text{for all } x_i \in U \subset \mathrm{Dom}\, M,\ x_i^* \in M(x_i),$$

It instead

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq c\|x_1^* - x_2^*\|^2 \quad \text{for all } x_i \in U \subset \mathrm{Dom}\, M,\ x_i^* \in M(x_i),$$

one says that M is c-cocoercive on U.

Theorem 0.1.1. *[2, 3, 5] Let C be a closed subset of H and $r \in]0, +\infty]$. Then the following assertions are equivalent.*

(a) The set C is r-prox-regular.

(b) For any $x, x' \in C$ and $v \in N^p_C(x)$ one has

$$\langle v, x' - x\rangle \le \frac{1}{2r}\|v\|\|x' - x\|^2.$$

(c) For any $x \in C$, any $v \in N^p_C(x) \cap \mathbb{B}$, and any nonnegative real number $t < r$ one has $x = P_C(x + tv)$.

(d) For any $x_i \in C, v_i \in N^p_C(x_i) \cap \mathbb{B}$ with $i = 1, 2$ one has

$$\langle v_1 - v_2, x_1 - x_2\rangle \ge -\frac{1}{r}\|x_1 - x_2\|^2,$$

that is, the set-valued mapping $N^p_C(\cdot) \cap \mathbb{B}$ is $1/r$-hypomonotone.

(e) For any positive $\gamma < 1$, for $u_i \in U^\gamma_r(C) := \{v \in H : 0 < d(v, C) < \gamma r\}$, and for $y_i \in (I + \gamma r\mathbb{B} \cap N^p_C(\cdot))^{-1}(u_i)$ with $i = 1, 2$, one has

$$\langle y_1 - y_2, u_1 - u_2\rangle \ge (1 - \gamma)\|y_1 - y_2\|^2,$$

that is, the set-valued operator $(I+\gamma r\mathbb{B}\cap N^p_C(\cdot))^{-1}$ is $(1-\gamma)$-cocoercive on the set $U^\gamma_r(C)$.

(f) For any positive $\gamma < 1$ the mapping P_C is well-defined on $U^\gamma_r(C)$ and Lipschitz continuous on $U^\gamma_r(C)$ with $(1 - \gamma)^{-1}$ as a Lipschitz constant, that is,

$$\|P_C(u_1) - P_C(u_2)\| \le (1 - \gamma)^{-1}\|u_1 - u_2\| \quad \text{for all } u_1, u_2 \in U^\gamma_r(C).$$

(g) For any positive $\gamma < 1$ the mapping P_C is well-defined on $U^\gamma_r(C)$ and

$$P_C(u) = (I + \gamma r\mathbb{B} \cap N^p_C(\cdot))^{-1}(u) \quad \text{for all } u \in U^\gamma_r(C).$$

(h) The mapping P_C is well-defined on $U_r(C) := \{v \in H : 0 < d(v, C) < r\}$ and locally Lipschitz continuous there.

(i) The function d^2_C is of class $\mathcal{C}^{1,1}$ on $U_r(C)$ and $\nabla d^2_C(u) = 2(u - P_C(u))$ for all $u \in U_r(C)$.

(j) The function d^2_C is of class $\mathcal{C}^1$ on $U_r(C)$.

(k) The function d^2_C is Fréchet differentiable on $U_r(C)$.

(l) $\partial_p d_C(u) \neq \emptyset$ *for all* $u \in U_r(C)$.

(m) $\partial_F d_C(u) \neq \emptyset$ *for all* $u \in U_r(C)$.

If C is weakly closed (which holds whenever H is finite dimensional), then one can add the following the list of equivalences:

(n) *The mapping P_C is well-defined on $U_r(C)$.*

The next theorem concerns the local prox-regularity.

Theorem 0.1.2. *[3] Let C be a closed subset of H and let $\bar{x} \in C$. Then the following assertions are equivalent.*

(a) *The set C is prox-regular at $\bar{x}$.*

(b) *There exist a neighborhood U of $\bar{x}$ and a real number $r > 0$ such that for all $x \in C \cap U$ and $v \in N^p_C(x) \cap \mathbb{B}$ one has*

$$\langle v, x' - x\rangle \leq \frac{1}{2r}\|x' - x\|^2 \quad \text{for all } x' \in C \cap U.$$

(c) *There exist a neighborhood U of $\bar{x}$ and a real number $r > 0$ such that for all $x \in C \cap U$ and $v \in N^p_C(x)$ one has*

$$\langle v, x' - x\rangle \leq \frac{1}{2r}\|v\|\|x' - x\|^2 \quad \text{for all } x' \in C \cap U.$$

(d) *There exist a neighborhood U of $\bar{x}$ and a real number $r > 0$ such that for any $x \in C \cap U$, any $v \in N^p_C(x) \cap \mathbb{B}$, and any nonnegative real number $t < r$ one has $x = P_C(x + tv)$.*

(e) *There exist a neighborhood U of $\bar{x}$ and a real number $r > 0$ such that for all $x_i \in C \cap U, v_i \in N^p_C(x_i) \cap \mathbb{B}$ with $i = 1, 2$ one has*

$$\langle v_1 - v_2, x_1 - x_2\rangle \geq -\frac{1}{r}\|x_1 - x_2\|^2,$$

that is, the set-valued mapping $N^p_C(\cdot) \cap \mathbb{B}$ is $1/r$-hypomonotone on U.

(f) *There exist a neighborhood U of $\bar{x}$ and a real number $\beta > 0$ such that P_C is well defined on U and β-cocoercive (hence monotone) there, that is,*

$$\langle P_C(u_1) - P_C(u_2), u_1 - u_2\rangle \geq \beta\|P_C(u_1) - P_C(u_2)\|^2 \quad \text{for all } u_1, u_2 \in U.$$

(g) *There exist a neighborhood U of $\bar{x}$ and a real number $\beta > 0$ such that P_C is well defined on U and Lipschitz continuous on U with $P_C = (I + \gamma r\mathbb{B} \cap N_C^p(\cdot))^{-1}$ there.*

(h) *The function d_C^2 is of class $\mathcal{C}^{1,1}$ on some neighborhood U of $\bar{x}$.*

(i) *The function d_C^2 is of class $\mathcal{C}^1$ on some neighborhood U of $\bar{x}$.*

(j) *The function d_C^2 is Fréchet differentiable on some neighborhood U of $\bar{x}$.*

(k) *There exist a neighborhood U of $\bar{x}$ such that $\partial_p d_C(u) \neq \emptyset$ for all $u \in U$.*

(l) *There exist a neighborhood U of $\bar{x}$ such that $\partial_F d_C(u) \neq \emptyset$ for all $u \in U$.*

If C is weakly closed (which holds whenever H is finite dimensional), then one can add the following the list of equivalences:

(m) *The mapping P_C is well-defined on U of $\bar{x}$.*

Prox-regular sets are proximally normally regular as stated in the following proposition.

Proposition 0.1.1. *Let C be a closed subset of H. If C is prox-regular at $\bar{u} \in C$, then for some neighborhood U of $\bar{u}$ one has the normal regularity*

$$N_C^p(u) = N_C^F(u) = N_C^L(u) = N_C^{Cl}(u) \quad \text{for all } u \in C \cap U.$$

0.1.1 Preservation of prox-regularity under operations

This section is related to the study of the preservation of prox-regularity under certain operations on sets.

To provide general sufficient conditions under which the prox-regularity of intersection or inverse image is preserved, we have to introduce the concept of normal cone property for intersection of finitely many sets or for inverse image set. Following [3], we say that a finite family of closed sets $(C_k)_{k=1}^m$ of H has the *normal cone intersection property* near a point $\bar{x} \in \bigcap\limits_{k=1}^{m} C_k$ with respect to a normal cone $N_{(\cdot)}(\cdot)$ if there exist some real $\beta > 0$ and some neighborhood U of $\bar{x}$ such that for all $x \in U \cap C_1 \cap \cdots \cap C_m$ we have

$$N_{\bigcap\limits_{k=1}^{m} C_k}(x) \cap \mathbb{B} \subset N_{C_1}(x) \cap \beta\mathbb{B} + \cdots + N_{C_m}(x) \cap \beta\mathbb{B}. \tag{3}$$

Let now $F : H \to Y$ be a mapping from H into another Hilbert space Y and let D be a subset of Y and $\bar{x} \in F^{-1}(D)$. In the same way, we say that the inverse image set $F^{-1}(D)$ has the *normal cone inverse image property* at

$\bar{x} \in F^{-1}(D)$ with respect to the normal cone $N_{(\cdot)}(\cdot)$ if there exist some real $\beta > 0$ and some neighborhood U of $\bar{x}$ such that for all $x \in U \cap F^{-1}(D)$

$$N_{F^{-1}(D)}(x) \cap \mathbb{B} \subset DF(x)^* \Big(N_D\big(F(x)\big) \cap \beta\mathbb{B} \Big). \tag{4}$$

Theorem 0.1.3. *[3] Let $(C_k)_{k=1}^m$ be a finite family of closed sets of H and let D be a closed set of Y.*

(a) If all the sets C_k are prox-regular at a point $\bar{x}$ of their intersection and if they have the normal cone intersection property near $\bar{x}$ with respect to the Fréchet normal cone, then their intersection set $\overset{m}{\underset{k=1}{\cap}} C_k$ is prox-regular at $\bar{x}$.

(b) If a mapping $F : H \to Y$ is of class $\mathcal{C}^{1,1}$ around a point $\bar{x} \in F^{-1}(D)$ and if the inverse image set $F^{-1}(D)$ has the normal cone inverse image property at $\bar{x}$ with respect to the Fréchet normal cone, then the inverse image set $F^{-1}(D)$ is prox-regular at $\bar{x}$.

As a consequence we have the useful corollary

Corollaire 0.1.1. *[3] Let C be a closed set of H which is prox-regular at $\bar{x} \in C$ and let $h : H \to \mathbb{R}^m$ be a mapping of class $\mathcal{C}^{1,1}$ near $\bar{x} \in M := \{x \in H : h(x) = 0\}$ and such that $Dh(\bar{x})$ onto. Assume that the only vector $\lambda = (\lambda_1, \cdots, \lambda_m)$ in $\mathbb{R}^m$ such that*

$$\sum_{i=1}^{m} \lambda_i \nabla h_i(\bar{x}) \in N_C^F(\bar{x})$$

is the null vector $\lambda = (0, \cdots, 0)$. Then $C \cap M$ is prox-regular at $\bar{x}$.

The case of a real-valued function h (that is, $m = 1$) is of particular importance.

Corollaire 0.1.2. *[3] Let C be a closed set of H which is prox-regular at $\bar{x} \in C$ and let $h : H \to \mathbb{R}$ be a real-valued function of class $\mathcal{C}^{1,1}$ near $\bar{x} \in M := \{x \in H : h(x) = 0\}$. If*

$$\nabla h(\bar{x}) \notin N_C^F(\bar{x}) \cup (-N_C^F(\bar{x})),$$

then $C \cap M$ is prox-regular at $\bar{x}$.

The next proposition consider the complement of the sublevel set of a semiconvex function. Recall that a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is σ-semiconvex on an open convex set U of the normed space X whenever

$$f\big(tx + (1-t)y\big) \leq tf(x) + (1-t)f(y) + \frac{1}{2}\sigma t(1-t)\|x-y\|^2$$

for all $t \in]0,1[$, $x, y \in U$.

Proposition 0.1.2. *[3] Let $g : H \to \mathbb{R}$ be a continuous function, $C = \{x \in H : g(x) \geq 0\}$, and $\bar{x} \in C$ with $g(\bar{x}) = 0$. Assume that g is $\mathcal{C}^{1,1}$ near $\bar{x}$ and semiconvex near $\bar{x}$. Assume also that $Dg(\bar{x})$ is non-null. Then the set C is prox-regular at $\bar{x}$. More precisely, if on open convex set U, with $\bar{x} \in U$, the function g is σ-semiconvex and Dg is γ-Lipschitz continuous, and if there is some real $\alpha > 0$ such that $\|Dg(x)\| \geq \alpha$ for all $x \in U \cap g^{-1}(0)$, then C is $\alpha^{-1}(\sigma + 2\gamma)$-prox-regular at every point $x \in U \cap C$.*

The next result is concerned with direct images of prox-regular sets. In this result $O_{r'}(C)$ denotes the set $O_{r'}(C) := \{x \in H : d_C(x) < r'\}$.

Corollaire 0.1.3. *[3] Let H and Y be Hilbert spaces, $C \subset H$ be a closed r-prox-regular set. Let $0 < r' < r$ and $f : O_{r'}(C) \to Y$ be a $\mathcal{C}^1$-mapping such that Df is Lipschitz continuous on $O_{r'}(C)$ with constant M, and $\sup\{\|Df(x)\| : x \in C\} \leq N$. Assume that f is one to one over C and such that f^{-1} (the inverse of the restriction $f_C : C \to f(C)$ with $f_C(x) = f(x)$ for all $x \in C$) is Lipschitz continuous over C with L as a Lipschitz constant. Set*

$$r_1 = \min\left\{\frac{r}{M}, L^{-2}\left(M + \frac{N}{r}\right)^{-1}\right\}.$$

Then $f(C)$ is closed and r_1-prox-regular.

0.2 Subsmooth sets

In this section, C will be a closed subset of the Banach space X.

First we begin by recalling that a subset C is *subsmooth* at $u_0 \in C$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $u_1, u_2 \in B(u_0, \delta) \cap C$ and all $u_i^* \in N_C^{Cl}(u_i) \cap \mathbb{B}_{X^*}$, $i = 1, 2$ we have

$$\langle u_1^* - u_2^*, u_1 - u_2 \rangle \geq -\varepsilon \|u_1 - u_2\|. \tag{5}$$

The set C is called *subsmooth*, if it is *subsmooth* at every $u_0 \in C$. Further, C is called *uniformly subsmooth*, if for every $\varepsilon > 0$ there exists $\delta > 0$, such that

(5) holds for all $u_1, u_2 \in C$ satisfying $\|u_1 - u_2\| < \delta$ and all $u_i^* \in N_C^{Cl}(u_i) \cap \mathbb{B}_{X^*}$, $i = 1, 2$.

Inspired by the notion of subsmoothness recalled above, we introduce the notions of uniform subsmoothness.

Definition 0.2.1. *Let E be a nonempty set. We say that a family $(C(t))_{t\in E}$ of closed sets of X is equi-uniformly subsmooth, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that (5) holds for any $t \in E$ and all $u_1, u_2 \in C(t)$ satisfying $\|u_1 - u_2\| < \delta$ and all $u_i^* \in N_{C(t)}^{Cl}(u_i) \cap \mathbb{B}_{X^*}$, $i \in \{1, 2\}$.*

Another concept in the line of (5) is related to the Clarke subdifferential of the distance function to the set C. The Clarke subdifferential recalled above takes a simpler from for a locally Lipschitz function. Indeed, it is known that the Clarke subdifferential of a locally Lipschitz continuous function $f : X \to \mathbb{R}$ at a point $u \in X$ is reduced the set

$$\partial_{Cl} f(u) := \left\{u^* \in X^* : \langle u^*, v\rangle \le f^0(u; v) \ \forall v \in X\right\},$$

where

$$f^0(u; v) := \limsup_{t\downarrow 0, y\to u} \frac{f(y + tv) - f(y)}{t}.$$

The above function $f^0(u; \cdot)$ is called the Clarke directional derivative of f at u. Recall that for any $u \in C$ we have

$$\partial_{Cl} d_C(u) \subset N_C(u) \cap \mathbb{B}_{X^*} \quad \text{and} \quad N_C(u) = \mathrm{cl}_{w^*}\Big(\mathbb{R}_+ \partial_{Cl} d_C(u)\Big),$$

where cl_{w^*} denotes the closure with respect to the $w(X^*, X)$-topology. Using the Clarke subdifferential of the distance function to the C in (5) intead of the truncated of the Clarke normal cone with the closed unit ball, we consider the following definition.

Definition 0.2.2. *We say that the set C (closed near $u_0 \in C$) is metrically subsmooth at u_0 when for every $\varepsilon > 0$ there exists some $\delta > 0$ such that (5) holds for all $u_1, u_2 \in B(u_0, \delta) \cap C$ and all $u_i^* \in \partial_{Cl} d_C(u_i)$, $i = 1, 2$. When the property holds at any u_0 in a closed set C we say that C is metrically subsmooth.*

The following result makes the connection between subsmoothness and other classical geometrical concepts.

Definition 0.2.3. *A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is subsmooth at $x_0 \in \mathrm{dom}\, f$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in B(x_0, \delta)$ with $x \in \mathrm{dom}\, \partial_C f$, $x^* \in \partial_C f(x)$*

$$f(y) \ge f(x) + \langle x^*, y - x\rangle - \varepsilon \|y - x\|.$$

Remark 0.2.1. *If f is of class $\mathcal{C}^1$ on an open set $U \subset X$, then it is subsmooth at any point of U.*

Proposition 0.2.1. *A function locally Lipschitz f is subsmooth at $x_0 \in$ $\operatorname{dom} f$ if, and only if, the set $\operatorname{epi} f$ is subsmooth at $u_0 = (x_0, f(x_0))$.*

Proposition 0.2.2. *[1] Let C be a closed subset of X. Then the following assertions hold:*

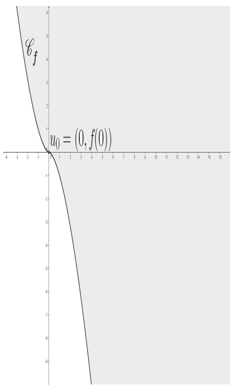
(a) Uniformly prox-regular sets are also uniformly subsmooth.

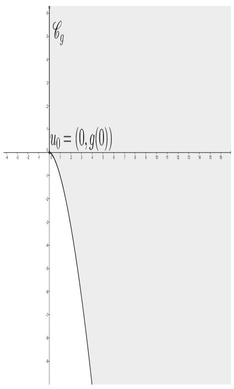
(b) Every prox-regular set C at u_0 is subsmooth at u_0.

(c) If C is subsmooth at u_0, then it is normally Fréchet regular at u_0, that is

$$N_C^F(u_0) = N_C^L(u_0) = N_C^{Cl}(u_0).$$

We consider the following functions:
$f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = -x^{5/3}$ and $g(x) = \begin{cases} -x^{5/3} & \text{if } x \geq 0 \\ \infty & \text{else.} \end{cases}$





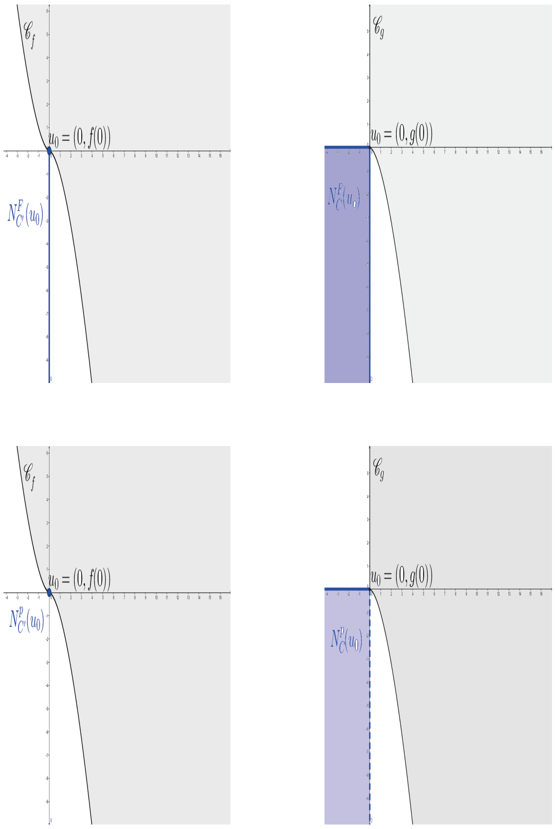


Figure 3: Two subsmooth sets C and C' which are not prox-regular.

Concerning the set $C := \operatorname{epi} f$, for $u_0 = (0, 0)$, we have the equalities

$$N^p_C(u_0) = \{(0,0)\} \text{ and } N^F_C(u_0) = \{0\}\times]-\infty, 0], \text{ or } N^p_C(u_0) \neq N^F_C(u_0),$$

hence C is not prox-regular at u_0 according to Proposition 0.1.1. The non prox-regular of the set $C' := \operatorname{epi} g$ can be seen throught the equalities

$$N^p_{C'}(u_0) = (]-\infty, 0]\times]-\infty, 0[) \cup \{(0,0)\} \text{ and } N^F_{C'}(u_0) =]-\infty, 0]\times]-\infty, 0],$$

so $N^p_{C'}(u_0) \neq N^F_{C'}(u_0)$, hence C' is not prox-regular at u_0.

Remark 0.2.2. *The converse of property (b) in the above proposition fails as shown by the following examples, in Figure 3, of sets C and C' which are subsmooth at $u_0 = (0, 0)$ but not prox-regular at u_0.*

We now characterize subsmoothness in terms of the Fréchet normal cone when X is a reflexive Banach space. In the following theorem we assume that U is an open subset of X and $C \cap U \neq \emptyset$.

Theorem 0.2.1. *[1] Let C be a closed subset of X. Then the following assertions are equivalent:*

(a) C is subsmooth on $C \cap U$;

(b) (5) holds at every point of $C \cap U$ with $N_C^L(\cdot) \cap \mathbb{B}$ in place of $N_C^{Cl}(\cdot) \cap \mathbb{B}$;

(c) (5) holds at every point of $C \cap U$ with $N_C^F(\cdot) \cap \mathbb{B}$ in place of $N_C^{Cl}(\cdot) \cap \mathbb{B}$;

(d) (5) holds at every point of $C \cap U$ with $\partial_{Cl} d_C(\cdot)$ in place of $N_C^{Cl}(\cdot) \cap \mathbb{B}$.

Remark 0.2.3. *By the theorem above, it is easily seen that C is subsmooth on $C \cap U$ if and only if C is metrically subsmooth on $C \cap U$.*

We recall the concept of Lewis' near radiality, C is called *nearly radial* at $u_0 \in C$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $u \in C \cap B(u_0, \delta)$ we have
$$d(K_C(u); u_0 - u) \leq \varepsilon \|u_0 - u\|,$$
where $K_C(u)$ is the Bouligand tangent cone to the set C at u, that is,
$$v \in K_C(u) \Longleftrightarrow \forall \delta > 0, \exists t \in]0, \delta[\text{ such that } \big(u + tB(v,\delta)\big) \cap C \neq \emptyset.$$
Further, if $T_C^{Cl}(u) = K_C(u)$ for all $u \in C$, we say that C is tangentially regular.

In the following result, we suppose that C is a closed set and U is an open subset of X such that $C \cap U \neq \emptyset$.

Theorem 0.2.2. *[1] If C is subssmooth on $C \cap U$, then it is tangentially regular on $C \cap U$ and nearly radial on $C \cap U$.*

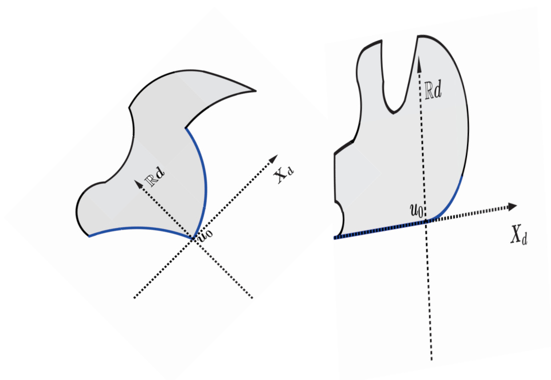


Figure 4: The sets which are epi-Lipschitzians at a certain boundary point.

We recall that a closed set C is said to be *epi-Lipschitz* at $u_0 \in C$ with respect to the direction $d \in X$ if there exist $\sigma > 0, \delta > 0$ such that for all $d' \in B(d, \sigma)$, all $u \in C \cap B(u_0, \sigma)$ and all $t \in]0, \delta[$ we have $u + td' \in C$. Therefore, every set C is epi-Lipschitz at every $u_0 \in \text{int}C$ (interior of C) with respect to any $d \in X$ and other hand, if $u_0 \in \text{bd}C$ (boundary of C), then C is epi-Lipschitz at u_0 with respect to some $d \neq 0$ if, and only if, the set C can be represented in a neighbourhood of u_0 as the epigraph of a Lipschitz continuous function f, which is called a locally Lipschitz representation of C at u_0, see Figure 4. This means that there exists a topological complement X_d of $\mathbb{R}d := \{td : t \in \mathbb{R}\}$ in X (that is, $X = X_d \oplus \mathbb{R}d$), a neighbourhood U of u_0 and a locally Lipschitz function $f : X_d \to \mathbb{R}$ such that

$$C \cap U = \{x \oplus sd : x \in X_d, f(x) \leq s\} \cap U.$$

Here X_d is endowed with the norm induced by the norm of X. We denote by $\pi : X \to X_d$ and $\rho : X \to \mathbb{R}$ the continuous linear mappings satisfying $u = \pi(u) \oplus \rho(u)d$ for all $u \in X$.

Let us recall that a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is called *approximatively convex* at u_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $u, v \in B(u_0, \delta)$ and $t \in]0, 1[$, we have

$$f(tu + (1-t)v) \leq tf(u) + (1-t)f(v) + \varepsilon t(1-t)\|u - v\|.$$

Theorem 0.2.3. *[1] Let X be a Banach space, let C be an epi-Lipschitz subset of X, and let $u_0 \in bdC$. Then the following statements are equivalent;*

(a) C is subsmooth at u_0.

(b) Every locally Lipschitz representation f of C at u_0 is approximately convex at $\pi(u_0)$.

(c) Some locally Lipschitz representation f of C at u_0 is approximately convex at $\pi(u_0)$.

Below we provide some sufficient conditions for subsmoothness of set-valued mapping.

Proposition 0.2.3. *[6] Suppose that G is defined by $G(u) = g(u) + C$ for all $u \in X$, where $g : X \to Y$ is a $\mathcal{C}^1$ mapping and C is a closed subset of Y. Let $(u, v) \in \text{gph}\, G$. If C is subsmooth at $v - g(u)$, then $\text{gph}\, G$ is subsmooth at (u, v).*

0.2.1 Preservation of subsmoothness under operations

Let $F : X \to Y$ be a mapping between X and another Banach space Y and let D be a subset of Y. Suppose that F is of class $\mathcal{C}^1$ near $u_0 \in C := F^{-1}(D)$ (here, F is assumed be of Class $\mathcal{C}^1$, while under the prox-regularity it is required that F is of class $\mathcal{C}^{1,1}$). Extending (4), we say that the inverse image of the set D by F, say $C := F^{-1}(D)$, has the *truncated normal cone inverse image property* near u_0 provided there are a positive constant β and a neighborhood U of u_0 sucht that

$$(6) \qquad N_C(u) \cap \mathbb{B}_{X^*} \subset DF(u)^*\Big(N_D\big(F(u)\big) \cap \beta\mathbb{B}_{Y^*}\Big) \quad \text{for all } u \in C \cap U,$$

where $DF(u)^*$ denotes the adjoint of the derivative mapping $DF(u)$ of F at u. Concerning the intersection of finitely many sets, we need, as for (3), to translate the condition in the inclusion above. Let $(C_i)_{i=1}^k$ be a finite system of sets of X and $u_0 \in \bigcap_{i=1}^{k} C_i$. We say that this system of sets satisfies the *truncated normal cone intersection property* near u_0 if there are a positive constant β and a neighborhood U of u_0 sucht that for all $u \in U \cap C_1 \cap \cdots \cap C_k$ we have

$$(7) \qquad N_{\bigcap_{i=1}^{k} C_i}(u) \cap \mathbb{B}_{X^*} \subset N_{C_1}(u) \cap (\beta\mathbb{B}_{Y^*}) + \cdots + N_{C_k}(u) \cap (\beta\mathbb{B}_{Y^*}).$$

Another important concept is related to the distance function to the set C and it does not require the subdifferentiability of the mapping F. We say that the mapping F is *metrically calm* at u_0 relative to the set D if there exist some constant $\beta > 0$ and some neighborhood U of u_0 sucht that

$$d_C(u) \leq \beta d_D\big(F(u)\big) \quad \text{for all } u \in U.$$

Theorem 0.2.4. *[4] Let $F : X \to Y$ be a mapping between Banach spaces X and Y and let $C := F^{-1}(D)$, where D is a subset of Y. Assume that F is of class $\mathcal{C}^1$ near $u_0 \in C$, that is, the derivative mapping $DF(\cdot)$ is continuous near u_0, and assume that D is closed near $F(u_0)$. The following hold.*

(a) If the set D is subsmooth at $F(u_0)$ and if the truncated normal cone inverse image property is satisfied for $F^{-1}(D)$ near u_0, then C is subsmooth at u_0.

(b) If the set D is metrically subsmooth at $F(u_0)$ and if the mapping F is metrically calm at u_0 with respect to the set D, then the set C is metrically subsmooth at u_0.

Theorem 0.2.5. *[4] Let $C_1, \cdots, C_k$ be a finite system of sets of X which are closed near $u_0 \in \bigcap_{i=1}^{k} C_i$. The following holds.*

- *If the sets $C_1, \cdots, C_k$ are subsmooth at u_0 and if the truncated normal cone intersection property is satisfied for these sets near u_0, then the intersection $\bigcap_{i=1}^{k} C_i$ is subsmooth at u_0.*

The next two theorems concern the uniform subsmoothness.

Theorem 0.2.6. *Let $F : X \to Y$ be a mapping between Banach spaces X and Y and let $C := F^{-1}(D)$, where D is a subset of Y. Assume that F is of class $\mathcal{C}^1$.*

- *If the set D is uniformly subsmooth and if the truncated normal cone inverse image property (6) is satisfied with the same real constant $\beta > 0$ for all $u \in F^{-1}(D)$, then C is uniformly subsmooth.*

Theorem 0.2.7. *Let $C_1, \cdots, \mathcal{C}_k$ be a finite system of sets of X. Suppose that the sets $C_1, \cdots, \mathcal{C}_k$ are uniformly subsmooth and that the truncated normal cone intersection property (7) is satisfied for these sets with the same real constant $\beta > 0$ for all $u \in \bigcap_{i=1}^{k} C_i$. Then the intersection $\bigcap_{i=1}^{k} C_i$ is uniformly subsmooth.*

The next two theorems concern a family of equi-uniformly subsmooth sets.

Theorem 0.2.8. *Let E be a nonempty set, let $F : X \to Y$ be a mapping between Banach spaces X and Y and let $C_t := F^{-1}(D_t)$ for any $t \in E$, where $(D_t)_{t\in E}$ is a family closed subsets of Y. Assume that the sets C_t are nonempty. Assume also that F is of class $\mathcal{C}^1$ and that the truncated normal cone inverse image property relative to a family sets $(D_t)_{t\in E}$ holds uniformly, that is, there exists some real constant $\beta > 0$ such that, for any $t \in E$, we have*

$$N_{C_t}(u) \cap \mathbb{B}_{X^*} \subset DF(u)^*\Big(N_{D_t}\big(F(u)\big) \cap \beta\mathbb{B}_{Y^*}\Big) \quad \text{for all } u \in C_t.$$

- *If the family $(D_t)_{t\in E}$ is equi-uniformly subsmooth, then $(C_t)_{t\in E}$ is equi-uniformly subsmooth.*

Theorem 0.2.9. *Let E be a nonempty set and let $(C_{1,t})_{t\in E}, \cdots, (C_{k,t})_{t\in E}$ be a finite system of families of sets of X such that for $i = 1, \cdots, k$ every family $(C_{i,t})_{t\in E}$ is equi-uniformly subsmooth. Suppose that $C_t := \bigcap_{i=1}^{k} C_{i,t}$ is* *nonempty for any $t \in E$. Suppose also that there is a real constant $\beta > 0$ such that for all $u \in C_t$*

$$N_{C_t}(u) \cap \mathbb{B}_{X^*} \subset N_{C_{1,t}}(u) \cap (\beta \mathbb{B}_{Y^*}) + \cdots + N_{C_{k,t}}(u) \cap (\beta \mathbb{B}_{Y^*}).$$

Then the family $(C_t)_{t \in E}$ is equi-uniformly subsmooth.

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Chapter 1

Subsmooth set and sweeping process

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Abstract. The class of subsmooth sets has been introduced in variational analysis in [1]. The subsmoothness property for a set corresponds to a variational behavior of *order one* of the set, while the prox-regularity property expresses a variational behavior of *order two*. The present paper establishes the existence of solution for perturbed differential inclusions defined by nonconvex and non prox-regular sweeping process associated with subsmooth sets.

Keyword: Differential inclusion; Sweeping process; Subdifferential; Normal cone; Prox-regular set; Subsmooth set.

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Introduction

Let H be a real Hilbert space, $T > 0$ be a positive real number, and $C : [0, T] \rightrightarrows H$ be a set-valued mapping with nonempty closed values moving in an absolutely continuous way. For any $x(0) = x_0 \in C(0)$, consider the differential inclusion

$$(1.1)\qquad \begin{cases} \dot{x}(t) \in -N_{C(t)}\big(x(t)\big) & \text{a.e } t \in [0,T] \\ x(t) \in C(t) & \forall t \in [0,T], \end{cases}$$

where $N_{C(t)}(\cdot)$ denotes a general normal cone to the set $C(t)$. This important problem of evolution has been introduced and studied in 1970, when the sets $C(t)$ are convex, by Moreau in the analysis of elastoplastic systems (see, [19, 20, 21]). In [4], C. Castaing introduced some new techniques from which many results can be derived, essentially the existence of a solution of (1.1) for $C(t) = S + \vartheta(t)$, where S is any fixed nonconvex closed subset of H and ϑ is a mapping with finite variation. Later, M. Valadier [24] dealt with sweeping processes associated with sets $C(t) = \mathbb{R}^n \backslash \operatorname{int}(K(t))$, where $K(t)$ are closed and convex sets. Condering the set-valued mapping G whose graph is closed and contains the graph of $(t,u) \mapsto N_{C(t)}(u) \cap \mathbb{B}$, he showed that the differential inclusion

$$\dot{x}(t) \in -G\big(t, x(t)\big), \quad x(0) = x_0 \in C(0)$$

admits at least a solution. Then, he obtained, in finite dimensional setting, existence of solution for (1.1) whenever the set-valued mapping $(t,u) \mapsto N_{C(t)}(u) \cap \mathbb{B}$ has a closed graph, where $N_{C(t)}(\cdot)$ is the Clarke normal cone. Moreover, in the finite dimensional context, many works have been realized when the sets $C(t)$ are nonconvex closed, as Benabdellah [2], Colombo and Goncharov [10], and Thibault [23].

The evolution problems associated with perturbed sweeping process began with the paper of Henry (see, [16]). Studying the planning procedures in mathematical economy, he introduced the differential inclusion

$$\dot{x}(t) \in \operatorname{Proj}_{T_C\left(x(t)\right)} G\big(x(t)\big), \quad x(0) = x_0 \in C,$$

where G is an upper semicontinuous set-valued mapping with nonempty compact convex values, C is a (nonmoving) nonempty closed convex set, and $T_C(\cdot)$ denotes the tangent cone to C and $\operatorname{Proj}_{T_C(x(t))}$ denotes the metric projection mapping onto the closed convex set $T_C(x(t))$. This differential inclusion has been also considered by B. Cornet [11, 12] with a Clarke tangentially regular set C, reducing the problem as in [16] to the existence of a solution of the differential inclusion

$$\dot{x}(t) \in -N_C\big(x(t)\big) + G\big(t, x(t)\big) \quad x(0) = x_0 \in C.$$

C. Castaing, T. X. Duc Ha and M. Valadier [7] and C. Castaing and M. D. P. Monteiro Marques [5] studied the sweeping process (1.1) with perturbations

$$
(1.2) \qquad \begin{cases} \dot{x}(t) \in -N_{C(t)}\big(x(t)\big) + G\big(t, x(t)\big) & \text{a.e } t \in [0,T] \\ \\ x(t) \in C(t) \quad \forall t \in [0,T] \\ \\ x(0) = x_0 \in C(0). \end{cases}
$$

in the cases where all the sets $C(t)$ are either convex or complements of open convex sets. The first general study of the differential inclusion (1.2) with general closed sets $C(t)$ moving in absolutely way in a finite dimensional setting has been realized by L. Thibault [23]. Later, several other papers dealt in the infinite dimensional Hilbert space H with the inclusion differential (1.2) under uniform prox-regularity assumptions, as the works of M. Bounkhel and L. Thibault [3], J. F. Edmond and L. Thibault in [14]

The main purpose of the present paper is to show how the subsmoothness property allows us to study the differential inclusion (1.2) in the general framework of infinite dimensional Hilbert space for nonconvex and non prox-regular sets $C(t)$. The subsmoothness of a set corresponds to a variational property of order one while the prox-regularity is a variational property of order two. Subsmooth sets are strongly connected with nearly radial sets of Lewis [17] and weakly regular sets of Jourani [15]. The plan of the paper is the following. We recall the needed concepts in the first section. In the second section, we prove the theorem of existence of solution of the differential inclusion (1.2).

1.1 Preliminaries

Throughout the paper, H stands for a real separable Hilbert space whose inner product is denoted by $\langle \cdot, \cdot \rangle$ and the associated norm by $\|\cdot\|$. The closed unit ball of H with center 0 will be denoted by $\mathbb{B}$ and $B(u,\eta)$ (respectively, $B[u,\eta]$) denotes the open (respectively, closed) ball of center $u \in H$ and radius $\eta > 0$. If I is a nonempty compact interval of $\mathbb{R}$, we will denote by $\mathcal{C}_H(I)$ the space of all continuous mappings from I to H. The norm of uniform convergence on $\mathcal{C}_H(I)$ will be denoted by $\|\cdot\|_\infty$, "a.e." denotes "for almost every" and $\dot{x}$ is the derivative of x.

Let C, C' be two subsets of H and let v be a vector in H, the real $d(v,C)$ or $d_C(v) := \inf\{\|v-u\| : u \in C\}$ is the distance of the point v from the set

C. We denote by

$$\text{Haus}(C, C') = \max\left\{\sup_{u\in C} d(u, C'), \sup_{v\in C'} d(v, C)\right\}$$

the Hausdorff distance between C and C'. For $v \in H$ the projection of v into $C \subset H$ is the set

$$\text{Proj}_{\,C}(v) := \{u \in C : d_C(v) = \|v - u\|\}.$$

This set is nonempty whenever C is *ball-compact*, that is, $C \cap r\mathbb{B}$ is compact for every real $r > 0$. Further, if $u \in \text{Proj}_{\,C}(v)$ then we have $v - u \in N^p_C(u)$ where $N^p_C(\cdot)$ denotes the proximal normal cone of C (see, [9]). If C is closed and convex, then $\text{Proj}_{\,C}(v)$ is a singleton and we will denote by $\text{proj}_{\,C}(v)$ the unique element of $\text{Proj}_{\,C}(v)$. For a nonempty interval $\mathcal{J}$ of $\mathbb{R}$, we recall that a set-valued mapping $F : \mathcal{J} \rightrightarrows H$ is called Lebesgue measurable if for each open set $U \subset H$ the set $F^{-1}(U) := \{t \in \mathcal{J} : F(t) \cap U \neq \emptyset\}$ is Lebesgue measurable. When the values of F are closed subsets of H, we know (see [6]) that the Lebesgue measurability of F is equivalent to the measurability of the graph of F, that is,

$$\text{gph}\, F \in \mathcal{L}(\mathcal{J}) \otimes \mathcal{B}(H),$$

where $\mathcal{L}(\mathcal{J})$ denotes the Lebesgue σ-field of $\mathcal{J}$, $\mathcal{B}(H)$ the Borel σ-field of H, and

$$\text{gph}\, F := \big\{(t, u) \in \mathcal{J} \times H : u \in F(t)\big\}.$$

For any subset C of H, $\overline{\text{co}}\, C$ stands for the closed convex hull of C, and $\sigma(\cdot, C)$ represents the support function of C, that is, for all $\xi \in H$,

$$\sigma(\xi, C) := \sup_{u\in C}\langle \xi, u\rangle.$$

If C is a nonempty subset of H, the Clarke normal cone $N(C; u)$ or $N_C(u)$ of C at $u \in C$ is defined by

$$N_C(u) = \{\xi \in H : \langle \xi, v\rangle \leq 0, \forall v \in T_C(u)\},$$

where the Clarke tangent cone $T(C; u)$ or $T_C(u)$ (see [8]) is defined as follows:

$$v \in T_C(u) \Leftrightarrow \begin{cases} \forall \varepsilon > 0, \exists \delta > 0 \text{ such that} \\ \\ \forall u' \in B(u, \delta) \cap C, \forall t \in]0, \delta[, (u' + tB(v, \varepsilon)) \cap C \neq \emptyset. \end{cases}$$

Equivalently, $v \in T_C(u)$ if and only if for any sequence $(u_n)_n$ of C converging to u and any sequence of positive reals $(t_n)_n$ converging to 0, there exists a sequence $(v_n)_n$ in H converging to v such that

$$u_n + t_n v_n \in C \text{ for all } n \in \mathbb{N}.$$

We put $N_C(u) = \emptyset$, whenever $u \notin C$. For any $\eta > 0$ we denote by $N_C^\eta(u)$ the truncated Clarke normal cone, that is,

$$N_C^\eta(u) = N_C(u) \cap \eta\mathbb{B}.$$

We typically denote by $f : H \to \mathbb{R} \cup \{+\infty\}$ a proper function (that is, f is finite at least at one point). The Clarke subdifferential $\partial f(u)$ of f at a point u (where f is finite) is defined by

$$\partial f(u) = \Big\{\xi \in H : (\xi, -1) \in N_{\operatorname{epi} f}\Big(\big(u, f(u)\big)\Big)\Big\},$$

where $\operatorname{epi} f$ denotes the epigraph of f, that is,

$$\operatorname{epi} f = \{(u, r) \in H \times \mathbb{R} : f(u) \leq r\}.$$

We also put $\partial f(u) = \emptyset$ if f is not finite at $u \in H$. If ψ_C denotes the indicator function of the set C, that is, $\psi_C(u) = 0$ if $u \in C$ and $\psi_C(u) = +\infty$ otherwise, then

$$\partial \psi_C(u) = N_C(u) \text{ for all } u \in H.$$

The Clarke subdifferential $\partial f(u)$ of a locally Lipschitz function f at u has also the other useful description

$$\partial f(u) = \{\xi \in H : \langle \xi, v\rangle \leq f^0(u, v), \forall v \in H\},$$

where

$$f^0(u, v) := \limsup_{(u',t)\to(u,0^+)} \frac{f(u' + tv) - f(u')}{t}.$$

The above function $f^0(u; \cdot)$ is called the Clarke directional derivative of f at u. The Clarke normal cone is known ([8]) to be related to the Clarke subdifferential of the distance function through the equality

$$N_C(u) = \mathrm{cl}_w(\mathbb{R}_+ \partial d_C(u)) \text{ for all } u \in C,$$

where $\mathbb{R}_+ := [0, \infty[$ and cl_w denotes the closure with respect to the weak topology of H. Further

$$\partial d_C(u) \subset N_C(u) \cap \mathbb{B} \quad \text{for all } u \in C. \tag{1.3}$$

The concept of Frchet subdifferential will be also needed. A vector $\xi \in H$ is said to be in the Frchet subdifferential $\partial_F f(u)$ of f at u (see [18]) provided that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $u' \in B(u, \delta)$ we have

$$\langle \xi, u' - u \rangle \leq f(u') - f(u) + \varepsilon \|u' - u\|.$$

It is known that we always have the inclusion

$$\partial_F f(u) \subset \partial f(u). \tag{1.4}$$

The Frchet normal cone of C at $u \in C$ is given by

$$N_C^F(u) = \partial_F \psi_C(u),$$

so the following inclusion always holds true

$$N_C^F(u) \subset N_C(u) \quad \text{for all } u \in C. \tag{1.5}$$

On the other hand, the Frchet normal cone is also related to the Frchet subdifferential of the distance function since the following relations hold true for all $u \in C$

$$N_C^F(u) = \mathbb{R}_+ \partial_F d_C(u)$$

and

$$\partial_F d_C(u) = N_C^F(u) \cap \mathbb{B}. \tag{1.6}$$

Another important property is

$$v - u \in N_C^F(u) \quad \text{hence also} \quad v - u \in N_C(u) \tag{1.7}$$

whenever $u \in \operatorname{Proj}_C(v)$, since $N_C^p(u) \subset N_C^F(u)$.

