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Structure chirale de la gravité quantique à boucles

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The Chiral Structure of Loop Quantum Gravity

Meinen Eltern

ABSTRACT

Loop gravity is a tentative theory to describe what happens at the Planck scale, the scale at which both general relativity and quantum theory become equally important. It comes in two versions. The canonical approach seeks to solve the Wheeler–DeWitt equation and find the physical states of the theory. Spinfoam gravity, on the other hand, takes a covariant path-integral representation to define the transition amplitudes of the theory. Both approaches use the same Hilbert space, but we do not know whether they actually define the same theory.

In this thesis, I will present four results, all of which lie in between the two approaches. We start with the classical theory. When Ashtekar first formulated Hamiltonian general relativity in terms of selfdual (complex Ashtekar) variables the ADM constraint equations turned into neat polynomials of the elementary fields. This was a huge simplification and eventually initiated the program of loop quantum gravity. For a number of technical reasons the complex variables have later been abandoned in favour of the $SU(2)$ Ashtekar–Barbero variables, and the simplification of the Hamiltonian constraint was lost again. These $SU(2)$ variables are usually derived from the Holst action, which contains the Barbero–Immirzi parameter as an additional coupling constant.

After the first introductory chapter, we will use the original selfdual connection to repeat the canonical analysis for the Holst action, while leaving the Barbero–Immirzi parameter untouched. The resulting constraint equations depend on this parameter, yet maintain a polynomial form. To guarantee that the metric is real, we have to introduce additional constraints. These reality conditions match the linear simplicity constraints of spinfoam gravity. They are preserved in time only if the spatial spin connection is torsionless, which appears as a secondary constraint in the canonical analysis. This is our first complex of results.

The next chapter is about the classical theory, and studies how to discretise gravity in terms of first-order holonomy-flux variables. The corresponding phase-space has a non-linear structure. Twistors allow to handle this non-linearity while working on a linear phase-space with canonical Darboux coordinates. This framework was originally introduced by Freidel and Speziale, but only for the case of $SU(2)$ Ashtekar–Barbero variables. Here, we develop the generalisation to $SL(2,\mathbb{C})$, that is we use twistors to parametrise the phase-space of selfdual holonomy-flux variables.

We will then discuss the spinfoam dynamics in terms of these twistorial variables, and develop a new Hamiltonian formulation of discretised gravity. This is based upon a continuum action adapted to a fixed simplicial discretisation of space-time. The action is a sum of the spinorial analogue of the topological "BF"-action and the reality conditions that guarantee the existence of a metric.

Chapter four studies the resulting quantum theory. Since the action is a polynomial in the spinors, canonical quantisation is rather straightforward. Transition amplitudes reproduce the EPRL (Engle–Pereira–Rovelli–Livine) spinfoam model. This is an interesting result, since it shows that spinfoam gravity can be derived from a classical action, with spinors as the fundamental configuration variables.

VORWORT

Von allen Grundkräften der Physik passt nur die Schwerkraft nicht zur Quantentheorie. Als schwächste aller fünf Wechselwirkungen (das sind die beiden Kernkräfte, die elektromagnetische Kraft und eben die Gravitation) spielt sie für die Physik des Mikrokosmos keine Rolle; die Gravitation beherrscht die Welt im Großen. Die allgemeine Relativitätstheorie liefert den mathematischen Rahmen. Einstein erklärt die Schwerkraft als Folge der geometrischen Struktur von Raum und Zeit: Genauso wie die Krümmung der Erdoberfläche den Kurs eines Flugzeugs bestimmt (von Wien nach Washington folgt der Pilot der kürzesten Verbindenden, einem Großkreis, keiner Geraden), genauso zwingt die Krümmung der Raumzeit die Erde auf ihre Bahn um die Sonne

Die Quantenmechanik spielt für die Umlaufbahnen der Planeten keine Rolle. Sie beschreibt die Physik im Kleinen. Ort und Impuls eines Teilchens lassen sich als komplementäre Variable nicht gleichzeitig scharf messen, sind als Zufallsgrößen unscharf verschmiert. Die Schrödingergleichung beschreibt diese Unschärfen als Wellenfeld in Raum und Zeit. *„Das Elektron trifft in zehn Minuten am Ort x ein.“* So ein Satz ist der Quantentheorie ganz unbekannt, wir sagen stattdessen: *„In zehn Minuten ist das Elektron mit $p(x)$-prozentiger Wahrscheinlichkeit am Orte x.“*

Den alten Streit um die Frage, ob es kleinste Teilchen gebe, oder die Welt aus einem stofflichen Kontinuum bestehe, beendet die Quantenmechanik mit einem salomonischen Urteil. Beides ist gleichermaßen wahr, und hängt von der Fragestellung ab. In dem einen Experiment enthüllt sich die Quantennatur der Welt: Angeregte Atome senden Lichtteilchen nur ganz bestimmter Farbe aus – das charakteristische Orange der Straßenlaternen kommt vom Natrium. Ein anderer Versuch zeigt die Kontinuumseigenschaften der Materie: Bei aufmerksamem Blick in eine Straßenlaterne kann man Beugungsringe sehen, wenn im Augapfel die Wellen des Natriumlichts an kleinen Hindernissen streuen.

Was hat das nun alles mit der Gravitation zu tun? Nach Einsteins allgemeiner Relativitätstheorie hat das Schwerefeld der Erde stets überall einen fest vorhersagbaren Wert. Für die Quantenwelt ist das nicht mehr so, hier sind nur mehr Wahrscheinlichkeitsaussagen möglich. Es kann nicht beides stimmen: Denn alle Materie folgt nicht nur den Regeln der Quantenmechanik, sondern koppelt in immer gleicher Weise an die Gravitation. Die Gesetze der Quantenmechanik müssen auch für die Schwerkraft gelten. Wie die Quantentheorie mit der Relativitätstheorie zu versöhnen sei, das weiß freilich niemand so genau. Trotz jahrzentelanger, teils recht phantastischer Bemühungen, fehlt uns noch immer eine Theorie der Quantengravitation.

Was können wir von einer Quantentheorie der Gravitation erwarten? Zunächst müsste sie alle bisherigen experimentellen Tests bestehen. Sie muss uns aber auch Fragen beantworten, die über unser bisheriges Verständnis weit hinausgehen: Was geschah beim Urknall? Was sind die Quanten des Gravitationsfeldes? Ist vielleicht die Geometrie der Raumzeit selbst gequantelt, gibt es gleichsam kleinste Raumatome? Was geschieht im Inneren eines schwarzen Lochs?

Meine Doktorarbeit beschäftigt sich mit nur einem Ansatz diese Fragen zu beantworten, mit der *loop quantum gravity* wie die Theorie auf Englisch heißt. An erster Stelle steht die Frage: Gelingt der Übergang zur bekannten Physik? Dafür braucht es geeignetes mathematisches Handwerkszeug. Meine Doktorbeit entwickelt solches Werkzeug, und untersucht den *klassichen Grenzfall* der Theorie. Ich kann zeigen, dass die Schleifentheorie im klassichen Limes zu einer Vielteilchentheorie wird. Die zugehörigen Punktteilchen bewegen sich allerdings nicht in Raum und Zeit, sondern leben in einem zweidimensionalen komplexen Vektorraum, im Raume der Spinoren.

RÉSUMÉ

La gravité quantique à boucles est une théorie candidate à la description unifiée de la relativité générale et de la mécanique quantique à l'échelle de Planck. Cette théorie peut être formulée de deux manières. L'approche canonique, d'une part, cherche à résoudre l'équation de Wheeler-DeWitt et à définir les états physiques. L'approche par les écumes de spins, d'autre part, a pour but de calculer les amplitudes de transition de la gravité quantique via une intégrale de chemin covariante. Ces deux approches s'appuient sur a même structure d'espace de Hilbert, mais la question de leur correspondance exacte reste un important problème ouvert à ce jour.

Dans ce travail de thèse, nous présentons quatre résultats en rapport avec ces deux approches. Après un premier chapitre introductif, le second chapitre concerne l'étude de la théorie classique. Historiquement, l'introduction des variables d'Ashtekar complexes (self-duales) dans la formulation hamiltonienne de la relativité générale fut motivée par l'obtention d'une contrainte scalaire polynomiale. Cette simplification drastique est à la base du programme de la gravité quantique à boucles. Pour un certain nombre de raisons techniques, ces variables complexes furent ensuite abandonnées au profit des variables d'Ashtekar-Barbero, pour lesquelles le groupe de jauge est $SU(2)$. Avec ce choix de variables réelles, la contrainte hamiltonienne n'est malheureusement plus polynomiale. La formulation en terme des variables $SU(2)$ réelles peut être obtenue à partir de l'action de Holst, qui contient le paramètre dit de Barbero-Immirzi comme constante de couplage additionnelle. Dans un premier temps, nous allons utiliser les variables d'Ashtekar complexes pour effectuer l'analyse canonique de l'action de Holst avec un paramètre de Barbero-Immirzi réel. Les contraintes qui découlent de cette analyse canonique dépendent de ce paramètre libre, et ont l'avantage d'être polynomiales. Afin de garantir que la métrique soit une quantité réelle, un ensemble de contraintes de réalité doivent être imposées. Il s'avère que ces conditions de réalité correspondent aux contraintes de simplicité linéaires utilisées pour la construction des modèles d'écumes de spins. Ces contraintes sont préservées par l'évolution hamiltonienne si et seulement si la connexion est sans torsion. Cette condition sur l'absence de torsion est en fait une contrainte secondaire de l'analyse canonique.

La second chapitre concerne également la théorie classique, mais s'intéresse à sa discrétisation en terme des variables de premier ordre dites holonomie-flux. L'espace des phases qui résulte de cette construction possède une structure non-linéaire. Le formalisme des twisteurs permet d'accommoder cette non-linéarité en travaillant sur un espace des phases linéaire paramétré par les coordonnées canoniques de Darboux. Ce formalisme fut introduit par Freidel et Speziale, mais uniquement dans le cas des variables $SU(2)$ d'Ashtekar-Barbero. Nous généralisons ce résultat au cas du groupe de Lorentz.

Nous étudions ensuite la dynamique en terme d'écumes de spins obtenue à partir de ces variables, et développons une nouvelle formulation hamiltonienne de la gravité discrétisée. Ce nouveau formalisme est obtenu en écrivant l'action de la théorie continue sur une discrétisation simpliciale de l'espace-temps fixée. L'action discrète ainsi obtenue est la somme de l'analogue en terme de spineurs d'une action topologique de type BF et des contraintes de réalité qui garantissent l'existence d'une métrique réelle. Cette action est polynomiale en terme des spineurs, ce qui permet de procéder à sa quantification canonique de manière relativement aisée.

Le dernier chapitre s'intéresse à la théorie quantique obtenue suivant cette procédure. Les amplitudes de transition reproduisent celles du modèle d'écume de spins

EPRL (Engle Pereira Rovelli Livine). Ce résultat est intéressant car il démontre que la formulation de la gravité quantique en termes d'écumes de spins peut être obtenue à partir d'une action classique écrite en terme de spineurs.

Contents

Preface

Two nights ago, on June 17th at 23:40, I saw a bright star moving fast from West to East in the sky over Marseille. Today I learnt it was the international space station (ISS). Artificial satellites are among the very few devices that bring the two scientific revolutions of the twentieth century together: The atomic clocks*, that will fly to the ISS in 2016, will work according to the principles of quantum theory, and general relativity will tell us how fast it runs. Yet, we do not need a quantum theory of gravity to understand time-measurements in low earth orbit.

In fact, for all everyday problems quantum theory and general relativity live in splendid isolation. This is no longer true once we reach the Planck scale, where we have to face questions that go beyond the two: What happened at the big bang? What is the final fate of an evaporating black hole? Are there quanta of space? What is the microscopic origin of Hawking radiation?

Today, theoretical physics lacks the unifying language to explore these questions, but there are promising research lines aiming for the goal: String theory, loop quantum gravity, causal dynamical triangulations, the asymptotic safety scenario, twistor theory, supergravity, non-commutative geometry and many other ideas seek to solve the trouble. By all measures, string theory, is the most successful and also the most ambitious proposal—it aims for a theory of everything. This thesis is about another approach, loop quantum gravity, which is its most prominent competitor.

I first studied loop quantum gravity in Austria, during a joined seminar of the University of Vienna and the Vienna University of Technology. Aichelburg and Balasin, gave an introduction to the theory, then the students had to present some selected papers. I spoke about the *Mathematical Structure of Loop Quantum Cosmology* by Ashtekar, Bojowald and Lewandowski [1].

The kinematics of the theory is very well understood, excitations of geometry can be neatly visualised as polyhedra glued among their facets. The area of a surface, angles, lengths and volumes turn into operators on a separable Hilbert space with discrete Planckian spectrum. For the dynamics the situation is different, our knowledge is fragmentary and incomplete. The spinfoam approach explores this gap, and seeks to define the transition amplitudes of the theory. It is the main focus of this thesis.

The most important result of my research concerns the mathematical foundations of the theory. I can show that there is a classical theory behind spinfoam gravity. This is a truncation of general relativity to a fixed discretisation of space-time, a version of first-order Regge calculus, with spinors as the fundamental configuration variables. The action is a one-dimensional integral over the edges of the discretisation. The spinors are canonical coordinates on the phase-space of the theory, and there is a Hamiltonian gen-

*The ACES (Atomic Clock Ensemble in Space) will consist of two atomic clocks on board the ISS, perform tests on special relativity, general relativity and search for time variations in the fine structure constant.

erating the evolution equations. This makes quantisation rather straight forward. The resulting transition amplitudes agree with those proposed by the spinfoam approach.

To develop this result, we need some preparations. The introduction is followed by chapter 2 that studies the Hamiltonian formulation of general relativity in first-order tetrad-connection variables. Although this is mostly a review, it contains some new insights, that are important for the rest of the work. I have distributed the main results of my thesis over the other two chapters. The classical aspects appear in chapter 3. This is where I derive the action of the theory. Chapter 4 develops the quantum theory. I have also added several supplements. They contain further results and some additional background material. The thesis ends with the conclusion and five appendices.

Over the course of my doctoral studies I wrote and contributed to several articles, all of which are listed below. This thesis collects only some of them, mainly those [P1-P4, P7] that are linked by the appearance of selfdual variables. If have also added parts of [P5] and [P6] to the thesis. These two articles do not mention complex variables, but they still fit very well into the thesis. Notice also, that all authors are ordered alphabetically, reflecting the impossibility to distinguish individual contributions, if ideas are shared and grow from lively debates among the collaborators.

Marseille, Summer 2013
W. M. W.

LIST OF PUBLICATIONS

[P1] Wolfgang M. Wieland: Hamiltonian Spinfoam Gravity, accepted for publication in *Class. Quantum Grav.* on 8 May 2013, `arXiv:1207.6348`

[P2] Simone Speziale and Wolfgang M. Wieland: Twistorial structure of loop-gravity transition amplitudes, *Phys. Rev. D* **86** (2012), 124023, `arXiv:1207.6348`

[P3] Wolfgang M. Wieland: Twistorial phase space for complex Ashtekar variables, *Class. Quantum Grav.* **29** (2012), 045007 (18pp), `arXiv:1107.5002`

This article has been selected by the Editorial Board of Classical and Quantum Gravity (CQG) to be one of the journal's Highlights of 2011-2012.

[P4] Wolfgang M. Wieland: Complex Ashtekar Variables and Reality Conditions for Holst's action, *Ann. Henri Poincaré* **13** (2012), 425–448, `arXiv:1012.1738`

[P5] Hal M. Haggard, Carlo Rovelli, Francesca Vidotto, and Wolfgang M. Wieland: The spin connection of twisted geometry, *Phys. Rev. D* **87** (2013), 024038, `arXiv:1211.2166`

[P6] Eugenio Bianchi and Wolfgang M. Wieland: Horizon energy as the boost boundary term in general relativity and loop gravity, (2012) `arXiv:1205.5325`

[P7] Wolfgang M. Wieland: Complex Ashtekar variables, the Kodama state and spinfoam gravity, (2011) `arXiv:1105.2330`

[P8] Eugenio Bianchi, Muxin Han, Elena Magliaro, Claudio Perini, Carlo Rovelli, and Wolfgang M. Wieland: Spinfoam fermions, (2011) `arXiv:1012.4719`

[P9] Herbert Balasin and Wolfgang M. Wieland: Simplified Hamiltonian constraint for a particular value of the Barbero–Immirzi parameter, (2009) `arXiv:0912.3443`

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State College, Fall 2013
W. M. W.

"Have we any good reason for thinking space-time is continuous? Do we know that, between one orbit and the next, other orbits are *geometrically* possible? Einstein has led us to think that the neighbourhood of matter makes space non-Euclidean; might it not also make it discontinuous? It is certainly rash to assume that the minute structure of the world resembles that which is found to suit large-scale phenomena, which may be only statistical averages. These considerations may serve as an introduction to the most modern theory of quantum mechanics, to which we must now turn our attention."
Bertrand Russell, *The Analysis of Matter*, (1927)

1
Introduction

1.1 THE PROBLEM

Quantum theory tells us nature is intrinsically probabilistic. The numbers computed from the Born rule do not reflect a lack of experimental precision, but teach us that there is a fundamental limit of how much we can now about a quantum system [2–4].

General relativity, on the other hand, is a deterministic field theory for a Lorentzian metric on a four-dimensional space-time manifold. The theory completes the revolution of Faraday and Maxwell: Fields are the only things existing. The points and coordinates of the space-time manifold lack a universal operational interpretation, they acquire a physical meaning only after having solved the equations of motion. What matters are events, and the relation among them [5].

Quantum field theory in curved space-time [6–8] is the closest the two theories get, but yet it rests upon the particular properties of a fixed background geometry. To build the Fock space, and speak about the particles as the quanta of the field, space-time must be asymptotically stationary at late or early times—but a general solution of Einstein's equations has no isometries whatsoever. Quantum states belong to hypersurfaces of constant time—but in general relativity there is no preferred foliation of space-time. Field operators separated by a space-like distance commute—but if the metric becomes an operator, and distances are no longer sharply defined, what does this mean for locality?

The lesson from general relativity is that gravity is not yet another field in Minkowski space-time, but gives geometry its very shape. Heisenberg, on the other hand, has taught us that quantum fluctuations are an inevitable consequence of $\hbar > 0$. If gravity is to be quantised, the minute structure of space-time itself should undergo quantum fluctuations. But if there is no absolute background structure with respect to which we can measure time and distance, what should these fluctuations ever refer to?

Beside these conceptual difficulties, there is the lack of a universal strategy of how to solve the trouble. The earliest idea was to use perturbation theory. If we expand the Einstein equations around a classical solution, they turn into the field equations for weak gravitational waves propagating at the speed of light. The quantum theory for the resulting spin two particle (the graviton) is however perturbatively non-renormalisable, which renders its high-energy limit physically meaningless [9, 10]. Additional symmetries may cure the problem. This was the hope of supersymmetry and conformal gravity. Neither of them work. In supergravity the divergences reappear at higher orders, while conformal gravity breaks unitarity [11–13]. String theory [14–17], on the other hand, gave the perturbative program new impetus. Gravity is no longer a fundamental field, but strings are the elementary building blocks of everything existing. The strings are moving in a fixed ten-dimensional background geometry, but this background is not arbitrary: The renormalisation-group equations imply the Einstein equations for the background metric coupled to a dilaton (a Brans–Dicke type of field) and a tower of additional fermionic and bosonic particles. Gravity only emerges at an effective level.

The non-perturbative approach takes another road to quantum gravity. Instead of adding further symmetries to improve the convergence of the perturbative expansion, this program argues for the symmetries that the theory already has at a non-perturbative level. This is the diffeomorphism group that generates finite coordinate transformations. Loop quantum gravity [5,18–21] is a particular realisation of this idea. It is the main subject of this thesis.

Beside string theory and loop quantum gravity, there is a zoo of many more well developed lines of research: Quantum Regge calculus and causal dynamical triangulations [22–25] propose a gravitational path-integral. Asymptotic safety studies the renormalisation group equations for generalised gravitational actions, and argues that the coupling constants approach a fix point for high energies [10, 26]. Hořava–Lifshitz gravity also tries to make sense out of ordinary perturbation theory: It abandons general covariance at short distance, and seeks to recover general relativity only at a macroscopic scale [27]. Non-commutative geometry turns the inertial coordinates of four-dimensional Minkowski space into operators hoping this would remove divergences at short distances [28]. Twistor theory [29–32] takes light rays as the fundamental building blocks; wherever two of them intersect this defines an event, which makes space-time itself a derived entity. Entropic gravity [33] is an approach where gravity is not a fundamental force, hence needs not to be quantised at all, but emerges as an effective description of some unknown microscopic theory. The books by Kiefer and Oriti [34, 35] give an excellent overview of many of the approaches mentioned.

Before we go into more details, let us first understand the scale of the problem.

1.2 THE SCALE OF THE PROBLEM

Combining the charge e and mass m_e of an electron with Planck's constant $\hbar$ we can build the Bohr radius

$$a_{\mathrm{B}} = \frac{\hbar^2}{e^2 m_e}. \tag{1.1}$$

This number has a universal meaning. It is the scale at which quantum mechanics prevents the electron from falling into the nucleus—the fundamental scale of atomic physics.

Combining Newton's constant G with the speed of light c and $\hbar$, we can build another length scale. This is the Planck length:

$$\ell_{\mathrm{P}} = \sqrt{\frac{8\pi\hbar G}{c^3}}. \tag{1.2}$$

Once again this constant is universal. It is the scale at which both quantum theory and general relativity become equally important. To understand how this scale emerges, let us study a simplified version of an argument that goes back to Bekenstein and Mukhanov [36, 37].

Consider two gravitating particles of mass m that are in a common orbit around another, separated by a distance $2r$. The total angular momentum of the system is

$$L = 2m\omega r^2, \tag{1.3}$$

where ω is the orbital frequency of each individual particle. Newton's law implies that for each particle the gravitational acceleration balances the centrifugal force, thus

$$m\omega^2 r = \frac{Gm^2}{4r^2}. \tag{1.4}$$

Let us now bring in Bohr's quantisation scheme for the angular momentum:

$$L = 2m\omega r^2 = \hbar j, \quad j \in \mathbb{N}_>. \tag{1.5}$$

This reveals, just as in atomic physics, that the spatial extension of the system is quantised. For the radius, the allowed values in powers of j are:

$$r = \frac{\hbar^2 j^2}{Gm^3}. \tag{1.6}$$

Putting $j = 1$ we reach the innermost radius allowed by quantum mechanics. This radius further shrinks as we increase the mass of the system. Can it become arbitrarily small? Probably not, and the reason is this:

Consider the velocity of the particles. Keeping $j = 1$ fixed, ωr grows quadratically with m, we have:

$$\omega r = \frac{Gm^2}{2\hbar}. \tag{1.7}$$

So far, we have only used Newtonian gravity and some aspects of quantum theory. But from Einstein's theory of special relativity, we know nothing can move faster than the speed of light. The particles at the innermost orbit reach this limit once m is of the order of the Planck mass:

$$m \approx \sqrt{\frac{\hbar c}{G}}. \tag{1.8}$$

At this point the system has found its minimal size, which is of the order of the Schwartzschild radius of a Planckian mass black hole. Going beyond this limit the system should disappear behind an event horizon.

Let us now repeat the construction for all other values of j. Just as before, the j-th orbit shrinks as we increase the mass of the system. From general relativity we know this process has a limit. There is no orbital motion beyond the event horizon. The event horizon appears where ωr is of the order of c. This implies, that the smallest possible orbit has a radius that scales linearly with the square root of j. In fact, it circumscribes a circle with an area of the order:

$$A_j \approx \frac{\hbar G}{c^3} j. \tag{1.9}$$

A similar formula appears in loop quantum gravity, where the area of a surface turns into an operator on the Hilbert space of the theory [38, 39]. For an elementary surface, the eigenvalues of this operator are given by the numbers:

$$A_j = \frac{8\pi\beta\hbar G}{c^3}\sqrt{j(j+1)}, \quad 2j \in \mathbb{N}_0, \tag{1.10}$$

where $\beta > 0$ is the Barbero–Immirzi parameter, which enters the theory as an additional coupling constant.

Although there is no obvious relation between the chain of arguments that has led us to (1.9) and the derivation of (1.10), I think, we can learn something important here.

In this section, we have reached the Planck scale by considering a bound gravitational system—the Newtonian analogue of a hydrogen atom. Calculating the energy levels of a hydrogen atom is a straight forward exercise in non-relativistic quantum mechanics. Yet, this turns into a difficult problem when it comes to quantum field theory (QFT); bound states are difficult for QFT. At this point quantum chromodynamics (QCD) often serves as a prototypical example: At high energies quarks are asymptotically free, and we can very well use ordinary perturbation theory to compute scattering amplitudes. At low energies quarks are however confined to form compound particles. These are bound states, and the S matrix is not so much the object of our interest anymore. In the confined phase we better use lattice-QCD, which is another kind of approximation.

If the physics of the Planck scale resembles what goes on in bound systems then perturbation theory may very well be the wrong tool to study quantum gravity. We should better use a non-perturbative approach. This is the perspective of loop quantum gravity, that does indeed share key features with lattice-QCD. But what is loop quantum gravity?

1.3 KEY CONCEPTS OF LOOP GRAVITY

Loop gravity [5, 18–20, 40] is a quantum theory of holonomies and fluxes. But what are holonomies and fluxes, and why should they play a major role in a quantum theory of gravity?

Holonomies measure the parallel transport along a line γ. We can define them for any gauge connection. In the prototypical example of electromagnetism (a $U(1)$ gauge theory) the holonomy is nothing but the exponential $h_\gamma[A] := \mathrm{e}^{\mathrm{i}\int_\gamma \mathrm{d}x^i A_i} \in U(1)$ of the vector potential A_i integrated over γ. For the definition of the fluxes, on the other hand, we have to first study the canonical structure. Looking at the Lagrangian we can identify the electric field E^i as the canonical momentum, and thus find the Poisson brackets: $\{E^i(\vec{x}), A_j(\vec{y})\} = \delta^i_j \delta^{(3)}(\vec{x}-\vec{y})$. The momentum variable defines a two-form,* which we can naturally smear over a two-dimensional surface obtaining the electric flux: $E[f] = \frac{1}{2}\int_f \mathrm{d}x^i \wedge \mathrm{d}x^j \epsilon_{ijk} E^k$. We can now choose paths $\gamma_1, \gamma_2, \dots$ and surfaces $f_1, f_2, \dots$ to arrive at a whole set of holonomies and fluxes. What makes these variables important for the quantisation program is that they close under the Poisson bracket, and thus form an algebra—the holonomy-flux algebra. If we go to the non-Abelian case this feature survives [41, 42]. In our prototypical example of a $U(1)$ gauge theory the only non-vanishing Poisson brackets are in fact: $\{E[f], h_\gamma\} = \mathrm{i}n(\gamma, f) h_\gamma$, where $n(\gamma, f)$ is the intersection number between γ and f.

For loop gravity the relevant vector potential was discovered by Ashtekar, Barbero and Immirzi [43–45]. They have shown that general relativity admits a canonical formulation on the phase space of an $SU(2)$ gauge theory. The canonical variables are the $SU(2)$ Ashtekar–Barbero connection ${A^{(\beta)i}}_a$, and its conjugate momentum—this is the densitised triad ${E_i}^a$. The fundamental Poisson brackets among these variables are: $\{{E_i}^a(p), {A^j}_b(q)\} = 8\pi G\beta/c^3 \delta^j_i \delta^a_b \delta^{(3)}(p, q)$, where the so-called Barbero–Immirzi parameter $\beta > 0$ enters the classical theory as a free number.

*If the configuration variable is a p-form then the momentum (in three spatial dimensions) must be a $(3-p)$-form.

What is the geometric interpretation of these variables? For the densitised triad the situation is simple. If $e^i{}_a$ denotes the cotriad* on a spatial slice, then the non-Abelian electric field of the theory is nothing but $E_i{}^a = \frac{1}{2}\epsilon_{ilm}\tilde{\eta}^{abc}e^l{}_b e^m{}_c$, where ϵ_{ijk} ($\tilde{\eta}^{abc}$) is the Levi-Civita tensor (density). The Ashtekar–Barbero connection $A^{(\beta)i}{}_a$, on the other hand, is a mixture of intrinsic and extrinsic data. The two ingredients are the intrinsic Levi-Civita connection $\Gamma^i{}_a$ on the spatial slice (itself functionally depending on the triad via the torsionless condition: $\partial_{[a}e^i{}_{b]} + \epsilon^i{}_{lm}\Gamma^l{}_{[a}e^m{}_{b]} = 0$) and the extrinsic curvature tensor K_{ab} (the second fundamental form). The Ashtekar–Barbero connection is the sum of these two terms weigthed by the Barbero–Immirzi parameter: $A^{(\beta)i}{}_a := \Gamma^i{}_a + \beta e^{ib}K_{ba}$.

Once we have a phase space we can try to run the program of canonical quantisation in order to define the kinematical** Hilbert space of the theory. In loop quantum gravity we do however not start from the continuous Poisson brackets $\{E_i{}^a(p), A^{(\beta)j}{}_b\} = \dots$, but rather look at the reduced phase space of holonomies and fluxes. This is motivated by the *loop assumption* [46, 47]:

> *The loop assumption:* Loop quantum gravity is a theory of locally flat connections. Curvature sits at one-dimensional defects. At the smallest scales, holonomies and fluxes can capture all gravitational degrees of freedom.

In a quantum theory the state of the system is represented by a complex valued functional of the configuration variable, in our case we take this to be the connection, and are thus searching for functionals $\Psi[A^{(\beta)i}{}_a]$. But if the curvature is concentrated on one-dimensional defects, it suffices to consider a collection of holonomies $h_{\gamma_1}, h_{\gamma_2}, \dots \in SU(2)$ to uniquely characterise the vector-potential up to gauge transformations (see [46, 48] and references therein). We can thus restrict ourselves to wave-functionals of the form: $\Psi_f[A^{(\beta)i}{}_a] = f(h_{\gamma_1}, h_{\gamma_1}, \dots)$, where f is a function on a number of copies of $SU(2)$. Functionals of this type are called *cylindrical*, we say in fact: A functional $\Psi_f[A^{(\beta)i}{}_a]$ of the connection is cylindrical with respect to a graph*** $\Gamma = (\gamma_1, \dots, \gamma_N)$ Γ (symbolically denoted by $\Psi \in \mathrm{Cyl}_\Gamma$) if there is a function f on N copies of $SU(2)$ such that $\Psi_f[A^{(\beta)i}{}_a] = f(h_{\gamma_1}, \dots, h_{\gamma_N})$, and we say $\Psi \in \mathrm{Cyl}$ if there is a graph Γ such that $\Psi \in \mathrm{Cyl}_\Gamma$. What is important about these functions is that between any two of them there is a natural inner product: if Ψ_f and $\Psi_{f'}$ are cylindrical with respect to the same graph we set:

$$\langle \Psi_f, \Psi_{f'} \rangle_\Gamma = \int_{SU(2)^N} d\mu_{\mathrm{Haar}}(U_1)\dots d\mu_{\mathrm{Haar}}(U_N)\overline{f(U_1, \dots, U_N)}f'(U_1, \dots, U_N), \quad (1.11)$$

where $d\mu_{\mathrm{haar}}(U)$ is the normalised Haar measure on the group. This inner product can be generalised to introduce an inner product on all of Cyl: For any two functions $\Psi, \Phi \in \mathrm{Cyl}$ we can always find a graph Γ large enough such that both Ψ and Φ are elements of Cyl_Γ. The symmetries of the Haar measure guarantee that the resulting number $\langle \Psi, \Phi \rangle_\Gamma$ is independent of the actual graph under consideration: $\langle \Psi, \Phi \rangle_\Gamma = \langle \Psi, \Phi \rangle_{\Gamma'}$, if $\Psi, \Phi \in \mathrm{Cyl}_\Gamma$ and also $\Psi, \Phi \mathrm{Cyl}_{\Gamma'}$. We can thus equip Cyl with a natural inner product and turn it into a Hilbert space [49].

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*This is an orthonormal basis in co-tangent space, $i = 1, 2, 3$ are internal indices, while $a, b, c, \dots$ are abstract indices on the spatial slice.

**The kinematical Hilbert space is only an auxiliary object needed to turn the classical constraints (the Gauß law, the vector and Hamiltonian constraints) into operators. Regaining the dynamics amounts to find the solution space of the constraints: Physical states are those that are annihilated by the constraints.

***A graph Γ is an ordered collection of piecewise differentiable oriented paths γ_i, $i = 1, \dots, N < \infty$.

To speak about physical states we also have to impose the constraints. For the so-called kinematical states this can be done with remarkable ease: The Gauß constraints restricts us to the $SU(2)$ gauge invariant subspace of Cyl, while the vector constraint identifies any two states that a diffemorphism can map into another. [19]

Penrose's spin network functions [50] form the most common orthonormal basis in the resulting Hilbert space. Excitations of geometry can neatly be visualised as fuzzy polyhedra glued among their facets [51–55]. Area, angles, length and volume turn into self-adjoint operators with a discrete Planckian spectrum [38, 39, 56–59]. The Hilbert space of a single tetrahedron may serve as a minimal example to illustrate the resulting quantum geometry. A classical tetrahedron is characterised by six numbers, e.g. the lengths of the six bones bounding the triangles. These numbers depend on the metric tensor, hence turn into operators once gravity is quantised. Yet, we cannot diagonlise all of them, simply because they do not commute among another. Therefore, a quantised tetrahedron lacks a true shape. If we make some of its geometrical properties sharp, others become fuzzy. The most common choice for a complete set of commuting operators consists of the four areas and one dihedral angle, or four areas and the volume of the tetrahedron.

For the dynamics the situation is different, no such clean physical picture is available. There are two ideas of how to define the dynamics of the theory. The first idea [19, 40] follows Dirac's program of canonical quantisation [60]. This uses the Hamiltonian formulation of the theory, which rests upon a spatio-temporal decomposition of the space-time manifold. Picking a time-coordinate breaks general covariance, only spatial diffeomorphisms remain manifest. Four-dimensional coordinate invariance is restored only dynamically by the Hamiltonian constraint. Its quantisation yields the Wheeler–DeWitt equation [61]. The second idea looks for a covariant path integral formulation. This comes under the name of spinfoam gravity [20, 62], which is the main focus of this thesis.

However these two approaches will ever manifest themselves, they should just be two ways to define the very same physical theory, and indeed, at least at a formal level, this it what happens [63] in the Wheeler–DeWitt theory: The path integral gives transition amplitudes that formally solve the Wheeler–DeWitt equation. Whether this is true also for loop gravity is one of the most important consistency checks for the theory. I cannot give a conclusive answer to this question, but I can show that spinfoam gravity comes from the canonical quantisation of a classical theory. This is a version of first-order Regge calculus [64], with spinors as the fundamental configuration variables. I will present this result in chapters 3 and 4. It should be a convincing evidence that spinors provide a universal language to bring the two sides of the theory together.

2
Hamiltonian general relativity

Classical mechanics follows from the quantum theory in the same way that geometrical optics is a consequence of the fundamental Maxwell equations. Schrödinger's key idea on the way to quantum theory was to reverse this logic. Reading the Hamilton–Jacobi equation as an eikonal approximation to the dynamics of the matter waves he could postulate his famous equation, and immediately calculate the energy spectrum of the hydrogen atom. This was one of the most impressive achievements of twentieth century's theoretical physics.

The program of canonical quantisation follows this insight, and start with the Hamiltonian formalism of the theory. Section 2.1 looks at the action. Working with first-order tetrad-connection variables we can replace the usual Einstein–Hilbert–Palatini action by the Holst action [65] without ever changing the classical dynamics of the theory. To make the variational principle well defined, we will specify the boundary conditions and study the boundary terms added to the action. In the presence of corners we will also need additional two-dimensional integrals, that must be added to the usual Gibbons–Hawking–York boundary term. [66–71] Section 2.2 looks at the three-plus-one split of the theory in terms of selfdual variables. Section 2.3 gives an application of the formalism thus developed. Studying the Hamilton–Jacobi equations of the theory, we will introduce a local notion of gravitational energy. The result will match what has been recently reported by Frodden, Gosh and Perez, who studied the gravitational energy as measured by a family of uniformly accelerated observers. [72, 73] This gives yet another independent derivation of their results. Section 2.4 introduces the Hamiltonian formalism in terms of complex Ashtekar variables [43]. The system of constraints contains both first- and second-class conditions. The first-class constraints generate the gauge symmetries of the theory, while the reality conditions are needed to guarantee the existence of a metric geometry.

This chapter has two supplements, in the first of which we will review the ADM (Arnowitt–Deser–Misner) formalism of general relativity in terms of metric variables [74]. We take a finite region of space-time, with the topology of a cylinder and study the appropriate boundary and corner terms. Repeating Witten's proof [75, 76] of the positivity of the ADM mass [77–79], we will then give a motivating example illustrating the power of spinorial methods [29, 30]. In fact, spinors will play a prominent role in the following chapters. The second supplement looks at the Kodama state [80–89], which is a formal solution of the quantised Hamiltonian constraint. Although it does probably not give the vacuum of the theory [90], this state has recently regained some attention. The idea is that it could be related to a certain deformation of the theory needed to introduce a cosmological constant [91–95].

2.1 THE HOLST ACTION AND ITS BOUNDARY TERMS

Loop quantum gravity rests upon the possibility to recover the ADM phase space in four space-time dimensions from the kinematical framework of an $\mathfrak{su}(2)$ (respectively $\mathfrak{sl}(2,\mathbb{C})$) Yang–Mills gauge theory. The underlying connection appears most naturally when starting from the Hamiltonian formulation emerging from the Holst action*. In terms of the cotetrad (also: covierbein) field η^α, and the $\mathfrak{so}(1,3)$-valued spin connection $\omega^\alpha{}_\beta$ we can write this action as the following:

$$\begin{aligned} S_{\text{Holst}}[\eta,\omega,n,z] = \frac{\hbar}{2\ell_{\text{P}}^2}\Big[\int_M \Big(\frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\eta^\alpha\wedge\eta^\beta\wedge\mathcal{F}^{\mu\nu}[\omega] - \frac{1}{\beta}\eta_\mu\wedge\eta_\nu\wedge\mathcal{F}^{\mu\nu}[\omega]+\\ -\frac{\Lambda}{12}\epsilon_{\alpha\beta\mu\nu}\eta^\alpha\wedge\eta^\beta\wedge\eta^\mu\wedge\eta^\nu\Big) - \int_{\partial M}\epsilon_{\alpha\beta\mu\nu}\eta^\alpha\wedge\eta^\beta\wedge n^\mu \mathcal{D}n^\nu+\\ -\int_S \epsilon_{\alpha\beta\mu\nu}n^\alpha z^\beta\,\eta^\mu\wedge\eta^\nu\,\Xi\Big] \equiv \frac{\hbar}{2\ell_{\text{P}}^2}\big(I_M + I_{\partial M} + I_S\big). \end{aligned} \tag{2.1}$$

Preparations Compared to the original paper [65] we have added a cosmological constant, and boundary terms needed to make the action functionally differentiable. Otherwise the variational principle remains obscure. This we will discuss in a minute. Before doing so, let us first clarify notation and terminology, which is further explained in the appendices.

Beside the cosmological constant Λ (with dimension of an inverse area), two more coupling constants appear, $0<\beta\in\mathbb{R}$ is the Barbero–Immirzi parameter and $\ell_{\text{P}} = \sqrt{8\pi\hbar G/c^3}$ is the reduced Planck length. We will see, the Barbero–Immirzi parameter does, however, not enter the classical equations of motion, but can only affect the quantum theory. Setting $\beta\to\infty$, brings the action back into a more familiar form.

The connection one-form $\omega^\alpha{}_\beta$ being dimensionless, this is also true for the curvature two-form $\mathcal{F}^\mu{}_\nu$; the cotetrad η^α, on the other hand, has dimensions of length, and therefore the whole expression has the correct dimensions of an action, i.e. dimensions of $\hbar$. We fix the sign conventions for both the metric $\eta_{\alpha\beta}$, used to move internal indices ($\alpha,\beta\dots\in\{0,1,2,3\}$), and the internal Levi-Civita tensor $\epsilon_{\alpha\beta\mu\nu}$ by setting $\eta_{00}=-1$, and $\epsilon_{0123}=1$. The action also contains the curvature of the $\mathfrak{so}(1,3)$ connection, defined by Cartan's second structure equation:

$$\mathcal{F}^\alpha{}_\beta[\omega] = \mathrm{d}\omega^\alpha{}_\beta + \omega^\alpha{}_\mu\wedge\omega^\mu{}_\beta. \tag{2.2}$$

We are considering a four-dimensional space-time region M, the boundary ∂M of which consists of two spatial regions Σ_0 and Σ_1 (with the topology of a three-dimensional ball) meeting at a two-sphere $S=\partial\Sigma_0=(\partial\Sigma_1)^{-1}$. Figure 2.1 gives an illustration of the lensoid geometry. The time-like normal of ∂M is n^a, written in internal space this becomes $n^\alpha = \eta^\alpha{}_a n^a$. We also take n^α on both Σ_0 and Σ_1 to be future oriented. The tangent space of S is two-dimensional, and so is its orthogonal complement $TS^\perp$. The internal vectors (n_0^α, z_0^α) are a basis in $TS^\perp$ (once mapped back by the tetrad towards ordinary tangent space); n_0^α is the future oriented normal to Σ_0 while z_0^α is perpendicular to S and points towards the outside of the three-dimensional hypersurface Σ_0. The same holds for the dyad (n_1^α, z_1^α); n_1^α is the future oriented normal of Σ_1, while z_1^α lies tangential to Σ_1 and is outwardly oriented (when looking from Σ_1). Again

*In fact it is rather misleading to call it that way. Holst though proving this action naturally leads to the $SU(2)$ Ashtekar–Barbero variables, did actually not introduce it first. This was done by Hojman et al. [96] already in the 1980. I'm grateful to Friedrich Hehl for pointing this out.

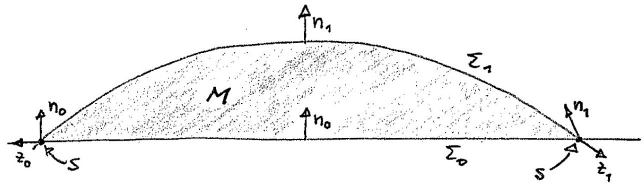

Figure 2.1: We are considering a lensoid region M in space-time, the boundary of which consists of two parts. The top and bottom Σ_1 and Σ_0 are spatial three-dimensional surfaces, their future oriented time-normal we call n^α. The two-dimensional corner S bounds both Σ_1 and Σ_0. It has space-like outwardly oriented normals z_0^α and z_1^α, that are tangential to Σ_0 and Σ_1 respectively.

figure 2.1 should further clarify the geometry. Let us also note that we can put the indices $(0,1)$ referring to the actual slice Σ_0 and Σ_1, wherever we want, and declare e.g. $n^1_\alpha = \eta_{\alpha\beta} n^\beta_1$. In a supplement to this chapter we will study a similar geometry of cylindrical shape, with the appropriate boundary and corner terms in the action, but this time in the metric formulation.

To make sense of the integrand, space-time must be orientable. Fixing an orientation in M also induces an orientation on the boundary. We say, the ordered triple $(x, y, z) \in T\Sigma_1 \times T\Sigma_1 \times T\Sigma_1$ is right-handed, if (n, x, y, z) is positively oriented in M, while on Σ_0 we do the opposite, and declare (x, y, z) to have positive orientation, only if $(-n, x, y, z)$ is positively oriented in M. For both cases n is the future oriented time-normal of the respective hypersurface. Finally, we also fix the orientation on the corner S, and declare $(x, y) \in TS \times TS$ to be positively oriented if (n, z, x, y) is positively oriented in M, with $z \in TS$ being a vector pointing outside of the spatial interior Σ_1 of S.

Let us now calculate the variation of the action, in order to eventually show two things. First of all we would like to recover the Einstein equations from the action (2.1), and see why the additional *Holst-term* $\propto \beta^{-1} \int_M \eta_\mu \wedge \eta_\nu \wedge \mathcal{F}^{\mu\nu}$ does not affect the equations of motion. Next, we should identify the boundary conditions needed for the variational principle to be well defined. This means, we have to prove the variations of the action vanish, if both equations of motion and boundary conditions are satisfied, i.e.:

$$\delta(S_{\text{Holst}})\big|_{\text{EOM+BC}} \stackrel{?}{=} 0. \tag{2.3}$$

Before we can actually show this, we have to explain two further elements appearing in the action. This is the exterior $\mathfrak{so}(1,3)$ covariant derivative $\mathcal{D}$ with respect to the spin connection $\omega^\alpha{}_\beta$, and the relative rapidity Ξ of the two spatial hypersurfaces. The covariant derivative acts on any tensor-valued* p-form $\varphi^{\alpha_1\alpha_2\cdots}$ as follows

$$\mathcal{D}\varphi^{\alpha_1\alpha_2\cdots} = \mathrm{d}\varphi^{\alpha_1\alpha_2\cdots} + \omega^{\alpha_1}{}_\beta \wedge \varphi^{\beta\alpha_2\cdots} + \omega^{\alpha_2}{}_\beta \wedge \varphi^{\alpha_1\beta\cdots} + \dots . \tag{2.4}$$

The rapidity Ξ, on the other hand, is nothing but the Minkowski inner product of the two respective normals, i.e.:

$$\operatorname{sh}\Xi = \eta_{\alpha\beta} n^\alpha_0 z^\beta_1 . \tag{2.5}$$

Equations of motion To show equivalence of the theory derived from the action (2.1) with general relativity let us study the variation of each term separately. We start with

*This refers to a tensor in internal space. Notice also, that our definition can immediately be generalised to mixed tensors with both covariant and contravariant indices in internal space.

the integral I_M over the bulk. This contains the curvature two-form. Looking at (2.2), and noting that the variation commutes with the exterior derivation, i.e. $\mathrm{d}\delta = \delta\mathrm{d}$, we soon find the variation of the field-strength to be:

$$\delta F^{\alpha}{}_{\beta} = \mathcal{D}\delta\omega^{\alpha}{}_{\beta}. \tag{2.6}$$

Calculations become more transparent when introducing the tensors

$$Q^{\alpha\beta}{}_{\mu\nu} = \frac{1}{2}\epsilon^{\alpha\beta}{}_{\mu\nu} - \frac{1}{\beta}\delta^{[\alpha}_{\mu}\delta^{\beta]}_{\nu}, \qquad O^{\alpha\beta}{}_{\mu\nu} = \frac{1}{2}\epsilon^{\alpha\beta}{}_{\mu\nu} + \frac{1}{\beta}\delta^{[\alpha}_{\mu}\delta^{\beta]}_{\nu}. \tag{2.7a}$$

One of which is the inverse of the other:

$$(OQ)^{\alpha\beta}{}_{\mu\nu} = (QO)^{\alpha\beta}{}_{\mu\nu} = Q^{\alpha\beta}{}_{\rho\sigma}O^{\rho\sigma}{}_{\mu\nu} = -\frac{1+\beta^2}{\beta^2}\delta^{[\alpha}_{\mu}\delta^{\beta]}_{\nu}. \tag{2.8}$$

With the help of Stoke's theorem we can perform a partial integration and eventually find the variation of the bulk term:

$$\begin{aligned}\delta I_M = \int_M \Big(2Q_{\alpha\beta\mu\nu}\delta\eta^{\alpha}\wedge\eta^{\beta}\wedge\mathcal{F}^{\mu\nu}[\omega] - \frac{\Lambda}{3}\epsilon_{\alpha\beta\mu\nu}\delta\eta^{\alpha}\wedge\eta^{\beta}\wedge\eta^{\mu}\wedge\eta^{\nu} + \\ - 2Q_{\alpha\beta\mu\nu}\mathcal{D}\eta^{\alpha}\wedge\eta^{\beta}\wedge\delta\omega^{\mu\nu}\Big) + 2\int_{\partial M} Q_{\alpha\beta\mu\nu}\eta^{\alpha}\wedge\eta^{\beta}\wedge\delta\omega^{\mu\nu}.\end{aligned} \tag{2.9}$$

The first integral gives the equations of motion in the bulk. We are in a first-order formalism, which implies that we can independently vary both the connection and the tetrad. Employing the inverse of Q, i.e. using equation (2.8), the variation of the connection in the bulk leads us to:

$$\mathcal{D}\eta^{[\alpha}\wedge\eta^{\beta]} = 0. \tag{2.10}$$

If the tetrad is non-degenerate, i.e. the volume element

$$d^4v_\eta := \frac{1}{4!}\epsilon_{\alpha\beta\mu\nu}\eta^{\alpha}\wedge\eta^{\beta}\wedge\eta^{\mu}\wedge\eta^{\nu} \neq 0 \tag{2.11}$$

does not vanish, which we always assume in the following, this implies the vanishing of torsion

$$\Theta^{\alpha} := \mathcal{D}\eta^{\alpha} = 0. \tag{2.12}$$

This can be seen as follows, let Θ^{α} be a Minkowski valued one-form such that $\Theta^{[\alpha}\wedge\eta^{\beta]} = 0$. Employing the antisymmetry of the wedge product this immediately yields

$$\Theta^{\alpha}\wedge\eta^{\mu}\wedge\eta^{\nu} = -\Theta^{\alpha}\wedge\eta^{\nu}\wedge\eta^{\mu} = -\Theta^{\nu}\wedge\eta^{\alpha}\wedge\eta^{\mu}. \tag{2.13}$$

Again using $\Theta^{[\alpha}\wedge\eta^{\beta]} = 0$ we thus get:

$$0 = \Theta^{[\alpha}\wedge\eta^{\mu]}\wedge\eta^{\nu} = -\Theta^{\nu}\wedge\eta^{[\alpha}\wedge\eta^{\mu]} = -\Theta^{\nu}\wedge\eta^{\alpha}\wedge\eta^{\mu}. \tag{2.14}$$

Therefore, $\Theta^{[\alpha}\wedge\eta^{\beta]} = 0$ implies also:

$$\Theta^{\mu}\wedge\eta^{\alpha}\wedge\eta^{\beta} = 0. \tag{2.15}$$

If the tetrad is non-degenerate this is the same as:

$$\Theta^{\mu} = 0. \tag{2.16}$$

We have thus already identified one of our equations of motion. This is the vanishing of torsion (2.12), that we can solve algebraically for the spin rotation coefficients $\omega^{\mu\nu}{}_{\alpha} := \omega^{\mu}{}_{\nu a}\eta_{\alpha}{}^{a}$ in terms of the tetrad and its first derivatives. The resulting connection induces the unique Levi-Civita derivative ∇, we write:

$$\text{if} : \mathcal{D}\eta^{\alpha} = 0 \Leftrightarrow \mathcal{D} = \nabla. \tag{2.17}$$

Next, we have to consider the variation of the tetrad. To this goal, let us first observe that the Holst modification of the action, i.e. the addition of the term $\beta^{-1}\eta_{\mu}\wedge\eta_{\nu}\wedge F^{\mu\nu}$ disappears if the torsion-free condition (2.12) is satisfied. The vanishing of torsion implies one of the Bianchi-identities:

$$\mathcal{D}\Theta^{\alpha} = \mathcal{D}^{2}\eta^{\alpha} = \mathcal{F}^{\alpha}{}_{\beta}\wedge\eta^{\beta} = 0. \tag{2.18}$$

Where we used the definition of the curvature, i.e. $\mathcal{D}^2 = F$. Equation (2.18) puts the Holst term to zero:

$$-\beta^{-1}\eta_{\mu}\wedge\eta_{\nu}\wedge\mathcal{F}^{\mu\nu} = 0. \tag{2.19}$$

Therefore, if the torsion free condition is satisfied, and the geometry is non-degenerate, we have

$$Q_{\alpha\beta\mu\nu}\eta^{\beta}\wedge\mathcal{F}^{\mu\nu}[\omega] = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\eta^{\beta}\wedge\mathcal{F}^{\mu\nu}[\omega]. \tag{2.20}$$

Looking back at (2.9), we can thus read off the equations of motion derived from the stationarity of the action, and get:

$$\epsilon_{\alpha\beta\mu\nu}\big(\eta^{\beta}\wedge\mathcal{F}^{\mu\nu}[\omega] - \frac{\Lambda}{3}\eta^{\beta}\wedge\eta^{\mu}\wedge\eta^{\nu}\big) = 0. \tag{2.21}$$

Decomposing the field strength into its components, i.e. setting $\mathcal{F}_{\mu\nu} = \frac{1}{2}\mathcal{F}_{\mu\nu\alpha\beta}\eta^{\alpha}\wedge\eta^{\beta}$, this can be put into the more familiar form:

$$\mathcal{F}^{\mu\alpha}{}_{\mu\beta}[\omega] - \frac{1}{2}\delta^{\alpha}_{\beta}\mathcal{F}^{\mu\nu}{}_{\mu\nu}[\omega] + \Lambda\delta^{\alpha}_{\beta} = 0. \tag{2.22}$$

In the absence of torsion—well imposed by one of our equations of motion, i.e. equation (2.12)—the field strength $\mathcal{F}_{\mu\nu\alpha\beta}[\omega,\eta]$ equals the Riemann curvature tensor $R_{\mu\nu\alpha\beta}[\eta]$, introducing the Ricci tensor $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$, together with the curvature scalar $R = R^{\mu}{}_{\mu}$, we have thus recovered the Einstein equations with a cosmological constant.

Boundary conditions The requirement for all remaining terms in δS_{Holst} to vanish on-shell*, will give us the missing boundary conditions. Before actually doing so, let us first introduce some additional structure. We define the three-dimensional internal metric, together with the intrinsic three-dimensional Levi-Civita tensor:

$$h_{\alpha\beta} := n_{\alpha}n_{\beta} + \eta_{\alpha\beta}, \qquad \epsilon_{\alpha\beta\mu} = n^{\nu}\epsilon_{\nu\alpha\beta\mu} \tag{2.23}$$

By the same argument that gave us the variation of the curvature (2.6) we can find that the covariant differential of the time-normal obeys:

$$\delta(\mathcal{D}n^{\mu}) = \mathcal{D}\delta n^{\mu} + \delta\omega^{\mu}{}_{\nu}n^{\nu} \tag{2.24}$$

*I.e. once the equations of motion are satisfied.

If em : $\partial M \to M$ is the canonical embedding of the three-dimensional boundary into M, and em* denotes the corresponding pull-back, we trivially have

$$n_\alpha \mathrm{em}^*\eta^\alpha = 0, \quad h_\beta{}^\alpha \mathrm{em}^*\eta^\beta = \mathrm{em}^*\eta^\alpha. \tag{2.25}$$

Using the decomposition of the identity to write $\delta\omega^{\mu\nu} = -n^\mu n_\rho \delta\omega^{\rho\nu} + h^\mu{}_\rho \omega^{\rho\nu}$ equation (2.25) thus leads us to:

$$\begin{aligned}\delta I_M\big|_{\mathrm{EOM}} &= \frac{1}{2}\int_{\partial M} \epsilon_{\alpha\beta\mu\nu}\eta^\alpha \wedge \eta^\beta \wedge \delta\omega^{\mu\nu} - \frac{1}{\beta}\int_{\partial M} \eta_\alpha \wedge \eta_\beta \wedge \delta\omega^{\alpha\beta} \\ &= -\int_{\partial M} \epsilon_{\alpha\beta\mu\nu}\eta^\alpha \wedge \eta^\beta \wedge n^\mu n_\rho \delta\omega^{\rho\nu} - \frac{1}{\beta}\int_{\partial M} \eta_\alpha \wedge \eta_\beta \wedge h^\alpha{}_\rho h^\beta{}_\sigma \delta\omega^{\rho\sigma}. \end{aligned} \tag{2.26}$$

We thus eventually get for the variation of the boundary term $I_{\partial M}$ as defined in (2.1) that:

$$\begin{aligned}\delta I_{\partial M} = &- 2\int_{\partial M} \epsilon_{\alpha\beta\mu\nu}\delta\eta^\alpha \wedge \eta^\beta \wedge\, n^\mu \mathcal{D}n^\nu - 2\int_{\partial M} \epsilon_{\alpha\beta\mu\nu}\eta^\alpha \wedge \eta^\beta \wedge\, \delta n^\mu \mathcal{D}n^\nu + \\ &+ 2\int_{\partial M} \epsilon_{\alpha\beta\mu\nu}\mathcal{D}\eta^\alpha \wedge \eta^\beta n^\mu \delta n^\nu - \int_{\partial M} \epsilon_{\alpha\beta\mu\nu}\eta^\alpha \wedge \eta^\beta \wedge\, n^\mu \delta\omega^\nu{}_\rho n^\rho + \\ &- \int_{\partial M} \mathrm{d}\big(\epsilon_{\alpha\beta\mu\nu}\eta^\alpha \wedge \eta^\beta\, n^\mu \delta n^\nu\big) \end{aligned} \tag{2.27}$$

The normal being normalised, we have

$$n_\alpha \delta n^\alpha = 0 = n_\alpha \mathcal{D}n^\alpha, \tag{2.28}$$

revealing both δn^α, and $\mathcal{D}n^\alpha$ are purely spatial in the internal index. This implies the second term in (2.27) vanishes. Summing (2.26) and (2.27), just as in (2.1) we see the first term in the second line of (2.26) cancels against the fourth term of (2.26). For the remaining terms to vanish, we have to demand boundary conditions on both the connection and the tetrad, a short moment of reflection reveals that

$$h^\mu{}_\alpha\, \mathrm{em}^*\delta\eta^\alpha = 0, \qquad h^\mu{}_\alpha h^\nu{}_\beta\, \mathrm{em}^*\delta\omega^{\alpha\beta} = 0, \tag{2.29}$$

does the job. We are thus left with a total derivative, and an additional term containing the variation of the normal. This is the third term in (2.27). Demanding stationarity also for the variations of the normal, implies the vanishing of the three-dimensional torsion

$$\mathrm{em}^*\big(\epsilon_{\alpha\beta\mu}\mathcal{D}\eta^\alpha \wedge \eta^\beta\big) = 0. \tag{2.30}$$

This equation holds already true if there is no four-dimensional torsion (2.12) anywhere appearing. So we do not learn anything new by varying n^α. The only term remaining is thus a total derivative. By Stoke's theorem this turns into an integral over the corner $S = \partial\Sigma_1 = \partial\Sigma_0^{-1}$ (where the exponent shall remind us of the respective orientation of the manifolds), and we get:

$$\begin{aligned}\delta\big(I_M + I_{\partial M}\big)\big|_{\mathrm{EOM+BC}} &= -\int_{\partial M} \mathrm{d}\big(\epsilon_{\alpha\beta\mu\nu}\eta^\alpha \wedge \eta^\beta\, n^\mu \delta n^\nu\big) = \\ &= -\int_S \big(\epsilon_{\alpha\beta\mu\nu}\eta^\alpha \wedge \eta^\beta\, n_1^\mu \delta n_1^\nu - \epsilon_{\alpha\beta\mu\nu}\eta^\alpha \wedge \eta^\beta\, n_0^\mu \delta n_0^\nu\big) \end{aligned} \tag{2.31}$$

Inserting the additional space-like normals z_0^α and z_1^α of S we can write this also as:

$$\delta\big(I_M + I_{\partial M}\big)\big|_{\mathrm{EOM+BC}} = \int_S \epsilon_{\mu\nu\alpha\beta} n_1^\mu z_1^\nu \eta^\alpha \wedge \eta^\beta \big(z_\rho^0 \delta n_0^\rho - z_\rho^1 \delta n_1^\rho\big)$$

where we've used the invariance of the ϵ-tensor under Lorentz transformation to get

$$\epsilon_{\mu\nu\alpha\beta} n_1^\mu z_1^\nu = \epsilon_{\mu\nu\alpha\beta} n_0^\mu z_0^\nu. \tag{2.32}$$

Corner term The only term that remains to study is the integral I_S over the two dimensional corner S. We will now show that this term cancels all variations of the action once both boundary conditions and equations of motion are satisfied. Let us first consider the variation of the corner term. The boundary conditions (2.29) demand all variations of the two dimensional volume element vanish, schematically:

$$\int_S \delta\big(\epsilon_{\alpha\beta\mu\nu}\eta^\alpha \wedge \eta^\beta n^\mu z^\nu\big)\dots\big|_{\mathrm{BC}} = 0. \tag{2.33}$$

The only possible variation of the two-dimensional corner integral can therefore only come from Ξ:

$$\delta I_S\big|_{\mathrm{BC}} = -\int_S \epsilon_{\alpha\beta\mu\nu} n^\alpha z^\beta \eta^\mu \wedge \eta^\nu \delta\Xi \tag{2.34}$$

We can readily compute this variation by writing one dyad in terms of the other:

$$\begin{pmatrix} n_1^\alpha \\ z_1^\alpha \end{pmatrix} = \begin{pmatrix} \mathrm{ch}\,\Xi & -\mathrm{sh}\,\Xi \\ -\mathrm{sh}\,\Xi & \mathrm{ch}\,\Xi \end{pmatrix} \begin{pmatrix} n_0^\alpha \\ z_0^\alpha \end{pmatrix} \tag{2.35}$$

Hence

$$\delta\,\mathrm{sh}\,\Xi = \mathrm{ch}\,\Xi\,\delta\Xi = \delta n^0_\alpha z_1^\alpha + n^0_\alpha \delta z_1^\alpha = \mathrm{ch}\,\Xi\,\delta n^0_\alpha z_0^\alpha + \mathrm{ch}\,\Xi\, n^1_\alpha \delta z_1^\alpha \tag{2.36}$$

If we compare the last line with equation (2.32) we see all variations of the action (2.1) vanish if both the boundary conditions (2.29) and the equations of motion, i.e. (2.22) and (2.12) are satisfied, and we have thus proven:

$$\begin{aligned} \delta\big(I_M + I_{\partial M} + I_S\big)\big|_{\mathrm{EOM+BC}} &= -\int_S \epsilon_{\alpha\beta\mu\nu} n_1^\alpha z_1^\beta \eta^\mu \wedge \eta^\nu \big(z^1_\rho \delta n_1^\rho + n^1_\rho \delta z_1^\rho\big) = \\ &= -\int_S \epsilon_{\alpha\beta\mu\nu} n_1^\alpha z_1^\beta \eta^\mu \wedge \eta^\nu\, \delta\big(z^1_\rho n_1^\rho\big) = 0 \end{aligned} \tag{2.37}$$

Summary Let us briefly recapitulate the last pages. Starting from the Holst action, augmented by the appropriate boundary and corner terms, we took both the connection and the tetrad to be kinematically independent. Indeed, the relation between the two is only given dynamically by the torsionless condition (2.12), algebraically fixing $\omega^\alpha{}_\beta$ in terms of the tetrad and its first derivatives. Then, for the variation of the action to vanish, it is not enough to impose the Einstein equations (2.22). We also need boundary conditions (2.29), that tell us to keep the intrinsic geometry of the three dimensional boundary surface fixed, while allowing for arbitrary variations of the extrinsic data.

