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par

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Titre :

# Convergence, interpolation, échantillonnage et bases de Riesz dans les espaces de Fock

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# Contents

# Résumé de la thèse

## 1 Interpolation, échantillonnage et convergence de la série de Lagrange dans l'espace de Fock

L'espace de Fock (standard) $\mathcal{F}^2_\alpha$ est l'ensemble des fonctions entières $f$ telles que

$$\|f\|_{\mathcal{F}^2_\alpha} := \int_{\mathbb{C}} |f(z)|^2 e^{-2\alpha|z|^2} dm(z) < \infty,$$

où $dm$ désigne la mesure de Lebesgue planaire.

De nombreux résultats sur l'espace de Fock et les opérateurs agissant sur cet espace sont présentés dans le livre récent de Zhu [26].

Le noyau reproduisant $\mathbf{k}_z$ de $\mathcal{F}^2_\alpha$,

$$\langle f, \mathbf{k}_z \rangle_{\mathcal{F}^2_\alpha} = f(z), \qquad f \in \mathcal{F}^2_\alpha, \quad z \in \mathbb{C},$$

est donné par

$$\mathbf{k}_\lambda(z) = e^{2\alpha z \overline{\lambda}}, \qquad z, w \in \mathbb{C}.$$

Soit $\Lambda$ une suite de points de $\mathbb{C}$. On dit que $\Lambda$ est séparée si

$$\inf_{\lambda,\lambda^* \in \Lambda} \{|\lambda - \lambda^*| \ : \lambda \neq \lambda^*\} > 0.$$

Notons que si $\Lambda$ est une suite séparée, alors il existe une constante $C_\Lambda > 0$ telle que

$$\sum_{\lambda \in \Lambda} |f(\lambda)|^2 e^{-2\alpha|\lambda|^2} \leq C_\Lambda \|f\|^2_{\mathcal{F}^2_\alpha}, \qquad f \in \mathcal{F}^2_\alpha.$$

La suite $\Lambda$ est dite d'échantillonnage pour $\mathcal{F}^2_\alpha$ s'il existe une constante $c_\Lambda > 0$ telle que

$$c_\Lambda^{-1} \|f\|^2_{\mathcal{F}^2_\alpha} \leq \sum_{\lambda \in \Lambda} |f(\lambda)|^2 e^{-2\alpha|\lambda|^2} \leq c_\Lambda \|f\|^2_{\mathcal{F}^2_\alpha}, \qquad f \in \mathcal{F}^2_\alpha.$$

La suite $\Lambda$ est dite d'interpolation pour $\mathcal{F}^2_\alpha$, si pour toute suite $v = (v_\lambda)_{\lambda \in \Lambda} \subset \mathbb{C}$ telle que

$$\sum_{\lambda \in \Lambda} |v_\lambda|^2 e^{-2\alpha|\lambda|^2} < \infty,$$

il existe $f \in \mathcal{F}^2_\alpha$ telle que $f(\lambda) = v_\lambda$ pour tout $\lambda \in \Lambda$.

Ces notions sont étroitement liées aux propriétés géométriques des familles correspondantes des noyaux reproduisants. Soit $\{e_\lambda\}_{\lambda\in\Lambda}$ une suite de vecteurs de l'espace de Hilbert $H$. On dit que $\{e_\lambda\}_{\lambda\in\Lambda}$ est un repère "frame" pour $H$ (see [23, Chapter 3]) s'il existe $C > 0$ tel que pour tout $h \in H$ on a

$$\frac{1}{C}\|h\|^2 \leq \sum_{\lambda\in\Lambda} |\langle h, e_\lambda\rangle|^2 \leq C\|h\|^2.$$

Par ailleurs, $\{e_\lambda\}_{\lambda\in\Lambda}$ est une base de Riesz dans son enveloppe linéaire fermée si pour un certain $C > 0$ et pour toute suite finie $\{a_\lambda\}$ nous avons

$$\frac{1}{C}\sum_{\lambda\in\Lambda} |a_\lambda|^2 \leq \Big\|\sum_{\lambda\in\Lambda} a_\lambda e_\lambda\Big\|^2 \leq C\sum_{\lambda\in\Lambda} |a_\lambda|^2.$$

Ainsi $\Lambda \in \mathbb{C}$ est d'interpolation si et seulement si la suite associée aux noyaux reproduisants normalisés est une base de Riesz dans son enveloppe linéaire fermée; $\Lambda$ est d'échantillonnage si et seulement si la suite associée aux noyaux reproduisants normalisés est un repère.

En 1992, Seip and Wallstén [20,25] ont obtenu une caractérisation complète des suites d'interpolation et d'échantillonnage pour l'espace. La caractérisation est basée sur une certaine notion de densité uniforme de type Beurling. Soit $D(z, r)$ le disque de centre $z$ et de rayon $r$. Soit $\Lambda$ une suite de points distincts de $\mathbb{C}$. La densité inférieure et supérieure de $\Lambda$ est donnée respectivement par

$$D^-(\Lambda) = \liminf_{r\to+\infty} \inf_{z\in\mathbb{C}} \frac{\mathrm{Card}(\Lambda \cap D(z, r))}{\pi r^2}$$

et

$$D^+(\Lambda) = \limsup_{r\to+\infty} \sup_{z\in\mathbb{C}} \frac{\mathrm{Card}(\Lambda \cap D(z, r))}{\pi r^2}.$$

**Théorème 1.** *[20, 25] La suite $\Lambda$ est d'échantillonnage pour $\mathcal{F}^2_\alpha$ si et seulement si $\Lambda$ est une union finie de suites séparées et contient une sous–suite $\Lambda'$ séparée telle que $D^-(\Lambda') > 2\alpha/\pi$.*

**Théorème 2.** *[20, 25] La suite $\Lambda$ est d'interpolation pour $\mathcal{F}^2_\alpha$ si et seulement si $\Lambda$ est séparée et $D^+(\Lambda) < 2\alpha/\pi$.*

Ainsi, si un ensemble $\Lambda$ est la densité critique ($D^+(\Lambda) = D^-(\Lambda) = 2\alpha/\pi$), alors $\Lambda$ n'est ni échantillonnage ni d'interpolation. Lyubarskii a introduit dans [11] la classe suivante d'ensembles suffisamment réguliers $\Lambda \in \mathbb{C}$ de densité critique (ou, plutôt, la classe de leurs fonctions génératrices):

**Définition 1.** *Soit $\gamma \in \mathbb{R}$, on dit qu'une fonction entière $S$ appartient à la classe $\mathcal{S}_\gamma$ si*

(1) *l'ensemble des zéros de $S$, notÃĹ $\Lambda$, est séparé et*

(2)
$$|S(z)| \asymp e^{\alpha|z|^2} \frac{\operatorname{dist}(z,\Lambda)}{(1+|z|)^\gamma}, \qquad z \in \mathbb{C}.$$

Pour ces ensembles réguliers $\Lambda$, les propriétés d'unicité et d'interpolation et la convergence des séries d'interpolation de Lagrange ont été étudiées par Lyubarskii dans [11] (voir aussi [13]). On dit qu'une suite $\Lambda$ est d'interpolation faible si pour tout $\lambda \in \Lambda$ il existe $f_\lambda \in \mathcal{F}^2_\alpha$ telle que $f_\lambda(\lambda) = 1$ et $f|\Lambda\backslash\{\lambda\} = 0$. Lyubarski a obtenu dans [11] le résultat suivant :

**Théorème 3.** *[11]*

- *$\Lambda$ est un ensemble d'unicité pour $\mathcal{F}^2_\alpha$ si et seulement si $\gamma \le 1$.*
- *$\Lambda$ est d'interpolation faible pour $\mathcal{F}^2_\alpha$ si et seulement si $\gamma > 0$.*

En particulier, pour $0 < \gamma \le 1$, la famille des noyaux reproduisants normalisés

$$\{\exp[2\alpha z\bar{\lambda} - \alpha|\lambda|^2]\}_{\lambda\in\Lambda}$$

est une famille complète et minimale de $\mathcal{F}^2_\alpha$. D'autre part, d'après les résultats de Seip–Wallstèn, cette famille n'est pas une base de Riesz de $\mathcal{F}^2_\alpha$. Autrement, $\Lambda$ serait une suite d'interpolation complète (d'interpolation et d'échantillonnage simultanément) dans $\mathcal{F}^2_\alpha$, ce qui est impossible. Ainsi, la série d'interpolation de Lagrange

$$\sum_{\lambda\in\Lambda} f(\lambda) \frac{S}{S'(\lambda)(\cdot - \lambda)}$$

ne converge pas inconditionnellement dans $\mathcal{F}^2_\alpha$. Cependant, on peut considérer la convergence dans un espace plus grand. Motivé par des applications d'analyse de Gabor, Lyubarskii et Seip [11, 13] ont obtenu le résultat suivant:

**Théorème 4.** *[11, 13] Soit $0 < \gamma < 1/2$, $S \in \mathcal{S}_\gamma$. Alors pour tout $f \in \mathcal{F}^2_\alpha$, on a*

$$\lim_{N\to\infty} \Big\| f - S \sum_{k=1}^{N} \frac{f(\lambda_k)}{S'(\lambda_k)(\cdot - \lambda_k)} \Big\|_{\alpha,1/2} = 0,$$

*où*

$$\mathcal{F}^2_{\alpha,1/2} = \Big\{ f \in \operatorname{Hol}(\mathbb{C}) : \|f\|^2_{\alpha,1/2} := \int_{\mathbb{C}} |f(z)|^2 e^{-2\alpha|z|^2}(1+|z|)^{-1} dm(z) < +\infty \Big\}.$$

## 2 Interpolation et échantillonnage dans les espaces de Fock à poids

Soit $\varphi$ une fonction sous–harmonique sur $\mathbb{C}$ tendant vers l'infini à l'infini. L'espace de Fock associé au poids $\varphi$ est donné par

$$\mathcal{F}^2_\varphi = \{f \in \mathrm{Hol}(\mathbb{C}) : \|f\|_{\varphi,2} := \int_{\mathbb{C}} |f(z)|^2 e^{-2\varphi(z)} dm(z) < \infty\}.$$

Pour un exemple inhabituel d'espaces de Fock pondérés par des poids non sous–harmoniques voir [19], ceci est motivé par des problèmes de physique quantique.

Soit à nouveau $\mathbf{k}_z$ le noyau reproduisant de $\mathcal{F}^2_\varphi$:

$$\langle f, \mathbf{k}_z \rangle_{\mathcal{F}^2_\varphi} = f(z), \qquad f \in \mathcal{F}^2_\varphi, \quad z \in \mathbb{C}.$$

Une suite $\Lambda \subset \mathbb{C}$ est dite *d'échantillonnage* pour $\mathcal{F}^2_\varphi$ si

$$\|f\|^2_\varphi \asymp \|f\|^2_{\varphi,\Lambda} := \sum_{\lambda\in\Lambda} \frac{|f(\lambda)|^2}{\mathbf{k}_\lambda(\lambda)}, \qquad f \in \mathcal{F}^2_\varphi.$$

Soit $\Lambda$ une suite de $\mathbb{C}$ et soit

$$\ell^2_{\varphi,\Lambda} = \{v = (v_\lambda)_{\lambda\in\Lambda} \quad : \sum_{\lambda\in\Lambda} \frac{|v_\lambda|^2}{\mathbf{k}_\lambda(\lambda)} < \infty\}.$$

La suite $\Lambda$ est dite *d'interpolation* pour $\mathcal{F}^2_\varphi$ si pour tout $v = (v_\lambda)_{\lambda\in\Lambda} \in \ell^2_{\varphi,\Lambda}$ il existe $f \in \mathcal{F}^2_\varphi$ telle que

$$v = f|\Lambda.$$

Les notions d'interpolation et d'échantillonnage peuvent être traduites en termes de propriétés géométriques des noyaux reproduisants (voir [23, Chapitre 3]). Par dualité, la famille des noyaux reproduisants normalisés $\{\mathbb{k}_\lambda := \mathbf{k}_\lambda / \| \mathbf{k}_\lambda \|_{\varphi,2}\}_{\lambda\in\Lambda}$ est une base de Riesz dans $\mathcal{F}^2_\varphi$ si et seulement si $\Lambda$ est une suite d'interpolation complète (d'échantillonnage et d'interpolation) de $\mathcal{F}^2_\varphi$.

De nombreux résultats sur les suites d'échantillonnage et d'interpolation dans des espaces de Fock pondérés ont été obtenus par

- Lyubarskii & Seip (1994): $\varphi(z) = |z|^2 h(\arg(z))$, où $h$ est une fonction convexe $2\pi$–periodique.
- Berndtsson & Ortega-Cerdà (1995) et Seip (1998): $\sup_{z\in\mathbb{C}} \Delta\varphi(z) < \infty$.
- Marco, Massaneda & Ortega-Cerdà (2003): $\Delta\varphi$ est une mesure doublante.
- Borichev & Lyubarskii (2010): $\varphi(z) = (\log|z|)^{1+\beta}$, $\beta > 0$.

Dans [4], Borichev, Dhuez et Kellay ont étudié le problème d'échantillonnage et d'interpolation pour $\mathcal{F}^2_\varphi$ lorsque le poids $\varphi$ croît plus vite que $|z|^2$. Plus précisément, soit $\varphi$ une fonction croissante sur $[0,+\infty)$, $\lim_{r\to\infty}\varphi(r)=\infty$. Nous supposons que $\varphi(z)=\varphi(|z|)$ est de classe $C^2$ lisse et sous–harmonique sur $\mathbb{C}$ et que la fonction $\rho$,

$$\rho(z) = (\Delta\varphi(z))^{-1/2},$$

décroît vers 0 à l'infini:

$$\rho'(r) \to 0, \qquad r \to \infty.$$

Ici

$$\Delta\varphi(r) = \varphi''(r) + \varphi'(r)/r, \qquad r > 0.$$

Des exemples typiques de tels poids sont: $\varphi(r) = r^p$, $p > 2$, et $\varphi(r) = e^r$.

Pour $z, w \in \mathbb{C}$, on introduit une notion de séparation, en posant

$$d_\rho(z,w) = \frac{|z-w|}{\min(\rho(z),\rho(w))}.$$

On dit qu'un sous-ensemble $\Lambda$ de $\mathbb{C}$ est $d_\rho$-séparé si

$$\inf_{\lambda\neq\lambda^*}\{d_\rho(\lambda,\lambda^*),\ \lambda,\lambda^*\in\Lambda\} > 0.$$

Les $d_\rho$-densitiés inférieure et supérieure de $\Lambda$ sont définies par

$$D^-_\rho(\Lambda) := \liminf_{R\to\infty}\liminf_{|z|\to\infty}\frac{\mathrm{Card}(\Lambda\cap D(z,R\rho(z))}{R^2},$$
$$D^+_\rho(\Lambda) := \limsup_{R\to\infty}\limsup_{|z|\to\infty}\frac{\mathrm{Card}(\Lambda\cap D(z,R\rho(z))}{R^2}.$$

Notons ici que

$$\|\,\mathbf{k}_\lambda\,\|^2_{\varphi,2} = \mathbf{k}_\lambda(\lambda) \asymp e^{2\varphi(\lambda)}\Delta\varphi(\lambda).$$

**Théorème 5.** *La suite $\Lambda$ est d'échantillonnage pour $\mathcal{F}^2_\varphi$ si et seulement si $\Lambda$ est union finie de suites $d_\rho$-séparées et $\Lambda$ contient une suite $d_\rho$-séparée $\Lambda'$ telle que $D^-_\rho(\Lambda) > 1/2$.*

**Théorème 6.** *La suite $\Lambda$ est d'interpolation pour $\mathcal{F}^2_\varphi$ si et seulement si elle est $d_\rho$-séparée et $D^+_\rho(\Lambda) < 1/2$.*

Des résultats similaires ont été obtenues pour l'espace Bergman à poids, voir [21] pour le cas classique, [4] pour les grands espaces de Bergman et [24] pour des petits espaces de Bergman.

Comme conséquence de [20], [25], [4], si l'espace de Fock à poids radial (régulier) contient l'espace de Fock classique, alors il ne possède pas de bases de Riesz de noyaux reproduisants.

# 3 Les résultats principaux

## 3.1 Chapitre 2

Soit $\varphi$ une fonction croissante sur $[0, +\infty)$, $\lim_{r\to\infty} \varphi(r) = \infty$. Dans cette partie nous supposons que le poids est radial $\varphi(z) = \varphi(|z|)$, $C^2$-lisse et strictement sous–harmonique sur $\mathbb{C}$. On pose $\rho(z) = (\Delta\varphi(z))^{-1/2}$. On supposera que $\varphi$ vérifie pour tout $C$ fixé

$$\rho(r + C\rho(r)) \asymp \rho(r), \qquad 0 < r < \infty.$$

Ceci est satisfait lorsque $\rho'(r) = o(1)$, $r \to \infty$. Comme exemple de ce type de poids, on peut prendre $\varphi(r) = r^a$, $a > 0$ ou $\varphi(r) = (\log r)^\beta$, $\beta > 1$.
Soient $z, w \in \mathbb{C}$. Comme précédemment, on pose $d_\rho(z, w) = |z - w|/\min(\rho(z), \rho(w))$ et on dit que $\Lambda \subset \mathbb{C}$ est $d_\rho$-séparé si $\inf_{\lambda\neq\lambda^*}\{d_\rho(\lambda, \lambda^*),\ \lambda, \lambda^* \in \Lambda\} > 0$.

Ensuite, nous introduisons une famille de sous-ensembles suffisamment réguliers $\Lambda$ dans $\mathbb{C}$ définis comme les ensembles des zéros pour des fonctions dans une classe spéciale

**Définition 2.** *Soit $\gamma \in \mathbb{R}$, on dit qu'une fonction entière $S$ appartient à la classe $\mathcal{S}_\gamma$ si*

(1) *l'ensemble des zéros de $S$, notÃĹ $\Lambda$, est $d_\rho$-séparé, et*

(2)
$$|S(z)| \asymp e^{\varphi(z)} \frac{d(z, \Lambda)}{\rho(z)} \frac{1}{(1 + |z|)^\gamma} \qquad z \in \mathbb{C}.$$

Pour la construction de telles fonctions pour des poids radiaux on peut consulter par exemple [4, 5, 12, 14].

Pour l'espace de Fock usuel ($\varphi(r) = r^2$), les classes $\mathcal{S}_\gamma$ ont été introduites par Lyubarskii dans [11]. Ces classes de fonctions sont des analogues de fonctions de type sinus pour l'espace de Paley–Wiener et leurs ensembles de zéros.

Un ensemble $\Lambda \subset \mathbb{C}$ est dit *ensemble d'interpolation faible* pour $\mathcal{F}^2_\varphi$ si pour tout $\lambda \in \Lambda$ il existe $f_\lambda \in \mathcal{F}^2_\varphi$ telle que t $f_\lambda(\lambda) = 1$ et $f_\lambda|\Lambda\setminus\{\lambda\} = 0$.

Un ensemble $\Lambda \subset \mathbb{C}$ est dit *ensemble d'unicité* pour $\mathcal{F}^2_\varphi$ si pour tout $f \in \mathcal{F}^2_\varphi$, $f|\Lambda = 0$ implique $f = 0$.

Il est clair que chaque suite d'échantillonnage pour $\mathcal{F}^2_\varphi$ est un ensemble d'unicité $\mathcal{F}^2_\varphi$ et chaque suite d'interpolation pour $\mathcal{F}^2_\varphi$ est un ensemble d'intrepolation faible pour $\mathcal{F}^2_\varphi$.

**Théorème 7.** *Soit $S \in \mathcal{S}_\gamma$, et soit $\Lambda$ l'ensemble des zéros de $S$. Alors*

($a$) *$\Lambda$ est un ensemble d'unicité si et seulement si $\gamma \le 1$,*

($b$) *$\Lambda$ est un ensemble d'interpolation faible si et seulement si $\gamma > 0$.*

La suite $\Lambda \subset \mathbb{C}$ est dite *d'interpolation complète* pour $\mathcal{F}^2_\varphi$ si elle est d'échantillonnage et d'interpolation pour $\mathcal{F}^2_\varphi$. Ceci est équivalent à dire que la famille correspondante des noyaux reproduisants normalisés est une base de Riesz de $\mathcal{F}^2_\varphi$

Les résultats de Seip et Wallstén montrent qu'il n'existe pas de suite d'échantillonnage et d'interpolation simultanément dans l'espace de Fock classique et donc il n'existe pas de base de Riesz de noyaux reproduisants normalisés dans l'espace Fock usuel. La situation change dans les petits espaces de Fock, lorsque la croissance est assez lente. Borichev et Lyubarskii [5] ont montré que lorsque $\varphi(r) = (\log r)^\beta$, $1 < \beta < 2$, il existe des bases de Riesz de noyaux reproduisants dans $\mathcal{F}^2_\varphi$. De plus (voir [5, 9, 23]), l'espace $\mathcal{F}^2_\varphi$ n'admet pas de bases de Riesz de noyaux reproduisants normalisés lorsque le poids (régulier) $\varphi$ satisfait $(\log r)^2 = o(\varphi(r))$, $r \to \infty$.

Par le Théorème 7, pour $0 < \gamma \leq 1$, la famille $\{\mathbf{k}_\lambda / \| \mathbf{k}_\lambda \|_{\varphi,2}\}_{\lambda\in\Lambda}$ est complète et minimale dans $\mathcal{F}^2_\varphi$. Donc la famille

$$\{S/[S'(\lambda)(\cdot - \lambda)]\}_{\lambda\in\Lambda}$$

forme un système biorthogonal et on associe âĂą toute $f \in \mathcal{F}^2_\varphi$ la série formelle (série d'interpolation de Lagrange )

$$f \sim \sum_{\lambda\in\Lambda} f(\lambda)\frac{S}{S'(\lambda)(\cdot - \lambda)}.$$

Cette série converge inconditionnellement dans $\mathcal{F}^2_\varphi$ si $\{\mathbf{k}_\lambda / \| \mathbf{k}_\lambda \|_{\varphi,2}\}_{\lambda\in\Lambda}$ est une base de Riesz dans $\mathcal{F}^2_\varphi$. Sinon ceci n'est plus valable et il est donc naturel de se poser la question de la convergence de cette série formelle lorsqu'on change la norme de l'espace.

Soit $\Lambda = \{\lambda_k\}$ l'ensemble des zéros de $S$ ordonnés de sorte que $|\lambda_k| \leq |\lambda_{k+1}|$, $k \geq 1$.

Comme dans Lyubarskii [11] et Lyubarskii–Seip [13] nous obtenons le résultat suivant

**Théorème 8.** *Soit* $0 \leq \beta \leq 1$, $\gamma + \beta \in (1/2, 1)$, *et soit* $S \in \mathcal{S}_\gamma$. *Si*

$$r^{1-2\beta} = O(\rho(r)), \qquad r \to +\infty, \tag{1}$$

*alors pour tout* $f \in \mathcal{F}^2_\varphi$, *on a*

$$\lim_{N\to\infty}\Big\| f - S\sum_{k=1}^{N} \frac{f(\lambda_k)}{S'(\lambda_k)(\cdot - \lambda_k)}\Big\|_{\varphi_\beta} = 0, \tag{2}$$

*où* $\varphi_\beta(r) = \varphi(r) + \beta\log(1 + r)$.

Lorsque $\beta = 1/2$, $\varphi(r) = r^2$, $\rho(r) \asymp 1$ on retrouve le résultat de Lyubarski-Seip [13, ThĂĹorĂŃme 10]. D'autre part, lorsque $\varphi(r) = (\log r)^a$, $1 < a \leq 2$, $r > 2$, $\rho(r) \asymp r(\log r)^{1-\frac{a}{2}}$, l'espace $\mathcal{F}^2_\varphi$ contient une base de Riesz de noyaux reproduisants normalisés [5]. De plus notre résultat montre que dans le cas où $r \lesssim \rho(r)$, avec $S \in \mathcal{S}_\gamma$, $\gamma \in (1/2, 1)$, la série d'interpolation converge déjà dans $\mathcal{F}^2_\varphi$.

Dans le cas où $0 < a \leq 2$, $\varphi(r) = r^a$, nous avons $\rho(r) \asymp r^{1-a/2}$, $r > 1$, et nous pouvons choisir $a/4 \leq \beta \leq 1$ avec le choix approprié de $\gamma$ comme dans le Théorème 8. Ainsi, plus nous nous rapprochons du poids $\phi(r) = (\log r)^2$, $\rho(r) \asymp r$, $r > 2$, et moins nous devons modifier la norme ($\beta$ devient plus petit) pour obtenir la convergence dans (2).

Maintenant, il est intéressant d'étudier l'optimalité de (1) dans le Théorème 8.

**Théorème 9.** *Soit* $0 < a \leq 2$, $\varphi(r) = r^a$, $r > 1$. *Si* $0 \leq \beta < a/4$, $\gamma \in \mathbb{R}$, *et* $S \in \mathcal{S}_\gamma$, *alors il existe* $f \in \mathcal{F}^2_\varphi$ *telle que*

$$\Big\| f - S \sum_{k=1}^{N} \frac{f(\lambda_k)}{S'(\lambda_k)(\cdot - \lambda_k)} \Big\|_{\varphi_\beta} \not\to 0, \qquad N \to \infty.$$

Ainsi, pour le poids puissance $\varphi(r) = r^a$, $0 < a \leq 2$, nous avons besoin de modifier la norme pour obtenir la convergence et la valeur critique de $\beta$ est $a/4$.