1.2 Sweeping process with subsmooth sets

We begin by recalling the concept of subsmoothness developed in [1]; it will be used to define the equi-uniformly subsmooth property for a family of closed sets of H.

1.2.1 Definition and elementary properties

Let C be a closed subset of H. The set C is known to be *prox-regular* at $u_0 \in C$ provided the following variational property of order two holds: There exist $r > 0$ and $\delta > 0$ such that for all $u_1, u_2 \in B(u_0, \delta) \cap C$ and all $\xi_i \in N_C(u_i) \cap \mathbb{B}$, $i = 1, 2$ we have

$$\langle \xi_1 - \xi_2, u_1 - u_2 \rangle \geq -\frac{1}{r} \|u_1 - u_2\|^2. \tag{1.8}$$

For several properties, characterizations and examples of such sets we refer the reader to [22].

Relaxing the inequality (1.8) in a variational property of order one, Aussel, Daniilidis and Thibault defined the concept of subsmooth sets as follows. The closed set $C \subset H$ is called *subsmooth* at $u_0 \in C$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $u_1, u_2 \in B(u_0, \delta) \cap C$ and all $\xi_i \in N_C(u_i) \cap \mathbb{B}$, $i = 1, 2$ we have

$$\langle \xi_1 - \xi_2, u_1 - u_2 \rangle \geq -\varepsilon \|u_1 - u_2\|. \tag{1.9}$$

The set C is called *subsmooth*, if it is *subsmooth* at every $u_0 \in C$. Further, C is called *uniformly subsmooth*, if for every $\varepsilon > 0$ there exists $\delta > 0$, such that (1.9) holds for all $u_1, u_2 \in C$ satisfying $\|u_1 - u_2\| < \delta$ and all $\xi_i \in N_C(u_i) \cap \mathbb{B}$, $i = 1, 2$.

For other variational properties of order one we refer to Lewis [17] where nearly radial sets are considered and to Jourani [15] where weakly regular sets are investigated. The connection between those two classes of sets and the class of subsmooth sets is studied in [13].

We next define the concept of equi-uniform subsmoothness, which will be basic to the rest of the paper.

Definition 1.2.1. *Let E be a nonempty set. We say that a family $(C(t))_{t \in E}$ of closed sets of H is equi-uniformly subsmooth, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that (1.9) holds for any $t \in E$ and all $u_1, u_2 \in C(t)$ satisfying $\|u_1 - u_2\| < \delta$ and all $\xi_i \in N_{C(t)}(u_i) \cap \mathbb{B}$, $i \in \{1, 2\}$.*

The following results will be used in the proof of The main theorem.

Lemma 1.2.1. *If a closed set C of H is subsmooth at $u \in C$, then*

$$\partial d_C(u) = \partial_F d_C(u)$$

and

$$N_C(u) = N_C^F(u).$$

Proof. Let $\varepsilon > 0$. From the subsmoothness of C at u, there exists some $\delta > 0$ such that for all $v \in C$ with $\|v - u\| < \delta$ and all $\xi \in N_C(u)$

$$\langle -\xi, v - u \rangle \geq -\varepsilon \|v - u\|.$$

Hence, we may write

$$\langle \xi, v - u \rangle \leq \varepsilon \|v - u\| \quad \text{for all } v \in B(u, \delta) \cap C.$$

This last inequality entails that $\xi \in N_C^F(u)$. This means $N_C(u) \subset N_C^F(u)$ hence $N_C(u) = N_C^F(u)$ since the reverse inclusion always holds true (see (1.5)). So, the second equality is established. Take now any $\xi \in \partial d_C(u)$. Then we have $\|\xi\| \leq 1$ and $\xi \in N_C(u)$ by (1.3). The equality proved above gives $\xi \in N_C^F(u)$, thus $\xi \in N_C^F(u) \cap \mathbb{B}$, which is equivalent to $\xi \in \partial_F d_C(u)$ according to (1.6). Consequently $\partial d_C(u) \subset \partial_F d_C(u)$ hence the equality $\partial d_C(u) = \partial_F d_C(u)$ holds true according to (1.4), completing the proof of the lemma. □

Lemma 1.2.2. *Let E be a metric space and let $(C(t))_{t \in E}$ be a family of nonempty closed sets of H which is equi-uniformly subsmooth and let a real $\eta > 0$. Let $Q \subset E$ and $s_0 \in \mathrm{cl} Q$. Then the following hold:*

(a) For all $(s, u) \in \mathrm{gph}\, C$ we have $\eta \partial d_{C(s)}(u) \subset \eta \mathbb{B}$;

(b) For any net $(s_j)_{j \in J}$ in Q converging to s_0, any net $(u_j)_{j \in J}$ converging to $u \in C(s_0)$ in $(H, \|\cdot\|)$ with $u_j \in C(s_j)$ and $d_{C(s_j)}(y) \underset{j \in J}{\to} 0$ for every $y \in C(s_0)$, and any net $(\zeta_j)_{j \in J}$ converging weakly to ζ in $(H, w(H, H))$ with $\zeta_j \in \eta \partial d_{C(s_j)}(u_j)$, we have $\zeta \in \eta \partial d_{C(s_0)}(u)$.

Proof. The assertion *(a)* being obvious according to (1.3), we have to show *(b)*. Let $\varepsilon > 0$. By Definition 1.2.1 choose $\delta > 0$ such that for all $s \in E$, $u_1, u_2 \in C(s)$ with $\|u_1 - u_2\| < \delta$ and all $\zeta_i \in N_{C(s)}(u_i) \cap \mathbb{B}$

$$\langle \zeta_1 - \zeta_2, u_1 - u_2 \rangle \geq -\varepsilon \|u_1 - u_2\|. \tag{1.10}$$

Fix any nets $(s_j)_{j \in J}$ in Q converging to s_0, $(u_j)_{j \in J}$ converging strongly to $u \in C(s_0)$ in H with $u_j \in C(s_j)$ and $d_{C(s_j)}(y) \underset{j \in J}{\to} 0$ for every $y \in C(s_0)$, where $(J, \preccurlyeq)$ is a directed preordered set. Fix also any net $(\zeta_j)_{j \in J}$ converging weakly to ζ in H such that $\zeta_j \in \eta \partial d_{C(s_j)}(u_j)$. Since $u_j \in C(s_j)$, the latter inclusion means $\eta^{-1} \zeta_j \in N_{C(s_j)}(u_j) \cap \mathbb{B}$ for all $j \in J$ (see (1.6) and Lemma 1.2.1). Fix $y \in B(u, \frac{\delta}{2}) \cap C(s_0)$. For each $n \in \mathbb{N}$ and each $j \in J$, choose some $y_{j,n} \in C(s_j)$ such that

$$\|y_{j,n} - y\| \leq d_{C(s_j)}(y) + \frac{1}{n}.$$

Endowing $J \times \mathbb{N}$ with the product preorder which is obviously directed, $(y_{j,n})_{(j,n)\in J\times\mathbb{N}}$ is a net in H. Since

$$d_{C(s_j)}(y) + \frac{1}{n} \underset{(j,n)\in J\times\mathbb{N}}{\longrightarrow} 0,$$

we have $\|y_{j,n} - y\| \underset{(j,n)\in J\times\mathbb{N}}{\longrightarrow} 0$, that is, $y_{j,n} \underset{(j,n)\in J\times\mathbb{N}}{\longrightarrow} y$ strongly in H and hence there exists $j_0 \in J$ and $n_0 \in \mathbb{N}$ such that for all $(j,n) \in J \times \mathbb{N}$ with $j \succcurlyeq j_0$ and $n \geq n_0$ we have $y_{j,n} \in B(u, \frac{\delta}{2})$. Put $u_{j,n} := u_j$ for all $(j,n) \in J \times \mathbb{N}$. Obviously $u_{j,n} \underset{(j,n)\in J\times\mathbb{N}}{\longrightarrow} u$ strongly in H (because $u_j \underset{j\in J}{\to} u$). So we may also suppose that $u_{j,n} \in B(u, \frac{\delta}{2})$ for all $(j,n) \in J \times \mathbb{N}$, with $j \succcurlyeq j_0$ and $n \geq n_0$. Thus, for all $(j,n) \in J \times \mathbb{N}$ with $j \succcurlyeq j_0$ and $n \geq n_0$ we have

$$\|y_{j,n} - u\| < \frac{\delta}{2} \text{ and } \|u_{j,n} - u\| < \frac{\delta}{2}.$$

Set $\zeta_{j,n} := \zeta_j$ and $s_{j,n} := s_j$ for all $(j,n) \in J \times \mathbb{N}$. The net $(s_{j,n})_{(j,n)\in J\times\mathbb{N}}$ converges to s_0 and the net $\zeta_{j,n})_{(j,n)\in J\times\mathbb{N}}$ converges weakly to ζ in H and $\eta^{-1}\zeta_{j,n} \in N_{C(s_{j,n})}(u_{j,n}) \cap \mathbb{B}$. Thanks to the latter inequalities above, for all $(j,n) \in J \times \mathbb{N}$ with $j \succcurlyeq j_0$ and $n \geq n_0$ we have $\|y_{j,n} - u_{j,n}\| < \delta$ with $y_{j,n}, u_{j,n} \in C(s_{j,n})$ and hence according to (1.10)

$$\langle 0 - \eta^{-1}\zeta_{j,n}, y_{j,n} - u_{j,n}\rangle \geq -\varepsilon\|y_{j,n} - u_{j,n}\|$$

or equivalently

$$\langle \eta^{-1}\zeta_{j,n}, y_{j,n} - u_{j,n}\rangle \leq \varepsilon\|y_{j,n} - u_{j,n}\|.$$

Since the net $(\eta^{-1}\zeta_{j,n})_{(j,n)\in J\times\mathbb{N}}$ is bounded (by the real number 1), we may pass to the limit to obtain

$$\langle \eta^{-1}\zeta, y - u\rangle \leq \varepsilon\|y - u\|$$

for all $y \in B(u, \frac{\delta}{2}) \cap C(s_0)$ and hence $\eta^{-1}\zeta \in N^F_{C(s_0)}(u)$. Further, $\eta^{-1}\zeta_{j,n} \in \mathbb{B}$ for all $(j,n) \in J \times \mathbb{N}$ and this ensures $\eta^{-1}\zeta \in \mathbb{B}$. Thus, $\eta^{-1}\zeta \in N^F_{C(s_0)}(u) \cap \mathbb{B}$, so $\eta^{-1}\zeta \in \partial_F d_{C(s_0)}(u) \subset \partial d_{C(s_0)}(u)$. The proof is complete. □

From Lemma 1.2.2 we easily deduce, thanks to properties of upper semi-continuous set-valued mappings (see [6]), the following proposition.

Proposition 1.2.1. *Let I be a nonempty interval of $\mathbb{R}$ and let $(C(t))_{t\in I}$ be a family of nonempty closed sets of H which is equi-uniformly subsmooth and let a real $\eta > 0$. Assume that there exists a nondecreasing continuous function $v : I \to \mathbb{R}_+$ such that, for any $y \in H$ and $s, t \in I$ with $s \leq t$,*

$$d\big(y, C(t)\big) \leq d\big(y, C(s)\big) + v(t) - v(s).$$

Then the following assertions hold:

(a) For all $(s,u) \in \operatorname{gph} C$ we have $\eta\partial d_{C(s)}(u) \subset \eta\mathbb{B}$;

(b) For any sequence $(s_n)_n$ in I converging to s with $s_n \geq s$, any sequence $(u_n)_n$ converging to $u \in C(s)$ with $u_n \in C(s_n)$, and any $\xi \in H$, we have

$$\limsup_{n\to\infty} \sigma\big(\xi, \eta\partial d_{C(s_n)}(u_n)\big) \leq \sigma\big(\xi, \eta\partial d_{C(s)}(u)\big).$$

Proof. Only *(b)* needs to be proved. Let $(s_n)_n$ and $(u_n)_n$ as in the statement. Fix any $\xi \in H$. Extracting a subsequence if necessary, we may suppose that

$$\limsup_{n\to\infty} \sigma\big(\xi, \eta\partial d_{C(s_n)}(u_n)\big) = \lim_{n\to\infty} \sigma\big(\xi, \eta\partial d_{C(s_n)}(u_n)\big).$$

For each n, chosse according to the weak compactness of $\eta\partial d_{C(s_n)}(u_n)$ some $\zeta_n \in \eta\partial d_{C(s_n)}(u_n)$ such that

$$\langle \xi, \zeta_n \rangle = \sigma\big(\xi, \eta\partial d_{C(s_n)}(u_n)\big).$$

Since $\|\zeta_n\| \leq \eta$ by *(a)*, a subsequence of $(\zeta_n)_n$ (that we do not relabel) converges weakly to some ζ in H. It results that

$$\langle \xi, \zeta \rangle = \limsup_{n\to\infty} \sigma\big(\xi, \eta\partial d_{C(s_n)}(u_n)\big). \tag{1.11}$$

Now, observe that for each $y \in C(s)$ that

$$0 \leq d(y, C(s_n)) \leq d(y, C(s)) + v(s_n) - v(s) = v(s_n) - v(s),$$

hence $d(y, C(s_n)) \to 0$ as $n \to \infty$ thanks to the right-hand continuity of v. We then apply Lemma 1.2.2 to obtain $\zeta \in \eta\partial d_{C(s)}(u)$. Combining the latter inclusion with (1.11) we see that

$$\limsup_{n\to\infty} \sigma\big(\xi, \eta\partial d_{C(s_n)}(u_n)\big) \leq \sigma\big(\xi, \eta\partial d_{C(s)}(u)\big),$$

which finishes the proof. □

1.2.2 Main results

Our existence theorem is started under the following assumptions.

Let $C : I \rightrightarrows H$ be a set-valued mapping. It is required to satisfy the following assumptions:

($\mathcal{H}_1$) For each $t \in I$, $C(t)$ is a nonempty ball-compact subset of H; there exists a nondecreasing absolutely continuous function $v : I \to \mathbb{R}_+$ such that, for any $y \in H$ and $s, t \in I$ with $s \le t$

$$d\big(y, C(t)\big) \le d\big(y, C(s)\big) + v(t) - v(s);$$

($\mathcal{H}_2$) The family $\big(C(t)\big)_{t\in I}$ is equi-uniformly subsmooth;

We consider also a set-valued mapping $\Gamma : I \times H \rightrightarrows H$ with nonempty closed convex values which is $\mathcal{L}(I) \otimes \mathcal{B}(H)-$measurable and upper semicontinuous with respect to $x \in H$ for almost all $t \in I$.

($\mathcal{H}_3$) The set-valued mapping Γ satisfies the growth condition

$$d\big(0, \Gamma(t, x)\big) \le \alpha(t)(1 + \|x\|)$$

for all $t \in I$ and all $x \in C([T_0, t]) := \bigcup_{T_0 \le s \le t} C(s)$, where $\alpha : I \to \mathbb{R}_+$ is an integrable function on I.

Theorem 1.2.1. *Let real numbers T_0 and T be fixed with $0 \le T_0 < T$. Assume that $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ above hold for the interval $I = [T_0, T]$. Then, there exists an absolutely continuous mapping $x : I \to H$ which is a solution on the whole interval I of the constrained differential inclusion*

$$(\mathcal{E}) \quad \begin{cases} \dot{x}(t) \in -N_{C(t)}\big(x(t)\big) + \Gamma\big(t, x(t)\big) \quad a.e.\ t \in I \\ x(t) \in C(t) \quad \forall t \in I \\ x(T_0) = x_0 \in C(T_0). \end{cases}$$

Proof. Fix $I := [T_0, T]$ throughout the proof. Observe first by ($\mathcal{H}_3$) that there is some $\alpha \in \mathrm{L}^1_{\mathbb{R}_+}(I)$ such that for all $(t, x) \in I \times C([T_0, t])$, we have

$$d\big(0, \Gamma(t, x)\big) \le \alpha(t)(1 + \|x\|). \tag{1.12}$$

I. We suppose in this first part I that

$$\int_{T_0}^{T} \alpha(s)ds \le \frac{1}{4}. \tag{1.13}$$

Let us put

$$r(x_0) := 2\bigg(\|x_0\| + \int_{T_0}^{T} |\dot{v}(s)|ds + 1\bigg).$$

We are going to construct a sequence of mappings (x_n) in $\mathcal{C}_H(I)$ which admits a subsequence which converges uniformly to a solution of $(\mathcal{E})$.

Step 1. Construction of the sequence (x_n).

For any integer $n \geq 1$, consider the partition of I defined by the points

$$t_i^n = T_0 + i\frac{T - T_0}{2^n}, (0 \leq i \leq 2^n).$$

Let $J = \{0, 1, \cdots, 2^n - 1\}$. Consider the mappings θ_n and $\hat{\theta}_n$ from the interval I of $\mathbb{R}$ into itself, defined by

$$\theta_n(t) = \begin{cases} t_{i+1}^n & \text{if} \quad t \in [t_i^n, t_{i+1}^n[, i \in J \\ \\ T & \text{if} \quad t = T, \end{cases} \tag{1.14}$$

$$\hat{\theta}_n(t) = \begin{cases} t_i^n & \text{if} \quad t \in]t_i^n, t_{i+1}^n], i \in J \\ \\ T_0 & \text{if} \quad t = T_0. \end{cases} \tag{1.15}$$

We have $|\theta_n(t) - t| < \frac{T - T_0}{2^n}$ and $|\hat{\theta}_n(t) - t| < \frac{T - T_0}{2^n}$, hence

$$\theta_n(t) \to t \text{ and } \hat{\theta}_n(t) \to t. \tag{1.16}$$

Put $x_0^n = x_0 \in C(t_0^n)$. Let f_0^n be the mapping from $[t_0^n, t_1^n]$ into H given by $f_0^n(t)$ as the element of minimal norm of $\Gamma(t, x_0^n)$, that is,

$$f_0^n(t) = \operatorname{Proj}_{\Gamma(t, x_0^n)}(0) \quad \text{for all } t \in [t_0^n, t_1^n].$$

The mapping f_0^n is measurable according to the measurability of the set-valued mapping $\Gamma(\cdot, x_0^n)$. Thanks to (1.12) we get

$$\|f_0^n(t)\| \leq \alpha(t)\big(1 + \|x_0^n\|\big) \quad \forall\, t \in [t_0^n, t_1^n]. \tag{1.17}$$

Since α is an integrable non-negative function on I, hence f_0^n is bounded by a function in $\mathrm{L}^1_{\mathbb{R}_+}(I)$. Thus $f_0^n \in \mathrm{L}^1_H(T_0, t_1^n)$.

The ball-compactness of $C(t)$ ensures that

$$\operatorname{Proj}_{C(t_1^n)}\Big(x_0^n + \int_{T_0}^{t_1^n} f_0^n(s)ds\Big) \neq \emptyset.$$

Then, we can choose a point x_1^n in $\operatorname{Proj}_{C(t_1^n)}\big(x_0^n + \int_{T_0}^{t_1^n} f_0^n(s)ds\big)$, hence $x_1^n \in C(t_1^n)$ and

$$\Big\|x_1^n - \Big(x_0^n + \int_{T_0}^{t_1^n} f_0^n(s)ds\Big)\Big\| = d\Big(x_0^n + \int_{T_0}^{t_1^n} f_0^n(s)ds, C(t_1^n)\Big).$$

So, according to $(\mathcal{H}_1)$ and the inclusion $x_0^n \in C(T_0)$, we have

$$\begin{aligned}\Big\|x_1^n \;&-\; \Big(x_0^n + \int_{T_0}^{t_1^n} f_0^n(s)ds\Big)\Big\| \\ &\leq\; d\Big(x_0^n + \int_{T_0}^{t_1^n} f_0^n(s)ds, C(T_0)\Big) + v(t_1^n) - v(T_0) \\ &\leq\; d\big(x_0^n, C(T_0)\big) + \Big\|\int_{T_0}^{t_1^n} f_0^n(s)ds\Big\| + v(t_1^n) - v(T_0) \\ &\leq\; \int_{T_0}^{t_1^n} \|f_0^n(s)\|ds + \int_{T_0}^{t_1^n} \dot{v}(s)ds.\end{aligned}$$

By (1.17), we obtain

$$\Big\|x_1^n - \Big(x_0^n + \int_{T_0}^{t_1^n} f_0^n(s)ds\Big)\Big\| \leq \int_{T_0}^{t_1^n} \Big(\alpha(s)\big(1 + \|x_0^n\|\big) + \dot{v}(s)\Big)ds.$$

Similarly as above, we choose a measurable mapping f_1^n from $[t_1^n, t_2^n]$ into H such that $f_1^n(t) \in \Gamma(t, x_1^n)$ for all $t \in [t_1^n, t_2^n]$. By (1.12), we have

$$\|f_1^n(t)\| \leq \alpha(t)\big(1 + \|x_1^n\|\big) \quad \forall t \in [t_1^n, t_2^n], \tag{1.18}$$

and this says in particular that f_1^n is integrable over $[t_1^n, t_2^n]$.

The ball-compactness of $C(t)$ ensures that

$$\operatorname{Proj}_{C(t_2^n)}\Big(x_1^n + \int_{t_1^n}^{t_2^n} f_1^n(s)ds\Big) \neq \emptyset.$$

Then, we can choose a point x_2^n in $\operatorname{Proj}_{C(t_2^n)}\big(x_1^n + \int_{t_1^n}^{t_2^n} f_1^n(s)ds\big)$, hence $x_2^n \in C(t_2^n)$ and

$$\Big\|x_2^n - \Big(x_1^n + \int_{t_1^n}^{t_2^n} f_1^n(s)ds\Big)\Big\| = d\Big(x_1^n + \int_{t_1^n}^{t_2^n} f_1^n(s)ds, C(t_2^n)\Big).$$

So, according to $(\mathcal{H}_1)$ and the inclusion $x_1^n \in C(t_1^n)$, we have

$$\begin{aligned}
\Big\| x_2^n - \Big(x_1^n \; &+ \; \int_{t_1^n}^{t_2^n} f_1^n(s)ds \Big) \Big\| \\
&\leq \; d\Big(x_1^n + \int_{t_1^n}^{t_2^n} f_1^n(s)ds, C(t_1^n) \Big) + v(t_2^n) - v(t_1^n) \\
&\leq \; d\big(x_1^n, C(t_1^n) \big) + \Big\| \int_{t_1^n}^{t_2^n} f_1^n(s)ds \Big\| + v(t_2^n) - v(t_1^n) \\
&\leq \; \int_{t_1^n}^{t_2^n} \| f_1^n(s) \| ds + \int_{t_1^n}^{t_2^n} \dot{v}(s)ds,
\end{aligned}$$

hence the inequality in (1.18) yields

$$\Big\| x_2^n - \Big(x_1^n + \int_{t_1^n}^{t_2^n} f_1^n(s)ds \Big) \Big\| \leq \int_{t_1^n}^{t_2^n} \Big(\alpha(s)(1 + \|x_1^n\|) + \dot{v}(s) \Big) ds.$$

By repeating the process, we define finite sequences (x_i^n) and measurable mappings (f_i^n) from $[t_i^n, t_{i+1}^n]$ into H with the following properties:

$$f_i^n(t) \in \Gamma(t, x_i^n) \text{ and } \|f_i^n(t)\| \leq \alpha(t)(1 + \|x_i^n\|) \quad \forall t \in [t_i^n, t_{i+1}^n]; \tag{1.19}$$

$$x_{i+1}^n \in \operatorname{Proj}_{C(t_{i+1}^n)} \Big(x_i^n + \int_{t_i^n}^{t_{i+1}^n} f_i^n(s)ds \Big); \tag{1.20}$$

$$\Big\| x_{i+1}^n - \Big(x_i^n + \int_{t_i^n}^{t_{i+1}^n} f_i^n(s)ds \Big) \Big\| \leq \int_{t_i^n}^{t_{i+1}^n} \Big(\alpha(s)\big(1 + \|x_i^n\|\big) + \dot{v}(s) \Big) ds. \tag{1.21}$$

Now, according to (1.21), and (1.19), we obtain

$$\begin{aligned}
\|x_{i+1}^n\| \;&\leq\; \|x_i^n\| + 2(1 + \|x_i^n\|) \int_{t_i^n}^{t_{i+1}^n} \alpha(s)ds + \int_{t_i^n}^{t_{i+1}^n} \dot{v}(s)ds \\
&\leq\; \|x_i^n\| + 2(1 + \max_{0 \leq j \leq 2^n} \|x_j^n\|) \int_{t_i^n}^{t_{i+1}^n} \alpha(s)ds + \int_{t_i^n}^{t_{i+1}^n} \dot{v}(s)ds.
\end{aligned}$$

Iterating it follows that

$$\begin{aligned}\|x_{i+1}^n\| &\leq \|x_0^n\| + 2(1 + \max_{0\leq j\leq 2^n}\|x_j^n\|)\sum_{k=0}^{i}\int_{t_k^n}^{t_{k+1}^n}\alpha(s)ds + \sum_{k=0}^{i}\int_{t_k^n}^{t_{k+1}^n}\dot{v}(s)ds \\ &\leq \|x_0^n\| + 2(1 + \max_{0\leq j\leq 2^n}\|x_j^n\|)\int_{T_0}^{T}\alpha(s)ds + \int_{T_0}^{T}\dot{v}(s)ds.\end{aligned}$$

This being true for all $0 \leq j \leq 2^n$, then

$$\max_{0\leq j\leq 2^n}\|x_j^n\| \leq \|x_0^n\| + 2(1 + \max_{0\leq j\leq 2^n}\|x_j^n\|)\int_{T_0}^{T}\alpha(s)ds + \int_{T_0}^{T}\dot{v}(s)ds.$$

It results, thanks to (1.13), that

$$\max_{0\leq j\leq 2^n}\|x_j^n\| \leq \|x_0^n\| + \frac{1}{2}(1 + \max_{0\leq j\leq 2^n}\|x_j^n\|) + \int_{T_0}^{T}\dot{v}(s)ds.$$

Consequently,

$$\max_{0\leq j\leq 2^n}\|x_j^n\| \leq 2\left(\|x_0^n\| + \frac{1}{2} + \int_{T_0}^{T}\dot{v}(s)ds\right) = r(x_0) - 1, \tag{1.22}$$

the equality being due to the definition of $r(x_0)$. Combining this with (1.19), we get

$$\|f_i^n(t)\| \leq \alpha(t)r(x_0). \tag{1.23}$$

For all $t \in [t_i^n, t_{i+1}^n[$ and all $i \in J$, let us set

$$f_n(t) := f_i^n(t). \tag{1.24}$$

Define $x_n : I \to H$ by

$$x_n(t) = x_i^n + \frac{\vartheta(t) - \vartheta(t_i^n)}{\vartheta(t_{i+1}^n) - \vartheta(t_i^n)}\left(x_{i+1}^n - x_i^n - \int_{t_i^n}^{t_{i+1}^n} f_i^n(s)ds\right) + \int_{t_i^n}^{t} f_i^n(s)ds \tag{1.25}$$

whenever $t \in [t_i^n, t_{i+1}^n]$ and $i \in J$, where

$$\vartheta(t) := \int_{T_0}^{t}\Big(r(x_0)\alpha(s) + \dot{v}(s)\Big)ds \quad \forall t \in I. \tag{1.26}$$

It follows from (1.21), (1.22) and (1.26), that

$$\left\|x_{i+1}^n - x_i^n - \int_{t_i^n}^{t_{i+1}^n} f_i^n(s)ds\right\| \leq \vartheta(t_{i+1}^n) - \vartheta(t_i^n). \tag{1.27}$$

For almost all $t \in [t_i^n, t_{i+1}^n]$, $i \in J$, we have

$$\dot{x}_n(t) = \frac{\dot{\vartheta}(t)}{\vartheta(t_{i+1}^n) - \vartheta(t_i^n)}\Big(x_{i+1}^n - x_i^n - \int_{t_i^n}^{t_{i+1}^n} f_i^n(s)ds\Big) + f_i^n(t) \tag{1.28}$$

and

$$\dot{\vartheta}(t) = r(x_0)\alpha(t) + \dot{v}(t). \tag{1.29}$$

We deduce from (1.24) and (1.27) that

$$\|\dot{x}_n(t) - f_n(t)\| \leq r(x_0)\alpha(t) + \dot{v}(t). \tag{1.30}$$

Claim: The mapping x_n is absolutely continuous over I.
For all $\tau, t \in [t_i^n, t_{i+1}^n]$, and $\tau < t$, we have

$$x_n(t) - x_n(\tau) = \frac{\vartheta(t) - \vartheta(\tau)}{\vartheta(t_{i+1}^n) - \vartheta(t_i^n)}\Big(x_{i+1}^n - x_i^n - \int_{t_i^n}^{t_{i+1}^n} f_i^n(s)ds\Big) + \int_\tau^t f_i^n(s)ds,$$

hence

$$\begin{aligned} \|x_n(t) &- x_n(\tau)\| \\ &\leq \frac{\vartheta(t) - \vartheta(\tau)}{\vartheta(t_{i+1}^n) - \vartheta(t_i^n)}\Big\|x_{i+1}^n - x_i^n - \int_{t_i^n}^{t_{i+1}^n} f_i^n(s)ds\Big\| + \int_\tau^t \|f_i^n(s)\|ds. \end{aligned}$$

We deduce from (1.23) and (1.27) that

$$\|x_n(t) - x_n(\tau)\| \leq \vartheta(t) - \vartheta(\tau) + r(x_0)\int_\tau^t \alpha(s)ds,$$

by (1.26), we get

$$\|x_n(t) - x_n(\tau)\| \leq \int_\tau^t \Big(2r(x_0)\alpha(s) + \dot{v}(s)\Big)ds. \tag{1.31}$$

This last inequality above holds for all $\tau, t \in [t_i^n, t_{i+1}^n]$ with $\tau < t$, hence the mapping x_n is absolutely continuous.

Thanks to (1.15), (1.19), (1.24), (1.25) and we obtain, by construction

$$f_n(t) \in \Gamma\Big(t, x_n\big(\hat{\theta}_n(t)\big)\Big) \quad \forall t \in I. \tag{1.32}$$

Further, by (1.23) and (1.24)

$$\|f_n(t)\| \leq r(x_0)\alpha(t) \quad \forall t \in I. \tag{1.33}$$

it follows through (1.30) that

$$\|\dot{x}_n(t)\| \leq 2r(x_0)\alpha(t) + \dot{v}(t). \tag{1.34}$$

According to (1.7) and (1.20), we have

$$x_i^n + \int_{t_i^n}^{t_{i+1}^n} f_i^n(s)ds - x_{i+1}^n \in N_{C(t_{i+1}^n)}\big(x_{i+1}^n\big).$$

This combined with (1.14), (1.24), (1.25) (1.28) and we obtain, by construction, for almost all $t \in I$ and for any n,

$$\dot{x}_n(t) - f_n(t) \in -N_{C\big(\theta_n(t)\big)}\Big(x_n\big(\theta_n(t)\big)\Big). \tag{1.35}$$

Step 2. Now, we proceed to prove that the sequence (x_n) admits a subsequence, which converges uniformly to a solution of $(\mathcal{E})$.

Observe first by $(\mathcal{H}_1)$ and (1.31) that for any $t \in [t_i^n, t_{i+1}^n]$

$$\begin{aligned} d\big(x_n(t), C(t)\big) &\leq \big\|x_n(t) - x_n(t_i^n)\big\| + d\big(x_n(t_i^n), C(t)\big) \\ &\leq \big\|x_n(t) - x_n(t_i^n)\big\| + d\big(x_n(t_i^n), C(t_i^n)\big) + v(t) - v(t_i^n) \\ &= \int_{t_i^n}^{t} \big(2r(x_0)\alpha(s) + \dot{v}(s)\big)ds + v(t) - v(t_i^n). \end{aligned}$$

Fix any $t \in I$. From the latter inequality and (1.15) it ensures that

$$d\big(x_n(t), C(t)\big) \leq 2\int_{\hat{\theta}_n(t)}^{t} \big(r(x_0)\alpha(s) + \dot{v}(s)\big)ds,$$

so, according to (1.16)

$$d\big(x_n(t), C(t)\big) \underset{n\to\infty}{\longrightarrow} 0.$$

This allows us to write

$$x_n(t) = c_n(t) + e_n(t) \text{ with } c_n(t) \in C(t) \text{ and } e_n(t) \to 0,$$

choose a real $\rho_1 > 0$ such that $\|e_n(t)\| \leq \rho_1$ for all n. By (1.31), we have also for all n

$$\|x_n(t)\| \leq \|x_0\| + \int_{T_0}^{t} \big(2r(x_0)\alpha(s) + \dot{v}(s)\big)ds =: \rho_2.$$

For $r := \rho_1 + \rho_2$, we obtain

$$c_n(t) \in C(t) \cap r\mathbb{B}$$

hence the set $\{c_n(t) : n \in \mathbb{N}\}$ is relatively compact in $(H, \|\cdot\|)$ thanks to the ball-compactness of $C(t)$. Using this and the convergence $e_n(t) \to 0$, we immediately see that the set $\{x_n(t) : n \in \mathbb{N}\}$ is relatively compact in $(H, \|\cdot\|)$. On the other hand, we observe that

$$\int_S \big(2r(x_0)\alpha(s) + |\dot{v}(s)|\big) ds \to 0 \quad \text{as} \quad \lambda(S) \to 0,$$

where λ denotes the Lebesgue measure. This is equivalent to saying that for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_S \big(2r(x_0)\alpha(s) + |\dot{v}(s)|\big) ds < \varepsilon$$

whenever $\lambda(S) < \delta$. It is then obvious to see through the latter inequality and through (1.31) that the sequence (x_n) is equi-continuous. Then it follows from Arzela-Ascoli's theorem that the sequence (x_n) admits a subsequence converging uniformly to some mapping $x \in \mathcal{C}_H(I)$.

Thanks to (1.34) and (1.33) the sequence $(\dot{x}_n)_n$ and $(f_n)_n$ are bounded by a function in $\mathrm{L}^1_{\mathbb{R}_+}(I)$. By extracting subsequences we may suppose that $f_n(\cdot) \to f(\cdot)$ and $\dot{x}_n(\cdot) \to u(\cdot)$, both convergences being obtained weakly in $\mathrm{L}^1_H(I)$, for some $f(\cdot)$ and some $u(\cdot)$ in $\mathrm{L}^1_H(I)$. Thus, for any $t \in I$,

$$x_n(t) = x_0 + \int_{T_0}^t \dot{x}_n(s) ds = x_0 + \int_{T_0}^T \dot{x}_n(s) \mathbb{1}_{[0,t]}(s) ds.$$

Since the sequence $(x_n(t))$ converges in H to $x(t)$, we may pass to the limit to obtain

$$x(t) = x_0 + \int_{T_0}^T u(s) \mathbb{1}_{[0,t]}(s) ds = x_0 + \int_{T_0}^t u(s) ds.$$

Consequently x is absolutely continuous with $\dot{x}(t) = u(t)$ for almost all $t \in I$ and hence

$$\dot{x}_n(\cdot) \to \dot{x}(\cdot) \text{ weakly in } \mathrm{L}^1_H(I). \tag{1.36}$$

Thanks to (1.16) and the uniform convergence of $x_n(\cdot)$ to $x(\cdot)$, we get $x_n(\theta_n(t))$ converges to $x(t)$ for each $t \in I$. Note also that, due to the fact that $d\big(x_n(t), C(t)\big)$ converges to 0 on I, we have $x(t) \in C(t)$ for all $t \in I$.