2.2 SELFDUAL VARIABLES AND THREE-PLUS-ONE SPLIT

The subject of this section shall be the spatio-temporal decomposition of the Holst action (2.1), that we have previously introduced. We will seek for this 3+1 split in terms of complex Askhtekar variables, which are geometrically natural when looking at the chiral aspects of space-time. The action, thus established, assumes a *canonical form*, which we say to stress that by just looking at the action, we can readily identify the Hamiltonian generating the time-evolution and the constraint equations. The following sections will then further explore this action, its constraint equations and the Poisson algebra they generate.

The selfdual action Let us start with some algebraic preparations to introduce the selfdual formulation of gravity. The selfdual projector

$$P^{\alpha\beta}{}_{\mu\nu} = \frac{1}{2}\big(\delta^{\alpha}_{[\mu}\delta^{\beta}_{\nu]} - \frac{\mathrm{i}}{2}\epsilon^{\alpha\beta}{}_{\mu\nu}\big) \tag{2.38}$$

maps any $\mathfrak{so}(1,3)$ element $\varphi^{\alpha}{}_{\beta}$ towards its selfdual component $P^{\alpha\beta}{}_{\mu\nu}\varphi^{\mu\nu}$ which is an element of the complexified Lie algebra. This projector is orthogonal to its complex conjugate, and we thus have:

$$P^{\alpha\beta}{}_{\rho\sigma}P^{\rho\sigma}{}_{\mu\nu} = P^{\alpha\beta}{}_{\mu\nu}, \quad P^{\alpha\beta}{}_{\rho\sigma}\bar{P}^{\rho\sigma}{}_{\mu\nu} = 0 \tag{2.39}$$

If we replace now all contractions between the curvature two-from $\mathcal{F}^{\alpha\beta}$ and $\eta_\alpha \wedge \eta_\beta$ appearing in the bulk contribution I_M to the action by the selfdual projector, we get the selfdual action of gravity. This is the following complex-valued functional of both connection and tetrad:

$$S_{\mathbb{C}}[\eta,\omega] = \int_M P_{\alpha\beta\mu\nu}\Sigma^{\alpha\beta}[\eta] \wedge \Big(\mathcal{F}^{\mu\nu}[\omega] - \frac{\Lambda}{6}\Sigma^{\mu\nu}[\eta]\Big) \tag{2.40}$$

Here, we have implicitly defined the Plebanski two-form, which is the infinitesimal area element

$$\Sigma_{\alpha\beta}[\eta] = \eta_\alpha \wedge \eta_\beta. \tag{2.41}$$

The bulk action I_M itself, is the real part of the selfdual action weighted by a complex-valued coupling constant containing the Barbero–Immirzi parameter $\beta \in \mathbb{R}_>$:

$$I_M = -\frac{\beta+\mathrm{i}}{\mathrm{i}\beta}S_{\mathbb{C}} + \frac{\beta-\mathrm{i}}{\mathrm{i}\beta}\bar{S}_{\mathbb{C}} \tag{2.42}$$

Omitting the complex conjugate, we would work with a complex action, which could cause troubles if we are interested to eventually quantise the theory. Now, both the area element $\Sigma^{\mu}{}_{\nu}$, and the field strength $\mathcal{F}^{\mu}{}_{\nu}$ are $\mathfrak{so}(1,3)$-valued two-forms. The universal cover of the Lorentz group being $SL(2,\mathbb{C})$, we can equally well use $\mathfrak{sl}(2,\mathbb{C})$-valued two-forms instead. The isomorphism between the two respective Lie algebras, including all missing definitions can be found in the appendix A.1. Accordingly, we introduce $\mathfrak{sl}(2,\mathbb{C})$-valued fields capturing all internal components of the Plebanski two-form, the field strength and the connection. We write

$$\Sigma = \Sigma^i\tau_i : \Sigma^i := \frac{1}{2}\epsilon_m{}^{in}\Sigma^m{}_n + \mathrm{i}\Sigma^i{}_0 \tag{2.43a}$$

$$\mathcal{F} = \mathcal{F}^i\tau_i : \mathcal{F}^i := \frac{1}{2}\epsilon_m{}^{in}\mathcal{F}^m{}_n + \mathrm{i}\mathcal{F}^i{}_0 \tag{2.43b}$$

$$\omega = \omega^i\tau_i : \omega^i := \frac{1}{2}\epsilon_m{}^{in}\omega^m{}_n + \mathrm{i}\omega^i{}_0 \tag{2.43c}$$

Here, we have implicitly introduced a complex basis $\{\tau_i = \frac{1}{2\mathrm{i}}\sigma_i\}_{i=1,2,3}$ in $\mathfrak{sl}(2,\mathbb{C})$, with σ_i are the usual Pauli matrices. The spatial Levi-Civita symbol ϵ_{ilm} (with $\epsilon_{123} = 1$) codes the corresponding structure constants, while the flat Euclidean metric δ_{ij} and its inverse move all these indices up and down. We can thus decompose any $\varphi \in \mathfrak{sl}(2,\mathbb{C})$ as $\varphi = \varphi^i\tau_i = \varphi_i\tau^i$, and call $\varphi^i \in \mathbb{C}^3$ its selfdual components.

Since the two respective Lie alegbras $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{so}(1,3)$ respective Lie algebras are isomorphic, the selfdual part of the curvature tensor is nothing but the curvature of the selfdual connection. Schematically $P\mathcal{F}[\omega] = \mathcal{F}[P\omega]$, or more explicitly:

$$\mathcal{F}^i[\omega] = \mathrm{d}\omega^i + \frac{1}{2}\epsilon^i{}_{lm}\omega^l \wedge \omega^m. \tag{2.44}$$

A few algebraic manipulations reveal the selfdual action (2.40) in terms of the $\mathfrak{sl}(2,\mathbb{C})$ valued two-forms Σ and $\mathcal{F}$, the result is:

$$S_{\mathbb{C}}[\eta,\omega] = \int_M \Sigma_i[\eta]\wedge\Big(\mathcal{F}^i[\omega]-\frac{\Lambda}{6}\Sigma^i[\eta]\Big) = -2\int_M \mathrm{Tr}\Big[\Sigma[\eta]\wedge\Big(\mathcal{F}[\omega]-\frac{\Lambda}{6}\Sigma[\eta]\Big)\Big]. \quad (2.45)$$

The next step is to actually decompose this action into its spatial and temporal components. This obviously needs two ingredients, which are space and time, and we will start with the former.

Choice of time. General relativity lacks any preferred clocks*; time is an additional concept, that we have to make a choice for. We thus select a time function $t: M \rightarrow [0,1]$, that foliates our lensoid region into equal time slices $\Sigma_{t_o} = \{p \in M | t(p) = t_o\}$, and strictly increases for all future oriented observers. The initial slice is Σ_0 while Σ_1 refers to the final hypersurface, figure 2.1 gives an illustration and some more details. The hypersurfaces all meet at the corner S, that is, we ask for $\partial\Sigma_t = S\,\forall t \in [0,1]$. Any two of them are diffeomorphic, thus suggesting to work with an abstract three-dimensional manifold Σ, and use a one-parameter family of embeddings,

$$\mathrm{em}_t : \Sigma \rightarrow \Sigma_t \subset M, \quad \forall t \in [0,1] \quad (2.46)$$

mapping space into space-time. To consistently integrate over Σ, we need an orientation, that we choose so, that it matches our conventions for Σ_1 that we have agreed on earlier; we say, $(x,y,z) \in T\Sigma\times T\Sigma\times T\Sigma$ is positively oriented, if $(n, \mathrm{em}_{t*}x, \mathrm{em}_{t*}y, \mathrm{em}_{t*}z)$ has positive orientation in M, where em_{t*} is the push-forward (the differential map of em_t from Σ towards Σ_t), and n denotes the future oriented surface normal of Σ_t. Sitting on one of these hypersurfaces, and asking how things evolve in t, we need to know how we can go to an infinitesimally close neighbouring surface $\Sigma_{t+\varepsilon}$. Again, there is no unique way to do so, simply because of general covariance. We just have to make a choice, which amounts to pick a future directed vector-field** $\boldsymbol{t} \in TM$, that "ticks" in the rate of t. By this we mean

$$\mathrm{d}t(\boldsymbol{t}) = \boldsymbol{t}\lrcorner\mathrm{d}t = 1. \quad (2.47)$$

In this line we have introduced the "hook"-notation, this is nothing but the interior product between a vector-field and a p-form resulting in $(p-1)$-form. Let us mention the definition, if φ is a p-form on M, and $V_1,\dots,V_p$ are vector-fields in M, $V_1\lrcorner\varphi$ is defined by saying $(V_1\lrcorner\varphi)(V_2,\dots,V_p) = \varphi(V_1,\dots,V_p)$.

Curvature and connection. Having established a time variable t, together with a flow $\boldsymbol{t}$ that we can use to move between neighbouring hypersurfaces, let us now better understand the geometry of those hypersurfaces. We thus seek to split any of our configuration variables into their temporal and spatial components, in order to identify those quantities that are intrinsically related to the spatial geometry itself.

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*Of course, once we know a physical line element it may select preferred observers, but this only comes after having solved the equations of motion. Another possibility is to deparametrise the theory and use certain matter fields as natural clock variables, cf. [97, 98]

**The reason why we do not use abstract indices, and not call this vector-field like everyone else [8] t^a instead, is very simple. In the following we will use abstract indices $a,b,c,\dots$ exclusively for the intrinsic geometry on the spatial slice. So far, we have not used any index notation for TM, and I do not find it useful to introduce it just for one single equation.

We thus call for any $\mathfrak{sl}(2,\mathbb{C})$-valued p-form φ^i on M the pullback $\mathrm{em}_t^*(\boldsymbol{t}\lrcorner\varphi^i)$, which is an $\mathfrak{sl}(2,\mathbb{C})$-valued $(p-1)$-form on Σ, its time component, while the spatial part—denote it f^i for the time being—defines a p-form on Σ:

$$f^i := \mathrm{em}_t^*(\varphi^i) \tag{2.48}$$

Next, we would like to introduce derivatives, and we start with velocities, that tell us how geometry changes when we pass from one t = const. slice to and infinitesimally close hypersurface. We define the time derivative of f^i in the most natural way one could think of, that is:

$$\dot{f}^i := \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{em}_t^*\big(\varphi^i\big) = \mathrm{em}_t^*\big(\mathcal{L}_{\boldsymbol{t}}\varphi^i\big), \tag{2.49}$$

The velocity $\dot{f}^i$ is therefore nothing but the four-dimensional Lie derivative of φ^i in the direction of $\boldsymbol{t}$, pulled back onto the spatial hypersurface Σ. Then we also need the induced covariant exterior derivative on Σ. We call this D, and introduce the $(p+1)$-form Df^i on Σ by setting:

$$Df^i = \mathrm{d}f^i + \epsilon^i{}_{lm}A^l \wedge f^m, \quad A^i = \mathrm{em}_t^*\omega^i \tag{2.50}$$

We can prove, that this definition of D matches all requirements needed for D to be an exterior covariant derivative on Σ. We now have covariant exterior derivatives on both Σ (i.e. D) and M (i.e. $\mathcal{D}$), and we should ask how to write the first in terms of the latter. If $f^i = \mathrm{em}_t^*\varphi^i$ is the projection of φ^i onto the spatial hypersurface a short moment of reflection provides us the desired relation:

$$Df^i := \mathrm{em}_t^*\big(\mathcal{D}\varphi^i\big) \tag{2.51}$$

This, together with the definition of the respective curvature tensors:

$$D^2 f^i = \epsilon^i{}_{lm}F^l \wedge f^m, \quad \mathcal{D}^2\varphi^i = \epsilon^i{}_{lm}\mathcal{F}^l \wedge \varphi^m \tag{2.52}$$

immediately implies that the pullback of the space-time connection coincides with the spatial curvature itself:

$$F^i = \mathrm{em}_t^*(\mathcal{F}^i) \tag{2.53}$$

Having identified the spatial part of the four-dimensional field strength, we now look at its temporal component, and get:

$$\begin{aligned}\boldsymbol{t}\lrcorner\mathcal{F}^i = \boldsymbol{t}\lrcorner\Big(\mathrm{d}\omega^i + \frac{1}{2}\epsilon^i{}_{lm}\omega^l \wedge \omega^m\Big) = \boldsymbol{t}\lrcorner\mathrm{d}\omega^i + \mathrm{d}(\boldsymbol{t}\lrcorner\omega^i) + \\ - \mathrm{d}(\boldsymbol{t}\lrcorner\omega^i) - \epsilon^i{}_{lm}\omega^l \wedge \boldsymbol{t}\lrcorner\omega^m = \mathcal{L}_{\boldsymbol{t}}\omega^i - \mathcal{D}\big(\boldsymbol{t}\lrcorner\omega^i\big).\end{aligned} \tag{2.54}$$

Introducing the time component

$$\Lambda^i = \mathrm{em}_t^*\big(\boldsymbol{t}\lrcorner\omega^i\big) \tag{2.55}$$

of the space-time connection, we see that the four-dimensional field strength measures the velocity of the connection. We have, in fact

$$\mathrm{em}_t^*\big(\boldsymbol{t}\lrcorner\mathcal{F}^i\big) = \dot{A}^i - D\Lambda^i. \tag{2.56}$$

Here we've also used the definition of the Lie derivative on M, i.e. the equation $\mathcal{L}_{\boldsymbol{t}}\varphi = \mathrm{d}(\boldsymbol{t}\lrcorner\varphi) + \boldsymbol{t}\lrcorner(\mathrm{d}\varphi)$, while for the intrinsic Lie-derivative on Σ we would use another symbol and write simply L_X instead.

The action in time gauge. To split the integral of the Lagrange density $\mathcal{L}$ (implicitly defined by (2.1)) over M into an integral over space and time, we proceed as follows:

$$\int_M \mathcal{L} = \int_0^1 \mathrm{d}t \int_\Sigma \mathrm{em}_t^*(\boldsymbol{t}\lrcorner\mathcal{L}) \tag{2.57}$$

We will do this in a certain gauge, that considerably simplifies calculations, and aligns the surface normal n^a of the $t = \text{const.}$ hypersurfaces Σ_t with the time-direction in internal Minkowski space, and we thus set:

$$n^\mu = \eta^\mu{}_a n^a = \delta^\mu_0. \tag{2.58}$$

This is equivalent to saying, that the spatial part of η^0 vanishes:

$$0 = \mathrm{em}_t^*\eta^0 \tag{2.59}$$

This is a rather innocent gauge condition, for we should think of the internal space as an abstractly given manifold lacking any preferred 0-direction whatsoever. In that case n^μ is just the only structure available to pick such a time-direction in internal space. The alignment of internal time direction and the hypersurface normal is not necessary, but considerably simplifies the Hamiltonian analysis. In fact, a number of authors have developed a Hamiltonian formalism without this gauge condition [99–104]. Classically both formalisms are equivalent, but subtleties could only arise once we go to the quantum theory. This has been argued most prominently by Sergei Alexandrov, e.g. in [105, 106].

Equation (2.59) gives four out of $4\times 3 = 12$ spatial components of η^μ. The remaining $3\times 3 = 9$ components define the cotriad on the spatial manifold through the pull-back:

$$e^i = \mathrm{em}_t^*\eta^i, \quad \text{(cotriad)} \tag{2.60}$$

Thus far, we have introduced the spatial components of η^μ, its temporal components define lapse and shift:

$$N = \mathrm{em}_t^*\big(\boldsymbol{t}\lrcorner\eta^0\big), \quad \text{(Lapse function)} \tag{2.61a}$$

$$N^i = \mathrm{em}_t^*\big(\boldsymbol{t}\lrcorner\eta^i\big) \quad \text{(Shift vector in internal space).} \tag{2.61b}$$

In the gauge defined by (2.58) the real and imaginary parts of the Ashtekar connection (2.50), correspond to the intrinsic $\mathfrak{so}(3)$ connection on the spatial slice, and the extrinsic curvature tensor $K_{ab} = K^i{}_a e_{ib}$ respectively. More explicitly:

$$A^i{}_a = \Gamma^i{}_a + \mathrm{i}K^i{}_a : \begin{cases} \Gamma^i = \dfrac{1}{2}\epsilon_m{}^{in}\mathrm{em}_t^*\omega^m{}_n & (\mathfrak{so}(3)\text{-connection}) \\ K^i = \mathrm{em}_t^*\omega^i{}_0 & \text{(extrinsic curvature)} \end{cases} \tag{2.62}$$

It is also useful to know about the spatio-temporal decomposition of the selfdual component of the Plebanski two-form (2.43a), the spatial part of which we call the densitised triad—the name should become clear in a moment. We define:

$$E_i := -\mathrm{em}_t^*(\Sigma_i) = \frac{1}{2}\epsilon_{ilm}e^l\wedge e^m \tag{2.63}$$

Geometrically, this is an $\mathfrak{su}(2)$-valued two-form. Now, the metric-independent Levi-Civita density* $\tilde{\eta}^{abc}$ allows us to map any two-form into a vector-valued density. We can thus write equation (2.63) equally well as

$$E_i{}^a = \frac{1}{2}\tilde{\eta}^{abc}\epsilon_{ilm}e^l{}_b e^m{}_c. \tag{2.64}$$

If the cotriad is invertible, there is a triad $e_i{}^a$ such that

$$e^i{}_a e_j{}^a = \delta^i_j. \tag{2.65}$$

In this case, the spatial volume element does not vanish:

$$d^3v_e = \frac{1}{3!}\epsilon_{ilm}e^i \wedge e^l \wedge e^m = \frac{1}{3!}\tilde{\eta}^{abc}\epsilon_{ilm}e^a{}_i e^l{}_b e^m{}_c, \tag{2.66}$$

and we can understand why we have called $E_i{}^a$ the densitised triad. The reason is, that (2.64) is now equivalent to:

$$E_i{}^a = d^3v_e e_i{}^a, \tag{2.67}$$

which is the triad weighted by the volume element, i.e. a density. Equation (2.63) defines the spatial part of the Plebanski two-form, for the temporal component our definitions of lapse function, i.e. (2.61a), and shift vector, i.e. (2.61b) both lead us to:

$$\mathrm{em}_t^*\big(\boldsymbol{t}\lrcorner\Sigma_i\big) = \epsilon_{min}N^m e^n + \mathrm{i}Ne_i \tag{2.68}$$

The set of equations giving the spatio-temporal decomposition of flux, i.e. both (2.68) and (2.63), and curvature, i.e. both (2.53) and (2.56), collect everything needed to split the selfdual action (2.45) into an integral over space and time:

$$\begin{aligned} S_{\mathbb{C}} = &\int_0^1 \mathrm{d}t \int_\Sigma \mathrm{em}_t^*\Big(\Sigma_i \wedge \boldsymbol{t}\lrcorner\mathcal{F}^i + \boldsymbol{t}\lrcorner\Sigma_i \wedge \mathcal{F}^i - \frac{\Lambda}{3}\boldsymbol{t}\lrcorner\Sigma_i \wedge \Sigma^i\Big) = \\ = &\int_0^1 \mathrm{d}t \int_\Sigma \Big(-\frac{1}{2}\tilde{\eta}^{abc}\epsilon_{ilm}e^l{}_b e^m{}_c\big(\dot{A}^i{}_a - D_a\Lambda^i\big) + \frac{1}{2}\tilde{\eta}^{abc}\epsilon_{min}N^m e^n{}_a F^i{}_{bc} + \\ &\qquad + \frac{\mathrm{i}}{2}N\tilde{\eta}^{abc}e_{ia}F^i{}_{bc} + \frac{\Lambda}{6}\tilde{\eta}^{abc}\epsilon_{ilm}\big(\epsilon^{kin}N_k e_{na} + \mathrm{i}Ne^i{}_a\big)e^l{}_b e^m{}_c\Big) \end{aligned} \tag{2.69}$$

We can bring this expression into a more recognisable form. If the triad is invertible, implicitly always assumed in the following, then we can switch between internal space $\mathbb{R}^3$ and tangent space $T\Sigma$ back and forth. We can thus map the shift vector (2.61b) towards tangent space and thus work with the quantity

$$N^a = e_i{}^a N^i \tag{2.70}$$

instead. We will also frequently use the rescaled lapse function $\underset{\sim}{N}$, which is a density of weight minus one, implicitly defined by:

$$\underset{\sim}{N} d^3v_e = N \tag{2.71}$$

*If $\{x^i\}$ are coordinates on the spatial slice this is defined as $\tilde{\eta}^{abc} = \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k \partial^a_i \partial^b_j \partial^c_k$, which transforms covariantly under changes of the coordinate system. There is also the inverse density $\underset{\sim}{\eta}_{abc}$ implicitly defined by setting $\tilde{\eta}^{abc}\underset{\sim}{\eta}_{def} = 3!\delta^a_{[d}\delta^b_e\delta^c_{f]}$.

Again, this definition of $\underset{\sim}{N}$, works only if the spatial volume element (2.66) is non-degenerate, i.e. the triad be invertible. Repeatedly using the epsilon-identity:

$$\tilde{\eta}^{abc}\epsilon_{ijk}(d^3v_e)^2 = 3!E_i{}^{[a}E_j{}^{b}E_k{}^{c]} \tag{2.72}$$

we can eventually bring (2.69) into its most common form:

$$\begin{aligned} S_{\mathbb{C}} = \int_0^1 \mathrm{d}t \int_\Sigma \Big[- E_i{}^a\big(\dot{A}^i{}_a - D_a\Lambda^i\big) + N^aF^i{}_{ab}E_i{}^b + \\ + \frac{\mathrm{i}}{2}\underset{\sim}{N}\big(\epsilon_i{}^{lm}E_l{}^aE_m{}^bF^i{}_{ab} + \frac{\Lambda}{3}\epsilon^{ilm}\underset{\sim}{\eta}_{abc}E_i{}^aE_j{}^bE_k{}^c\big)\Big] \end{aligned} \tag{2.73}$$

The simplicity of this action becomes apparent when looking at the equivalent expression (2.215) in the ADM formalism discussed in a supplement to this chapter. We will see writing down the ADM action requires both to know the inverse of the spatial metric $h_{ab} = e_{ia}e^i{}_b$, the square root of its determinant and also its spatial derivatives up to second order. Unlike that, the action discussed here contains only first derivatives of the connection, linearly appearing either in the symplectic potential $\propto E_i{}^a\dot{A}^i{}_a$ or the field strength $F^i{}_{ab}$, and nowhere else. Even more prominently, the corresponding Lagrangian is a polynomial of the configuration variables. This polynomial is of third order in the densitised triad $E_i{}^a$, while the Ashtekar connection itself appears only up to second order, and we see all the non-polynomiality of the ADM approach has been absorbed into a convenient choice of variables.

In this section, we have so far only been exploring the contribution to the action coming from the bulk, i.e. I_M in (2.1). The two remaining parts belong to the three dimensional boundary $\partial M = \Sigma_0 \cup \Sigma_1$, and the boundary $S = \partial\Sigma_1$ of the boundary itself. Employing the definitions of the densitised flux (2.64) and the extrinsic curvature (2.62) in time-gauge $n^\mu = \delta^\mu_0$, we can write the boundary and corner terms for our choice of variables as:

$$I_{\partial M} = -2\int_\Sigma E_i{}^aK^i{}_a\big|_{t=1} + 2\int_\Sigma E_i{}^aK^i{}_a\big|_{t=0} = -2\int_\Sigma E_i{}^aK^i{}_a\big|_{t=0}^1 \tag{2.74a}$$

$$I_S = -2\int_S E_iz^i\Xi \tag{2.74b}$$

Here, $z^i \in \mathbb{R}^3$ denotes the outwardly pointing normal z^a of Σ_t mapped into internal space, i.e.: $z^i = e^i{}_az^a$. Notice also the appearance of the Gibbons–Hawking–York boundary term in the first line, $E_i{}^aK^i{}_a$ is nothing but the trace of the extrinsic curvature tensor weighted by the three-dimensional volume element on the spatial slice.

The action (2.1) is a sum of the contribution from the bulk, that we have further split into its self- and antiselfdual parts, and the boundary terms for both the three-dimensional hypersurfaces Σ_0, Σ_1, and the two-dimensional corner S. We have:

$$S_{\text{Holst}} = \frac{\hbar}{2\ell_{\mathrm{P}}^2}\Big[-\frac{\beta+\mathrm{i}}{\mathrm{i}\beta}S_{\mathbb{C}} + \frac{\beta-\mathrm{i}}{\mathrm{i}\beta}\bar{S}_{\mathbb{C}} + I_{\partial M} + I_S\Big] \tag{2.75}$$

This action is functionally differentiable only if we respect the boundary conditions (2.29). Having agreed on time-gauge (2.58) these conditions reduce to the following requirements any variation of the elementary fields must fulfil:

$$\delta E_i{}^a\big|_{\partial M} = 0, \quad \delta\Gamma^i{}_a\big|_{\partial M} = 0, \tag{2.76}$$

The densitised flux diagonalises the spatial metric h_{ab} via $(d^3v_e)^2 h^{ab} = \delta^{ij} E_i{}^a E_j{}^b$, the $\mathfrak{su}(2)$ connection $\Gamma^i{}_a$, on the other hand, defines an intrinsic $\mathfrak{so}(3)$ connection on the spatial slice. We thus see the boundary conditions 2.29 require the intrinsic spatial geometry be fixed, while there be no restrictions on the variations of the extrinsic geometry coded by the extrinsic curvature tensor $K_{ab} = K^i{}_a e_{ib}$, and the additional multiplier fields N, N^a and Λ^i.

Concluding remarks. Summarising the last couple of pages, we took the action and performed a 3+1 split using selfdual variables. During this process, the action has (2.1) always remained the same, and the Barbero–Immirzi parameter has been left untouched. Working with complex Ashtekar variables $A^i{}_a = \Gamma^i{}_a + \mathrm{i}K^i{}_a$ does not amount to put $\beta = \mathrm{i}$ in the action.

2.3 LOCAL CORNER ENERGY

We saw, for the variation principle to be well defined the Holst action must acquire additional terms, one being the Gibbons–Hawking–York boundary term [66, 67] for first-order tetrad-connection variables [68], the other belonging to the two-dimensional corner, i.e. the *boundary of the boundary.* Having already explored the mathematical properties of this additional corner term, it is now time to ask for its physical role. Studying an accelerated observer close to the corner S (see figure 2.1 for an illustration), we will see, the two-dimensional boundary integral measures the local gravitational energy. The energy, thus uncovered, will match what has recently been studied in a series of pioneering articles by E. Frodden, A. Gosh and A. Perez, who boosted the understanding of thermodynamical properties of accelerated observers in both classical and quantum gravity [73, 107]. In this section we will present another look at these results rederived directly from the Hamilton–Jacobi equation of general relativity. This section is based upon the results partially published together with E. Bianchi [108].

First of all we must agree on some simplifying assumptions. We expand the metric $g_{ab} = {}^{(0)}g_{ab} + \varepsilon{}^{(1)}g_{ab} + \dots$ close to the corner S (that have the topology of a two-sphere) in powers of the ratio

$$\varepsilon = \frac{L}{\sqrt{A}} \ll 1 \tag{2.77}$$

of the two typical length scales of the problem; L is the proper distance from the corner S, and A is its area. Employing the principle of general covariance, we introduce a family of accelerated, static (non-rotating) observers that stay at fixed distance from the surface S, such that the line element assumes the asymptotic form of a two-dimensional Rindler metric plus the line element on the two-surface S. We can thus write:

$$ds^2 = -c^2 L^2 \mathrm{d}\Xi^2 + \mathrm{d}L^2 + \frac{A}{4\pi} d\sigma^2 + \mathcal{O}(\varepsilon) \tag{2.78}$$

Here we have introduced the observers' rapidity Ξ, defined just like in (2.5) given above, together with the induced two-dimensional metric $A/(4\pi)d\sigma^2$ on the corner. A typical example of such a geometry is given by the near-horizon approximation of the Schwartzschild space-time. In this case (using the standard Schwartzschild coordinates in the exterior region of the black-hole solution) $A = 16\pi M^2$ is the area of the horizon, $d\sigma^2$ equals the induced metric $\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2$ thereon, the observers rapidity is $\Xi = t/(4M)$, while the asymptotic expansion of the Newton potential in powers of ε yields $(1 - 2M/r) = L^2/(4M)^2(1 + \mathcal{O}(\varepsilon))$. There is, however, no need to restrict ourselves to this particular geometry, as shown for a wide class of black-hole solutions in reference [107].

Staying at a fixed distance L_o above the surface, the rapidity Ξ measures the observer's proper time τ according to

$$\mathrm{d}\tau = \frac{L_o(1+\mathcal{O}(\varepsilon))}{c}\mathrm{d}\Xi \approx \frac{L_o}{c}\mathrm{d}\Xi. \tag{2.79}$$

Here, and in the following "$\approx$" means equality up to terms of higher order in ε. Next, we match the time function* $t: M \to \mathbb{R}$ previously introduced with the proper time τ of the observer at the distance L_o. If $\gamma(\tau)$ is the observer's trajectory parametrised in proper time τ we thus ask for $t(\gamma(\tau)) = \tau$.

With a notion of time, that agrees with physical duration as measured by an accelerated observer, there should also come a notion of gravitational energy. The relation between time and energy becomes particularly clear when looking at the Hamilton–Jacobi equation and realising one as the conjugate of the other. Let us thus briefly recall those aspects of the Hamilton–Jacobi formalism that we will need in the following. Consider a one-dimensional mechanical system, that shall share with the general theory of relativity the absence of a preferred notion of time. The configuration variables be $q \in \mathbb{R}$, that measure location, and proper time τ. The canonical momenta be p and E respectively, where E stand for the energy. Call $S(q_f, q_i; \tau_f, \tau_i)$ Hamilton's principal function, i.e. the action

$$S = \int_0^1 \mathrm{d}t\big(p\dot{q} - \dot{\tau}H(p,q)\big) \tag{2.80}$$

evaluated on a solution of the equations of motion to the boundary value problem $q(t_i) = q_i$, $q(t_f) = q_f$, and $\tau(t_i) = \tau_i$, $\tau(t_f) = \tau_f$. Hamilton's principal function is a solution of the Hamilton–Jacobi equation:

$$p = \frac{\partial S(q, q_i; \tau, \tau_i)}{\partial q}, \qquad E = H\Big(q, \frac{\partial S(q, q_i; \tau, \tau_i)}{\partial q}\Big) = -\frac{\partial S(q, q_i; \tau, \tau_i)}{\partial \tau}, \tag{2.81}$$

The energy being conserved the solution of the Hamilton–Jacobi equation is only a function of the time interval $\tau_f - \tau_i$, and there is no dependence of $\tau_i + \tau_f$ therein. Performing a Legendre transformation, that amounts to keep the energy fixed while allowing for arbitrary variations of $\tau_f - \tau_i$, we can remove the τ-dependence in favour of an energy dependence, eventually revealing what is sometimes called the characteristic Hamilton function $S(q_f, q_i; E) = E \cdot (\tau_f - \tau_i) + S(q_f, q_i; \tau_f, \tau_i)$. Taking the derivative with respect to the energy we get the conjugate variable, which is the observer's proper time elapsed when passing from q_i to q_f:

$$\frac{\partial S(q_f, q_i; E)}{\partial E} = \tau_f - \tau_i. \tag{2.82}$$

Looking at the analogous equation for the gravitational action (2.1) we will now read off the observer's energy. Hamilton's principal function is the action evaluated on a solution of the equations of motion, its functional differentials define energy and time through equations (2.81) and (2.82). We thus need to study variations of the action around a solution of the equations of motion. Working in a first-order formalism these

*In the beginning of this chapter we have fixed t to the values $t = 0$, and $t = 1$ on the initial and final slices respectively, this restriction must now be relaxed for Ξ to assume arbitrary values in $\mathbb{R}$.

are the Einstein equations together with the torsion-free condition, i.e. equations (2.22) and (2.12) respectively. We look back at equations (2.26), (2.27) and (2.36) and see that we have already computed those variations explicitly, and thus readily find:

$$\delta\big(I_M + I_{\partial M} + I_S\big)\Big|_{\mathrm{EOM}} = -\frac{1}{\beta}\int_{\partial M} \eta_\alpha \wedge \eta_\beta \wedge \big(h^\alpha{}_\mu h^\beta{}_\nu \delta\omega^{\mu\nu}\big) \\ - \int_{\partial M} \epsilon_{\alpha\beta\mu\nu}\delta(\eta^\alpha \wedge \eta^\beta) \wedge n^\mu \mathcal{D} n^\nu - \int_S \delta(\epsilon_{\alpha\beta\mu\nu} n^\alpha z^\beta \eta^\mu \wedge \eta^\nu)\Xi \quad (2.83)$$

Let us rewrite this expression in a more compact form. Using time gauge (2.58) and employing our definitions for the densitised triad $E_i{}^a$ and for both the extrinsic curvature $K^i{}_a$ and the intrinsic $\mathfrak{so}(3)$-connection $\Gamma^i{}_a$ (collected in equations (2.63) and (2.62) respectively) we get:

$$\delta\big(I_M + I_{\partial M} + I_S\big)\Big|_{\mathrm{EOM}} = \frac{2}{\beta}\int_\Sigma E_i \wedge \delta\Gamma^i\Big|_{t_i}^{t_f} - 2\int_\Sigma \delta E_i \wedge K^i\Big|_{t_i}^{t_f} - 2\int_S \delta(E_i z^i)\Xi, \quad (2.84)$$

where we have also introduced the internal outwardly pointing normal $z^i = e^i{}_a z^a$ of the two-dimensional corner S. The first term is a total divergence. This becomes immediate when first looking at the functional differential of the pullback of the torsion-free equation (2.12) onto the spatial slice. In fact:

$$\mathcal{D}\eta^i = 0 \Rightarrow \mathfrak{D} e^i = \mathrm{d}e^i + \epsilon^i{}_{lm}\Gamma^l \wedge e^m = 0 \Rightarrow \mathfrak{D}\delta e^i + \epsilon^i{}_{lm}\delta\Gamma^l \wedge e^m = 0, \quad (2.85)$$

where have implicitly introduced the exterior covariant derivative $\mathfrak{D} = \mathrm{d} + [\Gamma, \cdot]$ on the spatial slice. Inserting (2.85) into the first term of (2.84), and once again using Stoke's theorem, we arrive at an integral over the two-dimensional corner:

$$\int_\Sigma E_i \wedge \delta\Gamma^i = \frac{1}{2}\int_\Sigma \epsilon_{ilm} e^i \wedge e^l \wedge \delta\Gamma^m = -\frac{1}{2}\int_\Sigma e^i \wedge \mathfrak{D}\delta e_i = \frac{1}{2}\int_S e^i \wedge \delta e_i \quad (2.86)$$

Let us also mention, that this additional corner term, often identified with a symplectic structure of a Chern–Simons connection, plays an important role in the semiclassical description of black-hole horizons in loop quantum gravity [109–112]. In our case we can drop this term, because the variations of the triad should be everywhere continuous. This in turn implies:

$$\int_S e^i \wedge \delta e_i\big|_{t_o} = \int_S e^i \wedge \delta e_i\big|_{t_1} \quad (2.87)$$

We have thus achieved to compute the functional differential of the Holst action evaluated on a solution of the equations of motion, and immediately see Hamilton's principal function is only a functional of the densitised triads, simply since no functional differential of the connection components ever appears:

$$\delta S_{\mathrm{Holst}}\Big|_{\mathrm{EOM}} = -\frac{\hbar}{\ell_{\mathrm{P}}^2}\Big[\int_\Sigma \delta E_i{}^a K^i{}_a\Big|_{t_i}^{t_f} + \int_S \delta(E_i z^i)\Xi\Big] \quad (2.88)$$

The last term is the desired expression, that we want to compare with (2.81) in order to read off the energy. This term consists of two elements, one being the rapidity Ξ, while the other measures the area A of the corner in terms of the flux of E through S by:

$$A = \int_S E_i z^i = \frac{1}{2}\int_S \epsilon_{ilm} z^i\, e^l \wedge e^m, \quad (2.89)$$

where z^i denotes again the outwardly pointing normal of $\partial\Sigma = S$ in three dimensional internal space. Next, we also need to better understand how the observer's rapidity Ξ can actually measure the elapsing proper time. Taking our simplifying assumption on the asymptotic behaviour of the metric at a distance L_o close to the corner S, i.e. employing equation (2.78), we find:

$$\Xi \approx c\frac{\tau_f - \tau_i}{L_o}. \tag{2.90}$$

And thus get

$$\frac{\delta S_{\text{Holst}}}{\delta A}\bigg|_{\text{EOM}} \approx -\frac{\hbar c}{\ell_{\mathrm{P}}^2}\frac{\tau_f - \tau_i}{L_o}. \tag{2.91}$$

Looking back at the defining equation for energy and time, i.e. equations (2.82, 2.81), we can identify the energy E_{bulk} of the gravitational field as measured by the local observer to be:

$$E_{\text{bulk}} \approx -\frac{\hbar c}{\ell_{\mathrm{P}}^2}\frac{A}{L_o}. \tag{2.92}$$

Consider now a process where the observer can exchange gravitational energy with the region beyond the surface S, call E_S the energy stored therein, and let δE_{bulk} and δE_S be the change of energy in the two respective regions. If we assume energy conservation this process must obey

$$\delta E_{\text{bulk}} + \delta E_S = 0. \tag{2.93}$$

If the observer moves without ever changing the distance from the corner we thus get

$$\delta E_S \approx \frac{c^4}{8\pi G}\frac{\delta A}{L_o} \tag{2.94}$$

This equation coincides with the local form of the first law of black-hole thermodynamics as introduced by Frodden, Gosh and Perez. It states that for an uniformly accelerated observer flying at fixed distance L_o above the surface S, any process resulting in an increase δA of the area of the surface is accompanied by a change δE_S of the energy stored behind the surface S. We can see in (4.37) below and reference [113] how this formula reappears also in the quantisation of the theory.

2.4 HAMILTONIAN ANALYSIS

The Hamiltonian formalism* splits the Euler–Lagrange equations of motion into two distinguished parts, each of which plays a geometrically different role on the auxiliary phase space $\mathcal{P}_{\text{aux}}$ of the theory. First of all, there are the evolution equations. Physical motions follow the Hamiltonian vector field $\mathfrak{X}_H = \{H, \cdot\} \in T\mathcal{P}_{\text{aux}}$ on phase space $\mathcal{P}_{\text{aux}}$, where $\{\cdot,\cdot\}$ denotes the Poisson bracket thereon. This gives the dynamics of the theory. Next, there are the constraint equations (let us call them $C_\mu = 0 = F_I$, $\mu = 1,\dots,n,\ I = 1,\dots,2m$ for the time being), which are those parts of the Euler–Lagrange equations that lack any time derivatives.

Within the set of constraint equations we can make a further distinction and separate first- (here: $C_\mu = 0$) and second-class (here: $F_I = 0$) constraints from another. The

*The book of Henneaux and Teitelboim [114] gives an excellent overview, a more condensed introduction can be found in the appendices of Thiemann's monograph [19].

first-class constraints are those that close* under the Poisson bracket, which is the same as to say that their Hamiltonian vector fields $\mathfrak{X}_{C_\mu}$ lie inside the constraint hypersurface $\mathcal{P}_o = \{x \in \mathcal{P}_{\text{aux}} : C_\mu(x) = 0 = F_I(x)\}$. This has important consequences, for they generate the gauge symmetries of the theory. Let us explain this in some more detail. The Hamiltonian is always linear in the first-class constraints, $H = H_o + \lambda^\mu C_\mu$, where λ^μ are the Lagrange multipliers enforcing the constraint equations. The equations of motion do not fix these gauge parameters λ^μ, and we can thus choose them as rather arbitrary functions on phase space. Changing them from λ_1^μ to $\lambda_1^\mu + \varepsilon\lambda_2^\mu$, amounts to add the constraint $\varepsilon\lambda_2^\mu C_\mu$ to the Hamiltonian. Continuously increasing ε moves the initial motion—a trajectory $x(t) = \exp(tX_H)[x_o] \in \mathcal{P}_{\text{aux}}$—along the Hamiltonian vector-field $\lambda^\mu \mathfrak{X}_{C_\mu}$ towards an infinitesimally neighbouring path $x_\varepsilon(t)$. Since the Hamiltonian vector field $\lambda_2^\mu X_{C_\mu} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} x_\varepsilon$ is tangential to the constraint hypersurface $\mathcal{P}_o$, this process can never take us out of $\mathcal{P}_o$. Therefore, given a point x on $\mathcal{P}_o$, all points within the fibre $[x] = \{y : y = \exp(\lambda^\mu X_{C_\mu})[x], \forall \lambda^\mu \in \mathbb{R}^n\}$ generated by the action of the first-class constraints must be identified as one and the same physical state $[x]$. The evolution equations, on the other hand, must preserve this bundle structure, that is, they must map states onto states and rigidly move the fibres around, which happens only once the Hamiltonian weakly commutes with all the constraints. Each first-class constraint projects the corresponding gauge orbit into a point, and removes, therefore, two unphysical degrees of freedom from the auxiliary phase space $\mathcal{P}_{\text{aux}}$.

The second-class constraints $F_I = 0$, on the other hand, are everything what is left in the set of constraint equations. They do not form an algebra or, more geometrically speaking, their Hamiltonian vector-fields always lie transversal to the constraint hypersurface. There is, however always an even number of them, as shown in e.g. [19]. Starting with an auxiliary phase-space $\mathcal{P}$ of $2N$ dimensions, equipped with n first- and $2m$ second-class constraints, we are thus left with $2(N - n - m)$ physical degrees of freedom. The resulting orbit-space $\mathcal{P}_{\text{phys}} = \{[x] : x \in \mathcal{P}_o\}$ carries a natural symplectic structure. The corresponding Poisson bracket, the Dirac bracket $\{\cdot,\cdot\}^*$, lives on this orbit space, but can actually be lifted to the full auxiliary phase-space $\mathcal{P}$ where it defines a degenerate symplectic form.

At this point, the Maxwell equations in vacuum often serve as a prototypical example. In the Hamiltonian framework, the electric field E^i is the conjugate momentum of the vector-potential A_i, and the only non-vanishing Poisson brackets of the elementary variables are $\{E^i(\vec{x}), A_j(\vec{y})\} = \delta^i_j \delta(\vec{x} - \vec{y})$. Half of the Maxwell equations are already solved by introducing the four-potential A_μ, and writing $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The remaining equations, are the evolution equation for the electric field $\partial_t E^i = \partial_l \partial^l A^i - \partial_l \partial^i A^l$, and the Gauß law $\partial_i E^i = 0$. Lacking any time-derivatives, this is a constraint equation. Its Hamiltonian vector-field generates the gauge symmetry of the theory. These are the transformations $A_i \to A_i - \partial_i \Lambda$. The Hamiltonian $H = \frac{1}{2}\int_{\mathbb{R}^3} d^3x(\vec{E}^2 + \vec{B}^2 - 2\partial_i E^i \Lambda)$, with $\Lambda = A_0$ being the time component of the four-vector potential A_μ, Poisson-commutes with the Gauß constraint, and measures the energy of the system.

2.4.1 Phase space, constraints and evolution equations

Looking at the action as it is written in equations (2.73, 2.75), that is after having already split all configuration variables into their spatio-temporal components, we can now readily introduce a Hamiltonian formulation of the Einstein equations in terms

*That is: $\forall I, \mu, \nu : \{C_\mu, C_\nu\} \approx 0 \approx \{C_\mu, F_I\}$, where "$\approx$" means here, and in everything that follows equality up to terms vanishing on the constraint hypersurface.

of first-order connection variables. We will achieve this reformulation without actually ever performing a Legendre transformation. Although this would certainly be possible, the very structure of the action (2.73) strongly suggests not do stoically apply the canonical algorithm to eventually find a Hamiltonian. The action (2.75) already assumes a Hamiltonian form, performing a singular Legendre transformation would introduce an unnecessarily large phase space containing canonical momenta associated to densitised lapse $\underset{\sim}{N}$, shift vector N^a, to Λ^i and to *both* the densitised triad $E_i{}^a$ an the $SL(2,\mathbb{C})$ connection $A^i{}_a$.

Our strategy will be different. Carefully looking at the action, we will just guess the corresponding phase-space in order to divide the Euler–Lagrange equations of motion into evolution equations generated by a Hamiltonian and a system of constraints, which are both of first- and second-class.

Let us start with the constraint equations. We take the variation with respect to the variables Λ^i, N, and N^a appearing in the action (2.75) and realise the following quantities must vanish:

$$\text{The Gauß constraint:}\qquad G_i[\Lambda^i] = -\int_\Sigma \big(\Lambda^i D_a \Pi_i{}^a + \text{cc.}\big) \overset{\text{EOM}}{=} 0, \tag{2.95a}$$

$$\text{The vector constraint:}\qquad H_a[N^a] = \int_\Sigma N^a \big(F^i{}_{ab}\Pi_i{}^b + \text{cc.}\big) \overset{\text{EOM}}{=} 0, \tag{2.95b}$$

$$\begin{aligned}\text{Hamiltonian constraint:}\qquad H[\underset{\sim}{N}] = -\frac{\ell_{\text{P}}^2}{\hbar}\int_\Sigma \underset{\sim}{N}\Big(&\frac{\beta}{\beta+\mathrm{i}}\epsilon_i{}^{lm}F^i{}_{ab}\Pi_l{}^a\Pi_m{}^b + \\ &+\frac{2\Lambda\ell_{\text{P}}^2}{3\hbar}\frac{\mathrm{i}\beta^2}{(\beta+\mathrm{i})^2}\epsilon^{ilm}\underset{\sim}{\eta}_{abc}\Pi_i{}^a\Pi_l{}^b\Pi_m{}^c + \text{cc.}\Big) \overset{\text{EOM}}{=} 0.\end{aligned} \tag{2.95c}$$

Where we have implicitly introduced the abbreviation

$$\Pi_i{}^a = +\frac{\hbar}{2\ell_{\text{P}}^2}\frac{\beta+\mathrm{i}}{\mathrm{i}\beta}E_i{}^a, \quad \bar{\Pi}_i{}^a = -\frac{\hbar}{2\ell_{\text{P}}^2}\frac{\beta-\mathrm{i}}{\mathrm{i}\beta}E_i{}^a.. \tag{2.96}$$

This shall be our momentum variable, it is an $\mathfrak{sl}(2,\mathbb{C})$-valued vector density on the spatial slice Σ. Notice, also, that the list of constraint equations is perfectly well defined for any given pair $(\Pi_i{}^a, A^j{}_b)$ of field configurations, not necessarily subject to the restrictions (2.96) given above. This is, in fact, how we shall think from now on of the constraint equations (2.95).

Working in the time gauge (2.58), the densistised triad $E_i{}^a$ as defined by (2.64) takes values in the Lie algebra of the rotation group. For the triad to be real its boost part must vanish, which turns (2.96) into the reality condition

$$C_i{}^a := \frac{\ell_{\text{P}}^2}{\mathrm{i}\hbar}\Big(\frac{\mathrm{i}\beta}{\beta+\mathrm{i}}\Pi_i{}^a + \frac{\mathrm{i}\beta}{\beta-\mathrm{i}}\bar{\Pi}_i{}^a\Big) \overset{\text{EOM}}{=} 0. \tag{2.97}$$

In the original approach of Ashtekar the Hamiltonian analysis started from a complex action, which was basically $\sqrt{-1}$ times the selfdual action introduced in above (2.40). Although this amounts to replace the Barbero–Immirzi parameter β in the action (2.1) by the imaginary unit, this does not allow us to say the reality conditions for the selfdual theory were nothing but the analytical continuation of (2.97) to imaginary values of β. In fact, there is quite some confusion in the literature as to what are the right reality conditions for the case of $\beta = \mathrm{i}$, and, I think, $\bar{\Pi}_i{}^a = 0$, would not be the correct choice [115, 116]. Looking at the action (2.1) and putting $\beta = \mathrm{i}$ we can identify the conjugate momentum of the connection to be $\Pi_i{}^a = \frac{\hbar}{\mathrm{i}\ell_{\text{P}}^2}E_i{}^a$. If we demand the triad to be real, i.e. $E_i{}^a = \bar{E}_i{}^a$, this implies the reality condition $\Pi_i{}^a + \bar{\Pi}_i{}^a = 0$, which coincides with equation (2.97) only in the limit $\beta \to \infty$. In the following we will, however, never work with a complexified Barbero–Immirzi parameter (in contrast to e.g. [115–118]), and always keep $\beta > 0$.

Introducing real and imaginary parts of $\Pi_i{}^a$, later corresponding to both boost and rotations, i.e setting $\hbar L_i{}^a = \Pi_i{}^a + \bar{\Pi}_i{}^a$, and $\mathrm{i}\hbar K_i{}^a = \Pi_i{}^a - \bar{\Pi}_i{}^a$, the reality conditions turn into:

$$C_i{}^a = \ell_{\mathrm{P}}^2 \frac{\beta}{\beta^2+1}\big(K_i{}^a + \beta L_i{}^a\big) = 0. \tag{2.98}$$

This is one of the defining equation of spinfoam gravity. In the next chapter, we will see, how this equation basically defines the transition amplitudes of the quantum theory.

Thus far, we have only introduced constraints, obtained by taking the variation of the action with respect to the Lagrange multipliers. The next step concerns the variations of both the connection and the densitised triad. This will give us the evolution equations together with an additional secondary constraint. The variational principle rests upon the boundary conditions, that depend on the topology of the boundary. We keep working in a space-time region of lensoid shape, and take the foliation previously introduced, as also sketched in figure 2.1. Therefore, the $t = \text{const.}$ hypersurfaces all meet at the corner S. This is possible only if the vector-field $\boldsymbol{t}$ following the flow of time vanishes at the corner. Looking back at the defining equations for lapse and shift, i.e. (2.61a, 2.61b) we see this implies:

$$N|_S = 0, \quad N^a|_S = 0, \tag{2.99}$$

which also agrees with our asymptotic expansion of the metric (2.78) around the corner. To further simplify the problem let us agree on a reasonable gauge-choice at the corner. The Lagrange multiplier Λ^i has both an imaginary and a real part:

$$\Lambda^i = \boldsymbol{t} \lrcorner \omega^i = \varphi^i + \mathrm{i}\xi^i. \tag{2.100}$$

The imaginary part represents a boost into the direction of ξ, while the real part describes an infinitesimal rotation around the φ^i-axis. We will later see, that φ^i is not subject to any further constraints but can be chosen freely, such that we can agree to put it to zero at the corner, i.e. we set

$$\varphi^i|_S = 0 \tag{2.101}$$

This gauge condition will also lead to $\dot{e}^i|_S = 0$ at the corner. Putting this time-derivative to zero is physically reasonable since the corner is the surface where all $t = \text{const.}$ hypersurfaces meet, i.e. where we should not expect any t-evolution to happen.

The variation of the action (2.73, 2.75) yields the Gauß law only after actually performing a partial integration, with the gauge condition (2.101) we find, in fact:

$$\int_\Sigma E_i{}^a D_a \Lambda^i = -\int_\Sigma D_a E_i{}^a \Lambda^i + \mathrm{i}\int_S E_i \xi^i \tag{2.102}$$

Inserting a decomposition of the identity in internal space, i.e. writing $\delta^i_j = x^i x_j + y^i y_j + z^i z_j$, where (x^i, y^i, z^i) are orthonormal vectors in $\mathbb{R}^3$ such that $z^a = e_i{}^a z^i$ is the outwardly pointing normal of S, we find, while also using time gauge (2.58) and the definition (2.43c) of the selfdual connection ω^i, that:

$$\int_S E_i \xi^i = \int_S E_i z^i\, z_j \xi^j = \int_S E_i z^i\, z_j \boldsymbol{t} \lrcorner \omega^j{}_0 = \int_S E_i z^i\, z_\mu \mathcal{D}_{\boldsymbol{t}} n^\mu \tag{2.103}$$

Repeating the calculation that has led us to the variation (2.36) of the rapidity Ξ we find $n_\mu \mathcal{D}_{\boldsymbol{t}} z^\mu$ equals $\dot{\Xi}$ and we eventually get:

$$\int_\Sigma E_i{}^a D_a \Lambda^i = -\int_\Sigma D_a E_i{}^a \Lambda^i - \mathrm{i}\int_S E_i z^i \dot{\Xi} = -\int_\Sigma D_a E_i{}^a \Lambda^i + \frac{\mathrm{i}}{2} I_S \tag{2.104}$$

Inserting this equation into the action (2.73) we see, the surface term $\int_S E_i \Lambda^i$ cancels the corner term I_S, and the Holst action (2.75) eventually turns into a sum of a symplectic potential, a boundary term, and the constraints.:

$$S_{\mathrm{Holst}}[E_i{}^a, A^i{}_a, \Lambda^i, \underset{\sim}{N}, N^a] = \int_0^1 \mathrm{d}t \Big[\int_\Sigma \big(\Pi_i{}^a \dot{A}^i{}_a + \mathrm{cc.}\big) - H^* \Big] - \frac{\hbar}{\ell_{\mathrm{P}}^2} \int_\Sigma E_i{}^a K^i{}_a \Big|_{t=0}^1 \tag{2.105}$$

The Hamiltonian H^*, on the other hand is just a sum over constraints:

$$H^* = G_i[\Lambda^i] + H_a[N^a] + H[\underset{\sim}{N}] \tag{2.106}$$

Notice, however, that we are still within the Lagrangian formalism, since $\Pi_i{}^a$ is not yet an independent variable but linearly related to the densitised triad via equation (2.96).

Taking variations of the action with respect to the triad $E_i{}^a$ and both the real and imaginary parts of the Ashtekar connection $A^i{}_a$, subject only to the boundary conditions (2.76), reveals the evolution equations of the theory. This eventually yields the following system of equations:

$$\frac{\beta + \mathrm{i}}{\mathrm{i}\beta} \dot{A}^i{}_a - \frac{\beta - \mathrm{i}}{\mathrm{i}\beta} \dot{\bar{A}}^i{}_a = \frac{2\ell_{\mathrm{P}}^2}{\hbar} \frac{\delta H^*}{\delta E_i{}^a} \tag{2.107a}$$

$$\dot{E}_i{}^a = -\frac{2\ell_{\mathrm{P}}^2}{\hbar} \frac{\mathrm{i}\beta}{\beta + \mathrm{i}} \frac{\delta H^*}{\delta A^i{}_a} = \frac{2\ell_{\mathrm{P}}^2}{\hbar} \frac{\mathrm{i}\beta}{\beta - \mathrm{i}} \frac{\delta H^*}{\delta E_i{}^a} \tag{2.107b}$$

Two observations are immediate to make. First of all, we see the time-derivative of the Ashtekar connection only appears in a peculiar combination involving both $A^i{}_a$ and its complex conjugate $\bar{A}^i{}_a$. The resulting quantity defines, surprisingly enough, again a connection*, this time the $\mathfrak{su}(2)$-Ashtekar–Barbero connection $A^{(\beta)i}{}_a$:

$$A^{(\beta)i}{}_a := \frac{\beta + \mathrm{i}}{2\mathrm{i}} A^i{}_a - \frac{\beta - \mathrm{i}}{2\mathrm{i}} \bar{A}^i{}_a = \Gamma^i{}_a + \beta K^i{}_a. \tag{2.108}$$

The second observation concerns the time derivative of the densitised triad. While the number of evolution equations for the connection may seem to be half too little, the opposite happens for the densitised triad. Looking at the second line, i.e. equation (2.107b), we see the velocity $\dot{E}_i{}^a$ must satisfy two independent equations. Subtracting the first one from the second reveals the secondary constraint:

$$T_i{}^a = \frac{\ell_{\mathrm{P}}^2}{\mathrm{i}\hbar} \frac{\mathrm{i}\beta}{\beta + \mathrm{i}} \frac{\delta H^*}{\delta A^i{}_a} + \mathrm{cc.} = 0. \tag{2.109}$$

This additional constraint, a result of the over-determination of the equations of motion for the densitised triad, is actually crucial, for it compensates the under-determination of the evolution equations for the connection.

This having said, we are now ready to turn towards the Hamiltonian formalism of the theory. We start by introducing the symplectic structure on the infinite-dimensional

*But only a spatial connection, as famously stressed by Alexandrov et al. [105, 119, 120].

auxiliary phase-space $\mathcal{P}_{\text{aux}}$ formed by pairs of field configurations $(\Pi_i{}^a, A^j{}_p)$ on Σ, where $A^i{}_a$ is the selfdual Ashtekar connection, while its conjugate momentum, denoted by $\Pi_i{}^a$, is an $\mathfrak{sl}(2,\mathbb{C})$-valued vector density of degree one (i.e. a Lie-algebra valued two-form). The only non-vanishing Poisson brackets of the elementary variables we define to be the following:

$$\{\Pi_i{}^a(p), A^j{}_b(q)\} = \delta^j_i \delta^a_b \delta^{(3)}(p,q) \tag{2.110a}$$

$$\{\bar{\Pi}_i{}^a(p), \bar{A}^j{}_b(q)\} = \delta^j_i \delta^a_b \delta^{(3)}(p,q) \tag{2.110b}$$

Next, we need a Hamiltonian flow compatible with the set of evolution equations (2.107a, 2.107b) derived from the Lagrangian framework. This must be done with care, as to account for the fact that the list (2.107) of evolution equations misses the time derivative of

$$\tilde{A}^{(\beta)i}{}_a = \Gamma^i{}_a - \beta^{-1} K^i{}_a = \frac{\beta + \mathrm{i}}{2\beta} A^i{}_a + \frac{\beta - \mathrm{i}}{2\beta} \bar{A}^i{}_a \tag{2.111}$$

We fix this yet unspecified velocity by an additional Lagrange multiplier $V^i{}_a$, which amounts to add the constraint (2.97) to the Hamiltonian:

$$H' := C_i{}^a[V^i{}_a] + G_i[\Lambda^i] + H_a[N^a] + H[\underset{\sim}{N}] = C_i{}^a[V^i{}_a] + H^* \tag{2.112}$$

Having a Hamiltonian we can define the time evolution of any (differentiable) functional $X : \mathcal{P}_{\text{aux}} \rightarrow \mathbb{C}$ on phase space as

$$\frac{\mathrm{d}}{\mathrm{d}t} X = \dot{X} = \frac{\mathrm{d}}{\mathrm{d}t} X = \{H', X\} \tag{2.113}$$

Although the auxiliary phase space is a complex infinite dimensional manifold, we do not want or need to restrict ourselves to functionals $X : \mathcal{P}_{\text{aux}} \rightarrow \mathbb{C}$ that are analytic (complex differentiable) on phase space. Allowing for non-analytic functionals is actually physically needed, since otherwise, it would be hard to make mathematical sense out of e.g. equation (2.97). Let us also use this moment to clarify again what the time derivative actually means geometrically. Having performed a spatio-temporal decomposition all our fields live on the fixed abstract spatial manifold Σ, and parametrically depend on the time variable chosen. On Σ, the derivation with respect to t is just an ordinary partial derivative, from a four-dimensional perspective, it is, however a Lie derivative into the direction of the time-flow vector-field $\boldsymbol{t}$ pulled back onto the $t = \text{const.}$ slice under consideration. The definition was explicitly given in (2.49) and further discussed in the surrounding lines.

