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Contrôlabilité de systèmes paraboliques linéaires couplés

THÈSE DE DOCTORAT

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À mon père,
qui me manque.

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Table des matières

Chapitre 1

Introduction

Dans cette thèse on étudie la contrôlabilité des systèmes d'équations de type chaleur. On rappelle dans un premier temps les résultats existant pour une seule équation de la chaleur d'une part, et pour les systèmes d'équations différentielles d'autre part. On donne ensuite les contributions apportées par cette thèse. Les principales thématiques abordées sont la contrôlabilité frontière des systèmes en dimension supérieure, la contrôlabilité avec un coefficient de couplage d'ordre 1 et la contrôlabilité avec un coefficient de couplage variable.

1.1 Équation de la chaleur et systèmes en dimension finie

On rappelle dans cette section ce qui est connu sur la contrôlabilité de l'équation de la chaleur et des systèmes d'équations différentielles en dimension finie. Cela permettra de bien comprendre la situation et les problèmes qui se posent pour les systèmes d'équations du type chaleur. Pour une introduction générale à la théorie du contrôle on renvoie aux livres [Cor07, Zab92, TW09].

1.1.1 Généralités sur la contrôlabilité de l'équation de la chaleur

Soit $\Omega \subset \mathbb{R}^N$ un ouvert borné connexe non-vide de classe C^2. On note $Q_T = (0,T) \times \Omega$ et $\Sigma_T = (0,T) \times \Gamma$ où $\Gamma = \partial\Omega$. On considère l'équation de la chaleur avec un terme source :

$$\begin{cases} \partial_t y - \Delta y = 1_\omega v & \text{dans } Q_T, \\ y = 0 & \text{sur } \Sigma_T, \\ y(0) = y_0 & \text{dans } \Omega, \end{cases} \tag{1.1}$$

où y est l'état, y_0 est la donnée initiale, v est le contrôle et l'ouvert non-vide $\omega \subset \Omega$ est la zone de contrôle.

On rappelle que pour tout $y_0 \in L^2(\Omega)$ et $v \in L^2(Q_T)$ il existe une unique solution (faible) $y \in C^0([0,T];L^2(\Omega))$ qui dépend continûment des données : il existe une constante $C > 0$ (indépendante de T) telle que

$$\|y\|_{C^0([0,T];L^2(\Omega))} \leq Ce^{CT}\left(\|y_0\|_{L^2(\Omega)} + \|v\|_{L^2(Q_T)}\right). \tag{1.2}$$

De plus, la régularité parabolique nous dit aussi que $y \in L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega))$ si $y_0 \in H_0^1(\Omega)$. On rappelle également qu'une des propriétés caractéristiques de l'équation de la chaleur est son effet à régulariser instantanément. Ainsi, pour tout ouvert non-vide $\mathcal{V} \subset \Omega$ qui n'intersecte pas ω, on a

$$y \in C^\infty((0,T] \times \mathcal{V}).$$

Contrôlabilité à zéro et contrôlabilité approchée

Les deux grands concepts en contrôlabilité dont nous aurons besoin sont les suivants :

Définition 1.

1. *On dit que l'équation* (1.1) *est contrôlable à zéro au temps T si, pour toute donnée initiale $y_0 \in L^2(\Omega)$, il existe un contrôle $v \in L^2(Q_T)$ tel que la solution associée $y \in C^0([0,T];L^2(\Omega))$ vérifie*

$$y(T) = 0.$$

2. *On dit que l'équation* (1.1) *est approximativement contrôlable au temps T si, pour toute donnée initiale $y_0 \in L^2(\Omega)$, toute cible $y_T \in L^2(\Omega)$, et toute précision $\epsilon > 0$, il existe un contrôle $v \in L^2(Q_T)$ tel que la solution associée $y \in C^0([0,T];L^2(\Omega))$ vérifie*

$$\|y(T) - y_T\|_{L^2(\Omega)} \leq \epsilon.$$

Lorsque l'équation (1.1) *est contrôlable (à zéro, approximativement) au temps T pour tout $T > 0$, on dira simplement qu'elle est contrôlable (à zéro, approximativement).*

Il existe en fait un autre concept de contrôlabilité où l'on cherche à atteindre exactement une cible y_T mais ce dernier ne se prête guère au modèle de la chaleur à cause de son effet régularisant mentionné plus haut (cela imposerait à la cible y_T d'être extrêmement régulière en dehors de ω). Notons que, lorsque l'équation (1.1) n'est pas contrôlable, il peut quand même être intéressant de caractériser les données initiales qui sont néanmoins contrôlables.

Il est très commode de reformuler les définitions précédentes sous forme d'inclusions d'image d'opérateurs. Introduisons les opérateurs linéaires

$$\begin{array}{rccc} F_T : & L^2(\Omega) & \longrightarrow & L^2(\Omega) \\ & y_0 & \longmapsto & \widehat{y}(T), \end{array} \qquad \begin{array}{rccc} G_T : & L^2(Q_T) & \longrightarrow & L^2(\Omega) \\ & v & \longmapsto & \overline{y}(T), \end{array}$$

où $\widehat{y}$ est la solution libre de l'équation (1.1), c'est-à-dire sans contrôle ($v = 0$), et $\overline{y}$ est la solution de l'équation (1.1) en partant de la donnée initiale $y_0 = 0$. Notons que ces opérateurs sont bien définis en vertu de la régularité de y et qu'ils sont également bornés grâce à la dépendance continue par rapport aux données.

Ainsi, il existe un contrôle v qui transfère la solution de l'équation (1.1) de y_0 à y_T au temps T si, et seulement si,

$$y_T \in F_T y_0 + \operatorname{Im} G_T.$$

On peut alors réécrire la définition 1 comme il suit :

1. L'équation (1.1) est contrôlable à zéro au temps T si

$$\operatorname{Im} F_T \subset \operatorname{Im} G_T. \tag{1.3}$$

2. L'équation (1.1) est approximativement contrôlable au temps T si

$$\overline{\operatorname{Im} G_T} = L^2(\Omega). \tag{1.4}$$

Système adjoint et dualité entre contrôlabilité et observabilité

On introduit le système adjoint à (1.1), c'est-à-dire l'équation rétrograde en temps

$$\begin{cases} -\partial_t z - \Delta z = 0 & \text{dans } Q_T, \\ z = 0 & \text{sur } \Sigma_T, \\ z(T) = z_T & \text{dans } \Omega. \end{cases} \tag{1.5}$$

On rappelle que pour tout $z_T \in L^2(\Omega)$ il existe une unique solution (faible) $z \in C^0([0,T]; L^2(\Omega))$ qui dépend continûment des données. On rappelle également que l'énergie de ce système croît à une vitesse exponentielle :

$$\|z(t_1)\|_{L^2(\Omega)} \leq C e^{-C(t_2-t_1)} \|z(t_2)\|_{L^2(\Omega)}, \quad \forall t_1, t_2 \in [0,T],\ t_1 \leq t_2. \tag{1.6}$$

En effectuant des intégrations par parties on obtient la relation fondamentale entre la solution y du problème initial (1.1) et la solution z du problème adjoint (1.5) :

$$\langle y(T), z_T \rangle_{L^2(\Omega)} - \langle y_0, z(0) \rangle_{L^2(\Omega)} = \int_0^T \langle v(t), 1_\omega z(t) \rangle_{L^2(\Omega)} \, dt. \tag{1.7}$$

Cette relation rend alors le calcul des adjoints des opérateurs G_T et F_T immédiat :

$$\begin{array}{rcccl} F_T^* & : & L^2(\Omega) & \longrightarrow & L^2(\Omega) \\ & & z_T & \longmapsto & z(0), \end{array} \qquad \begin{array}{rcccl} G_T^* & : & L^2(\Omega) & \longrightarrow & L^2(Q_T) \\ & & z_T & \longmapsto & 1_\omega z. \end{array} \tag{1.8}$$

Avec ces calculs et en passant au dual dans les inclusions (1.3) et (1.4), on obtient maintenant le théorème fondamental qui relie la notion de contrôlabilité à celle d'observabilité. C'est, dans de nombreuses situations, le point de départ pour commencer l'étude de la contrôlabilité en dimension infinie. Ce fait a été observé par Dolecki et Russell dans [DR77].

Théorème 2.

1. *L'équation* (1.1) *est contrôlable à zéro au temps T si, et seulement si, son système adjoint* (1.5) *est observable à zéro au temps T, c'est-à-dire si on a l'inégalité d'observabilité*

$$\exists C_T > 0, \forall z_T \in L^2(\Omega), \quad \|z(0)\|^2_{L^2(\Omega)} \leq C_T^2 \int_0^T \|1_\omega z(t)\|^2_{L^2(\Omega)}\,dt. \tag{1.9}$$

2. *L'équation* (1.1) *est approximativement contrôlable au temps T si, et seulement si, son système adjoint* (1.5) *est approximativement observable au temps T, c'est-à-dire si on a la propriété d'unicité*

$$\forall z_T \in L^2(\Omega), \quad (1_\omega z(t) = 0, \quad p.p.\ t \in (0,T)) \Longrightarrow z_T = 0.$$

Remarque 3. *Suite à ce théorème il est facile de voir que la contrôlabilité à zéro implique la contrôlabilité approchée si le système adjoint a la propriété d'unicité rétrograde, c'est-à-dire si*

$$z(0) = 0 \Longrightarrow z_T = 0.$$

Cette propriété est effectivement vérifiée pour le système adjoint (1.5)*, et plus généralement pour une large classe de systèmes paraboliques, voir [Ghi86].*

Notons que pour vérifier l'inégalité d'observabilité (1.9) il suffit de le faire pour des données z_T plus régulières (grâce à la dépendance continue par rapport aux données).

La constante C_T qui apparaît dans l'inégalité d'observabilité (1.9) est très liée au contrôle. La meilleure de ces constantes étant en fait le coût du contrôle (voir la section d'après). Pour l'instant, observons le fait suivant. Supposons qu'il existe une constante $K_T > 0$ telle que, pour toute donnée initiale y_0, il existe un contrôle v tel que

$$\|v\|_{L^2(Q_T)} \leq K_T \|y_0\|_{L^2(\Omega)}. \tag{1.10}$$

On verra que la contrôlabilité entraîne en fait automatiquement l'existence d'une telle constante. Alors, il découle facilement de (1.7) l'inégalité d'observabilité suivante :

$$\forall z_T \in L^2(\Omega), \quad \|z(0)\|^2_{L^2(\Omega)} \leq K_T^2 \int_0^T \|1_\omega z(t)\|^2_{L^2(\Omega)}\,dt. \tag{1.11}$$

La réciproque est en fait aussi vraie, à savoir que s'il existe une constante $K_T > 0$ telle que (1.11) ait lieu, alors, pour toute donnée initiale y_0, il existe un contrôle à zéro v avec (1.10). Il s'ensuit que la meilleure des constantes qui intervient dans (1.10) est également la meilleure des constantes dans (1.11), cette dernière étant appelée constante d'observabilité.

Contrôle à zéro optimal

A priori il n'y a pas unicité dans le choix du contrôle. Cependant on peut rechercher le "meilleur" contrôle au sens d'un certain critère, par exemple celui qui est de norme $L^2(Q_T)$

minimale. A y_0 fixé, si le problème (1.1) est contrôlable depuis y_0, alors l'ensemble des contrôles associés est un sous-espace affine fermé de $L^2(Q_T)$ (de direction $\ker G_T$) qui est donc non-vide. Le théorème de la projection sur un convexe fermé permet de voir alors que le problème de minimisation sous contrainte

$$\inf_v \|v\|_{L^2(Q_T)}, \quad v \text{ contrôle associé à } y_0,$$

admet une unique solution v_{opt} (la projection de 0 sur le convexe), appelé contrôle optimal.

Lorsque l'équation (1.1) est contrôlable à zéro au temps T, on peut donc définir l'application

$$y_0 \in L^2(\Omega) \longmapsto v_{\text{opt}} \in L^2(Q_T).$$

A l'aide du théorème du graphe fermé il est facile de voir que cette application linéaire est continue. On note alors K_T sa norme, qui est appelée coût du contrôle. L'estimation (1.10) a ainsi toujours lieu avec ce K_T. Il est facile de voir que l'application

$$T \in (0, +\infty) \longmapsto K_T$$

est décroissante et tend vers $+\infty$ quand T tend vers 0^+ (plus on se rapproche du temps initial, plus cela coûte cher de contrôler).

1.1.2 Contrôlabilité à zéro de l'équation de la chaleur

Pour l'équation de la chaleur (1.1), le problème de contrôlabilité à zéro est en fait complètement résolu. On rappelle ici différentes méthodes qui permettent d'obtenir ce résultat. Ces techniques seront également employées pour les systèmes d'équations dans la suite de ce manuscrit.

Théorème 4 ([LR95, FI96])**.** *L'équation de la chaleur* (1.1) *est contrôlable à zéro pour tout temps $T > 0$ et tout ouvert non-vide $\omega \subset \Omega$.*

La principale difficulté pour prouver ce théorème est lorsque la zone de contrôle ω est un sous-domaine strict de Ω. En effet, dans le cas où $\omega = \Omega$ il suffit de multiplier l'équation (1.5) par $(T-t)z$ et ensuite d'intégrer par parties pour obtenir l'inégalité d'observabilité (1.9) avec $C_T = 1/T$.

Contrôlabilité interne et contrôlabilité frontière

Par une astuce qui consiste à étendre le domaine, la contrôlabilité frontière est facilement déduite de la contrôlabilité interne. En effet, considérons l'équation de la chaleur avec un contrôle au bord

$$\begin{cases} \partial_t y - \Delta y = 0 & \text{dans } Q_T, \\ y = 1_\gamma v & \text{sur } \Sigma_T, \\ y(0) = y_0 & \text{dans } \Omega, \end{cases} \tag{1.12}$$

où $v \in L^2(\Sigma_T)$ est donc le contrôle et l'ouvert non-vide $\gamma \subset \partial\Omega$ est la zone de contrôle.

Comme la donnée au bord est très peu régulière, la notion de solution pour l'équation (1.12) est à prendre ici dans un sens plus faible, celui de la transposition (voir [LM68]). On sait alors que pour tout $y_0 \in H^{-1}(\Omega)$ et $v \in L^2(\Sigma_T)$, il existe une unique solution (par transposition) $y \in C^0([0,T]; H^{-1}(\Omega))$ qui dépend continûment des données.

Soit maintenant $\widetilde{\Omega}$ un ouvert borné connexe non-vide de classe C^2 tel que

$$\Omega \subset \widetilde{\Omega}, \quad \partial\Omega \cap \widetilde{\Omega} \subset\subset \gamma, \quad \widetilde{\Omega} \backslash \overline{\Omega} \neq \emptyset.$$

Soit $\omega \subset \widetilde{\Omega}\backslash\overline{\Omega}$ un ouvert non-vide. Pour $y_0 \in L^2(\Omega)$ on note $\overline{y_0}$ le prolongement de y_0 par zéro en dehors de Ω. D'après le théorème 4 on sait qu'il existe un contrôle à zéro $\widetilde{v} \in L^2(\widetilde{Q}_T)$ pour l'équation de la chaleur posée sur $\widetilde{\Omega}$:

$$\begin{cases} \partial_t \widetilde{y} - \Delta \widetilde{y} = 1_\omega \widetilde{v} & \text{dans } \widetilde{Q}_T, \\ \widetilde{y} = 0 & \text{sur } \widetilde{\Sigma}_T, \\ \widetilde{y}(0) = \overline{y_0} & \text{dans } \widetilde{\Omega}. \end{cases}$$

Alors, $y = \widetilde{y}_{|\Omega}$ est solution de l'équation 1.12 avec $v = \widetilde{y}_{|\partial\Omega} \in L^2(0,T; H^{1/2}(\Omega))$ et vérifie $y(T) = 0$, ce qui prouve la contrôlabilité à zéro de (1.12) (pour des données initiales $y_0 \in L^2(\Omega)$). □

Notons d'ores et déjà que ce procédé se révèlera impossible pour les systèmes d'équations, si on dispose de moins de contrôles que d'équations.

La méthode des moments

La méthode des moments a été utilisée dans [FR71] pour obtenir le tout premier résultat de contrôlabilité pour l'équation de la chaleur. C'est un résultat en dimension 1 et avec un contrôle frontière. Pour être plus précis, considérons

$$\begin{cases} \partial_t y - \partial_x^2 y = 0 & \text{dans } (0,T) \times (0,1), \\ y(t,0) = v(t), \quad y(t,1) = 0 & \text{sur } (0,T), \\ y(0) = y_0 & \text{dans } (0,1). \end{cases} \tag{1.13}$$

De même que dans le cas distribué où on avait la relation (1.7), on a ici

$$\langle y(T), z_T \rangle_{H^{-1},H_0^1} - \langle y_0, z(0) \rangle_{H^{-1},H_0^1} = \int_0^T v(t) \partial_x z(t,0)\, dt,$$

pour tout $z_T \in H_0^1(0,1)$, où z est toujours la solution du système adjoint (1.5). Il est alors facile de voir que l'équation (1.13) est contrôlable à zéro au temps T si, et seulement si, pour tout $y_0 \in H^{-1}(0,1)$, il existe un contrôle $v \in L^2(0,T)$ tel que

$$-\langle y_0, z(0) \rangle_{H^{-1},H_0^1} = \int_0^T v(t) \partial_x z(t,0)\, dt.$$

En décomposant maintenant y_0 et z en séries de Fourier dans la base de fonctions propres $\{\phi_k\}_{k\geq 1}$ de $-\partial_x^2$ (avec conditions au bord de Dirichlet), la relation précédente est équivalente à

$$c_k = \int_0^T v(T-t)e^{-\lambda_k t}\,dt, \quad \forall k \geq 1,$$

où on a posé $c_k = \frac{-e^{-\lambda_k T}}{\partial_x \phi_k(0)} \langle y_0, \phi_k \rangle_{H^{-1},H_0^1}$. C'est ce qu'on appelle un problème des moments. Pour le résoudre, on construit une famille $\{q_k\}_{k\geq 1} \subset L^2(0,T)$ biorthogonale à la famille des exponentielles $\left\{e^{-\lambda_k t}\right\}_{k\geq 1}$, c'est-à-dire telle que

$$\left\langle q_k, e^{-\lambda_l t} \right\rangle_{L^2(0,T)} = \delta_{kl},$$

et qui vérifie de plus l'estimation

$$\|q_k\|_{L^2(0,T)} \leq \rho_T e^{C\sqrt{\lambda_k}}, \quad \forall k \geq 1, \tag{1.14}$$

où $\rho_T > 0$ est une constante qui dépend de T. Une fois cette famille construite (c'est la tâche la plus difficile), il suffit de prendre v de la forme

$$v(t) = \sum_{k=1}^{+\infty} c_k q_k(T-t).$$

Vérifions que cette série converge bien dans $L^2(0,T)$. De l'estimation (1.14) et l'inégalité de Young $C\sqrt{\lambda_k} \leq \frac{\lambda_k T}{2} + \frac{C^2}{2T}$ on a

$$\begin{cases} \|q_k\|_{L^2(0,T)} \leq \rho_T e^{\frac{\lambda_k T}{2} + \frac{C^2}{2T}}, \\ |c_k| \leq \dfrac{1}{\sqrt{2}} e^{-\lambda_k T} \|y_0\|_{H^{-1}(0,1)}, \end{cases}$$

Ainsi,

$$\|v\|_{L^2(0,T)} \leq \frac{1}{\sqrt{2}} \rho_T e^{\frac{C^2}{2T}} \left(\sum_{k=1}^{+\infty} e^{-\frac{\lambda_k T}{2}} \right) \|y_0\|_{H^{-1}(0,1)}.$$

Comme $\lambda_k = k^2\pi^2$, cette série est comparable à l'intégrale de Gauss $\int_0^{+\infty} e^{-\frac{\pi^2 T}{2}x^2}\,dx$, qui vaut $\sqrt{1/(2\pi T)}$.

Par ailleurs, sachant qu'on peut en fait prendre $\rho_T = Ce^{C/T}$ (voir [Sei84]), cela fournit aussi une estimation du coût du contrôle en $Ce^{C/T}$.

Les inégalités de Carleman

Les inégalités de Carleman sont des inégalités à poids initialement introduites par Carleman [Car39] pour prouver la continuation unique d'un système d'équations aux dérivées partielles. Elles ont depuis lors subi de nombreux développements et elles se révèlent être des outils très efficaces pour prouver la contrôlabilité. Elles ont l'avantage d'être assez souples, en ce sens qu'elles sont invariables par perturbation de termes d'ordre 1 ou 0, et qu'on peut traiter le cas des coefficients variables, en t et x.

On suit la présentation de [FCG06]. Commençons par rappeler les poids qui interviennent dans ces inégalités. Soit $m > 1$ quelconque. On introduit les poids suivants :

$$\alpha(t,x) = \frac{e^{2\lambda m \max_{\overline{\Omega}} \beta} - e^{\lambda(m \max_{\overline{\Omega}} \beta + \beta(x))}}{t(T-t)}, \quad \varphi(t,x) = \frac{e^{\lambda(m \max_{\overline{\Omega}} \beta + \beta(x))}}{t(T-t)},$$

où la fonction $\beta \in C^2(\overline{\Omega})$ est telle que

$$\beta > 0 \text{ dans } \Omega, \quad \beta = 0 \text{ sur } \partial\Omega, \quad |\nabla \beta| > 0 \text{ dans } \overline{\Omega \backslash \omega}.$$

Pour l'existence d'une telle fonction, on renvoie à [FI96, Lemma 1.1].

Théorème 5 ([FI96])**.** *Il existe des constantes $C > 0$, $\lambda_0, s_0 \geq 1$ (indépendantes de T) telles que, pour toute solution z de l'équation*

$$\begin{cases} -\partial_t z - \Delta z = F & \textit{dans } Q_T, \\ z = 0 & \textit{sur } \Sigma_T, \\ z(T) = z_T & \textit{dans } \Omega, \end{cases} \tag{1.15}$$

avec $F \in L^2(Q_T)$ et $z_T \in H_0^1(\Omega)$, on ait

$$\begin{aligned} s^{-1} \iint_{Q_T} e^{-2s\alpha} \varphi^{-1} (|\partial_t z|^2 + |\Delta z|^2) \, dx \, dt + s\lambda^2 \iint_{Q_T} e^{-2s\alpha} \varphi \left|\nabla z\right|^2 \, dx \, dt \\ s^3 \lambda^4 \iint_{Q_T} e^{-2s\alpha} \varphi^3 \left|z\right|^2 \, dx \, dt \leq C \Bigg(s^3 \lambda^4 \iint_{(0,T)\times\omega} e^{-2s\alpha} \varphi^3 \left|z\right|^2 \, dx \, dt \\ + \iint_{Q_T} e^{-2s\alpha} \left|F\right|^2 \, dx \, dt \Bigg), \end{aligned} \tag{1.16}$$

pour tout $\lambda \geq \lambda_0$ et $s \geq (T + T^2) s_0$.

Cette inégalité de Carleman appliquée au système adjoint (1.5) donne en particulier

$$\iint_{Q_T} e^{-2s\alpha} \varphi^3 \left|z\right|^2 \, dx \, dt \leq C \iint_{(0,T)\times\omega} e^{-2s\alpha} \varphi^3 \left|z\right|^2 \, dx \, dt,$$

pour tout $\lambda \geq \lambda_0$ et $s \geq (T + T^2) s_0$. Sachant que les poids vérifient, pour une même constante $C > 0$ (indépendante de T),

$$\begin{cases} e^{-2s_0(T+T^2)\alpha} \varphi^3 \geq e^{-2C(1+1/T)} \dfrac{1}{T^6} & \text{sur } (T/4, 3T/4) \times \Omega, \\ e^{-2s_0(T+T^2)\alpha} \varphi^3 \leq e^{-C(1+1/T)} \dfrac{1}{T^6} & \text{sur } (0,T) \times \Omega, \end{cases}$$

où on a fixé λ assez grand pour la deuxième inégalité, on obtient donc

$$\iint_{(T/4,3T/4)\times\Omega} |z|^2 \, dx\, dt \leq Ce^{C/T} \iint_{(0,T)\times\omega} |z|^2 \, dx\, dt.$$

Par ailleurs, la croissance de l'énergie (1.6) donne

$$\|z(0)\|^2_{L^2(\Omega)} \leq \frac{C}{T} \iint_{(T/4,3T/4)\times\Omega} |z|^2 \, dx\, dt.$$

En combinant les deux dernières estimations on obtient ainsi l'inégalité d'observabilité (1.9) avec comme constante $C_T = Ce^{C/T}$ (on rappelle qu'il suffit de prouver cette dernière pour des données z_T régulières, ici $H^1_0(\Omega)$). □

Mentionnons également qu'il existe une version adaptée au contrôle frontière des inégalités de Carleman, (1.16) étant remplacée par

$$\begin{aligned} s^{-1} \iint_{Q_T} e^{-2s\alpha} \varphi^{-1} (|\partial_t z|^2 + |\Delta z|^2) \, dx\, dt + s\lambda^2 \iint_{Q_T} e^{-2s\alpha} \varphi \, |\nabla z|^2 \, dx\, dt \\ s^3\lambda^4 \iint_{Q_T} e^{-2s\alpha} \varphi^3 \, |z|^2 \, dx\, dt \leq C \Bigg(s\lambda \iint_{(0,T)\times\gamma} e^{-2s\alpha} \varphi \, |\partial_n z|^2 \, dx\, dt \\ + \iint_{Q_T} e^{-2s\alpha} \, |F|^2 \, dx\, dt \Bigg), \end{aligned}$$

pour d'autres fonctions poids β, α et φ.

La méthode de Lebeau-Robbiano

Soit $\lambda_1 \leq \lambda_2 \leq \ldots$ les valeurs propres (non nécessairement distinctes) de $-\Delta$, et soit $\{\phi_j\}_{j\geq 1}$ les fonctions propres associées. On note $E_J = \text{vect}\, \{\phi_j\}_{j\in [\![1,J]\!]}$ la somme des J premiers sous-espaces propres de $-\Delta$ (avec conditions au bord de Dirichlet).

La méthode de Lebeau-Robbiano pour établir la contrôlabilité de l'équation de la chaleur repose sur l'inégalité spectrale suivante, qui peut s'obtenir en utilisant des inégalités de Carleman ([LR95, LR07]) : il existe une constante $C > 0$ telle que, pour tout $J \geq 1$ et tout $(a_j)_{j\in[\![1,J]\!]} \in \mathbb{C}^J$, on ait

$$\sum_{j=1}^{J} |a_j|^2 \leq Ce^{C\sqrt{\lambda_J}} \int_\omega \left| \sum_{j=1}^{J} a_j \phi_j(x) \right|^2 dx. \tag{1.17}$$

De cette inégalité on peut en déduire une estimation du coût de la contrôlabilité partielle dans les espaces E_J. En effet, soit $z_T \in E_J$. On peut donc écrire $z_T = \sum_{j=1}^J \alpha_j \phi_j$ où les α_j sont des scalaires. Il s'ensuit que $z(t) = \sum_{j=1}^J e^{-\lambda_j(T-t)} \alpha_j \phi_j$. En appliquant l'inégalité de Lebeau-Robbiano (1.17) avec $a_j = e^{-\lambda_j(T-t)} \alpha_j$ on obtient

$$\|z(0)\|^2_{L^2(\Omega)} = \sum_{j=1}^{J} e^{-2\lambda_j T} |\alpha_j|^2 \leq \frac{1}{T} e^{C\sqrt{\lambda_J}} \int_0^T \|1_\omega z(t)\|^2_{L^2(\Omega)} \, dt, \quad \forall z_T \in E_J.$$

Par dualité on en déduit que, pour tout $y_0 \in L^2(\Omega)$, il existe un contrôle $v \in L^2(Q_T)$ tel que

$$\begin{cases} \Pi_{E_J} y(T) = 0, \\ \|v\|_{L^2(Q_T)} \le \dfrac{C}{\sqrt{T}} e^{C\sqrt{\lambda_J}} \|y_0\|_{L^2(\Omega)}. \end{cases} \tag{1.18}$$

L'idée est de maintenant construire le contrôle en deux temps. D'abord on contrôle les premières fréquences avec le coût obtenu dans (1.18). Ensuite, on laisse dissiper l'équation pour compenser le "mauvais coût" de l'étape précédente.

On écrit $[0,T) = \bigcup_{k=0}^{+\infty} [a_k, a_{k+1}]$, avec $a_0 = 0$, $a_{k+1} = a_k + 2T_k$, et $T_k = M2^{-k\rho}$, où $\rho \in (0, \frac{1}{N})$ et $M = \frac{T}{2}(1 - 2^{-\rho})$ a été choisi de sorte à ce que $2\sum_{k=0}^{+\infty} T_k = T$.

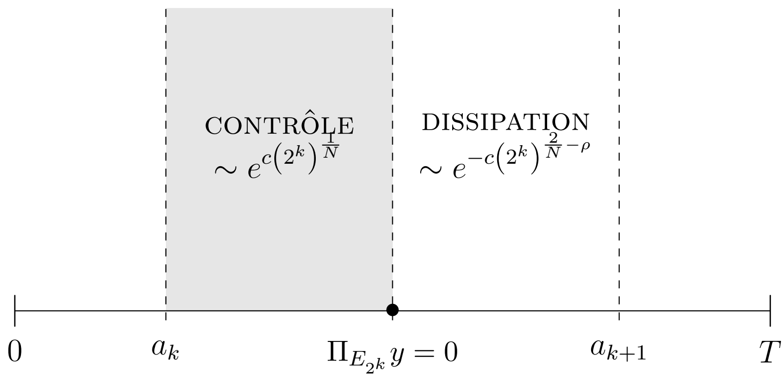


Figure 1.1 – Stratégie de Lebeau-Robbiano

Soit $y_0 \in L^2(\Omega)$ fixé. On commence par définir par itération des suites de contrôles partiels $(\overline{v^k})_{k \ge 0}$ et de solutions associées $(\overline{y^k})_{k \ge 0}$ de la manière suivante.

Pour la donnée initiale $\overline{y^{k-1}}(a_k)$ (resp. y_0 pour $k = 0$) on sait qu'il existe un contrôle partiel v^k tel que la solution associée y^k sur $(a_k, a_k + T_k)$, c'est-à-dire,

$$\begin{cases} \partial_t y^k - \Delta y^k = 1_\omega v^k & \text{dans } (a_k, a_k + T_k) \times \Omega, \\ y^k = 0 & \text{sur } (a_k, a_k + T_k) \times \partial\Omega, \\ y^k(a_k) = \overline{y^{k-1}}(a_k) & \text{dans } \Omega, \end{cases}$$

vérifie $\Pi_{E_{2^k}} y^k(a_k + T_k) = 0$. On pose alors

$$\overline{v^k}(t) = \begin{cases} v^k(t) & \text{si } t \in (a_k, a_k + T_k), \\ 0 & \text{si } t \in (a_k + T_k, a_{k+1}), \end{cases}$$

et on appelle $\overline{y^k}$ la solution associée sur (a_k, a_{k+1}). Comme $\overline{y^k}$ coïncide avec y^k sur $[a_k, a_k + T_k]$, on a aussi $\Pi_{E_{2^k}} \overline{y^k}(a_k + T_k) = 0$. Nous allons montrer que la fonction v définie par $v(t) = \overline{v^k}(t)$ pour $t \in (a_k, a_{k+1})$ appartient à $L^2(Q_T)$ et amène la solution correspondante y à zéro au temps T.

Dans ce qui suit, les constantes $C_1, C_2, \ldots$ peuvent dépendre de T (mais pas de k). On commence par une estimation sur l'intervalle $[a_k, a_k + T_k]$. Le coût du contrôle partiel (1.18), la formule de Weyl $\sqrt{\lambda_{2^k}} \underset{+\infty}{\sim} C_1 \left(2^k\right)^{\frac{1}{N}}$, et le choix de ρ donnent

$$\left\|\overline{v^k}\right\|_{L^2(a_k, a_k+T_k; L^2(\Omega))} \leq C_2 e^{C_2 2^{\frac{k}{N}}} \left\|\overline{y^{k-1}}(a_k)\right\|_{L^2(\Omega)}. \tag{1.19}$$

En utilisant la dépendance continue par rapport aux données (1.2), on a alors

$$\left\|\overline{y^k}(a_k + T_k)\right\|_{L^2(\Omega)} \leq C_3 e^{C_2 2^{\frac{k}{N}}} \left\|\overline{y^{k-1}}(a_k)\right\|_{L^2(\Omega)}.$$

Sur l'intervalle $[a_k + T_k, a_{k+1}]$ maintenant, on utilise le fait que $\Pi_{E_{2^k}} \overline{y^k}(a_k + T_k) = 0$ avec la dissipation parabolique et l'estimation $\lambda_{2^k+1} T_k \underset{+\infty}{\sim} C_1 \left(2^k + 1\right)^{\frac{2}{N}} 2^{-k\rho} \geq C_1 \left(2^k\right)^{\frac{2}{N} - \rho}$ pour obtenir

$$\left\|\overline{y^k}(a_{k+1})\right\|_{L^2(\Omega)} \leq C_4 e^{-C_5 \left(2^k\right)^{\frac{2}{N} - \rho}} \left\|\overline{y^k}(a_k + T_k)\right\|_{L^2(\Omega)}.$$

En combinant les deux estimations précédentes et par récurrence on a

$$\left\|\overline{y^k}(a_{k+1})\right\|_{L^2(\Omega)} \leq C_6 e^{\sum_{p=0}^{k} \left(-C_5 (2^p)^{\frac{2}{N} - \rho} + C_2 (2^p)^{\frac{1}{N}}\right)} \|y_0\|_{L^2(\Omega)}.$$

Sachant que $\rho < \frac{1}{N}$, il existe $p_0 \geq 0$ tel que, pour $k \geq p_0$, on a

$$\sum_{p=0}^{k} \left(-C_5 (2^p)^{\frac{2}{N} - \rho} + C_2 (2^p)^{\frac{1}{N}}\right) \leq C_8 - C_7 \sum_{p=p_0}^{k} (2^p)^{\frac{2}{N} - \rho} \leq C_8 - C_7 \left(2^k\right)^{\frac{2}{N} - \rho}.$$

Ainsi, pour $k \geq p_0$, on a

$$\left\|\overline{y^k}(a_{k+1})\right\|_{L^2(\Omega)} \leq C_9 e^{-C_7 \left(2^k\right)^{\frac{2}{N} - \rho}} \|y_0\|_{L^2(\Omega)}. \tag{1.20}$$

Montrons enfin avec ces estimations que la fonction v est bien un contrôle dans $L^2(Q_T)$. Les estimations (1.19) et (1.20) donnent

$$\|v\|^2_{L^2(Q_T)} = \sum_{k=0}^{+\infty} \left\|\overline{v^k}\right\|^2_{L^2(a_k, a_k+T_k; L^2(\Omega))} \leq C_{10} \left(C_{11} + \sum_{k=p_0+1}^{+\infty} e^{C_2 2^{\frac{k}{N}} - C_7 \left(2^{k-1}\right)^{\frac{2}{N} - \rho}}\right) \|y_0\|^2_{L^2(\Omega)}.$$

Ainsi, $v \in L^2(Q_T)$ et la solution correspondante $y \in C^0([0,T]; L^2(\Omega))$, qui coïncide sur chaque intervalle $[a_k, a_{k+1}]$ avec $\overline{y^k}$, vérifie grâce à (1.20)

$$\|y(a_{k+1})\|_{L^2(\Omega)} = \left\|\overline{y^k}(a_{k+1})\right\|_{L^2(\Omega)} \xrightarrow[k \to +\infty]{} 0 = \|y(T)\|_{L^2(\Omega)}.$$

1.1.3 Les systèmes d'équations en dimension finie

Considérons à présent le système d'équations différentielles ordinaires

$$\begin{cases} \dfrac{d}{dt}y = Ay + Bv \quad \text{dans } (0,T), \\ y(0) = y_0, \end{cases} \tag{1.21}$$

où y est toujours l'état, y_0 la donnée initiale, v le contrôle, et A, B sont des matrices réelles de tailles $n \times n$ et $n \times m$.

On rappelle que pour tout $y_0 \in \mathbb{R}^n$ et $v \in L^2(0,T;\mathbb{R}^m)$ il existe un unique $y \in C^0([0,T];\mathbb{R}^n)$ solution de (1.21) qui dépend continûment des données.

Les définitions de contrôlabilités restent inchangées. Cependant, sachant qu'un sous-espace vectoriel de dimension finie est toujours fermé, on voit qu'il n'y en fait pas lieu de distinguer toutes les notions de contrôlabilité en dimension finie. On dira donc simplement que le système (1.21) est contrôlable au temps T ou qu'il ne l'est pas.

On va maintenant rappeler deux critères très simples de contrôlabilité pour le système (1.21).

La condition de rang de Kalman

En dimension finie on peut en fait complètement calculer l'image de l'opérateur G_T :

Théorème 6 ([KFA69])**.** *On a*

$$\operatorname{Im} G_T = \operatorname{Im} [\, A : B \,]_n,$$

où $[\, A : B \,]_n = (B|AB|\cdots|A^{n-1}B)$ *est une matrice de taille* $n \times mn$.

En conséquence, on obtient que le système (1.21) est contrôlable au temps T si, et seulement si, la condition de rang de Kalman est vérifiée, c'est-à-dire,

$$\operatorname{rank} [\, A : B \,]_n = n. \tag{1.22}$$

Cette condition donne un critère algébrique vraiment simple. Par exemple, le système 2×2 avec un seul contrôle

$$\begin{cases} \dfrac{d}{dt}y_1 = a_{11}y_1 + a_{12}y_2 + v \quad \text{dans } (0,T), \\ \dfrac{d}{dt}y_2 = a_{21}y_1 + a_{22}y_2 \qquad \text{dans } (0,T), \end{cases}$$

est contrôlable si, et seulement si,

$$a_{21} \neq 0.$$

Cela montre que le coefficient de couplage a_{21} est le seul dont il faut vraiment se soucier.

Caractérisation de Fattorini - test de Hautus

Nous énonçons maintenant un critère dual à la condition de Kalman (1.22) :

Théorème 7 ([Fat66, Hau69])**.** *Le système* (1.21) *est contrôlable au temps T si, et seulement si,*

$$\ker(A^* - \theta) \cap \ker B^* = \{0\}\,, \quad \forall \theta \in \mathbb{C}. \tag{1.23}$$

La condition (1.23) n'est autre que la propriété de continuation unique dans les sous-espaces propres de A^*. Ce que dit donc ce théorème c'est qu'il suffit de vérifier cette propriété dans les sous-espaces propres de A^* pour qu'elle soit en fait vérifiée dans tout l'espace $\mathbb{R}^n$.

Ce théorème, dû à Fattorini, est en fait plus connu en dimension finie sous le nom de test de Hautus, bien que le papier de Fattorini [Fat66] soit antérieur à celui de Hautus [Hau69]. Qui plus est, contrairement au résultat de Hautus, celui de Fattorini ne se restreint pas seulement à la dimension finie, il reste vrai dans un cadre beaucoup plus général, dont nous reparlerons plus loin. Les preuves sont cependant bien différentes puisque Hautus donne une preuve directe de l'équivalence entre la caractérisation (1.23) et la condition de Kalman (1.22).

Remarque 8. *La condition de Fattorini peut également s'écrire sous forme de condition d'indépendance linéaire des familles $\{B^*V_{\theta_i,1}, \dots, B^*V_{\theta_i,m_i}\}$, $i \in [\![1,p]\!]$, où $V_{\theta_i,1}, \dots, V_{\theta_i,m_i}$ est une base du sous-espace propre de A^* associé à la valeur propre θ_i (qui est donc de dimension m_i). Cela fait ressortir une condition nécessaire sur le nombre de contrôle dont on doit disposer. En effet, pour avoir une chance de contrôler le système, il faut donc au moins que*

$$m \geq \max_{1\leq i\leq p} m_i.$$

1.2 Problématique des systèmes et plan du mémoire

On sait donc depuis [LR95, FI96] que, pour tout temps $T > 0$ et tout ouvert non-vide $\omega \subset \Omega$, l'équation de la chaleur est contrôlable à zéro avec un contrôle interne ou frontière. Comme on peut le voir, la contrôlabilité pour une équation est donc indépendante du temps de contrôle et de la zone de contrôle. De plus, la contrôlabilité interne et la contrôlabilité frontière sont deux propriétés équivalentes.

Pour les systèmes d'équations, il se peut très bien que tous ces résultats soient mis en défaut. Par exemple, l'astuce qui consiste à étendre le domaine pour déduire la contrôlabilité frontière de la contrôlabilité interne ne fonctionne plus lorsque l'on cherche à contrôler un système avec moins de contrôles que d'équations. Il s'avère même faux que la contrôlabilité avec un contrôle interne entraîne la contrôlabilité avec un contrôle frontière, cela a été établi dans [FCGBdT10]. Ce fût le premier résultat qui mît vraiment en opposition la contrôlabilité d'équations avec la contrôlabilité de systèmes. D'autre part, dans [AKBGBdT12], on voit même que la contrôlabilité de certains systèmes paraboliques peut n'avoir lieu qu'à partir d'un certain temps. Ceci est très surprenant pour des problèmes de nature parabolique (par opposition au cas hyperbolique). Enfin, dans [Oli13] et [BO13], on voit que la contrôlabilité peut également dépendre de la géométrie du domaine de contrôle et du nombre d'équations du système.

Outre le comportement très différent des systèmes d'équations, on rencontre également des problèmes techniques pour caractériser la contrôlabilité de ces derniers. Pour mieux comprendre cela, on va rappeler comment les inégalités de Carleman permettent de résoudre certains problèmes de contrôlabilité pour les systèmes d'équations. On pointe ensuite les difficultés à employer cette technique dans d'autres situations, qui seront au coeur du sujet de cette thèse.

Inégalités de Carleman pour les systèmes

Considérons dans un premier temps le système d'équations

$$\begin{cases} \partial_t y_1 - \Delta y_1 = 1_\omega v_1 & \text{dans } Q_T, \\ \partial_t y_2 - \Delta y_2 = a_{21}(t,x) y_1 + 1_\omega v_2 & \text{dans } Q_T, \\ \quad y_1 = y_2 = 0 & \text{sur } \Sigma_T, \end{cases} \tag{1.24}$$

où $v_1, v_2 \in L^2(Q_T)$ sont les contrôles et $a_{21} \in L^\infty(Q_T)$ est quelconque. On a donc autant de contrôles que d'équations.

Le système adjoint du système (1.24) est le suivant :

$$\begin{cases} -\partial_t z_1 - \Delta z_1 = a_{21}(t,x) z_2 & \text{dans } Q_T, \\ -\partial_t z_2 - \Delta z_2 = 0 & \text{dans } Q_T, \\ \quad z_1 = z_2 = 0 & \text{sur } \Sigma_T, \end{cases} \tag{1.25}$$

et l'inégalité d'observabilité associée s'écrit

$$\|z_1(0)\|^2_{L^2(\Omega)} + \|z_2(0)\|^2_{L^2(\Omega)} \leq C_T^2 \left(\int_0^T \|1_\omega z_1(t)\|^2_{L^2(\Omega)}\, dt + \int_0^T \|1_\omega z_2(t)\|^2_{L^2(\Omega)}\, dt \right). \tag{1.26}$$

On va utiliser les inégalités de Carleman pour prouver cette inégalité d'observabilité. On note $I(s,\lambda;z)$ le membre de gauche dans l'inégalité de Carleman (1.16), soit

$$I(s,\lambda;z) = s^{-1} \iint_{Q_T} e^{-2s\alpha} \varphi^{-1} (|\partial_t z|^2 + |\Delta z|^2)\, dx\, dt \\ + s\lambda^2 \iint_{Q_T} e^{-2s\alpha} \varphi\, |\nabla z|^2\, dx\, dt + s^3\lambda^4 \iint_{Q_T} e^{-2s\alpha} \varphi^3\, |z|^2\, dx\, dt.$$

On applique l'inégalité de Carleman (1.16) sur ω aux deux équations du système adjoint (1.25) pour obtenir

$$I(s,\lambda;z_1) + I(s,\lambda;z_2) \\ \leq C \left(s^3\lambda^4 \iint_{(0,T)\times\omega} e^{-2s\alpha}\varphi^3\, |z_1|^2\, dx\, dt + s^3\lambda^4 \iint_{(0,T)\times\omega} e^{-2s\alpha}\varphi^3\, |z_2|^2\, dx\, dt \\ + \iint_{Q_T} e^{-2s\alpha}\, |z_2|^2\, dx\, dt \right).$$

Pour s ou λ assez grand, on peut absorber le terme $\iint_{Q_T} e^{-2s\alpha} |z_2|^2 \, dx\, dt$ par $I(s, \lambda; z_2)$ et ainsi obtenir

$$\begin{aligned} & I(s,\lambda;z_1) + I(s,\lambda;z_2) \\ & \leq C \left(s^3\lambda^4 \iint_{(0,T)\times\omega} e^{-2s\alpha}\varphi^3 |z_1|^2 \, dx\, dt + s^3\lambda^4 \iint_{(0,T)\times\omega} e^{-2s\alpha}\varphi^3 |z_2|^2 \, dx\, dt \right). \end{aligned}$$

L'inégalité d'observabilité (1.26) s'en déduit alors de la même façon que pour l'équation de la chaleur. $\square$

Comme on peut le voir, la méthode présentée ci-dessus n'est pas spécifique à la structure particulière du système (1.24). Lorsque l'on a autant de contrôles que d'équations, on peut, plus généralement, considérer des systèmes avec une matrice de couplage pleine et il est également possible de rajouter des couplages d'ordre 1 à coefficients variables en t et x. La vraie problématique des systèmes sera donc de contrôler avec moins de contrôles que d'équations.

Contrôlabilité avec une seule force

On considère à présent le système d'équations

$$\begin{cases} \partial_t y_1 - \Delta y_1 = 1_\omega v & \text{dans } Q_T, \\ \partial_t y_2 - \Delta y_2 = a_{21}(t,x) y_1 & \text{dans } Q_T, \\ \quad y_1 = y_2 = 0 & \text{sur } \Sigma_T. \end{cases} \tag{1.27}$$

L'objectif est donc maintenant de contrôler ce système de 2 équations avec 1 contrôle. On suit ici la présentation de [CGR06] (voir aussi [AKBD06]) et on renvoie à [GBdT10] pour un cadre plus général.

On suppose que $a_{21} \in L^\infty(Q_T)$ vérifie l'hypothèse suivante : il existe un ouvert non-vide $\omega_1 \subset\subset \omega$ et $\epsilon > 0$ tels que

$$a_{21} \geq \epsilon \quad \text{dans } (0,T) \times \omega_1. \tag{1.28}$$

Comme on va le voir dans la démonstration qui suit, l'hypothèse (1.28) est cruciale (d'un point de vue technique). Sans cette hypothèse on ne sait pas comment utiliser les inégalités de Carleman pour la contrôlabilité des systèmes lorsque l'on dispose de moins de contrôles que d'équations.

Le système adjoint du système (1.27) est le suivant :

$$\begin{cases} -\partial_t z_1 - \Delta z_1 = a_{21}(t,x) z_2 & \text{dans } Q_T, \\ -\partial_t z_2 - \Delta z_2 = 0 & \text{dans } Q_T, \\ \quad z_1 = z_2 = 0 & \text{sur } \Sigma_T, \end{cases} \tag{1.29}$$

et l'inégalité d'observabilité associée s'écrit

$$\|z_1(0)\|^2_{L^2(\Omega)} + \|z_2(0)\|^2_{L^2(\Omega)} \leq C_T^2 \int_0^T \|1_\omega z_1(t)\|^2_{L^2(\Omega)} \, dt. \tag{1.30}$$

Soit $\omega_2 \subset\subset \omega_1$ un ouvert non-vide. On commence comme tout à l'heure par appliquer l'inégalité de Carleman (1.16) sur ω_2 aux deux équations du système adjoint (1.29) :

$$\begin{aligned} &I(s,\lambda;z_1) + I(s,\lambda;z_2) \\ &\leq C\left(s^3\lambda^4 \iint_{(0,T)\times\omega_2} e^{-2s\alpha}\varphi^3 |z_1|^2 \, dx\, dt + s^3\lambda^4 \iint_{(0,T)\times\omega_2} e^{-2s\alpha}\varphi^3 |z_2|^2 \, dx\, dt\right). \end{aligned}$$

Sauf que maintenant il reste encore à estimer le terme $s^3\lambda^4 \iint_{(0,T)\times\omega_2} e^{-2s\alpha}\varphi^3 |z_2|^2 \, dx\, dt$.

Soit $\xi \in C_0^\infty(\overline{\omega_1})$ une fonction de troncature vérifiant :

$$\xi = 1 \text{ dans } \omega_2, \quad 0 \leq \xi \leq 1 \text{ dans } \omega_1.$$

En utilisant l'hypothèse (1.28) on a

$$\epsilon s^3\lambda^4 \iint_{(0,T)\times\omega_2} e^{-2s\alpha}\varphi^3 |z_2|^2 \, dx\, dt \leq s^3\lambda^4 \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\varphi^3 \xi a_{21} |z_2|^2 \, dx\, dt$$

D'autre part, en multipliant par $s^3\lambda^4 e^{-2s\alpha}\varphi^3\xi z_2$ l'équation vérifiée par z_1 et en intégrant sur $(0,T)\times\omega_1$ on obtient

$$s^3\lambda^4 \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\varphi^3 \xi a_{21} |z_2|^2 \, dx\, dt = s^3\lambda^4 \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\varphi^3 \xi(-\partial_t z_1 z_2 - \Delta z_1 z_2)\, dx\, dt.$$

Considérons le terme $-s^3\lambda^4 \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\varphi^3\xi\Delta z_1 z_2 \, dx\, dt$ (l'autre se traitant de la même façon). On a

$$-s^3\lambda^4 \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\varphi^3\xi\Delta z_1 z_2 \, dx\, dt = -s^3\lambda^4 \iint_{(0,T)\times\omega_1} \Delta(e^{-2s\alpha}\varphi^3\xi z_2) z_1 \, dx\, dt,$$

où les termes de bord sont nuls grâce à la fonction de troncature ξ.

On sépare alors cette intégrale en trois autres, suivant $\Delta(e^{-2s\alpha}\varphi^3\xi z_2) = \Delta(e^{-2s\alpha}\varphi^3\xi)z_2 + 2\nabla(e^{-2s\alpha}\varphi^3\xi)\cdot\nabla z_2 + (e^{-2s\alpha}\varphi^3\xi)\Delta z_2$.

Estimons par exemple le premier terme $-s^3\lambda^4 \iint_{(0,T)\times\omega_1} \Delta(e^{-2s\alpha}\varphi^3\xi) z_2 z_1 \, dx\, dt$. Tout d'abord, l'estimation du poids

$$\left|\Delta(e^{-2s\alpha}\varphi^3\xi)\right| \leq Cs^2\lambda^2 e^{-2s\alpha}\varphi^5,$$

donne

$$-s^3\lambda^4 \iint_{(0,T)\times\omega_1} \Delta(e^{-2s\alpha}\varphi^3\xi) z_2 z_1 \, dx\, dt \leq Cs^5\lambda^6 \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\varphi^5 |z_2|\,|z_1| \, dx\, dt.$$

En utilisant maintenant l'inégalité de Young entre $\sqrt{\epsilon}\frac{1}{s\lambda}e^{-s\alpha}\frac{1}{\varphi}|z_1|$ et $\frac{1}{\sqrt{\epsilon}}s\lambda e^{-s\alpha}\varphi|z_2|$ on obtient

$$\begin{aligned} - s^3\lambda^4 \iint_{(0,T)\times\omega_1} \Delta(e^{-2s\alpha}\varphi^3\xi) z_2 z_1 \, dx\, dt \leq \frac{1}{\epsilon}Cs^7\lambda^8 \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\varphi^7 |z_2|^2 \, dx\, dt \\ + \epsilon C s^3\lambda^4 \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\varphi^3 |z_1|^2 \, dx\, dt, \end{aligned}$$

et pour ϵ assez petit on peut absorber le dernier terme.

En estimant de même tous les autres termes, on obtient finalement

$$I(s,\lambda;z_1) + I(s,\lambda;z_2) \leq Cs^7\lambda^8 \iint_{(0,T)\times\omega_1} e^{-2s\alpha}\varphi^7 |z_1|^2 \, dx\, dt.$$

Il est ensuite facile d'en déduire l'inégalité d'observabilité (1.30), comme précédemment. □

Dans la preuve, c'est l'hypothèse (1.28) qui nous a permis d'utiliser l'information fournie par l'équation où z_2 intervenait. Lorsque la zone de contrôle ne rencontre pas la zone de couplage (le cas modèle étant $a_{21} = 1_{\mathcal{O}}$ avec $\mathcal{O} \cap \omega = \emptyset$), on ne sait pas comment récupérer cette information. La contrôlabilité frontière des systèmes paraboliques se trouve actuellement dans la même impasse. On ne sait pas comment utiliser les inégalités de Carleman dans ce cas. Pour ces deux problèmes, il a fallu employer d'autres techniques [FCGBdT10, ABL12, Oli13, BBGBO13],...

Plan du mémoire

Les chapitres 2 à 5 sont les reprises dans l'ordre chronologique des articles [Oli12], [Oli13], [BBGBO13] et [BO13]. La suite de l'introduction est organisée en deux parties et résume les principaux résultats obtenus dans les articles cités ci-dessus. La première partie (section 1.3) est consacrée aux systèmes couplés par des coefficients constants et la seconde (section 1.4) aux systèmes en cascade avec des couplages variables ou d'ordre 1. On conclut l'introduction avec quelques perspectives et problèmes ouverts.

1.3 Contrôlabilité de systèmes à coefficients constants

1.3.1 Quelques résultats connus

On rappelle dans un premier temps deux résultats essentiels, issus de [AKBDGB09b] et [AKBGBdT11a], concernant la contrôlabilité interne et la contrôlabilité frontière en dimension 1 de systèmes à coefficients constants.