Step 3. Now, it remains to prove that $x(\cdot)$ is a solution of $(\mathcal{E})$.
Due to the fact that $(f_n(\cdot))$ and $(\dot{x}_n(\cdot))$ converge both weakly in $\mathrm{L}^1_H(I)$ to $f(\cdot)$ and $\dot{x}(\cdot)$ respectively, according to Mazur's lemma, there is a sequence $(z_n(\cdot), \phi_n(\cdot))_n$ which converges strongly in $\mathrm{L}^1_H(I)$ to $\big(\dot{x}(\cdot) - f(\cdot), f(\cdot)\big)$ with

$$z_n \in \mathrm{co}\,\{\dot{x}_k - f_k : k \geq n\} \text{ and } \phi_n \in \mathrm{co}\,\{f_k : k \geq n\},$$

for each $n \geq 1$. Extract a subsequence (that we dot not relabel) $(z_n(\cdot), \phi_n(\cdot))_n$ converging to $(\dot{x}(\cdot) - f(\cdot), f(\cdot))$ a.e.. This yields some fixed Lebesgue negligible set $N \subset I$ such that for each $t \in I \backslash N$ we have $(z_n(t), \phi_n(t))_n$ converges to $(\dot{x}(t) - f(t), f(t))$ and thus,

$$\dot{x}(t) - f(t) \in \bigcap_n \overline{\mathrm{co}}\,\big\{\dot{x}_k(t) - f_k(t) : k \geq n\big\} \tag{1.37}$$

$$f(t) \in \bigcap_n \overline{\mathrm{co}}\,\big\{f_k(t) : k \geq n\big\}. \tag{1.38}$$

Fix $t \in I \backslash N$ and for any $n \in \mathbb{N}$, using (1.30), (1.35) and putting $\eta := r(x_0)\alpha(t) + |\dot{v}(t)|$, we get by (1.6) and Lemma 1.2.1

$$\dot{x}_n(t) - f_n(t) \in -N^{\eta}_{C\left(\theta_n(t)\right)}\Big(x_n\big(\theta_n(t)\big)\Big) = -\eta \partial d_{C\left(\theta_n(t)\right)}\Big(x_n\big(\theta_n(t)\big)\Big).$$

Hence, by (1.37) and for all $\xi \in H$ we have

$$\langle \xi, \dot{x}(t) - f(t)\rangle \leq \sup_{k \geq n} \langle \xi, \dot{x}_k(t) - f_k(t)\rangle \leq \sup_{k \geq n} \sigma\bigg(\xi, -\eta \partial d_{C\left(\theta_k(t)\right)}\Big(x_k\big(\theta_k(t)\big)\Big)\bigg)$$

$$\langle \xi, \dot{x}(t) - f(t)\rangle \leq \lim_{n \to \infty} \sup_{k \geq n} \sigma\bigg(\xi, -\eta \partial d_{C\left(\theta_k(t)\right)}\Big(x_k\big(\theta_k(t)\big)\Big)\bigg)$$

thus

$$\langle \xi, \dot{x}(t) - f(t)\rangle \leq \limsup_{n \to \infty} \sigma\bigg(\xi, -\eta \partial d_{C\left(\theta_n(t)\right)}\Big(x_n\big(\theta_n(t)\big)\Big)\bigg)$$

or equivalently

$$\langle -\xi, -\dot{x}(t) + f(t)\rangle \leq \limsup_{n \to \infty} \sigma\bigg(-\xi, \eta \partial d_{C\left(\theta_n(t)\right)}\Big(x_n\big(\theta_n(t)\big)\Big)\bigg).$$

Since $x_n\big(\theta_n(t)\big) \in C\big(\theta_n(t)\big)$ along with $\theta_n(t) \to t$ and $x_n\big(\theta_n(t)\big) \to x(t) \in C(t)$ as $n \to \infty$, the latter inequality entails by Proposition 1.2.1 that

$$\langle -\xi, -\dot{x}(t) + f(t)\rangle \leq \sigma\Big(-\xi, \eta \partial d_{C(t)}\big(x(t)\big)\Big)$$

or equivalently

$$\langle \xi, \dot{x}(t) - f(t) \rangle \leq \sigma\Big(\xi, -\eta\partial d_{C(t)}\big(x(t)\big)\Big).$$

Since $\partial d_{C(t)}(x(t))$ is convex and closed for each $t \in I \setminus N$, we deduce that

$$\dot{x}(t) - f(t) \in -\eta\partial d_{C(t)}(x(t)) \subset -N_{C(t)}(x(t)). \tag{1.39}$$

It is not difficult to see that $f(t) \in \Gamma(t, x(t))$. Indeed, it result from (1.38) and (1.32) that for all $\xi \in H$

$$\langle \xi, f(t) \rangle \leq \sup_{k \geq n} \langle \xi, f_k(t) \rangle \leq \sup_{k \geq n} \sigma\left(\xi, \Gamma\Big(t, x_k\big(\hat{\theta}_k(t)\big)\Big)\right),$$

thus

$$\langle \xi, f(t) \rangle \leq \limsup_{n \to \infty} \sigma\left(\xi, \Gamma\Big(t, x_n\big(\hat{\theta}_n(t)\big)\Big)\right).$$

By the convergence of $x_n(\hat{\theta}_n(t))$ to $x(t)$ and the upper semicontinuity of $u \mapsto \sigma\big(\xi, \Gamma(t, u)\big)$, we have, for all $t \in I \setminus N$, for any $\xi \in H$,

$$\langle \xi, f(t) \rangle \leq \sigma\Big(\xi, \Gamma\big(t, x(t)\big)\Big).$$

As $\Gamma(t, x(t))$ is closed and convex, we conclude that, for all $t \in I \setminus N$,

$$f(t) \in \Gamma(t, x(t)).$$

This, along with (1.39), implies for all $t \in I \setminus N$

$$\dot{x}(t) \in -N_{C(t)}\big(x(t)\big) + \Gamma\big(t, x(t)\big),$$

and hence x is a solution of the constrained differential inclusion $(\mathcal{E})$.

II. In this second part II, we consider the case where

$$\int_{T_0}^{T} \alpha(s)ds > \frac{1}{4}.$$

Taking $\varepsilon = \frac{1}{4}$ there exists $\delta > 0$ such that for any Lebesgue measurable subset $S \subset [T_0, T]$ with $\lambda(S) < \delta$ we have $\int_S \alpha(s)ds \leq \frac{1}{4}$. Choose some integer $N \geq 1$ such that $\frac{T-T_0}{N} \leq \delta$ and consider a subdivision of I given by $T_0 < T_1 < \cdots < T_N = T$ with $T_i = T_0 + \frac{i(T-T_0)}{N}$ where $0 \leq i \leq N$. Of course, for any $0 \leq i \leq N-1$, we have

$$\int_{T_i}^{T_{i+1}} \alpha(s)ds \leq \frac{1}{4}.$$

Thanks to the part I, there are an absolutely continuous mapping $x_1 : [T_0, T_1] \to H$ and an integrable mapping $f_1 : [T_0, T_1] \to H$ such that $x_1(t) \in C(t)$ for all $t \in [T_0, T_1]$, $f_1(t) \in \Gamma(t, x_1(t))$ for almost all $t \in [T_0, T_1]$ and

$$\begin{cases} \dot{x}_1(t) \in -N_{C(t)}\big(x_1(t)\big) + f_1(t) \quad \text{a.e. } t \in [T_0, T_1] \\ \\ x_1(T_0) = x_0. \end{cases}$$

Likewise, according to the part I, there exists an absolutely continuous mapping $x_2 : [T_1, T_2] \to H$ and a integrable mapping $f_2 : [T_1, T_2] \to H$ such that $x_2(t) \in C(t)$ for all $t \in [T_1, T_2]$, $f_2(t) \in \Gamma(t, x_2(t))$ for almost all $t \in [T_1, T_2]$ and

$$\begin{cases} \dot{x}_2(t) \in -N_{C(t)}\big(x_2(t)\big) + f_1(t) \quad \text{a.e. } t \in [T_1, T_2] \\ \\ x_2(T_1) = x_1(T_1). \end{cases}$$

Inductively, there exists a finite sequence of absolutely continuous mappings $x_i : [T_{i-1}, T_i] \to H$ and a finite sequence of integrable mappings $f_i : [T_{i-1}, T_i] \to H$ with $1 \le i \le N$, such that $x_i(t) \in C(t)$ for all $t \in [T_{i-1}, T_i]$, $f_i(t) \in \Gamma(t, x_i(t))$ for almost all $t \in [T_{i-1}, T_i]$ and

$$\begin{cases} \dot{x}_i(t) \in -N_{C(t)}\big(x_i(t)\big) + f_i(t) \quad \text{a.e. } t \in [T_{i-1}, T_i] \\ \\ x_i(T_{i-1}) = x_{i-1}(T_{i-1}). \end{cases} \tag{1.40}$$

Now, let x, f be the mappings from I into H, defined by

$$x(t) := x_i(t) \quad \text{for all } t \in [T_{i-1}, T_i],$$

and

$$f(t) := f_i(t) \quad \text{for all } t \in]T_{i-1}, T_i]$$

where $1 \le i \le N$. Obviously, x is an absolutely continuous mapping such that $x(t) \in C(t)$ for all $t \in I$ and f is integrable over I with $f(t) \in \Gamma(t, x(t))$ for almost all $t \in I$. Therefore, by (1.40) we obtain

$$\dot{x}(t) \in -N_{C(t)}\big(x(t)\big) + f(t) \quad \text{a.e. } t \in I$$

hence

$$\begin{cases} \dot{x}(t) \in -N_{C(t)}\big(x(t)\big) + \Gamma\big(t, x(t)\big) \quad \text{a.e. } t \in I \\ \\ x(T_0) = x_1(T_0) = x_0. \end{cases}$$

The proof is then complete. □

Theorem 1.2.2. *Let real numbers T_0 and T be fixed with $0 \leq T_0 < T$. Assume that the hypothesis $(\mathcal{H}_2)$ and the following assertions hold for the interval $I = [T_0, T]$:*

$\mathcal{G}_1$ *For each $t \in I$, $C(t)$ is a nonempty compact subset of H and for an absolutely continuous function $v : I \to \mathbb{R}$ sucht that, for any $y \in H$ and $s, t \in I$*

$$|d\big(y, C(t)\big) - d\big(y, C(s)\big)| \leq |v(t) - v(s)|;$$

$\mathcal{G}_2$ *For any bounded subset S of H, there are α_S and β_S in $\mathrm{L}^1_{\mathbb{R}_+}(I)$ such that*

$$d\big(0, \Gamma(t,x)\big) \leq \alpha_S(t) + \beta_S(t)\|x\| \quad \textit{for all } (t,x) \in I \times S.$$

Then, there exist an absolutely continuous mapping $x : I \to H$ which is a solution on the whole interval I of the constrained differential inclusion $(\mathcal{E})$

Proof. On the one hand, by $(\mathcal{G}_2)$, for any bounded subset S of H, there are some $\alpha_S, \beta_S \in \mathrm{L}^1_{\mathbb{R}_+}(I)$ such that for all $(t,x) \in I \times S$, we have

$$d\big(0, \Gamma(t,x)\big) \leq \alpha_S(t) + \beta_S(t)\|x\|. \tag{1.41}$$

On the other hand, by $(\mathcal{G}_1)$ we have, for each $t \in I$,

$$\begin{aligned} |d\big(y, C(t)\big) - d\big(y, C(T_0)\big)| &\leq |v(t) - v(T_0)| \\ &\leq 2 \max_{s \in I} |v(s)|. \end{aligned}$$

Fixing any $t \in I$, we have, for all $y \in C(t)$,

$$d\big(y, C(T_0)\big) \leq 2 \max_{s \in I} |v(s)|,$$

which clearly implies that

$$\|y\| \leq \max_{y_0 \in C(T_0)} \|y_0\| + 2 \max_{s \in I} |v(s)|,$$

hence,

$$y \in \Big(\max_{y_0 \in C(T_0)} \|y_0\| + 2 \max_{s \in I} |v(s)| \Big) \mathbb{B}.$$

Consequently, for every $t \in I$,

$$C(t) \subset \Big(\max_{y_0 \in C(T_0)} \|y_0\| + 2 \max_{s \in I} |v(s)| \Big) \mathbb{B}.$$

Let a real $r \geq \max_{y_0 \in C(T_0)} \|y_0\| + 2\max_{s\in I} |v(s)|$, and by (1.41) let $\alpha(\cdot)$ and $\beta(\cdot)$ in $\mathrm{L}^1_{\mathbb{R}_+}(I)$ such that for all $(t,x) \in I \times r\mathbb{B}$ $d\big(0, \Gamma(t,x)\big) \leq \alpha_S(t) + \beta_S(t)\|x\|$. Then, for any $t \in I$ and $x \in C([T_0,t])$ we have

$$d\big(0, \Gamma(t,x)\big) \leq \alpha(t) + \beta(t)\|x\|.$$

We can apply Theorem 1.2.1 to obtain a solution of the constrained differential inclusion $(\mathcal{E})$. □

The results below are direct consequences of Theorem 1.2.1 and Theorem 1.2.2 respectively. Let two given set-valued mappings $C : [T_0, +\infty[\rightrightarrows H$ and $\Gamma : [T_0, +\infty[\times H \rightrightarrows H$, the latter being with nonempty closed convex values which is $\mathcal{L}([T_0,+\infty[) \otimes \mathcal{B}(H)$ − measurable and upper semicontinuous with respect to $x \in H$ for almost all $t \in [T_0, +\infty[$. They are required to satisfy the following assumptions:

$(\mathcal{H}_{1\infty})$ For each $t \in [T_0, +\infty[$, $C(t)$ is a nonempty ball-compact subset of H; there exists a nondecreasing locally absolutely continuous function $v : [T_0, +\infty[\to \mathbb{R}_+$ (that is, absolutely continuous on each compact subinterval of $[T_0, +\infty[$) such that, for any $y \in H$ and $s,t \in [T_0, +\infty[$ with $s \leq t$

$$d\big(y, C(t)\big) \leq d\big(y, C(s)\big) + v(t) - v(s);$$

$(\mathcal{H}_{2\infty})$ The family $\big(C(t)\big)_{t\in[T_0,+\infty[}$ is equi-uniformly subsmooth;

$(\mathcal{H}_{3\infty})$ The set-valued mapping Γ satisfies the growth condition

$$d\big(0, \Gamma(t,x)\big) \leq \alpha(t)(1 + \|x\|)$$

for all $t \in [T_0, +\infty[$ and all $x \in C([T_0,t]) := \bigcup\limits_{T_0 \leq s \leq t} C(s)$, where $\alpha : [T_0, +\infty[\to \mathbb{R}_+$ is a locally integrable function on $[T_0, +\infty[$ (that is, integrable on each compact subinterval of $[T_0, +\infty[$).

Corollaire 1.2.1. *Given a real number $T_0 \geq 0$. Assume that $\mathcal{H}_{1\infty}, \mathcal{H}_{2\infty}, \mathcal{H}_{3\infty}$ hold. Then, there exists a locally absolutely continuous mapping $x(\cdot)$ from $[T_0, +\infty[$ into H which is a solution on the whole interval $[T_0, +\infty[$ of the constrained differential inclusion*

$$(\mathcal{E}_\infty) \quad \begin{cases} \dot{x}(t) \in -N_{C(t)}\big(x(t)\big) + \Gamma\big(t, x(t)\big) \quad a.e\ t \in [T_0, +\infty[\\ x(t) \in C(t) \quad \forall t \in [T_0, +\infty[\\ x(T_0) = x_0 \in C(T_0). \end{cases}$$

Proof. We adapt the arguments of the part II in the proof of Theorem 1.2.1.

Put $T_k = T_0 + k$ for all $k \in \mathbb{N}$. From Theorem 1.2.1 there exists an absolutely continuous mapping $x^0 : [T_0, T_1] \to H$ and a integrable mapping $f^0 : [T_0, T_1] \to H$ such that $x^0(t) \in C(t)$ for all $t \in [T_0, T_1]$ and $f^0(t) \in \Gamma(t, x^0(t))$ for almost all $t \in [T_0, T_1]$ and

$$\begin{cases} \dot{x}^0(t) \in -N_{C(t)}\big(x^0(t)\big) + f^0(t) \quad \text{a.e. } t \in [T_0, T_1] \\ \\ x^0(T_0) = x_0. \end{cases}$$

Suppose $x^0, \cdots, x^{k-1}$ have been constructed such that, for $p = 0, \cdots, k-1$, $u^p : [T_p, T_{p+1}] \to H$ is an absolutely continuous, $f^p : [T_p, T_{p+1}] \to H$ is an integrable mapping with $f^p \in \Gamma(t, x^p(t))$ for almost all $t \in [T_p, T_{p+1}]$, $x^p(T_p) = x^{p-1}(T_p)$, $x^p \in C(t)$ for all $t \in [T_p, T_{p+1}]$ and

$$\dot{x}^p(t) \in -N_{C(t)}\big(x^p(t)\big) + f^p(t) \quad \text{a.e. } t \in [T_p, T_{p+1}].$$

Likewise, according to Theorem 1.2.1 again, there are an absolutely continuous mapping $x^k : [T_k, T_{k+1}] \to H$ and an integrable mapping $f^k : [T_k, T_{k+1}] \to H$ such that $x^k(t) \in C(t)$ for all $t \in [T_k, T_{k+1}]$, $f^k(t) \in \Gamma(t, x^k(t))$ for almost all $t \in [T_k, T_{k+1}]$ and

$$(1.42) \qquad \begin{cases} \dot{x}^k(t) \in -N_{C(t)}\big(x^k(t)\big) + f^k(t) \quad \text{a.e. } t \in [T_k, T_{k+1}] \\ \\ x^k(T_k) = x^{k-1}(T_k). \end{cases}$$

So, we obtain by induction x^k for all $k \in \{0\} \cup \mathbb{N}$ with the above properties.

Now, let x and f be two mappings from $[T_0, +\infty[$ into H, defined by

$$x(t) := x^k(t), \; f(t) := f^k(t) \quad \text{for all } t \in [T_k, T_{k+1}[\; \text{ with } k \in \{0\} \cup \mathbb{N}.$$

Obviously, the mapping x is locally absolutely continuous on $[T_0, +\infty[$ such that $x(t) \in C(t)$ for all $t \in [T_0, +\infty[$ and f is locally integrable on $[T_0, +\infty[$ with $f(t) \in \Gamma(t, x(t))$ for almost all $t \in [T_0, +\infty[$. Therefore, from (1.42) we obtain

$$\dot{x}(t) \in -N_{C(t)}\big(x(t)\big) + f(t) \quad \text{a.e. } t \in [T_0, +\infty[$$

thus,

$$\begin{cases} \dot{x}(t) \in -N_{C(t)}\big(x(t)\big) + \Gamma\big(t, x(t)\big) \quad \text{a.e. } t \in [T_0, +\infty[\\ \\ x(t) \in C(t) \; \forall t \in [T_0, +\infty[\\ \\ x(T_0) = x^0(T_0) = x_0. \end{cases}$$

The proof is then complete. $\square$

Corollaire 1.2.2. *Let real number $T_0 \geq 0$ be fixed. Assume that the hypothesis $(\mathcal{H}_{2\infty})$ and the following assertions hold:*

- *For each $t \in [T_0, +\infty[$, $C(t)$ is a nonempty compact subset of H and for a locally absolutely continuous function $v : [T_0, +\infty[\to \mathbb{R}$ sucht that, for any $y \in H$ and $s, t \in I$*

$$|d\big(y, C(t)\big) - d\big(y, C(s)\big)| \leq |v(t) - v(s)|;$$

- *For any bounded subset S of H, there are α_S and $\beta_S : [T_0, +\infty[\to \mathbb{R}_+$, which are locally integrable on $[T_0, +\infty[$ such that*

$$d\big(0, \Gamma(t, x)\big) \leq \alpha_S(t) + \beta_S(t)\|x\| \quad \text{for all } (t, x) \in [T_0, +\infty[\times S.$$

Then, there exist a locally absolutely continuous mapping $x : [T_0, +\infty[\to H$ which is a solution on the whole interval $[T_0, +\infty[$ of the constrained differential inclusion $(\mathcal{E}_\infty)$

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Chapter 2

Delay perturbed sweeping process with subsmooth sets

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Abstract. Recently, D. Aussel, A. Daniilidis and L. Thibault introduced a new class of sets, called subsmooth sets, in variational analysis (see, [1]). Subsmooth sets turn out to be naturally situated between the class of prox-regular sets and the classes of nearly radial sets and of weakly regular sets. The latter classes have been introduced by Lewis in 2002 and by Jourani in 2006, respectively. Motivated by the study of differential inclusions defined by nonconvex and non prox-regular sweeping process, we prove an existence of solutions, even in the presence of a delay, for perturbed differential inclusions governed by subsmooth sets.

Keyword: Subsmooth set; Differential inclusion; Sweeping process; Normal cone; Subdifferential

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Introduction

In this paper, our aim is the study of a nonconvex and non prox-regular perturbed sweeping process with time delay in an infinite dimensional Hilbert space, that is, the differential inclusion of the form

$$
(2.1) \qquad \begin{cases} \dot{x}(t) \in -N_{C(t)}\big(x(t)\big) + \Gamma\big(t, \Lambda(t)x\big) & \text{a.e } t \in [0,T] \\ x(t) \in C(t) \quad \text{for all } t \in [0,T], \\ x(s) = \varphi(s) \quad \forall s \in [-r,0], \end{cases}
$$

where $N_{C(t)}(\cdot)$ denotes a general normal cone to the set $C(t)$. Let us describe the elements and concepts involved in (2.1). Let H be a real Hilbert space, $T > 0$ be a real number, and C be a set-valued mapping from $[0,T]$ into H, with nonempty closed values moving in an absolutely continuous way. Given a finite delay $r \geq 0$, we consider the spaces $\mathcal{C}_0 := \mathcal{C}_H(-r,0)$ and $\mathcal{C}_T := \mathcal{C}_H(-r,T)$ endowed with the norm of the uniform convergence $\|\cdot\|_{\infty,0}$ and $\|\cdot\|_{\infty,T}$ respectively. With any $t \in [0,T]$, we associate the mapping $\Lambda(t)$ from $\mathcal{C}_T$ into $\mathcal{C}_0$ defined, for all $x \in \mathcal{C}_T$, by

$$
(2.2) \qquad \Lambda(t)x(s) := x(t+s) \quad \text{for all } s \in [-r,0].
$$

Let $\Gamma : [0,T] \times \mathcal{C}_0 \rightrightarrows H$ be a set-valued mapping with nonempty convex compact values satisfying the linear growth condition

$$
(2.3) \qquad \Gamma(t,\phi) \subset \alpha(t)(1 + \|\phi\|_{\infty,0})\mathbb{B} \quad \text{for all } (t,\phi) \in [0,T] \times \mathcal{C}_0,
$$

where $\alpha \in \mathrm{L}^1_{\mathbb{R}_+}(T_0,T)$ and $\mathbb{B}$ is the closed unit ball of H, and let φ be a fixed member of $\mathcal{C}_0$ such that $\varphi(0) \in C(0)$. A solution of (2.1) is a mapping $x : [-r,T] \to H$ which is absolutely continuous on $[0,T]$ with $x|_{[-r,0]} = \varphi$ and which satisfies the first inclusion of (2.1) for almost every $t \in [0,T]$ and the second inclusion for all $t \in [0,T]$.

Castaing and Monteiro Marques showed in [3] the existence of a solution of the above differential inclusion (2.1), under some conditions. Among others, Γ in [3] has all its values included in a fixed bounded set and C is Lipschitz and takes on convex compact values. Thibault [17] proved that, in the finite dimensional context, the problem above has always a solution for general subsets $C(t)$ and for Γ satisfying

$$
\Gamma(t,\phi) \subset \alpha(t)\mathbb{B} \quad \text{for all } (t,\phi) \in [0,T] \times \mathcal{C}_0,
$$

provided that $N_{C(t)}(x(t))$ is taken as the Clarke normal cone. Recently, in [4] Castaing, Salvadori and Thibault showed, in finite dimensional, the existence of a solution of (2.1) when the sets $C(t)$ are bounded and r-prox-regular

$(r > 0)$, with Γ satisfies (2.3). On the other hand, in the infinite dimensional setting, Bounkhel and Yarou [2] showed the existence of a solution for this differential inclusion when the set-valued mapping Γ has all its values contained in a fixed bounded set and the sets $C(t)$ are r-prox-regular and norm compact. Later, this problem has been studied by Edmond [9] with the case where $C(t)$ is bounded and r-prox-regular and Γ satisfies (2.3) with $\mathbb{B}$ replaced by a fixed compact set. In [10] and other existence result is proved when the sets $C(t)$ are r-prox-regular in an infinite dimensional Hilbert space and Γ is a single -valued mapping Lipschitz with respect to ϕ and satisfying (2.3).

The present paper provides an existence result for (2.1) in the infinite dimensional Hilbert setting where the sets $C(t)$ are supposed to be equi-subsmooth. The class of such sets is strictly bigger than of prox-regular sets (see [1]). It is also connected with the class of nearly radial sets of Lewis [12] and with the class of weakly regular sets of Jourani [11] (see [8]). The paper is structured as follows. In section 1, we give notation which will be used throughout the paper and we recall some definitions and results, in particular, on the Clarke (respectively, Fréchet) normal cone. In section 2, we prove the main theorem of the paper, that is, existence of solution of the differential inclusion (2.1) under the subsmoothness property of the sets $C(t)$ and under a relaxation of the assumption (2.3).

2.1 Preliminaries

Throughout the paper H is a real separable Hilbert space whose inner product is denoted by $\langle \cdot, \cdot \rangle$ and the associated norm by $\| \cdot \|$. The closed unit ball of H with center 0 will be denoted by $\mathbb{B}$ and $B(u, \eta)$ (respectively, $B[u, \eta]$) denotes the open (respectively, closed) ball of center $u \in H$ and radius $\eta > 0$. Given two reals $r, T > 0$, we will denote by $\mathcal{C}_T := \mathcal{C}_H(-r, T)$ (respectively, $\mathcal{C}_0 := \mathcal{C}_H(-r, 0)$) the space of all continuous mappings from $[-r, T]$ into H (respectively, $[-r, 0]$ into H). The norm of uniform convergence on $\mathcal{C}_T$ (respectively, $\mathcal{C}_0$) will be denoted by $\| \cdot \|_{\infty,T}$ (respectively, $\| \cdot \|_{\infty,0}$), "a.e" denotes "for almost every" and $\dot{x}$ is the derivative of x.

Let C, C' be two subsets of H and let v be a vector in H, the real $d(v, C)$ or $d_C(v) := \inf\{\|v - u\| : u \in C\}$ is the distance of the point v from the set C. We denote by

$$\mathrm{Haus}(C, C') = \max\left\{ \sup_{u \in C} d(u, C'), \sup_{v \in C'} d(v, C) \right\}$$

the Hausdorff distance between C and C'. For $v \in H$ the projection of v into

$C \subset H$ is the set

$$\operatorname{Proj}_C(v) := \{u \in C : d_C(v) = \|v - u\|\}.$$

This set is nonempty when C is ball-compact. Recall that a subset S of $(H, \|\cdot\|)$ is *ball-compact* provided that $S \cap r\mathbb{B}$ is compact in $(H, \|\cdot\|)$ for every real $r > 0$. Obviously any ball-compact set is norm closed, and in finite dimensions S is ball-compact if and only if it is closed. When $h \in \operatorname{Proj}_C(v)$, then we have $v - h \in N_C^p(h)$ where $N_C^p(\cdot)$ denotes the proximal normal cone of C (see, [6]).

For a nonempty interval $\mathcal{J}$ of $\mathbb{R}$, we recall that a set-valued mapping $F : \mathcal{J} \rightrightarrows H$ is called Lebesgue measurable if for each open set $U \subset H$ the set $F^{-1}(U) := \{t \in \mathcal{J} : F(t) \cap U \neq \emptyset\}$ is Lebesgue measurable. When the values of F are closed subsets of H, we know (see [5]) that the Lebesgue measurability of F is equivalent to the measurability of the graph of F, that is,

$$\operatorname{gph} F \in \mathcal{L}(\mathcal{J}) \otimes \mathcal{B}(H),$$

where $\mathcal{L}(\mathcal{J})$ denotes the Lebesgue σ-field of $\mathcal{J}$, $\mathcal{B}(H)$ the Borel σ-field of H, and

$$\operatorname{gph} F := \big\{(t, u) \in \mathcal{J} \times H : u \in F(t)\big\}.$$

For any subset C of H, $\overline{\operatorname{co}}\, C$ stands for the closed convex hull of C, and $\sigma(\cdot, C)$ represents the support function of C, that is, for all $\xi \in H$,

$$\sigma(\xi, C) := \sup_{u \in C} \langle \xi, u \rangle.$$

If C is a nonempty subset of H, the Clarke normal cone $N(C; u)$ or $N_C(u)$ of C at $u \in C$ is defined by

$$N_C(u) = \{\xi \in H : \langle \xi, v \rangle \leq 0, \forall v \in T_C(u)\},$$

where the Clarke tangent cone $T(C; u)$ or $T_C(u)$ (see [7]) is defined as follows:

$$v \in T_C(u) \Leftrightarrow \begin{cases} \forall \varepsilon > 0, \exists \delta > 0 \text{ such that} \\ \\ \forall u' \in B(u, \delta) \cap C, \forall t \in]0, \delta[, (u' + tB(v, \varepsilon)) \cap C \neq \emptyset. \end{cases}$$

Equivalently, $v \in T_C(u)$ if and only if for any sequence $(u_n)_n$ of C converging to u and any sequence of positive reals $(t_n)_n$ converging to 0, there exists a sequence $(v_n)_n$ in H converging to v such that

$$u_n + t_n v_n \in C \text{ for all } n \in \mathbb{N}.$$

We put $N_C(u) = \emptyset$, whenever $u \notin C$. For any $\eta > 0$ we denote by $N_C^\eta(u)$ the truncated Clarke normal cone, that is,

$$N_C^\eta(u) = N_C(u) \cap \eta\mathbb{B}.$$

We typically denote by $f : H \to \mathbb{R} \cup \{+\infty\}$ a proper function (that is, f is finite at least at one point). The Clarke subdifferential $\partial f(u)$ of f at a point u (where f is finite) is defined by

$$\partial f(u) = \Big\{\xi \in H : (\xi, -1) \in N_{\operatorname{epi} f}\Big(\big(u, f(u)\big)\Big)\Big\},$$

where $\operatorname{epi} f$ denotes the epigraph of f, that is,

$$\operatorname{epi} f = \{(u, r) \in H \times \mathbb{R} : f(u) \leq r\}.$$

We also put $\partial f(u) = \emptyset$ if f is not finite at $u \in H$. If ψ_C denotes the indicator function of the set C, that is, $\psi_C(u) = 0$ if $u \in C$ and $\psi_C(u) = +\infty$ otherwise, then

$$\partial \psi_C(u) = N_C(u) \text{ for all } u \in H.$$

The Clarke subdifferential $\partial f(u)$ of a locally Lipschitz function f at u has also the other useful description

$$\partial f(u) = \{\xi \in H : \langle \xi, v\rangle \leq f^0(u, v), \forall v \in H\},$$

where

$$f^0(u, v) := \limsup_{(u',t)\to(u,0^+)} \frac{f(u' + tv) - f(u')}{t}.$$

The above function $f^0(u; \cdot)$ is called the Clarke directional derivative of f at u. The Clarke normal cone is known ([7]) to be related to the Clarke subdifferential of the distance function through the equality

$$N_C(u) = \mathrm{cl}_w(\mathbb{R}_+ \partial d_C(u)) \text{ for all } u \in C,$$

where $\mathbb{R}_+ := [0, \infty[$ and cl_w denotes the closure with respect to the weak topology of H. Further

$$\partial d_C(u) \subset N_C(u) \cap \mathbb{B} \text{ for all } u \in C.$$

The concept of Fréchet subdifferential will be also needed. A vector $\xi \in H$ is said to be in the Fréchet subdifferential $\partial_F f(u)$ of f at u (see [14, 16]) provided that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $u' \in B(u, \delta)$ we have

$$\langle \xi, u' - u\rangle \leq f(u') - f(u) + \varepsilon\|u' - u\|.$$

It is known that we always have the inclusion

$$\partial_F f(u) \subset \partial f(u).$$

The Fréchet normal cone of C at $u \in C$ is given by

$$N_C^F(u) = \partial_F \psi_C(u),$$

so the following inclusion always holds true

$$N_C^F(u) \subset N_C(u) \quad \text{for all } u \in C.$$

On the other hand, the Fréchet normal cone is also related to the Fréchet subdifferential of the distance function since the following relations hold true for all $u \in C$

$$N_C^F(u) = \mathbb{R}_+ \partial_F d_C(u)$$

and

$$\partial_F d_C(u) = N_C^F(u) \cap \mathbb{B}. \tag{2.4}$$

Another important property is

$$v - u \in N_C^F(u) \quad \text{hence also} \quad v - u \in N_C(u) \tag{2.5}$$

whenever $u \in \operatorname{Proj}_C(v)$, since $N_C^p(u) \subset N_C^F(u)$.

2.2 Subsmoothness and variational inequality

This section is devoted to the study of a perturbed sweeping process whose perturbation is a set-valued mapping involving a delay. We first recall the definition of subsmooth sets in [1]. In this way we define the equi-uniformly subsmooth property for a family of closed sets of H.

Definition 2.2.1. *A closed set $C \subset H$ is called subsmooth at $u_0 \in C$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $u_1, u_2 \in B(u_0, \delta) \cap C$ and all $\xi_i \in N_C(u_i) \cap \mathbb{B}$, $i = 1, 2$ we have*

$$\langle \xi_1 - \xi_2, u_1 - u_2 \rangle \geq -\varepsilon \|u_1 - u_2\|. \tag{2.6}$$

The set C is called subsmooth, if it is subsmooth at every $u_0 \in C$.

We further say that C is uniformly subsmooth, if for every $\varepsilon > 0$ there exists $\delta > 0$, such that (2.6) holds for all $u_1, u_2 \in C$ satisfying $\|u_1 - u_2\| < \delta$ and all $\xi_i \in N_C(u_i) \cap \mathbb{B}$.

The class of subsmooth sets strictly contains that of prox-regular sets introduced in [16] and is connected with the class of nearly radial sets of [12] and with the class of weakly regular sets of [11], see [8].

Definition 2.2.2. *Let E be a nonempty set. We say that a family $(C(t))_{t\in E}$ of closed sets of H is equi-uniformly subsmooth, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that (2.6) holds for any $t \in E$ and all $u_1, u_2 \in C(t)$ satisfying $\|u_1 - u_2\| < \delta$ and all $\xi_i \in N_{C(t)}(u_i) \cap \mathbb{B}$.*

Given two set-valued mappings $C : [0,T] \rightrightarrows H$ and $\Gamma : [0,T] \times \mathcal{C}_0 \rightrightarrows H$, we shall deal with the differential inclusion defined as follows:

$$(\mathcal{E}_r) \quad \begin{cases} \dot{x}(t) \in -N_{C(t)}(x(t)) + \Gamma(t, \Lambda(t)x) \quad \text{a.e } t \in [0,T]; \\ \\ x(t) \in C(t) \quad \forall t \in \ [0,T]; \\ \\ x(\cdot) = \varphi(\cdot) \quad \text{in } [-r, 0], \end{cases}$$

where $\Lambda(t)$ is the mapping from $\mathcal{C}_T$ into $\mathcal{C}_0$ defined, for all $x \in \mathcal{C}_T$, by $\Lambda(t)x(s) := x(t+s)$ for all $s \in [-r, 0]$ and φ is a member of $\mathcal{C}_0$ such that $\varphi(0) \in C(0)$.

We are going to investigate the existence of solutions for the above differential inclusion. We call solution of $(\mathcal{E}_r)$ any mapping $x : [-r, T] \to H$ such that

1. for any $s \in [-r, 0]$, we have $x(s) = \varphi(s)$;
2. $x(t) \in C(t)$ for all $t \in [0,T]$;
3. the restriction $x|_{[0,T]}$ of x is absolutely continuous and its derivative satisfies the inclusion

$$\dot{x}(t) \in -N_{C(t)}\big(x(t)\big) + \Gamma\big(t, \Lambda(t)x\big) \quad \text{a.e } t \in [0,T].$$

The hypotheses concerning the set $C(t)$ and the set-valued mapping Γ with which we shall work are the following:

$(\mathcal{H}_1)$ For each $t \in [0,T]$, $C(t)$ is a nonempty ball-compact subset of H; the set $C(t)$ moves in an absolutely continuous way, that is, there exists a nondecreasing absolutely continuous function $v(\cdot) : [0,T] \to \mathbb{R}_+$ such that, for any $y \in H$ and $s, t \in [0,T]$

$$|d\big(y, C(t)\big) - d\big(y, C(s)\big)| \le |v(t) - v(s)|;$$

($\mathcal{H}_2$) The family $(C(t))_{t\in[0,T]}$ is equi-uniformly subsmooth;

($\mathcal{H}_3$) The set-valued mapping Γ, with nonempty convex closed values, is $\mathcal{L}([0,T])\otimes\mathcal{B}(\mathcal{C}_0)-$ measurable and upper semicontinuous with respect to $\phi\in\mathcal{C}_0$ for almost all $t\in[0,T]$ and, for some integrable nonnegative function $\alpha(\cdot)$ over $[0,T]$ such that.

$$d\big(0,\Gamma(t,\phi)\big)\leq\alpha(t)(1+\|\phi\|_{\infty,0})\quad\text{for all } t\in[0,T] \text{ and all } \phi\in\mathcal{C}_0.$$

Theorem 2.2.1. *Assume that* $(\mathcal{H}_1),(\mathcal{H}_2)$ *and* $(\mathcal{H}_3)$ *hold. Then, for any* φ *in* $\mathcal{C}_0$ *with* $\varphi(0)\in C(0)$*, the differential inclusion* $(\mathcal{E}_r)$ *has a solution.*

The following results will be used in the proof of Theorem 2.2.1.

Lemma 2.2.1. *[13] If a closed set* C *of* H *is subsmooth at* $u_0\in C$*, then*

$$\partial d_C(u_0)=\partial_F d_C(u_0)$$

and

$$N_C(u_0)=N_C^F(u_0).$$

Lemma 2.2.2. *[13] Let* E *be a metric space and let* $(C(t))_{t\in E}$ *be a family of nonempty closed sets of* H *which is equi-uniformly subsmooth and let a real* $\eta>0$*. Let* $Q\subset E$ *and* $s_0\in\mathrm{cl}Q$*. Then the following hold:*

(a) For all $(s,u)\in\mathrm{gph}\,C$ *we have* $\eta\partial d_{C(s)}(u)\subset\eta\mathbb{B}$*;*

(b) For any net $(s_j)_{j\in J}$ *in* Q *converging to* s_0*, any net* $(u_j)_{j\in J}$ *converging to* $u\in C(s_0)$ *in* $(H,\|\cdot\|)$ *with* $u_j\in C(s_j)$ *and* $d_{C(s_j)}(y)\underset{j\in J}{\to}0$ *for every* $y\in C(s_0)$ *, and any net* $(\zeta_j)_{j\in J}$ *converging weakly to* ζ *in* $(H,w(H,H))$ *with* $\zeta_j\in\eta\partial d_{C(s_j)}(u_j)$*, we have* $\zeta\in\eta\partial d_{C(s_0)}(u)$*.*

From Lemma 2.2.2 we easily deduce, thanks to properties of upper semicontinuous set-valued mappings (see [5]), the following proposition.

Proposition 2.2.1. *[13] Let* I *be a nonempty interval of* $\mathbb{R}$ *and let* $(C(t))_{t\in I}$ *be a family of nonempty closed sets of* H *which is equi-uniformly subsmooth and let a real* $\eta>0$*. Assume that there exists a continuous function* $v:I\to\mathbb{R}_+$ *such that, for any* $y\in H$ *and* $s,t\in I$ *with* $s\leq t$*,*

$$d\big(y,C(t)\big)\leq d\big(y,C(s)\big)+v(t)-v(s).$$

Then the following assertions hold:

(a) For all $(s,u)\in\mathrm{gph}\,C$ *we have* $\eta\partial d_{C(s)}(u)\subset\eta\mathbb{B}$*;*

(b) For any sequence $(s_n)_n$ in I converging to s with $s_n \geq s$, any sequence $(u_n)_n$ converging to $u \in C(s)$ with $u_n \in C(s_n)$, and any $\xi \in H$, we have

$$\limsup_{n\to\infty} \sigma\big(\xi, \eta\partial d_{C(s_n)}(u_n)\big) \leq \sigma\big(\xi, \eta\partial d_{C(s)}(u)\big).$$

Proof of Theorem 2.2.1.

I. We suppose that

$$\int_0^T \alpha(s)ds \, < \, \frac{1}{4}. \tag{2.7}$$

We are going to construct a sequence of mappings (x_n) in $\mathcal{C}_T$ which admits a subsequence which converges uniformly on $[-r, T]$ to a solution of $(\mathcal{E}_r)$.

Step 1. Construction of the sequence (x_n).

For any $t \in [0, T]$, consider the single-valued mapping $\hat{\Lambda}(t) : \mathcal{C}_t \to \mathcal{C}_0$ defined, for all $\xi \in \mathcal{C}_t := \mathcal{C}_H(-r, t)$ by

$$\hat{\Lambda}(t)\xi(s) := \xi(t+s) \quad \forall s \in [-r, 0].$$

Observe first by $(\mathcal{H}_3)$ that there is some $\alpha \in \mathrm{L}^1_{\mathbb{R}_+}(T_0, T)$ such that for all $(t, \phi) \in [0, T] \times \mathcal{C}_0$, we have

$$d\big(0, \Gamma(t, \phi)\big) \leq \alpha(t)\big(1 + \|\phi\|_{\infty,0}\big). \tag{2.8}$$

We will introduce a discretization, inspired by the one used in [3]. We define the mapping $u_0^n : [-r, 0] \to H$ by

$$u_0^n(s) = \varphi(s) \quad \forall\, s \in [-r, 0].$$

For any integer $n \geq 1$, consider the partition of $[0, T]$ defined by the points $t_j^n = j\frac{T}{n}$ $(j = 0, \cdots, n)$. For $t \in [0, T]$ and $J := \{1, \cdots, n\}$, we define the mappings

$$\theta_n(t) = \begin{cases} t_{j-1}^n & \text{if} \quad t \in]t_{j-1}^n, t_j^n], j \in J, \\ 0 & \text{if} \quad t = 0, \end{cases} \tag{2.9}$$

$$\hat{\theta}_n(t) = \begin{cases} t_j^n & \text{if} \quad t \in [t_{j-1}^n, t_j^n[, j \in J, \\ T & \text{if} \quad t = T. \end{cases} \tag{2.10}$$

Observe that for each $t \in [0,T]$, choosing j such that $t \in [t_{j-1}^n, t_j^n[$ if $t < T$ and $j = n$ if $t = T$, we have

$$|\hat{\theta}_n(t) - t| \leq |t_j^n - t_{j-1}^n| = \frac{T}{n}$$

and similarly, we have $|\theta_n(t) - t| \leq \frac{T}{n}$, then

$$\theta_n(t) \to t, \quad \hat{\theta}_n(t) \to t. \tag{2.11}$$

Put $u_0^n(t_0^n) = \varphi(t_0^n) =: p_0^n \in C(t_0^n)$. Let f_1^n be the mapping from $[t_0^n, t_1^n]$ into H given by $f_1^n(t)$ as the element of minimal norm of $\Gamma(t, \hat{\Lambda}(t_0^n)u_0^n)$, that is,

$$f_1^n(t) = \operatorname{Proj}_{\Gamma\left(t, \hat{\Lambda}(t_0^n)u_0^n\right)}(0) \quad \text{for all } t \in [t_0^n, t_1^n].$$

The mapping f_1^n is measurable according to the measurability of the set-valued mapping $\Gamma(\cdot, \hat{\Lambda}(t_0^n)u_0^n)$. Thanks to (2.8) we get

$$\|f_1^n(t)\| \leq (1 + \|\hat{\Lambda}(t_0^n)u_0^n\|_{\infty,0})\alpha(t) \quad \text{for all } t \in [t_0^n, t_1^n].$$

Since $\|\hat{\Lambda}(t_0^n)u_0^n\|_{\infty,0} = \|\varphi\|_{\infty,0}$, we obtain

$$\|f_1^n(t)\| \leq (1 + \|\varphi\|_{\infty,0})\alpha(t) \quad \text{for all } t \in [t_0^n, t_1^n]. \tag{2.12}$$

So, f_1^n is bounded by a function in $\mathrm{L}^1_{\mathbb{R}_+}(0,T)$, hence $f_1^n \in \mathrm{L}^1_H(t_0^n, t_1^n)$.