We can now readily check that the Hamilton equations

$$\dot{A}^i{}_a\big|_{C=0} = \big\{H', A^i{}_a\big\}\big|_{C=0}, \tag{2.114a}$$

$$\dot{\Pi}_i{}^a\big|_{C=0} = \big\{H', \Pi_i{}^a\big\}\big|_{C=0}, \tag{2.114b}$$

together with their complex conjugate reproduce the system (2.107) of evolution equations. The secondary constraint, on the other hand, i.e. equation (2.109) turns into a stability criterion. It demands the Hamiltonian flow must preserve the reality condition (2.97), and we thus have:

$$T_i{}^a = \dot{C}_i{}^a = \big\{H', C_i{}^a\big\}\big|_{C=0} = 0 \tag{2.115}$$

We can now also compute the time derivative of the connection (2.111) absent in our initial list (2.107) of evolution equations, and find:

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{A}^{(\beta)i}{}_a = \frac{\ell_{\mathrm{P}}^2}{\hbar}V^i{}_a + \{H^*, \tilde{A}^{(\beta)i}{}_a\} \tag{2.116}$$

This is, in fact, the only place where the additional Lagrange multiplier $V^i{}_a$ truly appears, which makes perfectly sense, since we have introduced it precisely to fix the yet undetermined velocity $\frac{\mathrm{d}}{\mathrm{d}t}\tilde{A}^{(\beta)i}{}_a$. We will later see how the secondary constraint (2.115) will give us $\tilde{A}^{(\beta)i}{}_a$ in terms of $E_i{}^a$ and $A^{(\beta)i}{}_a$ and eventually fixes the multiplier $V^i{}_a$ to the value $V^i{}_a = 0$.

This finishes the proof of compatibility between time evolution generated by H' and the corresponding Euler–Lagrange evolution equations (2.107, 2.109). It is hardly necessary to say what we shall do next. We will discuss the constraint algebra and check the stability of the constraint equations. This will reveal the gauge symmetries of the system. We will see, the Hamiltonian preserves the reality conditions only if two additional constraints are satisfied. First of all, the spatial projection $\mathfrak{D}e^i = \mathrm{d}e^i + \epsilon^i{}_{lm}\Gamma^l \wedge e^m$ of the four-dimensional torsion two-form $\Theta = \mathcal{D}\eta^i$ must vanish. Then we also find a restriction on the imaginary part on those components of the Lagrange multiplier Λ^i that generate boosts along the time direction. This, together with the Gauß-law $D_a\Pi_i{}^a = 0$ and the evolution equations for the triad amounts to set the four-dimensional torsion Θ to zero. The system of constraints will contain both first- and second-class constraints, solving the second-class constraints we will mention the Dirac bracket and comment on two strategies on how to actually quantise the theory.

2.4.2 Gauge transformations

Before we actually start calculating the Poisson algebra of the constraint equations (2.95), and check under which conditions the Hamiltonian H' preserves the reality conditions (2.97), let us first understand the kinematical symmetries of the system. First studying infinitesimal gauge transformations on the auxiliary phase space $\mathcal{P}_{\mathrm{aux}} \ni (A^i{}_a, \Pi_j{}^b)$ will drastically simplify our calculations later on.

Local-Lorentz transformations Contracting the Lie-algebra index with the standard basis $\tau_i = \frac{1}{2\mathrm{i}}\sigma_i$ let us first introduce the notation:

$$A_a = A^i{}_a\tau_i, \quad F_{ab} = F^i{}_{ab}\tau_i \quad \Pi^a = \Pi_i{}^a\tau^i \tag{2.117}$$

We have started with a theory of local inertial frames η^α, that locally define the four-dimensional line-element of space-time:

$$ds^2 = \eta_{\alpha\beta}\eta^\alpha \otimes \eta^\beta. \tag{2.118}$$

The Minkwoski metric $\eta_{\alpha\beta}$ being Lorentz invariant, we can introduce a local Lorentz transformation $g \in L_+^\uparrow \subset SO(1,3)$

$$\tilde{\eta}^\alpha = g_\beta{}^\alpha\eta^\beta \tag{2.119}$$

without ever changing ds^2. Here g smoothly attaches an element of the group of proper orthochronous Lorentz transformations to any point wherever the tetrad η^α is actually defined. The universal cover of the Lorentz group is $SL(2,\mathbb{C})$, allowing us to write

$$g^\alpha{}_\beta = g^A{}_B\bar{g}^{\bar{A}}{}_{\bar{B}}, \quad g \in SL(2,\mathbb{C}), \tag{2.120}$$

where we have taken advantage of the isomorphism between world-tensors and spinors and identified any Minkwoski index α with a pair of spinors $(A\bar{A})$, one (i.e. A) transforming under the fundamental representation of $SL(2,\mathbb{C})$, while the other (i.e. $\bar{A}$) transforms under the complex conjugate representation, appendix A.1 gives all missing definitions.

If we take the pull back of both the connection and the tetrad onto the spatial slice these transformations induce a gauge transformation ρ_g on phase-space. It acts according to:

$$\tilde{A}_a = \rho_g(A_a) = g^{-1}\partial_a g + g^{-1}Ag, \quad \tilde{\Pi}^a = \rho_g(\Pi^a) = g^{-1}\Pi^a g \tag{2.121}$$

We are interested in the infinitesimal version of these transformations, and thus define the derivation:

$$\text{for } \Lambda : U \subset \Sigma \to \mathfrak{sl}(2,\mathbb{C}), \quad \delta_\Lambda := \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\rho_{\exp(\varepsilon\Lambda)} \tag{2.122}$$

Where U is the neighbourhood where the fields are locally defined. A short calculation gives the desired result:

$$\delta_\Lambda A_a = D_a\Lambda \equiv D_a\Lambda^i\tau_i, \tag{2.123a}$$

$$\delta_\Lambda\Pi^a = -[\Lambda,\Pi^a] \equiv -[\Lambda,\Pi]_i{}^a\tau^i \equiv -\epsilon_{il}{}^m\Lambda^l\Pi_m{}^a\tau^i \tag{2.123b}$$

Diffeomorphisms The principle of general covariance states, that all observers, whether they are accelerated or inertial, must agree on the form of the fundamental laws of nature. Observers determine coordinate systems, and the principle of covariance thus requires, that all fundamental physical laws keep unchanged when writing one set of coordinates in terms of the other. This statement can be phrased without actually ever using coordinates, for it implies diffeomorphism invariance. If M and $\tilde{M}$ are two manifolds with metric and matter fields $\tilde{g}$ and $\tilde{\phi}$ subject to the Einstein equations on $\tilde{M}$, and $\varphi : M \to \tilde{M}$ be a diffeomorphism* then the pull-back $(\varphi^*\tilde{g}, \varphi^*\tilde{\phi})$ solves the Einstein equations on M.

The foliation of space-time into equal time slices Σ_t partially breaks diffeomorphism invariance. The phase-space by itself, only carries a representation of the spatial diffeomorphism group. Nevertheless, we can restore the four-dimensional diffeomorphism symmetry, but only after having actually solved the evolution equations. On phase space, the infinitesimal version δ_N of a diffeomorphism generated by a vector-field N on Σ is a Lie-derivative $L_N = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0}\exp(\varepsilon N)^* = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0}\exp(-\varepsilon N)_*$ into the direction of N. The Lie derivative acts on tensors, but our phase space variables, the Ashtekar connection $A^i{}_a$ and the momentum variable $\Pi_i{}^a$ carry an extra index living in the Lie algebra of the Lorentz group. This is a problem, for we need to declare how the Lie derivative should act on this index. To seek for a proper definition of $\delta_N A^i{}_a$ let us first ask for the properties this derivative ought to have. We can view the Lie derivative L_N of a tensor T as the infinitesimal difference between the pull-back $\tilde{T} = \exp(\varepsilon N)^*T$ and the original tensor T both evaluated at the same point. With the difference of two tensors again being a tensor, the Lie derivative maps tensors into tensors. On the other hand, given two connections A, and $\tilde{A}$ their difference $\Delta A = \tilde{A} - A$ does not

*A one-to-one map between M and $\tilde{M}$, such that both φ and φ^{-1} are continuous and differentiable.

define a connection, but transforms linearly under the adjoint representation of the gauge group. That is $\rho_g \Delta A = g^{-1} \Delta A g$. This suggests to require $\delta_N A^i{}_a$ must transform under the adjoint action of $SL(2,\mathbb{C})$. It turns out, that, in order to achieve the desired transformation property, it suffices to add an infinitesimal gauge transformation $\delta_\Lambda A$ (as defined in (2.123a)) with gauge parameter $\Lambda = -N^a A_a$ to the ordinary Lie derivative $L_N = \mathrm{d}N\lrcorner + N\lrcorner\mathrm{d}$ on the spatial manifold. Accordingly, we define the infinitesimal variation of the connection $\delta_N A$ under a diffeomorphism to be:

$$\delta_N A_a := L_N A_a - D_a\big(N^b A_b\big) = N^b F_{ba} \tag{2.124}$$

Notice the appearance of the curvature tensor, the proof of which simply repeats those steps that have already led us to equation (2.56).

To define the variation $\delta_N \Pi$ of the momentum variable without breaking gauge invariance we repeat the trick that has led us to the definition of $\delta_N A$. We thus add an infinitesimal gauge transformation generated by $-N^b A_b \in \mathfrak{sl}(2,\mathbb{C})$ to the ordinary Lie derivative, and thus get:

$$\delta_N \Pi^a = L_N \Pi^a + [N^b A_b, \Pi^a] \tag{2.125}$$

Writing $L_N \Pi^a + [N^b A_b, \Pi^a] = D_a(N^a \Pi^b - N^b \Pi^a) + N^b D_a \Pi^a$ we can realise the resulting expression has indeed the required property, and transforms under the adjoint representation of the gauge group.

The geometry of the gauge group. At this point the origin of the counter terms, that we have added to the ordinary Lie derivative L_N in both (2.124) and (2.125), must seem a bit dubious. To better understand their geometrical meaning let us see how they rather naturally appear when lifting the elementary fields onto a principal bundle over the spatial hypersurface with the gauge group $SL(2,\mathbb{C})$ as the standard fibre. This needs a little preparation to clarify the tools and language hence needed.*.

Locally, the fibre bundle looks like the Cartesian product of the gauge group $SL(2,\mathbb{C})$ and the three dimensional base manifold Σ. For any sufficiently small neighbourhood U of Σ, we can find a local trivialisation $s : U \times SL(2,\mathbb{C}) \to P$ that diffeomorphically maps $U \times SL(2,\mathbb{C})$ into the bundle. Each point $x \in U$ in the base manifold corresponds to a whole fibre when lifted up into the bundle. Indeed, the fibre over x is nothing but the image of $\{x\} \times SL(2,\mathbb{C})$ under the action of s. The gauge group has a natural right-action R_g on the fibres, which we can locally represent for any $(x,g) \in U \times SL(2,\mathbb{C})$ as $R_{g'}(x,g) = (x, gg')$. The map $\pi : P \to \Sigma$ that becomes $\pi(x,g) = x$ in any local trivialisation, is the canonical projection, and figure 2.2 gives an illustration of the geometry thus uncovered.

The right translation allows us to move along the fibres, its infinitesimal version gives the right invariant vector fields. In fact, for if $g = \exp(\Lambda) \in SL(2,\mathbb{C})$ we define the right invariant vector-field by setting:

$$X_\Lambda\Big|_p = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} R_{\exp(\varepsilon\Lambda)}(p) \in T_p P \tag{2.126}$$

In this language a connection $\boldsymbol{A}$ is a globally defined Lie algebra-valued one-form on the bundle, that has the two properties:

$$\forall \Lambda \in \mathfrak{sl}(2,\mathbb{C}) : X_\Lambda \lrcorner \boldsymbol{A} = \Lambda, \quad R_g^* \boldsymbol{A} = g^{-1} \boldsymbol{A} g \tag{2.127}$$

*The books of Isham, Bertlmann and Frankl give concise introductions to the theory of fibre bundles [121–123]

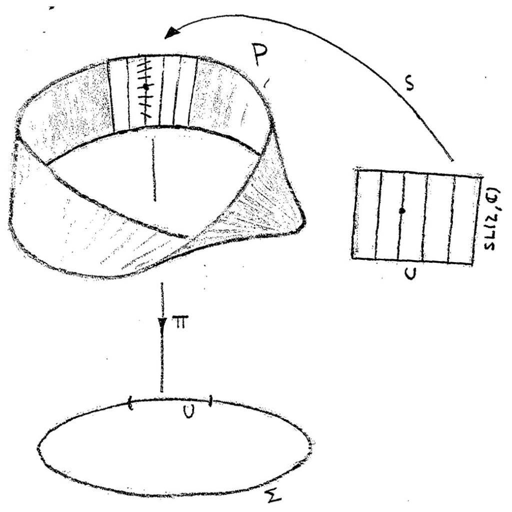

Figure 2.2: The kinematical symmetries can be best understood if lifted onto a principal bundle $P(\Sigma, SL(2,\mathbb{C}))$, with the three-dimensional hypersurface Σ as the base manifold, and $SL(2,\mathbb{C})$ as the standard fibre. In a sufficiently small neighbourhood $U \subset \Sigma$ the bundle locally looks like $U \times SL(2,\mathbb{C})$. The corresponding diffeomorphism lifting $U \times SL(2,\mathbb{C})$ into the bundle is s, while $\pi : P \to \Sigma$ denotes the canonical projection. The horizontal subspace HP of TP is the kernel of the $\mathfrak{sl}(2,\mathbb{C})$-valued connection one-form $\boldsymbol{A}$ on P.

Its pullback under the action of s defines its local representative A^s, which we can identify with the Ashtekar connection ${A^i}_a$ on the base manifold. Explicitly:

$$ {A^i}_a \equiv A^s = s^*\boldsymbol{A} \tag{2.128} $$

A family of n one-forms in a p-dimensional manifold defines a $(p-n)$-dimensional hypersurface in the tangent space of the manifold. Accordingly, we can define the horizontal subspace HP of the tangent space of the bundle. For any $p \in P$ this is the three-dimensional vector space:

$$ H_pP = \{V \in T_pP : V \lrcorner \boldsymbol{A}_p = \boldsymbol{A}_p(V) = 0\}, \tag{2.129} $$

which is isomorphic to the tangent space of the base manifold through $\pi_* H_pP = T_{\pi(p)}\Sigma$. If $\mathrm{pr}_H : T_pP \to H_p$ denotes the projector onto the horizontal subspace we can define the field strength as:

$$ \boldsymbol{F}(U,V) = (\mathrm{d}\boldsymbol{A})(\mathrm{pr}_H U, \mathrm{pr}_H V), \quad \boldsymbol{F} = \mathrm{d}\boldsymbol{A} + \boldsymbol{A} \wedge \boldsymbol{A} \tag{2.130} $$

Its local representative gives the curvature of the Ashtekar connection:

$$ {F^i}_{ab} \equiv F^s = s^*\boldsymbol{F} = \mathrm{d}A^s + A^s \wedge A^a \tag{2.131} $$

Finally, we can unambiguously lift any vector-(field) $N \in T_x\Sigma$ in the base manifold into a horizontal vector-(field) $N^\uparrow \in H_pP$ in the bundle, such that (i) $\pi(p) = x$, and (ii) $\pi_* N^\uparrow = N$.

Having said all this, let us now explore the geometric meaning of the variations of both the connection (2.124) and the momentum variable (2.125). We start with the variation $\delta_N A$ of the connection with respect to the vector field on the base manifold, as defined by (2.124) and find:

$$\begin{aligned}\delta_N A^i{}_a = N^b F^i{}_{ba} \equiv N \lrcorner F^s = s^*\big(N^\uparrow \lrcorner \boldsymbol{F}\big) = s^*\big(N^\uparrow \lrcorner \mathrm{d}\boldsymbol{A}\big) = \\ = s^*\big(N^\uparrow \lrcorner \mathrm{d}\boldsymbol{A} + \mathrm{d}(N^\uparrow \lrcorner \boldsymbol{A})\big) = s^*\big(L_{N^\uparrow}\boldsymbol{A}\big) \qquad (2.132)\end{aligned}$$

In other words, the counter-term introduced in (2.124) for $\delta_N A = L_N A + \dots$ to transform under the adjoint action of the gauge group, turns $\delta_N A$ into an ordinary Lie derivative on the bundle. If we lift $\delta_N A$ into the bundle it becomes the Lie derivative along the horizontal lift $N^\uparrow$ of N. For the momentum variable the situation is very similar.

First of all, we need to lift the momentum into the bundle. On the base manifold we can view $\Pi_i{}^a$ as a locally defined $\mathfrak{sl}(2,\mathbb{C})$-valued two-form* that transforms under the adjoint action of the group. On the bundle it is globally defined, and corresponds to a Lie-algebra valued two-form $\boldsymbol{\Pi}$ through the equations:

$$\Pi_i{}^a \equiv \Pi^s = s^*\boldsymbol{\Pi}, \qquad (2.133)$$

and its defining properties

$$X_\Lambda \lrcorner \boldsymbol{\Pi} = 0, \quad R_g^*\boldsymbol{\Pi} = g^{-1}\boldsymbol{\Pi} g. \qquad (2.134)$$

That is, the momentum is degenerate along the direction of the fibres, and transforms homogeneously under right translations.

If N is now a vector field in a neighbourhood U of the base manifold (see figure 2.2 for an illustration), this trivially** defines also a vectorfield on $U\times SL(2,\mathbb{C})$, and we can use the push-forward of $s : U \times \Sigma \rightarrow P$ to map it into the bundle. We decompose the resulting vector s_*N into its horizontal component $N^\uparrow$, and a component $N^\parallel$ parallel to the fibers. Using equation (2.127) we can read off the parallel component, and thus have:

$$s_*N = N^\uparrow + X_{\boldsymbol{A}(s_*N)} \equiv N^\uparrow + N^\parallel \qquad (2.135)$$

The transformation law (2.134) implies for a Lie derivative into the direction of the fibre that:

$$L_{X_\Lambda}\boldsymbol{\Pi} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} R_{\exp(\varepsilon\Lambda)}^*\boldsymbol{\Pi} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\big(\exp(-\varepsilon\Lambda)\boldsymbol{\Pi}\exp(\varepsilon\Lambda)\big) = -[\Lambda, \boldsymbol{\Pi}] \qquad (2.136)$$

This, together with (2.135), allows us to calculate the Lie derivative of the momentum variable $\boldsymbol{\Pi}$ into the direction of the horizontal lift $N^\uparrow$ of N. We get:

$$s^*\big(L_{N^\uparrow}\boldsymbol{\Pi}\big) = s^*\big(L_{s_*N}\boldsymbol{\Pi} - L_{N^\parallel}\boldsymbol{\Pi}\big) = L_N\Pi^s + \big[N\lrcorner A^s, \Pi^s\big] \equiv \delta_N \Pi_i{}^a \qquad (2.137)$$

This is the desired result. We see, that just as in (2.132), the counterterms added to the variation $\delta_N\Pi_i{}^a$ have a clean geometrical origin, for they result from the pullback of an ordinary Lie derivative on the fibre bundle down to the base manifold.

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*Remember: A vector valued density and a two form are the same thing through the equation: $\frac{1}{2}\tilde{\eta}^{abc}\Pi_{ibc} = \Pi_i{}^a$, where $\tilde{\eta}^{abc}$ is the Levi-Civita density.

**Given a vector N in U, this is a derivation that acts as $N[f] = N^\mu\partial_\mu f$ on a scalar $f : U \rightarrow M$. This is well defined also if f is a function on $U \times SL(2,\mathbb{C})$, which thus trivially extends N to $U \times SL(2,\mathbb{C})$.

Concluding remarks On the base manifold, the Lie derivatives $L_N A^i{}_a$, $L_N \Pi_i{}^a$ of the canonical variables are not defined globally, since L_N does not know how to act on the internal index i living in the Lie algebra. Adding appropriate counterterms to the ordinary Lie derivative, we found in equations (2.124) and (2.125) variations $\delta_N A^i{}_a$ and $\delta_N \Pi_i{}^a$, that are now globally defined, and transform under the adjoint representation of the local $SL(2,\mathbb{C})$ gauge symmetry. The variations, thus defined, have a clean geometrical interpretation once we go to the corresponding principal bundle, where δ_N becomes and ordinary Lie derivative $L_{N^\uparrow}$ into the direction of the horizontal lift $N^\uparrow$ of N. In fact, the infinitesimal gauge transformations δ_Λ, δ_N (for Λ and N denoting respectively an $\mathfrak{sl}(2,\mathbb{C})$-valued scalar, and an ordinary vector field on the base manifold) can be exponentiated yielding finite transformations. These are diffeomorphisms of the bundle, that preserve the fibers, i.e. map fibres onto fibres. Their projections (under the action of $\pi : P \rightarrow \Sigma$) define diffeomorphisms of the base manifold. Those that are mapped towards the identity are the pure gauge transformations generated by δ_Λ with $\Lambda \in \mathfrak{sl}(2,\mathbb{C})$.

2.4.3 First-class constraints

Within the list of constraints (2.95) we will now identify those that form a closed algebra. This algebra will have a clean geometrical meaning, for it contains the generators of the gauge symmetries that we have introduced above.

To simplify our calculations let us agree that the smearing functions Λ^i, N^a, and $\underset{\sim}{N}$ entering the definition of the constraints shall vanish at the boundary S of Σ. The derivation of the Poisson brackets further simplifies when first studying the functional differentials "$\mathbb{d}$" of the respective constraints. We start with the Gauß law, and get, after having dropped a surface term arising from a partial integration, that:

$$\begin{aligned}\mathbb{d}G_i[\Lambda^i] &= \int_\Sigma \Big(-\Lambda^i \epsilon_{il}{}^m \mathbb{d}A^l{}_a \Pi_m{}^a + D_a \Lambda^i \mathbb{d}\Pi_i{}^a \Big) + \text{cc.} = \\ &= \int_\Sigma \Big(-\delta_\Lambda(\Pi_i{}^a)\mathbb{d}A^i{}_a + \delta_\Lambda(A^i{}_a)\mathbb{d}\Pi_i{}^a \Big) + \text{cc.}\end{aligned} \tag{2.138}$$

Looking back at equation (2.123) we recognise the appearance of infinitesimal $\mathfrak{sl}(2,\mathbb{C})$ gauge transformations in this formula. This is a crucial observation, for it shows the Hamiltonian vector field of the Gauß constraint generates finite $SL(2,\mathbb{C})$ gauge transformations (those continuously connected to the identity) on phase-space. To be more precise, if $F : \mathcal{P}_{\text{aux}} \rightarrow \mathbb{R}$ is a functional on phase space, equation (2.138) implies:

$$\mathfrak{X}_{G_i[\Lambda^i]}\big|_{(\Pi,A)}[F] = \big\{G_i[\Lambda^i], F\big\}\big|_{(\Pi,A)} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} F\big[\rho_{\exp(\varepsilon\Lambda)}\Pi_j{}^b, \rho_{\exp(\varepsilon\Lambda)}A^i{}_a\big], \tag{2.139}$$

Taking the exponential we can identify the Hamiltonian vector field of the Gauß constraint with the generator of the desired transformations:

$$\exp\big(\mathfrak{X}_{G_i[\Lambda^i]}\big)\big|_{(\Pi,A)}[F] = F\big[\rho_{\exp(\Lambda)}\Pi_j{}^b, \rho_{\exp(\Lambda)}A^i{}_a\big] \tag{2.140}$$

For the vector constraint we proceed according to the very same strategy, and first calculate the functional differential of the constraint. The variation of the curvature two-form equals a total exterior covariant derivative

$$\mathbb{d}F^i{}_{ab} = D_a \mathbb{d}A^i{}_b - D_b \mathbb{d}A^i{}_b. \tag{2.141}$$

Performing a partial integration we can separate the covariant derivative from the functional differentials it acts on. This yields a surface term that vanishes due to our falloff conditions on the vector-valued smearing function N^a. A straight forward calculation thus gives:

$$\mathbb{d}H_a[N^a] = \int_\Sigma \Big(- D_a\big(N^a\Pi_i{}^b - N^b\Pi_i{}^a\big)\mathbb{d}A^i{}_a + N^a F^i{}_{ab}\mathbb{d}\Pi_i{}^b\Big) + \text{cc.} = 0 \qquad (2.142)$$

Comparing this equation with (2.124) we can write $N^a F^i{}_{ab}$ as the variation $\delta_N(A^i{}_a)$ of the connection under a spatial diffeomorphism horizontally lifted into the bundle. The analogous derivative for the momentum variable, i.e $\delta_N\Pi_i{}^a$ (defined as in (2.125)) is hidden in the first two terms of this equation. This follows from the following considerations:

$$\begin{aligned} D_a\big(N^a\Pi^b - N^b\Pi^a\big) &= \partial_a(N^a\Pi^b) - \partial_a N^b\Pi^a + \big[N^a A_a, \Pi^b\big] - N^b D_a\Pi^a \\ &= L_N\Pi^a + [N^a, \Pi^b] - N^b D_a\Pi^a = \delta_N\Pi^b - N^b D_a\Pi^a \end{aligned} \qquad (2.143)$$

where ∂_a is a partial derivative. We thus find the functional differential of the vector constraint to be:

$$\mathbb{d}H_a[N^a] = \int_\Sigma \Big(- \delta_N(\Pi_i{}^a)\mathbb{d}A^i{}_a + \delta_N(A^i{}_a)\mathbb{d}\Pi_i{}^b + N^a D_b\Pi_i{}^b\mathbb{d}A^i{}_a\Big) + \text{cc.} = 0 \quad (2.144)$$

Each term of this equation has again a clean geometric meaning. The first two of them, contain the pull back $\delta_N(\Pi_i{}^a)$ and $\delta_N(A^i{}_a)$ of the corresponding Lie derivatives in the bundle (2.132) and (2.137). They generate diffeomorphisms of the underlying principal bundle, that rigidly move the fibres around, i.e. they map fibres onto fibres. The last term, on the other hand, is proportional to the Gauß constraint, which vanishes on the constraint hypersurface, and could thus be dropped altogether.

Equations (2.138) and (2.144) uncover the geometric meaning of the Hamiltonian vector-fields of both the Gauß and vector constraint, as the generators of gauge symmetries of the theory. We can thus immediately deduce the Poisson brackets:

$$\big\{G_i[\Lambda^i], G_j[\mathrm{M}^j]\big\} = -G_i\big[[\Lambda, \mathrm{M}]^i\big] \qquad (2.145a)$$

$$\big\{G_i[\Lambda^i], H_a[N^a]\big\} = \big\{G_i[\Lambda^i], H[\underset{\sim}{N}]\big\} = 0 \qquad (2.145b)$$

$$\big\{H_a[M^a], H_a[N^b]\big\} = -H_a\big[[M,N]^a\big] - G_i\big[F^i(M,N)\big] \qquad (2.145c)$$

$$\big\{H_a[N^a], H[\underset{\sim}{N}]\big\} = -H[L_N\underset{\sim}{N}] - G_i\Big[\frac{\delta H[\underset{\sim}{N}]}{\delta\Pi_i{}^a}V^a\Big], \qquad (2.145d)$$

where we have used the commutators of Lie algebra elements $\Lambda, M \in \mathfrak{sl}(2,\mathbb{C})$: $[[\Lambda,\mathrm{M}]^i] = \epsilon^i{}_{lm}\Lambda^l\mathrm{M}^m$, and vector fields M, N: $[M,N]^a = M^b\partial_b N^a - N^b\partial_b M^a$.

We are now left to compute the Poisson bracket between two Hamiltonian constraints. This we do by first looking at the corresponding functional differential, which is readily computed to be:

$$\begin{aligned} \mathbb{d}H[\underset{\sim}{N}] = \int_\Sigma \underset{\sim}{N}\Big[&- \frac{2\ell_{\mathrm{P}}^2}{\hbar}\frac{\beta}{\beta+\mathrm{i}}\epsilon_i{}^{lm}F^i{}_{ab}\Pi_l{}^a\mathbb{d}\Pi_m{}^b + \\ &- 2\Lambda\Big(\frac{\ell_{\mathrm{P}}^2}{\hbar}\Big)^2\frac{\mathrm{i}\beta^2}{(\beta+\mathrm{i})^2}\epsilon^{ilm}\underset{\sim}{\eta}_{abc}\Pi_i{}^a\Pi_l{}^b\mathbb{d}\Pi_m{}^c\Big] + \\ &+ \int_\Sigma \frac{2\ell_{\mathrm{P}}^2}{\hbar}\frac{\beta}{\beta+\mathrm{i}}\mathbb{d}A^i{}_b\epsilon_i{}^{lm}D_a\big(\underset{\sim}{N}\Pi_l{}^a\Pi_m{}^b\big) + \text{cc.} \end{aligned} \qquad (2.146)$$

This allows us to find the desired Poisson bracket:

$$\begin{aligned}\big\{H[\underset{\sim}{M}],H[\underset{\sim}{N}]\big\}&=-\Big[\int_\Sigma \underset{\sim}{M}\Big(\frac{2\ell_{\mathrm{P}}^2}{\hbar}\frac{\beta}{\beta+\mathrm{i}}\epsilon_i{}^{lm}F^i{}_{ab}\Pi_l{}^a+2\Lambda\Big(\frac{\ell_{\mathrm{P}}^2}{\hbar}\Big)^2\frac{\mathrm{i}\beta^2}{(\beta+\mathrm{i})^2}\epsilon^{ilm}\underset{\sim}{\eta}_{abd}\Pi_i{}^d\Pi_l{}^a\Big)\cdot\\ &\quad\cdot\frac{2\ell_{\mathrm{P}}^2}{\hbar}\frac{\beta}{\beta+\mathrm{i}}\epsilon_m{}^{rs}D_c\big(\underset{\sim}{N}\Pi_r{}^c\Pi_s{}^b\big)-(\underset{\sim}{M}\leftrightarrow\underset{\sim}{N})\Big]+\mathrm{cc.}=\\ &=\Big(\frac{2\ell_{\mathrm{P}}^2}{\hbar}\Big)^2\frac{\beta^2}{(\beta+\mathrm{i})^2}\int_\Sigma\big(\underset{\sim}{M}\partial_c\underset{\sim}{N}-\underset{\sim}{N}\partial_c\underset{\sim}{M}\big)\Pi_l{}^a\Pi^{lc}F^i{}_{ab}\Pi_i{}^b+\mathrm{cc.}\end{aligned}\tag{2.147}$$

Notice, that we cannot write the right hand side of this equation as a linear combination of Gauß, vector and Hamiltonian constraint. This means, that on the auxiliary phase space, the algebra generated by G_i, H_a, and H does not close. This is, however, not so harmful, since, up to this point, we have not yet imposed the reality conditions $C_i{}^a=0$ on the momentum variable, as written in (2.97). If the momentum satisfies these additional reality conditions, we can introduce a densitised triad $E_i{}^a$, linearly related to the canonical momentum by equation (2.96), eventually revealing that

$$\big\{H[\underset{\sim}{M}],H[\underset{\sim}{N}]\big\}\Big|_{C=0}=-H_a\big[E_j{}^aE^{jb}\big(\underset{\sim}{M}\partial_c\underset{\sim}{N}-\underset{\sim}{N}\partial_c\underset{\sim}{M}\big)\big]\tag{2.148}$$

The right hand is proportional to the vector constraint, and we thus see the Poisson algebra generated by Gauß, vector and Hamiltonian constraint closes on the constraint hypersurface $C=0$. This does however not suffice to prove them to be of first-class, at least, not until we have also take the stability of the reality conditions $C_i{}^a=0$, i.e. the secondary constraint (2.115) into account. This we will now turn our attention to.

2.4.4 Second-class constraints

The Hamiltonian time-evolution must preserve the primary constraints (2.95) and (2.97). This includes the reality condition $C_i{}^a=0$, which leads us to the stability criterion (2.115). This equation demands that:

$$\begin{aligned}0\overset{!}{=}\dot{C}_i{}^a\big|_{C=0}&=\big\{H',C_i{}^a\big\}\big|_{C=0}=-\frac{1}{2\mathrm{i}}\epsilon_{il}{}^m(\Lambda^l-\bar{\Lambda}^l)E_m{}^a+\\ &\quad+\frac{1}{2\mathrm{i}}(D_b-\bar{D}_b)\big(N^bE_i{}^a-N^aE_i{}^b\big)+\frac{1}{2}\epsilon_i{}^{lm}(D_b+\bar{D}_b)\big(\underset{\sim}{N}E_l{}^bE_m{}^a\big)\\ &=-\epsilon_{il}{}^m\Big(\frac{1}{2\mathrm{i}}(\Lambda^l-\bar{\Lambda}^l)-N^bK^l{}_b-e^{lb}\partial_bN\Big)E_m{}^a+N\tilde{\eta}^{bac}\mathfrak{D}_be_{ic}.\end{aligned}\tag{2.149}$$

Here we have reintroduced the undensitised lapse $N=d^3v\underset{\sim}{N}$ and $\mathfrak{D}_a=\frac{1}{2}(D_a+\bar{D}_a)=\partial_a+[\Gamma_a,\cdot]$ defines yet another covariant derivative. In the last line of this equation both the expression within the big bracket and the $\mathfrak{D}e$ term must vanish independently. This can be seen as follows. Notice that the covariant derivative $\mathfrak{D}_a$ equals the intrinsic Levi-Civita derivative (i.e. the unique covariant derivative compatible with the triad $e^i{}_a$ on the spatial slice) up to a difference tensor $\Delta^i{}_a$. Using the fact that $\mathfrak{D}_aE_j{}^a=0$ provided both the Gauß constraint and the reality condition $C_i{}^a=0$ holds, it follows that $\Delta^i{}_a$ must be symmetric (in the sense of $\epsilon_{jlm}\Delta^l{}_bE^{mb}=0$). We can then reformulate equation (2.149) according to the following steps:

$$\begin{aligned}-\frac{1}{2\mathrm{i}}(\Lambda^j-\bar{\Lambda}^j)+N^aK^j{}_a+e^{ja}\partial_aN\Big|_{C,G=0}&\overset{!}{=}-\frac{1}{2}N(e^{ic}e^{jb}-e^{ib}e^{jc})\mathfrak{D}_be_{ic}\Big|_{C,G=0}=\\ &=-\frac{1}{2}N(e^{ic}e^{jb}-e^{ib}e^{jc})\epsilon_{ilm}\Delta^l{}_be^m{}_c\Big|_{C,G=0}=\\ &=\frac{1}{2}N\epsilon_{il}{}^j\Delta^l{}_be^{ib}\Big|_{C,G=0}=0.\end{aligned}\tag{2.150}$$

Since $\dot{C}_i{}^a$ must vanish we see the imaginary part of $\Lambda^i = \varphi^i + \mathrm{i}\xi^i$ (describing boosts along the flow of time) is completely fixed by the dynamics of the theory to the value

$$\xi^i = \frac{1}{2\mathrm{i}}\big(\Lambda^i - \bar{\Lambda}^i\big) = +N^a K^i{}_a + e^{ia}\partial_a N. \tag{2.151}$$

Since the condition $C_i{}^a = 0$ is invariant under internal rotations but does not remain valid if boosted, this restriction should not surprise us. For the real part φ^i of the Lagrange multiplier Λ^i we do however not get any restriction. It can be chosen freely, which is another way to say that there is a residual internal $\mathfrak{su}(2)$ gauge symmetry in the theory.

Comparing equation (2.151) with (2.150) reveals that $\mathfrak{D}e \equiv 2\mathfrak{D}_{[\mathfrak{a}}e^i{}_{b]}$ must vanish too. This is the three-dimensional torsionless equation. Equation (2.150) therefore implies an additional secondary constraint, that tells us that the difference tensor $\Delta^i{}_a$ must vanish:

$$\Delta^i{}_a = \frac{1}{2}\big(A^i{}_a - \bar{A}^i{}_a\big) - \overset{\mathrm{LC}}{\Gamma}{}^i{}_a[E] \stackrel{!}{=} 0., \tag{2.152}$$

where $\overset{\mathrm{LC}}{\Gamma}{}^i{}_a[E]$ denotes the Levi-Civita connection functionally depending on the densitised triad $E_i{}^a = d^3v_e e_i{}^a$. This equation is highly non polynomial [19] in E, the equivalent but technically different version

$$2T := De + \bar{D}e = 2\mathfrak{D}e \stackrel{!}{=} 0 \tag{2.153}$$

is much simpler to handle, since it just sets the spatial part of the four dimensional torsion two-form to zero.

Stability of $T = 0$. Next, we have to check if the Hamiltonian preserves the secondary constraint (2.153) that we have just introduced. To write down the torsion-free condition we need a triad, which is defined only if the reality condition (2.97) holds true. There is no triad on all of phase-space. To study the stability of the secondary constraint we shall calculate the Poisson bracket $\{H', T\}$, which requires the triad to be known, however, around the constraint hypersurface, and not just on the solution space itself. We thus need to extend the (densitised) triad away from the constraint hypersurface, and one possible way to do so is this:

$$E_i{}^a := \frac{\ell_{\mathrm{P}}^2}{\hbar}\Big(\frac{\mathrm{i}\beta}{\beta+\mathrm{i}}\Pi_i{}^a - \frac{\mathrm{i}\beta}{\beta-\mathrm{i}}\bar{\Pi}_i{}^a\Big) \tag{2.154}$$

which coincides with (2.96) once the reality condition (2.97) is satisfied. If time evolution perseveres all constraints, which we sill show in a minute, we could have chosen any other extension.

If we now calculate the desired Poisson bracket between the primary Hamiltonian (2.112) and the torsion free condition (2.153), we get

$$\dot{T}^i{}_{ab}\big|_{C,G,T=0} = \big\{H', D_{[a}e^i{}_{b]} + \mathrm{cc.}\big\}\big|_{C,G,T=0} = -\frac{\ell_{\mathrm{P}}^2}{\hbar}\frac{2\beta^2}{\beta^2+1}\epsilon^i{}_{lm}V^l{}_a e^m{}_b \stackrel{!}{=} 0. \tag{2.155}$$

Where $C, G, T = 0$ shall remind us that these equations hold provided torsion T vanishes and both reality condition and Gauß constraint are satisfied. Equation (2.155) does not vanish by itself; yet it must vanish and we thus get a restriction on the Lagrange multiplier $V^i{}_a$. An elementary algebraic manipulation yields:

$$\dot{T}^i{}_{ab}\big|_{C,G,T=0} = 0 \Leftrightarrow V^i{}_a = 0 \tag{2.156}$$

Stability of Gauß, vector and Hamiltonian constraint. Employing the results of the previous subsection, we can now readily check the stability of all remaining constraints. We have just seen, that the Hamiltonian preserves the secondary constraint $T = 0$ only if the Lagrange multiplier $V^i{}_a$ is set to zero. This restricts Dirac's primary Hamiltonian H', as defined by (2.112), to be just a sum of Gauß, vector and Hamiltonian constraints. We have already collected the corresponding Poisson brackets in (2.145) and (2.148), and immediately see, once $V^i{}_a = 0$ the Hamiltonian does indeed preserve all constraints of the theory.

Residual gauge symmetries. The Hamiltonian of the theory is a sum of constraints. Time evolution preserves the reality conditions (2.97) and (2.152) only if there are also restrictions on the Lagrange multipliers in front of the constraints. Lapse and shift determine the real part of the multiplier $\Lambda^i = \xi^i + \mathrm{i}\xi^i$ to the value (2.151), while $V^i{}_a$ must be put to zero. The remaining multipliers appearing in (2.112) give the residual gauge symmetries of the theory. The Gauß constraint $G_i[\varphi^i]$ generates internal rotations of the spatial triad (these are. those Lorentz transformations that leave the time internal direction n^μ (2.58) invariant), while the vector constraint $H_a[N^a]$ generates diffeomorphisms of the spatial slice (modulo internal gauge transformations). To recover the four-dimensional diffeomorphism symmetry we also need to include the Hamilton constraint that gives the diffeomorphisms along the time direction.

Dirac bracket an general strategy towards quantum theory. The rotational part of the Gauß law $\mathfrak{Re}(D_a\Pi_i{}^a)$ together with both the vector and the Hamilton constraint generate the gauge symmetries of the theory, and thus form the set of first class constraints. The remaining constraints, i.e. the reality conditions (2.97) and (2.152) are of second class. This follows from their mutual Poisson brackets:

$$\big\{C_i{}^a(p), C_j{}^b(q)\big\} = 0, \tag{2.157a}$$

$$\big\{C_i{}^a(p), \Delta^j{}_b(q)\big\} = 0, \tag{2.157b}$$

$$\big\{\Delta^i{}_a(p), \Delta^j{}_b(q)\big\} = \frac{\ell_{\mathrm{P}}^2}{\hbar}\frac{\beta}{\beta^2+1}\Big(\frac{\delta \overset{\mathrm{LC}}{\Gamma}[E]^j{}_b(q)}{\delta E_i{}^a(p)} - \frac{\delta \overset{\mathrm{LC}}{\Gamma}[E]^i{}_a(p)}{\delta E_j{}^b(q)}\Big) = 0 \tag{2.157c}$$

The first two of these equations are immediate to show, the last implicitly follows from (2.86), which proves that the Levi-Civita connection has a generating potential $\Phi[E]$:

$$\frac{\delta}{\delta E_i{}^a(p)}\Phi[E] \equiv \frac{\delta}{\delta E_i{}^a(p)}\int_\Sigma \overset{\mathrm{LC}}{\Gamma}[E]^j{}_b E_j{}^b = \overset{\mathrm{LC}}{\Gamma}[E]^i{}_a(p) + \text{surface-terms} \tag{2.158}$$

The solution space $\mathcal{P}^*$ of the reality conditions (2.97) and (2.152) carries a natural symplectic structure given by the Dirac bracket that extends as a degenerate symplectic form to the full auxiliary phase-space $\mathcal{P} \supset \mathcal{P}^*$. For any two functionals F, and G; defined on the full auxiliary phase space, the Dirac bracket is given by:

$$\big\{F, G\big\}^* = \big\{F, G\big\} - \frac{\hbar}{\ell_{\mathrm{P}}^2}\frac{\beta^2+1}{\beta}\int_{p\in\Sigma}\big\{C_i{}^a(p), F\big\}\big\{\Delta^i{}_a(p), G\big\} + \tag{2.159}$$

$$+\frac{\hbar}{\ell_{\mathrm{P}}^2}\frac{\beta^2+1}{\beta}\int_{p\in\Sigma}\big\{C_i{}^a(p), G\big\}\big\{\Delta^i{}_a(p), F\big\} \tag{2.160}$$

The $\mathfrak{su}(2)$ Ashtekar–Barbero connection $A^{(\beta)i}{}_a = \Gamma^i{}_a[E] + \beta K^i{}_a$ together with the densitised triad $E_i{}^a$ diagonalise the Dirac bracket according to the equations:

$$\big\{E_i{}^a(p), E_j{}^b(q)\big\}^*\Big|_{\mathcal{P}^*} = \big\{A^{(\beta)i}{}_a(p), A^{(\beta)j}{}_b(q)\big\}^*\Big|_{\mathcal{P}^*} = 0, \tag{2.161a}$$

$$\big\{E_i{}^a(p), A^{(\beta)j}{}_b(q)\big\}^*\Big|_{\mathcal{P}^*} = \frac{\beta\ell_{\mathrm{P}}^2}{\hbar}\delta_i^j\delta_b^a\delta^{(3)}(p,q), \tag{2.161b}$$

$$\big\{C_i{}^a(p),\,\cdot\,\big\}^*\Big|_{\mathcal{P}^*} = \big\{\Delta^i{}_a(p),\,\cdot\,\big\}^*\Big|_{\mathcal{P}^*} = 0. \tag{2.161c}$$

Since the reality conditions (2.97) and (2.152) imply that the Ashtekar–Barbero connection $A^{(\beta)i}{}_a$ together with the densistised triad $E_i{}^a = d^3v_e e_i{}^a$ already determine any point in $\mathcal{P}^*$, we can use them as canonical conjugate variables, and would thus recover the theory in its usual Hamiltonian formulation [19, 40].

The phase-space of $SU(2)$ Ashtekar–Barbero variables thus uncovered is often taken as the starting point for the canonical quantisation program, eventually revealing the kinematical Hilbert space of loop quantum gravity [19, 40]. We would then be left to quantise the generators of the residual gauge symmetries in order to find the kernel of the quantised Gauß, vector and Hamiltonian constraints, and study the physical states of the theory.

We could, however, also try to start from the bigger auxiliary phase space of selfdual variables. We would thus not solve the reality conditions classically, but follow the general ideas of Gupta and Bleuler [124, 125] and impose them directly in the quantum theory. Since the reduced phase-space of Ashtekar–Barbero variables only carries a representation of the rotation group, this would have the advantage of never explicitly breaking the local Lorentz symmetry.

The role of torsion At the end of this section, let us briefly see, how the four-dimensional torsion-free condition (2.12) reappears in the Hamiltonian formulation. Following the general strategy outlined in section 2.2 we perform a 3+1 split of the torsion two-form $\Theta^\alpha = \mathcal{D}\eta^\mu = 0$ and find*:

$$\mathrm{em}_t^*\big(\mathcal{D}\eta^0\big) = 0 \quad\Leftrightarrow\quad K^i\wedge e_i = 0 \Leftrightarrow K_{ij} = K_{ji} \tag{2.162a}$$

$$\mathrm{em}_t^*\big(\mathcal{D}\eta^i\big) = 0 \quad\Leftrightarrow\quad \mathfrak{D}e^i = 0 \tag{2.162b}$$

$$\mathrm{em}_t^*\big(\boldsymbol{t}\lrcorner\mathcal{D}\eta^0\big) = 0 \quad\Leftrightarrow\quad \xi^i e_i = K^i e_i + \mathrm{d}N \tag{2.162c}$$

$$\mathrm{em}_t^*\big(\boldsymbol{t}\lrcorner\mathcal{D}\eta^i\big) = 0 \quad\Leftrightarrow\quad \dot{e}^i = -\epsilon^i{}_{lj}\varphi^l e^j + \mathfrak{D}N^i + K^i N = \{H^*, e^i\}, \tag{2.162d}$$

Each of these equations appear in our canonical analysis. The first of them follows from the Gauß constraint, which implies through $C_i{}^a = 0$ and (2.96) that $D_a E_i{}^a = 0$. This equation has a real and an imaginary part. The imaginary part lacks any derivatives and reads $\mathfrak{Im}(D_a E_i{}^a) = \mathrm{i}\epsilon_i{}^{lm}K_{lb}E_m{}^b = \mathrm{i}\,d^3v_e\epsilon_i{}^{lm}K_{lm}$. This implies the symmetry of the extrinsic curvature tensor, i.e. $K_{ij} = K_{ji}$. Equations (2.162b) and (2.162c) arise differently. They are needed to guarantee the stability of the reality condition (2.97), that lead us to both (2.151) and (2.152), which is equivalent to (2.153). Finally there is equation (2.162d) which is nothing but the evolution equation for the triad. In summary, the four-dimensional torsionless condition separates into three different equations. Its components either are restrictions on the Lagrange multipliers,

*Here we are using the language of differential forms to simplify our notation: $e^i \equiv e^i{}_a$, $K^i \equiv K^i{}_a$ and $\mathfrak{D}e \equiv 2\mathfrak{D}_{[a}e^i{}_{b]}$.

impose a secondary constraint on phase-space, or define an evolution equation for the spatial triad.

2.5 SUMMARY

This chapter looked at the Hamiltonian formalism of general relativity in terms of first order tetrad-connection variables. Starting from the Holst action with a cosmological constant we first studied the variations of the action while keeping the intrinsic three-geometry of the boundary fixed. We took a lensoid region in space-time, and saw that except of the usual Gibbons–Hawking–York boundary term (in terms of first-order variables) there is also another corner term needed.

This additional corner terms appears, in fact, rather generically whenever there is a discontinuity in the boundary, e.g. at the two dimensional surface where both the initial and the final slice meet.

Next, there was section 2.2 looking at the spatio-temporal decomposition of the action. We rewrote the action in terms of complex Ashtekar variables, while keeping the Barbero–Immirzi parameter unchanged. The Barbero–Immirzi parameter has in fact two different roles in the theory. First of all, it appears as an additional coupling constant in the action, but then it also measures the relative strength between the spin connection $\Gamma[E]$ and the $\mathfrak{su}(2)$ Ashtekar–Barbero connection $A^{(\beta)} = \Gamma[E]+\beta K$. Choosing the Barbero–Immirzi parameter to be real valued and positive, does however not force us to work with the phase-space of $\mathfrak{su}(2)$ Ashtekar–Barbero variables. Geometrically, the complex Ashtekar variables are a better choice: First of all, they transform linearly under local Lorentz transformations, while the Ashtekar–Barbero variables do not. Secondly, written in selfdual variables, the constraint equations simplify—including the Hamiltonian constraint, (2.95c) which turns into a polynomial of the canonical variables.

Section 2.3 gave an application. Studying the Hamilton–Jacobi equations of the theory we could introduce a notion of energy as measured by a family of uniformly accelerated observers close to the corner. We recovered the local version of the first-law of thermodynamics as introduced by Frodden, Gosh and Perez, and thus gave an independent derivation of their results. [73, 107]

Section 2.4 introduced the phase-space of the theory. Working with complex Asktekar variables we found additional reality conditions (2.97) needed for the metric to be real. These reality conditions coincide with the linear simplicity constraints [126,127] of spinfoam gravity. We will see, in the next chapters, that this observation will open the possibility to formulate both spinfoam gravity and the canonical formulation of loop quantum gravity on equal footing. We studied the algebra of constraints and found both first and second-class constraints to be imposed. The first-class constraints generate the gauge symmetries of the theory. The Gauß constraint is responsible for internal Lorentz transformations, while the vector constraint generates the spatial diffeomorphism symmetry. The Hamilton constraint, on the other hand, generates diffeomorphisms along the time-flow vector-field. The total Hamiltonian is a sum of constraints, it vanishes and time-evolution is nothing but a gauge transformation on phase-space. Consequently, the time variable with respect to which we introduced the Hamiltonian, has no physical meaning whatsoever.

Then there are also the second-class constraints. The Hamiltonian preserves the reality conditions (2.97) only if (i) the spatial projection of the four-dimensional torsion two-form vanishes and (ii) the boost component of the Lagrange multiplier in front of the Gauß constraint is fixed in terms of lapse and shift to the value (2.151). Introducing the Dirac bracket we realised the Ashtekar–Barbero variables as canonical coordinates on the solution space of the second-class constraints. These variables are sometimes presented as if they were the only reasonable starting point for the quantisation program underlying the loop theory. One of the main objectives of this thesis is to argue that this is not true, and that we can also work with variables transforming more covariantly under the local $SL(2,\mathbb{C})$ symmetry group of general relativity.

The remaining part of this chapter contains two supplements, the first of which reviews the Hamiltonian formulation of general relativity in terms of metric variables [8, 74]. We will study the boundary value problem for a region of cylindrical shape and introduce the appropriate boundary and corner terms. Repeating Witten's proof [75] of the positivity of the ADM mass [77–79], this supplement also gives a motivating example for the use of spinors in general relativity. Spinors will play a prominent role in the rest of this work when studying the chiral aspects of loop quantum gravity. The second supplement concerns the Kodama state [80–89], which was one of the first solutions of the Hamiltonian constraint proposed. Although it does probably not describe the vacuum state of the theory, it has recently regained some attention when trying to add a cosmological constant to the theory [91–95].

SUPPLEMENT:
ADM FORMALISM AND WITTEN'S PROOF

This supplement gives an introduction to the ADM (Arnowitt, Deser, Misner [8, 74]) formulation of general relativity. It was the first consistent Hamiltonian description of the dynamics of the theory, and boosted the case for a background independent quantisation. It has, however one major disadvantage.

Only classical matter (electromagentic waves, classical fluids, dust, solids, and all related things) fit into this formalism. Hinging upon a metric, the ADM framework cannot account for fermions. We can, in fact, write the Dirac equation in curved spacetime, only when introducing local inertial frames, that is a vierbein (a tetrad) [128].

Nevertheless, it is also important to know about the metric formalism, and recognise that in the absence of fermions both formulations are actually equivalent. The ADM formulation is also particularly well suited to introduce a certain notion of energy and momentum for the gravitational field itself [74, 77, 78]. This will, in fact, be a main topic of this section, which splits into three parts. The first of which studies the original Einstein–Hilbert action augmented by the appropriate boundary and corner terms [66–71]. Next, we perform a 3+1 split and recover the ADM Hamiltonian together with the ADM energy-momentum vector. The last part presents Witten's proof of the positivity of the ADM mass. [75, 76, 79]. This we do, also as a motivating example for what follows in the rest of this book. We will see spinors not only play an important role in classical general relativity, but appear even more prominently when exploring the chiral structure of loop quantum gravity.

Action and boundary terms

To study the ADM Hamiltonian formulation of general relativity, we start from the Einstein–Hilbert action supplemented by the appropriate boundary terms. These consist of the famous Gibbon–Hawking–York boundary term, essentially the trace of the extrinsic curvature tensor, together with additional corner terms belonging to the two-dimensional "corners" [66–71]. Here we are considering a boundary of cylindrical topology, as illustrated in figure 2.3. The action explicitly reads:

$$S_{\mathrm{EH}}[g_{ab}, n_a, z_a] = \frac{c^3}{16\pi G}\Big[\int_M d^4v\, R - 2\int_{\Sigma_1} d^3v\, K + 2\int_{\Sigma_0} d^3v\, K + 2\int_Z d^3v\, K + \\ - 2\int_{S_1} d^2v\, \Xi + 2\int_{S_0} d^2v\, \Xi\Big]. \tag{2.163}$$

Preparations. Here M is the four dimensional space-time manifold, the boundary of which consists of three parts $\partial M = \Sigma_0 \cup \Sigma_1 \cup Z$. The cylindrical shell Z, is a three-dimensional time-like hypersurface, it has topology $[0,1] \times S$, where S is a two dimensional space-like surface with the topology of a sphere. The remaining parts, Σ_0, and Σ_1 are three dimensional space-like hypersurfaces respectively, think of them as being homotopic to a ball in $\mathbb{R}^3$. We introduce a time function $t : M \to [0,1]$ to foliate M as $M = [0,1] \times \Sigma$, and define the time-slices $\Sigma_t = \{t\} \times \Sigma$ together with its boundary $\partial\Sigma_t = S_t$, and we thus also have $Z = [0,1] \times \Sigma$.

The four-, three- and two-dimensional volume elements $d^n v$ are defined* from the Levi-Civita tensor ε_{abcd} as follows:

$$d^4v = \frac{1}{4!}\varepsilon_{abcd}\tilde{\eta}^{abcd}, \tag{2.164a}$$

$$d^3v_a = \frac{1}{3!}\varepsilon_{abcd}\tilde{\eta}^{bcd}, \tag{2.164b}$$

$$d^2v_{ab} = \frac{1}{2!}\varepsilon_{abcd}\tilde{\eta}^{cd}. \tag{2.164c}$$

If n_a be the future oriented time-normal of Σ_t, and z_a denote the outwardly pointing normal of Z, we can define the intrinsic volume elements on both Σ_t and Z as follows:

$$d^3v = \begin{cases} d^3v_a n^a & \text{on } \Sigma_t, \\ d^3v_a z^a & \text{on } Z, \end{cases} \tag{2.165}$$

and equally for the two dimensional volume element:

$$d^2v := d^2v_{ab} n^a z^b. \tag{2.166}$$

Further clarifying notation $R = R^a{}_{bac} g^{cd}$ is the Ricci scalar of the metric tensor g_{ab}, while K is the trace of the extrinsic curvature tensor:

$$K = \begin{cases} \nabla_a n^a & \text{on } \Sigma_t, \\ \nabla_a z^a & \text{on } Z. \end{cases} \tag{2.167}$$

We conlcude this paragraph by giving the definition of the rapidity angle Ξ as:

$$\operatorname{sh}\Xi = -n_a z^a. \tag{2.168}$$

*Here $\tilde{\eta}^{ab\dots}$ denote the n-dimensional metric-independent Levi-Civita densities, that in any coordiante system $\{x^\mu\}$ are defined as follows: $\tilde{\eta}^{ab\dots} := \partial^a_\mu \partial^b_\nu \dots \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \dots$, with the circumflex stressing that we are looking at a density and not a tensor.

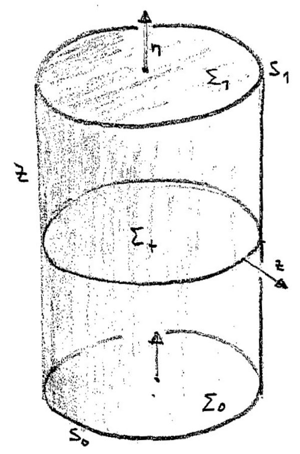

Figure 2.3: We are considering a cylindrical region M in space-time, the boundary of which consists of three parts. The top and bottom Σ_1 and Σ_0 are spatial three-dimensional surfaces, their future oriented time-normal we call n^a. The remaining part of the cylinder's boundary is the outer shelf Z, with space-like outwardly pointing normal z_a. Both Σ_1 and Σ_0 have a boundary themselves, these are the two-dimensional surfaces S_1 and S_0 respectively.

Fixing the orientation. To integrate the n-forms $d^n v$ we have to choose an orientation. On Σ_0 (and Σ_1) we say an ordered triple $(u^a, v^a, w^a) \in T\Sigma_{0,1} \times T\Sigma_{0,1} \times T\Sigma_{0,1}$ of tangent vectors in $T\Sigma_{0,1}$ is positively oriented if the quadruple (n^a, u^a, v^a, w^a) is positively oriented in M, while on Z we declare (u^a, v^a, w^a) to have positive orientation if (z^a, u^a, v^a, w^a) is positively oriented. On S_t, on the other hand, we say (u^a, v^a) have positive orientation if (n^a, z^a, u^a, v^a) is positively oriented with respect to M.

Variations and boundary conditions. For the moment let us neglect the corner terms belonging to the boundary of both Σ_1, and Σ_0. We so study only the variation:

$$\delta S' := \delta\Big[\int_M d^4v\, R + 2\int_{\partial M} d^3v\, \varepsilon K\Big] =? \tag{2.169}$$

Here, we have simplified our notation to treat all three components of the boundary $\partial M = \Sigma_0 \cup \Sigma_1 \cup Z$ at the same time, while the function $\varepsilon : \partial M \to \{\pm 1\}$, tells us where we actually are. We set $\varepsilon(p) = 1$ for both $p \in Z$ and $p \in \Sigma_0$, while $\varepsilon(p) = -1$ only on the space-like hyper-surfaces $\Sigma_1 \ni p$. We also define the normal v_a to the boundary as:

$$v_a = \begin{cases} n_a & \text{on } \Sigma_{0,1}, \\ z_a & \text{on } Z. \end{cases} \tag{2.170}$$

We can now introduce the induced metric on ∂M according to

$$h_{ab} = -s\, v_a v_b + g_{ab}, \quad s = v_a v^a \in \{\pm 1\}, \tag{2.171}$$

allowing us to define the extrinsic curvature tensor of the boundary:

$$K_{ab} = h_a{}^c \nabla_c v_b. \tag{2.172}$$

The vanishing of torsion implies:

$$K_{ab} = K_{ba}, \tag{2.173}$$

most easily seen when writing the normal as a gradient of some scalar constant along the boundary. Regarding variations we use the following notation. If $g^{(\varepsilon)}_{ab}$, and $v^{(\varepsilon)}_a$ are smooth one-parameter families of our configuration data, we can set for any function $\mathcal{F}$ of these fields:

$$\delta\big[\mathcal{F}[g_{ab}, v_a]\big] := \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \mathcal{F}[g^{(\varepsilon)}_{ab}, v^{(\varepsilon)}_a], \tag{2.174}$$

provided the existence of the differential is granted. We can thus define the covector $\delta v_a \in T^*\partial M$ together with the tensors δg_{ab}, and δh_{ab} as the variations

$$\delta v_a := \delta[v_a], \quad \delta g_{ab} := \delta[g_{ab}], \quad \delta h_{ab} := \delta[h_{ab}]. \tag{2.175}$$

This definition, and the usual rules of how to move indices, imply

$$\delta g^{ab} = g^{ac} g^{bd} \delta_{cd} = -\delta[g^{ab}], \tag{2.176}$$

together with:

$$\delta[v^a] = -\delta g^{ab} v_b + \delta v^a. \tag{2.177}$$

Equations (2.176) and (2.176) are often the source of troubles, and one should keep them in mind when going through the calculations presented below.