Contrôlabilité interne. Considérons le système

$$\begin{cases} \partial_t y - D\Delta y = Ay + 1_\omega Bv & \text{dans } Q_T, \\ y = 0 & \text{sur } \Sigma_T, \end{cases} \tag{1.31}$$

où $D, A \in \mathcal{M}_n(\mathbb{R}), B \in \mathcal{M}_{n\times m}(\mathbb{R})$ sont des matrices constantes et D est diagonalisable avec des valeurs propres réelles strictement positives. Dans ce qui suit, on note λ_k, $k \geq 1$, les fonctions propres distinctes de $-\Delta$ (avec condition de Dirichlet homogène).

En utilisant les inégalités de Carleman, on peut obtenir (voir [AKBDGB09b]) le résultat suivant.

Théorème 9 ([AKBDGB09b]). *Le système* (1.31) *est contrôlable à zéro au temps T si, et seulement si,*

$$\operatorname{rang}\left[\, -\lambda_k D + A \,:\, B \,\right]_n = n, \quad \forall k \geq 1. \tag{1.32}$$

• Cette condition est indépendante du temps de contrôle T et de la zone de contrôle ω.

• Il est intéressant de remarquer [AKBDGB09b, Remark 1.1] que même sans couplage ($A = 0$), le système (1.31) peut quand même être contrôlable grâce à la matrice de diffusion D. En effet, si B est un vecteur dont tous les coefficients sont non nuls, et que les valeurs propres de D sont toutes distinctes, alors le système (1.31) est contrôlable pour $A = 0$.

• La difficulté de la preuve vient essentiellement de la matrice de diffusion D. Dans le cas où D est la matrice identité, il suffit d'effectuer le changement de variable fourni par la condition $\operatorname{rang}\left[\, A \,:\, B \,\right]_n = n$ pour se ramener à un système pour lequel le résultat de [GBdT10] s'applique, voir [AKBDGB09a] pour plus de détails.

Contrôlabilité frontière en dimension 1. Dans [FCGBdT10], puis [AKBGBdT11a], les auteurs généralisent la méthode des moments de Fattorini et Russell [FR71, FR75] pour caractériser la contrôlabilité frontière à zéro en dimension 1 du système suivant :

$$\begin{cases} \partial_t y - \partial_x^2 y = Ay & \text{dans } (0,T) \times (0,1), \\ y(t,0) = Bv(t), \quad y(t,1) = 0 & \text{sur } (0,T), \end{cases} \tag{1.33}$$

où $A \in \mathcal{M}_n(\mathbb{R}), B \in \mathcal{M}_{n\times m}(\mathbb{R})$ sont des matrices constantes.

Théorème 10 ([AKBGBdT11a]). *Le système* (1.33) *est contrôlable à zéro au temps T si, et seulement si,*

$$\operatorname{rang}\left[\, A_k \,:\, C_k \,\right]_{nk} = nk, \quad \forall k \geq 1, \tag{1.34}$$

où on a noté

$$A_k = \begin{pmatrix} -\lambda_1 + A & 0 & \cdots & \cdots & 0 \\ 0 & -\lambda_2 + A & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -\lambda_k + A \end{pmatrix} \in \mathcal{M}_{nk}(\mathbb{R}), \quad C_k = \begin{pmatrix} B \\ B \\ \vdots \\ \vdots \\ B \end{pmatrix} \in \mathcal{M}_{nk\times m}(\mathbb{R}). \tag{1.35}$$

• Cette condition de rang (1.34), tout comme (1.32) en fait, provient de la dimension finie. C'est d'ailleurs la raison pour laquelle elle est également appelée condition de Kalman. Il est facile de voir que c'est en effet une condition nécessaire en regardant la contrôlabilité approchée dans la somme des k premiers sous-espaces propres de l'opérateur $\Delta + A^*$ (qui sont de dimension finie).

• Ce résultat à été initialement prouvé dans le cas de deux équations $n = 2$ et un contrôle $m = 1$ dans [FCGBdT10]. Ce dernier est d'ailleurs le tout premier résultat concernant la contrôlabilité frontière de systèmes paraboliques.

• La même condition caractérise également la contrôlabilité approchée ([AKBGBdT11a, Theorem 6.1]).

• Pour $k = 1$, la condition (1.34) donne $\operatorname{rang}[\,A\,:\,B\,]_n = n$, ce qui n'est autre que la condition qui caractérise la contrôlabilité interne de ce même système (théorème 9). Ainsi, la contrôlabilité frontière implique la contrôlabilité interne pour ces systèmes, alors que le contraire n'est pas vrai en général. Cela est à mettre en contraste avec les résultats concernant les équations, pour lesquelles ces deux propriétés sont équivalentes.

• Soient $\{\theta_i\}_{i\in[\![1,p]\!]} \subset \mathbb{C}$ les valeurs propres distinctes de la matrice A^*. A priori la condition de rang (1.34) doit être vérifiée pour toutes les fréquences k mais il suffit en fait qu'elle soit satisfaite pour une seule k_0, qui est la première pour laquelle la propriété

$$-\lambda_k + \theta_i \neq -\lambda_{k'} + \theta_{i'}, \quad \forall k' \geq 1, \forall i, i' \in [\![1,p]\!], \quad k' \neq k,\, i' \neq i, \tag{1.36}$$

est vérifiée pour tout $k \geq k_0$ [AKBGBdT11a, Corollary 3.3].

• Dans le cas particulier où on cherche à contrôler le système par une seule force ($m = 1$), la condition devient plus simple à exprimer [AKBGBdT11a, Proposition 3.4]. En effet, dans ce cas (1.34) est vérifiée si, et seulement si, $k_0 = 1$ et la condition de Kalman est satisfaite, soit :

1. $-\lambda_k + \theta_i \neq -\lambda_{k'} + \theta_{i'}$ pour tout $k, k' \geq 1$ et $i, i' \in [\![1,p]\!]$ tels que $k' \neq k$ et $i' \neq i$,
2. $\operatorname{rang}[\,A\,:\,B\,]_n = n$.

On retrouve alors la condition telle qu'elle est énoncée dans [FCGBdT10] pour deux équations et un contrôle.

• On peut également considérer le cas où le contrôle agit sur l'autre partie du bord, cela ne change pas la condition (1.34). Par contre, si on met un contrôle $B_1 v_1$ sur une partie et un autre contrôle $B_2 v_2$ sur l'autre partie, alors la matrice C_k doit être remplacée par

$$C_k = \begin{pmatrix} B_1 & -B_2 \\ B_1 & B_2 \\ B_1 & -B_2 \\ \vdots & \vdots \\ B_1 & (-1)^k B_2 \end{pmatrix}$$

et la condition (1.34) reste alors inchangée. Cette alternance de signe qui apparaît provient en fait de la dérivée normale des fonctions propres de l'opérateur $-\partial_x^2$. Même si on est en dimension 1, on voit que la géométrie peut donc jouer un rôle dans la contrôlabilité des systèmes paraboliques.

1.3.2 Contrôles agissant sur différentes parties du domaine ou de sa frontière

Dans le chapitre 2 on s'intéresse au problème de contrôlabilité suivant : trouver des contrôles à zéro internes $u_j \in L^2(0,T;L^2(0,1))$, $j \in [\![1,m]\!]$, et frontières $v_i \in L^2(0,T)^{m_i}$, $i \in [\![1,2]\!]$, pour le système

$$\begin{cases} \partial_t y - \partial_x^2 y = Ay + 1_{\omega_1} D_1 u_1 + \ldots + 1_{\omega_m} D_m u_m & \text{dans } (0,T)\times(0,1), \\ y(t,0) = B_1 v_1(t), \quad y(t,1) = B_2 v_2(t) & \text{sur } (0,T). \end{cases} \tag{1.37}$$

Dans (1.37), $A \in \mathcal{M}_n(\mathbb{R}), B_i \in \mathcal{M}_{n\times m_i}(\mathbb{R})$, $i \in [\![1,2]\!]$, sont des matrices constantes et $D_j \in \mathbb{R}^n$, $j \in [\![1,m]\!]$, sont des vecteurs constants (le cas où les D_j sont des matrices se réduit à cette configuration, quitte à considérer plusieurs fois la même zone de contrôle ω_j).

Bien évidemment si l'une des conditions (1.32) ou (1.34) est vérifiée, le système (1.37) est contrôlable à zéro. Cependant, il se peut très bien que ces deux conditions ne soient pas vérifiées et que, pourtant, le système soit quand même contrôlable. C'est tout l'intérêt de ce travail. On décrit une telle situation après l'énoncé du théorème.

Théorème 11 ([Oli12])**.** *Le système* (1.37) *est contrôlable à zéro au temps T si, et seulement si,*

$$\operatorname{rang}\,[\,A_k \,:\, C_k\,]_{nk} = nk, \quad \forall k \geq 1,$$

où A_k est défini comme en (1.35) *et*

$$C_k = \begin{pmatrix} B_1 & -B_2 & D & 0 & \cdots & \cdots & 0 \\ B_1 & B_2 & 0 & D & \ddots & & \vdots \\ B_1 & -B_2 & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & 0 \\ B_1 & (-1)^k B_2 & 0 & \cdots & \cdots & 0 & D \end{pmatrix} \in \mathcal{M}_{nk\times(m_1+m_2+mk)}(\mathbb{R}),$$

avec $D = (D_1|\cdots|D_m) \in \mathcal{M}_{n\times m}(\mathbb{R})$.

- Lorsqu'il n'y a que des contrôles internes ou frontières, on retrouve les conditions des théorèmes 9 et 10, cela se lit facilement sur la condition de rang. Ce théorème a donc pour but d'unifier ces deux résultats.
- De même que pour le théorème 10 il suffit de vérifier cette condition seulement pour une fréquence k_0 (et qui est définie de la même façon).
- Lorsqu'il n'y a que des contrôles internes ($B_1 = B_2 = 0$), le résultat est valable en dimension quelconque (voir Corollary 2.5 au chapitre 2).

• Afin d'illustrer les propos précédant l'énoncé du théorème, considérons le jeu de données suivant :

$$A = \begin{pmatrix} 2 & 6 & 2 \\ 4 & 0 & -2 \\ 2 & -3/2 & 2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad B_2 = 0, \quad D = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{1.38}$$

Puisque $\operatorname{rang}[\,A : D\,]_3 = 2 \neq 3$ et $\operatorname{rang}[\,A : B\,]_3 = 2 \neq 3$, on ne peut contrôler seulement avec des contrôles internes ou des contrôles frontières d'après les résultats des théorèmes 9 et 10. Par contre on a $\operatorname{rang}[\,A_1 : C_1\,]_3 = 3$, et les valeurs propres de A^* sont $-5, 3$ et 6 ce qui montre que la première fréquence pour laquelle la propriété (1.36) est vérifiée est $k_0 = 1$. D'après le théorème précédent, le système (1.37) avec (1.38) est contrôlable.

• On va prouver le théorème 11 pour cet exemple particulier, cela donne une bonne idée de la preuve générale. Dans un premier temps, on voit que la matrice A est équivalente à la matrice suivante

$$\widetilde{A} = \begin{pmatrix} 6 & 0 & 0 \\ 6 & 0 & 15 \\ -2 & 1 & -2 \end{pmatrix}$$

grâce au changement de base

$$P = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & -2 \\ 0 & 1 & 2 \end{pmatrix},$$

qui n'est autre que la matrice $(B_1|D|AD)$. Ainsi, la contrôlabilité du système initial (1.37) est équivalente à la contrôlabilité du système

$$\begin{cases} \partial_t \widetilde{y} - \partial_x^2 \widetilde{y} = \begin{pmatrix} 6 & 0 & 0 \\ 6 & 0 & 15 \\ -2 & 1 & -2 \end{pmatrix} \widetilde{y} + 1_{\omega_1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u & \text{dans } (0,T) \times (0,1), \\ \widetilde{y}(t,0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} v(t), \quad \widetilde{y}(t,1) = 0 & \text{sur } (0,T). \end{cases}$$

Comme on peut le voir, la matrice de changement de base P a été construite de sorte à ce que, dans la nouvelle base, la première équation soit indépendante des composantes $\widetilde{y}_2$ et $\widetilde{y}_3$. On peut donc se servir du contrôle au bord v pour amener dans un premier temps la première composante $\widetilde{y}_1$ à zéro au temps $T/2$ (par exemple). Il reste alors à contrôler sur $(T/2,T)$ le système réduit suivant :

$$\begin{cases} \partial_t \begin{pmatrix} \widetilde{y}_2 \\ \widetilde{y}_3 \end{pmatrix} - \partial_x^2 \begin{pmatrix} \widetilde{y}_2 \\ \widetilde{y}_3 \end{pmatrix} = \begin{pmatrix} 0 & 15 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \widetilde{y}_2 \\ \widetilde{y}_3 \end{pmatrix} + 1_{\omega_1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} u & \text{dans } (T/2,T)\times(0,1), \\ \begin{pmatrix} \widetilde{y}_2 \\ \widetilde{y}_3 \end{pmatrix}(t,0) = \begin{pmatrix} \widetilde{y}_2 \\ \widetilde{y}_3 \end{pmatrix}(t,1) = 0 & \text{sur } (T/2,T), \end{cases}$$

ce qui est faisable étant donné que la condition de rang de Kalman version contrôle distribué est vérifiée. □

1.3.3 Contrôlabilité frontière en dimension $N > 1$

On a vu qu'en dimension 1, la méthode des moments permet de caractériser la contrôlabilité par la condition (1.34). En dimension supérieure, il n'existe que très peu de résultats [ABL12, AB12]. Dans [AB12] est obtenu le premier résultat de contrôlabilité à zéro en dimension quelconque pour le système en cascade

$$\begin{cases} \partial_t y_1 - \Delta y_1 = 0 & \text{dans } Q_T, \\ \partial_t y_2 - \Delta y_2 = a_{21}(x) y_1 & \text{dans } Q_T, \\ y_1 = 1_\gamma v, \quad y_2 = 0 & \text{sur } \Sigma_T, \end{cases} \tag{1.39}$$

où $a_{21} \in L^\infty(\Omega)$ et l'ouvert non-vide $\gamma \subset \partial\Omega$ est la zone de contrôle.

Théorème 12 ([AB12])**.** *Soit Ω de classe C^∞. On suppose que $a_{21} \geq 0$ sur Ω et qu'il existe $\epsilon > 0$ et un ouvert non-vide $\mathcal{O} \subset \Omega$ tels que que $a_{21} \geq \epsilon$ sur $\mathcal{O}$. On suppose de plus que $\mathcal{O}$ et γ satisfont la condition géométrique de contrôle des ondes. Alors, le système* (1.39) *est contrôlable à zéro au temps T.*

Il est en fait établi pour un système d'équations des ondes et les propriétés de contrôlabilité sont ensuite transférées sur le système (1.39) via la méthode de transmutation. Cela requiert donc en particulier la condition géométrique de contrôle des ondes sur les zones de contrôle et de couplage, hypothèse a priori moins naturelle pour les systèmes paraboliques. Rappelons tout de même que cette condition géométrique est automatiquement vérifiée en dimension 1.

1.3.3.1 Contrôlabilité approchée

Dans le chapitre 3 on utilise le théorème de Fattorini pour obtenir de nouveaux résultats de contrôlabilité frontière en dimension $N > 1$ pour le système

$$\begin{cases} \partial_t y - \Delta y = Ay & \text{dans } Q_T, \\ y = 1_\gamma B v & \text{sur } \Sigma_T. \end{cases} \tag{1.40}$$

Une condition spectrale suffisante. Dans la section 3.2.1 on obtient notamment que la condition (1.36) de [FCGBdT10] est en fait une condition suffisante en dimension quelconque :

Théorème 13 ([Oli13]). *On suppose que* $\ker(\theta_i - A^*) \cap \ker B^* = \{0\}$ *pour tout* $i \in [\![1, p]\!]$. *Si*

$$-\lambda_k + \theta_i \neq -\lambda_{k'} + \theta_{i'}, \tag{1.41}$$

pour tout $k, k' \geq 1$ *et* $i, i' \in [\![1, p]\!]$ *tels que* $k' \neq k$ *et* $i' \neq i$, *alors, le système* (1.40) *est approximativement contrôlable.*

• La condition (1.41) n'est en général pas nécessaire, sauf dans le cas très particulier où l'on cherche à contrôler le système avec une seule force ($m = 1$) en dimension $N = 1$ (voir secton 3.2.4). Lorsque la situation $-\lambda_k + \theta_i = -\lambda_{k'} + \theta_{i'}$ se produit il faut alors vérifier une propriété d'unicité sur les fonctions propres de l'opérateur $\Delta + A^*$ associées à ces indices, ce qui peut être une tâche difficile.

• Une application intéressante de ce théorème est lorsque la matrice A^* n'a qu'une seule valeur propre puisqu'alors l'hypothèse (1.41) est automatiquement vérifiée. Ainsi, le système (1.40) avec

$$A = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ a_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n-1} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

est approximativement contrôlable si, et seulement si,

$$a_{i,i-1} \neq 0, \quad \forall i \in [\![2, n]\!].$$

Étude sur un domaine rectangulaire. Dans la section 3.2.5 on regarde ce qu'il peut se passer quand le domaine Ω est un rectangle : $\Omega = (0, X_1) \times (0, X_2)$,

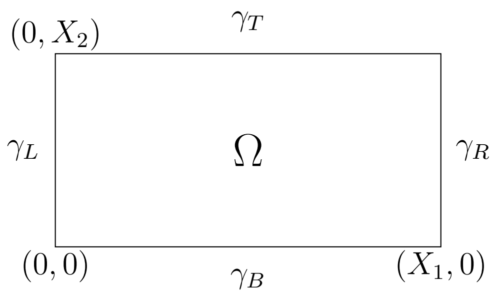


FIGURE 1.2 – Domaine Ω

où γ_L, γ_R, γ_T et γ_B sont les faces du rectangle (mais on peut tout aussi bien ne considérer qu'une partie de ces faces).

Théorème 14 ([Oli13]). *Supposons vérifiée la condition nécessaire* $\ker(\theta_i - A^*) \cap \ker B^* = \{0\}$ *pour tout* $i \in [\![1, p]\!]$. *Si* $\gamma = \gamma_L \cup \gamma_T$ *et* $n = 2$, *alors le système* (1.40) *est approximativement contrôlable au temps* T.

• On peut interpréter ce théorème de la façon suivante. Le fait d'avoir deux directions γ_L et γ_T semble "créer" un contrôle supplémentaire. Tout se passe alors comme si l'on avait deux contrôles pour deux équations, ce qui suffit pour contrôler. Ceci n'est pas le cas lorsque l'on considère deux faces parallèles $\gamma = \gamma_L \cup \gamma_R$ puisque la contrôlabilité est réduite à la contrôlabilité d'un système de dimension 1 (qui peut être mise en défaut), voir Theorem 3.19. De plus, pour $n > 2$ on peut trouver des cas où le système (1.40) n'est pas approximativement contrôlable sur $\gamma = \gamma_L \cup \gamma_T$ (voir Theorem 3.21), ce qui semble renforcer cette idée. On voit donc que non seulement la géométrie de la zone de contrôle γ peut jouer un rôle important, mais que c'est tout aussi bien le cas pour le nombre d'équations n.

1.3.3.2 Contrôlabilité à zéro dans des domaines cylindriques

Dans le chapitre 4 on obtient un résultat de contrôlabilité à zéro par un contrôle frontière en dimension quelconque du système

$$\begin{cases} \partial_t y - \Delta y = Ay & \text{dans } Q_T, \\ y = 1_\gamma B v & \text{sur } \Sigma_T, \end{cases} \tag{1.42}$$

mais dans des domaines Ω ayant une géométrie particulière, à savoir qu'ils sont de la forme

$$\Omega = \Omega_1 \times \Omega_2,$$

avec $\Omega_i \subset \mathbb{R}^{N_i}$, $i \in [\![1, 2]\!]$, des ouverts bornés connexes non-vides réguliers.

Théorème 15 ([BBGBO13]). *Soit* $\gamma_1 \subset \partial\Omega_1$ *un ouvert non-vide. Supposons que le système posé sur* Ω_1

$$\begin{cases} \partial_t y^1 - \Delta_{x_1} y^1 = Ay^1 & \textit{dans } (0,T) \times \Omega_1, \\ y^1 = 1_{\gamma_1} B v^1 & \textit{sur } (0,T) \times \partial\Omega_1, \end{cases}$$

soit contrôlable à zéro pour tout temps $T > 0$, *avec en plus l'estimation suivante sur le coût du contrôle associé* $C_T^{\Omega_1}$:

$$C_T^{\Omega_1} \leq C e^{C/T}, \quad \forall T > 0.$$

Alors, pour tout ouvert non-vide $\omega_2 \subset \Omega_2$, *le système* (1.42) *posé sur* $\Omega = \Omega_1 \times \Omega_2$ *est contrôlable à zéro pour tout temps* $T > 0$ *sur le domaine de contrôle* $\gamma = \gamma_1 \times \omega_2$.

• Dans le cas où l'on cherche à contrôler sur la totalité d'une des faces de $\Omega_1 \times \Omega_2$, c'est-à-dire avec $\omega_2 = \Omega_2$, la preuve de ce théorème est assez simple. Elle peut être établie en utilisant une décomposition de Fourier dans la direction Ω_2 comme déjà indiqué dans [Mil05], et cela ne requiert en aucun cas l'estimation du coût du contrôle. La nouveauté dans ce théorème réside donc surtout dans le fait qu'on puisse considérer comme zone de contrôle un sous-domaine strict de la frontière.

Coût du contrôle en dimension 1. Le deuxième résultat établi dans le chapitre 4 est l'estimation du coût du contrôle en $e^{C/T}$ pour le système (1.33) en dimension 1. Cela fournit un exemple important auquel on peut appliquer le théorème 15.

La construction décrite dans [AKBGBdT11a] ne permettait pas d'obtenir d'estimation du coût contrôle et a donc dû être revue. Afin d'obtenir ce facteur $e^{C/T}$, on suit ainsi une approche légèrement différente en pistant la dépendance en temps T de toutes les constantes qui interviennent.

Théorème 16 ([BBGBO13])**.** *Sous l'hypothèse que la condition de rang* (1.34) *est vérifiée, pour tout* $T > 0$ *et* $y_0 \in H^{-1}(0,1)^n$ *il existe un contrôle à zéro* $v \in L^2(0,T)^m$ *pour le système* (1.33) *qui vérifie de plus l'estimation*

$$\|v\|_{L^2(0,T)^m} \leq Ce^{C/T}\|y_0\|_{H^{-1}(0,1)^n}.$$

• Combinant les théorèmes 15 et 16 on obtient donc que le système (1.42) posé sur $\Omega = (0,1) \times \Omega_2$, est contrôlable à zéro au temps T sur $\gamma = \{0\} \times \omega_2$ si, et seulement si,

$$\operatorname{rang}\,[\,A_k\,:\,C_k\,]_{nk} = nk, \quad \forall k \geq 1,$$

où A_k et C_k sont définis comme dans (1.35).

On peut penser par exemple à la contrôlabilité sur un cylindre où la zone de contrôle est située sur l'une de ses bases :

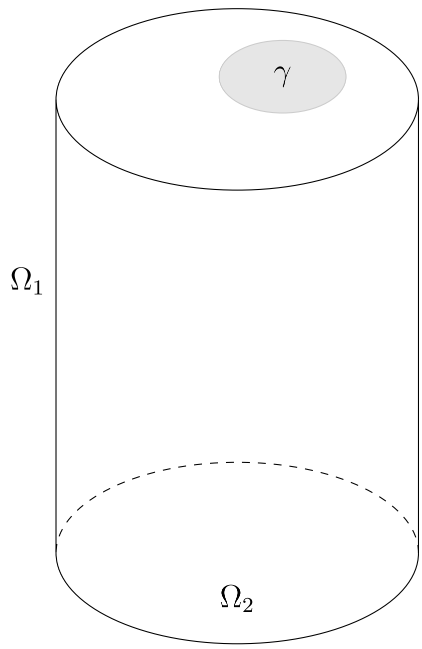


FIGURE 1.3 – Configuration géométrique typique

1.4 Contrôlabilité approchée de systèmes en cascade

La théorème de Fattorini est le point de départ de tous les résultats qui suivent. Les systèmes en cascades ayant une bonne structure spectrale, la caractérisation de Fattorini permet de donner des critères de contrôlabilité approchée assez simples, et qui sont de plus nécessaires et suffisants. Ce type de systèmes fût étudié pour la première fois dans [dT00] dans le cadre du contrôle insensibilisant.

On rappelle que le théorème 13 constitue également un résultat de contrôlabilité pour les systèmes en cascade (à coefficients constants).

1.4.1 Couplages d'ordre 1

1.4.1.1 Contrôlabilité interne

Dans la section 3.4 du chapitre 3 on apporte de nouveaux résultats pour la contrôlabilité interne du système en cascade suivant, où le couplage est assuré par un terme constant d'ordre 1,

$$\begin{cases} \partial_t y_1 - \Delta y_1 = 1_\omega v & \text{dans } Q_T, \\ \partial_t y_2 - \Delta y_2 = G_{21} \cdot \nabla y_1 & \text{dans } Q_T, \\ y_1 = y_2 = 0 & \text{sur } \Sigma_T. \end{cases} \tag{1.43}$$

Dans (1.43), $G_{21} \in \mathbb{R}^N$ est un vecteur constant non-nul et $\omega \subset \Omega$ est la zone de contrôle.

Il existe très peu de résultats concernant ce type de système. De plus, aucun ne semble apporter de réponse complète, une restriction sur la dimension N ou une condition géométrique sur ω est nécessaire pour les appliquer.

Théorème 17 ([Gue07]). *Le système* (1.43) *est contrôlable à zéro en dimension 1.*

Théorème 18 ([BCGdT13]). *Si $\partial\omega \cap \partial\Omega$ est non-vide, alors le système* (1.43) *est contrôlable à zéro (en dimension quelconque).*

Le premier de ces résultats est une conséquence de [Gue07, Theorem 4]. On a donc un résultat positif en dimension 1 ou bien lorsque que la zone de contrôle ω touche le bord du domaine. Ces hypothèses semblent cependant seulement techniques. Dans la section 3.4 on démontre le résultat de contrôlabilité suivant :

Théorème 19 ([Oli13]). *Le système* (1.43) *est approximativement contrôlable (en dimension quelconque et quel que soit ω).*

1.4.1.2 Contrôlabilité frontière en dimension 1

Concernant la contrôlabilité frontière avec un couplage d'ordre 1 il n'existe, à ma connaissance, aucun résultat, pas même en dimension 1 et à coefficients constants. Dans la section 3.3.2 on obtient une caractérisation complète de la contrôlabilité approchée en dimension 1 pour le système à coefficients variables suivant :

$$\begin{cases} \partial_t y_1 - \partial_x^2 y_1 = 0 & \text{dans } (0,T)\times(0,1), \\ \partial_t y_2 - \partial_x^2 y_2 = G_{21}(x)\partial_x y_1 + a_{21}(x) y_1 & \text{dans } (0,T)\times(0,1), \\ y_1 = 1_{\{0\}} v, \quad y_2 = 0 & \text{sur } (0,T)\times\{0,1\}\,, \end{cases} \tag{1.44}$$

où $G_{21} \in W^{1,\infty}(0,1)$ et $a_{21} \in L^\infty(0,1)$.

Théorème 20 ([Oli13]). *Le système* (1.44) *est approximativement contrôlable si, et seulement si,*

$$\int_0^1 \left(-\frac{1}{2}G'_{21}(x) + a_{21}(x)\right)(\phi_k(x))^2 dx \neq 0, \quad \forall k \geq 1, \tag{1.45}$$

où on rappelle que les ϕ_k sont les fonctions propres du Laplacien en dimension 1 sur $(0,1)$.

• Avec ce théorème on peut vérifier simplement si le système (1.44) est approximativement contrôlable ou non. Par exemple, dès que la fonction $-\frac{1}{2}G'_{21} + a_{21}$ a un signe constant (strictement positive ou négative), on sait que la réponse est affirmative.

• En comparant avec le théorème 19 obtenu précédemment, on observe une réelle différence entre contrôlabilité interne et contrôlabilité au bord pour ces systèmes. En effet, on voit facilement grâce à la condition (1.45) que dans le cas où G_{21} est constant et $a_{21} = 0$ le système (1.44) n'est pas approximativement contrôlable par le bord (alors qu'il l'est avec un contrôle interne !).

• Lorsque $G_{21} = 0$, (1.45) est également une condition nécessaire et suffisante de contrôlabilité à zéro [GB].

1.4.2 Contrôlabilité interne à coefficients variables en dimension 1

Dans ce qui suit $\Omega = (0,1)$. Au chapitre 5 on s'intéresse à la contrôlabilité interne des systèmes

$$\begin{cases} \partial_t y - \partial_x^2 y = A(x) y + 1_\omega B v & \text{dans } Q_T, \\ \qquad y = 0 & \text{sur } \Sigma_T, \end{cases} \tag{1.46}$$

où les matrices $A(x)$ et B sont d'une des formes suivantes :

(1.46-i) $A(x) = \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (contrôlabilité d'un système 2×2).

(1.46-ii) $A(x) = \begin{pmatrix} 0 & 0 & 0 \\ a_{21}(x) & 0 & 0 \\ a_{31}(x) & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ (contrôlabilité simultanée de plusieurs systèmes 2×2).

$$\text{(1.46-iii)} \quad A(x) = \begin{pmatrix} 0 & 0 & 0 \\ a_{21}(x) & 0 & 0 \\ 0 & a_{32}(x) & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ (contrôlabilité d'un système } 3 \times 3).$$

On rappelle que dans le cas des coefficients constants le problème est complètement résolu depuis [AKBDGB09b] (théorème 9) et que les systèmes (1.46-i) et (1.46-iii) sont alors contrôlable à zéro, tandis que le système (1.46-ii) ne l'est pas.

D'autre part, lorsque les couplages a_{21} et a_{32} sont localisés dans une zone commune de ω, on sait ([GBdT10]) que la contrôlabilité à zéro des systèmes (1.46-i) et (1.46-iii) est vérifiée (d'ailleurs, en dimension quelconque).

Quand cette condition n'est plus assurée, le problème reste assez ouvert. Bien que l'on connaisse depuis [KdT10] des conditions suffisantes pour que le système (1.46-i) soit approximativement contrôlable même si le couplage est localisé en dehors de ω, on va voir que ces dernières ne sont pas, en général, nécessaires. On aborde ici le problème avec une approche différente, et on obtient des conditions nécessaires et suffisantes de contrôlabilité approchée pour ce système.

En dimension 1 ([RdT11]), ou en dimension supérieure mais avec la condition géométrique de contrôle des ondes ([AB12]), le système (1.46-i) peut même être contrôlable à zéro. Notons que dans ces deux résultats, une hypothèse de signe intervient. On montre dans ce qui suit que cette dernière n'est en aucun cas nécessaire (au moins pour la contrôlabilité approchée), et qu'elle cache d'autant plus des situations très intéressantes. En particulier, la géométrie de la zone de contrôle peut jouer un rôle.

Enfin, concernant le système (1.46-ii), il n'existe, à ma connaissance, aucun résultat (le résultat de [Mau13] ne s'applique pas dans ce cadre).

Notations. Pour énoncer les résultats qui suivent on introduit quelques notations.

• Pour tout $k \geq 1$, on note $\widetilde{\phi}_k$ une solution de l'équation $-\partial_x^2 \widetilde{\phi}_k - \lambda_k \widetilde{\phi}_k = 0$ sur $\Omega = (0,1)$ qui vérifie de plus $\widetilde{\phi}_k(0) \neq 0$ et $\widetilde{\phi}_k(1) \neq 0$ (par exemple $\widetilde{\phi}_k(x) = \cos(k\pi x)$ convient). Les résultats qui suivent ne dépendent pas du choix particulier des $\widetilde{\phi}_k$.

• On note $\mathcal{C}\left(\overline{\Omega\backslash\omega}\right)$ l'ensemble des composantes connexe de $\overline{\Omega\backslash\omega}$. Pour tout $C \in \mathcal{C}\left(\overline{\Omega\backslash\omega}\right)$ et $f \in L^2(\Omega)$, on définit le vecteur $M_k(f, C) \in \mathbb{R}^2$ par

$$M_k(f,C) = \begin{pmatrix} \int_C f\phi_k\,dx \\ 0 \end{pmatrix} \text{ si } C \cap \partial\Omega \neq \emptyset, \qquad M_k(f,C) = \begin{pmatrix} \int_C f\phi_k\,dx \\ \int_C f\widetilde{\phi}_k\,dx \end{pmatrix} \text{ si } C \cap \partial\Omega = \emptyset.$$

Enfin, pour $f \in L^2(\Omega)$ on définit la famille de vecteurs de $\mathbb{R}^2$ suivante :

$$\mathcal{M}_k(f,\omega) = (M_k(f,C))_{C \in \mathcal{C}\left(\overline{\Omega\backslash\omega}\right)} \in (\mathbb{R}^2)^{\mathcal{C}\left(\overline{\Omega\backslash\omega}\right)}.$$

Par exemple, si ω est connexe, on a $1 \leq \operatorname{card}(\overline{\Omega\backslash\omega}) \leq 2$ et, pour tout $C \in \mathcal{C}\left(\overline{\Omega\backslash\omega}\right)$ on a

$$M_k(f, C) = \begin{pmatrix} \int_C f\phi_k \, dx \\ 0 \end{pmatrix}.$$

On verra cependant qu'il peut être très intéressant de ne pas seulement se restreindre au seul cas où ω est connexe.

Un résultat de continuation unique avec un terme source. Tous les résultats qui suivent sont basés sur la caractérisation suivante (prouvée dans la section 5.2.3) de la propriété de continuation unique pour une équation elliptique avec un second membre :

Théorème 21 ([BO13])**.** *Soit $F \in L^2(\Omega)$ et ω un sous-ensemble ouvert non-vide de Ω. Soit $k \geq 1$ fixé. Il existe une solution $u \in \mathcal{D}(-\partial_x^2)$ au problème*

$$\begin{cases} -\partial_x^2 u - \lambda_k u = F & \text{dans } \Omega, \\ u = 0 & \text{dans } \omega, \end{cases} \tag{1.47}$$

si, et seulement si,

$$\begin{cases} F = 0 & \text{dans } \omega, \\ \mathcal{M}_k(F, \omega) = 0. \end{cases} \tag{1.48}$$

Application à la contrôlabilité simultanée de plusieurs systèmes 2×2. L'un des nouveaux résultats que l'on obtient au chapitre 5 concerne le système (1.46) lorsque la matrice $A(x)$ est de la forme suivante :

$$A(x) = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ a_{21}(x) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x) & 0 & \cdots & 0 \end{pmatrix}. \tag{1.49}$$

Pour cette structure particulière on peut voir dans un premier temps (section 5.3.1) que l'on peut toujours se ramener au cas où aucun des supports des fonctions de couplages $a_{i1}(x)$, $i \in [\![2, n]\!]$, n'intersecte la zone de contrôle ω, soit

$$a_{i1} 1_\omega = 0, \quad \forall i \in [\![2, n]\!]. \tag{1.50}$$

Dans la section 5.3.2 on prouve alors le résultat suivant.

Théorème 22 ([BO13])**.** *Supposons que la matrice $A(x)$ est de la forme* (1.49) *et que les couplages sont tels que* (1.50) *est vérifié.*

Alors, le système (1.46) *est approximativement contrôlable si, et seulement si,*

$$\forall k \geq 1, \quad \operatorname{rang}\ \{\mathcal{M}_k(a_{21}\phi_k, \omega), \dots, \mathcal{M}_k(a_{n1}\phi_k, \omega)\} = n - 1.$$

• La condition de rang doit être interprétée dans l'espace vectoriel $(\mathbb{R}^2)^{\mathcal{C}\left(\overline{\Omega\backslash\omega}\right)}$. Une conséquence directe de cette condition est qu'il faut au moins un certain nombre de composantes de $\overline{\Omega\backslash\omega}$ pour espérer contrôler le système. En effet, on voit qu'il est nécessaire d'avoir

$$2\,\mathrm{card}\,\mathcal{C}\left(\overline{\Omega\backslash\omega}\right) \geq n-1.$$

Lorsque ω est connexe on a donc absolument aucune chance de contrôler un système de la forme (1.49) avec 6 équations.

Exemple avec un seul système 2×2**.** A l'aide du théorème 22 on donne dans la section 5.3.3.1 des conditions nécessaires ou suffisantes très simples pour la contrôlabilité approchée des systèmes 2×2 suivant :

$$\begin{cases} \partial_t y_1 - \partial_x^2 y_1 = 1_\omega v & \text{dans } Q_T, \\ \partial_t y_2 - \partial_x^2 y_2 = a_{21}(x) y_1 & \text{dans } Q_T, \\ \quad y_1 = y_2 = 0 & \text{sur } \Sigma_T. \end{cases} \tag{1.51}$$

Théorème 23 ([BO13])**.** *Soit* $\mathcal{O}_2 = \mathrm{supp}\,(a_{21})$ *le support de* a_{21}*.*

1. *Si* $\mathcal{O}_2 \cap \omega \neq \emptyset$*, alors le système* (1.51) *est approximativement contrôlable.*
2. *Supposons maintenant que* $\mathcal{O}_2 \cap \omega = \emptyset$*.*
   (a) *Si le coefficient de couplage* a_{21} *vérifie*

$$\int_0^1 a_{21}(\phi_k)^2\, dx \neq 0, \quad \forall k \geq 1, \tag{1.52}$$

   alors le système (1.51) *est approximativement contrôlable.*

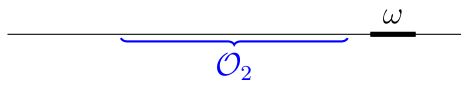
   (b) *Si le système* (1.51) *est approximativement contrôlable et* $\mathcal{O}_2$ *est entièrement inclus dans une seule composante connexe de* $\overline{\Omega\backslash\omega}$ *qui touche le bord de* Ω*, alors* (1.52) *est vérifiée.*

• Ce théorème ne couvre cependant pas toutes les situations. Prenons par exemple le coefficient de couplage suivant

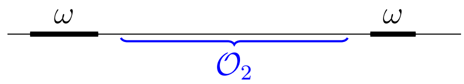
$$a_{21}(x) = \left(x - \frac{1}{2}\right) 1_{\mathcal{O}_2}(x), \quad \mathcal{O}_2 = \left(\frac{1}{4}, \frac{3}{4}\right),$$

pour lequel on peut vérifier que la condition (1.52) s'avère fausse, et considérons les deux configurations géométriques de la figure 1.4 pour ω.

Alors, on peut voir en utilisant le théorème 23 que le système correspondant n'est pas approximativement contrôlable dans le cas de la figure 1.4a alors qu'il est approximativement contrôlable dans le cas de la figure 1.4b (en revenant à la caractérisation donnée dans le théorème 22). On renvoie à la section 5.3.3.1 pour la preuve de ces faits et de nombreuses autres applications.



(a) ω connexe



(b) ω non-connexe

FIGURE 1.4 – Deux géométries pour l'étude d'un système 2×2

Systèmes 3×3. Dans la section 5.3.3.2 on détaille également le cas de la contrôlabilité simultanée de deux systèmes 2×2. On y voit notamment que la position des domaines de couplages permet de savoir rapidement si la contrôlabilité est assurée ou non. Enfin, dans la section 5.4, on traite aussi les systèmes en cascade 3×3 et on obtient un résultat qui se rapproche du théorème 23 pour les systèmes 2×2.

1.5 Perspectives et problèmes ouverts

Cette thèse s'est attardée sur certains problèmes de contrôlabilité, mais il reste encore beaucoup de problèmes ouverts très intéressants. On va en mentionner quelques uns, mais tout d'abord, si on devait résumer la différence entre la contrôlabilité d'une équation et celle d'un système d'équations, on pourrait dire que la contrôlabilité des systèmes dépend vraiment de tous les paramètres du système :

– la nature du contrôle (distribué ou frontière [FCGBdT10]),
– la géométrie de la zone de contrôle ([Oli13, BO13]),
– le nombre de contrôles ([Oli13, BO13]),
– le temps de contrôle ([AKBDGB09a, AKBGBdT12]),
– la position des domaines de couplages par rapport à celle de la zone de contrôle ([KdT10, RdT11, BO13]),
– la nature même du couplage (par exemple s'il a un signe,... [BO13]),
– et même l'ordre du couplage ([Oli13]).

Temps minimal sur un rectangle ? Un premier problème auquel on aimerait donner une réponse est le suivant. On suppose que $\Omega \subset \mathbb{R}^2$ est comme dans la figure 1.2 et on considère sur ce domaine le système 2×2 suivant :

$$\begin{cases} \partial_t y - D\Delta y = Ay & \text{dans } Q_T, \\ y = 1_\gamma Bv & \text{sur } \Sigma_T, \end{cases} \tag{1.53}$$

avec

$$D = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

On peut alors établir le même résultat que le théorème 14, à savoir que si $\gamma = \gamma_L \cup \gamma_T$ contient donc deux directions, et si le système n'a que $n = 2$ équations, alors le système (1.53) est approximativement contrôlable. D'autre part, il est connu qu'en dimension 1, ce même système peut être contrôlable seulement à partir d'un certain temps (voir [AKBGBdT12]). La question que l'on peut alors se poser est de savoir si ce résultat persiste dans notre cas, où si on peut obtenir un résultat plus fort de contrôlabilité pour tout temps. L'idée générale étant toujours la même : la géométrie de la zone de contrôle semble ajouter un contrôle et on se retrouve alors dans une situation comparable à celle d'un système 2×2 avec 2 contrôles à notre disposition, ce dernier étant contrôlable (pour tout temps). Cela illustrerait encore davantage l'importance de la géométrie de la zone de contrôle dans les problèmes de contrôlabilité de systèmes paraboliques.

Contrôlabilité des systèmes non-autonomes. Un autre problème ouvert intéressant mais qui semblent dépasser le cadre de cette thèse concerne la contrôlabilité des systèmes d'équations à coefficients variables en t seulement. Pour ces derniers, les inégalités de Carleman peuvent être utilisées pour obtenir une condition de Kalman pour la contrôlabilité distribuée à zéro ([AKBDGB09a]). En revanche, la contrôlabilité frontière des systèmes non-autonomes est complètement ouverte. Aucune technique actuelle ne permet de donner de réponse, même pour un "simple" système en cascade de 2 équations et même si on s'intéresse seulement à la contrôlabilité approchée.

Matrice de couplage pleine. La contrôlabilité frontière des systèmes à coefficients variables en x, sans aucune structure supposée sur la matrice de couplage $A(x)$ est également ouverte. L'approche spectrale s'annonce ici plus difficile et les autres techniques semblent inopérationnelles.

Chapitre 2

Null-controllability for some linear parabolic systems with controls acting on different parts of the domain and its boundary

Ce chapitre est la reprise de l'article [Oli12], publié dans Mathematics of Control, Signals, and Systems.

Abstract. In this work we study the null-controllability properties of linear parabolic systems with constant coefficients in the case where several controls are acting on different distributed subdomains and/or on the boundary. We prove a Kalman rank condition in the one-dimensional case. In the case where only distributed controls are considered we also establish related results such as a Carleman estimate.

Keywords : Kalman rank condition ; Boundary controllability ; Distributed controllability ; Carleman estimate

2.1 Introduction

Let $n \in \mathbb{N}^*$, $n_D, n_B \in \mathbb{N}$, be respectively the number of equations, the number of distributed controls and the number of boundary controls we will consider. Let $\Omega \subset \mathbb{R}^N$ ($N \in \mathbb{N}^*$) be a bounded connected open set with boundary $\partial\Omega$ regular enough. For every $T > 0$ we denote $Q_T = (0,T) \times \Omega$ and $\Sigma_T = (0,T) \times \partial\Omega$. Let $\omega_1, \dots, \omega_{n_D}$ be given non empty open subsets of Ω (possibly disjoint) and let $\Gamma_1, \dots, \Gamma_{n_B}$ be given non empty open subsets of $\partial\Omega$. We consider the following type of $n \times n$ parabolic system :

$$\begin{cases} \partial_t y = \Delta y + Ay + D_1 u_1(t,x) 1_{\omega_1}(x) + \ldots + D_{n_D} u_{n_D}(t,x) 1_{\omega_{n_D}}(x) \text{ in } Q_T, \\ y = B_1 v_1(t,x) 1_{\Gamma_1}(x) + \ldots + B_{n_B} v_{n_B}(t,x) 1_{\Gamma_{n_B}}(x) \text{ on } \Sigma_T, \end{cases} \tag{2.1}$$

where y is the state, $A \in \mathcal{M}_n(\mathbb{R})$ is a coupling matrix. For every $i \in \{1, \dots, n_D\}$, $D_i \in \mathbb{R}^n$ and $u_i \in L^2(Q_T)$ is a *distributed control* acting on ω_i. For every $j \in \{1, \dots, n_B\}$, $B_j \in \mathbb{R}^n$ and $v_j \in L^2(\Sigma_T)$ is a *boundary control* acting on Γ_j.

Let us recall that, for every $T > 0$, for every $y_0 \in L^2(\Omega; \mathbb{R}^n)$, $u_i \in L^2(Q_T)$, $i \in \{1, \dots, n_D\}$, and $v_j \in L^2(\Sigma_T)$, $j \in \{1, \dots, n_B\}$, there exists a unique solution to (2.1) $y \in L^2(Q_T; \mathbb{R}^n) \cap C^0([0,T]; H^{-1}(\Omega; \mathbb{R}^n))$, defined by transposition, which satisfies $y(0) = y_0$ (see for instance [FCGBdT10, Appendix] for more details).

Let be given $y_0 \in L^2(\Omega; \mathbb{R}^n)$ and $T > 0$, it will be said that system (2.1) is *null-controllable on* $(0,T)$ *from the state* y_0 if there exists $u_i \in L^2(Q_T)$ for every $i \in \{1, \dots, n_D\}$ and there exists $v_j \in L^2(\Sigma_T)$ for every $j \in \{1, \dots, n_B\}$ such that the corresponding solution to (2.1) with $y(0) = y_0$ satisfies $y(T) = 0$. Let be given $T > 0$, it will be said that system (2.1) is *null-controllable on* $(0,T)$ if for every $y_0 \in L^2(\Omega; \mathbb{R}^n)$ system (2.1) is null-controllable on $(0,T)$ from the state y_0. It will be said that system (2.1) is *null-controllable* if for every $T > 0$ system (2.1) is null-controllable on $(0,T)$.

Let us recall that the scalar operator $-\Delta$ with homogeneous Dirichlet boundary condition admits a sequence of eigenvalues $\{\lambda_k\}_{k \in \mathbb{N}^*} \subset \mathbb{R}_+^*$ such that the associated sequence of normalized eigenfunctions $\{\phi_k\}_{k \in \mathbb{N}^*}$ is a Hilbert basis of $L^2(\Omega)$.

notations For any $q, N_1, N_2 \in \mathbb{N}^*$, for any matrix $A \in \mathcal{M}_{N_1}(\mathbb{R})$, $B \in \mathcal{M}_{N_1 \times N_2}(\mathbb{R})$, we denote $[A|B]$ the matrix whose first columns are those of A and the following ones are those of B and we define

$$
\begin{gathered}
[A : B]_q = \left[B|AB|\cdots|A^{q-1}B\right] \in \mathcal{M}_{N_1 \times N_2 q}(\mathbb{R}), \quad ((B))_q = \begin{bmatrix} B & 0 & \cdots & \cdots & 0 \\ 0 & B & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & B \end{bmatrix} \in \mathcal{M}_{N_1 q \times N_2 q}(\mathbb{R}), \\
(B)_q = \begin{bmatrix} B \\ B \\ \vdots \\ \vdots \\ B \end{bmatrix} \in \mathcal{M}_{N_1 q \times N_2}(\mathbb{R}), \quad (B)_q^{\pm} = \begin{bmatrix} -B \\ B \\ \vdots \\ \vdots \\ (-1)^q B \end{bmatrix} \in \mathcal{M}_{N_1 q \times N_2}(\mathbb{R}), \\
\mathcal{A}_q = \begin{bmatrix} A_1 & 0 & \cdots & \cdots & 0 \\ 0 & A_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & A_q \end{bmatrix} \in \mathcal{M}_{N_1 q}(\mathbb{R}) \text{ with } A_k = -\lambda_k I + A \in \mathcal{M}_{N_1}(\mathbb{R}).
\end{gathered}
\tag{2.2}
$$

Note that

$$\operatorname{rank}\left(([\,A\,:\,B\,]_n)\right)_q = q \operatorname{rank}[\,A\,:\,B\,]_n, \quad \forall q \in \mathbb{N}^*. \tag{2.3}$$

In controllability theory of linear ordinary differential systems there exists a complete characterization of controllability, this is the so-called *Kalman rank condition* (see for instance [TW09, Corollary 1.4.10]), that is to say, if $A \in \mathcal{M}_n(\mathbb{R})$ and $B \in \mathcal{M}_{n\times m}(\mathbb{R})$ ($n, m \in \mathbb{N}^*$), then the linear ordinary differential system $y' = Ay + Bu$ is controllable if and only if

$$\operatorname{rank}[\,A\,:\,B\,]_n = n.$$

To give an appropriate condition of this type in the framework of linear parabolic systems has been a subject of several research. Recently in [AKBDGB09b] a Kalman rank condition has been proved for the distributed null-controllability with only one control region : the authors proved that the system

$$\begin{cases} \partial_t y = \Delta y + Ay + Du_1(t,x)1_{\omega_1}(x) \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \end{cases}$$

is null-controllable if and only if

$$\operatorname{rank}[\,A\,:\,D\,]_n = n, \tag{2.4}$$

see [AKBDGB09b, Theorem 1.1] and [AKBDGB09b, Proposition 2.2]. In fact they proved a more general Kalman rank condition for linear parabolic systems with different coefficients in front of the operator $-\Delta$, for more details see [AKBDGB09b].

In [FCGBdT10] the authors gave a necessary and sufficient condition for the boundary null-controllability in the one-dimensional case and for two equations (see [FCGBdT10, Theorem 1.1]). Through this theorem they also showed that the Kalman rank condition for distributed null-controllability is a necessary condition for the boundary null-controllability but it is not sufficient. The result of [FCGBdT10] has been improved in [AKBGBdT11a] where the authors proved a new Kalman rank condition for boundary null-controllability of $n \times n$ linear parabolic system (still in the one-dimensional case though), that is : the system

$$\begin{cases} \partial_t y = \partial_{xx}^2 y + Ay \text{ in } (0,T) \times (0,1), \\ y(t,0) = B_1 v_1(t), \quad y(t,1) = B_2 v_2(t) \text{ on } (0,T), \end{cases}$$

is null-controllable if and only if

$$\operatorname{rank}\left[\,\mathcal{A}_q\,:\,\left((B_1)_q \middle| (B_2)_q^{\pm}\right)\,\right]_{nq} = nq, \quad \forall q \in \mathbb{N}^*, \tag{2.5}$$

see [AKBGBdT11a, Theorem 6.3].

We also point out reference [AKBDGB09a] where the authors worked on the case of regular time-dependent matrices $A = A(t)$ and $D = D(t)$ (with one distributed control) and they proved that the system

$$\begin{cases} \partial_t y = \Delta y + A(t)y + D(t)u_1(t,x)1_{\omega_1}(x) \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \end{cases}$$

is null-controllable if

$$\exists t_0 \in [0,T], \quad \operatorname{rank} \mathcal{K}(t_0) = n, \tag{2.6}$$

where

$$\mathcal{K}(t) = [\mathcal{K}_0(t)|\ldots|\mathcal{K}_{n-1}(t)]$$

with

$$\begin{cases} \mathcal{K}_0(t) &= D(t) \\ \mathcal{K}_i(t) &= A(t)\mathcal{K}_{i-1}(t) - \dfrac{d}{dt}\mathcal{K}_{i-1}(t), \quad \forall i \in \{1,\ldots,n-1\} \end{cases}$$

see [AKBDGB09a, Theorem 1.2].

Finally, let us mention that the null-controllability properties of linear parabolic systems have been also studied in the case of space varying coefficients and one distributed control force. However few results are known, even for the distributed controllability. Sufficient conditions to the distributed null-controllability are given in [AKBD06], [AKBDK05], [GBPG06],[Gue07] and [dT00] for systems of two equations and see [GBdT10] for $n \times n$ systems. To our knowledge [dT00] is also the first result using Carleman estimates for two coupled parabolic equations. Concerning the boundary null-controllability of parabolic systems, let us mention [ABL12] where the authors prove a result without any restriction on the dimension (but under some geometric condition, see [ABL12] for more details).

In the present work we try to give an overview of the controllability properties of systems like (2.1). One of the main task of this work will be to prove a Kalman rank condition for system (2.1) which will then generalize the previously known Kalman conditions (2.4) and (2.5).

2.2 Statements of the results

2.2.1 Main result

The first and main result of this work concerns system (2.1) in the case $N = 1$. As a consequence it is equivalent to consider the following system :

$$\begin{cases} \partial_t y = \partial^2_{xx} y + Ay + D_1 u_1(t,x)1_{\omega_1}(x) + \ldots + D_{n_D} u_{n_D}(t,x)1_{\omega_{n_D}}(x) \text{ in } Q_T, \\ y(t,0) = B^L_1 v_1(t) + \cdots + B^L_{n_L} v_{n_L}(t), \quad y(t,1) = B^R_1 w_1(t) + \cdots + B^R_{n_R} w_{n_R}(t) \text{ on } (0,T), \end{cases} \tag{2.7}$$

with $n_B \leq n_L + n_R \leq 2n_B$, and where we take $\Omega = (0,1)$ for the sake of simplicity. Let us denote $D = [D_1|D_2|\cdots|D_{n_D}] \in \mathcal{M}_{n\times n_D}(\mathbb{R})$, $B^L = \left[B_1^L|\cdots|B_{n_L}^L\right]$ and $B^R = \left[B_1^R|\cdots|B_{n_R}^R\right]$. Then, the result reads :

Theorem 2.1 (Kalman rank condition)**.** *System (2.7) is null-controllable if and only if*

$$\operatorname{rank}\left[\ \left[\mathcal{A}_q\ :\ \left(\left(B^L\right)_q\middle|\left(B^R\right)_q^{\pm}\right)\right]_{nq}\ \middle|\ \left(\left([A:D]_n\right)\right)_q\ \right] = nq, \quad \forall q\in\mathbb{N}^*, \tag{2.8}$$

(where we used the notations introduced above).