The ball-compactness of $C(t)$ ensures that

$$\operatorname{Proj}_{C(t_1^n)}\Big(p_0^n + \int_{t_0^n}^{t_1^n} f_1^n(s)ds\Big) \neq \emptyset.$$

Then, we can choose a point p_1^n in $\operatorname{Proj}_{C(t_1^n)}\big(p_0^n + \int_{t_0^n}^{t_1^n} f_1^n(s)ds\big)$, hence $p_1^n \in C(t_1^n)$ and

$$\Big\|p_1^n - \Big(p_0^n + \int_{t_0^n}^{t_1^n} f_1^n(s)ds\Big)\Big\| = d\Big(p_0^n + \int_{t_0^n}^{t_1^n} f_1^n(s)ds, C(t_1^n)\Big).$$

So, according to $(\mathcal{H}_1)$ and the inclusion $p_0^n \in C(t_0^n)$, we have

$$\begin{aligned}
\Big\|p_1^n - \Big(p_0^n &+ \int_{t_0^n}^{t_1^n} f_1^n(s)ds\Big)\Big\| \\
&\le d\Big(p_0^n + \int_{t_0^n}^{t_1^n} f_1^n(s)ds, C(t_0^n)\Big) + v(t_1^n) - v(t_0^n) \\
&\le d\big(p_0^n, C(t_0^n)\big) + \Big\|\int_{t_0^n}^{t_1^n} f_1^n(s)ds\Big\| + v(t_1^n) - v(t_0^n) \\
&\le \int_{t_0^n}^{t_1^n} \|f_1^n(s)\|ds + \int_{t_0^n}^{t_1^n} \dot{v}(s)ds.
\end{aligned}$$

By (2.12), it follows that

$$\Big\|p_1^n - \Big(p_0^n + \int_{t_0^n}^{t_1^n} f_1^n(s)ds\Big)\Big\| \le \int_{t_0^n}^{t_1^n} \Big((1 + \|\varphi\|_{\infty,0})\alpha(s) + \dot{v}(s)\Big)ds.$$

For all $t \in [t_0^n, t_1^n]$, we define

$$z_1^n(t) = p_0^n + \frac{\vartheta_1^n(t) - \vartheta_1^n(t_0^n)}{\vartheta_1^n(t_1^n) - \vartheta_1^n(t_0^n)}\Big(p_1^n - p_0^n - \int_{t_0^n}^{t_1^n} f_1^n(s)ds\Big) + \int_{t_0^n}^{t} f_1^n(s)ds,$$

where

$$\vartheta_1^n(t) := \int_{t_0^n}^{t} \Big((1 + \|\varphi\|_{\infty,0})\alpha(s) + \dot{v}(s)\Big)ds.$$

Note that $z_1^n(t_0^n) = \varphi(t_0^n) = p_0^n$ and $z_1^n(t_1^n) = p_1^n$. Let us consider the mapping $u_1^n(\cdot) : [-r, t_1^n] \to H$ defined by

$$u_1^n(t) = \begin{cases} \varphi(t) & \text{if} \quad t \in [-r, t_0^n] \\ z_1^n(t) & \text{if} \quad t \in [t_0^n, t_1^n], \end{cases}$$

and let us observe that u_1^n is continuous on $[-r, t_1^n]$ since $z_1^n(t_0^n) = \varphi(t_0^n)$ and z_1^n is obviously continuous.

Similarly as above, we choose a measurable mapping f_2^n from $[t_1^n, t_2^n]$ into H such that $f_2^n(t) \in \Gamma(t, \hat{\Lambda}(t_1^n)u_1^n)$ for all $t \in [t_1^n, t_2^n]$. By (2.8), we have

$$\|f_2^n(t)\| \le (1 + \|\hat{\Lambda}(t_1^n)u_1^n\|_{\infty,0})\alpha(t) \quad \text{for all } t \in [t_1^n, t_2^n],$$

and we observe that

$$\begin{aligned}\|\hat{\Lambda}(t_1^n)u_1^n\|_{\infty,0} &= \sup_{s\in[-r,0]} \|u_1^n(s+t_1^n)\| = \sup_{s\in[-r+t_1^n,t_1^n]} \|u_1^n(s)\| \\ &\leq \sup_{s\in[-r,t_1^n]} \|u_1^n(s)\| = \|u_1^n\|_{\infty,t_1^n}.\end{aligned}$$

Thus, for all $t \in [t_1^n, t_2^n]$ we have

$$\|f_2^n(t)\| \leq (1+\|u_1^n\|_{\infty,t_1^n})\alpha(t), \tag{2.13}$$

and this says in particular that f_2^n is integrable on $[t_1^n, t_2^n]$.

Again, due to the ball-compactness of $C(t)$, we have

$$\mathrm{Proj}_{\,C(t_2^n)}\Big(p_1^n + \int_{t_1^n}^{t_2^n} f_2^n(s)ds\Big) \neq \emptyset.$$

Then, there exists a point p_2^n in $\mathrm{Proj}_{\,C(t_2^n)}\big(p_1^n + \int_{t_1^n}^{t_2^n} f_2^n(s)ds\big)$, hence $p_2^n \in C(t_2^n)$ and

$$\Big\|p_2^n - \Big(p_1^n + \int_{t_1^n}^{t_2^n} f_2^n(s)ds\Big)\Big\| = d\Big(p_1^n + \int_{t_1^n}^{t_2^n} f_2^n(s)ds, C(t_2^n)\Big).$$

So, according to $(\mathcal{H}_1)$ and the inclusion $p_1^n \in C(t_1^n)$, we have

$$\begin{aligned}\Big\|p_2^n - \Big(p_1^n &+ \int_{t_1^n}^{t_2^n} f_2^n(s)ds\Big)\Big\| \\ &\leq d\Big(p_1^n + \int_{t_1^n}^{t_2^n} f_2^n(s)ds, C(t_1^n)\Big) + v(t_2^n) - v(t_1^n) \\ &\leq d\big(p_1^n, C(t_1^n)\big) + \Big\|\int_{t_1^n}^{t_2^n} f_2^n(s)ds\Big\| + v(t_2^n) - v(t_1^n) \\ &\leq \int_{t_1^n}^{t_2^n} \|f_2^n(s)\|ds + \int_{t_1^n}^{t_2^n} \dot{v}(s)ds.\end{aligned}$$

Taking (2.13) into account, it follows that

$$\Big\|p_2^n - \Big(p_1^n + \int_{t_1^n}^{t_2^n} f_2^n(s)ds\Big)\Big\| \leq \int_{t_1^n}^{t_2^n} \Big((1+\|u_1^n\|_{\infty,t_1^n})\alpha(s) + \dot{v}(s)\Big)ds. \tag{2.14}$$

As previously, for each $t \in [t_1^n, t_2^n]$ we put

$$z_2^n(t) = p_1^n + \frac{\vartheta_2^n(t) - \vartheta_2^n(t_1^n)}{\vartheta_2^n(t_2^n) - \vartheta_2^n(t_1^n)}\Big(p_2^n - p_1^n - \int_{t_1^n}^{t_2^n} f_2^n(s)ds\Big) + \int_{t_1^n}^{t} f_2^n(s)ds$$

where

$$\vartheta_2^n(t) := \int_{t_0^n}^{t} \Big(\big(1 + \|u_1^n\|_{\infty, t_1^n}\big)\alpha(s) + \dot{v}(s) \Big) ds \quad \forall t \in [t_0^n, t_2^n],$$

so that $z_2^n(t_1^n) = z_1^n(t_1^n) = p_1^n$. We consider the mapping $u_2^n : [-r, t_2^n] \to H$ by

$$u_2^n(t) = \begin{cases} \varphi(t) & \text{if} \quad t \in [-r, 0] \\ \\ z_i^n(t) & \text{if} \quad t \in [t_{i-1}^n, t_i^n], \ (i = 1, 2). \end{cases}$$

We observe that the restriction of u_2^n to $[-r, t_1^n]$ coincides with u_1^n and z_2^n is the restriction of u_2^n on $[t_1^n, t_2^n]$. Further, we have $z_2^n(t_1^n) = u_1^n(t_1^n)$, thus u_2^n is continuous on $[-r, t_2^n]$.

By repeating the process, we obtain the sequences (p_j^n), $(z_j^n(\cdot))$, $(\vartheta_j^n(\cdot))$, mappings $(u_j^n(\cdot))$ continuous on $[-r, t_j^n]$ and mappings $(f_j^n(\cdot))$ integrable on $[t_j^n, t_{j+1}^n]$, satisfying for $j \in J$ the following properties :

$$f_j^n(t) \in \Gamma(t, \hat{\Lambda}(t_{j-1}^n) u_{j-1}^n) \quad \forall t \in [t_{j-1}^n, t_j^n]; \tag{2.15}$$

$$p_j^n \in \operatorname{Proj}_{C(t_j^n)} \Big(p_{j-1}^n + \int_{t_{j-1}^n}^{t_j^n} f_j^n(s) ds \Big); \tag{2.16}$$

$$\begin{aligned} z_j^n(t) &= p_{j-1}^n + \frac{\vartheta_j^n(t) - \vartheta_j^n(t_{j-1}^n)}{\vartheta_j^n(t_j^n) - \vartheta_j^n(t_{j-1}^n)} \Big(p_j^n - p_{j-1}^n - \int_{t_{j-1}^n}^{t_j^n} f_j^n(s) ds \Big) \\ &\quad + \int_{t_{j-1}^n}^{t} f_j^n(s) ds \quad \forall t \in [t_{j-1}^n, t_j^n] \\ &\quad \text{and } z_j^n(t_{j-1}^n) = z_{j-1}^n(t_{j-1}^n) = p_{j-1}^n, z_j^n(t_j^n) = p_j^n; \end{aligned} \tag{2.17}$$

$$\vartheta_j^n(t) := \int_0^t \Big(\big(1 + \|u_{j-1}^n\|_{\infty, t_{j-1}^n}\big)\alpha(s) + |\dot{v}(s)| \Big) ds \quad \forall t \in [0, t_j^n]; \tag{2.18}$$

$$u_j^n(t) = \begin{cases} \varphi(t) & \text{if} \quad t \in [-r, 0] \\ \\ z_i^n(t) & \text{if} \quad t \in [t_{i-1}^n, t_i^n], \ (1 \le i \le j) \end{cases} \tag{2.19}$$

$$\|\hat{\Lambda}(t_j^n) u_j^n\|_{\infty, 0} \le \|u_j^n\|_{\infty, t_j^n}; \tag{2.20}$$

$$\|f_j^n(t)\| \le \big(1 + \|u_{j-1}^n\|_{\infty, t_{j-1}^n}\big)\alpha(t); \tag{2.21}$$

$$(2.22)\quad \Big\|p_j^n - p_{j-1}^n - \int_{t_{j-1}^n}^{t_j^n} f_j^n(s)ds\Big\| \leq \int_{t_{j-1}^n}^{t_j^n} \Big(\big(1+\|u_{j-1}^n\|_{\infty, t_{j-1}^n}\big)\alpha(s) + |\dot{v}(s)|\Big)ds.$$

Now, let us define $x_n : [-r, T] \to H$ and $f_n : [0, T[\to H$ by

$$(2.23)\qquad x_n(t) := u_n^n(t) \quad \text{for all } t \in [-r, T],$$

$$(2.24)\qquad f_n(t) := f_j^n(t) \quad \text{for all } t \in [t_{j-1}^n, t_j^n[, j \in J.$$

By construction x_n is continuous on $[-r, T]$. Let us establish that x_n is absolutely continuous on $[0, T]$. It clearly suffices to show that z_j^n is absolutely continuous on $[t_{j-1}^n, t_j^n], j \in J$. Indeed, for any $\tau, t \in [t_{j-1}^n, t_j^n]$, and $\tau < t$, we have

$$z_j^n(t) - z_j^n(\tau) = \frac{\vartheta_j^n(t) - \vartheta_j^n(\tau)}{\vartheta_j^n(t_j^n) - \vartheta_j^n(t_{j-1}^n)}\Big(p_j^n - p_{j-1}^n - \int_{t_{j-1}^n}^{t_j^n} f_j^n(s)ds\Big) + \int_\tau^t f_j^n(s)ds,$$

which ensures that

$$\|z_j^n(t) - z_j^n(\tau)\| \leq \frac{\vartheta_j^n(t) - \vartheta_j^n(\tau)}{\vartheta_j^n(t_j^n) - \vartheta_j^n(t_{j-1}^n)}\Big\|p_j^n - p_{j-1}^n - \int_{t_{j-1}^n}^{t_j^n} f_j^n(s)ds\Big\| + \int_\tau^t \|f_j^n(s)\|ds.$$

By (2.22) and (2.18)

$$(2.25)\qquad \Big\|p_j^n - p_{j-1}^n - \int_{t_{j-1}^n}^{t_j^n} f_j^n(s)ds\Big\| \leq \vartheta_j^n(t_j^n) - \vartheta_j^n(t_{j-1}^n).$$

It results from (2.18) and (2.21), that

$$\|z_j^n(t) - z_j^n(\tau)\| \leq \int_\tau^t \Big(2\big(1 + \|u_{j-1}^n\|_{\infty, t_{j-1}^n}\big)\alpha(s) + |\dot{v}(s)|\Big)ds.$$

This last inequality above holds for all $\tau, t \in [t_{j-1}^n, t_j^n], j \in J$ with $\tau < t$, hence the mappings z_j^n are absolutely continuous.

According to (2.9), (2.15) and (2.24), we have, by construction,

$$f_n(t) \in \Gamma\big(t, \hat{\Lambda}(\theta_n(t))u_{n-1}^n\big) \quad \forall t \in [0, T[.$$

Using the mapping $\Lambda(t) : \mathcal{C}_T \to \mathcal{C}_0$ defined in (2.2), it results that

$$(2.26)\qquad f_n(t) \in \Gamma\big(t, \Lambda(\theta_n(t))x_n\big) \quad \forall t \in [0, T].$$

Thanks to (2.26), (2.8) and (2.20), we obtain

$$\|f_n(t)\| \leq (1 + \|x_n\|_{\infty,T})\alpha(t) \quad \forall t \in [0,T]. \tag{2.27}$$

For any $t \in [t_{j-1}^n, t_j^n], j \in J$, we have

$$\begin{aligned} x_n(t) = p_{j-1}^n + \frac{\vartheta_j^n(t) - \vartheta_j^n(t_{j-1}^n)}{\vartheta_j^n(t_j^n) - \vartheta_j^n(t_{j-1}^n)} \Big(p_j^n - p_{j-1}^n - \int_{t_{j-1}^n}^{t_j^n} f_j^n(s)ds \Big) \\ + \int_{t_{j-1}^n}^{t} f_j^n(s)ds, \end{aligned} \tag{2.28}$$

hence for almost all $t \in [t_{j-1}^n, t_j^n]$, $j \in J$, we get

$$\dot{x}_n(t) = \frac{\dot{\vartheta}_j^n(t)}{\vartheta_j^n(t_j^n) - \vartheta_j^n(t_{j-1}^n)} \Big(p_j^n - p_{j-1}^n - \int_{t_{j-1}^n}^{t_j^n} f_j^n(s)ds \Big) + f_j^n(t). \tag{2.29}$$

For each $j \in J$, taking (2.25) and (2.24) into account, it follows that for almost every $t \in [t_{j-1}^n, t_j^n]$

$$\|\dot{x}_n(t) - f_n(t)\| \leq \dot{\vartheta}_j^n(t),$$

and thanks to the equality $\dot{\vartheta}_j^n(t) = (1 + \|u_{j-1}^n\|_{\infty, t_{j-1}^n})\alpha(t) + |\dot{v}(t)|$ (see (2.18)), we obtain for almost every $t \in [t_{j-1}^n, t_j^n]$

$$\|\dot{x}_n(t) - f_n(t)\| \leq (1 + \|u_{j-1}^n\|_{\infty, t_{j-1}^n})\alpha(t) + |\dot{v}(t)|.$$

Since u_{j-1}^n is the restriction of x_n to $[-r, t_{j-1}^n]$ we deduce that for almost every $t \in [t_{j-1}^n, t_j^n]$

$$\|\dot{x}_n(t) - f_n(t)\| \leq (1 + \|x_n\|_{\infty,T})\alpha(t) + |\dot{v}(t)|.$$

Consequently, for almost every $t \in [0,T]$ we have

$$\|\dot{x}_n(t) - f_n(t)\| \leq (1 + \|x_n\|_{\infty,T})\alpha(t) + |\dot{v}(t)|. \tag{2.30}$$

Referring to (2.27), it results that

$$\|\dot{x}_n(t)\| \leq 2(1 + \|x_n\|_{\infty,T})\alpha(t) + |\dot{v}(t)|.$$

As x_n is absolutely continuous on $[0,T]$, it follows that, for any $t \in [0,T]$,

$$\|x_n(t) - x_n(0)\| \leq \int_0^t \Big(2\big(1 + \|x_n\|_{\infty,T}\big)\alpha(s) + |\dot{v}(s)| \Big) ds$$

hence

$$\|x_n(t)\| \le \|\varphi(0)\| + \int_0^T \Big(2\big(1+\|x_n\|_{\infty,T}\big)\alpha(s) + |\dot{v}(s)|\Big)ds$$

and thus

$$\|x_n\|_{\infty,T} \le \|\varphi\|_{\infty,0} + \int_0^T \Big(2\big(1+\|x_n\|_{\infty,T}\big)\alpha(s) + |\dot{v}(s)|\Big)ds.$$

Referring to (2.7), we have

$$\|x_n\|_{\infty,T} \le L-1 \tag{2.31}$$

where

$$L := \Big(1-2\int_0^T \alpha(s)ds\Big)^{-1}\Big(\|\varphi\|_{\infty,0} + \int_0^T |\dot{v}(s)|ds + \frac{1}{2}\Big) + 1.$$

Then, (2.27) entails

$$\|f_n(t)\| \le L\alpha(t). \tag{2.32}$$

Note that, by (2.30) and (2.31), for almost all $t \in [0,T]$,

$$\|\dot{x}_n(t) - f_n(t)\| \le L\alpha(t) + |\dot{v}(t)|. \tag{2.33}$$

We have also

$$\|\dot{x}_n(t)\| \le 2L\alpha(t) + |\dot{v}(t)|. \tag{2.34}$$

We observe by (2.5) and (2.16) that

$$p_j^n - p_{j-1}^n - \int_{t_{j-1}^n}^{t_j^n} f_j^n(s)ds \in -N_{C(t_j^n)}(p_j^n),$$

hence, by (2.28) and (2.29) we have $\dot{x}_n(t) - f_j^n(t) \in -N_{C(t_j^n)}\big(x_n(t_j^n)\big)$. It results from (2.10) and (2.24) that, by construction, for almost all $t \in [0,T]$ and for any n,

$$\dot{x}_n(t) - f_n(t) \in -N_{C\left(\hat{\theta}_n(t)\right)}\Big(x_n\big(\hat{\theta}_n(t)\big)\Big). \tag{2.35}$$

Step 2. Now, we proceed to prove that the sequence $(x_n)_n$ admits a subsequence, which converges uniformly to a solution of $(\mathcal{E}_r)$.

Denote by y_n the restriction of x_n to $[0,T]$, that is, $y_n := x_n|_{[0,T]}$. Observe first by $(\mathcal{H}_1)$ and (2.34) that for every $t \in [0,T]$

$$\begin{aligned} d_{C(t)}(y_n(t)) = d_{C(t)}(x_n(t)) &\leq \|x_n(t) - x_n(t_j^n)\| + d_{C(t)}(x_n(t_j^n)) \\ &\leq \int_{t_j^n}^t \|\dot{x}_n(s)\| ds + |v(t) - v(t_j^n)| \\ &\leq 2\int_{t_j^n}^t \big(L\alpha(s) + |\dot{v}(s)|\big) ds, \end{aligned}$$

so by (2.10)

$$d_{C(t)}\big(y_n(t)\big) \leq 2\int_{\hat{\theta}_n(t)}^t \big(L\alpha(s) + |\dot{v}(s)|\big) ds.$$

It results from (2.11) and this last inequality above that

$$d_{C(t)}\big(y_n(t)\big) \underset{n\to\infty}{\longrightarrow} 0.$$

This combined with (2.31) yields $y_n(t) \in C(t) \cap r\mathbb{B}$ and ensures that the set $\{y_n(t),\, n \in \mathbb{N}\}$ is relatively compact in H, in view of hypothesis $(\mathcal{H}_1)$. Since x_n is absolutely continuous on $[0,T]$ we may write according to (2.34), for any $t, \tau \in [0,T]$ with $\tau < t$,

$$\|y_n(t) - y_n(\tau)\| = \|x_n(t) - x_n(\tau)\| \leq \int_\tau^t \big(2L\alpha(s) + |\dot{v}(s)|\big) ds. \tag{2.36}$$

Observe that

$$\int_S \big(2L\alpha(s) + |\dot{v}(s)|\big) ds \to 0 \quad \text{as} \quad \lambda(S) \to 0,$$

where λ denotes the Lebesgue measure. This is equivalent to saying that for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_S \big(2L\alpha(s) + |\dot{v}(s)|\big) ds < \varepsilon$ whenever $\lambda(S) < \delta$. It is then obvious to see through the latter inequality and through (2.36) that the sequence $(y_n)_n$ is equi-continuous on $[0,T]$. Then it follows from Arzela-Ascoli theorem that the sequence $(y_n)_n$ admits a subsequence, still denoted by $(y_n)_n$ for simplicity, converging uniformly in $\mathcal{C}_H(0,T)$ to some mapping $y \in \mathcal{C}_H(0,T)$. Define $x \in \mathcal{C}_T$ by putting

$$\begin{cases} x(t) = y(t) & \text{for all } t \in [0,T], \\ x(t) = \varphi(t) & \text{for all } t \in [-r,0], \end{cases}$$

we also see the sequence $(x_n)_n$ converges uniformly on $[-r,T]$ to x. Moreover, thanks to (2.34) and (2.32), the sequences $(\dot{y}_n)_n$ and $(f_n)_n$ are bounded by a function in $\mathrm{L}^1_{\mathbb{R}_+}(0,T)$. By extracting subsequences we may suppose that $f_n \to f$ and $\dot{y}_n \to u$, both convergences being obtained weakly in $\mathrm{L}^1_H(0,T)$. Thus, for any $t \in [0,T]$,

$$x_n(t) = \varphi(0) + \int_0^t \dot{y}_n(s)ds = \varphi(0) + \int_0^T \dot{y}_n(s)\mathbb{1}_{[0,t]}(s)ds.$$

Since the sequence $(x_n(t))$ converges in H to $x(t)$, we may pass to the limit to obtain

$$x(t) = \varphi(0) + \int_0^T u(s)\mathbb{1}_{[0,t]}(s)ds = \varphi(0) + \int_0^t u(s)ds.$$

Consequently x is absolutely continuous on $[0,T]$, with $\dot{x}(t) = u(t)$ for almost all $t \in [0,T]$ and hence y is absolutely continuous on $[0,T]$ and

$$\dot{y}_n \to \dot{y} \text{ weakly in } \mathrm{L}^1_H(0,T). \tag{2.37}$$

Thanks to (2.11) and the uniform convergence of $(x_n)_n$ to x, we get $x_n(\theta_n(t))$ converges to $x(t)$ for each $t \in [0,T]$. Note also that, due to the fact that $d_{C(t)}(x_n(t))$ converges to 0 on $[0,T]$, we have $x(t) \in C(t)$ for all $t \in [0,T]$.

Claim: $\Lambda(\theta_n(t))x_n$ converges to $\Lambda(t)x$.

First, let us denote the modulus of continuity of a function g defined on an interval I of $\mathbb{R}$ by

$$\omega(g, I, \varepsilon) := \sup\{\|g(t) - g(s)\| : \, s,t \in J, \, |t-s| \le \varepsilon\}.$$

Then

$$\begin{aligned} \|\Lambda(\theta_n(t))x_n - \Lambda(t)x_n\|_{\infty,0} &= \sup_{\tau\in[-r,0]} \|x_n(\theta_n(t)+\tau) - x_n(t+\tau)\| \\ &\le \omega(x_n, [-r,T], \tfrac{T}{n}) \\ &\le \omega(\varphi, [-r,0], \tfrac{T}{n}) + \omega(x_n, [0,T], \tfrac{T}{n}). \end{aligned}$$

Considering $\varrho(t) = \displaystyle\int_0^t \big(2L\alpha(s) + |\dot{v}(s)|\big)ds$ we deduce from the latter inequality and (2.36) that

$$\|\Lambda(\theta_n(t))x_n - \Lambda(t)x_n\|_{\infty,0} \le \omega(\varphi, [-r,0], \tfrac{T}{n}) + \omega(\varrho, [0,T], \tfrac{T}{n})$$

Since φ and ϱ are uniformly continuous on $[-r,0]$ and $[0,T]$ respectively, then,

$$\|\Lambda(\theta_n(t))x_n - \Lambda(t)x_n\|_{\infty,0} \underset{n\to\infty}{\longrightarrow} 0;$$

and since the uniform convergence of x_n to x on $[-r,T]$ implies $\Lambda(t)x_n$ converges uniformly to $\Lambda(t)x$ on $[-r,0]$, we deduce that

$$\Lambda(\theta_n(t))x_n \to \Lambda(t)x \quad \text{in } \mathcal{C}_0. \tag{2.38}$$

Step 3. Now, it remains to prove that x is a solution of $(\mathcal{E}_r)$.

Due to the fact that $(f_n)_n$ and $(\dot{y}_n)_n$ converge both weakly in $\mathrm{L}^1_H(0,T)$ to f and $\dot{y}$ respectively, according to Mazur's lemma, there is a sequence $(z_n,\phi_n)_n$ which converges strongly in $\mathrm{L}^1_{H\times H}(0,T)$ to $(\dot{y}-f,f)$ with

$$z_n \in \mathrm{co}\,\{\dot{y}_k - f_k : k \geq n\} \text{ and } \phi_n \in \mathrm{co}\,\{f_k : k \geq n\},$$

for each $n \geq 1$. Extract a subsequence (that we dot not relabel) $(z_n,\phi_n)_n$ converging to $(\dot{y}-f,f)$ a.e, that is, there exists some fixed Lebesgue negligible set $N \subset [0,T]$ such that for each $t \in [0,T]\backslash N$ we have $(z_n(t),\phi_n(t))_n$ converges to $(\dot{y}(t)-f(t),f(t))$ or equivalently to $(\dot{x}(t)-f(t),f(t))$. Therefore, for each $t \in [0,T]\backslash N$,

$$\dot{x}(t) - f(t) \in \bigcap_n \overline{\mathrm{co}}\,\big\{\dot{x}_k(t) - f_k(t) : k \geq n\big\} \tag{2.39}$$

$$f(t) \in \bigcap_n \overline{\mathrm{co}}\,\big\{f_k(t) : k \geq n\big\}. \tag{2.40}$$

Fix $t \in [0,T]\backslash N$ and for all $n \in \mathbb{N}$, using (2.33), (2.35) and putting $\eta := L\alpha(t) + |\dot{v}(t)|$, we get by (2.4) and Lemma 2.2.1

$$\dot{x}_n(t) - f_n(t) \in -N^{\eta}_{C\left(\hat{\theta}_n(t)\right)}\Big(x_n\big(\hat{\theta}_n(t)\big)\Big) = -\eta\partial d_{C\left(\hat{\theta}_n(t)\right)}\Big(x_n\big(\hat{\theta}_n(t)\big)\Big).$$

Hence, by (2.39) and for all $\xi \in H$ we have

$$\langle \xi, \dot{x}(t) - f(t)\rangle \leq \sup_{k\geq n}\langle \xi, \dot{x}_k(t) - f_k(t)\rangle \leq \sup_{k\geq n}\sigma\Bigg(\xi, -\eta\partial d_{C\left(\hat{\theta}_k(t)\right)}\Big(x_k\big(\hat{\theta}_k(t)\big)\Big)\Bigg)$$

$$\langle \xi, \dot{x}(t) - f(t)\rangle \leq \lim_{n\to\infty}\sup_{k\geq n}\sigma\Bigg(\xi, -\eta\partial d_{C\left(\hat{\theta}_k(t)\right)}\Big(x_k\big(\hat{\theta}_k(t)\big)\Big)\Bigg)$$

thus

$$\langle \xi, \dot{x}(t) - f(t)\rangle \leq \limsup_{n\to\infty}\sigma\Bigg(\xi, -\eta\partial d_{C\left(\hat{\theta}_n(t)\right)}\Big(x_n\big(\hat{\theta}_n(t)\big)\Big)\Bigg)$$

or equivalently

$$\langle -\xi, -\dot{x}(t) + f(t)\rangle \leq \limsup_{n\to\infty} \sigma\Big(-\xi, \eta\partial d_{C\left(\hat{\theta}_n(t)\right)}\Big(x_n\big(\hat{\theta}_n(t)\big)\Big)\Big).$$

Since $x_n\big(\hat{\theta}_n(t)\big) \in C\big(\hat{\theta}_n(t)\big)$ and $x(t) \in C(t)$, the latter inequality entails by Proposition 2.2.1 that

$$\langle -\xi, -\dot{x}(t) + f(t)\rangle \leq \sigma\Big(-\xi, \eta\partial d_{C(t)}\big(x(t)\big)\Big)$$

or equivalently

$$\langle \xi, \dot{x}(t) - f(t)\rangle \leq \sigma\Big(\xi, -\eta\partial d_{C(t)}\big(x(t)\big)\Big).$$

Since $\partial d_{C(t)}(x(t))$ is convex and closed for each $t \in [0,T]\setminus N$, we deduce that

$$\dot{x}(t) - f(t) \in -\eta\partial d_{C(t)}(x(t)) \subset -N_{C(t)}(x(t)). \tag{2.41}$$

It is not difficult to see that $f(t) \in \Gamma(t, \Lambda(t)x)$. Indeed, it result from (2.40) and (2.26) that for all $\xi \in H$

$$\langle \xi, f(t)\rangle \leq \sup_{k\geq n}\langle \xi, f_k(t)\rangle \leq \sup_{k\geq n} \sigma\Big(\xi, \Gamma\Big(t, \Lambda\big(\theta_k(t)\big)x_k\Big)\Big),$$

thus

$$\langle \xi, f(t)\rangle \leq \limsup_{n\to\infty} \sigma\Big(\xi, \Gamma\Big(t, \Lambda\big(\theta_n(t)\big)x_n\Big)\Big).$$

Due to (2.38) and to the upper semicontinuity of $\phi \mapsto \sigma\big(\xi, \Gamma(t,\phi)\big)$, we have, for all $t \in [0,T]\setminus N$, for any $\xi \in H$,

$$\langle \xi, f(t)\rangle \leq \sigma\Big(\xi, \Gamma\big(t, \Lambda(t)x\big)\Big).$$

As $\Gamma(t, \Lambda(t)x)$ is closed and convex, we conclude that, for all $t \in [0,T]\setminus N$,

$$f(t) \in \Gamma(t, \Lambda(t)x).$$

This, along with (2.41), implies for all $t \in [0,T]\setminus N$

$$\dot{x}(t) \in -N_{C(t)}\big(x(t)\big) + \Gamma\big(t, \Lambda(t)x\big),$$

and hence x is a solution of the constrained differential inclusion $(\mathcal{E}_r)$.

II. Case where
$$\int_0^T \alpha(s)ds \geq \frac{1}{4}.$$
Taking $\varepsilon = \frac{1}{4}$ there exists $\delta > 0$ such that for any Lebesgue measurable subset $S \subset [0,T]$ with $\lambda(S) < \delta$ we have $\int_S \alpha(s)ds < \frac{1}{4}$. Choose some integer $N \geq 1$ such that $\frac{T}{N} < \delta$ and consider a subdivision of $[0,T]$ given by $0 = T_0 < T_1 < \cdots < T_N = T$ with $T_i = i\frac{T}{N}$ where $i \in \{0, \cdots, N\}$. Of course, for any $i \in \{0, \cdots, N-1\}$, we have

$$\int_{T_i}^{T_{i+1}} \alpha(s)ds < \frac{1}{4}. \tag{2.42}$$

We have $\int_{T_0}^{T_1} \alpha(s)ds < \frac{1}{4}$, in view of (2.42). The part I ensures the existence of a mapping $x_1 : [-r, T_1] \to H$ absolutely continuous on $[0, T_1]$ such that

$$\begin{cases} x_1(s) = \varphi(s) \quad \text{for all } s \in [-r, 0], \\ x_1(t) \in C(t) \quad \text{for all } t \in [0, T_1], \\ \dot{x}_1(t) \in -N_{C(t)}\big(x_1(t)\big) + \Gamma\big(t, \Lambda(t)x_1\big) \quad \text{a.e. } t \in [0, T_1]. \end{cases} \tag{2.43}$$

Let us define the function $\tilde{\alpha}_1$ from $\left[0, \frac{T}{N}\right]$ into $\mathbb{R}_+$ by
$$\tilde{\alpha}_1(t) := \alpha(t + T_1),$$
so by (2.42), we have
$$\int_0^{\frac{T}{N}} \tilde{\alpha}_1(s)ds < \frac{1}{4}.$$

Consider the set-valued mappings $\tilde{\Gamma}_1 : \left[0, \frac{T}{N}\right] \times \mathcal{C}_0 \rightrightarrows H$ and $\tilde{C}_1 : \left[0, \frac{T}{N}\right] \rightrightarrows H$ defined respectively by

$$\tilde{\Gamma}_1(t, \psi) := \Gamma(t + T_1, \psi), \quad \tilde{C}_1(t) := C(t + T_1). \tag{2.44}$$

Obviously, by $(\mathcal{H}_3)$ for any $(t, \psi) \in \left[0, \frac{T}{N}\right] \times \mathcal{C}_0$ we have
$$d\big(0, \tilde{\Gamma}_1(t, \psi)\big) \leq \tilde{\alpha}_1(t)\Big(1 + \|\psi\|_{\infty,0}\Big).$$
Consider also the single-valued mapping $\tilde{\varphi}_1 : [-r, 0] \to H$ defined by

$$\tilde{\varphi}_1(s) := x_1(s + T_1). \tag{2.45}$$

and note that $\tilde{\varphi}_1 \in \mathcal{C}_0$ along with $\tilde{\varphi}_1(0) = x_1(T_1) \in C(T_1)$, so $\tilde{\varphi}_1(0) \in \tilde{C}_1(0)$ by definition of $\tilde{C}_1$. Likewise, according to the part I, there is a mapping $\tilde{x}_1 : \left[-r, \frac{T}{N}\right] \to H$ which is absolutely continuous on $\left[0, \frac{T}{N}\right]$ and such that

$$(2.46) \qquad \begin{cases} \tilde{x}_1(s) = \tilde{\varphi}_1(s) \text{ for all } s \in [-r, 0] \\ \\ \tilde{x}_1(t) \in \tilde{C}_1(t) \text{ for all } t \in \left[0, \frac{T}{N}\right] \end{cases}$$

$$(2.47) \quad \dot{\tilde{x}}_1(t) \in -N_{\tilde{C}_1(t)}\big(\tilde{x}_1(t)\big) + \tilde{\Gamma}_1\big(t, \Lambda(t)\tilde{x}_1\big) \quad \text{a.e. } t \in \left[0, \frac{T}{N}\right].$$

Putting

$$x_2(t) = \begin{cases} x_1(t) & \text{if} \quad t \in [-r, T_1], \\ \\ \tilde{x}_1(t - T_1) & \text{if} \quad t \in [T_1, T_2], \end{cases}$$

it results from (2.44) and (2.47) that

$$(2.48) \qquad \dot{x}_2(t) \in -N_{C(t)}\big(x_2(t)\big) + \Gamma\big(t, \Lambda(t - T_1)\tilde{x}_1\big) \quad \text{a.e. } t \in [T_1, T_2].$$

Claim: $\Lambda(t - T_1)\tilde{x}_1 = \Lambda(t)x_2$ for every $t \in [T_1, T_2]$.
Fix $t \in [T_1, T_2]$. For any $s \in [-r, 0]$, we observe: if $s \le T_1 - t$, we obtain

$$\tilde{x}_1(t - T_1 + s) = \tilde{\varphi}_1(t - T_1 + s),$$

$$\tilde{\varphi}_1(t - T_1 + s) = x_2(t + s),$$

where the first equality follows from (2.46) and the second equality follows from (2.45) and by the definition of x_2. On the other hand if $T_1 - t \le s \le 0$ it follows from the definition of x_2 that $\tilde{x}_1(t - T_1 + s) = x_2(t + s)$. So, for any $s \in [-r, 0]$, we see that

$$\Lambda(t - T_1)\tilde{x}_1(s) = \tilde{x}_1(t - T_1 + s) = \begin{cases} x_2(t + s) & \text{if} \quad s \le T_1 - t, \\ \\ x_2(t + s) & \text{if} \quad T_1 - t \le s \le 0, \end{cases}$$

thus

$$\Lambda(t - T_1)\tilde{x}_1(s) = \Lambda(t)x_2(s).$$

This assures that

$$\Lambda(t - T_1)\tilde{x}_1 = \Lambda(t)x_2,$$

and justifies the claim.

The equality of the claim combined with (2.48) yields

$$\dot{x}_2(t) \in -N_{C(t)}\big(x_2(t)\big) + \Gamma\big(t, \Lambda(t)x_2\big) \quad \text{a.e. } t \in [T_1, T_2].$$

According to (2.43) and to the latter inclusion, we obtain

$$\text{(2.49)} \qquad \begin{cases} x_2(s) = \varphi(s) \quad \text{for all } s \in [-r, 0], \\ \\ x_2(t) \in C(t) \quad \text{for all } t \in [0, T_2], \\ \\ \dot{x}_2(t) \in -N_{C(t)}\big(x_2(t)\big) + \Gamma\big(t, \Lambda(t)x_2\big) \quad \text{a.e. } t \in [0, T_2]. \end{cases}$$

Now, suppose that (2.49) holds for $2, 3, \cdots, i$ with $i \leq N-1$. As above, define $\tilde{\alpha}_i : \left[0, \frac{T}{N}\right] \to \mathbb{R}_+$ by

$$\tilde{\alpha}_i(t) := \alpha(t + T_i),$$

and note by (2.42) that

$$\int_0^{\frac{T}{N}} \tilde{\alpha}_i(s) ds < \frac{1}{4}.$$

Take the set-valued mappings $\tilde{\Gamma}_i : \left[0, \frac{T}{N}\right] \times \mathcal{C}_0 \rightrightarrows H$ and $\tilde{C}_i : \left[0, \frac{T}{N}\right] \rightrightarrows H$ with

$$\text{(2.50)} \qquad \tilde{\Gamma}_i(t, \psi) := \Gamma(t + T_i, \psi), \quad \tilde{C}_i(t) := C(t + T_i).$$

The assumption $(\mathcal{H}_3)$ gives, for all $t \in \left[0, \frac{T}{N}\right]$ and $\psi \in \mathcal{C}_0$,

$$d\big(0, \tilde{\Gamma}_i(t, \psi)\big) \leq \tilde{\alpha}_i(t)\Big(1 + \|\psi\|_{\infty,0}\Big).$$

Define the mapping $\tilde{\varphi}_i : [-r, 0] \to H$ by

$$\text{(2.51)} \qquad \tilde{\varphi}_i(s) := x_i(s + T_i),$$

and observe that $\tilde{\varphi}_i \in \mathcal{C}_0$ and $\tilde{\varphi}_i(0) = x_i(T_i) \in C(T_i)$, that is, $\tilde{\varphi}_i(0) \in \tilde{C}_i(0)$.