Next, we also need the variation of the Levi-Civita connection ∇_a, which is the unique metric compatible and torsionless derivative on M. Given some fixed vector-field V^a on M, we have $\delta[V^a] = 0$, and the linearity of the covariant derivative implies that there always is a difference tensor $\delta\Gamma^a{}_{bc}$ allowing us to write:

$$\delta\big[\nabla_a V^b\big] = \nabla_a V^b + \delta\Gamma^b{}_{ca} V^c. \tag{2.178}$$

We can determine the difference tensor* $\delta\Gamma^b{}_{ca}$ through the variation of the connection annihilating the metric $\nabla_a g_{bc} = 0$. This yields

$$\nabla_a g_{ab} = 0 \Rightarrow \nabla_a \delta g_{bc} = \delta\Gamma_{bca} + \delta\Gamma_{cba}. \tag{2.179}$$

The vanishing of torsion implies $\delta\Gamma_{bca} = \delta\Gamma_{bac}$ eventually fixing all components of the difference tensor $\delta\Gamma_{abc}$ to be:

$$\delta\Gamma_{abc} = \frac{1}{2}\big(\nabla_c \delta g_{ab} + \nabla_b \delta g_{ca} - \nabla_a \delta g_{bc}\big). \tag{2.180}$$

Then, we also need the variation of the curvature scalar appearing in the action. The defining equation $\nabla_{[a}\nabla_{b]} V^c = 2R^c{}_{dab} V^d$ of the Riemann tensor yields:

$$\delta\big[R^c{}_{dab}\big] = \nabla_a \delta\Gamma^c{}_{db} - \nabla_b \delta\Gamma^c{}_{da}. \tag{2.181}$$

For the Ricci scalar we thus get a total four-divergence:

$$\delta[R] = \nabla_a \delta\Gamma^{ab}{}_b - \nabla_a \delta\Gamma^{ba}{}_b. \tag{2.182}$$

*One could think of this tensor just as the variation $\delta[\cdot]$ of the Christoffel symbols $\Gamma^\mu{}_{\nu\alpha}$ in some coordinate system $\{x^\mu\}$. We then see this variation does indeed define a proper tensor, because the difference of two connection coefficients always transforms linearly.

The variation of the respective volume elements is the last piece missing in order to eventually compute (2.169). To this goal, let us observe that the derivative of the determinant of a matrix $X_{\alpha\beta}$ with respect to its components must always obey:

$$\frac{\partial \det X_{\alpha\beta}}{\partial X_{\mu\nu}} X_{\rho\nu} = \delta^{\mu}_{\rho} \det X_{\alpha\beta}. \tag{2.183}$$

With this equation we find the variation of the volume elements to be:

$$\delta[d^4 v] = \frac{1}{2} d^4 v \, g^{ab} \delta g_{ab}, \tag{2.184a}$$

$$\mathrm{em}^* \delta[d^3 v] = \frac{1}{2} \mathrm{em}^* d^3 v \, g^{ab} \delta h_{ab}, \tag{2.184b}$$

where em^* denotes the pullback onto the boundary ∂M.

We have now everything collected, that we need to compute the variation of the action S', and what we find is this:

$$\begin{aligned}
\delta[S'] &= \delta\Big[\int_M d^4 v\, R + 2\int_{\partial M} d^3 v\, \varepsilon K\Big] = \\
&= -\int_M d^4 v\, \delta g_{ab}\big(R^{ab} - \tfrac{1}{2} g^{ab} R\big) + \int_M d^4 v \nabla_a \big(\delta\Gamma^{ab}{}_b - \delta\Gamma^{ba}{}_b\big) + \\
&\quad + 2\int_{\partial M} d^3 v\, \varepsilon \big(\tfrac{1}{2}\delta h_{ab} g^{ab} K - \nabla_a \delta g^{ab} v_b - \delta g^{ab} \nabla_a v_b + \\
&\quad + \nabla_a \delta v^a + \delta \Gamma^a{}_{ba} v^b\big).
\end{aligned} \tag{2.185}$$

Remembering the four-dimensional Gauß law

$$\begin{aligned}
\int_M d^4 v \nabla_a V^a &= \int_{\partial M} d^3 v_a V^a = \int_{\partial M} d^3 v\, \varepsilon v_a V^a = \\
&= -\int_{\Sigma_1} d^3 v\, n_a V^a + \int_{\Sigma_0} d^3 v\, n_a V^a + \int_Z d^3 v\, z_a V^a,
\end{aligned} \tag{2.186}$$

we can replace the four-divergence, that came from the variation of the Ricci scalar (2.182), by an integral over the boundary ∂M of M. Demanding stationarity of the action, we see, the only surviving term in the bulk, i.e. the interior of M, implies the vacuum Einstein equations

$$G_{ab} = R_{ab} - \tfrac{1}{2} g_{ab} R = 0. \tag{2.187}$$

Looking at the remaining terms of (2.185) belonging to the three dimensional surface ∂M we can read off the boundary conditions. For, any variation of the action (2.163) preserving the boundary conditions must vanish provided the equations of motion hold true.

To this goal we have first to further simplify (2.185). The boundary integral coming from the variation of the Ricci tensors combines with the terms already present at ∂M to eventually give:

$$\delta[S']\Big|_{\mathrm{EOM}} = \int_{\partial M} d^3 v \varepsilon \Big[- v_a \nabla_b \delta g^{ab} + 2\nabla_a \delta v^a + \delta h^a{}_a K - 2\delta g^{ab} \nabla_a v_b \Big]. \tag{2.188}$$

In addition to the four-dimensional Gauß theorem this step also involved equation (2.180). We now have to further simplify equation (2.188). First of all we recognise that

$$\delta h_{ab} g^{ab} = \delta h_{ab} h^{ab}. \tag{2.189}$$

Then we also need that:

$$\delta g^{ab}\nabla_a v_b = \delta h_{ab}K^{ab} + s\delta g_{ab}v^a v^c \nabla_c v^b, \tag{2.190}$$

where $s = v^a v_a$ is again the signature of the respective boundary component Σ_0, Σ_1 and Z in question. We now want to try best to get rid of the derivative terms appearing in (2.188). We do this by introducing the induced three-dimensional covariant derivative $\mathfrak{D}_a$ on ∂M. This derivative is three-metric compatibel, i.e. $\mathfrak{D}_a h_{bc} = 0$ and torsionless. If V^a is vector tangential to ∂M, $\mathfrak{D}_a V^b$ is defined by projecting the four-dimensional derivation down to ∂M:

$$\mathfrak{D}_a V^b := h^b{}_{b'} h_a{}^{a'} \nabla_{a'} V^{b'}. \tag{2.191}$$

By linearity this definition extends to any tensors intrinsically defined on ∂M. To remove as many derivatives as possible, we absorb them into three-divergences which we can eventually turn into integrals over the corners S_0 and S_1. The simplest three-divergences that we can construct from both δz_a and δg_{ab} are:

$$\mathfrak{D}_a\big(h^{ab}\delta g_{bc}v^c\big) = -sK\delta g_{ab}v^a v^b + v_a\nabla_b\delta g^{ab} - sv^a v^b v^c \nabla_a \delta g_{bc} + K^{ab}\delta g_{ab}, \tag{2.192a}$$

$$\mathfrak{D}_a\big(h^{ab}\delta v_b\big) = -sKv^a\delta v_a - sv^a v^b \nabla_a \delta v_b + \nabla_a \delta v^a. \tag{2.192b}$$

what we also need is

$$v^a v^b v^c \nabla_a \delta h_{bc} = -2v^a \nabla_a v^b v^c \delta g_{bc} + 2v^a \nabla_a v^b \delta v_b, \tag{2.193}$$

in order to eventually get:

$$\delta[S']\Big|_{\text{EOM}} = \int_{\partial M} d^3v\varepsilon\Big[- D_a(h^{ab}\delta g_{bc}v^c) + 2D_a(h^{ab}\delta v_b) + \tag{2.194}$$

$$+ \delta g_{ab}(h^{ab}K - K^{ab}) - 2sv^a\nabla_a v^b \delta v_b\Big]. \tag{2.195}$$

The first two terms are total three-divergences, we can thus write them as just two-dimensional integrals over S_1 and S_0, that will eventually cancel the variation of the corner terms in the action (2.163). This we will see later. For the moment, let us consider the last two terms of (2.195). If we want the variation of the action to vanish, we have to demand the boundary conditions

$$h^{ac}h^{bd}\delta g_{cd}\big|_{\partial M} = 0, \tag{2.196a}$$

$$v^c \nabla_c v^a \delta v_a\big|_{\partial M} = 0. \tag{2.196b}$$

Instead of (2.196b) one often uses an even stronger condition:

$$\delta v_a \propto v_a. \tag{2.197}$$

This we will however not need in the following.

Briefly summarising what we have done so far, we have seen, once we solve the Einstein equations, the boundary conditions (2.196a) and (2.196b) imply the only remaining term in the variation (2.169) is an integral over the two-dimensional corners S_0 and S_1. We will now show that the action (2.163) is functionally diferentiable, that is the contributions from the corners cancel the variations of the rapidity angles Ξ appearing in the definition of the action (2.163).

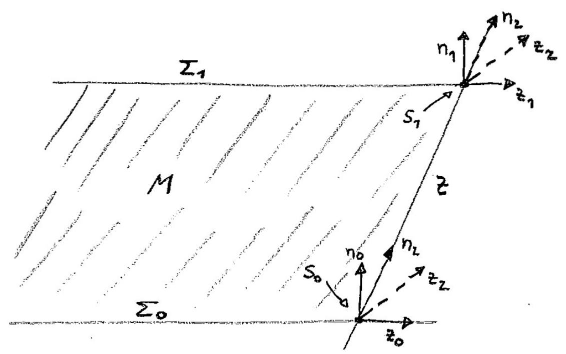

Figure 2.4: At each of the two two-dimensional corners, S_0 (resp. S_1) we introduce two pairs of normals, (n_0^a, z_0^a) (resp. (n_1^a, z_1^a)) and (n_2^a, z_2^a). All time-like vectors are future oriented, while z_0^a, z_1^a and z_2^a all point outside the bounding shell Z. The vector n_2^a lies tangential to Z, and z_2^a is its normal; n_0^a (resp. n_1^a), on the other hand, are future oriented normals of Σ_0 (resp. Σ_1), while z_0^a (z_1^a) is tangential to Σ_0 (Σ_1) but normal to S_0 (S_1).

Four different normal-vectors enter the boundary integral (2.195) at the two respective corners, and we introduce the following notation* to label them unambiguously. The vector n_0^a (n_1^a) is the future oriented time normal of Σ_0 (Σ_1), while z_0^a (z_1^a) is the spatial (outwardly pointing) normal of S_0 (S_1) orthogonal to n_0^a (n_1^a). On the cylindric shell n_2^a is a future oriented tangent vector orthogonal to $S_t = \partial\Sigma_t$, while z_2^a is its outwardly pointing normal. Figure 2.4 should further clarify the notation.

We introduce the rapidity angles

$$\mathrm{sh}\,\Xi\big|_{\Sigma_0} = -n^0_a z_2^a, \quad \mathrm{sh}\,\Xi\big|_{\Sigma_1} = -n^1_a z_2^a, \tag{2.198}$$

in accordance with our definition (2.168). We can therefore express at both S_0 and S_1 one dyade (n_i^a, n_i^a) in terms of the other, e.g. at S_1:

$$n_2^a = \mathrm{ch}\,\Xi\; n_1^a + \mathrm{sh}\,\Xi\; z_1^a, \tag{2.199a}$$
$$z_2^a = \mathrm{sh}\,\Xi\; n_1^a + \mathrm{ch}\,\Xi\; z_1^a. \tag{2.199b}$$

In the following we will, in fact, only look at the contributions from S_1, the corner term at S_0 is found from the expressions at S_0 by simply replacing the index "1" by "0". We indicate this procedure by writing $(0 \leftrightarrow 1)$.

Again using Gauß's theorem, this time for the three-dimensional manifolds Σ_0, Σ_1 and Z, we find, once boundary conditions (BC), and the Einstein equations (EOM) are satisfied, that the variation of the action (2.195) turns into the following integral over the corners:

$$\begin{aligned}\delta[S']\big|_{\text{EOM+BC}} &= \int_{\partial M} d^3v\varepsilon\Big[-\mathfrak{D}_a\big(h^{ab}\delta g_{bc}v^c\big) + 2\mathfrak{D}_a\big(h^{ab}\delta v_b\big)\Big] = \\ &= \int_{\Sigma_1} d^2v\Big[z^1_a n^1_b\delta g^{ab} - z^2_a n^2_b\delta g^{ab} - 2z_1^a\delta n^1_a + 2z_2^a\delta n^2_a\Big] - (0 \leftrightarrow 1).\end{aligned} \tag{2.200}$$

Notice that the integral of the total three-divergence over the closed region ∂M in the first line of this equation does *not* to vanish, this just happens so, simply because the normal v^a of ∂M is not continous across the corner.

*Indices $0, 1, 2$ refer to the three-dimensional surfaces Σ_0, Σ_1 and the cylindric shell Z respectively, we put them wherever there is enough space left, i.e $n^2_a = g_{ab}n_2^a$.

In (2.200) there still appears the variation of the metric δg_{ab}. We can remove this term, and replace it by variations of the normals, by first looking at the following identity:

$$z^1_a n^1_b \delta g^{ab} = -z^1_a n^1_b \delta[g^{ab}] = \delta z^1_a n^a_1 + \delta n^1_a z^a_1, \tag{2.201}$$

which follows from the orthogonality of n^a_1 and z^a_1. Equation (2.200) thus simplifies to:

$$\delta[S']\big|_{\text{EOM+BC}} = \int_{S_1} d^2v \left[\delta z^1_a n^a_1 - \delta n^1_a z^a_1 - \delta z^2_a n^a_2 + \delta n^2_a z^a_z\right] - (0 \leftrightarrow 1). \tag{2.202}$$

Where, again the covectors δn^i_a, δz^i_a have to be understood as our fundamental variations in the sense of equation (2.174) and (2.175). That is, e.g.:

$$\delta z^i_a = \delta[z^i_a]. \tag{2.203}$$

Let us now turn towards the other half of the problem, that is the variation of the corner terms as they originally stand in the action (2.163). The integrals over S_0 and S_1 consist of two elements. There appear the rapidity angles Ξ next to the two-dimensional volume element d^2v as defined in (2.166); but the boundary conditions (2.196a) require the pullback of the variation $\delta[d^2v]$ onto $S_{1,2}$ must vanish. Therefore, we only have to ask for the variation of Ξ, which we can easily compute to be:

$$\delta[\operatorname{sh}\Xi]\big|_{S_1} = \operatorname{ch}\Xi\,\delta[\Xi]\big|_{S_1} = -\delta n^1_a z^a_2 - \delta z^2_a n^a_1 + \delta g^{ab} n^1_a z^2_b. \tag{2.204}$$

Employing equation (2.199), we can write one dyade in terms of the other, such that:

$$\delta[\operatorname{sh}\Xi] = -\operatorname{sh}\Xi\,\delta n^1_a n^a_1 - \operatorname{ch}\Xi\,\delta n^1_a z^a_1 - \operatorname{ch}\Xi\,\delta z^2_a n^a_2 + \operatorname{sh}\Xi\,\delta z^2_a z^a_2 + \delta g^{ab} n^1_a z^2_b. \tag{2.205}$$

Since we eventually want to compare this with (2.202), we need to get again rid of the δg^{ab} term. The vectors (n^a_i, z^a_i) are normalised and orthogonal, this, together with equation (2.199) implies:

$$\delta g^{ab} n^1_a z^2_b = -\delta[g^{ab}] n^1_a \big(n^1_b \operatorname{sh}\Xi + z^1_b \operatorname{ch}\Xi\big) = 2\delta n^1_a n^a_1 \operatorname{sh}\Xi + \operatorname{ch}\Xi\big(\delta n^1_a z^a_1 + \delta z^1_a n^a_1\big). \tag{2.206}$$

The same argument, this time repeated for n^1_a written in terms of n^2_a and z^2_a yields:

$$\delta g^{ab} n^1_a z^2_b = -2\delta z^2_a z^a_2 \operatorname{sh}\Xi + \operatorname{ch}\Xi\big(\delta n^2_a z^a_2 + \delta z^2_a n^a_2\big). \tag{2.207}$$

Summing the last two equations, and inserting the resulting expression for $\delta g^{ab} n^1_a z^2_b$ into (2.205), we eventually find the variation of the rapidity parameter:

$$\delta[\Xi]\big|_{S_1} = \frac{1}{2}\big(\delta z^1_a n^a_1 - \delta n^1_a z^a_1 + \delta n^2_a z^a_2 - \delta z^2_a n^a_2\big). \tag{2.208}$$

We can thus anwer our initial problem (2.169), and say:

$$\delta[S']\big|_{\text{EOM+BC}} = 2\int_{S_1} d^2v\,\delta\Xi - 2\int_{S_0} d^2v\,\delta\Xi. \tag{2.209}$$

Therefore, the action (2.163), that is the Einstein–Hilbert action with its appropriate boundary and corner terms, is stationary around the equations of motion, if the boundary conditions (2.196) are respected, in other words:

$$\delta[S_{\text{EH}}]\big|_{\text{EOM+BC}} = 0. \tag{2.210}$$

Conclusion. Equation (2.210) is often summarised by stating the integral over the Ricci scalar is functionally differentiable, only if we add additional three- and two-dimensional boundary terms. These additional terms in the action consist not only of the usual Gibbons–Hawking–York term [66, 67] for the three-dimensional boundary. There are further terms needed, taht belong to the two dimensional corners, which form, so to speak, the boundary of the boundary itself [69–71]. This is a genuine feature of the Lorentzian theory. Given a the three-dimensional boundary of a compact four-dimensional region in space-time, its normal must somewhere cross the light-cone, and this is where a discontinuity generically appears*. The additional corner terms are then introduced in order to take care of this discontinuity.

The ADM Hamiltonian

The Hamiltonian formalism is about time-evolution. Looking at the action (2.163) we can, however, not identify any distinguished time coordinate. Indeed, general relativity does not have a preferred notion of time, which is one of the most prominent consequences of the principle of general covariance.

For that reason, we just pick one specific choice of time $t : M \to [0, 1]$. We have already introduced a convenient notation earlier, when studying the cylindrical space-time region $M = [0, 1] \times \Sigma$, sliced into $t = \text{const.}$ hypersurfaces $\Sigma_t = \{t\} \times \Sigma$. Figure 2.3 gives again an illustration of the geometry, that we are going to study in more detail now.

Let us first look at the future oriented normal n^a of the equal-time slices Σ_t. These are the "level"-surfaces of the time-function. Their normal must therefore be proportional to the gradient of t, and we thus know that there is a function, the lapse function $N : M \to \mathbb{R}$, such that

$$n_a = -N\nabla_a t, \quad N > 0. \tag{2.211}$$

Next, we are choosing a "clock", that follows** a future oriented vector field t^a. If this clock shall "tick" such that it actually measures t, we must have:

$$t^a\nabla_a t = 1. \tag{2.212}$$

In other words, we could think of t^a as a derivative associated to the t-coordinate:

$$t^a = \Big(\frac{\partial}{\partial t}\Big)^a. \tag{2.213}$$

We can decompose this vector-field into a component tangential to Σ_t, this we call the shift vector N^a, and an orthonormal part. Equations (2.211) and (2.212) imply, that the normal component must be proportional to the lapse function, and we can thus write:

$$t^a = Nn^a + N^a. \tag{2.214}$$

*In principle we could put corners wherever we want. This would however make even more corner terms necessary. We could also try the opposite, and smear out the boundary to smoothly cross the light cone. This is, however, only possible if we regularise the Gibbons–Hawking–York term at the moment where the normal becomes momentarily null [115, 129]

**This is not a clock in any strict physical sense, the vector-field t^a must not follow any physical trajectory, and t^a could in principle be space-like.

The action that we are considering is (2.163), let us just agree on one simplifying assumption: The hypersurfaces of constant time shall intersect the cylindrical shell Z orthonormally. The corner terms thus disappear and our action becomes

$$\text{if} \quad n_a z^a = 0: \quad S_{\mathrm{EH}}[g_{ab}, n_a, z_a] = \frac{c^3}{16\pi G}\Big[\int_M d^4v\, R - 2\int_{\Sigma_1} d^3v\, K + \\ + 2\int_{\Sigma_1} d^3v\, K + \int_Z d^3v\, Q\Big]. \tag{2.215}$$

Compared to (2.163) we have also slightly changed our notation to better distinguish the extrinsic curvature K_{ab} on the spatial slices Σ_t from the extrinsic curvature Q_{ab} on the outer shell Z. Otherwise our notation would become a little too cumbersome here. For the sake of clarity, let us also repeat the definitions of the extrinsic curvature tensors on both Σ_t and Z:

$$K_{ab} = h_a{}^c \nabla_c n_b, \quad Q_{ab} = q_a{}^c \nabla_c z_b, \tag{2.216}$$

where $h_{ab} = n_a n_b + g_{ab}$, and $q_{ab} = -z_a z_b + g_{ab}$ are the intrinsic three-dimensional metric tensors on Σ_t, and Z respectively. Here, just as in the last subsection, z^a is the outwardly pointing normal of the outer shell $Z = [0,1] \times S$, while $S_t = \{t\} \times S = \partial\Sigma_t$.

The next step is to actually decompose all configuration variables into parts intrinsically defined on the spatial slice, and additional extrinsic quantities coding the infinitesimal change of the intrinsic geometry when following the time-flow vector-field t^a. We start with the integrations measures (2.164) themselves. We find, with our conventions for the choice of orientation on M, Σ_t, $Z = [0,1] \times \Sigma$, and $S_t = \partial\Sigma_t$ as defined in the paragraph just below of equation (2.168) that

$$\int_M d^4v \ldots = +\int_0^1 \mathrm{d}t \int_{\Sigma_t} d^3v \ldots \tag{2.217a}$$

$$\int_Z d^3v \ldots = -\int_0^1 \mathrm{d}t \int_{S_t} d^2v \ldots \tag{2.217b}$$

Then we also have to decompose the integrands. First of all we are looking at the trace Q of the extrinsic curvature tensor of the outer shell Z. We can use the three dimensional covariant Levi-Civita derivative $\mathfrak{D}_a$ on the spatial hypersurfaces of equal time, defined just as in equation (2.191), to write:

$$Q^a{}_a = \nabla_a z^a = h_a{}^b \nabla_b z^a - n_a n^b \nabla_b z^a = \mathfrak{D}_a z^a + n^b \nabla_b n_a z^a \tag{2.218}$$

We are now left to split the four dimensional Rici scalar into its extrinsic and intrinsically spatial parts. This famously employs the Gauß–Codazzi relations linking the intrinsic three-dimensional curvature with the four dimensional curvature. Let $V^a \in T\Sigma_t$ be a vector-field tangential to a $t =$ const. hypersurface. We define the three-dimensional Riemann curvature tensor:

$$\mathfrak{D}_a \mathfrak{D}_b V^c - \mathfrak{D}_b \mathfrak{D}_a V^c = {}^{(3)}R^c{}_{dab} V^d. \tag{2.219}$$

A short-calculation relates the three-dimensional curvature to the four-dimensional Riemann tensor R_{abcd}. We eventually get:

$${}^{(3)}R_{cdab} = h_c{}^{c'} h_d{}^{d'} h_a{}^{a'} h_b{}^{b'} R_{c'd'a'b'} - K_{ca} K_{db} + K_{cb} K_{da} \tag{2.220}$$

Contracting the appropriate indices yields the three-dimensional Ricci scalar

$$^{(3)}R = R + 2n^a n^b R_{ab} - (K^a{}_a)^2 + K_{ab}K^{ab} \tag{2.221}$$

The $R_{ab}n^a n^b$ term is still a little disturbing, because there is the four-dimensional Ricci tensor, while we are rather looking for the properties of the three-dimensional geometry. We thus turn this term in a sum over two pieces, one being quadratic in the extrinsic curvature while the other is just a total divergence. In fact, using the defining property of the Riemann tensor

$$\nabla_a \nabla_b n^c - \nabla_b \nabla_a n^c = R^c{}_{dab} n^d, \tag{2.222}$$

we eventually get

$$R_{ab}n^a n^b = \nabla_a\big(n^b \nabla_n n^a - n^a \nabla_b n^b\big) + (K^a{}_a)^2 - K_{ab}K^{ab}. \tag{2.223}$$

Combining (2.221) and (2.223) allows us to write the four dimensional Ricci scalar as:

$$R = -(K)^2 + K_{ab}K^{ab} + {}^{(3)}R - 2\nabla_a\big(n^b \nabla_b n^a - n^a K\big) \tag{2.224}$$

Inserting this equation into our original action (2.215), while also using the 3+1 decomposition of the extrinsic curvature (2.218) on the outer shell, and taking care of both the volume elements (2.217) and the orientation the four- and three-dimensional manifolds, we eventually get:

$$\begin{aligned} S_{\mathrm{EH}} =& \frac{c^3}{16\pi G}\Big[\int_0^1 \mathrm{d}t \int_{\Sigma_t} d^3v\, N\Big(K_{ab}K^{ab} - K^2 + {}^{(3)}R - 2\nabla_a\big(n^b\nabla_n n^a - n^a K\big)\Big) \\ & - 2\int_{\Sigma_1} d^3v\, K + 2\int_{\Sigma_0} d^3v\, K \\ & - 2\int_0^1 \mathrm{d}t \int_{\Sigma_t} d^2v\, N\big(\mathfrak{D}_a z^a - n^b \nabla_b n_a z^a\big)\Big] \end{aligned} \tag{2.225}$$

Almost all boundary terms cancel through the-four dimensional Gauß law (2.186), which leads us to:

$$\begin{aligned} S_{\mathrm{EH}} = \frac{c^3}{16\pi G}\Big[\int_0^1 \mathrm{d}t \int_{\Sigma_t} d^3v\, N\big(K_{ab}K^{ab} - K^2 + {}^{(3)}R\big) \\ - 2\int_0^1 \mathrm{d}t \int_{S_t} d^2v\, N \mathfrak{D}_a z^a\Big]. \end{aligned} \tag{2.226}$$

This is the ADM action, which is the original action (2.215) written in terms of the intrinsic metric $h_{ab} = n_a n_b + g_{ab}$ and the lapse function N and the shift vector N^a (2.214).

Next, we are introducing the canonical momentum of the spatial metric. We can find it from the differential of the Lagrangian with respect to the "velocity" of the configuration variable, that is the time derivative $\dot{h}_{ab}$ of the spatial metric. There is a geometrically clean way of defining this velocity in a coordiante invariant way. This definition takes the four-dimensional Lie-derivative $\mathcal{L}_t$ of the spatial metric along the time-flow vector-field t^a and projects it back onto the spatial hypersurface. Following this prescription we thus set

$$\dot{h}_{ab} := h_a{}^{a'} h_b{}^{b'} \mathcal{L}_t h_{a'b'}. \tag{2.227}$$

Next, we will link this velocity to the quantities we already know. Inserting the decomposition of the time-flow vector-field into lapse and shift, that is using (2.214), we find the extrinsic curvature K_{ab} measures—up to an infinitesimal diffeomorphism generated by the shift vector— right that velocity:

$$K_{ab} = \frac{1}{2N}\big(\dot{h}_{ab} - L_N h_{ab}\big). \tag{2.228}$$

The last term on the right hand side is the intrinsic three-dimensional Lie-derivative $L_N h_{ab}$ of the spatial metric—the infinitesimal diffeomorphism just mentioned. With $\mathfrak{D}_a$ again denoting the intrinsic three-dimensional covariant derivative this term equates to

$$L_N h_{ab} = \mathfrak{D}_a N_b + \mathfrak{D}_b N_a. \tag{2.229}$$

At this step it is useful to introduce the Wheeler–DeWitt metric (a tensor density of weight one), and its inverse of weight minus one:

$$\tilde{G}^{abcd} = \frac{d^3v}{2}\big(h^{ac}h^{bd} + h^{ad}h^{bc} - 2h^{ab}h^{cd}\big), \tag{2.230a}$$

$$\underset{\sim}{G}_{abcd} = \frac{(d^3v)^{-1}}{2}\big(h_{ac}h_{bd} + h_{ad}h_{bc} - h_{ab}h_{cd}\big). \tag{2.230b}$$

The powers $(d^3v)^{\pm}$ should just remind us, in a coordinate invariant fashion, of these quantities being tensor densities. Variation of the action with repsect to $\dot{h}_{ab}$ reveals the ADM momentum

$$\tilde{\pi}^{ab} = \tilde{G}^{abcd}K_{cd} = d^3v\big(K^{ab} - h^{ab}K\big) \tag{2.231}$$

together with the symplectic structure. The only non-vanishing Poisson brackets are

$$\big\{\tilde{\pi}^{ab}(p), h_{cd}(q)\big\} = \frac{16\pi G}{c^3} h^{(a}{}_c h^{b)}{}_d \tilde{\delta}^{(3)}(p,q), \tag{2.232}$$

where $\tilde{\delta}^{(3)}(p,q)$ is the three dimensional Dirac-density on the spatial slice. After performing a partial integration to replace $\mathfrak{D}_a N_b \tilde{\pi}^{ab}$ by $N_a \mathfrak{D}_b \tilde{\pi}^{ab}$ the action assumes the Hamiltonian form

$$\begin{aligned} S_{\mathrm{EH}} = \frac{c^3}{16\pi G}\Big[\int_0^1 \mathrm{d}t \int_{\Sigma_t} \Big(\dot{h}_{ab}\tilde{\pi}^{ab} + 2N_a\mathfrak{D}_b\tilde{\pi}^{ab} - N\big(\underset{\sim}{G}_{abcd}\tilde{\pi}^{ab}\tilde{\pi}^{cd} - d^3v\,{}^{(3)}R\big)\Big) \\ - 2\int_0^1 \mathrm{d}t \int_{S_t} d^2v\big(N\mathfrak{D}_a z^a + z_a N_b(K^{ab} - h^{ab}K)\big)\Big] \end{aligned} \tag{2.233}$$

This action lacks any derivatives of lapse and shift. The variation of these *Lagrange multipliers* thus leads us to the following constraint equations:

$$\mathcal{H} = \frac{c^3}{16\pi G}\int_{\Sigma_t} M\big(\underset{\sim}{G}_{abcd}\tilde{\pi}^{ab}\tilde{\pi}^{cd} - d^3v\,{}^{(3)}R\big) \overset{!}{=} 0, \qquad H[N] = \int_{\Sigma_t} N\mathcal{H} \tag{2.234a}$$

$$\mathcal{H}_a := -\frac{c^3}{8\pi G} h_{ca}\mathfrak{D}_b\tilde{\pi}^{cb} \overset{!}{=} 0, \qquad H_a[N^a] = \int_{\Sigma_t} N^a\mathcal{H}_a \tag{2.234b}$$

We call $\mathcal{H}$ and $\mathcal{H}_a$ the Hamiltonian and diffeomorphism (vector constraint) respectively. Remember also our boundary conditions (2.196a). They require any variation of lapse and shift must vanish on the cylindrical shell Z. That is

$$\delta N\big|_Z = 0 = \delta N^a\big|_Z. \tag{2.235}$$

Therefore the integral over the cylindrical shell appearing in the action (2.215) does neither contribute to the Hamiltonian nor the vector constraint.

Looking at the evolution equations for h_{ab} and K_{ab}, we find that they are generated by the ADM Hamiltonian:

$$\dot{h}_{ab} = \{H^{\mathrm{ADM}}, h_{ab}\} \tag{2.236}$$

This Hamiltonian is a sum of the constraints (2.234) and the ADM energy:

$$H^{\mathrm{ADM}} = H[N] + H_a[N^a] + E^{\mathrm{ADM}} \tag{2.237}$$

The first two terms vanish if the Einstein equations are satisfied. In this case only the last term, thus measuring the total energy of the gravitating system, can survive. We can easily read this energy off the action (2.5), it is the integral

$$E^{\mathrm{ADM}} = -P_a^{\mathrm{ADM}}[t^a] = \frac{c^3}{8\pi G}\int_{S_t} d^2v\big(N\mathfrak{D}_a z^a + z_a N_b(K^{ab} - h^{ab}K)\big), \tag{2.238}$$

over the two-dimensional boundary $S_t = \partial\Sigma_t$ of a $t=$ const. hypersurface. Here, we have also implicitly introduced the ADM four-momentum P_a^{ADM} contracted with the time-flow vectorfield t^a. We will see, that the energy as it stands in (2.238), does actually diverge as we approach spatial infinity. This will make a regularisation necessary, which will eventually give us the ADM energy in its most recognisable form. Once we have this expression, we will then repeat Witten's proof of why this energy is always positive.

Asymptotic form of the ADM energy. It is important to know how equation (2.238) behaves for an asymptotically flat geometry at spatial infinity. This will lead us to the most common form of the ADM energy.

A space-time is said to be asymptotically flat, if there is an inertial coordinate system $\{x^\mu\}_{\mu=0,1,2,3}$ at spatial infinity, such that the metric and its derivatives scale as

$$g_{\mu\nu} = \eta_{\mu\nu} + f_{\mu\nu} + \mathcal{O}(r^{-2}), \qquad \begin{cases} f_{\mu\nu} = \mathcal{O}(r^{-2}) \\ \partial_\alpha f_{\mu\nu} = \mathcal{O}(r^{-1}) \end{cases} \tag{2.239}$$

where, $r = \sqrt{\delta_{ij}x^i x^j}$ is the radial coordinate. Calculations simplify when introducing an orthonormal frame, that is a cotetrad $e^\mu{}_a$. Looking at (2.239) we find the cotetrad scales asymptotically as:

$$\eta^\mu{}_a = \mathrm{d}x^\mu_a + \tfrac{1}{2}f^\mu{}_a + \mathcal{O}(r^{-2}) \tag{2.240}$$

Here, and in the following indices are moved by the flat background structures. These are the Minkowski metric $\eta_{\mu\nu}$, the flat cotetrad $\mathrm{d}x^\mu_a$ and its inverse ∂^a_μ. The spin rotation coefficients of the perturbed cotetrad $\eta^\mu{}_a$ take the asymptotic form

$$\omega^\mu{}_{\nu\alpha} = \frac{1}{2}\big(\partial_\nu f_\alpha{}^\mu - \partial^\mu f_{\alpha\nu}\big) + \mathcal{O}(r^{-3}). \tag{2.241}$$

The extrinsic curvature is one of these coefficients:

$$K_{ij} = \omega_{i0j} = \frac{1}{2}\big(\partial_0 f_{ij} - \partial_i f_{j0}\big) + \mathcal{O}(r^{-3}) \tag{2.242}$$

Inserting this expansion into the expression of the ADM momentum (2.238) we eventually find

$$P_i^{\mathrm{ADM}} = -\frac{c^3}{16\pi G}\lim_{r\to\infty}\int_{S_r^2} d^2x_k\big(\partial_0 f^k{}_i - \partial^k f_{i0} - \delta_i^k\partial_0 f^l{}_l + \delta_i^k\partial_l f^l{}_0\big), \tag{2.243}$$

where S_r^2 is the $r = \sqrt{x_i x^i} =$ const. sphere bounding the spatial hypersurface, and $d^2x_k = \mathrm{d}\cos\vartheta\,\mathrm{d}\varphi\, r^2\partial_k r$ is the integration measure of the sphere in the flat background geometry.

For the ADM energy itself the situation is a bit more tricky. Our definition (2.238) diverges for $r\to\infty$. The divergences stems from the zeroth order, that is the contribution from flat space-time at infinity. The calculation is immediate, the zeroth order is:

$$-\,{}^{(0)}P_0^{\mathrm{ADM}} = \lim_{r\to\infty}\int_{S^2} d^2\Omega r^2\partial_i\Big(\frac{x^i}{\sqrt{x^j x_j}}\Big) = \infty \tag{2.244}$$

This diverging term, does, however, not contain any of the gravitational degrees of freedom at infinity, since we easily see, there is no f_{ab} appearing here. If we substract it from the Hamiltonian , dynamics remains unchanged and we we do not loose anything. This means that we are only interested in the variation δE^{ADM} of the ADM energy around the flat background at infinity. We now calculate this variation. As a first step we write the energy in terms of tetrads:

$$-\,P_0^{\mathrm{ADM}} = \frac{c^3}{8\pi G}\int_{S_r} d^2v\mathfrak{D}_a z^a = \frac{c^3}{8\pi G}\int_{S_r}\epsilon_{\alpha\beta\mu\nu}n^\alpha z^\beta\nabla z^\mu\wedge e^\nu \tag{2.245}$$

Here n^α, and z^α are the two internal normals of the two-dimensional $r=$ const. sphere, e.g. $n^\alpha = e^\alpha{}_a n^a$, while ∇ is the exterior covariant derivative. A short calculation reveals all variations of these normals vanish around the flat background. If we also assume the variation should preserve the intrinsic two-dimensional metric of the sphere, and take these as our boundary conditions, then we can only take the variation with respect to the connection

$$-\,\delta P_0^{\mathrm{ADM}}\big|_{\mathrm{BC}} = \frac{c^3}{8\pi G}\int_{S_r}\epsilon_{\alpha\beta\mu\nu}n^\alpha z^\beta z^\nu\delta\omega^\mu{}_\nu\wedge\mathrm{d}x^\nu \tag{2.246}$$

This variation $\delta\omega^\mu{}_{\nu\alpha}$ is just the difference between the spin rotation coefficients corresponding to the two cotetrads $\mathrm{d}x^\mu_\alpha$ and $\eta^\mu{}_a$ at infinity. Since, the inertial coordinate system is, ipso facto non-rotating (we could also say instead $\mathrm{d}\wedge\mathrm{d}x^\mu_\alpha = 0$) this difference must equal to equation (2.241). In other words

$$\delta\omega_{\mu\nu\alpha} = \omega_{\mu\nu\alpha} - \underbrace{0}_{\text{contribution from }\mathrm{d}x^\mu} \tag{2.247}$$

A little algebra gives

$$-\,\delta P_0^{\mathrm{ADM}}\big|_{\mathrm{BC}} = \frac{c^3}{8\pi G}\int_{S_r} d^2x_i\omega^{ki}{}_k = \frac{c^3}{16\pi G}\int_{S_r} d^2x_i\big(\partial^i f^k{}_k - \partial_k f^{ki}\big), \tag{2.248}$$

where $d^2x_k = \mathrm{d}\cos\vartheta\,\mathrm{d}\varphi\, r^2\partial_k r$ again denotes the two-dimensional volume element of the sphere with respect to the flat background geometry. If the shift vector vanishes at infinity, we thus get the familiar expression of the regularised ADM energy

$$E^{\mathrm{ADM}}\big|_{\mathrm{reg}} = -P_0^{\mathrm{ADM}}\big|_{\mathrm{reg}} = \frac{c^3}{16\pi G}\lim_{r\to\infty}\int_{S_r} d^2x_i\big(\partial^i f^k{}_k - \partial_k f^{ki}\big) \tag{2.249}$$

In the next subsection, we will sketch Witten's proof of why this energy is always positive. Before we go into this, let us first comment on the first attempts to quantise the theory starting from the action (2.163).

Wheeler–DeWitt theory. The Hamiltonian theory that we have just reviewed, led to the first attempts towards a background independent quantum theory of the gravitational field. The strategy was fairly simple, the momentum variables got replaced by functional derivatives, acting on complex-valued functionals $\Psi[h_{ab}]$ of the spatial metric:

$$\tilde{\pi}^{ab}(p)\Psi[h_{ab}] = -\frac{16\mathrm{i}\pi\hbar G}{c^3}\frac{\delta}{\delta h_{ab}(p)}\Psi[h_{ab}]. \tag{2.250}$$

The wave-functional $\Psi[h_{ab}]$ was to describe the quantum state of the gravitational field. For Ψ to be a physical state additional conditions need to be statisfied. The classical constraints (2.234) must be turned into operators annihilating the physical states Ψ_{phys}, schematically:

$$\hat{\mathcal{H}}\Psi_{\mathrm{phys}} = 0, \quad \hat{\mathcal{H}}_a\Psi_{\mathrm{phys}} = 0 \tag{2.251}$$

The first of these constraints is now famously called the Wheeler–DeWitt equation. It is however badly ill-defined. Beside the notorious ordering ambiguities in the kinetic term, and the troubles when multiplying operator-valued distributions at a point, what makes things really difficult are all the non-polynomial terms appearing in the Hamiltonian constraint. Following this approach, one would have to deal with square roots and the inverse of totally unbound operators. Furthermore no Hilbert space, where these operators could at least formally be defined, could have ever been constructed. One would have needed a measure in the infinite-dimensional "superspace" of all three dimensional geometries. For these technical troubles, actual calculations were mostly done in symmetry-reduced settings, where only some gravitational degrees of freedom (e.g. the scale factor of the universe) are quantised.

In the next two chapters, I will try to convince the reader, that loop quantum gravity can go much beyond the Wheeler–DeWitt theory, and may hopefully reveal some truth about what goes on at the Planck scale.

Witten's proof of the positivity of the ADM energy

To prove the positivity of the ADM mass Witten starts from the three-dimensional Dirac equation on an asymptotically flat Cauchy hypersurface Σ. This equation states

$$\not{D}\chi^A := \sigma^A{}_{Ba}\nabla^a\chi^B = 0. \tag{2.252}$$

The underlying hypersurface belongs to a four-dimensional space-time manifold that solves the Einstein equations with some rather unspecified matter content. The only restriction on the energy-momentum tensor is the dominant energy condition. This is the requirement that for any observer following a time-like vector field n^a, the momentum vector $-T_{ab}n^b$ should always point towards the future, (i.e. any observer should never see matter moving faster than light).

Preparations. Let us stop here for a moment in order to explain the various elements of this equation. The spinor-field χ^A takes values in $\mathbb{C}^2$, $A, B, C, \dots$ are left-handed indices, for their right-handed counterparts we put a macron and write $\bar{A}, \bar{B}, \bar{C}, \dots$. The spinor indices are moved by the epsilon tensor according to the definitions collected in appendices A.2 and A.3. The four-dimensional Levi-Civita derivative is ∇_a (both metric compatible, and torsionless), while $\sigma^A{}_{Ba}$ is a matrix-valued one-form in $T^*\Sigma$, generalising the Euclidean Pauli matrices to the spatial slice under consideration. We can construct these matrices most easily from the four-dimensional soldering forms (the Infeld–van der Waerden symbols)

$$\sigma^{A\bar{A}}{}_a = \eta^\beta{}_a \sigma^{A\bar{A}}{}_\beta, \tag{2.253}$$

that enter e.g. the definition of the selfdual part of the Plebanski two-form $\Sigma_{\alpha\beta} = \eta_\alpha \wedge \eta_\beta$. The seldual part of the Plebanski two-form is a basis in $\mathfrak{sl}(2,\mathbb{C})$, we can find it from the wedge product of the soldering forms:

$$\Sigma^A{}_{Bab} = -\frac{1}{2}\sigma^{A\bar{C}}{}_{[a}\bar{\sigma}_{\bar{C}Bb]}. \tag{2.254}$$

Here, the anti-symmetrisation should act on space-time indices only. The Pauli matrices on the spatial slice Σ are now nothing but the "electric" (i.e. the time-space) component of the selfdual generators (2.254). We set

$$\sigma^A{}_{Ba} := -2\Sigma^A{}_{Bab} n^b, \tag{2.255}$$

where n^a is the future oriented time-normal orthogonal to the spatial three surface Σ. These matrices have a number of important properties, they are traceless, purely spatial, i.e. $\sigma^A{}_{Ba} n^a = 0$, and obey the Pauli identity

$$\sigma^A{}_{Ca}\sigma^C{}_{Bb} = h_{ab}\delta^A_B + \mathrm{i}\varepsilon_{ab}{}^c\sigma^A{}_{Bc}, \tag{2.256}$$

where $h_{ab} = n_a n_b + g_{ab}$ again denotes the induced metric on the spatial slice, while $\varepsilon_{abc} = \varepsilon_{dabc} n^d$ is the three dimensional Levi-Civita tensor. These properties can most easily be proven by considering an orthonormal frame and using the explicit matrix representation (A.2) of the Infeld–van der Waerden forms. If equation (2.256) holds in one frame, it must be true in any frame, simply because the soldering forms (2.253) transform covariantly under local Lorentz transformations. We can then also show:

$$\Sigma^A{}_{Bcd} h^c{}_a h^d{}_b = \frac{1}{2\mathrm{i}}\varepsilon_a{}^c{}_b \sigma^A{}_{Bc}, \tag{2.257}$$

which follows from the selfduality of the generators (2.254) and our definition of the Pauli matrices (2.255). The time-normal picks a Hermitian metric, with respect to which we can define the Hermitian conjugation:

$$\delta_{A\bar{A}} := \sigma_{A\bar{A}}{}^a n_a, \quad \chi^\dagger_A := \delta_{A\bar{A}}\bar{\chi}^A \tag{2.258}$$

Then, we also have

$$\sigma^A{}_{Ba} = h_a{}^b \sigma^{A\bar{B}}{}_b \delta_{B\bar{B}} = -h_a{}^b \bar{\sigma}_{\bar{A}Bb}\delta^{A\bar{A}}, \tag{2.259}$$

which implies the Hermiticity of the Pauli matrices with respect to the metric just defined:

$$\sigma^A{}_{Ba} = \delta^{A\bar{A}}\bar{\sigma}^{\bar{B}}{}_{\bar{A}a}\delta_{B\bar{B}} = (\sigma^\dagger)^A{}_{Ba} \tag{2.260}$$

If the connection is metric compatible and torsionfree, the soldering-form is covariantly constant:

$$\nabla_b \sigma^{A\bar{A}}{}_a = 0, \tag{2.261}$$

where the covariant derivative must "act" on all three indices*. For the Pauli matrices the situation is different. The time normal appears linearly in the defining equation (2.255), resulting in the covaraint derivative to be:

$$\nabla_a \sigma^A{}_{Bb} = -2\Sigma^A{}_{Bbc} \nabla_a n^c. \tag{2.262}$$

Sketch of Witten's proof. With these preparations we are ready for the actual proof of the positivity of the ADM mass. If χ^A is a solution of (2.252), also the squared Dirac operator $\not{D}$ must annihilate the spinor-field χ^A. In other words:

$$\begin{aligned} 0 = \not{D}^2 \chi^A &= \sigma^A{}_D{}^a \nabla_a \big(\sigma^D{}_B{}^b \nabla_b \chi^B\big) = \sigma^A{}_D{}^a \sigma^D{}_B{}^b \nabla_a \nabla_b \chi^B - 2\sigma^A{}_D{}^a \Sigma^D{}_B{}^{bc} K_{ac} \nabla_b \chi^B \\ &= h^{ab} \nabla_a \nabla_b \chi^A + \mathrm{i}\varepsilon^{abc} \sigma^A{}_{Bc} \nabla_a \nabla_b \chi^B + \sigma^A{}_{Da} \sigma^D{}_{Bb} K^{ab} n^c \nabla_c \chi^B + \\ &\quad - 2\sigma^A{}_D{}^a \Sigma^D{}_{Bb'c'} h^{bb'} h^{cc'} K_{ac} \nabla_b \chi^B. \end{aligned} \tag{2.263}$$

Here, we used the matrix identities (2.255, 2.257) and also split the covariant derivative $\nabla_b \chi^B$ into its spatial and temporal components. The extrinsic curvature tensor $K_{ab} := h_a{}^c \nabla_c n_b$ appeared, since the Pauli matrices are purely spatial. This allowed us to write $\sigma^A{}_B{}^a \nabla_a n_b = \sigma^A{}_B{}^a h_a{}^c \nabla_c n_b = \sigma^A{}_{Ba} K^a{}_b$.

We can further simplify equation (2.263). With the appearance of the antisymmetric ε^{abc} tensor we can introduce the commutator of two covariant derivatives. This yields the $\mathfrak{sl}(2,\mathbb{C})$-valued curvature two-form $R^A{}_{Bab}$:

$$\nabla_{[a} \nabla_{b]} \chi^A = \frac{1}{2} R^A{}_{Bab} \chi^B. \tag{2.264}$$

This is the field strength of the selfdual connection:

$$R^A{}_B = \mathrm{d}\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B, \tag{2.265}$$

We can build the $\mathfrak{sl}(2,\mathbb{C})$-valued selfdual connection in several ways, for instance as follows: If $\omega^\mu{}_{\nu a}$ are the spin rotation coefficients, i.e. the components of the $\mathfrak{so}(1,3)$ Lorentz connection, themselves determined by the equation for the vanishing of torsion

$$\nabla \eta^\mu = \mathrm{d}\eta^\mu + \omega^\mu{}_\nu \wedge \eta^\nu = 0, \tag{2.266}$$

then, the $\mathfrak{sl}(2,\mathbb{C})$-connection is nothing but the selfdual part of the spin connection, i.e.:

$$\omega^A{}_B = \frac{1}{2} \Sigma^A{}_{B\mu\nu} \omega^{\mu\nu}. \tag{2.267}$$

This relation survives one level higher, when looking at the respective curvature tensors. Indeed, the field-strength of the selfdual connection equals the selfdual part of the Riemann curvature tensor, in other words

$$R^A{}_{Bab} \chi^B = \frac{1}{2} \Sigma^A{}_{Bcd} R^{cd}{}_{ab} \chi^B. \tag{2.268}$$

*In a coordinate system $\{x^\mu\}$ the covariant derivative acts as $\nabla_\mu \sigma^{A\bar{A}}{}_\alpha = \partial_\mu \sigma^{A\bar{A}}{}_\alpha - \Gamma^\beta{}_{\alpha\mu} \sigma^{A\bar{A}}{}_\beta + \omega^A{}_{C\mu} \sigma^{C\bar{A}}{}_\alpha + \bar{\omega}^{\bar{A}}{}_{\bar{C}\mu} \sigma^{A\bar{C}}{}_\alpha$, where $\Gamma^\alpha{}_{\beta\mu}$ are the Christoffel symbols and $\omega^A{}_{B\mu}$ ($\bar{\omega}^{\bar{A}}{}_{\bar{B}\mu}$) are the selfdual (anti-sefldual) part of the spin rotation coefficients.

Thus inserting the curvature tensor, we can now algebraically simplify the term involving the commutator of two covaraint derivatives in (2.263) according to:

$$\begin{aligned}
\varepsilon^{abc}\sigma^{A}{}_{Dc}\Sigma^{D}{}_{Bef}R^{ef}{}_{ab} = -2\varepsilon^{abc}\sigma^{A}{}_{Dc}\Sigma^{D}{}_{Bef}n^{f}n_{d}R^{ed}{}_{ab}+ \\
+\varepsilon^{abc}\sigma^{A}{}_{Dc}\Sigma^{D}{}_{Bef}h^{e}{}_{e'}h^{f}{}_{f'}R^{e'f'}{}_{ab} = \\
= \varepsilon^{abc}\sigma^{A}{}_{Dc}\sigma^{D}{}_{Be}n_{d}R^{ed}{}_{ab} + \frac{1}{2\mathrm{i}}\varepsilon^{abc}\varepsilon_{edf}\sigma^{A}{}_{Dc}\sigma^{D}{}_{B}{}^{d}R^{ef}{}_{ab} = \\
= \mathrm{i}\sigma^{A}{}_{Bf}\varepsilon_{ce}{}^{f}\varepsilon^{cab}n_{d}R^{ed}{}_{ab} + \frac{1}{2\mathrm{i}}\varepsilon^{abc}\varepsilon_{ecf}\delta^{A}_{B}R^{ef}{}_{ab} + \frac{1}{2}\varepsilon^{abc}\varepsilon_{edf}\varepsilon_{c}{}^{dg}\sigma^{A}{}_{Bg}R^{ef}{}_{ab} = \\
= 2\mathrm{i}\sigma^{A}{}_{B}{}^{a}h_{b}{}^{c}n_{d}R^{bd}{}_{ca} + \mathrm{i}\delta^{A}_{B}h_{a}{}^{b}h_{c}{}^{d}R^{ac}{}_{bd} = \\
= 2\mathrm{i}\sigma^{A}{}_{B}{}^{a}n^{b}G_{ab} + 2\mathrm{i}\delta^{A}_{B}n^{a}n^{b}G_{ab},
\end{aligned} \tag{2.269}$$

where we have used Bianchi identity $R_{a[bcd]} = 0$, together with the matrix identities (2.255, 2.256, 2.257), and the determinant formula $\varepsilon^{abc}\varepsilon_{def} = 3!h^{[a}_{d}h^{b}_{e}h^{c]}_{f}$. Notice the crucial appearance of the Einstein-tensor $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$ in the last line. This allows us to introduce the energy-momentum content of the underlying space-time geometry. Employing the Einstein equations

$$G_{ab} = \frac{8\pi G}{c^3}T_{ab}, \tag{2.270}$$

and after having defined both energy density ϵ, and momentum-flux p_a of the matter content according to

$$\epsilon = T_{ab}n^{a}n^{b}, \quad p_a = -h_{a}{}^{b}T_{bc}n^{c}, \tag{2.271}$$

we thus get:

$$\varepsilon^{abc}\sigma^{A}{}_{Dc}\Sigma^{D}{}_{Bef}R^{ef}{}_{ab} = \frac{16\pi\mathrm{i}G}{c^3}\big(-\sigma^{A}{}_{Ba}p^{a} + \delta^{A}_{B}\epsilon\big) \tag{2.272}$$

Next we will simplify the two remaining parts of (2.263). Using the symmetry $K_{ab} = K_{ba}$ of the extrinsic curvature tensor, together with $\not{\mathfrak{D}}\chi^{A} = 0$ and the algebraic properties of the Pauli matrices, we eventually get:

$$\sigma^{A}{}_{C}{}^{a}\sigma^{C}{}_{B}{}^{b}K_{ab}n^{c}\nabla_{c}\chi^{B} = K^{a}{}_{a}n^{b}\nabla_{b}\chi^{A} \tag{2.273a}$$

$$-2\sigma^{A}{}_{C}{}^{a}\Sigma^{C}{}_{B}{}^{b'c'}h^{b}{}_{b'}h^{c}{}_{c'}K_{ac}\nabla_{b}\chi^{B} = -\sigma^{A}{}_{Ba}K^{ab}\nabla_{b}\chi^{B}. \tag{2.273b}$$

Reinserting (2.272) and (2.273) into (2.263) we have:

$$\begin{aligned}
0 = \not{\mathfrak{D}}^{2}\chi^{A} = h^{ab}\nabla_{a}\nabla_{b}\chi^{A} - \frac{4\pi G}{c^3}\big(-\sigma^{A}{}_{Ba}p^{a} + \delta^{A}_{B}\varepsilon\big)\chi^{B}+ \\
+K^{a}{}_{a}n^{b}\nabla_{b}\chi^{A} - \sigma^{A}{}_{Ba}K^{ab}\nabla_{b}\chi^{B}.
\end{aligned} \tag{2.274}$$

In the next step, Witten integrates this equation against the Hermitian conjugate $\chi^{\dagger}_{A}$ over the spatial slice. The resulting expression turns into an integral over the two-dimensional boundary $S = \partial\Sigma$. This boundary integral stems from the following three-divergence:

$$\begin{aligned}
\mathfrak{D}_{a}\big(h^{ab}\delta_{A\bar{A}}\bar{\chi}^{\bar{A}}\nabla_{b}\chi^{A}\big) = \delta_{A\bar{A}}h^{ab}\nabla_{b}\bar{\chi}^{\bar{A}}\nabla_{a}\chi^{A} + K^{a}{}_{a} \qquad n^{b}\nabla_{b}\chi^{A}+ \\
+\sigma_{A\bar{A}}{}^{a}K^{ab}\bar{\chi}^{\bar{A}}\nabla_{b}\chi^{A},
\end{aligned} \tag{2.275}$$

where we have used the intrinsic three-dimensional covariant derivative $\mathfrak{D}_a$, as defined in (2.191). With Gauß's theorem, the integral of (2.273) contracted with $\chi^\dagger_A$ thus gives:

$$0 = \int_\Sigma d^3v \chi^\dagger_A \not{\mathfrak{D}}^2 \chi^A = \int_{\partial\Sigma} d^2v_a h^{ab} \chi^\dagger_A \nabla_b \chi^A + \\ - \int_\Sigma d^3v \delta_{A\bar{B}} h^{ab} \nabla_b \bar{\chi}^{\bar{B}} \nabla_a \chi^A - \frac{4\pi G}{c^3} \int_\Sigma d^3v \chi^\dagger_A \big(\epsilon \delta^A_B - p^a \sigma^A{}_{Ba}\big) \chi^B \tag{2.276}$$

Here d^3v is the metricial volume element on Σ, while $d^2v_a = d^2vz_a$ is the induced integration measure on $\partial\Sigma$, with $z_a \in T^*\Sigma$ being the outwardly pointing normal of $\partial\Sigma$. See equations (2.164), and figure 2.3 for further details.

Equation (2.276) is the key to Witten's proof of the positivity of the ADM mass. If the dominant energy condition [8] holds, the last term of (2.276) is always negative. With the hypersurface Σ being spacelike, h_{ab} and $\delta_{A\bar{A}}$ are positive definite, and therefore also the second term has a definite sign. We thus have:

$$\int_{\partial\Sigma} d^2v_a h^{ab} \chi^\dagger_A \nabla_b \chi^A \geq 0. \tag{2.277}$$

It remains to show, that the left hand side equates to the ADM energy. If compared to the original proof, we present here a simplified, less rigorous argument.

We assumed that the metric is asymptotically flat. This allows us to introduce inertial Minkowski coordinates x^μ, $\mu = 0,1,2,3$, around spatial infinity, that be asymptotically aligned to the spatial hypersuface Σ, that is Σ should approach an $x^0 = \text{const.}$ hypersurface. Moreover, its boundary $\partial\Sigma$ be a two a dimensional sphere S^2_r of constant $r = \sqrt{\delta_{ij}x^ix^j}$ reaching spatial infinity for $r \to \infty$. We will now asymptotically solve (2.252) in order to evaluate (2.277) for $r \to \infty$. We do this by common methods of ordinary perturbation theory, without, however, rigorously proving the mathematical consistency of the whole procedure. We assume, the following asymptotic behaviour of our elementary fields

$$\chi^A = {}^{(0)}\chi^A + \varepsilon\, {}^{(1)}\chi^A + \mathcal{O}(\varepsilon^2) \tag{2.278a}$$

$$\eta^\alpha{}_a = {}^{(0)}\eta^\alpha{}_a + \varepsilon{}^{(1)}\eta^\alpha{}_a + \mathcal{O}(\varepsilon^2) \tag{2.278b}$$

The parameter $0 < \varepsilon \ll 1$ in this perturbative expansion is the variable to solve (2.252) order by order, we could think of this "bookkeeping device" as the ratio $\varepsilon = M/L$ of the typical length scales of the problem, where M is the mass (in units of length) of the gravitating source and L measures the spatial distance from the source. For the tetrad, the zeroth order equals the differentials of the inertial coordinate system at spatial infinity. Looking back at (2.240), we can thus introduce the abbreviations

$${}^{(0)}\eta^\alpha{}_a = \mathrm{d}x^\alpha_a, \quad \varepsilon{}^{(1)}\eta^\alpha{}_a = f^\alpha{}_a \tag{2.279}$$

We now also need to know how the derivatives of our elementary variables scale along the radial coordinate. This is done in great detail in Witten's original work. Here we take the asymptotic behaviour of the fields as an additional input. In agreement with the original paper we set:

$$\partial_\alpha {}^{(i)}\chi^A = \mathcal{O}(r^{-1}), \quad \partial_\alpha {}^{(i)}\eta^\mu{}_\nu = \mathcal{O}(r^{-1}) \tag{2.280}$$

Further clarifying notation, ∂_a is the covariant derivative with respect to the flat background metric $\eta_{ab} = \eta_{\alpha\beta}\mathrm{d}x^\alpha_a \mathrm{d}x^\beta_b$ at spatial infinity. Moreover, here and in what follows, four(three)-dimensional indices are always raised and lowered by the flat background structures. These are the Minkowski (Euclidean) metric $\eta_{\alpha\beta}$ (δ_{ij}) at infinity, the inertial cotetrad $\mathrm{d}x^\mu_a$ and its inverse ∂^a_α. For e.g. the σ-matrices we can thus write the expansion:

$$\sigma^A{}_{B\mu}\eta^\mu{}_a = \sigma^A{}_{Bi}\mathrm{d}x^i_a + \mathcal{O}(\varepsilon) = \mathcal{O}(\varepsilon), \tag{2.281}$$

where σ_i are just the ordinary Pauli matrices of $\mathbb{R}^3$. We then also need the spin rotation coefficients. Expanding around $\varepsilon = 0$ we get:

$$\omega_{\alpha\beta\mu} = \frac{1}{2}\big(\partial_\beta f_{\mu\alpha} - \partial_\alpha f_{\mu\beta}\big) + \mathcal{O}(\varepsilon^2), \tag{2.282}$$

We are now ready to formally solve equation (2.252) order by order in the dimensionless parameter ε. The zeroth order of equation (2.252) is easily identified to be:

$$\sigma^A{}_B{}^i\partial_i{}^{(0)}\chi^B = 0 \tag{2.283}$$

Looking at a Fourier expansion $\chi(x^i) \propto \exp(\mathrm{i}k_ix^i)$ it follows from $(\sigma^ik_i)^2 = \vec{k}^2\mathbb{1}$ that the only solution of (2.283) is a spinor constant in space:

$$^{(0)}\chi^A(x^i) = \varepsilon^A \tag{2.284}$$

For the next non-vanishing order in ε we have:

$$\varepsilon\sigma^A{}_B{}^i\partial_i{}^{(1)}\chi^B + \sigma^A{}_B{}^i\omega^B{}_{Ci}\,{}^{(0)}\chi^C = 0, \tag{2.285}$$

where $\omega^A{}_{B\mu}$ is the selfdual part of (2.282), just as defined in (2.267). We introduce the three dimensional Christoffel symbols $\Gamma^i{}_j$, together with the extrinsic curvature*,

$$\Gamma^i{}_j = \frac{1}{2}\epsilon_l{}^{im}\omega^l{}_{mi} \tag{2.286a}$$
$$K_{ij} = K_{ji} = \omega^0{}_{ij}, \tag{2.286b}$$

and can write the selfdual connection in terms of complex Ashtekar variables:

$$\omega^A{}_{Bi} = \frac{1}{2\mathrm{i}}\sigma^A{}_{Bl}\big(\Gamma^l{}_i + \mathrm{i}K^l{}_i\big) = \frac{1}{2\mathrm{i}}\sigma^A{}_{Bl}A^l{}_i. \tag{2.287}$$

Equation (2.285) allows us to express the radial derivative in terms of the zeroth order spinor ε^A and additional derivatives on the sphere:

$$\partial_r{}^{(1)}\chi^A = -\partial^i_r\sigma^A{}_{Bi}\big(\sigma^B{}_{Cj}q^{jk}\partial_k{}^{(1)}\chi^C + \varepsilon^{-1}\sigma^A{}_B{}^k\omega^B{}_{Ck}\varepsilon^C\big) \tag{2.288}$$

where, $q_{ij} = \delta_{ij} - \mathrm{d}r_i\mathrm{d}r_j$ denotes the two-dimensional induced metric on the sphere.