## 3.1 Chapitre 3

Dans [5], Borichev–Lyubarskii ont montré l'existence d'une suite *d'interpolation complète* pour $\mathcal{F}^2_\varphi$ (d'interpolation et d'échantillonnage pour $\mathcal{F}^2_\varphi$), avec $\varphi(r) = (\log^+ r)^2$.

**Théorème 10.** *Soit* $\varphi(z) = (\log|z|)^2$ *et soit* $\Gamma = \{\gamma_n\}_{n\geq 0}$ *avec* $\gamma_n = e^{\frac{n+1}{2}} e^{i\theta_n}$ *où les* $\theta_n$ *sont des réels arbitraires. Alors* $\{\mathbb{k}_{\gamma_n}\}_{n\geq 0}$ *est une base de Riesz* $\mathcal{F}^2_\varphi$.

Voir aussi [2, 16] pour d'autres résultats dans cette direction. Notons que lorsque $\varphi(r) \gg (\log^+ r)^2$, il n'existe pas de suite d'interpolation complète pour $\mathcal{F}^2_\varphi$ [4,14,18,20,23].

L'objectif principal du chapitre 3 est l'étude de l'interpolation et de l'échantillonnage pour $\mathcal{F}^2_\varphi$ avec $\varphi(r) = \alpha(\log^+ r)^2$. Notons que le cas des petits espaces de Bergman a été récemment étudiÃĹ par Seip [24].

Selon [5, Lemme 2.7], dans notre cas, le noyau reproduisant admet l'estimation suivante:

$$\| \mathbf{k}_z \|^2_{\varphi,2} = \mathbf{k}_z(z) \asymp \frac{e^{2\varphi(z)}}{1 + |z|^2}, \qquad z \in \mathbb{C}. \tag{3}$$

Notre principal résultat est une caractérisation des suites d'interpolation complètes en fonction de leur écart par rapport à la suite Borichev–Lyubarskii. Nous obtenons une caractérisation des bases de Riesz du même type que celle du Théorème 1/4 d'Avdonin–Kadets et sa généralisation [1] dans le cas des espaces de Paley–Wiener.

**Théorème 11.** *Soient* $\varphi(r) = \alpha(\log^+ r)^2$ *avec* $\alpha > 0$, $\Gamma = \{\gamma_n\}_{n\geq 0} = \{e^{\frac{n+1}{2\alpha}}\}_{n\geq 0}$ *et* $\Lambda = \{\lambda_n\}$ *avec* $\lambda_n = \gamma_n e^{\delta_n} e^{i\theta_n}$, $|\lambda_n| \leq |\lambda_{n+1}|$ *et* $\theta_n \in \mathbb{R}$. *La famille* $\{\mathbb{k}_\lambda\}_{\lambda\in\Lambda}$ *est une base de Riesz de* $\mathcal{F}^2_\varphi$ *si et seulement si les trois conditions suivantes sont satisfaites*

1. *$\Lambda$ est $d_\rho$ séparé ,*

2. $(\delta_n) \in \ell^\infty$,

3. *Il existe $N \geq 1$ et $\delta > 0$ tels que*

$$\sup_n \frac{1}{N} \Big| \sum_{k=n+1}^{n+N} \delta_k \Big| \leq \delta < \frac{1}{4\alpha}.$$

Nous étudions également le cas $p = \infty$. L'espace de Fock à poids correspondant est défini par

$$\mathcal{F}_\varphi^\infty = \{f \in \mathrm{Hol}(\mathbb{C}) : \|f\|_{\varphi,\infty} := \sup_{z\in\mathbb{C}} |f(z)|e^{-\varphi(z)} < \infty\}.$$

La suite $\Lambda$ est dite d'échantillonnage $\mathcal{F}_\varphi^\infty$, s'il existe $L > 0$ telle que

$$\|f\|_{\varphi,\infty} \leq L\|f\|_{\varphi,\infty,\Lambda} := L \sup_{\lambda\in\Lambda} |f(\lambda)|e^{-\varphi(\lambda)}, \qquad f \in \mathcal{F}_\varphi^\infty. \tag{4}$$

On notera par $L_\varphi(\Lambda)$ (la constante d'échantillonnage) le plus petit des $L$ satisfaisant (4),

La suite $\Lambda$ est dite d'interpolation si pour toute suite $v \in \ell^\infty_{\varphi,\Lambda}$ il existe une fonction $f \in \mathcal{F}_\varphi^\infty$ telle que

$$v = f|\Lambda.$$

Le Théorème 11 nous permet de déduire plusieurs conditions de densité pour l'interpolation et l'échantillonnage dans $\mathcal{F}_\varphi^p$, $p = 2, \infty$. Avant d'énoncer ces résultats, nous avons besoin de plus de notations.

Soit $\mathcal{A}(r, R)$ la couronne limitée par les cercles de centre 0 et de rayons respectifs $0 < r < R$, $A(r, R) := \{z \in \mathbb{C} : r \leq |z| < R\}$. Pour une suite $\Lambda$ $d_\rho$-séparée on définit la densité inférieure et supérieure respectivement par

$$D^-(\Lambda) = \liminf_{R\to+\infty} \liminf_{r\to+\infty} \frac{\mathrm{Card}(\Lambda \cap \mathcal{A}(r, Rr))}{\log R}$$

et

$$D^+(\Lambda) = \limsup_{R\to+\infty} \limsup_{r\to+\infty} \frac{\mathrm{Card}(\Lambda \cap \mathcal{A}(r, Rr))}{\log R}.$$

Ces densités ne changent pas quand on enlève ou on ajoute un nombre fini de points à la suite.

Nous sommes maintenant en mesure de formuler nos résultats.

**Théorème 12.** *(Echantillonnage, $p = \infty$) Soit $\varphi(r) = \alpha(\log^+ r)^2$, $\alpha > 0$,*

*(i) Toute suite $d_\rho$ séparée $\Lambda$ avec $D^-(\Lambda) > 2\alpha$ est d'échantillonnage pour $\mathcal{F}_\varphi^\infty$.*

(ii) *Si la suite $\Lambda$ est d'échantillonnage pour $\mathcal{F}^{\infty}_{\varphi}$ alors elle contient une sous-suite $d_\rho$-séparée $\widetilde{\Lambda}$ avec $D^{-}(\widetilde{\Lambda}) \geq 2\alpha$.*

**Théorème 13.** *(Echantillonnage, $p = 2$.) Soit $\varphi(r) = \alpha(\log^+ r)^2$, $\alpha > 0$,*

(i) *Toute suite $d_\rho$ séparée $\Lambda$ avec $D^{-}(\Lambda) > 2\alpha$ est d'échantillonage pour $\mathcal{F}^2_{\varphi}$.*

(ii) *Si $\Lambda$ est d'échantillonnage pour $\mathcal{F}^2_{\varphi}$ alors elle est union finie de suites $d_\rho$-séparararées et $\Lambda$ contient une suite $d_\rho$ séparée $\widetilde{\Lambda}$ telle que $D^{-}(\widetilde{\Lambda}) \geq 2\alpha$.*

**Théorème 14.** *(Interpolation $p = 2, \infty$) Let $\varphi(r) = \alpha(\log^+ r)^2$, $\alpha > 0$,*

(i) *Toute suite $d_\rho$ séparée $\Lambda$ avec $D^{+}(\Lambda) < 2\alpha$ est d'interpolation pour $\mathcal{F}^p_{\varphi}$, $p = 2, \infty$.*

(ii) *Si la suite $\Lambda$ est d'interpolation pour $\mathcal{F}^p_{\varphi}$, $p = 2, \infty$. alors elle est $d_\rho$-séparée et $D^{+}(\Lambda) \leq 2\alpha$.*

Pour $p = 2, \infty$, le résultat suivant montre que si la densité est critique alors deux situations principales peuvent se produire: il existe des suites d'interpolation complète et il en existe qui ne sont ni d'interpolation ni d'échantillonnage. Cela implique notamment qu'il n'y a pas de caractérisation de densité pour des suites d'échantillonnage ou d'interpolation pour $p = 2, \infty$.

**Théorème 15.** *(i) Il existe une suite $d_\rho$- séparée $\Gamma$ telle que $D^{+}(\Gamma) = D^{-}(\Gamma) = 2\alpha$, et elle est d'interpolation et d'échantillonage pour $\mathcal{F}^2_{\varphi}$.*

*(ii) Il existe une suite $d_\rho$- séparée $\Gamma$ telle que $D^{+}(\Gamma) = D^{-}(\Gamma) = 2\alpha$, et elle n'est ni d'interpolation ni d'échantillonage pour $\mathcal{F}^2_{\varphi}$.*

*(iii) Il existe une suite $d_\rho$- séparée $\Gamma$ telle que $D^{+}(\Gamma) = D^{-}(\Gamma) = 2\alpha$, et elle est d'interpolation et d'échantillonage pour $\mathcal{F}^{\infty}_{\varphi}$.*

*(iv) Il existe une suite $d_\rho$- séparée $\Gamma$ telle que $D^{+}(\Gamma) = D^{-}(\Gamma) = 2\alpha$, et elle n'est ni interpolation ni déchantillonage pour $\mathcal{F}^{\infty}_{\varphi}$.*

Le résultat (i) est dû à Borichev et Lyubarskii [5], ils ont construit une suite $\Gamma = \Gamma_\alpha$ qui est d'interpolation complète (les noyaux reproduisants correspondants forment une base de Riesz). Le cas (iii) lorsque $p = \infty$ s'obtient à partir de $\Gamma$ en ajoutant juste un point. Les cas (ii) et (iv) correspondent à certaines perturbations de la suite construite par Borichev–Lyubarskii.

# Chapter 1

# Introduction

## 1.1 Interpolation, sampling and convergence of Lagrange series in the standard Fock space

The (standard) Fock space $\mathcal{F}^2_\alpha$ consits of all entire functions $f$ such that

$$\|f\|_{\mathcal{F}^2_\alpha} := \int_{\mathbb{C}} |f(z)|^2 e^{-2\alpha|z|^2} dm(z) < \infty,$$

where $dm$ is area measure.

For numerous results on the Fock space and operators acting on this space see the recent book of Zhu [26].

The reproducing kernel $\mathbf{k}_z$ of $\mathcal{F}^2_\alpha$,

$$\langle f, \mathbf{k}_z \rangle_{\mathcal{F}^2_\alpha} = f(z), \qquad f \in \mathcal{F}^2_\alpha, \quad z \in \mathbb{C},$$

is given by

$$\mathbf{k}_\lambda(z) = e^{2\alpha z \overline{\lambda}}, \qquad z, w \in \mathbb{C}.$$

Let $\Lambda$ be a sequence of point in $\mathbb{C}$. We say that $\Lambda$ is separated if

$$\inf_{\lambda, \lambda^* \in \Lambda} \{|\lambda - \lambda^*| \ : \lambda \neq \lambda^*\} > 0.$$

Note that if $\Lambda$ is separated sequence, then there exists a positive constant $C_\Lambda$ such that

$$\sum_{\lambda \in \Lambda} |f(\lambda)|^2 e^{-2\alpha|\lambda|^2} \leq C_\Lambda \|f\|^2_{\mathcal{F}^2_\alpha}, \qquad f \in \mathcal{F}^2_\alpha.$$

We say that the sequence $\Lambda$ is a sampling sequence for $\mathcal{F}^2_\alpha$ if there exists a constant $c_\Lambda > 0$ such that

$$c_\Lambda^{-1} \|f\|^2_{\mathcal{F}^2_\alpha} \leq \sum_{\lambda \in \Lambda} |f(\lambda)|^2 e^{-2\alpha|\lambda|^2} \leq c_\Lambda \|f\|^2_{\mathcal{F}^2_\alpha}, \qquad f \in \mathcal{F}^2_\alpha.$$

The sequence $\Lambda$ is called interpolation sequence for $\mathcal{F}^2_\alpha$, if for every sequence $v = (v_\lambda)_{\lambda\in\Lambda} \subset \mathbb{C}$ such that

$$\sum_{\lambda\in\Lambda} |v_\lambda|^2 e^{-2\alpha|\lambda|^2} < \infty,$$

there exists a function $f \in \mathcal{F}^2_\alpha$ such that $f(\lambda) = v_\lambda$ for all $\lambda \in \Lambda$.

These notions are closely related to geometric properties of the corresponding families of reproducing kernels. Let $\{e_\lambda\}_{\lambda\in\Lambda}$ be a sequence of vectors in a Hilbert space $H$. We say that $\{e_\lambda\}_{\lambda\in\Lambda}$ is a frame in $H$ (see [23, Chapter 3]) if for some $C > 0$ and for every $h \in H$ we have

$$\frac{1}{C}\|h\|^2 \le \sum_{\lambda\in\Lambda} |\langle h, e_\lambda\rangle|^2 \le C\|h\|^2.$$

Furthermore, $\{e_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis in its closed linear span if for some $C > 0$ and for each finite sequence $\{a_\lambda\}$ we have

$$\frac{1}{C}\sum_{\lambda\in\Lambda} |a_\lambda|^2 \le \Big\|\sum_{\lambda\in\Lambda} a_\lambda e_\lambda\Big\|^2 \le C\sum_{\lambda\in\Lambda} |a_\lambda|^2.$$

Then a set $\Lambda \in \mathbb{C}$ is interpolating if and only if the sequence of the associated normalized reproducing kernels is a Riesz basis in its closed linear span; $\Lambda$ is sampling if and only if the sequence of the associated normalized reproducing kernels is a frame.

In 1992, Seip and Wallstén [20, 25] characterized interpolating and sampling sequences in the Fock space. The characterizations are based on a certain notion of Beurling-type asymptotic uniform density. Denote by $D(z, r)$ the disc of radius $r$ centered at $z$. Let $\Lambda$ be a sequence of distinct point in $\mathbb{C}$. The lower and upper densities of $\Lambda$ are defined by

$$D^-(\Lambda) = \liminf_{r\to+\infty} \inf_{z\in\mathbb{C}} \frac{\mathrm{Card}(\Lambda \cap D(z, r))}{\pi r^2}$$

and

$$D^+(\Lambda) = \limsup_{r\to+\infty} \sup_{z\in\mathbb{C}} \frac{\mathrm{Card}(\Lambda \cap D(z, r))}{\pi r^2}.$$

**Theorem 1.1.1.** *[20, 25] A discrete set $\Lambda$ is a set of sampling for $\mathcal{F}^2_\alpha$ if and only if $\Lambda$ is a finite union of separated sequences and contains a separated sequence $\Lambda'$ for which $D^-(\Lambda') > 2\alpha/\pi$.*

**Theorem 1.1.2.** *[20, 25] A discrete set $\Lambda$ is a set of interpolation for $\mathcal{F}^2_\alpha$ if and only if $\Lambda$ is separated and $D^+(\Lambda) < 2\alpha/\pi$.*

Thus, if a set $\Lambda$ is of critical density ($D^+(\Lambda) = D^-(\Lambda) = 2\alpha/\pi$), then it is neither sampling nor interpolating. Lyubarskii introduced in [11] the following class of sufficiently regular sets $\Lambda \in \mathbb{C}$ of critical density (or, rather, of their generating functions):

**Definition 1.1.3.** *Given $\gamma \in \mathbb{R}$, we say that an entire function $S$ belongs to the class $\mathcal{S}_\gamma$ if*

(1) *the zero set $\Lambda$ of $S$ is separated, and*

(2)
$$|S(z)| \asymp e^{\alpha|z|^2}\frac{\operatorname{dist}(z,\Lambda)}{(1+|z|)^\gamma}, \qquad z \in \mathbb{C}.$$

For such regular sets $\Lambda$, the properties of being uniqueness/interpolation sets and the convergence of the Lagrange interpolation series was studied by Lyubarskii in [11] (see also [13]). The results obtained there are as follows.

**Theorem 1.1.4.** *[11]*

- *$\Lambda$ is a uniqueness set for $\mathcal{F}^2_\alpha$ if and only if $\gamma \le 1$.*
- *$\Lambda$ is a weak interpolation set for $\mathcal{F}^2_\alpha$ if and only if $\gamma > 0$.*

Here weak interpolation means that for every $\lambda \in \Lambda$ there exists $f_\lambda \in \mathcal{F}^2_\alpha$ such that $f_\lambda(\lambda) = 1$ and $f|\Lambda\backslash\{\lambda\} = 0$.

In particular, for $0 < \gamma \le 1$, the family of normalized reproducing kernels

$$\{\exp[2\alpha z\bar{\lambda} - \alpha|\lambda|^2]\}_{\lambda\in\Lambda}$$

is a complete minimal family in $\mathcal{F}^2_\alpha$. At the same time, by the results of Seip–Wallstèn, this family is not a Riesz basis in $\mathcal{F}^2_\alpha$. (Otherwise $\Lambda$ would be a complete interpolating sequence (simultaneously interpolating and sampling) in $\mathcal{F}^2_\alpha$, which is impossible). Thus the Lagrange interpolation series

$$\sum_{\lambda\in\Lambda} f(\lambda)\frac{S}{S'(\lambda)(\cdot-\lambda)}$$

do not converge unconditionally in $\mathcal{F}^2_\alpha$. However, one can consider convergence in a larger space. Motivated by Gabor analysis applications, Lyubarskii and Seip [11, 13] obtained the following result:

**Theorem 1.1.5.** *[11, 13] Let $0 < \gamma < 1/2$, $S \in \mathcal{S}_\gamma$. Then for every $f \in \mathcal{F}^2_\alpha$, we have*

$$\lim_{N\to\infty}\Big\|f - S\sum_{k=1}^{N}\frac{f(\lambda_k)}{S'(\lambda_k)(\cdot-\lambda_k)}\Big\|_{\alpha,1/2} = 0,$$

*where*

$$\mathcal{F}^2_{\alpha,1/2} = \Big\{f \in \operatorname{Hol}(\mathbb{C}) : \|f\|^2_{\alpha,1/2} := \int_{\mathbb{C}}|f(z)|^2e^{-2\alpha|z|^2}(1+|z|)^{-1}dm(z) < +\infty\Big\}.$$

## 1.2 Interpolation and sampling in weighted Fock spaces

Given a subharmonic function $\varphi$ tending to infinity at infinity we define the corresponding weighted Fock space by

$$\mathcal{F}^2_\varphi = \{f \in \mathrm{Hol}(\mathbb{C}) : \|f\|_{\varphi,2} := \int_{\mathbb{C}} |f(z)|^2 e^{-2\varphi(z)} dm(z) < \infty\}.$$

(For a physically motivated unusual example of Fock spaces defined by non-subharmonic weights see [19]).

Again we denote by $\mathbf{k}_z$ the reproducing kernel of $\mathcal{F}^2_\varphi$:

$$\langle f, \mathbf{k}_z\rangle_{\mathcal{F}^2_\varphi} = f(z), \qquad f \in \mathcal{F}^2_\varphi, \quad z \in \mathbb{C}.$$

A sequence $\Lambda \subset \mathbb{C}$ is called *sampling* for $\mathcal{F}^2_\varphi$ if

$$\|f\|^2_\varphi \asymp \|f\|^2_{\varphi,\Lambda} := \sum_{\lambda\in\Lambda} \frac{|f(\lambda)|^2}{\mathbf{k}_\lambda(\lambda)}, \qquad f \in \mathcal{F}^2_\varphi.$$

Let $\Lambda$ be a sequence of points in $\mathbb{C}$ and let

$$\ell^2_{\varphi,\Lambda} = \{v = (v_\lambda)_{\lambda\in\Lambda} \quad : \sum_{\lambda\in\Lambda} \frac{|v_\lambda|^2}{\mathbf{k}_\lambda(\lambda)} < \infty\}.$$

The sequence $\Lambda$ is called *interpolating* for $\mathcal{F}^2_\varphi$ if for every $v = (v_\lambda)_{\lambda\in\Lambda} \in \ell^2_{\varphi,\Lambda}$ there exists $f \in \mathcal{F}^2_\varphi$ such that

$$v = f|\Lambda.$$

Once again, interpolation means that the sequence of the associated normalized reproducing kernels is a Riesz basis in its closed linear span and sampling means that this sequence is a frame. Standard duality arguments show that the system $\{\mathbb{k}_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis in $\mathcal{F}^2_\varphi$ if and only if $\Lambda$ is a complete interpolating (sampling and interpolating) sequence for $\mathcal{F}^2_\varphi$.

Numerous results on sampling and interpolation in weighted Fock spaces were obtained by

• Lyubarskii & Seip (1994): $\varphi(z) = |z|^2 h(\arg(z))$, where $h$ is a $2\pi$–periodic convex function.

• Berndtsson & Ortega-Cerdà (1995) and Seip (1998): $\sup_{z\in\mathbb{C}} \Delta\varphi(z) < \infty$.

• Marco, Massaneda & Ortega-Cerdà (2003): $\Delta\varphi$ is a doubling measure.

• Borichev & Lyubarskii (2010): $\varphi(z) = (\log|z|)^{1+\beta}$, $\beta > 0$.

In [4], Borichev–Dhuez–Kellay studied the sampling and interpolation problem for $\mathcal{F}^2_\varphi$ when the weights $\varphi$ grow more rapidly than $|z|^2$. More precisely, let $\varphi$ be an increasing function defined on $[0, +\infty)$, $\lim_{r\to\infty} \varphi(r) = \infty$. We assume that the radial weight $\varphi(z) = \varphi(|z|)$ is $C^2$-smooth and strictly subharmonic on $\mathbb{C}$, that the function $\rho$,

$$\rho(z) = (\Delta\varphi(z))^{-1/2},$$

decreases to 0 at infinity and that

$$\rho'(r) \to 0, \qquad r \to \infty.$$

Here

$$\Delta\varphi(r) = \varphi''(r) + \varphi'(r)/r, \qquad r > 0.$$

Typical examples of such weights are $\varphi(r) = r^p$, $p > 2$, and $\varphi(r) = e^r$.

Given $z, w \in \mathbb{C}$, we define a scaled distance function

$$d_\rho(z, w) = \frac{|z - w|}{\min(\rho(z), \rho(w))}.$$

We say that a subset $\Lambda$ of $\mathbb{C}$ is $d_\rho$-separated if

$$\inf_{\lambda\neq\lambda^*} \{d_\rho(\lambda, \lambda^*),\ \lambda, \lambda^* \in \Lambda\} > 0.$$

The lower and upper $d_\rho$-densities of $\Lambda$ are defined by

$$D_\rho^-(\Lambda) := \liminf_{R\to\infty} \liminf_{|z|\to\infty} \frac{\operatorname{Card}(\Lambda \cap D(z, R\rho(z)))}{R^2},$$
$$D_\rho^+(\Lambda) := \limsup_{R\to\infty} \limsup_{|z|\to\infty} \frac{\operatorname{Card}(\Lambda \cap D(z, R\rho(z)))}{R^2}.$$

Note that here

$$\| \mathbf{k}_\lambda \|_{\varphi,2}^2 = \mathbf{k}_\lambda(\lambda) \asymp e^{2\varphi(\lambda)}\Delta\varphi(\lambda).$$

**Theorem 1.2.1.** *A set $\Lambda$ is a sampling set for $\mathcal{F}_\varphi^2$ if and only if $\Lambda$ is a finite union of $d_\rho$-separated sets and $\Lambda$ contains a $d_\rho$-separated subset $\Lambda'$ such that $D_\rho^-(\Lambda) > 1/2$.*

**Theorem 1.2.2.** *A set $\Lambda$ is an interpolation set for $\mathcal{F}_\varphi^2$ if and only if it is $d_\rho$-separated and $D_\rho^+(\Lambda) < 1/2$.*

(Similar results hold for the (weighted) Bergman space, see [21] for the classical case, [4] for large Bergman spaces, and [24] for small Bergman spaces).

As a consequence of [20], [25], [4], if a (regular) radial weighted Fock space contains the standard one, then it does not possess Riesz bases of reproducing kernels.

## 1.3 Main results

### 1.3.1 Chapter 2

Let $\varphi$ be an increasing function defined on $[0,+\infty)$, $\lim_{r\to\infty}\varphi(r)=\infty$. In this part of our work we assume that the radial weight $\varphi(z)=\varphi(|z|)$ is $C^2$-smooth and strictly subharmonic on $\mathbb{C}$, and set $\rho(z)=(\Delta\varphi(z))^{-1/2}$. One more condition on $\varphi$ is that for every fixed $C$ we have

$$\rho(r+C\rho(r))\asymp\rho(r),\qquad 0<r<\infty.$$

In particular, this holds if $\rho'(r)=o(1)$, $r\to\infty$. Typical $\varphi$ are power functions, $\varphi(r)=r^a$, $a>0$, and $\varphi(r)=(\log r)^\beta$, $\beta>1$.
Given $z,w\in\mathbb{C}$, we set as before $d_\rho(z,w)=|z-w|/\min(\rho(z),\rho(w))$ and we say that a subset $\Lambda\subset\mathbb{C}$ is $d_\rho$–separated if $\inf_{\lambda\neq\lambda^*}\{d_\rho(\lambda,\lambda^*),\ \lambda,\lambda^*\in\Lambda\}>0$.

Next, we introduce a family of sufficiently regular subsets $\Lambda$ in $\mathbb{C}$ defined as the zero sets for functions in a special class.