Remark 2.2. 1. *Theorem 2.1 contains both Kalman conditon (2.4) for distributed null-controllability and Kalman condition (2.5) for boundary null-controllability, see (2.3).*

2. *We can also reformulate condition (2.8) as follows :*

$$\operatorname{rank}\left[\mathcal{A}_q\ :\ \left(\left(B^L\right)_q\middle|\left(B^R\right)_q^{\pm}\middle|\left((D)\right)_q\right)\right]_{nq} = nq, \quad \forall q\in\mathbb{N}^*. \tag{2.9}$$

This is due to the equalities $\operatorname{rank}\left[\mathcal{A}_q\ :\ ((D))_q\right]_{nq} = \operatorname{rank}\left(([\,A\ :\ D\,]_n)\right)_q$ *for all* $q\in\mathbb{N}^*$*. This characterization will be used to prove Theorem 2.1.*

3. *Let us observe that condition (2.8) is only algebraic. In particular it does not depend on* $\omega_1,\ldots,\omega_{n_D}$*.*

4. *Condition (2.8) is checkable thanks to the following fact : to check condition (2.8) is equivalent to check it for a particular* $q = q_0$ *which is such that*

$$\mu_i - \mu_j \neq \lambda_k - \lambda_l, \quad \forall k,l\in\mathbb{N}^* \text{ with } k>q_0 \text{ and } l\neq k, \quad \forall i,j\in\{1,\ldots,n\}, \tag{2.10}$$

where $\{\mu_k\}_{k\in\{1,\ldots,n\}}\subset\mathbb{C}$ *is the set of the eigenvalues of* A*. To prove this fact one can adapt the proof of [AKBGBdT11a, Corollary 3.3], by using the characterization (2.9). Moreover one can see that such a* q_0 *does always exist, see [AKBGBdT11a, Proposition 3.2] for instance.*

Let us illustrate the last item of Remark 2.2 through the following example :

Example 2.3. *Let* $T>0$ *and* $\omega\subset(0,1)$ *a non empty open subset. Consider the following* 3×3 *one-dimensional parabolic system :*

$$\begin{cases} \left.\begin{array}{l} \partial_t y_1 = \partial_{xx}^2 y_1 + 2y_1 + 6y_2 + 2y_3, \\ \partial_t y_2 = \partial_{xx}^2 y_2 + 4y_1 - 2y_3, \\ \partial_t y_3 = \partial_{xx}^2 y_3 + 2y_1 - 3/2 y_2 + 2y_3 + u(t,x)1_\omega(x), \end{array}\right\} \text{in } (0,T)\times(0,1), \\ \left.\begin{array}{l} y_1(t,0) = v(t), \quad y_1(t,1) = 0, \\ y_2(t,0) = 0 \quad y_2(t,1) = 0, \\ y_3(t,0) = 0 \quad y_3(t,1) = 0, \end{array}\right\} \text{on } (0,T), \end{cases} \tag{2.11}$$

so that

$$A = \begin{bmatrix} 2 & 6 & 2 \\ 4 & 0 & -2 \\ 2 & -3/2 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad B_L = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad B_R = 0.$$

We can see that if only one control is acting then this system is not null-controllable. Indeed we have $\operatorname{rank}[\,A : D\,]_3 = 2 \neq 3$ *and* $\operatorname{rank}[\,A : B\,]_3 = 2 \neq 3$ *so the distributed and boundary Kalman conditions fail. Nevertheless we have* $\operatorname{rank}[\,[\,A : B\,]_3 \mid [\,A : D\,]_3\,] = 3$ *and the eigenvalues of* A *are* $-5, 3$ *and* 6 *so that condition (2.10) is satisfied for* $q_0 = 1$ *and thus, by the previous remark, condition (2.8) is also satisfied.*

We make the following remark about this example, this gives a good idea of the proof of Theorem 2.1 :

Remark 2.4. *Observe that in fact the matrix* A *of Example 2.3 is equivalent to the matrix*

$$C = \begin{bmatrix} 6 & 0 & 0 \\ 6 & 0 & 15 \\ -2 & 1 & -2 \end{bmatrix}$$

through the following change of basis :

$$P = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix} = [B|D|AD] \qquad \textit{(with the notations of Example 2.3)}\ .$$

As a consequence the null-controllability of system (2.11) is equivalent to the null-controllability of the system

$$\begin{cases} \partial_t z = \partial_{xx}^2 z + \begin{bmatrix} 6 & 0 & 0 \\ 6 & 0 & 15 \\ -2 & 1 & -2 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t,x)1_\omega(x) \ \textit{in}\ (0,T)\times(0,1), \\ z(t,0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} v(t), \quad z(t,1) = 0 \ \textit{on}\ (0,T), \end{cases}$$

And we can see that we can lead the first component of this system to zero at time $T/2$ (for instance) by using only the boundary control v. Then, it remains to prove that the system

$$\begin{cases} \partial_t \hat{z} = \partial_{xx}^2 \hat{z} + \begin{bmatrix} 0 & 15 \\ 1 & -2 \end{bmatrix} \hat{z} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t,x) 1_\omega(x) \text{ in } \left(\frac{T}{2}, T\right) \times (0,1), \\ \hat{z}(t,0) = 0, \quad \hat{z}(t,1) = 0 \text{ on } \left(\frac{T}{2}, T\right), \end{cases}$$

is null-controllable, which can be done by checking the distributed Kalman condition.

2.2.2 More results in the case $n_B = 0$

Let us now consider the case without boundary control with an arbitrary space dimension N, that is

$$\begin{cases} \partial_t y = \Delta y + Ay + D_1 u_1(t,x) 1_{\omega_1}(x) + \ldots + D_{n_D} u_{n_D}(t,x) 1_{\omega_{n_D}}(x) \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma_T. \end{cases} \tag{2.12}$$

From the proof of Theorem 2.1 we will see that in fact Theorem 2.1 still holds without any restriction on N if we have no boundary controls :

Corollary 2.5. *Let $N \geq 1$ be arbitrary. Then, system (2.12) is null-controllable if and only if*

$$\operatorname{rank} [\, A \,:\, D \,]_n = n. \tag{2.13}$$

On the other hand, when the Kalman condition (2.13) is not fulfilled it is possible to characterize the states that can be driven to 0 :

Proposition 2.6. *Assume that $N \geq 1$ and $\operatorname{rank} [\, A \,:\, D \,]_n < n$. Then, system (2.12) is null-controllable on $(0,T)$ from the state y_0 for every $T > 0$ if and only if*

$$y_0 \in L^2 \left(\Omega; \operatorname{span}[\, A \,:\, D \,]_n\right).$$

This can be proved by extending the arguments given in [AKBDGB09b, Theorem 1.5].

We also can also extend the Kalman rank condition for time-dependent matrices (2.6) : let us consider the system

$$\begin{cases} \partial_t y = \Delta y + A(t) y + D_1(t) u_1(t,x) 1_{\omega_1}(x) + \ldots + D_{n_D}(t) u_{n_D}(t,x) 1_{\omega_{n_D}}(x) \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma_T. \end{cases} \tag{2.14}$$

where $A \in \mathcal{C}^{n-1}([0,T]; \mathcal{M}_n(\mathbb{R}))$ and $D_i \in \mathcal{C}^n([0,T]; \mathbb{R}^n)$.

In this case we have

Theorem 2.7. *If there exists $t_0 \in [0,T]$ such that*

$$\exists t_0 \in [0,T], \quad \operatorname{rank} \mathcal{K}(t_0) = n, \tag{2.15}$$

where

$$\mathcal{K}(t) = [\mathcal{K}_0(t)|\ldots|\mathcal{K}_{n-1}(t)]$$

with

$$\left\{ \begin{array}{lcl} \mathcal{K}_0(t) & = & D(t) = [D_1(t)|\ldots|D_{n_D}(t)] \\ \mathcal{K}_i(t) & = & A(t)\mathcal{K}_{i-1}(t) - \dfrac{d}{dt}\mathcal{K}_{i-1}(t), \quad \forall i \in \{1,\ldots,n-1\} \end{array} \right.$$

then the system (2.14) is null-controllable at time T.

This theorem will be proved thanks to a Carleman estimate for cascade systems, see Theorem 2.13 below.

2.3 The Kalman rank condition

Recall that all along this section we assume that $N = 1$ and the system considered is (2.7). In fact, for the sake of simplicity of the notations we will consider

$$\left\{ \begin{array}{l} \partial_t y = \partial_{xx}^2 y + Ay + D_1 u_1(t,x) 1_{\omega_1}(x) + \ldots + D_{n_D} u_{n_D}(t,x) 1_{\omega_{n_D}}(x) \text{ in } Q_T, \\ y(t,0) = B_1 v_1(t), \quad y(t,1) = B_2 v_2(t) \text{ on } (0,T). \end{array} \right. \tag{2.16}$$

2.3.1 Some known results

Before starting the proof of Theorem 2.1 let us recall for convenience the following results :

Proposition 2.8. *Let be given $A \in \mathcal{M}_n(\mathbb{R})$ and $B \in \mathcal{M}_{n\times m}(\mathbb{R})$ ($n, m \in \mathbb{N}^*$). We have*

$$\forall V \in \mathbb{R}^n, \quad \Big(\forall t \geq 0, \quad B^* e^{tA^*} V = 0\Big) \Longrightarrow V = 0$$

if and only if

$$\operatorname{rank}\,[\,A\,:\,B\,]_n = n$$

if and only if

$$\ker(A^* - \theta I) \cap \ker B^* = \{0\}, \quad \forall \theta \in \mathbb{C}. \tag{2.17}$$

Condition (2.17) is the so-called *Hautus test*. For a proof see for instance [TW09, Chapter 1]. To state the second result we need to define the *adjoint system* of (2.16) :

$$\left\{ \begin{array}{l} -\partial_t \Phi = \partial_{xx}^2 \Phi + A^* \Phi \text{ in } Q_T, \\ \Phi(t,0) = 0, \quad \Phi(t,1) = 0 \text{ on } (0,T). \end{array} \right. \tag{2.18}$$

The introduction of system (2.18) is of interest thanks to the following proposition, which gives a characterization of the null-controllability of system (2.16) through an inequality on its adjoint system (2.18) :

Proposition 2.9 (Observability inequality)**.** *Let be given $T > 0$. System (2.16) is null-controllable on $(0,T)$ if and only if there exists $C > 0$ such that for every $\Phi^T \in H_0^1(\Omega;\mathbb{R}^n)$ the solution Φ to the adjoint system (2.18) with $\Phi(T) = \Phi^T$ satisfies*

$$\begin{aligned} ||\Phi(0)||^2_{H_0^1(\Omega;\mathbb{R}^n)} \leq C\Bigg(& \sum_{i=1}^{n_D} \int_0^T \int_{\omega_i} |D_i^*\Phi(t,x)|^2 \, dx \, dt \\ & + \int_0^T |B_1^*\partial_x\Phi(t,0)|^2 \, dt + \int_0^T |B_2^*\partial_x\Phi(t,1)|^2 \, dt \Bigg), \end{aligned} \tag{2.19}$$

For a proof see for instance [FCGBdT10, Appendix]. Inequality (2.19) is called *observability inequality.*

2.3.2 Proof of Theorem 2.1

The key point of the proof is to do an appropriate change of basis thanks to the hypothesis (2.8). In this new basis, the matrix A becomes a block upper triangular matrix C ; and B_1, B_2 and D become such that one control is acting on each diagonal block of C. Note that this technique, firstly used in [AKBDGB09b], is very specific to the fact that the coefficients before the operator $-\partial^2_{xx}$ are the same on every single equation. In a second time we will check that every diagonal block of C satisfies the appropriate Kalman condition (boundary or distributed). And as a consequence, taking also advantage of the fact that the last block of C is decoupled from the upper ones, we can start to control the last block in a time before T and lead to zero at this time the components associated to this block ; this allows us to iterate the process for the remaining blocks and finally lead every component to zero at time T.

Proof. **Step 1** Under the condition (2.8) we start to construct a basis in which the matrices A, D and B_1, B_2 has the desired structure. We have

Lemma 2.10. *Assume that condition (2.8) holds. Then, there exists $r_D \in \{0,\dots,n_D\}$, $D_{i_1},\dots,D_{i_{r_D}} \in \{D_k\}_{1\leq k\leq n_D}$ and $s_1,\dots,s_{r_D} \in \{1,\dots,n\}$ such that for every $q \in \mathbb{N}^*$ there exists $r_B \in \{0,1,2\}$, $\mathsf{B}_{\mathsf{j}_1},\dots,\mathsf{B}_{\mathsf{j}_{\mathsf{r}_\mathsf{B}}} \in \left\{(B_1)_q, (B_2)_q^{\pm}\right\}$ and $\tilde{s}_1,\dots\tilde{s}_{r_B} \in \{1,\dots,nq\}$, such that*

$$P_q = \left[\, P_q^D \,\middle|\, P_q^B \,\right] \in \mathcal{M}_{nq}(\mathbb{R})$$

is invertible, where we have denoted

$$P_q^D = \left(\left(\left[[\,A\,:\,D_{i_1}\,]_{s_1}\middle|\cdots\middle|[\,A\,:\,D_{i_{r_D}}\,]_{s_{r_D}}\right]\right)\right)_q \text{ and } P_q^B = \left[[\,\mathcal{A}_q\,:\,\mathsf{B}_{\mathsf{j}_1}\,]_{\tilde{s}_1}\middle|\cdots\middle|[\,\mathcal{A}_q\,:\,\mathsf{B}_{\mathsf{j}_{\mathsf{r}_\mathsf{B}}}\,]_{\tilde{s}_{r_B}}\right].$$

Moreover for every $k \in \{1,\dots,r_D\}$,

$$A^{s_k}D_{i_k} \in \operatorname{span}\left[[\,A\,:\,D_{i_1}\,]_{s_1}\middle|\cdots\middle|[\,A\,:\,D_{i_k}\,]_{s_k}\right].$$

Proof. **Step 1** We assume that $D \neq 0$ otherwise the result stated by Theorem 2.1 is already known (see [AKBGBdT11a]). Thus there exists $D_{i_1} \in \{D_k\}_{1\leq k\leq n_D}$ such that $D_{i_1} \neq 0$. We set

$$s_1 = \text{rank}\left(D_{i_1}, AD_{i_1}, \dots, A^{n-1}D_{i_1}\right)$$

so that $\text{rank}\,(D_{i_1}, AD_{i_1}, \dots, A^{s_1-1}D_{i_1}) = s_1$. If $s_1 = n$ then the proof ends here by taking $P_q = (([D_{i_1}|AD_{i_1}|\dots|A^{s_1-1}D_{i_1}]))_q$. If $s_1 < n$ we check if there exists $D_{i_2} \in \{D_k\}_{1\leq k\leq n_D} \setminus \{D_{i_1}\}$ such that $(D_{i_1}, AD_{i_1}, \dots, A^{s_1-1}D_{i_1}, D_{i_2})$ is linearly independent. If this is not the case we go to step 2. But if such a D_{i_2} exists we set

$$s_2 = \text{rank}\left(D_{i_1}, AD_{i_1}, \dots, A^{s_1-1}D_{i_1}, D_{i_2}, AD_{i_2}, \dots, A^{n-1}D_{i_2}\right) - s_1$$

so that $\text{rank}\,(D_{i_1}, AD_{i_1}, \dots, A^{s_1-1}D_{i_1}, D_{i_2}, AD_{i_2}, \dots, A^{s_2-1}D_{i_2}) = s_1 + s_2$. If $s_1 + s_2 = n$ the proof ends. If $s_1 + s_2 < n$ then we continue the previous process. This stops when we have found a rank $r_D \in \{1, \dots, n\}$, $i_1, \dots, i_{r_D} \in \{1, \dots, n_D\}$ and $s_1, \dots, s_{r_D} \in \{1, \dots, n\}$ such that

$$\left(D_{i_1}, AD_{i_1}, \dots, A^{s_1-1}D_{i_1}, D_{i_2}, AD_{i_2}, \dots, A^{s_2-1}D_{i_2}, \dots, A^{s_{r_D}-1}D_{i_{r_D}}\right) \tag{2.20}$$

is linearly independent and such that every element of $\{D_k\}_{1\leq k\leq n_D} \setminus \{D_{i_1}, \dots, D_{i_{r_D}}\}$ belongs to the space spanned by the family (2.20). As said before if $\sum_{k=1}^{r_D} s_k = n$ the proof ends (and let us remark that in this case system (2.16) is null-controllable with distributed controls alone). If this is not the case :

Step 2 Thanks to condition (2.8) there exists $\mathsf{B}_{\mathsf{j}_1} \in \left\{(B_1)_q, (B_2)_q^{\pm}\right\}$ and $\hat{s} \in \{1, \dots, nq\}$ such that

$$\left(\left(\left(D_{i_1}, AD_{i_1}, \dots, A^{s_1-1}D_{i_1}, D_{i_2}, AD_{i_2}, \dots, A^{s_2-1}D_{i_2}, \dots, A^{s_{r_D}-1}D_{i_{r_D}}\right)\right)_q, \mathcal{A}_q^{\hat{s}-1}\mathsf{B}_{\mathsf{j}_1}\right)$$

is linearly independent. One can check that this necessary implies that the familly

$$\left(\left(D_{i_1}, AD_{i_1}, \dots, A^{s_1-1}D_{i_1}, D_{i_2}, AD_{i_2}, \dots, A^{s_2-1}D_{i_2}, \dots, A^{s_{r_D}-1}D_{i_{r_D}}\right)_q, \mathsf{B}_{\mathsf{j}_1}\right)$$

is also linearly independent. We set

$$\begin{aligned}\tilde{s}_1 = \ \ &\text{rank}\left(\left(\left(D_{i_1}, \dots, A^{s_1-1}D_{i_1}, \dots, D_{i_{r_D}}, \dots, A^{s_{r_D}-1}D_{i_{r_D}}\right)\right)_q, \mathsf{B}_{\mathsf{j}_1}, \dots, \mathcal{A}_q^{nq-1}\mathsf{B}_{\mathsf{j}_1}\right) \\ &- \sum_{k=1}^{r_D} s_k q.\end{aligned}$$

If $\tilde{s}_1 + \sum_{k=1}^{r_D} s_k q = nq$ we have done. If this is not the case, thanks to condition (2.8) we can find $\mathsf{B}_{\mathsf{j}_2} \in \left\{(B_1)_q, (B_2)_q^{\pm}\right\} \setminus \left\{\mathsf{B}_{\mathsf{j}_1}\right\}$ such that

$$\left(\left(\left(D_{i_1}, \dots, A^{s_1-1}D_{i_1}, \dots, D_{i_{r_D}}, \dots, A^{s_{r_D}-1}D_{i_{r_D}}\right)\right)_q, \mathsf{B}_{\mathsf{j}_1}, \dots, \mathcal{A}_q^{\tilde{s}_1-1}\mathsf{B}_{\mathsf{j}_1}, \mathsf{B}_{\mathsf{j}_2}\right)$$

is linearly independent. We iterate the same process and finally obtain the result. $\square$

We apply Lemma 2.10 and for the sake of simplicity of the notations we will treat one case, that is $r_B = 2$ and $\mathsf{B}_{\mathsf{j}_1} = (B_1)_q$, $\mathsf{B}_{\mathsf{j}_2} = (B_2)_q^{\pm}$. For $q = 1$ we obtain that $P_1 \in \mathcal{M}_n(\mathbb{R})$ is invertible and

$$P_1^{-1}AP_1 = C = \begin{bmatrix} C_1 & \times & \cdots & \cdots & \times \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & C_{r_D} & \times \\ 0 & \cdots & \cdots & 0 & K \end{bmatrix} \text{ where } C_i = \begin{bmatrix} 0 & 0 & \cdots & 0 & \times \\ 1 & \ddots & & \vdots & \vdots \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & \times \end{bmatrix} \in \mathcal{M}_{s_i}(\mathbb{R})$$

and $K \in \mathcal{M}_{\tilde{s}_B}(\mathbb{R})$ with $\tilde{s}_B = \sum_{k=1}^{2} \tilde{s}_k$. Moreover for every $l \in \{1, \dots, r_D\}$ we have

$$P_1 e_{S_l} = D_{i_l} \text{ and } P_1 e_{\tilde{S}_1} = B_1, \quad P_2 e_{\tilde{S}_2} = -B_2$$

where we denote $S_k = 1 + \sum_{r=1}^{k-1} s_r$, $\tilde{S}_l = S_{r_D+1} + \sum_{r=1}^{l-1} \tilde{s}_r$ and the vector e_j denotes the real vector of $\mathbb{R}^n$ with 1 on its j-th component and 0 elsewhere.

Let us now remark that if the system

$$\begin{cases} \partial_t z = \partial_{xx}^2 z + Cz + e_{S_1}\hat{u}_{i_1}(t,x)1_{\omega_{i_1}}(x) + \ldots + e_{S_{r_D}}\hat{u}_{i_{r_D}}(t,x)1_{\omega_{i_{r_D}}}(x) \text{ in } Q_T, \\ z(t,0) = e_{\tilde{S}_1}\hat{v}_1(t), \quad z(t,1) = -e_{\tilde{S}_2}\hat{v}_2(t) \text{ on } (0,T), \end{cases} \tag{2.21}$$

is null-controllable, then system (2.16) is also null-controllable by doing the change of variables $z = P_1^{-1}y$ and then taking for all $l \in \{1, \dots, n_D\}$ $u_l = \hat{u}_{i_k}$ if there exists $k \in \{1, \dots, r_D\}$ such that $l = i_k$, $u_l = 0$ otherwise, and taking for all $l \in \{1, 2\}$ $v_l = \hat{v}_l$. So let us now prove that the system (2.21) is null-controllable :

Step 2 We rewrite the solution z of system (2.21) as follow :

$$z = \begin{bmatrix} z_1 \\ \vdots \\ z_{r_D} \\ z_B \end{bmatrix}$$

where $z_B \in \mathbb{R}^{\tilde{s}_B}$ and $z_i \in \mathbb{R}^{s_i}$ for all $i \in \{1, \dots, r_D\}$.

Now we look at the system satisfied by z_B and observe that it is independent of $z_1, \dots z_{r_D}$:

$$\begin{cases} \partial_t z_B = \partial_{xx}^2 z_B + K z_B \text{ in } Q_T, \\ z_B(t,0) = e_1\hat{v}_1(t), \quad z_B(t,1) = -e_{\tilde{s}_1}\hat{v}_2(t) \text{ on } (0,T). \end{cases} \tag{2.22}$$

Assume for the moment that K satisfies the boundary Kalman condition

$$\operatorname{rank}\left[\mathcal{K}_q \,:\, \left((e_1)_q \middle| (-e_{\tilde{s}_1})_q^{\pm}\right)\right]_{\tilde{s}_B q} = \tilde{s}_B q, \quad \forall q \in \mathbb{N}^*, \tag{2.23}$$

where we recall that

$$\mathcal{K}_q = \begin{bmatrix} K_1 & 0 & \cdots & \cdots & 0 \\ 0 & K_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & K_q \end{bmatrix} \in \mathcal{M}_{\tilde{s}_B q}(\mathbb{R}) \text{ and } K_k = -\lambda_k I + K \in \mathcal{M}_{\tilde{s}_B}(\mathbb{R}).$$

Then, we deduce that the system (2.22) is null-controllable. In particular, let a time $T_B \in (0,T)$ be given, then there exist controls $\hat{\hat{v}}_1, \hat{\hat{v}}_2 \in L^2(0,T_B)$ such that $z_B(T_B) = 0$ in Ω. We choose

$$\hat{v}_1(t) = \begin{cases} \hat{\hat{v}}_1(t) \text{ if } t \in (0,T_B), \\ 0 \text{ otherwise.} \end{cases} \qquad \hat{v}_2(t) = \begin{cases} \hat{\hat{v}}_2(t) \text{ if } t \in (0,T_B), \\ 0 \text{ otherwise.} \end{cases}$$

as controls, and one can see that $z_B(t) = 0$ in Ω for all $t \geq T_B$. As a consequence $\hat{z}$ defined by

$$\hat{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_{r_D} \end{bmatrix}$$

satisfies

$$\begin{cases} \partial_t \hat{z} = \partial^2_{xx}\hat{z} + \hat{C}\hat{z} + e_{S_1}\hat{u}_{i_1}(t,x)1_{\omega_{i_1}}(x) + \ldots + e_{S_{r_D}}\hat{u}_{i_{r_D}}(t,x)1_{\omega_{i_{r_D}}}(x) \text{ in } Q_{(T_B,T)}, \\ \hat{z}(t,0) = 0, \quad \hat{z}(t,1) = 0 \text{ on } (T_B,T), \end{cases}$$

with

$$\hat{C} = \begin{bmatrix} C_1 & \times & \cdots & \cdots & \times \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \times \\ 0 & \cdots & \cdots & 0 & C_{r_D} \end{bmatrix}.$$

And since for all $i \in \{1,\ldots,r_D\}$ the distributed Kalman condition $\operatorname{rank}[\,C_i \,:\, e_i\,]_{s_i} = s_i$ is satisfied we can iterate this process and this will lead the result.

As a consequence it remains to prove that the condition (2.23) is satisfied :

Step 3 In fact, condition (2.23) holds if and only if for all $q \in \mathbb{N}^*$ the Hautus test holds (see Proposition 2.8) :

$$\ker\left(\mathcal{K}_q^* - \theta I\right) \cap \ker\left((e_1)_q \middle| (-e_{\tilde{s}_1})_q^{\pm}\right)^* = \{0\}, \quad \forall \theta \in \mathbb{C}. \tag{2.24}$$

To prove (2.24) we will use condition (2.8), and to this aim let us then first reformulate (2.24) in terms of the original data of the problem, that is A, D, B_1 and B_2. This is done through the following lemma :

Lemma 2.11. *Assume that condition (2.8) holds. We have the following equivalences :*

1. *For all $q \in \mathbb{N}^*$ the Hautus test (2.24) holds.*
2. *For all $q \in \mathbb{N}^*$, for all $\theta \in \mathbb{C}$, for all $s \in \mathbb{N}^*$ and for all $V^1, \dots, V^s \in \mathbb{R}^{\tilde{s}_B q}$ linearly independent vectors of* $\ker\left(\mathcal{K}_q^* - \theta I\right)$, *the set*
$$\left\{ \quad \left((e_1)_q \middle| (-e_{\tilde{s}_1})_q^{\pm}\right)^* V^k \quad \right\}_{1 \le k \le s}$$
is linearly independent in $\mathbb{R}^{2q}$.
3. *For all $q \in \mathbb{N}^*$, for all $\theta \in \mathbb{C}$, for all $s \in \mathbb{N}^*$ and for all $W^1, \dots, W^s \in \mathbb{R}^{nq}$ linearly independent vectors of* $\ker\left(\mathcal{A}_q^* - \theta I\right) \cap \ker\left(\left(D_{i_1}|\cdots|D_{i_{r_D}}\right)\right)_q^*$, *the set*
$$\left\{ \quad \left(\mathsf{B}_{\mathsf{j}_1} \middle| \mathsf{B}_{\mathsf{j}_2}\right)^* W^k \quad \right\}_{1 \le k \le s}$$
is linearly independent in $\mathbb{R}^{2q}$.

And those conditions are implied by the following one : for all $q \in \mathbb{N}^*$ we have

$$\ker\left(\mathcal{A}_q^* - \theta I\right) \cap \ker\left(\left(D_{i_1}|\cdots|D_{i_{r_D}}\right)\right)_q^* \cap \ker\left(\mathsf{B}_{\mathsf{j}_1} \middle| \mathsf{B}_{\mathsf{j}_2}\right)^* = \{0\}, \quad \forall \theta \in \mathbb{C}. \tag{2.25}$$

Proof. One can see that item 1 is equivalent to item 2 (see for instance [AKBGBdT11a, Proposition 3.1]) and that condition (2.25) implies item 3. Thus let us prove that item 2 and item 3 are equivalent. Let us fix $q \in \mathbb{N}^*$ and $\theta \in \mathbb{C}$. We define a bijective linear map Φ by :

$$\begin{array}{rccc} \Phi & : & \ker\left(\mathcal{K}_q^* - \theta I\right) & \longrightarrow & \ker\left(\mathcal{A}_q^* - \theta I\right) \cap E \\ & & V = \begin{bmatrix} V_1 \\ \vdots \\ V_q \end{bmatrix} & \longmapsto & \begin{bmatrix} \tilde{\Phi}(V_1) \\ \vdots \\ \tilde{\Phi}(V_q) \end{bmatrix} \end{array}$$

where

$$\tilde{\Phi}(V_l) = (P_1^*)^{-1} \begin{bmatrix} 0 \\ V_l \end{bmatrix} \in \mathbb{R}^n,$$

and

$$E = \left\{ W = \begin{bmatrix} W_1 \\ \vdots \\ W_q \end{bmatrix} \in \mathbb{R}^{nq} \text{ such that } \forall l \in \{1, \dots, q\}, \quad W_l = (P_1^*)^{-1} \begin{bmatrix} 0 \\ \times \end{bmatrix} \right\}.$$

One can check that in fact $E = \ker(P_q^D)^*$ and thus $\ker\left(\mathcal{A}_q^* - \theta I\right) \cap E = \ker\left(\mathcal{A}_q^* - \theta I\right) \cap \ker\left(\left(D_{i_1}|\cdots|D_{i_{r_D}}\right)\right)_q^*$. Moreover, for all $s \in \mathbb{N}^*$ and all $\{\alpha_k\}_{1\le k\le s} \subset \mathbb{R}$ we have

$$\begin{aligned}
\sum_{k=1}^{s} \alpha_k \Big(\mathsf{B}_{\mathsf{j}_1}\Big|\mathsf{B}_{\mathsf{j}_2}\Big)^* \Phi(V^k) &= \sum_{k=1}^{s} \alpha_k \left(\sum_{l=1}^{q} \Big(B_1\Big|(-1)^l B_2\Big)^* \Phi_l(V_l^k)\right) \\
&= \sum_{k=1}^{s} \alpha_k \left(\sum_{l=1}^{q} \Big(e_{\tilde{S}_1}\Big|(-1)^{l+1} e_{\tilde{S}_2}\Big)^* P_1^* \Phi_l(V_l^k)\right) \\
&= \sum_{k=1}^{s} \alpha_k \left(\sum_{l=1}^{q} \Big(e_{\tilde{S}_1}\Big|(-1)^{l+1} e_{\tilde{S}_2}\Big)^* \begin{bmatrix} 0 \\ V_l^k \end{bmatrix}\right) \\
&= \sum_{k=1}^{s} \alpha_k \left(\sum_{l=1}^{q} \Big(e_1\Big|(-1)^{l+1} e_{\tilde{s}_1}\Big)^* V_l^k\right) \\
&= \sum_{k=1}^{s} \alpha_k \Big((e_1)_q\Big|(-e_{\tilde{s}_1})_q^{\pm}\Big)^* V^k.
\end{aligned}$$

Combining those two facts the claim is proved.

□

As a consequence of Lemma 2.11 it is sufficient to prove that (2.25) is true, and in fact it is a consequence of the Hautus test, and hypothesis (2.8) :

Lemma 2.12. *Assume that condition (2.8) holds. Then, for all $q \in \mathbb{N}^*$ we have*

$$\ker\left(\mathcal{A}_q^* - \theta I\right) \cap \ker\left(\left(D_{i_1}|\cdots|D_{i_{r_D}}\right)\right)_q^* \cap \ker\Big(\mathsf{B}_{\mathsf{j}_1}\Big|\mathsf{B}_{\mathsf{j}_2}\Big)^* = \{0\}, \quad \forall \theta \in \mathbb{C}. \tag{2.26}$$

Proof. (2.26) can be rewritten as

$$\ker\left(\mathcal{A}_q^* - \theta I\right) \cap \ker\left(\mathsf{B}_{\mathsf{j}_1}\Big|\mathsf{B}_{\mathsf{j}_2}\Big|\left(\left(D_{i_1}|\cdots|D_{i_{r_D}}\right)\right)_q\right)^* = \{0\}, \quad \forall \theta \in \mathbb{C}.$$

which is equivalent to (by the Hautus test)

$$\operatorname{rank}\left[\mathcal{A}_q \,:\, \left(\mathsf{B}_{\mathsf{j}_1}\Big|\mathsf{B}_{\mathsf{j}_2}\Big|\left(\left(D_{i_1}|\cdots|D_{i_{r_D}}\right)\right)_q\right)\right]_{nq} = nq,$$

and this last formulation is also equivalent to

$$\operatorname{rank}\left[\Big[\mathcal{A}_q \,:\, \Big(\mathsf{B}_{\mathsf{j}_1}\Big|\mathsf{B}_{\mathsf{j}_2}\Big)\Big]_{nq}\Big|\left(\left(\Big[A \,:\, \Big(D_{i_1}|\cdots|D_{i_{r_D}}\Big)\Big]_n\right)\right)_q\right] = nq, \tag{2.27}$$

in the same way as condition (2.8) is equivalent to condition (2.9) (see Remark 2.2, item 2). Now thanks to Lemma 2.10 we can see that (2.27) holds (observe that we have more powers of $\mathcal{A}_q$ and A in (2.27)).

□

Let us now prove the necessary part of Theorem 2.1 :
Step 4 Suppose that there exists $q_0 \in \mathbb{N}^*$ such that

$$\mathrm{rank}\left[\ \left[\mathcal{A}_{q_0}\ :\ \left((B_1)_{q_0}\middle|(B_2)^{\pm}_{q_0}\right)\right]_{nq_0}\ \middle|\ (([\,A\ :\ D\,]_n))_{q_0}\ \right] < nq_0.$$

Thanks to the other characterization of condition (2.8) (see Remark 2.2, item 2) this means we have

$$\mathrm{rank}\left[\mathcal{A}_{q_0}\ :\ \left((B_1)_{q_0}\middle|(B_2)^{\pm}_{q_0}\middle|((D))_{q_0}\right)\right]_{nq_0} < nq_0. \tag{2.28}$$

Thus, there exists $\Psi^T \in \mathbb{R}^{nq_0}$ with $\Psi^T \neq 0$ such that $\Psi(t) = e^{\mathcal{A}^*_{q_0}(T-t)}\Psi^T$ satisfies (see Proposition 2.8)

$$\left((B_1)_{q_0}\middle|(B_2)^{\pm}_{q_0}\middle|((D))_{q_0}\right)^*\Psi(t) = 0, \quad \forall t \in [0,T]. \tag{2.29}$$

Let us write $\Psi(t)$ as follow :

$$\begin{aligned}
\Psi(t) = e^{\mathcal{A}^*_{q_0}(T-t)}\Psi^T \ &= \begin{bmatrix} e^{A_1^*(T-t)} & 0 & \cdots & \cdots & 0 \\ 0 & e^{A_2^*(T-t)} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & e^{A^*_{q_0}(T-t)} \end{bmatrix} \begin{bmatrix} \Psi_1^T \\ \Psi_2^T \\ \vdots \\ \vdots \\ \Psi_{q_0}^T \end{bmatrix} \\
&= \begin{bmatrix} e^{(-\lambda_1 I + A^*)(T-t)}\Psi_1^T \\ e^{(-\lambda_2 I + A^*)(T-t)}\Psi_2^T \\ \vdots \\ \vdots \\ e^{(-\lambda_{q_0} I + A^*)(T-t)}\Psi_{q_0}^T \end{bmatrix} = \begin{bmatrix} \Psi_1(t) \\ \Psi_2(t) \\ \vdots \\ \vdots \\ \Psi_{q_0}(t) \end{bmatrix}, \quad \forall t \in [0,T].
\end{aligned}$$

Thus (2.29) gives

$$\begin{bmatrix} B_1^* & B_1^* & \cdots & \cdots & \cdots & B_1^* \\ -B_2^* & B_2^* & \cdots & \cdots & \cdots & (-1)^{q_0}B_2^* \\ D^* & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & D^* & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & & \ddots & \vdots \\ \vdots & & \ddots & & \ddots & 0 \\ 0 & \cdots & \cdots & & 0 & D^* \end{bmatrix} \begin{bmatrix} \Psi_1(t) \\ \Psi_2(t) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \Psi_{q_0}(t) \end{bmatrix} = 0, \quad \forall t \in [0,T],$$

i.e.

$$\forall t \in [0,T], \quad \begin{cases} \sum_{k=1}^{q_0} B_1^* \Psi_k(t) = 0, \\ \sum_{k=1}^{q_0} (-1)^k B_2^* \Psi_k(t) = 0, \\ D^* \Psi_k(t) = 0, \quad \forall k \in \{1, \dots, q_0\}. \end{cases} \tag{2.30}$$

Let us now prove that the observability inequality (2.19) fails. To this aim we define $\Phi^T \in H_0^1(\Omega; \mathbb{R}^n)$ by

$$\Phi^T = \sum_{k=1}^{q_0} \lambda_k{}^{-1/2} \Psi_k^T \phi_k.$$

and let Φ be the solution to

$$\begin{cases} -\partial_t \Phi = \partial_{xx}^2 \Phi + A^* \Phi \text{ in } Q_T, \\ \Phi = 0 \text{ on } \Sigma_T, \\ \Phi(T) = \Phi^T \text{ in } \Omega, \end{cases}$$

i.e.

$$\begin{aligned} \Phi(t) = \sum_{k=1}^{q_0} e^{(-\lambda_k I + A^*)(T-t)} & \begin{bmatrix} \left\langle \Phi_1^T, \phi_k \right\rangle_{L^2} \\ \vdots \\ \left\langle \Phi_n^T, \phi_k \right\rangle_{L^2} \end{bmatrix} \phi_k \\ & = \sum_{k=1}^{q_0} \lambda_k{}^{-1/2} e^{(-\lambda_k I + A^*)(T-t)} \Psi_k^T \phi_k = \sum_{k=1}^{q_0} \lambda_k{}^{-1/2} \Psi_k(t) \phi_k. \end{aligned} \tag{2.31}$$

From (2.30) we obtain

$$\forall t \in (0,T), \forall i \in \{1, \dots, n_D\}, \quad D_i^* \Phi(t) = \sum_{k=1}^{q_0} \lambda_k{}^{-1/2} D_i^* \Psi_k(t) \phi_k = 0 \text{ in } \Omega,$$

and since the space dimension is $N = 1$ we also obtain

$$\forall t \in (0,T), \quad B_1^* \partial_x \Phi(t,0) = B_1^* \sum_{k=1}^{q_0} \Psi_k(t) \underbrace{\lambda_k{}^{-1/2} \partial_x \phi_k(0)}_{=1} = 0,$$

and

$$\forall t \in (0,T), \quad B_2^* \partial_x \Phi(t,1) = B_2^* \sum_{k=1}^{q_0} \Psi_k(t) \underbrace{\lambda_k{}^{-1/2} \partial_x \phi_k(1)}_{=(-1)^k} = 0.$$

Finally let us remark that $\Phi(0) \neq 0$ since $\Psi^T \neq 0$ (see (2.31)). As a consequence the observability inequality (2.19) fails and so does the null-controllability of (2.16). $\square$

2.4 The Carleman estimate

Recall that all along this section, no boundary controls are considered and the space dimension N is arbitrary. The aim is to prove Theorem 2.7 and as said before this latter is a consequence of a Carleman estimate.

2.4.1 A Carleman estimate for cascade systems

Let us introduce the framework in which the Carleman estimate will be established. We consider this time the following $n \times n$ parabolic system :

$$\begin{cases} \partial_t y = \Delta y + Cy + e_{S_1} u_1(t,x) 1_{\omega_1}(x) + \ldots + e_{S_r} u_r(t,x) 1_{\omega_r}(x) \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \end{cases} \tag{2.32}$$

where $r \in \{1, \ldots, n\}$ is the number of controls, $u_1 \in L^2(Q_T), \ldots, u_r \in L^2(Q_T)$ are the controls and where $C = C(t,x) \in L^\infty(Q_T; \mathcal{M}_n(\mathbb{R}))$ is a matrix with the following block cascade type structure :

$$C = \begin{bmatrix} C_{11} & \times & \times & \cdots & \times \\ 0 & C_{22} & \times & \cdots & \times \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \times \\ 0 & \cdots & \cdots & 0 & C_{rr} \end{bmatrix} \text{ with } C_{jj} = \begin{bmatrix} c^j_{11} & c^j_{12} & c^j_{13} & \cdots & c^j_{1s_j} \\ c^j_{21} & c^j_{22} & c^j_{23} & \cdots & c^j_{2s_j} \\ 0 & c^j_{32} & c^j_{33} & \cdots & c^j_{3s_j} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & c^j_{s_j s_{j-1}} & c^j_{s_j s_j} \end{bmatrix}, \tag{2.33}$$

where $s_j \in \mathbb{N}$ is the size of the bloc C_{jj} (in particular we have $\sum_{j=1}^r s_j = n$) and where $S_1, \ldots, S_{r+1}$ are such that one control is exerted on each first equation of a block, this reads $S_i = 1 + \sum_{j=1}^{i-1} s_j$ for all $i \in \{1, \ldots, r+1\}$.

Those notations in mind, we have

Theorem 2.13. *Assume that for every $j \in \{1, \ldots, r\}$ there exists a nonempty open subset $\tilde{\omega}_j \subset \omega_j$ and $\underline{c}^j > 0$ such that for every $i \in \{1, \ldots, s_j\}$ we have*

$$c^j_{i+1,i} \geq \underline{c}^j \text{ or } -c^j_{i+1,i} \geq \underline{c}^j \text{ on } (0,T) \times \tilde{\omega}_j. \tag{2.34}$$

Then, there exist functions $\beta_1, \ldots, \beta_r \in C^2(\overline{\Omega})$ such that $0 < \beta_1 < \ldots < \beta_r$, there exists $C > 0$, $s_0 > 0$ and $l_0 > 0$ such that, for every $\Phi^T \in L^2(\Omega; \mathbb{R}^n)$, the solution Φ to the system

$$\begin{cases} -\partial_t \Phi = \Delta \Phi + C^* \Phi \text{ in } Q_T, \\ \Phi = 0 \text{ on } \Sigma_T, \\ \Phi(T) = \Phi^T \text{ in } \Omega, \end{cases} \tag{2.35}$$

satisfies

$$\sum_{j=1}^{r}\sum_{i=S_j}^{S_{j+1}-1}\mathcal{J}_j(3(S_{j+1}-i),\Phi_i)\leq C\sum_{j=1}^{r}\iint_{(0,T)\times\omega_j}(s\varphi)^{l_0}e^{-2s\eta_j}|\Phi_{S_j}|^2,$$

for all $s\geq s_0$. Here we have denoted

$$\mathcal{J}_j(d,q)=\iint_{Q_T}(s\varphi)^{d-2}e^{-2s\eta_j}\left|\nabla q\right|^2+\iint_{Q_T}(s\varphi)^{d}e^{-2s\eta_j}\left|q\right|^2, \tag{2.36}$$

and $\varphi(t)=(t(T-t))^{-1}$, $\eta_j(t,x)=\beta_j(x)\varphi(t)$ for $j\in\{1,\dots,r\}$.

Proof. We adapt the proof of [GBdT10, Theorem 1.1], but we consider different weight functions on each block and we use the fact that we still can choose such functions in an ordered way. First, let us rewrite system (2.35) on each block as follows :

$$\forall j\in\{1,\dots,r\},\quad\left\{\begin{array}{l}-\partial_t\Phi_i=\Delta\Phi_i+\sum\limits_{k=1}^{i}c_{ki}\Phi_k+c_{i+1,i}\Phi_{i+1}\text{ in }Q_T,\quad\forall i\in\{S_j,\dots S_{j+1}-2\}\\[2ex] -\partial_t\Phi_{S_{j+1}-1}=\Delta\Phi_{S_{j+1}-1}+\sum\limits_{k=1}^{S_{j+1}-1}c_{k,S_{j+1}-1}\Phi_k\text{ in }Q_T,\\[2ex] \Phi_i=0\text{ on }\Sigma_T,\quad\forall i\in\{S_j,\dots,S_{j+1}-1\}.\end{array}\right. \tag{2.37}$$

where we rewrote for convenience $C=(c_{ij})_{1\leq i,j\leq n}$. And let us recall the following Carleman estimate for one single parabolic equation (see [IY03, Lemma 2.3]) :

Lemma 2.14. *Let $\omega\subset\Omega$ be a non-empty open subset. For every $\underline{\beta}>0$, there exists a function $\beta\in C^2(\overline{\Omega})$ such that $\beta>\underline{\beta}$ and such that, for every $d\in\mathbb{R}$, there exist $C>0$, $s_0>0$ such that, for every $\psi^T\in L^2(\Omega)$ and every $f\in L^2(Q_T)$, the solution ψ to*

$$\left\{\begin{array}{l}-\partial_t\psi=\Delta\psi+f(t,x)\ \textit{in}\ Q_T,\\ \psi=0\ \textit{on}\ \Sigma_T,\\ \psi(T)=\psi^T\ \textit{in}\ \Omega,\end{array}\right.$$

satisfies

$$\iint_{Q_T}(s\varphi)^{d-2}e^{-2s\eta}\left|\nabla\psi\right|^2+\iint_{Q_T}(s\varphi)^{d}e^{-2s\eta}\left|\psi\right|^2$$
$$\leq C\left(\iint_{(0,T)\times\omega}(s\varphi)^{d}e^{-2s\eta}\left|\psi\right|^2+\iint_{Q_T}(s\varphi)^{d-3}e^{-2s\eta}\left|f\right|^2\right)$$

for all $s\geq s_0$. Where $\eta(t,x)=\beta(x)\varphi(t)$.

To start the proof of Theorem 2.13, let be given $\tilde{\tilde{\omega}}_1 \subset\subset \tilde{\omega}_1, \ldots, \tilde{\tilde{\omega}}_p \subset\subset \tilde{\omega}_p$. For every $j \in \{1, \ldots, r\}$ we apply Lemma 2.14 with $\omega = \tilde{\tilde{\omega}}_j$ which enables us to construct functions $\beta_1, \ldots, \beta_r \in C^2(\overline{\Omega})$ such that $0 < \beta_1 < \ldots < \beta_r$ and such that each function Φ_i, $S_j \leq i \leq S_{j+1} - 1$, satisfies, due to the particular structure of C^* (see (2.37)) :

$$\begin{aligned}\forall i \neq S_{j+1} - 1, \quad \mathcal{J}_j(3(S_{j+1} - i), \Phi_i) \leq \frac{1}{2} C_1 \Bigg(& \mathcal{L}_j \left(\tilde{\omega}_j; 3(S_{j+1} - i), \Phi_i\right) \\ & + \mathcal{J}_j(3(S_{j+1} - 1 - i), \Phi_{i+1})) + \sum_{k=1}^{i} \mathcal{J}_j(3(S_{j+1} - 1 - i), \Phi_k) \Bigg),\end{aligned}$$

and

$$\mathcal{J}_j(3, \Phi_{S_{j+1}-1}) \leq \frac{1}{2} C_1 \left(\mathcal{L}_j \left(\tilde{\omega}_j; 3(S_{j+1} - i), \Phi_{S_{j+1}-1}\right) + \sum_{k=1}^{S_{j+1}-1} \mathcal{J}_j(0, \Phi_k) \right),$$

for s large enough, where here and in what follows we denote

$$\mathcal{L}_k(\omega; d, q) = \iint_{(0,T)\times\omega} (s\varphi)^d e^{-2s\eta_k} |q|^2.$$

Summing over i this gives

$$\begin{aligned}\sum_{i=S_j}^{S_{j+1}-1} {C_1}^{i-S_j} \mathcal{J}_j(3(S_{j+1} - i), \Phi_i) \leq & \sum_{i=S_j}^{S_{j+1}-1} \frac{1}{2} {C_1}^{i+1-S_j} \Bigg(\mathcal{L}_j \left(\tilde{\omega}_j; 3(S_{j+1} - i), \Phi_i\right) \\ & + \sum_{k=1}^{i} \mathcal{J}_j(3(S_{j+1} - 1 - i), \Phi_k) \Bigg).\end{aligned}$$

Summing over j this leads to

$$\begin{aligned}\sum_{j=1}^{r} \sum_{i=S_j}^{S_{j+1}-1} \mathcal{J}_j(3(S_{j+1} - i), \Phi_i) \leq \; & C_2 \Bigg(\sum_{j=1}^{r} \sum_{i=S_j}^{S_{j+1}-1} \mathcal{L}_j \left(\tilde{\omega}_j; 3(S_{j+1} - i), \Phi_i\right) \\ & + \sum_{j=1}^{r} \sum_{i=S_j}^{S_{j+1}-1} \sum_{k=1}^{i} \mathcal{J}_j(3(S_{j+1} - 1 - i), \Phi_k) \Bigg).\end{aligned} \tag{2.38}$$

Now let us remark that the last term in the right hand-side of (2.38) can be seen as the sum of two terms :

$$\begin{aligned}\sum_{j=1}^{r} \sum_{i=S_j}^{S_{j+1}-1} \sum_{k=1}^{i} \mathcal{J}_j(3(S_{j+1} - 1 - i), \Phi_k) = \; & \sum_{j=1}^{r} \sum_{i=S_j}^{S_{j+1}-1} \mathcal{J}_j(3(S_{j+1} - 1 - i), \Phi_i) \\ & + \sum_{j=1}^{r} \sum_{i=S_j}^{S_{j+1}-1} \sum_{k=1}^{i-1} \mathcal{J}_j(3(S_{j+1} - 1 - i), \Phi_k),\end{aligned}$$

and for s large enough the first one can be absorbed by the left hand-side of (2.38), because $S_{j+1} - 1 - i < S_{j+1} - i$, and the second one can also be absorbed by the left hand-side of (2.38) since the functions β_j have been constructed such that

$$0 < \beta_1 < \beta_2 < \ldots < \beta_r,$$

so that, for every $d_1, d_2 \geq 0$, there exists $C > 0$ such that

$$(s\varphi)^{d_1} e^{-2s\eta_j} \leq C(s\varphi)^{d_2} e^{-2s\eta_{j-1}}, \quad \forall j \in \{2, \dots, r\}$$

for s large enough. Thus we finally obtain

$$\sum_{j=1}^{r} \sum_{i=S_j}^{S_{j+1}-1} \mathcal{J}_j(3(S_{j+1}-i), \Phi_i) \leq C_3 \left(\sum_{j=1}^{r} \sum_{i=S_j}^{S_{j+1}-1} \mathcal{L}_j \left(\tilde{\tilde{\omega}}_j; 3(S_{j+1}-i), \Phi_i \right) \right) \tag{2.39}$$

for s large enough. To pursue we use the following lemma (see [GBdT10, Section 4]) :

Lemma 2.15. *Assume that (2.34) holds. Then, for every $\epsilon > 0$, $j \in \{1, \dots r\}$, $i \in \{S_j + 1, \dots, S_{j+1} - 1\}$ and $l \in \mathbb{N}$ and every open sets $\mathcal{O}_0, \mathcal{O}_1$ such that $\tilde{\tilde{\omega}}_j \subset \mathcal{O}_1 \subset\subset \mathcal{O}_0 \subset \tilde{\omega}_j$, there exist $C > 0$, $s_0 > 0$ and $l_1(j,l), \dots, l_{i-1}(j,l) \in \mathbb{N}$ such that the solution Φ to (2.35) satisfies*

$$\begin{aligned} \forall i \neq S_{j+1}-1, \quad \mathcal{L}_j(\mathcal{O}_1; l, \Phi_i) \leq\ & \epsilon \Big(\mathcal{J}_j(3(S_{j+1}-i), \Phi_i) + \mathcal{J}_j(3(S_{j+1}-1-i), \Phi_{i+1}) \Big) \\ & + C \sum_{k=1}^{i-1} \mathcal{L}_j(\mathcal{O}_0; l_k(j,l), \Phi_k). \end{aligned}$$

and

$$\mathcal{L}_j(\mathcal{O}_1; l, \Phi_{S_{j+1}-1}) \leq \epsilon \mathcal{J}_j(3, \Phi_{S_{j+1}-1}) + C \sum_{k=1}^{S_{j+1}-2} \mathcal{L}_j(\mathcal{O}_0; l_k(j,l), \Phi_k).$$

for all $s \geq s_0$.

For each $j \in \{1, \dots, r\}$ let be given $\tilde{\tilde{\omega}}_j \subset\subset \mathcal{O}_{j,1} \subset\subset \dots \subset\subset \mathcal{O}_{j,s_j-1} \subset\subset \tilde{\omega}_j$. Applying Lemma 2.15 to $i = S_{j+1} - 1$, $\mathcal{O}_1 = \tilde{\tilde{\omega}}_j$, $\mathcal{O}_0 = \mathcal{O}_{j,1}$, $l = 3(S_{j+1} - i) = l_j^1$ and $\epsilon = \frac{1}{2C_3}$, we have

$$\mathcal{L}_j(\tilde{\tilde{\omega}}_j; l_j^1, \Phi_{S_{j+1}-1}) \leq \frac{1}{2C_3} \mathcal{J}_j(3, \Phi_{S_{j+1}-1}) + C_4 \sum_{k=1}^{S_{j+1}-2} \mathcal{L}_j(\mathcal{O}_{j,1}; l_k(j, l_j^1), \Phi_k).$$

Back to (2.39) we obtain

$$\sum_{j=1}^{r} \sum_{i=S_j}^{S_{j+1}-1} \mathcal{J}_j(3(S_{j+1}-i), \Phi_i) \leq C_5 \left(\sum_{j=1}^{r} \sum_{k=1}^{S_{j+1}-2} \mathcal{L}_j(\mathcal{O}_{j,1}; \max\left\{3(S_{j+1}-k), l_k(j, l_j^1)\right\}, \Phi_k) \right).$$

Applying now Lemma 2.15 to $i = S_{j+1} - 2$, $\mathcal{O}_1 = \mathcal{O}_{j,1}$, $\mathcal{O}_0 = \mathcal{O}_{j,2}$, $l = \max\left\{6, l_{S_{j+1}-2}(j, l_j^1)\right\} = l_j^2$ and $\epsilon = \frac{1}{2C_5}$, we obtain

$$\begin{aligned} & \sum_{j=1}^{r} \sum_{i=S_j}^{S_{j+1}-1} \mathcal{J}_j(3(S_{j+1}-i), \Phi_i) \\ & \qquad \leq C_6 \left(\sum_{j=1}^{r} \sum_{k=1}^{S_{j+1}-3} \mathcal{L}_j(\mathcal{O}_{j,2}; \max\left\{\max\left\{3(S_{j+1}-k), l_k(j, l_j^1)\right\}, l_k(j, l_j^2)\right\}, \Phi_k) \right). \end{aligned}$$

Iterating the process leads to the estimate

$$\sum_{j=1}^{r}\sum_{i=S_j}^{S_{j+1}-1}\mathcal{J}_j(3(S_{j+1}-i),\Phi_i)$$
$$\leq C_7\left(\sum_{j=1}^{r}\sum_{k=1}^{S_j}\mathcal{L}_j(\mathcal{O}_{j,s_j-1};\max\left\{3(S_{j+1}-k),l_k(j,l_j^1),\ldots,l_k(j,l_j^{s_j-1})\right\},\Phi_k)\right),$$

which can be rewriten as follow by separating the term $k=S_j$

$$\sum_{j=1}^{r}\sum_{i=S_j}^{S_{j+1}-1}\mathcal{J}_j(3(S_{j+1}-i),\Phi_i)\leq C_7\left(\sum_{j=1}^{r}\mathcal{L}_j(\mathcal{O}_{j,s_j-1};\max\left\{3s_j,l_{S_j}(j,l_j^1),\ldots,l_{S_j}(j,l_j^{s_j-1})\right\},\Phi_{S_j})\right.$$
$$\left.+\sum_{j=1}^{r}\sum_{k=1}^{S_j-1}\mathcal{L}_j(\mathcal{O}_{j,s_j-1};\max\left\{3(S_{j+1}-k),l_k(j,l_j^1),\ldots,l_k(j,l_j^{s_j-1})\right\},\Phi_k)\right).\quad(2.40)$$

Let us now recall that we have chosen $\beta_1<\beta_2<\ldots<\beta_r$ so that the last term in the right-hand side of (2.40) can be absorbed, for s large enough, by the term of the left-hand side of (2.40), no matter what the power of s in those terms are.