Let us now study the asymptotic expansion of (2.277). We find the whole expression starts at first order in ε

$$\int_{\partial\Sigma} d^2v_ah^{ab}\delta_{A\bar{A}}\bar{\chi}^{\bar{A}}\nabla_b\chi^B = \int_{S^2_r} d^2x\delta_{A\bar{A}}{}^{(0)}\bar{\chi}^{\bar{A}}\partial^i_r\big(\varepsilon\partial_i{}^{(1)}\chi^A + \omega^A{}_{Bi}{}^{(0)}\chi^B\big) + \\ + \mathcal{O}(\varepsilon^2), \tag{2.289}$$

*We can always choose our Lorentz frame such, that for both the physical metric and the asymptotically flat geometry the surface normal n^a of Σ is always aligned to δ^μ_0, i.e. $n^\mu = e^\alpha{}_an^a = \delta^\alpha_0 = \mathrm{d}x^\alpha_an^a$. In this case $\omega^0{}_{ij}$ agrees with the extrinsic curvature of the spatial hypersurface. This is implicitly assumed in the following.

Notice also that the spin rotation coefficients (2.282) are of order ε, and therefore they do indeed contribute to the first non-vanishing term in the asymptotic expansion of (2.289). Moreover, d^2x denotes the volume element on S_r^2 with respect to the asymptotically flat background geometry. We now want to get rid of the radial derivative of ${}^{(1)}\chi^A$ appearing in the first term on the right hand side of equation (2.289). This we do by employing equation (2.288), and find after some algebraic manipulations involving the Pauli identities (2.256) that:

$$\int_{S_r^2} d^2x \delta_{A\bar{A}} {}^{(0)}\bar{\chi}^{\bar{A}} \partial_r^i \big(\varepsilon \partial_i {}^{(1)}\chi^A + \omega^A{}_{Bi} {}^{(0)}\chi^B\big) = \int_{S_r^2} d^2x \varepsilon_A^\dagger \partial_r^i \Big[-\varepsilon \mathrm{i} \epsilon_i{}^{jk} \sigma^A{}_{Bj} \partial_k {}^{(1)}\chi^B + \\ - \frac{1}{2} \epsilon_{ilm} A^{ml} \varepsilon^A - \frac{\mathrm{i}}{2} \epsilon_i{}^{ln} \epsilon_{nm}{}^k \sigma^A{}_{Bk} A^m{}_l \varepsilon^B \Big] \tag{2.290}$$

Let us study this expression a bit more carefully. We see the first term on the right hand side is an integral of the total exterior derivative of the one-form $\varepsilon_A^\dagger \sigma^A{}_{Bk} {}^{(1)}\chi^B \mathrm{d}x^k$ over the two-dimensional surface $S_r^2 = \partial\Sigma$. By Stoke's theorem this integral vanishes by itself simply because S_r^2 is closed and has no boundary. With some little algebra (2.290) now further simplifies:

$$\int_{S_r^2} d^2v_a \delta_{A\bar{B}} \bar{\chi}^{\bar{B}} h^{ab} \nabla_b \chi^A = \int_{S_r^2} d^2x \varepsilon_A^\dagger \partial_r^i \Big[-\frac{1}{2} \epsilon_{ilm} \big(\Gamma^{ml} + \mathrm{i} \underbrace{K^{ml}}_{K_{ij}=K_{ji}}\big) \varepsilon^A + \\ - \frac{\mathrm{i}}{2} \sigma^A{}_{Bk} \big(\underbrace{\Gamma_i{}^k}_{\to \text{boundary term}} + \mathrm{i} K_i{}^k - \delta_i^k \underbrace{\Gamma^m{}_m}_{=0} - \mathrm{i} \delta_i^k K^m{}_m \big) \varepsilon^B \Big] + \mathcal{O}(\varepsilon^2) \tag{2.291}$$

All terms decorated with small braces vanish by the mechanism indicated underneath, e.g. the expression containing the anti-symmetrisation of K_{ij} disappears since the extrinsic curvature tensor is symmetric, while $\partial_r^i \Gamma_i{}^k = \frac{1}{2} \partial_r^i \epsilon_{min} \partial^n f^{km}$ contracted with the constant three vector $\varepsilon^\dagger \sigma_k \varepsilon$ is an exterior derivative. This derivative cannot contribute to the overall integral again through Stoke's theorem, since the two-dimensional boundary S_r is closed. For the trace $\Gamma^m{}_m$, on the other hand, we see this equates to zero, simply by looking at its defining equation (2.286b). In summary, the only surviving terms are:

$$\begin{aligned} \int_{S_r^2} d^2v_a \delta_{A\bar{B}} \bar{\chi}^{\bar{B}} h^{ab} \nabla_b \chi^A &= -\frac{1}{2} \int_{S_r^2} d^2x^i \varepsilon_A^\dagger \partial_r^i \Big[\epsilon_{ilm} \Gamma^{ml} \delta_B^A + \\ &\quad - \sigma^A{}_{Bk} \big(K_i{}^k - \delta_i^k K^m{}_m \big) \Big] \varepsilon^B + \mathcal{O}(\varepsilon^2) \\ &= \frac{1}{4} \int_{S_r^2} d^2x \varepsilon_A^\dagger \Big[\delta_B^A \big(\partial_i f_l{}^l - \partial_l f^l{}_i \big) + \\ &\quad + \sigma^A{}_{Bk} \big(\partial_0 f_i{}^k - \partial_i f^k{}_0 - \delta_i^k (\partial_0 f^l{}_l - \partial_l f^l{}_0) \big) \Big] \varepsilon^B + \mathcal{O}(\varepsilon^2) \end{aligned} \tag{2.292}$$

Here $d^2x^i = d^2x \partial_r^i$ is the two-dimensional volume element on the sphere, multiplied by the surface normal. The first term is on the right is of order ε, and all higher orders have been neglected. This equation, when looking back at (2.276) gives the positivity of the ADM mass. Comparing the two terms of equation (2.292) with the spatial and temporal components of the regularised ADM four-momentum at spatial infinity, as defined by equations (2.243) and (2.249), we get the desired inequality:

$$\lim_{r\to\infty} \int_{S_r^2} d^2v_a h^{ab} \delta_{A\bar{A}} \bar{\chi}^{\bar{A}} \nabla_b \chi^A = \frac{4\pi G}{c^3} P_a^{\mathrm{ADM}} [\sigma_{A\bar{A}}{}^a \bar{\varepsilon}^{\bar{A}} \varepsilon^A]_{\mathrm{reg}} = \\ = \int_\Sigma d^3v \delta_{A\bar{A}} h^{ab} \nabla_a \bar{\chi}^{\bar{B}} \nabla_b \chi^A - \frac{4\pi G}{c^3} \int_\Sigma d^3v \bar{\chi}^{\bar{A}} \chi^A \sigma_{A\bar{A}b} n_a T^{ab} \geq 0 \tag{2.293}$$

Here we have used our assumptions on the asymptotic behaviour of the elementary fields (2.278) and their derivatives (2.280), implying only the first order in ε survives when going to $r = L \to \infty$. This is not an assumption in Witten's original proof, which was much more careful on this very essential step.

Conclusions Because $\varepsilon^a = -\sigma_{A\bar{A}}{}^a \bar{\varepsilon}^{\bar{A}} \varepsilon^A$ defines an arbitrary future pointing null vector, equation (2.293) proves, that the regularised ADM four-momentum (2.249, 2.243) defines a future-oriented four-vector sitting in the asymptotically flat background. In other words, the ADM mass 2.249 is always positive. For, the first integrand in the second line, i.e the term $\delta_{A\bar{A}} h^{ab} \nabla_a \bar{\chi}^{\bar{A}} \nabla_b \chi^A$, is always greater or equal to zero, while the second term is positive by our assumption, i.e. the dominant energy condition. From (2.293) we can also prove that the ADM energy vanishes only in empty Minkowski space. This can be seen as follows. If the ADM energy equates to zero, both the first and the second term of the second line on the right hand side of (2.293) must vanish individually. The first term disappears only if:

$$h_a{}^b \nabla_b \chi^A \big|_p = 0, \quad \forall p \in \Sigma \tag{2.294}$$

Taking the commutator of two such derivatives, and projecting the resulting quantity back onto the spatial slice immediately leads us to:

$$h_a{}^{a'} h_b{}^{b'} R^{cd}{}_{a'b'} \Sigma^A{}_{Bcd} \chi^B \big|_p = 0 \Rightarrow h_a{}^{a'} h_b{}^{b'} R_{cda'b'} \big|_p = 0 \quad \forall p \in \Sigma \tag{2.295}$$

Here, the left hand side implies what is written on the right, because we can choose for any $p \in \Sigma$ the value $\chi^A(p) \in \mathbb{C}^2$ freely: for each $\varepsilon^A \in \mathbb{C}^2$ we can always find a solution χ^A of (2.252) such that $\chi^A(p) = \varepsilon^A$. Next, the ADM energy is conserved in time. This means that the value of the ADM energy does not change under deformations of the spatial hypersurface. Therefore, (2.295) holds for any hypersurface, and we find that the ADM energy vanishes, only if the geometry is flat, i.e. $R_{abcd} = 0$. If the geometry is flat, Einstein's equations imply the absence of matter, i.e. $T_{ab} = 0$. Therefore the ADM mass vanishes only in flat Minkowski space without matter. This concludes our review on Witten's beautiful analysis of the ADM four-momentum.

SUPPLEMENT: THE KODAMA STATE

We are now going to study the Kodama state [80–89] in selfdual variables. It formally solves the Hamiltonian constraint and was thus thought to describe a possible vacuum of the theory. Witten [90] soon realised this cannot work. Nevertheless this state may still play some role in the quantum theory. In fact, the Kodama state has regained some attraction, when a number of authors conjectured [91–95] that a q-deformation [139] of loop quantum gravity would automatically add a cosmological constant to the theory.

To introduce the Kodama state we have to understand some elementary properties of the Chern–Simons functional of the Lorentz group. This will be our first task. The second is to see why we could think the Kodama state solved the Hamiltonian constraint, and why it yet does deserve some attention.

The Chern–Simons functional of the Lorentz connection

To define the $SL(2,\mathbb{C})$ Chern–Simons functional of the selfdual Ashtekar connection we set:

$$Y[A] = \int_\Sigma \mathrm{Tr}\big(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A\big). \tag{2.296}$$

We calculate the functional differential and find

$$\mathbb{d}Y[A] = -\int_{\partial\Sigma} \mathrm{Tr}\big(A \wedge \mathbb{d}A\big) + 2\int_\Sigma \mathrm{Tr}\big(\mathbb{d}A \wedge F\big). \tag{2.297}$$

If Σ is closed the boundary term vanishes and we immediately see that the Chern–Simons functional is a generating potential for the "magnetic" field, i.e.

$$\frac{\delta Y[A]}{\delta A^i{}_a(p)} = -\frac{1}{2}\tilde{\eta}^{abc} F_{ibc}\Big|_p \equiv -B_i{}^a(p), \tag{2.298}$$

where $\tilde{\eta}^{abc}$ is again the Levi-Civita density. If Σ is open, or has a finite boundary, we can achieve (2.298) only by requiring additional boundary conditions. Take $\Sigma = \mathbb{R}^3$ as an example. The boundary term in (2.297) disappears if we restrict ourselves to variations of the connection that vanish at spatial infinity. To be more precise, we demand that $\lim_{r\to\infty} \mathrm{em}_r^* \mathbb{d}A = 0$, where em_r denotes the canonical embedding of the $|\vec{x}| = r = \text{const.}$ two-sphere into $\mathbb{R}^3$.

Gauge invariance Since the integrand of (2.296) manifestly breaks gauge invariance, we have to check what happens when using the transformed connection

$$A^g = g^{-1}\mathrm{d}g + g^{-1}Ag, \tag{2.299}$$

where $g : SL(2,\mathbb{C}) \to \Sigma$ denotes the gauge element. Performing a partial integration we find

$$Y[A^g] = Y[A] - \int_{\partial\Sigma} \mathrm{Tr}\big(\mathrm{d}g g^{-1} \wedge A\big) - \frac{1}{3}\int_\Sigma \mathrm{Tr}\big(g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g\big). \tag{2.300}$$

If we restrict ourselves either to the case of $\Sigma \simeq S_3$ or demand appropriate boundary conditions* on g, this implies the difference $Y[A^g] - Y[A]$ equals the winding number:

$$n(g) := \frac{1}{24\pi^2}\int_\Sigma \mathrm{Tr}\big(g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g\big). \tag{2.301}$$

The winding number $n(g) \in \mathbb{Z}$ measures how of often the map $g : S_3 \to SL(2,\mathbb{C})$ wraps $SL(2,\mathbb{C})$ around S^3. How can we understand that $n(g)$ defines a natural number? Reference [123] proofs $n(g) \in \mathbb{Z}$ for an $SU(2)$ gauge symmetry. We can immediately generalise this proof to allow for $SL(2,\mathbb{C})$ gauge transformations. The argument goes as follows. First of all one needs to check that $n(g)$ is a topological invariant. This means that if we can continuously deform the map g into g', then we have $n(g) = n(g')$. Next we note that any Lorentz transformation can be written as a product of a rotation and a boost, and we can thus write any gauge transformation as $g = U\exp(\frac{1}{2}\Xi^i\sigma_i)$, where $U : \Sigma \to SU(2)$, and $\Xi^i : \Sigma \to \mathbb{R}^3$. This decomposition implicitly shows that we can continuously deform Ξ^i to the null vector $\Xi^i = 0$, hence $n(g) = n(U) \in \mathbb{Z}$.

*Possible boundary conditions require that the Maurer–Cartan form vanishes at infinity, i.e. $\lim_{r\to\infty} \mathrm{em}_r^* g^{-1}\mathrm{d}g = 0$ where em_r again denotes the embedding of the $|\vec{x}| = r = \text{const.}$ two-sphere into $\mathbb{R}^3$

We have seen the Chern–Simons functional transforms homogeneously only under small gauge transformations, i.e. those that are connected to the identity. For generic gauge elements $g : \Sigma \rightarrow SL(2,\mathbb{C})$ we find instead:

$$Y[A^g] - Y[A] = -8\pi^2 n(g), \quad n(g) \in \mathbb{Z}. \tag{2.302}$$

Next, there is also the complex conjugate connection $\bar{A}^i{}_a = \Gamma^i{}_a - \mathrm{i}K^i{}_a$, for which equation (2.302) remains unchanged, we have:

$$Y[\bar{A}^g] - Y[\bar{A}] = -8\pi^2 n(\bar{g}) = -8\pi^2\overline{n(g)} = -8\pi^2 n(g). \tag{2.303}$$

Where $\bar{g}$ equals the gauge transformation in the complex conjugate representation of the group, i.e. the right handed representation. Notice that the respective winding numbers are the same, i.e. $n(g) = n(\bar{g})$.

The Kodama state as a formal solution of the Hamiltonian constraint

Following the general ideas of [80, 82, 86, 130] we are now going to reconstruct the Kodama state for chiral variables. This is a *formal* solution of all *first-class* constraints (2.95) of the theory. Because it does not impose the reality conditions (2.97, 2.152) this functional does however not describe a proper quantum state of gravity.

We are working in a "position" representation, and take complex-valued functional $\Psi[A]$ of the connection to describe the quantum states of gravity. The connection acts by multiplication, while the canonical Poisson commutation relations (2.110) tell us, that the momentum becomes a functional derivative:

$$\Pi_i{}^a(p) \longrightarrow \frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta A^i{}_a(p)}, \qquad \bar{\Pi}_i{}^a(p) \longrightarrow \frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta \bar{A}^i{}_a(p)}. \tag{2.304}$$

Let us now turn our attention to the constraints (2.95). Physical states are those complex valued functionals of the connection that are annihilated by the quantised constraints. Knowing that both the vector constraint and the Gauß law generate the kinematical symmetries of the theory, they have an unambiguous quantisation. In fact, the quantised Gauß law becomes the infinitesimal generator of the gauge symmetries (2.121) while the vector constraint generates diffeomorphisms lifted upstairs into the underlying $SL(2,\mathbb{C})$ principal bundle. For the Hamiltonian constraint the situation is far more complicated. We have to actually quantise the constraint and find the kernel of the operator. Choosing a rather naive ordering, we are led to the particularly simple proposal:

$$\begin{aligned}\hat{H}[\underset{\sim}{N}] := \hbar\ell_{\mathrm{P}}^2 \int_\Sigma \underset{\sim}{N}\Big(&\frac{\beta}{\beta+\mathrm{i}}\epsilon^{ilm}\underset{\sim}{\eta}_{abc}\frac{\delta}{\delta A^l{}_b}\frac{\delta}{\delta A^m{}_c}\Big(B_i{}^a + \frac{2\Lambda\ell_{\mathrm{P}}^2}{3}\frac{\beta}{\beta+\mathrm{i}}\frac{\delta}{\delta A^i{}_a}\Big) + \\ &+ \frac{\beta}{\beta-\mathrm{i}}\epsilon^{ilm}\underset{\sim}{\eta}_{abc}\frac{\delta}{\delta \bar{A}^l{}_b}\frac{\delta}{\delta \bar{A}^m{}_c}\Big(\bar{B}_i{}^a - \frac{2\Lambda\ell_{\mathrm{P}}^2}{3}\frac{\beta}{\beta-\mathrm{i}}\frac{\delta}{\delta \bar{A}^i{}_a}\Big)\Big)\end{aligned} \tag{2.305}$$

where $B_i{}^a$ denotes the magnetic field, implicitly defined in (2.298). Notice that the ordering is rather bad. The momentum does not commute wit the magnetic field. Hence we arrive at an operator that is not even formally self-adjoined, i.e. there is no way to expect that $\hat{H}[\underset{\sim}{N}] = \hat{H}[\underset{\sim}{N}]^\dagger$.

There is, of course, a purpose for secretly choosing such a particularly bad ordering. Looking back at the Chern–Simons functional, and equation (2.298), we see, that only for this ordering, the Kodama state

$$\Omega[A] = \exp\Big(+\frac{3}{2\Lambda\ell_{\mathrm{P}}^2}\frac{\beta+\mathrm{i}}{\beta}Y[A] - \frac{3}{2\Lambda\ell_{\mathrm{P}}^2}\frac{\beta-\mathrm{i}}{\beta}Y[\bar{A}]\Big) \in \big\{z\in\mathbb{C}\big||z|=1\big\} \tag{2.306}$$

solves the quantised Hamiltonian constraint, i.e. $\hat{H}[\underset{\sim}{N}]\Omega = 0$. Notice also, that the value of $\Omega[A]$ is always confined to the unite circle, i.e. $\Omega[A] \in S_1$. Let us also emphasise that this state, i.e. the Kodama state generalised to arbitrary values of the Barbero–Immirzi parameter, has originally been studied by Randono et al. in [86–88].

Both diffeomorphisms and small gauge transformation cannot change the value of the Kodama state, with (2.121) we find e.g. that $\Psi[A] = \Psi[\rho_{\exp(\Lambda)}A]$. This means that the Kodama state not only solves the Hamiltonian constraint (in a particularly bad ordering), but that is also lies in the kernel of both the Gauß and the vector constraint, and thus solves all first-class constraints of the theory (2.95).

The Chern–Simons functional transforms (2.302) inhomogeneously under large gauge transformations, and for generic values of Λ so does the Kodama state:

$$\begin{aligned}\Omega[A^g] &= \Omega[A]\exp\Big(-\frac{12\pi^2}{\Lambda\ell_{\mathrm{P}}^2}\frac{\beta+\mathrm{i}}{\beta}n(g) - \mathrm{cc.}\Big) = \\ &= \Omega[A]\exp\Big(-\frac{24\mathrm{i}\pi^2}{\Lambda\beta\ell_{\mathrm{P}}^2}n(g)\Big).\end{aligned} \tag{2.307}$$

If we wish to interpret the Kodama state as a genuine “wave function” it should better be single-valued, which is possible only if the product $\Lambda\beta\ell_{\mathrm{P}}^2$ of the elementary parameters is restricted to discrete values, in other words:

$$\Lambda = \frac{12\pi}{\beta\ell_{\mathrm{P}}^2}\frac{1}{n}, \quad n \in \mathbb{Z} - \{0\} \tag{2.308}$$

implying that

$$\Omega[A] = \exp\Big(\frac{n}{8\pi}(\beta+\mathrm{i})Y[A] - \mathrm{cc.}\Big). \tag{2.309}$$

is single-valued. Notice the natural appearance of the minimal length scale $(\ell_{\mathrm{LQG}})^2 = \beta\ell_{\mathrm{P}}^2$ of loop quantum gravity in the formula for the cosmological constant (2.308). Inserting the present value of the cosmological constant we get a relation between the level n in the exponent and the Barbero–Immizi parameter:

$$\hbar\Lambda\hbar G c^3 = \frac{3}{2}\frac{1}{\beta n} \approx 3\cdot 10^{-122}. \tag{2.310}$$

Several arguments suggest β to be of order one, which means that the level n must be incredibly large, and the Kodama state $\Omega[A]$ would be a rapidly oscillating phase on the affine space of $SL(2,\mathbb{C})$ connections.

The Kodama state is a formal solution of the Hamiltonian constraint, and also solves both the Gauß law and the vector constraint. Does this mean it is a viable candidate for the vacuum state of gravity? Probably not, and the reasons are as follows.

No rigorous Hilbert-space. First of all we do not have a Hilbert space. Introducing momentum operators (2.304) and declaring the connection should act by multiplication, i.e. $(\hat{A}^i_a\Psi)[A] = A^i{}_a\Psi[A]$ does not suffice to arrive at a sensible quantum theory. We also need an inner product between quantum states $\Psi[A]$ and $\Psi'[A]$. Otherwise we could never speak about probabilities and transition amplitudes, which is what quantum theory is actually all about.

Naive ordering. Next, the ordering that we have chosen in the definition of the Hamiltonian constraint is rather bad. Even without ever specifying the Hilbert space, we have to expect the operator $\hat{H}[\underset{\sim}{N}]$ as defined by (2.305) is not self adjoined, because the momentum does not commute with the field-strength.

Missing second-class constraints. The strongest argument why we should discard the Kodama state as a viable candidate for the vacuum of the theory has to do with the second-class constraints. In fact, the Kodama state solves none of them, neither (2.97) nor (2.152). This can be seen most easily for the condition on the momentum variable, i.e. the linear simplicity constraint (2.97). In the continuum we can impose this condition strongly, its quantisation becomes:

$$\hat{C}_i{}^a = -\ell_{\mathrm{P}}^2\Big(\frac{\mathrm{i}\beta}{\beta+\mathrm{i}}\frac{\delta}{\delta A^i{}_a} + \frac{\mathrm{i}\beta}{\beta-\mathrm{i}}\frac{\delta}{\delta \bar{A}^i{}_a}\Big). \tag{2.311a}$$

Looking back at the definition of the $\mathfrak{su}(2)$ Ashtekar–Barbero connection (2.108), and employing the chain rule, we can see any state that is only a functional of the Ashtekar–Barbero connection solves this constraint:

$$\Psi[A] = \Psi[\Gamma + \mathrm{i}K] = \psi[A^{(\beta)}] = \psi[\Gamma + \beta K] \Leftrightarrow \hat{C}_i{}^a\Psi = 0. \tag{2.312}$$

The Kodama state depends however on all components of the $\mathfrak{sl}(2,\mathbb{C})$ connection, and does therefore not solve the reality condition (2.97) on the momentum variable. For the torsionless condition we can employ a similar argument, and conclude that the Kodama state does not solve any of the second-class constraints (2.152) and (2.97).

Just one state. Even if we would allow ourselves to ignore all of these problems, we would have still found just one single solution, and it is far from obvious how we should ever extract any reasonable physics from one state only.

Nevertheless, there is a positive argument in favour of the Kodama state. The Kodama state is nothing but a solution to the classical constraint:

$$\mathcal{B}_i{}^a := B_i{}^a + \frac{2\Lambda\ell_{\mathrm{P}}^2}{3\hbar}\frac{\mathrm{i}\beta}{\beta+\mathrm{i}}\Pi_i{}^a \stackrel{!}{=} 0 \tag{2.313}$$

If the linear simplicity constraint (2.97) holds true this condition becomes $3B_i{}^a + \Lambda E_i{}^a = 0$. If the connection is torsionless, this condition implies that we are working with a maximally symmetric space-time. In other words, if we are interested to recover the de Sitter solution in quantum gravity it is equation (2.313) that we should look at. What is now also important to know, is that $\mathcal{B}$ weakly commutes with all first-class constraints:

$$\big\{\mathcal{B}_i{}^a, H[\underset{\sim}{N}]\big\}\big|_{\mathcal{B}=0} = 0, \quad \big\{\mathcal{B}_i{}^a, H_b[N^b]\big\}\big|_{\mathcal{B}=0} = 0, \quad \big\{\mathcal{B}_i{}^a, G_j[\Lambda^j]\big\}\big|_{\mathcal{B}=0} = 0 \tag{2.314}$$

What does these equations mean for the quantum theory? If a set of operators weakly commutes, this means, that we can, in principle, diagonalise them simultaneously. It thus makes sense to search for those states that solve, at least locally, equation (2.313) in the quantum theory.

We can use this observation to give a proposal for how to account for a cosmological constant in spinfoam gravity. To sketch the general idea, let me give a snapshot of the mathematical structure of the theory.

Loop quantum gravity has a well defined Hilbertspace $\mathcal{H}_{\mathrm{LQG}}$, quantum states $\psi \in \mathcal{H}_{\mathrm{LQG}}$ are spin network functions, that are square integrable functions on $SU(2)^N$, $N < \infty$. Transition amplitudes are built from the Feynman amplitudes of the individual scattering processes. The amplitude for any such elementary scattering formally reads:

$$Z[\psi] = \int_{\mathcal{A}_{SL(2,\mathbb{C})}} \mathcal{D}[A]\,(Y\psi)[A] \tag{2.315}$$

The integral is over the affine space of $SL(2,\mathbb{C})$ connections, $\mathcal{D}[A]$ is an $SL(2,\mathbb{C})$ invariant integration measure, and the *Dupuis–Livine map* Y [106, 131–134] sends any $SU(2)$ spin network function, into an $SL(2,\mathbb{C})$ spin network function, that we can view as a functional of the $SL(2,\mathbb{C})$ space-time connection A. The uniform integration over all connections imposes that the geometry be locally flat. This is where the fundamental *loop assumption* has actually entered.

We can formally write this amplitude as

$$Z[\psi] = \langle Y^\dagger 1|\psi\rangle, \tag{2.316}$$

where the empty state $\langle 1|$ is the vacuum of the theory [49]. We could now replace this "bra" by the Kodama state, and would thus arrive at an amplitude of the following form:

$$Z_\Lambda[\psi] = \int_{\mathcal{A}_{SL(2,\mathbb{C})}} \mathcal{D}[A]\, \overline{\Omega[A]}(Y\psi)[A]. \tag{2.317}$$

If (2.316) describes the elementary Feynman amplitudes for locally flat geometries, then (2.317) implies the geometry locally looks like de Sitter space. In analogy with (2.316) and (2.309) we can say:

$$Z_\Lambda[\psi] = \langle Y^\dagger \Omega|\psi\rangle. \tag{2.318}$$

In this understanding, the Kodama state does not represent a physical state of the theory, but determines the vertex amplitude of each individual scattering process. The proposed amplitude coincides with the Chern–Simons expectation value [135] of $SL(2,\mathbb{C})$ holonomies in some unitary representations of the Lorentz group. For compact gauge groups this functional has been well explored (e.g. [136–138]), much less is known for the non-compact case. The conjecture is [91–95], that the spinfoam model gets deformed, with $SL(2,\mathbb{C})$ turning into a quantum group [139] $SL_q(2,\mathbb{C})$.

3
The discretised theory

Loop quantum gravity [5, 18–20, 40, 140] comes in two versions. The historically first of which provides a canonical quantisation of general relativity, and seeks to solve the Wheeler–DeWitt equation [18,19,141]. The second, we call it spinfoam gravity [20,142], proposes a covariant path integral. Both approaches share [143, 144] their kinematical structure—the Hilbert space with observables representing, area, angles, volume and parallel transport [38, 56, 57]. Unfortunately, we know very little about how far this relation extends beyond kinematics, eventually revealing a solid framework also for the dynamics of the theory [**?**, **?**]. Indications supporting this idea have only come from the symmetry reduced framework of loop quantum cosmology [145–147].

The problem of how the canonical theory and spinfoam gravity can fit together is not only of mathematical significance. First of all, it is a consistency check for the theory. If loop gravity were a fully developed theory, it should come in different formulations, providing specific advantages and simplifications, although being mutually equivalent. On top of that, a framework large enough to contain both Lagrangian and Hamiltonian dynamics should allow us to answer some of the most pressing questions in the field: What is the classical theory underlying loop gravity, and does it reproduce general relativity? Does loop gravity contain torsion, and are there any secondary constraints missing? Is there a local notion of energy, and what can we use it for?

This chapter achieves such a unifying classical framework in the simplified setting of a fixed discretisation. In the first four sections, we will, in fact, develop a Hamiltonian formulation of spinfoam gravity, adapted to a simplicial decomposition of space-time. This discretisation consists of triangles, tetrahedra and four-simplices glued together. Calculations will heavily use the twistorial framework of loop quantum gravity, recently developed by Dupuis, Freidel, Livine, Speziale, Tambornino and myself [148–153]. These spinorial variables do not change the physical content of loop gravity, but offer a new perspective on how to look at the covariant aspects of the theory.

In the first four sections we will introduce spinors for loop gravity (section 3.1), perform the constraint analysis (section 3.2) and study the geometrical interpretation of the equations of motion (sections 3.2 and 3.3). Working with complex variables we have to deal with reality conditions, which guarantee that the metric is real. The Hamiltonian time evolution must preserve these constraint equations, a condition that may need additional secondary constraints to be fulfilled. We will, however, not find any secondary constraints, but only get restrictions on the Lagrange multipliers. This comes as a surprise. For, as we know from the last chapter, the torsionless condition (2.12) turns into a secondary constraint (2.152) needed to preserve the reality conditions.

The first four sections give the main result of the chapter, yet there are some more topics beside. Section 3.4 deals with torsion, stresses its significance for the Minkowski theorem [154] in Minkowski space, and defines the torsionless connection for twisted geometries [155] (twisted geometries are relevant for loop gravity, they generalise Regge geometries [64] to allow further discontinuities in the metric tensor). Another topic concerns the relation to the canonical theory. In this thesis we are mostly using selfdual variables, while canonical loop gravity favours $SU(2)$ Ashtekar–Barbero variables. Section 3.5 gives the relation between the two. There are also two more supplements, one dealing with the general properties of the $SL(2,\mathbb{C})$ parallel transport, the other speaking about the geometry of a four-simplex bounded by five space-like tetrahedra.

This chapter consists almost entirely of the classical aspects of my recent publication [156], but I have also added what I found relevant from [157] and [149].

3.1 SELFDUAL TWO-FORMS, SPINORS AND REALITY CONDITIONS FOR LOOP GRAVITY

This section develops the tools necesarry for the rest of the chapter. It is based upon the publications [99, 149, 150, 156, 158]. Section 3.1.1 reviews complex Ashtekar variables and their corresponding holonomy-flux algebra, section 3.1.2 develops the twistorial parametrisation of the phase space of holonomy-flux variables, while section 3.1.3 gives the reality conditions.

3.1.1 Complex Ashtekar variables

The spinfoam approach seeks to define transition amplitudes for loop quantum gravity. It starts from the following topological action:

$$S[\Sigma, A] = \frac{\mathrm{i}\hbar}{\ell_{\mathrm{P}}^2}\frac{\beta+\mathrm{i}}{\beta}\int_{\mathcal{M}} \Sigma_{AB}\wedge F^{AB}(A) + \text{cc.}, \tag{3.1}$$

which is the "BF-action" [21] expressed in selfdual variables. Here $\ell_{\mathrm{P}}^2 = 8\pi\hbar G/c^3$ is the Planck area, β is the Barbero–Immirzi parameter, $F = \mathrm{d}A + \frac{1}{2}[A,A]$ denotes the curvature of the selfdual connection, $\Sigma^A{}_B$ is an $\mathfrak{sl}(2,\mathbb{C})$ valued two-form, and the antisymmetric ϵ-tensor* moves the spinor indices $A, B, C, \dots \in \{0,1\}$. These indices transform under the fundamental $(\frac{1}{2},0)$ representation of $SL(2,\mathbb{C})$. We use "cc." to denote the complex conjugate of everything preceding (including the pre-factor $\mathrm{i}(\beta + \mathrm{i})\dots$), and so the $(0,\frac{1}{2})$ representation also appears. Indices transforming under this complex conjugate representation carry an overbar, we write $\bar{A}, \bar{B}, \bar{C}, \dots$. Working with a closed manifold, we do not have to worry about boundary terms that are otherwise needed [112, 115, 159].

This action shares the symplectic structure of general relativity, but the dynamics is trivial. Indeed, performing a 3+1 split $\mathcal{M} = \mathbb{R}\times\mathcal{S} \ni (t,p)$, we find the symplectic structure of complex Ashtekar variables:

$$\big\{\Pi_i{}^a(p), A^j{}_b(q)\big\} = \delta_i^j\delta_b^a\tilde{\delta}(p,q) = \big\{\bar{\Pi}_i{}^a(p), \bar{A}^j{}_b(q)\big\}, \tag{3.2}$$

here indices i, j, k running from 1 to 3 refer to the standard basis** in $\mathfrak{sl}(2,\mathbb{C})$, $\tilde{\delta}$ is the Dirac-delta density on the spatial hypersurface $\mathcal{S}_t = \{t\}\times\mathcal{S}$, and $a, b, c, \dots$ are abstract indices on the spatial slice. The Ashtekar connection $A^i{}_a = \Gamma^i{}_a + \mathrm{i}K^i{}_a$ is the pullback of the selfdual connection onto the spatial hypersurfaces; in general relativity

*The ϵ-tensor lowers indices as $v_A = v^B\epsilon_{BA} \in \mathbb{C}^{2*}$, while its inverse raises them by $v^A = \epsilon^{AB}v_B \in \mathbb{C}^2$; the inverse is implicitly defined by putting $\epsilon_{AC}\epsilon^{BC} = \epsilon_A{}^B = \delta_A^B$.

**Given any $\phi \in \mathfrak{sl}(2,\mathbb{C})$ we write $\phi = \phi^i\tau_i$, where $\sigma_i = 2\mathrm{i}\tau_i$ are the Pauli matrices.

its real and imaginary parts (Γ and K respectively) correspond to the intrinsic $\mathfrak{su}(2)$ connection and the extrinsic curvature of the spatial hypersurface. The momentum conjugate is linearly related to Σ, we have:

$$\Pi_i{}^a = -\frac{\hbar}{\ell_{\mathrm{P}}^2}\frac{\beta+\mathrm{i}}{2\mathrm{i}\beta}\Sigma_i{}^a = -\frac{\hbar}{\ell_{\mathrm{P}}^2}\frac{\beta+\mathrm{i}}{4\mathrm{i}\beta}\tilde{\eta}^{abc}\Sigma_{ibc}, \tag{3.3}$$

where $\tilde{\eta}^{abc}$ is the Levi-Civita density on the spatial hypersurface*.

The continuum Poisson brackets behave too singularly to perform a background-independent quantisation. Therefore, we introduce a reduced phase space of smeared variables. We can define the smeared variables most elegantly when we consider a cellular decomposition of the $t = \text{const.}$ hypersurface. In this thesis, we restrict to triangulations, and thus divide the spatial manifold into tetrahedra glued among bounding faces. Generalisations to arbitrary cellular decompositions exist and have been studied elsewhere, e.g. in [42].

Within the spatial manifold, the faces, the oriented triangles $\tau_1, \tau_2, \dots, \tau_L$, are the duals of oriented links $\gamma_1, \gamma_2, \dots \gamma_L$. To smear the connection, we take a link and study the $SL(2,\mathbb{C})$ parallel transport between adjacent tetrahedra. We are thus led to the holonomy:

$$h[\tau] = \mathrm{Pexp}\big(-\int_\gamma A\big) \in SL(2,\mathbb{C}). \tag{3.4}$$

The momentum variable Π defines a two-form; parallel transported into the frame of a tetrahedron, we can naturally smear it over an adjacent triangle τ obtaining the gravitational flux:

$$\Pi^A{}_B[\tau] \equiv \Pi[\tau] = \int_{p\in\tau} \big(h_{\delta(p\rightarrow\gamma(0))}\big)^A{}_C \Pi^C{}_D(p)\big(h^{-1}_{\delta(p\rightarrow\gamma(0))}\big)^D{}_B \in \mathfrak{sl}(2,\mathbb{C}), \tag{3.5}$$

where $h_{\delta(p\rightarrow\gamma(0))}$ is an $SL(2,\mathbb{C})$ holonomy connecting $p \in \tau$ with the source point $\gamma(0)$. The underlying path $\delta(p \rightarrow \gamma(0))$ consist of two parts, the first of which lies inside τ and goes from $p \in \tau$ towards the intersection point $\tau \cap \gamma$, whereas the second part goes from the intersection point along γ towards the source $\gamma(0)$.

The continuum Poisson brackets (3.2) induce commutation relations among holonomies and fluxes; variables belonging to different triangles commute, while for a single link we get the commutation relations of $T^*SL(2,\mathbb{C})$:

$$\big\{\Pi_i, \Pi_j\big\} = -\epsilon_{ij}{}^k\Pi_k, \quad \big\{\Pi_i, h\big\} = -h\tau_i, \quad \big\{h^A{}_B, h^C{}_D\big\} = 0. \tag{3.6}$$

There are also the Poisson brackets of the anti-selfdual variables, which are nothing but the complex conjugate of the former variables. Moreover, just as in (3.2), the two sectors of opposite chirality commute. Since τ carries an orientation let us also mention the quantities:

$$h[\tau^{-1}] = h[\tau]^{-1}, \quad \underset{\sim}{\Pi}[\tau] := \Pi[\tau^{-1}] = -h[\tau]\Pi[\tau]h[\tau]^{-1}. \tag{3.7}$$

Before we go on let us make one more observation. The definition of the flux (3.5) depends on the underlying family of paths $\delta(p \rightarrow \gamma(0))$ chosen. It is therefore quite remarkable [42] that this dependence drops out of the Poisson algebra (3.6), and leads us to the phase space $T^*SL(2,\mathbb{C})$.

*Notice a subtlety in our notation: If $\Sigma^A{}_B = \Sigma^i{}_{ab}\tau^A{}_{Bi}$ is geometric, hence comes from a tetrad we would find a relative minus sign between the definitions of $\Sigma_i{}^a$ and $E_i{}^a$ through: $\Sigma_i{}^a = \frac{1}{2}\tilde{\eta}^{abc}\Sigma_{ibc} = \frac{1}{2}\tilde{\eta}^{abc}\epsilon_m{}^i{}_n e^m{}_b e^n{}_c = -E_i{}^a$, in accordance with (2.64).

3.1.2 Spinors for loop gravity

The phase space of loop gravity on a graph, $T^*SL(2,\mathbb{C})^L$, allows for a description in terms of spinors. This framework will become important for the rest of this thesis; it is useful for us since it embeds the nonlinear phase space $T^*SL(2,\mathbb{C})^L$ into a vector space with canonical Darboux coordinates.

The flux defines an $\mathfrak{sl}(2,\mathbb{C})$ element, which is traceless, and we can thus always find a pair of diagonalising spinors (the proof can be found in [29], which together with [150], can serve as the main reference of this section):

$$\Pi[\tau]_{AB} = -\frac{1}{2}\omega_{(A}\pi_{B)} = -\frac{1}{4}\big(\omega_A\pi_B + \omega_B\pi_A\big). \tag{3.8}$$

With the flux evaluated in the frame of the source $\gamma(0)$, also π and ω belong to the initial point. If we parallel transport them to the target point we get another pair of spinors:

$$\underset{\sim}{\pi}^A = h[\tau]^A{}_B\omega^B, \quad \underset{\sim}{\omega}^A = h[\tau]^A{}_B\pi^B. \tag{3.9}$$

These secondary spinors diagonalise the flux in the frame of the final point:

$$\Pi_{AB}[\tau^{-1}] = \underset{\sim}{\Pi}_{AB}[\tau] = \frac{1}{2}\underset{\sim}{\omega}_{(A}\underset{\sim}{\pi}_{B)}. \tag{3.10}$$

Since the spinors π and ω often come as a pair, it is useful to introduce the twistor:

$$Z := (\bar{\pi}_{\bar{A}}, \omega^A) = (\bar{\mathbb{C}}^2)^* \oplus \mathbb{C}^2 =: \mathbb{T}. \tag{3.11}$$

If π and ω be linearly independent, that is

$$\pi_A\omega^A \neq 0, \tag{3.12}$$

they form a basis in $\mathbb{C}^2$. We can safely agree on this constraint because it holds true unless the triangle represents a null surface, and we are working with spatial hypersurfaces anyhow. In this case, equation (3.9) gives the holonomy in a certain basis, which uniquely fixes this $SL(2,\mathbb{C})$ element. We can now reverse the construction, start with a pair $(\underset{\sim}{Z}, Z)$ of twistors and attach them to the source and target point respectively. Inverting equations (3.9) and (3.8) we then recover both holonomy and flux in terms of spinors. The holonomy explicitly reads

$$h[\tau]^A{}_B = \frac{\underset{\sim}{\omega}^A\pi_B - \underset{\sim}{\pi}^A\omega_B}{\sqrt{\pi\omega}\sqrt{\underset{\sim}{\pi}\underset{\sim}{\omega}}}, \quad \text{with: } \pi\omega := \pi_A\omega^A. \tag{3.13}$$

For the holonomy to have unit determinant it must preserve ϵ_{AB}. Within our spinorial framework this immediately turns into the *area-matching constraint*

$$C[\underset{\sim}{Z}, Z] = \underset{\sim}{\pi}_A\underset{\sim}{\omega}^A - \pi_A\omega^A = 0. \tag{3.14}$$

The space of spinors on a link can be equipped with a locally $SL(2,\mathbb{C})$ invariant symplectic structure. We set

$$\big\{\pi^A, \omega^B\big\} = \epsilon^{AB} = -\big\{\underset{\sim}{\pi}^A, \underset{\sim}{\omega}^B\big\}, \tag{3.15a}$$

$$\big\{\bar{\pi}^{\bar{A}}, \bar{\omega}^{\bar{B}}\big\} = \bar{\epsilon}^{\bar{A}\bar{B}} = -\big\{\underset{\sim}{\bar{\pi}}^{\bar{A}}, \underset{\sim}{\bar{\omega}}^{\bar{B}}\big\}, \tag{3.15b}$$

and can prove [150] that on the constraint hypersurface $C = 0$, these Poisson brackets induce the commutation relations of $T^*SL(2,\mathbb{C})$ for the flux (3.8) and the holonomy

(3.13). The parametrisations (3.13) and (3.8) are not unique. There are in fact two symmetries; one of which is discrete, the other is continuous. The discrete symmetry simultaneously exchanges π with ω, and $\underset{\sim}{\pi}$ with $\underset{\sim}{\omega}$, while the Hamiltonian vector field of C generates the conformal symmetry

$$(\omega, \pi; \underset{\sim}{\omega}, \underset{\sim}{\pi}) \mapsto (z\omega, z^{-1}\pi; z\underset{\sim}{\omega}, z^{-1}\underset{\sim}{\pi}), \quad z \in \mathbb{C} - \{0\}, \tag{3.16}$$

leaving both (3.13) and (3.8) invariant. We perform the symplectic quotient with respect to these symmetries and eventually get $T^*SL(2,\mathbb{C})$ removed from all its null configurations $(\Pi, h) \in T^*SL(2,\mathbb{C}) : \Pi_{AB}\Pi^{AB} = 0$. The proof is again in [150].

3.1.3 Reality conditions

The action (3.1) defines a topological theory. We recover general relativity in terms of first-order variables only if we impose constraints on Σ_{AB}. We want Σ_{AB} to be geometrical, that is to represent an infinitesimal area element. In the continuum theory this means Σ_{AB} should be the selfdual part of the Plebanski two-form $\Sigma_{\alpha\beta} = \eta_\alpha \wedge \eta_\beta$ (where η^α is the tetrad and $\alpha = 0, \dots, 3$ are internal Minkowski indices).

The Hamiltonian flow preserves these *simplicity constraints* [126, 127, 160, 161] only if the space-time connection is torsionless (which can be shown in many ways, as in section 2.4, or in references [99, 100, 158]). In the Hamiltonian framework, equations of motion either are constraints (possibly both on Lagrangian multipliers and the phase space variables) or evolution equations. Conversely, the torsionless condition of the space-time connection splits into three distinct parts (see equation (2.162), but also references [158, 162]). Using time gauge (2.58) (thus aligning the internal normal to the hypersurface normal, that is setting $e^0 = N\mathrm{d}t$), we found in (2.151) that the Lagrange multiplier* $\Lambda^i = \mathfrak{Im}(A^i(\partial_t))$ is determined by the lapse (N) and the shift (N^a) to have the value $\Lambda^i = N^a K^i{}_a + e^{ia}\partial_a N$. The second part of the torsionless condition represents an evolution equation for the spatial triad, whereas the last and most important part gives a second-class constraint among the phase space variables. This is equation(2.152) that states:

$$A^i{}_a + \mathrm{cc.} = 2\Gamma^i{}_a(e), \tag{3.17}$$

where $\Gamma(e)$ is the spatial Levi-Civita connection functionally depending on the triad.

In the discrete theory the situation is different. We do not have a continuous tetrad, and we cannot define the continuous simplicity constraints $\Sigma_{\alpha\beta} - \eta_\alpha \wedge \eta_\beta = 0$ directly. Instead we have smeared variables on a triangulation of a $t = \text{const.}$ spatial hypersurface. However, the physical meaning of the simplicity constraints is clear, they guarantee that the two-form $\Sigma_{\alpha\beta}$ is geometric, hence defines a plane in internal space. For our smeared variables we can demand something similar: $\Sigma_{\alpha\beta}[\tau]$, that is the two-form $\Sigma_{\alpha\beta}$ smeared over the triangle τ in the frame of the tetrahedron at $\gamma(0)$, should define a spatial plane in internal Minkowski space. This is true if there is a time-like vector n^α such that:

$$\Sigma_{\alpha\beta}[\tau]n^\beta \equiv -\Sigma_{AB}[\tau]\bar{\epsilon}_{\bar{A}\bar{B}}n^{B\bar{B}} - \mathrm{cc.} = 0. \tag{3.18}$$

These are the linear simplicity constraints [126], which are reality conditions on the momentum variables (see equations (2.97, 2.98) and reference [158]). Geometrically, the vector n^α represents the normal to the tetrahedron the triangle τ is seen from. This normal should thus be the same for all four triangles meeting at a tetrahedron.

*In the following lines ∂_t denotes the time-flow vectorfield.

A detailed discussion of the geometric origin of the simplicity constraints can be found in the appendix of reference [126].

In terms of spinors, equation (3.18) turns into two independent constraints [158]. One is real, the other is complex, and there are thus three real constraints to solve:

$$D[\omega,\pi]=\frac{\mathrm{i}}{\beta+\mathrm{i}}\pi_A\omega^A+\mathrm{cc.}=0, \tag{3.19a}$$

$$F_n[\omega,\pi]=n^{A\bar{A}}\pi_A\bar{\omega}_{\bar{A}}=0. \tag{3.19b}$$

The constraint $D=0$, is locally Lorentz invariant and guarantees the area of τ is real; $F_n=0$, on the other hand, is preserved only under spatial rotations, and tells us the null-vector $m^\alpha\equiv\omega^A\bar{\pi}^{\bar{A}}$ in complexified Minkowski space lies orthogonal to n^α. If we were to work in the time gauge, we would align the normal n^α with $n^\alpha_o=\delta^\alpha_0$ and the matrix $n^{A\bar{A}}$ would turn into

$$n^{A\bar{A}}=\frac{\mathrm{i}}{\sqrt{2}}\delta^{A\bar{A}}\equiv n_o^{A\bar{A}}, \tag{3.20}$$

where $\delta^{A\bar{A}}$ is the identity matrix. In this gauge the real and imaginary parts of the momentum

$$\Pi_i=\frac{1}{2}\big(L_i+\mathrm{i}K_i\big) \tag{3.21}$$

generate rotations and boosts relative to n^α, and the reality conditions (3.19) turn into:

$$\frac{1}{\beta+\mathrm{i}}\Pi_i+\mathrm{cc.}=\frac{1}{\beta^2+1}\big(K_i+\beta L_i\big)=0. \tag{3.22}$$

Whether or not spinfoam gravity misses the secondary constraints, and forgets about equation (3.17), raises some of the most pressing and strongly debated [103, 105, 163–165] questions in our field. This debate concerns two separate issues: (i) do we correctly impose the linear simplicity constraints, i.e. equation (3.22), and (ii) are there any further constraints missing? The spinorial framework of loop quantum gravity will allow us to study the first part of this question. In the next section we are going to prove that spinfoam gravity correctly solves equation (3.22) without missing any secondary constraints. This sounds promising for the model [126, 127], yet it does certainly not prove it right. We may still miss additional conditions on top of the linear constraints (3.22). This is, in fact, the second question to be asked. Although we cannot give a conclusive answer, we still learn something important about the nature of the problem: If there were additional constraints missing, they cannot arise from the stability of the simplicity constraints under the time evolution.

3.2 HAMILTONIAN DYNAMICS FOR SPINFOAM GRAVITY

This section introduces a continuous formulation of the dynamics on a fixed two-complex. We will check if the equations of motion preserve the reality conditions (in whatever form, i.e. (3.22) or (3.19), preferred), and study the equations of motion. They will, in fact, immediately prove the curvature smeared over a spinfoam face does not vanish, hence the model carries curvature. To achieve these claims, and properly answer the issue of the secondary constraints we need an action, or even better to find a suitable Hamiltonian framework. And this is what we are going to do first.

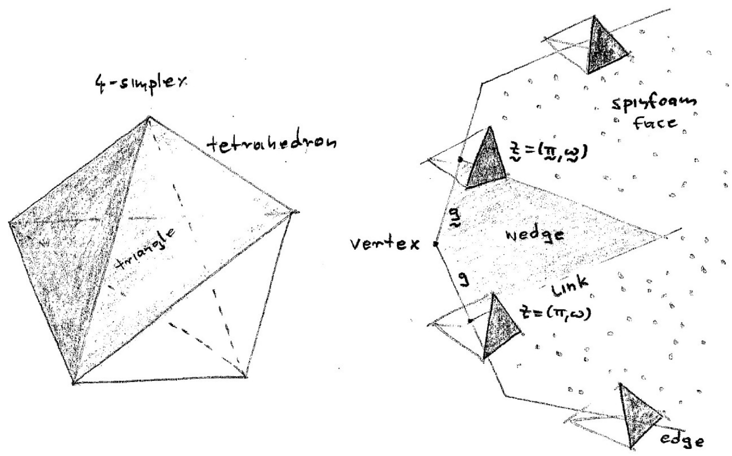

Figure 3.1: A four-simplex consists of five tetrahedra glued among their triangles. Its dual we call a *vertex*. Each triangle belongs to many four-simplices (vertices), but a tetrahedron can only be in two of them. The surface dual to a triangle is a *spinfoam face*, it touches all adjacent four-simplices. An edge, a tetrahedron's dual, connects two vertices. The part of a spinfoam face lying inside a given four-simplex, we call a *spinfoam wedge*, the boundary of which has two parts. The first one consists of edges, enters the bulk, and passes through the vertex. The second part belongs to the boundary of the four-simplex, we call it a link.

3.2.1 The discrete action on a spinfoam wedge

We start with the topological action (3.1) discretised over a simplicial decomposition of the four-dimensional space-time manifold $\mathcal{M}$, and use our spinors to parametrise the action. First steps towards this task have already been reported in [149]. The elementary building blocks are four-simplices glued among the bounding tetrahedra. All tetrahedra consist of four triangles, each of which is dual to a spinfoam face. The part of a spinfoam face belonging to a four-simplex we call a wedge w, for the corresponding dual triangle (in the frame of a tetrahedron) we write τ_w. Figure 3.1 should further clarify the geometry. Here we are committing ourselves to simplicial discretisations, which we do for technical convenience; generalisation to cellular decompositions should be found along the lines of [166].

Let $F^{AB}[w]$ be the curvature tensor integrated over the wedge, and $\Sigma_{AB}[\tau_w]$ be the two-form Σ_{AB} smeared over the dual triangle (in the frame of a tetrahedron), while w^{-1} and τ^{-1} denote the oppositely oriented surfaces. The remaining ambiguity concerns the relative orientation $\epsilon(\tau_w, w)$ between the two surfaces, which we take to be one*.

*If the tangent vectors (x, y) are positively oriented in τ_w, and the pair (t, z) is positively oriented in w, the relative orientation $\epsilon(\tau_w, w)$ is the orientation of the quadruple (x, y, t, z).

We discretise the topological action over each four-simplex and find a sum over wedges:

$$
\begin{aligned}
S^{\mathrm{top}}[\Sigma, A] &= \frac{\mathrm{i}\hbar}{\ell_{\mathrm{P}}^2}\frac{\beta+\mathrm{i}}{\beta}\int_M \Sigma_{AB}\wedge F^{AB}(A) + \mathrm{cc.}\\
&\approx \frac{\mathrm{i}\hbar}{\ell_{\mathrm{P}}^2}\frac{\beta+\mathrm{i}}{2\beta}\sum_{w:\mathrm{wedges}}\big(\Sigma_{AB}[\tau_w]F^{AB}[w] + \Sigma_{AB}[\tau_w^{-1}]F^{AB}[w^{-1}]\big) + \mathrm{cc.}\\
&\equiv \sum_{w:\mathrm{wedges}} S_w^{\mathrm{top}}.
\end{aligned}
\tag{3.23}
$$

For small curvature we can replace $F^{AB}[w]$ by the holonomy around the loop ∂w bounding the wedge:

$$
h^{AB}[\partial w] = \mathrm{Pexp}\big(-\int_{\partial w} A\big)^{AB} \approx -\epsilon^{AB} + \int_w F^{AB} =: -\epsilon^{AB} + F^{AB}[w]. \tag{3.24}
$$

Within this approximation we will now rewrite everything in terms of our spinorial variables. For the flux the situation is simple. The triangle belongs to a tetrahedron in the boundary of the four-simplex. Boundary variables are part of the original phase space $T^*SL(2,\mathbb{C})$ of complex Ashtekar variables, and we can thus use our spinorial parametrisation (3.8) to write:

$$
\Sigma_{AB}[\tau_w] = \frac{\ell_{\mathrm{P}}^2}{\hbar}\frac{\mathrm{i}\beta}{\beta+\mathrm{i}}\omega_{(A}\pi_{B)}. \tag{3.25}
$$

For the holonomy around the wedge we have to be more careful. The boundary of the wedge consists of two parts, one of which enters the bulk. The first part, connecting the two adjacent tetrahedra $\mathcal{T}$ and $\underset{\sim}{\mathcal{T}}$, lies in the boundary of the four-simplex. The corresponding holonomy $h[\tau_w]$ is again contained in the phase space of $T^*SL(2,\mathbb{C})$, and we can thus take the spinorial parametrisation (3.13) to write the group element. The second part enters the bulk, and we need additional $SL(2,\mathbb{C})$ elements g and $\underset{\sim}{g}$ that give the parallel transport from the center of the four-simplex towards $\mathcal{T}$ and $\underset{\sim}{\mathcal{T}}$ respectively. These additional holonomies are not part of our phase space of complex Ashtekar variables, instead they are Lagrange multipliers, which should become clear once we discuss the Gauß law. Gluing the two holonomies together we find the parallel transport around the boundary of the wedge starting at the tetrahedron $\mathcal{T}$:

$$
h^{AB}[\partial w] = \big(g\underset{\sim}{g}^{-1}\big)^A{}_C h^{CB}[\tau_w] = \big(g\underset{\sim}{g}^{-1}\big)^A{}_C \frac{\underset{\sim}{\omega}^C\pi^B - \underset{\sim}{\pi}^C\omega^B}{\sqrt{\underset{\sim}{\pi}\underset{\sim}{\omega}}\sqrt{\pi\omega}}. \tag{3.26}
$$

We thus find the contribution to the discretised action from a single wedge to be:

$$
S_w^{\mathrm{top}} = -\frac{1}{2}M_w\big(g\underset{\sim}{g}^{-1}\big)^{AB}\big(\omega_A\underset{\sim}{\pi}_B + \pi_A\underset{\sim}{\omega}_B\big) + \mathrm{cc.}, \tag{3.27}
$$

where we have introduced the quantity:

$$
M_w = \frac{1}{2}\Big(\frac{\sqrt{\pi\omega}}{\sqrt{\underset{\sim}{\pi}\underset{\sim}{\omega}}} + \frac{\sqrt{\underset{\sim}{\pi}\underset{\sim}{\omega}}}{\sqrt{\pi\omega}}\Big). \tag{3.28}
$$

This normalisation equates to one once we go to the solution space of the area-matching constraint (3.14), where the action (3.27) turns into a simple bilinear of the spinors. The reality conditions (3.19) also decompose into bilinears of the spinors, and this is the reason why the spinorial parametrisation of loop quantum gravity will be so useful.

3.2.2 The continuum action on a wedge

The action (3.27) just introduced admits a straight forward continuum limit. To show this, we split the wedge w into smaller wedges $w_1, \dots, w_N$, introduce $N-1$ additional spinorial variables $(\omega^{(i)}, \pi^{(i)})$ together with group elements $g^{(i)} \in SL(2,\mathbb{C})$ that represent the parallel transport from the vertex to the i-th discretisation step at the boundary of the spinfoam face. For the i-th wedge the action (3.27) becomes:

$$S^{\text{top}}_{w_i} = -\frac{1}{2} M_{w_i} \big(g^{(i)} (g^{(i+1)})^{-1}\big)^{AB} \big(\omega^{(i)}_A \pi^{(i+1)}_B + \pi^{(i)}_A \omega^{(i+1)}_B\big) + \text{cc.}, \tag{3.29}$$

and we also have the boundary conditions $(\omega^{(1)}, \pi^{(1)}, g^{(1)}) = (\omega, \pi, g)$; $(\omega^{(N+1)}, \pi^{(N+1)}, g^{(N+1)}) = (\underset{\sim}{\omega}, \underset{\sim}{\pi}, \underset{\sim}{g})$. We now take the continuum limit $N \to \infty$. Put $\varepsilon = N^{-1}$, set for all variables $f(\varepsilon(i-1)) := f^{(i)}$, and choose the quantity $t = \varepsilon(i-1)$ as our natural continuous time variable. To obtain the continuum limit we perform an expansion in ε implicitly assuming all quantities $(\pi(t), \omega(t), g(t))$ are differentiable in the parameter time t.