**Definition 1.3.1.** *Given $\gamma\in\mathbb{R}$, we say that an entire function $S$ belongs to the class $\mathcal{S}_\gamma$ if*

(1) *the zero set $\Lambda$ of $S$ is $d_\rho$-separated, and*

(2)
$$|S(z)|\asymp e^{\varphi(z)}\frac{d(z,\Lambda)}{\rho(z)}\frac{1}{(1+|z|)^\gamma},\qquad z\in\mathbb{C}.$$

For constructions of such functions in radial weighted Fock spaces see, for example, [4,5,12,14].

In the standard Fock spaces ($\varphi(r)=r^2$) the classes $\mathcal{S}_\gamma$ were introduced by Lyubarskii in [11]. They are analogs of the sine type functions for the Paley–Wiener space, and their zero sets include rectangular lattices and their perturbations.

A set $\Lambda\subset\mathbb{C}$ is called *a weak interpolation set* for $\mathcal{F}^2_\varphi$ if for every $\lambda\in\Lambda$ there exists $f_\lambda\in\mathcal{F}^2_\varphi$ such that $f_\lambda(\lambda)=1$ and $f_\lambda|\Lambda\setminus\{\lambda\}=0$.

A set $\Lambda\subset\mathbb{C}$ is called *a uniqueness set* for $\mathcal{F}^2_\varphi$ if for every $f\in\mathcal{F}^2_\varphi$, the relation $f|\Lambda=0$ implies that $f=0$.

It is obvious that each sampling sequence for $\mathcal{F}^2_\varphi$ is a set of uniqueness for $\mathcal{F}^2_\varphi$ and each interpolation sequence for $\mathcal{F}^2_\varphi$ is a weak interpolation set for $\mathcal{F}^2_\varphi$.

**Theorem 1.3.2.** *Let $\phi$ and $\rho$ be as above. Given $S\in\mathcal{S}_\gamma$, consider its zero set $\Lambda$. Then*

($a$) *$\Lambda$ is a uniqueness set for $\mathcal{F}^2_\varphi$ if and only if $\gamma\leq 1$,*

($b$) *$\Lambda$ is a weak interpolation set for $\mathcal{F}^2_\varphi$ if and only if $\gamma>0$.*

The sequence $\Lambda \subset \mathbb{C}$ is said to be *complete interpolating* for $\mathcal{F}^2_\varphi$ if it is simultaneously interpolating and sampling for $\mathcal{F}^2_\varphi$. This is equivalent to the property that the corresponding family of the normalized reproducing kernels is a Riesz basis in $\mathcal{F}^2_\varphi$.

The results of Seip and Wallstén show that there are no sequences which are simultaneously interpolating and sampling, and hence there are no unconditional or Riesz bases of (normalized) reproducing kernels in the unweighted case. The situation changes in small Fock spaces when the weight increases slowly. Borichev and Lyubarskii [5] have shown that for $\varphi(r) = (\log r)^\beta$, $1 < \beta \le 2$, there exist Riesz bases of reproducing kernels in $\mathcal{F}^2_\varphi$. Furthermore (see [5, 9, 23]), the space $\mathcal{F}^2_\varphi$ does not admit Riesz bases of the (normalized) reproducing kernels for regular $\varphi$ such that $(\log r)^2 = o(\varphi(r))$, $r \to \infty$.

By Theorem 1.3.2, for $0 < \gamma \le 1$, the family $\{\mathbf{k}_\lambda\}_{\lambda\in\Lambda}$ is a complete minimal family in $\mathcal{F}^2_\varphi$. Thus the family

$$\Big\{S/[S'(\lambda)(\cdot-\lambda)]\Big\}_{\lambda\in\Lambda}$$

is its biorthogonal system and we associate to any $f \in \mathcal{F}^2_\varphi$ the (formal) Lagrange interpolation series

$$f \sim \sum_{\lambda\in\Lambda} f(\lambda)\frac{S}{S'(\lambda)(\cdot-\lambda)}.$$

This series converges unconditionally in $\mathcal{F}^2_\varphi$ if $\{\mathbb{k}_\lambda = \mathbf{k}_\lambda/\|\mathbf{k}_\lambda\|\}_{\lambda\in\Lambda}$ is a Riesz basis in $\mathcal{F}^2_\varphi$. Otherwise, it does not necessarily converge unconditionally in $\mathcal{F}^2_\varphi$, and it is natural to ask whether this series admits a summation method if we modify (slightly) the norm of the space.

Denote by $\Lambda = \{\lambda_k\}$ the zero sequence of $S$ ordered in such a way that $|\lambda_k| \le |\lambda_{k+1}|$, $k \ge 1$. By analogy to Lyubarskii [11] and Lyubarskii–Seip [13], we obtain the following result:

**Theorem 1.3.3.** *Let $0 \le \beta \le 1$, $\gamma+\beta \in (1/2, 1)$, and let $S \in \mathcal{S}_\gamma$. Suppose that*

$$r^{1-2\beta} = O(\rho(r)), \qquad r \to +\infty. \tag{1.1}$$

*Then for every $f \in \mathcal{F}^2_\varphi$ we have*

$$\lim_{N\to\infty}\Big\| f - S\sum_{k=1}^{N}\frac{f(\lambda_k)}{S'(\lambda_k)(\cdot-\lambda_k)}\Big\|_{\varphi_\beta} = 0, \tag{1.2}$$

*where $\varphi_\beta(r) = \varphi(r) + \beta\log(1+r)$.*

The result corresponding to $\beta = 1/2$, $\varphi(r) = r^2$, $\rho(r) \asymp 1$ is contained in [13, Theorem 10]. On the other hand, in the case $\varphi(r) = (\log r)^a$, $1 < a \le 2$, we have $\rho(r) \asymp r(\log r)^{1-\frac{a}{2}}$, $r > 2$, and the space $\mathcal{F}^2_\varphi$ contains Riesz bases of (normalized) reproducing kernels [5].

Furthermore, our theorem shows that in the case $r \lesssim \rho(r)$, $r > 1$, when $S \in \mathcal{S}_\gamma$, $\gamma \in (1/2, 1)$, the interpolation series converges already in $\mathcal{F}^2_\varphi$.

In the case $0 < a \le 2$, $\varphi(r) = r^a$, we have $\rho(r) \asymp r^{1-a/2}$, $r > 1$, and we can choose $a/4 \le \beta \le 1$ with appropriate $\gamma$ as in Theorem 1.3.3. Thus, the closer we are to $\phi(r) = (\log r)^2$, $\rho(r) \asymp r$, $r > 2$, the less we should modify the norm (by the smaller $\beta$) to get convergence in (1.2).

Now, it is interesting to find out how sharp is condition (1.1) in Theorem 1.3.3.

**Theorem 1.3.4.** *Let $0 < a \le 2$, $\varphi(r) = r^a$, $r > 1$. If $0 \le \beta < a/4$, $\gamma \in \mathbb{R}$, and $S \in \mathcal{S}_\gamma$, then there exists $f \in \mathcal{F}^2_\varphi$ such that*

$$\Big\| f - S \sum_{k=1}^{N} \frac{f(\lambda_k)}{S'(\lambda_k)(\cdot - \lambda_k)} \Big\|_{\varphi_\beta} \not\to 0, \qquad N \to \infty.$$

Thus, for the power weights $\varphi(r) = r^a$, $0 < a \le 2$, we really need to modify the norm to get the convergence, and the critical value of $\beta$ is $a/4$.

### 1.3.2 Chapter 3

In [5], Borichev–Lyubarskii have shown the existence of *complete interpolating* sequences in $\mathcal{F}^2_\varphi$ (simultaneously interpolating and sampling for $\mathcal{F}^2_\varphi$), for $\varphi(r) = (\log^+ r)^2$.

**Theorem 1.3.5.** *Let $\varphi(z) = (\log|z|)^2$ and let $\Gamma = \{\gamma_n\}_{n\ge 0}$ with $\gamma_n = e^{\frac{n+1}{2}} e^{i\theta_n}$ where $\theta_n$ are arbitrary real numbers. Then $\{\mathbb{k}_{\gamma_n}\}_{n\ge 0}$ is a Riesz basis in $\mathcal{F}^2_\varphi$.*

(See also [2,16] for other results in this direction). Note that for $\varphi(r) \gg (\log^+ r)^2$, there are no complete interpolating sequences for $\mathcal{F}^2_\varphi$ [4,14,18,20,23].

The main focus of the Chapter 3 is on interpolation and sampling for $\mathcal{F}^2_\varphi$ when $\varphi(r) = \alpha(\log^+ r)^2$. (Note that the case of small Bergman spaces has recently been considered by Seip [24]).

According to [5, Lemma 2.7], in our case, the reproducing kernel admits the following estimate:

$$\| \mathbf{k}_z \|^2_{\varphi,2} = \mathbf{k}_z(z) \asymp \frac{e^{2\varphi(z)}}{1+|z|^2}, \qquad z \in \mathbb{C}. \tag{1.3}$$

Our central result is a characterization of complete interpolating sequences in terms of their deviation from the Borichev-Lyubarskii sequence. This is in the spirit of the Kadets–Ingham 1/4-theorem and its generalizations. Note that Avdonin discusses a similar situation in the case of the Paley–Wiener space [1] where he proves the sufficiency part of the corresponding statement.

**Theorem 1.3.6.** *Let $\varphi(r) = \alpha(\log^+ r)^2$, $\alpha > 0$. Furthermore, let $\Gamma = \{\gamma_n\}_{n\geq 0} = \{e^{\frac{n+1}{2\alpha}}\}_{n\geq 0}$ and $\Lambda = \{\lambda_n\}$ with $\lambda_n = \gamma_n e^{\delta_n} e^{i\theta_n}$, $|\lambda_n| \leq |\lambda_{n+1}$, $\theta_n \in \mathbb{R}$. Then $\{\Bbbk_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis for $\mathcal{F}^2_\varphi$ if and only if the following three conditions hold.*

1. *$\Lambda$ is $d_\rho$-separated,*
2. *$(\delta_n) \in \ell^\infty$,*
3. *there exist $N \geq 1$ and $\delta > 0$ such that*

$$\sup_n \frac{1}{N}\Big|\sum_{k=n+1}^{n+N} \delta_k\Big| \leq \delta < \frac{1}{4\alpha}.$$

We also investigate the case $p = \infty$. The corresponding weighted Fock space is defined by

$$\mathcal{F}^\infty_\varphi = \{f \in \mathrm{Hol}(\mathbb{C}) : \|f\|_{\varphi,\infty} := \sup_{z\in\mathbb{C}} |f(z)|e^{-\varphi(z)} < \infty\}.$$

A sequence $\Lambda \subset \mathbb{C}$ is called sampling for $\mathcal{F}^\infty_\varphi$, if there exists $L > 0$ such that

$$\|f\|_{\varphi,\infty} \leq L\|f|\Lambda\|_{\varphi,\infty,\Lambda} := L\sup_{\lambda\in\Lambda} |f(\lambda)|e^{-\varphi(\lambda)}, \qquad f \in \mathcal{F}^\infty_\varphi. \tag{1.4}$$

We denote by $L_\varphi(\Lambda)$ the least $L$ satisfying (1.4), called the sampling constant.

The sequence $\Lambda$ is called interpolating if for every sequence $v = (v_\lambda)_{\lambda\in\Lambda} \in \ell^\infty_{\varphi,\Lambda}$, i.e. $\|v\|_{\varphi,\infty,\Lambda} < \infty$, there is a function $f \in \mathcal{F}^\infty_\varphi$ such that

$$v = f|\Lambda.$$

We derive from our theorem on Riesz bases several density conditions for interpolation and sampling in $\mathcal{F}^p_\varphi$, $p = 2, \infty$. Before stating these results, we need more notation.

Let $\mathcal{A}(r, R)$ be the annulus centered at the origin with inner and outer radii $r$ and $R$: $\mathcal{A}(r, R) := \{z \in \mathbb{C} : r \leq |z| < R\}$. For a $d_\rho$-separated squence $\Lambda$ we define the lower and upper densities respectively by

$$D^-(\Lambda) = \liminf_{R\to+\infty} \liminf_{r\to+\infty} \frac{\mathrm{Card}(\Lambda \cap \mathcal{A}(r, Rr))}{\log R}$$

and

$$D^+(\Lambda) = \limsup_{R\to+\infty} \limsup_{r\to+\infty} \frac{\mathrm{Card}(\Lambda \cap \mathcal{A}(r, Rr))}{\log R}.$$

These densities do not change when we remove or add a finite number of points to a sequence.

We are now in a position to formulate our density results.

**Theorem 1.3.7.** *(Sampling, $p = \infty$.) Let $\varphi(r) = \alpha(\log^+ r)^2$, $\alpha > 0$. Then*

*(i) Every $d_\rho$-separated sequence $\Lambda$ with $D^-(\Lambda) > 2\alpha$ is a set of sampling for $\mathcal{F}^\infty_\varphi$.*

*(ii) If the sequence $\Lambda$ is a set of sampling for $\mathcal{F}^\infty_\varphi$, then it contains a $d_\rho$-separated subsequence $\widetilde{\Lambda}$ with $D^-(\widetilde{\Lambda}) \geq 2\alpha$.*

**Theorem 1.3.8.** *(Sampling, $p = 2$.) Let $\varphi(r) = \alpha(\log^+ r)^2$, $\alpha > 0$. Then*

*(i) Every $d_\rho$-separated sequence $\Lambda$ with $D^-(\Lambda) > 2\alpha$ is a set of sampling for $\mathcal{F}^2_\varphi$.*

*(ii) If the sequence $\Lambda$ is a set of sampling for $\mathcal{F}^2_\varphi$, then it is a finite union of $d_\rho$-separated subsequences, and $\Lambda$ contains a $d_\rho$-separated sequence $\widetilde{\Lambda}$ such that $D^-(\widetilde{\Lambda}) \geq 2\alpha$.*

**Theorem 1.3.9.** *(Interpolation, $p = 2, \infty$.) Let $\varphi(r) = \alpha(\log^+ r)^2$, $\alpha > 0$. Then*

*(i) Every $d_\rho$-separated sequence $\Lambda$ with $D^+(\Lambda) < 2\alpha$ is a set of interpolation for $\mathcal{F}^p_\varphi$, $p = 2, \infty$.*

*(ii) If the sequence $\Lambda$ is a set of interpolation for $\mathcal{F}^p_\varphi$, $p = 2, \infty$, then it is $d_\rho$-separated and $D^+(\Lambda) \leq 2\alpha$.*

For $p = 2, \infty$, the following result shows that if the density is critical then two key situations may occur: that of complete interpolating sequences and that of sequences which are neither interpolating nor sampling. This in particular implies that there is no density characterization for sampling or interpolating sequences for $p = 2, \infty$.

**Theorem 1.3.10.** *(i) There exists a $d_\rho$-separated sequence $\Gamma$ such that $D^+(\Gamma) = D^-(\Gamma) = 2\alpha$, which is sampling and interpolating for $\mathcal{F}^2_\varphi$.*

*(ii) There exists a $d_\rho$-separated sequence $\Gamma$ such that $D^+(\Gamma) = D^-(\Gamma) = 2\alpha$, which is neither sampling nor interpolating for $\mathcal{F}^2_\varphi$.*

*(iii) There exists a $d_\rho$-separated sequence $\Gamma$ such that $D^+(\Gamma) = D^-(\Gamma) = 2\alpha$, which is sampling and interpolating for $\mathcal{F}^\infty_\varphi$.*

*(iv) There exists a $d_\rho$-separated sequence $\Gamma$ such that $D^+(\Gamma) = D^-(\Gamma) = 2\alpha$, which is neither sampling nor interpolating for $\mathcal{F}^\infty_\varphi$.*

The result in (i) is due to Borichev–Lyubarskii [5] who construct a sequence $\Gamma = \Gamma_\alpha$ which is complete interpolating (the corresponding reproducing kernels form an unconditional basis). Its counterpart for $p = \infty$ is obtained from $\Gamma_\alpha$ by adding just one point. The sequences yielding (ii) and (iv) corresponds to slightly perturbed versions of the Borichev–Lyubarskii sequence.

# Chapter 2

# Convergence of Lagrange interpolation series in the Fock spaces

## 2.1 Introduction and main results.

In this paper we study the weighted Fock spaces $\mathcal{F}^2_\varphi$

$$\mathcal{F}^2_\varphi = \Big\{ f \in \mathrm{Hol} : \|f\|^2_\varphi = \int_{\mathbb{C}} |f(z)|^2 e^{-2\varphi(|z|)}\, dm(z) < \infty \Big\};$$

here $dm$ is area measure and $\varphi$ is an increasing function defined on $[0, +\infty)$, $\lim_{r\to\infty} \varphi(r) = \infty$. We assume that the radial weight $\varphi(z) = \varphi(|z|)$ is $C^2$ smooth and strictly subharmonic on $\mathbb{C}$, and set

$$\rho(z) = (\Delta\varphi(z))^{-1/2},$$

so that $\Delta\varphi(r) = \varphi''(r) + \varphi'(r)/r$ $(r > 0)$. One more condition on $\varphi$ is that for every fixed $C$ we have

$$\rho(r + C\rho(r)) \asymp \rho(r), \qquad 0 < r < \infty.$$

(The notation $A \asymp B$ means that there is a constant $C$ independent of the relevant variables such that $C^{-1}B \le A \le CB$.) In particular, this holds if $\rho'(r) = o(1)$, $r \to \infty$. The function $\rho$ plays the rôle of a scaling parameter, see the definition of $d_\rho$ below.

Typical $\varphi$ are power functions,

$$\varphi(r) = r^a, \quad a > 0.$$

For such functions $\varphi$ we have

$$\rho(r) \asymp r^{1-a/2}, \qquad r > 1.$$

Furthermore, if

$$\varphi(r) = (\log r)^2,$$

then

$$\rho(r) \asymp r, \qquad r > 1.$$

Given $z, w \in \mathbb{C}$, we define a scaled distance function

$$d_\rho(z, w) = \frac{|z - w|}{\min(\rho(z), \rho(w))}.$$

We say that a subset $\Lambda$ of $\mathbb{C}$ of is $d_\rho$–separated if

$$\inf_{\lambda \neq \lambda^*} \{d_\rho(\lambda, \lambda^*),\ \lambda, \lambda^* \in \Lambda\} > 0.$$

Next, we introduce a family of sufficiently regular subsets $\Lambda$ in $\mathbb{C}$ defined as the zero sets for the functions in a special class.

**Definition 2.1.1.** *Given $\gamma \in \mathbb{R}$, we say that an entire function $S$ belongs to the class $\mathcal{S}_\gamma$ if*

(1) *the zero set $\Lambda$ of $S$ is $d_\rho$-separated, and*

(2)
$$|S(z)| \asymp e^{\varphi(z)} \frac{d(z, \Lambda)}{\rho(z)} \frac{1}{(1 + |z|)^\gamma}, \qquad z \in \mathbb{C}.$$

For constructions of such functions in radial weighted Fock spaces see, for example, [4, 5, 12, 14].

In the standard Fock spaces ($\varphi(r) = r^2$) the classes $\mathcal{S}_\gamma$ were introduced by Lyubarskii in [11]. They are analogs of the sine type functions for the Paley–Wiener space, and their zero sets include rectangular lattices and their perturbations.

**Definition 2.1.2.** *A set $\Lambda \subset \mathbb{C}$ is called a weak interpolation set for $\mathcal{F}^2_\varphi$ if for every $\lambda \in \Lambda$ there exists $f_\lambda \in \mathcal{F}^2_\varphi$ such that $f_\lambda(\lambda) = 1$ and $f_\lambda|\Lambda \backslash \{\lambda\} = 0$.*

*A set $\Lambda \subset \mathbb{C}$ is called a uniqueness set for $\mathcal{F}^2_\varphi$ if for every $f \in \mathcal{F}^2_\varphi$, the relation $f|\Lambda = 0$ implies that $f = 0$.*

**Theorem 2.1.3.** *Let $\phi$ and $\rho$ be as above. Given $S \in \mathcal{S}_\gamma$, consider its zero set $\Lambda$. Then*

($a$) *$\Lambda$ is a uniqueness set for $\mathcal{F}^2_\varphi$ if and only if $\gamma \leq 1$,*

($b$) *$\Lambda$ is a weak interpolation set for $\mathcal{F}^2_\varphi$ if and only if $\gamma > 0$.*

Denote by $\mathbf{k}_z$ be the reproducing kernel of $\mathcal{F}^2_\varphi$:

$$\langle f, \mathbf{k}_z, \rangle_{\mathcal{F}^2_\varphi} = f(z), \qquad f \in \mathcal{F}^2_\varphi, \quad z \in \mathbb{C}.$$

The sequence $\Lambda \subset \mathbb{C}$ is called *sampling* for $\mathcal{F}^2_\varphi$ if

$$\|f\|^2_\varphi \asymp \|f\|^2_{\varphi,\Lambda} := \sum_{\lambda\in\Lambda} \frac{|f(\lambda)|^2}{\mathbf{k}_\lambda(\lambda)}, \qquad f \in \mathcal{F}^2_\varphi,$$

and *interpolating* for $\mathcal{F}^2_\varphi$ if for every $v = (v_\lambda)_{\lambda\in\Lambda}$ with $\|v\|_{\varphi,\Lambda} < \infty$ there exists $f \in \mathcal{F}^2_\varphi$ such that

$$v = f|\Lambda.$$

It is obvious that each sampling sequence for $\mathcal{F}^2_\varphi$ is a set of uniqueness for $\mathcal{F}^2_\varphi$ and each interpolation sequence for $\mathcal{F}^2_\varphi$ is a weak interpolation set for $\mathcal{F}^2_\varphi$.

The sequence $\Lambda \subset \mathbb{C}$ is called *complete interpolating* sequence for $\mathcal{F}^2_\varphi$ if it is simultaneously interpolating and sampling for $\mathcal{F}^2_\varphi$.

Let $\mathbb{k}_\lambda = \mathbf{k}_\lambda / \| \mathbf{k}_\lambda \|_{\varphi,2}$ be the normalized reproducing kernel at $\lambda$. Let $\Lambda \subset \mathbb{C}$. We say that $\{\mathbb{k}_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis in $\mathcal{F}^2_\varphi$ if it is complete and for some $C > 0$ and each finite sequence $\{a_\lambda\}$ we have

$$\frac{1}{C}\sum_{\lambda\in\Lambda} |a_\lambda|^2 \le \Big\| \sum_{\lambda\in\Lambda} a_\lambda \, \mathbb{k}_\lambda \Big\|^2_\varphi \le C \sum_{\lambda\in\Lambda} |a_\lambda|^2.$$

Note that in $\mathcal{F}^2_\varphi$, interpolation and sampling can be expressed in terms of geometric properties of reproducing kernels: interpolation means that the sequence of the associated reproducing kernels is a Riesz basis in its closed linear span; sampling means that this sequence is a frame (see [23, Chapter 3]). Standard duality arguments show that the system $\{\mathbb{k}_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis in $\mathcal{F}^2_\varphi$ if and only if $\Lambda$ is a complete interpolating sequence for $\mathcal{F}^2_\varphi$.

In 1992 Seip and Wallstén [20,25] characterized interpolating and sampling sequences in these spaces when $\varphi(r) = r^2$. Their results show that there are no sequences which are simultaneously interpolating and sampling, and hence there are no unconditional or Riesz bases in this situation. The situation changes in small Fock spaces when the weight increases slowly. Borichev and Lybarskii [5] have shown that for $\varphi(r) = (\log r)^2$ there exist Riesz bases in $\mathcal{F}^2_\varphi$. Furthermore ( [5,9,23]), the space $\mathcal{F}^2_\varphi$ does not admit Riesz bases of the (normalized) reproducing kernels for regular $\varphi$, $(\log r)^2 = o(\varphi(r))$, $r \to \infty$.

By Theorem 2.1.3, for $0 < \gamma \le 1$ the family $\{\mathbb{k}_\lambda\}_{\lambda\in\Lambda}$ is a complete minimal family in $\mathcal{F}^2_\varphi$. Thus the family

$$\Big\{S/[S'(\lambda)(\cdot - \lambda)]\Big\}_{\lambda\in\Lambda}$$

is the biorthogonal system and we associate to any $f \in \mathcal{F}^2_\varphi$ the formal (Lagrange interpolation) series

$$f \sim \sum_{\lambda\in\Lambda} f(\lambda) \frac{S}{S'(\lambda)(\cdot - \lambda)}.$$

This series converges unconditionally in $\mathcal{F}^2_\varphi$ if $\{\mathbb{k}_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis in $\mathcal{F}^2_\varphi$. Otherwise, it does not necessarily converge unconditionally in $\mathcal{F}^2_\varphi$, and it is natural to ask whether this series admits a summation method if we modify (slightly) the norm of the space.

Denote by $\Lambda = \{\lambda_k\}$ the zero sequence of $S$ ordered in such a way that $|\lambda_k| \le |\lambda_{k+1}|$, $k \ge 1$. Similarly to Lyubarskii [11] and Lyubarskii–Seip [13], we obtain the following result:

**Theorem 2.1.4.** *Let $0 \le \beta \le 1$, $\gamma + \beta \in (1/2, 1)$, and let $S \in \mathcal{S}_\gamma$. Suppose that*

$$r^{1-2\beta} = O(\rho(r)), \qquad r \to +\infty. \tag{2.1}$$

*Then for every $f \in \mathcal{F}^2_\varphi$ we have*

$$\lim_{N\to\infty}\Big\| f - S\sum_{k=1}^{N} \frac{f(\lambda_k)}{S'(\lambda_k)(\cdot - \lambda_k)}\Big\|_{\varphi_\beta} = 0, \tag{2.2}$$

*where $\varphi_\beta(r) = \varphi(r) + \beta\log(1+r)$.*

The result corresponding to $\beta = 1/2$, $\varphi(r) = r^2$, $\rho(r) \asymp 1$ is contained in [13, Theorem 10]. On the other hand, in the case $\varphi(r) = (\log r)^a$, $1 < a \le 2$, $\rho(r) \asymp r(\log r)^{1-\frac{a}{2}}$, $r > 2$, the space $\mathcal{F}^2_\varphi$ contains Riesz bases of (normalized) reproducing kernels [5]. Furthermore, our theorem shows that in the case $r \lesssim \rho(r)$, $r > 1$, when $S \in \mathcal{S}_\gamma$, $\gamma \in (1/2, 1)$, the interpolation series converges already in $\mathcal{F}^2_\varphi$. (The notation $A \lesssim B$ means that there is a constant $C$ independent of the relevant variables such that $A \le CB$.)