□

2.4.2 Proof of Theorem 2.7

All the work is based on to the previous Carleman estimate. Indeed, following the ideas of [AKBDGB09a] we can construct a change of basis thanks to condition (2.15) which leads to a cascade system (see [AKBDGB09a, Lemma 4.1]) with possibly controls acting on different subdomains. Applying the previous Carleman estimate we deduce the result.

2.5 Further results and comments

1. Untill now we looked at the null-controllability properties for systems where the coefficients in front of the operator $-\Delta$ were the same on every equation but we can also consider systems where those coefficients are different ; let us consider

$$\begin{cases}\partial_t y = J\Delta y + Ay + D_1u_1(t,x)1_{\omega_1}(x)+\ldots+D_{n_D}u_{n_D}(t,x)1_{\omega_{n_D}}(x) \text{ in } Q_T,\\ y=0 \text{ on } \Sigma_T,\end{cases}\quad(2.41)$$

with $J\in\mathcal{M}_n(\mathbb{R})$ such that J is diagonalizable with positive eigenvalues. Then there exists also a Kalman rank condition, which is a simple extension of the one proved in [AKBD06] :

Theorem 2.16. *Under the previous assumption on J, system (2.41) is null-controllable if and only if*

$$\operatorname{rank}\left[\,-\lambda_k J + A \,:\, D\,\right]_n = n, \quad \forall k \in \mathbb{N}^*. \tag{2.42}$$

When J is the identity matrix condition (2.42) is indeed equivalent to condition (2.13) since $\operatorname{rank}\left[\,-\lambda_k I + A \,:\, D\,\right]_n = \operatorname{rank}\left[\,A \,:\, D\,\right]_n$ for all $k \in \mathbb{N}^*$.

Theorem 2.16 can be proved by slightly changing the proof of Theorem 1.1 of [AKBD06]. Indeed, with the notations of [AKBD06], changing the definition of $\mathcal{K}$ one can see that Theorem 1.1 is still a consequence of Theorem 1.3 and Theorem 1.3 is still a consequence of Theorem 1.2 and Theorem 2.1. The proof of Theorem 2.1 remains unchanged and the proof of Theorem 1.2 can be adapted to our case by applying Theorem 3.2 to $\phi = D_i^* \varphi$ instead of $\phi = (B^*\varphi)_i$.

2. All the results of section 2.2.2 and Theorem 2.16 remain true if we replace the operator $-\Delta$ by more general elliptic operators $-R$, of the form

$$\begin{cases} Ry = \displaystyle\sum_{i,j=1}^{N} \partial_i \left(r_{ij}(x) \partial_j y \right), \\ r_{ij} \in W^{1,\infty}(\Omega), \quad r_{ij} = r_{ji} \text{ in } \Omega, \quad \forall i,j \in \{1,\dots,N\}, \\ \exists\, \underline{r} > 0, \quad \displaystyle\sum_{i,j=1}^{N} r_{ij}\xi_i\xi_j \geq \underline{r}|\xi|^2 \text{ in } \Omega, \quad \forall \xi \in \mathbb{R}^N. \end{cases}$$

3. In this paper we used a particular strategy which does not apply to many other problems. For instance the case of space varying coefficients can not be analyzed the same way. Indeed we did a change of variable so we used the fact that Δ commutes with P_1.

As our result is based on the one of [AKBGBdT11a], the N-dimensional case with $N > 1$ is still open.

Chapitre 3

Boundary approximate controllability of some linear parabolic systems

Ce chapitre est la reprise de l'article [Oli13], qui a été soumis.

Abstract. This paper focuses on the boundary approximate controllability of two classes of linear parabolic systems, namely a system of n heat equations coupled through constant terms and a 2×2 cascade system coupled by means of a first order partial differential operator with space-dependent coefficients.

For each system we prove a sufficient condition in any space dimension and we show that this condition turns out to be also necessary in one dimension with only one control. For the system of coupled heat equations we also study the problem on rectangle, and we give characterizations depending on the position of the control domain. Finally, we exhibit a cascade system for which the distributed controllability holds whereas the boundary controllability does not.

The method relies on a general characterization due to H.O. Fattorini.

Keywords : Parabolic systems ; Boundary controllability ; Distributed controllability ; Hautus test.

3.1 Introduction

The controllability of parabolic systems is a difficult problem. While Carleman estimates have been successfully used to prove the distributed null-controllability of some linear parabolic systems (e.g. [AKBDGB09b], [GBdT10], [Gue07], [Mau13], [BCGdT13]), there are still many cases where these estimates appear to be of no help. An example of such situation is when the control domain and the coupling domain do not meet each other ([ABL12], [RdT11]). The boundary controllability is another of these situations and requires new techniques to be solved. In [FCGBdT10] and [AKBGBdT11a], the authors developped the method of moments of H.O. Fattorini and D.L. Russell to establish a characterization of the boundary null-controllability in dimension 1 for a system of n coupled heat equations. In [ABL12], the authors used transmutation techniques to obtain a boundary null-controllability result in any dimension for a system of 2 heat equations, with a particular coupling. Finally, in [KdT10] the authors proved the boundary approximate controllability of a cascade system of 2 heat equations in any dimension by developping the solution into Fourier series. To the author knowledge, these results are the only ones concerning the boundary controllability of linear parabolic systems of heat-type. For more details, a good account on actual methods and recent open problems for the distributed or boundary controllability of linear parabolic systems we refer to the survey [AKBGBdT11b].

In the present work we are interested in the boundary approximate controllability of two classes of linear parabolic systems introduced in [FCGBdT10] and [KdT10]. More precisely, the first system we study is the following [1]

$$\left\{\begin{array}{rcll} \partial_t y & = & \overrightarrow{\Delta} y + Ay & \text{in } (0,T)\times\Omega. \\ y & = & 1_\gamma Bg & \text{on } (0,T)\times\partial\Omega. \\ y(0) & = & y_0 & \text{in } \Omega. \end{array}\right. \tag{3.1}$$

where $T>0$, Ω is a bounded open subset of $\mathbb{R}^N$, assumed regular enough, y is the state, y_0 is the initial data, A and B are $n\times n$ and $n\times m$ constant matrices with complex coefficients, g is the control, to be searched in $L^2(0,T;L^2(\partial\Omega)^m)$ - so that in fact we have m controls - and $\gamma\subset\partial\Omega$ is the control domain.

First of all, let us recall some basic facts about this kind of systems and their controllability properties :

1. System (3.1) is well-posed in the following sense : for every $y_0\in H^{-1}(\Omega)^n$ and $g\in L^2(0,T;L^2(\partial\Omega)^m)$, there exists a unique solution defined by transposition $y\in C^0([0,T];H^{-1}(\Omega)^n)\cap L^2(0,T;L^2(\Omega)^n)$ that depends continuously on the initial data y_0 and the control g.
2. System (3.1) is said to be approximately controllable at time T if for every $y_0, y_1\in H^{-1}(\Omega)^n$ and every $\epsilon>0$, there exists a control $g\in L^2(0,T;L^2(\partial\Omega)^m)$ such that the corresponding solution y satisfies

$$\|y(T)-y_1\|_{H^{-1}(\Omega)^n}\leq\epsilon.$$

   We say that system (3.1) is approximately controllable if it is approximately controllable at time T for every $T>0$.
3. It is nowadays well-known that the controllability has a dual concept called observability and that they are linked by the following result : system (3.1) is approximately

1. $\overrightarrow{\Delta}$ denotes the vectorial Laplacian, in constrast with Δ for the scalar Laplacian.

controllable at time T if and only if its adjoint system [2]

$$\begin{cases} \partial_t z = \overrightarrow{\Delta} z + A^* z & \text{in } (0,T)\times\Omega. \\ z = 0 & \text{on } (0,T)\times\partial\Omega. \\ z(0) = z_0 & \text{in } \Omega. \end{cases}$$

is approximately observable at time T, that is it verifies the following unique continuation property

$$\forall z_0 \in H_0^1(\Omega)^n, \quad \Big(B^* 1_\gamma \partial_n z(t) = 0 \text{ for a.e. } t\in(0,T)\Big) \Longrightarrow z_0 = 0. \tag{3.2}$$

The boundary controllability problem for (3.1) has been introduced in [FCGBdT10]. In this paper, the authors proved a necessary and sufficient condition for this system to be null-controllable, and the same condition also characterizes the approximate controllability, see [FCGBdT10, Theorem 1.1] and [FCGBdT10, Theorem 5.2]. We point out that this work has been done in 1D and for 2 equations. A generalization to the case of n equations can be found in [AKBGBdT11a], still in 1D. To the author knowledge, the only result that can be applied to system (3.1) in any dimension is [ABL12, Corollary 2.2], but the matrix A has to have a very particular structure and it requires a geometric condition on γ.

In this paper we will provide conditions for the approximate controllability of this system in several interesting particular cases, see the sections 3.2.1 to 3.2.5 below. Some results are already known but we give new and simpler proofs.

The second system we deal with is the following

$$\begin{cases} \partial_t y_1 = \Delta y_1 & \text{in } (0,T)\times\Omega. \\ \partial_t y_2 = \Delta y_2 + G(x)\cdot\nabla y_1 + a(x) y_1 & \text{in } (0,T)\times\Omega. \\ y_1 = 1_\gamma g, \quad y_2 = 0 & \text{on } (0,T)\times\partial\Omega. \\ y_1(0) = y_{1,0}, \quad y_2(0) = y_{2,0} & \text{in } \Omega. \end{cases} \tag{3.3}$$

where $G \in W^{1,\infty}(\Omega)^N$, $a \in L^\infty(\Omega)$, and g is still the control, but this time we only have one control : $g \in L^2(0,T;L^2(\partial\Omega))$.

The interest in the controllability of such systems started with [KdT10]. In this paper the authors gave sufficient conditions for the approximate controllability.

In the present work we bring a new point of view to treat this problem. This allows us to recover the result of [KdT10] and also to provide a necessary and sufficient condition in the 1D case.

2. Since the data are more regular and the system is autonomous, the solution can be taken in the sense of semigroups : $z(t) = \mathcal{S}(t) z_0$, where $\mathcal{S}(t)$ is the semigroup generated on $L^2(\Omega)^n$ by the operator $\overrightarrow{\Delta} + A^*$ with domain $H^2(\Omega)^n \cap H_0^1(\Omega)^n$. Let us recall that, for $z_0 \in H_0^1(\Omega)^n$, we have $z \in C^0([0,T];H_0^1(\Omega)^n) \cap L^2(0,T;H^2(\Omega)^n \cap H_0^1(\Omega)^n)$.

The main tool to achieve our goals will be the use of a theorem of H.O. Fattorini. In fact, in 1966, H.O. Fattorini gave an interesting characterization of the approximate controllability under a general abstract framework. In his paper [Fat66] he proved that, under some reasonable assumptions, the only observation of the eigenfunctions completely characterizes the approximate controllability. Actually, this theorem has been proved for bounded observation operators but it can easily be generalized to the case of relatively bounded observation operators as follows :

Theorem 3.1. *Let H and U be some complex Hilbert spaces. Assume that $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset H \longrightarrow H$ generates a strongly continuous semigroup $\mathcal{S}(t)$ on H, has a compact resolvent, and the system of root vectors of its adjoint $\mathcal{A}^*$ is complete in H. Let $\mathcal{C} : \mathcal{D}(\mathcal{C}) \subset H \longrightarrow U$ be relatively bounded with respect to $\mathcal{A}$. Then, we have the property*

$$\forall z_0 \in \mathcal{D}(\mathcal{A}), \quad \Big(\mathcal{C}\mathcal{S}(t)z_0 = 0 \textit{ for a.e. } t \in (0,+\infty)\Big) \Longrightarrow z_0 = 0, \tag{3.4}$$

if and only if

$$\ker(s-\mathcal{A}) \cap \ker \mathcal{C} = \{0\}, \quad \forall s \in \mathbb{C}.$$

We give a proof of this theorem (which slightly changes from the one of [Fat66]) in the appendix 3.5.

Remark 3.2. *Note that the condition* $\ker(s-\mathcal{A}) \cap \ker \mathcal{C} = \{0\}$ *can also be formulated as*

$$\forall z_0 \in \ker(s-\mathcal{A}), \quad \Big(\mathcal{C}\mathcal{S}(t)z_0 = 0 \textit{ for a.e. } t \in (0,+\infty)\Big) \Longrightarrow z_0 = 0.$$

We use the first formulation because it is more eloquent, see Remark 3.4 below, but the most important is that Theorem 3.1 states that, in order to verify the property (3.4) it is enough to do so only on the eigenspaces of $\mathcal{A}$.

Remark 3.3. *We will see that the operators $\mathcal{A}$ we consider generate an analytic semigroup. For instance, for the first system we shall apply Theorem 3.1 to $\mathcal{A} = \overrightarrow{\Delta} + A^*$. It follows from this property that $z(\cdot) = \mathcal{S}(\cdot)z_0$ is analytic in time and has a regularizing effect ($\mathcal{S}(t)z_0 \in \mathcal{D}(\mathcal{A}^\infty)$ as soon as $t > 0$, even for $z_0 \in H$). This allows us to replace in (3.4) the interval $(0,+\infty)$ by any interval $(0,T)$, $T > 0$, and to take the data z_0 in any space that at least contains $\mathcal{D}(\mathcal{A}^\infty)$. This shows that (3.2) and (3.4) are equivalent properties. In partictular, we see that the approximate controllability of our systems is independent of the time of control T.*

Remark 3.4. *When $H = \mathbb{C}^n$ and $U = \mathbb{C}^m$, $\mathcal{A} = A^*$ and $\mathcal{C} = B^*$ (where A and B are still contant matrices) this theorem can be used to prove that the ordinary differential system*

$$\begin{cases} \dfrac{d}{dt}y &= Ay + Bg \quad \textit{in } (0,T). \\ y(0) &= y_0 \end{cases}$$

is controllable[3] *if and only if*

$$\ker(s - A^*) \cap \ker B^* = \{0\}\,, \quad \forall s \in \mathbb{C}.$$

This characterization is nowadays known as the Hautus test (despite it has been proved earlier by H.O. Fattorini). M.L.J Hautus gave a direct proof of the equivalence with another characterization, the well-known Kalman rank condition (see [Hau69, Theorem 1', §2])

$$\operatorname{rank}(B|AB|A^2B|\cdots|A^{n-1}B) = n.$$

Finally, let us mention the recent work [BT12] where the authors also extended the theorem of [Fat66] in view of the stabilizability of some other parabolic systems.

Notations We denote by $\{-\lambda_l\}_l$ the distinct Dirichlet eigenvalues of Δ on Ω. For each l, we denote by $\{\phi_{l,m}\}_m$ an orthonormal basis in $L^2(\Omega)$ of the eigenspace of Δ associated with the eigenvalue $-\lambda_l$, and by m_l the dimension of this eigenspace. It can be verified that all the following results are independent of the choice of the basis $\{\phi_{l,m}\}_m$.

In section 3.3, we use the notation $\mathcal{P}_{\lambda_l}$ for the orthogonal projection in $L^2(\Omega)$ on the eigenspace of Δ associated with $-\lambda_l$, that is $\mathcal{P}_{\lambda_l}u = \sum_{m=1}^{m_l} \langle u, \phi_{l,m}\rangle_{L^2(\Omega)}\phi_{l,m}$, for $u \in L^2(\Omega)$.

In sections 3.2.3, 3.2.4 and 3.3.2, we consider the 1D case. In particular $m_l = 1$ so that, for commodity, we simply use the notation ϕ_l instead of $\phi_{l,1}$.

In section 3.2.5, we use the notation $-\lambda_l^{X_1}$ (resp. $-\lambda_l^{X_2}$) to emphasize that this is the eigenvalues corresponding to the domain $\Omega = (0, X_1)$ (resp. $\Omega = (0, X_2)$), and we denote by $\phi_l^{X_1}$ (resp. $\phi_l^{X_2}$) a corresponding eigenfunction.

3.2 Results for the first system

We start by applying Theorem 3.1 to the operators

$$\mathcal{A} = \overrightarrow{\Delta} + A^*, \quad \mathcal{D}(\mathcal{A}) = H^2(\Omega)^n \cap H_0^1(\Omega)^n,$$

and

$$\mathcal{C} = B^* 1_\gamma \partial_n, \quad \mathcal{D}(\mathcal{C}) = H^2(\Omega)^n \cap H_0^1(\Omega)^n.$$

By a perturbation argument we can check that $\mathcal{A}$ generates an analytic semigroup on $L^2(\Omega)^n$, has a compact resolvent and the system of root vectors of $\mathcal{A}^*$ is complete in $L^2(\Omega)^n$ (using, for instance, the Keldysh's perturbation theorem, see [Mar88, Theorem 4.3, Chapter I, §4]), so that it satisfies the required hypothesis. On the other hand, the operator $\mathcal{C}$ is clearly relatively bounded with respect to $\mathcal{A}$.

Thus, system (3.1) is approximately controllable (at some time or at any time, see Remark 3.3) if and only if

$$\ker(s - (\overrightarrow{\Delta} + A^*)) \cap \ker(B^* 1_\gamma \partial_n) = \{0\}\,, \quad \forall s \in \mathbb{C}. \tag{3.5}$$

To describe the spectral elements of $\overrightarrow{\Delta} + A^*$ we introduce the following notations :

3. In finite dimension all the notions of controllability are equivalent.

Notations We denote by $\{\theta_i\}_i \subset \mathbb{C}$ the distinct eigenvalues of the matrix A^* and, for each i, by $\{w_{i,j}\}_j \subset \mathbb{C}^n$ a basis of $\ker(\theta_i - A^*)$.

In view of section 3.2.3, we also denote by m_i the dimension of $\ker(\theta_i - A^*)$ and we define

$$P_i = \left(\; w_{i,1} | \cdots | w_{i,m_i} \; \right).$$

One can check that all the following results do not depend on the choice of the basis $\{w_{i,j}\}_j$.

These notations in mind, it is not difficult to see that the spectrum of $\overrightarrow{\Delta} + A^*$ is

$$\sigma\left(\overrightarrow{\Delta} + A^*\right) = \{-\lambda_l + \theta_i\}_{l,i},$$

and its eigenspaces are

$$\ker(s - (\overrightarrow{\Delta} + A^*)) = \operatorname{span}\{w_{i,j}\phi_{l,m}\}_{\substack{i,j,l,m \\ -\lambda_l+\theta_i=s}}.$$

As we can see, the spectral structure of the operator $\overrightarrow{\Delta} + A^*$ is somehow separated into a scalar differential part and a vectorial algebraic part. Moreover, the operator $\mathcal{C}$ we consider is $\mathcal{C} = B^* 1_\gamma \partial_n$, and $1_\gamma \partial_n$ acts on the scalar differential part while B^* acts on the vectorial algebraic part (recall that B is a constant matrix). In this particular situation we have good hopes to obtain an easier characterization than condition (3.5). This is what establish the results in the following sections.

Remark 3.5. *We shall emphasize that the eigenvalues $-\lambda_l + \theta_i$ are not necessarily distinct. All along this work, for an eigenvalue $s \in \sigma\left(\overrightarrow{\Delta} + A^*\right)$, we will denote by $l_1^s, \ldots, l_{r_s}^s$ and $i_1^s, \ldots, i_{r_s}^s$ (with possibly $r_s = 1$) all the distinct indices such that*

$$s = -\lambda_{l_1^s} + \theta_{i_1^s} = \ldots = -\lambda_{l_{r_s}^s} + \theta_{i_{r_s}^s}.$$

Note that $r_s < +\infty$ since there is a finite number of θ_i.

As as result, any $u \in \ker(s - (\overrightarrow{\Delta} + A^*))$ has a writing of the form

$$u = \sum_{k=1}^{r_s} \sum_{j,m} \alpha_{k,j,m} w_{i_k^s,j} \phi_{l_k^s,m},$$

for some $\alpha_{k,j,m} \in \mathbb{C}$.

Since we will always reason at s fixed, we will omit the dependence with respect to s during the proofs (for the sake of clarity), though we will keep this notation in the statements of the results.

3.2.1 A sufficient condition

As noticed in Remark 3.5 it may happen that some eigenvalue s can be written as $s = -\lambda_l + \theta_i = -\lambda_{l'} + \theta_{i'}$ with $i' \neq i$ and $l' \neq l$. This phenomenon of "resonance" is a consequence of the coupling (the matrix A) and as a result is specific to the fact that we study a system, in contrast with a single equation. We will see that all the difficulties will precisely come from this point. This fact has been highlighted for the very first time in [FCGBdT10]. The following theorem shows that, when there is no phenomenon of resonance, the controllability is simply reduced to an algebraic condition, whatever the space dimension N and the control domain γ are.

Theorem 3.6. *Assume that for every eigenvalue $s \in \sigma\left(\overrightarrow{\Delta} + A^*\right)$ we have $r_s = 1$. Then, the ND system*

$$\begin{cases} \partial_t y &= \overrightarrow{\Delta} y + Ay \quad \text{in } (0,T) \times \Omega. \\ y &= 1_\gamma B g \quad \text{on } (0,T) \times \partial\Omega. \\ y(0) &= y_0 \quad \text{in } \Omega. \end{cases}$$

is approximately controllable if and only if

$$\ker(\theta_i - A^*) \cap \ker B^* = \{0\}, \quad \forall i. \tag{3.6}$$

In general, the assumption of this theorem is not a necessary condition, except in some very particular cases, see Theorem 3.16 in section 3.2.4.

Remark 3.7. *Condition (3.6) is nothing but the condition of Theorem 3.1 on the algebraic part of the system (see also Remark 3.4). We would also expect to require the similar condition concerning the scalar differential part, namely*

$$\ker(-\lambda_l - \Delta) \cap \ker(1_\gamma \partial_n) = \{0\}, \quad \forall l, \tag{3.7}$$

but actually this condition is always fulfilled, see [MMS68, Lemma], so that it is implicitly hidden in the theorem (and this will be used in the proof). This condition corresponds to the approximate controllability of the heat equation from the boundary.

Remark 3.8. *An easy but nontheless interesting consequence of Theorem 3.6 is when A^* has only one eigenvalue. In this case the assumption is naturally satisfied. This permits for* *instance to easily prove that a ND cascade system*

$$\begin{cases} \partial_t y &= \overrightarrow{\Delta} y + \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ a_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n-1} & 0 \end{pmatrix} y & \text{in } (0,T)\times\Omega. \\ y &= 1_\gamma \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} g & \text{on } (0,T)\times\partial\Omega. \\ y(0) &= y_0 & \text{in } \Omega. \end{cases}$$

is approximately controllable if and only if $a_{i,i-1} \neq 0$ *for every* $2 \leq i \leq n$.

Theorem 3.6 is a straightforward consequence of the following two lemma. The first lemma shows that condition (3.6) is always a necessary condition for the approximate controllability of system (3.1), while the second lemma shows that this condition is also enough to "control" the eigenvalues s for which $r_s = 1$. Since we assume that there are only such eigenvalues, Theorem 3.6 will be proved.

Lemma 3.9. *If system (3.1) is approximately controllable, then (3.6) holds.*

Lemma 3.10. *Assume that (3.6) holds. Then, for any eigenvalue* $s \in \sigma\left(\overrightarrow{\Delta} + A^*\right)$ *such that* $r_s = 1$*, we have*

$$\ker(s - (\overrightarrow{\Delta} + A^*)) \cap \ker(B^* 1_\gamma \partial_n) = \{0\}\,.$$

Proof of Lemma 3.9. Let $w \in \ker(\theta_i - A^*) \cap \ker B^*$. Let $\lambda \in \sigma(\Delta)$. Taking any nonzero $\phi \in \ker(\lambda - \Delta)$ we see that $u = \phi w$ belongs to $\ker(\lambda + \theta_i - (\overrightarrow{\Delta} + A^*)) \cap \ker(B^* 1_\gamma \partial_n)$, so that $u = 0$ by assumption, and thus also $w = 0$. □

Proof of Lemma 3.10. Let $s \in \sigma\left(\overrightarrow{\Delta} + A^*\right)$ with $r_s = 1$, $u \in \ker(s - (\overrightarrow{\Delta} + A^*)) \cap \ker(B^* 1_\gamma \partial_n)$. Since $r_s = 1$, u writes

$$u = \sum_{j,m} \alpha_{j,m} w_{i_1,j} \phi_{l_1,m}$$

for some $\alpha_{j,m} \in \mathbb{C}$. Let us set $\beta_j = 1_\gamma \partial_n \left(\sum_m \alpha_{j,m} \phi_{l_1,m}\right) \in L^2(\partial\Omega)$ so that we have

$$B^* \left(\sum_j \beta_j w_{i_1,j}\right) = 0.$$

Since $\sum_j \beta_j w_{i_1,j} \in \ker(\theta_{i_1} - A^*)$ we can use (3.6) to obtain $\sum_j \beta_j w_{i_1,j} = 0$. Using the linear independance of $\{w_{i_1,j}\}_j$ we deduce that $\beta_j = 0$ for every j, that is

$$1_\gamma \partial_n \left(\sum_m \alpha_{j,m} \phi_{l_1,m} \right) = 0, \quad \forall j.$$

Since $\sum_m \alpha_{j,m} \phi_{l_1,m} \in \ker(-\lambda_{l_1} - \Delta)$, using now (3.7) gives $\sum_m \alpha_{j,m} \phi_{l_1,m} = 0$. Thanks to the linear independance of $\{\phi_{l_1,m}\}_m$ we conclude that $\alpha_{j,m} = 0$ for every j, m, that is $u = 0$. □

3.2.2 As many controls as equations

As a second result we recover the known fact (see [FCGBdT10, Theorem 5.3]) that we can control the system from the boundary if we put as many controls as equations. In this particular case, the coupling becomes inconsequential (the matrix A can even be $A = 0$, that is no coupling at all). This situation can be understood as n uncoupled equations with one control for each. This result has been obtained in [FCGBdT10] by means of a Carleman estimate but we provide here an alternative proof, which is also simpler in our case.

Theorem 3.11. *The ND system*

$$\begin{cases} \partial_t y &= \overrightarrow{\Delta} y + Ay & \text{in } (0,T) \times \Omega. \\ y &= 1_\gamma B g & \text{on } (0,T) \times \partial\Omega. \\ y(0) &= y_0 & \text{in } \Omega. \end{cases}$$

is approximately controllable if we assume that

$$\ker B^* = \{0\}.$$

Proof. Let s be an eigenvalue of $\overrightarrow{\Delta} + A^*$ and $u \in \ker(s - (\overrightarrow{\Delta} + A^*)) \cap \ker(B^* 1_\gamma \partial_n)$. Then, u writes

$$u = \sum_{k=1}^{r} \sum_{j,m} \alpha_{k,j,m} w_{i_k,j} \phi_{l_k,m}$$

for some $\alpha_{k,j,m} \in \mathbb{C}$. Since $u \in \ker(B^* 1_\gamma \partial_n)$ and $\ker B^* = \{0\}$ by assumption, we have

$$\sum_{k,j} \left(\sum_m \alpha_{k,j,m} 1_\gamma \partial_n \phi_{l_k,m} \right) w_{i_k,j} = 0.$$

By the linear independence of $\{w_{i,j}\}_{i,j}$ we obtain

$$1_\gamma \partial_n \left(\sum_m \alpha_{k,j,m} \phi_{l_k,m} \right) = 0, \quad \forall k, \forall j.$$

Since $\sum_m \alpha_{k,j,m} \phi_{l_k,m} \in \ker(-\lambda_{l_k} - \Delta)$ we deduce that $\sum_m \alpha_{k,j,m} \phi_{l_k,m} = 0$ (using (3.7)), and by the linear independence of $\{\phi_{l,m}\}_m$ it follows that $\alpha_{k,j,m} = 0$ for every k, j, m, that is $u = 0$. □

3.2.3 The 1D case

The 1D case is a very particular situation because the boundary is reduced to two points, $\{0\}$ and $\{L\}$, if $\Omega = (0, L)$. In particular, only three possibilities arise for γ, namely $\gamma = \{0\}$, $\gamma = \{L\}$ or $\gamma = \{0\} \cup \{L\}$. We will study these three cases.

The results of this section have already been obtained in [AKBGBdT11a], with another formulation though, and a different proof.

We start with the case $\gamma = \{0\}$ (we refer to the beginning of section 3.2 for the notations) :

Theorem 3.12. *The 1D system*

$$\begin{cases} \partial_t y &=& \overrightarrow{\Delta} y + Ay & \text{in } (0,T)\times(0,L). \\ y &=& 1_{\{0\}} Bg & \text{on } (0,T)\times\{0,L\}. \\ y(0) &=& y_0 & \text{in } (0,L). \end{cases}$$

is approximately controllable if and only if, for every $s \in \sigma\left(\overrightarrow{\Delta} + A^*\right)$, we have

$$\operatorname{rank}\left(B^* P_{i_1^s} | \cdots | B^* P_{i_{r_s}^s}\right) = \sum_{k=1}^{r_s} m_{i_k}.$$

Proof. Let $u \in \ker(s - (\overrightarrow{\Delta} + A^*)) \cap \ker B^* 1_{\{0\}} \partial_n$, where $s \in \sigma\left(\overrightarrow{\Delta} + A^*\right)$. We know that u writes

$$u = \sum_{k=1}^{r} \sum_{j,m} \alpha_{k,j,m} w_{i_k,j} \phi_{l_k}$$

for some $\alpha_{k,j,m} \in \mathbb{C}$ and we have

$$\sum_{k=1}^{r} \sum_{j=1}^{m_{i_k}} \alpha_{k,j} B^* w_{i_k,j} \phi'_{l_k}(0) = 0.$$

This implies that $\alpha_{k,j} = 0$ for every k, j if and only if the matrix

$$\begin{pmatrix} \phi'_{l_1}(0) B^* P_{i_1} & \Big| & \cdots & \Big| & \phi'_{l_r}(0) B^* P_{i_r} \end{pmatrix}$$

has full rank, that is

$$\operatorname{rank}\begin{pmatrix} \phi'_{l_1}(0) B^* P_{i_1} & \Big| & \cdots & \Big| & \phi'_{l_r}(0) B^* P_{i_r} \end{pmatrix} = \sum_{k=1}^{r} m_{i_k}.$$

To conclude it remains to observe that

$$\operatorname{rank}\begin{pmatrix} \phi'_{l_1}(0) B^* P_{i_1} & \Big| & \cdots & \Big| & \phi'_{l_r}(0) B^* P_{i_r} \end{pmatrix} \\ = \operatorname{rank}\begin{pmatrix} B^* P_{i_1} & \Big| & \cdots & \Big| & B^* P_{i_r} \end{pmatrix}$$

since $\phi'_l(0) \neq 0$ for every l. □

The same result holds if we consider $\gamma = \{L\}$ instead of $\gamma = \{0\}$. When γ is the whole boundary, that is $\gamma = \{0\} \cup \{L\}$, we have the following characterization :

Theorem 3.13. *The 1D system*

$$\begin{cases} \partial_t y &= \overrightarrow{\Delta} y + Ay & \text{in } (0,T)\times(0,L). \\ y &= Bg & \text{on } (0,T)\times\{0,L\}. \\ y(0) &= y_0 & \text{in } (0,L). \end{cases}$$

is approximately controllable if and only if, for every $s \in \sigma\left(\overrightarrow{\Delta} + A^*\right)$, we have

$$\operatorname{rank}\left(\begin{array}{c|c|c} \phi'_{l_1^s}(0)B^*P_{i_1^s} & \cdots & \phi'_{l_{r_s}^s}(0)B^*P_{i_{r_s}^s} \\ \phi'_{l_1^s}(L)B^*P_{i_1^s} & & \phi'_{l_{r_s}^s}(L)B^*P_{i_{r_s}^s} \end{array}\right) = \sum_{k=1}^{r_s} m_{i_k}.$$

The proof is the same as the proof of Theorem 3.12.

Remark 3.14. *Further to these two theorems, we see that it may happen that system (3.1) is controllable with a control acting on both parts of the boundary whereas it is not controllable if the control only acts on one part. Indeed, let us consider on $\Omega = (0,\pi)$ the system described by*

$$A = \begin{pmatrix} 0 & -4 \\ 1 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We recall that the eigenvalues of Δ on $(0,\pi)$ are $-\lambda_l = -l^2$ and the corresponding eigenfunctions are $\phi_l(x) = \sqrt{\frac{2}{\pi}}\sin(lx)$. We can check that

$$\sigma(A^*) = \{\theta_1 = 1, \theta_2 = 4\},$$

$$\ker(\theta_1 - A^*) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}, \quad \ker(\theta_2 - A^*) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 4 \end{pmatrix}\right\}.$$

Since $r_s = 1$ for every eigenvalue $s \in \sigma\left(\overrightarrow{\Delta} + A^*\right)$ except $s = -\lambda_1 + \theta_1 = -\lambda_2 + \theta_2$, it is not difficult to check that the condition of Theorem 3.13 is fulfilled, whereas the one of Theorem 3.12 is not.

3.2.4 Only one control : $m = 1$

Another interesting situation is when we try to control system (3.1) with only one control. This corresponds to $m = 1$, so that the matrix B is in fact a (column) vector.

Remark 3.15. *It is not difficult to see that the condition* $\ker(\theta_i - A^*) \cap \ker B^* = \{0\}$ *is equivalent to (see the beginning of section 3.2 for the notations)*

$$\operatorname{rank} B^* P_i = m_i.$$

Thus, when B^* *has now only one line, we necessarily have*

$$m_i = 1.$$

In such a case, note also that P_i *is reduced to* $w_{i,1}$*, so that* $B^* P_i$ *is a scalar, and* $\operatorname{rank} B^* P_i = 1$ *then simply means that this scalar is not zero.*

Let us come back to the 1D case. We can always assume that $\ker(\theta_i - A^*) \cap \ker B^* = \{0\}$ for every i since it is a necessary condition (see Lemma 3.9). According to Remark 3.15, we then know that $m_i = 1$ and $B^* P_i$ is a nonzero scalar for every i. Thus, for every $s \in \sigma\left(\overrightarrow{\Delta} + A^*\right)$, we have

$$\begin{cases} \operatorname{rank}\left(B^* P_{i_1^s} | \cdots | B^* P_{i_{r_s}^s}\right) = \operatorname{rank}(1|\cdots|1), \\ \displaystyle\sum_{k=1}^{r_s} m_{i_k} = r_s. \end{cases}$$

As a result, in this particular case which is $m = 1$, Theorem 3.12 becomes

Theorem 3.16. *Assume that* $m = 1$*. The 1D system*

$$\begin{cases} \partial_t y &= \overrightarrow{\Delta} y + Ay & \text{in } (0,T) \times (0,L). \\ y &= 1_{\{0\}} B g & \text{on } (0,T) \times \{0,L\}. \\ y(0) &= y_0 & \text{in } (0,L). \end{cases}$$

is approximately controllable if and only if the following two conditions hold :

1. $\ker(\theta_i - A^*) \cap \ker B^* = \{0\}$ *for every* i.
2. *For every eigenvalue* $s \in \sigma\left(\overrightarrow{\Delta} + A^*\right)$ *we have* $r_s = 1$.

This result is historically the first relevant difference between distributed and boundary controllability for parabolic systems (these properties are equivalent for the heat equation for instance). This has been proved in [FCGBdT10]. Moreover, this also shows that if this system is controllable with a boundary control then it is also controllable with a distributed control (recall that the distributed controllability of this system is characterized by only the first condition, see [AKBDGB09b]). We insist on the fact that this is a result in 1D; except in the framework of [ABL12], the problem is open in higher space dimension.

We have a similar result for Theorem 3.13 when $m = 1$:

Theorem 3.17. *Assume that $m = 1$. The 1D system*

$$\begin{cases} \partial_t y &= \overrightarrow{\Delta} y + Ay & \text{in } (0,T) \times (0,L). \\ y &= Bg & \text{on } (0,T) \times \{0,L\}. \\ y(0) &= y_0 & \text{in } (0,L). \end{cases}$$

is approximately controllable if and only if the following two conditions hold :

1. $\ker(\theta_i - A^*) \cap \ker B^* = \{0\}$ *for every i.*
2. *For every eigenvalue $s \in \sigma\left(\overrightarrow{\Delta} + A^*\right)$, either $r_s = 1$, either $r_s = 2$ and in this latter case :*

$$\operatorname{rank} \begin{pmatrix} \phi'_{l_1^s}(0) & \phi'_{l_2^s}(0) \\ \phi'_{l_1^s}(L) & \phi'_{l_2^s}(L) \end{pmatrix} = 2.$$

Finally, let us give a result in any dimension when $m = 1$.

Theorem 3.18. *Assume that $m = 1$. The ND system*

$$\begin{cases} \partial_t y &= \overrightarrow{\Delta} y + Ay & \text{in } (0,T) \times \Omega. \\ y &= 1_\gamma Bg & \text{on } (0,T) \times \partial\Omega. \\ y(0) &= y_0 & \text{in } \Omega. \end{cases}$$

is approximately controllable if and only if the following two conditions hold :

1. $\ker(\theta_i - A^*) \cap \ker B^* = \{0\}$ *for every i.*
2. *For every $s \in \sigma\left(\overrightarrow{\Delta} + A^*\right)$, we have*

$$\left(\ker(-\lambda_{l_1^s} - \Delta) + \ldots + \ker(-\lambda_{l_{r_s}^s} - \Delta)\right) \cap \ker(1_\gamma \partial_n) = \{0\}.$$

Note that the second condition is relevant only for s with $r_s > 1$, see (3.7).

Proof of Theorem 3.18. Let $s \in \sigma\left(\overrightarrow{\Delta} + A^*\right)$ and let $u \in \ker(s - (\overrightarrow{\Delta} + A^*)) \cap \ker(B^* 1_\gamma \partial_n)$. We know that u writes

$$u = \sum_{k=1}^{r} \sum_{j,m} \alpha_{k,j,m} w_{i_k,j} \phi_{l_k,m}$$

for some, $\alpha_{k,j,m} \in \mathbb{C}$ and we have

$$1_\gamma \partial_n \left(\sum_k \sum_m \beta_{k,m} \phi_{l_k,m} \right) = 0,$$

where $\beta_{k,m} = \sum_j \alpha_{k,j,m} B^* w_{i_k,j}$.

Since B^* is a row vector, $\beta_{k,m}$ is a scalar, so that we can use the second condition and obtain that $\sum_m \beta_{k,m}\phi_{l_k,m} = 0$ for every k. By the linear indepedence of $\{\phi_{l,m}\}_m$ we obtain that $\beta_{k,m} = 0$ for every k, m, that is

$$B^*\left(\sum_j \alpha_{k,j,m} w_{i_k,j}\right) = 0, \quad \forall k, m.$$

Using now the first condition this gives $\sum_j \alpha_{k,j,m} w_{i_k,j} = 0$ and it follows that $\alpha_{k,j,m} = 0$ for every k, j, m, that is $u = 0$.

Let us now show that these conditions are also necessary. We only prove it for the second condition since it is already known for the first one, see Lemma 3.9.

Let $\phi = \phi_{l_1} + \ldots + \phi_{l_r}$, with $\phi_l \in \ker(-\lambda_l - \Delta)$, be such that $1_\gamma \partial_n \phi = 0$. For every k, let w_{i_k} be any eigenvector of A^* associated with θ_{i_k}. We know that $B^* w_i$ is a scalar and $B^* w_i \neq 0$ thanks to the first condition (we have just recalled that it is a necessary condition). Thus, we can define

$$u = \frac{1}{B^* w_{i_1}} w_{i_1}\phi_{l_1} + \ldots + \frac{1}{B^* w_{i_r}} w_{i_r}\phi_{l_r}.$$

We can see that $u \in \ker(s - (\overrightarrow{\Delta} + A^*)) \cap \ker(B^* 1_\gamma \partial_n)$ so that $u = 0$ by assumption. It follows from the linear independence of $\{w_i\}_i$ that $\phi_{l_k} = 0$ for every k, that is $\phi = 0$. $\square$

3.2.5 On a rectangular domain

In this section we still consider system (3.1) but the domain Ω is now a rectangle

$$\Omega = (0, X_1) \times (0, X_2).$$

We denote the faces of our rectangle by γ_L, γ_R, γ_T and γ_B, as on Figure 3.1 :

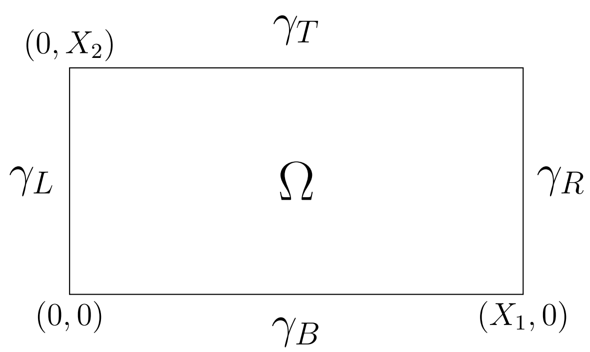


FIGURE 3.1 – Domain Ω for section 3.2.5.

The goal of this section is to prove several results about the boundary controllability of system (3.1), by discussing on the geometric position of γ.

Theorem 3.19. *If $\gamma = \gamma_L$, then system (3.1) is approximately controllable if and only if so is the following 1D system*

$$\begin{cases} \partial_t y &= \overrightarrow{\Delta}_{x_1} y + Ay \quad \text{in } (0,T)\times(0,X_1). \\ y &= 1_{\{0\}} Bg \quad \text{on } (0,T)\times\{0,X_1\}. \\ y(0) &= y_0 \quad \text{in } (0,X_1). \end{cases} \tag{3.8}$$

If $\gamma = \gamma_L \cup \gamma_R$ then system (3.1) is approximately controllable if and only if so is the following 1D system

$$\begin{cases} \partial_t y &= \overrightarrow{\Delta}_{x_1} y + Ay \quad \text{in } (0,T)\times(0,X_1). \\ y &= Bg \quad \text{on } (0,T)\times\{0,X_1\}. \\ y(0) &= y_0 \quad \text{in } (0,X_1). \end{cases}$$

We recall that the controllability of these 1D systems has been studied in sections 3.2.3 and 3.2.4.

For the heat equation, a similar result has been established in [Mil05] for the null-controllability when γ is one of the faces of $\partial\Omega$.

We consider next the case of two consecutive faces, with for instance $\gamma = \gamma_L \cup \gamma_T$.

Theorem 3.20. *If $\gamma = \gamma_L \cup \gamma_T$ and $\ker(\theta_i - A^*) \cap \ker B^* = \{0\}$ for every i, then system (3.1) is approximately controllable for $n = 2$.*

The geometry of γ (including two different directions γ_L and γ_T) is such that in some sense it "creates" an additional control. Thus, everything happens as if we had two controls for two equations and we can expect the controllability to hold, as it is showed in section 3.2.2. Theorem 3.19 shows that this is not true if we pick two parallel faces $\gamma = \gamma_L \cup \gamma_R$. When more equations are considered, the following counter-example strengthen this point of view.

Theorem 3.21. *Even if $\gamma = \gamma_L \cup \gamma_T$ and $\ker(\theta_i - A^*) \cap \ker B^* = \{0\}$ for every i, system (3.1) may be not approximately controllable when $n \geq 4$.*

Remark 3.22. *It is worth mentioning that, in all the previous statements, we can replace γ_L (resp. γ_R, γ_T, γ_B) by a nonempty open part of it. This is easily seen in the following proofs by using the analyticity of the 1D eigenfunctions of Δ.*

The main ingredient that will make the proofs work is the following. The (not necessarily distinct) eigenvalues of $\Delta = \Delta_{x_1} + \Delta_{x_2}$ on $(0,X_1)\times(0,X_2)$ are

$$-\Lambda_{p,q} = -\lambda_p^{X_1} - \lambda_q^{X_2},$$

and the corresponding eigenfunctions are

$$\Phi_{p,q}(x,y) = \phi_p^{X_1}(x)\phi_q^{X_2}(y), \quad x \in (0,X_1), y \in (0,X_2).$$

In this case the dimension m_l of the eigenspace of Δ associated with $-\lambda_l$ is exactly the number of distinct couples of indices (p,q) such that $\Lambda_{p,q} = \lambda_l$. We denote by $(p_l^1, q_l^1), \ldots, (p_l^{m_l}, q_l^{m_l})$ all such indices. Note that for every m we necessarily have

$$p_l^m \neq p_l^{m'} \quad \text{and} \quad q_l^m \neq q_l^{m'}, \quad \forall m' \neq m. \tag{3.9}$$

Indeed, it follows from the definition of these indices and the form of $\Lambda_{p,q}$ that, if $p_l^m = p_l^{m'}$, then we also have $q_l^m = q_l^{m'}$, which is excluded by definition.

Proof of Theorem 3.19. Let us consider the case $\gamma = \gamma_L$; the proof for $\gamma = \gamma_L \cup \gamma_R$ relies on the same kind of arguments. We also only prove that, if system (3.8) is approximately controllable, then so is system (3.1), the converse being easier.

Let $s \in \sigma\left(\overrightarrow{\Delta} + A^*\right)$ and $u \in \ker(s - (\overrightarrow{\Delta} + A^*)) \cap \ker(B^* 1_{\gamma_L} \partial_n)$. With the notations previously introduced u then writes

$$u = \sum_{k=1}^{r} \sum_{m=1}^{m_{l_k}} \left(\sum_j \alpha_{k,j,m} w_{i_k,j} \right) \phi_{p_{l_k}^m}^{X_1} \phi_{q_{l_k}^m}^{X_2},$$

for some $\alpha_{k,j,m} \in \mathbb{C}$, and we have

$$\sum_{k=1}^{r} \sum_{m=1}^{m_{l_k}} \beta_{k,m} \phi_{q_{l_k}^m}^{X_2}(y) = 0, \quad \forall y \in (0, X_2), \tag{3.10}$$

where we have set

$$\beta_{k,m} = -\gamma_{k,m} \left(\phi_{p_{l_k}^m}^{X_1} \right)'(0), \quad \gamma_{k,m} = B^* \left(\sum_j \alpha_{k,j,m} w_{i_k,j} \right).$$

Note that we can always assume that $\ker(\theta_i - A^*) \cap \ker B^* = \{0\}$ for every i since it is a necessary condition (see Lemma 3.9) for both systems (3.1) and (3.8). As a result, to prove that $u = 0$ it is equivalent to show that $\gamma_{k,m} = 0$ (or $\beta_{k,m} = 0$) for every k, m.

For convenience we assume that $r = 2$. Thus (3.10) becomes

$$\sum_{m=1}^{m_{l_1}} \beta_{1,m} \phi_{q_{l_1}^m}^{X_2}(y) + \sum_{m=1}^{m_{l_2}} \beta_{2,m} \phi_{q_{l_2}^m}^{X_2}(y) = 0, \quad \forall y \in (0, X_2).$$

Using the linear independence of $\left\{\phi_q^{X_2}\right\}_q$ in $L^2(0, X_2)$, two cases may happen. For some given m, if $q_{l_1}^m \neq q_{l_2}^{m'}$ for every m' then, taking also (3.9) into account, we obtain $\beta_{1,m} = 0$. On the other hand, if there exists m' such that $q_{l_1}^m = q_{l_2}^{m'}$, then this m' is unique thanks to (3.9) and we obtain that $\beta_{1,m} + \beta_{2,m'} = 0$, that is

$$-\left(\gamma_{1,m} \phi_{p_{l_1}^m}^{X_1} + \gamma_{2,m'} \phi_{p_{l_2}^{m'}}^{X_1} \right)'(0) = 0.$$

Since $q_{l_1}^m = q_{l_2}^{m'}$ we have
$$-\lambda^{X_1}_{p^m_{l_1}} + \theta_{i_1} = -\lambda^{X_1}_{p^{m'}_{l_2}} + \theta_{i_2},$$
and the assumption that system (3.8) is approximately controllable permits to conclude that $\gamma_{1,m} = \gamma_{2,m'} = 0$.

Thus, in both situations $\gamma_{1,m} = 0$, and it follows that $\gamma_{2,m} = 0$ (when $r > 2$ we reason by induction). □

Proof of Theorem 3.20. Since we assume that $n = 2$, the matrix A^* has at most two distinct eigenvalues. If A^* has only one eigenvalue then we already know that the system is approximately controllable, see Remark 3.8 in section 3.2.1. Let us then assume that A^* has two distinct eigenvalues
$$\theta_{i_1} \neq \theta_{i_2}. \tag{3.11}$$

With the same notations as in the proof of Theorem 3.19, let us show that is it not possible to have
$$\begin{cases} \displaystyle\sum_{m=1}^{m_{l_1}} \beta_{1,m} \phi^{X_2}_{q^m_{l_1}} + \sum_{m=1}^{m_{l_2}} \beta_{2,m} \phi^{X_2}_{q^m_{l_2}} = 0, \\ \displaystyle\sum_{m=1}^{m_{l_1}} \delta_{1,m} \phi^{X_1}_{p^m_{l_1}} + \sum_{m=1}^{m_{l_2}} \delta_{2,m} \phi^{X_1}_{p^m_{l_2}} = 0, \end{cases}$$
with $\gamma_{k,m} \neq 0$ for every k, m, where
$$\delta_{k,m} = \gamma_{k,m} \left(\phi^{X_2}_{q^m_{l_k}}\right)' (X_2).$$

From the first equation we see that the sets $\left\{q^m_{l_1}\right\}_{1\le m\le m_{l_1}}$ and $\left\{q^{m'}_{l_2}\right\}_{1\le m'\le m_{l_2}}$ are in bijection. Indeed, if there exists m such that $q^m_{l_1} \neq q^{m'}_{l_2}$ for every m' then, using the linear independence of $\left\{\phi^{X_2}_q\right\}_q$ in $L^2(0, X_2)$, we obtain $\beta_{1,m} = \gamma_{1,m} = 0$. Since the same fact holds for the second equation, the sets $\left\{p^m_{l_1}\right\}_{1\le m\le m_{l_1}}$ and $\left\{p^{m'}_{l_2}\right\}_{1\le m'\le m_{l_2}}$ are also in bijection.

As a consequence, denoting $M = m_{l_1} = m_{l_2}$, we have
$$\sum_{m=1}^{M} \lambda^{X_2}_{q^m_{l_1}} = \sum_{m'=1}^{M} \lambda^{X_2}_{q^{m'}_{l_2}}, \quad \sum_{m=1}^{M} \lambda^{X_1}_{p^m_{l_1}} = \sum_{m'=1}^{M} \lambda^{X_1}_{p^{m'}_{l_2}},$$
so that
$$\sum_{m=1}^{M} \Lambda_{p^m_{l_1}, q^m_{l_1}} = \sum_{m'=1}^{M} \Lambda_{q^{m'}_{l_2}, q^{m'}_{l_2}}.$$

Let us denote by S this common value. Since $-\Lambda_{p^m_{l_1}, q^m_{l_1}} + \theta_{i_1} = -\Lambda_{q^m_{l_2}, q^m_{l_2}} + \theta_{i_2}$ for every m, if we sum we obtain
$$-S + M\theta_{i_1} = -S + M\theta_{i_2},$$
and thus
$$\theta_{i_1} = \theta_{i_2},$$
a contradiction with our assumption (3.11). □

Proof of Theorem 3.21. We provide an example of system with 4 equations for which the condition $\ker(\theta_i - A^*) \cap \ker B^* = \{0\}$ holds for every i and that is not approximately controllable on $\gamma = \gamma_L \cup \gamma_T$. This example can easily be generalized to the case $n > 4$.

We take $X_1 = X_2 = \pi$, so that the eigenvalues of Δ are simply

$$-\Lambda_{p,q} = -p^2 - q^2,$$

and we choose

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 120 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -10 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We can check that

$$\sigma(A^*) = \{\theta_1 = -8, \theta_2 = -5, \theta_3 = 3, \theta_4 = 0\}.$$
$$\ker(\theta_i - A^*) \cap \ker B^* = \{0\}, \quad i = 1, 2, 3, 4.$$

Now, observe that we have the relation

$$-\Lambda_{1,1} + \theta_1 = -\Lambda_{2,1} + \theta_2 = -\Lambda_{2,3} + \theta_3 = -\Lambda_{1,3} + \theta_4 = -10.$$

In view of this relation we define

$$u = \phi_{1,1} - \frac{1}{2}\phi_{2,1} + \frac{1}{6}\phi_{2,3} - \frac{1}{3}\phi_{1,3}.$$

Clearly $u \neq 0$. Let us show that however $\partial_n u = 0$ on $\gamma_L \cup \gamma_T$. This will contradict Theorem 3.18.