It results from part I again that there exists a mapping $\tilde{x}_i(\cdot) : \left[-r, \frac{T}{N}\right] \to H$ which is absolutely continuous on $\left[0, \frac{T}{N}\right]$ and such that

$$\text{(2.52)} \qquad \begin{cases} \tilde{x}_i(s) = \tilde{\varphi}_i(s) \text{ for all } s \in [-r, 0], \\ \\ \tilde{x}_i(t) \in \tilde{C}_i(t) \text{ for all } t \in \left[0, \frac{T}{N}\right] \end{cases}$$

$$(2.53) \quad \dot{\tilde{x}}_i(t) \in -N_{\tilde{C}_i(t)}\big(\tilde{x}_i(t)\big) + \tilde{\Gamma}_i\big(t, \Lambda(t)\tilde{x}_i\big) \quad \text{a.e. } t \in \left[0, \frac{T}{N}\right].$$

In an analogous way as above, put

$$x_{i+1}(t) = \begin{cases} x_i(t) & \text{if} \quad t \in [-r, T_i], \\ \\ \tilde{x}_i(t - T_i) & \text{if} \quad t \in [T_i, T_{i+1}]. \end{cases}$$

From (2.50) and (2.52), it follows that

$$(2.54) \qquad x_{i+1}(t) \in C(t) \quad \text{for all } t \in [T_i, T_{i+1}].$$

It follows also from (2.50) and (2.53) that

$$(2.55) \quad \dot{x}_{i+1}(t) \in -N_{C(t)}\big(x_{i+1}(t)\big) + \Gamma\big(t, \Lambda(t - T_i)\tilde{x}_i\big) \quad \text{a.e. } t \in [T_i, T_{i+1}].$$

Claim: $\Lambda(t - T_i)\tilde{x}_i = \Lambda(t)x_{i+1}$ for every $t \in [T_i, T_{i+1}]$.
Fix $t \in [T_i, T_{i+1}]$. For any $s \in [-r, 0]$ we observe : if $s \le T_i - t$, then from (2.52), $\tilde{x}_i(t - T_i + s) = \tilde{\varphi}_i(t - T_i + s)$ and from (2.51) and by the definition of x_{i+1}, $\tilde{\varphi}_i(t - T_i + s) = x_{i+1}(t + s)$. On the other hand, if $T_i - t \le s \le 0$, $\tilde{x}_i(t - T_i + s) = x_{i+1}(t + s)$, thanks to the definition of x_{i+1}. So, for any $s \in [-r, 0]$, we see that

$$\Lambda(t - T_i)\tilde{x}_i(s) = \tilde{x}_i(t - T_i + s) = \begin{cases} x_{i+1}(t + s) & \text{if} \quad s \le T_i - t, \\ \\ x_{i+1}(t + s) & \text{if} \quad T_i - t \le s \le 0, \end{cases}$$

thus

$$\Lambda(t - T_i)\tilde{x}_i(s) = \Lambda(t)x_{i+1}(s).$$

Consequently,

$$\Lambda(t - T_i)\tilde{x}_i = \Lambda(t)x_{i+1},$$

as stated in the claim.

It follows from (2.55) that

$$\dot{x}_{i+1}(t) \in -N_{C(t)}\big(x_{i+1}(t)\big) + \Gamma\big(t, \Lambda(t)x_{i+1}\big) \quad \text{a.e. } t \in [T_i, T_{i+1}],$$

and this combined with the induction property (2.49) at the step i and with (2.54)

$$(2.56) \quad \begin{cases} x_{i+1}(s) = \varphi(s) \quad \text{for all } s \in [-r, 0] \\ \\ x_{i+1}(t) \in C(t) \quad \text{for all } t \in [0, T_{i+1}] \\ \\ \dot{x}_{i+1}(t) \in -N_{C(t)}\big(x_{i+1}(t)\big) + \Gamma\big(t, \Lambda(t)x_{i+1}\big) \quad \text{a.e. } t \in [0, T_{i+1}]. \end{cases}$$

Therefore, this ensures that (2.56) holds by induction for $0, 1, \cdots, N$. Consequently, we obtain a solution $x := x_N$ on the whole interval $[-r, T]$. The proof is then complete. $\square$

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Chapter 3

Nonconvex sweeping process with a moving set depending on the state

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Abstract. Recently, a great advance has been made in the study of sweeping process variational inequalities with the papers [1, 8, 9, 19] where, for a prox-regular moving set depending both on the time and on the state, several existence results are provided. Those authors also studied, the case where such a differential inclusion is perturbed by a multimapping. The present paper establishes the existence of solutions for such perturbed differential inclusions in a more general context.

Keyword : Differential inclusion; Sweeping process; Normal cone; Prox-regular set; Subdifferential

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Introduction

The general class of differential inclusion known as the sweeping process, has been introduced and thoroughly studied in the 70s period by J. J. Moreau in a series of seminal papers [25, 26, 27, 28]. That differential inclusion can be expressed in the form

$$(\mathcal{I}) \qquad \begin{cases} \dot{u}(t) \in -N_{C(t)}\big(u(t)\big) \quad \text{a.e } \ t \in [0,T], \\ \\ u(0) = u_0 \in C(0), \end{cases}$$

where $C(t)$ is a closed convex set moving in an absolutely continuous way (or with a bounded variation) in an infinite dimensional Hilbert space H and $N_{C(t)}(\cdot)$ denotes the usual normal cone. This evolution differential inclusion corresponds to several important mechanical problems (see [23, 28]). In the context of nonconvex sets $C(t)$, new techniques have been found from which one can derive several results; in particular, Castaing showed in finite dimensions the existence of a solution when $C(t)$ is the form $C(t) = S + \vartheta(t)$, where S is any fixed closed subset of H and ϑ is a mapping with finite variation. Valadier's method in [33] yields, but still in the finite dimensional setting, the existence of a solution whenever the graph of the multimapping $(t,u) \mapsto N_{C(t)}(u) \cap \mathbb{B}$ is closed and $N_{C(t)}(\cdot)$ is the Clarke normal cone. An important example in [33] with such a closedness property corresponds to the complement of the interior of a convex set moving in an absolutely continuous way. For some other contributions see also [5, 18, 34]. Recently Benabdellah [2], Colombo and Goncharov [12], and Thibault [32] have proved that, in the finite dimensional context, the problem $(\mathcal{I})$ above has always a solution when $N_{C(t)}(\cdot)$ is the Clarke normal cone. For the case where the sets $C(t)$ are (uniformly) prox-regular and move with a bounded variation we refer the reader to the paper [16] by J. F. Edmond and L. Thibault.

In the same 70s period, Henry [20] introduced for the study of planning procedures in mathematical economy the differential inclusion

$$\begin{cases} \dot{u}(t) \in \operatorname{Proj}_{T_C\left(u(t)\right)}\Big(G\big(u(t)\big)\Big) \quad \text{a.e } \ t \in [0,T] \\ \\ u(0) = u_0 \in C, \end{cases}$$

where $G(\cdot)$ is an upper semicontinuous multimapping with nonempty compact convex values, C is a (nonmoving) nonempty closed convex set, $T_C(\cdot)$ is the tangent cone to C and $\operatorname{Proj}_{T_C}(\cdot)$ denotes the metric projection mapping onto the closed convex set $T_C(\cdot)$. Later, Cornet (see [14] and [15]), as in

Henry [20], reduced the last inclusion above to the existence of a solution for the following problem

$$\begin{cases} \dot{u}(t) \in -N_C\big(u(t)\big) + G\big(t, u(t)\big) \quad \text{a.e } \ t \in [0, T] \\ \\ u(0) = u_0 \in C, \end{cases}$$

which is a particular case of the differential inclusion $(\mathcal{I})$ perturbed by a multimapping $G : [0, T] \times H \rightrightarrows H$, that is, the differential inclusion

$$(\mathcal{II}) \qquad \begin{cases} \dot{u}(t) \in -N_{C(t)}\big(u(t)\big) + G\big(t, u(t)\big) \quad \text{a.e } t \in [0, T] \\ \\ u(0) = u_0 \in C(0). \end{cases}$$

Significant progress concerning the differential inclusion $(\mathcal{II})$ has been made in the finite dimensional setting by Castaing, Duc Ha, and Valadier [7] and by Castaing and Monteiro Marques [5] (see also the references therein), under the assumption of convexity of $C(t)$ or of its complement. The case of general closed sets $C(t)$ moving in an absolutely continuous way in a finite dimensional setting has been studied for $(\mathcal{II})$ by Thibault [32]. Several other works can also be found in the references in [7]. Recently in the infinite dimensional Hilbert space H, Bounkhel and Thibault [4] and Edmond and Thibault [17] showed the existence of a solution of $(\mathcal{II})$ when the sets $C(t)$ are r-prox-regular ($r > 0$) and move in an absolutely continuous way. The mapping $G(\cdot, \cdot)$ was required to have all its values included in a fixed compact subset. For the study of $(\mathcal{I})$ with r-prox-regular subsets $C(t)$ of the infinite dimensional Hilbert space H, we refer to [12].

In all the aforementioned works, the sets $C(t)$ do not depend on the state $u(t)$. The first work dealing with a moving set $C(t, x)$ depending on the time and the state has been made in [22] under the convexity assumption for $C(t, x)$. Recently, N. Chemetov and M. D. P. Monteiro Marques [9], established the first results concerning the situation where the moving set $C(t, x)$, depending both on the time and on the state, is nonconvex. Given a single valued mapping $G : [0, T] \times H \to H$ of Carathéodory type (that is, measurable in t and continuous in x), they studied the differential inclusion

$$(\mathcal{II}') \qquad \begin{cases} \dot{u}(t) \in -N_{C\left(t, u(t)\right)}\big(u(t)\big) + G\big(t, u(t)\big) \quad \text{a.e } t \in [0, T] \\ \\ u(0) = u_0 \in C(0, u_0), \end{cases}$$

for a constraint multimapping $C : [0, T] \times H \rightrightarrows H$ with nonconvex prox-regular values which are ball-compact. Associating with each absolutely continuous mapping $y : [0,T] \to H$, with $y(0) = u_0$, the unique solution $\phi(y)$ of the time-dependent sweeping process (with unknown mapping u)

$$\dot{u}(t) \in -N_{C\left(t,y(t)\right)}\big(u(t)\big) + G\big(t, y(t)\big) \quad \text{with } u(0) = u_0 \in C\big(0, y(0)\big),$$

the solution of $(\mathcal{I}\mathcal{I}')$ is obtained in [9] by applying the Schauder fixed point theorem to an appropriate compact convex subset of the space of continuous mappings from $[0,T]$ into H. In [8], by means of a generalized version of the Schauder fixed point theorem from [21, 29], C. Castaing, A. G Ibrahim and M. Yarou provided another approach allowing them to prove, the existence of a solution when $G \equiv \{0\}$ and $C(t,x)$ is prox-regular and ball-compact; with the same approach they also obtained an existence result (even in the presence of a delay) when G is a convex-valued multimapping bounded on $[0,T] \times \mathcal{C}_H(-r,0)$ ($\mathcal{C}_H(-r,0)$ denotes the space of all continuous mappings from $[-r,0]$ to H), and $C(t,x)$ is convex and ball-compact. D. Azzam, S. Izza and L. Thibault [1] obtained, in finite dimensions, a solution for $(\mathcal{I}\mathcal{I}')$, with a multimapping G, via a reduction to an unconstrained differential inclusion. In [19], assuming that the prox-regular sets $C(t,x)$ are contained in a fixed compact set of H and using (without a fixed point theorem) the scheme $u_0^n = u_0$, $u_{i+1}^n = \mathrm{Proj}_{\,C(t_{i+1}^n, u_i^n)}(u_i^n - \frac{T}{2^n} g_i^n)$ with $g_i^n \in G(t_i^n, u_i^n)$ (where $t_i^n := i\frac{T}{2^n}$, $i = 0, \cdots, 2^n - 1$), T. Haddad established the existence of solution of $(\mathcal{I}\mathcal{I}')$ with a multimapping G with compact convex values.

Our main purpose in this paper is to study, in the same setting of infinite dimensional Hilbert space H, the perturbed sweeping process $(\mathcal{I}\mathcal{I}')$, and to show how the approach in [8] can be adapted to yield the existence of solution for $(\mathcal{I}\mathcal{I}')$ with prox-regular sets $C(t,x)$ and a multimapping G with (unnecessarily bounded) closed convex values. For that adaption, a result on the Hölder continuity (with respect to the Hausdorff distance) of the metric projection to prox-regular set is required. The paper is organized as follows: in the next section, we introduce notation which will be used and recall several concepts of nonsmooth and variational analysis which are involved throughout the paper. The second section gives the behaviour of the metric projection mapping onto prox-regular set. The last section is devoted to the proof of the theorem of existence of a solution of the differential inclusion $(\mathcal{I}\mathcal{I}')$ with a nonconvex prox-regular set $C(t,x)$.

3.1 Preliminaries and Notation

Throughout the paper H is a Hilbert space whose inner product is denoted by $\langle \cdot, \cdot \rangle$ and the associated norm by $\| \cdot \|$. The closed unit ball of H with center 0 will be denoted by $\mathbb{B}$ and $B(u,\eta)$ (respectively, $B[u,\eta]$) denotes the open (respectively, closed) ball of center $u \in H$ and radius $\eta > 0$. Given a real $T > 0$, we will denote by $\mathcal{C}_H(0,T)$ the space of all continuous mappings from $[0,T]$ to H, "a.e" denotes "for almost all" and $\dot{u}$ is the derivative of u.

Let C, C' be two subsets of H and let v be a vector in H, the real $d(v,C)$ or $d_C(v) := \inf\{\|v-u\| : u \in C\}$ is the distance of the point v from the set C. We denote by

$$\mathrm{Haus}(C,C') = \max\left\{\sup_{u\in C} d(u,C'), \sup_{v\in C'} d(v,C)\right\}$$

the Hausdorff distance between C and C'. Let us denote, for $r > 0$ and $\gamma \in]0,1[$, by $U_r^\gamma(C)$ (respectively, by $E_r^\gamma(C)$) the open tube around the set C (respectively, the open enlargement of the set C), that is,

$$U_r^\gamma(C) := \{v \in H : 0 < d(v,C) < \gamma r\},$$

respectively,

$$E_r^\gamma(C) := \{v \in H : d(v,C) < \gamma r\}.$$

We need first to recall some notation and definitions that will be used in all the paper. For any subset C of H, $\overline{\mathrm{co}}\, C$ stands for the closed convex hull of C, and $\sigma(\cdot, C)$ represents the support function of C, that is, for all $\xi \in H$,

$$\sigma(\xi, C) := \sup_{u\in C}\langle \xi, u\rangle.$$

If C is a nonempty subset of H, the Clarke normal cone $N(C;u)$ or $N_C(u)$ of C at $u \in C$ is defined by

$$N_C(u) = \{\xi \in H : \langle \xi, v\rangle \leq 0, \forall v \in T_C(u)\},$$

where the Clarke tangent cone $T(C;u)$ or $T_C(u)$ (see [10]) is defined as follows:

$$v \in T_C(u) \Leftrightarrow \begin{cases} \forall \varepsilon > 0, \exists \delta > 0 \text{ such that} \\ \\ \forall u' \in B(u,\delta) \cap C, \forall t \in]0,\delta[, (u' + tB(v,\varepsilon)) \cap C \neq \emptyset. \end{cases}$$

Equivalently, $v \in T_C(u)$ if and only if for any sequence $(u_n)_n$ of C converging to u and any sequence of positive reals $(t_n)_n$ converging to 0, there exists a sequence $(v_n)_n$ in H converging to v such that

$$u_n + t_n v_n \in C \text{ for all } n \in \mathbb{N}.$$

We put $N_C(u) = \emptyset$, whenever $u \notin C$.

We typically denote by $f : H \to \mathbb{R} \cup \{+\infty\}$ a proper function (that is, f is finite at least at one point). The Clarke subdifferential $\partial f(u)$ of f at a point u (where f is finite) is defined by

$$\partial f(u) = \Big\{\xi \in H : (\xi, -1) \in N_{\operatorname{epi} f}\Big(\big(u, f(u)\big)\Big)\Big\},$$

where $\operatorname{epi} f$ denotes the epigraph of f, that is,

$$\operatorname{epi} f = \{(u, r) \in H \times \mathbb{R} : f(u) \leq r\}.$$

We also put $\partial f(u) = \emptyset$ if f is not finite at $u \in H$. If ψ_C denotes the indicator function of the set C, that is, $\psi_C(u) = 0$ if $u \in C$ and $\psi_C(u) = +\infty$ otherwise, then

$$\partial \psi_C(u) = N_C(u) \text{ for all } u \in H.$$

The Clarke subdifferential $\partial f(u)$ of a locally Lipschitz function f at u has also the other useful description

$$\partial f(u) = \{\xi \in H : \langle \xi, v\rangle \leq f^0(u, v), \forall v \in H\},$$

where

$$f^0(u, v) := \limsup_{(u',t)\to(u,0^+)} \frac{f(u' + tv) - f(u')}{t}.$$

The above function $f^0(u; \cdot)$ is called the Clarke directional derivative of f at u. The Clarke normal cone is known ([10]) to be related to the Clarke subdifferential of the distance function through the equality

$$N_C(u) = \operatorname{cl}_w(\mathbb{R}_+ \partial d_C(u)) \text{ for all } u \in C,$$

where $\mathbb{R}_+ := [0, \infty[$ and cl_w denotes the closure with respect to the weak topology of H. Further

$$\partial d_C(u) \subset N_C(u) \cap \mathbb{B} \text{ for all } u \in C.$$

We will also need the concept of proximal subgradient. A vector $\xi \in H$ is a proximal subgradient of f at u (see, [11, 24, 31]) if there exist some constant real number $\sigma \geq 0$ and some $\delta > 0$ such that

$$\langle \xi, v - u\rangle \leq f(v) - f(u) + \sigma\|v - u\|^2 \text{ for all } v \in B(u, \delta).$$

The set $\partial_p f(u)$ of all proximal subgradients of f at u is the proximal subdifferential of f at u. If $f(u)$ is not finite we put $\partial_p f(u) = \emptyset$. It is known that we always have the inclusion

$$\partial_p f(u) \subset \partial f(u).$$

The proximal normal cone of C at $u \in C$ is given by

$$N_C^p(u) = \partial_p \psi_C(u),$$

so the following inclusion always holds true

$$N_C^p(u) \subset N_C(u) \text{ for all } u \in C.$$

On the other hand, the proximal normal cone enjoys a geometrical characterization (see, [11]) given by the equality

$$N_C^p(u) = \{\xi \in H : \exists \rho > 0 \text{ s.t. } u \in \operatorname{Proj}_C(u + \rho\xi)\},$$

where

$$\operatorname{Proj}_C(v) := \{u \in C : d(v, C) = \|v - u\|\}$$

is the set of nearest points of v in C. When this set has a unique point, we will use the notation $\mathrm{P}_C(v)$. For $u \in C$, the proximal cone is also related to the distance function to C via the equalities (see, [11, 3])

$$N_C^p(u) = \mathbb{R}_+ \partial_p d_C(u)$$

and

$$N_C^p(u) \cap \mathbb{B} = \partial_p d_C(u).$$

3.2 Metric projection onto prox-regular set

First we begin by recalling that, for a given $r \in]0, +\infty]$, a subset C of the Hilbert space H is (uniformly) r-prox-regular (see, [30]) if for any $u \in C$ and for any $\xi \in N_C(u)$ with $\|\xi\| < 1$, then u is the unique nearest point of $u + r^{-1}\xi$ in C.

The following Theorem provides some properties of the proximal and Clarke subdifferentials of the function distance $d_C(\cdot)$ when the set C is r-prox-regular. It also summarizes some important consequences of the prox-regularity property which will be needed in the sequel of the paper. For the proof of these results we refer the reader to [30, 4].

Theorem 3.2.1. *Let C be a nonempty closed subset in the Hilbert space H and let $r > 0$. If the subset C is r-prox-regular, then the following hold:*

a) For any point v in the open enlargement $E_r^\gamma(C)$, the mapping $\mathrm{P}_C(v)$ exists and is continuous;

b) For any $v \in U_r^\gamma(c)$ and $y = \mathrm{P}_C(v)$ we have $y \in \operatorname{Proj}_C\Big(y + r\frac{v-y}{\|v-y\|}\Big)$;

c) The Clarke and the proximal subdifferentials of $d_C(\cdot)$ coincide at all points $v \in E_r^\gamma(C)$;

d) The Clarke and the proximal normal cone to C coincide at all points $u \in C$.

We will also need the following lemma from [4].

Lemma 3.2.1. *Let $r > 0$. Assume that $C(t)$ is r-prox-regular for all $t \in [0, T]$ and that there exists an absolutely continuous function $\vartheta : [0, T] \to \mathbb{R}$ such that*

$$|d(x, C(t)) - d(x, C(t))| \le |\vartheta(t) - \vartheta(s)| \quad \textit{for all } s, t \in [0, T].$$

Then, for any given $0 < \delta < r$, the following holds:

> *For any $s \in [0, T]$, any sequence $(u_n)_n$ converging to $u \in C(s) + (r - \delta)\mathbb{B}$ in $(H, \|\cdot\|)$ ($(u_n)_n$ is not necessarily in $C(s_n)$), any sequence $(s_n)_n$ in $[0, T]$ converging to s and any sequence $(\zeta_n)_n$ converging weakly to ζ in $(H, w(H, H))$ with $\zeta_n \in \partial d_{C(s_n)}(u_n)$, we have $\zeta \in \partial d_{C(s)}(u)$.*

For several other important geometric concepts of regularity in nonsmooth analysis, we refer to [3, 10, 13]. Consider now the behaviour of $\mathrm{P}_C(u)$ with respect to the r-prox-regular set C when we endow the space of r-prox-regular sets with the Hausdorff distance.

Theorem 3.2.2. *Let C and C' be r-prox-regular sets of the Hilbert space H for a constant $r > 0$ and let $\gamma \in]0, 1[$. Then for all $u \in U_r^\gamma(C)$ and $v \in U_r^\gamma(C')$ we have*

$$\|\mathrm{P}_C(u) - \mathrm{P}_{C'}(v)\| \le (1-\gamma)^{-1}\|u - v\| + \sqrt{\frac{2\gamma r}{1-\gamma}}\Big(\mathrm{Haus}\,(C, C')\Big)^{1/2}.$$

Proof. Let $u \in U_r^\gamma(C)$, $v \in U_r^\gamma(C')$ be fixed. Put $x := \mathrm{P}_C(u)$ and $y := \mathrm{P}_{C'}(v)$ (note that the projections exist according to Theorem 3.2.1). Put also $h := \mathrm{Haus}\,(C, C')$ and observe that $d(y, C) \le h$ because $y \in C'$. Suppose for a moment that $x \ne u$ and $y \ne v$. By that Theorem 3.2.1 we obtain $x \in \mathrm{Proj}\,_C(x + r\frac{u-x}{\|u-x\|})$ and $y \in \mathrm{Proj}\,_{C'}(y + r\frac{v-y}{\|v-y\|})$, which entails for all $z \in C$

$$\left\|x + r\frac{u-x}{\|u-x\|} - x\right\| \le \left\|x + r\frac{u-x}{\|u-x\|} - z\right\|$$

$\Leftrightarrow$

$$r - \|z - y\| \le \left\|x + r\frac{u-x}{\|u-x\|} - z\right\| - \|z - y\|$$

thus,

$$r - h \leq r - d(y, C) \leq \left\| x + r\frac{u-x}{\|u-x\|} - z \right\| - \|z - y\| \leq \left\| x + r\frac{u-x}{\|u-x\|} - y \right\|.$$

We deduce that

$$r^2 - 2rh \leq r^2 - 2rh + h^2 \leq \|x-y\|^2 + \frac{2r}{\|u-x\|}\langle u-x, x-y\rangle + r^2,$$

and the inequality between the first and the third member is equivalent to

$$\|u-x\|(\|x-y\|^2 + 2rh) \geq 2r\langle u-x, y-x\rangle.$$

This gives without the restriction $x \neq u$ that

$$\gamma r(\|x-y\|^2 + 2rh) \geq 2r\langle u-x, y-x\rangle.$$

Likewise we have

$$\gamma r(\|x-y\|^2 + 2rh) \geq 2r\langle v-y, x-y\rangle.$$

Adding both inequalities we obtain

$$\gamma(\|x-y\|^2 + 2rh) \geq \langle v-u, x-y\rangle + \|x-y\|^2$$

$\Leftrightarrow$

$$2\gamma rh + \langle u-v, x-y\rangle \geq (1-\gamma)\|x-y\|^2$$

thus

$$\frac{2\gamma rh}{1-\gamma} + \frac{1}{1-\gamma}\|u-v\|\|x-y\| \geq \|x-y\|^2.$$

This yields

$$\begin{aligned}\left(\|x-y\| - \frac{1}{2(1-\gamma)}\|u-v\|\right)^2 &\leq \frac{2\gamma rh}{1-\gamma} + \left(\frac{1}{2(1-\gamma)}\|u-v\|\right)^2 \\ &\leq \left(\sqrt{\frac{2\gamma rh}{1-\gamma}} + \frac{1}{2(1-\gamma)}\|u-v\|\right)^2,\end{aligned}$$

hence

$$\|x-y\| - \frac{1}{2(1-\gamma)}\|u-v\| \leq \sqrt{\frac{2\gamma rh}{1-\gamma}} + \frac{1}{2(1-\gamma)}\|u-v\|,$$

and this translates the desired inequality of the theorem. $\square$

The result can be applied to r-prox-regular moving set $C(t,u)$ satisfying

$$|d(x, C(t,u)) - d(x, C(t,v))| \leq L\|u - v\| \tag{3.1}$$

whenever $t \in [0,T]$ and $x, u, v \in H$ where L is some real constant with $L \in [0,1[$; indeed the latter inequality is equivalent to

$$\mathrm{Haus}\,(C(t,u), C(t,v)) \leq L\|u - v\|.$$

This application has as an immediate consequence the following result.

Corollaire 3.2.1. *Let $C(t,u)$ be r-prox-regular moving sets of the Hilbert space H for a constant $r > 0$ which satisfy (3.1), and let $\gamma \in]0,1[$. Then, for all $u, v \in H$ and $x \in U_r^\gamma(C(t,u)) \cap U_r^\gamma(C(t,v))$, we have*

$$\|\mathrm{P}_{C(t,u)}(x) - \mathrm{P}_{C(t,v)}(x)\| \leq \sqrt{\frac{2\gamma r L}{1-\gamma}}\|u - v\|^{1/2}.$$

Consider now for each $(t,x) \in [0,T] \times H$ fixed, the mapping ϕ from $\mathrm{Dom}\,\mathrm{P}_{C(t)}$ defined by $u \mapsto \mathrm{P}_{C(t,u)}(x)$. Thus, Corollary 3.2.1 above establishes the local Hölder continuity of ϕ on $U_r^\gamma(C(t,u))$ whenever the variable set $C(t,u)$ is r-prox-regular.

3.3 Existence of solution of the general perturbed sweeping process differential inclusion

We shall deal with two multimappings $C : [0,T] \times H \rightrightarrows H$ with nonempty closed values and $G : [0,T] \times H \rightrightarrows H$ with nonempty closed convex values. They are required to satisfy the following assumptions:

($\mathcal{H}_1$) The multimapping G is scalarly upper semicontinuous with respect to both variables (that is, for each $y \in H$ the function $(t,u) \to \sigma(y, G(t,u))$ is upper semicontinuous) and, for some real $\alpha > 0$

$$d\big(0, G(t,u)\big) \leq \alpha$$

for all $t \in [0,T]$ and $u \in H$ with $u \in C(t,u)$;

($\mathcal{H}_2$) For each $t \in [0,T]$ and each $u \in H$, the sets $C(t,u)$ are r-prox-regular for some constant $r > 0$;

($\mathcal{H}_3$) There are real constants $L_1 > 0$, $L_2 \in]0,1[$ such that, for all $t, s \in [0,T]$ and $x, y, u, v \in H$

$$\big|d\big(x, C(t,u)\big) - d\big(y, C(s,v)\big)\big| \le \|x-y\| + L_1|t-s| + L_2\|u-v\|;$$

($\mathcal{H}_4$) For any bounded subset $A \subset H$, the set $C([0,T] \times A)$ is relatively ball-compact, that is, the intersection of $C([0,T] \times A)$ with any closed ball of H is relatively compact in H.

Remark 3.3.1. *Note that the multimapping is scalarly upper semicontinuous whenever it is $\|\cdot\| \times$ weak upper semicontinuous in the usual sense, that is, for every $(t_0, x_0) \in [0,T] \times H$ and every weak open set $W \supset G(t_0, x_0)$ there exists some $\|\cdot\|$-neighbohood V of (t_0, x_0) such that*

$$W \supset G(t,x) \text{ for all } (t,x) \in V.$$

Theorem 3.3.1. *Assume that H is a Hilbert space, that $(\mathcal{H}_1 - \mathcal{H}_4)$ hold. Then, for any $u_0 \in H$ with $u_0 \in C(0, u_0)$, there exists a Lipschitz continuous mapping $u : [0,T] \to H$ such that*

$$(\mathcal{D}) \quad \begin{cases} \dot{u}(t) \in -N_{C\left(t,u(t)\right)}\big(u(t)\big) + G\big(t, u(t)\big) \quad \text{a.e } t \in [0,T], \\ u(t) \in C\big(t, u(t)\big) \ \forall t \in [0,T], \\ u(t) = u_0 + \int_0^t \dot{u}(s)ds \ \forall t \in [0,T], \end{cases}$$

that is, $u(\cdot)$ is a Lipschitz solution of the differential inclusion $(\mathcal{D})$ with $\|\dot{u}(t)\| \le \frac{L_1+2\alpha}{1-L_2}$ a.e. $t \in [0,T]$.

Proof. We will construct a sequence of absolutely continuous mappings $(u_n(\cdot))$ which has a subsequence converging pointwise to a solution of $(\mathcal{D})$.

Consider some integer $p \ge 1$ such that

$$\frac{T}{p} < \frac{r(1-L_2)}{2\big(\alpha(1+3L_2) + L_1(1+L_2)\big)}. \tag{3.2}$$

For each integer $n \ge 1$, we consider the partition of $[0,T]$ by the points $t_k^n = k\frac{T}{p^n}$, $k = 0, 1, \cdots, p^n$. For each $(t,x) \in [0,T] \times H$ denote by $g(\cdot,\cdot)$ the element of minimal norm of the closed convex set $G(t,x)$ of H, that is,

$$g(t,x) = \mathrm{P}_{G(t,x)}(0).$$

Put $x_0^n := u_0 \in C(t_0^n, u_0)$.

Step 1. We construct $x_0^n, x_1^n, \cdots, x_{p^n}^n$ in H such that for each $k = 0, 1, \cdots, p^n - 1$, the following inclusions hold

$$x_{k+1}^n \in C(t_{k+1}^n, x_{k+1}^n) \tag{3.3}$$

$$x_k^n + \frac{T}{p^n} g(t_k^n, x_k^n) - x_{k+1}^n \in N_{C(t_{k+1}^n, x_{k+1}^n)}(x_{k+1}^n), \tag{3.4}$$

along with the inequality

$$\|x_{k+1}^n - x_k^n\| \le \frac{L_1 + 2\alpha}{1 - L_2} \frac{T}{p^n}. \tag{3.5}$$

Observe first by $(\mathcal{H}_1)$ that $\|g(t_0^n, u_0)\| \le \alpha$. Then, for any $v \in B(u_0, 2\frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n})$, we have

$$\begin{aligned}
& d\Big(u_0 + \frac{T}{p^n} g(t_0^n, u_0), C(t_1^n, v)\Big) \\
& \quad \le d\Big(u_0 + \frac{T}{p^n} g(t_0^n, u_0), C(t_0^n, u_0)\Big) + L_1 |t_1^n - t_0^n| + L_2 \|v - u_0\| \\
& \quad \le \|\frac{T}{p^n} g(t_0^n, u_0)\| + L_1 \frac{T}{p^n} + 2L_2 \frac{L_1 + 2\alpha}{1 - L_2} \frac{T}{p^n} \\
& \quad \le \Big(\alpha + L_1 + 2L_2 \frac{L_1 + 2\alpha}{1 - L_2}\Big) \frac{T}{p^n} \\
& \quad = \frac{\alpha(1 + 3L_2) + L_1(1 + L_2)}{1 - L_2} \frac{T}{p^n} \\
& \quad < \frac{1}{2} r \quad \text{according to (3.2).}
\end{aligned}$$

Since $C(t_1^n, v)$ is r-prox-regular, Theorem 3.2.1 guarantees, for every $v \in B\big(u_0, 2\frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}\big)$, that

$$\phi_1(v) := \mathrm{P}_{C(t_1^n, v)}\Big(u_0 + \frac{T}{p^n} g\big(t_0^n, u_0\big)\Big) \tag{3.6}$$

is well defined. Taking into account Corollary 3.2.1, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ we see that the mapping $\phi_1 : B(u_0, 2\frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}) \to H$ is locally Hölder continuous. Further, for all $v \in B[u_0, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}]$, we have $\phi_1(v) \in B[u_0, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}]$. Indeed, for any such v it follows from the definition of $\phi_1(v)$ and from $(\mathcal{H}_3)$ (as above) that

$$\begin{aligned}
\|\phi_1(v) - u_0\| &\leq \|\phi_1(v) - (u_0 + \frac{T}{p^n} g(t_0^n, u_0))\| + \frac{T}{p^n}\|g(t_0^n, u_0)\| \\
&= d\Big(u_0 + \frac{T}{p^n} g(t_0^n, u_0), C(t_1^n, v)\Big) + \frac{T}{p^n}\|g(t_0^n, u_0)\| \\
&\leq d\Big(u_0 + \frac{T}{p^n} g(t_0^n, u_0), C(t_0^n, u_0)\Big) + L_1|t_1^n - t_0^n| \\
&+ L_2\|v - u_0\| + \frac{T}{p^n}\|g(t_0^n, u_0)\| \\
&\leq \frac{T}{p^n}\|g(t_0^n, u_0)\| + L_1|t_1^n - t_0^n| + L_2\|v - u_0\| + \frac{T}{p^n}\|g(t_0^n, u_0)\| \\
&\leq \Big(2\alpha + L_1 + L_2\frac{L_1 + 2\alpha}{1 - L_2}\Big)\frac{T}{p^n} = \frac{L_1 + 2\alpha}{1 - L_2}\frac{T}{p^n}.
\end{aligned}$$

Consequently, for all $v \in B[u_0, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}]$

$$\phi_1(v) \in C\Big(t_1^n, B\Big[u_0, \frac{L_1 + 2\alpha}{1 - L_2}\frac{T}{p^n}\Big]\Big) \bigcap B\Big[u_0, \frac{L_1 + 2\alpha}{1 - L_2}\frac{T}{p^n}\Big],$$

then by $(\mathcal{H}_4)$, the set $\phi_1\big(B[u_0, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}]\big)$ is relatively compact. So, the mapping ϕ_1 is continuous from the closed convex set $B[u_0, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}]$ into itself and the range of $B[u_0, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}]$ by ϕ_1 is relatively compact. We may then apply to the mapping ϕ_1 the extended Schauder fixed point theorem established in [21] or [29] to obtain some $x_1^n \in B[u_0, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}]$ such that $x_1^n = \phi_1(x_1^n)$.This ensures in particular

$$x_1^n \in C(t_1^n, x_1^n) \ \text{ and } \ \|x_1^n - x_0^n\| \leq \frac{L_1 + 2\alpha}{1 - L_2}\frac{T}{p^n}$$

and by (3.6)

$$u_0 + \frac{T}{p^n} g(t_0^n, u_0) - x_1^n \in N_{C(t_1^n, x_1^n)}(x_1^n).$$

Now, suppose that, for $0, 1, \cdots, k+1$, with $k + 1 \leq p^n - 1$ the points $x_0^n, x_1^n, \cdots, x_{k+1}^n$ have been constructed so that properties (3.3), (3.4) and (3.5) hold true. By construction

$$x_{k+1}^n \in C(t_{k+1}^n, x_{k+1}^n)$$

and hence according to $(\mathcal{H}_1)$

$$\|g(t_{k+1}^n, x_{k+1}^n)\| \leq \alpha.$$

Further, as above for any $v \in B(x^n_{k+1}, 2\frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n})$, we have

$$
\begin{aligned}
&d\Big(x^n_{k+1} + \frac{T}{p^n}g(t^n_{k+1}, x^n_{k+1}), C(t^n_{k+2}, v)\Big) \\
&\le d\Big(x^n_{k+1} + \frac{T}{p^n}g(t^n_{k+1}, x^n_{k+1}), C(t^n_{k+1}, x^n_{k+1})\Big) + L_1|t^n_{k+2} - t^n_{k+1}| + L_2\|v - x^n_{k+1}\| \\
&\le \|\frac{T}{p^n}g(t^n_{k+1}, x^n_{k+1})\| + L_1\frac{T}{p^n} + 2L_2\frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n} \\
&\le \Big(\alpha + L_1 + 2L_2\frac{L_1+2\alpha}{1-L_2}\Big)\frac{T}{p^n} = \frac{\alpha(1+3L_2) + L_1(1+L_2)}{1-L_2}\frac{T}{p^n} \\
&< \frac{1}{2}r \quad \text{according to (3.2).}
\end{aligned}
$$

The r-prox-regularity of $C(t^n_{k+2}, v)$ ensures by Theorem 3.2.1 that

$$
\phi_{k+2}(v) := \mathrm{P}_{C(t^n_{k+2},v)}\Big(x^n_{k+1} + \frac{T}{p^n}g\big(t^n_{k+1}, x^n_{k+1}\big)\Big) \tag{3.7}
$$

is well defined. Thus, in an analogous way as above, ϕ_{k+2} from $B(x^n_{k+1}, 2\frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n})$ into H is locally Hölder continuous and for all $v \in B[x^n_{k+1}, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}]$ we have

$$
\phi_{k+2}(v) \in C\bigg(t^n_{k+2}, B\Big[x^n_{k+1}, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}\Big]\bigg) \bigcap B\Big[x^n_{k+1}, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}\Big]. \tag{3.8}
$$

Indeed,

$$
\begin{aligned}
\|\phi_{k+2}(v) - x^n_{k+1}\| &\le \|\phi_{k+2}(v) - (x^n_{k+1} + \frac{T}{p^n}g(t^n_{k+1}, x^n_{k+1}))\| + \frac{T}{p^n}\|g(t^n_{k+1}, x^n_{k+1})\| \\
&= d\Big(x^n_{k+1} + \frac{T}{p^n}g(t^n_{k+1}, x^n_{k+1}), C(t^n_{k+2}, v)\Big) + \frac{T}{p^n}\|g(t^n_{k+1}, x^n_{k+1})\| \\
&\le d\Big(x^n_{k+1} + \frac{T}{p^n}g(t^n_{k+1}, x^n_{k+1}), C(t^n_{k+1}, x^n_{k+1})\Big) + L_1|t^n_{k+2} - t^n_{k+1}| \\
&+ L_2\|v - x^n_{k+1}\| + \frac{T}{p^n}\|g(t^n_{k+1}, x^n_{k+1})\| \\
&\le \frac{T}{p^n}\|g(t^n_{k+1}, x^n_{k+1})\| + L_1|t^n_{k+2} - t^n_{k+1}| + L_2\|v - x^n_{k+1}\| \\
&+ \frac{T}{p^n}\|g(t^n_{k+1}, x^n_{k+1})\| \\
&\le \Big(2\alpha + L_1 + L_2\frac{L_1+2\alpha}{1-L_2}\Big)\frac{T}{p^n} = \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}.
\end{aligned}
$$

This justifies the inclusion (3.8), then by $(\mathcal{H}_4)$, the set $\phi_{k+2}(B[x^n_{k+1}, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}])$ is relatively compact. So, the mapping ϕ_{k+2} is continuous from the closed convex set $B[x^n_{k+1}, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}]$ into itself and the range of $B[x^n_{k+1}, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}]$ by ϕ_{k+2} is relatively compact. We may then apply to the mapping ϕ_{k+2} the extended Schauder fixed point theorem established in [21] or [29] to obtain some $x^n_{k+2} \in B[x^n_{k+1}, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}]$ such that $x^n_{k+2} = \phi_{k+2}(x^n_{k+2})$. This ensures in particular

$$x^n_{k+2} \in C(t^n_{k+2}, x^n_{k+2}) \text{ and } \|x^n_{k+2} - x^n_{k+1}\| \le \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}$$

and by (3.7)

$$x^n_{k+1} + \frac{T}{p^n} g(t^n_{k+1}, x^n_{k+1}) - x^n_{k+2} \in N_{C(t^n_{k+2}, x^n_{k+2})}(x^n_{k+2}).$$

Therefore, the construction of $x^n_0, x^n_1, \cdots, x^n_{p^n}$ is achieved by induction such that properties (3.3), (3.4) and (3.5) for $k = 0, 1, \cdots, p^n - 1$ are satisfied.