In the case $0 < a \le 2$, $\varphi(r) = r^a$, $\rho(r) \asymp r^{1-a/2}$, $r > 1$, we can choose $a/4 \le \beta \le 1$ with appropriate $\gamma$ as in Theorem 2.1.4. Thus, the closer we are to $\phi(r) = (\log r)^2$, $\rho(r) \asymp r$, $r > 2$, the less we should modify the norm (by the smaller $\beta$) to get convergence in (2.2). Now, it is interesting to find out how sharp is condition (2.1) in Theorem 2.1.4.

**Theorem 2.1.5.** *Let $0 < a \le 2$, $\varphi(r) = r^a$, $r > 1$. If $0 \le \beta < a/4$, $\gamma \in \mathbb{R}$, and $S \in \mathcal{S}_\gamma$, then there exists $f \in \mathcal{F}^2_\varphi$ such that*

$$\Big\| f - S\sum_{k=1}^{N} \frac{f(\lambda_k)}{S'(\lambda_k)(\cdot - \lambda_k)}\Big\|_{\varphi_\beta} \not\to 0, \qquad N \to \infty.$$

Thus, for the power weights $\varphi(r) = r^a$, $0 < a \le 2$, we really need to modify the norm to get the convergence, and the critical value of $\beta$ is $a/4$.

## 2.2 Proofs.

### 2.2.1 Proof of Theorem 2.1.3

(a) If $\gamma > 1$ then $\mathcal{S}_\gamma \subset \mathcal{F}^2_\varphi$ and $S|\Lambda = 0$. Hence $\Lambda$ is not a uniqueness set.

If $\gamma \leq 1$, then $\mathcal{S}_\gamma \cap \mathcal{F}^2_\varphi = \emptyset$. Suppose that there exists $g \in \mathcal{F}^2_\varphi$ such that $g|\Lambda = 0$. Then $g = FS$ for an entire function $F$, and we have

$$\int_{\mathbb{C}} |F(w)|^2 |S(w)|^2 e^{-2\varphi(w)}\, dm(w) < \infty. \tag{2.3}$$

Given $\Omega \subset \mathbb{C}$, denote

$$\mathcal{I}[\Omega] = \int_\Omega |F(w)|^2 \frac{d^2(w,\Lambda)}{(1+|w|)^{2\gamma}\rho^2(w)}\, dm(w).$$

By (2.3), we have

$$\mathcal{I}[\mathbb{C}] < \infty.$$

Denote by $D(z,r)$ the disc of radius $r$ centered at $z$. Let

$$\Omega_\varepsilon = \bigcup_{\lambda\in\Lambda} D(\lambda, \varepsilon\rho(\lambda)),$$

where $\varepsilon$ is such that the discs $D(\lambda, 2\varepsilon\rho(\lambda))$ are pairwise disjoint. We have

$$\mathcal{I}[\mathbb{C}] = \mathcal{I}[\mathbb{C}\backslash\Omega_{2\varepsilon}] + \sum_{\lambda\in\Lambda} \mathcal{I}[D(\lambda, 2\varepsilon\rho(\lambda))\backslash D(\lambda,\varepsilon\rho(\lambda))] + \sum_{\lambda\in\Lambda} \mathcal{I}[D(\lambda,\varepsilon\rho(\lambda))].$$

It is clear that

$$\mathcal{I}[\mathbb{C}\backslash\Omega_{2\varepsilon}] \geq c_1 \int_{\mathbb{C}\backslash\Omega_{2\varepsilon}} \frac{|F(w)|^2}{(1+|w|)^{2\gamma}}\, dm(w).$$

On the other hand,

$$\int_{D(\lambda,\varepsilon\rho(\lambda))} |F(w)|^2\, dm(w) \leq c_2 \int_{D(\lambda,2\varepsilon\rho(\lambda))\backslash D(\lambda,\varepsilon\rho(\lambda))} |F(w)|^2\, dm(w),$$

and, hence,

$$\mathcal{I}[D(\lambda, 2\varepsilon\rho(\lambda))\backslash D(\lambda,\varepsilon\rho(\lambda))] \geq c_3 \mathcal{I}[D(\lambda,\varepsilon\rho(\lambda))].$$

Therefore

$$\int_{\mathbb{C}} \frac{|F(w)|^2}{(1+|w|)^{2\gamma}}\, dm(w) < \infty,$$

the function $F$ is constant, and $g = cS$. Since $\mathcal{S}_\gamma \cap \mathcal{F}^2_\varphi = \emptyset$, we get a contradiction. Statement (a) is proved.

(b) Let $\gamma > 0$. Set

$$f_\lambda(z) = \frac{S(z)}{S'(\lambda)(z-\lambda)}, \qquad \lambda \in \Lambda.$$

It is obvious that $f_\lambda \in \mathcal{F}^2_\varphi$, $f_\lambda|(\Lambda\backslash\{\lambda\}) = 0$ and $f_\lambda(\lambda) = 1$. Hence $\Lambda$ is a weak interpolation set. If $\gamma \leq 0$, $\lambda \in \Lambda$, then, by (a), $\Lambda\backslash\{\lambda\}$ is a uniqueness set for $\mathcal{F}^2_\varphi$. Therefore, $\Lambda$ is not a weak interpolation set for $\mathcal{F}^2_\varphi$. $\square$

### 2.2.2 Proof of Theorem 2.1.4

We follow the scheme of proof proposed in [11,13] and concentrate mainly on the places where the proofs differ. We need some auxiliary notions and lemmas. The proof of the first lemma is the same as in [4, Lemma 4.1].

**Lemme 2.2.1.** *For every $\delta > 0$, there exists $C > 0$ such that for functions $f$ holomorphic in $D(z, \delta\rho(z))$ we have*

$$|f(z)|^2 e^{-2\varphi(z)} \le \frac{C}{\rho(z)^2} \int_{D(z,\delta\rho(z))} |f(w)|^2 e^{-2\varphi(w)}\, dm(w).$$

**Definition 2.2.2.** *A simple closed curve $\gamma = \{r(\theta)e^{-i\theta},\ \theta \in [0, 2\pi]\}$ is called $K$–bounded if $r$ is $C^1$-smooth and $2\pi$-periodic on the real line and $|r'(\theta)| \le K$, $\theta \in \mathbb{R}$.*

Let $\gamma \in \mathbb{R}$, $S \in \mathcal{S}_\gamma$, and let $\Lambda = \{\lambda_k\}$ be the zero set of $S$ ordered in such a way that $|\lambda_k| \le |\lambda_{k+1}|$, $k \ge 1$. We can construct a sequence of numbers $R_N \to \infty$ and a sequence of contours $\Gamma_N$ such that

1. $\Gamma_N = R_N \gamma_N$, where $\gamma_N$ are $K$-bounded with $K > 0$ independent of $N$.
2. $d_\rho(\Lambda, \Gamma_N) \ge \varepsilon$ for some $\varepsilon > 0$ independent of $N$.
3. $\{\lambda_k\}_1^N$ lie inside $\Gamma_N$ and $\{\lambda_k\}_{N+1}^\infty$ lie outside $\Gamma_N$.
4. $\Gamma_N \subset \{z : R_N - \rho(R_N) < |z| < R_N + \rho(R_N)\}$.

Indeed, for some $0 < \varepsilon < 1$ the discs $D_k = D(\lambda_k, \varepsilon\rho(\lambda_k))$ are disjoint. For some $\delta = \delta(\varepsilon) > 0$ we have

$$\varepsilon\rho(\lambda_k) > 4\delta\rho(\lambda_N), \qquad \big||\lambda_k| - |\lambda_N|\big| < \delta\rho(\lambda_N).$$

Fix $\psi \in C_0^\infty[-1, 1]$, $0 < \psi < 1$, such that $\psi > 1/2$ on $[-1/2, 1/2]$.

Put

$$\Xi = \Big\{k : \big||\lambda_k| - |\lambda_N|\big| < \frac{1}{4}\delta\rho(\lambda_N)\Big\},$$

denote $\lambda_k = r_k e^{i\theta_k}$, $k \in \Xi$, and set

$$r(\theta) = 1 + \sum_{k\in\Xi} s_k \frac{\delta\rho(\lambda_N)}{|\lambda_N|} \psi\Big(\frac{|\lambda_N|}{\delta\rho(\lambda_N)}(\theta - \theta_k)\Big),$$

where $s_k = 1$, $k \le N$, $s_k = -1$, $k > N$.

Finally, set

$$\gamma_N = \Big\{r(\theta)e^{i\theta},\ \theta \in [0, 2\pi]\Big\}.$$

**Lemme 2.2.3.**
$$R_N\rho(R_N)\int_{\gamma_N}|f(R_N\zeta)|^2e^{-2\varphi(R_N\zeta)}|d\zeta|\to 0,\qquad N\to\infty.$$

*Proof.* Set
$$C_N=\bigcup_{\zeta\in\gamma_N}D(R_N\zeta,\rho(R_N\zeta)).$$
Since
$$\rho(R_N\zeta)\asymp\rho(R_N),\qquad \zeta\in\gamma_N,$$
by Lemma 2.2.1 we have
$$\begin{aligned}
&R_N\rho(R_N)\int_{\gamma_N}|f(R_N\zeta)|^2e^{-2\varphi(R_N\zeta)}|d\zeta|\\
&\qquad\lesssim R_N\rho(R_N)\int_{\gamma_N}\Big[\frac{1}{\rho(R_N\zeta)^2}\int_{D(R_N\zeta,\rho(R_N\zeta))}|f(w)|^2e^{-2\varphi(w)}\,dm(w)\Big]|d\zeta|\\
&\qquad\asymp\frac{R_N}{\rho(R_N)}\int_{C_N}|f(w)|^2e^{-2\varphi(w)}\Big(\int_{\gamma_N}\chi_{D(R_N\zeta,\rho(R_N\zeta))}(w)|d\zeta|\Big)\,dm(w)\\
&\qquad\lesssim\int_{C_N}|f(w)|^2e^{-2\varphi(w)}\,dm(w)\to 0,\qquad N\to\infty.
\end{aligned}$$
□

## Proof of Theorem 2.1.4

Let $\chi_N(z)=1$ if $z$ lies inside $\Gamma_N$ and 0 otherwise. Put
$$\Sigma_N(z,f)=S(z)\sum_{k=1}^{N}\frac{f(\lambda_k)}{S'(\lambda_k)(z-\lambda_k)},$$
and set
$$I_N(z,f)=\frac{1}{2\pi i}\int_{\Gamma_N}\frac{f(\zeta)}{S(\zeta)(z-\zeta)}d\zeta$$
The Cauchy formula gives us that
$$I_N(z,f)=\sum_{k=1}^{N}\frac{f(\lambda_k)}{S'(\lambda_k)(z-\lambda_k)}-\chi_N(z)\frac{f(z)}{S(z)},\qquad z\notin\Gamma_N.$$
Hence,
$$\Sigma_N(z,f)-f(z)=S(z)I_N(z,f)+(\chi_N(z)-1)f(z),$$

and to complete the proof of the theorem, it remains only to verify that

$$||SI_N(\cdot, f)||_{\varphi_\beta} \to 0, \qquad N \to \infty.$$

Let $\omega$ be a Lebesgue measurable function such that

$$\int_0^\infty \int_0^{2\pi} |\omega(re^{it})|^2 e^{-2\varphi(r)}(1+r)^{-2\beta} r dr dt \leq 1, \tag{2.4}$$

and let

$$J_N(f,\omega) = \int_0^\infty \int_0^{2\pi} \omega(re^{it}) S(re^{it}) I_N(re^{it}, f) e^{-2\varphi(r)}(1+r)^{-2\beta} r dr dt.$$

By duality, it remains to show that

$$\sup |J_N(f,\omega)| \to 0, \qquad N \to \infty,$$

where the supremum is taken over all $\omega$ satisfying (2.4).

We have

$$\begin{aligned}2\pi i J_N(f,\omega) &= \int_{\Gamma_N} \frac{f(\zeta)}{S(\zeta)} \int_{\mathbb{C}} \frac{\omega(z)S(z)}{z-\zeta} e^{-2\varphi(z)}(1+|z|)^{-2\beta} dm(z) d\zeta \\ &= \int_{\Gamma_N} \frac{f(\zeta)}{S(\zeta)} \int_{\mathbb{C}} \frac{\phi(z)}{z-\zeta}(1+|z|)^{-\beta-\gamma} dm(z) d\zeta,\end{aligned}$$

where

$$\phi(z) = [\omega(z)e^{-\varphi(z)}(1+|z|)^{-\beta}][S(z)e^{-\varphi(z)}(1+|z|)^{\gamma}].$$

Note that

$$\int_{\mathbb{C}} |\phi(z)|^2 \, dm(z) \leq C.$$

Set

$$\psi(z) = R_N \phi(R_N z),$$

(here $\psi$ depends on $N$). We have

$$\int_{\mathbb{C}} |\psi(z)|^2 \, dm(z) \leq C.$$

Changing the variables $z = R_N w$ and $\zeta = R_N \eta$, we get

$$2\pi i J_N(f,\omega) = R_N \int_{\gamma_N} \frac{f(R_N\eta)}{S(R_N\eta)} \int_{\mathbb{C}} \frac{\psi(w)}{w-\eta}(1+R_N|w|)^{-\beta-\gamma} dm(w) d\eta.$$

Consider the operators

$$T_N(\psi)(\eta) = \int_{\mathbb{C}} \frac{\psi(w)}{w-\eta} |w|^{-\beta-\gamma} dm(w), \qquad \psi \in L^2(\mathbb{C}, dm(w)).$$

Since $\gamma+\beta\in(1/2,1)$, by [13, Lemma 13], the operators $T_N$ are bounded from $L^2(\mathbb{C},dm(w))$ into $L^2(\gamma_N)$ and
$$\sup_N \|T_N\|<\infty.$$
Hence, by Lemma 2.2.3 and by the property $r^{1-2\beta}=O(\rho(r))$, $r\to\infty$, we get
$$\begin{aligned}
J_N(f,\omega) &\lesssim R_N^{1-\beta-\gamma}\Big|\int_{\gamma_N}\frac{f(R_N\eta)}{S(R_N\eta)}T_N(\psi)(\eta)d\eta\Big| \\
&\lesssim R_N^{1-\beta}\int_{\gamma_N}|f(R_N\eta)|e^{-\varphi(R_N\eta)}|T_N(\psi)(\eta)|\,|d\eta| \\
&\lesssim \Big(R_N\rho(R_N)\int_{\gamma_N}|f(R_N\eta)|^2e^{-2\varphi(R_N\eta)}|d\eta|\Big)^{1/2} \\
&\qquad\times\|T_N\|\|\psi\|_{L^2(\gamma_N)}\to 0,\qquad N\to\infty.
\end{aligned}$$
This completes the proof. □

### 2.2.3 Proof of Theorem 2.1.5

It suffices to find $f\in\mathcal{F}^2_\varphi$ and a sequence $N_k$ such that (in the notations of the proof of Theorem 2.1.4)
$$A_k=\Big\|S\chi_{N_k}\int_{\Gamma_{N_k}}\frac{f(\zeta)}{S(\zeta)(\cdot-\zeta)}\,d\zeta\Big\|_{\varphi_\beta}\not\to 0,\qquad k\to\infty. \tag{2.5}$$
We follow the method of the proof of [13, Theorem 11]. Let us write down the Taylor series of $S$:
$$S(z)=\sum_{n\ge0}s_nz^n.$$
Since $S\in\mathcal{S}_\gamma$, by Cauchy's inequality, we have
$$\begin{aligned}
|s_n| &\lesssim \inf_{r>0}e^{r^a}\frac{r}{r^{1-\frac{a}{2}}}\frac{1}{(1+r)^\gamma}r^{-n} \\
&\lesssim \inf_{r>0}e^{r^a}r^{-n-\gamma+\frac{a}{2}} \\
&\lesssim \exp\Big(-\frac{n}{a}\ln\frac{n}{ae}-\frac{\gamma}{a}\ln n\Big),\qquad n>0.
\end{aligned}$$
Choose $0<\varepsilon<\frac{a}{2}-2\beta$. Given $R>0$ consider
$$S_R=\sum_{|n-aR^a|<R^{\frac{a}{2}+\varepsilon}}s_nz^n.$$
Then for every $n$ we have
$$|S(z)-S_R(z)|e^{-|z|^a}=O(|z|^{-n}),\qquad \big||z|-R\big|<\rho(R),\qquad R\to\infty.$$

Next we use that for some $c > 0$ independent of $n$, we have

$$\int_0^\infty r^{2n+1}e^{-2r^a}\,dr \le c\int_{|r-(\frac{n}{a})^{1/a}|<n^{(1/a)-(1/2)}} r^{2n+1}e^{-2r^a}\,dr.$$

Therefore,

$$\begin{aligned}
\|S_R\|_\varphi^2 &= \sum_{|n-aR^a|<R^{\frac{a}{2}+\varepsilon}} \pi|s_n|^2\int_0^\infty r^{2n+1}e^{-2r^a}\,dr \\
&\le \sum_{|n-aR^a|<R^{\frac{a}{2}+\varepsilon}} c|s_n|^2\int_{|r-(\frac{n}{a})^{1/a}|<n^{(1/a)-(1/2)}} r^{2n+1}e^{-2r^a}\,dr \\
&\le \sum_{|n-aR^a|<R^{\frac{a}{2}+\varepsilon}} c|s_n|^2\int_{|r-R|<c_1R^{1-\frac{a}{2}+\varepsilon}} r^{2n+1}e^{-2r^a}\,dr \\
&\le \sum_{n\ge 0} c|s_n|^2\int_{|r-R|<c_1R^{1-\frac{a}{2}+\varepsilon}} r^{2n+1}e^{-2r^a}\,dr \\
&= c\int_{|r-R|<c_1R^{1-\frac{a}{2}+\varepsilon}} \sum_{n\ge 0}|s_n|^2 r^{2n+1}e^{-2r^a}\,dr \\
&= c\int_{\left||z|-R\right|<c_1R^{1-\frac{a}{2}+\varepsilon}} |S(z)|^2e^{-2|z|^a}\,dm(z) \le c_2R^{2-\frac{a}{2}+\varepsilon-2\gamma}.
\end{aligned}$$

Fix $\varkappa$ such that

$$1-\frac{a}{4}+\frac{\varepsilon}{2}-\gamma < \varkappa < 1-\beta-\gamma.$$

Choose a sequence $N_k$, $k \ge 1$, such that for $R_k = |\lambda_{N_k}|$ we have $R_{k+1} > 2R_k$, $k \ge 1$, and

$$\left|e^{-|z|^a}\sum_{m\ne k} S_{R_m}(z)R_m^{-\varkappa}\right| \le \frac{1}{|z|^{\gamma+1}}, \qquad \left||z|-R_k\right| < \rho(R_k), \quad k \ge 1.$$

Set

$$f = \sum_{k\ge 1} S_{R_k}R_k^{-\varkappa}.$$

Then $f \in \mathcal{F}_\varphi^2$, and

$$\frac{f}{S} = R_k^{-\varkappa} + O(R_k^{-1-\varkappa}) \quad \text{on } \Gamma_{N_k}, \quad k \to \infty.$$

Hence,

$$\left|S(z)\int_{\Gamma_{N_k}} \frac{f(\zeta)}{S(\zeta)(z-\zeta)}\,d\zeta\right| \ge cR_k^{-\varkappa}\frac{e^{|z|^a}}{(1+|z|)^\gamma}, \qquad |z| < \frac{R_k}{2},$$

and finally

$$A_k \ge cR_k^{-\varkappa}\Big(\int_0^{R_k/2} \frac{r^{1-2\beta}\,dr}{(1+r)^{2\gamma}}\Big)^{1/2} \to \infty, \qquad k \to \infty.$$

This proves (2.5) and thus completes the proof of the theorem. □

# Chapter 3

# Sampling, interpolation and Riesz bases in small Fock spaces

## 3.1 Introduction and main results.

Interpolation and sampling problems are well studied objects. Complete results for corresponding sequences are known for broad classes of spaces of analytic functions. We refer the reader to the monograph by Seip for an account on these problems [23]. Two prominent examples here are the Fock spaces and the Bergman spaces. For these, interpolating and sampling sequences have been studied by Seip in the classical situation. More recently, a series of results was obtained for weighted versions of these spaces. In particular, when the weight makes the space to become significantly smaller than the standard one, the geometric properties of sampling/interpolation sequences change significantly. Seip showed that in small Bergman spaces, locally, interpolating sequences look like interpolating sequences in Hardy spaces [24]. In small Fock spaces, Borichev and Lyubarskii recently exhibited Riesz bases [5]. In this paper we will investigate further the situation of small Fock spaces. Our main focus is on the Hilbertian situation $p = 2$. As it turns out, no density characterization can be expected in this situation for interpolation or sampling. There are actually sequences which are simultaneously interpolating and sampling, also called complete interpolating sequences. Note that complete interpolating sequences necessarily have critical density. We also provide sequences with critical density which are neither interpolating nor sampling for $p = 2, \infty$.

The central result of this paper is a characterization of complete interpolating sequences when $p = 2$ which is in the spirit of the $1/4$ Kadets–Ingham theorem in the Paley–Wiener space and its more general version of Avdonin. Using this characterization, we deduce sufficient density conditions for interpolation and sampling.

We also would like to emphasize the connection between these spaces and the de Branges spaces. The complete interpolating sequence introduced by Borichev–Lyubarskii in [5] defines a generating function $G$ which, when $p = 2$, allows to identify the Fock spaces we are interested in with the de Branges space $H(G)$. Consequently, the measure $dx/|G(x)|^2$ is a sampling measure for our Fock spaces, which proves a conjecture raised in [16]. Note that in [15], the authors consider sampling and interpolation in the class of de Branges spaces for which the phase function defines a doubling measure. Our space corresponds to the situation when the phase function is locally but not globally doubling so that their results apparently do not apply here. Still it can be observed that these authors obtain a similar kind of density characterization as ours when $p = 2$ (at least for real sequences they consider).

Finally, we discuss some results for $p = \infty$. The theorem on Riesz bases allow us also to deduce analogous density conditions as in the case $p = 2$. Also it is possible to construct complete interpolating sequences in this situation.

We now introduce the necessary notation. Let $\varphi \in C^2(\mathbb{C} \setminus \{0\})$ be a subharmonic radial function, $\varphi(z) = \varphi(|z|)$, such that $\varphi(r) \nearrow +\infty$, $r \to +\infty$.

Let us introduce

$$\rho(z) = (\Delta\varphi(z))^{-1/2};$$

$\Delta\varphi(r) = \varphi''(r) + \varphi'(r)/r$, $r > 0$. In what follows we assume that for some $\eta > 0$ and for $|c| < \eta$,

$$\rho(x + c\rho(x)) \asymp \rho(x), \qquad 0 < x < \infty.$$

(this condition is for instance satisfied when $\rho'(x) = O(1)$, $x \to +\infty$). We associate with $\rho$ a "distance" (a semimetric):

$$d_\rho(z, w) = \frac{|z - w|}{1 + \min(\rho(z), \rho(w))}, \qquad z, w \in \mathbb{C}.$$

Note that when $z, w$ are in a fixed disk, this distance is comparable to euclidean distance.