Taking into account that $\left(\phi_p^{X_1}\right)'(0) = p\sqrt{\frac{2}{\pi}}$, for $x_2 \in (0, \pi)$ we have

$$\begin{aligned} -\partial_{x_1} u(0, x_2) = &- \underbrace{\left(\left(\phi_1^{X_1}\right)'(0) - \frac{1}{2}\left(\phi_2^{X_1}\right)'(0)\right)}_{=0} \phi_1^{X_2}(x_2) \\ &- \underbrace{\left(\frac{1}{6}\left(\phi_2^{X_1}\right)'(0) - \frac{1}{3}\left(\phi_1^{X_1}\right)'(0)\right)}_{=0} \phi_3^{X_2}(x_2), \end{aligned}$$

so that indeed $\partial_n u = 0$ on γ_L.

In the same way, for $x_1 \in (0, \pi)$ we have

$$\begin{aligned} \partial_{x_2} u(x_1, \pi) = &\ \phi_1^{X_1}(x_1) \underbrace{\left(\left(\phi_1^{X_2}\right)'(\pi) - \frac{1}{3}\left(\phi_3^{X_2}\right)'(\pi)\right)}_{=0} \\ &+ \phi_2^{X_1}(x_1) \underbrace{\left(-\frac{1}{2}\left(\phi_1^{X_2}\right)'(\pi) + \frac{1}{6}\left(\phi_3^{X_2}\right)'(\pi)\right)}_{=0}, \end{aligned}$$

and thus $\partial_n u = 0$ also on γ_T. $\square$

3.3 Results for the second system

We now turn out to the results concerning the second system

$$\begin{cases} \partial_t y_1 = \Delta y_1 & \text{in } (0,T)\times\Omega. \\ \partial_t y_2 = \Delta y_2 + G(x)\cdot\nabla y_1 + a(x)y_1 & \text{in } (0,T)\times\Omega. \\ y_1 = 1_\gamma g, \quad y_2 = 0 & \text{on } (0,T)\times\partial\Omega. \\ y_1(0) = y_{1,0}, \quad y_2(0) = y_{2,0} & \text{in } \Omega. \end{cases}$$

As mentionned in the introduction, it is known that this system is approximately controllable at time T if and only if its adjoint system

$$\begin{cases} \partial_t z_1 = \Delta z_1 - G(x)\cdot\nabla z_2 + (a(x) - \operatorname{div} G(x))z_2 & \text{in } (0,T)\times\Omega. \\ \partial_t z_2 = \Delta z_2 & \text{in } (0,T)\times\Omega. \\ z_1 = 0, \quad z_2 = 0 & \text{on } (0,T)\times\partial\Omega. \\ z_1(0) = z_{1,0}, \quad z_2(0) = z_{2,0} & \text{in } \Omega. \end{cases}$$

has the unique continuation property

$$\forall z_{1,0}, z_{2,0} \in H_0^1(\Omega), \quad \Big(1_\gamma \partial_n z_1(t) = 0 \text{ for a.e. } t\in(0,T)\Big) \Longrightarrow z_{1,0} = z_{2,0} = 0.$$

For commodity, let us denote

$$\mathcal{Q} = -G(x)\cdot\nabla + (a(x) - \operatorname{div} G(x)).$$

Thus, we want to apply Theorem 3.1 to the operators

$$\mathcal{A} = \begin{pmatrix} \Delta & \mathcal{Q} \\ 0 & \Delta \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}) = H^2(\Omega)^2 \cap H_0^1(\Omega)^2,$$

and

$$\mathcal{C} = \begin{pmatrix} 1_\gamma\partial_n & 0 \end{pmatrix}, \quad \mathcal{D}(\mathcal{C}) = H^2(\Omega)^2 \cap H_0^1(\Omega)^2.$$

Again, by using a perturbation argument we can check that $\mathcal{A}$ generates an analytic semi-group and indeed satisfies the hypothesis of Theorem 3.1. The operator $\mathcal{C}$ is the same as for the first system we studied, taking $n = 2$ and $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

As a consequence, this system is approximately controllable if and only if

$$\ker \begin{pmatrix} s-\Delta & -\mathcal{Q} \\ 0 & s-\Delta \end{pmatrix} \cap \ker\begin{pmatrix} 1_\gamma\partial_n & 0 \end{pmatrix} = \{0\}, \quad \forall s\in\mathbb{C}. \tag{3.12}$$

It is not difficult to see that the spectrum of $\mathcal{A}$ is

$$\sigma(\mathcal{A}) = \{-\lambda_l\}_l ,$$

(we refer to the introduction for the notations) and that its eigenspaces can be decomposed as follows :

$$\ker(-\lambda_l - \mathcal{A}) = U_l \oplus V_l,$$

with

$$U_l = \left\{ \begin{pmatrix} u \\ 0 \end{pmatrix} \right\}_{u \in \ker(-\lambda_l - \Delta)}, \quad V_l = \left\{ \begin{pmatrix} \mathcal{S}_l \mathcal{Q} v \\ v \end{pmatrix} \right\}_{v \in \ker(-\lambda_l - \Delta) \cap \ker \mathcal{P}_{\lambda_l} \mathcal{Q}},$$

where $\mathcal{S}_l : f \in \ker \mathcal{P}_{\lambda_l} \longmapsto u \in \ker \mathcal{P}_{\lambda_l}$ with u the unique solution (in $\ker \mathcal{P}_{\lambda_l}$) of the equation $(-\lambda_l - \Delta)u = f$.

3.3.1 A sufficient condition

The following theorem is, in some sense, the analogue of Theorem 3.6 in section 3.2.1. This also recovers [KdT10, Theorem 1.5].

Theorem 3.23. *Assume that*

$$\ker(-\lambda_l - \Delta) \cap \ker \mathcal{P}_{\lambda_l} \mathcal{Q} = \{0\} , \quad \forall l. \tag{3.13}$$

Then, the ND system

$$\begin{cases} \partial_t y_1 = \Delta y_1 & \textit{in } (0,T) \times \Omega. \\ \partial_t y_2 = \Delta y_2 + G(x) \cdot \nabla y_1 + a(x) y_1 & \textit{in } (0,T) \times \Omega. \\ y_1 = 1_\gamma g, \quad y_2 = 0 & \textit{on } (0,T) \times \partial\Omega. \\ y_1(0) = y_{1,0}, \quad y_2(0) = y_{2,0} & \textit{in } \Omega. \end{cases}$$

is approximately controllable.

Remark 3.24. *Condition (3.13) can be reformulated into the following rank condition :*

$$\operatorname{rank} \begin{pmatrix} \langle \mathcal{Q}\phi_{l,1}, \phi_{l,1} \rangle_{L^2(\Omega)} & \cdots & \langle \mathcal{Q}\phi_{l,1}, \phi_{l,m_l} \rangle_{L^2(\Omega)} \\ \vdots & & \vdots \\ \langle \mathcal{Q}\phi_{l,m_l}, \phi_{l,1} \rangle_{L^2(\Omega)} & \cdots & \langle \mathcal{Q}\phi_{l,m_l}, \phi_{l,m_l} \rangle_{L^2(\Omega)} \end{pmatrix} = m_l, \quad \forall l. \tag{3.14}$$

Proof of Theorem 3.23. The assumption (3.13) means that $V_l = \{0\}$ for every l, so that $\ker(-\lambda_l - \mathcal{A}) = U_l$ for every l. Thus, any $w \in \ker(-\lambda_l - \mathcal{A}) \cap \ker \mathcal{C}$ writes $w = \begin{pmatrix} u \\ 0 \end{pmatrix}$ for some $u \in \ker(-\lambda_l - \Delta)$ and satisfies

$$Cw = 1_\gamma \partial_n u = 0.$$

This implies $u = 0$ (see (3.7)) and thus also $w = 0$. □

3.3.2 The 1D case

As for Theorem 3.16 in section 3.2.4, condition (3.13) turns out to be also necessary in 1D :

Theorem 3.25. *The 1D system*

$$\begin{cases} \partial_t y_1 = \Delta y_1 & \textit{in } (0,T)\times(0,L). \\ \partial_t y_2 = \Delta y_2 + G(x)\cdot\nabla y_1 + a(x)y_1 & \textit{in } (0,T)\times(0,L). \\ y_1 = 1_{\{0\}}g, \quad y_2 = 0 & \textit{on } (0,T)\times\{0,L\}. \\ y_1(0) = y_{1,0}, \quad y_2(0) = y_{2,0} & \textit{in } (0,L). \end{cases} \tag{3.15}$$

is approximately controllable if and only if

$$\int_0^L \left(-\frac{1}{2}G'(x) + a(x)\right)|\phi_l(x)|^2\,dx \neq 0, \quad \forall l. \tag{3.16}$$

Remark 3.26. *This theorem provides an easy condition to check whether this system is approximately controllable or not. For instance, for $a = 0$ and G constant, the corresponding system is not approximately controllable.*

Proof of Theorem 3.25. Since $m_l = 1$ for every l $(N = 1)$, condition (3.14) now reads as

$$\langle Q\phi_l, \phi_l\rangle_{L^2(\Omega)} \neq 0, \quad \forall l,$$

which gives the same condition than (3.16) after an integration by part on the gradient term.

From Theorem 3.23 we already know that this condition is sufficient. Let us prove that it is also necessary in our case.

Assume that this condition does not hold or, equivalently, that (3.13) does not hold. Then, for some l, there exists at least one eigenfunction of $\mathcal{A}$ associated with $-\lambda_l$ in U_l, say $\begin{pmatrix} u \\ 0 \end{pmatrix}$, and another one in V_l, say $\begin{pmatrix} \mathcal{S}_l\mathcal{Q}v \\ v \end{pmatrix}$. If $(\mathcal{S}_l\mathcal{Q}v)'(0) = 0$ then condition (3.12) already fails. On the other hand, if $(\mathcal{S}_l\mathcal{Q}v)'(0) \neq 0$, then condition (3.12) also fails because of the following relation

$$\left(\frac{1}{u'(0)}u - \frac{1}{(\mathcal{S}_l\mathcal{Q}v)'(0)}\mathcal{S}_l\mathcal{Q}v\right)'(0) = 0.$$

$\square$

3.4 Further results : distributed controllability

All along this work we were interested in the boundary controllability problem but let us mention that the method also works for distributed controllability. For instance, we can recover the result of [AKBDGB09b] concerning system (3.1). We can also obtain the following result :

Theorem 3.27. *Let ω be a nonempty open subset of Ω. Assume that Ω is connected and G and a are real analytic functions in Ω. Then, the ND system*

$$\begin{cases} \partial_t y_1 = \Delta y_1 + 1_\omega g & \text{in } (0,T)\times\Omega. \\ \partial_t y_2 = \Delta y_2 + G(x)\cdot\nabla y_1 + a(x) y_1 & \text{in } (0,T)\times\Omega. \\ y_1 = 0, \quad y_2 = 0 & \text{on } (0,T)\times\partial\Omega. \\ y_1(0) = y_{1,0}, \quad y_2(0) = y_{2,0} & \text{in } \Omega. \end{cases}$$

is approximately controllable if and only if

$$\ker(-\lambda_l - \Delta) \cap \ker \mathcal{Q} = \{0\}, \quad \forall l, \tag{3.17}$$

where we recall that $\mathcal{Q} = -G(x)\cdot\nabla + (a(x) - \operatorname{div} G(x))$.

To the author knowledge, [BCGdT13], [Gue07] and [KdT10] are the only works for the distributed null and approximate controllability of this system. However in these papers, even for the case $a = 0$ and G constant, a geometric assumption or a particular form of G is required (see [BCGdT13, Theorem 2.1] and [Gue07, Theorem 4]), while [KdT10, Theorem 1.5] can not be applied.

Proof of Theorem 3.27. This time the observation operator is $\mathcal{C} = \begin{pmatrix} 1_\omega & 0 \end{pmatrix}$ (it is a bounded operator on $L^2(\Omega)^2$). Let $w \in \ker(-\lambda_l - \mathcal{A}) \cap \ker \mathcal{C}$. Thus, w writes $w = \begin{pmatrix} u \\ 0 \end{pmatrix} + \begin{pmatrix} \mathcal{S}_l \mathcal{Q} v \\ v \end{pmatrix}$ for some $u \in \ker(-\lambda_l - \Delta)$ and $v \in \ker(-\lambda_l - \Delta) \cap \ker \mathcal{P}_{\lambda_l}\mathcal{Q}$, and satisfies

$$1_\omega (u + \mathcal{S}_l \mathcal{Q} v) = 0. \tag{3.18}$$

Since v is an analytic function, so is $\mathcal{Q}v$. Thus, $\mathcal{S}_l\mathcal{Q}v$ is an analytic function as solution of an elliptic partial differential equation with analytic data (see for instance [Hör64, Theorem 7.5.1]). Note that u is also analytic. Thus, (3.18) is equivalent to

$$u + \mathcal{S}_l \mathcal{Q} v = 0.$$

This implies that $u = 0$ since $u = -\mathcal{S}_l\mathcal{Q}v \in \ker \mathcal{P}_{\lambda_l}$ and $u \in \ker(-\lambda_l - \Delta) = \operatorname{Im} \mathcal{P}_{\lambda_l}$. Thus, $\mathcal{S}_l\mathcal{Q}v = -u = 0$ and it follows that $\mathcal{Q}v = 0$ (see the definition of $\mathcal{S}_l$). This implies $v = 0$ if and only if the hypothesis holds. $\square$

Let us illustrate this result with $a = 0$ and $G \neq 0$ constant. This means that we take $\mathcal{Q} = -G \cdot \nabla$. We can verify that this $\mathcal{Q}$ satisfies (3.17) since

$$\forall u \in H_0^1(\Omega), \quad G \cdot \nabla u = 0 \text{ in } \Omega \Longrightarrow u = 0.$$

Indeed, set $v(x) = e^{G \cdot x} u(x)$. We have $v \in H_0^1(\Omega)$ and (using the hypothesis on u)

$$G \cdot \nabla v = |G|^2 v.$$

Multiplying this equality by v and integrating by parts we obtain

$$|G|^2 \int_\Omega |v(x)|^2 \, dx = 0.$$

This implies that $v = 0$ and thus also $u = 0$.

It follows from Theorem 3.27 that the ND system

$$\begin{cases} \partial_t y_1 = \Delta y_1 + 1_\omega g & \text{in } (0,T) \times \Omega. \\ \partial_t y_2 = \Delta y_2 + G \cdot \nabla y_1 & \text{in } (0,T) \times \Omega. \\ y_1 = 0, \quad y_2 = 0 & \text{on } (0,T) \times \partial\Omega. \\ y_1(0) = y_{1,0}, \quad y_2(0) = y_{2,0} & \text{in } \Omega. \end{cases}$$

is approximately controllable.

Remark 3.28. *This establishes another difference between distributed and boundary controllability for parabolic systems. Indeed, let us recall that we have seen that this same system in 1D with a boundary control is not approximately controllable, see Remark 3.26.*

3.5 Proof of Theorem 3.1

For the sake of completeness we give here the proof of Theorem 3.1. We recall that this proof is just adapted from the one in [Fat66] in order to deal with relatively bounded observation operators.

Let us recall the notations and assumptions. H and U are complex Hilbert spaces, $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset H \longrightarrow H$ generates a strongly continuous semigroup on H, has a compact resolvent, the system of root vectors of $\mathcal{A}^*$ is complete in H, and $\mathcal{C} : \mathcal{D}(\mathcal{C}) \subset H \longrightarrow U$ is relatively bounded with respect to $\mathcal{A}$.

We denote by $\rho(\mathcal{A})$ the resolvent set of our closed linear operator $\mathcal{A}$ and, for $\lambda \in \rho(\mathcal{A})$, $\mathcal{R}(\lambda; \mathcal{A}) = (\lambda - \mathcal{A})^{-1}$ the resolvent operator.

Since $\mathcal{A}$ has a compact resolvent, its spectrum $\sigma(\mathcal{A}) = \mathbb{C} \backslash \rho(\mathcal{A})$ consists in a sequence of isolated points, say $\{\mu_j\}_j$. In particular $\rho(\mathcal{A})$ is path connected. We have $\sigma(\mathcal{A}^*) = \overline{\sigma(\mathcal{A})} = \{\overline{\mu_j}\}_j$.

Let now $C_j \subset \rho(\mathcal{A})$ be a positive-oriented small circle enclosing μ_j and such that no other eigenvalue than μ_j lies inside this circle. For every j we define the spectral projection

$$\begin{array}{rccl} \mathcal{P}_{\mu_j}: & H & \longrightarrow & H \\ & u & \longmapsto & \displaystyle\frac{1}{2\pi i}\int_{C_j} \mathcal{R}(\xi;\mathcal{A})\, u\, d\xi. \end{array}$$

The operator $\mathcal{P}_{\mu_j}$ is a bounded linear operator and one can prove that the range of this operator is exactly the root subspace of $\mathcal{A}$ associated with μ_j, i.e. $\ker(\mu_j - \mathcal{A})^{\tau_j}$, where τ_j is the smallest indice k such that $\ker(\mu_j - \mathcal{A})^{k+1} = \ker(\mu_j - \mathcal{A})^k$. For a proof of this fact we refer to [DS71, 2 Lemma, Chapter XIX]. A computation shows that

$$\mathcal{P}^*_{\mu_j} = \frac{1}{2\pi i}\int_{\overline{C_j}} \mathcal{R}(\xi;\mathcal{A}^*)\; d\xi,$$

where $\overline{C_j}$ is the circle centered in $\overline{\mu_j}$ with the same radius as C_j. Since there are no eigenvalue of $\mathcal{A}^*$ except $\overline{\mu_j}$ inside the circle $\overline{C_j}$, the range of this operator is exactly the root subspace of $\mathcal{A}^*$ associated with $\overline{\mu_j}$.

Let us now recall some properties of semigroups. Since $\mathcal{A}$ generates a strongly continuous semigroup $\mathcal{S}(t)$ on H, we know that there exists $M > 0$ and $\omega_0 \in \mathbb{R}$ such that

$$\|\mathcal{S}(t)\|_{\mathcal{L}(H)} \leq M e^{\omega_0 t}, \quad \forall t \geq 0.$$

Moreover, for every $z_0 \in \mathcal{D}(\mathcal{A})$, we have $\mathcal{S}(t)z_0 \in \mathcal{D}(\mathcal{A})$ with $\mathcal{A}\mathcal{S}(t)z_0 = \mathcal{S}(t)\mathcal{A}z_0$ and the map $t \in [0,+\infty) \longmapsto \mathcal{S}(t)z_0 \in \mathcal{D}(\mathcal{A})$ is continuous. Finally, the resolvent set $\rho(A)$ contains the halfplane $\{\lambda \in \mathbb{C} \,|\, \mathfrak{Re}\,\lambda > \omega_0\}$. For a proof of these facts we refer to [EN00, Proposition 5.5, Chapter I], [EN00, Lemma 1.3, Chapter II] and [EN00, Theorem 1.10, Chapter II], respectively.

Lemma 3.29 (Corollary 2.2 of [Fat66]). *Let $z_0 \in \mathcal{D}(\mathcal{A})$ be fixed. The three following properties are equivalent :*

1. $\mathcal{C}\mathcal{S}(t)z_0 = 0$ *for a.e.* $t \in (0,+\infty)$.
2. $\mathcal{C}\mathcal{R}(\lambda;\mathcal{A})\, z_0 = 0$ *for every* $\lambda \in \mathbb{C}$ *with* $\mathfrak{Re}\,\lambda > \omega_0$.
3. $\mathcal{C}\mathcal{R}(\lambda;\mathcal{A})\, z_0 = 0$ *for every* $\lambda \in \rho(\mathcal{A})$.

Proof. Recall that the resolvent of an operator can be represented as the Laplace transform of the semigroup it generates for $\mathfrak{Re}\,\lambda > \omega_0$ (see for instance [EN00, Theorem 1.10, Chapter II]) :

$$\begin{aligned} \mathcal{R}(\lambda;\mathcal{A})\, z_0 &= \int_0^{+\infty} e^{-\lambda t}\mathcal{S}(t)z_0\, dt \\ &= \lim_{j\to+\infty}\int_0^{j} e^{-\lambda t}\mathcal{S}(t)z_0\, dt \quad \text{(limit in } H). \end{aligned}$$

Actually, this limit can also be considered in the sense of $\mathcal{D}(\mathcal{A})$. Indeed,

$$\int_0^{j} e^{-\lambda t}\mathcal{S}(t)z_0\, dt \in \mathcal{D}(\mathcal{A}),$$

with

$$\mathcal{A}\int_0^j e^{-\lambda t}\mathcal{S}(t)z_0\,dt = \int_0^j e^{-\lambda t}\mathcal{A}\mathcal{S}(t)z_0\,dt = \int_0^j e^{-\lambda t}\mathcal{S}(t)\mathcal{A}z_0\,dt,$$

$$\xrightarrow[j\to+\infty]{} \mathcal{R}\left(\lambda;\mathcal{A}\right)\mathcal{A}z_0.$$

Since $\mathcal{C}$ is bounded on $\mathcal{D}\left(\mathcal{A}\right)$ we obtain

$$\mathcal{C}\mathcal{R}\left(\lambda;\mathcal{A}\right)z_0 = \int_0^{+\infty} e^{-\lambda t}\mathcal{C}\mathcal{S}(t)z_0\,dt, \quad \mathfrak{Re}\,\lambda > \omega_0.$$

It is now clear that 1. implies 2. while the converse follows from the injectivity of the Laplace transform.

The remaining equivalence is a consequence of the analytic continuation of the resolvent. □

Lemma 3.30 (Proposition 3.1 of [Fat66])**.** *Let $z_0 \in \mathcal{D}\left(\mathcal{A}\right)$ be fixed. If the third point of Lemma 3.29 holds, then $\mathcal{C}\mathcal{R}\left(\lambda;\mathcal{A}\right)\mathcal{P}_{\mu_j}z_0 = 0$ for every $\lambda \in \rho(\mathcal{A})$ and every j.*

Proof. Let $\lambda \in \rho(\mathcal{A})$ lies outside the circle C_j.

The first resolvent equation $(\lambda-\xi)\mathcal{R}\left(\lambda;\mathcal{A}\right)\mathcal{R}\left(\xi;\mathcal{A}\right) = \mathcal{R}\left(\xi;\mathcal{A}\right) - \mathcal{R}\left(\lambda;\mathcal{A}\right)$ gives

$$\begin{aligned}
\mathcal{R}\left(\lambda;\mathcal{A}\right)\mathcal{P}_{\mu_j}z_0 &= \mathcal{R}\left(\lambda;\mathcal{A}\right)\frac{1}{2\pi i}\int_{C_j}\mathcal{R}\left(\xi;\mathcal{A}\right)z_0\,d\xi \\
&= \frac{1}{2\pi i}\int_{C_j}\mathcal{R}\left(\lambda;\mathcal{A}\right)\mathcal{R}\left(\xi;\mathcal{A}\right)z_0\,d\xi \\
&= -\frac{1}{2\pi i}\int_{C_j}\frac{\mathcal{R}\left(\xi;\mathcal{A}\right)-\mathcal{R}\left(\lambda;\mathcal{A}\right)}{\xi-\lambda}z_0\,d\xi \\
&= -\frac{1}{2\pi i}\int_{C_j}\frac{\mathcal{R}\left(\xi;\mathcal{A}\right)}{\xi-\lambda}z_0\,d\xi + \left(\frac{1}{2\pi i}\int_{C_j}\frac{1}{\xi-\lambda}\,d\xi\right)\mathcal{R}\left(\lambda;\mathcal{A}\right)z_0.
\end{aligned}$$

Since λ lies outside C_j, the second integrand is analytic in some disk enclosing C_j and thus, by Cauchy's theorem, the second integral is zero. This gives

$$\mathcal{R}\left(\lambda;\mathcal{A}\right)\mathcal{P}_{\mu_j}z_0 = -\frac{1}{2\pi i}\int_{C_j}\frac{\mathcal{R}\left(\xi;\mathcal{A}\right)}{\xi-\lambda}z_0\,d\xi.$$

Once again this integral can be taken in $\mathcal{D}\left(\mathcal{A}\right)$. Thus, applying $\mathcal{C}$ we have

$$\mathcal{C}\mathcal{R}\left(\lambda;\mathcal{A}\right)\mathcal{P}_{\mu_j}z_0 = -\frac{1}{2\pi i}\int_{C_j}\frac{\mathcal{C}\mathcal{R}\left(\xi;\mathcal{A}\right)}{\xi-\lambda}z_0\,d\xi.$$

Using the assumption we obtain $\mathcal{C}\mathcal{R}\left(\lambda;\mathcal{A}\right)\mathcal{P}_{\mu_j}z_0 = 0$ for every such λ, and thus, by analytic continuation, for every $\lambda \in \rho(\mathcal{A})$. □

We are now ready to prove Theorem 3.1. Let us just introduce a last definition for commodity. For a subspace $E \subset H$ invariant under $\mathcal{S}(t)$, we say that the pair $(\mathcal{A},\mathcal{C})$ is observable in E if

$$\forall z_0 \in E, \quad \Big(\mathcal{C}\mathcal{S}(t)z_0 = 0 \text{ for a.e. } t \in (0,+\infty)\Big) \Longrightarrow z_0 = 0.$$

Proof of Theorem 3.1. We will prove that the following properties are equivalent :

1. The pair $(\mathcal{A},\mathcal{C})$ is observable in every eigenspace of $\mathcal{A}$.
2. The pair $(\mathcal{A},\mathcal{C})$ is observable in every root subspace of $\mathcal{A}$.
3. The pair $(\mathcal{A},\mathcal{C})$ is observable in $\mathcal{D}(\mathcal{A})$.

It is adapted from Corollary 3.2 and Corollary 3.3 of [Fat66]. We recall that the first condition is equivalent to : $\ker(s-\mathcal{A}) \cap \ker \mathcal{C} = \{0\}$ for every $s \in \mathbb{C}$ (see Remark 3.2).

The scheme of the proof is 1. $\Longrightarrow$ 2. $\Longrightarrow$ 3. $\Longrightarrow$ 1. (the last implication is obvious).

Assume that the pair $(\mathcal{A},\mathcal{C})$ is observable in every eigenspace. If z_0 belongs to the root subspace of $\mathcal{A}$ associated with μ_j, then $\mathcal{S}(t)z_0$ is a polynomial in t, up to a factor $e^{\mu_j t}$:

$$\mathcal{S}(t)z_0 = e^{\mu_j t} p_j(t),$$

with

$$p_j(t) = \sum_{\sigma=0}^{\tau_j-1} a_{j,\sigma} t^{\sigma}, \quad a_{j,\sigma} = \frac{(-1)^{\sigma}}{\sigma!}(\mu_j - \mathcal{A})^{\sigma} z_0.$$

This can be seen using the uniqueness of the solution to the evolution equation satisfied by $\mathcal{S}(\cdot)z_0$. Thus, the identity $\mathcal{C}\mathcal{S}(\cdot)z_0 = 0$ reads

$$\mathcal{C}(\mu_j - \mathcal{A})^{\sigma} z_0 = 0, \quad 0 \le \sigma \le \tau_j - 1.$$

In particular for $\sigma = \tau_j - 1$ we have

$$\mathcal{C}(\mu_j - \mathcal{A})^{\tau_j - 1} z_0 = 0.$$

Now, recall that z_0 lies in the root subspace $\ker(\mu_j - \mathcal{A})^{\tau_j}$, so that

$$(\mu_j - \mathcal{A})^{\tau_j - 1} z_0 \in \ker(\mu_j - \mathcal{A}).$$

Thus, the assumption implies that

$$(\mu_j - \mathcal{A})^{\tau_j - 1} z_0 = 0. \tag{3.19}$$

Taking this time $\sigma = \tau_j - 2$ we have

$$\mathcal{C}(\mu_j - \mathcal{A})^{\tau_j - 2} z_0 = 0,$$

and from (3.19) we know that

$$(\mu_j - \mathcal{A})^{\tau_j - 2} z_0 \in \ker(\mu_j - \mathcal{A}),$$

so that the assumption gives

$$(\mu_j - \mathcal{A})^{\tau_j - 2} z_0 = 0.$$

Iterating this process we obtain $z_0 = 0$.

Assume now that the pair $(\mathcal{A}, \mathcal{C})$ is observable in every root subspace and let $z_0 \in \mathcal{D}(\mathcal{A})$ be such that $\mathcal{C}\mathcal{S}(t) z_0 = 0$. Applying Lemma 3.29 and Lemma 3.30 we obtain that $\mathcal{C}\mathcal{S}(t)\mathcal{P}_{\mu_j} z_0 = 0$ for a.e. $t \in (0, +\infty)$ and every j. By assumption we deduce that $\mathcal{P}_{\mu_j} z_0 = 0$ for every j, that is, $z_0 \in \left(\operatorname{Im} \mathcal{P}^*_{\mu_j}\right)^{\perp}$ for every j. Since the system of root vectors of $\mathcal{A}^*$ is assumed to be complete in H, we conclude that $z_0 = 0$. $\square$

Chapitre 4

Sharp estimates of the one-dimensional boundary control cost for parabolic systems and application to the N-dimensional boundary null-controllability in cylindrical domains

Ce chapitre est la reprise de l'article [BBGBO13], qui est un travail en collaboration avec A. Benabdallah, F. Boyer et M. González-Burgos. Il a été soumis.

Abstract. In this paper we consider the boundary null-controllability of a system of n parabolic equations on domains of the form $\Omega = (0,\pi) \times \Omega_2$ with Ω_2 a smooth domain of $\mathbb{R}^{N-1}$, $N > 1$. When the control is exerted on $\{0\} \times \omega_2$ with $\omega_2 \subset \Omega_2$, we obtain a necessary and sufficient condition that completely characterizes the null-controllability. This result is obtained through the Lebeau-Robbiano strategy and require an upper bound of the cost of the one-dimensional boundary null-control on $(0,\pi)$. This latter is obtained using the moment method and it is shown to be bounded by $Ce^{C/T}$ when T goes to 0^+.

Keywords : Parabolic systems ; Boundary Controllability ; Biorthogonal families ; Kalman rank condition.

4.1 Introduction

The controllability of systems of n partial differential equations by $m < n$ controls is a relatively recent subject. We can quote [LZ98], [dT00], [BN02] among the first works.

More recently in [AKBDGB09b], with fine tools of partial differential equations, the so-called Kalman rank condition, which characterizes the controllability of linear systems in finite dimension, has been generalized in view of the distributed null-controllability of some classes of linear parabolic systems. On the other hand, while for scalar problems the boundary controllability is known to be equivalent to the distributed controllability, it has been proved in [FCGBdT10] that this is no more the case for systems. This reveals that the controllability of systems is much more subtle. In [AKBGBdT12], it is even shown that a minimal time of control can appear if the diffusion is different on each equation, which is quite surprising for a system possessing an infinite speed of propagation. It is important to emphasize that the previous quoted results concerning the boundary controllability were established in space dimension one. They used the moment method, generalizing the works of [FR71, FR75] concerning the boundary controllability of the one-dimensional scalar heat equation. We refer to [AKBGBdT11b] for more details and a survey on the controllability of parabolic systems.

In higher space dimension the boundary controllability of parabolic systems remains widely open and it is the main purpose of this article to give some partial answers. To our knowledge, the only results on this issue are the one of [ABL12] and [AB12]. Let us also mention [Oli13] for related questions for the approximate controllability problem. In [ABL12, AB12] the results for parabolic systems are deduced from the study of the boundary control problem of two coupled wave equations using transmutation techniques. As a result there are some geometric constraints on the control domain. We will see that this restriction is not necessary.

In the present work, we focus on the boundary null-controllability of the following n coupled parabolic equations by m controls in dimension $N > 1$

$$\begin{cases} \partial_t y = \Delta y + Ay & \text{in } (0,T)\times\Omega, \\ \quad y = 1_\gamma Bv & \text{on } (0,T)\times\partial\Omega, \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \tag{4.1}$$

in the case where the domain Ω has a Cartesian product structure

$$\Omega = \Omega_1 \times \Omega_2,$$

where $\Omega_i \subset \mathbb{R}^{N_i}$, $i = 1,2$ are bounded open regular domains. In (4.1), $T > 0$ is the control time, the non-empty relative subset $\gamma \subset \partial\Omega$ is the control domain, y is the state, y_0 is the initial data, $A \in \mathcal{M}_n(\mathbb{R})$ and $B \in \mathcal{M}_{n\times m}(\mathbb{R})$ are constant matrices and v is the boundary control.

Under appropriate assumptions we show that the controllability of System (4.1) is reduced to the controllability of the same system posed on Ω_1 (see Theorem 4.4 below). The proof is based on the method of Lebeau-Robiano [LR95]. This strategy (already used in a different framework in [BDR07]) requires an estimate of the cost of the N_1-dimensional control with respect to the control time when $T \to 0^+$.

In a second part, we establish that the cost of the one-dimensional null-control on $(0,T)$ is bounded by $Ce^{C/T}$, for some $C > 0$, as $T \to 0^+$ (see Theorem 4.6 below). This is the second main result of this paper and this also shows that our first result above can be applied at least in the case $N_1 = 1$. The demonstration of this result follows the approach of [FR71] and [Mil04] (for the scalar case). It requires to take back the proofs contained in [AKBGBdT11a]. In the scalar case, [Sei84] (see also [FCZ00]) gave a similar estimate of the cost of the boundary control of the heat equation, which is known to be optimal thanks to the work [Güi85].

Note finally, that the extension of the present results to more general domains Ω in $\mathbb{R}^N$ as well as the study of the case with different diffusion coefficient on each equation remain open problems.

4.1.1 Reminders and notations

Let us first recall that System (4.1) is well-posed in the sense that, for every $y_0 \in H^{-1}(\Omega)^n$ and $v \in L^2(0,T;L^2(\partial\Omega)^m)$, there exists a unique solution $y \in C^0([0,T];H^{-1}(\Omega)^n) \cap L^2(0,T;L^2(\Omega)^n)$, defined by transposition. Moreover, this solution depends continuously on the initial data y_0 and the control v. More precisely,

$$\|y\|_{C^0([0,T];H^{-1}(\Omega)^n)} \leq Ce^{CT}\left(\|y_0\|_{H^{-1}(\Omega)^n} + \|v\|_{L^2(0,T;L^2(\partial\Omega)^m)}\right), \tag{4.2}$$

where here and all along this work $C > 0$ denotes a generic positive constant that may change line to line but which does not depend on T nor y_0. We shall also use sometimes the notations C', C'', and so on.

Let us now precise the concept of controllability we will deal with in this paper. We say that System (4.1) is null-controllable at time T if for every $y_0 \in H^{-1}(\Omega)^n$, there exists a control $v \in L^2(0,T;L^2(\partial\Omega)^m)$ such that the corresponding solution y satisfies

$$y(T) = 0.$$

In such a case, it is well-known that there exists $C_T > 0$ such that

$$\|v\|_{L^2(0,T;L^2(\partial\Omega)^m)} \leq C_T\|y_0\|_{H^{-1}(\Omega)^n}, \quad \forall y_0 \in H^{-1}(\Omega)^n. \tag{4.3}$$

The infimum of the constants C_T satisfying (4.3) is called the cost of the null-control at time T.

Remark 4.1. *Even if it means replacing $y(t)$ by $e^{-\mu t}y(t)$ and A by $A-\mu$, with $\mu > 0$, we can assume without loss of generality that the matrix A is stable : all its eigenvalues have a negative real part.*

Finally, let us recall the well-known duality between controllability and observability.

Theorem 4.2. *Let E be a closed subspace of $H_0^1(\Omega)^n$ and set $E^{-1} = -\Delta E \subset H^{-1}(\Omega)^n$. Let us denote Π_E (resp. $\Pi_{E^{-1}}$) the orthogonal projection on E (resp. E^{-1}). Let $C_T > 0$ be fixed. For every $y_0 \in E^{-1}$ there exists a control $v \in L^2(0,T;L^2(\partial\Omega)^m)$ such that*

$$\begin{cases} \Pi_{E^{-1}}y(T) = 0, \\ \|v\|_{L^2(0,T;L^2(\partial\Omega)^m)} \leq C_T\|y_0\|_{H^{-1}(\Omega)^n}, \end{cases}$$

where y is the corresponding solution to (4.1), *if and only if*

$$\|\Pi_E z(0)\|^2_{H^1_0(\Omega)^n} \leq C_T^2 \int_0^T \|1_\gamma B^* \partial_n z(t)\|^2_{L^2(\partial\Omega)^m} dt, \quad \forall z_T \in E,$$

where z is the solution to the adjoint system

$$\begin{cases} -\partial_t z &= \Delta z + A^* z & \text{in } (0,T) \times \Omega, \\ z &= 0 & \text{on } (0,T) \times \partial\Omega, \\ z(T) &= z_T & \text{in } \Omega. \end{cases} \tag{4.4}$$

Notations We gather here some standard notations that we shall use all along this paper. For any real numbers $a < b$ we denote $[\![a,b]\!] = [a,b] \cap \mathbb{Z}$. For $z \in \mathbb{C}$, $\Re(z)$ and $\Im(z)$ denote the real and imaginary part of z. Finally, $x \in \mathbb{R} \mapsto \lfloor x \rfloor \in \mathbb{Z}$ denotes the floor function.

4.1.2 Main results

4.1.2.1 Boundary controllability for a multidimensional parabolic system

The first main achievement of this work is the following.

Theorem 4.3. *Let $\omega_2 \subset \Omega_2$ be a non-empty open subset and take $\Omega_1 = (0,\pi)$. Then, System* (4.1) *is null-controllable at time T on $\gamma = \{0\} \times \omega_2$ if and only if*

$$\operatorname{rank}(B_k | A_k B_k | A_k^2 B_k | \cdots | A_k^{nk-1} B_k) = nk, \quad \forall k \geq 1, \tag{4.5}$$

where we have introduced the notations

$$A_k = \begin{pmatrix} -\lambda_1 + A & 0 & \cdots & \cdots & 0 \\ 0 & -\lambda_2 + A & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -\lambda_k + A \end{pmatrix} \in \mathcal{M}_{nk}(\mathbb{R}), \quad B_k = \begin{pmatrix} B \\ B \\ \vdots \\ \vdots \\ B \end{pmatrix} \in \mathcal{M}_{nk \times m}(\mathbb{R}). \tag{4.6}$$

One may think to a cylindrical domain where the control domain is a subset of the top or bottom face (see Figure 4.1).

This result will be otained as a corollary of some other theorems that are important results too. The first one is the following and it should be connected with [Fat75] and [Mil05].

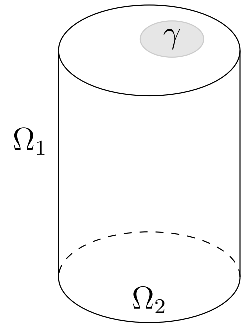


FIGURE 4.1 – Typical geometric situation

Theorem 4.4. *Let $\gamma_1 \subset \partial\Omega_1$ be a non-empty relative subset. Assume that the following N_1-dimensional system*

$$\begin{cases} \partial_t y^1 &= \Delta_{x_1} y^1 + Ay^1 & \text{in } (0,T)\times\Omega_1, \\ y^1 &= 1_{\gamma_1} Bv^1 & \text{on } (0,T)\times\partial\Omega_1, \\ y^1(0) &= y_0^1 & \text{in } \Omega_1, \end{cases} \tag{4.7}$$

is null-controllable for any time $T > 0$, with in addition the following bound for the control cost $C_T^{\Omega_1}$

$$C_T^{\Omega_1} \leq Ce^{C/T}, \quad \forall T > 0. \tag{4.8}$$

Then, for any non-empty open set $\omega_2 \subset \Omega_2$, the N-dimensional System (4.1) is null-controllable at any time $T > 0$ on the control domain $\gamma = \gamma_1 \times \omega_2$.

Remark 4.5. *The converse of Theorem 4.4 also holds. More precisely, if the N-dimensional System* (4.1) *is null-controllable at time T, then the N_1-dimensional System* (4.7) *is also null-controllable at time T. This can be proved using a Fourier decomposition in the direction of Ω_2.*

It is worth mentioning that, such a decomposition also shows that, when $\omega_2 = \Omega_2$, the proof of Theorem 4.4 is much simpler and it does not need the control cost estimate (4.8)*. Moreover, the domain Ω_2 can even be unbounded in this case.*

4.1.2.2 Estimate of the control cost for a 1D boundary controllability problem

The second result of this paper provides an important example where Theorem 4.4 can be successfully applied.

More precisely, we show that the assumption (4.8) on the short time behavior of the control cost actually holds in the 1D case for the following system if we assume the rank condition (4.5)

$$\begin{cases} \partial_t y = \partial_x^2 y + Ay & \text{in } (0,T)\times(0,\pi), \\ y(t,0) = Bv(t),\ y(t,\pi) = 0 & \text{in } (0,T), \\ y(0) = y_0 & \text{in } (0,\pi). \end{cases} \tag{4.9}$$

We recall that it has been established in [AKBGBdT11a] that System (4.9) is null-controllable at time $T > 0$ if and only if the rank condition (4.5) holds.

However, in the above-mentioned reference, no estimate on the control cost is provided. This is the next goal of the present paper, to give a more precise insight into the proof of the controllability result for System (4.9) that allows a precise estimate of the control cost as a function of T.

Theorem 4.6. *Assume that the rank condition (4.5) holds. Then, for every $T > 0$ and $y_0 \in H^{-1}(0,\pi)^n$ there exists a null-control $v \in L^2(0,T)^m$ for System (4.9) which, in addition, satisfies*

$$\|v\|_{L^2(0,T)^m} \leq Ce^{C/T}\|y_0\|_{H^{-1}(0,\pi)^n}.$$

This theorem, combined with Theorem 4.4 and Remark 4.5 give a proof of Theorem 4.3.

4.1.2.3 Bounds on biorthogonal families of exponentials

The proof of Theorem 4.6 is mainly based on the existence of a suitable biorthogonal family of time-dependent exponential functions. The construction provided in [AKBGBdT11a] does not allow to estimate the control cost. That is the reason why we propose here a slightly different approach which is the key to obtain the factor $e^{C/T}$. This abstract result, which is interesting in itself and potentially useful in other situations, can be formulated as follows.

Theorem 4.7. *Let $\{\Lambda_k\}_{k\geq 1} \subset \mathbb{C}$ be a sequence of complex numbers with the following properties*

($\mathcal{H}_1$) $\Lambda_k \neq \Lambda_n$ *for all* $k, n \in \mathbb{N}$ *with* $k \neq n$.

($\mathcal{H}_2$) $\Re(\Lambda_k) > 0$ *for every* $k \geq 1$.

($\mathcal{H}_3$) For some $\beta > 0$,

$$|\Im(\Lambda_k)| \leq \beta\sqrt{\Re(\Lambda_k)}, \quad \forall k \geq 1.$$

($\mathcal{H}_4$) $\{\Lambda_k\}_{k\geq 1}$ *is non-decreasing in modulus*

$$|\Lambda_k| \leq |\Lambda_{k+1}|, \quad \forall k \geq 1.$$

($\mathcal{H}_5$) $\{\Lambda_k\}_{k\geq 1}$ *satisfies the following gap condition : for some* $\rho, q > 0$,

$$\begin{cases} |\Lambda_k - \Lambda_n| \geq \rho\left|k^2 - n^2\right|, & \forall k, n : |k-n| \geq q. \\ \inf\limits_{k\neq n : |k-n|<q} |\Lambda_k - \Lambda_n| > 0. \end{cases}$$

($\mathcal{H}_6$) For some $p, \alpha > 0$,

$$\left| p\sqrt{r} - \mathcal{N}(r) \right| \leq \alpha, \quad \forall r > 0, \tag{4.10}$$

where $\mathcal{N}$ is the counting function associated with the sequence $\{\Lambda_k\}_{k\geq 1}$, that is the function defined by

$$\mathcal{N}(r) = \#\left\{k : |\Lambda_k| \leq r\right\}, \quad \forall r > 0. \tag{4.11}$$

Then, there exists $T_0 > 0$ such that, for every $\eta \geq 1$ and $0 < T < T_0$, we can find a family of $\mathbb{C}$-valued functions

$$\{\varphi_{k,j}\}_{k\geq 1, j\in[\![0,\eta-1]\!]} \subset L^2(-T/2, T/2)$$

biorthogonal[1] *to $\{e_{k,j}\}_{k\geq 1, j\in[\![0,\eta-1]\!]}$, where for every $t \in (-T/2, T/2)$,*

$$e_{k,j}(t) = t^j e^{-\Lambda_k t},$$

with in addition

$$\|\varphi_{k,j}\|_{L^2(-T/2,T/2)} \leq Ce^{C\sqrt{\Re(\Lambda_k)+\frac{C}{T}}}, \tag{4.12}$$

for any $k \geq 1$, $j \in [\![0, \eta-1]\!]$.

4.2 Boundary null-controllability on product domains

4.2.1 Settings and preliminary remarks

Let $\lambda_j^{\Omega_1}$ (resp. $\lambda_j^{\Omega_2}$), $j \geq 1$, be the Dirichlet eigenvalues of the Laplacian on Ω_1 (resp. Ω_2), and let $\phi_j^{\Omega_1}$ (resp. $\phi_j^{\Omega_2}$) be the corresponding normalized eigenfunction.

Let us introduce the (closed) subspaces of $H_0^1(\Omega)^n$ on which we will establish the partial observability later on (section 4.2.2)

$$E_J = \left\{ \sum_{j=1}^{J} \left\langle u, \phi_j^{\Omega_2} \right\rangle_{L^2(\Omega_2)} \phi_j^{\Omega_2} \,\middle|\, u \in H_0^1(\Omega)^n \right\} \subset H_0^1(\Omega)^n, \quad J \geq 1,$$

where the notation $\sum_{j=1}^{J} \langle u, \phi_j^{\Omega_2} \rangle_{L^2(\Omega_2)} \phi_j^{\Omega_2}$ is used to mean the function

$$(x_1, x_2) \in \Omega \longmapsto \sum_{j=1}^{J} \langle u(x_1, \cdot), \phi_j^{\Omega_2} \rangle_{L^2(\Omega_2)} \phi_j^{\Omega_2}(x_2).$$

We then define the "dual" spaces of E_J

$$E_J^{-1} = -\Delta E_J \subset H^{-1}(\Omega)^n, \quad J \geq 1.$$

Let us recall that we denote by Π_{E_J} (resp. $\Pi_{E_J^{-1}}$) the orthogonal projection in $H_0^1(\Omega)^n$ (resp. $H^{-1}(\Omega)^n$) onto E_J (resp. E_J^{-1}). It is not difficult to see that we have the relation $\Pi_{E_J^{-1}}(-\Delta u) = -\Delta \Pi_{E_J} u$ for any $u \in H_0^1(\Omega)^n$.

1. that is $\langle \varphi_{k,j}, e_{l,\nu} \rangle_{L^2(-T/2,T/2)} = \int_{-T/2}^{T/2} \varphi_{k,j}(t) \overline{e_{l,\nu}}(t)\, dt = \delta_{kl}\delta_{j\nu}$.

Lemma 4.8. *For any $u \in H_0^1(\Omega)^n$, we have*

$$u = \sum_{j=1}^{+\infty} \langle u, \phi_j^{\Omega_2} \rangle_{L^2(\Omega_2)} \phi_j^{\Omega_2}.$$

It follows from this lemma that $\Pi_{E_J} u = \sum_{j=1}^{J} \langle u, \phi_j^{\Omega_2} \rangle_{L^2(\Omega_2)} \phi_j^{\Omega_2}$ for any $u \in H_0^1(\Omega)^n$.

Proof of Lemma 4.8 Let us show that the sequence $\{S_J u\}_{J \geq 1}$ defined by

$$S_J u = \sum_{j=1}^{J} \langle u, \phi_j^{\Omega_2} \rangle_{L^2(\Omega_2)} \phi_j^{\Omega_2},$$

is a Cauchy sequence of $H_0^1(\Omega)^n$. For any $J > K \geq 1$ we have

$$\begin{aligned} \| S_J u - S_K u \|^2_{H_0^1(\Omega)^n} &= \left\| \sum_{j=K+1}^{J} \left\langle u, \phi_j^{\Omega_2} \right\rangle_{L^2(\Omega_2)} \phi_j^{\Omega_2} \right\|^2_{H_0^1(\Omega)^n} \\ &= \sum_{j=K+1}^{J} \left\| \left\langle u, \phi_j^{\Omega_2} \right\rangle_{L^2(\Omega_2)} \right\|^2_{H_0^1(\Omega_1)^n} + \sum_{j=K+1}^{J} \lambda_j^{\Omega_2} \left\| \left\langle u, \phi_j^{\Omega_2} \right\rangle_{L^2(\Omega_2)} \right\|^2_{L^2(\Omega_1)^n} \end{aligned}$$

Using Lebesgue's dominated convergence theorem it is not difficult to see that these terms go to zero as $J, K \to +\infty$. As a result $S_J u \xrightarrow[J \to +\infty]{H_0^1} v$ for some $v \in H_0^1(\Omega)^n$. In particular, $\langle v, \phi_k^{\Omega_1} \phi_j^{\Omega_2} \rangle_{L^2(\Omega)} = \langle u, \phi_k^{\Omega_1} \phi_j^{\Omega_2} \rangle_{L^2(\Omega)}$ for every $j, k \geq 1$, and it follows that $v = u$.

4.2.2 Partial observability

One of the key points to make use of the Lebeau-Robbiano strategy is the estimate of the cost of the partial observabilities on the approximation subspaces. This will be used for the active control phase.

Proposition 4.9. *Let Ω_2 be of class C^2. Assume that System (4.7) is controllable at time T with cost $C_T^{\Omega_1}$. Then,*

$$\| \Pi_{E_J} z(0) \|^2_{H_0^1(\Omega)^n} \leq C (C_T^{\Omega_1})^2 e^{C\sqrt{\lambda_J^{\Omega_2}}} \int_0^T \| 1_{\gamma_1 \times \omega_2} B^* \partial_n z(t) \|^2_{L^2(\partial\Omega)^m} dt, \quad \forall z_T \in E_J, \quad (4.13)$$

where z is the solution to the adjoint system (4.4).

By Theorem 4.2 we deduce that

Corollary 4.10. *For any $J \geq 1, y_0 \in E_J^{-1}$, there exists a control $v(y_0) \in L^2(0, T; L^2(\partial\Omega)^m)$ with*

$$\| v(y_0) \|_{L^2(0,T;L^2(\partial\Omega)^m)} \leq C \, (C_T^{\Omega_1}) e^{C\sqrt{\lambda_J^{\Omega_2}}} \| y_0 \|_{H^{-1}(\Omega)^n}, \quad (4.14)$$

such that the solution y to system (4.1) *satisfies*

$$\Pi_{E_J^{-1}} y(T) = 0.$$

Proof of Proposition 4.9 Let $z_T \in E_J$ so that

$$z_T(x_1, x_2) = \sum_{j=1}^{J} z_T^j(x_1)\phi_j^{\Omega_2}(x_2),$$

for some $z_T^j \in H_0^1(\Omega_1)^n$. Let z be the solution of (4.4), the adjoint system of (4.1), associated with z_T. Thus,

$$z(t, x_1, x_2) = \sum_{j=1}^{J} z^j(t, x_1)\phi_j^{\Omega_2}(x_2),$$

where z^j is the solution to

$$\begin{cases} -\partial_t z^j &= \left(\Delta_{x_1} - \lambda_j^{\Omega_2}\right) z^j + A^* z^j & \text{in } (0,T)\times\Omega_1, \\ z^j &= 0 & \text{on } (0,T)\times\partial\Omega_1, \\ z^j(T) &= z_T^j & \text{in } \Omega_1. \end{cases}$$

Note that $\Pi_{E_J} z(0) = z(0)$. A computation of $\|z(0)\|^2_{H_0^1(\Omega)^n}$ gives

$$\|z(0)\|^2_{H_0^1(\Omega)^n} = \sum_{j=1}^{J} \left\|z^j(0)\right\|^2_{H_0^1(\Omega_1)^n} + \sum_{j=1}^{J} \lambda_j^{\Omega_2} \left\|z^j(0)\right\|^2_{L^2(\Omega_1)^n}.$$

Using the Poincaré inequality we obtain,

$$\|z(0)\|^2_{H_0^1(\Omega)^n} \leq C\lambda_J^{\Omega_2} \sum_{j=1}^{J} \left\|z^j(0)\right\|^2_{H_0^1(\Omega_1)^n}. \tag{4.15}$$

Observe now that $z^j(t) = e^{-(T-t)\lambda_j^{\Omega_2}}\psi(t)$, where ψ is the solution to the adjoint system of (4.7) associated with z_T^j. Thus, using the assumption that (4.7) is controllable with cost $C_T^{\Omega_1}$, we obtain by Theorem 4.2 that

$$\left\|z^j(0)\right\|^2_{H_0^1(\Omega_1)^n} \leq (C_T^{\Omega_1})^2 \int_0^T \left\|1_{\gamma_1} B^* \partial_{n_1} z^j(t)\right\|^2_{L^2(\partial\Omega_1)^m} dt,$$

where n_1 denotes the unit outward normal vector of Ω_1. Combined to (4.15), this gives

$$\|z(0)\|^2_{H_0^1(\Omega)^n} \leq C(C_T^{\Omega_1})^2 \lambda_J^{\Omega_2} \int_0^T \sum_{j=1}^{J} \left\|1_{\gamma_1} B^* \partial_{n_1} z^j(t)\right\|^2_{L^2(\partial\Omega_1)^m} dt.$$

Let us denote by B_k the kth column of B. Applying the Lebeau-Robbiano's spectral inequality [LR95] (see also [LR07, Section 3.A] [2])

$$\sum_{j=1}^{J} |a_j|^2 \leq Ce^{C\sqrt{\lambda_J^{\Omega_2}}} \int_{\omega_2} \left|\sum_{j=1}^{J} a_j \phi_j^{\Omega_2}(x_2)\right|^2 dx_2$$

2. and [TT11, Theorem 1.5] when Ω_2 is a rectangular domain.

to the sequence of scalars $a_j = B_k^* \partial_{n_1} z^j(t,\sigma_1)$, $\sigma_1 \in \partial\Omega_1$ being fixed, and summing over $1 \leq k \leq m$, this gives

$$\sum_{j=1}^{J} \left| B^* \partial_{n_1} z^j(t,\sigma_1) \right|_{\mathbb{C}^n}^2 \leq C e^{C\sqrt{\lambda_J^{\Omega_2}}} \int_{\omega_2} \left| \sum_{j=1}^{J} B^* \partial_{n_1} z^j(t,\sigma_1) \phi_j^{\Omega_2}(x_2) \right|_{\mathbb{C}^n}^2 dx_2.$$

To conclude it only remains to integrate over γ_1 and observe that

$$n(\sigma) = \begin{pmatrix} n_1(\sigma_1) \\ 0 \end{pmatrix} \text{ for } \sigma = (\sigma_1, x_2) \in \partial\Omega_1 \times \Omega_2.$$

4.2.3 Dissipation along the direction Ω_2

The other point of the Lebeau-Robbiano strategy relies on the natural dissipation of the system when no control is exerted (the passive phase). For our purpose, we need an exponential dissipation in the direction Ω_2.