Step 2. Construction of $u_n(\cdot)$.
For any $t \in [t^n_k, t^n_{k+1}]$ with $k = 0, 1, \cdots, p^n - 1$, put

$$u_n(t) := \frac{t^n_{k+1} - t}{t^n_{k+1} - t^n_k} x^n_k + \frac{t - t^n_k}{t^n_{k+1} - t^n_k} x^n_{k+1}.$$

Thus, for almost all $t \in [t^n_k, t^n_{k+1}]$,

$$\dot{u}_n(t) = -\frac{x^n_k}{t^n_{k+1} - t^n_k} + \frac{x^n_{k+1}}{t^n_{k+1} - t^n_k} = -\frac{p^n}{T}(x^n_k - x^n_{k+1}).$$

By construction, (3.3), (3.4), (3.5) and the latter equalities give

$$u_n(t^n_{k+1}) \in C\big(t^n_{k+1}, u_n(t^n_{k+1})\big) \tag{3.9}$$

$$-\dot{u}_n(t) \in N_{C\left(t^n_{k+1}, u_n(t^n_{k+1})\right)}\big(u_n(t^n_{k+1})\big) - g\big(t^n_k, u_n(t^n_k)\big) \text{ a.e } t \in [t^n_k, t^n_{k+1}[\tag{3.10}$$

with

$$\|\dot{u}_n(t)\| = \frac{p^n}{T}\|x^n_k - x^n_{k+1}\| \le \frac{L_1+2\alpha}{1-L_2} := M. \tag{3.11}$$

Put

$$\delta_n(t) := \begin{cases} t^n_k & \text{if} \quad t \in [t^n_k, t^n_{k+1}[\\ t^n_{p^n-1} & \text{if} \quad t = T, \end{cases}$$

and

$$\theta_n(t) := \begin{cases} t_{k+1}^n & \text{if} \quad t \in [t_k^n, t_{k+1}^n[\\ T & \text{if} \quad t = T. \end{cases}$$

Observe that for each $t \in [0, T]$, choosing k such that $t \in [t_k^n, t_{k+1}^n[$ if $t < T$ and $k = p^n - 1$ if $t = T$, we have

$$|\delta_n(t) - t| \leq |t_{k+1}^n - t_k^n| = \frac{T}{p^n}, \text{ so } \delta_n(t) \to t \text{ as } n \to +\infty,$$

and similarly $\theta_n(t) \to t$ as $n \to +\infty$. Further, for each $t \in [t_k^n, t_{k+1}^n[$, the definitions of $\delta_n(\cdot)$ and $\theta_n(\cdot)$ combined with (3.9) and (3.10) yield

$$u_n\big(\theta_n(t)\big) \in C\Big(\theta_n(t), u_n\big(\theta_n(t)\big)\Big) \tag{3.12}$$

$$-\dot{u}_n(t) \in N_{C\Big(\theta_n(t), u_n\big(\theta_n(t)\big)\Big)}\Big(u_n\big(\theta_n(t)\big)\Big) - g\Big(\delta_n(t), u_n\big(\delta_n(t)\big)\Big) \text{ a.e } t \in [0, T] \tag{3.13}$$

Step 3. Convergence of a subsequence of $(u_n(\cdot))$ to some absolutely continuous mapping $u(\cdot)$.
For each $k = 0, 1, \cdots, p^n - 1$, it results from (3.5) that

$$\|x_{k+1}^n - u_0\| \leq \|x_{k+1}^n - x_k^n\| + \cdots + \|x_1^n - x_0^n\| \leq (k+1)\frac{L_1 + 2\alpha}{1 - L_2}\frac{T}{p^n},$$

so

$$\|x_{k+1}^n\| \leq \|u_0\| + \frac{L_1 + 2\alpha}{1 - L_2}T := \beta.$$

Fix any $t \in [0, T]$ and consider, for any infinite subset $N \subset \mathbb{N}$, the sequence $(u_n(t))_{n \in N}$. It follows from (3.12) that $u_n(\theta_n(t)) \in C(\theta_n(t), u_n(\theta_n(t))) \cap \beta\mathbb{B}$, which implies that $u_n(\theta_n(t)) \in C([0, T] \times \beta\mathbb{B}) \cap \beta\mathbb{B}$. By $(\mathcal{H}_4)$ the sequence $(u_n(\theta_n(t)))$ is relatively compact, so there is an infinite subset $N_0 \subset N$ such that $(u_n(\theta_n(t)))_{n \in N_0}$ converges to some vector $l(t) \in H$. Putting $h_n(t) := u_n(\theta_n(t)) - u_n(t)$ for all $n \in N_0$, by (3.11), we obtain

$$\|h_n(t)\| \leq \int_t^{\theta_n(t)} \|\dot{u}_n(s)\| ds \leq M(\theta_n(t) - t) \underset{n \to \infty}{\longrightarrow} 0.$$

Then, $(u_n(t))_{n \in N_0}$ converges to $l(t)$, thus the set $\{u_n(t) : n \in \mathbb{N}\}$ is relatively compact in H. The sequence $(u_n)_{n \in \mathbb{N}}$ being in addition equicontinuous according to (3.11), this sequence $(u_n)_{n \in \mathbb{N}}$ is relatively compact in $\mathcal{C}_H(0, T)$, so we can extract a subsequence of $(u_n)_{n \in \mathbb{N}}$ (that we do not relabel) which converges uniformly to u on $[0, T]$. By the inequality (3.11) again there is a subsequence of $(\dot{u}_n)_{n\in\mathbb{N}}$ (that we do not relabel) which converges $w(\mathrm{L}^1_H, \mathrm{L}^\infty_H)$ in $\mathrm{L}^1_H(0,T)$ to a mapping $w \in \mathrm{L}^1_H(0,T)$ with $\|w(t)\| \le M$ a.e. $t \in [0,T]$. Fixing $t \in [0,T]$ and taking any $\xi \in H$, the above weak convergence in $\mathrm{L}^1_H(0,T)$ yields

$$\lim_{n\to\infty} \int_0^T \langle \mathbb{1}_{[0,t]}(s)\xi, \dot{u}_n(s)\rangle ds = \int_0^T \langle \mathbb{1}_{[0,t]}(s)\xi, w(s)\rangle ds,$$

or equivalently

$$\lim_{n\to\infty} \langle \xi, u_0 + \int_0^t \dot{u}_n(s)ds\rangle = \langle \xi, u_0 + \int_0^t \dot{w}(s)ds\rangle.$$

This means, for each $t \in [0,T]$, that $u_n(t) \underset{n\to\infty}{\longrightarrow} u_0 + \int_0^t w(s)ds$ weakly in H. Since the sequence $(u_n(t))_{n\in\mathbb{N}}$ also converges strongly to $u(t)$ in H, it ensures that $u(t) = u_0 + \int_0^t w(s)ds$, so the mapping $u(\cdot)$ is absolutely continuous on $[0,T]$ with $\dot{u} = w$. The mapping $u(\cdot)$ is even Lipschitz on $[0,T]$ with M as a Lipschitz constant therein.

Step 4. We show now that $u(\cdot)$ is a solution of $(\mathcal{D})$.
Put

$$z_n(t) := g(\delta_n(t), u_n(\delta_n(t))) \text{ for all } t \in [0,T],$$

and observe that z_n is a step mapping. Since $\|g(\delta_n(t), u_n(\delta_n(t)))\| \le \alpha$ for all $n \in \mathbb{N}$ and $t \in [0,T]$, we may suppose (taking a subsequence if necessary) that the sequence $(z_n(\cdot))_n$ converges $w(\mathrm{L}^1_H, \mathrm{L}^\infty_H)$ in $\mathrm{L}^1_H(0,T)$ to a mapping $z(\cdot) \in \mathrm{L}^1_H(0,T)$ with $\|z(t)\| \le \alpha$ a.e $t \in [0,T]$.

For all $t \in [0,T]$ we have $u(t) \in C(t, u(t))$. Indeed, by $(\mathcal{H}_3)$ and (3.11)

$$\begin{aligned} d\Big(u_n(t), C\big(t, u(t)\big)\Big) & \\ &\le \|u_n(t) - u_n\big(\theta_n(t)\big)\| + L_1|t-\theta_n(t)| + L_2\|u(t) - u_n\big(\delta_n(t)\big)\| \\ &\le (M+L_1)|t-\theta_n(t)| + L_2M|\delta_n(t)-t| + L_2\|u(t)-u_n(t)\| \end{aligned}$$

then,

$$d(u_n(t), C(t,u(t))) \underset{n\to\infty}{\longrightarrow} 0, \text{ so } d(u(t), C(t,u(t))) = 0 \text{ and } u(t) \in C(t,u(t)).$$

Further, from the inequality $\|\dot{u}_n(t) - z_n(t)\| \le M + \alpha =: \gamma$ a.e. and from the inclusion (3.13) it follows for a.e. $t \in [0,T]$ that

$$\begin{aligned} -\dot{u}_n(t) + z_n(t) &\in N_{C\left(\theta_n(t), u_n\left(\theta_n(t)\right)\right)}\Big(u_n\big(\theta_n(t)\big)\Big) \bigcap \gamma\mathbb{B} \\ &= \gamma \partial d_{C\left(\theta_n(t), u_n\left(\theta_n(t)\right)\right)}\Big(u_n\big(\theta_n(t)\big)\Big), \end{aligned} \tag{3.14}$$

$$(3.15)\qquad z_n(t) \in G\Big(\delta_n(t), u_n\big(\delta_n(t)\big)\Big).$$

Since $(-\dot{u}_n + z_n, z_n)_n$ converges weakly in $\mathrm{L}^1_{H\times H}(0,T)$ to $(-\dot{u} + z, z)$, by Mazur theorem, there are

$$(3.16)\qquad \xi_n \in \mathrm{co}\,\{-\dot{u}_q + z_q : q \geq n\} \text{ and } \zeta_n \in \mathrm{co}\,\{z_q : q \geq n\}$$

such that $(\xi_n, \zeta_n)_n$ converges strongly in $\mathrm{L}^1_{H\times H}(0,T)$ to $(-\dot{u}+z, z)$. Extracting a subsequence if necessary we suppose that $(\xi_n(\cdot), \zeta_n(\cdot))_n$ converges a.e. to $(-\dot{u}(\cdot) + z(\cdot), z(\cdot))$, then there is a Lebesgue negligible set $S \subset [0,T]$ such that for every $t \in [0,T]\backslash S$ on one hand $(\xi_n(t), \zeta_n(t)) \to (-\dot{u}(t) + z(t), z(t))$ strongly in H and on the other hand the inclusions (3.14) and (3.15) hold true for every integer n as well as the inclusions

$$-\dot{u}(t)+z(t) \in \bigcap_n \overline{\mathrm{co}}\,\Big\{-\dot{u}_q(t)+z_q(t) : q \geq n\Big\} \text{ and } z(t) \in \bigcap_n \overline{\mathrm{co}}\,\Big\{z_q(t) : q \geq n\Big\}.$$

It results from (3.14) and (3.15) that for any $n \in \mathbb{N}$, any $t \in [0,T]\backslash S$, and for any $y \in H$

$$(3.17)\qquad \big\langle y, -\dot{u}_n(t) + z_n(t)\big\rangle \leq \sigma\Bigg(y, \gamma\partial d_{C\left(\theta_n(t), u_n\left(\theta_n(t)\right)\right)}\Big(u_n\big(\theta_n(t)\big)\Big)\Bigg)$$

and

$$(3.18)\qquad \big\langle y, z_n(t)\big\rangle \leq \sigma\Bigg(y, G\Big(\delta_n(t), u_n\big(\delta_n(t)\big)\Big)\Bigg).$$

Further, for each $n \in \mathbb{N}$ and any $t \in [0,T]\backslash S$, from (3.16) we have

$$\big\langle y, \xi_k(t)\big\rangle \leq \sup_{q\geq n} \big\langle y, -\dot{u}_q(t) + z_q(t)\big\rangle \text{ for all } k \geq n$$

and

$$\big\langle y, \zeta_k(t)\big\rangle \leq \sup_{q\geq n} \big\langle y, z_q(t)\big\rangle \text{ for all } k \geq n$$

and taking the limit in both inequalities as $k \to +\infty$ gives through (3.17) and (3.18)

$$\begin{aligned}\big\langle y, -\dot{u}(t) + z(t)\big\rangle &\leq \sup_{q\geq n} \big\langle y, -\dot{u}_q(t) + z_q(t)\big\rangle \\ &\leq \sup_{q\geq n} \sigma\Bigg(y, \gamma\partial d_{C\left(\theta_q(t), u_q\left(\theta_q(t)\right)\right)}\Big(u_q\big(\theta_q(t)\big)\Big)\Bigg)\end{aligned}$$

and

$$\Big\langle y, z(t)\Big\rangle \leq \sup_{q\geq n} \Big\langle y, z_q(t)\Big\rangle \leq \sup_{q\geq n} \sigma\Bigg(y, G\Big(\delta_q(t), u_q\big(\delta_q(t)\big)\Big)\Bigg),$$

which ensures that

$$\Big\langle y, -\dot{u}(t) + z(t)\Big\rangle \leq \limsup_{n\to+\infty} \sigma\Bigg(y, \gamma\partial d_{C\Big(\theta_n(t), u_n\big(\theta_n(t)\big)\Big)}\Big(u_n\big(\theta_n(t)\big)\Big)\Bigg)$$

and

$$\Big\langle y, z(t)\Big\rangle \leq \limsup_{n\to+\infty} \sigma\Bigg(y, G\Big(\delta_n(t), u_n\big(\delta_n(t)\big)\Big)\Bigg).$$

According to $(\mathcal{H}_3)$ and Lemma 3.2.1, the multimapping $(t, u, x) \to \partial d_{C(t,u)}(x)$ takes on weakly compact convex values and is upper semicontinuous from $[0, T] \times H \times H$ into $(H, w(H, H))$, hence for each $y \in H$ the real-valued function $\sigma(y, \gamma\partial d_{C(\cdot,\cdot)}(\cdot))$ is upper semicontinuous on $[0, T] \times H \times H$. Further, $\sigma(y, G(\cdot, \cdot))$ is also upper semicontinuous on $[0, T] \times H$ by assumption $(\mathcal{H}_1)$. It follows that, for every $t \in [0, T]\backslash S$ and every $y \in H$,

$$\Big\langle y, -\dot{u}(t) + z(t)\Big\rangle \leq \sigma\Big(y, \gamma\partial d_{C\big(t,u(t)\big)}\big(u(t)\big)\Big)$$

and

$$\Big\langle y, z(t)\Big\rangle \leq \sigma\Big(y, G\big(t, u(t)\big)\Big),$$

which ensures that $-\dot{u}(t) + z(t) \in \gamma\partial d_{C(t,u(t))}(u(t))$ and $z(t) \in G(t, u(t))$, consequently

$$\dot{u}(t) \in -N_{C\big(t,u(t)\big)}\big(u(t)\big) + z(t) \text{ a.e.}$$

$$z(t) \in G\big(t, u(t)\big) \text{ a.e.}$$

with

$$\|\dot{u}(t) - z(t)\| \leq \gamma.$$

The proof is complete. □

The next theorem proves on the whole interval $\mathbb{R}_+ := [0, +\infty[$, the existence of solution to the above evolution problem . In the result of Theorem 3.3.1, the solution is Lipschitz on the interval $[0, T]$, but in the theorem below, the solution is locally Lipschitz on $\mathbb{R}_+$.

Theorem 3.3.2. *Let $G : \mathbb{R}_+ \times H \rightrightarrows H$ be a multimapping which is scalarly upper semicontinuous with respect to both variables. Assume that H is a Hilbert space and that $(\mathcal{G}_1 - \mathcal{G}_4)$ below hold:*

($\mathcal{G}_1$) *There exists a non-negative function* $\beta(\cdot) \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+)$ *such that*

$$d\big(0, G(t,u)\big) \leq \beta(t)$$

for all $t \in \mathbb{R}_+$ *and* $u \in H$ *with* $u \in C(t,u)$*;*

($\mathcal{G}_2$) *For each* $t \in \mathbb{R}_+$ *and each* $u \in H$*, the sets* $C(t,u)$ *are nonempty closed in* H *and* r*-prox-regular for some constant* $r > 0$*;*

($\mathcal{G}_3$) *There are real constants* $L_1 > 0$, $L_2 \in]0,1[$ *such that, for all* $t, s \in \mathbb{R}_+$ *and* $x, y, u, v \in H$

$$\Big|d\big(x, C(t,u)\big) - d\big(y, C(s,v)\big)\Big| \leq \|x-y\| + L_1|t-s| + L_2\|u-v\|;$$

($\mathcal{G}_4$) *For any real* $T > 0$ *and any bounded subset* $A \subset H$*, the set* $C([0,T]\times A)$ *is ball-compact, that is, the intersection of* $C([0,T]\times A)$ *with any closed ball of* H *is relatively compact in* H*.*

Then, given $u_0 \in H$ *with* $u_0 \in C(0,u_0)$*, there exists a mapping* $u : \mathbb{R}_+ \to H$ *which is locally Lipschitz continuous on* $\mathbb{R}_+$ *and satisfies*

$$(\mathcal{D}_{\mathbb{R}_+}) \quad \begin{cases} \dot{u}(t) \in -N_{C\left(t,u(t)\right)}\big(u(t)\big) + G\big(t, u(t)\big) \quad \text{a.e } t \in \mathbb{R}_+, \\[2ex] u(t) \in C\big(t, u(t)\big) \quad \forall t \in \mathbb{R}_+, \\[2ex] u(t) = u_0 + \int_0^t \dot{u}(s)ds \quad \forall t \in \mathbb{R}_+. \end{cases}$$

Proof. Put $T_k = k$ for all $k \in \{0\} \cup \mathbb{N}$. It will suffice to prove that Theorem 3.3.1 applies on each interval $[T_k, T_{k+1}]$.

According to assumptions $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$ we have $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4$ hold on the interval $[T_0, T_1]$. Since $u_0 \in C(T_0, u_0)$, by Theorem 3.3.1 there exists a Lipschitz continuous mapping $u^0 : [T_0, T_1] \to H$ such that

$$\begin{cases} \dot{u}^0(t) \in -N_{C\left(t,u^0(t)\right)}\big(u^0(t)\big) + G\big(t, u^0(t)\big) \quad \text{a.e } \ t \in [T_0, T_1], \\[2ex] u^0(t) \in C\big(t, u^0(t)\big) \ \forall t \in [T_0, T_1], \\[2ex] u^0(T_0) = u_0. \end{cases}$$

Suppose $u^0, \cdots, u^{k-1}$ have been constructed such that, for $l = 0, \cdots, k-1$, $u^l : [T_l, T_{l+1}] \to H$ is Lipschitz continuous, $u^l(T_l) = u^{l-1}(T_l)$, $u^l(t) \in C(t, u^l(t))$ for all $t \in [T_l, T_{l+1}]$ and

$$\dot{u}^l(t) \in -N_{C\left(t,u^l(t)\right)}\big(u^l(t)\big) + G\big(t, u^l(t)\big) \quad \text{a.e } \ t \in [T_l, T_{l+1}].$$

In an analogous way as above, the hypotheses $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$ ensure that $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4$ hold on the interval $[T_k, T_{k+1}]$ and we have $u^{k-1}(T_k) \in C(T_k, u^{k-1}(T_k))$. It follows from Theorem 3.3.1 that there is a Lipschitz continuous mapping $u^k : [T_k, T_{k+1}] \to H$ such that

$$(3.19) \qquad \begin{cases} \dot{u}^k(t) \in -N_{C\left(t,u^k(t)\right)}\big(u^k(t)\big) + G\big(t, u^k(t)\big) \quad \text{a.e } t \in [T_k, T_{k+1}], \\ \\ u^k(t) \in C\big(t, u^k(t)\big) \ \forall t \in [T_k, T_{k+1}], \\ \\ u^k(T_k) = u^{k-1}(T_k). \end{cases}$$

So, we obtain by induction u^k for all $k \in \{0\} \cup \mathbb{N}$ with the above properties.

Let $u : \mathbb{R}_+ \to H$ be the mapping defined by

$$u(t) := u^k(t) \qquad \text{for all } t \in [T_k, T_{k+1}[\text{ with } k \in \{0\} \cup \mathbb{N}.$$

It is easily seen that u is locally Lipschitz continuous on $\mathbb{R}_+$. Therefore, it results from (3.19) that

$$\begin{cases} \dot{u}(t) \in -N_{C\left(t,u(t)\right)}\big(u(t)\big) + G\big(t, u(t)\big) \quad \text{a.e } t \in \mathbb{R}_+, \\ \\ u(t) \in C\big(t, u(t)\big) \ \forall t \in \mathbb{R}_+, \\ \\ u(0) = u^0(T_0) = u_0. \end{cases}$$

This proves the theorem. $\square$

The corollaries below are direct consquences of Theorem 3.3.1 and Theorem 3.3.2 respectively.

Corollaire 3.3.1. *Let $G : [0, T] \times H \rightrightarrows H$ be a multimapping which is scalarly upper semicontinuous with respect to both variables. Assume that H is a finite dimensional Euclidean space and that the assumptions below hold:*

- *There exists a positive real number α such that*

$$d\big(0, G(t, u)\big) \leq \alpha$$

  for all $t \in [0, T]$ and $u \in H$ with $u \in C(t, u)$;

- *For each $t \in [0, T]$ and each $u \in H$, the sets $C(t, u)$ are nonempty closed in H and r-prox-regular for some constant $r > 0$;*

- *There are real constants* $L_1 > 0$, $L_2 \in]0,1[$ *such that, for all* $t, s \in [0,T]$ *and* $x, y, u, v \in H$

$$\big|d\big(x, C(t,u)\big) - d\big(y, C(s,v)\big)\big| \leq \|x - y\| + L_1|t - s| + L_2\|u - v\|.$$

Then, given $u_0 \in H$ *with* $u_0 \in C(0, u_0)$*, there exists a mapping* $u : [0,T] \to H$ *which is Lipschitz continuous on* $[0,T]$ *and satisfies* $(\mathcal{D})$*. Further, we have* $\|\dot{u}(t)\| \leq \frac{L_1 + 2\alpha}{1 - L_2}$ *a.e.* $t \in [0,T]$.

Corollaire 3.3.2. *Let* $G : \mathbb{R}_+ \times H \rightrightarrows H$ *be a multimapping which is scalarly upper semicontinuous with respect to both variables. Assume that* H *is a finite dimensional Euclidean space and that the following assumptions hold:*

- *There exists a non-negative function* $\beta(\cdot) \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+)$ *such that*

$$d\big(0, G(t,u)\big) \leq \beta(t)$$

  for all $t \in \mathbb{R}_+$ *and* $u \in H$ *with* $u \in C(t,u)$*;*

- *For each* $t \in \mathbb{R}_+$ *and each* $u \in H$*, the sets* $C(t,u)$ *are nonempty closed in* H *and* r*-prox-regular for some constant* $r > 0$*;*

- *There are real constants* $L_1 > 0$, $L_2 \in]0,1[$ *such that, for all* $t, s \in \mathbb{R}_+$ *and* $x, y, u, v \in H$

$$\big|d\big(x, C(t,u)\big) - d\big(y, C(s,v)\big)\big| \leq \|x - y\| + L_1|t - s| + L_2\|u - v\|.$$

Then, given $u_0 \in H$ *with* $u_0 \in C(0, u_0)$*, there exists a mapping* $u : \mathbb{R}_+ \to H$ *which is locally Lipschitz continuous on* $\mathbb{R}_+$ *and satisfies* $(\mathcal{D}_{\mathbb{R}_+})$

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Chapter 4

Delay perturbation of sweeping process with prox-regular sets

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Abstract. For a general class of nonconvex sets, we discuss the existence of solutions for an evolution equation with delay of the form $\dot{u}(t) \in -N_{C(t,u(t))}(u(t)) + G(t, \Lambda(t)u)$ a.e in $[0, T]$. This is done with the Clarke normal cone $N_{C(t,u(t))}(u(t))$ of $C(t, u(t))$ at $u(t)$, under the assumption that the closed sets $C(t, x)$ are prox-regular and that $G : [0, T] \times \mathcal{C}_0 \rightrightarrows H$ is a set-valued mapping taking closed convex values and scalarly upper semicontinuous with respect to both variables.

Keyword : Differential inclusion; Sweeping process; Normal cone; Prox-regular set; Subdifferential

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Introduction

In this article we consider the evolution problem in a Hilbert space H

$$(\mathcal{D}_r) \qquad \begin{cases} \dot{u}(t) \in -N_{C\left(t,u(t)\right)}\big(u(t)\big) + G\big(t, \Lambda(t)u\big) \quad \text{a.e } t \in [0,T] \\ \\ u(t) \in C\big(t, u(t)\big) \quad \forall t \in [0,T] \\ \\ u(s) = \varphi(s) \; \forall s \in [-r, 0]; \; u(0) = u_0 \in C(0, u_0), \end{cases}$$

where $r > 0$ is a finite delay, $G : [0,T] \times \mathcal{C}_0 \rightrightarrows H$ is a set-valued mapping taking closed convex values, $C : [0,T] \times H \rightrightarrows H$ is a set-valued mapping with nonempty closed values and φ is an element of $\mathcal{C}_0$ with $\varphi(0) = u_0 \in C(0, u_0)$. Here $N_{C\left(t,u(t)\right)}(\cdot)$ denotes a normal cone to the set $C(t, u(t))$ and $\mathcal{C}_0 := \mathcal{C}_H(-r, 0)$ is the Banach space of all continuous mappings from $[-r, 0]$ to H equipped with the norm of uniform convergence $\|\cdot\|_{\infty,0}$. For any $t \in [0,T]$, the mapping $\Lambda(t)$ from $\mathcal{C}_T := \mathcal{C}_H(-r, T)$ into $\mathcal{C}_0$ is given by $\Lambda(t)u(s) = u(t+s)$ for all $s \in [-r, 0]$ and $u \in \mathcal{C}_T$. By a solution of $(\mathcal{D}_r)$ we mean a mapping $u : [-r, T] \to H$ such that its restriction on $[-r, 0]$ is equal to φ and its restriction to $[0,T]$ is absolutely continuous, that is, $u(t) = u_0 + \int_0^t \dot{u}(s)ds$, for all $t \in [0,T]$ with $\dot{u} \in \mathrm{L}^1_H(0,T)$, and such that the conditions in $(\mathcal{D}_r)$ are satisfied. Such perturbed both time-dependent and state-dependent sweeping processes with delay have been studied in the paper of C. Castaing, A. G Ibrahim and M. Yarou [8]; their approach strongly uses the convexity and ball-compactness assumption for $C(t,x)$ and G is bounded on $[0,T] \times \mathcal{C}_0$. We refer to [1, 8, 9, 22] for other works related to both time-dependent and state-dependent sweeping processes but without delay. We must also say that non perturbed sweeping processes have been introduced by J. J. Moreau [25, 26, 28]

Our main purpose in this paper is to prove existence result for $(\mathcal{D}_r)$ when C has prox-regular values. The paper is organized as follows. In section 1, we give notation which will be used throughout the paper and we recall some definitions and results. Section 2 is devoted to prove the existence of solution for $(\mathcal{D}_r)$.

4.1 Preliminaries and fundamental results

Throughout the paper H is a Hilbert space whose inner product is denoted by $\langle \cdot, \cdot \rangle$ and the associated norm by $\|\cdot\|$. The closed unit ball of H with center 0 will be denoted by $\mathbb{B}$ and $B(u, \eta)$ (respectively, $B[u, \eta]$) denotes the open (respectively, closed) ball of center $u \in H$ and radius $\eta > 0$. Given a real $T > 0$, we will denote by $\mathcal{C}_H(0,T)$ the space of all continuous mappings from $[0,T]$ to H, "a.e" denotes "for almost every where" and $\dot{u}$ is the derivative of u.

Let C, C' be two subsets of H and let v be a vector in H, the real $d(v,C)$ or $d_C(v) := \inf\{\|v-u\| : u \in C\}$ is the distance of the point v from the set C. We denote by

$$\mathrm{Haus}(C,C') = \max\left\{\sup_{u\in C} d(u,C'), \sup_{v\in C'} d(v,C)\right\}$$

the Hausdorff distance between C and C'. Let us denote, for $\rho > 0$ and $\gamma \in]0,1[$, by $U_\rho^\gamma(C)$ (respectively, by $E_\rho^\gamma(C)$) the open tube around the set C (respectively, the open enlargement of the set C), that is,

$$U_\rho^\gamma(C) := \{v \in H : 0 < d(v,C) < \gamma\rho\},$$

respectively,

$$E_\rho^\gamma(C) := \{v \in H : d(v,C) < \gamma\rho\}.$$

We need first to recall some notation and definitions that will be used in all the paper. For any subset C of H, $\overline{\mathrm{co}}\, C$ stands for the closed convex hull of C, and $\sigma(\cdot, C)$ represents the support function of C, that is, for all $\xi \in H$,

$$\sigma(\xi, C) := \sup_{u\in C}\langle \xi, u\rangle.$$

If C is a nonempty subset of H, the Clarke normal cone $N(C;u)$ or $N_C(u)$ of C at $u \in C$ is defined by

$$N_C(u) = \{\xi \in H : \langle \xi, v\rangle \le 0, \forall v \in T_C(u)\},$$

where the Clarke tangent cone $T(C;u)$ or $T_C(u)$ (see [10]) is defined as follows:

$$v \in T_C(u) \Leftrightarrow \begin{cases} \forall \varepsilon > 0, \exists \delta > 0 \text{ such that} \\ \\ \forall u' \in B(u,\delta)\cap C, \forall t \in]0,\delta[, (u' + tB(v,\varepsilon)) \cap C \neq \emptyset. \end{cases}$$

Equivalently, $v \in T_C(u)$ if and only if for any sequence $(u_n)_n$ of C converging to u and any sequence of positive reals $(t_n)_n$ converging to 0, there exists a sequence $(v_n)_n$ in H converging to v such that

$$u_n + t_n v_n \in C \text{ for all } n \in \mathbb{N}.$$

We put $N_C(u) = \emptyset$, whenever $u \notin C$.

We typically denote by $f : H \to \mathbb{R} \cup \{+\infty\}$ a proper function (that is, f is finite at least at one point). The Clarke subdifferential $\partial f(u)$ of f at a point u (where f is finite) is defined by

$$\partial f(u) = \Big\{\xi \in H : (\xi, -1) \in N_{\mathrm{epi}\, f}\Big(\big(u, f(u)\big)\Big)\Big\},$$

where $\mathrm{epi}\, f$ denotes the epigraph of f, that is,

$$\mathrm{epi}\, f = \{(u, \rho) \in H \times \mathbb{R} : f(u) \leq \rho\}.$$

We also put $\partial f(u) = \emptyset$ if f is not finite at $u \in H$. If ψ_C denotes the indicator function of the set C, that is, $\psi_C(u) = 0$ if $u \in C$ and $\psi_C(u) = +\infty$ otherwise, then

$$\partial \psi_C(u) = N_C(u) \text{ for all } u \in H.$$

The Clarke subdifferential $\partial f(u)$ of a locally Lipschitz function f at u has also the other useful description

$$\partial f(u) = \{\xi \in H : \langle \xi, v\rangle \leq f^0(u, v), \forall v \in H\},$$

where

$$f^0(u, v) := \limsup_{(u',t)\to(u,0^+)} \frac{f(u' + tv) - f(u')}{t}.$$

The above function $f^0(u; \cdot)$ is called the Clarke directional derivative of f at u. The Clarke normal cone is known ([10]) to be related to the Clarke subdifferential of the distance function through the equality

$$N_C(u) = \mathrm{cl}_w(\mathbb{R}_+ \partial d_C(u)) \text{ for all } u \in C,$$

where $\mathbb{R}_+ := [0, \infty[$ and cl_w denotes the closure with respect to the weak topology of H. Further

$$\partial d_C(u) \subset N_C(u) \cap \mathbb{B} \text{ for all } u \in C.$$

We will also need the concept of proximal subgradient. A vector $\xi \in H$ is a proximal subgradient of f at u (see, [11, 24, 32]) if there exist some constant real number $\sigma \geq 0$ and some $\delta > 0$ such that

$$\langle \xi, v - u\rangle \leq f(v) - f(u) + \sigma\|v - u\|^2 \text{ for all } v \in B(u, \delta).$$

The set $\partial_p f(u)$ of all proximal subgradients of f at u is the proximal subdifferential of f at u. If $f(u)$ is not finite we put $\partial_p f(u) = \emptyset$. It is known that we always have the inclusion

$$\partial_p f(u) \subset \partial f(u).$$

The proximal normal cone of C at $u \in C$ is given by

$$N_C^p(u) = \partial_p \psi_C(u),$$

so the following inclusion always holds true

$$N_C^p(u) \subset N_C(u) \text{ for all } u \in C.$$

On the other hand, the proximal normal cone enjoys a geometrical characterization (see, [11]) given by the equality

$$N_C^p(u) = \{\xi \in H : \exists \rho > 0 \text{ s.t. } u \in \operatorname{Proj}_C(u + \rho\xi)\},$$

where

$$\operatorname{Proj}_C(v) := \{u \in C : d(v, C) = \|v - u\|\}$$

is the set of nearest points of v in C. When this set has a unique point, we will use the notation $\mathrm{P}_C(v)$. For $u \in C$, the proximal cone is also related to the distance function to C via the equalities (see, [11, 3])

$$N_C^p(u) = \mathbb{R}_+ \partial_p d_C(u)$$

and

$$N_C^p(u) \cap \mathbb{B} = \partial_p d_C(u). \tag{4.1}$$

Now, we begin by recalling that, for a given $\rho \in]0, +\infty]$, a subset C of the Hilbert space H is (uniformly) ρ-prox-regular (see [31]) if for any $u \in C$ and for any $\xi \in N_C(u)$ with $\|\xi\| < 1$, then u is the unique nearest point of $u + \rho^{-1}\xi$ in C.

The following proposition summarize some important consequences of the prox-regularity property which will be needed in the sequel of the paper. For the proof of these results we refer the reader to [31].

Theorem 4.1.1. *Let C be a nonempty closed subset in the Hilbert space H and let $\rho > 0$. If the subset C is ρ-prox-regular, then the following hold:*

a) For any point v in the open enlargement $E_\rho^\gamma(C)$, the mapping $\mathrm{P}_C(v)$ exists and is continuous;

b) For any $v \in U_\rho^\gamma(c)$ and $y = \mathrm{P}_C(v)$ we have $y \in \operatorname{Proj}_C\Big(y + \rho \frac{v-y}{\|v-y\|}\Big)$;

c) The Clarke and the proximal subdifferentials of $d_C(\cdot)$ coincide at all points $v \in E_\rho^\gamma(C)$;

d) The Clarke and the proximal normal cone to C coincide at all points $u \in C$.

We will also need the following lemma from [4].

Lemma 4.1.1. *Let $\rho > 0$. Assume that $C(t)$ is ρ-prox-regular for all $t \in [0,T]$ and that there exists an absolutely continuous function $\vartheta : [0,T] \to \mathbb{R}$ such that*

$$|d(x, C(t)) - d(x, C(t))| \leq |\vartheta(t) - \vartheta(s)| \quad \text{for all } s, t \in [0,T].$$

Then, for any given $0 < \delta < \rho$, the following holds:

For any $s \in [0,T]$, any sequence $(u_n)_n$ converging to $u \in C(s)+(\rho-\delta)\mathbb{B}$ in $(H, \|\cdot\|)$ ($(u_n)_n$ is not necessarily in $C(s_n)$), any sequence $(s_n)_n$ in $[0,T]$ converging to s and any sequence $(\zeta_n)_n$ converging weakly to ζ in $(H, w(H,H))$ with $\zeta_n \in \partial d_{C(s_n)}(u_n)$, we have $\zeta \in \partial d_{C(s)}(u)$.

The following results, recently established in [29], will play an important role in the proof of our main result.

Theorem 4.1.2. *Let C and C' be ρ-prox-regular sets of the Hilbert space H for a constant $\rho > 0$ and let $\gamma \in]0,1[$. Then for all $u \in U_\rho^\gamma(C)$ and $v \in U_\rho^\gamma(C')$ we have*

$$\|\mathrm{P}_C(u) - \mathrm{P}_{C'}(v)\| \leq (1-\gamma)^{-1}\|u - v\| + \sqrt{\frac{2\gamma\rho}{1-\gamma}}\Big(\mathrm{Haus}\,(C, C')\Big)^{1/2}.$$

The result can be applied to ρ-prox-regular moving set $C(t,u)$ satisfying

$$|d(x, C(t,u)) - d(x, C(t,v))| \leq L\|u - v\| \tag{4.2}$$

whenever $t \in [0,T]$ and $x, u, v \in H$ where L is some real constant with $L \in [0,1[$; indeed the latter inequality is equivalent to

$$\mathrm{Haus}\,(C(t,u), C(t,v)) \leq L\|u - v\|.$$

This application has as an immediate consequence the following result.

Corollaire 4.1.1. *Let $C(t,u)$ be ρ-prox-regular moving sets of the Hilbert space H for a constant $\rho > 0$ which satisfy (4.2), and let $\gamma \in]0,1[$. Then, for all $u, v \in H$ and $x \in U_\rho^\gamma(C(t,u)) \cap U_\rho^\gamma(C(t,v))$, we have*

$$\|\mathrm{P}_{C(t,u)}(x) - \mathrm{P}_{C(t,v)}(x)\| \leq \sqrt{\frac{2\gamma\rho L}{1-\gamma}}\|u - v\|^{1/2}.$$

Consider now for each $(t,x) \in [0,T] \times H$ fixed, the mapping ϕ from $\mathrm{Dom}\,\mathrm{P}_{C(t)}$ defined by $u \mapsto \mathrm{P}_{C(t,u)}(x)$. Thus, Corollary 4.1.1 above establishes the local Hölder continuity of ϕ on $U_\rho^\gamma(C(t,u))$ whenever the variable set $C(t,u)$ is ρ-prox-regular.

4.2 Perturbed nonconvex sweeping process with delay

Let two given set-valued mappings $C : [0,T] \times H \rightrightarrows H$ with nonempty closed values and $G : [0,T] \times \mathcal{C}_0 \rightrightarrows H$ with nonempty closed convex values. Suppose that they satisfy the following assumptions:

($\mathcal{H}_1$) The set-valued mapping G is scalarly upper semicontinuous with respect to both variables (that is, for each $y \in H$ the function $(t, \varphi) \to \sigma(y, G(t, \varphi))$ is upper semicontinuous) and, for some real $\alpha > 0$

$$d\big(0, G(t, \varphi)\big) \leq \alpha$$

for all $t \in [0,T]$ and $\varphi \in \mathcal{C}_0$;

($\mathcal{H}_2$) For each $t \in [0,T]$ and each $u \in H$, the sets $C(t,u)$ are nonempty closed in H and ρ-prox-regular for some constant $\rho > 0$;

($\mathcal{H}_3$) There are real constants $L_1 > 0$, $L_2 \in]0,1[$ such that, for all $t, s \in [0,T]$ and $x, y, u, v \in H$

$$\Big|d\big(x, C(t,u)\big) - d\big(y, C(s,v)\big)\Big| \leq \|x - y\| + L_1|t - s| + L_2\|u - v\|;$$

($\mathcal{H}_4$) For any bounded subset $A \subset H$, the set $C([0,T] \times A)$ is relatively ball-compact, that is, the intersection of $C([0,T] \times A)$ with any closed ball of H is relatively compact in H.

Theorem 4.2.1. *Assume that H is a Hilbert space, that $(\mathcal{H}_1 - \mathcal{H}_4)$ hold. Then, for any $\varphi \in \mathcal{C}_0$ and $u_0 \in H$ with $\varphi(0) = u_0 \in C(0, u_0)$, the differential inclusion*

$$(\mathcal{D}_r) \quad \begin{cases} \dot{u}(t) \in -N_{C\big(t, u(t)\big)}\big(u(t)\big) + G\big(t, \Lambda(t)u\big) & \text{a.e } t \in [0,T], \\ u(t) \in C\big(t, u(t)\big) \quad \forall t \in [0,T], \\ u = \varphi \ \textit{in} \ [-r, 0], \end{cases}$$

has at least one solution $u : [-r,T] \to H$, which is continuous on $[-r,T]$ and Lipschitz on $[0,T]$ with $\|\dot{u}(t)\| \leq \frac{L_1 + 2\alpha}{1 - L_2}$ a.e $t \in [0,T]$.

Proof. Fix an integer $p \geq 1$ and suppose, without loss of generality, that

$$T < p\rho(1 - L_2)\Big(2\alpha(1 + 3L_2) + 2L_1(1 + L_2)\Big)^{-1}. \tag{4.3}$$

We are going to construct a sequence of mappings $(u_n(\cdot))$ in $\mathcal{C}_H(-r,T)$ which has a subsequence converging pointwise to a solution of $(\mathcal{D}_r)$.