Let us first look at the normalisation M_{w_i}. Putting $E(t) = E^{(i)} = \epsilon^{AB} \pi^{(i)}_A \omega^{(i)}_B$ we have

$$M_{w_i} = \frac{1}{2}\left(\frac{\sqrt{E(t)}}{\sqrt{E(t+\varepsilon)}} + \frac{\sqrt{E(t+\varepsilon)}}{\sqrt{E(t)}}\right) = 1 + \mathcal{O}(\varepsilon^2). \tag{3.30}$$

We see the first nonvanishing order is quadratic. For the holonomies, on the other hand, the expansion contains a linear term:

$$g^{(i+1)} (g^{(i)})^{-1} = \text{Pexp}\Big(-\int_{\varepsilon(i-1)}^{\varepsilon i} \mathrm{d}t\, A_{e(t)}(\partial_t)\Big) = \mathbb{1} - \varepsilon A_{e(t)}(\gamma_t) + \mathcal{O}(\varepsilon^2). \tag{3.31}$$

Here e is the path (the "edge") bounding the spinfoam face, and t is the associated coordinate. Next, we need to study the product of the spinors. We find

$$\begin{aligned}\omega^{(i)}_A \pi^{(i+1)}_B &= \omega_A\big(\varepsilon(i-1)\big)\pi_B(\varepsilon i) = \omega_A(t)\pi_B(t+\varepsilon) = \\ &= \omega_A(t)\pi_B(t) + \varepsilon \omega_A(t) \dot{\pi}_B(t) + \mathcal{O}(\varepsilon^2).\end{aligned} \tag{3.32}$$

Combining (3.32) and (3.31) we get the expansion of the bilinear appearing in the action:

$$\begin{aligned}-\big(g^{(i)}(g^{(i+1)})^{-1}\big)^{AB} \omega^{(i)}_A \pi^{(i+1)}_B &= \epsilon^{AB} \omega_A(t)\pi_B(t) + \varepsilon \epsilon^{AB} \omega_A(t) \dot{\pi}_B(t) + \\ &\quad - \varepsilon A^{AB}_{e(t)}(\dot{e}) \omega_B(t) \pi_A(t) + \mathcal{O}(\varepsilon^2) = \\ &= \omega_A(t)\pi^A(t) + \varepsilon \omega_A(t) D_{\partial_t} \pi^A(t) + \mathcal{O}(\varepsilon^2).\end{aligned} \tag{3.33}$$

The same is true for the second part of (3.29) with ω and π exchanged. Moreover, $\mathcal{D}_{\partial_t} \pi^B = \dot{\pi}^B + A^B{}_C(\partial_t)\pi^C$ denotes the covariant derivative, being the infinitesimal version of the bulk holonomies $g \in SL(2,\mathbb{C})$. Putting all the pieces together, the zeroth order cancels and we find that the Lagrangian starts at linear order in epsilon:

$$S^{\text{top}}_{w_i} = \frac{\varepsilon}{2}\Big(\omega_A(t)\mathcal{D}_{\partial_t}\pi^A(t) + \pi_A(t)\mathcal{D}_{\partial_t}\omega^A(t)\Big) + \mathcal{O}(\varepsilon^2) + \text{cc.} \tag{3.34}$$

Summing the contributions from all infinitesimal wedges $w_1, w_2, \dots, w_N$ and taking the limit $N \to \infty$ we are left with a line integral:

$$S^{\text{top}}_{w} = \frac{1}{2}\int_0^1 \mathrm{d}t \Big(\omega_A(t)\mathcal{D}_{\partial_t}\pi^A(t) + \pi_A(t)\mathcal{D}_{\partial_t}\omega^A(t)\Big) + \text{cc.} \tag{3.35}$$

This action, being nothing but a covariant symplectic potential, generates trivial equations of motion, which just tell us ω and π are parallel along the edge $e \subset \partial w$:

$$\mathcal{D}_{\partial_t}\omega^A = 0 = \mathcal{D}_{\partial_t}\pi^A. \tag{3.36}$$

What is more important, concerns the area-matching condition. On each infinitesimal wedge w_i equation (3.14) becomes:

$$C = \epsilon^{AB}\pi_A^{(i+1)}\omega_B^{(i+1)} - \epsilon^{AB}\pi_A^{(i)}\omega_B^{(i)} = E(t+\varepsilon) - E(t). \tag{3.37}$$

Therefore, when taking the continuum limit in the sense of this section, the area-matching condition turns into the conservation law:

$$\dot{E} = \frac{\mathrm{d}}{\mathrm{d}t}(\pi_A\omega^A) = \mathcal{D}_{\partial_t}(\pi_A\omega^A) \stackrel{!}{=} 0. \tag{3.38}$$

We can now see the area-matching constraint is satisfied just because the equations of motion (3.36) guarantee $\dot{E} = 0$ holds for all times.

3.2.3 The constrained continuum action on an edge

In the following, we will rearrange the sum over wedges to find the contribution to the total action from a single edge e. Each edge carries a unit time-like four-vector n^α representing the internal (future pointing) Minkowski normal of the tetrahedron dual to the edge. We have started from a discrete model, where we know this normal only somewhere in the middle of the edge (say at parameter time $t = t_o = \frac{1}{2}$). Employing local Lorentz invariance we can always put that normal into the canonical gauge, i.e.:

$$n^{A\bar{A}}(t = t_o = \tfrac{1}{2}) = \frac{\mathrm{i}}{\sqrt{2}}\delta^{A\bar{A}} = n_o^{A\bar{A}}. \tag{3.39}$$

But now that we have a continuous action we need this normal all along the edge. We achieve this by using one of the key assumptions of spinfoam gravity, that states the geometry be locally flat. This implies the normal is covariantly constant along the edge. To be more precise, if we take the edge to be parametrised by our coordinate $t \in [0,1]$, we assume

$$\forall t \in (0,1) : \mathcal{D}_{\partial_t} n^\alpha = 0. \tag{3.40}$$

Notice the boundary values 0 and 1 are excluded here, reflecting the fact that (3.40) holds locally but cannot be achieved all around the spinfoam face. In fact a number of edges, say e_1, e_2, $\dots e_N$, bound a spinfoam face, along each of which we can introduce a continuous time variable $t_1 \in (0,1]$, $t_2 \in (1,2]$, $\dots t_N \in (N-1,N)$ along the lines of the last section. Wherever two edges meet, that is on a spinfoam vertex, a discontinuity may arise, measured by the angle Ξ_i:

$$\mathrm{ch}(\Xi_i) = -\lim_{\varepsilon \searrow 0} n_\alpha(i-\varepsilon)n^\alpha(i+\varepsilon). \tag{3.41}$$

Let us now come back to the main issue of this section, the action for an edge. At every edge four triangles τ_I with $I = 1, \dots 4$ meet, for each of which we introduce spinors* $(\omega^{(I)}, \pi^{(I)})$. The topological action on an edge becomes (after having shifted the boundaries to 0 and 1 again):

$$S_e^{\mathrm{top}} = \frac{1}{2}\sum_{I=1}^{4}\int_0^1 \mathrm{d}t\Big(\omega_A^{(I)}\mathcal{D}_{\partial_t}\pi^A_{(I)} + \pi_A^{(I)}\mathcal{D}_{\partial_t}\omega^A_{(I)}\Big) + \mathrm{cc}. \tag{3.42}$$

*In the following we will keep the index (I) only when it is strictly necessary, to further simplify our notation we will also put this index wherever there is "enough" space, e.g. $\pi^A_{(I)} = \epsilon^{AB}\pi_B^{(I)}$.

Before we add the reality conditions (3.19) to this action, let us first discuss the last constraint missing, i.e. the Gauß law. We find it from the variation of the edge action (3.42) with respect to the selfdual connection $A^A{}_B$ contracted with the tangent vector $\partial_t = \dot{e}$ of the edge. The resulting $\mathfrak{sl}(2,\mathbb{C})$-valued Lagrange multiplier

$$\Phi^{AB}(t) := A^{AB}_{e(t)}(\partial_t) \tag{3.43}$$

appears linearly in the covariant derivative $D_{\partial_t}\pi^A = \dot{\pi}^a + \Phi^A{}_B\pi^B$. The variation of the action (3.42) with respect to this Lagrange multiplier leads us to the Gauß constraint

$$G_{AB} = -\frac{1}{2}\sum_{I=1}^{4}\pi^{(I)}_{(A}\omega^{(I)}_{B)} = \sum_{I=1}^{4}\Pi^{(I)}_{AB} = \sum_{I=1}^{4}\Pi^{(I)}_i\tau_{AB}{}^i = 0. \tag{3.44}$$

When introducing real and imaginary parts of $2\Pi_i = L_i + \mathrm{i}K_i$ corresponding to boosts and rotations with respect to the standard normal (3.40) the Gauß law turns into:

$$\sum_{I=1}^{4} L^{(I)}_i = 0 = \sum_{I=1}^{4} K^{(I)}_i. \tag{3.45}$$

If we now remember the reality conditions imply (2.98, 3.22) the combination $K_i + \beta L_i$ vanishes on all triangles, we can see that not all the constraints are independent. If we impose the reality conditions for the three triangles $I = 1, 2, 3$, the Gauß law immediately implies them for the fourth.

We impose the reality conditions on the triangles through additional Lagrange multipliers $z_{(I)} : [0,1] \to \mathbb{C}$ and $\lambda_{(I)} : [0,1] \to \mathbb{R}$ in the action, and thus get for the action on an edge:

$$S_e = \frac{1}{2}\sum_{I=1}^{4}\int_0^1 \mathrm{d}t\Big(\omega^{(I)}_A\mathcal{D}_{\partial_t}\pi^A_{(I)} + \pi^{(I)}_A\mathcal{D}_{\partial_t}\omega^A_{(I)} + \\ - 2z_{(I)}F_n[\pi_{(I)},\omega_{(I)}] - \lambda_{(I)}D[\pi_{(I)},\omega_{(I)}]\Big) + \text{cc.}. \tag{3.46}$$

Once we have an action, we should discuss the equations of motion. This involves two steps. To begin with, in section 3.2.4, we are going to study the constraints and perform the Dirac algorithm [60]. Then we also have to study the evolution equations, and ask for their geometric interpretation, which we are going to do in sections 3.2.5, 3.3 and 3.4.

3.2.4 Dirac analysis of all constraints

Let us turn to whether the equations of motion preserve the constraints. We do this in the Hamiltonian picture. With the Lagrangian (3.46), linear in the time derivatives, we can immediately find the Hamiltonian, which is itself constrained to vanish. If we introduce the *primary Hamiltonian*

$$H'[\pi,\omega](t) = z(t)F_{n(t)}[\pi,\omega] + \frac{\lambda(t)}{2}D[\pi,\omega] + \text{cc.}, \tag{3.47}$$

we can write the evolution equations in the most covariant way possible:

$$\mathcal{D}_{\partial_t}\omega^A = \{H',\omega^A\}, \quad \mathcal{D}_{\partial_t}\pi^A = \{H',\pi^A\}. \tag{3.48}$$

The canonical commutation relations are $\{\pi^A, \omega^B\} = \epsilon^{AB}$, and $\mathcal{D}_{\partial_t}$ is again the covariant $\mathfrak{sl}(2,\mathbb{C})$ derivative $\mathcal{D}_{\partial_t}\pi^A = \partial_t \pi^A + \Phi^A{}_B \pi^B$, with $\Phi^A{}_B$ being the selfdual connection contracted with the tangent vector of the edge—just as defined in (3.43).

To prove that the Hamiltonian vector field preserves the constraints we discuss each of them separately.

(i) stability of the area-matching constraint $\dot{E} = 0$, and of $D = 0$. The area-matching constraint $\dot{E} = 0$ guarantees the area of a triangle is the same seen from all tetrahedra it belongs to. The Hamiltonian vector field of $E = \pi_A \omega^A$ acts as follows:

$$\{E, \pi^A\} = -\pi^A, \quad \{E, \omega^A\} = \omega^A, \quad \{E, \bar{\pi}^{\bar{A}}\} = 0 = \{E, \bar{\omega}^{\bar{A}}\}. \tag{3.49}$$

We thus easily get

$$\dot{E} = \mathcal{D}_{\partial_t} E = \{H', E\} = z(t)F - \bar{z}(t)\bar{F} \propto 0, \tag{3.50}$$

where $\propto$ means equality up to constraints. Since $D = \mathrm{i}E/(\beta + \mathrm{i}) + \mathrm{cc.}$, hence linear in E, equation (3.50) also implies that the reality condition $D = 0$ holds for all times:

$$\dot{D} = \{H', D\} \propto 0. \tag{3.51}$$

Therefore, the Hamiltonian time evolution along a spinfoam edge preserves both the area-matching constraint (3.14, 3.38) and the Lorentz invariant part $D = 0$ of the simplicity constraints $K_i + \beta L_i = 0$.

(ii) stability of $F_n = 0$. Before we explore under which conditions our primary Hamiltonian (3.47) is compatible with the constraint $F_n = 0$, let us first recall* all solutions of the reality conditions $F_n = 0 = D$. They are parametrised by a real number $J \neq 0$, and tell us the momentum π is proportional to $\bar{\omega}$. We find, in fact,

$$\pi_A = -\mathrm{i}\sqrt{2}(\beta + \mathrm{i})J \frac{n_{A\bar{A}}\bar{\omega}^{\bar{A}}}{\|\omega\|_n^2}, \tag{3.52}$$

with the $SU(2)$ norm $\|\omega\|_n^2 = -\mathrm{i}\sqrt{2} n_{A\bar{A}}\omega^A\bar{\omega}^{\bar{A}}$. Notice that we can always assume $J > 0$. We have mentioned, in the lines shortly above equation (3.16), that there is a discrete symmetry simultaneously exchanging all π and ω spinors. Since

$$J = \frac{\pi_A \omega^A}{\beta + \mathrm{i}}, \tag{3.53}$$

a transformation exchanging π and ω, maps J into $-J$, hence $J > 0$ without loss of generality.

The quantity J parametrising the solutions of the reality conditions also has a clean geometrical interpretation. It measures the area $A[\tau]$ of the triangle τ under consideration. A short calculation gives the precise relation:

$$A^2[\tau] = \Sigma^i[\tau]\Sigma_i[\tau] = -2\Sigma^{AB}\Sigma_{AB} = \Big(\frac{\beta \ell_{\mathrm{P}}^2 J}{\hbar}\Big)^2. \tag{3.54}$$

*We will derive these solutions explicitly in chapter 3.5, further details can be found in references [150] and [149].

We are now ready to come back to our original problem, and show how the Hamiltonian can preserve the reality conditions. Since the normal is covariantly constant, we get for the time evolution of $F_n = 0$ the equation:

$$\dot{F}_n = \frac{\mathrm{d}}{\mathrm{d}t}\big(n^{A\bar{A}}\pi_A\bar{\omega}_{\bar{A}}\big) = \mathcal{D}_{\partial_t}\big(n^{A\bar{A}}\pi_A\bar{\omega}_{\bar{A}}\big) = \big\{H', F_n\big\} \propto \bar{z}(t)\big\{\bar{F}_n, F_n\big\}. \tag{3.55}$$

We calculate the missing Poisson bracket in the gauge where $n^\alpha = n^\alpha_o$ and find:

$$\begin{aligned}\big\{\bar{F}_{n_o}, F_{n_o}\big\} &= \frac{1}{2}\delta_{A\bar{A}}\delta_{B\bar{B}}\big\{\bar{\pi}^{\bar{A}}\omega^A, \pi^B\bar{\omega}^{\bar{B}}\big\} = \frac{1}{2}\delta_{A\bar{A}}\delta_{B\bar{B}}\Big[\bar{\epsilon}^{\bar{A}\bar{B}}\omega^A\pi^B + \epsilon^{AB}\bar{\pi}^{\bar{A}}\bar{\omega}^{\bar{B}}\Big] = \\ &= -\frac{1}{2}\big(\pi_A\omega^A - \mathrm{cc.}\big) = -\mathrm{i}J.\end{aligned} \tag{3.56}$$

The result being manifestly $SL(2,\mathbb{C})$ invariant we can conclude that

$$\dot{F}_n \propto -\mathrm{i}\bar{z}(t)J. \tag{3.57}$$

We have assumed the area of the triangle does not vanish, hence $J \neq 0$. This implies the Hamiltonian flow preserves the constraint $F_n = 0$ only if we put the Lagrange multiplier $z(t)$ to zero. Reinserting this restriction on the Lagrange multiplier into the primary Hamiltonian (3.47) we get the secondary Hamiltonian

$$H'' = \lambda(t)D[\pi,\omega]. \tag{3.58}$$

(iii) stability of the Gauß law. The secondary Hamiltonian (3.58) generates the edge-evolution compatible with the simplicity constraints for one pair of spinors. There are, however, four of these pairs per edge—one twistor $Z = (\bar{\pi}_{\bar{A}}, \omega^A)$ for each adjacent triangle. The Gauß law* is an example of an observable depending on all of them. Its time evolution is governed by the physical Hamiltonian, which is the sum over the secondary Hamiltonians (3.58) of all four triangles:

$$H^{\mathrm{phys}} = \sum_{I=1}^{4}\lambda_{(I)}(t)D[\pi_{(I)},\omega_{(I)}]. \tag{3.59}$$

The Hamiltonian has this simple form, just because the action for an edge (3.46) splits into a sum over adjacent triangles, without any "interaction-terms" appearing. Since the Hamiltonian vector field of the constraint $D = 0$ acts as

$$\mathfrak{X}_D[\omega^A] = \{D,\omega^A\} = \frac{\mathrm{i}}{\beta+\mathrm{i}}\omega^A, \quad \mathfrak{X}_D[\pi^A] = \{D,\pi^A\} = -\frac{\mathrm{i}}{\beta+\mathrm{i}}\pi^A, \tag{3.60}$$

we immediately get for any choice of λ, that the Gauß constraint is covariantly preserved:

$$\mathcal{D}_{\partial_t}G_{AB} = \big\{H^{\mathrm{phys}}, G_{AB}\big\} = 0. \tag{3.61}$$

The partial derivative, on the other hand, vanishes weakly:

$$\frac{\mathrm{d}}{\mathrm{d}t}G_{AB} \propto 0. \tag{3.62}$$

which follows from the commutation relations of the Lorentz algebra:

$$\{L_i, L_j\} = -\epsilon_{ij}{}^l L_l, \quad \{L_i, K_j\} = -\epsilon_{ij}{}^l K_l, \quad \{K_i, K_j\} = +\epsilon_{ij}{}^l L_l. \tag{3.63}$$

*The Gauß law follows from the stationarity of the action (3.46) under variations of Φ^{AB} (3.43). Since the time normals depend on Φ^{AB} through (3.40), and linearly appear in the F_n-term of the action (3.46), this adds a term to the Gauß law (3.44) which is linear in the multipliers $z_{(I)}$. We will later prove that all $z_{(I)}$ must vanish, hence $G_{AB} = 0$ as in (3.44).

First- and second-class constraints. We got the constraint equations on an edge by varying the Lagrange multipliers $\Phi \in \mathfrak{sl}(2,\mathbb{C})$, $z_{(I)} \in \mathbb{C}$ and $\lambda_{(I)} \in \mathbb{R}$ in the action (3.46). If we want to quantise the theory we have to compute the constraint algebra and identify first-class and second-class constraints therein. The set of constraints consists of both the rotational and boost part of the Gauß law, together with the simplicity constraints on the triangles. Only some of these constraints are independent: If we impose the simplicity constraints $K_i + \beta L_i = 0$ on three triangles only, the Gauß law $\sum_{I=1}^4 K_i^{(I)} = \sum_{I=1}^4 L_i^{(I)} = 0$ implies them on the fourth. These constraints can be rearranged to treat all triangles equally. We can then impose just the rotational part of the Gauß law, and require the simplicity constraints (3.19) on all four triangles. This rearrangement leads us to the following system of constraints:

$$G_i^{\mathrm{rot}} = \sum_{I=1}^{4} L_i[\pi_{(I)}, \omega_{(I)}] \overset{!}{=} 0, \tag{3.64a}$$

$$D_{(I)} \equiv D[\pi_{(I)}, \omega_{(I)}] \overset{!}{=} 0, \quad \forall I = 1, \dots, 4, \tag{3.64b}$$

$$F_{(I)} \equiv F_{n_o}[\pi_{(I)}, \omega_{(I)}] \overset{!}{=} 0, \quad \forall I = 1, \dots, 4. \tag{3.64c}$$

Notice, that in a general gauge, where the time normal does not assume the canonical form $n^\alpha = n_o^\alpha$, we must boost the constraints into the direction of n^α. We would then work with the constraints

$$G_{i(n)}^{\mathrm{rot}} := \exp\big(\mathfrak{X}_{G_i^{\mathrm{boost}}}\eta^i\big) G_i^{\mathrm{rot}}, \quad \text{and} \quad F_n = \exp\big(\mathfrak{X}_{G_i^{\mathrm{boost}}}\eta^i\big) F_{n_o} \tag{3.65}$$

instead. Here, $\mathfrak{X}_{G_i^{\mathrm{boost}}} = \{G_i^{\mathrm{boost}}, \cdot\}$ denotes the Hamiltonian vector field of the boost part of the Gauß law and the generic normal n has been parametrised as:

$$(n^0, n^i) = \Big(\mathrm{ch}(|\eta|),\ \mathrm{sh}(|\eta|)\frac{\eta^i}{|\eta|}\Big), \quad \text{where:} \quad |\eta| = \sqrt{\delta_{ij}\eta^i\eta^j}. \tag{3.66}$$

All of the constraints (3.64) are preserved by the physical Hamiltonian generating the time evolution along an edge, e.g. $\mathcal{D}_{\partial_t} G_{i(n)}^{\mathrm{rot}} = \{H^{\mathrm{phys}}, G_{i(n)}^{\mathrm{rot}}\} \propto 0$. To identify first- and second-class constraints within this set, we have to study their mutual Poisson brackets. We find:

$$\{G_i^{\mathrm{rot}}, G_j^{\mathrm{rot}}\} = -\epsilon_{ij}{}^{l} G_{l(n)}^{\mathrm{rot}}, \quad \{G_i^{\mathrm{rot}}, D_{(I)}\} = 0, \quad \{G_i^{\mathrm{rot}}, F_{(I)}\} = 0 = \{G_i^{\mathrm{rot}}, \bar{F}_{(I)}\}, \tag{3.67a}$$

$$\{D_{(I)}, F_{(J)}\} = -\frac{2\mathrm{i}\beta}{\beta^2+1}\delta_{IJ} F_{(I)}, \quad \{D_{(I)}, \bar{F}_{(J)}\} = \frac{2\mathrm{i}\beta}{\beta^2+1}\delta_{IJ}\bar{F}_{(I)}, \tag{3.67b}$$

$$\{F_{(I)}, \bar{F}_{(J)}\} = \mathrm{i}\delta_{IJ}\mathfrak{Im}(\pi_A^{(I)}\omega_{(I)}^A) = \mathrm{i}\delta_{IJ}\mathfrak{Im}(E_{(I)}). \tag{3.67c}$$

The set of first-class constraints consists of the rotational component of the Gauß law, attached to each edge, together with the Lorentz invariant simplicity constraint $D = 0$, attached to each triangle. The constraint $F_n = 0$ is second class and generates an additional $\mathfrak{su}(2)$ algebra. This becomes more explicit once we define the ladder operators $J_\pm$ together with the generator J_3:

$$J_- = J_1 - \mathrm{i}J_2 =: -\sqrt{2}\bar{F}_{n_o}, \quad J_+ = J_1 + \mathrm{i}J_2 := -\sqrt{2}F_{n_o}, \quad J_3 := \mathfrak{Im}(E), \tag{3.68}$$

with the Poisson bracket of the rotation group:

$$\{J_i, J_k\} = -\epsilon_{ij}{}^{k} J_k. \tag{3.69}$$

In our case J_- and J_+ are constrained to vanish, while $J_3 \neq 0$, reflects the fact that the constraints form a second-class system. A last comment on the time gauge (3.39): if we want to relax this condition, little will happen, the constraints get boosted as in (3.65), but the structure constants appearing in the constraint algebra (3.67) remain the same.

3.2.5 Solving the equations of motion for the spinors

In this section we will solve the equations of motion for the spinors. For any triangle adjacent to the edge it is the physical Hamiltonian (3.58) that generates the time evolution of the corresponding spinors:

$$\mathcal{D}_{\partial_t}\omega^A = \{H'', \omega^A\}, \quad \mathcal{D}_{\partial_t}\pi^A = \{H'', \pi^A\}. \tag{3.70}$$

We thus get the following equations of motion:

$$\big(\mathcal{D}_{\partial_t}\omega^A\big)(t) = \dot{\omega}^A(t) + \Phi^A{}_B(t)\omega^B(t) = \lambda(t)\big\{D, \omega^A\big\}_t = +\frac{\mathrm{i}}{\beta+\mathrm{i}}\lambda(t)\omega^A(t), \tag{3.71a}$$

$$\big(\mathcal{D}_{\partial_t}\pi^A\big)(t) = \dot{\pi}^A(t) + \Phi^A{}_B(t)\pi^B(t) = \lambda(t)\big\{D, \pi^A\big\}_t = -\frac{\mathrm{i}}{\beta+\mathrm{i}}\lambda(t)\pi^A(t), \tag{3.71b}$$

with $\Phi^A{}_B$ defined in (3.43). We introduce the parallel transport between time t and t' along the edge

$$U(t,t') = \mathrm{Pexp}\big(-\int_t^{t'} \mathrm{d}s\Phi(s)\big) \in SL(2,\mathbb{C}), \tag{3.72}$$

and use it to write down the general solution of the equations of motion. We get

$$\omega^A(t) = \exp\Big[+\frac{\mathrm{i}}{\beta+\mathrm{i}}\int_0^t \mathrm{d}s\lambda(s)\Big]U^A{}_B(0,t)\omega^A(0), \tag{3.73a}$$

$$\pi^A(t) = \exp\Big[-\frac{\mathrm{i}}{\beta+\mathrm{i}}\int_0^t \mathrm{d}s\lambda(s)\Big]U^A{}_B(0,t)\pi^A(0). \tag{3.73b}$$

Closing the loop. In (3.73) we found the solution of the equations of motion for the spinors on an edge. The spinors represent the flux (3.5) through a triangle seen from the frame of the tetrahedron dual to the edge. But the triangle belongs to many tetrahedra, hence many edges. These edges, e_1, e_2, $\dots e_N$, bound the spinfoam face dual to the triangle. Just as we have done above, we can introduce a continuum time variable $t_1 \in (0,1]$, $t_2 \in (1,2]$, $\dots t_N \in (N-1, N)$ for each of these edges, and study the field of spinors $Z = (\bar{\pi}_{\bar{A}}, \omega^A) : [0,N) \ni t \mapsto \bar{\mathbb{C}}^{2*} \otimes \mathbb{C}^2$ along the edge. This field describes the triangle dual to the spinfoam face in the frame of the various edges. To guarantee $Z(t)$ describes, for all $t \in [0,N)$, the same triangle we need boundary conditions:

$$\forall i \in \{1,\dots,N-1\} : \lim_{\varepsilon\searrow 0}\omega^A(i+\varepsilon) = \lim_{\varepsilon\searrow 0}\omega^A(i-\varepsilon), \quad \lim_{\varepsilon\searrow 0}\pi^A(i+\varepsilon) = \lim_{\varepsilon\searrow 0}\pi^A(i-\varepsilon), \tag{3.74a}$$

and also

$$\lim_{\varepsilon\searrow 0}\omega^A(\varepsilon) = \lim_{\varepsilon\searrow 0}\omega^A(N-\varepsilon), \quad \lim_{\varepsilon\searrow 0}\pi^A(\varepsilon) = \lim_{\varepsilon\searrow 0}\pi^A(N-\varepsilon). \tag{3.74b}$$

Using the evolution equations (3.73) we find that the boundary conditions (3.74) turn into a constraint on the holonomy once we close the loop around the spinfoam face.

This happens as follows: We invert (3.73) and solve it for the holonomy $U(0, N)$ in terms of the quadruple of spinors $(Z(0), Z(N))$. Inserting the boundary conditions we get:

$$U^A{}_B(0,N) = \mathrm{Pexp}\big(-\int_0^N \mathrm{d}s\Phi(s)\big)^A{}_B = \\ = (\pi\omega)^{-1}\Big(\mathrm{e}^{-\frac{\mathrm{i}}{\beta+\mathrm{i}}\Lambda}\omega^A(0)\pi_B(0) - \mathrm{e}^{+\frac{\mathrm{i}}{\beta+\mathrm{i}}\Lambda}\pi^A(0)\omega_B(0)\Big), \tag{3.75}$$

where $\pi\omega = \pi_A\omega^A$, and we have introduced the quantity

$$\Lambda = \int_0^N \mathrm{d}t\lambda(t), \tag{3.76}$$

which we can write equally well as

$$\exp\Big(-\frac{2\Lambda}{\beta^2+1}\Big) = \frac{\|U(0,N)\omega(0)\|_n^2}{\|\omega(0)\|_n^2}. \tag{3.77}$$

Equation (3.75) is an interesting result. First of all it tells us that the holonomy around a spinfoam face cannot be a generic $SL(2,\mathbb{C})$ element but preserves the flux through the triangle dual to the spinfoam face, i.e.:

$$U^A{}_C(0,N)U^B{}_D(0,N)\omega^{(C}(0)\pi^{D)}(0) = \omega^{(A}(0)\pi^{B)}(0). \tag{3.78}$$

The same constraint also appears in Regge calculus [64, 167–170], but there is a major difference. In Regge calculus the holonomy (3.75) is further constrained to be a pure boost, here it is neither a boost, nor a rotation, but a four-screw, i.e. a combination of a rotation and a boost in the direction of the rotation axis. This feature reappears in the quantum theory [171–174], and calls for a more careful analysis.

In the next two sections we will further delve in the geometry of the spinfoam face and prove that λ is a measure of both extrinsic and intrinsic curvature.

3.3 EXTRINSIC CURVATURE

We are now going to calculate the extrinsic curvature smeared along a link connecting two adjacent tetrahedra. This will give us a better understanding of the Lagrange multiplier λ appearing in the action (3.46). We will indeed prove that it is a measure of the extrinsic curvature smeared along a link.

Let us consider first two points labelled by coordinates t and t' on the boundary of the spinfoam face. Take the holonomy $h(t, t')$ along the link connecting the two respective tetrahedra sitting at t and t', we use (3.13) and thus have:

$$h^A{}_B(t,t') = \frac{\underset{\sim}{\omega}^A\pi_B - \underset{\sim}{\pi}^A\omega_B}{\sqrt{\pi\omega}\sqrt{\underset{\sim}{\pi}\underset{\sim}{\omega}}}, \tag{3.79}$$

where we have introduced the abbreviation

$$(\underset{\sim}{\omega}, \underset{\sim}{\pi}, \omega, \pi) = (\omega(t'), \pi(t'), \omega(t), \pi(t)). \tag{3.80}$$

At this point let us stress again that links and edges have to be carefully distinguished. Edges enter the bulk of four-simplices, whereas links belong to the three-dimensional boundary of the four-simplex, see figure 3.1.

The two tetrahedra are embedded into the four-dimensional manifold with normals $n = n(t)$ and $\underset{\sim}{n} = n(t')$. The extrinsic curvature smeared over the link between the two is measured by the angle [149]:

$$\begin{aligned}\mathrm{ch}(\Xi(t,t')) &= -\underset{\sim}{n}_{A\bar{A}} h^A{}_B(t,t')\bar{h}^{\bar{A}}{}_{\bar{B}}(t,t') n^{B\bar{B}} = \\ &= -\frac{1}{|\pi\omega|^2}\underset{\sim}{n}_{A\bar{A}}\big(\underset{\sim}{\omega}^A\pi_B - \underset{\sim}{\pi}^A\omega_B\big)\big(\bar{\underset{\sim}{\omega}}^{\bar{A}}\bar{\pi}_{\bar{B}} - \bar{\underset{\sim}{\pi}}^{\bar{A}}\bar{\omega}_{\bar{B}}\big)n^{B\bar{B}} = \\ &= \frac{1}{2}\Big(\frac{\|\omega\|^2_n}{\|\underset{\sim}{\omega}\|^2_{\underset{\sim}{n}}} + \frac{\|\underset{\sim}{\omega}\|^2_n}{\|\omega\|^2_{\underset{\sim}{n}}}\Big).\end{aligned} \tag{3.81}$$

This equation gives the angle up to a sign, we remove the remaining ambiguity, just as in reference [175], by defining

$$\mathrm{e}^{\Xi(t,t')} := \frac{\|\omega(t)\|^2_{n(t)}}{\|\omega(t')\|^2_{n(t')}}. \tag{3.82}$$

There are now two important cases to distinguish. In the first one, t and t' lie on the same edge. The normal is parallel along the edge, hence transported by the holonomy according to

$$n^{A\bar{A}}(t') = U^A{}_B(t,t')\bar{U}^{\bar{A}}{}_{\bar{B}}(t,t')n^{B\bar{B}}(t). \tag{3.83}$$

With this equation the normals cancel from the definition of the angle (3.156), and we find the following.

$$\text{If } t, t' \text{ belong to the same edge:}\quad \mathrm{e}^{\Xi(t,t')} = \frac{\|\omega(t)\|^2_{n(t)}}{\|\omega(t')\|^2_{n(t')}} = \mathrm{e}^{-\frac{2}{\beta^2+1}\int_t^{t'}\mathrm{d}s\lambda(s)}. \tag{3.84}$$

In the second case t and t' belong to neighbouring tetrahedra, with normals $n(t)$ and $n(t')$ to be distinguished. Assume the two tetrahedra meet at the i-th vertex, that is at coordinate value $t = i$. We thus get

$$\mathrm{e}^{\Xi(t,t')} = \mathrm{e}^{-\frac{2}{\beta^2+1}\int_t^{t'}\mathrm{d}s\lambda(s)}\frac{\|U(t,i)^{-1}\omega(i)\|^2_{n(t)}}{\|U(i,t')\omega(i)\|^2_{n(t')}}. \tag{3.85}$$

In the middle of each edge we have chosen time gauge (3.39), hence:

$$\forall i : n^{A\bar{A}}(\tfrac{2i+1}{2}) = \frac{\mathrm{i}}{\sqrt{2}}\delta^{A\bar{A}}. \tag{3.86}$$

We compute the angle between adjacent tetrahedra, as introduced in (3.41), and get:

$$\mathrm{e}^{\Xi_i} := \lim_{\varepsilon\searrow 0}\mathrm{e}^{\Xi(i-\varepsilon,i+\varepsilon)} = \frac{\|U(i-\varepsilon,i)^{-1}\omega(i)\|^2_{n(i-\varepsilon)}}{\|U(i,i+\varepsilon)\omega(i)\|^2_{n(i+\varepsilon)}} = \frac{\|g_{e_i^{\text{target}}}\omega(i)\|^2}{\|g^{\text{source}}_{e_{i+1}}\omega(i)\|^2}, \tag{3.87}$$

with the norm $\|\omega\|^2 = \delta_{A\bar{A}}\omega^A\bar{\omega}^{\bar{A}}$, and the abbreviations:

$$g^{\text{target}}_{e_i} = U(i, \tfrac{2i-1}{2}),\quad g^{\text{source}}_{e_i} = U(i-1, \tfrac{2i-1}{2}). \tag{3.88}$$

These group elements belong to the final and initial point of the edges; $g^{\text{target}}_{e_i}$, for example, is the $SL(2,\mathbb{C})$ holonomy along the i-th half-edge going from the vertex v_i towards the center of the edge at parameter time $t = \frac{2i-1}{2}$. These bulk holonomies play an important role in the asymptotic analysis of the spinfoam amplitude [175,176], which is why we have introduced them here explicitly.

Do the Ξ-angles just considered define proper observables? There are two gauge symmetries to take care off, the $SU(2)$ transformations generated by G_i^{rot} (3.64) (or rather $G_{i(n)}^{\mathrm{rot}}$ for the more general case), and the scaling transformations generated by the Hamiltonian vector field (3.60) of D. Since the $SU(2)$ norm is, by definition, rotational invariant, and the angles are a function of those, they are certainly $SU(2)$ invariant too. But $\Xi(t,t')$ transforms nontrivially under D. We have in fact:

$$\|\omega\|_n^2 \mapsto \exp(\varepsilon\mathfrak{X}_D)\big[\|\omega\|_n^2\big] = \mathrm{e}^{\frac{2}{\beta^2+1}\varepsilon}\|\omega\|_n^2. \tag{3.89}$$

Since ε may locally be an arbitrary continuous function of t, the gauge transformation shifts the integral over the Lagrange multiplier to a new value:

$$\int_t^{t'} \mathrm{d}s\lambda(s) \mapsto \varepsilon(t') - \varepsilon(t) + \int_t^{t'} \mathrm{d}s\lambda(s), \quad \text{thus:} \quad \lambda \mapsto \dot{\varepsilon} + \lambda. \tag{3.90}$$

We see $\Xi(t,t')$ is generally not D-invariant and does not define a proper observable. Nevertheless there is a gauge invariant quantity, that we can build out of λ. The overall angle, as defined in (3.76) is an observable. This is true, simply because we are working with periodic boundary conditions (3.74) that require periodicity $\varepsilon(0) = \varepsilon(N)$ of the gauge parameter.

We can make the gauge invariance of Λ even more obvious. Notice first that any transformation generated by D cannot change the angles (3.41, 3.87) between adjacent tetrahedra. Consider next the boundary conditions (3.74). They imply all angles $\Xi(\frac{2i-1}{2}, \frac{2i+1}{2})$ sum up to zero when going around a spinfoam face. This means

$$1 = \mathrm{e}^{\sum_{i=1}^N \Xi_i}\mathrm{e}^{-\frac{2}{\beta^2+1}\int_0^N \mathrm{d}t\lambda(t)}, \quad \text{thus} \quad \sum_{i=1}^N \Xi_i = \frac{2}{\beta^2+1}\Lambda. \tag{3.91}$$

The last identity gives Λ in terms of the angles Ξ_i between the normals of adjacent tetrahedra. In the next section we prove this quantity is proportional to the curvature tensor smeared over the spinfoam face, revealing a close analogy with Regge calculus. This proportionality will be exact and not an approximation.

3.4 INTRINSIC CURVATURE

The previous sections revealed a Hamiltonian generating the time evolution along a spinfoam edge. We have seen this Hamiltonian preserves the constraint equations—the Gauß law together with the simplicity constraints—once the Lagrange multiplier in front of the second-class constraint $F_n = 0$ vanishes. Both Gauß's law and the simplicity constraints have a well explored physical interpretation, they guarantee all triangles represent spatial planes in internal Minkowski space that close to form a tetrahedron [51,52]. Knowing the geometric interpretation of the constraints, what do the evolution equations tell us? Do they also have a clean physical interpretation? In this section we will explore this questions, and show that the equations of motion for the spinors probe the curvature smeared over a wedge. For this we need some preparations and study first how the holonomy changes under variations of the path.

Be $\gamma_\varepsilon : s \in [0,1] \mapsto \gamma_\varepsilon(s) \in \mathcal{S}$ an ε-parameter family of paths, piecewise differentiable in both ε and s. We can now take two derivatives obtaining the tangent vector $\gamma'_\varepsilon(s) =$

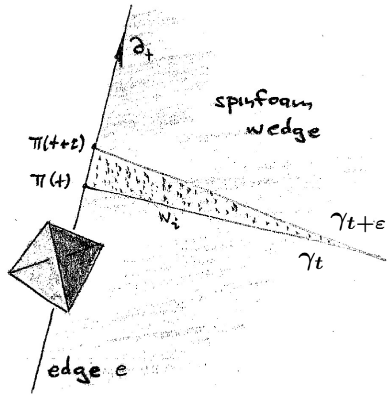

Figure 3.2: Going from t to $t+\varepsilon$ we can probe an infinitesimal wedge, the boundary of which has two parts. The first part belongs to the edge and has a tangent vector ∂_t. The second part (the triangular line in the picture) is a link inside the wedge, itself split into two halfs. Its "upper" part we call $\gamma_{t+\varepsilon}$, while the lower half is γ_t, putting them together determines $\pi(t+\varepsilon)$: The spinor $\pi(t+\varepsilon)$ is the parallel transport of $\pi(t)$ along the connecting link $\gamma_{t+\varepsilon}^{-1}\circ\gamma_t$.

$\frac{\mathrm{d}}{\mathrm{d}s}\gamma_\varepsilon(s)\in T_{\gamma_\varepsilon(s)}\mathcal{S}$ and the variation $\delta\gamma_\varepsilon(s)=\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\gamma_\varepsilon(s)\in T_{\gamma_\varepsilon(s)}\mathcal{S}$. For $\varepsilon=0$ we write, e.g. $\delta\gamma(s):=\delta\gamma_{\varepsilon=0}(s)$. From the defining differential equation of the holonomy, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}s}h_{\gamma_\varepsilon(s)}=-A_{\gamma_\varepsilon(s)}(\gamma_\varepsilon')h_{\gamma_\varepsilon(s)},\tag{3.92}$$

we can get the variation of the parallel transport at $\varepsilon=0$. We just need to differentiate equation (3.92) with respect to ε, multiply everything by $h^{-1}_{\gamma_\varepsilon(s)}$ and integrate the resulting quantity against $\int_0^1\mathrm{d}s$. Performing a partial integration we then get the variation of the holonomy

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}h_{\gamma_\varepsilon(1)}=-A_{\gamma(1)}(\delta\gamma)h_{\gamma(1)}+h_{\gamma(1)}A_{\gamma(0)}(\delta\gamma)+&\\+\int_0^1\mathrm{d}s h_{\gamma(1)}h^{-1}_{\gamma(s)}F_{\gamma(s)}(\gamma',\delta\gamma)h_{\gamma(t)}.&\end{aligned}\tag{3.93}$$

A more detailed proof can be found in the first supplement to this chapter. Let us now see how the equations of motion for the spinors define such a variation. In our original continuum limit, discussed in section 3.1.2, the quadruple $(\pi(t+\varepsilon),\omega(t+\varepsilon),\pi(t),\omega(t))$ probe the $SL(2,\mathbb{C})$ holonomy-flux variables on an infinitesimal wedge w_i. The spinors parametrise the holonomy along the link $\gamma_{t+\varepsilon}^{-1}\circ\gamma_t$ connecting the tetrahedra at t and $t+\varepsilon$ by:

$$\pi^A(t+\varepsilon)=\big(h^{-1}_{\gamma_{t+\varepsilon}(1)}h_{\gamma_t(1)}\big)^A{}_B\pi^B,\quad\omega^A(t+\varepsilon)=\big(h^{-1}_{\gamma_{t+\varepsilon}(1)}h_{\gamma_t(1)}\big)^A{}_B\omega^B.\tag{3.94}$$

This is just equation (3.9) written in terms of the continuous variables on an edge. The underlying path $\gamma_{t+\varepsilon}^{-1}\circ\gamma_t$ is defined as follows: Three lines bound the infinitesimal wedge, the first goes along the edge, from t towards $t+\varepsilon$. The second part is γ_t entering the spinfoam face starting at time t. The last part $\gamma_{t+\varepsilon}^{-1}$ closes the loop; it goes from inside the spinfoam face towards the edge at $t+\varepsilon$. Figure 3.2 gives an illustration of the geometry. Using equation (3.93) we can now take the covariant derivative of (3.94) to find:

$$\mathcal{D}_{\partial_t}\pi^A(t) = F^A{}_B(t)\pi^B(t), \quad \mathcal{D}_{\partial_t}\omega^A(t) = F^A{}_B(t)\omega^B(t), \tag{3.95}$$

where we have introduced the curvature smeared over an infinitesimal wedge. More explicitly

$$F^A{}_B(t) := \int_0^1 \mathrm{d}s \left[h^{-1}_{\gamma_t(s)} F_{\gamma_t(s)}\Big(\frac{\mathrm{d}}{\mathrm{d}s}\gamma_t(s), \frac{\mathrm{d}}{\mathrm{d}t}\gamma_t(s)\Big) h_{\gamma_t(s)}\right]^A{}_B \in \mathfrak{sl}(2,\mathbb{C}). \tag{3.96}$$

Notice that equation (3.95) has the structure of a deviation equation, with the deviation vector replaced by a spinor. If we now compare (3.95) with our equations of motion as they appear in (3.71) we can read off the smeared curvature $F(t)$. What we find is that:

$$F^{AB}(t) = \frac{\mathrm{i}}{\beta+\mathrm{i}}\frac{2\lambda}{\pi\omega}\omega^{(A}(t)\pi^{B)}(t), \tag{3.97}$$

We can now go even further and smear the curvature tensor all along the spinfoam face f. The curvature having two free indices, this has to be done in a certain frame. We can reach this frame by additional holonomies along the family of paths $\{\gamma_t\}_{t\in(0,N)}$. These paths map the spinors attached to the boundary of the spinfoam face towards its center—this is the point where all the wedges come together. Indices referring to the frame at the center of the spinfoam face we denote by $A_o, B_o, \dots$. In this frame, the only t-dependence of the integrand is in $\lambda(t)$, and we can immediately perform the t-integration to arrive at:

$$\int_f F^{A_oB_o} = \frac{1}{2}\int_0^N \mathrm{d}t F^{A_oB_o}(t) = \frac{\Lambda}{\beta+\mathrm{i}}\frac{\Sigma^{A_oB_o}[\tau]}{A[\tau]}. \tag{3.98}$$

The factor of one half appears since every infinitesimal wedge has a triangular shape, $A[\tau]$ is the area (3.54) of the triangle, while Λ denotes the integral of λ along the boundary of the spinfoam face (see (3.76)). With (3.91) we can see this integral is nothing but the sum of the angles between adjacent tetrahedra at all the vertices the triangle belongs to. We could thus say the curvature smeared over a spinfoam face is proportional to the "deficit angle" $\sum_{i=1}^N \Xi_i$ collected when going around a spinfoam face. Although this sounds very much like Regge calculus, there are two subtle differences appearing. First, and most importantly, the curvature smeared over the spinfoam face does not represent a pure boost as in Regge calculus, but instead a four-screw, which is a combination of a rotation and a boost into the direction that the rotation goes around. The relative strength between these two components is measured by the Barbero–Immirzi parameter, which may be an important observation when we ask for the classical role of that parameter. The second difference is more technical. In Regge calculus, curvature is distributional, and concentrated on the triangles of the simplicial decomposition. Here it is not, but continuously spread over all wedges.

3.5 TORSION AND TWISTED GEOMETRIES

3.5.1 The role of torsion for the discretised theory

During the last sections we developed a Hamiltonian formalism of the spinfoam dynamics along an edge. The constraint equations must hold for all times, which leads to restrictions on the Lagrange multipliers in front of the second-class constraints. In fact, the multiplier z imposing $F_n = 0$ just vanishes. However no secondary constraints appear. This should come as a surprise to us. In the continuum, time evolution preserves the reality conditions only if additional secondary constraints hold true (see for instance [99, 100, 158]). Together with the restrictions on the Lagrange multipliers, and the evolution equations for the triad, they force the Lorentz connection to be torsionless, i.e. $\Theta^\alpha = \mathcal{D}\eta^\alpha = 0$.

Let us now ask where the torsionless condition can show up in a discrete theory of gravity. Torsion is a two-form, which suggests to smear it over the "natural" two-dimensional structures appearing. These are the triangles τ, each of which is bounded by three lines forming the "bones" of the spatial triangulation. With the covariant version of Stoke's theorem the integral over the triangle turns into a sum over the bones $b \in \partial\tau$ bounding the surface:

$$\Theta^\alpha := \mathcal{D}\eta^\alpha = 0 \Rightarrow \sum_{b\in\partial\tau} \eta^\alpha[b] = 0. \tag{3.99}$$

Here $\eta^\alpha[b]$ denotes the tetrad smeared over a bone parallel transported into the frame at the center of the triangle τ. Despite its simple looking from, this equation becomes rather awkward when entering loop gravity. Our elementary building blocks are area-angle variables—fluxes $\Sigma[\tau]$ smeared over triangles τ. We do not have the length-angle variables of the tetrad formalism at our disposal, and we are thus unable to probe equation (3.99) directly. The tetrads are in fact complicated functions that require invertability of the fluxes—a highly nontrivial condition in a discrete theory of gravity.

But assuming (3.99) holds true, we can deduce equations more suitable for area-angle variables. Consider first the covariant exterior derivative of the Plebanski two-form $\Sigma_{\alpha\beta} = \eta_\alpha \wedge \eta_\beta$. This is a three-form constrained to vanish due to (3.99). We can integrate this three-form over any tetrahedron, and obtain—again using the non-Abelian Stoke's theorem—Gauß's law:

$$\mathcal{D}(\eta^\alpha \wedge \eta^\beta) = \mathcal{D}\Sigma^{\alpha\beta} = 0 \Rightarrow \sum_{\tau\in\partial\mathcal{T}} \Sigma^{\alpha\beta}[\tau] = 0. \tag{3.100}$$

For any tetrahedron, the sum of the fluxes through the bounding triangles must vanish (with the fluxes parallel transported into the center of the tetrahedron). This is just the Gauß constraint, that we have already found in (3.45), and therefore parts of the torsionless condition are already satisfied. We can play this trick one more time, arriving at yet another torsional constraint.

The vanishing of torsion implies the exterior covariant derivative of the volume three-form $\eta_\mu \wedge \eta_\nu \wedge \eta_\rho$ must vanish. This defines a four-form, the integral of which must vanish for any four-dimensional region. We take a four-simplex surrounding a vertex v; it is bounded by tetrahedra $\mathcal{T} \in \partial v$ equipped with normals $n^\alpha[\mathcal{T}]$ in the frame of the center of the four-simplex. Assume all normals are future oriented, and let $\varepsilon[\mathcal{T}] \in \{-1, 1\}$ be the sign needed for the vector $\varepsilon[\mathcal{T}]n^\alpha[\mathcal{T}]$ to point outwards the four-simplex v. In the discrete theory, the integral of the covariant exterior derivative of the volume three-form turns into a sum over the tetrahedra bounding the integration domain:

$$-\frac{1}{3!}\mathcal{D}(\epsilon_{\alpha\mu\nu\rho}\eta^\mu \wedge \eta^\nu \wedge \eta^\rho) = \mathcal{D}(n_\alpha d^3\mathrm{vol}_n) = 0 \Rightarrow \sum_{\mathcal{T}\in\partial v} \varepsilon[\mathcal{T}] n_\alpha[\mathcal{T}]\,{}^3\mathrm{vol}[\mathcal{T}] = 0. \tag{3.101}$$

Here ${}^3\mathrm{vol}[\mathcal{T}]$ denotes the volume of the tetrahedron $\mathcal{T}$, a quantity that we can write fully in terms of fluxes:

$${}^3\mathrm{vol}[\mathcal{T}] = \frac{\sqrt{2}}{3}\sqrt{\left|\epsilon_{ijk}\Sigma^i[\tau_1]\Sigma^j[\tau_2]\Sigma^k[\tau_3]\right|}. \tag{3.102}$$

Therefore, the *four-dimensional closure constraint* (3.101) fits well into the area-angle calculus of spinfoam gravity. Reference [51] has already discussed this constraint, what is new here, is the torsional interpretation we gave to it.

We have mentioned above that the torsional constraint (3.100) is already satisfied, due to Gauß's law. Its rotational components $\sum_{I=1}^{4} L_i^{(I)} = 0$ appear as first class constraints, while the boost parts $\sum_{I=1}^{4} K_i^{(I)} = 0$ vanish weakly, i.e. they are of second class. The situation is similar for the four-dimensional closure constraint (3.101) just mentioned: Given a solution of *all* the equations of motion (these are the Gauß law, the simplicity constraints together with the evolution equations for the spinors), the four-dimensional closure constraint is automatically fulfilled. In our language, the proof [175] of this statement would prominently employ the evolution equations (3.71) and therefore equation (3.101) holds in the weakest possible way: Only if we solve all the equations of motion we find that the four-dimensional closure constraint is satisfied. In quantum theory, we recover this constraint only at the saddle point, and we may need to impose this condition more strongly.

A naive argument taken from the Euclidean theory supports the idea that the constraint (3.101) holds yet too weakly: If we set the Barbero–Immirzi parameter appearing in the Euclidean spinfoam model [126,127] equal to ± 1 the amplitudes collapse into a topological theory with vanishing left-handed (right-handed) curvature. The classical theory, derived from the Euclidean Holst action admits however non-trivial solutions that are curved, even if the Barbero–Immirzi parameter equals the critical values ± 1. Therefore, at least the Euclidean model misses an additional constraint that may very well be related to the four-dimensional closure constraint introduced in this section.

The torsional equations (3.99), (3.100) and (3.101) have an important geometrical interpretation provided by Minkowski's theorem [154]. The Minkowski theorem holds in any dimension N, irrespective of the metric signature*. It states that given a number of covectors $v^1, \dots v^M$, $M > N$ that close to zero, there exists a unique N-dimensional convex polytope in $\mathbb{R}^N$, bounded by $N-1$-dimensional facets normal to $v^1, \dots v^M$, with their volume given by the magnitude of $v^1, \dots v^M$. The role of the Minkowski theorem for the three-dimensional geometry is well explored, [51,52,55,148,177–180]. The hope is, that the conservation law (3.101) provides the geometry of the spinfoam vertices, just as the Gauß law (3.100) uncovered the geometry at the nodes of the spin network functions. This is a question that lies outside the scope of the present work; a rigorous analysis of the role torsion plays for the geometry of the four-simplex is missing, and we leave this task open for work to come.

3.5.2 The Minkowksi theorem in Minkowski space

In this section we will prove that Minkowski's theorem holds irrespective of the metric signature. To this goal, let us first recall the Minkowski theorem in $\mathbb{R}^4$. We choose Cartesian coordinates (X^0, X^1, X^2, X^3), and introduce the Euclidean metric:

$$ds^2 = \delta_{\mu\nu} \mathrm{d}X^\mu \mathrm{d}X^\nu = (\mathrm{d}X^0)^2 + (\mathrm{d}X^1)^2 + (\mathrm{d}X^2)^2 + (\mathrm{d}X^3)^2. \tag{3.103}$$

*The metric plays actually little role in the Minkowski theorem, a point that has so far been largely ignored to my knowledge. In the next section, we will in fact prove that the Minkowski theorem also holds in Minkowski space.

Consider now a set of N positive numbers $V(1),\dots,V(N)$ and covectors $n_\mu(1),\dots,$ $n_\mu(N)$ normalised to one ($\delta^{\mu\nu}$ is the inverse metric): $\delta^{\mu\nu}n_\mu(i)n_\nu(i)=1$. Suppose that (i) the normals span all of $\mathbb{R}^4$ and (ii) close to zero if we weight them by $V(i)$:

$$\operatorname{span}\big\{n^\mu(i)\big\}_{i=1,\dots,N}=\mathbb{R}^4,\quad\text{and:}\quad\sum_{i=1}^N V(i)n_\mu(i)=0.\tag{3.104}$$

The Minkowski theorem states that (i) and (ii) is both sufficient and necessary to reconstruct a four-dimensional convex polytope $P\subset\mathbb{R}^4$ out of this data. Its boundary ∂P splits into N three-dimensional polytopes T_i, with their respective three-volumina given by $\{V(i)\}_{i=1,\dots,N}$ while their outwardly oriented four-normals are given by $\{n^\mu(i)\}_{i=1,\dots,N}$ (with the Euclidean metric (3.103) and its inverse moving the indices). The resulting polytope is unique up to rigid translations in $\mathbb{R}^4$. We can remove the translational symmetry by demanding that the center of mass lies at the origin:

$$\int_P d^4X\,X^\mu=0,\tag{3.105}$$

where $d^4X=\frac{1}{4!}\epsilon_{\alpha\beta\mu\nu}\mathrm{d}X^\alpha\wedge\dots\wedge\mathrm{d}X^\nu$ and $\epsilon_{0123}=1$. Notice also that we should view the polytope P simply as a pointset $P\subset\mathbb{R}^4$. Its boundary $\partial P=\bigcup_{i=1}^N T_i$ is the union of the three-dimensional polytopes T_i. We call them the facets of P.

We will now show that the metric (3.103) plays little role in the reconstruction of P from the volumes and normals of the bounding facets. In fact the *pseudo-normals*

$$V_\mu(i)=V(i)n_\mu(i),\tag{3.106}$$

together with the four-volume element d^4X are the only ingredients needed to reconstruct the polytope.

To understand why this is true, let us first define the following covectors attached to each bounding polytope:

$$V_\mu[T_i]:=\frac{1}{3!}\int_{T_i}\epsilon_{\mu\nu\rho\sigma}\mathrm{d}X^\nu\wedge\mathrm{d}X^\rho\wedge\mathrm{d}X^\sigma.\tag{3.107}$$

Notice that no metric structure enters the definition of these covectors, the only ingredient is the four-dimensional volume element (which is in one-to-one correspondence with the ϵ-tensor $\epsilon_{0123}=1$). It is immediate to see that any such covector $V_\mu[T_i]$ annihilates the tangent space of T_i: for if Z^μ be tangent to T_i: $V_\mu[T_i]Z^\mu=0$, hence $V_\mu[T_i]\propto n_\mu(i)$. The proportionality is given by the volume, and therefore $V_\mu[T_i]=V(i)n_\mu(i)$. This follows* from the determinant formula:

$$\epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}\epsilon_{\beta_1\beta_2\beta_3\beta_4}=\sum_{\pi\in S_4}\operatorname{sign}(\pi)\delta_{\alpha_1\beta_{\pi(1)}}\delta_{\alpha_2\beta_{\pi(2)}}\delta_{\alpha_3\beta_{\pi(3)}}\delta_{\alpha_4\beta_{\pi(4)}},\tag{3.108}$$

where S_4 is the group of permutations of four elements. In other words, the volume $V(i)$ of T_i determines the magnitude of $V_\mu[T_i]$ according to

$$V_\mu[T_i]=V_\mu(i)=V(i)n_\mu(i),\tag{3.109}$$

*The proof is simple: Multiply $V_\mu[T_i]$ by the normal $n^\mu(i)$ and write the resulting intgeral as $V_\mu[T_i]n^\mu(i)=\int_{T_i}d^3x\sqrt{\big(\epsilon_{\alpha\beta\mu\nu}n^\alpha(i)\frac{\partial X^\beta}{\partial x^1}\frac{\partial X^\mu}{\partial x^2}\frac{\partial X^\nu}{\partial x^3}\big)^2}$, where $\{x^1,x^2,x^3\}$ are positively oriented coordinates in T_i. If we then employ (3.108) we get the volume as the integral of the square root of the determinant of the induced metric on T_i.

where $V(i)$ equals the Euclidean volume of the three-dimensional bounding polytopes. Let us make the dependence of (3.109) on the metric tensor more explicit, we thus write:

$$V(i) = \mathrm{Vol}[T_i, \delta] := \int_{T_i} d^3x \sqrt{\det\big(\delta(\partial_i, \partial_j)\big)} = \sqrt{\delta^{\mu\nu} V_\mu[T_i] V_\nu[T_i]} \tag{3.110}$$

where we have introduced (positively oriented) coordiantes $\{x^1, x^2, x^3\}$ on T_i, and $\delta(\partial_i, \partial_j) = \delta_{\mu\nu} \frac{\partial X^\mu}{\partial x^i} \frac{\partial X^\nu}{\partial x^j}$. We can now also see that $n_\mu(i)$ is indeed the metrical normal vector of T_i:

$$n_\mu(i) = n_\mu[T_i, \delta] = \frac{V_\mu[T_i]}{\mathrm{Vol}[T_i, \delta]}. \tag{3.111}$$

So far, we have done nothing new. Now we should ask the crucial question: How does the reconstruction of the polytope out of normals and volumes depend on the metric tensor—given two metric tensors $\delta_{\alpha\beta}$ and $\tilde{\delta}_{\alpha\beta}$ would we still get the same polytope? The answer is yes provided the two metrics induce the same four-dimensional volume element, i.e. $\det\delta = \det\tilde{\delta} = 1$. This can be seen as follows.

Let us start again with a set of N covectors $V_\mu(i)$ $i = 1, \dots, N$ that span the algebraic <dual of $\mathbb{R}^4$ and close to zero. Using the metric $\delta_{\mu\nu}$ we can separate $V_\mu(i) = V(i) n_\mu(i)$ into its magnitude $V(i)$ and its normal direction $n_\mu(i)$: $\delta^{\mu\nu} n_\mu(i) n_\nu(i) = 1$. We can now use this data to reconstruct a polytope P bounded by facets T_i and centered at the origin.

Suppose now that we would have used another Euclidean metric $\tilde{\delta}_{\mu\nu} = \delta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu$ with $\det\Lambda > 0$ without loss of generality. We would then write $V_\mu(i) = \tilde{V}(i)\tilde{n}_\mu(i)$, and use this splitting (together with the metric $\tilde{\delta}_{\alpha\beta}$) to reconstruct the corresponding polytope $\tilde{P}$ now bounded by three dimensional facets that we call $\tilde{T}_i$. Their normals are $\tilde{n}_\mu(i) : \tilde{\delta}^{\mu\nu}\tilde{n}_\mu(i)\tilde{n}_\nu(i) = 1$, while $\tilde{V}(i)$ gives the volume of $\tilde{T}_i$ as measured by $\tilde{\delta}_{\alpha\beta}$. Again we assume both P and $\tilde{P}$ to be centered at the origin.