Proposition 4.11. *If there is no control on (t_0,t_1) (i.e. $v = 0$ on (t_0,t_1)) and the corresponding solution y of System* (4.1) *satisfies*

$$\Pi_{E_J^{-1}} y(t_0) = 0,$$

then we have the following dissipation estimate

$$\|y(t)\|_{H^{-1}(\Omega)^n} \leq C e^{-\lambda_{J+1}^{\Omega_2}(t-t_0)} \|y(t_0)\|_{H^{-1}(\Omega)^n}, \quad \forall t \in (t_0,t_1).$$

Proof. Let $y(t_0) = -\Delta \tilde{y}_0$, $\tilde{y}_0 \in H_0^1(\Omega)^n$. The assumption $\Pi_{E_J^{-1}} y(t_0) = 0$ translates into $\Pi_{E_J} \tilde{y}_0 = 0$.

Let $\tilde{y}$ be the solution in $H_0^1(\Omega)^n$ to

$$\begin{cases} \partial_t \tilde{y} &= \Delta \tilde{y} + A\tilde{y} & \text{in } (t_0,t_1) \times \Omega, \\ \tilde{y} &= 0 & \text{on } (t_0,t_1) \times \partial\Omega, \\ \tilde{y}(t_0) &= \tilde{y}_0 & \text{in } \Omega. \end{cases}$$

Since the matrix A is constant, we can check that

$$y = -\Delta \tilde{y} \quad \text{in } (t_0,t_1) \times \Omega,$$

and thus

$$\|y(t)\|_{H^{-1}(\Omega)^n} = \|\tilde{y}(t)\|_{H_0^1(\Omega)^n}, \quad \|y(t_0)\|_{H^{-1}(\Omega)^n} = \|\tilde{y}_0\|_{H_0^1(\Omega)^n}.$$

As a consequence it only remains to prove the dissipation for regular data, namely

$$\|\tilde{y}(t)\|_{H_0^1(\Omega)^n} \leq C e^{-\lambda_{J+1}^{\Omega_2}(t-t_0)} \|\tilde{y}_0\|_{H_0^1(\Omega)^n}, \quad \forall t \in (t_0,t_1),$$

for $\tilde{y}_0$ such that $\Pi_{E_J}\tilde{y}_0 = 0$ i.e. of the form (see Lemma 4.8)

$$\tilde{y}_0 = \sum_{j=J+1}^{+\infty} \tilde{y}_{0,j}\phi_j^{\Omega_2}, \quad \tilde{y}_{0,j} = \left\langle \tilde{y}_0, \phi_j^{\Omega_2} \right\rangle_{L^2(\Omega_2)^n} \in H_0^1(\Omega_1)^n.$$

Since $\Pi_{E_J}\tilde{y}_0 = 0$ and A is constant, we have $\Pi_{E_J}\tilde{y}(t) = 0$ for every $t \in (t_0, t_1)$ and as a result the following Poincaré inequality holds

$$\lambda_{J+1}^{\Omega_2}\|\tilde{y}(t)\|^2_{L^2(\Omega)^n} \leq \|\nabla\tilde{y}(t)\|^2_{L^2(\Omega)^n} \quad \forall t \in (t_0, t_1).$$

Combined to Young's inequality this leads to

$$\lambda_{J+1}^{\Omega_2}\|\nabla\tilde{y}(t)\|^2_{L^2(\Omega)^n} \leq 4\|\Delta\tilde{y}(t)\|^2_{L^2(\Omega)^n} \quad \text{for a.e. } t \in (t_0, t_1).$$

Using now standard energy estimates and the fact that the matrix A is constant and stable (see Remark 4.1), we finally obtain the desired dissipation

$$\|\tilde{y}(t)\|_{H_0^1(\Omega)^n} \leq Ce^{-\lambda_{J+1}^{\Omega_2}(t-t_0)}\|\tilde{y}_0\|_{H_0^1(\Omega)^n}.$$

$\square$

4.2.4 Lebeau-Robbiano time procedure

We are now ready to prove Theorem 4.4.

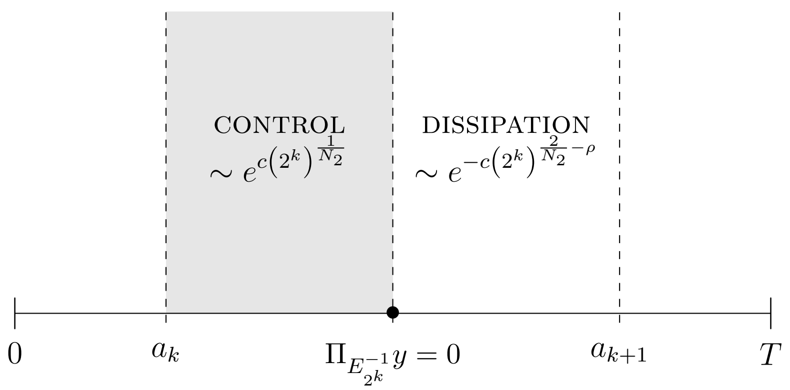
Let $y_0 \in H^{-1}(\Omega)^n$ be fixed. Let us decompose the interval $[0,T)$ as follows

$$[0,T) = \bigcup_{k=0}^{+\infty}[a_k, a_{k+1}],$$

with

$$a_0 = 0, \quad a_{k+1} = a_k + 2T_k, \quad T_k = M2^{-k\rho}.$$

where $\rho \in (0, \frac{1}{N_2})$ and $M = \frac{T}{2}(1-2^{-\rho})$ has been determined to ensure that $2\sum_{k=0}^{+\infty} T_k = T$.



We define the control v and the corresponding solution y piecewisely and by induction as follows

$$v(t) = \begin{cases} v\left(\Pi_{E_{2^k}^{-1}} y(a_k)\right)(t) & \text{if } t \in (a_k, a_k + T_k), \\ 0 & \text{if } t \in (a_k + T_k, a_{k+1}). \end{cases}$$

Let us show that v belongs to $L^2(0,T;L^2(\partial\Omega)^m)$ and steers y to 0 at time T.

Step 1 : Estimate on the interval $[a_k, a_k + T_k]$ From the continuous dependence with respect to the data (4.2) and since $T_k \leq T$ we know that

$$\|y(a_k + T_k)\|_{H^{-1}(\Omega)^n} \leq C\left(\|y(a_k)\|_{H^{-1}(\Omega)^n} + \|v\|_{L^2(a_k,a_k+T_k;L^2(\partial\Omega)^m)}\right). \tag{4.16}$$

Using the estimate of the cost of the control (4.14) we have

$$\|v\|_{L^2(a_k,a_k+T_k;L^2(\partial\Omega)^m)} \leq C C_{T_k}^{\Omega_1} e^{C\sqrt{\lambda_{2^k}^{\Omega_2}}} \left\|\Pi_{E_{2^k}^{-1}} y(a_k)\right\|_{H^{-1}(\Omega)^n}$$

and since $\left\|\Pi_{E_{2^k}^{-1}}\right\|_{\mathcal{L}(H^{-1})} \leq 1$, this gives

$$\|v\|_{L^2(a_k,a_k+T_k;L^2(\partial\Omega)^m)} \leq C C_{T_k}^{\Omega_1} e^{C\sqrt{\lambda_{2^k}^{\Omega_2}}} \|y(a_k)\|_{H^{-1}(\Omega)^n}.$$

Using now the estimate of $C_T^{\Omega_1}$ with respect to T (assumption (4.8)), this leads to

$$\|v\|_{L^2(a_k,a_k+T_k;L^2(\partial\Omega)^m)} \leq c e^{c\left(\frac{1}{T_k} + \sqrt{\lambda_{2^k}^{\Omega_2}}\right)} \|y(a_k)\|_{H^{-1}(\Omega)^n}.$$

On the other hand, Weyl's asymptotic formula states that

$$\sqrt{\lambda_{2^k}^{\Omega_2}} \underset{+\infty}{\sim} C\left(2^k\right)^{\frac{1}{N_2}}$$

and (by the choice of ρ)

$$\frac{1}{T_k} = \frac{1}{M} 2^{k\rho} \leq C 2^{\frac{k}{N_2}},$$

so that

$$\|v\|_{L^2(a_k,a_k+T_k;L^2(\partial\Omega)^m)} \leq C e^{C 2^{\frac{k}{N_2}}} \|y(a_k)\|_{H^{-1}(\Omega)^n}. \tag{4.17}$$

Combined to (4.16) this yields

$$\begin{aligned} \|y(a_k + T_k)\|_{H^{-1}(\Omega)^n} &\leq C\left(1 + e^{C 2^{\frac{k}{N_2}}}\right) \|y(a_k)\|_{H^{-1}(\Omega)^n} \\ &\leq C e^{C 2^{\frac{k}{N_2}}} \|y(a_k)\|_{H^{-1}(\Omega)^n}. \end{aligned} \tag{4.18}$$

Step 2 : Estimate on the interval $[a_k+T_k, a_{k+1}]$ Since $\Pi_{E^{-1}_{2^k}} y(a_k+T_k) = 0$, the dissipation (Proposition 4.11) gives

$$\|y(a_{k+1})\|_{H^{-1}(\Omega)^n} \leq Ce^{-\lambda^{\Omega_2}_{2^k+1} T_k} \|y(a_k+T_k)\|_{H^{-1}(\Omega)^n}. \tag{4.19}$$

Step 3 : Final estimate From (4.19) and (4.18) we deduce

$$\|y(a_{k+1})\|_{H^{-1}(\Omega)^n} \leq Ce^{-\lambda^{\Omega_2}_{2^k+1} T_k + C2^{\frac{k}{N_2}}} \|y(a_k)\|_{H^{-1}(\Omega)^n}.$$

By induction we obtain

$$\|y(a_{k+1})\|_{H^{-1}(\Omega)^n} \leq Ce^{\sum_{p=0}^k \left(-\lambda^{\Omega_2}_{2^p+1} T_p + C2^{\frac{p}{N_2}}\right)} \|y_0\|_{H^{-1}(\Omega)^n}.$$

Since

$$\lambda^{\Omega_2}_{2^p+1} T_p \underset{+\infty}{\sim} C(2^p+1)^{\frac{2}{N_2}} 2^{-p\rho} \geq C'(2^p)^{\frac{2}{N_2}-\rho},$$

we obtain

$$\|y(a_{k+1})\|_{H^{-1}(\Omega)^n} \leq Ce^{\sum_{p=0}^k \left(-C'(2^p)^{\frac{2}{N_2}-\rho} + C(2^p)^{\frac{1}{N_2}}\right)} \|y_0\|_{H^{-1}(\Omega)^n}.$$

Since $\rho < \frac{1}{N_2}$, there exists a $p_0 \geq 1$ such that

$$-C'(2^p)^{\frac{2}{N_2}-\rho} + C(2^p)^{\frac{1}{N_2}} \leq -C''(2^p)^{\frac{2}{N_2}-\rho}, \quad \forall p \geq p_0. \tag{4.20}$$

It follows that, for $k \geq p_0$, we have

$$\sum_{p=0}^k \left(-C'(2^p)^{\frac{2}{N_2}-\rho} + C(2^p)^{\frac{1}{N_2}}\right) \leq C''' - C'' \sum_{p=p_0}^k (2^p)^{\frac{2}{N_2}-\rho} \leq C''' - C''\left(2^k\right)^{\frac{2}{N_2}-\rho}.$$

So that, finally,

$$\|y(a_{k+1})\|_{H^{-1}(\Omega)^n} \leq Ce^{-C\left(2^k\right)^{\frac{2}{N_2}-\rho}} \|y_0\|_{H^{-1}(\Omega)^n}. \tag{4.21}$$

Step 4 : The function v **is a control** Estimates (4.17) and (4.21) show that the function v is in $L^2(0,T;L^2(\partial\Omega))$:

$$\|v\|^2_{L^2(0,T:L^2(\partial\Omega)^m)} = \sum_{k=0}^{+\infty} \|v\|^2_{L^2(a_k,a_k+T_k:L^2(\partial\Omega)^m)} \leq C \underbrace{\left(\sum_{k=0}^{+\infty} e^{C2^{\frac{k}{N_2}} - C'\left(2^k\right)^{\frac{2}{N_2}-\rho}}\right)}_{<+\infty \text{ by } (4.20)} \|y_0\|^2_{H^{-1}(\Omega)^n}.$$

Moreover, estimate (4.21) also shows that the function v is indeed a control :

$$\|y(a_{k+1})\|_{H^{-1}(\Omega)^n} \xrightarrow[k\to+\infty]{} 0 = \|y(T)\|_{H^{-1}(\Omega)^n}.$$

4.3 Cost of the one-dimensional boundary null-control

We prove here Theorem 4.6 assuming Theorem 4.7 is proved (see the next section). All along this part we shall use the notations of [AKBGBdT11a].

4.3.1 Arrangement and properties of the eigenvalues

Let us first recall that the Dirichlet eigenvalues of the Laplacian $-\partial_x^2$ on $(0,\pi)$ (with domain $H^2(0,\pi)\cap H_0^1(0,\pi)$) are $\lambda_k = k^2$, $k\geq 1$.

We denote by $\{\mu_l\}_{l\in[\![1,p]\!]}\subset\mathbb{C}$ the set of distinct eigenvalues of A^*. For $l\in[\![1,p]\!]$, we denote the dimension of the eigenspace of A^* associated with μ_l by n_l and the size of its Jordan chains by $\tau_{l,j}$, $j\in[\![1,n_l]\!]$. In [AKBGBdT11a, Case 2, p. 583], it is shown that we can always assume that $\tau_{l,j}=\tau_l$ is independent of j. Finally, we set $\widehat{n}=\max_{l\in[\![1,p]\!]} n_l$.

We assume that the set $\{\mu_l\}_{l\in[\![1,p]\!]}$ is arranged in the following (non unique) way

$$\forall l\in[\![1,p-1]\!],\quad \begin{cases}\Re(\mu_l)\geq\Re(\mu_{l+1}),\\ |\mu_l|\leq|\mu_{l+1}| \text{ if } \Re(\mu_l)=\Re(\mu_{l+1}).\end{cases}\tag{4.22}$$

We should point out that in [AKBGBdT11a, page 562], it is assumed that $\{\mu_l\}_{l\in[\![1,p]\!]}$ is ordered in such a way that $\widehat{n}=n_1$. Actually, this is only used for commodity and the same reasoning holds if we take $\widehat{n}$ instead of n_1.

Let us now recall that the eigenvalues of the operator $\partial_x^2+A^*$ (with domain $H^2(0,\pi)^n\cap H_0^1(0,\pi)^n$) are given by $-\lambda_k+\mu_i$, $k\geq 1$ and $i\in[\![1,p]\!]$. Moreover, there exists $k_0\geq 1$ such that

$$-\lambda_k+\mu_i\neq-\lambda_l+\mu_j,\tag{4.23}$$

for every $k\geq k_0$, $l\geq 1$, $l\neq k$, and $i,j\in[\![1,p]\!]$ with $i\neq j$ (see [AKBGBdT11a, Proposition 3.2]).

From (4.22), we see that there exists $k_1\geq 1$ large enough so that

$$2\lambda_{k_1}\left(\Re(\mu_l)-\Re(\mu_{l+1})\right)+|\mu_{l+1}|^2-|\mu_l|^2\geq 0,$$

for every $l\in[\![1,p-1]\!]$. Therefore, we deduce that

$$|\lambda_k-\mu_l|\leq|\lambda_k-\mu_{l+1}|,\tag{4.24}$$

for every $k\geq k_1$ and $l\in[\![1,p-1]\!]$.

Finally, let $k_2\geq 1$ be large enough so that

$$1+|\lambda_k-\mu_i|\leq|\lambda_{k+1}-\mu_j|,\tag{4.25}$$

for every $k\geq k_2$ and $i,j\in[\![1,p]\!]$ with $i\neq j$, which is always possible since $\lambda_k=k^2$.

We set

$$K_0=\max\{k_0,k_1,k_2\}.$$

To this K_0 we associate $\widetilde{p} \geq 1$, the number of distinct eigenvalues of the matrix $A^*_{K_0}$ defined in (4.6). Let $\{\gamma_\ell\}_{\ell \in [\![1,\widetilde{p}]\!]} \subset \{-\lambda_k + \mu_l\}_{k \in [\![1,K_0]\!], l \in [\![1,p]\!]}$ be the set of distinct eigenvalues of $A^*_{K_0}$ arranged in such a way that $|\gamma_\ell| \leq |\gamma_{\ell+1}|$ for every $\ell \in [\![1, \widetilde{p}-1]\!]$.

For $\ell \in [\![1,\widetilde{p}]\!]$, the dimension of the eigenspace of $A^*_{K_0}$ associated with γ_ℓ is denoted by N_ℓ, and the size of its Jordan chains by $\widetilde{\tau}_{\ell,j}$, $j \in [\![1, N_\ell]\!]$. Since we assumed that $\tau_{l,j} = \tau_l$ it follows that $\widetilde{\tau}_{\ell,j} = \widetilde{\tau}_\ell$ is also independent of j. Finally, we set $\widehat{N} = \max_{\ell \in [\![1,\widetilde{p}]\!]} N_\ell$.

We choose to arrange the eigenvalues $\{\Lambda_k\}_{k \geq 1} \subset \mathbb{C}$ of the operator $-(\Delta + A^*)$ as follows :

$$\begin{cases} \Lambda_\ell = -\gamma_\ell, \quad \text{for } \ell \in [\![1,\widetilde{p}]\!], \\ \Lambda_{\widetilde{p}+i} = \lambda_{K_0+j} - \mu_l, \quad \text{with} \quad j = \left\lfloor \frac{i-1}{p} \right\rfloor + 1 \text{ and } l = i - \left\lfloor \frac{i-1}{p} \right\rfloor p, \quad \text{for } i \geq 1. \end{cases}$$

Observe that the sequence $\{\Lambda_k\}_{k \geq 1}$ satisfies the assumptions $(\mathcal{H}_1)$-$(\mathcal{H}_5)$ of Theorem 4.7 :

- $(\mathcal{H}_1)$ follows from (4.23).
- $(\mathcal{H}_2)$ holds because the matrix A is stable (see Remark 4.1).
- $(\mathcal{H}_3)$ is clear since $|\Im(\Lambda_k)| \leq \max_{l \in [\![1,p]\!]} |\Im(\mu_l)|$ and $\Re(\Lambda_k) \geq \lambda_1 - \max_{l \in [\![1,p]\!]} \Re(\mu_l)$ (which is positive since A^* is stable).
- $(\mathcal{H}_4)$ is a consequence of (4.24) and (4.25).
- Finally, let us show that $(\mathcal{H}_5)$ holds for q large enough. Let $k = \widetilde{p} + i_k$ and $n = \widetilde{p} + i_n$ (the case $k \leq \widetilde{p}$ or $n \leq \widetilde{p}$ is simpler). Let j_k, j_n and l_k, l_n be such that $\Lambda_k = \lambda_{K_0+j_k} - \mu_{l_k}$ and $\Lambda_n = \lambda_{K_0+j_n} - \mu_{l_n}$. We have

$$\begin{aligned} |\Lambda_n - \Lambda_k|^2 = |\lambda_{K_0+j_k} - \lambda_{K_0+j_n} + \mu_{l_n} - \mu_{l_k}|^2 &\geq \Big| |\lambda_{K_0+j_k} - \lambda_{K_0+j_n}| - |\mu_{l_n} - \mu_{l_k}| \Big|^2 \\ &\geq |\lambda_{K_0+j_k} - \lambda_{K_0+j_n}|^2 - 2\,|\lambda_{K_0+j_k} - \lambda_{K_0+j_n}|\,|\mu_{l_n} - \mu_{l_k}| + |\mu_{l_n} - \mu_{l_k}|^2 . \end{aligned}$$

Let us denote $m = \min_{\substack{1 \leq l,l' \leq p \\ l \neq l'}} |\mu_l - \mu_{l'}|$, $M = \max_{\substack{1 \leq l,l' \leq p \\ l \neq l'}} |\mu_l - \mu_{l'}|$, $d = |j_k - j_n|$, $s = j_k + j_n$ and $x = d(s + 2K_0)$. Thus,

$$|\Lambda_n - \Lambda_k|^2 \geq x^2 - 2Mx + m.$$

On the other hand, since $|i_k - i_n| < p(|j_k - j_n| + 1)$ and $i_k + i_n \leq p(j_k + j_n) + 2$, we have

$$\left|k^2 - n^2\right|^2 = |i_k - i_n|^2 (i_k + i_n + 2\widetilde{p})^2 \leq p^2 (d+1)^2 (sp + 2 + 2\widetilde{p})^2$$

By assumption $d, s \to +\infty$, so that

$$\left|k^2 - n^2\right|^2 \leq C d^2 (s + 2K_0)^2 = Cx^2.$$

Taking for instance $\rho = 1/\sqrt{2C}$ and x large enough we obtain the first property of $(\mathcal{H}_5)$. The second property is actually satisfied for any q.

The counting function We recall that the counting function $\mathcal{N}$ associated with the sequence $\{\Lambda_k\}_{k\geq 1}$ is given by

$$\mathcal{N}(r) = \#\left\{k : |\Lambda_k| \leq r\right\}, \quad \forall r > 0.$$

This function $\mathcal{N}$ is piecewise constant and non-decreasing on the interval $[0, +\infty)$. Thanks to $(\mathcal{H}_5)$ we have $\lim_{k\to+\infty} |\Lambda_k| = +\infty$, so that $\mathcal{N}(r) < +\infty$ for every $r \in [0, +\infty)$ and $\lim_{r\to+\infty} \mathcal{N}(r) = +\infty$. Moreover, $(\mathcal{H}_4)$ shows that, for every $r > 0$, we have

$$\mathcal{N}(r) = n \Longleftrightarrow \left(|\Lambda_n| \leq r \text{ and } |\Lambda_{n+1}| > r\right), \tag{4.26}$$

so that, in particular, we have

$$\sqrt{\left|\Lambda_{\mathcal{N}(r)}\right|} \leq \sqrt{r} < \sqrt{\left|\Lambda_{\mathcal{N}(r)+1}\right|}.$$

On the other hand, from the very definition of Λ_k for $k > \widetilde{p}$, we have

$$\left(\frac{\mathcal{N}(r)}{p} + \widetilde{K_0}\right)^2 - M \leq \left|\Lambda_{\mathcal{N}(r)}\right| \leq \left(\frac{\mathcal{N}(r)}{p} + \widetilde{\widetilde{K_0}}\right)^2 + M, \quad \text{for any } r \text{ s.t. } \mathcal{N}(r) > \widetilde{p},$$

where $M = \max_{l\in[\![1,p]\!]} |\mu_l|$, $\widetilde{K_0} = K_0 - \frac{\widetilde{p}+1}{p} + 1$ and $\widetilde{\widetilde{K_0}} = \widetilde{K_0} + 1$. Combining the two previous estimates, it is not difficult to obtain the last assumption $(\mathcal{H}_6)$ of Theorem 4.7.

4.3.2 The moment problem

In [AKBGBdT11a] it has been proved (Proposition 5.1) that, under the assumption (4.5), System (4.9) is null-controllable at time T if for every $q \in [\![1, \widehat{N}]\!]$ there exists a solution $u_q \in L^2(0,T)$ to the moments problem

$$\left\{\begin{array}{l} \displaystyle\int_0^T \frac{t^\nu}{\nu!} e^{\overline{\gamma_\ell} t}\, u_q(t)\, dt = c_{\ell,\nu,q}(y_0;T), \quad \forall \ell \in [\![1,\widetilde{p}]\!], \forall \nu \in [\![0,\widetilde{\tau}_\ell - 1]\!], \\ \displaystyle\int_0^T \frac{t^\sigma}{\sigma!} e^{(-\lambda_k+\overline{\mu_l})t}\, u_q(t)\, dt = d^k_{l,\sigma,q}(y_0;T), \quad \forall k > K_0, \forall l \in [\![1,p]\!], \forall \sigma \in [\![0,\tau_l - 1]\!], \end{array}\right. \tag{4.27}$$

where $c_{\ell,\nu,q}$ and $d^k_{l,\sigma,q}$ are given in [AKBGBdT11a, Proposition 5.1]. The precise definition of those terms is not really important here, however we recall that they satisfy the following estimates (see [AKBGBdT11a, Equations (49) and (52)])

$$\begin{aligned} |c_{\ell,\nu,q}(y_0;T)| &\leq C\left\|e^{A^*_{K_0}T}\right\|_{\mathcal{M}_{nK_0}(\mathbb{R})} \|y_0\|_{H^{-1}(0,\pi)^n} \\ &\leq Ce^{CT}\|y_0\|_{H^{-1}(0,\pi)^n}, \end{aligned} \tag{4.28}$$

and

$$\begin{aligned} \left|d^k_{l,\sigma,q}(y_0;T)\right| &\leq \frac{C}{k}\left\|e^{(-\lambda_k+A^*)T}\right\|_{\mathcal{M}_n(\mathbb{R})} \left|\langle y_0, \phi_k\rangle_{H^{-1},H^1_0(0,\pi)}\right|_{\mathbb{C}^n} \\ &\leq Ce^{CT}\frac{\sqrt{\lambda_k}}{k}e^{-\lambda_k T}\|y_0\|_{H^{-1}(0,\pi)^n}. \end{aligned} \tag{4.29}$$

The control $v(t)$ is then given as a linear combination of $u_q(T-t)$, $q \in [\![1, \widehat{N}]\!]$, and as a result satisfies

$$\|v\|_{L^2(0,T)^m} \le C \max_{q\in[\![1,\widehat{N}]\!]} \|u_q\|_{L^2(0,T)}. \tag{4.30}$$

Assume for the moment that Theorem 4.7 is proved. Let $T_0 > 0$ be the time given by Theorem 4.7 and set

$$\eta = \max\{\tau_l, \widetilde{\tau}_\ell, \mid l \in [\![1,p]\!],\ \ell \in [\![1,\widetilde{p}]\!]\}.$$

For $T < T_0$ we can then introduce the biorthogonal family $\{\varphi_{k,j}\}_{k\ge1, j\in[\![0,\eta-1]\!]} \subset L^2(-T/2, T/2)$ associated with the sequence $\{\Lambda_k\}_{k\ge1}$. As we need to work on the interval $(-T/2, T/2)$, we perform the change of variable $s = t - \frac{T}{2}$ in (4.27) and obtain

$$\begin{cases} \displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{\nu!}\left(s+\frac{T}{2}\right)^\nu e^{\overline{\gamma_\ell} s}\, u_q\left(s+\frac{T}{2}\right) ds = e^{-\frac{T}{2}\overline{\gamma_\ell}} c_{\ell,\nu,q}(y_0;T), \quad \forall \ell \in [\![1,\widetilde{p}]\!], \forall \nu \in [\![0, \widetilde{\tau}_\ell - 1]\!], \\ \displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{\sigma!}\left(s+\frac{T}{2}\right)^\sigma e^{(-\lambda_k+\overline{\mu_l})s}\, u_q\left(s+\frac{T}{2}\right) ds = e^{-(-\lambda_k+\overline{\mu_l})\frac{T}{2}} d^k_{l,\sigma,q}(y_0;T), \quad \begin{cases} \forall k > K_0, \\ \forall l \in [\![1,p]\!], \\ \forall \sigma \in [\![0, \tau_l - 1]\!]. \end{cases} \end{cases}$$

Using the binomial formula $\left(s+\frac{T}{2}\right)^J = \sum_{j=0}^J \binom{J}{j} s^{J-j}\left(\frac{T}{2}\right)^j$ we finally have

$$\begin{cases} \displaystyle\sum_{j=0}^{\nu}\binom{\nu}{j}\left(\frac{T}{2}\right)^j \int_{-\frac{T}{2}}^{\frac{T}{2}} s^{\nu-j} e^{\overline{\gamma_\ell} s}\, u_q\left(s+\frac{T}{2}\right) ds = \widehat{c_{\ell,\nu,q}}(y_0;T), \quad \forall \ell \in [\![1,\widetilde{p}]\!], \forall \nu \in [\![0, \widetilde{\tau}_\ell - 1]\!], \\ \displaystyle\sum_{j=0}^{\sigma}\binom{\sigma}{j}\left(\frac{T}{2}\right)^j \int_{-\frac{T}{2}}^{\frac{T}{2}} s^{\sigma-j} e^{(-\lambda_k+\overline{\mu_l})s}\, u_q\left(s+\frac{T}{2}\right) ds = \widehat{d^k_{l,\sigma,q}}(y_0;T), \quad \begin{cases} \forall k > K_0, \\ \forall l \in [\![1,p]\!], \\ \forall \sigma \in [\![0, \tau_l - 1]\!]. \end{cases} \end{cases}$$

with

$$\widehat{c_{\ell,\nu,q}}(y_0;T) = \nu! e^{-\frac{T}{2}\overline{\gamma_\ell}} c_{\ell,\nu,q}(y_0;T), \quad \widehat{d^k_{l,\sigma,q}}(y_0;T) = \sigma! e^{-(-\lambda_k+\overline{\mu_l})\frac{T}{2}} d^k_{l,\sigma,q}(y_0;T). \tag{4.31}$$

For $T < T_0$, a solution to the moments problem (4.27) is then given for every $t \in (0,T)$ by (note that $-\lambda_k + \mu_l = \Lambda_{\widetilde{p}+(k-K_0-1)p+l}$ for $k > K_0$)

$$u_q(t) = \sum_{\ell=1}^{\widetilde{p}} \sum_{\nu=0}^{\widetilde{\tau}_\ell - 1} \widehat{\widehat{c_{\ell,\nu,q}}}(y_0;T)\varphi_{\ell,\nu}\left(t-\frac{T}{2}\right) \\ + \sum_{k>K_0}\sum_{l=1}^{p}\sum_{\sigma=0}^{\tau_l-1} \widehat{\widehat{d^k_{l,\sigma,q}}}(y_0;T)\varphi_{\widetilde{p}+(k-K_0-1)p+l,\sigma}\left(t-\frac{T}{2}\right),$$

provided that u_q lies in $L^2(0,T)$ (see below), and where $\widehat{\widehat{c_{\ell,\nu,q}}}$ and $\widehat{\widehat{d^k_{l,\sigma,q}}}$ solve the triangular systems

$$P(T)\begin{pmatrix}\widehat{\widehat{c_{\ell,0,q}}}\\ \vdots \\ \widehat{\widehat{c_{\ell,\widetilde{\tau_\ell}-1,q}}}\end{pmatrix}=\begin{pmatrix}\widehat{c_{\ell,0,q}}\\ \vdots \\ \widehat{c_{\ell,\nu,q}}\end{pmatrix},\quad Q(T)\begin{pmatrix}\widehat{\widehat{d^k_{l,0,q}}}\\ \vdots \\ \widehat{\widehat{d^k_{l,\tau_l-1,q}}}\end{pmatrix}=\begin{pmatrix}\widehat{d^k_{l,0,q}}\\ \vdots \\ \widehat{d^k_{l,\tau_l-1,q}}\end{pmatrix},$$

where the coefficients of $P(T)$ and $Q(T)$ are respectively given for $i \geq j$ by $p_{ij}(T) = \binom{i-1}{j-1}\left(\frac{T}{2}\right)^{i-j}$, $q_{ij}(T) = \binom{i-1}{j-1}\left(\frac{T}{2}\right)^{i-j}$ and $p_{ij}(T) = q_{ij}(T) = 0$ otherwise. Observe that

$$\left\|P(T)^{-1}\right\|_{\mathcal{M}_{\widetilde{\tau_\ell}-1}(\mathbb{R})} \leq CT^{\widetilde{\tau_\ell}-1},\quad \left\|Q(T)^{-1}\right\|_{\mathcal{M}_{\tau_l-1}(\mathbb{R})} \leq CT^{\tau_l-1}.$$

From this, the definition (4.31) of $\widehat{c_{\ell,\nu,q}}$ and $\widehat{d^k_{l,\sigma,q}}$, and the estimates (4.28) and (4.29) of $c_{\ell,\nu,q}$ and $d^k_{l,\sigma,q}$, we obtain

$$\left|\widehat{\widehat{c_{\ell,\nu,q}}}(y_0;T)\right| \leq CT^{\widetilde{\tau_\ell}-1}\left|e^{-\frac{T}{2}\overline{\gamma_\ell}}\right| e^{CT}\|y_0\|_{H^{-1}(0,\pi)^n} \leq Ce^{CT}\|y_0\|_{H^{-1}(0,\pi)^n}, \tag{4.32}$$

and

$$\begin{aligned}\left|\widehat{\widehat{d^k_{l,\sigma,q}}}(y_0;T)\right| &\leq CT^{\tau_l-1}\left|e^{-(-\lambda_k+\overline{\mu_l})\frac{T}{2}}\right|\frac{\sqrt{\lambda_k}}{k}e^{CT}e^{-\lambda_k T}\|y_0\|_{H^{-1}(0,\pi)^n},\\ &\leq Ce^{CT}\frac{\sqrt{\lambda_k}}{k}e^{-\lambda_k\frac{T}{2}}\|y_0\|_{H^{-1}(0,\pi)^n}.\end{aligned}\tag{4.33}$$

It remains to prove that $u_q \in L^2(0,T)$ and to estimate its norm with respect to T and y_0. This is actually thanks to the estimate (4.12) that this latter can be achieved. Indeed, using also (4.32) and (4.33) we have

$$\begin{aligned}\|u_q\|_{L^2(0,T)} &\leq Ce^{CT}\sum_{\ell=1}^{\widetilde{p}} e^{C\sqrt{-\Re(\gamma_\ell)}+\frac{C}{T}}\|y_0\|_{H^{-1}(0,\pi)^n},\\ &\quad +Ce^{CT}\sum_{k>K_0}\frac{\sqrt{\lambda_k}}{k}e^{-\lambda_k\frac{T}{2}}\sum_{l=1}^{p}e^{C\sqrt{\lambda_k-\Re(\mu_l)}+\frac{C}{T}}\|y_0\|_{H^{-1}(0,\pi)^n},\\ &\leq Ce^{CT+\frac{C}{T}}\left(1+\sum_{k>K_0}\frac{\sqrt{\lambda_k}}{k}e^{-\lambda_k\frac{T}{2}+C\sqrt{\lambda_k}}\right)\|y_0\|_{H^{-1}(0,\pi)^n}.\end{aligned}\tag{4.34}$$

Let us now estimate the series. Young's inequality gives

$$C\sqrt{\lambda_k} \leq \lambda_k\frac{T}{4}+\frac{C^2}{T},$$

for every $k \geq 1$ and $T > 0$, so that

$$-\lambda_k\frac{T}{2}+C\sqrt{\lambda_k} \leq -\lambda_k\frac{T}{4}+\frac{C^2}{T}.$$

Thus, using also that $\lambda_k = k^2$, we obtain

$$\sum_{k>K_0} \frac{\sqrt{\lambda_k}}{k} e^{-\lambda_k \frac{T}{2} + C\sqrt{\lambda_k}} \leq e^{\frac{C}{T}} \sum_{k\geq 0} e^{-k^2 \frac{T}{4}}.$$

A comparison with the Gauss integral gives

$$\sum_{k\geq 0} e^{-k^2 \frac{T}{4}} \leq 2\sqrt{\frac{4\pi}{T}} \leq Ce^{\frac{C}{T}}.$$

Coming back to (4.34) we then have

$$\|u_q\|_{L^2(0,T)} \leq Ce^{CT+\frac{C}{T}} \|y_0\|_{H^{-1}(0,\pi)^n}.$$

Finally, (4.30) gives, for every $T < T_0$,

$$\|v\|_{L^2(0,T)} \leq Ce^{\frac{C}{T}} \|y_0\|_{H^{-1}(0,\pi)^n}.$$

Thus, when $T < T_0$ we have obtained a null-control to System (4.9) which satisfies the desired estimate. The case $T \geq T_0$ is actually reduced to the previous one. Indeed, any continuation by zero of a control on $(0, T_0/2)$ is a control on $(0, T)$ and the estimate follows from the decrease of the cost with respect to the time.

4.4 Biorthogonal families to complex matrix exponentials.

This section is devoted to the proof of Theorem 4.7.

4.4.1 Idea of the proof

For any $\eta \geq 1$ and T small enough (depending on η), we have to construct a family $\{\varphi_{k,j}\}_{k\geq 1, j\in [\![0,\eta-1]\!]}$ in $L^2(-T/2, T/2)$ such that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \varphi_{k,j}(t) t^\nu e^{-\overline{\Lambda_l} t}\, dt = \delta_{kl}\delta_{j\nu},$$

for every $k, l \geq 1$ and $j, \nu \in [\![0, \eta - 1]\!]$, with in addition the following bound

$$\|\varphi_{k,j}\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)} \leq Ce^{C\sqrt{\Re(\Lambda_k)}+\frac{C}{T}},$$

for any $k \geq 1$ and $j \in [\![0, \eta - 1]\!]$.

The idea is to use the Fourier transform with the help of the Paley-Wiener theorem (see [Rud74, Theorem 19.3]) that we recall here.

Theorem 4.12. *Let Φ be an entire function of exponential type $T/2$ (that is $|\Phi(z)| \leq Ce^{\frac{T}{2}|z|}$ for all $z \in \mathbb{C}$ [3]) such that*

$$\|\Phi\|^2_{L^2(-\infty,+\infty)} = \int_{-\infty}^{+\infty} |\Phi(x)|^2 \, dx < +\infty.$$

Then, there exists $\varphi \in L^2(-T/2, T/2)$ such that

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{T}{2}}^{\frac{T}{2}} \varphi(t) e^{itz} \, dt, \quad \forall z \in \mathbb{C}. \tag{4.35}$$

Moreover, the Plancherel theorem gives

$$\|\varphi\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)} = \|\Phi\|_{L^2(-\infty,+\infty)}.$$

Observe that the function in (4.35) is infinitely derivable on $\mathbb{C}$ with, for every $\nu \in [\![0, \eta - 1]\!]$,

$$\Phi^{(\nu)}(z) = \frac{i^\nu}{\sqrt{2\pi}} \int_{-\frac{T}{2}}^{\frac{T}{2}} \varphi(t) t^\nu e^{itz} \, dt, \quad \forall z \in \mathbb{C}.$$

Thus, Theorem 4.7 will be proved if we manage to build suitable entire functions as stated in the following result.

Theorem 4.13. *Assume that the sequence $\{\Lambda_k\}_{k\geq 1} \subset \mathbb{C}$ satisfies the assumptions ($\mathcal{H}_1$)-($\mathcal{H}_6$).*

There exists $T_0 > 0$ such that, for any $\eta \geq 1$ and $0 < T < T_0$, there exists a family $\{\Phi_{k,j}\}_{k\geq 1, j \in [\![0,\eta-1]\!]}$ of entire functions of exponential type $T/2$ satisfying

$$\Phi^{(\nu)}_{k,j}(i\overline{\Lambda_l}) = \frac{i^\nu}{\sqrt{2\pi}} \delta_{kl} \delta_{j\nu}, \quad \forall k, l \geq 1, \quad \forall j, \nu \in [\![0, \eta - 1]\!], \tag{4.36}$$

and

$$\|\Phi_{k,j}\|_{L^2(-\infty,+\infty)} \leq Ce^{C\sqrt{\Re(\Lambda_k)} + \frac{C}{T}}, \tag{4.37}$$

for any $k \geq 1$ and $j \in [\![0, \eta - 1]\!]$.

Remark 4.14. *A sequence $\{\Lambda_k\}_{k\geq 1} \subset \mathbb{C}$ satisfies the assumptions ($\mathcal{H}_1$)-($\mathcal{H}_6$) if and only if so does the sequence $\left\{\overline{\Lambda_k}\right\}_{k\geq 1}$. For this reason, and commodity, we will prove Theorem 4.13 for the sequence $\left\{\overline{\Lambda_k}\right\}_{k\geq 1}$.*

3. Here and only here, C may even depend on T without affecting the result.

4.4.2 Proof of Theorem 4.13

Some preliminary remarks It is interesting to point out some properties of the sequence $\{\Lambda_k\}_{k\geq 1}$ which can be deduced from assumptions $(\mathcal{H}_3)$, $(\mathcal{H}_4)$ and $(\mathcal{H}_6)$.

1. First, under assumptions $(\mathcal{H}_4)$ and $(\mathcal{H}_6)$ we have that
$$\sum_{k\geq 1} \frac{1}{|\Lambda_k|} < +\infty. \tag{4.38}$$
Indeed, using that $\mathcal{N}$ is piecewise constant and non-decreasing on the interval $[0, +\infty)$, we can write
$$\sum_{k\geq 1} \frac{1}{|\Lambda_k|} = \int_{|\Lambda_1|^-}^{+\infty} \frac{1}{r}\, d\mathcal{N}(r) = \int_{|\Lambda_1|}^{+\infty} \frac{1}{r^2}\mathcal{N}(r)\, dr$$
$$\leq \int_{|\Lambda_1|}^{+\infty} \frac{\alpha + p\sqrt{r}}{r^2}\, dr = \frac{\alpha}{|\Lambda_1|} + \frac{2p}{\sqrt{|\Lambda_1|}} < +\infty.$$

2. Then, from assumption $(\mathcal{H}_3)$ we can also deduce the following behavior of the sequence $\{\Lambda_k\}_{k\geq 1}$
$$|\Lambda_k| - \Re(\Lambda_k) \leq \beta\sqrt{\Re(\Lambda_k)} \quad \text{and} \quad |\Lambda_k| \leq C\Re(\Lambda_k), \quad \forall k \geq 1. \tag{4.39}$$
Indeed, one has
$$|\Lambda_k|^2 = \Re(\Lambda_k)^2 + \Im(\Lambda_k)^2 \leq \Re(\Lambda_k)^2 + \beta^2\Re(\Lambda_k) \leq \left(\Re(\Lambda_k) + \beta\sqrt{\Re(\Lambda_k)}\right)^2.$$

Let us now introduce the complex functions given, for every $z \in \mathbb{C}$, by
$$f(z) = \prod_{k\geq 1}\left(1 - \frac{z}{\Lambda_k}\right), \quad f_n(z) = \prod_{\substack{k\geq 1\\ k\neq n}}\left(1 - \frac{z}{\Lambda_k}\right). \tag{4.40}$$

Thanks to (4.38), the previous products are uniformly convergent on compact sets of $\mathbb{C}$ and therefore f and f_n are entire functions. Moreover, the zeros of f and f_n are exactly $\{\Lambda_k\}_{k\geq 1}$ and $\{\Lambda_k\}_{k\neq n}$ and they are zeros of multiplicity 1 (recall that the Λ_k are distinct by $(\mathcal{H}_1)$). For a proof of these facts we refer to [Rud74, Theorem 15.4].

On the other hand, let us fix $d = p\pi + 2$. For any $\tau > 0$ such that $\tau < d^2/2$ we define the real positive sequence $\{a_n\}_{n\geq 0}$ given by
$$a_n = \frac{d^2}{\tau^2} + \frac{4\,(n^2-1)}{d^2}, \quad \forall n \geq 0, \tag{4.41}$$
To this sequence we associate a complex function M defined by
$$M(z) = \prod_{n\geq 1} \frac{\sin(z/a_n)}{z/a_n}, \quad \forall z \in \mathbb{C}. \tag{4.42}$$

Since
$$\left|\frac{\sin(z)}{z}\right| \leq e^{|z|}, \quad \forall z \in \mathbb{C},$$
and $a_n \underset{+\infty}{\sim} Cn^2$, the previous product is uniformly convergent on compact sets of $\mathbb{C}$ and M is an entire function of exponential type $\tau_M > 0$, where
$$\tau_M = \sum_{n\geq 1} \frac{1}{a_n} < +\infty. \tag{4.43}$$
More precisely, M satisfies
$$|M(z)| \leq e^{\tau_M |z|}, \quad \forall z \in \mathbb{C}. \tag{4.44}$$
Observe that there is no constant in front of the term $e^{\tau_M|z|}$. This point will be very important in the sequel (see the proof of Proposition 4.15 in the appendix 4.5) to obtain estimates with constants C that do not depend on τ (which will play the role of T, see below). Note also that M has only real zeros since $\{a_n\}_{n\geq 1}$ is a real sequence. Finally, we will often use that $\tau_M < \tau$. This fact is proved in Lemma 4.17 in the appendix 4.5.

Proof of Theorem 4.13 We follow some techniques developed in [AKBGBdT11a] (see in particular Lemma 4.4 in this reference).

Set $T_0 = d^2$ and, for any $0 < T < T_0$, set $\tau = \frac{T}{2\eta}$, in such a way that the condition $\tau < d^2/2$ holds. The function M defined above will then correspond to this value of τ.

Let us consider the functions
$$\begin{cases} \Phi_k(z) = \dfrac{1}{\eta!}\left[W_k(z)\right]^\eta, \quad W_k(z) = \dfrac{f(-iz)}{-if'(\Lambda_k)} \dfrac{M(z+\Im(\Lambda_k))}{M(i\Re(\Lambda_k))}, \\ \widetilde{\Phi}_k(z) = \dfrac{1}{\eta!}\left[\widetilde{W}_k(z)\right]^\eta \quad \widetilde{W}_k(z) = \dfrac{f_k(-iz)}{-if'(\Lambda_k)} \dfrac{M(z+\Im(\Lambda_k))}{M(i\Re(\Lambda_k))}, \end{cases} \tag{4.45}$$
defined for every $z \in \mathbb{C}$ and $k \geq 1$.

Let us already give some estimates for the functions W_k, $\widetilde{W}_k$ (and as result also for Φ_k and $\widetilde{\Phi}_k$) that will be used later :

Proposition 4.15. *Assume that the sequence $\{\Lambda_k\}_{k\geq 1}$ satisfies the assumptions ($\mathcal{H}_1$)-($\mathcal{H}_6$), and let $\tau < d^2/2$. Then, for any $k \geq 1$ and $z \in \mathbb{C}$,*
$$|W_k(z)| + |\widetilde{W}_k(z)| \leq e^{C\sqrt{|z|}+\tau_M(|z|-\Re(\Lambda_k))+C\sqrt{\Re(\Lambda_k)}+\frac{C}{\tau}}. \tag{4.46}$$
On the other hand, for any $k \geq 1$ and $x \in \mathbb{R}$,
$$|W_k(x)| + |\widetilde{W}_k(x)| \leq e^{-\sqrt{|x|}+C\sqrt{\Re(\Lambda_k)}+\frac{C}{\tau}}. \tag{4.47}$$

The proof of this rather technical proposition is given in the appendix 4.5. For now, let us continue with the proof of Theorem 4.13.

Since the function M only has real zeros, all the functions introduced in (4.45) are well-defined and they are entire functions. For every $l \geq 1$, $i\Lambda_l$ is a simple zero of the function W_k since Λ_l is a simple zero of f and $i\Lambda_l + \Im(\Lambda_k)$ is not a zero of M ($\Im[i\Lambda_l + \Im(\Lambda_k)] = \Re(\Lambda_l) \neq 0$ by $(\mathcal{H}_2)$). Thus, we deduce that, for every $l \geq 1$, $i\Lambda_l$ is a zero of Φ_k with exact multiplicity η, i.e.,

$$\Phi_k^{(\eta)}(i\Lambda_l) = [W_k'(i\Lambda_l)]^\eta \neq 0 \quad \text{and} \quad \Phi_k^{(\nu)}(i\Lambda_l) = 0, \quad \forall k, l \geq 1, \quad \forall \nu \in [\![0, \eta - 1]\!].$$

Observe that, in particular $\Phi_k^{(\eta)}(i\Lambda_k) = 1$. At this point, the function $\Phi_{k,j} = \Phi_k$ then satisfies (4.36) for $l \neq k$.

For any $k \geq 1$, $j \in [\![0, \eta - 1]\!]$ and $z \in \mathbb{C}$, let us now set

$$f_{k,j}(z) = \frac{\Phi_k(z)}{(z - i\Lambda_k)^{\eta - j}} = \left(\frac{-1}{i\Lambda_k}\right)^\eta \widetilde{\Phi}_k(z)(z - i\Lambda_k)^j.$$

Note that, for $x \in \mathbb{R}$, we deduce from (4.47), $(\mathcal{H}_4)$ and (4.39), that

$$|f_{k,j}(x)| \leq Ce^{-\frac{\eta}{2}\sqrt{|x|} + C\sqrt{\Re(\Lambda_k)} + \frac{C}{\tau}}. \tag{4.48}$$

From the properties of the function Φ_k, we get

$$\begin{cases} f_{k,j}^{(\nu)}(i\Lambda_l) = 0, & \forall l \geq 1 \text{ with } l \neq k, \ \forall \nu \in [\![0, \eta - 1]\!], \\ f_{k,j}^{(\nu)}(i\Lambda_k) = 0, & \forall \nu \in [\![0, j - 1]\!], \\ f_{k,j}^{(j+r)}(i\Lambda_k) = \dfrac{(j + r)!}{(\eta + r)!}\Phi_k^{(\eta + r)}(i\Lambda_k), & \forall r \geq 0. \end{cases} \tag{4.49}$$

We look now for $\Phi_{k,j}$ in the following form

$$\Phi_{k,j}(z) = p(z) f_{k,j}(z),$$

with p a polynomial function of degree $\eta - j - 1$ which depends on k, j (for simplicity, this dependance is omitted in the notation).

As a consequence of inequality (4.46) and the fact that $\tau_M < \tau$, the function $\Phi_{k,j}$ is an entire function of exponential type $\eta\tau = T/2$. [4]

In view of (4.49), if we simply take $p = 1$, then the relations (4.36) are satisfied for $l \neq k$ and $l = k$ if $\nu < j$. Thus, in order to get (4.36), we have to choose p such that $\Phi_{k,j}^{(j)}(i\Lambda_k) = \frac{i^j}{\sqrt{2\pi}}$ and $\Phi_{k,j}^{(j+r)}(i\Lambda_k) = 0$ for $r \in [\![1, \eta - j - 1]\!]$, that is

$$\begin{cases} p(i\Lambda_k) = \dfrac{i^j}{\sqrt{2\pi}} \dfrac{1}{f_{k,j}^{(j)}(i\Lambda_k)} = \dfrac{i^j}{\sqrt{2\pi}} \dfrac{\eta!}{j!}, \\ \displaystyle\sum_{\ell=0}^{r-1} a_{r\ell} p^{(\ell)}(i\Lambda_k) + p^{(r)}(i\Lambda_k) = 0, \quad \forall r \in [\![1, \eta - j - 1]\!], \end{cases} \tag{4.50}$$

4. the constant C such that $|\Phi_{k,j}(z)| \leq Ce^{\eta\tau|z|}$ for every $z \in \mathbb{C}$ depends on k, j, τ, etc... but this is not important as mentioned earlier.

where

$$a_{r\ell} = \frac{\binom{j+r}{l}}{\binom{j+r}{r}} \frac{f_{k,j}^{(j+r-\ell)}(i\Lambda_k)}{f_{k,j}^{(j)}(i\Lambda_k)} = \frac{r!\eta!}{\ell!(\eta+r-\ell)!} \Phi_k^{(\eta+r-\ell)}(i\Lambda_k), \tag{4.51}$$

for every $r \in [\![1, \eta - j - 1]\!]$ and $\ell \in [\![0, r-1]\!]$ (they are well-defined since $f_{k,j}^{(j)}(i\Lambda_k) \neq 0$).

These relations allow us to compute $p^{(r)}(i\Lambda_k)$ for every $r \in [\![0, \eta - j - 1]\!]$ and thus completely determine p which is then given by

$$p(z) = \sum_{r=0}^{\eta-j-1} \frac{p^{(r)}(i\Lambda_k)}{r!}(z - i\Lambda_k)^r.$$

In order to get the bound (4.37) for $\Phi_{k,j}$, let us prove some estimates of the polynomial p previously constructed. If we set $P = \left(p^{(r)}(i\Lambda_k)\right)_{r\in[\![0,\eta-j-1]\!]} \in \mathbb{C}^{\eta-j}$, then we can rewrite the identities in (4.50) as a linear system of the form $\mathbf{A}P = \mathbf{B}$ with

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ a_{10} & 1 & \ddots & & \vdots \\ a_{20} & a_{21} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_{\eta-j-1,0} & a_{\eta-j-1,1} & \cdots & a_{\eta-j-1,\eta-j-2} & 1 \end{pmatrix} \in \mathcal{M}_{\eta-j}(\mathbb{R}), \ \mathbf{B} = \begin{pmatrix} \frac{i^j}{\sqrt{2\pi}} \frac{\eta!}{j!} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{C}^{\eta-j},$$

and $a_{r\ell}$ given in (4.51). Again, following [AKBGBdT11a, Eq. (31),p. 570], it is possible to show

$$|P|_{\mathbb{C}^{\eta-j}} \leq C \left(\sum_{\substack{r\in[\![1,\eta-j-1]\!] \\ \ell\in[\![0,r]\!]}} \left| \Phi_k^{(\eta+r-\ell)}(i\Lambda_k) \right|^2 \right)^{\frac{\eta-j-1}{2}}. \tag{4.52}$$

Finally, let us estimate $|\Phi_k^{(\eta+r-\ell)}(i\Lambda_k)|$, for $r \in [\![1, \eta-j-1]\!]$ and $\ell \in [\![0, r]\!]$. Since Φ_k is an entire function, we can write

$$\Phi_k^{(m)}(i\Lambda_k) = \frac{m!}{2i\pi} \int_{|z-i\Lambda_k|=1} \frac{\Phi_k(z)}{(z-i\Lambda_k)^{m+1}}\, dz, \quad \forall m \geq 0,$$

so that

$$\left| \Phi_k^{(m)}(i\Lambda_k) \right| \leq C \sup_{z:|z-i\Lambda_k|=1} |\Phi_k(z)|.$$

Using inequality (4.46), the fact that $|z| \leq 1 + |\Lambda_k|$ for z such that $|z - i\Lambda_k| = 1$, inequalities (4.39), and the fact that $\tau_M < d^2/2$, we obtain

$$\left| \Phi_k^{(m)}(i\Lambda_k) \right| \leq Ce^{C\sqrt{\Re(\Lambda_k)}+\frac{C}{\tau}}, \quad \forall k \geq 1, \ \forall m \geq 0.$$

Going back to (4.52), we get

$$|P|_{\mathbb{C}^{\eta-j}} \leq Ce^{C\sqrt{\Re(\Lambda_k)}+\frac{C}{\tau}}.$$

Recall that the vector P contains the coefficients $p^{(r)}(i\Lambda_k)$ of the polynomial p. Thus, using that $|z|^r/r! \leq Ce^{\frac{\eta}{4}\sqrt{|z|}}$ for any $r \in [\![0, \eta]\!]$, and using (4.39), we obtain

$$|p(z)| \leq Ce^{\frac{\eta}{4}\sqrt{|z|}+C\sqrt{\Re(\Lambda_k)}+\frac{C}{\tau}}, \quad \forall z \in \mathbb{C}.$$

Combining the previous estimate, written for $x \in \mathbb{R}$, and (4.48) we deduce the expected bound (4.37) for $\Phi_{k,j} = pf_{k,j}$.