The sequence $\Lambda \subset \mathbb{C}$ is said to be $d_\rho$-separated if there is $d_\Lambda > 0$ such that

$$\inf\{d_\rho(z, w) : z, w \in \Lambda, z \neq w\} \geq d_\Lambda.$$

In [5] Borichev–Lyubarskii have shown the existence of *complete interpolating* sequences in $\mathcal{F}^2_\varphi$ (simultaneously interpolating and sampling for $\mathcal{F}^2_\varphi$, see precise definitions below) when $\varphi(r) = (\log^+ r)^2$ and

$$\mathcal{F}^2_\varphi = \{f \in \mathrm{Hol}(\mathbb{C}) : \|f\|^2_{\varphi,2} := \int_{\mathbb{C}} |f(z)|^2 e^{-2\varphi(z)} dm(z) < \infty\}.$$

(See also [2, 16] for other results in this direction). In order to define sampling and interpolating sequences for $\mathcal{F}^2_\varphi$, we consider first $\mathbf{k}_z$, the reproducing kernel of $\mathcal{F}^2_\varphi$:

$$\langle f, \mathbf{k}_z \rangle_{\mathcal{F}^2_\varphi} = f(z), \qquad f \in \mathcal{F}^2_\varphi, \quad z \in \mathbb{C}.$$

According to [5, Lemma 2.7], the kernel admits the following estimate:

$$\| \mathbf{k}_z \|^2_{\varphi,2} = \mathbf{k}_z(z) \asymp \frac{e^{2\varphi(z)}}{1+|z|^2}, \qquad z \in \mathbb{C}. \tag{3.1}$$

The sequence $\Lambda \subset \mathbb{C}$ is called sampling for $\mathcal{F}^2_\varphi$ if

$$\|f\|^2_{\varphi,2} \asymp \|f|\Lambda\|^2_{\varphi,2,\Lambda} := \sum_{\lambda\in\Lambda} \frac{|f(\lambda)|^2}{\mathbf{k}_\lambda(\lambda)}, \quad f \in \mathcal{F}^2_\varphi,$$

and interpolating if for every $v = (v_\lambda)_{\lambda\in\Lambda} \in \ell^2_{\varphi,\Lambda}$ , i.e. $\|v\|_{\varphi,2,\Lambda} < \infty$ there exists $f \in \mathcal{F}^2_\varphi$ such that

$$v = f|\Lambda.$$

Let $\mathbb{k}_\lambda = \mathbf{k}_\lambda / \| \mathbf{k}_\lambda \|_{\varphi,2}$ be the normalized reproducing kernel at $\lambda$. Let $\Lambda \subset \mathbb{C}$. We say that $\{\mathbb{k}_\lambda\}_{\lambda\in\Lambda}$ is a Riesz sequence in $\mathcal{F}^2_\varphi$ if for some $C > 0$ and for each finite sequence $\{a_\lambda\}$, we have

$$\frac{1}{C}\sum_{\lambda\in\Lambda} |a_\lambda|^2 \le \Big\| \sum_{\lambda\in\Lambda} a_\lambda \mathbb{k}_\lambda \Big\|^2_{2,\varphi} \le C \sum_{\lambda\in\Lambda} |a_\lambda|^2,$$

and a Riesz basis if it is also complete. It is well known that $\Lambda$ is interpolating if and only if $\{\mathbb{k}_\lambda\}_{\lambda\in\Lambda}$ is a Riesz sequence, and $\Lambda$ is complete interpolating if and only if $\{\mathbb{k}_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis in $\mathcal{F}^2_\varphi$. Note that for $\varphi(r) \gg (\log^+ r)^2$, there are no complete interpolating sequences for $\mathcal{F}^2_\varphi$ [4, 14, 18, 20, 23].

The main focus of this paper is on interpolation and sampling for $\mathcal{F}^2_\varphi$ when $\varphi(r) = \alpha(\log^+ r)^2$. Note that the case of small Bergman spaces has recently been considered by Seip [23].

Our central result is a characterization of complete interpolating sequences in terms of their deviation from the Borichev–Lyubarskii sequence. This is in the spirit of the Kadets–Ingham 1/4-theorem and its generalizations. Note that Avdonin discusses a similar situation in the case of the Paley–Wiener space [1].

**Theorem 3.1.1.** *Let $\alpha > 0$, $\varphi(r) = \alpha(\log^+ r)^2$, let $\Gamma = \{\gamma_n\}_{n\ge 0} = \{e^{\frac{n+1}{2\alpha}}\}_{n\ge 0}$ and let $\Lambda = \{\lambda_n\}$ with $\lambda_n = \gamma_n e^{\delta_n} e^{i\theta_n}$, $|\lambda_n| \le |\lambda_{n+1}|$, $\theta_n \in \mathbb{R}$. Then $\{\mathbb{k}_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis for $\mathcal{F}^2_\varphi$ if and only if the following three conditions hold.*

*(a) $\Lambda$ is $d_\rho$-separated,*

*(b) $(\delta_n) \in \ell^\infty$,*

*(c) there exists $N > 1$ and $\delta > 0$ such that*

$$\sup_n \frac{1}{N}\Big| \sum_{k=n+1}^{n+N} \delta_k \Big| \le \delta < \frac{1}{4\alpha}.$$

It is interesting whether one can deduce this theorem from the results of [2] on the boundedness of the weighted Hilbert transform for measures with sparse supports. Of course, to apply [2, Theorem 1.1] in one direction one should modify this theorem, replacing sparse sequences $\Gamma$ by finite unions of such sequences.

We also deal with the case $p = \infty$. The corresponding weighted Fock space is defined by

$$\mathcal{F}^\infty_\varphi = \{f \in \mathrm{Hol}(\mathbb{C}) : \|f\|_{\varphi,\infty} := \sup_{z\in\mathbb{C}} |f(z)|e^{-\varphi(z)} < \infty\}.$$

A sequence $\Lambda \subset \mathbb{C}$ is called sampling for $\mathcal{F}^\infty_\varphi$, if there exists $L > 0$ such that

$$\|f\|_{\varphi,\infty} \le L\|f|\Lambda\|_{\varphi,\infty,\Lambda} := L\sup_{\lambda\in\Lambda} |f(\lambda)|e^{-\varphi(\lambda)}, \qquad f \in \mathcal{F}^\infty_\varphi. \tag{3.2}$$

We denote by $L_\varphi(\Lambda)$ the smallest $L$ satisfying (3.2), called the sampling constant.

A sequence $\Lambda$ is called interpolating if for every sequence $v = (v_\lambda)_{\lambda\in\Lambda}$ in $\ell^\infty_{\varphi,\Lambda}$, i.e. such that $\|v\|_{\varphi,\infty,\Lambda} < \infty$, there is a function $f \in \mathcal{F}^\infty_\varphi$ such that

$$v = f|\Lambda.$$

We derive from Theorem 3.1.1 several density conditions for interpolation and sampling in $\mathcal{F}^p_\varphi$, $p = 2, \infty$. Before stating these results, we need some more notation.

Let $\mathcal{A}(r, R)$ be the annulus centered at the origin with inner and outer radii $r$ and $R$: $\mathcal{A}(r, R) := \{z \in \mathbb{C} : r \le |z| < R\}$. For a $d_\rho$-separated sequence $\Lambda$ we define the lower and upper densities respectively by

$$D^-(\Lambda) = \liminf_{R\to+\infty} \liminf_{r\to+\infty} \frac{\mathrm{Card}(\Lambda \cap \mathcal{A}(r, Rr))}{\log R}$$

and

$$D^+(\Lambda) = \limsup_{R\to+\infty} \limsup_{r\to+\infty} \frac{\mathrm{Card}(\Lambda \cap \mathcal{A}(r, Rr))}{\log R}.$$

These densities do not change when we remove or add a finite number of points to a sequence.

We are now in a position to formulate our density results.

**Theorem 3.1.2.** *(Sampling, $p = \infty$.) Let $\varphi(r) = \alpha(\log^+ r)^2$, $\alpha > 0$. Then*

*(i) every $d_\rho$-separated sequence $\Lambda$ with $D^-(\Lambda) > 2\alpha$, is a set of sampling for $\mathcal{F}^\infty_\varphi$;*

*(ii) if the sequence $\Lambda$ is a set of sampling for $\mathcal{F}^\infty_\varphi$ then it contains a $d_\rho$-separated subsequence $\widetilde{\Lambda}$ with $D^-(\widetilde{\Lambda}) \ge 2\alpha$.*

**Theorem 3.1.3.** *(Sampling, $p = 2$.) Let $\varphi(r) = \alpha(\log^+ r)^2$, $\alpha > 0$. Then*

*(i) every $d_\rho$-separated sequence $\Lambda$ with $D^-(\Lambda) > 2\alpha$, is a set of sampling for $\mathcal{F}^2_\varphi$;*

*(ii) if the sequence $\Lambda$ is a set of sampling for $\mathcal{F}^2_\varphi$, then it is a finite union of $d_\rho$-separated subsequences and $\Lambda$ contains a $d_\rho$-separated sequence $\widetilde{\Lambda}$ such that $D^-(\widetilde{\Lambda}) \geq 2\alpha$.*

**Theorem 3.1.4.** *(Interpolation, $p = 2, \infty$.) Let $\varphi(r) = \alpha(\log^+ r)^2$, $\alpha > 0$. Then*

*(i) every $d_\rho$-separated sequence $\Lambda$ with $D^+(\Lambda) < 2\alpha$ is a set of interpolation for $\mathcal{F}^p_\varphi$, $p = 2, \infty$;*

*(ii) if the sequence $\Lambda$ is a set of interpolation for $\mathcal{F}^p_\varphi$, $p = 2, \infty$ then it is a $d_\rho$-separated sequence with $D^+(\Lambda) \leq 2\alpha$.*

For $p = 2, \infty$ the following result shows that when the density is critical then two key situations may occur: that of complete interpolating sequences, and sequences which are neither interpolating nor sampling. This in particular implies that there is no density characterization for sampling or interpolating sequences for $p = 2, \infty$.

**Theorem 3.1.5.** *(i) There exists a $d_\rho$-separated sequence $\Gamma$ such that $D^+(\Gamma) = D^-(\Gamma) = 2\alpha$, which is sampling and interpolating for $\mathcal{F}^2_\varphi$.*

*(ii) There exists a $d_\rho$-separated sequence $\Gamma$ such that $D^+(\Gamma) = D^-(\Gamma) = 2\alpha$, which is neither sampling nor interpolating for $\mathcal{F}^2_\varphi$.*

*(iii) There exists a $d_\rho$-separated sequence $\Gamma$ such that $D^+(\Gamma) = D^-(\Gamma) = 2\alpha$, which is sampling and interpolating for $\mathcal{F}^\infty_\varphi$.*

*(iv) There exists a $d_\rho$-separated sequence $\Gamma$ such that $D^+(\Gamma) = D^-(\Gamma) = 2\alpha$, which is neither sampling nor interpolating for $\mathcal{F}^\infty_\varphi$.*

The result in (i) is due to Borichev–Lyubarskii [5] who construct a sequence $\Gamma^2$ which is complete interpolating (the corresponding reproducing kernels form an unconditional basis). Its counterpart for $p = \infty$ is obtained from $\Gamma^2$ just by adding one point. The sequences yielding (ii) and (iv) corresponds to slightly perturbed versions of the Borichev–Lyubarskii sequence.

The paper is organized as follows. In the next section, we present some elementary results on sampling and interpolation in our spaces. Some of them follow from a Bernstein type inequality that we will also give in this section. It is an interesting remark that we can consider our Fock spaces as subspaces of a suitable $H^\infty$ from which we deduce that half lines are sampling for $p = \infty$. Furthermore, we show that the lower density of a zero sequence has to be less than or equal to the critical density. Section 3.3 is devoted to the proof of an Ingham–Kadets type result (Theorem 3.1.1), which allows us to deduce the density results on sampling and interpolation in Section 3.4. In the short Section 3.5 we prove Theorem 3.1.5.

A final word on notation: $A \lesssim B$ means that there is a constant $C$ independent of the relevant variables such that $A \leq CB$. We write $A \asymp B$ if both $A \lesssim B$ and $B \lesssim A$.

## 3.2 Preliminary results

### 3.2.1 $d_\rho$-separated sequences

Recall that $\varphi(r) = \alpha(\log^+ r)^2$, so that $\rho(r) = r/\sqrt{2\alpha}$, $r \geq 1$. Hence

$$d_\rho(z, w) = \sqrt{2\alpha}\frac{|z-w|}{\sqrt{2\alpha} + \min(|z|, |w|)}, \qquad |z| \geq 1,\ |w| \geq 1.$$

For $\beta < 1$ and $0 \neq \lambda \in \mathbb{C}$ the ball corresponding to this distance is given by

$$D_\rho(\lambda, \beta) := \{z \in \mathbb{C} : d_\rho(z, \lambda) < \beta\}.$$

When $\beta$ is small, $D_\rho(\lambda, \beta)$ is comparable to a euclidean disk $D(\lambda, q|\lambda|)$ with a suitable constant $q$ depending on $\beta$. From this we deduce that $\Lambda$ is $d_\rho$-separated if and only if there exists $c > 0$ such that the euclidean disks $D(\lambda, c|\lambda|)$, $\lambda \in \Lambda$ are disjoint.

A central tool in our discussion is the following Bernstein type result whose proof can be found in [4].

**Lemme 3.2.1.** *Let $f$ be a holomorphic function.*

*(i) If $\|f\|_{\varphi,\infty} = 1$, then for every $c > 0$ there exists $0 < \beta < 1$ such that whenever $|f(z_0)|e^{-\varphi(z_0)} \geq c$ for some $z_0 \in \mathbb{C}$, then for every $z \in D_\rho(z_0, \beta)$ we have $|f(z)|e^{-\varphi(z)} \geq \frac{c}{2}e^{-\alpha\pi^2}$.*

*(ii) If $0 < \beta < \beta_0$, then for $z \in \mathbb{C}$ with $d_\rho(z, z_0) \leq \beta$ we have $\big||f(z)|e^{-\varphi(z)} - |f(z_0)|e^{-\varphi(z_0)}\big| \lesssim d_\rho(z, z_0)\max_{D_\rho(z_0,\beta)}|f|e^{-\varphi}$.*

*(iii)* $|f(z)|e^{-\varphi(z)} \lesssim \dfrac{1}{|z|^2}\displaystyle\int_{D_\rho(z,\beta)} |f(w)|e^{-\varphi(w)}dm(w).$

From Lemma 3.2.1 we can deduce the following immediate corollaries (proofs can be found for instance in [4, 8, 23]).

**Corollaire 3.2.2.** *If $\Lambda$ is sampling for $\mathcal{F}^p_\varphi$, $p = 2, \infty$ then there exists a $d_\rho$-separated sequence $\widetilde{\Lambda} \subset \Lambda$ which is sampling for $\mathcal{F}^p_\varphi$.*

**Corollaire 3.2.3.** *Every set of interpolation for $\mathcal{F}^p_\varphi$, $p = 2, \infty$ is $d_\rho$-separated.*

**Corollaire 3.2.4.** *Suppose $\Lambda$ is sampling for $\mathcal{F}^\infty_\varphi$. If $\delta > 0$ is sufficiently small (depending on $L_\varphi(\Lambda)$), then every $\widetilde{\Lambda} \subset \mathbb{C}$ with*

$$d_\rho(\Lambda, \widetilde{\Lambda}) = \sup_{z\in\Lambda} \inf_{w\in\widetilde{\Lambda}} d_\rho(z, w) < \delta,$$

*is sampling with sampling constant majorized by an expression depending only on $\delta$.*

**Corollaire 3.2.5.** *Let $\Lambda = \{\lambda_k\}_k$ be a separated sampling sequence with*

$$\frac{1}{C}\|f\|_{2,\varphi} \le \|f|\Lambda\|_{2,\varphi,\Lambda},$$

*for some $C \ge 1$. Then there is a $\delta > 0$ and $C' > 0$ such that for every $\widetilde{\Lambda} = \{\tilde{\lambda}_k\}_k$ with*

$$d_\rho(\lambda_k, \tilde{\lambda}_k) \le \delta,$$

*we have*

$$\frac{1}{C'}\|f\|_{2,\varphi} \lesssim \|f|\widetilde{\Lambda}\|_{2,\varphi,\widetilde{\Lambda}}.$$

The key sequence for our considerations is the following reference sequence introduced by Borichev–Lyubarskii in [5]:

$$\Gamma = \Gamma_\alpha = \{e^{\frac{n+1}{2\alpha}} e^{i\theta_n}\}_{n\ge 0}, \qquad \theta_n \in \mathbb{R},$$

and which turns actually out to be a complete interpolating sequence (the corresponding sequence of reproducing kernels is a Riesz basis). Let us recall the estimates they obtain.

**Lemme 3.2.6.** *[5]. Let $\varphi(r) = \alpha(\log^+ r)^2$ and $\Gamma = \Gamma_\alpha$. The product*

$$G(z) = \prod_{\gamma\in\Gamma}\left(1 - \frac{z}{\gamma}\right)$$

*converges uniformly on compact sets in $\mathbb{C}$ and satisfies*

$$|G(z)| \asymp e^{\varphi(z)} \frac{\operatorname{dist}(z, \Gamma)}{1 + |z|^{3/2}}, \qquad z \in \mathbb{C},$$

*where the constants are independent of the choice of $\theta_n$. Here* $\operatorname{dist}(z, \Gamma)$ *denotes the euclidean distance between $z$ and $\Gamma$. Also*

$$|G'(\gamma)| \asymp \frac{e^{\varphi(\gamma)}}{1 + |\gamma|^{3/2}}, \quad \gamma \in \Gamma.$$

**Lemme 3.2.7.** *Let $\Lambda \subset \mathbb{C}$. Then*

$$\|f\|_{2,\varphi,\Lambda} \leq c(\Lambda)\|f\|_{2,\varphi}, \qquad f \in \mathcal{F}^2_\varphi, \tag{3.3}$$

*if and only if $\Lambda$ is a finite union of $d_\rho$–separated subsets.*

*Proof.* If $\Lambda$ is a finite union of $d_\rho$–separated subsets, then (3.3) follows from Lemma 3.2.1 (iii). In the opposite direction, let $\Gamma$ and $G$ be as in Lemma 3.2.6, and put

$$G_\gamma(z) = \frac{G(z)}{(z-\gamma)G'(\gamma)}\frac{e^{\varphi(\gamma)}}{\gamma}.$$

Then

$$|G_\gamma(z)| \asymp \frac{\gamma^{1/2}}{1+|z|^{3/2}}e^{\varphi(z)}\frac{\mathrm{dist}(z,\Gamma)}{|z-\gamma|}.$$

The function $G_\gamma$ belongs to $\mathcal{F}^2_\varphi$ and $\sup_\gamma \|G_\gamma\|_{2,\varphi} \lesssim 1$ (see [5, Proof of Theorem 2.5]). Hence, by Lemma (3.2.6), we have

$$\begin{aligned} 1 \gtrsim \|G_\gamma\|_{2,\varphi} &\gtrsim \|G_\gamma\|_{2,\varphi,\Lambda} \geq \sum_{\lambda\in\Lambda\cap D_\rho(\gamma,\beta)} |G_\gamma(\lambda)|^2 e^{-2\varphi(\lambda)}(1+|\lambda|^2) \\ &\geq c\mathrm{Card}(\Lambda\cap D_\rho(\gamma,\beta)) \end{aligned}$$

(note that $\mathrm{dist}(\gamma,\Lambda) \simeq |\lambda-\gamma|$ since $\Gamma$ is separated and we can choose $\beta$ such that $D_\rho(\gamma,\beta)$ stays sufficiently far from $\Gamma \setminus \{\gamma\}$). So

$$\sup_{\gamma\in\Gamma} \mathrm{Card}(\Lambda\cap D_\rho(\gamma,\beta)) < \infty$$

for arbitrary real numbers $\theta_n$ (we can pick $\beta$ such that that $D_\rho(\gamma,\beta)$ covers $\mathbb{C}$ when $\gamma$ runs through $\Gamma$ and $\theta_n$ through $\mathbb{R}$). Hence $\Lambda$ is a finite union of $d_\rho$–separated sequence. □

**Lemme 3.2.8.** *If $\Lambda$ is $d_\rho$-separated and sampling for $\mathcal{F}^\infty_{(1+\varepsilon)\varphi}$ for some $\varepsilon > 0$, then $\Lambda$ is sampling for $\mathcal{F}^2_\varphi$.*

*Proof.* We are going to use the same Beurling duality argument as in [14] (see also [3,23]). Let

$$\mathcal{F}^{\infty,0}_\varphi = \{f \in \mathcal{F}^\infty_\varphi : \lim_{|z|\to\infty} |f(z)|e^{-(1+\varepsilon)\varphi(z)} = 0\}.$$

By the sampling property, the operator family $T_z : \{f(\lambda)\}_{\lambda\in\Lambda} \longmapsto e^{-(1+\varepsilon)\varphi(z)}f(z)$, $z \in \mathbb{C}$, is uniformly bounded from $\{f|\Lambda : f \in \mathcal{F}^{\infty,0}_\varphi\} \subset c_0$ to $\mathbb{C}$. Hence, by duality, there exists a family $(g(z,\lambda))_{\lambda\in\Lambda}$ such that

$$e^{-(1+\varepsilon)\varphi(z)}f(z) = \sum_{\lambda\in\Lambda} e^{-(1+\varepsilon)\varphi(\lambda)}f(\lambda)g(z,\lambda), \qquad f \in \mathcal{F}^{\infty,0}_{(1+\varepsilon)\varphi}.$$

and $\sup_z \sum_\lambda |g(z,\lambda)| < \infty$. Let now $\Gamma = \Gamma_{\varepsilon\alpha} = \{e^{\frac{n+1}{2\alpha\varepsilon}} e^{i\theta_n}\}$, and consider the function $G \in \mathcal{F}^\infty_{\varepsilon\varphi}$ of Lemma 3.2.6 vanishing on $\Gamma$. When $e^{\frac{n+1}{2\alpha\varepsilon}} \le |z| \le e^{\frac{n+2}{2\alpha\varepsilon}}$, let $\gamma_z = e^{\frac{n+2}{2\alpha\varepsilon}} e^{i\theta_n}$, so that

$$e^{-1/(2\alpha\varepsilon)} \le |z/\gamma_z| \le 1.$$

Set

$$P_z(w) = \frac{G(w)}{(w-\gamma_z)G'(\gamma_z)} \frac{w^2}{z^2}.$$

For $w \in \mathbb{C}$ we have

$$|P_z(w)| \asymp e^{\varepsilon(\varphi(w)-\varphi(\gamma_z))} \frac{|w|^{1/2}}{|z|^{1/2}} \frac{\operatorname{dist}(w,\Gamma)}{|w-\gamma_z|} \lesssim e^{\varepsilon(\varphi(w)-\varphi(z))} \frac{|w|^{1/2}}{|z|^{1/2}} \frac{\operatorname{dist}(w,\Gamma)}{|w-\gamma_z|}. \tag{3.4}$$

Given $f \in \mathcal{F}^2_\varphi$, by Lemma 3.2.1 (iii) and (3.4) we have $w \mapsto f(w)P_z(w) \in \mathcal{F}^{\infty,0}_{(1+\varepsilon)\varphi}$ and hence

$$e^{-(1+\varepsilon)\varphi(z)} f(z)P_z(z) = \sum_{\lambda\in\Lambda} e^{-(1+\varepsilon)\varphi(\lambda)} f(\lambda)P_z(\lambda)g(z,\lambda).$$

Since $|P_z(z)| \asymp 1$, again by (3.4) we obtain that

$$|f(z)|e^{-\varphi(z)} \lesssim \sum_{\lambda\in\Lambda} |f(\lambda)|e^{-\varphi(\lambda)} \frac{|\lambda|^{1/2}}{|z|^{1/2}} \frac{\operatorname{dist}(\lambda,\Gamma)}{|\lambda-\gamma_z|} |g(z,\lambda)|.$$

Since $\sum_\lambda |g(z,\lambda)| < \infty$, Hölder's inequality and (3.1) give us that

$$\begin{aligned}
|f(z)|^2 e^{-2\varphi(z)} &\lesssim \sum_{\lambda\in\Lambda} |f(\lambda)|^2 e^{-2\varphi(\lambda)} \frac{|\lambda|}{|z|} \frac{\operatorname{dist}(\lambda,\Gamma)^2}{|\lambda-\gamma_z|^2} \sum_{\lambda\in\Lambda} |g(z,\lambda)|^2 \\
&\lesssim \sum_{\lambda\in\Lambda} \frac{|f(\lambda)|^2}{\mathbf{k}_\lambda(\lambda)} \frac{1}{|\lambda||z|} \frac{\operatorname{dist}(\lambda,\Gamma)^2}{|\lambda-\gamma_z|^2} \Big(\sum_{\lambda\in\Lambda} |g(z,\lambda)|\Big)^2 \\
&\lesssim \sum_{\lambda\in\Lambda} \frac{|f(\lambda)|^2}{\mathbf{k}_\lambda(\lambda)} \frac{1}{|\lambda||z|} \frac{\operatorname{dist}(\lambda,\Gamma)^2}{|\lambda-\gamma_z|^2}.
\end{aligned} \tag{3.5}$$

It remains to verify that

$$I(\lambda) = \frac{1}{|\lambda|} \int_{\mathbb{C}} \frac{\operatorname{dist}(\lambda,\Gamma)^2}{|\lambda-\gamma_z|^2|z|} dm(z) < \infty \tag{3.6}$$

uniformly in $\lambda$, since by (3.5), we then obtain the sampling inequality

$$\|f\|^2_{2,\varphi} \lesssim \sup_\lambda I(\lambda) \|f\|^2_{2,\varphi,\Lambda}.$$

We will now show (3.6). Since $d(\lambda,\Gamma) \le |\lambda-\gamma_z|$ and $|\gamma_z| \asymp |z|$, we have

$$\frac{1}{|\lambda|} \int_{z:\,|\gamma_z|<2|\lambda|} \frac{\operatorname{dist}(\lambda,\Gamma)^2}{|\lambda-\gamma_z|^2|z|} dm(z) \lesssim \frac{1}{|\lambda|} \int_{|z|\lesssim|\lambda|} \frac{1}{|z|} dm(z) \lesssim 1.$$

If $|\gamma_z| \geq 2|\lambda|$, then $|\lambda - \gamma_z| \geq |\gamma_z|/2 \asymp |z|$ and $\operatorname{dist}(\lambda, \Gamma) \lesssim |\lambda|$, so that

$$\int_{z:\, 2|\lambda| \leq |\gamma_z|} \frac{1}{|\lambda||z|} \frac{\operatorname{dist}(\lambda, \Gamma)^2}{|\lambda - \gamma_z|^2} dm(z) \lesssim |\lambda| \int_{|\lambda| \lesssim |z|} \frac{1}{|z|^3} dm(z) \lesssim 1.$$

Hence (3.6) is established and the proof is complete. □

In the remaining part of this section we make two more elementary observations for sampling and interpolation in $\mathcal{F}^\infty_\varphi$.