Step 1. Construction of the sequence $(u_n)_n$
For any $t \in [0,T]$, consider the single-valued mapping $\hat{\Lambda}(t) : \mathcal{C}_t \to \mathcal{C}_0$ defined, for all $\xi \in \mathcal{C}_t := \mathcal{C}_H(-r,t)$ by

$$\hat{\Lambda}(t)\xi(s) := \xi(t+s) \quad \forall s \in [-r,0].$$

For each integer $n \geq 1$, we partition $[0,T]$ by the points

$$t_k^n = k\frac{T}{p^n}, \ k = 0,1,\cdots,p^n.$$

For each $(t,\varphi) \in [0,T] \times \mathcal{C}_0$ denote by $g(t,\varphi)$ the element of minimal norm of the closed convex set $G(t,\varphi)$ of H, that is,

$$g(t,\varphi) = \mathrm{P}_{G(t,\varphi)}(0).$$

Put $x_0^n := u_0 \in C(t_0^n, u_0)$ and $u_0^n(t) = \varphi(t)$ for all $t \in [-r, t_0^n]$. For any $v \in B(u_0, 2\frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n})$, from $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$ we get

$$\begin{aligned}
&d\Big(u_0 + \frac{T}{p^n}g(t_0^n, \hat{\Lambda}(t_0^n)u_0^n), C(t_1^n, v)\Big) \\
&\quad \leq d\Big(u_0 + \frac{T}{p^n}g(t_0^n, \hat{\Lambda}(t_0^n)u_0^n), C(t_0^n, u_0)\Big) + L_1|t_1^n - t_0^n| + L_2\|v - u_0\| \\
&\quad \leq \Big\|\frac{T}{p^n}g(t_0^n, \hat{\Lambda}(t_0^n)u_0^n)\Big\| + L_1\frac{T}{p^n} + 2L_2\frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n} \\
&\quad \leq \Big(\alpha + L_1 + 2L_2\frac{L_1+2\alpha}{1-L_2}\Big)\frac{T}{p^n} = \frac{\alpha(1+3L_2) + L_1(1+L_2)}{1-L_2}\frac{T}{p^n},
\end{aligned}$$

and hence it results according to (4.3) that

$$d\Big(u_0 + \frac{T}{p^n}g(t_0^n, \hat{\Lambda}(t_0^n)u_0^n), C(t_1^n, v)\Big) < \frac{1}{2}\rho.$$

By the ρ-prox-regularity assumption, Theorem 4.1.1 guarantees, for every $v \in B\big(u_0, 2\frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}\big)$, that

$$\phi_1(v) := \mathrm{P}_{C(t_1^n, v)}\Big(u_0 + \frac{T}{p^n}g\big(t_0^n, \hat{\Lambda}(t_0^n)u_0^n\big)\Big) \tag{4.4}$$

is well defined. It results from Corollary 4.1.1, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ that the mapping $\phi_1 : B\big(u_0, 2\frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}\big) \to H$ is locally Hölder continuous. Further, for all $v \in B\big[u_0, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}\big]$, we have $\phi_1(v) \in B\big[u_0, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}\big]$. Indeed, for any such v it follows from the definition of $\phi_1(v)$ and from $(\mathcal{H}_3)$ (as above) that

$$\begin{aligned}
\|\phi_1(v) - x_0^n\| &\le \Big\|\phi_1(v) - \big(x_0^n + \frac{T}{p^n} g(t_0^n, \hat{\Lambda}(t_0^n)u_0^n)\big)\Big\| + \frac{T}{p^n}\|g(t_0^n, \hat{\Lambda}(t_0^n)u_0^n)\| \\
&= d\Big(x_0^n + \frac{T}{p^n} g(t_0^n, \hat{\Lambda}(t_0^n)u_0^n), C(t_1^n, v)\Big) + \frac{T}{p^n}\|g(t_0^n, \hat{\Lambda}(t_0^n)u_0^n)\| \\
&\le d\Big(x_0^n + \frac{T}{p^n} g(t_0^n, \hat{\Lambda}(t_0^n)u_0^n), C(t_0^n, x_0^n)\Big) + \frac{T}{p^n}\|g(t_0^n, \hat{\Lambda}(t_0^n)u_0^n)\| \\
&+ L_2\|v - x_0^n\| + L_1|t_1^n - t_0^n| \\
&\le 2\frac{T}{p^n}\|g(t_0^n, \hat{\Lambda}(t_0^n)u_0^n)\| + L_1|t_1^n - t_0^n| + L_2\|v - x_0^n\| \\
&\le \Big(2\alpha + L_1 + L_2\frac{L_1+2\alpha}{1-L_2}\Big)\frac{T}{p^n} = \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}.
\end{aligned}$$

Since this holds for any $v \in B\big[u_0, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}\big]$ we deduce

$$\phi_1(v) \in C\Big(t_1^n, B\Big[u_0, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}\Big]\Big) \bigcap B\Big[u_0, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}\Big],$$

and the set $\phi_1\big(B[u_0, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}]\big)$ is relatively compact, in view of hypothesis $(\mathcal{H}_4)$. So, the mapping ϕ_1 is continuous from the closed convex set $B\big[u_0, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}\big]$ into itself and the range of $B\big[u_0, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}\big]$ by ϕ_1 is relatively compact. The extended Schauder fixed point theorem (established in [21] or [30]), applied to the mapping ϕ_1, implies that there exists $x_1^n \in B[u_0, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}]$ such that $x_1^n = \phi_1(x_1^n)$. Consequently, we obtain

$$x_1^n \in C(t_1^n, x_1^n) \;\text{ and }\; \|x_1^n - u_0\| \le \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}$$

and by (4.4)

$$u_0 + \frac{T}{p^n} g(t_0^n, \hat{\Lambda}(t_0^n)u_0^n) - x_1^n \in N_{C(t_1^n, x_1^n)}(x_1^n).$$

Define $\chi_1^n : [t_0^n, t_1^n] \to H$ by

$$\chi_1^n(t) := \frac{t_1^n - t}{t_1^n - t_0^n} x_0^n + \frac{t - t_0^n}{t_1^n - t_0^n} x_1^n \quad \text{ if } t \in [t_0^n, t_1^n].$$

and consider $u_1^n : [-r, t_1^n] \to H$ given by

$$u_1^n(t) := \begin{cases} \chi_1^n(t) & \text{if} \quad t \in [t_0^n, t_1^n] \\ \varphi(t) & \text{if} \quad t \in [-r, 0]. \end{cases}$$

As above, for any $v \in B\big(x_1^n, 2\frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}\big)$, we observe by $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$ that

$$\begin{aligned}
&d\Big(x_1^n + \frac{T}{p^n} g(t_1^n, \hat{\Lambda}(t_1^n)u_1^n), C(t_2^n, v)\Big) \\
&\qquad \le d\Big(x_1^n + \frac{T}{p^n} g(t_1^n, \hat{\Lambda}(t_1^n)u_1^n), C(t_1^n, x_1^n)\Big) + L_1|t_2^n - t_1^n| + L_2\|v - x_1^n\| \\
&\qquad \le \Big\|\frac{T}{p^n} g(t_1^n, \hat{\Lambda}(t_1^n)u_1^n)\Big\| + L_1\frac{T}{p^n} + 2L_2\frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n} \\
&\qquad \le \Big(\alpha + L_1 + 2L_2\frac{L_1+2\alpha}{1-L_2}\Big)\frac{T}{p^n} = \frac{\alpha(1+3L_2)+L_1(1+L_2)}{1-L_2}\frac{T}{p^n},
\end{aligned}$$

which combined with (4.3) yields

$$d\Big(x_1^n + \frac{T}{p^n} g(t_1^n, \hat{\Lambda}(t_1^n)u_1^n), C(t_2^n, v)\Big) < \frac{1}{2}\rho.$$

By the ρ-prox-regularity assumption, Theorem 4.1.1 ensures, for every $v \in B\big(x_1^n, 2\frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}\big)$, that

$$\phi_2(v) := \mathrm{P}_{C(t_2^n, v)}\Big(x_1^n + \frac{T}{p^n} g\big(t_1^n, \hat{\Lambda}(t_1^n)u_1^n\big)\Big) \tag{4.5}$$

is well defined. It results from Corollary 4.1.1, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ that the mapping $\phi_2 : B\big(x_1^n, 2\frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}\big) \to H$ is locally Hölder continuous. Further, for all $v \in B\big[x_1^n, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}\big]$, we have $\phi_2(v) \in B\big[x_1^n, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}\big]$. Indeed, for any such v it follows from the definition of $\phi_2(v)$, from $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$ (as above) that

$$\begin{aligned}
\|\phi_2(v) - x_1^n\| &\le \Big\|\phi_2(v) - \big(x_1^n + \frac{T}{p^n} g(t_1^n, \hat{\Lambda}(t_1^n)u_1^n)\big)\Big\| + \frac{T}{p^n}\|g(t_1^n, \hat{\Lambda}(t_1^n)u_1^n)\| \\
&= d\Big(x_1^n + \frac{T}{p^n} g(t_1^n, \hat{\Lambda}(t_1^n)u_1^n), C(t_2^n, v)\Big) + \frac{T}{p^n}\|g(t_1^n, \hat{\Lambda}(t_1^n)u_1^n)\| \\
&\le d\Big(x_1^n + \frac{T}{p^n} g(t_1^n, \hat{\Lambda}(t_1^n)u_1^n), C(t_1^n, x_1^n)\Big) + \frac{T}{p^n}\|g(t_1^n, \hat{\Lambda}(t_1^n)u_1^n)\| \\
&\quad + L_2\|v - x_1^n\| + L_1|t_2^n - t_1^n| \\
&\le 2\frac{T}{p^n}\|g(t_1^n, \hat{\Lambda}(t_1^n)u_1^n)\| + L_1|t_2^n - t_1^n| + L_2\|v - x_1^n\| \\
&\le \Big(2\alpha + L_1 + L_2\frac{L_1+2\alpha}{1-L_2}\Big)\frac{T}{p^n} = \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}.
\end{aligned}$$

Since this holds for any $v \in B\big[x_1^n, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}\big]$ we deduce

$$\phi_2(v) \in C\Big(t_2^n, B\Big[x_1^n, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}\Big]\Big) \bigcap B\Big[x_1^n, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}\Big],$$

and the set $\phi_2\big(B[x_1^n, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}]\big)$ is relatively compact, in view of hypothesis $(\mathcal{H}_4)$. So, the mapping ϕ_2 is continuous from the closed convex set $B\big[x_1^n, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}\big]$ into itself and the range of $B\big[x_1^n, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}\big]$ by ϕ_2 is relatively compact. The extended Schauder fixed point theorem (established in [21] or [30]), applied to the mapping ϕ_2, implies that there exists $x_2^n \in B[x_1^n, \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}]$ such that $x_2^n = \phi_2(x_2^n)$. Consequently, we obtain

$$x_2^n \in C(t_2^n, x_2^n) \text{ and } \|x_2^n - x_1^n\| \le \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}$$

and by (4.5)

$$x_1^n + \frac{T}{p^n} g(t_1^n, \hat{\Lambda}(t_1^n)u_1^n) - x_2^n \in N_{C(t_2^n, x_2^n)}(x_2^n).$$

Define $\chi_2^n : [t_1^n, t_2^n] \to H$ by

$$\chi_2^n(t) := \frac{t_2^n - t}{t_2^n - t_1^n} x_1^n + \frac{t - t_1^n}{t_2^n - t_1^n} x_2^n \quad \text{if } t \in [t_1^n, t_2^n].$$

and define also $u_2^n : [-r, t_2^n] \to H$ by

$$u_2^n(t) := \begin{cases} \chi_2^n(t) & \text{if} \quad t \in [t_1^n, t_2^n] \\ u_1^n(t) & \text{if} \quad t \in [-r, t_1^n]. \end{cases}$$

By repeating the process, for $k = 0, 1, \cdots, p^n - 1$, we obtain $(x_k^n)_{k=0}^{p^n}$ in H, $(\chi_k^n(\cdot))_{k=1}^{p^n}$ with $\chi_k^n : [t_{k-1}^n, t_k^n] \to H$ and $(u_k^n(\cdot))_{k=0}^{p^n}$ with $u_k^n : [-r, t_k^n] \to H$ such that the following properties hold:

$$x_{k+1}^n \in C(t_{k+1}^n, x_{k+1}^n) \text{ and } \|x_{k+1}^n - x_k^n\| \le \frac{L_1+2\alpha}{1-L_2}\frac{T}{p^n}, \tag{4.6}$$

$$x_k^n + \frac{T}{p^n} g(t_k^n, \hat{\Lambda}(t_k^n)u_k^n) - x_{k+1}^n \in N_{C(t_{k+1}^n, x_{k+1}^n)}(x_{k+1}^n), \tag{4.7}$$

$$\chi_{k+1}^n : [t_k^n, t_{k+1}^n] \to H \text{ with}$$

$$\chi_{k+1}^n(t) := \frac{t_{k+1}^n - t}{t_{k+1}^n - t_k^n} x_k^n + \frac{t - t_k^n}{t_{k+1}^n - t_k^n} x_{k+1}^n \quad \text{if } t \in [t_k^n, t_{k+1}^n],$$

$$u_{k+1}^n : [-r, t_{k+1}^n] \to H \text{ with}$$

$$u_{k+1}^n(t) := \begin{cases} \chi_{k+1}^n(t) & \text{if} \quad t \in [t_k^n, t_{k+1}^n] \\ u_k^n(t) & \text{if} \quad t \in [-r, t_k^n]. \end{cases}$$

Now, let us define the sequence of mappings $(u_n(\cdot))_n$ from $[-r, T]$ into H with $u_n : [-r, T] \to H$ given by

$$u_n(t) := u^n_{p^n}(t) \quad \text{for all } t \in [-r, T].$$

Thus, for almost all $t \in [t^n_k, t^n_{k+1}]$ and $k = 0, 1, \cdots, p^n - 1$,

$$\dot{u}_n(t) = -\frac{x^n_k}{t^n_{k+1} - t^n_k} + \frac{x^n_{k+1}}{t^n_{k+1} - t^n_k} = -\frac{p^n}{T}(x^n_k - x^n_{k+1}).$$

This combined with (4.6) and (4.7), by construction, yields

$$u_n(t^n_{k+1}) \in C\big(t^n_{k+1}, u_n(t^n_{k+1})\big)$$

$$-\dot{u}_n(t) \in N_{C\left(t^n_{k+1}, u_n(t^n_{k+1})\right)}\big(u_n(t^n_{k+1})\big) - g\big(t^n_k, \hat{\Lambda}(t^n_k)u^n_k\big) \text{ a.e } t \in [t^n_k, t^n_{k+1}[$$

with, according to (4.6),

$$\|\dot{u}_n(t)\| = \frac{p^n}{T}\|x^n_k - x^n_{k+1}\| \leq \frac{L_1 + 2\alpha}{1 - L_2} := M. \tag{4.8}$$

Putting

$$\delta_n(t) := \begin{cases} t^n_k & \text{if} \quad t \in [t^n_k, t^n_{k+1}[\\ t^n_{p^n-1} & \text{if} \quad t = T, \end{cases}$$

and

$$\theta_n(t) := \begin{cases} t^n_{k+1} & \text{if} \quad t \in [t^n_k, t^n_{k+1}[\\ T & \text{if} \quad t = T, \end{cases}$$

we obtain

$$u_n\big(\theta_n(t)\big) \in C\Big(\theta_n(t), u_n\big(\theta_n(t)\big)\Big) \tag{4.9}$$

$$-\dot{u}_n(t) \in N_{C\left(\theta_n(t), u_n\left(\theta_n(t)\right)\right)}\Big(u_n\big(\theta_n(t)\big)\Big) - g\Big(\delta_n(t), \hat{\Lambda}\big(\delta_n(t)\big)u^n_{p^n-1}\Big) \text{ a.e } t \in [0, T[.$$

Using the mapping $\Lambda(t) : \mathcal{C}_H([-r, T]) \to \mathcal{C}_H([-r, 0])$ defined in the introduction, it is easily seen that, for any $\xi \in \mathcal{C}_T$ and any $t \in [0, T]$,

$$\Lambda(t)\xi = \hat{\Lambda}(t)\xi_{|[-r,t]}$$

where $\xi_{|[-r,t]}$ denotes the restriction of ξ to $[-r, t]$. The last inclusion above can then be written as

$$-\dot{u}_n(t) \in N_{C\left(\theta_n(t), u_n\left(\theta_n(t)\right)\right)}\Big(u_n\big(\theta_n(t)\big)\Big) - g\Big(\delta_n(t), \Lambda\big(\delta_n(t)\big)u_n\Big) \text{ a.e } t \in [0, T]. \tag{4.10}$$

Step 2. Convergence of a subsequence of $(u_n(\cdot))$ to some mapping $u(\cdot)$ absolutely continuous on $[0,T]$.
For any $t \in [0,T]$, the sequences $\theta_n(t)$ and $\delta_n(t)$ converge to t. Indeed, for each $t \in [0,T]$, choosing k such that $t \in [t_k^n, t_{k+1}^n[$ if $t < T$ and $k = p^n - 1$ if $t = T$, we have

$$|\theta_n(t) - t| \le |t_{k+1}^n - t_k^n| = \frac{T}{p^n}, \text{ so } \theta_n(t) \to t \text{ as } n \to +\infty,$$

and similarly $\delta_n(t) \to t$ as $n \to +\infty$.

For each $k = 0, 1, \cdots, p^n - 1$, it results from the inequality in (4.6) that

$$\|x_{k+1}^n - u_0\| \le \|x_{k+1}^n - x_k^n\| + \cdots + \|x_1^n - x_0^n\| \le (k+1)\frac{L_1 + 2\alpha}{1 - L_2}\frac{T}{p^n},$$

so

$$\|x_{k+1}^n\| \le \|u_0\| + \frac{L_1 + 2\alpha}{1 - L_2} T := \beta.$$

Consider $v_n := u_{n|[0,T]}$, that is, $v_n(t) = u_n(t)$ for all $t \in [0,T]$. Fix any $t \in [0,T]$ and consider, for any infinite subset $N \subset \mathbb{N}$, the sequence $(v_n(t))_{n\in N}$. It follows from (4.9) that $v_n(\theta_n(t)) \in C(\theta_n(t), v_n(\theta_n(t))) \cap \beta\mathbb{B}$, which implies that $v_n(\theta_n(t)) \in C([0,T] \times \beta\mathbb{B}) \cap \beta\mathbb{B}$. By $(\mathcal{H}_4)$ the sequence $(v_n(\theta_n(t)))$ is relatively compact, so there is an infinite subset $N_0 \subset N$ such that $(v_n(\theta_n(t)))_{n\in N_0}$ converges to some vector $l(t) \in H$. Putting $h_n(t) := v_n(\theta_n(t)) - v_n(t)$ for all $n \in N_0$, by (4.8), we obtain

$$\|h_n(t)\| \le \int_t^{\theta_n(t)} \|\dot v_n(s)\| ds \le M(\theta_n(t) - t) \underset{n\to\infty}{\longrightarrow} 0.$$

Then, $(v_n(t))_{n\in N_0}$ converges to $l(t)$, thus the set $\{v_n(t) : n \in \mathbb{N}\}$ is relatively compact in H. The sequence $(v_n(\cdot))_{n\in\mathbb{N}}$ being in addition equicontinuous according to (4.8), this sequence $(v_n(\cdot))_{n\in\mathbb{N}}$ is relatively compact in $\mathcal{C}_H(0,T)$, so we can extract a subsequence of $(v_n(\cdot))_{n\in\mathbb{N}}$ (that we do not relabel) which converges uniformly to some mapping $v(\cdot)$ on $[0,T]$. By the inequality (4.8) again there is a subsequence of $(\dot v_n)_{n\in\mathbb{N}}$ (that we do not relabel) which converges $w(\mathrm{L}^1_H, \mathrm{L}^\infty_H)$ in $\mathrm{L}^1_H(0,T)$ to a mapping $w \in \mathrm{L}^1_H(0,T)$ with $\|w(t)\| \le M$ a.e. $t \in [0,T]$. Fixing $t \in [0,T]$ and taking any $y \in H$, the above weak convergence in $\mathrm{L}^1_H(0,T)$ yields

$$\lim_{n\to\infty} \int_0^T \langle \mathbb{1}_{[0,t]}(s) y, \dot v_n(s)\rangle ds = \int_0^T \langle \mathbb{1}_{[0,t]}(s) y, w(s)\rangle ds,$$

or equivalently

$$\lim_{n\to\infty} \langle y, u_0 + \int_0^t \dot v_n(s) ds\rangle = \langle y, u_0 + \int_0^t \dot w(s) ds\rangle.$$

This means, for each $t \in [0,T]$, that $v_n(t) \underset{n\to\infty}{\longrightarrow} u_0 + \int_0^t w(s)ds$ weakly in H. Since the sequence $(v_n(t))_{n\in\mathbb{N}}$ also converges strongly to $v(t)$ in H, it ensures that $v(t) = u_0 + \int_0^t w(s)ds$, so the mapping $v(\cdot)$ is absolutely continuous on $[0,T]$ with $\dot{v} = w$. The mapping $v(\cdot)$ is even Lipschitz on $[0,T]$ with M as a Lipschitz constant therein, since the convex set $\{\zeta \in \mathrm{L}^1_H(0,T) : \|\zeta(t)\| \leq H\}$ is norm closed in $\mathrm{L}^1_H(0,T)$, and hence weakly closed.

Further, since $u_n(t) = \varphi(t)$ for all $t \in [-r,0]$, putting $u(t) = \varphi(t)$ if $t \in [-r,0]$ and $u(t) = v(t)$ if $t \in [0,T]$, we see that $(u_n(\cdot))_n$ converges uniformly on $[-r,T]$ to $u(\cdot)$ and $u(\cdot)$ is continuous on $[-r,T]$ and Lipschitz on $[0,T]$.

Step 3. Let us prove that $u(\cdot)$ is a solution of $(\mathcal{D}_r)$.
Claim: For any $t \in [0,T]$, $(\Lambda\big(\delta_n(t)\big)u_n)_n$ converges uniformly to $\Lambda(t)u$ in $\mathcal{C}_0$. Fix any $t \in [0,T]$. We can write, for every $s \in [-r,0]$

$$\begin{aligned}
&\|u_n(\delta_n(t)+s) - u(t+s)\| \\
&\leq \|u_n\big(\delta_n(t)+s\big) - u\big(\delta_n(t)+s\big)\| + \|u\big(\delta_n(t)+s\big) - u(t+s)\| \\
&\leq \sup_{\tau\in[-r,T]} \|u_n(\tau) - u(\tau)\| + M|t - \delta_n(t)|,
\end{aligned}$$

so we have

$$\|\Lambda\big(\delta_n(t)\big)u_n - \Lambda(t)u\|_{\mathcal{C}_0} \leq \|u_n - u\|_{\mathcal{C}_T} + M|t - \delta_n(t)|.$$

Since $u_n(\cdot)_n$ converges uniformly to $u(\cdot)$ on $[-r,T]$, we deduce

$$\|\Lambda\big(\delta_n(t)\big)u_n - \Lambda(t)u\|_{\mathcal{C}_0} \underset{n\to+\infty}{\longrightarrow} 0,$$

which justifies the claim.

According to $(\mathcal{H}_1)$, we have $\big\|g\big(\delta_n(t), \Lambda\big(\delta_n(t)\big)u_n\big)\big\| \leq \alpha$ for all $n \in \mathbb{N}$ and $t \in [0,T]$, then putting $z_n(t) := g\big(\delta_n(t), \Lambda\big(\delta_n(t)\big)u_n\big)$ for all $t \in [0,T]$, we may assume (taking a subsequence if necessary) that the sequence $(z_n(\cdot))$ converges $w(\mathrm{L}^1_H, \mathrm{L}^\infty_H)$ in $\mathrm{L}^1_H(0,T)$ to a mapping $z(\cdot) \in \mathrm{L}^1_H(0,T)$ with $\|z(t)\| \leq \alpha$ a.e $t \in [0,T]$.

For all $t \in [0,T]$ we have $u(t) \in C(t,u(t))$. Indeed, since $u_n(\theta_n(t)) \in C(t, u_n(\theta_n(t)))$, the assumption $(\mathcal{H}_3)$ ensures that

$$\begin{aligned}
&d\Big(u_n(t), C\big(t,u(t)\big)\Big) \\
&\qquad \leq \|u_n(t) - u_n(\theta_n(t))\| + L_1|t - \theta_n(t)| + L_2\|u(t) - u_n(\theta_n(t))\| \\
&\qquad \leq (M + L_1 + ML_2)|t - \theta_n(t)| + L_2\|u(t) - u_n(t)\|
\end{aligned}$$

then,

$$d(u_n(t), C(t, u(t))) \underset{n\to\infty}{\longrightarrow} 0, \text{ so } d(u(t), C(t, u(t))) = 0 \text{ and } u(t) \in C(t, u(t)).$$

Further, from the inequality $\|\dot{u}_n(t) - z_n(t)\| \le M + \alpha := \gamma$ a.e. and from the inclusion (4.10) it follows for a.e. $t \in [0, T]$ that

$$(4.11) \qquad \begin{aligned} -\dot{u}_n(t) + z_n(t) &\in N_{C\left(\theta_n(t), u_n\left(\theta_n(t)\right)\right)}\Big(u_n\big(\theta_n(t)\big)\Big) \bigcap \gamma\mathbb{B} \\ &= \gamma \partial d_{C\left(\theta_n(t), u_n\left(\theta_n(t)\right)\right)}\Big(u_n\big(\theta_n(t)\big)\Big), \quad \text{see (4.1)} \end{aligned}$$

$$(4.12) \qquad z_n(t) \in G\Big(\delta_n(t), \Lambda\big(\delta_n(t)\big)u_n\Big).$$

It follows from $(-\dot{u}_n + z_n, z_n)_n$ converges weakly in $\mathrm{L}^1_{H\times H}(0,T)$ to $(-\dot{u} + z, z)$ and by Mazur theorem there are

$$(4.13) \qquad \xi_n \in \operatorname{co}\{-\dot{u}_q + z_q : q \ge n\} \text{ and } \zeta_n \in \operatorname{co}\{z_q : q \ge n\}$$

such that $(\xi_n, \zeta_n)_n$ converges strongly in $\mathrm{L}^1_{H\times H}(0,T)$ to $(-\dot{u} + z, z)$. Extracting a subsequence if necessary we suppose that $(\xi_n(\cdot), \zeta_n(\cdot))_n$ converges a.e. to $(-\dot{u}(\cdot) + z(\cdot), z(\cdot))$, then there is a Lebesgue negligible set $S \subset [0, T]$ such that, for every $t \in [0, T]\backslash S$, on one hand $(\xi_n(t), \zeta_n(t)) \to (-\dot{u}(t) + z(t), z(t))$ strongly in H and on the other hand the inclusions (4.11) and (4.12) hold true for all integer n as well as the inclusions

$$-\dot{u}(t) + z(t) \in \bigcap_n \overline{\operatorname{co}}\Big\{-\dot{u}_q(t) + z_q(t) : q \ge n\Big\} \text{ and } z(t) \in \bigcap_n \overline{\operatorname{co}}\Big\{z_q(t) : q \ge n\Big\}.$$

It results from (4.11) and (4.12) that for any $n \in \mathbb{N}$, any $t \in [0, T]\backslash S$, and for any $y \in H$

$$(4.14) \qquad \big\langle y, -\dot{u}_n(t) + z_n(t)\big\rangle \le \sigma\bigg(y, \gamma\partial d_{C\left(\theta_n(t), u_n\left(\theta_n(t)\right)\right)}\Big(u_n\big(\theta_n(t)\big)\Big)\bigg)$$

and

$$(4.15) \qquad \big\langle y, z_n(t)\big\rangle \le \sigma\bigg(y, G\Big(\delta_n(t), \Lambda\big(\delta_n(t)\big)u_n\Big)\bigg).$$

Further, for each $n \in \mathbb{N}$ and any $t \in [0, T]\backslash S$, from (4.13) we have

$$\big\langle y, \xi_k(t)\big\rangle \le \sup_{q\ge n} \big\langle y, -\dot{u}_q(t) + z_q(t)\big\rangle \text{ for all } k \ge n$$

and

$$\big\langle y, \zeta_k(t)\big\rangle \leq \sup_{q\geq n} \big\langle y, z_q(t)\big\rangle \text{ for all } k \geq n$$

and taking the limit in both inequalities as $k \to +\infty$ gives through (4.14) and (4.15)

$$\begin{aligned}\big\langle y, -\dot{u}(t) + z(t)\big\rangle &\leq \sup_{q\geq n} \big\langle y, -\dot{u}_q(t) + z_q(t)\big\rangle \\ &\leq \sup_{q\geq n} \sigma\bigg(y, \gamma\partial d_{C\left(\theta_q(t), u_q\left(\theta_q(t)\right)\right)}\Big(u_q\big(\theta_q(t)\big)\Big)\bigg)\end{aligned}$$

and

$$\big\langle y, z(t)\big\rangle \leq \sup_{q\geq n} \big\langle y, z_q(t)\big\rangle \leq \sup_{q\geq n} \sigma\Big(y, G\Big(\delta_q(t), \Lambda\big(\delta_q(t)\big)u_q\Big)\Big),$$

which ensures that

$$\big\langle y, -\dot{u}(t) + z(t)\big\rangle \leq \limsup_{n\to+\infty} \sigma\bigg(y, \gamma\partial d_{C\left(\theta_n(t), u_n\left(\theta_n(t)\right)\right)}\Big(u_n\big(\theta_n(t)\big)\Big)\bigg)$$

and

$$\big\langle y, z(t)\big\rangle \leq \limsup_{n\to+\infty} \sigma\Big(y, G\Big(\delta_n(t), \Lambda\big(\delta_n(t)\big)u_n\Big)\Big).$$

According to $(\mathcal{H}_3)$ and Lemma 4.1.1 the set-valued mapping $(t, u, x) \to \partial d_{C(t,u)}(x)$, taking on weakly compact convex values, is upper semicontinuous from $[0, T] \times H \times H$ into $(H, w(H, H))$, hence for each $y \in H$ the real-valued function $\sigma(y, \gamma\partial d_{C(\cdot,\cdot)}(\cdot))$ is upper semicontinuous on $[0, T] \times H \times H$. Further, $\sigma(y, G(\cdot, \cdot))$ is also upper semicontinuous on $[0, T] \times \mathcal{C}_H(-r, 0)$ by assumption $(\mathcal{H}_1)$. It follows that, for every $t \in [0, T]\backslash S$ and every $y \in H$,

$$\big\langle y, -\dot{u}(t) + z(t)\big\rangle \leq \sigma\Big(y, \gamma\partial d_{C\left(t, u(t)\right)}\big(u(t)\big)\Big)$$

and

$$\big\langle y, z(t)\big\rangle \leq \sigma\Big(y, G\big(t, \Lambda(t)u\big)\Big),$$

which ensures that $-\dot{u}(t) + z(t) \in \gamma\partial d_{C(t,u(t))}(u(t))$ and $z(t) \in G(t, \Lambda(t)u)$, consequently

$$\dot{u}(t) \in -N_{C\left(t, u(t)\right)}\big(u(t)\big) + z(t) \text{ a.e } t \in [0, T]$$

$$z(t) \in G\big(t, \Lambda(t)u\big) \text{ a.e } t \in [0, T]$$

with

$$\|\dot{u}(t) - z(t)\| \leq \gamma \text{ a.e } t \in [0, T].$$

The proof is complete. $\square$

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Chapter 5

Perturbed sweeping process with a subsmooth set depending on the state

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Abstract. The class of subsmooth sets introduced in [1] strictly contains the class of closed convex sets and the class of prox-regular sets. The present paper is concerned with the study of perturbed differential sweeping process inclusions where the moving set is nonconvex and non prox-regular and depends both on the time and on the state. We prove the existence of solution under the subsmoothness of the moving set.

Keyword : Differential inclusion; Sweeping process; Normal cone; Prox-regular set; Subsmooth set; Subdifferential

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Introduction

In this paper, given a Hilbert space H, we discuss the existence of solution of the evolution process differential inclusion of the form

$$(\mathcal{D}) \qquad \begin{cases} \dot{u}(t) \in -N_{C\left(t,u(t)\right)}\big(u(t)\big) + G\big(t, u(t)\big) \quad \text{a.e } t \in [0,T] \\ \\ u(t) \in C\big(t, u(t)\big) \quad \forall t \in [0,T] \\ \\ u(0) = u_0 \in C(0, u_0). \end{cases}$$

In $(\mathcal{D})$, $C : [0,T] \times H \rightrightarrows H$ is a multimapping with nonempty closed values and $G : [0,T] \times H \rightrightarrows H$ is a multimapping with nonempty closed convex values, and $N_{C\left(t,u(t)\right)}(\cdot)$ denotes a normal cone to the set $C(t, u(t))$. As stated, the set $C(t,x)$ depends both on the time t and on the state x. Such differential inclusions have been introduced, for a time-dependent set, in the form

$$(\mathcal{SP}) \qquad \begin{cases} \dot{u}(t) \in -N_{K(t)}\big(u(t)\big) \quad \text{a.e } t \in [0,T] \\ \\ u(t) \in K(t) \quad \forall t \in [0,T] \\ \\ u(0) = u_0 \in K(0). \end{cases}$$

by J. J. Moreau [17, 18, 19] who called $(\mathcal{SP})$ a sweeping procee because of the mechanical interpretation (see, [17, 18, 19]).

The first work devoted to the inclusion $(\mathcal{D})$ has been realized by M. Kunze and M. D. P. Monteiro Marques [15] with $G \equiv \{0\}$ and $C(t,x)$ convex for all $t \in [0,T]$ and all $x \in H$. In [8], G is a (single-valued) mapping measurable with respect to the first variable and continuous with respect to the second one. Associating with each absolutely continuous mapping $y : [0,T] \to H$, with $y(0) = u_0$, the unique solution $\phi(y)$ of the time-dependent sweeping process (with unknown mapping u)

$$\dot{u}(t) \in -N_{C\left(t,y(t)\right)}\big(u(t)\big) + G\big(t, y(t)\big) \quad \text{with } u(0) = u_0 \in C\big(0, y(0)\big),$$

N. Chemetov and M. D. P. Monteiro Marques, by applying the classical Schauder fixed point theorem, proved the existence of solution of $(\mathcal{D})$, for nonconvex prox-regular and ball-compact sets $C(t,x)$ moving in a Lipschitz way. To be more precise, in [8], it is assumed that there exists an absolutely continuous function $\vartheta : [0,T]H$, which is monotone increasing, and a constant $L_2 \in]0,1[$, such that

$$|d\big(x, C(t,u)\big) - d\big(y, C(s,v)\big)| \leq \|x - y\| + \vartheta(t)\vartheta(s) + L_2\|u - v\|$$

for all $t, s \in [0, T]$ with $s < t$ and $x, y, u, v \in H$. Recently, in [6] C. Castaing, A. G Ibrahim and M. Yarou obtained, under the prox-regularity and ball-compactness assumption for $C(t, x)$, the existence of solution for $(\mathcal{D})$ when $G \equiv \{0\}$ via another method applying a generalized version of the Schauder fixed point theorem from [14, 22]. On the other hand, with $G \not\equiv \{0\}$ and $C(t, x)$ convex and ball-compact using a careful adaptation of their method, they also showed in the same paper [6] an existence result for $(\mathcal{D})$ with delay, that is, G is an upper semicontinuous and bounded multimapping defined on $[0, T] \times \mathcal{C}_H(-r, 0)$ and taking on weakly compact convex values of H; by $\mathcal{C}_H(-r, 0)$ we denote with $r > 0$ the space of all continuous mappings from $[-r, 0]$ to H. We refer to D. Azzam-Laouir, S. Izza and L. Thibault [2] for a reduction approach of $(\mathcal{D})$ to an unconstrained differential inclusion when $C(t, x)$ is prox-regular, G is a multimapping, and H is finite dimensional. J. Noel and L. Thibault proved the existence of a solution for $(\mathcal{D})$ in the Hilbert setting when $C(t, x)$ is a ball-compact prox-regular set and G is a multimapping; the method in [21] is based upon a result on the Hölder property of the metric projection to prox-regular set with respect to the Hausdorff distance. With the sets $C(t, x)$ prox-regular and contained in a fixed compact set and through the scheme

$$u_0^n = u_0 \; u_{i+1}^n = \operatorname{Proj}_{C(t_{i+1}^n, u_i^n)}(u_i^n - \frac{T}{2^n} g_i^n)$$

$$\text{with } g_i^n \in G(t_i^n, u_i^n) \text{ where } t_i^n := i\frac{T}{2^n},\; i = 0, \cdots, 2^n - 1,$$

T. Haddad [13] gave another approach which yields to a proof of existence in the Hilbert setting for $(\mathcal{D})$ without application of any fixed point theorem.

In the present paper, using ideas from [13] and [21] we provided a new constructive proof of existence of solution for $(\mathcal{D})$ when the sets $C(t, x)$ are ball-compact and subsmooth. The method also allows us to relax the growth conditions on the multimapping G which are assumed in [6, 13, 8]. The class of subsmooth sets introduced in [1] strictly contains the class of closed convex sets and the class of prox-regular sets. In the first section, we recall some preliminaries and we prove an upper semicontinuity result which will be used in our developement. The second section is devoted to the aforementioned constructive proof (using no fixed-point theorem) of the differential inclusion $(\mathcal{D})$ governed by subsmooth sets $C(t, x)$.

5.1 Preliminaries

Throughout the paper H is a Hilbert space whose inner product is denoted by $\langle \cdot, \cdot \rangle$ and the associated norm by $\|\cdot\|$. The closed unit ball of H with center

0 will be denoted by $\mathbb{B}$ and $B(u,\eta)$ (respectively, $B[u,\eta]$) denotes the open (respectively, closed) ball of center $u \in H$ and radius $\eta > 0$. Given a real $T > 0$, we will denote by $\mathcal{C}_H(0,T)$ the space of all continuous mappings from $[0,T]$ to H, "a.e." denotes "for almost every where" and $\dot{u}$ is the derivative of u. Let C, C' be two subsets of H and let v be a vector in H, the real $d(v,C)$ or $d_C(v) := \inf\{\|v-u\| : u \in C\}$ is the distance of the point v from the set C. We denote by

$$\operatorname{Haus}(C,C') = \max\Big\{\sup_{u\in C} d(u,C'), \sup_{v\in C'} d(v,C)\Big\}$$

the Hausdorff distance between C and C'. For $v \in H$ the projection of v into $C \subset H$ is the set

$$\operatorname{Proj}_C(v) := \{u \in C : d_C(v) = \|v-u\|\}.$$

This set is nonempty when C is ball-compact. Recall that a subset S of $(H, \|\cdot\|)$ is *ball-compact* provided that $S \cap r\mathbb{B}$ is compact in $(H,\|\cdot\|)$ for every real $r > 0$. Obviously any ball-compact set is norm closed, and in finite dimensions S is ball-compact if and only if it is closed. When $h \in \operatorname{Proj}_C(v)$, then we have $v - h \in N_C^p(h)$ where $N_C^p(\cdot)$ denotes the proximal normal cone of C (see, [9]).

For a nonempty interval $\mathcal{J}$ of $\mathbb{R}$, we recall that a multimapping $F : \mathcal{J} \rightrightarrows H$ is called Lebesgue measurable if for each open set $U \subset H$ the set $F^{-1}(U) := \{t \in \mathcal{J} : F(t) \cap U \neq \emptyset\}$ is Lebesgue measurable. When the values of F are closed subsets of H, we know (see [5]) that the Lebesgue measurability of F is equivalent to the measurability of the graph of F, that is,

$$\operatorname{gph} F \in \mathcal{L}(\mathcal{J}) \otimes \mathcal{B}(H),$$

where $\mathcal{L}(\mathcal{J})$ denotes the Lebesgue σ-field of $\mathcal{J}$, $\mathcal{B}(H)$ the Borel σ-field of H, and

$$\operatorname{gph} F := \big\{(t,u) \in \mathcal{J} \times H : u \in F(t)\big\}.$$

For any subset C of H, $\overline{\operatorname{co}}\, C$ stands for the closed convex hull of C, and $\sigma(\cdot, C)$ represents the support function of C, that is, for all $\xi \in H$,

$$\sigma(\xi, C) := \sup_{u\in C}\langle \xi, u\rangle.$$

If C is a nonempty subset of H, the Clarke normal cone $N(C;u)$ or $N_C(u)$ of C at $u \in C$ is defined by

$$N_C(u) = \{\xi \in H : \langle \xi, v\rangle \leq 0, \forall v \in T_C(u)\},$$

where the Clarke tangent cone $T(C;u)$ or $T_C(u)$ (see [10]) is defined as follows:

$$v \in T_C(u) \Leftrightarrow \begin{cases} \forall \varepsilon > 0, \exists \delta > 0 \text{ such that} \\ \\ \forall u' \in B(u,\delta) \cap C, \forall t \in]0,\delta[, (u' + tB(v,\varepsilon)) \cap C \neq \emptyset. \end{cases}$$

Equivalently, $v \in T_C(u)$ if and only if for any sequence $(u_n)_n$ of C converging to u and any sequence of positive reals $(t_n)_n$ converging to 0, there exists a sequence $(v_n)_n$ in H converging to v such that

$$u_n + t_n v_n \in C \text{ for all } n \in \mathbb{N}.$$

We put $N_C(u) = \emptyset$, whenever $u \notin C$. For any $\eta > 0$ we denote by $N_C^\eta(u)$ the truncated Clarke normal cone, that is,

$$N_C^\eta(u) = N_C(u) \cap \eta\mathbb{B}.$$

We typically denote by $f : H \to \mathbb{R} \cup \{+\infty\}$ a proper function (that is, f is finite at least at one point). The Clarke subdifferential $\partial f(u)$ of f at a point u (where f is finite) is defined by

$$\partial f(u) = \Big\{\xi \in H : (\xi, -1) \in N_{\operatorname{epi} f}\Big((u, f(u))\Big)\Big\},$$

where $\operatorname{epi} f$ denotes the epigraph of f, that is,

$$\operatorname{epi} f = \{(u,r) \in H \times \mathbb{R} : f(u) \leq r\}.$$

We also put $\partial f(u) = \emptyset$ if f is not finite at $u \in H$. If ψ_C denotes the indicator function of the set C, that is, $\psi_C(u) = 0$ if $u \in C$ and $\psi_C(u) = +\infty$ otherwise, then

$$\partial \psi_C(u) = N_C(u) \text{ for all } u \in H.$$

The Clarke subdifferential $\partial f(u)$ of a locally Lipschitz function f at u has also the other useful description

$$\partial f(u) = \{\xi \in H : \langle \xi, v\rangle \leq f^0(u,v), \forall v \in H\},$$

where

$$f^0(u,v) := \limsup_{(u',t) \to (u,0^+)} \frac{f(u' + tv) - f(u')}{t}.$$

The above function $f^0(u;\cdot)$ is called the Clarke directional derivative of f at u. The Clarke normal cone is known ([10]) to be related to the Clarke subdifferential of the distance function through the equality

$$N_C(u) = \operatorname{cl}_w(\mathbb{R}_+ \partial d_C(u)) \text{ for all } u \in C,$$

where $\mathbb{R}_+ := [0, \infty[$ and cl_w denotes the closure with respect to the weak topology of H. Further

$$\partial d_C(u) \subset N_C(u) \cap \mathbb{B} \quad \text{for all } u \in C. \tag{5.1}$$

The concept of Fréchet subdifferential will be also needed. A vector $\xi \in H$ is said to be in the Fréchet subdifferential $\partial_F f(u)$ of f at u (see [16]) provided that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $u' \in B(u, \delta)$ we have

$$\langle \xi, u' - u \rangle \leq f(u') - f(u) + \varepsilon \|u' - u\|.$$

It is known that we always have the inclusion

$$\partial_F f(u) \subset \partial f(u).$$

The Fréchet normal cone of C at $u \in C$ is given by

$$N_C^F(u) = \partial_F \psi_C(u),$$

so the following inclusion always holds true

$$N_C^F(u) \subset N_C(u) \quad \text{for all } u \in C.$$

On the other hand, the Fréchet normal cone is also related to the Fréchet subdifferential of the distance function since the following relations hold true for all $u \in C$

$$N_C^F(u) = \mathbb{R}_+ \partial_F d_C(u)$$

and

$$\partial_F d_C(u) = N_C^F(u) \cap \mathbb{B}. \tag{5.2}$$

Another important property is

$$v - u \in N_C^F(u) \quad \text{hence also} \quad v - u \in N_C(u)$$

whenever $u \in \mathrm{Proj}\,_C(v)$, since $N_C^p(u) \subset N_C^F(u)$.