4.5 Proof of Proposition 4.15

We start with another property satisfied by the sequence $\{\Lambda_k\}_{k\geq 1}$, namely that it behaves as k^2.

Lemma 4.16. *Under assumptions ($\mathcal{H}_4$), ($\mathcal{H}_5$) and ($\mathcal{H}_6$), we have*

$$Ck \leq \sqrt{|\Lambda_k|} \leq C'k, \quad \forall k \geq 1. \tag{4.53}$$

The second lemma was often used.

Lemma 4.17. *Let $\tau < d^2/2$. For the function M given by* (4.42) *we have $\tau_M < \tau$ (where τ_M is given in (4.43)).*

The next lemma are devoted to give bounds of every terms involved in the definitions (4.45) of W_k and $\widetilde{W}_k$.

Lemma 4.18. *Under assumption ($\mathcal{H}_6$) we have, for every $z \in \mathbb{C}$ and $n \geq 1$,*

$$\log|f(z)| \leq (d-1)\sqrt{|z|} + C, \quad \log|f_n(z)| \leq (d-1)\sqrt{|z|} + C,$$

where f and f_n are defined in (4.40).

Lemma 4.19. *Under assumptions ($\mathcal{H}_4$), ($\mathcal{H}_5$) and ($\mathcal{H}_6$) we have, for every $n \geq 1$,*

$$\log|f'(\Lambda_n)| \geq -C\sqrt{|\Lambda_n|},$$

where f is defined in (4.40).

Lemma 4.20. *Let $\tau < d^2/2$. The function M given by* (4.42) *satisfies*

$$M(0) = 1, \quad \log|M(x)| \leq -d\sqrt{|x|} + \frac{C}{\tau}, \quad \forall x \in \mathbb{R}. \tag{4.54}$$

Lemma 4.21. *Let $\tau < d^2/2$. The function M given by* (4.42) *satisfies*

$$\log|M(iy)| \geq 0, \quad \forall y \in \mathbb{R}, \tag{4.55}$$

and also

$$\log|M(iy)| \geq \tau_M|y| - C\sqrt{|y|} - \frac{C}{\tau}, \quad \forall y \in \mathbb{R}. \tag{4.56}$$

Proof of Proposition 4.15 Let us recall the definition of W_k :

$$W_k(z) = \frac{f(-iz)}{-if'(\Lambda_k)} \frac{M(z + \Im(\Lambda_k))}{M(i\Re(\Lambda_k))}.$$

From Lemma 4.18 and 4.19 and $|\Lambda_k| \leq C\Re(\Lambda_k)$ (see (4.39)) we deduce that

$$\left| \frac{f(-iz)}{-if'(\Lambda_k)} \right| \leq e^{(d-1)\sqrt{|z|} + C\sqrt{\Re(\Lambda_k)}}. \tag{4.57}$$

On the other hand, from inequality (4.56) of Lemma 4.21 and using (4.44) we can also infer

$$\left| \frac{M(z + \Im(\Lambda_k))}{M(i\Re(\Lambda_k))} \right| \leq e^{\tau_M |z| + \tau_M (|\Im(\Lambda_k)| - \Re(\Lambda_k)) + C\sqrt{\Re(\Lambda_k)} + \frac{C}{\tau}}.$$

Note that $\tau_M |\Im(\Lambda_k)| \leq C\sqrt{\Re(\Lambda_k)}$ thanks to $(\mathcal{H}_3)$ and $\tau_M < d^2/2$. Thus, putting both inequalities together we deduce estimate (4.46) for the function W_k.

Let us now take $x \in \mathbb{R}$. Applying inequality (4.54) of Lemma 4.20 and, this time, inequality (4.55) of Lemma 4.21, we arrive to

$$\left| \frac{M(x + \Im(\Lambda_k))}{M(i\Re(\Lambda_k))} \right| \leq e^{-d\sqrt{|x|} + d\sqrt{|\Im(\Lambda_k)|} + \frac{C}{\tau}}.$$

Note that $\sqrt{|\Im(\Lambda_k)|} \leq C\sqrt{\Re(\Lambda_k)}$ by (4.39). Thus, the previous inequality together with (4.57) (written for $x \in \mathbb{R}$) provide the estimate (4.47) for $W_k(x)$, with x real.

The same reasoning provide the estimate for $\widetilde{W}_k$.

Proof of Lemma 4.16 The lower bound easily follows from $(\mathcal{H}_5)$ by taking $n = 1$.

To prove the upper bound, let us first observe that, for any k and n such that $|\Lambda_k| = |\Lambda_n|$, we have, using $(\mathcal{H}_4)$,

$$\left| \Re(\Lambda_k)^2 - \Re(\Lambda_n)^2 \right| = \left| \Im(\Lambda_k)^2 - \Im(\Lambda_n)^2 \right| \leq \beta^2 \left(\Re(\Lambda_k) + \Re(\Lambda_n) \right),$$

so that

$$|\Re(\Lambda_k) - \Re(\Lambda_n)| \leq \beta^2.$$

It follows that (using $(\mathcal{H}_4)$ again)

$$|\Lambda_k - \Lambda_n| \leq |\Re(\Lambda_k) - \Re(\Lambda_n)| + |\Im(\Lambda_k) - \Im(\Lambda_n)| \leq \beta^2 + 2\beta\sqrt{|\Lambda_k|}.$$

By using $(\mathcal{H}_5)$, and the fact that $k + n \geq k$, we obtain

$$|k - n| \leq \max \left\{ q, \frac{\beta^2 + 2\beta\sqrt{|\Lambda_k|}}{\rho k} \right\}.$$

Note that if k is such that $\frac{\beta^2+2\beta\sqrt{|\Lambda_k|}}{\rho k} \leq q$ then $\sqrt{|\Lambda_k|} \leq \left(\frac{q\rho}{2\beta}\right) k$ and we are done. Let us then deal with the k such that $\frac{\beta^2+2\beta\sqrt{|\Lambda_k|}}{\rho k} > q$.

Applying the previous estimate with $n = \mathcal{N}(|\Lambda_k|)$ (which indeed satisfies $|\Lambda_n| = |\Lambda_k|$ by (4.26) and $(\mathcal{H}_4)$), we deduce that

$$\mathcal{N}(|\Lambda_k|) \leq k + |\mathcal{N}(|\Lambda_k|) - k| \leq k + \frac{\beta^2 + 2\beta\sqrt{|\Lambda_k|}}{\rho k},$$

and by $(\mathcal{H}_6)$ we finally obtain

$$p\sqrt{|\Lambda_k|} \leq \alpha + \mathcal{N}(|\Lambda_k|) \leq k + \frac{\beta^2 + 2\beta\sqrt{|\Lambda_k|}}{\rho k}.$$

For k large enough, we obtain

$$\frac{p}{2}\sqrt{|\Lambda_k|} \leq k + \frac{\beta^2}{\rho k} \leq \left(1 + \frac{\beta^2}{\rho}\right) k,$$

and the lemma is proved.

Proof of Lemma 4.17 For the proof we will follow some ideas from [FR71] and [Mil04] (see also [Red77]). Let us consider the counting function N associated with the sequence $\{a_n\}_{n\geq 1}$ given by (4.41) :

$$N(r) = \#\{n \geq 1 : a_n \leq r\}.$$

Observe that the sequence $\{a_n\}_{n\geq 0}$ can be written as

$$a_n = a_0 + \frac{n^2}{A^2}, \quad \forall n \geq 1, \quad \text{with} \quad A = \frac{d}{2} \quad \text{and} \quad a_0 = \frac{d^2}{\tau^2} - \frac{4}{d^2},$$

and that $a_0 > 0$ since we assumed that $\tau < d^2/2$. Thus, $N(r) = 0$ for $r < a_1$, and

$$N(r) = \lfloor A\sqrt{r - a_0} \rfloor, \quad \forall r \geq a_1,$$

where we recall that $\lfloor \cdot \rfloor$ is the floor function. Note that

$$A\sqrt{r} - A\sqrt{a_0} \leq N(r) \leq A\sqrt{r}, \quad \forall r \geq 0.$$

These remarks in mind, we have

$$\begin{aligned} \tau_M = \sum_{n\geq 1} \frac{1}{a_n} &= \int_{a_1^-}^{+\infty} \frac{1}{r}\, dN(r) = \int_{a_1^-}^{+\infty} \frac{N(r)}{r^2}\, dr \leq \int_{a_1}^{+\infty} \frac{A\sqrt{r - a_0}}{r^2}\, dr \\ &< A\int_{a_1}^{+\infty} \frac{\sqrt{r}}{r^2}\, dr = \frac{2A}{\sqrt{a_1}} = \tau, \end{aligned}$$

where the last inequality is strict since $a_0 \neq 0$.

Proof of Lemma 4.18 Given $z \in \mathbb{C}$, one has

$$\log|f(z)| \leq \sum_{k\geq 1} \log\left(1 + \frac{|z|}{|\Lambda_k|}\right) = \int_{|\Lambda_1|^-}^{+\infty} \log\left(1 + \frac{|z|}{t}\right) d\mathcal{N}(t).$$

Taking into account $\lim_{t\to+\infty} \mathcal{N}(t)/t = 0$ (consequence of $(\mathcal{H}_6)$) an integration by parts gives

$$\int_{|\Lambda_1|^-}^{+\infty} \log\left(1 + \frac{|z|}{t}\right) d\mathcal{N}(t) = \int_{|\Lambda_1|^-}^{+\infty} \frac{|z|}{t(|z|+t)} \mathcal{N}(t)\, dt.$$

After the change of variable $t = |z|s$, we obtain

$$\int_{|\Lambda_1|^-}^{+\infty} \frac{|z|}{t(|z|+t)} \mathcal{N}(t)\, dt = \int_{|\Lambda_1|^-/|z|}^{+\infty} \frac{\mathcal{N}(|z|s)}{s(s+1)}\, ds.$$

From $(\mathcal{H}_6)$, we conclude that

$$\begin{aligned}\int_{|\Lambda_1|^-/|z|}^{+\infty} \frac{\mathcal{N}(|z|s)}{s(s+1)}\, ds &\leq p\sqrt{|z|} \int_{|\Lambda_1|^-/|z|}^{+\infty} \frac{1}{\sqrt{s}(s+1)}\, ds + \alpha \int_{|\Lambda_1|^-/|z|}^{+\infty} \frac{1}{s(s+1)}\, ds \\ &\leq p\pi\sqrt{|z|} + \alpha \log\left(1 + \frac{|z|}{|\Lambda_1|}\right).\end{aligned}$$

Since the function $z \in \mathbb{C} \longmapsto \alpha \log(1 + |z|/\,|\Lambda_1|) - \sqrt{|z|}$ is bounded on $\mathbb{C}$, the lemma is proved.

Repeating the arguments, we obtain the same estimate for f_n.

Proof of Lemma 4.19 For proving the result we are going to follow some ideas from [LK71] and [FR75] (see also [FCGBdT10]).

Firstly, note that

$$f'(\Lambda_n) = -\frac{1}{\Lambda_n} \prod_{k\neq n} \left(1 - \frac{\Lambda_n}{\Lambda_k}\right), \quad \forall n \geq 1. \tag{4.58}$$

Given $n \geq 1$, let us introduce the sets

$$S_1(n) = \{k \neq n : |\Lambda_k| \leq 2|\Lambda_n|\} \quad \text{and} \quad S_2(n) = \{k : |\Lambda_k| > 2|\Lambda_n|\}.$$

and the infinite product

$$\mathcal{P}_n = \prod_{k\neq n} \left|1 - \frac{\Lambda_n}{\Lambda_k}\right|. \tag{4.59}$$

Let us give a lower bound for the product $\mathcal{P}_n$. To this end, we split this product into two parts using the sets $S_1(n)$ and $S_2(n)$:

1. From the definition of $S_1(n)$ and using $(\mathcal{H}_5)$, we can write

$$\prod_{k\in S_1(n)}\left|1-\frac{\Lambda_n}{\Lambda_k}\right| = \prod_{\substack{k\in S_1(n)\\ |k-n|\geq q}}\left|\frac{\Lambda_k-\Lambda_n}{\Lambda_k}\right|\prod_{\substack{k\in S_1(n)\\ |k-n|<q}}\left|\frac{\Lambda_k-\Lambda_n}{\Lambda_k}\right|$$

$$\geq \prod_{\substack{k\in S_1(n)\\ |k-n|\geq q}}\frac{\rho}{2}\frac{|k-n|(k+n)}{|\Lambda_n|}\prod_{\substack{k\in S_1(n)\\ |k-n|<q}}\frac{1}{2}\frac{A}{|\Lambda_n|},$$

where

$$A=\inf_{k\neq n:|k-n|<q}|\Lambda_k-\Lambda_n|>0.$$

It follows that

$$\prod_{k\in S_1(n)}\left|1-\frac{\Lambda_n}{\Lambda_k}\right|\geq \prod_{k\in S_1(n)}\frac{\rho}{2}\frac{|k-n|(k+n)}{|\Lambda_n|}\prod_{\substack{k\in S_1(n)\\ |k-n|<q}}\frac{A}{\rho|k-n|(k+n)}.$$

Since

$$\prod_{\substack{k\in S_1(n)\\ |k-n|<q}}\frac{A}{\rho|k-n|}\geq\left(\frac{A}{\rho q}\right)^{2q-1},\qquad \prod_{\substack{k\in S_1(n)\\ |k-n|<q}}\frac{1}{k+n}\geq\frac{1}{(2n+q-1)^{2q-1}},\quad \forall n\geq 1,$$

we deduce that

$$\prod_{\substack{k\in S_1(n)\\ |k-n|<q}}\frac{A}{|k-n|(k+n)}\geq\frac{C}{(2n+q-1)^{2q-1}}.$$

As $|\Lambda_n|\geq Cn^2$ for every $n\geq 1$ (see (4.53)), we obtain

$$\prod_{\substack{k\in S_1(n)\\ |k-n|<q}}\frac{A}{|k-n|(k+n)}\geq\frac{C}{|\Lambda_n|^{\frac{2q-1}{2}}}.$$

Let us define $r_n=\#\{k\in S_1(n):k<n\}$ and $s_n=\#\{k\in S_1(n):k>n\}$. From (4.53), we deduce that $k+n\geq C\sqrt{|\Lambda_n|}$ for any $n,k\geq 1$. Thus, for any $n\geq 1$,

$$\prod_{k\in S_1(n)}\left|1-\frac{\Lambda_n}{\Lambda_k}\right|\geq C\,|\Lambda_n|^{-q-\frac{1}{2}}\,r_n!\left(\frac{\rho\gamma_2}{2|\Lambda_n|^{1/2}}\right)^{r_n}s_n!\left(\frac{\rho\gamma_2}{2|\Lambda_n|^{1/2}}\right)^{s_n}=C\,|\Lambda_n|^{-q-\frac{1}{2}}\,\mathcal{P}_n^{(1)}\mathcal{P}_n^{(2)}. \tag{4.60}$$

Let us argue with $\mathcal{P}_n^{(1)}$. A similar reasoning will provide a lower bound for $\mathcal{P}_n^{(2)}$. Observe that there exists two constants $c_0,c_1>0$ such that

$$r!\geq c_0\left(\frac{r}{e}\right)^r,\quad\forall r\geq 1,$$

and

$$-c_1 = \inf_{s>0} s(\log s).$$

We can then write

$$\begin{aligned}\mathcal{P}_n^{(1)} = r_n! \left(\frac{\rho\gamma_2}{2|\Lambda_n|^{1/2}}\right)^{r_n} &\geq c_0 \left(\frac{\rho\gamma_2 r_n}{2e|\Lambda_n|^{1/2}}\right)^{r_n} \\ &= c_0 \exp\left[\frac{2e|\Lambda_n|^{1/2}}{\rho\gamma_2}\left(\frac{\rho\gamma_2 r_n}{2e|\Lambda_n|^{1/2}}\right)\log\left(\frac{\rho\gamma_2 r_n}{2e|\Lambda_n|^{1/2}}\right)\right] \geq c_0 \exp\left(-\frac{2ec_1}{\rho\gamma_2}|\Lambda_n|^{1/2}\right).\end{aligned}$$

Putting this inequality (and the similar one for the product $\mathcal{P}_n^{(2)}$) in (4.60) we obtain

$$\prod_{k\in S_1(n)} \left|1 - \frac{\Lambda_n}{\Lambda_k}\right| \geq e^{-C\sqrt{|\Lambda_n|}-C}, \quad \forall n \geq 1. \tag{4.61}$$

2. Let us now estimate the product (4.59) for $k \in S_2(n)$ that we denote by $\mathcal{P}_n^{(3)}$. Let $c_2 > 0$ be such that

$$\log(1-s) \geq -c_2 s, \quad \forall s \in [0, 1/2]. \tag{4.62}$$

Observe that, for $k \in S_2(n)$ one has $|\Lambda_n|/|\Lambda_k| \leq 1/2$, so that we can use (4.62) to obtain

$$\begin{aligned}\log \mathcal{P}_n^{(3)} &\geq \sum_{k\in S_2(n)} \log\left(1 - \frac{|\Lambda_n|}{|\Lambda_k|}\right) \geq -c_2|\Lambda_n| \sum_{k\in S_2(n)} \frac{1}{|\Lambda_k|} = -c_2|\Lambda_n| \int_{2|\Lambda_n|} \frac{1}{r} d\mathcal{N}(r) \\ &= -c_2|\Lambda_n| \left(-\frac{\mathcal{N}(2|\Lambda_n|)}{2|\Lambda_n|} + \int_{2|\Lambda_n|^-} \frac{\mathcal{N}(r)}{r^2} dr\right) \geq -c_2|\Lambda_n| \int_{2|\Lambda_n|^-} \frac{\mathcal{N}(r)}{r^2} dr \\ &\geq -c_2|\Lambda_n| \int_{2|\Lambda_n|^-} \frac{\alpha + p\sqrt{r}}{r^2} dr = -c_2|\Lambda_n| \left(\frac{\alpha}{2|\Lambda_n|} + \frac{2p}{\sqrt{2|\Lambda_n|}}\right) \\ &= -\frac{\alpha c_2}{2} - \sqrt{2} p c_2 |\Lambda_n|^{1/2}.\end{aligned}$$

Putting (4.61) and this last inequality in (4.59), we deduce

$$\mathcal{P}_n = \prod_{k\neq n} \left|1 - \frac{\Lambda_n}{\Lambda_k}\right| \geq e^{-C\sqrt{|\Lambda_n|}-C}, \quad \forall n \geq 1,$$

Since $|\Lambda_n| \geq |\Lambda_1|$ for every $n \geq 1$ (see $(\mathcal{H}_4)$) we finally have

$$\mathcal{P}_n \geq e^{-C\sqrt{|\Lambda_n|}}, \quad \forall n \geq 1.$$

This inequality and formula (4.58) provide the desired estimate. This ends the proof.

Proof of Lemma 4.20 For the proof we will follow some ideas from [FR71] and [Mil04] (see also [Red77]). Let us first consider again the counting function N associated with the sequence $\{a_n\}_{n\geq 1}$ given by (4.41) :

$$N(r) = \#\{n \geq 1 : a_n \leq r\}.$$

Observe again that the sequence $\{a_n\}_{n\geq 0}$ can be written as

$$a_n = a_0 + \frac{n^2}{A^2}, \quad \forall n \geq 1, \quad \text{with} \quad A = \frac{d}{2} \quad \text{and} \quad a_0 = \frac{d^2}{\tau^2} - \frac{4}{d^2}, \tag{4.63}$$

and that $a_0 > 0$ since we assumed that $\tau < d^2/2$. Thus, $N(r) = 0$ for $r < a_1$, and

$$N(r) = \lfloor A\sqrt{r - a_0} \rfloor, \quad \forall r \geq a_1, \tag{4.64}$$

We will often use that

$$A\sqrt{r} - A\sqrt{a_0} \leq N(r) \leq A\sqrt{r}, \quad \forall r \geq 0.$$

Let us prove the inequality (4.54). Observe that M is an even function. So, we will show (4.54) for $x \in (0, +\infty)$. From the definition (4.42) of M, one has

$$\log|M(x)| = \sum_{n\geq 1} \log\left|\frac{\sin(x/a_n)}{x/a_n}\right| = \int_{a_1^-}^{+\infty} g\left(\frac{x}{r}\right) dN(r),$$

here

$$g(s) = \log\left|\frac{\sin(s)}{s}\right|, \quad s \in \mathbb{R}.$$

– Since, g is non increasing on $[0, 1)$, for any $x \in [0, a_1]$, we have

$$\log|M(x)| \leq \log|M(0)| = 0 \leq -d\sqrt{x} + d\sqrt{a_1} \leq -d\sqrt{x} + \frac{d^2}{\tau},$$

which gives the claim in that case.

– Assume now that $x > a_1$. We write

$$\log|M(x)| = \sum_{a_n \leq x} g(x/a_n) + \sum_{a_n > x} g(x/a_n) \equiv I + J.$$

Since g is negative and non increasing on $[0, 1]$, the second sum J can be bounded as follows

$$\begin{aligned} J &\leq \sum_{2x \geq a_n > x} g(x/a_n) \leq -|g(1/2)|(N(2x) - N(x)) \\ &\leq -|g(1/2)|(A\sqrt{2x - a_0} - 1 - A\sqrt{x - a_0}) = |g(1/2)| - A|g(1/2)|\frac{x}{\sqrt{2x - a_0} + \sqrt{x - a_0}} \\ &\leq |g(1/2)| - A\frac{|g(1/2)|}{\sqrt{2} + 1}\sqrt{x}. \end{aligned}$$

In the first sum I, we use the inequality $g(s) \leq -\log s$ for any $s \geq 0$, to get

$$\begin{aligned}
I &\leq -\sum_{a_n \leq x} \log(x/a_n) = \int_{a_1^-}^{x} \log\left(\frac{r}{x}\right) dN(r) = -\int_{a_1}^{x} \frac{N(r)}{r}\, dr \\
&\leq \int_{a_1}^{x} \frac{1 - A\sqrt{r-a_0}}{r}\, dr = \log(x/a_1) - A\left(\int_{a_1}^{x} \frac{1}{\sqrt{r-a_0}}\, dr - a_0 \int_{a_1}^{x} \frac{1}{r\sqrt{r-a_0}}\, dr\right) \\
&\leq \log(x/a_1) - 2A\sqrt{x-a_0} + 2A\sqrt{a_1-a_0} + A\sqrt{a_0}\int_1^{+\infty} \frac{1}{r\sqrt{r-1}}\, dr \\
&\leq -2A\sqrt{x} + c_1 A\sqrt{a_0} + \log(x) + 2,
\end{aligned}$$

with $c_1 = 2 + \int_1^{+\infty} \frac{1}{r\sqrt{r-1}}\, dr$.
Combining the two estimates gives

$$\log|M(x)| \leq -A\left(2 + \frac{|g(1/2)|}{1+\sqrt{2}}\right)\sqrt{x} + \log x + c_1 A\sqrt{a_0} + 2 + |g(1/2)|.$$

Observe now that $a_0 \leq d^2/\tau^2$, that $2A = d$ and that the function

$$x \in [0,+\infty[\mapsto -A\frac{|g(1/2)|}{1+\sqrt{2}}\sqrt{x} + \log(x) + 2 + |g(1/2)|,$$

is bounded by some number $c_2 > 0$ depending only on $A = d/2$. We finally get the inequality

$$\log|M(x)| \leq -d\sqrt{x} + \frac{c_1 d^2}{2\tau} + c_2,$$

which gives the claim by using that $1 \leq \frac{d^2}{2\tau}$.

Proof of Lemma 4.21 We start by observing that

$$\frac{\sin(iy)}{iy} = \frac{\sinh y}{y} \geq 1, \quad \forall y \in \mathbb{R}.$$

As a consequence, we obtain $M(iy) \geq 1$, for any $y \in \mathbb{R}$. Thus, we immediately get (4.55).

We will now obtain the proof of (4.56) by adapting the proof of Lemma 6.3 of [FR71] to the sequence $\{a_n\}_{n\geq 1}$ given by (4.41). We set $c_0 = \log\sqrt{3} > 0$.

– Assume first that $|y|/c_0 \leq a_1$. Then, by using (4.55), we get

$$\log|M(iy)| \geq 0 \geq \tau_M|y| - \tau_M c_0 a_1 = \tau_M|y| - \tau_M c_0 \frac{d^2}{\tau^2} \geq \tau_M|y| - c_0\frac{d^2}{\tau}, \quad \forall \frac{|y|}{c_0} \leq a_1 = \frac{d^2}{\tau^2},$$

and the claim is proved in that case.

– Assume now that $|y|/c_0 \geq a_1$. Observe that

$$\frac{\sin(iy)}{iy} = \frac{1}{2}\left(\frac{e^y - e^{-y}}{y}\right) = \frac{e^{|y|} - e^{-|y|}}{2|y|} = e^{|y|}\frac{1 - e^{-2|y|}}{2|y|}, \quad \forall y \neq 0.$$

Thus, using the definitions (4.42) and (4.43) of M and τ_M, we have

$$\log|M(iy)| = \sum_{n\geq 1}\frac{|y|}{a_n} + \sum_{n\geq 1}\log\left(\frac{1-e^{-2|y|/a_n}}{2|y|/a_n}\right) = \tau_M|y| + \mathcal{I}, \tag{4.65}$$

where the sequence $\{a_n\}_{n\geq 1}$ is given by (4.41).
In order to bound the series $\mathcal{I}$, we will use the inequalities

$$\frac{1-e^{-2y}}{2y} \geq e^{-2y}, \quad \forall y>0, \quad \text{and} \quad \frac{1-e^{-2y}}{2y} \geq \frac{1}{3y}, \quad \forall y \geq \log\sqrt{3} = c_0.$$

So, for $y\in\mathbb{R}$ with $|y|/c_0 \geq a_1$, one has,

$$\mathcal{I} = \sum_{n\geq 1}\log\left(\frac{1-e^{-2|y|/a_n}}{2|y|/a_n}\right) \geq -\sum_{\substack{n\geq 1\\ a_n>|y|/c_0}}\frac{2|y|}{a_n} + \sum_{\substack{n\geq 1\\ a_n\leq|y|/c_0}}\log\left(\frac{a_n}{3|y|}\right) \equiv \mathcal{I}_1+\mathcal{I}_2\,. \tag{4.66}$$

– Let us first bound from below $\mathcal{I}_1$ in the expression (4.66). One has

$$\mathcal{I}_1 = -\sum_{\substack{n\geq 1\\ a_n>|y|/c_0}}\frac{2|y|}{a_n} = -2|y|\int_{a_{n_0}^-}^{+\infty}\frac{dN(r)}{r} \geq -2|y|\int_{|y|/c_0}^{+\infty}\frac{dN(r)}{r},$$

where $n_0\geq 1$ is the smallest integer such that $a_{n_0} > |y|/c_0$ and $N(\cdot)$ is the counting function associated to the sequence $\{a_n\}_{n\geq 1}$ (see (4.63) and (4.64)). Integrating by parts, we obtain :

$$\begin{aligned}\mathcal{I}_1 &\geq -2|y|\left[\frac{1}{r}N(r)\Big|_{|y|/c_0}^{+\infty} + \int_{|y|/c_0}^{+\infty}\frac{N(r)}{r^2}\,dr\right] \geq -2|y|A\int_{|y|/c_0}^{+\infty}\frac{\sqrt{r-a_0}}{r^2}\,dr\\ &\geq -2A|y|\int_{|y|/c_0}^{+\infty} r^{-3/2}\,dr,\end{aligned}$$

that is to say,

$$\mathcal{I}_1 \geq -4c_0^{1/2}A\sqrt{|y|}, \quad \forall\frac{|y|}{c_0} > a_1\,. \tag{4.67}$$

– Let us deal with the second term $\mathcal{I}_2$ in (4.66) for $|y|$ satisfying $a_1 < |y|/c_0$. Using that for any $r\in[a_1,|y|/c_0]$ one has $r<3|y|$ ($c_0=\log\sqrt{3}$), we can write

$$\mathcal{I}_2 = \sum_{\substack{n\geq 1\\ a_n\leq|y|/c_0}}\log\left(\frac{a_n}{3|y|}\right) = \int_{a_1^-}^{a_{n_1}}\log\left(\frac{r}{3|y|}\right)dN(r) \geq \int_{a_1^-}^{|y|/c_0}\log\left(\frac{r}{3|y|}\right)dN(r),$$

where $n_1\geq 1$ is the largest integer such that $a_{n_1}\leq |y|/c_0$.

Again, integrating by parts, we deduce

$$\begin{aligned}
\mathcal{I}_2 &\geq N(r)\log\left(\frac{r}{3|y|}\right)\Bigg|_{a_1^-}^{|y|/c_0} - \int_{a_1^-}^{|y|/c_0} \frac{N(r)}{r}\,dr \\
&\geq -\log(3c_0)N(|y|/c_0) - A\int_{a_1^-}^{|y|/c_0} \frac{\sqrt{r-a_0}}{r}\,dr \\
&\geq -\log(3c_0)N(|y|/c_0) - A\int_0^{|y|/c_0} \frac{1}{\sqrt{r}}\,dr \geq -A\left(2+\log(3c_0)\right)c_0^{-1/2}\sqrt{|y|}.
\end{aligned}$$

In view of (4.65) and (4.66), this last inequality together with (4.67) provide ($c_0 = \log\sqrt{3}$)

$$\log|M(iy)| \geq \tau_M|y| - c_1 d\sqrt{|y|},$$

with $c_1 = (1 + 2c_0 + \log(3c_0)/2)c_0^{-1/2}$.

Owing to the previous calculations, we finally obtain the inequality (4.56). This ends the proof.

Chapitre 5

Approximate controllability conditions for some linear 1D parabolic systems with space-dependent coefficients

Ce chapitre est la reprise de l'article [BO13], qui est un travail en collaboration avec F. Boyer, et qui a été soumis.

Abstract. In this article we are interested in the controllability with one single control force of parabolic systems with space-dependent zero-order coupling terms. We particularly want to emphasize that, surprisingly enough for parabolic problems, the geometry of the control domain can have an important influence on the controllability properties of the system, depending on the structure of the coupling terms.

Our analysis is mainly based on a criterion given by Fattorini in [Fat66] and systematically used in [Oli13] for instance, that reduces the problem to the study of a unique continuation property for elliptic systems. We provide several detailed examples of controllable and non-controllable systems. This work gives theoretically justifications of some numerical observations described in [Boy13].

Keywords : Parabolic systems ; Distributed controllability ; Geometric condition ; Unique continuation ; Hautus test.

5.1 Introduction

This paper deals with the controllability properties at time $T > 0$ of the following class of 1D linear parabolic systems

$$\begin{cases} \partial_t y + \mathscr{L} y = A(x)y + 1_\omega B v & \text{in } (0,T)\times\Omega, \\ y(0) = y_0 & \text{in } \Omega. \end{cases} \tag{5.1}$$

Here, the domain is $\Omega = (0,1)$, $y \in \mathcal{C}^0([0,T], L^2(\Omega)^n)$ is the state, $y_0 \in L^2(\Omega)^n$ the initial data, $A(x)$ is a $n \times n$ real matrix with entries in $L^\infty(\Omega)$, B is a constant vector in $\mathbb{R}^n$ and $v \in L^2((0,T) \times \Omega)$ is the (scalar-valued) control which is acting only on the control domain ω, a non-empty open subset of Ω. The diffusion operator $\mathscr{L} = \mathcal{L}\,\mathrm{Id}$ operates on vector-valued functions component-wise through the scalar elliptic operator $\mathcal{L}$ defined by

$$\mathcal{L} = -\partial_x \left(\gamma(x)\partial_x \cdot\right) + \gamma_0(x)\,\cdot, \tag{5.2}$$

with domain $\mathcal{D}(\mathcal{L}) = \{u \in H_0^1(\Omega),\ \mathcal{L}u \in L^2(\Omega)\}$ corresponding to homogeneous Dirichlet boundary condition. The coefficients of $\mathcal{L}$ are supposed to satisfy standard assumptions $\gamma, \gamma_0 \in L^\infty(\Omega)$, with $\inf_\Omega \gamma > 0$.

This is an important class of systems that can be considered as "toy models" to understand how the structure of the coupling terms can influence the behavior of a controlled system with a few number of controls. In the case where $A(x) = A$ is constant, it is shown in [AKBDGB09a] that (5.1) is null-controllable if and only if the Kalman rank condition between matrices A and B holds. This result is thus independent of the control domain ω and of the operator $\mathcal{L}$ (and is actually true in any space dimension).

The situation is more complex for systems with space-dependent coupling coefficients in which case there exist only few controllability results [GBdT10, KdT10, RdT11, ABL12, AB12, Mau13, Oli13, BCGdT13]. Most of them are still partial and deal with systems of 2 equations. In [GBdT10], the null-controllability was established for $n\times n$ systems with some structure assumption on the coupling and under the crucial hypothesis that the control domain ω intersects the support $\mathcal{O}$ of the coupling terms. The structural assumption was removed in [BCGdT13] and [Mau13], however with some other technical hypothesis, still in the case $\omega \cap \mathcal{O} \neq \emptyset$. On the other hand, approximate controllability in the case where the coupling term only acts away from the control domain, that is $\omega \cap \mathcal{O} = \emptyset$, was proved for a cascade system with non-negative coupling terms in [KdT10]. In the same framework, the null-controllability was then obtained in the one-dimensional case in [RdT11], and then in any dimension in [AB12] under a geometric condition on the control and the coupling domains, though. These restrictions come from the geometric control condition (GCC) for the wave-type systems that are used in these works to deduce results for parabolic systems through the transmutation method.

We will see in this paper that the geometry of the control domain ω will play an important role in the study of those systems, even though the GCC is automatically satisfied in 1D ; for instance we shall provide examples of systems which are controllable for some choices of ω but not controllable for other choices. This is not usual in the parabolic framework.

Let us also underline that the results in [KdT10], [RdT11] and [AB12] require some sign conditions for the coupling terms. To the authors knowledge there is no available result in the literature in the case $\omega \cap \mathcal{O} = \emptyset$ without such a sign assumption. However, it is worth mentionning that the proof of sufficient controllability conditions given in [KdT10] still holds without this sign assumption, see Section 5.3.3.1. This is another achievement of the present paper to provide necessary and sufficient conditions in the general case, that is without *a priori* assumptions on the sign of the coupling terms.

Last but not least, we also investigate the case of some $n \times n$ systems with $n > 2$ that do not enter the framework of [GBdT10] and [Mau13].

The notion we deal with in this is paper is the one of approximate controllability (which is weaker than null-controllability [Ghi86]), that can be stated as follows : For every $\epsilon > 0$ and $y_0, y_T \in L^2(\Omega)^n$, find a control $v \in L^2(0,T;L^2(\Omega))$ such that the solution y of (5.1) satisfies

$$\|y(T) - y_T\|_{L^2(\Omega)^n} \leq \epsilon.$$

Since B is a non-trivial constant vector, and $\mathscr{L} = \mathcal{L}\,\mathrm{Id}$, we see that a simple linear change of unknowns let us transform the system into the case where $B = (1, 0, \dots, 0)^*$, the first vector of the canonical basis of $\mathbb{R}^n$ (in this work we denote by M^* the transpose of any matrix M). This means that the direct action of the control v only concerns the first component of the system.

We are particularly interested in the study of the system under the following structural assumptions on the coupling terms :

1. Controllability of a 2×2 cascade system (section 5.3.3.1)

$$A(x) = \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix}. \tag{5.3}$$

2. Simultaneous controllability of two 2×2 cascade systems (section 5.3.3.2)

$$A(x) = \begin{pmatrix} 0 & 0 & 0 \\ a_{21}(x) & 0 & 0 \\ a_{31}(x) & 0 & 0 \end{pmatrix}. \tag{5.4}$$

3. Controllability of a 3×3 cascade system (section 5.4)

$$A(x) = \begin{pmatrix} 0 & 0 & 0 \\ a_{21}(x) & 0 & 0 \\ 0 & a_{32}(x) & 0 \end{pmatrix}. \tag{5.5}$$

Finally, in section 5.5 we give some examples and counter-examples of simultaneous controllability for an uncoupled 2×2 system ($A \equiv 0$) with different diffusion on each equation, that is when the operator $\mathscr{L}$ is not anymore of the form $\mathscr{L} = \mathcal{L}\,\mathrm{Id}$ (but still diagonal).

5.2 Unique continuation criterion for triangular systems

5.2.1 The Fattorini theorem

The adjoint system of (5.1) is

$$\begin{cases} -\partial_t q + \mathscr{L} q = A(x)^* q & \text{in } (0,T)\times\Omega, \\ q(T) = q_F & \text{in } \Omega, \end{cases} \tag{5.6}$$

and it is well known (see for instance [Cor07, Theorem 2.43]) that the approximate controllability at time $T > 0$ of (5.1) is equivalent to the unique continuation property for the adjoint parabolic system : there is no non-trivial solutions of (5.6) such that $B^*q = 0$ on $(0,T)\times\omega$.

However, Fattorini proved in [Fat66] that, for such systems, this parabolic unique continuation property is actually equivalent to an elliptic unique continuation property which is much easier to handle (and which does not depend on T).

Theorem 5.1 ([Fat66, Corollary 3.3]). *System* (5.1) *is approximately controllable at time* $T > 0$*, if and only for any* $s \in \mathbb{C}$ *and any* $u \in \mathcal{D}(\mathscr{L})$ *we have*

$$\left.\begin{aligned} \mathscr{L}u - A(x)^*u = su \quad & \textit{in } \Omega \\ B^*u = 0 \quad & \textit{in } \omega \end{aligned}\right\} \Longrightarrow u = 0. \tag{5.7}$$

This means that the analysis of the approximate controllability for the original system can be reduced to a careful study of the eigenfunctions (associated with the eigenvalue s) of the underlying elliptic operator

$$\mathscr{A} = \mathscr{L} - A(x)^*.$$

In the theory of ordinary differential system, this controllability condition is also known as the Hautus test. The characterization given by Fattorini has been recently developped and used in [BT12] and [Oli13] for the study of some other parabolic systems.

Note that, for the particular systems studied in the present paper (excepted in Section 5.5), B^*u is nothing but the first component of u. Thus, the study of the approximate controllability of all the systems considered in Sections 5.3 and 5.4 reduces to the following question : does it exist an eigenfunction of $\mathscr{A}$ whose first component is identically zero on the control domain ω ?

In all the cases considered, we observe that for any $x \in \Omega$, $A(x)$ is strictly lower triangular. Thus, the eigenvalues of the operator $\mathscr{A}$ are simply the $\{\lambda_k\}_{k\geq 1}$ where λ_k is the kth eigenvalue of $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$, the corresponding eigenfunction being denoted by ϕ_k. Indeed, assume that u is an eigenfunction of $\mathscr{A}$ associated with an eigenvalue $s \in \mathbb{C}$ and let $i \geq 1$ be the higher index for which u_i is not identically zero. Writing the ith component of the equation $\mathscr{A}u = su$, leads to

$$su_i = \mathcal{L}u_i - \sum_{j>i} a_{ji}(x)u_j = \mathcal{L}u_i,$$

so that s is an eigenvalue of $\mathcal{L}$ and finally $s = \lambda_k$ for some $k \geq 1$.

Moreover, we observe that the first component u_1 of u solves an equation of the following form

$$\mathcal{L}u_1 - \lambda_k u_1 = F \text{ in } (0,T) \times \Omega, \tag{5.8}$$

where F can be computed as a function of the other components of u and the entries in $A(x)$ as we shall see below.

As a starting point of the analysis we are thus led to study necessary and sufficient conditions on the source term F ensuring that (5.8) does not have any solution u_1 which identically vanishes on the control domain ω.

5.2.2 Notations

For any $k \geq 1$, let $\widetilde{\phi}_k$ be any solution of the ordinary differential equation $\mathcal{L}\widetilde{\phi}_k - \lambda_k \widetilde{\phi}_k = 0$ which satisfies $\widetilde{\phi}_k(0) \neq 0$. Observe that ϕ_k and $\widetilde{\phi}_k$ are linearly independent, and that $\widetilde{\phi}_k \notin \mathcal{D}(\mathcal{L})$ since it does not satisfy the Dirichlet boundary condition. In the case $\mathcal{L} = -\partial_x^2$, one can choose for instance $\widetilde{\phi}_k(x) = \cos(k\pi x)$. Obviously, one can check that all the results given in this paper do not depend on the particular choice of $\widetilde{\phi}_k$ satisfying the above properties.

We denote by $\mathcal{C}\left(\overline{\Omega\backslash\omega}\right)$ the set of all connected components of $\overline{\Omega\backslash\omega}$, and for every $C \in \mathcal{C}\left(\overline{\Omega\backslash\omega}\right)$ and $f \in L^1(\Omega)$, we define the vector $M_k(f, C) \in \mathbb{R}^2$ by

$$M_k(f,C) = \begin{cases} \begin{pmatrix} \int_C f\phi_k \, dx \\ 0 \end{pmatrix} & \text{if } C \cap \partial\Omega \neq \emptyset, \\ \begin{pmatrix} \int_C f\phi_k \, dx \\ \int_C f\widetilde{\phi}_k \, dx \end{pmatrix} & \text{if } C \cap \partial\Omega = \emptyset. \end{cases} \tag{5.9}$$

Then, for any $f \in L^1(\Omega)$ we define the following family of vectors of $\mathbb{R}^2$

$$\mathcal{M}_k(f,\omega) = (M_k(f,C))_{C \in \mathcal{C}\left(\overline{\Omega\backslash\omega}\right)} \in (\mathbb{R}^2)^{\mathcal{C}\left(\overline{\Omega\backslash\omega}\right)}.$$

We will frequently use the fact that, for any $u \in \mathcal{D}(\mathcal{L})$, we have $u, \gamma\partial_x u \in \mathcal{C}^0(\overline{\Omega})$. Moreover, in order to simplify a little the notation, we shall write v' (resp. v'') instead of $\partial_x v$ (resp. $\partial_x^2 v$) for functions v depending only on the 1D variable x.

5.2.3 Unique continuation for a 1D non-homogeneous scalar problem

We establish necessary and sufficient conditions for a non-homogeneous scalar problem to have a solution which vanishes identically on a given subset of the domain. As we will see below, this is the main tool for analysing the elliptic unique continuation property for eigenfunctions of $\mathscr{A}$.

Theorem 5.2. *Let $F \in L^2(\Omega)$ and ω be a non-empty open subset of Ω. Let $k \geq 1$ be fixed. There exists a solution $u \in \mathcal{D}(\mathcal{L})$ to the following problem*

$$\begin{cases} \mathcal{L}u - \lambda_k u = F & \text{in } \Omega, \\ u = 0 & \text{in } \omega, \end{cases} \tag{5.10}$$

if and only if

$$\begin{cases} F = 0 & \text{in } \omega, \\ \mathcal{M}_k(F, \omega) = 0. \end{cases} \tag{5.11}$$

Proof. Let us perform a preliminary computation. Let $[\alpha, \beta] \subset [0, 1]$ and $u \in \mathcal{D}(\mathcal{L})$ be a solution of $\mathcal{L}u - \lambda_k u = F$.

Let $v \in L^2(\Omega)$ be any distribution solution of the ordinary differential equation $\mathcal{L}v - \lambda_k v = 0$. We multiply by v the equation satisfied by u and we perform two integration by parts to get

$$\begin{aligned} \int_\alpha^\beta F v \, dx = & - \Big[(\gamma u')(\beta) v(\beta) - u(\beta)(\gamma v')(\beta) \Big] \\ & + \Big[(\gamma u')(\alpha) v(\alpha) - u(\alpha)(\gamma v')(\alpha) \Big], \end{aligned} \tag{5.12}$$

This formula will be used in the sequel with $v = \phi_k$ and $v = \widetilde{\phi}_k$. We can now turn to the proof of the claimed equivalence.

$\Rightarrow$ Assume that there exists a u satisfying (5.10).

- Since $u = 0$ in ω, it is clear from the equation that $F = 0$ on ω. Moreover, by continuity, u and $\gamma u'$ are identically 0 on $\overline{\omega}$.
- Let $C = [\alpha, \beta]$ be a connected component of $\overline{\Omega \backslash \omega}$. Observe that α (resp. β) necessarily belongs either to $\overline{\omega}$ or to $\partial\Omega$, and that

  $$\begin{cases} \alpha \in \partial\Omega \Longrightarrow u(\alpha) = 0 \text{ and } \phi_k(\alpha) = 0, \\ \alpha \in \overline{\omega} \Longrightarrow u(\alpha) = 0 \text{ and } \gamma u'(\alpha) = 0. \end{cases}$$

  Therefore, in both cases, we have $u(\alpha) = 0$ and $\phi_k(\alpha)(\gamma u')(\alpha) = 0$, the same being true for when one changes α into β.
  It follows from (5.12) with $v = \phi_k$ that

  $$\int_C F \phi_k \, dx = 0,$$

  which proves the claim.
- Assume additionally that the connected component C is such that $C \cap \partial\Omega = \emptyset$. As we have seen above, in that case we have $u(\alpha) = u(\beta) = (\gamma u')(\alpha) = (\gamma u')(\beta) = 0$. Therefore, (5.12) with $v = \widetilde{\phi}_k$ immediately gives that

  $$\int_C F \widetilde{\phi}_k \, dx = 0.$$

$\Leftarrow$ Since $\mathcal{M}_k(F,\omega) = 0$, we can sum all the integrals corresponding to the various connected components to obtain that $\int_{\overline{\Omega\setminus\omega}} F\phi_k\,dx = 0$. Using that $F = 0$ on ω, we conclude that $\int_\Omega F\phi_k\,dx = 0$. This orthogonality condition implies the existence of at least one solution $u_0 \in \mathcal{D}(\mathcal{L})$ of the non-homogeneous equation

$$\mathcal{L}u_0 - \lambda_k u_0 = F, \text{ in } \Omega.$$

Actually, any solution of this problem has the form $u = u_0 + \mu\phi_k$, $\mu \in \mathbb{R}$. We will show that we can find a μ such that this function u vanishes identically on ω.

– We first show that one can choose μ in such a way that there exists a point $x_0 \in \overline{\omega}$ satisfying

$$u(x_0) = (\gamma u')(x_0) = 0. \tag{5.13}$$

  – Assume first that $\overline{\omega} \cap \partial\Omega \neq \emptyset$ and for instance that $0 \in \overline{\omega}$. Thanks to the Dirichlet boundary condition we have $u(0) = 0$ and we just need to impose $(\gamma u')(0) = 0$, that is $(\gamma u_0')(0) + \mu(\gamma\phi_k')(0) = 0$. This determines μ in a unique way since $(\gamma\phi_k')(0) \neq 0$ and gives $x_0 = 0$.
  – In the case where $\overline{\omega} \cap \partial\Omega = \emptyset$, we denote by $[0,\beta]$ the connected component of $\overline{\Omega\setminus\omega}$ containing 0. By assumption

$$\int_0^\beta F\phi_k\,dx = 0. \tag{5.14}$$

  Since $F = 0$ in ω, we can replace β in this formula by $\beta + \delta$ with $\delta > 0$ small enough such that $]\beta, \beta+\delta] \subset \omega$ and $\phi_k(\beta+\delta) \neq 0$ (the zeros of the eigenfunction ϕ_k are isolated).
  We can then fix the parameter μ in such a way that

$$u(\beta+\delta) = u_0(\beta+\delta) + \mu\phi_k(\beta+\delta) = 0.$$

  It follows from (5.12) with $v = \phi_k$, (5.14) (with the upper bound $\beta+\delta$ instead of β), and from the boundary condition satisfied by u and ϕ_k at 0, that

$$0 = (\gamma u')(\beta+\delta)\phi_k(\beta+\delta) - u(\beta+\delta)(\gamma\phi_k')(\beta+\delta).$$

  Since u vanishes at $\beta+\delta$, but ϕ_k does not, we deduce that

$$(\gamma u')(\beta+\delta) = 0.$$

  Therefore u and $(\gamma u')$ vanish at the same point $x_0 = \beta+\delta$ in ω.

– The parameter μ is now fixed and we know that there is a x_0 such that (5.13) holds.
  We want to show that $u = 0$ on ω. By contradiction, we assume that there is a $x_1 \in \omega$, such that $u(x_1) \neq 0$. Without loss of generality we assume for instance that $x_0 < x_1$. Observe that $[x_0, x_1] \cap \left(\overline{\Omega\setminus\omega}\right)$ is a (possibly empty) union of connected

components of $\overline{\Omega \backslash \omega}$ and that none of them touches the boundary of Ω. Since $F = 0$ in ω, and $\mathcal{M}_k(F, \omega) = 0$, we deduce that

$$0 = \int_{x_0}^{x_1} F\phi_k \, dx = \int_{x_0}^{x_1} F\widetilde{\phi}_k \, dx.$$

Using (5.12) with $v = \phi_k$ (resp. with $v = \widetilde{\phi}_k$) and (5.13), we get

$$\begin{cases} 0 = (\gamma u')(x_1)\phi_k(x_1) - u(x_1)(\gamma\phi_k')(x_1), \\ 0 = (\gamma u')(x_1)\widetilde{\phi}_k(x_1) - u(x_1)(\gamma\widetilde{\phi}_k')(x_1). \end{cases}$$

Since the Wronskian matrix

$$\begin{pmatrix} \phi_k(x_1) & -(\gamma\phi_k')(x_1) \\ \widetilde{\phi}_k(x_1) & -(\gamma\widetilde{\phi}_k')(x_1) \end{pmatrix},$$

is invertible (recall that ϕ_k and $\widetilde{\phi}_k$ are two independant solutions of the second order differential equation $\mathcal{L}v - \lambda_k v = 0$) we deduce that $u(x_1) = (\gamma u')(x_1) = 0$ which is a contradiction.

□

5.3 Simultaneous controllability of several 2×2 cascade systems

In this section we are interested in the controllability of system (5.1) when the matrix $A(x)$ is of the following form

$$A(x) = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ a_{21}(x) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x) & 0 & \cdots & 0 \end{pmatrix}. \tag{5.15}$$

In this system, the distributed control v only acts on the first component y_1 and this component serves itself as a simultaneous control for the other components through the coupling terms $a_{21}, \ldots, a_{n1}$.

5.3.1 Reduction

Observe first that we can always reorder the unknowns y_k and the entries a_{k1}, for $2 \le k \le n$, in such a way that for some $p \in \{2, ..., n+1\}$

$$\begin{cases} \mathrm{Span}\,(a_{21}1_\omega, ..., a_{n1}1_\omega) = \mathrm{Span}\,(a_{p1}1_\omega, ..., a_{n1}1_\omega)\,, \\ a_{p1}1_\omega, ..., a_{n1}1_\omega \text{ are linearly independent,} \end{cases} \tag{5.16}$$

where, conventionally, the case $p = n+1$ is the one where $a_{k1} = 0$ in ω for any $2 \leq k \leq n$ in which case both properties are obvious.

By using (5.16), we can write

$$a_{i1} 1_\omega = \sum_{j=p}^{n} \alpha_{ij} a_{j1} 1_\omega, \quad \forall i \in \{2, ..., p-1\},$$

for some $\alpha_{ij} \in \mathbb{R}$. We perform now the (revertible) change of unknowns $y \to \tilde{y}$ defined by

$$\begin{cases} \tilde{y}_i = y_i - \displaystyle\sum_{j=p}^{n} \alpha_{ij} y_j, \quad \forall i \in \{2, ..., p-1\}, \\ \tilde{y}_i = y_i, \quad \forall i \in \{1\} \cup \{p, ..., n\}. \end{cases}$$

It is easily verified that $\tilde{y}$ solves a system of the same form as (5.1), with a new coupling matrix, still referred to as $A(x)$, which satisfies

$$\begin{cases} a_{i1} = 0, \text{ on } \omega, \quad \forall i \in \{2, ..., p-1\}, \\ a_{p1} 1_\omega, ..., a_{n1} 1_\omega, \quad \text{are linearly independent.} \end{cases} \tag{5.17}$$

Finally, since the change of variable is invertible, we observe that the controllability of the original system for y is equivalent to the one of the new system for $\tilde{y}$.

Therefore, from now on we shall assume that (5.17) holds and we introduce the following reduced system of size $p-1$

$$\begin{cases} \partial_t \hat{y} + \mathscr{L} \hat{y} = \widehat{A}(x) \hat{y} + 1_\omega B v & \text{in } (0,T) \times \Omega, \\ \hat{y}(0) = \hat{y}_0 & \text{in } \Omega, \end{cases} \tag{5.18}$$

where $\widehat{A}(x)$ is the $(p-1) \times (p-1)$ matrix defined by

$$\widehat{A}(x) = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ a_{21}(x) & 0 & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{p-1,1}(x) & 0 & \cdots & 0 \end{pmatrix}. \tag{5.19}$$

Proposition 5.3. *Assume that* (5.17) *holds, then the following statements are equivalent.*

1. *System* (5.1) *is approximately controllable for any initial data* $y_0 \in L^2(\Omega)^n$.
2. *System* (5.18) *is approximately controllable for any initial data* $\hat{y}_0 \in L^2(\Omega)^{p-1}$.