**Proposition 3.2.9.** *If $\Lambda$ is sampling for $\mathcal{F}^\infty_\varphi$ and $\Lambda \setminus \{\lambda\}$ is a zero set for $\mathcal{F}^\infty_\varphi$, then $\Lambda \setminus \{\lambda\}$ is interpolating.*

*Proof.* By assumption there exists a function $f \in \mathcal{F}^\infty_\varphi$ vanishing on $\Lambda \setminus \{\lambda\}$, and $f(\lambda) = 1$. For $\mu \in \Lambda \setminus \{\lambda\}$ define the entire function

$$g_\mu(z) = \begin{cases} f(z)\frac{z-\lambda}{z-\mu} & \text{if } z \neq \mu \\ f'(\mu)(\mu - \lambda) & \text{if } z = \mu, \end{cases}$$

which is in $\mathcal{F}^\infty_\varphi$. Clearly $g_\mu$ vanishes on $\{\lambda\} \cup \Lambda \setminus \{\mu\}$.

Pick now a finite sequence $(v_\mu)_{\mu \neq \lambda}$ and set

$$f_v(z) = \sum_{\mu \neq \lambda} v_\mu \frac{g_\mu(z)}{g_\mu(\mu)}.$$

By construction $f_v \in \mathcal{F}^\infty_\varphi$ as a finite sum of functions in $\mathcal{F}^\infty_\varphi$, and $f_v$ interpolates $v_\mu$ in $\mu \neq \lambda$. Let us estimate the norm of $f_v$. Observe that $f_v(\lambda) = 0$ since $g_\mu(\lambda) = 0$ for every $\mu \neq \lambda$. Using the fact that $\Lambda$ is sampling we have

$$\|f_v\|_{\varphi,\infty} \asymp \sup_{\mu \in \Lambda} |f_v(\mu)| e^{-\varphi(\mu)} = \sup_{\mu \in \Lambda \setminus \{\lambda\}} |f_v(\mu)| e^{-\varphi(\mu)} = \sup_{\mu \in \Lambda \setminus \{\lambda\}} |v_\mu| e^{-\varphi(\mu)} = \|v\|_{\varphi,\infty,\Lambda \setminus \{\lambda\}}.$$

Hence we can define a bounded interpolation operator $v \longmapsto f_v$ and $\Lambda \setminus \{\lambda\}$ is an interpolating sequence. □

Here is a dual result:

**Proposition 3.2.10.** *Suppose $\Lambda$ is an interpolating sequence for $\mathcal{F}^\infty_\varphi$ and $\Lambda \cup \{\lambda\}$ is not interpolating for a $\lambda \notin \Lambda$. Then $\Lambda \cup \{\lambda\}$ is sampling for $\mathcal{F}^\infty_\varphi$.*

*Proof.* Suppose first that $\Lambda$ is an interpolating and uniqueness sequence (in view of Theorem 3.1.5 such sequences do exist). Then $R : \mathcal{F}^\infty_\varphi \longrightarrow \ell^\infty_{\varphi,\Lambda}$, $f \longmapsto f|\Lambda$ is onto and injective. Thus $R$ is an isomorphism and $\Lambda$ is sampling (and interpolating). Clearly the bigger sequence $\Lambda \cup \{\lambda\}$ is also sampling for $\mathcal{F}^\infty_\varphi$.

Suppose now that $\Lambda$ is an interpolating sequence which is not a uniqueness set. Then there exists $h$ vanishing on $\Lambda$ and such that $h(\lambda) = 1$ which implies that $\Lambda \cup \{\lambda\}$ is interpolating. □

The reader might have observed that we have proved a slightly stronger result: if $\Lambda$ is interpolating but $\Lambda \cup \{\lambda\}$ is not, then $\Lambda$ is a uniqueness (and thus sampling) sequence.

**Proposition 3.2.11.** *Every half-line starting from the origin is sampling for $\mathcal{F}_\varphi^\infty$.*

*Proof.* Pick $f \in \mathcal{F}_{\varphi,\infty}^\infty$ with $\|f\|_\varphi = 1$. Define

$$F(z) = f(z)e^{-\alpha(\operatorname{Log} z)^2},$$

cutting the plane at the positive real axis. Then $F$ is an analytic function in $\mathbb{C} \setminus \mathbb{R}_*^+$. Moreover,

$$|F(z)| = |f(z)|e^{-\alpha(\log^2|z|-(\arg z)^2)} \asymp |f(z)|e^{-\alpha(\log|z|)^2}. \tag{3.7}$$

Hence $F \in H^\infty(\mathbb{C} \setminus \mathbb{R}_*^+)$ implying that

$$\sup_{z\in\mathbb{C}} |f(z)|e^{-\alpha(\log_+|z|)^2} \asymp \sup_{z\in\mathbb{C}\setminus\mathbb{R}_*^+} |F(z)| = \sup_{z\in\mathbb{R}_+} |F(z)| = \|f\|_{\varphi,\infty,\mathbb{R}_+},$$

which proves the claim. □

### 3.2.2 De Branges spaces

In order to investigate the Hilbertian counterpart of the above result we identify the Fock space with a de Branges space. Let $G$ be the generating function associated with the Borichev–Lyubarskii sequence $\Gamma$ (with $\theta_n = -\pi/2$). Recall that the de Branges space associated with $G$ is given by

$$\subset H(G) := \{f \text{ entire} : f/G \in H^2(\mathbb{C}^+) \text{ and } f^*/G \in H^2(\mathbb{C}^+)\},$$

where $f^*(z) = \overline{f(\bar{z})}$, and $H(G)$ is normed by

$$\|f\|_{H(G)}^2 := \int_{\mathbb{R}} \left|\frac{f(x)}{G(x)}\right|^2 dx, \quad f \in H(G).$$

We have the following result

**Proposition 3.2.12.** *Let $G$ be the generating function associated with the Borichev–Lyubarskii sequence $\Gamma$ (with $\theta_n = -\pi/2$). Then the space $\mathcal{F}_\varphi^2$ is a de Branges space:*

$$\mathcal{F}_\varphi^2 = H(G).$$

*Proof.* Let $G$ be the canonical product with zero set $\Gamma = \{-ie^{\frac{n+1}{2\alpha}}\}_{n\geq 0}$. We know that the normalized reproducing kernels $\{\mathbb{k}_\gamma\}_{\gamma\in\Gamma} = \{\mathbf{k}_\gamma / \| \mathbf{k}_\gamma \|_{\varphi,2}\}_{\gamma\in\Gamma}$ form a Riesz basis in $\mathcal{F}^2_\varphi$. Then the biorthogonal family

$$\frac{\| \mathbf{k}_\gamma \|_{\varphi,2}}{G'(\gamma)} \cdot \frac{G(z)}{z-\gamma}, \qquad \gamma \in \Gamma,$$

is a Riesz basis in $\mathcal{F}^2_\varphi$. By the formula (3.1) and the estimate for $|G'(\gamma)|$ in Lemma 3.2.6 we conclude that the above biorthogonal system is of the form $a_\gamma |\gamma|^{1/2} \cdot G(z)/(z-\gamma)$, where $|a_\gamma| \asymp 1$. Hence, any function in the space $\mathcal{F}^2_\varphi$ can be written as

$$F(z) = \sum_{\gamma\in\Gamma} c_\gamma |\gamma|^{1/2} \cdot \frac{G(z)}{z-\gamma} \ , \tag{3.8}$$

where $\{c_\gamma\} \in \ell^2$ and $\|F\|_{\varphi,2} \asymp \|\{c_\gamma\}\|_{\ell^2}$. Writing for simplicity $\gamma_n = -iy_n$, we have

$$\frac{F(z)}{G(z)} = \sum_n \frac{c_n y_n^{1/2}}{z+iy_n} \ ,$$

and the series converges in the Hardy space $H^2 = H^2(\mathbb{C}^+)$, since $\gamma$ is an interpolating sequence. Analogously, if we put $\Theta = G^*/G$, we get

$$\frac{F^*(z)}{G(z)} = \sum_n \bar{c}_n y_n^{1/2} \frac{\Theta(z)}{z-iy_n},$$

again the series converges in $H^2$, since $\Theta$ is an interpolating Blaschke product (with zeros $iy_n$). We conclude that $F/G$ and $F^*/G$ are in $H^2$. Conversely, any function in $H(G)$ can be written as a series of the form (3.8), since the functions $G(z)/(z-\gamma)$ form a Riesz basis in $H(G)$ whenever the zero set of $G$ is an interpolating sequence. □

It is also possible to have the comparison with the integral over the positive or negative rays, instead of the real line. In order to see this note that in the case when $c_n \in \mathbb{R}$ for the function

$$f(z) = \frac{F(z)}{G(z)} = \sum_n \frac{c_n y_n^{1/2}}{z+iy_n},$$

we have $f(-t) = -\overline{f(t)}$ and in this case there is just the equality of the integrals over the positive and the negative rays.

As a consequence we get the analogous result to Proposition 3.2.11 in the case $p = 2$.

**Corollaire 3.2.13.** *Let $G$ be the generating function associated with the Borichev–Lyubarskii sequence $\Gamma$ (with $\theta_n = -\pi/2$). Then the measure $dx/|G(x)|^2$ is sampling on $\mathbb{R}_+$, $\mathbb{R}_-$ or $\mathbb{R}$ for $\mathcal{F}^2_\varphi$: for every $f \in \mathcal{F}^2_\varphi$,*

$$\|f\|^2_{\varphi,2} \asymp \int_{\mathbb{R}_+} \left|\frac{f(x)}{G(x)}\right|^2 dx \asymp \int_{\mathbb{R}_-} \left|\frac{f(x)}{G(x)}\right|^2 dx \asymp \int_{\mathbb{R}} \left|\frac{f(x)}{G(x)}\right|^2 dx.$$

This proves the conjecture appearing in equation (4.3.1) in [16].

### 3.2.3 Density results

**Lemme 3.2.14.** *If $\Lambda$ is $d_\rho$-separated then*

$$D^-(\Lambda) \le D^+(\Lambda) < \infty.$$

*Proof.* As already mentioned in the beginning of Section 3.2, when $\Lambda$ is $d_\rho$-separated, then there exists $c$ such that the euclidean disks $D(\lambda, c|\lambda|)$, $\lambda \in \Lambda$, are disjoint. A standard argument, based for instance on the consideration of the euclidean area of $\mathcal{A}(x, Rx)$ and that of the disks $D(\lambda, c|\lambda|)$, $\lambda \in \Lambda \cap \mathcal{A}(x, Rx)$, shows that this implies in particular that for fixed $\eta > 0$ every annulus $\mathcal{A}(x, \eta x)$ contains a uniformly bounded number of points of $\Lambda$ (this number depends on $\eta$):

$$\mathrm{Card}(\Lambda \cap \mathcal{A}(x, \eta x)) \le M, \quad x > 0.$$

Suppose now that $R > R_0$ and $r > r_0$ with $R_0, r_0$ big enough. Let $N$ be the least integer such that $r\eta^N \ge rR$ so that $N \simeq \log R/\log\eta$. Then

$$\mathrm{Card}(\Lambda \cap \mathcal{A}(r, Rr)) = \sum_{\lambda \in \Lambda \cap \mathcal{A}(r,Rr)} 1 \le \sum_{n=1}^{N} \mathrm{Card}(\Lambda \cap \mathcal{A}(r\eta^{n-1}, r\eta^n)) \lesssim \frac{M}{\log\eta} \log R.$$

□

**Proposition 3.2.15.** *If $\Lambda$ is a zero sequence for $\mathcal{F}_\varphi^\infty$ then*

$$\liminf_{R\to+\infty} \frac{\mathrm{Card}(\Lambda \cap D(0,R))}{\log R} \le 2\alpha. \tag{3.9}$$

*Proof.* Suppose there is a function $g$ that vanishes on $\Lambda$ with

$$\liminf_{R\to+\infty} \frac{\mathrm{Card}(\Lambda \cap D(0,R))}{\log R} > 2\alpha. \tag{3.10}$$

Assuming $g(0) \neq 0$ (otherwise divide by a suitable power of $z$ which does not change the other zeros of $g$ and gives a function still in $\mathcal{F}_\varphi^\infty$), Jensen's formula yields for every $R > 0$,

$$\sum_{\lambda\in\Lambda:|\lambda|<R} \log\frac{R}{|\lambda|} = \frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\theta})|\,d\theta - \log|g(0)| \le \alpha(\log R)^2 + C.$$

Denote now by $n_g(R)$ the number of zeros of $g$ in $R\mathbb{D}$. Then

$$\sum_{\lambda\in\Lambda:|\lambda|<R} \log\frac{R}{|\lambda|} = \int_0^R \frac{n_g(t)}{t}\,dt.$$

From (3.10) we deduce that for $\varepsilon > 0$ small enough there exists $R_0 > 0$ such that for every $R \geq R_0$,

$$n_g(R) = \mathrm{Card}(\Lambda \cap R\mathbb{D}) \geq 2\alpha(1+\varepsilon)\log R.$$

Then for every $R \geq R_0$,

$$\sum_{\lambda\in\Lambda:|\lambda|<R} \log\frac{R}{|\lambda|} \geq \int_{R_0}^{R} \frac{n_g(t)}{t}dt \geq 2\alpha(1+\varepsilon)\int_{R_0}^{R}\frac{\log t}{t}dt \geq 2\alpha(1+\varepsilon)\left(\frac{(\log R)^2}{2} - \frac{(\log R_0)^2}{2}\right).$$

It follows that

$$\alpha(1+\varepsilon)\Big((\log R)^2 - (\log R_0)^2\Big) \leq \alpha(\log R)^2 + C$$

which is impossible when $R$ is big. □

We may deduce the following corollary.

**Corollaire 3.2.16.** *If $\Lambda$ satsifies* (3.9) *then*

$$D^-(\Lambda) \leq 2\alpha.$$

*Proof.* By contraposition, suppose that $D^-(\Lambda) > 2\alpha$. Then there are $R_0$ and $r_0$ such that for every $R > R_0$ and $r > r_0$ we have

$$\frac{\mathrm{Card}(\Lambda \cap \mathcal{A}(r, Rr))}{\log R} \geq 2(1+\varepsilon)\alpha,$$

for a suitable fixed $\varepsilon$. Set $\eta = \max(R_0, r_0)$ and let $x = \eta^{N+\kappa} \in [\eta^N, \eta^{N+1})$ ($\kappa \in [0,1)$) be big, then

$$\begin{aligned}\frac{\mathrm{Card}(\Lambda \cap D(0,x))}{\log x} &\geq \frac{\sum_{k=1}^{N-1}\mathrm{Card}(\Lambda \cap \mathcal{A}(\eta^k, \eta^{k+1}))}{\log x}\\ &\geq \frac{2(1+\varepsilon)\alpha(N-1)\log\eta}{(N+\kappa)\log\eta}\\ &\longrightarrow 2(1+\varepsilon)\alpha, \quad N\to\infty,\end{aligned}$$

i.e. $\Lambda$ does not satisfy (3.9). □

The two preceding results together team up in:

**Corollaire 3.2.17.** *If $\Lambda$ is a zero sequence for $\mathcal{F}_\varphi^\infty$, then*

$$D^-(\Lambda) \leq 2\alpha.$$

## 3.3 Proof of the result on Riesz bases

"$\Longleftarrow$": We use Bari's Theorem [17, p.132]. As in the introduction, let $\mathbf{k}_\lambda$ be the reproducing kernel of $\mathcal{F}^2_\varphi$ and let $\mathbb{k}_\lambda = \mathbf{k}_\lambda / \| \mathbf{k}_\lambda \|_{\varphi,2}$ be the normalized kernel at $\lambda$. Let $F$ be an entire function with simple zeros at each $\lambda \in \Lambda$

$$F(z) := \prod_{n\geq 0}\Big(1-\frac{z}{\lambda_n}\Big), \qquad z \in \mathbb{C}$$

and set

$$g_\lambda(z) = \frac{F(z)}{F'(\lambda)(z-\lambda)}\| \mathbf{k}_\lambda \|_{\varphi,2}, \qquad z \in \mathbb{C}.$$

If the functions $g_\lambda$ are in $\mathcal{F}^2_\varphi$, then the family $\{g_\lambda\}_{\lambda\in\Lambda}$ is biorthogonal to $\mathcal{K}_\Lambda := \{\mathbb{k}_\lambda\}_{\lambda\in\Lambda}$. Hence to show that $\mathcal{K}_\Lambda$ is Riesz basis it suffices to prove

(i) $F/(\cdot - \lambda) \in \mathcal{F}^2_\varphi$ for $\lambda \in \Lambda$.

(ii) $\mathcal{K}_\Lambda$ is complete : $\bigvee\{\mathbb{k}_\lambda, \;\; \lambda \in \Lambda\} = \mathcal{F}^2_\varphi$

(iii) $\displaystyle\sum_{\lambda\in\Lambda} \big|\langle f, \mathbb{k}_\lambda\rangle\big|^2 \lesssim \|f\|^2_{\varphi,2}$

(iv) $\displaystyle\sum_{\lambda\in\Lambda} \big|\langle f, g_\lambda\rangle\big|^2 \lesssim \|f\|^2_{\varphi,2}$

To prove (i), let $|z| = e^t$ with $|\lambda_{n-1}| \leq |z| \leq |\lambda_n|$ and suppose that $d(z,\Lambda) = |z-\lambda_{n-1}|$. Let $m \in \mathbb{N}$ be such that

$$\frac{m}{2\alpha} - \frac{1}{4\alpha} \leq t < \frac{m}{2\alpha} + \frac{1}{4\alpha}.$$

Then $|m-n|$ is bounded uniformly in $|z|$, and $\log|\lambda_s| - t$ is bounded uniformly in $z$ and $s$ between $m$ and $n$. We use that for $d_\rho$-separated sequences the behavior of the function $F$ is essentially given by the first $n$ terms. We have

$$\begin{aligned}
\log|F(z)| &= \sum_{0\leq k\leq n-2} \log\frac{|z|}{|\lambda_k|} + \log|1-\frac{z}{\lambda_{n-1}}| + O(1)\\
&= \sum_{0\leq k\leq n-1}\Big(t-\frac{k+1}{2\alpha}\Big) + \log\operatorname{dist}(z,\Lambda) - t - \sum_{0\leq k\leq n-1}\delta_k + O(1)\\
&= \sum_{0\leq k\leq m-1}\Big(t-\frac{k+1}{2\alpha}\Big) + \log\operatorname{dist}(z,\Lambda) - t - \sum_{0\leq k\leq m-1}\delta_k + O(1)\\
&= mt - \frac{m(m+1)}{4\alpha} - \sum_{0\leq k\leq m-1}\delta_k + \log\operatorname{dist}(z,\Lambda) - t + O(1)\\
&= \alpha t^2 - \frac{3}{2}t + \log\operatorname{dist}(z,\Lambda) - \sum_{0\leq k\leq m-1}\delta_k + O(1), \qquad t\to\infty.
\end{aligned}$$

Next, if $m = lN + r$, $0 \le r < N$, then

$$\Big|\sum_{k=0}^{m-1} \delta_k\Big| \le \sum_{j=0}^{l-1}\Big|\sum_{i=0}^{N-1} \delta_{jN+i}\Big| + \Big|\sum_{i=0}^{r-1} \delta_{lN+i}\Big| \le \frac{lN}{4\alpha} + O(1) = 2\alpha\delta t + O(1).$$

Therefore, for some $\eta > 0$,

$$e^{\varphi(z)}\frac{\operatorname{dist}(z,\Lambda)}{(1+|z|)^{2-\eta}} \lesssim |F(z)| \lesssim e^{\varphi(z)}\frac{\operatorname{dist}(z,\Lambda)}{(1+|z|)^{1+\eta}}, \qquad z \in \mathbb{C}. \tag{3.11}$$

This proves (i).

Next we pass to property (iii). By assumption, $\Lambda$ is $d_\rho$-separated, and so by Lemma 3.2.1 we have

$$\sum_{\lambda\in\Lambda} \left|\langle f, \mathbb{k}_\lambda\rangle\right|^2 = \sum_{\lambda\in\Lambda} \frac{|f(\lambda)|^2}{\|\,\mathbf{k}_\lambda\,\|^2_{\varphi,2}} \lesssim \|f\|^2_{\varphi,2}.$$

Let us turn to (ii). By Lemma 3.2.6 we have

$$|G(iy)| \asymp \frac{e^{\varphi(y)}}{\sqrt{y}}, \qquad y > 1.$$

If $f \in \mathcal{F}^2_\varphi$ then by Lemma 3.2.1 (iii), $|f(z)| = o(e^{\varphi(z)}/(1+|z|))$ and so

$$|f(iy)/G(iy)| = o(y^{-1/2}), \qquad y \to +\infty.$$

Let $y = e^t$ and let $n$ be such that

$$\frac{n}{2\alpha} - \frac{1}{4\alpha} \le t < \frac{n}{2\alpha} + \frac{1}{4\alpha}.$$

Then, as before,

$$\frac{|F(iy)|}{|G(iy)|} \asymp \frac{\prod_{k=0}^{n-1}|y/\lambda_k|}{\prod_{k=0}^{n-1}|y/\gamma_k|} = \exp\left(-\sum_{k=0}^{n-1}\delta_k\right) \lesssim \sqrt{y}, \qquad y > 1.$$

Therefore

$$\frac{1}{\sqrt{y}} \lesssim \frac{|F(iy)|}{|G(iy)|}, \qquad y > 1,$$

and $F \notin \mathcal{F}^2_\varphi$. If $\mathcal{K}_\Lambda$ is not complete, then there exists $f \in \mathcal{F}^2_\varphi$ vanishing on $\Lambda$. So $f = FS$ for some entire function $S$. Since $|f(iy)|/|G(iy)| = o(1/\sqrt{y})$, $|S(iy)| = O(1)$. Since $f \in \mathcal{F}^2_\varphi$ and using (3.11), we get $S \in \mathcal{F}^\infty_{2\varphi}$. Since $i\mathbb{R}$ is sampling for $S \in \mathcal{F}^\infty_{2\varphi}$ by Proposition 3.2.11 the function $S$ is bounded on $\mathbb{C}$ and hence constant. Consequently, $f = cF$, and we reach a contradiction. Statement (ii) is proved.

It remains to show (iv). We use again the fact that

$$|F(\gamma_n)| \asymp \frac{\operatorname{dist}(\gamma_n, \Lambda)}{\gamma_n} \prod_{0\le k\le n-1} \frac{\gamma_n}{|\lambda_k|}$$

and that

$$|F'(\lambda_n)| \asymp \frac{1}{|\lambda_n|} \prod_{0\le k\le n-1} \left|\frac{\lambda_n}{\lambda_k}\right|.$$

Since the family $\{\mathbb{k}_\gamma\}_{\gamma\in\Gamma}$ is a Riesz basis [5], we can write

$$f = \sum_{m\ge 0} a_m \, \mathbb{k}_{\gamma_m}, \qquad (a_m)_m \in \ell^2,$$

and the sum in (iv) becomes

$$\sum_{n\ge 0} \Big| \sum_{m\ge 0} a_m \underbrace{\frac{F(\gamma_m)}{F'(\lambda_n)(\gamma_m - \lambda_n)} \cdot \frac{\|\mathbf{k}_{\lambda_n}\|_{\varphi,2}}{\|\mathbf{k}_{\gamma_m}\|_{\varphi,2}}}_{A_{n,m}} \Big|^2.$$

It remains to check that the matrix $[A_{n,m}]$ is bounded.

• We have $|\lambda_n| = \gamma_n e^{\delta_n} = e^{\frac{n+1}{2\alpha}+\delta_n}$ and

$$\begin{aligned} |A_{n,n}| &\asymp \frac{\gamma_n^n}{|\lambda_n^n|} \cdot \frac{\gamma_n e^{\alpha(\log\lambda_n)^2}}{\lambda_n e^{\alpha(\log\gamma_n)^2}} = e^{-(n+1)\delta_n + \alpha[(\log\gamma_n+\delta_n)^2 - (\log\gamma_n)^2]} \\ &= e^{2\alpha\delta_n(-\frac{n+1}{2\alpha}+\log\gamma_n)} e^{-\delta_n^2} \asymp 1. \end{aligned}$$

• If $m < n$, then $|\gamma_m - \lambda_n| \asymp |\lambda_n|$. Hence,

$$|A_{n,m}| \asymp \frac{\operatorname{dist}(\gamma_m, \Lambda)}{|\lambda_m|} \frac{\gamma_m^m}{|\lambda_n|^n} \cdot \left( \prod_{m\le k\le n-1} |\lambda_k| \right) \cdot \frac{\gamma_m e^{\alpha(\log\lambda_n)^2}}{|\lambda_n| e^{\alpha(\log\gamma_m)^2}} = \frac{\operatorname{dist}(\gamma_m, \Lambda)}{|\lambda_m|} e^{c(m,n)},$$

where

$$\begin{aligned} c(m,n) &= \frac{(m+1)^2}{2\alpha} - \frac{(n+1)^2}{2\alpha} - (n+1)\delta_n + \sum_{k=m}^{n-1} \Big( \frac{k+1}{2\alpha} + \delta_k \Big) \\ &\quad + \frac{(n+1)^2}{4\alpha} + (n+1)\delta_n - \frac{(m+1)^2}{4\alpha} \\ &= -\frac{n-m}{4\alpha} + \sum_{k=m}^{n-1} \delta_k + O(1). \end{aligned}$$

• If $m > n$, then $|\gamma_m - \lambda_n| \asymp |\gamma_m|$. Hence

$$|A_{n,m}| \asymp \frac{\operatorname{dist}(\gamma_m, \Lambda)}{|\lambda_m|} \frac{\gamma_m^m}{|\lambda_n|^n} \left( \prod_{n\le k\le m-1} \frac{1}{|\lambda_k|} \right) \cdot \frac{e^{\alpha(\log\lambda_n)^2}}{e^{\alpha(\log\gamma_m)^2}} = \frac{\operatorname{dist}(\gamma_m, \Lambda)}{|\lambda_m|} e^{d(m,n)},$$

where

$$\begin{aligned} d(m,n) &= \frac{m(m+1)}{2\alpha} - \frac{n(n+1)}{2\alpha} - n\delta_n - \sum_{k=n}^{m-1}\Big(\frac{k+1}{2\alpha} + \delta_k\Big) \\ &\quad + \frac{(n+1)^2}{4\alpha} + (n+1)\delta_n + \alpha\delta_n^2 - \frac{(m+1)^2}{4\alpha} \\ &= -\frac{m-n}{4\alpha} - \sum_{k=n}^{m-1}\delta_k + O(1). \end{aligned}$$

Hence, for $n \neq m$ we have

$$|A_{n,m}| \asymp \frac{\operatorname{dist}(\gamma_m, \Lambda)}{|\lambda_m|} \exp\Big(-\frac{|n-m|}{4\alpha} + \sum_{k=m}^{n-1}\delta_k + O(1)\Big), \tag{3.12}$$

where the sum is taken with the negative sign if $m > n$.