Consider now the determinant formula (3.108) which is now modified only by an overall factor proportional to the determinant of Λ:

$$\epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}\epsilon_{\beta_1\beta_2\beta_3\beta_4} = \frac{1}{\det\Lambda^2} \sum_{\pi\in S_4} \mathrm{sign}(\pi) \tilde{\delta}_{\alpha_1\beta_{\pi(1)}} \tilde{\delta}_{\alpha_2\beta_{\pi(2)}} \tilde{\delta}_{\alpha_3\beta_{\pi(3)}} \tilde{\delta}_{\alpha_4\beta_{\pi(4)}}, \tag{3.112}$$

We can thus repeat the argument that has led us to (3.109) in order to find:

$$V_\mu[T_i] = V_\mu(i) = \det\Lambda\, V_\mu[\tilde{T}_i] \tag{3.113}$$

Looking back at the definition of the volume and the normal (i.e. equations (3.110) and (3.111)) we see immediately that:

$$\mathrm{Vol}[T_i, \delta] = \det\Lambda\, \mathrm{Vol}[\tilde{T}_i, \delta], \quad \text{and:} \quad n_\mu[\tilde{T}_i, \delta] = n_\mu[T_i, \delta] \tag{3.114}$$

We have thus found two convex polytopes P and $\tilde{P}$ (both centered at the origin). With respect to the original $\delta_{\alpha\beta}$-metric any two bounding facets T_i and $\tilde{T}_i$ have identical normals, while the volumina coincide only if $\det\Lambda = 1$. In this case the two data are the same, and the uniqueness of the reconstruction guarantees that the two polytopes are the same, hence $P = \tilde{P}$ provided $\det\Lambda = 1$.

Let us make an intermediate summary: Taking N covectors $V_\mu(i)$ that span all of $\mathbb{R}^{4*}$ and close to zero we can reconstruct a unique convex polytope in $\mathbb{R}^4$ centered at the origin. We then introduce an auxiliary metric $\delta_{\alpha\beta}$ with $\det\delta = 1$ to facilitate the Minkowski-reconstruction of the polytope from these covectors. The result of the construction is independent of the metric chosen: We could have picked any other Euclidean metric $\tilde{\delta}_{\alpha\beta}$ that satisfies $\det\tilde{\delta} = 1$ and would have found yet the same polytope.

This tell us a lot for the Minkowski theorem in Minkowski space $(\mathbb{R}^4, \eta_{\alpha\beta})$. Again we look at a set of N covectors $V_\mu(i)$ that span all of $\mathbb{R}^{4*}$ (i.e. the dual of $\mathbb{R}^4$) and close to zero. We can now pick a future oriented time-like normal and construct the Euclidean metric $\delta_{\mu\nu} = 2T_\mu T_\nu + \eta_{\mu\nu}$. The next step is to use this Euclidean metric to run the reconstruction algorithm. We end up with a unique convex* polytope centered at the origin. This polytope is bounded by three-dimensional polytopes T_i. We compute the conormals $V_\mu[T_i]$ (as defined by (3.107)) and find again $V_\mu(i) = V_\mu[T_i]$. The resulting polytope P is independent of the metric chosen: If we chose another future oriented time normal $\tilde{T}_\mu$ we would use another metric $\tilde{\delta}_{\mu\nu} = 2\tilde{T}_\mu\tilde{T}_\nu + \eta_{\mu\nu}$. The two normals are related by a Lorentz transformation: $\Lambda^\alpha{}_\beta \in SO(1,3) : \tilde{T}_\alpha = \Lambda^\beta{}_\alpha T_\beta$, and so are the two metrics: $\tilde{\delta}_{\alpha\beta} = \delta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta$. But $\det\Lambda = 1$ implies $P = \tilde{P}$, the two polyhedra are the same and thus independent of the metric chosen.

We are now only left to understand the metrical interpretation of these conormals. This is again uncovered by the determinant formula: In fact, equation (3.108) also holds in Minkowski space, it just picks a minus sign:

$$\epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}\epsilon_{\beta_1\beta_2\beta_3\beta_4} = -\sum_{\pi\in S_4}\mathrm{sign}(\pi)\eta_{\alpha_1\beta_{\pi(1)}}\eta_{\alpha_2\beta_{\pi(2)}}\eta_{\alpha_3\beta_{\pi(3)}}\eta_{\alpha_4\beta_{\pi(4)}}. \tag{3.115}$$

This equation implies that the magnitude of the pseudo-normals measures the *Lorentzian* three-volume of T_i:

$$\eta^{\mu\nu}V_\mu(i)V_\nu(i) = \varepsilon_i\mathrm{Vol}[T_i,\eta]^2 \tag{3.116}$$

where $\eta^{\mu\nu}$ is the inverse Minkowski metric, $\mathrm{Vol}[T_i,\eta]$ measures the Lorentzian three-volume of T_i, and $\varepsilon_i = \pm 1$ for if T_i is a space-like (time-like) three-surface. In the case of T_i being null $V_\mu(i)$ is a null-vector, hence $\mathrm{Vol}[T_i,\eta] = 0$, and we thus have for all three cases:

$$\mathrm{Vol}[T_i,\eta] = \int_{T_i}\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3\sqrt{|\det\eta(\partial_i,\partial_j)|} = \sqrt{\left|\eta^{\mu\nu}V_\mu[T_i]V_\nu[T_i]\right|}, \tag{3.117}$$

with $\eta(\partial_i,\partial_j) = \eta_{\alpha\beta}\frac{\partial X^\alpha}{\partial x^i}\frac{\partial X^\beta}{\partial x^j}$ and $\{x^1,x^2,x^3\}$.

Let us summarise this result: Given a set of N covectors $V_\mu(i) \in \mathbb{R}^{4*}$ that close to zero and span all of $\mathbb{R}^{4*}$, we can construct a convex polytope $P \subset \mathbb{R}^4$ unique up to rigid translations. This polytope is bounded by N three-dimensional polytopes T_i such that $V_\mu(i) = \frac{1}{3!}\int_{T_i}\epsilon_{\mu\nu\alpha\beta}\mathrm{d}X^\mu\wedge\dots\wedge X^\beta$. These covectors acquire a geometrical interpretation only if we introduce a metric (say a Lorentz metric $\eta_{\alpha\beta}$): In this case the magnitude $\eta^{\mu\nu}V_\mu(i)V_\nu(i)$ measures the metrical volume of T_i while the vector $V^\mu(i) \propto n_\mu(i)$ points into the direction perpendicular to T_i.

3.5.3 The spin connection for twisted geometries

In the first part of this section we studied the role of torsion for discrete geometries. Employing the Minkowski reconstruction theorem (generalised to Minkowski space in

*Note that we do not need a metric to speak about convexity: A set $P \in \mathbb{R}^4$ is said to be convex if for any two points $X^\mu, Y^\mu \in P$ also all elements of the connecting line $tX^\mu + (1-t)Y^\mu$ with $t \in [0,1]$ are points in P.

subsection 3.5.2) we realised torsion guarantees the geometricity of the elementary building blocks: If there is no torsion in a four-simplex, then each bone bounding a triangle has a unique length, and all bones close to form a triangle. Every individual four-simplex is geometric, but if we ask how the elementary tetrahedra glue across neighbouring four-simplices we may find a discontinuity. This is what happens in loop gravity. When looking at a spatial slice, these discontinuities induce a so-called twisted geometry, which were discovered first in the pioneering articles [51, 54, 148]. In this section we compute the torsionless connection for twisted geometries, but first of all let ius explain what we actually mean by a twisted geometry.

A twisted geometry is a generalisation of a three-dimensional Regge geometry. It is an oriented three-dimensional simplicial complex (a triangulation), equipped with a flat Euclidean metric in each tetrahedron, together with the condition that for any two tetrahedra sharing a triangle both metrics agree on the area bivector in between.* The definition of the area bivectors is as follows. In a locally flat region we can find inertial coordinates $(x^1, x^2, x^3) = (x, y, z)$ to write the area bivector of an oriented triangle τ as the surface integral:

$$E_i[\tau] = \frac{1}{2}\int_\tau \epsilon_{ijk}\mathrm{d}x^i \wedge \mathrm{d}x^k. \tag{3.118}$$

Any triangle bounds two tetrahedra, and we thus have two metrics to compute its shape. Three numbers determine the shape of a triangle—for example its area and two angles. Twisted geometries preserve the area, but the angles may change across the triangle. If we only match the areas, we get a twisted geometry, if in addition we also match the angles between any two bones, we further reduce to a Regge geometry.

In loop quantum gravity the semi-classic limit leaves us with a twisted geometry. The fundamental phase-space variables are the holonomies of the Ashtekar–Barbero connection along the links between adjacent tetrahedra, and the area bivectors between.

In the continuum, the underlying connection ${A^{(\beta)i}}_a$ neatly splits into two parts. The Ashtekar connection is, in fact, nothing but the spin connection ${\Gamma^i}_a[e]$ shifted by the extrinsic curvature tensor ${K^i}_a$: ${A^{(\beta)i}}_a = {\Gamma^i}_a + \beta {K^i}_a$, where β is the Barbero-Immirzi parameter. The extrinsic curvature tensor ${K^i}_a$ depends on the embedding of the spatial slice into the space-time manifold. The spin connection, on the other hand, is fully determined by the intrinsic geometry through Cartan's first structure equation, namely the condition of vanishing three-torsion.

As pointed out in [148,149], there is no such clean separation of extrinsic and intrinsic contributions for the discrete theory, because the Cartan equation requires continuity of the triad across the triangle. For this reason, a definition of the spin connection for twisted geometries has been an open task just until recently. The solution was found in [157], which I published together with Hal Haggard, Carlo Rovelli and Francesca Vidotto. In the following I will briefly report on this result, thus providing a definition of ${\Gamma^i}_a[e]$ that remains meaningful for a twisted geometry.

In a twisted geometry there is a discontinuity of the metric across a triangle. There is one flat metric from the "left", and another one from the "right", each of which induce

*This definition is slightly stronger than the one emerging from the classical limit of loop quantum gravity, since it fixes the full triangulation and not just its dual graph. Also, the definition given here refers only to the *intrinsic* geometry. The full definition of the twisted geometry that appears in quantum gravity includes also the *extrinsic* curvature, which plays no role here. Finally, for simplicity we restrict our attention to triangulations, but the results presented extend to generic cellular decompositions (and therefore to polyhedra other than tetrahedra).

the same bivector on the triangle. This bivector $E_i[\tau]$ is nothing but the area of the triangle weighted by its normal:

$$E_i[\tau] = A[\tau] n_i[\tau]. \tag{3.119}$$

The triangle has both a unique area and normal. If the normal is the same from the two sides, the discontinuity of the metric can only be in the induced metric on the plane of the triangle. This is a two-dimensional metric, thus described by three numbers. Three numbers determine a triangle, and the area is one of them. The discontinuity must therefore be in the two remaining degrees of freedom, that describe the shape of the triangle up to an overall scaling.

The triangle looks different from the two sides, yet its area is the same. Given two triangles that have the same area there is a linear change of coordinates that map one to the other. If we embed the two triangles into $\mathbb{R}^2$ this transformation must be an $SL(2, \mathbb{R})$ element. We can thus use an element $e \in SL(2, \mathbb{R})$ to parametrise the discontinuity of the geometry at the plane of the triangle.

Let us now choose a coordinate system $\{x, y, z\}$ covering the two tetrahedra and align it to the triangle: the triangle should rest at $z = 0$, while ∂_z should be its normal vector. The geometry is locally flat, and we can thus always choose this coordinate system such that it is inertial in the "left" tetrahedron. This means that $e^i = \mathrm{d}x^i$ is a cotriad on the left hand side tetrahedron (i.e. for $z < 0$). There is no discontinuity in the normal direction, and we can thus always find a triad on the right hand side that has the constant form

$$e^1 = e^1{}_x \, \mathrm{d}x + e^1{}_y \, \mathrm{d}y, \quad e^2 = e^2{}_x \, \mathrm{d}x + e^2{}_y \, \mathrm{d}y, \quad e^3 = \mathrm{d}z. \tag{3.120}$$

The area is preserved across the triangle, and we thus have the condition $\det e = 1$, where e is the matrix

$$e = \begin{pmatrix} e^1{}_x & e^1{}_y & 0 \\ e^2{}_x & e^2{}_y & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.121}$$

This is an $SL(3, \mathbb{R})$ element, or, more specifically it is in the $SL(2, \mathbb{R})$ upper block diagonal subgroup of $SL(3, \mathbb{R})$.

The geometrical interpretation of these groups is straightforward: e is the linear transformation that sends a triangle with the dimensions given by the left metric into the triangle with the dimensions given by the right. In other words, e is the linear transformation that makes the two triangles match. Since the triangle is two dimensional, we can always restrict this linear transformation to an element of $SL(2, \mathbb{R})$.

Once we have a triad e^i, we can look at Cartan's first structure equation:

$$\mathrm{d}e^i + \epsilon^i{}_{jk} \, \omega^j \wedge e^k = 0. \tag{3.122}$$

For a given triad, there is a unique solution for $\omega^i{}_a = \Gamma^i{}_a[e]$, which in turn defines the torsionless spin connection. For a twisted geometry there is a discontinuity in the cotriad, and Cartan's first structure equation does not make sense any longer. To define the spin connection for a twisted geometry we need a regularisation. We therefore introduce a smeared cotriad which is now continuous all across the triangle, but depends on a regulator Δ. The limit of $\Delta \to 0$ brings us back to a discontinuous triad, and defines, through Cartan's first structure equation, the torsionless spin connection for a twisted connection.

For this purpose, let us first look at the region $\mathbb{R}^2 \times [0, \Delta]$ around the triangle. We are now searching for a continuous cotriad $e(z)$ such that $e(0) = \mathbb{1}$ is the cotriad in the left triangle, while $e(\Delta) = e$ gives the cotriad (3.121) in the right triangle. In the limit of $\Delta \searrow 0$ this region shrinks to the plane of the triangle, and there appears a discontinuity in the metric. Once we have a continuous triad, this defines the spin connection, and we can compute the parallel transport $U(e) \in SO(3)$ across the triangle. The limit $\Delta \searrow 0$ then defines the holonomy across the triangle for a twisted geometry. The only missing ingredient is to choose the actual function $e(z)$ interpolating between the two sides of the triangle. This function cannot be arbitrary, for there is a highly nontrivial condition: The resulting parallel transport must transform homogeneously once we rotate the frames at either side of the triangle. In other words:

$$U(R_s e R_t^{-1}) = R_s U(e) R_t^{-1} \tag{3.123}$$

for any $R_s, R_t \in SO(3)$. Looking at the polar decomposition of e

$$e = \exp(A)\exp(S) \tag{3.124}$$

where A is antisymmetric and S is symmetric, we can find an$<$ interpolating triad

$$e(z) = \exp(zA)\exp(zS), \tag{3.125}$$

that satisfies equation (3.123) for all $R_s, R_t \in SO(3)$. This defines a *continuous* triad joining the two tetrahedra, differentiable* in $(0, \Delta)$. We can now compute the spin connection and take the limit $\Delta \searrow 0$. This defines a torsionless spin connection on the twisted geometry.

We will now compute this connection explicitly. From the last equation, we have

$$\mathrm{d}e^i = \big(A + \exp(zA)S\exp(-zA)\big)^i{}_j \,\mathrm{d}z \wedge e^j. \tag{3.126}$$

Inserting this into the Cartan equation (and lowering an index) we have

$$\big(A + \exp(zA)S\exp(-zA)\big)_{ij}\,\mathrm{d}z \wedge e^j = -\epsilon_{ijk}\ \omega^j \wedge e^k. \tag{3.127}$$

The solution of this equation is given by

$$\omega^i = \omega^i{}_j\ e^j, \tag{3.128}$$

where

$$\omega^i{}_j = -\epsilon^{ikl}\big(A + \exp(zA)S\exp(-zA)\big)_{jk} e_l{}^z + \frac{1}{2}\epsilon^{klm} A_{kl} e_m{}^z \delta^i_j, \tag{3.129}$$

and $e_i{}^z$ for $i = 1, 2, 3$ are matrix elements of the triad.

What is relevant for us here is only the holonomy of the connection along the transversal direction. Consider a path γ crossing the region at constant x and y. The holonomy of $\omega = \omega^i{}_j e^j \otimes \tau_i$ along this path is given by

$$U \;=\; \mathrm{Pexp}\Big(-\int_\gamma \omega\Big) = \mathrm{Pexp}\Big(-\int_0^\Delta \omega(\partial_z)\mathrm{d}z\Big). \tag{3.130}$$

*This triad has a discontinuity in its first derivatives at $z \in \{0, \Delta\}$, since for $z < 0$ ($z > \Delta$) $e(z)$ assumes the constant form $e(z) = \mathbb{1}$ ($e(z) = e$). This discontinuity is, however, of little physical importance, since we can always smooth it out by a chance of coordinates $z \to \tilde{z}(z)$. Our resulting holonomy has, however, a coordinate invariant definition, and does therefore not depend on this choice.

Observe now that $\omega^i{}_j$ is block-diagonal, and its entries $\omega^i{}_j e^j{}_z = \omega^i(\partial_z)$ in the third column are determined only by the last term in (3.129). Therefore,

$$\omega^k(\partial_z) = \frac{1}{2}\epsilon^{kij} A_{ij} \tag{3.131}$$

So that

$$U = \exp A \tag{3.132}$$

that is, the holonomy is precisely the orthogonal matrix in the polar decomposition of e. For the explicit form of the polar decomposition, we have then that

$$U(e) = e(e^T e)^{-1/2}, \tag{3.133}$$

where e^T is the transpose of e. Since $U(e)$ is independent of the size of the interpolating region, taking the limit $\Delta \to 0$ is immediate. The resulting distributional torsionless spin connection is concentrated on the face $\tau : (\sigma^1, \sigma^2) \mapsto x^a(\sigma)$ and is given by

$$\Gamma^i{}_{ja} = -A^i{}_j \, d\tau_a = -\vartheta \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^i{}_j d\tau_a. \tag{3.134}$$

where ϑ is an angle, and the distributional one-form of the triangle is defined by

$$d\tau_a(x) \equiv \int_\tau d^2\sigma \, \frac{\partial x^b}{\partial \sigma^1} \frac{\partial x^c}{\partial \sigma^2} \, \underset{\sim}{\eta}_{abc} \, \tilde{\delta}(x - x(\sigma)), \tag{3.135}$$

where $\{\sigma^1, \sigma^2\}$ are coordinates in the plane of the triangle, and $\underset{\sim}{\eta}_{abc}$ is the inverse Levi-Civita density. From this expression it is easy to verify that (3.123) is satisfied.

In this section we defined the torsionless spin connection for a twisted geometry, and computed the corresponding holonomy along the link dual to the triangle. In our inertial coordinate system, the resulting parallel transport is a function of a single angle ϑ, as defined in (3.134). This angle is one of the three numbers parametrising the twist in the metric across the triangle. We can write it fully in terms of the bivectors of the two adjacent tetrahedra. The original paper, reference [157] contains the explicit expression. This angle is therefore a function solely of the intrinsic discrete data on the spatial hypersurface. The same happens in the continuum theory, where the triad fully determines the torsionless spin connection.

This result should clarify some confusion in the literature. It has often been argued (see for instance [163, 164]) that the distributional nature of twisted geometries hints at the presence of torsion. This section proved this intuition wrong. A metric by itself does not define torsion. Torsion is a property of a metric-compatible connection, and thus requires a connection on top of a metric. There is only one connection which is both metric compatible and torsionless. This is the spin connection, and we saw that the torsionless condition defines a metric compatible connection also for the case of twisted geometries.

3.6 FROM SPINORS DOWN TO ASHTEKAR–BARBERO VARIABLES

How can we know that our spinorial framework, which we have advocated in the first part of this chapter, is actually compatibel with loop gravity in terms of $SU(2)$ Ashtekar–Barbero variables? To explore this issue, we will have to study the reality conditions (2.97), write them in terms of the spinorial representation, and solve them. The result will reduce twistors down to $SU(2)$ spinors, with the emergence of the $SU(2)$ holonomy of the β-dependent Ashtekar–Barbero connection. The following chapter is based on what I have published together with Simone Speziale in [149].

3.6.1 Solving the linear simplicity constraints

We start with the selfdual fluxes (2.96) and discretise the reality condition (2.97) in terms of both the source and target variables of each link. This yields

$$\forall\tau: \quad \begin{cases} \dfrac{1}{\beta+\mathrm{i}}\Pi_i[\tau] + \mathrm{cc.} = 0 \Leftrightarrow \Pi_i[\tau] = -\mathrm{e}^{\mathrm{i}\vartheta}\bar{\Pi}_i[\tau], \\ \dfrac{1}{\beta+\mathrm{i}}\underset{\sim}{\Pi}_i[\tau] + \mathrm{cc.} = 0 \Leftrightarrow \underset{\sim}{\Pi}_i[\tau] = -\mathrm{e}^{\mathrm{i}\vartheta}\underset{\sim}{\bar{\Pi}}_i[\tau], \end{cases} \tag{3.136}$$

where we have introduced the angle:

$$\mathrm{e}^{\mathrm{i}\vartheta} = \frac{\beta+\mathrm{i}}{\beta-\mathrm{i}}, \quad \vartheta = \cot\tfrac{\vartheta}{2}. \tag{3.137}$$

Defining the $\mathfrak{sl}(2,\mathbb{C})$ Lie algebra elements $\Pi[\tau] = \Pi_i[\tau]\tau^i$, where $\{2\mathrm{i}\tau_i = \sigma_i\}_{i=1,2,3}$ are the three Pauli matrices, we can write this as:

$$\Pi[\tau] = \mathrm{e}^{i\vartheta}\Pi^\dagger[\tau], \qquad \underset{\sim}{\Pi}[\tau] = \mathrm{e}^{i\vartheta}\underset{\sim}{\Pi}^\dagger[\tau], \tag{3.138}$$

with the Hermitian conjugate taken according to

$$(\Pi^\dagger)^A{}_B = \delta^{A\bar{A}}\delta_{B\bar{B}}\bar{\Pi}^{\bar{B}}{}_{\bar{A}}. \tag{3.139}$$

In the spinorial parametrisation, the first equation in (3.136) reads

$$-2\Pi_{AB} = \omega_{(A}\pi_{B)} = -\mathrm{e}^{\mathrm{i}\vartheta}\delta_{A\bar{A}}\delta_{B\bar{B}}\bar{\omega}^{(\bar{A}}\bar{\pi}^{\bar{B})}. \tag{3.140}$$

It apparently gives two equivalent decompositions of Π_{AB} in terms of spinors and their complex conjugate. But the decomposition of a symmetric bispinor is unique [29] up to exchange and complex rescaling of the constituents, therefore $\bar{\pi}_{\bar{A}}$ and ω^A must be linearly related.* Furthermore, part of the complex rescaling is fixed by the phase appearing explicitly in (3.140), leaving only the freedom to real rescalings. Hence, we can parametrise the solutions as

$$\pi_A = r\mathrm{e}^{\mathrm{i}\frac{\vartheta}{2}}\delta_{A\bar{A}}\bar{\omega}^{\bar{A}}, \qquad \omega_A = -\frac{1}{r}\mathrm{e}^{\mathrm{i}\frac{\vartheta}{2}}\delta_{A\bar{A}}\bar{\pi}^{\bar{A}}, \qquad r \in \mathbb{R} - \{0\}. \tag{3.141}$$

The matching of left and right geometries as implied by (3.136) immediately translates into the left and right spinors being proportional. The same conclusion holds in a general gauge, with a generic normal replacing the identity matrix, as in (3.20). Remarkably, the simplicity equations then take up the same form as Penrose's incidence relation. It would be intriguing to explore the existence of a deeper connection between these two notions. That simplicity implies proportionality of the spinors is a key result, and was also derived in [181]. It means that a *simple* twistor, i.e. a twistor satisfying the simplicity constraints, is determined by a single spinor, plus a real number, the meaning of which will become clear below.

By contractions with ω and π, equation (3.140) can be conveniently separed in two parts,

$$D = \frac{\mathrm{i}}{\beta+\mathrm{i}}\omega^A\pi_A + \mathrm{cc.} = 0, \qquad F = \frac{\mathrm{i}}{\sqrt{2}}\delta^{A\bar{A}}\pi_A\bar{\omega}_{\bar{A}} = n^{A\bar{A}}\pi_A\bar{\omega}_{\bar{A}} = 0. \tag{3.142}$$

*The other possibility, a linear relation between ω^A and $\bar{\omega}^{\bar{A}}$ (and π_A) and $\bar{\pi}_{\bar{A}}$ would only yield degenerate solutions, for, if $\omega_A = z\delta_{A\bar{A}}\bar{\omega}^{\bar{A}}$ with $z \in \mathbb{C}$, we also have $0 = \omega_A\omega^A = z\|\omega\|^2$, thus $\omega^A = 0$.

Here, D is real and Lorentz-invariant, while F is complex but only $SU(2)$ invariant. Following the literature, we will refer to D as the *diagonal* simplicity constraint, and F as *off-diagonal*. The simplicity constraints D, F, along with $\underset{\sim}{D}$, $\underset{\sim}{F}$ for the tilded spinors, and the area matching condition C (3.14), form a system of constraints on the link space $\mathbb{T} \times \mathbb{T} = \mathbb{T}^2 \cong \mathbb{C}^8$. The algebra can be easily checked to give

$$\begin{aligned} &\{D, F\} = -\frac{2\mathrm{i}\beta}{\beta^2+1}F, \quad \{F, \bar{F}\} = \mathrm{i}\,\mathfrak{Im}(\pi\omega), \\ &\{C, D\} = 0, \quad \{C, F\} = -F = -\{\bar{C}, F\}, \end{aligned} \tag{3.143}$$

and the same for tilded quantities.

The system lacks the discretised version of the torsionless condition (2.152) that may reappear once we also take the dynamics into account. We will come back to this point below, because it should play an important role in the identification of the extrinsic curvature. For the moment we do not have any secondary constraints, and study only the system of simplicity constraints by itself. Looking at (3.143) we conclude that the diagonal simplicity constraints D and $\underset{\sim}{D}$ are of first class, as well as C, whereas F and $\underset{\sim}{F}$ are second class. That some constraints are second class even in the absence of secondary constraints is a well-known consequence of the non-commutativity (3.6) of the fluxes.

The first class constraints generate orbits inside the constraint hypersurface. The orbits of C are given* in (3.16), whereas those generated by the diagonal simplicity constraints are found from

$$\{D, \omega^A\} = \frac{\mathrm{i}}{\beta+\mathrm{i}}\omega^A, \qquad \{D, \pi_A\} = -\frac{\mathrm{i}}{\beta+\mathrm{i}}\pi_A. \tag{3.144}$$

We also remark that the system is *reducible*, since only three of the four constraints D, $\underset{\sim}{D}$ $\mathfrak{Re}(C)$ and $\mathfrak{Im}(C)$ are linearly independent. We thus have three independent first class constraints, and two, complex, second class constraints. The reduced phase space has $16 - 3 \times 2 - 2 - 2 = 6$ real dimensions, and we will now prove it to be $T^*SU(2)$. To that end, it is convenient to treat separately the area matching and the simplicity constraints, the order being irrelevant. There are two convenient choices of independent constraints, depending on the order in which one solves them. If solving the set $S = \{D, \underset{\sim}{D}, F, \underset{\sim}{F}\}$ of simplicity constraints first, we use the constraints

$$C_{\mathrm{red}} = \frac{C}{\beta+\mathrm{i}} + \mathrm{cc.}, \quad \underbrace{D, \quad \underset{\sim}{D}, \quad F, \quad \underset{\sim}{F}}_{S}. \tag{3.145}$$

If instead we solve C first, we can take

$$\mathfrak{Re}(C), \quad \mathfrak{Im}(C), \quad \underbrace{B := D + \underset{\sim}{D}, \quad F, \quad \underset{\sim}{F}}_{S_{\mathrm{red}}}. \tag{3.146}$$

The situation is summarised in figure 3.3.

Let us proceed solving the simplicity constraints first. For the untilded quantities, (3.141) solves all four S constraints, however the expression is not D-gauge-invariant. For each half-link, gauge-invariant quantities live on the reduced space $\mathbb{T}/\!\!/S \simeq \mathbb{C}^2$,

*If $\mathfrak{X}_C = \{C, \cdot\}$ denotes the Hamiltonian vector field, we have $\exp(z\mathfrak{X}_C)[\omega^A] = \mathrm{e}^z\omega^A$ for $z \in \mathbb{C}$.

$$
\begin{tikzcd}
\mathbb{T}^2 \arrow[rr, "S"] \arrow[d, "C"] & & \mathbb{C}^2 \times \mathbb{C}^2 \arrow[d, "C_{\text{red}}"] \\
T^*SL(2,\mathbb{C}) \arrow[rr, "S_{\text{red}}"] & & T^*SU(2)
\end{tikzcd}
$$

Figure 3.3: Primary constraint structures between twistor and holonomy-flux spaces. S and C schematically denote the simplicity and area matching constraints, and arrows include division by gauge orbits, when relevant.

and are parametrised by a single spinor, say $z^A \in \mathbb{C}^2$. Since the simplicity constraints introduce a Hermitian metric, we have a norm $\|\omega\|^2 = \delta_{A\bar{A}}\omega^A\bar{\omega}^{\bar{A}}$, and use it to define

$$J = \frac{\|\omega\|^2}{\sqrt{1+\beta^2}}\, r, \tag{3.147}$$

which satisfies $\{D, J\} = 0$. In terms of J, equation (3.141) gives

$$\pi_A = (\beta + \mathrm{i})\frac{J}{\|\omega\|^2}\delta_{A\bar{A}}\bar{\omega}^{\bar{A}}, \qquad \pi\omega = (\beta + \mathrm{i})J. \tag{3.148}$$

Then, the reduced spinor, which is now D-gauge-invariant, can be taken to be

$$z^A = \sqrt{2J}\frac{\omega^A}{\|\omega\|^{\mathrm{i}\beta+1}}, \qquad \|z\| = \sqrt{2J}. \tag{3.149}$$

Since we are assuming $\pi_A\omega^A \neq 0$, this implies $J \neq 0$. We can further always assume $J > 0$: In the case $J < 0$, we can flip the sign by simultaneously exchanging π with ω and $\underset{\sim}{\pi}$ with $\underset{\sim}{\omega}$. The transformation

$$(\pi_A, \omega^A; \underset{\sim}{\pi}_A, \underset{\sim}{\omega}^A) \to (\omega_A, \pi^A; \underset{\sim}{\omega}_A, \underset{\sim}{\pi}^A) \tag{3.150}$$

is, in fact, a symmetry of our spinorial parametrisation (3.8, 3.10) and (3.13). Hence, selecting the sign of J removes this additional $\mathbb{Z}_2$ symmetry of the reduction.

The same results apply to the tilded quantities. The reduced space $\mathbb{T}^2/\!\!/S \simeq \mathbb{C}^2 \times \mathbb{C}^2$ is parametrised by the following spinors,

$$z^A = \sqrt{2J}\frac{\omega^A}{\|\omega\|^{\mathrm{i}\beta+1}}, \qquad \underset{\sim}{z}^A = \sqrt{2\underset{\sim}{J}}\frac{\underset{\sim}{\omega}^A}{\|\underset{\sim}{\omega}\|^{\mathrm{i}\beta+1}}. \tag{3.151}$$

Notice that they transform linearly under rotations, but not under boosts: they are $SU(2)$ spinors, the Lorentzian structures being partially eliminated by the gauge-choice needed to define the linear simplicity constraints.

To get the Dirac brackets for the reduced $SU(2)$ spinors, we introduce the embedding em_S of the $D = \underset{\sim}{D} = F = \underset{\sim}{F} = 0$ constraint hypersurface into the original twistorial phase space, and compute the pullback of the symplectic potential. This gives

$$
\begin{aligned}
\mathrm{em}_S^*\Theta &= \mathrm{em}_S^*\big(\pi_A \mathrm{d}\omega^A - \underset{\sim}{\pi}_A \mathrm{d}\underset{\sim}{\omega}^A + \mathrm{cc.}\big) \\
&= \mathrm{em}_S^*\Bigg[\beta(J + \underset{\sim}{J})\,\mathrm{d}\,\ln\left(\frac{\|\omega\|}{\|\tilde{\omega}\|}\right) + \beta(J - \underset{\sim}{J})\,\mathrm{d}\,\ln(\|\omega\|\|\underset{\sim}{\omega}\|) + \\
&\qquad + \left(\mathrm{i}\frac{J}{\|\omega\|^2}\delta_{A\bar{A}}\bar{\omega}^{\bar{A}}\mathrm{d}\omega^A - \mathrm{i}\frac{\underset{\sim}{J}}{\|\underset{\sim}{\omega}\|^2}\delta_{A\bar{A}}\bar{\underset{\sim}{\omega}}^{\bar{A}}\mathrm{d}\underset{\sim}{\omega}^A + \mathrm{cc.}\right)\Bigg] \\
&= \frac{\mathrm{i}}{2}\delta_{A\bar{A}}\big(\bar{z}^{\bar{A}}\mathrm{d}z^A - \bar{\underset{\sim}{z}}^{\bar{A}}\mathrm{d}\underset{\sim}{z}^A - \mathrm{cc.}\big).
\end{aligned} \tag{3.152}
$$

The induced Dirac brackets are the canonical brackets of four harmonic oscillators,

$$\{\bar{z}^{\bar{A}}, z^{A}\}_{\mathrm{D}} = -\mathrm{i}\delta^{\bar{A}A} = -\{\underset{\sim}{\bar{z}}^{\bar{A}}, \underset{\sim}{z}^{A}\}_{\mathrm{D}}. \tag{3.153}$$

This reduction is illustrated in the top horizontal line of figure 3.3. The next step is to implement the area-matching condition. As anticipated, part of $C = 0$ is automatically satisfied on the surface of $D = \underset{\sim}{D} = 0$. Using (3.151), the independent part C_{red} can be seen to give the real-valued $SU(2)$ version of the area-matching condition introduced in [148], that is

$$C_{\mathrm{red}} = \|z\|^2 - \|\underset{\sim}{z}\|^2 = 0. \tag{3.154}$$

The gauge orbits generated by C_{red} are $U(1)$ phase transformations $z^A \mapsto \mathrm{e}^{\mathrm{i}\varphi} z^A$, for some angle φ. As proven in [148], canonical variables on the reduced phase space $(\mathbb{C}^2 \times \mathbb{C}^2)/\!\!/C_{\mathrm{red}}$ are $SU(2)$ holonomies and fluxes, satisfying their canonical Poisson algebra. We are thus left with the phase space $T^*SU(2)$, with its elements $(U, \Sigma) \in SU(2) \times \mathfrak{su}(2)$ parametrised as in reference [148], according to.*

$$U^A{}_B(z, \underset{\sim}{z}) = \frac{\underset{\sim}{z}^A \delta_{B\bar{B}} \bar{z}^{\bar{B}} + \delta^{A\bar{A}} \underset{\sim}{\bar{z}}_{\bar{A}} z_B}{\|z\| \|\underset{\sim}{z}\|}, \qquad \Sigma_{AB}(z, \underset{\sim}{z}) = \frac{\beta \ell_{\mathrm{P}}^2}{\hbar} \frac{\mathrm{i}}{2} z_{(A} \delta_{B)\bar{B}} \bar{z}^{\bar{B}}. \tag{3.155}$$

This proves that the symplectic reduction of $(\bar{\pi}_A, \omega^A; \underset{\sim}{\bar{\pi}}_{\bar{A}}, \underset{\sim}{\omega}^A) \in \mathbb{T}^2$ by the area-matching and simplicity constraints gives $T^*SU(2)$.

Let us conclude this section with two important remarks. The first is the identification of an Abelian pair of canonically conjugated variables on $T^*SL(2, \mathbb{C})$. We introduce the quantity

$$\Xi := 2 \ln \left(\frac{\|\omega\|}{\|\underset{\sim}{\omega}\|} \right). \tag{3.156}$$

On the constraint hypersurface $C = 0$ we then find that

$$\{\mathfrak{Re}(\pi_A \omega^A), \Xi\} = 1. \tag{3.157}$$

Also, from (3.148) we know that:

$$\beta J = \mathfrak{Re}(\pi_A \omega^A), \quad \text{thus:} \quad \{J, \Xi\} = \beta^{-1}, \tag{3.158}$$

where we have implicitly extended J to a function on all of phase-space $\mathbb{T}^2$. This conjugated pair corresponds to the (oriented) area and the rapidity associated with the dual face τ. In fact, from (3.3), the squared area equals

$$A^2[\tau] := \delta^{ij} \Sigma_i[\tau] \Sigma_j[\tau] = \frac{\ell_{\mathrm{P}}^4 \beta^2}{\hbar^2} J^2. \tag{3.159}$$

As for the rapidity, it is defined by the scalar product between the time-like normals of the two three-cells sharing the face, that is n and $\underset{\sim}{n}$. These are both related to the identity matrix by the time gauge (3.20). The non-trivial information is then carried

*With respect to the literature [53,182], we have added the dimensional coefficients of the physical flux induced from the action. Also, the holonomy appearing here does not flip the spinors along the link, consistently with the definition of h. The alternative choice is to swap $\underset{\sim}{\omega}$ and $\underset{\sim}{\pi}$ in (3.9), which also affects the canonical commutation relations (3.15a).

by the SO(1,3) holonomy $\Lambda(h_\gamma)$ between the two, needed to evaluate the scalar product in the same frame. A short calculation gives

$$\begin{aligned}\underset{\sim}{n}_\alpha \Lambda(h_\gamma)^\alpha{}_\beta n^\beta &= \underset{\sim}{n}_{A\bar{A}} h_\gamma{}^A{}_B \bar{h}_\gamma{}^{\bar{A}}{}_{\bar{B}} n^{B\bar{B}} \\ &= -\frac{1}{2}\frac{1}{|\pi\omega|^2}\delta_{A\bar{A}}\delta^{B\bar{B}}\big(\underset{\sim}{\omega}^A\pi_B - \underset{\sim}{\pi}^A\omega_B\big)\big(\bar{\underset{\sim}{\omega}}^{\bar{A}}\bar{\pi}_{\bar{B}} - \bar{\underset{\sim}{\pi}}^{\bar{A}}\bar{\underset{\sim}{\omega}}_{\bar{B}}\big) = \\ &= -\frac{1}{2}\left(\frac{\|\underset{\sim}{\omega}\|^2}{\|\omega\|^2} + \frac{\|\omega\|^2}{\|\underset{\sim}{\omega}\|^2}\right) = -\mathrm{ch}(\Xi),\end{aligned} \tag{3.160}$$

valid on the constraint surface (3.145). The rapidity between three-cells describes the extrinsic curvature in Regge calculus, therefore this Abelian pair captures a scalar part of the ADM Poisson brackets, as we'll make clearer in the next section.

The second remark concerns the orbits generated by B (as defined in (3.146)). Let us look on how $T^*SL(2,\mathbb{C})$ lies, as a quotient space, inside of $\mathbb{T}^2$. From (3.146), we see that on the space reduced by $C = 0$, that is $T^*SL(2,\mathbb{C})$, the independent simplicity constraints are $B = F = \underset{\sim}{F} = 0$. Since F is complex while B is real, these equations remove five out of twelve dimensions of $T^*SL(2,\mathbb{C})$, and thus characterise a seven-dimensional quotient space embedded in $\mathbb{T}^2$. From the previous construction, we know that six dimensions are spanned by the $SU(2)$ holonomy-flux variables, or equivalently by the $SU(2)$ spinors reduced by (3.154). Since

$$\{B, z^A\} = 0 = \{B, \underset{\sim}{z}^A\}, \quad \{B, \Xi\} = \frac{4}{1+\beta^2}, \tag{3.161}$$

the seventh dimension spreads along the orbits of B, every one of which can be parametrised by the angle $\Xi \in \mathbb{R}$. Accordingly, we denote the constraint surface T_Ξ, and $T_\Xi \simeq T^*SU(2) \times \mathbb{R}$. This means that a pair of simple twistors, solutions of the area-matching and the simplicity constraints, are parametrised by the $SU(2)$ spinors, plus the rapidity angle.

On T_Ξ, the Lorentz fluxes already coincide with the $\mathfrak{su}(2)$ Lie algebra elements introduced in (3.155), providing a discrete counterpart of the continuum equation (3.3). For the Lorentz holonomy we find, plugging (3.151) and (3.156) into (3.13),

$$h_{\mathrm{red}}{}^A{}_B \equiv h^A{}_B\big|_{F=0} = \frac{\mathrm{e}^{-\frac{1}{2}(\mathrm{i}\beta+1)\Xi}\underset{\sim}{z}^A\delta_{B\bar{B}}\bar{z}^{\bar{B}} + \mathrm{e}^{\frac{1}{2}(\mathrm{i}\beta+1)\Xi}\delta^{A\bar{A}}\bar{\underset{\sim}{z}}_{\bar{A}}z_B}{\|z\|\|\underset{\sim}{z}\|}. \tag{3.162}$$

This is still a completely general $SL(2,\mathbb{C})$ group element. If we now choose the specific $\Xi = 0$ section through the orbits of B, it reduces to an $SU(2)$ holonomy, and coincides with the B-invariant holonomy U (3.155). The constraint hypersurface T_Ξ plays an important role, because there we can distinguish the reduced Lorentz holonomy (3.162) from the $SU(2)$ holonomy (3.155). The difference is captured by the orbits of the diagonal simplicity constraint.

3.6.2 Ashtekar–Barbero holonomy and extrinsic curvature

Consider now the constraint hypersurface T_Ξ, and the two holonomies $U(z,\underset{\sim}{z})$ and $h_{\mathrm{red}}(z,\ \underset{\sim}{z},\ \Xi)$, as defined in (3.155) and (3.162). While h_{red} describes the Lorentzian parallel transport, we will now show that the $SU(2)$ holonomy $U(z,\underset{\sim}{z})$ equals the holonomy of the real-valued Ashtekar–Barbero connection $A^{(\beta)i}{}_a = \Gamma^i{}_a + \beta K^i{}_a$ (here $\Gamma^i{}_a$

and $K^i{}_a$ are the real and imaginary components of the selfdual $SL(2,\mathbb{C})$ connection $A^i{}_a = \Gamma^i{}_a + \mathrm{i}K^i{}_a$). We will thus prove that

$$U(z,\underset{\sim}{z}) = U_\gamma := \mathrm{Pexp}\Big(-\int_\gamma \Gamma + \beta K\Big). \tag{3.163}$$

This identification is very important for the spinfoam formalism, and the understanding of the relation between covariant and canonical structures. It is needed to match the boundary states appearing in spinfoam models with the $SU(2)$ spin network states found from the canonical approach, see e.g. the discussions in [132–134, 183].

To prove (3.163), let us first recall (see equation (3.4)) that h is a left-handed group element corresponding to the parallel transport by the left-handed part of the Lorentz connection, $A = \Gamma + \mathrm{i}K$, where Γ represents the intrinsic covariant three-derivative. This three-derivative defines the $SU(2)$ parallel transport

$$G_\gamma := \mathrm{Pexp}\Big(-\int_\gamma \Gamma^i\tau_i\Big) \in SU(2). \tag{3.164}$$

The intrinsic and extrinsic contributions to the holonomies can be disentangled via an "interaction picture" for the path-ordered exponentials,[*]

$$\begin{aligned} h_\gamma &= \mathrm{Pexp}\Big(-\int_\gamma \Gamma + \mathrm{i}K\Big) = \\ &= G_\gamma\, \mathrm{Pexp}\Big(-\mathrm{i}\int_0^1 \mathrm{d}t\, G^{-1}_{\gamma(t)} K_{\gamma(t)}(\dot\gamma) G_{\gamma(t)}\Big) \equiv G_\gamma\, V_K, \end{aligned} \tag{3.165}$$

$$\begin{aligned} U_\gamma &= \mathrm{Pexp}\Big(-\int_\gamma \Gamma + \beta K\Big) = \\ &= G_\gamma\, \mathrm{Pexp}\Big(-\beta\int_0^1 \mathrm{d}t\, G^{-1}_{\gamma(t)} K_{\gamma(t)}(\dot\gamma) G_{\gamma(t)}\Big) \equiv G_\gamma\, V_K^\beta. \end{aligned} \tag{3.166}$$

Both holonomies provide maps $\mathbb{C}^2 \mapsto \mathbb{C}^2$ between tilded and untilded spinors, but while h transports the covariant ω^A-spinors, U transports the reduced spinors z^A. Let us introduce a short-hand ket notation,

$$|0\rangle \equiv \frac{z^A}{\|z\|}, \quad |1\rangle \equiv \frac{\delta^{A\bar A}\bar z_{\bar A}}{\|z\|}, \quad |\underset{\sim}{0}\rangle \equiv \frac{\underset{\sim}{z}^A}{\|\underset{\sim}{z}\|}, \quad |\underset{\sim}{1}\rangle \equiv \frac{\delta^{A\bar A}\bar{\underset{\sim}{z}}_{\bar A}}{\|\underset{\sim}{z}\|}. \tag{3.167}$$

The holonomies can be thus characterised as the unique solutions to the equations

$$|\underset{\sim}{0}\rangle = e^{(\mathrm{i}\beta+1)\Xi/2} h|0\rangle = U|0\rangle, \qquad |\underset{\sim}{1}\rangle = e^{(-\mathrm{i}\beta+1)\Xi/2}(h^\dagger)^{-1}|1\rangle = U|1\rangle. \tag{3.168}$$

Next, we recall that the source and target generators of the Lorentz algebra are related via the holonomy, see (3.7). This relation, together with the simplicity constraints, implies that

$$\Pi = \mathrm{e}^{\mathrm{i}\vartheta}\Pi^\dagger = -\mathrm{e}^{\mathrm{i}\vartheta}(h^{-1}\underset{\sim}{\Pi} h)^\dagger = -h^\dagger \underset{\sim}{\Pi}(h^{-1})^\dagger = h^\dagger h \Pi (h^\dagger h)^{-1}. \tag{3.169}$$

[*]This can be explicitly proven by looking at the defining differential equation for the holonomy, which admits a unique solution for the inital conditions $U_{\gamma(0)} = \mathbb{1} = h_{\gamma(0)}$. It is the same type of equality that appears in the interaction picture used in time-dependent perturbation theory, with Γ being the free Hamiltonian, and K the potential.

We see that the simplicity constraints automatically lead to a certain “alignment” between the holonomy and the generators, that immediately translates into an equation for the spinors:

$$(h^\dagger h)^A{}_B\omega^B = \mathrm{e}^{-\Xi}\omega^A, \qquad (h^\dagger h)^A{}_B\pi^B = \mathrm{e}^{\Xi}\pi^A, \tag{3.170}$$

with Ξ given in (3.156). Inserting (3.165) in (3.170), we find

$$V_K^\dagger V_K|0\rangle = \mathrm{e}^{-\Xi}|0\rangle, \qquad V_K^\dagger V_K|1\rangle = \mathrm{e}^{+\Xi}|1\rangle. \tag{3.171}$$

For small extrinsic curvature, we have that $V_K > 0$ and $V_K^\dagger = V_K$ such that this eigenvalue equation has just one solution, given by*

$$V_K = \mathrm{e}^{-\Xi/2}|0\rangle\langle 0| + \mathrm{e}^{\Xi/2}|1\rangle\langle 1|. \tag{3.172}$$

Within the same approximation, we also have

$$V_K^\beta = \mathrm{e}^{\mathrm{i}\beta\Xi/2}|0\rangle\langle 0| + \mathrm{e}^{-\mathrm{i}\beta\Xi/2}|1\rangle\langle 1|. \tag{3.173}$$

Finally, using the interaction picture in (3.169), as well as properties (3.172) and (3.173), we find

$$\begin{aligned} U|0\rangle &= e^{+(\mathrm{i}\beta+1)\Xi/2} h|0\rangle = G V_K^\beta|0\rangle, \\ U|1\rangle &= e^{(-\mathrm{i}\beta+1)\Xi/2} (h^\dagger)^{-1}|1\rangle = G V_K^\beta|1\rangle, \end{aligned} \tag{3.174}$$

and since $|0\rangle$ and $|1\rangle$ are a complete basis, this proves the desired result (3.163).

The above equation provides a discrete counterpart to $A^{(\beta)i}{}_a = \Gamma^i{}_a + \beta K^i{}_a = A^i{}_a + (\beta - \mathrm{i})K^i{}_a$, with Ξ playing the role of the extrinsic curvature. Notice also that from the linearised form of (3.172), and the continuum interpretation of V_K, we deduce

$$\Xi \approx \int_0^1 \mathrm{d}s\, R^{(\mathrm{ad})}(G^{-1}_{\gamma(s)})^i{}_j K^j_{\gamma(s)}(\dot\gamma) n_i[\tau], \tag{3.175}$$

where $R^{(\mathrm{ad})}(G)^i{}_j \in SO(3)$ is the $SU(2)$ element G in the adjoint representation. That is, the rapidity approximates the extrinsic curvature smeared over the dual link, projected down onto the direction $n^i[\tau]$ normal to the surface. As anticipated earlier, the canonical pairing (3.158) between Ξ and the area $A[\tau]$ nicely describes the scalar part of the ADM phase space of general relativity, where flux $E_i{}^a$ and extrinsic curvature $K^i{}_a$ are canonical conjugated [19].

We conclude that the $SU(2)$ spinors z and $\underset{\sim}{z}$ obtained from the symplectic reduction parametrise holonomies and fluxes of the $SU(2)$ Ashtekar–Barbero variables. To prove this identification, it has been necessary to work on the covariant phase space, or at least on the constraint hypersurface $T_\Xi \cong T^*SU(2) \times \mathbb{R}$, where we could disentangle extrinsic and intrinsic parts of the $SU(2)$ holonomy. Therefore, to have a full geometric meaning, the $SU(2)$ variables need to be embedded in T_Ξ. This should not come as a surprise: from the continuum theory we know that one needs to embed the Ashtekar–Barbero connection into the space of Lorentzian connections in order to distinguish intrinsic from extrinsic contributions, and that the secondary constraints provide the embedding.

*The solution is exact if the extrinsic curvature is covariantly constant along the link, i.e. $G^{-1}_{\gamma(t)} K_{\gamma(t)}(\dot\gamma) G_{\gamma(t)}$ is t-independent.

Let us discuss this in more details. In the continuum theory, the Ashtekar–Barbero variables, $(E, A^{(\beta)} = \Gamma + \beta K)$, are canonical coordinates on the reduced phase space, but are well-defined everywhere as functions on the original phase space through equations (2.108) and (2.154). Then, solving the secondary constraints gives $\Gamma = \Gamma[E]$, and provides a specific embedding (schematically $A^{(\beta)} \mapsto \Gamma[E] + \mathrm{i}\beta^{-1}(A^{(\beta)} - \Gamma[E])$) of the $SU(2)$ variables into the original phase space. If one forgets about secondary constraints, and treats the linear primary constraints as a first-class system, one ends up with a quotient space of orbits $A^{(\beta)} = \text{const.}$ intersecting the constraint hypersurface transversally.* Then, restoring the secondary constraints provides a non-trivial section, i.e. a gauge-fixing through these orbits, that is an embedding mapping any pair $(E, A^{(\beta)})$ towards a point $(\Pi = \frac{\hbar}{2\ell_{\mathrm{P}}^2}\frac{\beta+\mathrm{i}}{\mathrm{i}\beta}E, A = \Gamma[E] + \mathrm{i}K)$ in the original phase-space (remember (2.96) and (2.152)). Such treatment of second-class constraints resonates with the gauge-unfixing ideas [184, 185] recently applied to the framework of loop quantum gravity in [186, 187].

At the discrete level we do not know the correct representation of the secondary constraints, but, I think, there are two possibilities.

The first possibility is that we are indeed missing additional secondary constraints. In this case it is reasonable to assume, that they have the same effect on the constraint algebra as in the continuum, making B second class. Solving them, which should not be possible link by link but require the knowledge of the whole graph structure, would provide a non-trivial section** through the orbits (3.161) of B. This section is given by the rapidity as a non-local function $\Xi_\tau(z_{\tau_1}, z_{\tau_2}, \dots)$ where for each link (τ is the dual triangle) the angle Ξ_τ is determined by spinors $z_{\tau_1}, z_{\tau_2}, \dots$ all over the graph. This idea can be made explicit with the ubiquitous example of the flat four-simplex. In this case, the ten lengths of the bones ℓ_b define a metric geometry. Then, all spinors are functions of these data (modulo gauges), and in particular, for each link, there is a function $\Xi_t = \Xi_t(\ell_b)$ that gives the rapidity in terms of the lengths of the bones $\{\ell_b\}$. Hence, on the graph phase space $T_\Xi{}^L$ there is a functional dependence $\Xi_\tau(z_{\tau_1}, z_{\tau_2}, \dots)$ between the ten dihedral angles and the twenty spinors, which provides the desired non-trivial section of the bundle $T_\Xi{}^L$. Concerning the explicit form of the secondary constraints, it has been suggested in [163, 169, 188] that they should impose the shape matching conditions, thus reducing twisted geometries to Regge gravity.

In my opinion, in the light of the results of section 3.5.3, this conjecture cannot any longer be justified. In the continuum theory, the secondary constraints (2.152) imply the vanishing of the three-dimensional torsion two-form. Section 3.5.3 shows that there is a torsionless spin connection also if the shapes of the triangles do not match across adjacent tetrahedra. The torsionless equation has therefore nothing to do with the shape matching conditions. Reversing this argument, we see that we cannot expect the shape matching conditions to appear as secondary constraints in the Dirac analysis of the discrete theory.

The second possibility is that there are indeed no secondary constraints missing in the discrete theory. At first, this statement seems utterly wrong, since we know from

*This happens because the Hamiltonian flow of second-class constraints always points away from the constraint hypersurface. In the notation of chapter 2 the relevant Poisson brackets are $\{C_i{}^a(p), A^{(\beta)j}{}_b(q)\} = 0$, where $C_i{}^a(p)$ are the analogue of the linear simplicity constraints for the continuum theory, as defined in (2.97). We thus see that we can use the $SU(2)$ Ashtekar–Barbero connection to label the orbits of the linear simplicity constraints.

**The trivial section being $\Xi = 0$.

the continuum theory that there are secondary constraints, and without them we do not get general relativity. Yet it is also true that the distinction between secondary and primary constraints is only accidentally, and just depends on the chronological details of the Dirac constraint analysis. Equations of motion that appear as secondary constraints for some choice of canonical variables may play another role once we go to a different Hamiltonian formulation.

A large part of this chapter, essentially all of section 3.2, and parts of section 3.3 and 3.4 developed a new Hamiltonian formulation underlying spinfoam gravity. Hamiltonian mechanics always requires some choice of time. Our choice was very different from the global time parameter appearing in the usual ADM formulation of general relativity (remember chapter 2). In fact, our Hamiltonian generated the evolution in the t-variable parametrising the edges of the symplicial discretisation. Results obtained from the ADM approach do therefore not easily translate to our framework. And indeed, the Hamiltonian that we found preserved the simplicity constraints without the need of additional secondary constraints. The distinction between secondary and primary constraints does however not tell us much by itself. Let us therefore once again remind ourselves of the physical content of the secondary constraints in the continuum, and then ask what this means for the discrete theory.

In the continuum, the physical role of the secondary constraints is clear. They imply the vanishing of the three-dimensional torsion two-form (remember equations (2.162)). In the last section, section 3.5, we studied the role of torsion for the discrete theory. First of all we found that torsion implies the geometricity of the elementary building blocks:* If there is no torsion, all bones in the triangulation close to form triangles, the triangles close to form tetrahedra, and at each spinfoam vertex five tetrahedra meet and form a four-simplex. But this is only a local result, it does not imply geometricity across triangles. In fact, in a twisted geometry the shape of a triangle can change when passing from one bounding tetrahedron to the next, while torsion still vanishes. This was the result of section 3.5.3 where we defined the torsionless spin connection for twisted geometries.

The crucial question to be asked is therefore not whether there are secondary constraints or not, but rather if spinfoam gravity correctly imposes the vanishing of torsion. In section 3.5.2 we have related the vanishing of torsion to the geometricity of each four-simplex. If we now remember the result of Barrett et al. [175], that proves** the geometricity of each spinfoam vertex in the semi-classical limit, this suggests that spinfoam gravity does indeed correctly imposes the vanishing of torsion without missing any further constraints.

3.7 SUMMARY

The first section, section 3.1 gave a general review. We started with the topological "BF"-theory in selfdual variables. This action has trivial equations of motion, but it is important for us because it has the same phase-space as general relativity—only the dynamics is different. In fact, adding constraints that impose the geometricicty of the fluxes brings us back to general relativity. They guarantee the existence of a tetrad η_α, and restrict the two-form field $\Sigma_{\alpha\beta}$ to be $\Sigma_{\alpha\beta} = \eta_\alpha \wedge \eta_\beta$.

*This was a result of Minkowski's theorem generalised to Minkowski space, i.e. of section 3.5.2.

**The proof is based upon the possibility to uniquely reconstruct a four-simplex out of the fluxes $\Sigma_{\alpha\beta}[\tau] = \int_\tau \eta_\alpha \wedge \eta_\beta$ through its triangles. We will repeat this reconstruction theorem in a supplement attached to this chapter.

The next step was to study the discretisation of the theory on a simplical decomposition of space time. We introduced holonomy flux variables for the Lorentz group. The Poisson brackets of the continuum theory induce commutation relations for holonomies and fluxes. On each link of the discretisation the holonomies and fluxes form, in fact, the canonical phase space of the cotangent bundle $T^*SL(2,\mathbb{C})$. In this phase-space the momenta do not commute (3.6), but twistors allow to handle this noncommutativity while working on a complex vector space with canonical Darboux coordinates. On each link there are two twistors, one attached to the initial point, the other belonging to the final point, and together they simultaneously parametrise the holonomy and the flux.

Next, there was section 3.2, where we studied the dynamics. We first discretised the topological "BF"-action in terms of holonomies and fluxes. We then took our spinorial framework to simplify the action. Performing a continuum limit on the edges of the four-dimensional discretisation we were left with a one-dimensional action for the spinors on a spinfoam face(3.35). Introducing additional Lagrange multipliers we added the spinorial version of the linear simplicity constraints to the action.

Then we studied the equations of motion for the spinors. We found they can easily be integrated, the only trouble being the periodic boundary conditions, that imply a constraint on the holonomy along the loop bounding the spinfoam face, i.e. equation (3.75). This parallel transport is neither a pure boost, as in Regge calculus, nor a rotation, but a combination of both, with the Barbero–Immirzi parameter measuring the relative strength. Nevertheless, there are key similarities with Regge calculus. If parallel transported along the bounding loop, the flux through the triangle dual to the spinfoam face is mapped into itself, while the curvature (3.98) is a function of the deficit angles between adjacent tetrahedra (3.91).

In this model only the Gauß constraint couples the spinorial variables belonging to different wedges. One could, of course, think of many more possible interactions between neighbouring wedges. In fact, additional interaction terms should naturally arise once we study the continuum limit and go to an ever finer triangulation. Whether a constraint is of first- or second-class depends, however, on all terms in the action; adding additional terms could therefore easily spoil our conclusions. So what is the relevance of this edge dynamics, and why do we not consider all possible interactions at once? The answer is simple. We are aiming at a general framework for first order Regge calculus, and on the way towards this goal we try to keep the dynamics on the elementary building blocks as simple as possible. More general interactions will be studied once this model is fully understood.

Section 3.5 studied the role of torsion in the discrete theory. We saw, the vanishing of torsion not only implies the Gauß law for each tetrahedron, but also an additional constraint (3.101) on each vertex of the simplicial decomposition. This constraint demands that on every four-simplex the outwardly pointing normals of the bounding tetrahedra weighted by their volumes sum up to zero. This four-dimensional closure constraint is fulfilled only once we go to the solution space of all the equations of motion. In quantum theory this torsional condition thus holds in the weakest possible way: Only at the saddle point of the spinfoam amplitude [175] we would see the bounding tetrahedra close to form a 4-simplex. We argued that this may be yet too weak, and that the four-dimensional closure constraint (3.101) could be imposed more strongly. We studied the Minkowski's theorem in Minkowski space, and saw that this four-dimensional closure constraint actually suffices to reconstruct a geometric four-simplex out of the volume weighted four-normals of the bounding tetrahedra. Section 3.5 concluded with an analysis of the torsionless condition for twisted geometries, and also gave the reduction from the twistorial phase space down to the original framework of $SU(2)$ Ashtekar–Barbero variables.

In summary, this chapter introduced a canonical formulation of spinfoam gravity adapted to a simplicial discretisation of space-time. This framework should be of general interest, as it provides a solid foundation where different models could fruitfully be compared. An alternative Hamiltonian description for general dynamical systems on discrete manifolds has been introduced recently by Dittrich and Höhn [189]. In their model time is discrete, hence difference equations replace the Hamilton equations, while there is still a clean notion of canonical momenta, gauge symmetries, observables, first- and second-class constraints. Our model should be seen as lying in between this theory and the full continuum limit: The spinors are continuous fields, yet they are not living in space-time itself, but are supported only on the edges of the discretisation.

This chapter closes with two supplements. The first supplement studies the holonomy of the selfdual connection, its functional differential and its variation under deformations of the underlying path. The last supplement studies a four-simplex bounded by three spatial tetrahedra, this supplement also reviews the anaylsis of Barrett, on how the simplicity constraints guaranetee the geometricity of a four-simplex (see refrence [175, 190–192] for further reading).