Proof. 1.⇒2. This is obvious since (5.18) is a subsystem of (5.1).

2.⇒1. Assume that (5.1) is not approximately controllable. The criterion given by Fattorini (Theorem 5.1) implies that (5.7) is not true. Therefore, there exists a non-trivial $u \in \mathcal{D}(\mathscr{L})$ which satisfies, for some $k \geq 1$,

$$\begin{cases} \mathscr{L}u - A(x)^* u = \lambda_k u & \text{in } \Omega, \\ u_1 = 0 & \text{in } \omega. \end{cases}$$

Observe that, from the particular structure of $A(x)^*$, $u = (u_1, \ldots, u_n)^*$ has necessarily the following form

$$u = \begin{pmatrix} u_1 \\ \delta_2 \phi_k \\ \vdots \\ \delta_n \phi_k \end{pmatrix},$$

with $\delta_i \in \mathbb{R}$ for $i = 2, ..., n$ and that u_1 solves

$$\mathcal{L}u_1 - \lambda_k u_1 = \left(\sum_{i=2}^{n} \delta_i a_{i1} \right) \phi_k.$$

Since u_1 vanishes on ω as well as a_{i1} for $i = 2, ..., p-1$ (from Assumption (5.17)), we deduce that

$$\left(\sum_{i=p}^{n} \delta_i a_{i1} \right) \phi_k = 0, \quad \text{almost everywhere in } \omega.$$

Since $\phi_k \neq 0$ almost everywhere (its zeros are isolated), it follows that

$$\sum_{i=p}^{n} \delta_i a_{i1} 1_\omega = 0.$$

By (5.17), the functions $a_{i1} 1_\omega$, $i = p, ..., n$ are linearly independent so that $\delta_i = 0$ for any $i = p, ..., n$.
Coming back to the equation satisfied by u_1, we get

$$\mathcal{L}u_1 - \lambda_k u_1 = \left(\sum_{i=2}^{p-1} \delta_i a_{i1} \right) \phi_k = \left(\sum_{i=2}^{p-1} a_{i1} u_i \right).$$

It follows that the reduced vector $\hat{u}(x) = (u_1(x), ..., u_{p-1}(x))^* \in \mathbb{R}^{p-1}$ is a non-trivial eigenfunction of the reduced adjoint system

$$\mathscr{L}\hat{u} - \widehat{A}(x)^* \hat{u} = \lambda_k \hat{u},$$

that satisfies

$$\hat{u}_1 = u_1 = 0, \quad \text{in } \omega.$$

From Theorem 5.1, this is in contradiction with the approximate controllability of the reduced system (5.18).

□

The conclusion of this discussion is that for the study of the approximate controllability of System (5.1) with $A(x)$ given by (5.15), we can always assume that all the supports of the coupling functions $a_{i1}(x)$, $i = 2, ..., n$ do not intersect the control domain ω, that is

$$a_{i1} 1_\omega = 0, \quad \forall i \in \{2, ..., n\}. \tag{5.20}$$

5.3.2 Necessary and sufficient approximate controllability conditions

The main result of this section is the following

Theorem 5.4. *Consider the matrix* $A(x)$ *as defined in* (5.15) *and assume that* (5.20) *holds.*

Then, System (5.1) is approximately controllable if and only if

$$\forall k \geq 1, \quad \text{rank } \{\mathcal{M}_k (a_{21}\phi_k, \omega), \dots, \mathcal{M}_k (a_{n1}\phi_k, \omega)\} = n - 1.$$

Remark 5.5. *In this formula the rank condition is understood in the (possibly infinite dimensional) vector space* $(\mathbb{R}^2)^{\mathcal{C}(\overline{\Omega\backslash\omega})}$.

In the usual case where $\overline{\Omega\backslash\omega}$ *has a finite number of connected components, this condition can be more classically written in a matrix formulation.*

Remark 5.6. *The first conclusion that the rank condition above let us draw is that there is a minimal number of connected components of* $\overline{\Omega\backslash\omega}$ *that are required to have a chance to control the system. Recall that the goal is to be able to control all the* n *components of the solution with only one control* v.

More precisely, we see that it is necessary (but not at all sufficient) to have $2 \,\text{card}\, \mathcal{C}\left(\overline{\Omega\backslash\omega}\right) \geq n - 1$ *for the approximate controllability to be possible. Observe that, if the system is not controllable, it is of course useless to split the control domain* ω *into smaller parts : this will actually increase the number of connected components of* $\overline{\Omega\backslash\omega}$ *but without adding non-trivial terms in the rank condition, because of* (5.20).

Looking more attentively at the rank condition we see that, for instance, one can not hope to control a 3×3 *system (resp. a* 4×4 *system) of this form if* ω *is an interval that touches the boundary (resp. that does not touch the boundary). A more detailed description of such examples is given in Section 5.3.3.2.*

Proof. We use the criterion of Fattorini (Theorem 5.1) and we study whether or not (5.7) holds. As we have already seen in Section 5.2.1, the only non-trivial case is the one where $s = \lambda_k$ for some $k \geq 1$, in which case a solution u of $\mathscr{L}u - A(x)^* u = \lambda_k u$ can be written

$$u = \begin{pmatrix} u_1 \\ \delta_2 \phi_k \\ \vdots \\ \delta_n \phi_k \end{pmatrix},$$

with $\delta_i \in \mathbb{R}$, $i = 2, ..., n$ and $u_1 \in \mathcal{D}(\mathcal{L})$ satisfying

$$\mathcal{L}u_1 - \lambda_k u_1 = \left(\sum_{i=2}^{n} \delta_i a_{i1}\right) \phi_k.$$

From Theorem 5.2, and since by assumption all the a_{i1} vanish on ω, such a solution u exists and satisfies $u_1 = 0$ in ω, if and only if

$$\mathcal{M}_k \left(\sum_{i=2}^{n} \delta_i a_{i1} \phi_k, \omega\right) = 0. \tag{5.21}$$

On the other hand, note that $u = 0$ if and only if $\delta_2 = \cdots = \delta_n = 0$ and $u_1 = 0$ on ω. This follows from the unique continuation for a single parabolic equation (see for instance [Miz58], [FI96] and [AE08], depending on the regularity required for the diffusion coefficient γ).

In summary, (5.7) holds if and only if (5.21) implies $\delta_2 = ... = \delta_n = 0$. Clearly,

$$\mathcal{M}_k \left(\sum_{i=2}^{n} \delta_i a_{i1} \phi_k, \omega\right) = \sum_{i=2}^{n} \delta_i \mathcal{M}_k \left(a_{i1} \phi_k, \omega\right),$$

and thus the approximate controllability is equivalent to the linear independence of the vectors $(\mathcal{M}_k (a_{i1}\phi_k, \omega))_{2 \leq i \leq n}$, for any $k \geq 1$, which gives exactly the claim. $\square$

5.3.3 Some applications

5.3.3.1 A single 2×2 cascade system

Let us study the following simplest example of system concerned by the previous analysis

$$\begin{cases} \partial_t y_1 + \mathcal{L}y_1 = 1_\omega v & \text{in } (0,T) \times \Omega, \\ \partial_t y_2 + \mathcal{L}y_2 = a_{21}(x) y_1 & \text{in } (0,T) \times \Omega, \end{cases} \tag{5.22}$$

which corresponds to the case (5.3). Depending on the assumptions on the coupling term a_{21} different results can be obtained. A first result in this direction is the following.

Theorem 5.7. *Let us denote the support of a_{21} by $\mathcal{O}_2$.*

1. *If $\mathcal{O}_2 \cap \omega \neq \emptyset$, then System (5.22) is approximately controllable.*
2. *Assume now that $\mathcal{O}_2 \cap \omega = \emptyset$.*
   (a) *If the coupling coefficient a_{21} satisfies*

   $$\int_0^1 a_{21} (\phi_k)^2 \, dx \neq 0, \quad \forall k \geq 1, \tag{5.23}$$

   then System (5.22) is approximately controllable.

   (b) *If System (5.22) is approximately controllable and $\mathcal{O}_2$ is entirely included in a connected component of $\overline{\Omega \backslash \omega}$ that touches the boundary $\partial\Omega$, then (5.23) holds.*

Remark 5.8.
- *In the first situation it can be proved that System* (5.22) *is even null-controllable in this case (see for instance [GBdT10]), but the proof is much longer and technical.*
- *With* (5.23) *we recover the (sufficient) condition of [KdT10, Theorem 1.5]. It is easy to see that this condition is fulfilled if* a_{21} *has a sign on* Ω *: for instance* $a_{21} \not\equiv 0$ *and* $a_{21} \geq 0$ *almost everywhere on* Ω*. Actually, under this sign assumption, the null-controllability of this system is known (see [RdT11, Theorem 5]).*
- *The geometric configuration required in the last point* (2b) *holds in particular if* $\mathcal{O}_2$ *and* ω *are two disjoint intervals. Condition* (5.23) *is however not necessary in general, see the examples below.*

Proof (of Theorem 5.7).

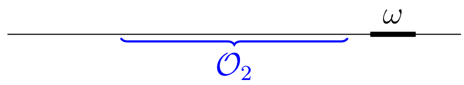
1. If a_{21} is not identically zero on ω, we deduce from Proposition 5.3 (with $p = n = 2$) that the approximate controllability of (5.22) is equivalent to the one of the scalar parabolic equation
$$\partial_t \hat{y} + \mathcal{L}\hat{y} = 1_\omega v$$
with Dirichlet boundary condition. This kind of scalar heat equation is known to be approximately controllable (see the references given in the proof of Theorem 5.4) and thus, we obtain that (5.22) is also approximately controllable.
2. Assume now that $\mathcal{O}_2 \cap \omega = \emptyset$. In this case, the rank condition in Theorem 5.4 simply reduces to the property
$$\mathcal{M}_k\left(a_{21}\phi_k, \omega\right) \neq 0, \quad \forall k \geq 1. \tag{5.24}$$
   (a) In particular, if we assume that (5.23) holds, then, for any $k \geq 1$, there exists at least one connected component C of $\overline{\Omega\backslash\omega}$ such that
$$\int_C a_{21}(\phi_k)^2\, dx \neq 0.$$
   This shows that the first component of $M_k\left(a_{21}\phi_k, C\right)$ is not zero and thus Condition (5.24) holds and System (5.22) is approximately controllable.

   (b) Let C be the connected component of $\overline{\Omega\backslash\omega}$ that contains $\mathcal{O}_2$. Since by assumption C touches the boundary of Ω, we have $\mathcal{M}_k\left(a_{21}\phi_k, \omega\right) \neq 0$ if and only if $\int_C a_{21}(\phi_k)^2\, dx \neq 0$. On the other hand, since $\mathcal{O}_2 \subset C$, we have $\int_0^1 a_{21}(\phi_k)^2\, dx = \int_C a_{21}(\phi_k)^2\, dx$ and the claim is proved.

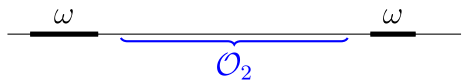
We will now investigate some examples (not necessarily under the assumptions of the previous theorem though).

1. In the first example we consider a coupling coefficient a_{21} that vanishes in ω and does not have a constant sign in $\overline{\Omega\backslash\omega}$. We will provide in particular some controllable systems for which (5.23) fails. Up to our knowledge our analysis is the first available result in this framework.

   We will study two slightly different situations depending on the geometry of the control domain ω, as shown in Figure 5.1.



(a) ω is connected



(b) ω is not connected

FIGURE 5.1 – Two geometries for the study of a 2×2 system

– For some $\alpha \in \mathbb{R}$, we consider (see Figure 5.1a)

$$\omega \subset \left(\frac{3}{4}, 1\right), \quad a_{21}(x) = (x - \alpha)1_{\mathcal{O}_2}(x), \quad \mathcal{O}_2 = \left(\frac{1}{4}, \frac{3}{4}\right).$$

In this case, we are in the framework of Theorem 5.7 and, as a result, the approximate controllability holds if and only if (5.23) holds. If, for any $k \geq 1$, we set

$$\alpha_k = \frac{\int_{\mathcal{O}_2} x\phi_k^2(x)\,dx}{\int_{\mathcal{O}_2} \phi_k^2(x)\,dx},$$

then we obtain that

$$\text{(5.22) is approximately controllable} \Longleftrightarrow \alpha \notin \{\alpha_k\}_{k\geq 1}.$$

As an illustration, in the case $\mathcal{L} = -\partial_x^2$, we have $\phi_k(x) = \sin(k\pi x)$ and a direct computation shows that $\alpha_k = 1/2$ for any $k \geq 1$. Therefore, the system is approximately controllable if and only if $\alpha \neq 1/2$.
To our knowledge, no (positive or negative) null-controllability result is available for this system. However, the numerical results given in [Boy13] in a similar case seem to suggest that it is possible that null-controllability does not hold in general when approximate controllability holds.

– With the same choice of a_{21} and $\mathcal{O}_2$, we consider now the case where

$$\omega \cap \mathcal{O}_2 = \emptyset, \omega \cap \left(\frac{3}{4}, 1\right) \neq \emptyset, \text{ and } \omega \cap \left(0, \frac{1}{4}\right) \neq \emptyset,$$

as shown in Figure 5.1b.
For $\alpha \notin \{\alpha_k\}_{k\geq 1}$, the controllability result obtained above immediately imply the approximate controllability of the system in this new framework.
However, for $\alpha \in \{\alpha_k\}_{k\geq 1}$ it may happen that the system is approximately controllable with this new choice of the control domain ω despite it is not approximately controllable for the previous choice of ω. Indeed, we observe that the only connected component of $\overline{\Omega\backslash\omega}$ that plays a role in the problem does not touch the boundary $\partial\Omega$ anymore. Therefore, in the rank condition given in Theorem 5.4, the second

components in $\mathcal{M}_k\,(a_{21}\phi_k, \omega)$ are no more trivial (see (5.9)) and we have, if $\alpha = \alpha_k$ for some $k \geq 1$,

$$(5.22) \text{ is approximately controllable} \iff \int_{\mathcal{O}_2} (x - \alpha_k)\phi_k\widetilde{\phi}_k \, dx \neq 0.$$

This new condition is not explicit in general, but for instance in the case $\mathcal{L} = -\partial_x^2$ we have $\phi_k(x) = \sin(k\pi x)$ and $\widetilde{\phi}_k(x) = \cos(k\pi x)$ we can check that (recall that $\alpha = 1/2$ is the only interesting value here)

$$\begin{aligned}\int_C a_{21}(x)\phi_k(x)\widetilde{\phi}_k(x)\, dx &= k\pi \int_{1/4}^{3/4} (x - 1/2)\sin(k\pi x)\,\cos(k\pi x)\; dx \\ &= \begin{cases} \frac{-1}{8}(-1)^{k/2} & \text{if } k \text{ is even,} \\ \frac{-1}{4k\pi}(-1)^{(k-1)/2} & \text{if } k \text{ is odd.} \end{cases}\end{aligned}$$

As a consequence, those integrals are never equal to zero and the approximate controllability of the system is proved in this case for any value of α.
It is worth mentioning that for this example the null-controllability of the system remains an open problem (it seems that there is no result available in this direction as soon as the coupling function a_{21} does not have a constant sign).

2. Let us go back to the geometry of Figure 5.1a, in the particular case $\mathcal{L} = -\partial_x^2$ and $\alpha = \frac{1}{2}$.
We have seen that System (5.22) is not approximately controllable in that case, which means that there is at least one initial data y_0 that can not be steered as close to zero as we would like to.
Actually, we can obtain a more precise result in that case since we have seen that, when $\alpha = \frac{1}{2}$, the integrals $\int\limits_C (x - \frac{1}{2})\phi_k^2\, dx$ vanish for every $k \geq 1$ (and not only for one value of k). More precisely, we will identify a set of an infinite number of necessary conditions that should be satisfied by the initial data y_0 in order for the system to be approximately controllable from y_0.
Using that $\mathcal{M}_k\,(a_{21}\phi_k, \omega) = 0$ for any $k \geq 1$, so that from Theorem 5.2, we deduce the existence (and uniqueness) of a function denoted by ψ_k which satisfies

$$\begin{cases} -\psi_k'' - \lambda_k\psi_k = a_{21}\phi_k & \text{in } \Omega, \\ \psi_k = 0 & \text{in } \omega. \end{cases} \tag{5.25}$$

Proposition 5.9. *Let $y_0 = (y_{0,1}, y_{0,2})^* \in L^2(\Omega)^2$. If System (5.22) is approximately controllable from the initial data y_0, we have*

$$\langle y_{0,1}, \psi_k \rangle_{L^2(\Omega)} + \langle y_{0,2}, \phi_k \rangle_{L^2(\Omega)} = 0, \quad \forall k \geq 1. \tag{5.26}$$

Proof. We introduce the set Q_F of the non-observable adjoint states defined as follows

$$Q_F = \Big\{ q_F \in L^2(\Omega)^2, \;\; \text{s.t. the solution of (5.6) satisfies } 1_\omega B^* q(t) = 0, \; \forall t \Big\}.$$

In the present case, we recall that $B^*q = q_1$ is the first component of q.
It is proved in [Boy13, Proposition 1.17] that our system is approximately controllable at time T from a given initial data y_0 if and only if

$$\langle y_0, q(0)\rangle_{L^2(\Omega)} = 0, \quad \text{for any solution } q \text{ of (5.6) with } q_F \in Q_F. \tag{5.27}$$

By construction of ψ_k, the vector $q_F = (\psi_k, \phi_k)^*$ belongs to Q_F for any $k \geq 1$, and the associated solution of the adjoint problem (5.6) is nothing but

$$q(t) = e^{-\lambda_k(T-t)} q_F.$$

It follows that, if the system is controllable from y_0, we necessarily should have

$$\begin{aligned} 0 = \langle y_0, q(0)\rangle_{L^2(\Omega)} &= e^{-\lambda_k T}\langle y_0, q_F\rangle_{L^2(\Omega)} \\ &= e^{-\lambda_k T}\left(\langle y_{0,1}, \psi_k\rangle_{L^2(\Omega)} + \langle y_{0,2}, \phi_k\rangle_{L^2(\Omega)}\right), \end{aligned}$$

for any $k \geq 1$, and the proof is complete. $\square$

Remark 5.10. *It follows from this proposition that the set of initial data for which System* (5.22) *is approximately controllable is a closed subspace of $L^2(\Omega)^2$ of infinite codimension.*
However, we observe that this set is not trivial. Indeed, let us consider an initial data of the form $y_0 = (y_{0,1}, y_{0,2})^*$ with $y_{0,1}$ supported in ω then we have

$$(5.22) \textit{ is approximately controllable from } y_0 \iff y_{0,2} = 0.$$

- $\Rightarrow$ *Since for any $k \geq 1$, $\psi_k = 0$ in ω and $y_{0,1}$ is supported in ω, the first term in* (5.26) *automatically vanishes. Therefore, we have $\langle y_{0,2}, \phi_k\rangle_{L^2(\Omega)} = 0$ for any $k \geq 1$ which leads to $y_{0,2} = 0$.*
- $\Leftarrow$ *We use the characterisation* (5.27) *which reduces to $\langle y_{0,1}, q_1(0)\rangle_{L^2(\Omega)} = 0$ for any $q_F \in Q_F$, since $y_{0,2} = 0$. But this new condition is automatically satisfied because $y_{0,1}$ is supported in ω and $q_1(0) = 0$ in ω by definition of Q_F.*

3. Let us give another example where the controllability conditions are slightly more complex. Our aim is to emphasize that the notion of approximate controllability is very sensitive to the coupling terms in the system ; in some sense we can say that it is not a stable notion with respect to the coefficients of the equation under study.
The situation we consider is the following (see Figure 5.2)

$$\begin{cases} \mathcal{L} = -\partial_x^2, \quad \text{so that } \phi_k(x) = \sin(k\pi x), \forall k \geq 1, \\ \omega \subset (\beta, 1), \quad a_{21} = 1_{\mathcal{O}_2} - 1_{\mathcal{O}'_2}, \\ \text{with } \ \mathcal{O}_2 = [\alpha, \alpha + L], \mathcal{O}'_2 = [\alpha + d, \alpha + d + L], \end{cases} \tag{5.28}$$

for some fixed $\beta \in (0,1)$ and $L, d, \alpha \geq 0$ such that $\alpha + L + d \leq \beta$. Therefore, the coupling term a_{21} takes the values 1 and -1 on two intervals of the same length and its support does not touch the control domain ω.

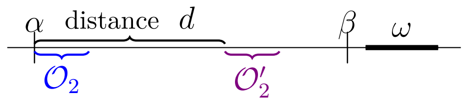


FIGURE 5.2 – The geometry for example (5.28)

There is again one single connected component of $\overline{\Omega\backslash\omega}$, that we denote by C, that plays a role in the controllability, and this latter one touches the boundary $\partial\Omega$. Thus, we are in the framework of Theorem 5.7. Let us compute

$$\begin{aligned}\int_C a_{21}(x)\sin^2(k\pi x)\,dx &= \int_\alpha^{\alpha+L}\sin^2(k\pi x)\,dx - \int_{\alpha+d}^{\alpha+d+L}\sin^2(k\pi x)\,dx\\ &= \frac{-1}{k\pi}\sin(k\pi L)\sin(k\pi d)\sin(k\pi(2\alpha+d+L)).\end{aligned}$$

As a conclusion, System (5.22) is approximately controllable if and only if

$$L \notin \mathbb{Q}, \quad d \notin \mathbb{Q}, \quad 2\alpha + d + L \notin \mathbb{Q}.$$

Fix for instance $L, \alpha \geq 0$ such that $L \notin \mathbb{Q}$, $2\alpha + L \in \mathbb{Q}$ and $\alpha + L < \beta$. Then, for any $d \in [0, \beta - \alpha - L]$, we have

$$\text{System (5.22) is approximately controllable} \iff d \notin \mathbb{Q}.$$

5.3.3.2 Simultaneous control of 2×2 cascade systems

In this section we study the controllability properties of the following 3×3 one-dimensional system,

$$\begin{cases}\partial_t y_1 + \mathcal{L}y_1 = 1_\omega v & \text{in } (0,T)\times\Omega,\\ \partial_t y_2 + \mathcal{L}y_2 = a_{21}(x)y_1 & \text{in } (0,T)\times\Omega,\\ \partial_t y_3 + \mathcal{L}y_3 = a_{31}(x)y_1 & \text{in } (0,T)\times\Omega,\end{cases} \tag{5.29}$$

which corresponds to the case (5.4). Observe that there is no direct interaction between y_2 and y_3 so that the problem can be understood as follows : find a single control $v \in L^2(0,T;L^2(\Omega))$ which simultaneously drives near zero at time T the solutions of the two 2×2 subsystems for (y_1, y_2) in the one hand and for (y_1, y_3) in the other hand.

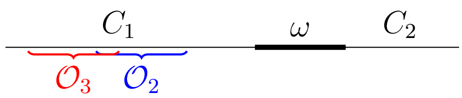
We recall that we can always assume that the coupling terms a_{21} and a_{31} identically vanish on ω, see section 5.3.1. Let us denote by $\mathcal{O}_2$ and $\mathcal{O}_3$ the supports of a_{21} and a_{31}, respectively.

We will illustrate the controllability properties of the system in various situations depending on the geometric configuration of the coupling domains $\mathcal{O}_2$, $\mathcal{O}_3$ and of the control domain ω.

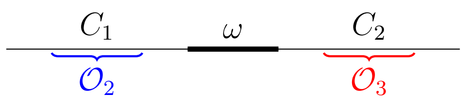
1. We assume first that ω is connected.

In such case there is at most two connected components in $\overline{\Omega \backslash \omega}$, say C_1 and C_2, and they necessarily touch the boundary. Theorem 5.4 then states that system (5.29) is approximately controllable if and only if

$$\operatorname{rank}\begin{pmatrix} M_k\left(a_{21}\phi_k, C_1\right) & M_k\left(a_{31}\phi_k, C_1\right) \\ M_k\left(a_{21}\phi_k, C_2\right) & M_k\left(a_{31}\phi_k, C_2\right) \end{pmatrix} = 2, \quad \forall k \geq 1. \tag{5.30}$$



(a) Coupling terms in the same connected component of $\overline{\Omega \backslash \omega}$



(b) Coupling terms in different connected components of $\overline{\Omega \backslash \omega}$

FIGURE 5.3 – Various geometric situations for the 3×3 system

– First case : $\mathcal{O}_2$ and $\mathcal{O}_3$ are included in the same connected component of $\overline{\Omega \backslash \omega}$, see Figure 5.3a. We see that system (5.29) can not be approximately controllable (whether the supports of a_{21} and a_{31} intersect each other or not). Indeed, (5.30) cannot be true because

$$\operatorname{rank}\begin{pmatrix} M_k\left(a_{21}\phi_k, C_1\right) & M_k\left(a_{31}\phi_k, C_1\right) \\ 0 & 0 \end{pmatrix} \leq 1, \ \forall k \geq 1,$$

since C_1 touches the boundary $\partial\Omega$ and thus there is only one row in this 4×2 matrix which can be non-trivial, see (5.9).

– Second case : $\mathcal{O}_2$ and $\mathcal{O}_3$ are included in two different connected components of $\overline{\Omega \backslash \omega}$, see Figure 5.3b.
Here, we have

$$\operatorname{rank}\begin{pmatrix} M_k\left(a_{21}\phi_k, C_1\right) & 0 \\ 0 & M_k\left(a_{31}\phi_k, C_2\right) \end{pmatrix} = 2 \iff \begin{cases} M_k\left(a_{21}\phi_k, C_1\right) \neq 0, \\ M_k\left(a_{31}\phi_k, C_2\right) \neq 0. \end{cases}$$

Thus, the approximate controllability of system (5.29) in this case is equivalent to the approximate controllability of the two 2×2 systems

$$\begin{cases} \partial_t \hat{y}_1 + \mathcal{L}\hat{y}_1 = 1_\omega \hat{v} & \text{in } (0,T) \times \Omega, \\ \partial_t \hat{y}_2 + \mathcal{L}\hat{y}_2 = a_{21}(x)\hat{y}_1 & \text{in } (0,T) \times \Omega, \end{cases}$$

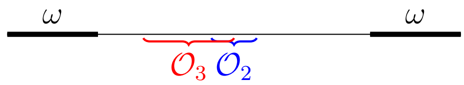


FIGURE 5.4 – The case of a non-connected control domain

and

$$\begin{cases} \partial_t \tilde{y}_1 + \mathcal{L}\tilde{y}_1 = 1_\omega \tilde{v} & \text{in } (0,T)\times\Omega, \\ \partial_t \tilde{y}_3 + \mathcal{L}\tilde{y}_3 = a_{31}(x)\tilde{y}_1 & \text{in } (0,T)\times\Omega, \end{cases}$$

Of course, it is not required here that the controls $\hat{v}$ and $\tilde{v}$ are the same. Actually, by a direct argument we can even prove that the null-controllability of system (5.29) is equivalent to the null-controllability of these 2×2 systems. Indeed, let $\omega = (a,b) \subset\subset \Omega = (0,1)$ and take $\mathcal{L} = -\partial_x^2$ for simplicity. Let $\alpha, \beta \in C^\infty(\overline{\Omega})$ be smooth cut-off functions satisfying

$$\begin{cases} \alpha = 1 \text{ in } (0,a), \\ \alpha = 0 \text{ in } (b,1), \end{cases} \qquad \begin{cases} \beta = 0 \text{ in } (0,a), \\ \beta = 1 \text{ in } (b,1). \end{cases}$$

If $\hat{v}$ and $\tilde{v}$ are null-controls for the 2×2 systems above, we define the control v by

$$v = \alpha\hat{v} + \beta\tilde{v} + (\partial_x^2\alpha)\hat{y} + 2(\partial_x\alpha)(\partial_x\hat{y}) + (\partial_x^2\beta)\tilde{y} + 2(\partial_x\beta)(\partial_x\tilde{y}),$$

It is clear that v belongs to $L^2(\Omega)$ and is supported in ω. On the other hand we can check that $y_1 = \alpha\hat{y}_1 + \beta\tilde{y}_1$, $y_2 = \hat{y}_2$ and $y_3 = \tilde{y}_3$ so that $y_1(T) = y_2(T) = y_3(T) = 0$.

2. Let us consider now a case where ω is not connected. More precisely, we choose $\omega = (0,\alpha) \cup (\beta,1)$ with, for instance, $\alpha < 1/2 < \beta$. In that case, $\overline{\Omega\backslash\omega}$ has also one single connected component C but C does not touch the boundary of Ω. In order to make the computations explicit, we set $\mathcal{L} = -\partial_x^2$.
   We take $a_{21} = 1_{\mathcal{O}_2}$, $a_{31} = 1_{\mathcal{O}_3}$ where $\mathcal{O}_2 =]1/2-\delta_2, 1/2+\delta_2[$ is an interval centered at $1/2$ and $\mathcal{O}_3$ is another interval $\mathcal{O}_3 =]\alpha_3 - \delta_3, \alpha_3+\delta_3[$. They are chosen in such a way that $\mathcal{O}_2 \cap \omega = \mathcal{O}_3 \cap \omega = \emptyset$, see Figure 5.4. The controllability rank condition given by Theorem 5.4 then writes

$$\operatorname{rank}\begin{pmatrix} \int\limits_{\mathcal{O}_2} \sin(k\pi x)^2\, dx & \int\limits_{\mathcal{O}_3} \sin(k\pi x)^2\, dx \\ \int\limits_{\mathcal{O}_2} \sin(k\pi x)\cos(k\pi x)\, dx & \int\limits_{\mathcal{O}_3} \sin(k\pi x)\cos(k\pi x)\, dx \end{pmatrix} = 2, \quad \forall k \geq 1.$$

Using the symmetry of $\mathcal{O}_2$ with respect to $1/2$, we see that

$$\int_{\mathcal{O}_2} \sin(k\pi x)\cos(k\pi x)\, dx = 0.$$

Since $\int_{\mathcal{O}_2} \sin(k\pi x)^2 \, dx > 0$, if follows that the system is controllable if and only if $\int_{\mathcal{O}_3} \sin(k\pi x) \cos(k\pi x) \, dx \neq 0$ for any $k \geq 1$. A straightforward computation shows that

$$\int_{\mathcal{O}_3} \sin(k\pi x) \cos(k\pi x) \, dx = \frac{\sin(2k\pi\delta_3) \sin(2k\pi\alpha_3)}{2k\pi},$$

so that we conclude that

The system is approximately controllable $\Longleftrightarrow$ $\alpha_3 \notin \mathbb{Q}$ and $\delta_3 \notin \mathbb{Q}$.

Let us draw a kind of summary of the previous discussion when $a_{21} = 1_{\mathcal{O}_2}$ and $a_{31} = 1_{\mathcal{O}_3}$ are the characteristic functions of intervals that do not intersect ω :

- In the situation of Figure 5.3a, System (5.29) is never approximately controllable.
- In the situation of Figure 5.3b, System (5.29) is always approximately controllable.
- In the situation of Figure 5.4, the approximate controllability of System (5.29), depends on the precise size and position of the intervals $\mathcal{O}_2$ and $\mathcal{O}_3$.

5.4 Controllability of a 3×3 cascade system

In this section, we are interested in the controllability properties of the following system

$$\begin{cases} \partial_t y_1 + \mathcal{L} y_1 = 1_\omega v & \text{in } (0,T) \times \Omega, \\ \partial_t y_2 + \mathcal{L} y_2 = a_{21}(x) y_1 & \text{in } (0,T) \times \Omega, \\ \partial_t y_3 + \mathcal{L} y_3 = a_{32}(x) y_2 & \text{in } (0,T) \times \Omega. \end{cases} \tag{5.31}$$

This system has a cascade structure since the control v only acts on the first component of the solution which itself has an influence on the second component y_2 through the coupling term $a_{21} y_1$, and finally y_2 also acts on the third component through another coupling term $a_{32} y_2$.

For simplicity, we assume all along this section that there exists a non-empty open set $\mathcal{O}_2 \subset \Omega$ such that

$$a_{21} \neq 0 \text{ a.e. in } \mathcal{O}_2, \quad a_{21} = 0 \text{ a.e. in } \Omega \setminus \mathcal{O}_2. \tag{5.32}$$

This is a (weak) regularity assumption which holds for instance if a_{21} is piecewise continuous and not identically zero.

Remark 5.11. *A first necessary condition for the approximate controllability of System* (5.31) *is the approximate controllability of the subsystem* (5.22)*, which has been studied in Section 5.3.3.1.*

The following result give additional necessary and sufficient conditions that allows a quite simple analysis of the approximate controllability of System (5.31). Under particular assumptions on the coupling coefficients, we see that the study of the controllability for the 3×3 system (5.31) reduces to the study of the controllability of some 2×2 systems. This should be connected with Theorem 5.7 for 2×2 systems.

Theorem 5.12. *Assume that the* 2×2 *subsystem* (5.22) *is approximately controllable and let* a_{21} *satisfy* (5.32).

1. *Assume that* $\mathcal{O}_2 \cap \omega \neq \emptyset$.
   (a) *If the* 2×2 *system*

$$\begin{cases} \partial_t y_2 + \mathcal{L} y_2 = 1_{\mathcal{O}_2 \cap \omega} \hat{v} & \text{in } (0,T) \times \Omega, \\ \partial_t y_3 + \mathcal{L} y_3 = a_{32}(x) y_2 & \text{in } (0,T) \times \Omega, \end{cases} \tag{5.33}$$

   is approximately controllable, then System (5.31) *is itself approximately controllable.*

   (b) *If System* (5.31) *is approximately controllable and* $\mathcal{O}_2 \subset \omega$, *then System* (5.33) *is approximately controllable.*

2. *Assume now that* $\mathcal{O}_2 \cap \omega = \emptyset$.
   (a) *If the coupling coefficient* a_{32} *satisfies*

$$\int_0^1 a_{32} (\phi_k)^2 \, dx \neq 0, \quad \forall k \geq 1, \tag{5.34}$$

   then, System (5.31) *is approximately controllable.*

   (b) *If System* (5.31) *is approximately controllable and* $\mathcal{O}_2$ *is entirely included in a connected component of* $\overline{\Omega \backslash \omega}$ *that touches the boundary* $\partial\Omega$, *then* (5.34) *holds.*

Let us consider some basic examples of applications of this result.

– Assumption (5.34) is for example fulfilled if a_{32} has a constant sign on Ω and is not identically zero. Combining this result with the discussion in Section 5.3.3.1, we deduce that our 3×3 system is approximately controllable if a_{32} and a_{21} both have constant signs (not necessarily the same sign) on Ω, and are non-identically zero. This situation is illustrated numerically in [Boy13, Sect 4.4.2].
– However, observe that the sign condition for a_{32} is not necessary for the previous corollaries to apply. For instance, as we have seen in the item #1 of Section 5.3.3.1, (5.34) also holds for any $k \geq 1$ in the case where $\mathcal{L} = -\partial_x^2$ and $a_{32} = (x-\alpha) 1_{(1/4,3/4)}(x)$ for $\alpha \neq 1/2$.
– Finally, consider the case where $\mathcal{L} = -\partial_x^2$, $\omega = (1/2, 1)$, $a_{21}(x) = 1_{(0,1/2)}(x)$ and $a_{32}(x) = x - 1/2$. Here the coupling domain $\mathcal{O}_2 = (0, 1/2)$ and ω are two disjoint intervals. Since a straightforward computation shows that (5.34) fails for any $k \geq 1$, we can apply Theorem 5.12 and see that the system is not approximately controllable. This result is also numerically illustrated in [Boy13, Section 4.4.2]
  Observe that the subsystem satisfied by (y_1, y_2) is approximately controllable. The lack of controllability is thus a consequence of the structure of the coupling term a_{32} between the second and third components. Note also that a_{32} was however supported everywhere.

The proof of Theorem 5.12 relies on the following characterization.

Proposition 5.13. *Assume that the* 2×2 *subsystem* (5.22) *is approximately controllable and let* a_{21} *satisfy* (5.32). *Then, System* (5.31) *is* **not** *approximately controllable if and only if there exists* $k \geq 1$ *and* $v \in \mathcal{D}(\mathcal{L})$ *such that*

$$\begin{cases} \mathcal{L}v - \lambda_k v = a_{32}(x)\phi_k & \textit{in } \Omega, \\ v = 0 & \textit{in } \mathcal{O}_2 \cap \omega, \\ \mathcal{M}_k(a_{21}v, \omega) = 0. \end{cases} \tag{5.35}$$

Proof (of Proposition 5.13). From Theorem 5.1, System (5.31) is **not** approximately controllable if and only if there exists $k \geq 1$ and $u = (u_1, u_2, u_3)^* \in \mathcal{D}(\mathscr{L})$ with $u \neq 0$ such that

$$\begin{cases} \mathcal{L}u_1 - \lambda_k u_1 = a_{21}u_2 & \text{in } \Omega, \\ \mathcal{L}u_2 - \lambda_k u_2 = a_{32}u_3 & \text{in } \Omega, \\ \mathcal{L}u_3 - \lambda_k u_3 = 0 & \text{in } \Omega, \\ u_1 = 0 & \text{in } \omega. \end{cases}$$

Clearly, $u_3 = \delta\phi_k$ for some $\delta \in \mathbb{R}$. Moreover, we have $\delta \neq 0$. Indeed, if we assume that $\delta = 0$ then (u_1, u_2) is not trivial and satisfies

$$\begin{cases} \mathcal{L}u_1 - \lambda_k u_1 = a_{21}u_2 & \text{in } \Omega, \\ \mathcal{L}u_2 - \lambda_k u_2 = 0 & \text{in } \Omega, \\ u_1 = 0 & \text{in } \omega, \end{cases}$$

and this is a contradiction with the approximate controllability of the subsystem (5.22), by Theorem 5.1.

Thus, under this assumption, System (5.31) is **not** approximately controllable if and only if there exists $k \geq 1$, $\delta \neq 0$ and $u_1, u_2 \in \mathcal{D}(\mathcal{L})$ such that

$$\begin{cases} \mathcal{L}u_1 - \lambda_k u_1 = a_{21}u_2 & \text{in } \Omega, \\ \mathcal{L}u_2 - \lambda_k u_2 = \delta a_{32}\phi_k & \text{in } \Omega, \\ u_1 = 0 & \text{in } \omega. \end{cases}$$

Using Theorem 5.2 this is equivalent to the existence of $k \geq 1$, $\delta \neq 0$ and $u_2 \in \mathcal{D}(\mathcal{L})$ such that

$$\begin{cases} \mathcal{L}u_2 - \lambda_k u_2 = \delta a_{32}\phi_k & \text{in } \Omega, \\ a_{21}u_2 = 0 & \text{in } \omega, \\ \mathcal{M}_k(a_{21}u_2, \omega) = 0. \end{cases}$$

Finally, by definition of $\mathcal{O}_2$ (see (5.32)), we have $a_{21}u_2 = 0$ almost everywhere in ω if and only if $u_2 = 0$ almost everywhere in $\mathcal{O}_2 \cap \omega$. This proves the proposition with $v = u_2/\delta$.

We turn out to the proof of Theorem 5.12.

Proof (of Theorem 5.12). We use the characterization of Proposition 5.13.

1. Assume first that $\mathcal{O}_2 \cap \omega \neq \emptyset$. Note that this condition automatically implies the approximate controllability of the 2×2 subsystem (5.22) by Theorem 5.7.

(a) Looking at the first two equations of (5.35) and using Theorem 5.1, it is not difficult to see that, if System (5.31) is not approximately controllable, then the 2×2 system (5.33) is not approximately controllable either.

(b) When $\mathcal{O}_2 \subset \omega$, the third condition $\mathcal{M}_k(a_{21}v, \omega) = 0$ of (5.35) is always fulfilled since, in the one hand $a_{21} = 0$ almost everywhere in $\Omega \setminus \mathcal{O}_2$ (by (5.32)) and in the other hand $v = 0$ almost everywhere in $\mathcal{O}_2 \cap \omega = \mathcal{O}_2$. It follows from Theorem 5.1 that System (5.31) is approximately controllable if and only if so is the 2×2 system (5.33).

2. Assume now that $\mathcal{O}_2 \cap \omega = \emptyset$.

(a) The orthogonality condition (5.34) is necessary for the existence of a solution to the first equation of (5.35). Thus, System (5.31) is approximately controllable if this latter one fails.

(b) It follows in particular from the assumption on $\mathcal{O}_2$ that $\mathcal{O}_2 \cap \omega = \emptyset$. Thus, the second equation of (5.35) is now empty. On the other hand, let $k \geq 1$ be such that

$$\int_0^1 a_{32}(\phi_k)^2 \, dx = 0.$$

Then, we know that the first equation in (5.35) admits an infinite number of solutions of the form $v = v_0 + \alpha\phi_k$, $\alpha \in \mathbb{R}$, where $v_0 \in \mathcal{D}(\mathcal{L})$ is the unique solution of this equation that satisfies $\langle v_0, \phi_k \rangle_{L^2(\Omega)} = 0$. Let C be the connected component of $\overline{\Omega \backslash \omega}$ that contains $\mathcal{O}_2$. Since by assumption C touches the boundary of Ω, and by (5.9), we have $\mathcal{M}_k(a_{21}v, \omega) = 0$ if and only if $\int_C a_{21} v \phi_k \, dx = 0$. It remains to prove that we can choose α such that

$$\int_C a_{21} v \phi_k \, dx = \int_C a_{21} v_0 \phi_k \, dx + \alpha \int_C a_{21} (\phi_k)^2 \, dx = 0.$$

In particular, it is enough to prove that $\int_C a_{21}(\phi_k)^2 \, dx \neq 0$. By assumption the 2×2 subsystem (5.22) is approximately controllable, and $a_{21} = 0$ almost everywhere in $\omega \subset \Omega \setminus \mathcal{O}_2$. Thus, $\mathcal{M}_k(a_{21}\phi_k, \omega) \neq 0$ (by Theorem 5.4), so that $\int_C a_{21}(\phi_k)^2 \, dx \neq 0$ (same reasoning as above), and the claim is proved.

5.5 Simultaneous control of uncoupled systems

In this section we still study systems of the general form (5.1) but in a slightly different framework compared to the previous sections.

Since we are mainly going to deal with examples, we restrict ourselves to the case $n = 2$ for simplicity. We assume here that $B = (b_1, b_2)^*$ is any vector in $\mathbb{R}^2$, that the coupling terms satisfy $A(x) = 0$ for any $x \in \Omega$ and that the (diagonal) operator $\mathscr{L}$ is given by

$$\mathscr{L} = \begin{pmatrix} \mathcal{L}_1 & 0 \\ 0 & \mathcal{L}_2 \end{pmatrix}, \tag{5.36}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are two possibly different elliptic operators. Hence, the system we are interested in writes

$$\begin{cases} \partial_t y + \mathscr{L} y = 1_\omega B v & \text{in } (0,T)\times\Omega, \\ y(0) = y_0 & \text{in } \Omega. \end{cases} \tag{5.37}$$

We assume that $b_1 \neq 0$ and $b_2 \neq 0$, because if it is not the case, the controllability of (5.37) clearly fails. Observe that the controllability also fails if $\mathcal{L}_1 = \mathcal{L}_2$.

In the case where the operators $\mathcal{L}_i$ are different but multiples of the same operator $\mathcal{L}$ the following null-controllability result was proved in [AKBDGB09b, Remark 1.1].

Theorem 5.14. *Let $\mathcal{L}$ be an elliptic operator as defined in the introduction* (5.2) *and ω a non-empty open subset in Ω. For $i = 1, 2$, we set $\mathcal{L}_i = d_i \mathcal{L}$ for some $d_i > 0$, we define $\mathscr{L}$ by* (5.36) *and we suppose given $B = (b_1, b_2)^*$ with $b_1 \neq 0$ and $b_2 \neq 0$. Then,*

$$(5.37)\textit{is null-controllable} \Longleftrightarrow d_1 \neq d_2.$$

We are interested here in studying some examples where the operators $\mathcal{L}_i$ are different but not proportional to a given elliptic operator ; this appears to be a more intricate problem. The strategy is still based on the unique continuation criterion given by Fattorini and is therefore restricted to the approximate controllability property.

We will assume that $\mathcal{L}_1 = -\partial_x^2$ and that $\mathcal{L}_2 = -\partial_x\left(\gamma(x)\partial_x \cdot\right)$ for some $\gamma \in L^\infty(\Omega)$ and $\inf_\Omega \gamma > 0$.

In this framework, Theorem 5.1 says that the system is approximately controllable if and only if, for any $s \in \mathbb{C}$ we have

$$\left.\begin{aligned} \mathcal{L}_1 u_1 &= s u_1 \quad \text{in } \Omega \\ \mathcal{L}_2 u_2 &= s u_2 \quad \text{in } \Omega \\ b_1 u_1 + b_2 u_2 &= 0 \qquad \text{in } \omega \end{aligned}\right\} \Longrightarrow u_1 = u_2 = 0, \quad \forall u_1 \in \mathcal{D}\left(\mathcal{L}_1\right), u_2 \in \mathcal{D}\left(\mathcal{L}_2\right).$$

However, since $b_i \neq 0$, this condition is equivalent to

$$\left.\begin{aligned} \mathcal{L}_1 u_1 &= s u_1 \quad \text{in } \Omega \\ \mathcal{L}_2 u_2 &= s u_2 \quad \text{in } \Omega \\ u_1 &= u_2 \quad\;\; \text{in } \omega \end{aligned}\right\} \Longrightarrow u_1 = u_2 = 0, \quad \forall u_1 \in \mathcal{D}\left(\mathcal{L}_1\right), u_2 \in \mathcal{D}\left(\mathcal{L}_2\right). \tag{5.38}$$

Of course, if $\mathcal{L}_1$ and $\mathcal{L}_2$ have no common eigenvalues then this condition is automatically satisfied and the system is approximately controllable. If $\mathcal{L}_1$ and $\mathcal{L}_2$ have a common eigenvalue, we have to analyse if the corresponding eigenfunctions can coincide on the control domain ω.

Let us look more precisely at two different examples.

- Example 1 : the diffusion coefficients are equal in the control domain ω.
  More precisely, we set $\omega = (0, 1/2)$ and we assume that γ is piecewise constant

$$\gamma(x) = \begin{cases} 1, & \text{for } x \in (0, 1/2) = \omega, \\ \gamma_2, & \text{for } x \in (1/2, 1), \end{cases}$$

with $\gamma_2 > 0$.
Let $\lambda_k = k^2\pi^2$, $k \geq 1$ be an eigenvalue of $\mathcal{L}_1 = -\partial_x^2$ and $u_1(x) = \sin(k\pi x)$ be the associated eigenfunction. An eigenfunction u_2 of $\mathcal{L}_2$ for the same eigenvalue and which coincides with u_1 in ω has necessarily the following form

$$u_2(x) = \begin{cases} \sin(k\pi x), & \text{for } x \in (0, 1/2) \\ \delta \sin\left(\frac{k\pi}{\sqrt{\gamma_2}}(x-1)\right), & \text{for } x \in (1/2, 1), \end{cases}$$

and, δ should be determined in order to satisfy the following transmission conditions at $x = 1/2$

$$\begin{cases} -\delta \sin\left(\dfrac{k\pi}{2\sqrt{\gamma_2}}\right) = \sin\left(\dfrac{k\pi}{2}\right), \\ \dfrac{\delta}{\sqrt{\gamma_2}} \cos\left(\dfrac{k\pi}{2\sqrt{\gamma_2}}\right) = \cos\left(\dfrac{k\pi}{2}\right). \end{cases}$$

– If $k = 2p$ is even, the existence of a δ satisfying those equations is equivalent to

$$\sqrt{\gamma_2} = \frac{p}{q}, \quad \text{for some } q \in \mathbb{N}^*.$$

– If $k = 2p + 1$ is odd, the existence of δ is equivalent to

$$\sqrt{\gamma_2} = \frac{2p+1}{2q+1}, \quad \text{for some } q \in \mathbb{N}^*.$$

The conclusion of this study is that, the system is approximately controllable if and only if $\sqrt{\gamma_2} \notin \mathbb{Q}$.

– Example 2 : The non-controllability situations that we underlined in Example 1 seem to be the consequence of the fact that the diffusion coefficients of the two operators $\mathcal{L}_1$ and $\mathcal{L}_2$ coincide in the control domain ω. However, we want to show here that we can construct an example of a non-controllable system of the same kind even if the diffusion coefficients are completely different for the two operators.
We first choose $0 < \alpha < 1/4$, and the control domain $\omega = (0, \alpha)$. We set

$$\beta = \frac{\sin(2\pi\alpha)}{\sin(\pi\alpha)}, \quad C = \frac{\sin(\pi\alpha)\cos(2\pi\alpha)}{2\sin(2\pi\alpha)} - \cos(\pi\alpha).$$

We consider now the following definition of the diffusion coefficient that defines the operator $\mathcal{L}_2$

$$\gamma(x) = \begin{cases} 1 + \frac{C}{\cos(\pi x)}, & \text{for } x \in (0, \alpha) = \omega, \\ \frac{1}{4}, & \text{for } x \in (\alpha, 1). \end{cases}$$

Observe that, even if $C < 0$, we still have $\inf \gamma > 0$. A straightforward computation shows that the function u_2 defined by

$$u_2(x) = \begin{cases} \sin(\pi x), & \text{for } x \in (0, \alpha) = \omega, \\ \beta \sin(2\pi x), & \text{for } x \in (\alpha, 1), \end{cases}$$

is an eigenfunction of $\mathcal{L}_2$ associated with the eigenvalue π^2 which obviously coincides with $\sin(\pi x)$ on ω.
As a consequence, with this particular choice of the diffusion coefficient, the parabolic system under study is not approximately controllable.

5.6 Conclusion and perspectives

In this paper, we have given some easily checkable necessary and sufficient conditions for the approximate controllability of some 1D coupled parabolic systems with space-dependent coefficients. These conditions have been illustrated on many simple examples to show that the controllability issue for those systems can be an intricate problem depending on the geometry of the control domain and of the characteristics of the coupling terms in the system.

Finally, we observe that some of our examples can be extended to simple Cartesian geometries but the study of the general multi-dimensional systems is far from being straightforward and is still widely open.

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Résumé

Dans cette thèse on s'intéresse à la contrôlabilité de deux classes de systèmes paraboliques linéaires.

On étudie dans un premier temps la contrôlabilité à zéro de systèmes à coefficients constants en dimension 1 lorsque les contrôles agissent sur différentes parties du domaine ou de sa frontière. Un changement de variable adapté et des résultats antérieurs basés sur les inégalités de Carleman et la méthode des moments permettent d'obtenir une caractérisation complète du problème. On voit notamment que le système peut être contrôlable sans qu'il soit forcément possible de contrôler avec seulement des contrôles internes ou frontières.

On regarde ensuite avec le théorème de Fattorini la contrôlabilité frontière approchée de ces systèmes en dimension quelconque. On obtient une condition de résonance des valeurs propres qui se révèle suffisante. On considère également le problème sur un rectangle et on démontre que les systèmes de 2 équations sont toujours contrôlables si la zone de contrôle contient 2 directions.

Dans un autre travail sur les systèmes à coefficients constants, on obtient une estimation du coût du contrôle frontière à zéro en dimension 1 à l'aide de la méthode des moments. On combine ensuite ce résultat avec la méthode de Lebeau-Robbiano pour montrer que la contrôlabilité frontière à zéro dans des domaines cylindrique est réduite à la contrôlabilité frontière à zéro en dimension 1, étant connu que cette dernière est caractérisée par la condition de rang de Kalman.

On étudie ensuite la contrôlabilité approchée de systèmes en cascade avec un couplage d'ordre 1. On prouve que la contrôlabilité interne avec un couplage constant à toujours lieu, quel que soit la dimension et la zone de contrôle. On établit d'autre part une caractérisation de la contrôlabilité frontière en dimension 1 avec un couplage variable.

Enfin, dans une dernière partie on s'intéresse à la contrôlabilité interne approchée de systèmes en cascade à coefficients variables en dimension 1. En utilisant le théorème de Fattorini et la structure des systèmes étudiés on est ramené à établir une caractérisation de la propriété de continuation unique pour une équation elliptique non-homogène. A l'aide de la caractérisation alors obtenue on montre en particulier comment la géométrie de la zone de contrôle peut influencer la contrôlabilité des systèmes.

Abstract

This thesis focuses on the controllability of two classes of linear parabolic systems.

We start with the study of the null-controllability of systems with constant coefficients in dimension 1 where the controls are acting on different parts of the domain or its boundary. A suitable change of variable and some previous results based on Carleman estimates and the method of moments lead to a complete caracterization of the problem. In particular, we see that the system can be controllable whereas it is not if we allow only distributed or boundary controls.

With the help of the theorem of Fattorini we then look at the boundary approximate controllability of these systems in any dimension. We obtain a resonance condition that is sufficient. We also consider the problem on a rectangular domain and we show that it is always controllable for a system of 2 equations if we assume that the control domain contains 2 directions.

In another work on the systems with constant coefficients, we obtain an estimate of the boundary null-control cost in dimension 1, using the method of moments. We then combine this result with the Lebeau-Robbiano strategy to show that the boundary null-controllability in cylindrical domains is reduced to the boundary null-controllability in dimension 1, this latter being caracterized by the Kalman rank condition.

We then study the approximate controllability of cascade systems with a first order coupling term. We prove the distributed controllability when the coupling is constant, whatever the dimension and control domain are. On the other hand, we establish a caracterisation of the boundary controllability in dimension 1 for space-dependent couplings.

Last, we investigate the distributed approximate controllability of cascade systems with space-dependent coefficients in dimension 1. Using the theorem of Fattorini and the structure of the systems under study we are lead to characterize the unique continuation property for a non-homogeneous elliptic equation. With the help of the caracterization then obtained we show in particular how the geometry of the control domain can affect the controllability properties of systems.

Mots clés : systèmes paraboliques ; contrôlabilité interne ; contrôlabilité frontière ; continuation unique ; théorème de Fattorini ; test de Hautus ; condition de Kalman ; inégalités de Carleman.

Classification AMS : 93B05, 93B07, 93C20, 35K05, 93C05.