Thus, the matrix $[A_{n,m}]$ is bounded and the statement (iv) is proved.

"$\Longrightarrow$":

(a) By Corollary 3.2.3, $\Lambda$ is $d_\rho$-separated.

(b) Suppose that $\delta_n \notin \ell^\infty$. Then there exists an infinite subsequence of indices $\mathcal{N} = \{n_k\}$ such that for each $k$ there exists $m_k$ such that $d_\rho(\lambda_{n_k}, \gamma_{m_k}) \lesssim 1$, but $|n_k - m_k| \to \infty$ as $k \to \infty$. Note that $\mathcal{F}^2_\varphi$ is rotation invariant, and so $\{e^{i\theta}\lambda_n\}$ also is a complete interpolating sequence for any $\theta \in \mathbb{R}$. Thus, we may assume without loss of generality that

$$d_\rho(\lambda_{n_k}, \Gamma) \asymp 1,$$

that is, $|\lambda_{n_k} - \gamma| \geq c|\lambda_{n_k}|$, $\gamma \in \Gamma$ (with constants independent of $k$) and also that

$$d_\rho(\gamma_{m_k}, \Lambda) \asymp 1.$$

To simplify the notations we write $n$ and $m$ in place of $n_k$ and $m_k$, keeping in mind that $|m-n| \gg 1$. Let us write $|\lambda_n| = \gamma_m e^{\delta_{nm}}$ with $|\delta_{nm}| \lesssim 1$. Moreover, we may assume that $|\delta_{nm}| > 1$, $\delta_{nm}(m-n) < 0$. Otherwise, we may replace $m$ by $m \pm m_0$ for some sufficiently large $m_0$ that can be chosen to be independent of $k$.

Now let us consider

$$A_{n,m} = (g_{\lambda_n}, \Bbbk_{\gamma_m}) = \frac{F(\gamma_m)}{F'(\lambda_n)(\gamma_m - \lambda_n)} \cdot \frac{\|\mathbf{k}_{\lambda_n}\|_{\varphi,2}}{\|\mathbf{k}_{\gamma_m}\|_{\varphi,2}}.$$

Let us estimate $F(\gamma_m)$ and $F'(\lambda_n)$:

$$|F(\gamma_m)| \asymp \prod_{k=1}^{n}\Big|1 - \frac{\gamma_m}{\lambda_k}\Big| \cdot \prod_{k=n+1}^{\infty}\Big|1 - \frac{\gamma_m}{\lambda_k}\Big| \asymp \frac{\gamma_m^n}{|\lambda_1\lambda_2\ldots\lambda_n|}.$$

Here we use the facts that $\operatorname{dist}(\gamma_m, \Lambda) \le c|\gamma_m|$, $|\lambda_n| \asymp \gamma_m$ and that $\Lambda$ is $d_\rho$-separated. Analogously,

$$|F'(\lambda_n)| \asymp \frac{\lambda_n^{n-2}}{|\lambda_1\lambda_2\dots\lambda_{n-1}|}.$$

Thus, using the estimate (3.1) for the norm of the reproducing kernel as well as the estimates $\gamma_m \asymp |\lambda_n| \asymp |\gamma_m - \lambda_n|$, we get

$$|A_{n,m}| \asymp \frac{\gamma_m^n}{|\lambda_n|^n} e^{\alpha(\log^2|\lambda_n| - \log^2 \gamma_m)}.$$

Note that $\log \gamma_m = \frac{m+1}{2\alpha}$ and $\log|\lambda_n| = \frac{m+1}{2\alpha} + \delta_{nm}$. Then we have

$$\begin{aligned}\log|A_{n,m}| &= n\frac{m+1}{2\alpha} - n\Big(\frac{m+1}{2\alpha} + \delta_{nm}\Big) - \frac{(m+1)^2}{4\alpha} + \alpha\Big(\frac{m+1}{2\alpha} + \delta_{nm}\Big)^2 + O(1)\\ &= -\delta_{nm}(n-m) + O(1) > |n-m| + O(1).\end{aligned}$$

Thus, $|A_{n,m}| \gtrsim e^{|n-m|}$. Repeating this estimate for each $k$ (note that all asymptotic estimates $\asymp$ and $\gtrsim$ hold uniformly with respect to $k$) we conclude that $\lim_{k\to\infty} |A_{n_k,m_k}| \to \infty$. Since $A_{m,n} = (g_{\lambda_n}, \Bbbk_{\gamma_m})$, we conclude that $\|g_{\lambda_{n_k}}\|_{\varphi,2} \to \infty$, $k \to \infty$, and so $\{\Bbbk_{\lambda_n}\}$ is not uniformly minimal (in particular, it is not a Riesz basis).

(c) If for arbitrarily large $N$ we have

$$\sup_n \frac{1}{N}\Big|\sum_{k=n+1}^{n+N} \delta_k\Big| > \frac{1}{4\alpha},$$

then, replacing, if necessary, $\Lambda$ by $e^{i\theta}\Lambda$, and using (3.12) we obtain that the matrix $[A_{n,m}]$ in unbounded.

Now, suppose that for every $N \ge 1$,

$$\sup_n \frac{1}{N}\Big|\sum_{k=n+1}^{n+N} \delta_k\Big| = \frac{1}{4\alpha}.$$

Then for every $N$ there exists $M = M(N)$ such that

$$\begin{cases} \Big|\sum_{k=M+1}^{M+N} \delta_k\Big| \ge \frac{N}{4\alpha} - 1\\ \Big|\sum_{k=M+1}^{M+K} \delta_k\Big| \le \frac{K}{4\alpha}. \end{cases}$$

with $N/3 \le K \le 2N/3$. Then

$$\Big|\sum_{k=M+K+1}^{M+N} \delta_k\Big| \ge \frac{N-K}{4\alpha} - 1.$$

Again by (3.12) we have

$$|A_{M+K,M+N}| + |A_{M+N,M+K}| \geq \exp\Big(\Big|\sum_{k=M+K+1}^{M+N} \delta_k\Big| - \frac{1}{4\alpha}(N-K)\Big) \asymp 1, \qquad N/3 \leq K \leq 2N/3,$$

and the matrix $[A_{n,m}]$ in not bounded.

## 3.4 Proof of the density results

### 3.4.1 Sufficient conditions, $p = 2$

First we deduce the sufficient conditions of Theorems 3.1.3 and 3.1.4 from Theorem 3.1.1 when $p = 2$. These conditions follow immediately from the next theorem that gives a more precise information in the Hilbertian situation.

**Theorem 3.4.1.** *Let $\varphi(r) = \alpha(\log^+ r)^2$ and let $\Lambda$ be a $\rho$-separated sequence. Then*
(*i*) *if $D^+(\Lambda) < 2\alpha$, then $\Lambda$ is a subset of some complete interpolating sequence in $\mathcal{F}^2_\varphi$;*
(*ii*) *if $D^-(\Lambda) > 2\alpha$, then $\Lambda$ contains a complete interpolating sequence in $\mathcal{F}^2_\varphi$.*

For the sake of simplicity, we choose $\alpha = 1/2$. Recall that with our choice of $\alpha$ the set $\Gamma = \{\gamma_n\} = \{e^{n+1}\}_{n\in\mathbb{N}}$ becomes a complete interpolating sequence for the space.

*Proof of (i).* It follows from the condition $D^+(\Lambda) < 1$ that for sufficiently large $M > 0$, every annulus

$$A_m = \Big\{z : e^{Mm+\frac{1}{2}} < |z| < e^{M(m+1)+\frac{1}{2}}\Big\}, \qquad m \geq 0,$$

contains at most $M - 1$ points from $\Lambda$. Fix such $M$. Furthermore, there exists an $\eta > 0$ such that each $A_m$ contains an annulus $B_m$ of width $\eta$ which contains no points of $\Lambda$.

Our goal is to add some sequence $\Lambda'$ to $\Lambda$ so that the new sequence $\Lambda \cup \Lambda'$ could be written as $\gamma_n e^{\delta_n} e^{i\theta_n}$ and for some $N$ we would have

$$\sup_n \frac{1}{N}\Big|\sum_{k=n}^{n+N} \delta_k\Big| \leq \delta < \frac{1}{4\alpha} = \frac{1}{2}. \tag{3.13}$$

Let us denote the points from $\Lambda \cap A_m$ by $\lambda^m_1, \ldots, \lambda^m_{l_m}$ (we of course assume that $\lambda_l$ are ordered so that the modulus is nondecreasing), and let us associate with each of them some point from $\Gamma \cap A_m$. E.g., let us write

$$\lambda^m_l = e^{Mm+l} e^{\delta_{Mm+l}} e^{i\theta_{Mm+l}}, \qquad 1 \leq l \leq l_m.$$

In each annulus $A_m$ we still have at least one point from $\Gamma \cap A_m$ to which nothing is associated.

We now take a large number $N$ (the choice will be specified later) and consider the groups of the annuli $A_m$, namely put

$$\tilde{A}_k = \bigcup_{m=kN+1}^{kN+N} A_m, \qquad k \geq 0.$$

Now in the whole group of annuli $\tilde{A}_k$ there are at least $N$ free points of $\Gamma$ to which we need to assign some element of the sequence $\Lambda'$ that we want to construct. We will do this in such a way that for any $k$ we have

$$\left| \sum_{n=(kN+1)M+1}^{(kN+N+1)M} \delta_n \right| \leq CM, \tag{3.14}$$

for some absolute constant $C$ whence for sufficiently large $N$, (3.13) will be satisfied. Thus, from now on, $k$ will be fixed.

We use an idea from the paper [22] by Seip (page 173). The points of $\Lambda' \cap \tilde{A}_m$ will be chosen within the annuli $B_m$ (of the width $\eta$). Note that we can even put all missing points in one annulus $B_m$, if we want, and still have $\rho$-separation, but, of course, the separation constant will depend on $\eta$, $M$ and $N$ and may be rather small. Let us consider all possible sequences $\Lambda' \subset \cup_{m=kN+1}^{kN+N} B_m$ with separation constants uniformly bounded away from zero, and let us write the elements of $\Lambda \cup \Lambda'$ as $\gamma_n e^{\delta_n} e^{i\theta_n}$. Note that for any $m$ and $n = Mm+1, \ldots, Mm+l_m$ the values $\delta_n$ are already fixed. Moreover, since for these $n$ the corresponding $\lambda$-s are in the same annulus $A_m$ we have $|\delta_n| \leq M$, whence

$$-M^2 N \leq \sum_{m=kN+1}^{kN+N} \sum_{n=Mm+1}^{Mm+l_m} \delta_n \leq M^2 N.$$

Now assume that we chose all the points of $\Lambda'$ in the annulus $B_{kN+1}$ (the smallest of all $B_m$ in our group). Then for

$$kNM + jM + l_{kN+j} + 1 < n \leq kNM + (j+1)M, \qquad 2 \leq \geq N-1,$$

we have

$$\delta_n \leq -(j-1)M,$$

whence (using the fact that we have at least $N$ free indices in each $A_m$)

$$\sum_{j=2}^{N-1} \sum_{n=kNM+jM+l_{kN+j}+1}^{kNM+(j+1)M} \delta_n \leq -\sum_{j=2}^{N-1} (j-1)MN \leq -\frac{MN^2}{3},$$

when $N$ is sufficiently large. Thus, with this choice of $\Lambda'$ we have

$$\sum_{n=(kN+1)M+1}^{(kN+N+1)M} \delta_n \le -\frac{MN^2}{3} + O(M^2N) < 0,$$

if $N \gg M$.

Analogously, if we choose all the points of $\Lambda'$ in the annulus $B_{kN+N}$ (the largest of all $B_m$ in our group), we will have $\delta_n \ge (N-2-j)M$ for $kNM+jM+l_{kN+j}+1 < n \le kNM+(j+1)M$, $0 \le j \le N-3$, whence

$$\sum_{n=(kN+1)M+1}^{(kN+N+1)M} \delta_n \ge \frac{MN^2}{3} - O(M^2N) > 0.$$

Finally, note that if two choices of $\Lambda'$ coincide up to one point which is in some $B_m$ for one choice and which is in $B_{m+1}$ for the other choice, then the corresponding sums

$$\sum_{n=(kN+1)M+1}^{(kN+N+1)M} \delta_n$$

considered for these two choices of $\Lambda'$ will differ by at most $2M$. Since the two configurations of $\Lambda'$ described above may be obtained from the other by changing only one point and moving it to a neighboring annulus $B_n$, we conclude that there exists some intermediate choice of $\Lambda'$ with the property (3.14) (with $C = 2$). □

*Proof of (ii).* The idea is the same and so we may omit some details. Let $M, N$ be as above, but now we assume that each $A_m$ contains at least $M+1$ points for some fixed $M$. Let us assume that $N \gg M$ and choose $j_0 \in \mathbb{N}$ so that $3j_0M < N \le 3(j_0+1)M$.

For $j_0 \le j \le N - j_0$ and $kNM + jM + 1 \le n \le kNM + (j+1)M$ we choose in an arbitrary way $\lambda_n \in \Lambda \cap A_j$ and write them as

$$\gamma_n e^{\delta_n} e^{i\theta_n}.$$

Then $|\delta_n| \le M$ and

$$-(N-2)M^2 \le \sum_{j=j_0}^{N-j_0} \sum_{n=kNM+jM+1}^{kNM+(j+1)M} \delta_n \le (N-2)M^2.$$

Note that we did not assign any point from $\Lambda$ to $n$-s in the first and in the last interval, namely, for $kNM + 1 \le n \le kNM + j_0M$ and for $kNM + (N - j_0)M + 1 \le n \le kNM + NM$.

Recall that we still have $N$ free points of $\Lambda$ in each $A_m$. Now consider two choices of $\lambda_n$ for these values of $n$. For the first choice let us assign some points $\lambda_n \in \Lambda \cap A_j$ to

$kNM + jM + 1 \le n \le kNM + (j+1)M$ and $N - j_0 \le j \le N - 1$. However, for $kNM + 1 \le n \le kNM + j_0M$ let us choose $j_0M$ points $\lambda_n$ in $\cup_{j>2N/3} A_j$. This is possible, since we have at least $N/3 > j_0M$ free points from $\Lambda$ in $\cup_{j>2N/3} A_j$. Then, for $kNM + 1 \le n \le kNM + j_0M$, we have

$$\delta_n \le -\frac{MN}{3},$$

and hence,

$$\begin{aligned}\sum_{n=(kN+1)M+1}^{(kN+N+1)M} \delta_n &= \sum_{j=0}^{j_0-1} \sum_{n=kNM+jM+1}^{kNM+(j+1)M} \delta_n + \sum_{j=j_0}^{N-1} \sum_{n=kNM+jM+1}^{kNM+(j+1)M} \delta_n \\ &\le -\frac{j_0M^2N}{3} + O(M^2N) < 0.\end{aligned}$$

Analogously, choosing the points $\lambda_n \in \Lambda \cap A_j$ for $kNM + jM + 1 \le n \le kNM + (j+1)M$ and $0 \le j \le j_0 - 1$, and taking $\lambda_n$ in $\cup_{j<N/3} A_j$ for $kNM + (N - j_0)M + 1 \le n \le kNM + NM$, we see that the corresponding sum of $\delta_n$ is positive.

The proof is completed as in (i): each configuration $\Lambda_n$ may be obtained from the other by changing exactly on point at each step, and, moreover, these points can be chosen at the distance (with respect to the logarithm) at most $2M$. Thus, the corresponding sum will be at most $4M$ for some choice of $\{\lambda_n\} \subset \Lambda$. □

### 3.4.2 Necessary conditions for sampling/interpolation, $p = 2, \infty$, Theorems 3.1.2, 3.1.3 and 3.1.4

To obtain the necessary conditions for the sequence to be sampling/interpolating, we use the technique developed by Ramanathan and Steger [14, 18]. We follow the scheme of proof proposed in [14] and concentrate mainly on the places where the proofs differ.

**Lemme 3.4.2.** *Let $\varepsilon > 0$. Assume $\Lambda$ to be interpolating for $\mathcal{F}^p_{(1-\varepsilon)\varphi}$, $p = 2, \infty$, and $\mathcal{S}$ to be sampling for $\mathcal{F}^2_\varphi$ and $d_\rho$-separated. Then for small $\delta > 0$ we have for sufficiently big $R$,*

$$(1 - \delta^2)\mathrm{Card}(\Lambda \cap \mathcal{A}(x, Rx)) \le \mathrm{Card}(\mathcal{S} \cap \mathcal{A}(\delta x, Rx/\delta)).$$

*Proof.* Let $p = 2$. Since $\Lambda$ is interpolating for $\mathcal{F}^2_{(1-\varepsilon)\varphi}$, for every $\lambda \in \Lambda$ there exists $f_\lambda \in \mathcal{F}^2_{(1-\varepsilon)\varphi}$, such that $f_\lambda(\lambda) = 1$, $f_\lambda|\Lambda\setminus\{\lambda\} = 0$ and $\|f_\lambda\|_{(1-\varepsilon)\varphi,2} \lesssim |\lambda| e^{-(1-\varepsilon)\varphi(\lambda)}$. By (3.1),

$$|f_\lambda(z)| = |\langle f_\lambda, \mathbf{k}_z\rangle| \lesssim e^{(1-\varepsilon)(\varphi(z)-\varphi(\lambda))}|\lambda|/(1 + |z|). \tag{3.15}$$

Let $G$ be the function from Lemma 3.2.6 associated to $\Gamma := \Gamma_{\varepsilon\alpha} = \{e^{\frac{n+1}{2\varepsilon\alpha}} e^{i\theta_n} : n \ge 0\}$, and let $\gamma_\lambda \in \Gamma$ be a point such that $\mathrm{dist}(\lambda, \Gamma) = |\lambda - \gamma_\lambda|$. Then $|\lambda| \asymp \gamma_\lambda$. With an appropriate choice of $\theta_n$ we can assume that $d_\rho(\Lambda,\Gamma)>0$. Define

$$\kappa(z,\lambda) := \begin{cases} f_\lambda(z)\dfrac{G(z)}{z-\gamma_\lambda}\dfrac{\lambda-\gamma_\lambda}{G(\lambda)}\dfrac{z}{\lambda}\|\mathbf{k}_\lambda\|_{\varphi,2} & \text{if } z\in\mathbb{C}\setminus\{\lambda\},\\ G'(\lambda)\dfrac{\lambda-\gamma_\lambda}{G(\lambda)}\|\mathbf{k}_\lambda\|_{\varphi,2} & \text{if } z=\lambda. \end{cases}$$

By construction, $\kappa(\cdot,\lambda)\in\mathcal{F}^2_\varphi$, $\lambda\in\Lambda$, and the system $\{\kappa(\cdot,\lambda)\}_{\lambda\in\Lambda}$ is biorthogonal to $\{\mathbb{k}_\lambda\}_{\lambda\in\Lambda}$. Moreover $\|\kappa(\cdot,\lambda)\|_{\varphi,2}$ is uniformly bounded. To verify this, it suffices to estimate this norm on the Borichev–Lyubarskii sampling sequence $\Gamma_\alpha$ with $\operatorname{dist}(\gamma,\Gamma_{\varepsilon a})\asymp|\gamma|$, $\gamma\in\Gamma_a$, using Lemma 3.2.6 and (3.15):

$$\begin{aligned}
\|\kappa(\cdot,\lambda)\|^2_{\varphi,2} &\asymp \sum_{\gamma\in\Gamma_\alpha}|\kappa(\gamma,\lambda)|^2e^{-2\varphi(\gamma)}(1+|\gamma|^2)\\
&\asymp \sum_{\gamma\in\Gamma_\alpha}|f_\lambda(\gamma)|^2\frac{e^{2\varepsilon\varphi(\gamma)}\operatorname{dist}^2(\gamma,\Gamma_{\varepsilon\alpha})}{(1+|\gamma|^3)|\gamma-\gamma_\lambda|^2}\,\frac{|\lambda-\gamma_\lambda|^2(1+|\lambda|^3)}{e^{2\varepsilon\varphi(\lambda)}\operatorname{dist}^2(\lambda,\Gamma_{\varepsilon\alpha})}\times\\
&\qquad\times\frac{e^{2\varphi(\lambda)}}{1+|\lambda|^2}\left|\frac{\gamma}{\lambda}\right|^2e^{-2\varphi(\gamma)}(1+|\gamma|^2)\\
&\lesssim \sum_{\gamma\in\Gamma_{\varepsilon\alpha}}\frac{\operatorname{dist}^2(\gamma,\Gamma_\varepsilon)}{|\gamma-\gamma_\lambda|^2}\,\frac{1+|\lambda|}{1+|\gamma|}\\
&\lesssim \sum_{\gamma\in\Gamma_{\varepsilon\alpha},|\gamma|\le|\lambda|}\frac{1+|\gamma|}{1+|\lambda|}+\sum_{\gamma\in\Gamma_{\varepsilon\alpha},|\gamma|>|\lambda|}\frac{1+|\lambda|}{1+|\gamma|}.
\end{aligned}$$

Both sums are majorized by the sum of a geometric progression, and, hence, are uniformly bounded.

Let now $\{\widetilde{k}(\cdot,s)_{s\in\mathcal{S}}\}$ be the dual frame (see, for example, [6]) for $\{\mathbf{k}_s/\|\mathbf{k}_s\|_{\varphi,2}\}$ in $\mathcal{F}^2_\varphi$. Consider the following finite dimensional subspaces of $\mathcal{F}^2_\varphi$:

$$W_{\mathcal{S}}=\{\widetilde{k}(\cdot,s): s\in\mathcal{S}\cap\mathcal{A}(\delta x,Rx/\delta)\}\quad\text{and}\quad W_\Lambda=\{\kappa(\cdot,\lambda):\lambda\in\Lambda\cap\mathcal{A}(x,Rx)\}.$$

We define $P_{\mathcal{S}}$ and $P_\Lambda$ as the orthogonal projections of $\mathcal{F}^2_\varphi$ onto $W_{\mathcal{S}}$ and $W_\Lambda$, respectively. Consider the operator $T=P_\Lambda P_{\mathcal{S}}$ defined from $W_\Lambda$ to $W_\Lambda$. We have

$$\operatorname{tr}(T)=\sum_{\lambda\in\Lambda\cap\mathcal{A}(x,Rx)}\langle T\kappa(\cdot,\lambda),P_\Lambda\mathbb{k}_\lambda\rangle=\sum_{\lambda\in\Lambda\cap\mathcal{A}(x,Rx)}\langle T\kappa(\cdot,\lambda),\mathbb{k}_\lambda\rangle.$$

Hence

$$\begin{aligned}
&\operatorname{Card}(\Lambda\cap\mathcal{A}(x,Rx))[1-\sup_\lambda\|P_{\mathcal{S}}(\kappa(\cdot,\lambda))-\kappa(\cdot,\lambda)\|_{\varphi,2}]\\
&\qquad\le\operatorname{tr}(T)\le\operatorname{rank}P_{\mathcal{S}}\le\operatorname{Card}(\mathcal{S}\cap\mathcal{A}(\delta x,Rx/\delta)).
\end{aligned}\tag{3.16}$$

It remains to verify that $\|P_{\mathcal{S}}(\kappa(\cdot,\lambda)) - \kappa(\cdot,\lambda)\|_{\varphi,2}$ are small for sufficiently small $\delta$ independently of $\lambda$. By Lemma 3.2.6 we have

$$\frac{|\kappa(s,\lambda)|^2}{\|\,\mathbf{k}_s\,\|^2_{\varphi,2}} \lesssim \frac{|\lambda|}{1+|s|}\frac{d(s,\Gamma_{\varepsilon\alpha})^2}{|s-\gamma_\lambda|^2}. \tag{3.17}$$

Since $\mathcal{S}$ is sampling and $d_\rho$-separated, and since $\lambda \in \mathcal{A}(x,Rx)$, we have

$$\begin{aligned}\|P_{\mathcal{S}}(\kappa(\cdot,\lambda)) - \kappa(\cdot,\lambda)\|_{\varphi,2} &\lesssim \sum_{s\notin\mathcal{A}(\delta x, Rx/\delta)} \frac{|\kappa(s,\lambda)|^2}{\|\,\mathbf{k}_s\,\|^2_{\varphi,2}} \\ &\lesssim \sum_{s\notin\mathcal{A}(\delta x, Rx/\delta)} \frac{|\gamma_\lambda|}{1+|s|}\frac{d(s,\Gamma_{\varepsilon\alpha})^2}{|s-\gamma_\lambda|^2} \\ &\lesssim \sum_{|s|\le|\gamma_\lambda|,\, s\notin\mathcal{A}(\delta x, Rx/\delta)} \frac{|s|}{|\gamma_\lambda|} + \sum_{|s|\ge|\gamma_\lambda|,\, s\notin\mathcal{A}(\delta x, Rx/\delta)} \frac{|\gamma_\lambda|}{|s|}.\end{aligned}$$

For $R$ big enough, we can suppose $\gamma_\lambda \in \mathcal{A}(x,Rx)$. Since $\mathcal{S}$ is separated, each annulus $\mathcal{A}_k := \mathcal{A}(2^k, 2^{k+1})$ contains a uniformly bounded number of points of $\mathcal{S}$. Let $M$ be an upper bound of these numbers. Let $N, K$ be such that $2^N \le \delta x \le 2^{N+1}$ and $2^K \le Rx/\delta \le 2^{K+1}$. Then an estimate analogous to that above yields:

$$\begin{aligned}\sum_{|s|\le|\gamma_\lambda|,\, s\notin\mathcal{A}(\delta x, Rx/\delta)} \frac{|s|}{|\gamma_\lambda|} &\lesssim \frac{1}{|\gamma_\lambda|}\sum_{k=1}^{N}\sum_{s\in\mathcal{A}_k}|s| \lesssim \frac{M2^N}{|\gamma_\lambda|} \lesssim \delta, \\ \sum_{|s|\ge|\gamma_\lambda|,\, s\notin\mathcal{A}(\delta x, Rx/\delta)} \frac{|\gamma_\lambda|}{|s|} &\lesssim |\gamma_\lambda|\sum_{k\ge K}\sum_{s\in\mathcal{A}_k}\frac{1}{|s|} \lesssim \frac{M|\gamma_\lambda|}{2^K} \lesssim \delta,\end{aligned}$$

and we are done.