The next lemma 5.1.2, recently established in [20], will play an important role in the proof of our main results.

We recall firstly the definition of subsmooth sets (see, [1]). Let C be a closed subset of H. We say that C is *subsmooth at* $u \in C$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle \xi_1 - \xi_2, u_1 - u_2 \rangle \geq -\varepsilon \|u_1 - u_2\|. \tag{5.3}$$

whenever $u_1, u_2 \in B(u, \delta) \cap C$ and $\xi_i \in N_C(u_i) \cap \mathbb{B}$, $i = 1, 2$. The set C is *subsmooth*, if it is *subsmooth* at each point of C. We further say that C is *uniformly subsmooth*, if for every $\varepsilon > 0$ there exists $\delta > 0$, such that (5.3) holds for all $u_1, u_2 \in C$ satisfying $\|u_1 - u_2\| < \delta$ and all $\xi_i \in N_C(u_i) \cap \mathbb{B}$.

Definition 5.1.1. *Let $\{C(t,v):\ (t,v)\in[0,T]\times H\}$ be a family of closed sets of H. This family is called equi-uniformly subsmooth, if for every $\varepsilon>0$, there exists $\delta>0$ such that (5.3) holds for each $(t,v)\in[0,T]\times H$, and all $u_1,u_2\in C(t,v)$ satisfying $\|u_1-u_2\|<\delta$ and all $\xi_i\in N_{C(t,v)}(u_i)\cap\mathbb{B}$.*

Lemma 5.1.1. *[20] If a closed set C of H is subsmooth at $u_0\in C$, then*

$$\partial d_C(u_0)=\partial_F d_C(u_0)$$

and

$$N_C(u_0)=N_C^F(u_0).$$

Lemma 5.1.2. *[20] Let E be a metric space and let $(C(q))_{q\in E}$ be a family of nonempty closed sets of H which is equi-uniformly subsmooth and let a real $\eta>0$. Let $Q\subset E$ and $q_0\in \mathrm{cl}Q$. Then the following hold:*

(a) For all $(q,u)\in\operatorname{gph}C$ we have $\eta\partial d_{C(q)}(u)\subset\eta\mathbb{B}$;

(b) For any net $(q_j)_{j\in J}$ in Q converging to q_0, any net $(u_j)_{j\in J}$ converging to $u\in C(q_0)$ in $(H,\|\cdot\|)$ with $u_j\in C(q_j)$ and $d_{C(q_j)}(y)\underset{j\in J}{\to}0$ for every $y\in C(q_0)$, and any net $(\zeta_j)_{j\in J}$ converging weakly to ζ in $(H,w(H,H))$ with $\zeta_j\in\eta\partial d_{C(q_j)}(u_j)$, we have $\zeta\in\eta\partial d_{C(q_0)}(u)$.

Through Lemma 5.1.2 we can, using some ideas in [20], establish the following partial upper semicontinuity property.

Proposition 5.1.1. *Let $\{C(t,v):\ (t,v)\in[0,T]\times H\}$ be a family of nonempty closed sets of H which is equi-uniformly subsmooth and let a real $\eta>0$. Assume that there exist real constants $L_1>0$ and $L_2>0$ such that, for any $x,y,u,v\in H$ and $s,t\in[0,T]$*

$$|d\big(x,C(t,u)\big)-d\big(y,C(s,v)\big)|\le\|x-y\|+L_1|t-s|+L_2\|u-v\|.$$

Then the following assertions hold:

(a) For all $(s,v,y)\in\operatorname{gph}C$ we have $\eta\partial d_{C(s,v)}(y)\subset\eta\mathbb{B}$;

(b) For any sequence $(s_n)_n$ in $[0,T]$ converging to s, any sequence $(v_n)_n$ converging to v, any sequence $(y_n)_n$ converging to $y\in C(s,v)$ with $y_n\in C(s_n,v_n)$, and any $\xi\in H$, we have

$$\limsup_{n\to\infty}\sigma\big(\xi,\eta\partial d_{C(s_n,v_n)}(y_n)\big)\le\sigma\big(\xi,\eta\partial d_{C(s,v)}(y)\big).$$

Proof. The proof will be a careful adaptation of techniques of the proof of Proposition 1.2.1 in [20]. We only have to prove *(b)*. Let $(s_n)_n$, $(v_n)_n$ and $(y_n)_n$ as in the statement. Extracting a subsequence if necessary, we may suppose that

$$\limsup_{n\to\infty} \sigma\big(\xi, \eta\partial d_{C(s_n,v_n)}(y_n)\big) = \lim_{n\to\infty} \sigma\big(\xi, \eta\partial d_{C(s_n,v_n)}(y_n)\big).$$

For any n, the weak compactness of $\eta\partial d_{C(s_n,v_n)}(y_n)$ ensures the existence of some $\zeta_n \in \eta\partial d_{C(s_n,v_n)}(y_n)$ such that

$$\langle \xi, \zeta_n\rangle = \sigma\big(\xi, \eta\partial d_{C(s_n,v_n)}(y_n)\big).$$

Since $\|\zeta_n\| \leq \eta$ by *(a)*, a subsequence of $(\zeta_n)_n$ (that we do not relabel) converges weakly to some ζ in H. It results that

$$\langle \xi, \zeta\rangle = \limsup_{n\to\infty} \sigma\big(\xi, \eta\partial d_{C(s_n,v_n)}(y_n)\big). \tag{5.4}$$

Now, observe, for each $z \in C(s, v)$, that

$$0 \leq d\big(z, C(s_n, v_n)\big) \leq d\big(z, C(s, v)\big) + L_1|s_n - s| + L_2\|v_n - v\|.$$

Since $(v_n)_n$ and $(s_n)_n$ converge to v and s respectively, it follows that $d\big(z, C(s_n, v_n)\big)$ converge to 0. We then apply Lemma 5.1.2 to obtain $\zeta \in \eta\partial d_{C(s,v)}(y)$. The latter inclusion combined with (5.4) yields

$$\limsup_{n\to\infty} \sigma\big(\xi, \eta\partial d_{C(s_n,v_n)}(y_n)\big) \leq \sigma\big(\xi, \eta\partial d_{C(s,v)}(y)\big),$$

This complete the proof. □

5.2 Subsmoothnes and variational inequality

We show in this section under reasonable assumptions that there always exists a solution for variational evolution differential inclusion governed by subsmooth set.

We shall be dealing with two multimappings $G : [0, T] \times H \rightrightarrows H$ with nonempty weakly compact convex values and $C : [0, T] \times H \rightrightarrows H$ with nonempty values. They are required to satisfy the following assumptions:

$(\mathcal{H}_1)$ The multimapping G is scalarly upper semicontinuous with respect to both variables (that is, for each $y \in H$ the function $(t, u) \to \sigma(y, G(t, u))$ is upper semicontinuous) and, for some real $\alpha > 0$

$$d\big(0, G(t, u)\big) \leq \alpha$$

for all $t \in [0, T]$ and $u \in H$ with $u \in C(t, u)$;

($\mathcal{H}_2$) For each $t \in [0,T]$ and each $u \in H$, the sets $C(t,u)$ are nonempty and equi-uniformly subsmooth;

($\mathcal{H}_3$) There are real constants $L_1 > 0$, $L_2 \in]0,1[$ such that, for all $t,s \in [0,T]$ and $x,y,u,v \in H$
$$\big|d\big(x, C(t,u)\big) - d\big(y, C(s,v)\big)\big| \leq \|x-y\| + L_1|t-s| + L_2\|u-v\|.$$

($\mathcal{H}_4$) For any bounded subset $A \subset H$, the set $C([0,T]\times A)$ is relatively ball-compact, that is, the intersection of $C([0,T]\times A)$ with any closed ball of H is relatively compact in H.

Of course the inequality condition in ($\mathcal{H}_3$) is equivalent to
$$\big|d\big(x, C(t,u)\big) - d\big(x, C(s,v)\big)\big| \leq L_1|t-s| + L_2\|u-v\|$$
for all $t,s \in [0,T]$ and $x,u,v \in H$.

Theorem 5.2.1. *Assume that H is a Hilbert space, that $\mathcal{H}_1, \cdots, \mathcal{H}_4$ hold. Then, for any $u_0 \in H$ with $u_0 \in C(0,u_0)$, there exists a Lipschitz continuous solution $u : [0,T] \to H$ of the differential inclusion*
$$(\mathcal{D}) \quad \begin{cases} \dot{u}(t) \in -N_{C\left(t,u(t)\right)}\big(u(t)\big) + G\big(t,u(t)\big) \quad a.e\ \ t \in [0,T], \\ \\ u(t) \in C\big(t,u(t)\big)\ \forall t \in [0,T], \end{cases}$$
with $\|\dot{u}(t)\| \leq \frac{L_1+2\alpha}{1-L_2}$ a.e $t \in [0,T]$.

Proof. For each integer $n \geq 1$, we consider the partition of $[0,T]$ by the points
$$t_k^n = k\frac{T}{n},\ k = 0,1,\cdots,n.$$
For each $(t,x) \in [0,T]\times H$ denote by $g(t,x)$ the element of minimal norm of the closed convex set $G(t,x)$ of H, that is,
$$g(t,x) = \operatorname{Proj}_{G(t,x)}(0).$$
Put $x_0^n := u_0 \in C(t_0^n, u_0)$.

Step 1. We construct $x_0^n, x_1^n, \cdots, x_n^n$ in H such that for each $k = 0,1,\cdots,n-1$, the following inclusions hold
$$x_{k+1}^n \in C(t_{k+1}^n, x_k^n) \tag{5.5}$$

$$x_k^n + \frac{T}{n} g(t_k^n, x_k^n) - x_{k+1}^n \in N_{C(t_{k+1}^n, x_k^n)}(x_{k+1}^n), \tag{5.6}$$

along with the inequality $\|x_1^n - x_0^n\| \leq (L_1 + 2\alpha)\frac{T}{n}$ and for $k = 1, \cdots, n-1$

$$\|x_{k+1}^n - x_k^n\| \leq (L_1 + 2\alpha)\frac{T}{n} + L_2 \|x_{k-1}^n - x_{k-2}^n\|. \tag{5.7}$$

The ball-compactness of $C(t_1^n, x_0^n)$ ensures that we can choose

$$x_1^n \in \operatorname{Proj}_{C(t_1^n, x_0^n)}\Big(x_0^n + \frac{T}{n} g(t_0^n, x_0^n)\Big)$$

and hence

$$x_1^n \in C(t_1^n, x_0^n)$$

$$x_0^n + \frac{T}{n} g(t_0^n, x_0^n) - x_1^n \in N_{C(t_1^n, x_0^n)}(x_1^n).$$

On the other hand, using $\|g(t_0^n, x_0^n)\| \leq \alpha$, in view of hypothesis $(\mathcal{H}_1)$ we have

$$\begin{aligned}
\|x_1^n - x_0^n\| &\leq \Big\| x_1^n - \big(x_0^n + \frac{T}{n} g(t_0^n, x_0^n)\big)\Big\| + \Big\|\frac{T}{n} g(t_0^n, x_0^n)\Big\| \\
&= d\Big(x_0^n + \frac{T}{n} g(t_0^n, x_0^n), C(t_1^n, x_0^n)\Big) + \Big\|\frac{T}{n} g(t_0^n, x_0^n)\Big\| \\
&\leq d\Big(x_0^n + \frac{T}{n} g(t_0^n, x_0^n), C(t_0^n, x_0^n)\Big) + L_1 |t_1^n - t_0^n| + \Big\|\frac{T}{n} g(t_0^n, x_0^n)\Big\| \\
&\leq 2\Big\|\frac{T}{n} g(t_0^n, x_0^n)\Big\| + L_1 \frac{T}{n} \\
&\leq \Big(L_1 + 2\alpha\Big)\frac{T}{n}. \qquad (5.8)
\end{aligned}$$

Now, suppose that, for $0, 1, \cdots, k$, with $k \leq n-1$ the points $x_0^n, x_1^n, \cdots, x_k^n$ have been constructed, so that properties (5.5), (5.6) and (5.7) hold true. Since $C(t_{k+1}^n, x_k^n)$ is ball-compact, then we can find

$$x_{k+1}^n \in \operatorname{Proj}_{C(t_{k+1}^n, x_k^n)}\Big(x_k^n + \frac{T}{n} g(t_k^n, x_k^n)\Big),$$

and therefore,

$$x_{k+1}^n \in C(t_{k+1}^n, x_k^n),$$

$$x_k^n + \frac{T}{n} g(t_k^n, x_k^n) - x_{k+1}^n \in N_{C(t_{k+1}^n, x_k^n)}(x_{k+1}^n).$$

By $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$, we get

$$\begin{aligned}\|x_{k+1}^n - x_k^n\| \leq & \Big\|x_{k+1}^n - \big(x_k^n + \frac{T}{n}g(t_k^n, x_k^n)\big)\Big\| + \Big\|\frac{T}{n}g(t_k^n, x_k^n)\Big\| \\ & = d\Big(x_k^n + \frac{T}{n}g(t_k^n, x_k^n), C(t_{k+1}^n, x_k^n)\Big) + \Big\|\frac{T}{n}g(t_k^n, x_k^n)\Big\| \\ & \leq d\Big(x_k^n + \frac{T}{n}g(t_k^n, x_k^n), C(t_k^n, x_{k-1}^n)\Big) + \Big\|\frac{T}{n}g(t_k^n, x_k^n)\Big\| \\ & + L_1|t_{k+1}^n - t_k^n| + L_2\|x_k^n - x_{k-1}^n\| \\ & \leq 2\alpha\frac{T}{n} + L_1\frac{T}{n} + L_2\|x_k^n - x_{k-1}^n\|.\end{aligned}$$

The finite sequence $x_0^n, x_1^n \cdots, x_n^n$ satisfying (5.5), (5.6) and (5.7) is then contructed by induction.

Fix any $k = 1, \cdots, n-1$. We observe that

$$\begin{aligned}\|x_{k+1}^n - x_k^n\| &\leq 2\alpha\frac{T}{n} + L_1\frac{T}{n} + L_2\|x_k^n - x_{k-1}^n\| \\ &\leq 2\alpha\frac{T}{n} + L_1\frac{T}{n} + L_2\Big(2\alpha\frac{T}{n} + L_1\frac{T}{n} + L_2\|x_{k-1}^n - x_{k-2}^n\|\Big) \\ &= 2\alpha\frac{T}{n}(1+L_2) + L_1\frac{T}{n}(1+L_2) + L_2^2\|x_{k-1}^n - x_{k-2}^n\|,\end{aligned}$$

thus, we deduce

$$\begin{aligned}\|x_{k+1}^n - x_k^n\| \leq 2\alpha\frac{T}{n}(1 + L_2 + L_2^2 + \cdots + L_2^{k-1}) + L_1\frac{T}{n}(1 + L_2 + L_2^2 + \cdots + L_2^{k-1}) \\ + L_2^k\|x_1^n - x_0^n\| = \frac{T}{n}(2\alpha + L_1)(1 + L_2 + L_2^2 + \cdots + L_2^{k-1}) + L_2^k\|x_1^n - x_0^n\|.\end{aligned}$$

It follows from (5.8) that

$$\|x_{k+1}^n - x_k^n\| \leq (2\alpha + L_1)\big(1 + L_2 + L_2^2 + \cdots + L_2^k\big)\frac{T}{n},$$

and since $L_2 < 1$, we deduce

$$\|x_{k+1}^n - x_k^n\| \leq \frac{2\alpha + L_1}{1 - L_2}\frac{T}{n}, \tag{5.9}$$

and the latter inequality still holds true for $k = 0$ according to (5.8). Further for $k = 1, \cdots, n-1$,

$$\begin{aligned}\|x_{k+1}^n\| &\leq \|x_{k+1}^n - x_k^n\| + \|x_k^n - x_{k-1}^n\| + \cdots + \|x_1^n - x_0^n\| + \|x_0^n\| \\ &\leq \|u_0\| + \frac{2\alpha + L_1}{1 - L_2}(k+1)\frac{T}{n} \\ &\leq \|u_0\| + \frac{2\alpha + L_1}{1 - L_2}T =: \beta. \qquad (5.10)\end{aligned}$$

Step 2. Construction of $u_n(\cdot)$.
For any $t \in [t_k^n, t_{k+1}^n]$ with $k = 0, 1, \cdots, n-1$, put

$$u_n(t) := \frac{t_{k+1}^n - t}{t_{k+1}^n - t_k^n} x_k^n + \frac{t - t_k^n}{t_{k+1}^n - t_k^n} x_{k+1}^n.$$

Thus, for almost all $t \in [t_k^n, t_{k+1}^n]$,

$$\dot{u}_n(t) = -\frac{x_k^n}{t_{k+1}^n - t_k^n} + \frac{x_{k+1}^n}{t_{k+1}^n - t_k^n} = -\frac{n}{T}(x_k^n - x_{k+1}^n).$$

By construction, (5.5), (5.6), (5.7) and the latter equalities give

$$u_n(t_{k+1}^n) \in C\big(t_{k+1}^n, u_n(t_k^n)\big) \tag{5.11}$$

$$-\dot{u}_n(t) \in N_{C\left(t_{k+1}^n, u_n(t_k^n)\right)}\big(u_n(t_{k+1}^n)\big) - g\big(t_k^n, u_n(t_k^n)\big) \text{ a.e } t \in [t_k^n, t_{k+1}^n[\tag{5.12}$$

with (by (5.9))

$$\|\dot{u}_n(t)\| = \frac{n}{T}\|x_k^n - x_{k+1}^n\| \leq \frac{L_1 + 2\alpha}{1 - L_2} =: M. \tag{5.13}$$

Put

$$\delta_n(t) := \begin{cases} t_k^n & \text{if} \quad t \in [t_k^n, t_{k+1}^n[\\ t_{n-1}^n & \text{if} \quad t = T, \end{cases}$$

and

$$\theta_n(t) := \begin{cases} t_{k+1}^n & \text{if} \quad t \in [t_k^n, t_{k+1}^n[\\ T & \text{if} \quad t = T. \end{cases}$$

Observe that for each $t \in [0, T]$, choosing k such that $t \in [t_k^n, t_{k+1}^n[$ if $t < T$ and $k = n-1$ if $t = T$, we have

$$|\delta_n(t) - t| \leq |t_{k+1}^n - t_k^n| = \frac{T}{p^n}, \text{ so } \delta_n(t) \to t \text{ as } n \to +\infty,$$

and similarly $\theta_n(t) \to t$ as $n \to +\infty$. Further, for each $t \in [t_k^n, t_{k+1}^n[$, the definitions of $\delta_n(\cdot)$ and $\theta_n(\cdot)$ combined with (5.11) and (5.12) yield

$$u_n\big(\theta_n(t)\big) \in C\Big(\theta_n(t), u_n\big(\delta_n(t)\big)\Big) \tag{5.14}$$

$$-\dot{u}_n(t) \in N_{C\Big(\theta_n(t), u_n\big(\delta_n(t)\big)\Big)}\Big(u_n\big(\theta_n(t)\big)\Big) - g\Big(\delta_n(t), u_n\big(\delta_n(t)\big)\Big) \text{ a.e } t \in [0,T]. \tag{5.15}$$

Step 3. Convergence of a subsequence of $(u_n(\cdot))$ to some absolutely continuous mapping $u(\cdot)$.
Fix any $t \in [0,T]$ and consider, for any infinite subset $N \subset \mathbb{N}$, the sequence $(u_n(t))_{n\in N}$. It follows from (5.10) and (5.14) that

$$u_n\big(\theta_n(t)\big) \in C\Big(\theta_n(t), u_n\big(\delta_n(t)\big)\Big) \cap \beta\mathbb{B},$$

which implies that $u_n(\theta_n(t)) \in C([0,T] \times \beta\mathbb{B}) \cap \beta\mathbb{B}$. By $(\mathcal{H}_4)$ the sequence $(u_n(\theta_n(t)))$ is relatively compact, so there is an infinite subset $N_0 \subset N$ such that $(u_n(\theta_n(t)))_{n\in N_0}$ converges to some vector $l(t) \in H$. Putting $h_n(t) := u_n(\theta_n(t)) - u_n(t)$ for all $n \in N_0$, by (5.13), we obtain

$$\|h_n(t)\| \le \int_t^{\theta_n(t)} \|\dot{u}_n(s)\| ds \le M(\theta_n(t) - t) \underset{n\to\infty}{\longrightarrow} 0.$$

Then, $(u_n(t))_{n\in N_0}$ converges to $l(t)$, thus the set $\{u_n(t) : n \in \mathbb{N}\}$ is relatively compact in H. The sequence $(u_n)_{n\in\mathbb{N}}$ being in addition equicontinuous according to (5.13), this sequence $(u_n)_{n\in\mathbb{N}}$ is relatively compact in $\mathcal{C}_H(0,T)$, so we can extract a subsequence of $(u_n)_{n\in\mathbb{N}}$ (that we do not relabel) which converges uniformly to u on $[0,T]$. By the inequality (5.13) again there is a subsequence of $(\dot{u}_n)_{n\in\mathbb{N}}$ (that we do not relabel) which converges $w(\mathrm{L}^1_H, \mathrm{L}^\infty_H)$ in $\mathrm{L}^1_H(0,T)$ to a mapping $w \in \mathrm{L}^1_H(0,T)$ with $\|w(t)\| \le M$ a.e. $t \in [0,T]$. Fixing $t \in [0,T]$ and taking any $\xi \in H$, the above weak convergence in $\mathrm{L}^1_H(0,T)$ yields

$$\lim_{n\to\infty} \int_0^T \langle \mathbb{1}_{[0,t]}(s)\xi, \dot{u}_n(s)\rangle ds = \int_0^T \langle \mathbb{1}_{[0,t]}(s)\xi, w(s)\rangle ds,$$

or equivalently

$$\lim_{n\to\infty} \langle \xi, u_0 + \int_0^t \dot{u}_n(s) ds\rangle = \langle \xi, u_0 + \int_0^t \dot{w}(s) ds\rangle.$$

This means, for each $t \in [0,T]$, that $u_n(t) \underset{n\to\infty}{\longrightarrow} u_0 + \int_0^t w(s)ds$ weakly in H. Since the sequence $(u_n(t))_{n\in\mathbb{N}}$ also converges strongly to $u(t)$ in H, it ensures that $u(t) = u_0 + \int_0^t w(s)ds$, so the mapping $u(\cdot)$ is absolutely continuous on $[0,T]$ with $\dot{u} = w$. The mapping $u(\cdot)$ is even Lipschitz on $[0,T]$ with M as a

Lipschitz constant therein.

Step 4. We show now that $u(\cdot)$ is a solution of $(\mathcal{D})$.
Let $z_n(t) := g(\delta_n(t), u_n(\delta_n(t)))$ for all $t \in [0,T]$. Since

$$\|g(\delta_n(t), u_n(\delta_n(t)))\| \leq \alpha \quad \text{for all } n \in \mathbb{N} \text{ and } t \in [0,T],$$

we may suppose (taking a subsequence if necessary) that the sequence $(z_n(\cdot))$ converges $w(\mathrm{L}^1_H, \mathrm{L}^\infty_H)$ in $\mathrm{L}^1_H(0,T)$ to a mapping $z(\cdot) \in \mathrm{L}^1_H(0,T)$ with $\|z(t)\| \leq \alpha$ a.e $t \in [0,T]$.

For all $t \in [0,T]$ we have $u(t) \in C(t, u(t))$. Indeed, by $(\mathcal{H}_3)$ and (5.13)

$$\begin{aligned} d\Big(u_n(t), C\big(t, u(t)\big)\Big) & \\ &\leq \|u_n(t) - u_n\big(\theta_n(t)\big)\| + L_1|t - \theta_n(t)| + L_2\|u(t) - u_n\big(\delta_n(t)\big)\| \\ &\leq (M + L_1)|t - \theta_n(t)| + L_2 M|\delta_n(t) - t| + L_2\|u(t) - u_n(t)\| \end{aligned}$$

then,

$$d(u_n(t), C(t, u(t))) \underset{n\to\infty}{\longrightarrow} 0, \text{ so } d(u(t), C(t, u(t))) = 0 \text{ and } u(t) \in C(t, u(t)).$$

Further, from the inequality $\|\dot{u}_n(t) - z_n(t)\| \leq M + \alpha := \gamma$ a.e. and from the inclusion (5.15) it follows for a.e. $t \in [0,T]$ that

$$\begin{aligned} -\dot{u}_n(t) + z_n(t) &\in N_{C\left(\theta_n(t), u_n\left(\delta_n(t)\right)\right)}\Big(u_n\big(\theta_n(t)\big)\Big) \bigcap \gamma\mathbb{B} \\ &= \gamma \partial d_{C\left(\theta_n(t), u_n\left(\delta_n(t)\right)\right)}\Big(u_n\big(\theta_n(t)\big)\Big), \end{aligned} \tag{5.16}$$

$$z_n(t) \in G\Big(\delta_n(t), u_n\big(\delta_n(t)\big)\Big). \tag{5.17}$$

Since $(-\dot{u}_n + z_n, z_n)_n$ converges weakly in $\mathrm{L}^1_{H\times H}(0,T)$ to $(-\dot{u} + z, z)$, by Mazur theorem, there are

$$\xi_n \in \operatorname{co}\{-\dot{u}_q + z_q : q \geq n\} \text{ and } \zeta_n \in \operatorname{co}\{z_q : q \geq n\} \tag{5.18}$$

such that $(\xi_n, \zeta_n)_n$ converges strongly in $\mathrm{L}^1_{H\times H}(0,T)$ to $(-\dot{u} + z, z)$. Extracting a subsequence if necessary we suppose that $(\xi_n(\cdot), \zeta_n(\cdot))_n$ converges a.e. to $(-\dot{u}(\cdot) + z(\cdot), z(\cdot))$, then there is a Lebesgue negligible set $S \subset [0,T]$ such that for every $t \in [0,T] \backslash S$ on one hand $(\xi_n(t), \zeta_n(t)) \to (-\dot{u}(t) + z(t), z(t))$

strongly in H and on the other hand the inclusions (5.16) and (5.17) hold true for every integer n as well as the inclusions

$$-\dot{u}(t)+z(t) \in \bigcap_n \overline{\mathrm{co}}\Big\{-\dot{u}_q(t)+z_q(t) : q \geq n\Big\} \text{ and } z(t) \in \bigcap_n \overline{\mathrm{co}}\Big\{z_q(t) : q \geq n\Big\}.$$

It results from (5.16) and (5.17) that for any $n \in \mathbb{N}$, any $t \in [0,T]\backslash S$, and for any $y \in H$

$$\Big\langle y, -\dot{u}_n(t) + z_n(t)\Big\rangle \leq \sigma\bigg(y, \gamma\partial d_{C\left(\theta_n(t), u_n\left(\delta_n(t)\right)\right)}\Big(u_n\big(\theta_n(t)\big)\Big)\bigg) \tag{5.19}$$

and

$$\Big\langle y, z_n(t)\Big\rangle \leq \sigma\bigg(y, G\Big(\delta_n(t), u_n\big(\delta_n(t)\big)\Big)\bigg). \tag{5.20}$$

Further, for each $n \in \mathbb{N}$ and any $t \in [0,T]\backslash S$, from (5.18) we have

$$\Big\langle y, \xi_k(t)\Big\rangle \leq \sup_{q \geq n}\Big\langle y, -\dot{u}_q(t) + z_q(t)\Big\rangle \text{ for all } k \geq n$$

and

$$\Big\langle y, \zeta_k(t)\Big\rangle \leq \sup_{q \geq n}\Big\langle y, z_q(t)\Big\rangle \text{ for all } k \geq n$$

and taking the limit in both inequalities as $k \to +\infty$ gives through (5.19) and (5.20)

$$\begin{aligned}\Big\langle y, -\dot{u}(t) + z(t)\Big\rangle &\leq \sup_{q \geq n}\Big\langle y, -\dot{u}_q(t) + z_q(t)\Big\rangle \\ &\leq \sup_{q \geq n}\sigma\bigg(y, \gamma\partial d_{C\left(\theta_q(t), u_q\left(\delta_q(t)\right)\right)}\Big(u_q\big(\theta_q(t)\big)\Big)\bigg)\end{aligned}$$

and

$$\Big\langle y, z(t)\Big\rangle \leq \sup_{q \geq n}\Big\langle y, z_q(t)\Big\rangle \leq \sup_{q \geq n}\sigma\bigg(y, G\Big(\delta_q(t), u_q\big(\delta_q(t)\big)\Big)\bigg),$$

which ensures that

$$\Big\langle y, -\dot{u}(t) + z(t)\Big\rangle \leq \limsup_{n \to +\infty}\sigma\bigg(y, \gamma\partial d_{C\left(\theta_n(t), u_n\left(\delta_n(t)\right)\right)}\Big(u_n\big(\theta_n(t)\big)\Big)\bigg)$$

and

$$\Big\langle y, z(t)\Big\rangle \leq \limsup_{n \to +\infty}\sigma\bigg(y, G\Big(\delta_n(t), u_n\big(\delta_n(t)\big)\Big)\bigg).$$

According to $(\mathcal{H}_3)$ and Proposition 5.1.1, the multimapping $\partial d_{C(t,u)}(x)$ takes on weakly compact convex values and is upper semicontinuous from $[0,T] \times H \times H$ into $(H, w(H,H))$, hence for each $y \in H$ the real-valued function $\sigma(y, \gamma\partial d_{C(\cdot,\cdot)}(\cdot))$ is upper semicontinuous on $[0,T] \times H \times H$. Further, $\sigma(y, G(\cdot,\cdot))$ is also upper semicontinuous on $[0,T] \times H$ by assumption $(\mathcal{H}_1)$. It follows that, for every $t \in [0,T] \backslash S$ and every $y \in H$,

$$\big\langle y, -\dot{u}(t) + z(t)\big\rangle \leq \sigma\Big(y, \gamma\partial d_{C\big(t,u(t)\big)}\big(u(t)\big)\Big)$$

and

$$\big\langle y, z(t)\big\rangle \leq \sigma\Big(y, G\big(t, u(t)\big)\Big),$$

which ensures that $-\dot{u}(t) + z(t) \in \gamma\partial d_{C(t,u(t))}(u(t))$ and $z(t) \in G(t, u(t))$, consequently

$$\dot{u}(t) \in -N_{C\big(t,u(t)\big)}\big(u(t)\big) + z(t) \text{ a.e.}$$

$$z(t) \in G\big(t, u(t)\big) \text{ a.e.}$$

with

$$\|\dot{u}(t) - z(t)\| \leq \gamma.$$

The proof is complete. $\square$

It is worth mentioning that the next theorem proves the existence of solution on the whole interval $\mathbb{R}_+ := [0, +\infty[$. Nevertheless, the assumptions $\mathcal{H}_1, \cdots, \mathcal{H}_4$ are replaced by $\mathcal{G}_1, \cdots, \mathcal{G}_4$ when the time describes $\mathbb{R}_+$.

Theorem 5.2.2. *Let $G : \mathbb{R}_+ \times H \rightrightarrows H$ be a multimapping which is scalarly upper semicontinuous with respect to both variables. Assume that H is a Hilbert space, that $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$ below hold:*

$(\mathcal{G}_1)$ *There exists a non-negative function $\beta(\cdot) \in \mathrm{L}^{\infty}_{\mathrm{loc}}(\mathbb{R}_+)$ such that*

$$d\big(0, G(t,u)\big) \leq \beta(t)$$

for all $t \in \mathbb{R}_+$ and $u \in H$ with $u \in C(t,u)$;

$(\mathcal{G}_2)$ *For each $t \in \mathbb{R}_+$ and each $u \in H$, the sets $C(t,u)$ are nonempty closed in H and equi-uniformly subsmooth;*

$(\mathcal{G}_3)$ *There are real constants $L_1 > 0$, $L_2 \in]0,1[$ such that, for all $t, s \in \mathbb{R}_+$ and $x, y, u, v \in H$*

$$\Big|d\big(x, C(t,u)\big) - d\big(y, C(s,v)\big)\Big| \leq \|x-y\| + L_1|t-s| + L_2\|u-v\|.$$

($\mathcal{G}_4$) *For any real $\tau > 0$ and any bounded subset $A \subset H$, the set $C([0,\tau] \times A)$ is relatively ball-compact.*

Then, given $u_0 \in H$ with $u_0 \in C(0, u_0)$, there exists a mapping $u : \mathbb{R}_+ \to H$ which is locally Lipschitz continuous on $\mathbb{R}_+$ and satisfies

$$(\mathcal{D}_{\mathbb{R}_+}) \quad \begin{cases} \dot{u}(t) \in -N_{C\big(t,u(t)\big)}\big(u(t)\big) + G\big(t, u(t)\big) \quad \text{a.e } t \in \mathbb{R}_+, \\[2ex] u(t) \in C\big(t, u(t)\big) \ \forall t \in \mathbb{R}_+, \\[2ex] u(t) = u_0 + \int_0^t \dot{u}(s)ds \ \forall t \in \mathbb{R}_+. \end{cases}$$

Proof. Put $T_k = k$ for all $k \in \{0\} \cup \mathbb{N}$. It will suffice to prove that Theorem 5.2.1 applies on each interval $[T_k, T_{k+1}]$.

According to assumptions $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$ we have $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4$ hold on the interval $[T_0, T_1]$. Since $u_0 \in C(T_0, u_0)$, by Theorem 5.2.1 there exists a Lipschitz continuous mapping $u^0 : [T_0, T_1] \to H$ such that

$$\begin{cases} \dot{u}^0(t) \in -N_{C\big(t,u^0(t)\big)}\big(u^0(t)\big) + G\big(t, u^0(t)\big) \quad \text{a.e } \ t \in [T_0, T_1], \\[2ex] u^0(t) \in C\big(t, u^0(t)\big) \ \forall t \in [T_0, T_1], \\[2ex] u^0(T_0) = u_0. \end{cases}$$

Suppose $u^0, \cdots, u^{k-1}$ have been constructed such that, for $l = 0, \cdots, k-1$, $u^l : [T_l, T_{l+1}] \to H$ is Lipschitz continuous, $u^l(T_l) = u^{l-1}(T_l)$, $u^l(t) \in C(t, u^l(t))$ for all $t \in [T_l, T_{l+1}]$ and

$$\dot{u}^l(t) \in -N_{C\big(t,u^l(t)\big)}\big(u^l(t)\big) + G\big(t, u^l(t)\big) \quad \text{a.e } \ t \in [T_l, T_{l+1}].$$

In an analogous way as above, the hypotheses $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$ ensure that $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4$ hold on the interval $[T_k, T_{k+1}]$ and we have $u^{k-1}(T_k) \in C(T_k, u^{k-1}(T_k))$. It follows from Theorem 5.2.1 that there is a Lipschitz continuous mapping $u^k : [T_k, T_{k+1}] \to H$ such that

$$(5.21) \quad \begin{cases} \dot{u}^k(t) \in -N_{C\big(t,u^k(t)\big)}\big(u^k(t)\big) + G\big(t, u^k(t)\big) \quad \text{a.e } \ t \in [T_k, T_{k+1}], \\[2ex] u^k(t) \in C\big(t, u^k(t)\big) \ \forall t \in [T_k, T_{k+1}], \\[2ex] u^k(T_k) = u^{k-1}(T_k). \end{cases}$$

So, we obtain by induction u^k for all $k \in \{0\} \cup \mathbb{N}$ with the above properties.

Let $u : \mathbb{R}_+ \to H$ be the mapping defined by

$$u(t) := u^k(t) \quad \text{for all } t \in [T_k, T_{k+1}[\text{ with } k \in \{0\} \cup \mathbb{N}.$$

It is easily seen that u is locally Lipschitz continuous on $\mathbb{R}_+$. Therefore, it results from (5.21) that

$$\begin{cases} \dot{u}(t) \in -N_{C\left(t,u(t)\right)}\big(u(t)\big) + G\big(t, u(t)\big) \quad \text{a.e } t \in \mathbb{R}_+, \\ \\ u(t) \in C\big(t, u(t)\big) \ \forall t \in \mathbb{R}_+, \\ \\ u(0) = u^0(T_0) = u_0. \end{cases}$$

This proves the theorem. □

As a direct consequences of Theorem 5.2.1 and Theorem 5.2.2 we obtain:

Corollaire 5.2.1. *Let $G : [0,T] \times H \rightrightarrows H$ be a multimapping which is scalarly upper semicontinuous with respect to both variables. Assume that H is a finite dimensional Euclidean space and that the assumptions below hold:*

- *There exists a positive real number α such that*

  $$d\big(0, G(t,u)\big) \le \alpha$$

  for all $t \in [0,T]$ and $u \in H$ with $u \in C(t,u)$;

- *For each $t \in [0,T]$ and each $u \in H$, the sets $C(t,u)$ are nonempty closed in H and equi-uniformly subsmooth;*

- *There are real constants $L_1 > 0$, $L_2 \in]0,1[$ such that, for all $t, s \in [0,T]$ and $x, y, u, v \in H$*

  $$\big|d\big(x, C(t,u)\big) - d\big(y, C(s,v)\big)\big| \le \|x - y\| + L_1|t - s| + L_2\|u - v\|.$$

Then, given $u_0 \in H$ with $u_0 \in C(0,u_0)$, there exists a mapping $u : [0,T] \to H$ which is Lipschitz continuous on $[0,T]$ and satisfies $(\mathcal{D})$. Further, we have $\|\dot{u}(t)\| \le \frac{L_1+2\alpha}{1-L_2}$ a.e. $t \in [0,T]$.

Corollaire 5.2.2. *Let $G : \mathbb{R}_+ \times H \rightrightarrows H$ be a multimapping which is scalarly upper semicontinuous with respect to both variables. Assume that H is a finite dimensional Euclidean space and that the following assumptions hold:*

- *There exists a non-negative function* $\beta(\cdot) \in \mathrm{L}^{\infty}_{\mathrm{loc}}(\mathbb{R}_+)$ *such that*

$$d\big(0, G(t,u)\big) \leq \beta(t)$$

for all $t \in \mathbb{R}_+$ *and* $u \in H$ *with* $u \in C(t,u)$*;*

- *For each* $t \in \mathbb{R}_+$ *and each* $u \in H$*, the sets* $C(t,u)$ *are nonempty closed in* H *and equi-uniformly subsmooth;*

- *There are real constants* $L_1 > 0$*,* $L_2 \in]0,1[$ *such that, for all* $t, s \in \mathbb{R}_+$ *and* $x, y, u, v \in H$

$$\Big| d\big(x, C(t,u)\big) - d\big(y, C(s,v)\big) \Big| \leq \|x-y\| + L_1|t-s| + L_2\|u-v\|.$$

Then, given $u_0 \in H$ *with* $u_0 \in C(0,u_0)$*, there exists a mapping* $u : \mathbb{R}_+ \to H$ *which is locally Lipschitz continuous on* $\mathbb{R}_+$ *and satisfies* $(\mathcal{D}_{\mathbb{R}_+})$*.*

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