Now let $p=\infty$. Since $\Lambda$ is interpolating for $\mathcal{F}^\infty_{(1-\varepsilon)\varphi}$, then for every $\lambda\in\Lambda$, there exists $f_\lambda \in \mathcal{F}^\infty_{(1-\varepsilon)\varphi}$, such that $f_\lambda(\lambda)=1$, $f_\lambda|\Lambda\backslash\{\lambda\}=0$ and $\|f_\lambda\|_{(1-\varepsilon)\varphi,\infty} \lesssim e^{-(1-\varepsilon)\varphi(\lambda)}$. Note that $|f_\lambda(z)| \le e^{(1-\varepsilon)(\varphi(z)-\varphi(\lambda))}$. Set

$$\kappa(z,\lambda) = f_\lambda(z)\frac{G(z)}{z-\gamma_\lambda}\frac{\lambda-\gamma_\lambda}{G(\lambda)}\|\,\mathbf{k}_\lambda\,\|_{\varphi,2}, \qquad z\in\mathbb{C}.$$

Again by Lemma 3.2.6 we have (3.17) and applying the same argument as above, we get our result. □

• Let us first deduce from this lemma the necessary condition for sampling in Theorem 3.1.3. Suppose that $\Lambda$ is a sampling sequence for $\mathcal{F}^2_\varphi$. By Corollary 3.2.2 and Lemma 3.2.7, $\Lambda$ is a finite union of $d_\rho$-separated subsets and contains a $d_\rho$-separated subset $\Lambda^*$

which is also a sampling set for $\mathcal{F}^2_\varphi$. Let $\Gamma = \Gamma_{(1-\varepsilon)\alpha} = \{e^{\frac{n+1}{2(1-\varepsilon)\alpha}} e^{i\theta_n} : n \geq 0\}$ for some $\varepsilon > 0$. We have $D^-(\Gamma) = D^+(\Gamma) = 2(1-\varepsilon)\alpha$. By [5, Theorem 2.8], $\Gamma$ is an interpolating sequence for $\mathcal{F}^2_{(1-\varepsilon)\varphi}$ and the comparison Lemma 3.4.2 gives $D^-(\Lambda^*) \geq 2\alpha$.

• The necessary condition for sampling in Theorem 3.1.2 follows from Corollary 3.2.2, Lemma 3.2.8, and the necessary condition for sampling in Theorem 3.1.3.

• Next we consider the necessary condition for interpolation in Theorem 3.1.4. Consider $\Gamma = \Gamma_{(1+\varepsilon)\alpha} = \{e^{\frac{n+1}{2(1+\varepsilon)\alpha}} e^{i\theta_n} : n \geq 0\}$ for some $\varepsilon > 0$. We have $D^-(\Gamma) = D^+(\Gamma) = 2(1+\varepsilon)\alpha$ and by [5, Theorem 2.8], $\Gamma$ is a sampling sequence for $\mathcal{F}^2_{(1+\varepsilon)\varphi}$. If $\Lambda$ is an interpolating sequence for $\mathcal{F}^p_\varphi$, $p = 2, \infty$, then by Lemma 3.4.2, comparing the densities between interpolating and sampling sequences, we obtain $D^+(\Lambda) \leq 2\alpha$.

### 3.4.3 Sufficient conditions, $p = \infty$

**Lemme 3.4.3.** *If $\Lambda$ is $d_\rho$-separated and $D^+(\Lambda) < 2\alpha$, then $\Lambda$ is interpolating for $\mathcal{F}^\infty_\varphi$.*

*Proof.* We know that the sequence $\Lambda$ is interpolating for $\mathcal{F}^2_{(1-\varepsilon)\varphi}$ for small $\varepsilon > 0$. Without loss of generality, $0 \notin \Lambda$. As in the proof of Lemma 3.4.2, there exist functions $f_\lambda$ with $f_\lambda(\mu) = \delta_{\lambda\mu}$, $\lambda, \mu \in \Lambda$ such that

$$|f_\lambda(z)| \lesssim e^{(1-\varepsilon)(\varphi(z)-\varphi(\lambda))}|\lambda|/(1+|z|).$$

Now, let $\Gamma = \{e^{(n+1)/2\varepsilon\alpha}\}_{n\geq 1}$ be the standard Borichev–Lyubarskii sequence. Let $G$ be the associated generating function which satisfies

$$|G(z)| \asymp e^{\varepsilon\varphi(z)} \frac{\operatorname{dist}(z,\Gamma)}{1+|z|^{3/2}}, \quad z \in \mathbb{C}.$$

Pick a finite $v = (v_\lambda)_\lambda$, $\|v\|_{\infty,\varphi,\Lambda} \leq 1$ and set

$$f(z) = \sum_{\lambda\in\Lambda} f_\lambda(z) v_\lambda \frac{G(z)}{z-\gamma_\lambda} \frac{\lambda-\gamma_\lambda}{G(\lambda)} \frac{z^2}{\lambda^2},$$

where $\gamma_\lambda$ is the point of $\Gamma$ closest to $\lambda$ so that $|\lambda - \gamma_\lambda| = \operatorname{dist}(\lambda, \Gamma)$. Let $n \leq \log|z| < n+1$. Then

$$\begin{aligned}
|f(z)|e^{-\varphi(z)} &\lesssim \sum_{\lambda\in\Lambda} \frac{\operatorname{dist}(z,\Gamma)}{|z-\gamma_\lambda|} \frac{(1+|\lambda|^{1/2})}{|(1+|z|^{1/2})} \\
&= \sum_{1\leq k\leq n+1,\, \lambda\in\mathcal{A}(e^k,e^{k+1})} \frac{\sqrt{|\lambda|}}{\sqrt{|z|}} + \sum_{k\geq n+2,\, \lambda\in\mathcal{A}(e^k,e^{k+1})} \frac{|z|}{|\lambda|} \frac{\sqrt{|\lambda|}}{\sqrt{|z|}} \\
&\lesssim \sum_{k\geq 0} e^{-k/2}.
\end{aligned}$$

The sum is uniformly bounded in $z$ which proves the claim. □

**Lemme 3.4.4.** *If $\Lambda$ is $d_\rho$-separated and $D^-(\Lambda) > 2\alpha$, then $\Lambda$ is sampling for $\mathcal{F}^\infty_\varphi$.*

*Proof.* Suppose that $\Lambda$ is $d_\rho$-separated and

$$D^-(\Lambda) > 2\alpha.$$

If $\Lambda$ is not sampling, then there exist $f_n \in \mathcal{F}^\infty_\varphi$ such that $\|f_n\|_{\varphi,\infty} = 1$, $\|f_n\|_{\varphi,\infty,\Lambda} \to 0$ and a sequence $(w_n)_n$ with $|f_n(w_n)|e^{-\varphi(w_n)} \geq 1/2$. Two cases may occur: the sequence $(w_n)_n$ is either (A) bounded or (B) unbounded.

(A): In this case $(w_n)_n$ admits a subsequence converging to some $w \in \mathbb{C}$ and hence there is a subsequence of $(f_n)$ which converges uniformly on compact subsets to a non zero $f \in \mathcal{F}^\infty_\varphi$ since $f(w) \neq 0$. Since $f|\Lambda = 0$ this contradicts to Corollary 3.2.17.

(B): In view of Proposition 3.2.11 we can assume $w_n \in (0, +\infty)$. Passing to a subsequence, we can assume that $\lim_{n\to\infty} w_n = +\infty$.

We choose $N_n$ such that $2\alpha \log w_n \leq N_n < 1 + 2\alpha \log w_n$ and consider the functions

$$g_n(z) = f_n(zw_n)z^{-N_n}e^{-\varphi(w_n)}.$$

Then

$$|g_n(z)|e^{-\alpha \log^2 |z|} \leq e^{\varphi(zw_n)-\alpha \log^2 |z| - N_n \log |z| - \varphi(w_n)} = \exp[\log |z|(2\alpha \log w_n - N_n)],$$

and hence,

$$|g_n(z)| \leq 2e^{\alpha \log^2 |z|}, \qquad \frac{1}{2} \leq |z| < \infty.$$

Furthermore,

$$|g_n(1)| \geq \frac{1}{2}.$$

Passing to the limit along a subsequence, we obtain a function $g$ holomorphic in $\mathbb{C} \setminus \{0\}$ such that

$$|g(z)| \leq 2e^{\alpha \log^2 |z|}, \qquad \frac{1}{2} \leq |z| < \infty.$$

Its zero set $\widetilde{\Lambda}$ is $d_\rho$-separated and for some $B > 1$,

$$\frac{\mathrm{Card}(\widetilde{\Lambda} \cap \mathcal{A}(r, Br))}{\log B} \geq \eta > 2\alpha, \qquad r \geq 1. \tag{3.18}$$

Given $R > 1$, set

$$G_R(z) = \prod_{g(w)=0,\, 1<|w|<R} \Big(1 - \frac{z}{w}\Big).$$

Since $\widetilde{\Lambda}$ is $d_\rho$-separated and (3.18) is fulfilled we have

$$\min_{\overline{\mathbb{D}}} |G_R| \geq h > 0$$

and for every $A > 0$,

$$\liminf_{R\to\infty} \max_{\theta\in[0,2\pi]} \frac{R^A e^{\alpha \log^2 R}}{|G_R(Re^{i\theta})|} = 0.$$

Choose an integer $A$ such that

$$\frac{2^{-A} e^{\alpha \log^2 2}}{h} < \frac{1}{2},$$

and fix $R > 1$ such that

$$\max_{\theta\in[0,2\pi]} \frac{R^A e^{\alpha \log^2 R}}{|G_R(Re^{i\theta})|} < \frac{1}{2}.$$

An application of the maximum principle to the function $z \mapsto g(z)z^A$ in the annulus $1/2 < |z| < R$ gives a contradiction. The lemma is proved. □

## 3.5 Proof of Theorem 3.1.5

We start by constructing a slightly perturbed version of the function from Lemma 3.2.6 (using essentially the same argument).

**Lemme 3.5.1.** *Let $\alpha > 0$. There exists a $d_\rho$-separated sequence $\Gamma$ such that the upper and lower densities of this sequence are equal to $2\alpha$, the infinite product*

$$E(z) = \prod_{\gamma\in\Gamma} \Big(1 - \frac{z}{\gamma}\Big) \tag{3.19}$$

*converges on every compact subset of $\mathbb{C}$, and satisfies the following estimates:*

$$|E(z)| \lesssim e^{\alpha(\log^+|z|)^2} \frac{d(z,\Gamma)}{1+|z|^{3/2}}, \qquad z \in \mathbb{C}, \tag{3.20}$$

$$\limsup_{|z|\to\infty} |z|^{1/2} |E(z)| e^{-\alpha(\log^+|z|)^2} > 0, \tag{3.21}$$

$$\liminf_{|\gamma|\to\infty,\, \gamma\in\Gamma} |\gamma|^{3/2} |E'(\gamma)| e^{-\alpha(\log^+|\gamma|)^2} = 0, \tag{3.22}$$

*and for any entire function $F \neq 0$, we have*

$$\int_{\mathbb{C}} |E(z)F(z)|^2 e^{-2\alpha(\log^+|z|)^2}\, dm(z) = \infty, \tag{3.23}$$

*Proof.* For every $t > 0$ there exists a unique $m$ such that

$$\frac{1}{\alpha}\Big(m - \frac{1}{2}\Big) \le t < \frac{1}{\alpha}\Big(m + \frac{1}{2}\Big).$$

We consider $z$ with $|z| = e^t$. Then

$$\begin{aligned}\log|E(z)| &= \sum_{|\gamma|\le e^m,\,\gamma\in\Gamma} \log\Big|1-\frac{z}{\gamma}\Big| + O(1)\\ &= \sum_{|\gamma|\le e^{m-1},\,\gamma\in\Gamma} \big(t-\log|\gamma|\big) + \log\operatorname{dist}(z,\Gamma) - t + O(1).\end{aligned} \tag{3.24}$$

For a rapidly growing sequence $\{n_k\}$ we set

$$\Gamma = \Big\{\exp\frac{n}{2\alpha}\Big\}_{n\ge1,\,n\ne n_k,3n_k} \bigcup \Big\{\exp\frac{4n_k\pm1}{4\alpha}\Big\}_{k\ge1}$$

Then $\Gamma$ is $d_\rho$-separated, of density $2\alpha$, and $E$ converges on compact subsets of $\mathbb{C}$. Estimates (3.20)–(3.22) and assertion (3.23) follow from (3.24). □

## Proof of Theorem 3.1.5

(i) is the Borichev–Lyubarskii result [5].

(iii) Again let $\Gamma = \{\gamma_n\}_{n\ge1} = \{e^{\frac{n+1}{2\alpha}}\}_{n\ge0}$ and set $\widetilde{\Gamma} = \Gamma\cup\{0\}$. Let $G$ be the associated function of Lemma 3.2.6 vanishing exactly on $\Gamma$. For every finite sequence $v = (v_\lambda)_{\lambda\in\widetilde{\Gamma}}$, we construct the corresponding interpolating function and estimate its norm:

$$f_v(z) = v_0\frac{G(z)}{G(0)} + \sum_{\gamma\in\Gamma} v_\gamma \frac{G(z)}{G'(\gamma)(z-\gamma)}\frac{z}{\gamma}.$$

As above,

$$\begin{aligned}|f_v(z)| &\lesssim |v_0|e^{\varphi(z)}\frac{\operatorname{dist}(z,\Gamma)}{1+|z|^{3/2}} + \sum_{\gamma\in\Gamma}|v_\gamma|e^{\varphi(z)}\frac{\operatorname{dist}(z,\Gamma)}{1+|z|^{3/2}}\frac{|z|}{\gamma}\frac{1+|\gamma|^{3/2}}{e^{\varphi(\gamma)}|z-\gamma|}\\ &\lesssim |v_0|e^{\varphi(z)} + e^{\varphi(z)}\|v\|_{\infty,\varphi,\Lambda}\sum_{\gamma\in\Gamma}\frac{\operatorname{dist}(z,\Gamma)}{|z-\gamma|}\frac{1+\gamma^{1/2}}{1+|z|^{1/2}}\\ &\lesssim \|v\|_{\infty,\varphi,\Gamma}e^{\varphi(z)}, \quad z\in\mathbb{C},\end{aligned}$$

and we deduce that $\widetilde{\Gamma}$ is an interpolating sequence for which we can construct a linear operator of interpolation.

Clearly $\widetilde{\Gamma}$ is $d_\rho$–separated and $D^+(\widetilde{\Gamma}) = D^-(\widetilde{\Gamma}) = 2\alpha$.

We now show that $\widetilde{\Gamma}$ is a uniqueness sequence. For this, let $f\in\mathcal{F}^\infty_\varphi$ vanish on $\widetilde{\Gamma}$. Consider the holomorphic function $g = f/(zG)$. Then

$$|g(z)| \lesssim \frac{1+\sqrt{|z|}}{\operatorname{dist}(z,\Gamma)},$$

and, by the maximum modulus principle, $g = 0$.

As a conclusion the sequence $\widetilde{\Gamma}$ is an interpolating sequence which is also a uniqueness sequence and thus a sampling sequence.

(ii) Pick the sequence $\Gamma$ and the function $E$ from Lemma 3.5.1. In view of Theorem 3.1.1, this sequence cannot be complete interpolating for $\mathcal{F}^2_\varphi$ since the points $\exp \frac{4n_k \pm 1}{4\alpha}$ are shifted too far away from their places $(\exp \frac{n_k}{2\alpha}, \exp \frac{3n_k}{2\alpha})$ in the reference sequence. By (3.23), $\Gamma$ is a uniqueness set $\mathcal{F}^2_\varphi$, and by (3.20), the functions $E_\gamma := E/E'(\gamma)(\cdot - \gamma)$, $\gamma \in \Gamma$, are in $\mathcal{F}^2_\varphi$.

Let us prove that $\Gamma$ is neither sampling nor interpolating for $\mathcal{F}^2_\varphi$.

Suppose first that it is sampling. Then for every finite sequence $v = (v_\gamma)_{\gamma\in\Gamma} \in \ell^2_{\varphi,\Gamma}$ the function $f_v = \sum_\gamma v_\gamma E_\gamma$ would interpolate $v$ on $\Gamma$, and by the sampling property of $\Gamma$,

$$\|f_v\|^2_{\varphi,2} \asymp \|f_v|\Gamma\|^2_{\varphi,2,\Gamma} = \sum_\gamma |v_\gamma|^2 e^{-2\varphi(\gamma)}(1 + |\gamma|^2) = \|v\|_{\varphi,2,\Gamma}.$$

In other words, the interpolation operator $v \longmapsto f_v$ would be continuous from $\ell^2_{\varphi,\Gamma}$ to $\mathcal{F}^2_\varphi$, and the sequence would be interpolating. Since it was supposed to be sampling it would thus be complete interpolating, and we would get a contradiction.

Suppose next that it is interpolating. Since $\Gamma$ is a uniqueness sequence, it is complete interpolating, and again we obtain a contradiction.

(iv) We use again the sequence $\Gamma$ and the function $E$ from Lemma 3.5.1. Set $\widetilde{\Gamma} = \Gamma \cup \{0\}$, $\widetilde{E}(z) = zE(z)$. By (3.21), $\widetilde{\Gamma}$ is a uniqueness set $\mathcal{F}^2_\varphi$, and by (3.20), the functions $E_\gamma := E/E'(\gamma)(\cdot - \gamma)$, $\gamma \in \Gamma$, and $E_0 := E$ are in $\mathcal{F}^\infty_\varphi$.

Suppose that $\widetilde{\Gamma}$ is a complete interpolating sequence for $\mathcal{F}^\infty_\varphi$. Then

$$|E_\gamma(0)| \lesssim \|E_\gamma\|_{\varphi,\infty} \lesssim \|E_\gamma|\Gamma\|_{\varphi,\infty,\Gamma} = e^{-\varphi(\gamma)},$$

and we obtain a contradiction to (3.22).

The rest of the proof is analogous to that in part (ii). □

# Bibliography


[1] S. A. Avdonin, *On the question of Riesz bases of exponential functions in $L^2$*, Vestnik Leningrad. Univ. No. 13 Mat. Meh. Astronom. Vyp. 3 (1974), 5–12, in Russian; English translation in Vestnik Leningrad Univ. Math. **7** (1979) 203–211.

[2] Y. Belov, T. Mengestie, K. Seip, *Discrete Hilbert transforms on sparse sequences*, Proc. London Math. Soc. **103** (2011) 73–105.

[3] A. Beurling, *The Collected Works of Arne Beurling 2, Harmonic Analysis (L. Carleson, P. Malliavin, J. Neuberger, J. Wermer, eds.)*, Birkhäuser Boston Inc., Boston, MA, 1989.

[4] A. Borichev, R. Dhuez, K. Kellay, *Sampling and interpolation in large Bergman and Fock spaces*, J. Funct. Anal. **242** (2007) 563–606.

[5] A. Borichev, Yu. Lyubarskii, *Riesz bases of reproducing kernels in Fock type spaces*, Journal of the Institute of Mathematics of Jussieu **9** (2010) 449–461.

[6] O. Christensen, *Frames and bases. An introductory course*, Birkhäuser, Boston, 2008.

[7] J. B. Garnett, *Bounded analytic functions*. Revised first edition. Graduate Texts in Mathematics, 236. Springer, New York, 2007.

[8] H. Hedenmalm, B. Korenblum, K. Zhu, *Theory of Bergman Spaces*, Springer–Verlag, New York, 2000.

[9] K. Isaev, R. Yulmukhametov, *On the nonexistence of unconditional bases of exponentials in Bergman spaces on non-polygonal domains*, Izv. Ross. Akad. Nauk Ser. Mat. **71** (2007) 69–90 (Russian); English translation in Izv. Math. **71** (2007) 1145–1166.

[10] M. Kadets, *The exact value of the Paley–Wiener constant*, Sov. Math. Dokl. **5** (1964), 559–561.

[11] Yu. Lyubarskii, *Frames in the Bargmann space of entire functions, in Entire and Subharmonic Functions*, Adv. Soviet Math. **11** (1992) 167–180, American Mathematical Society, Providence, RI.

[12] Yu. Lyubarskii, E. Malinnikova, *On approximation of subharmonic functions,* J. d'Anal. Math. **83** (2001) 121–149.

[13] Yu. Lyubarskii, K. Seip, *Convergence and summability of Gabor expansions at the Nyquist density*, J. Fourier Anal. Appl. **5** (1999) 127–157.

[14] N. Marco, X. Massaneda, J. Ortega-Cerdà, *Interpolating and sampling sequences for entire functions*, Geom. Funct. Anal. **13** (2003) 862–914.

[15] J. Marzo, S. Nitzan, J.-F. Olsen, *Sampling and interpolation in de Branges spaces with doubling phase*, preprint.

[16] T. Mengestie, *Two weight discrete Hilbert transforms and systems of reproducing kernels,* PhD thesis, Norwegian University of Science and Technology, 2011.

[17] N. K. Nikolski, *Treatise on the shift operators. Spectral Function Theory,* Springer–Verlag, Berlin, 1986.

[18] J. Ortega-Cerdà, K. Seip, *Beurling-type density theorems for weighted $L^p$ spaces of entire functions,* J. Anal. Math. 75 (1998) 247–266.

[19] G. Rozenblum, N. Shirokov, *Entire functions in weighted $L_2$ and zero modes of the Pauli operator with non-signdefinite magnetic field*, Cubo **12** (2010) 115–132.

[20] K. Seip, *Density theorems for sampling and interpolation in the Bargmann–Fock space I*, J. Reine Angew. Math. **429** (1992) 91–106.

[21] K. Seip, *Beurling type density theorems in the unit disk*, Invent. Math. **113** (1993) 21–39.

[22] K. Seip, *A simple construction of exponential bases in $L^2$ of the union of several intervals*, Proc. Edinburgh Math. Soc. **38** (1995) 171–177.

[23] K. Seip, *Interpolating and sampling in spaces of analytic functions*, American Mathematical Society, Providence, 2004.

[24] K. Seip, *Interpolation and sampling in small Bergman spaces*, Collect. Math. **64** (2013) 61–72.

[25] K. Seip, R. Wallstén, *Density theorems for sampling and interpolation in the Bargmann–Fock space. II*, J. Reine Angew. Math. **429** (1992) 107–113.

[26] K. Zhu, *Analysis on Fock spaces*, Graduate Texts in Mathematics, 263, Springer, New York, 2012.

**Abstract.** *We study the uniqueness sets, the weak interpolation sets, and convergence of the Lagrange interpolation series in radial weighted Fock spaces. We study also sampling, interpolation and Riesz bases in small radial weighted Fock spaces*

**Resumé.** *Nous étudions le problème d'unicité, de l'interpolation faible et de la convergence de la série d'interpolation de Lagrange dans les espaces de Fock pondérés par des poids radiaux. Nous étudions aussi les suites d'échatillonnage, d'interpolation et les bases de Riesz dans les petit espaces de Fock.*