SUPPLEMENT: THE HOLONOMY

The holonomy of a connection defines the parallel transport along the manifold. Here, we restrict ourselves to the complex, i.e. $\mathfrak{sl}(2,\mathbb{C})$-valued Ashtekar connection, but we could easily generalise this supplement to allow for any other local gauge group.

We now say, a spinor field* V^A on the base manifold Σ is parallel along $X \in T\Sigma$, if it is covariantly constant in the direction of X, i.e.:

$$D_X V^A = X^a D_a V^A = X^a \partial_a V^A + A^A{}_{Ba} X^a V^B = 0. \tag{3.176}$$

This definition makes sense, also if we know the spinor V^A only on a one-dimensional path $\gamma : [0,1] \to \Sigma$: If $V^A(t) \in \mathbb{C}^2$ denotes the spinor at the point $\gamma(t) \in \Sigma$, we say it is parallel along γ, provided that:

$$\frac{\mathrm{d}}{\mathrm{d}t} V^A(t) = -A^A{}_{Ba}\big|_{\gamma(t)} \dot{\gamma}^a(t) V^B(t) \equiv -A_{\gamma(t)}(\dot{\gamma})^A{}_B V^B(t). \tag{3.177}$$

The initial value $V^A(t=0) = V^A_o$ uniquely determines $V^A(t)$ for all other $t \in [0,1]$. Since the differential equation (3.177) is linear in V^A, the superposition principle holds, and the map relating V^A_o with $V^A(t)$ is linear, i.e.:

$$V^A(t) = h_{\gamma(t)}[A]^A{}_B V^B_o. \tag{3.178}$$

This defines the holonomy $h_{\gamma(t)}[A]$, that provides the parallel translation between the two endpoints. The holonomy along an oriented path γ is, in fact, the unique solution of the following system of ordinary differential equations:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} h_{\gamma(t)}[A]^A{}_B = -A_{\gamma(t)}(\dot{\gamma})^A{}_C h_{\gamma(t)}[A]^C{}_B,\\ &\text{to the initial condition: } h_{\gamma(t=0)}[A]^A{}_B = \delta^A_B. \end{aligned} \tag{3.179}$$

*$A \in \{0,1\}$, with V^A taking values in $\mathbb{C}^2$.

Notice also that the holonomy is an element of the gauge group $SL(2,\mathbb{C})$, which follows immediately from $\epsilon_{AB}h^A{}_C h^B{}_D = \epsilon_{CD}\det h$ by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\epsilon_{AB}h_{\gamma(t)}[A]^A{}_C h_{\gamma(t)}[A]^B{}_D\Big) = 0, \quad \text{and} \quad h_{\gamma(t=0)}[A]^A{}_B = \delta^A_B, \tag{3.180}$$

where ϵ_{AB} is the two-dimensional anti-symmetric tensor. We can now iteratively solve (3.179) by a Dyson series. The resulting expression gives the holonomy as the path-ordered exponential of the connection:

$$\begin{aligned} h_{\gamma(t)}[A] = \mathrm{Pexp}\Big(-\int_{\gamma(t)} A\Big) = \mathbb{1} - \int_0^t \mathrm{d}s A(s) + \\ + \int_0^t \mathrm{d}s_2 \int_0^{s_2} \mathrm{d}s_1 A(s_2)A(s_1) \pm \dots, \quad \text{with: } A(t) = A_{\gamma(t)}(\dot{\gamma}). \end{aligned} \tag{3.181}$$

Elementary properties The holonomy is a functional of the connection, which is a gauge dependent quantity. So how does the holonomy change under $SL(2,\mathbb{C})$ gauge transformations? If $\rho_g A = g^{-1}\mathrm{d}g + g^{-1}Ag$ denotes the gauge transformed connection, just as in equation (2.121) above, we can check that the transformed holonomy

$$(g\circ\gamma)^{-1}(t)h_\gamma[A](g\circ\gamma)(0) \tag{3.182}$$

solves the defining differential equation with the connection A replaced by $\rho_g A$. If we now remember the holonomy as the unique solution of the defining differential equation (3.179) we can immediately write down the desired transformation property:

$$h_{\gamma(t)}[\rho_g A] = (g\circ\gamma)^{-1}(t)h_\gamma[A](g\circ\gamma)(0). \tag{3.183}$$

By the very same argument we find the behaviour under diffeomorphisms. If $\phi:\Sigma\rightarrow\Sigma$ is a diffeomorphism, and $\phi^*: T^*_p\Sigma\rightarrow T^*_{\phi^{-1}(p)}\Sigma$ is the corresponding pull-pack, then the holonomy transforms as:

$$h_{\gamma(t)}[\phi^* A] = h_{\phi(\gamma(t))}[A]. \tag{3.184}$$

This also implies that the holonomy does not change under a reparametrisation of the path γ.

Next, there are the algebraic properties of the holonomy under reorientation and composition, i.e.

$$h_{\gamma^{-1}} = h_\gamma^{-1}, \quad h_{\beta\circ\alpha} = h_\beta h_\alpha. \tag{3.185}$$

Here, we wrote γ^{-1} for the oppositely oriented path, explicitly defined by:

$$\gamma^{-1}:[0,1]\rightarrow\Sigma:\gamma^{-1}(t) = \gamma(1-t), \tag{3.186}$$

while the missing definition for glueing two paths α and β meeting at $\alpha(0)=\beta(1)$ is:

$$(\alpha\circ\beta)(t) = \begin{cases} \beta(2t), t\in[0,\tfrac{1}{2}), \\ \alpha(2t-1), t\in[\tfrac{1}{2},1]. \end{cases} \tag{3.187}$$

Once again, equations (3.185) follow from the holonomy being the unique solution of its defining differential equation (3.179).

Functional differentials The holonomy is a functional of both the connection and the underlying path. We can therefore consider two independent functional differentials. Let us start with the variation of the path, which is more difficult to calculate. We introduce a smooth one-parameter family $\{\gamma_\varepsilon : [0,1] \ni t \mapsto \gamma_\varepsilon(t) \in \Sigma\}_{\varepsilon\in[0,1]}$ of smooth paths γ_ε, such that t and ε span a two-dimensional surface. At each point $\gamma_\varepsilon(t) \in \Sigma$ there are now two independent tangent vectors, that we call $\dot\gamma_\varepsilon$ and $\delta\gamma_\varepsilon$ respectively. In a local coordinate system $\{x^\mu\}_{\mu=1,2,3}$ around γ_ε these vectors look like this:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}x^\mu(\gamma_\varepsilon(t)) = \delta\gamma^\mu_\varepsilon(t), \qquad \frac{\mathrm{d}}{\mathrm{d}t}x^\mu(\gamma_\varepsilon(t)) = \dot\gamma^\mu_\varepsilon(t). \tag{3.188}$$

The two tangent vectors commute, i.e.:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}. \tag{3.189}$$

We are interested in the variation around $\varepsilon = 0$, and thus write $\delta\gamma = \delta\gamma_{\varepsilon=0}$, and equally:

$$\delta h_\gamma := \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} h_{\gamma_\varepsilon(1)}. \tag{3.190}$$

We take the variation of the defining differential equation (3.179) and get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}h_{\gamma_\varepsilon(t)} = -\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big(A_{\gamma_\varepsilon(t)}(\dot\gamma_\varepsilon)\Big)h_{\gamma_\varepsilon(t)} - A_{\gamma_\varepsilon(t)}(\dot\gamma_\varepsilon)\frac{\mathrm{d}}{\mathrm{d}\varepsilon}h_{\gamma_\varepsilon}. \tag{3.191}$$

Next, we multiply this equation by $h^{-1}_{\gamma_\varepsilon(t)}$ from the left, and integrate from 0 to 1. A partial integration yields:

$$\begin{aligned}\int_0^1 \mathrm{d}t h^{-1}_{\gamma_\varepsilon(t)}\frac{\mathrm{d}^2}{\mathrm{d}t\mathrm{d}\varepsilon}h_{\gamma_\varepsilon(t)} &= h^{-1}_{\gamma_\varepsilon(t)}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}h_{\gamma_\varepsilon(t)}\Big|_{t=0}^1 - \int_0^1 \mathrm{d}t\frac{\mathrm{d}}{\mathrm{d}t}h^{-1}_{\gamma_\varepsilon(t)}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}h_{\gamma_\varepsilon(t)} = \\ &= h^{-1}_{\gamma_\varepsilon(1)}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}h_{\gamma_\varepsilon(1)} - \int_0^1 \mathrm{d}t h_{\gamma_\varepsilon(t)}A_{\gamma_\varepsilon(t)}(\dot\gamma_\varepsilon)\frac{\mathrm{d}}{\mathrm{d}\varepsilon}h_{\gamma_\varepsilon(t)}.\end{aligned} \tag{3.192}$$

In the last line we used that $\delta h_{\gamma(0)} = \delta\mathbb{1} = 0$ and also $\dot h^{-1}_{\gamma(t)} = h^{-1}_\gamma A_{\gamma(t)}(\dot\gamma)$. Equation (3.192) together with (3.191) yields:

$$h^{-1}_{\gamma_\varepsilon(1)}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}h_{\gamma_\varepsilon(1)} = -\int_0^1 \mathrm{d}t h^{-1}_{\gamma_\varepsilon(t)}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big(A_{\gamma_\varepsilon(t)}(\dot\gamma_\varepsilon)\Big)h_{\gamma_\varepsilon(t)}. \tag{3.193}$$

Introducing local coordiantes $\{x^\mu\}_{\mu=1,2,3}$ as in (3.188), we perform a partial integration and use $\dot h_{\gamma(t)} = -A_{\gamma(t)}(\dot\gamma)h_{\gamma(t)}$ and $\dot h^{-1}_{\gamma(t)} = h^{-1}_{\gamma(t)}A_{\gamma(t)}(\dot\gamma)$ to arrive at the following:

$$\begin{aligned}h^{-1}_{\gamma_\varepsilon(1)}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}h_{\gamma_\varepsilon(1)} &= -\int_0^1 \mathrm{d}t h^{-1}_{\gamma_\varepsilon(t)}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big(\big(A_\mu\circ\gamma_\varepsilon\big)(t)\dot\gamma^\mu_\varepsilon(t)\Big)h_{\gamma_\varepsilon(t)} \\ &= -\int_0^1 \mathrm{d}t h^{-1}_{\gamma_\varepsilon(t)}\Big(\big(A_\mu\circ\gamma_\varepsilon\big)(t)\frac{\mathrm{d}}{\mathrm{d}t}\delta\gamma^\mu_\varepsilon(t) + \big(\partial_\nu A_\mu\circ\gamma_\varepsilon\big)(t)\delta\gamma^\nu_\varepsilon(t)\dot\gamma^\mu_\varepsilon(t)\Big)h_{\gamma_\varepsilon(t)} = \\ &= -h^{-1}_{\gamma_\varepsilon(t)}A_{\gamma_\varepsilon(t)}(\delta\gamma)h_{\gamma_\varepsilon(t)}\Big|_{t=0}^1 - \int_0^1 \mathrm{d}t h^{-1}_{\gamma_\varepsilon(t)}\Big(\big(\partial_\nu A_\mu\circ\gamma_\varepsilon\big)(t)\delta\gamma^\nu_\varepsilon(t)\dot\gamma^\mu_\varepsilon(t) + \\ &\qquad - \big(\partial_\nu A_\mu\circ\gamma_\varepsilon\big)(t)\dot\gamma^\nu_\varepsilon(t)\delta\gamma^\mu_\varepsilon(t) + A_{\gamma_\varepsilon(t)}(\dot\gamma_\varepsilon)A_{\gamma_\varepsilon(t)}(\delta\gamma_\varepsilon) + \\ &\qquad - A_{\gamma_\varepsilon(t)}(\delta\gamma_\varepsilon)A_{\gamma_\varepsilon(t)}(\dot\gamma_\varepsilon)\Big)h_{\gamma_\varepsilon(t)} = \\ &= -h^{-1}_{\gamma_\varepsilon(t)}A_{\gamma_\varepsilon(t)}(\delta\gamma)h_{\gamma_\varepsilon(t)}\Big|_{t=0}^1 - \int_0^1 \mathrm{d}t h^{-1}_{\gamma_\varepsilon(t)}\big(F_{\nu\mu}\circ\gamma_\varepsilon\big)(t)\delta\gamma^\nu_\varepsilon(t)\dot\gamma^\mu_\varepsilon(t)h_{\gamma_\varepsilon(t)}.\end{aligned} \tag{3.194}$$

In the last line we have introduced the field strength $F = \mathrm{d}A + \frac{1}{2}[A, A]$ of the selfdual connection. Setting $\varepsilon = 0$, $t = 1$, and writing $h_\gamma \equiv h_{\gamma(1)}$ we can summarise the last equation by saying:

$$h_\gamma^{-1}\delta h_\gamma[A] = -h_\gamma^{-1}A_{\gamma(1)}(\delta\gamma)h_\gamma + A_{\gamma(0)}(\delta\gamma^a) - \int_0^1 \mathrm{d}t h_{\gamma(t)}^{-1}F_{\gamma(t)}(\delta\gamma, \dot{\gamma})h_{\gamma(t)}. \quad (3.195)$$

This gives the infinitesimal change of the holonomy under variations of the underlying path. Equation (3.195) plays an important role in loop quantum gravity, for it allows to approximate the curvature tensor by the holonomy. Consider a small oriented two-dimensional surface α_ε, that be given in some local coordinate system $\{x^\mu\}_{\mu=1,2,3}$ as the square $(x^1, x^2) \in [0, \varepsilon] \times [0, \varepsilon]$. We can then use (3.195) to calculate the first non-vanishing term of the Taylor expansion of the holonomy $h_{\partial\alpha_\varepsilon}$ around $\varepsilon = 0$. It is of quadratic order in ε, such that the first two terms of the Taylor series are:

$$h_{\partial\alpha_\varepsilon} = \mathbb{1} + \underbrace{\int_{q\in\alpha_\varepsilon} h_{p\to q}^{-1}F_q h_{p\to q}}_{\mathcal{O}(\varepsilon^2)} + \mathcal{O}(\varepsilon^3), \quad (3.196)$$

where $h_{p\to q}$ is a family of holonomies connecting the point $p = \partial\alpha(0)$ with the points $q \in \alpha$ in the interior of the plaquette.

The holonomy depends not only on the underlying path but is also a functional of the connection. We can thus ask for the differential with respect to $A^i{}_a$. Repeating the steps that have led us to (3.193) we get:

$$h_\gamma^{-1}[A]\mathbf{d}h_\gamma[A] = -\int_0^1 \mathrm{d}t h_{\gamma(t)}^{-1}[A]\mathbf{d}A_{\gamma(t)}(\dot{\gamma})h_{\gamma(t)}[A]. \quad (3.197)$$

Which is the same as to say:

$$h_\gamma^{-1}[A]\frac{\delta}{\delta A^i{}_a(p)}h_\gamma[A] = -\int_0^1 \mathrm{d}t h_{\gamma(t)}^{-1}[A]\tau_i h_{\gamma(t)}[A]\dot{\gamma}^a(t)\delta^{(3)}(\gamma(t), p). \quad (3.198)$$

This equation contains all the essential information* to prove the commutation relations of the holonomy-flux algebra given in (3.6).

SUPPLEMENT:
THE GEOMETRY OF A FOUR-SIMPLEX

In this supplement we are going to explore the geometry of a four-simplex. We will see the Gauß law and the linear simplicity constraints together imply its geometricity. Geometricity means that we can introduce a metric compatible with the fluxes, and speak e.g. about the unique length of the bones bounding the triangles of the four-simplex. This is the reconstruction, or shape-matching problem.

*A subtlety arises from the delta-function $\delta^{(3)}(p, \gamma(t))$, which can lead to ill-defined expressions of the form of $\int_0^1 \mathrm{d}t\delta(t)f(t)$. Introducing the smearing $\int_0^1 \mathrm{d}t\delta(t)f(t) := \lim_{\varepsilon\searrow 0}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\mathrm{d}\varepsilon\int_\varepsilon^1 \mathrm{d}t\delta(t)f(t)$ yields the "natural" value $\int_0^1 \mathrm{d}t\delta(t)f(t) = \frac{1}{2}f(0)$.

Terminology Before we go into the reconstruction problem, let us first define some terminology, and get a feeling for how a four-simplex actually looks like (see also figure 3.4). Given its corners $X_1^\mu, \ldots, X_5^\mu \in \mathbb{R}^4$, we can view the four-simplex as the set of points $\Delta = \{X^\mu \in \mathbb{R}^4 | X^\mu = \sum_{i=1}^5 t_i X_i^\mu, \sum_{i=1}^5 t_i = 1, t_i > 0\}$. There are thus $5 \times 4 = 20$ numbers determining a four-simplex. If we now identify any two four-simplices related by a Poincaré transformation, we see there are only ten of them left: Ten numbers define a four-simplex up to Poincaré transformations. A four-simplex contains several sub-simplices (see again figure 3.4): There are the corners already introduced above, and also bones, triangles and tetrahedra. The bones are the geodesic lines connecting any two of the corners, they are given by the vectors:

$$b^\mu(ij) = X_j^\mu - X_i^\mu. \tag{3.199}$$

Two bones $b^\mu(kl)$ and $b^\mu(lm)$ span an oriented triangle τ_{ij}: If $(ijklm)$ is an even permutation of (12345), we declare the pair $(b^\mu(kl), b^\mu(lm))$ to have positive orientation in τ_{ij}. The triangles τ_{ij}, τ_{ik}, τ_{il}, and τ_{im} bound a tetrahedron, we call it T_i, and v_i is its volume, while $n^\mu(i)$ denotes its normal. In the following we restrict ourselves to the case where $n^\mu(i)$ is a future oriented time-like vector, i.e. $n^\mu(i) n^\nu(i) \eta_{\mu\nu} = -1$, $n^0(i) > 0$. Extensions to allow for more general geometries should be sought along the following lines. We also assume the four-simplex be non-degenerate, which means that its four-volume V should not vanish. If $\{x^\mu\}_{\mu=0,\ldots,3}$ are inertial coordinates in Minkowski space this is the condition that:

$$V = \frac{1}{4!} \int_\Delta \varepsilon_{\alpha\beta\mu\nu} \mathrm{d}x^\alpha \wedge \cdots \wedge \mathrm{d}x^\nu \overset{!}{\neq} 0. \tag{3.200}$$

Because of their prominent role in loop quantum gravity, we are most interested in the fluxes through the triangles. We compute them as the integrals:

$$\Sigma_{\alpha\beta}(ij) = \int_{\tau_{ij}} \mathrm{d}x_\alpha \wedge \mathrm{d}x_\beta = -\Sigma_{\alpha\beta}(ji). \tag{3.201}$$

Stoke's theorem implies for any tetrahedron the fluxes sum up to zero:

$$\forall i : \sum_{j : j \neq i} \Sigma_{\alpha\beta}(ij) = 0. \tag{3.202}$$

This is nothing but Gauß's law. If $n^\alpha(i)$ denotes the normal of the i-th tetrahedron we also have the condition:

$$\forall i, j : \Sigma_{\alpha\beta}(ij) n^\alpha(i) = 0, \tag{3.203}$$

which is the linear simplicity constraint that has appeared at several occasions in our theory as a reality condition on phase-space.

Reconstruction of a four-simplex from fluxes We have thus seen that for every tetrahedron bounding a four-simplex, the four fluxes sum up to zero and lie perpendicular to the tetrahedron's normal. We now show that the opposite is also true. Given five future oriented normals $n^\alpha(i)$, that span all of $\mathbb{R}^4$, and ten bivectors $\Sigma_{\alpha\beta}(ij) = -\Sigma_{\alpha\beta}(ji)$ subject to both the Gauß law, i.e. equation (3.202), and the linear simplicity constraint, i.e. equation (3.203), implies that there is a non-degenerate four-simplex Δ, such that the bivectors $\Sigma_{\alpha\beta}(ij)$ are the fluxes through its triangles and the vectors $n^\alpha(i)$ are the normals of the tetrahedra. The resulting four-simplex is unique up to rigid translations and the inversion $X^\mu \mapsto -X^\mu$ at the origin.

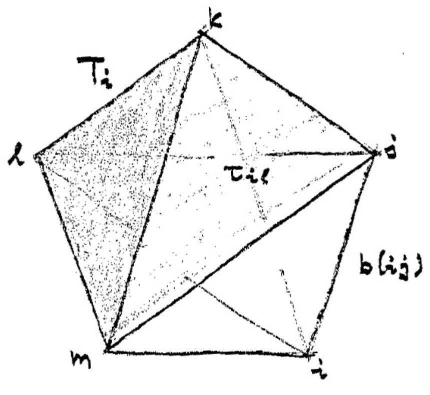

Figure 3.4: A four simplex in Minkowski space consists of five corners $X_1, \dots, X_5$, and contains several subsimplices. There are ten bones $b(ij)$, ten triangles τ_{ij}, and five tetrahedra T_i.

The first step is to realise that equation (3.203) implies that the fluxes have the following form:

$$\Sigma_{\alpha\beta}(ij) = \frac{a(ij)}{\operatorname{sh}\Xi_{ij}} \epsilon_{\alpha\beta\mu\nu} n^\mu(i) n^\nu(j), \tag{3.204}$$

where $a(ij) > 0$ is the area of the triangle and $\Xi_{ji} = \Xi_{ij} \in \mathbb{R}$ is the rapidity between the two adjacent normals, i.e.:

$$\operatorname{ch}\Xi_{ij} = -n_\mu(i) n^\mu(j). \tag{3.205}$$

Let us now introduce some simplifying abbreviations. First of all we define the three-dimensional volume elements $\epsilon_{\alpha\beta\mu}(i)$ together with the three-metric $h_{\alpha\beta}(i)$ by setting

$$\epsilon_{\alpha\beta\mu}(i) := \epsilon_{\nu\alpha\beta\mu} n^\nu(i), \quad h_{\alpha\beta}(i) = n_\alpha(i) n_\beta(i) + \eta_{\alpha\beta}. \tag{3.206}$$

We can now write the fluxes in the i-th tetrahedron as the spatial pseudo-vectors:

$$\Sigma_\beta(ij) := \frac{1}{2} \epsilon_{\mu\beta\nu}(i) \Sigma^{\mu\nu}(ij). \tag{3.207}$$

The three-dimensional volume $v_i > 0$ of the i-th tetrahedron is the wedge product of three bounding triangles:

$$\pm\frac{9}{2} v_i^2 = \epsilon_{\alpha\beta\mu}(i) \Sigma^\alpha(ij) \Sigma^\beta(ik) \Sigma^\mu(il). \tag{3.208}$$

Here, and in everything what follows, we take $(ijklm)$ to be an even permutation of (12345). The sign of (3.208) gives the relative orientation of the surfaces. We say, in fact, the triple $(\tau_{ij}, \tau_{ik}, \tau_{il})$ is positively (negatively) oriented in T_i, if (3.208) is positive (negative). Equation (3.204) brings the volume formula into the form:

$$\pm\frac{9}{2} v_i^2 = \epsilon^{\alpha\beta\mu\nu} n_\alpha(i) n_\beta(j) n_\mu(k) n_\nu(l) \frac{a(ij)a(ik)a(il)}{\operatorname{sh}\Xi_{ij} \operatorname{sh}\Xi_{ik} \operatorname{sh}\Xi_{il}} \tag{3.209}$$

A similar formula exists for the four-volume V (as defined by (3.200)) of the polytope, it is:

$$\pm V = \frac{1}{4!} \epsilon^{\alpha\beta\mu\nu} \Sigma_{\alpha\beta}(ij) \Sigma_{\mu\nu}(lm). \tag{3.210}$$

The closure constraint (3.202) guarantees that V does not change if we would have used any other pair of triangles. The sign of (3.210) gives the relative orientation, the pair (τ_{ij}, τ_{lm}) has positive orientation from the four-dimensional perspective, if (3.210) is positive, otherwise it has negative orientation. If we now substitute (3.204) for the fluxes, the volume formula turns into:

$$\mp V = \frac{1}{3!}\epsilon^{\alpha\beta\mu\nu} n_\alpha(i) n_\beta(j) n_\mu(l) n_\nu(m) \frac{a(ij)a(lm)}{\operatorname{sh}\Xi_{ij}\operatorname{sh}\Xi_{lm}} \tag{3.211}$$

We combine the formulae for the three- and four-volume, i.e. equations (3.208) and (3.210) so as to get the following expression for the rapidity:

$$\frac{\operatorname{sh}\Xi_{ij}}{a(ij)} = \frac{4}{3}\frac{V}{v_i v_j}\varepsilon(ij), \quad \varepsilon(ij) = \operatorname{sign}\Xi_{ij}. \tag{3.212}$$

This implies for the ratio between any two volumina that:

$$\frac{v_j}{v_k} = \frac{\operatorname{sh}\Xi_{ik}}{\operatorname{sh}\Xi_{ij}}\frac{a(ij)}{a(ik)}\varepsilon(ij)\varepsilon(ik). \tag{3.213}$$

So far, we have defined the three-volume of the tetrahedra (3.208) and the four volume (3.210) without actually ever constructing the underlying four-simplex. This is the only task left. We substitute our solution (3.204) of the simplicity constraints (3.203) into the Gauß law and get:

$$\forall i : \sum_{j:j\neq i}\Sigma_{\alpha\beta}(ij) = \sum_{j:j\neq i}\frac{a(ij)}{\operatorname{sh}\Xi_{ij}}\epsilon_{\alpha\beta\mu\nu}n^\mu(i)n^\nu(j) = 0.$$

The last identity is the same as to say:

$$\forall i : \exists\lambda_i : \sum_{j:j\neq i}\frac{a(ij)}{\operatorname{sh}\Xi_{ij}}n^\mu(j) = \lambda_i n^\mu(i). \tag{3.214}$$

With (3.212) we can write this as:

$$\forall i : \exists\mu_i : \sum_{j:j\neq i}\varepsilon(ij)v_j n^\mu(j) + \mu_i n^\mu(i) = 0 \tag{3.215}$$

If we now use our assumption that any quadruple of normals be linearly independent in $\mathbb{R}^4$ (i.e. employ the non-degeneracy of the geometry), we find all μ_i must be proportional to the three-volume v_i by a universal sign:

$$\exists\sigma \in \{-1, 1\} : \forall i : \mu_i = \sigma v_i \tag{3.216}$$

We can then also find a set of numbers $\varepsilon_i \in \{-1, 1\}$, (again proven by the non-degeneracy of the geometry) such that:

$$\varepsilon(ij) = \sigma\varepsilon_i\varepsilon_j. \tag{3.217}$$

The numbers ε_i are unique up to a global sign. Equation (3.215) now simplifies to the following:

$$\sum_i \varepsilon_i v_i n^\mu(i) = 0. \tag{3.218}$$

This condition is the four-dimensional analogue of Gauß's law (3.202). The Minkowski theorem in four dimensions, as discussed in section 3.5.2, guarantees that there is a corresponding four-simplex consisting of five tetrahedra with normals $n^\alpha(i)$ and volume v_i. This four-simplex is unique up to rigid translations and the reflection $X^\mu \mapsto -X^\mu$ at the origin. The discrete ambiguity arises because we have never specified whether the normals are outwardly oriented or point inside the four-simplex.

We can now compute the fluxes through the triangles (3.201) of the four-simplex thus constructed. Do they match those that were our initial data, subject only to the constraints (3.202) and (3.203)? The answer is yes, and the proof is to actually build the four-simplex in question.

The first step is to look at the bones. A bone $b^\mu(il)$ connects the i-th corner with the l-th. From the perspective of the j-th tetrahedron, this bone follows the intersection of the triangles τ_{jm} and τ_{jk}. (And $(ijklm)$ be again an even permutation of (12345).) Inserting our parametrisation of the fluxes, i.e. equation (3.204), this means:

$$b^\mu(il) \propto \epsilon^{\mu\alpha\beta}(j)\Sigma_\alpha(jm)\Sigma_\beta(jk) = \epsilon^{\mu\beta\alpha\nu} n_\alpha(j) n_\nu(m) n_\beta(k) \frac{a(jm)}{\operatorname{sh}\Xi_{jm}} \frac{a(jk)}{\operatorname{sh}\Xi_{jk}}. \tag{3.219}$$

Forming a triangle, the bones must close to zero:

$$b^\mu(il) + b^\mu(lm) + b^\mu(mi) = 0. \tag{3.220}$$

This suggests the following ansatz for the bone $b^\mu(il)$:

$$b^\mu(il) = \lambda \varepsilon_j \varepsilon_m \varepsilon_k \frac{v_j v_m v_k}{V^2} \epsilon^{\mu\nu\alpha\beta} n_\nu(j) n_\alpha(m) n_\beta(k). \tag{3.221}$$

Where λ is an overall normalisation yet to be calculated. Equation (3.218) implies that for each triangle the bones close to zero. We find the value of λ by demanding that their wedge product gives the fluxes through the triangles:

$$\Sigma_\alpha(jk) = \frac{\sigma}{2} \epsilon_{\alpha\mu\nu}(j) b^\mu(il) b^\nu(lm). \tag{3.222}$$

This fixes λ to the value $\pm\frac{3}{2^3}$, and we can eventually write the bones in terms of the fluxes as:

$$b^\mu(il) = \pm\frac{2}{3}\frac{\varepsilon_j}{v_j} \epsilon^{\mu\alpha\beta}(j)\Sigma_\alpha(jm)\Sigma_\beta(jk). \tag{3.223}$$

This concludes the reconstruction: Picking a point in Minkowski space, we fix one corner, i.e. we set $X_1^\mu = 0$, and find all the others via $b^\mu(ij) = X_j^\mu - X_i^\mu$. We can now also identify the residual gauge symmetries. Moving X_1^μ around we realise the translational symmetry, while flipping the sign in equation (3.223) corresponds to the inversion $X_i^\mu \mapsto -X_i^\mu$ at the origin.

Concluding remarks The relation with loop gravity is the following. We have not had metric variables to begin with, but have started from the Plebanski two-form and the connection smeared over triangles and links. The Plebanski two-form is, in fact, integrated over the triangles in the frame of the tetrahedron:

$$\Sigma_{\alpha\beta}[\tau_{ij}] = \int_{p\in\tau_{ij}} h(p \to T_i)_\alpha{}^\mu h(p \to T_i)_\beta{}^\nu \Sigma_{\mu\nu}(p), \tag{3.224}$$

where $h(p \to T_i) \in L_+^\uparrow$ is a Lorentz holonomy mapping any point in the triangle τ_{ij} towards the centre of the tetrahedron T_i.

Variables attached to different tetrahedra belong to different frames. If we want to compare them, we need a map into a common frame. This is the bulk holonomy $g_i \in L_+^\uparrow$ from the centre of the four-simplex (i.e. the vertex) to the centre of the i-th tetrahedron (i.e. the node). The resulting variables are in one to one correspondence with the fluxes and normals of this supplement, that is:

$$\Sigma_{\alpha\beta}(ij) = (g_i)^{\mu}{}_{\alpha}(g_i)^{\nu}{}_{\beta}\Sigma_{\mu\nu}[\tau_{ij}], \quad n_\alpha(i) = (g_i)^{\mu}{}_{\alpha} n^{\alpha}(0), \tag{3.225}$$

where $n^\alpha(0) = \delta^\alpha_0$, which is the time-gauge condition: In the frame of each tetrahedron we have chosen a Lorentz gauge aligning its future oriented time normal with $n^\alpha(0)$.

We have thus seen, the Gauß law together with the linear simplicity constraints impose the geometricity of the underlying four-simplex. In particular, the length $\ell(ij)$ of each bone $b^\mu(ij)$ turns into a unique function of the fluxes (implicitly given by (3.223)). This function is needed to explore the relation between loop quantum gravity and classical Regge calculus.

4
Quantum theory

The last chapter gave a Hamiltonian formulation of the dynamics of spinfoam gravity. On each edge we have introduced four twistorial fields $Z_{(I)} : [0,1] \ni t \mapsto (\bar{\pi}^{(I)}_{\bar{A}}(t), \omega^A_{(I)}(t)) \in \mathbb{T}$, together with Lagrange multipliers $\lambda_{(I)}$ and $\Phi^A{}_B(t)$. These variables must satisfy certain constraints: the simplicity constraints (3.19), the Gauß law (3.45), together with the boundary conditions (3.2.5) needed to close the spinfoam face. The system has a finite number of degrees of freedom (for each edge, there are four twistor fields, and a set of Lagrange multipliers), time evolution is governed by a Hamiltonian (itself constrained to vanish), and the spinors are canonical (Darboux) coordinates covering all of phase space. These key observations make quantisation rather straight forward.

There are two sections in this chapter. First of all, we are going to study the kinematical structure for the quantum theory, and solve the quantised constraint equations. The second part, section 4.2 concerns the dynamics, and defines the amplitudes for the case of a manifold without a boundary. This chapter collects results from my recent publications; it is entirely based upon the articles [149, 150] and in particular [156].

4.1 CANONICAL QUANTISATION AND PHYSICAL STATES

The space of twistors $\mathbb{T} \ni Z = (\bar{\pi}_{\bar{A}}, \omega^A)$ equipped with the Poisson brackets (3.15a) can readily be quantised. Taking a Schrödinger representation, the "position" operators $\hat{\omega}^A$ and $\hat{\bar{\omega}}^{\bar{A}}$ act by multiplication. We thus work on the Hilbert space $L^2(\mathbb{C}^2, d^4\omega)$, with the canonical $SL(2,\mathbb{C})$-invariant integration measure:

$$d^4\omega = \frac{1}{16}\big(\mathrm{d}\omega_A \wedge \mathrm{d}\omega^A \wedge \mathrm{cc.}\big). \tag{4.1}$$

This Hilbert space is an auxiliary object, introduced just to have the proper arena to define the constraints. Its elements are nonanalytic functions of the ω^A-spinor. This becomes important when studying the momentum operator, that acts as a gradient in $\mathbb{C}^2$:

$$\big(\hat{\pi}_A f\big)(\omega) = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\omega^A} f(\omega), \quad \big(\hat{\bar{\pi}}_{\bar{A}} f\big)(\omega) = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\bar{\omega}^{\bar{A}}} f(\omega). \tag{4.2}$$

On this Hilbert space the $SL(2,\mathbb{C})$ action

$$\big(\mathcal{D}(g)f\big)(\omega^A) = f\big((g^{-1})^A{}_B\omega^B\big) \tag{4.3}$$

is unitary, but reducible. Irreducible subspaces are spanned by distributions, that are homogenous in the spinors [193, 194]. There are two quantum numbers $\rho \in \mathbb{R}$ and

$k \in \frac{\mathbb{Z}}{2}$ labelling the irreducible subspaces $\mathcal{H}_{(\rho,k)}$. These quantum numbers parametrise the homogeneity weights according to:

$$\forall z \in \mathbb{C} - \{0\}, f^{(\rho,k)} \in \mathcal{H}_{(\rho,k)} : f^{(\rho,k)}(z\omega^A) = z^{-k-1+\mathrm{i}\rho}\bar{z}^{+k-1+\mathrm{i}\rho} f^{(\rho,k)}(z\omega^A). \tag{4.4}$$

This implies the important relations:

$$\omega^A \frac{\partial}{\partial \omega^A} f^{(\rho,k)}(\omega^A) = (-k-1+\mathrm{i}\rho) f^{(\rho,k)}_{jm}, \tag{4.5a}$$

$$\bar{\omega}^{\bar{A}} \frac{\partial}{\partial \bar{\omega}^{\bar{A}}} f^{(\rho,k)}(\omega^A) = (+k-1+\mathrm{i}\rho) f^{(\rho,k)}_{jm}. \tag{4.5b}$$

We introduce the canonical quantisation of boosts and rotations:

$$\hat{\Pi}_i = -\tau^{AB}{}_i \hat{\omega}_A \hat{\pi}_B, \quad \hat{L}_i = \hat{\Pi}_i + \text{hc.}, \quad \hat{K}_i = -\mathrm{i}\hat{\Pi}_i + \text{hc.} \tag{4.6}$$

Notice no ordering ambiguity is appearing here, simply because the basis elements $\tau^A{}_{Bi}$ are traceless $\tau^A{}_{Ai} = 0$. We can work with a distributional basis on our auxiliary Hilbert space $L^2(\mathbb{C}^2, d^4\omega)$, that simultaneously diagonalises $\hat{L}_3$, $\hat{L}^2 = \hat{L}^i\hat{L}_i$, together with the two Casimirs $\hat{L}^i\hat{K}_i$ and $\hat{L}^2 - \hat{K}^2$ of the Lorentz group. The action of the Casimirs is in fact:

$$\big(\hat{L}^2 - \hat{K}^2 + 2\mathrm{i}\hat{L}_i\hat{K}^i\big) f^{(\rho,k)}_{jm} = -\hbar^2(\rho^2 - k^2 + 1 + 2\mathrm{i}\rho k) f^{(\rho,k)}_{jm}, \tag{4.7}$$

which follows from (4.5) by expressing $\hat{\Pi}_i\hat{\Pi}^i$ in terms of $\hat{\omega}^A\hat{\pi}_A$. Additional quantum numbers are spins $j = 0, \frac{1}{2}, 1, \dots$ and the eigenvalues $m = -j, \dots j$ of $\hat{L}_3$. The canonical basis reads explicitly:

$$f^{(\rho,k)}_{jm}(\omega^A) = \sqrt{\frac{2j+1}{\pi}} \|\omega\|^{2(\mathrm{i}\rho - 1)} R^{(j)}\big(U^{-1}(\omega^A)\big)^k{}_m, \tag{4.8}$$

with

$$U(\omega^A) = \frac{1}{\|\omega\|^2}\begin{pmatrix} \omega^0 & -\bar{\omega}^{\bar{1}} \\ \omega^1 & \bar{\omega}^{\bar{0}} \end{pmatrix} \in SU(2), \tag{4.9}$$

and $R^j(U)^m{}_n = \langle j, m|U|j, n\rangle$ being the Wigner matrix of the j-th irreducible $SU(2)$ representation. The basis vectors obey the generalised orthogonality relations

$$\big\langle f^{(\rho,k)}_{jm}, f^{(\rho',k')}_{j'm'}\big\rangle_{\mathbb{C}^2} = \int_{\mathbb{C}^2} d^4\omega \overline{f^{(\rho,k)}_{jm}(\omega^A)} f^{(\rho',k')}_{j'm'}(\omega^A) = \pi^2\delta(\rho-\rho')\delta_{kk'}\delta_{jj'}\delta_{mm'}. \tag{4.10}$$

We thus have a direct integral:

$$L^2(\mathbb{C}^2, d^4\omega) = \frac{1}{\pi}\int_{\mathbb{R}}^{\oplus} \mathrm{d}\rho \bigoplus_{k\in\mathbb{Z}} \mathcal{H}_{(\rho,k)}. \tag{4.11}$$

We are now ready to discuss the quantisation of the constraints. For the Lorentz invariant part of the reality conditions we have to choose an ordering. We define it with the momentum and position operators in the following order:

$$\hat{D} = \frac{\hbar}{\beta+\mathrm{i}}\omega^A\frac{\partial}{\partial\omega^A} - \frac{\hbar}{\beta-\mathrm{i}}\Big(\bar{\omega}^{\bar{A}}\frac{\partial}{\partial\bar{\omega}^{\bar{A}}} + 2\Big) \equiv \frac{\hbar}{\beta+\mathrm{i}}\omega^A\frac{\partial}{\partial\omega^A} + \text{hc.} \tag{4.12}$$

This operator is diagonal on the homogenous functions. With the action of the Euler operators (4.5) we get immediately:

$$\hat{D} f^{(\rho,k)}_{jm} = \frac{2\hbar}{\beta^2+1}\big(\rho - \beta(k+1)\big) f^{(\rho,k)}_{jm}. \tag{4.13}$$

The solution space of this constraint is non-normalisable in $L^2(\mathbb{C}^2, d^4\omega)$, simply because we cannot integrate the homogenous functions along the rays $\omega^A(z) = z\omega^A(0)$ (with $z \in \mathbb{C}$). We can, however, introduce a surface integral removing this divergence. Take the Hamiltonian vector field $\mathfrak{X}_D$, defined for any scalar function f as $\mathfrak{X}_D[f] = \{D, f\}$, and consider the interior product $\mathfrak{X}_D \lrcorner d^4\omega$. This defines a nondegenerate three-form on the space of orbits (3.60) generated by D, that we are using to define an inner product:

$$\langle f, f' \rangle_{\mathbb{C}^2/D} = \frac{\beta^2+1}{2\pi} \int_{\mathbb{C}^2/D} (\mathfrak{X}_D \lrcorner d^4\omega)\, \overline{f} f'. \tag{4.14}$$

To evaluate this integral we need to choose a gauge section, that embeds the three-dimensional surface $\mathbb{C}^2/D$ into $\mathbb{C}^2$. The inner product is independent of this choice only if the Lie derivative of the integrand vanishes:

$$\mathcal{L}_{\mathfrak{X}_D}\big(d^4\omega\, \overline{f} f'\big) = 0. \tag{4.15}$$

If both f and f' are in the kernel of $\hat{D}$ this condition holds true, and we arrive at a well defined inner product. In that case, we can deform the integration domain in the direction of $\mathfrak{X}_D$, that is along the orbits generated by D, without changing the integral. For our basis vectors we get in fact:

$$\begin{aligned}\big\langle f^{(\beta(k+1),k)}_{jm}, f^{(\beta(k'+1),k')}_{j'm'} \big\rangle_{\mathbb{C}^2/D} &= \frac{\beta^2+1}{2\pi} \int_{\mathbb{C}^2/D} (\mathfrak{X}_D \lrcorner d^4\omega)\, \overline{f^{(\beta(k+1),k)}_{jm}}\, f^{(\beta(k'+1),k')}_{j'm'} \\ &= \delta_{kk'}\delta_{jj'}\delta_{mm'},\end{aligned} \tag{4.16}$$

implicitly showing the integral (4.14) defines a nondegenerate inner product on the kernel of $\hat{D}$. We are now left with the remaining $F_n = 0$ constraint. Knowing the classical constraint generates an additional $\mathfrak{su}(2)$ algebra (remember equation (3.68)) it is not hard to see that the operators

$$\hat{F}_n = n^{A\bar{A}} \hat{\bar{\pi}}_{\bar{A}} \hat{\omega}_A, \quad \hat{\bar{F}}_n = \hat{F}_n^\dagger = n^{A\bar{A}} \hat{\pi}_A \hat{\bar{\omega}}_{\bar{A}}, \tag{4.17}$$

act as creation and annihilation operators for the quantum number k, more explicitly:

$$\hat{F}_{n_o} f^{(\rho,k)}_{jm} = -\frac{\hbar}{\sqrt{2}} \sqrt{(j-k)(j+k+1)}\, f^{(\rho,k+1)}_{jm}, \tag{4.18a}$$

$$\hat{F}^\dagger_{n_o} f^{(\rho,k)}_{jm} = -\frac{\hbar}{\sqrt{2}} \sqrt{(j+k)(j-k+1)}\, f^{(\rho,k-1)}_{jm}. \tag{4.18b}$$

Here we have chosen time gauge where the normal $n_o^{A\bar{A}}$ assumes the form of (3.20). Unless $j = 0 = k$ we cannot find states in the kernel of both $\hat{F}_n$ and its Hermitian conjugate, which reflects F_n is of second class. We proceed with Gupta and Bleuler [124, 125] and impose only one half of the constraint. The kernel of $\hat{F}_n$ is spanned by states $k = j$:

$$\hat{F}_{n_o} f^{(\rho,j)}_{jm} = 0, \tag{4.19}$$

while $k = -j$ labels the states in the kernel of its Hermitian conjugate:

$$\hat{F}^\dagger_{n_o} f^{(\rho,-j)}_{jm} = 0. \tag{4.20}$$

We can restrict ourselves to only one of these two possibilities. This is motivated as follows. The quantum number k is an eigenvalue of the operator $\hat{\omega}^A\hat{\pi}_A + \hat{\pi}_A\hat{\omega}^A - \text{hc.}$, we have

$$\big[\hat{\omega}^A\hat{\pi}_A + \hat{\pi}_A\hat{\omega}^A - \text{hc.}\big] f^{(\rho,k)}_{jm} = 4\mathrm{i}\hbar k f^{(\rho,k)}_{jm}. \tag{4.21}$$

This operator represents the following classical quantity on the solution space of the simplicity constraints:

$$2\omega^A\pi_A - \text{cc.} = 4\mathrm{i}J. \tag{4.22}$$

Just below equation (3.53) we argued that we can always assume $J > 0$ thereby removing the discrete symmetry exchanging ω and π. If we agree on the constraint $J > 0$ also in quantum theory we can discard the solution (4.20) and just work with (4.19). The solution space of the simplicity constraints $\mathcal{H}_{\text{simpl}}$ is then restricted to the kernel of the operators $\hat{D}$ and $\hat{F}_{n_o}$, it is spanned by the basis vectors $f^{(\beta(j+1),j)}_{jm}$, and we so have

$$\mathcal{H}_{\text{simpl}} = \overline{\text{span}}\big\{f^{(\beta(j+1),j)}_{jm} : j = 0, \tfrac{1}{2}, \dots ; m = -j, \dots, j\big\}. \tag{4.23}$$

The Gauß constraint is the last remaining constraint to impose. As mentioned in above we only need to solve the rotational part of the Gauß law, which appears as a first-class constraint in our set of constraints (3.64). The canonical quantisation of the classical constraint becomes the operator

$$\hat{G}^{\text{rot}}_i = \sum_{i=1}^{4} \hat{L}^{(I)}_i \tag{4.24}$$

on $\bigotimes^4 L^2(\mathbb{C}^2, d^4\omega)$, with e.g.:

$$\hat{L}^{(1)}_i = \hat{L}_i \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}. \tag{4.25}$$

The constraint generates rotations leaving invariant the four-normal n_o, e.g. for $f \in \bigotimes^4 L^2(\mathbb{C}^2, d^4\omega)$:

$$\begin{aligned}&\big(\exp(-\tfrac{\mathrm{i}}{\hbar}\hat{G}^{\text{rot}}_i\varphi^i)f\big)(\omega_{(1)}, \dots, \omega_{(1)}) = f(U\omega_{(1)}, \dots, U\omega_{(4)}),\\ &\text{with:}\quad U = \exp(\varphi^i\tau_i) \in SU(2).\end{aligned} \tag{4.26}$$

The kernel of all constraints, the Hilbert space $\mathcal{H}_{\text{phys}}$, is thus given by the $SU(2)$ invariant part of $\bigotimes^4 \mathcal{H}_{\text{simpl}}$, that is:

$$\mathcal{H}_{\text{phys}} = \Big(\bigotimes^4 \mathcal{H}_{\text{simpl}}\Big)/SU(2). \tag{4.27}$$

The general state in this Hilbert space looks like this:

$$\begin{aligned}\Psi(\omega_{(1)}, \omega_{(2)}, \omega_{(3)}, \omega_{(4)}) = \sum_{m_1=-j_1}^{j_1} \dots \sum_{j,\dots,j_4} \sum_{m_4=-j_4}^{j_4} I^{m_1\dots m_4}(j_1, \dots, j_4)\\ \cdot f^{(\beta(j_1+1),j_1)}_{j_1 m_1}(\omega_{(1)}) f^{(\beta(j_2+1),j_2)}_{j_2 m_2}(\omega_{(2)}) f^{(\beta(j_3+1),j_3)}_{j_3 m_3}(\omega_{(3)}) f^{(\beta(j_4+1),j_4)}_{j_4 m_4}(\omega_{(4)}),\end{aligned} \tag{4.28}$$

with $I^{m_1\dots m_4}(j_1, \dots, j_4)$ being an intertwiner, which is an element of the spin zero component of the tensor product of $SU(2)$ representations of spins $j_1, \dots, j_4$. The defining property of an intertwiner is that it be $SU(2)$ invariant:

$$\begin{aligned}\forall U \in SU(2) : \sum_{n_1=-j_1}^{j_1} \dots \sum_{n_4=-j_4}^{j_4} I^{n_1\dots n_4}(j_1, \dots, j_4) R^{(j_1)}(U)^{m_1}{}_{n_1} \cdots\\ \cdots \quad R^{(j_4)}(U)^{m_4}{}_{n_4} = I^{m_1\dots m_4}.\end{aligned} \tag{4.29}$$

Before we go on to the next section, let us comment on how to relax time gauge. For any two normals n and n_o on the upper hyperboloid $\eta_{\alpha\beta}n^\alpha n^\beta = -1,\ n^0 > 0$ there is a proper orthochronous Lorentz transformation that sends one to the other. Let g_n be the corresponding $SL(2,\mathbb{C})$ element such that:

$$n^{A\bar{A}} = g_{nB}{}^{A}\bar{g}_{n\bar{B}}{}^{\bar{A}} n_o^{B\bar{B}}. \tag{4.30}$$

We then also have:

$$\hat{F}_n = \mathcal{D}(g_n)^{-1}\hat{F}_{n_o}\mathcal{D}(g_n). \tag{4.31}$$

Any vector in the kernel of $\hat{F}_n$ can be constructed from its preimage in $\mathcal{H}_{\mathrm{simpl}}$. The vectors

$$f^{(\beta(j+1),j)}_{jm(n)} := \mathcal{D}(g_n)^{-1} f^{(\beta(j+1),j)}_{jm} \tag{4.32}$$

are in fact an orthonormal basis in the kernel of $\hat{F}_n$. More importantly, the constraints $\hat{F}_n$ and $\hat{F}_n^\dagger$ weakly vanish. All matrix elements of $\hat{F}_n$ with respect to states $f^{(\beta(j+1),j)}_{jm(n)}$ and $f^{(\beta(j'+1),j)}_{j'm'(n')}$ equate to zero:

$$\big\langle f^{(\beta(j+1),j)}_{jm(n)}, \hat{F}_n f^{(\beta(j'+1),j')}_{j'm'(n')}\big\rangle_{\mathbb{C}^2} = 0 = \big\langle f^{(\beta(j+1),j)}_{jm(n)}, \hat{F}_{n'} f^{(\beta(j'+1),j')}_{j'm'(n')}\big\rangle_{\mathbb{C}^2}. \tag{4.33}$$

As before, we are now left to impose the rotational part of the Gauß constraint. But we have left time gauge, and the $SU(2)$-Gauß law becomes boosted to:

$$\hat{G}^{\mathrm{rot}}_{i(n)} = \mathcal{D}(g_n)^{-1}\hat{G}^{\mathrm{rot}}_i\mathcal{D}(g_n). \tag{4.34}$$

The general solution $\Psi_{(n)}$ of all constraints can thus easily be found from (4.28) by just performing a unitary transformation:

$$\Psi_{(n)}(\omega_{(1)},\dots,\omega_{(4)}) = \big(\mathcal{D}(g_n)^{-1}\Psi\big)(\omega_{(1)},\dots,\omega_{(4)}) = \Psi(g_n\omega_{(1)},\dots,g_n\omega_{(4)}). \tag{4.35}$$

These states are nothing but the spinorial equivalent of Levine's projected spin network states [131, 195].

4.2 LOCAL SCHRÖDINGER EQUATION AND SPINFOAM AMPLITUDE

To begin with, consider only the evolution of the quantum states along a single edge. As in the classical part of the paper we can align the space-time normal in the middle of the edge to the canonical choice, that is we go to the time gauge (3.39) at $t = t_o = \frac{1}{2}$.

Classically, the Hamilton function governs the time evolution along an edge. Any function $O_t : \mathbb{T} \to \mathbb{R}$ on the phase space of a single triangle evolves according to

$$\frac{\mathrm{d}}{\mathrm{d}t}O_t = \Big\{\big(\Phi^{AB}(t)\pi_A\omega_B + \mathrm{cc.}\big) + \lambda(t)D, O_t\Big\}. \tag{4.36}$$

With $\Phi^{AB}(t)$ again being the selfdual connection contracted with the tangent vector of the edge, as defined in (3.43). When going to the quantum theory the Hamiltonian function becomes an operator defining the Schrödinger equation. With hc. denoting the Hermitian conjugate of everything preceding it reads

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\psi_t = \big(\Phi^{AB}(t)\hat{\pi}_A\hat{\omega}_B + \mathrm{hc.}\big)\psi_t + \lambda(t)\hat{D}\psi_t. \tag{4.37}$$

This is an important intermediate result. The Hamiltonian on the right hand side agrees with what Bianchi has reported in his thermodynamical considerations of spinfoam gravity [113]. If we restrict Φ to be a boost in the direction orthogonal to the triangle, we end up with the "boost-Hamiltonian" [108], that becomes the energy of a locally accelerated observer [72] once we are in the semi-classical regime.

At $t = t_o$ we are in the time gauge, physical states are annihilated by $\hat{F}_{n_o}$, and lie in the kernel of $\hat{D}$, such that our initial condition becomes

$$\psi_{t=t_o} = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} c^{jm} f_{jm}^{(\beta(j+1),j)}. \tag{4.38}$$

The last part of (4.37) vanishes on the physical Hilbert space, implying the Hamiltonian acts as an infinitesimal Lorentz transformation. We have:

$$\begin{aligned}\big(\Phi^{AB}\hat{\pi}_A\hat{\omega}_B + \text{hc.}\big)f(\omega^A) &= \mathrm{i}\hbar\big(\Phi^A{}_B\omega^B\frac{\partial}{\partial\omega^A} + \bar{\Phi}^{\bar{A}}{}_{\bar{B}}\bar{\omega}^{\bar{B}}\frac{\partial}{\partial\bar{\omega}^{\bar{A}}}\big)f(\omega^A) = \\ &= \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} f\big(\exp(\varepsilon\Phi)^A{}_B\omega^B\big).\end{aligned} \tag{4.39}$$

We can thus trivially integrate the Schrödinger equation to find:

$$\psi_t(\omega^A) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} c^{jm} f_{jm}^{(\beta(j+1),j)}\big(U^A{}_B(t,t_o)\omega^B\big) = \big(\mathcal{D}(U(t_o,t))\psi_{t_o}\big)(\omega^A). \tag{4.40}$$

Where we have inserted the $SL(2,\mathbb{C})$ holonomy along the edge, introduced in (3.72), and $\mathcal{D}$ is the unitary $SL(2,\mathbb{C})$ action as defined in (4.3). With the normal parallel along the edge, hence transported by the holonomy as in (3.83), equation (4.33) implies that the constraint $F_n = 0$ holds weakly for all times, i.e.:

$$\forall f \in \mathrm{kern}(\hat{D}), t \in [0,N) : \langle f, \hat{F}_{n(t)}\psi_t\rangle_{\mathbb{C}^2} = 0 = \langle f, \hat{F}^\dagger_{n(t)}\psi_t\rangle_{\mathbb{C}^2}. \tag{4.41}$$

Consider now the process where the spinor is "scattered" from $t_o = \frac{1}{2}$ into $t_1 = \frac{3}{2}$ passing through a vertex. Since we are both at t_o and t_1 in the canonical gauge (3.86), we can take as initial and final states

$$\psi_{t_1}^{\text{final}} = f_{j'm'}^{(\beta(j'+1),j')}, \quad \psi_{t_o}^{\text{initial}} = f_{jm}^{(\beta(j+1),j)}. \tag{4.42}$$

The corresponding transition amplitude is:

$$A(\psi_{t_o}^{\text{initial}} \to \psi_{t_1}^{\text{final}}) = \Big\langle f_{j'm'}^{(\beta(j'+1),j)}, \mathcal{D}(U(t_o,t_1)) f_{jm}^{(\beta(j+1),j)}\Big\rangle_{\mathbb{C}^2/D}, \tag{4.43}$$

which vanishes unless $j = j'$ due to (4.16). With j being the quantisation of J, as defined in (3.53), we see, also in quantum theory, the area of the triangle is preserved when going around the spinfoam face. This is the quantum theoretical version of the area-matching constraint $\dot{E} = 0$ introduced in (3.38).

We are now going to close the edges to form a loop, obtaining the amplitude* for a spinfoam face f. The boundary of the spinfoam face passes through vertices $v_1, \dots, v_N$ lying between edges $\{e_i\}_{i=1,\dots N}$ that go from the vertex v_{i-1} towards the i-th. By going

*In quantum mechanics, the analogue of what we are calculating here, is the "partition" function $Z(\mathrm{i}t) = \mathrm{Tr}(\mathrm{e}^{-\mathrm{i}t\hat{H}})$.

around the spinfoam face we will see N processes of the form of (4.43) happening. We write the elementary amplitude for the scattering process (4.43) at the i-th vertex in the condensed form of

$$\langle jm_{i+1}|g_{e_i,e_{i+1}}|jm_i\rangle = \Big\langle f^{(\beta(j+1),j)}_{jm_{i+1}}, \mathcal{D}(U(\tfrac{2i-1}{2}, \tfrac{2i+1}{2}))f^{(\beta(j+1),j)}_{jm_i}\Big\rangle_{\mathbb{C}^2/D}, \tag{4.44}$$

where we used the abbreviations

$$g_{e_i,e_{i+1}} = g^{\mathrm{source}}_{e_{i+1}}(g^{\mathrm{target}}_{e_i})^{-1}, \quad g^{\mathrm{target}}_{e_i} = U(i, \tfrac{2i-1}{2}), \quad g^{\mathrm{source}}_{e_i} = U(i-1, \tfrac{2i-1}{2}). \tag{4.45}$$

Here $i = N+1$ has everywhere to be identified with $i = 1$. We obtain the amplitude Z_f, i.e. the "partition" function for a spinfoam face f, by summing the product of the amplitudes for each individual process (4.43) over the orthonormal basis at the edges, that is we have to trace over spins j and $m_{i=1\dots N} = -j, \dots, j$. The resulting quantity depends parametrically on the edge holonomies $g_{e_i,e_{i+1}}$ as follows:

$$Z_f(\underline{g}) \equiv Z_f(g_{e_0,e_1}, \dots, g_{e_{N-1},e_0}) = \sum_{j=0}^{\infty}\sum_{m_1=-j}^{j}\dots\sum_{m_N=-j}^{j}\prod_{i=1}^{N}\langle jm_{i+1}|g_{e_i,e_{i+1}}|jm_i\rangle. \tag{4.46}$$

This expression gives us the amplitude for a single spinfoam face, and agrees with what we know from the EPRL model. To arrive at the full amplitude for the discretised space-time maniold we have to glue the contributions from the individual spinfoam faces. We take the product of all Z_f over all faces f appearing in the two-complex and integrate over the free gauge parameters left. These are the edge holonomies g^{source}_e and g^{target}_e. To ensure local Lorentz invariance [132], this measure must be invariant under $SL(2,\mathbb{C})$ transformations at the vertices. The simplest choice for such a measure just takes the $SL(2,\mathbb{C})$ Haar measure dg^{source}_e and dg^{target}_e at both the source and target points of each edge. The resulting spinfoam amplitude for the underlying discretised manifold is in exact agreement with the EPRL model.

Alternatives and ambiguities The most important ambiguity in this construction concerns the glueing of the individual face amplitudes to form the full spinfoam amplitude. Local Lorentz invariance alone does not fix the integration measure for the bulk holonomies g^{source}_e and g^{target}_e to be just a certain power of the Haar measure of $SL(2,\mathbb{C})$. Indeed, we can easily define a measure which is locally Lorentz invariant but does not agree with the choices made by the EPRL model.

To this goal, consider first the four-dimensional closure constraint (3.101), that we found in section 3.5 from discretising the four-dimensional torsion-free condition on a simplicial complex. For the EPRL model the additional torsional constraint (3.101) holds in the weakest possible sense. Only once we go to the semi-classical limit the bounding tetrahedra close to form a geometric four-simplex.* If we now want to impose this additional torsional constraint more strongly, we can achieve this by simply inserting an additional delta function at each spinfoam vertex, and would thus arrive at the following modification of the EPRL amplitude:

$$\begin{aligned}Z = \prod_{e:\mathrm{edges}}\int_{SL(2,\mathbb{C})} dg^{\mathrm{source}}_e \int_{SL(2,\mathbb{C})} dg^{\mathrm{target}}_e \sum_{\eta_e\in\{-1,1\}} \\ \prod_{v:\mathrm{vertices}} \delta_{\mathbb{R}^4}\Big(\sum_{\mathcal{T}\in v}\varepsilon[\mathcal{T},\eta_{e(\mathcal{T})}]n^\alpha[\mathcal{T}]\,{}^3\widehat{\mathrm{vol}}[\mathcal{T}]\Big)\prod_{f:\mathrm{faces}} Z_f(\underline{g}).\end{aligned} \tag{4.47}$$

*This follows from the asymptotic analysis of [175] and the possibility to reconstruct a four-simplex out of its fluxes as shown in the second supplement of chapter 3.

Here $n^\alpha[\mathcal{T}]$ is the normal of the tetrahedron $\mathcal{T}$ parallel transported into the center of the four-simplex, ${}^3\widehat{\mathrm{vol}}[\mathcal{T}]$ denotes the quantisation [38,56] of its volume (3.102), $e(\mathcal{T})$ is the edge dual to $\mathcal{T}$, while $\varepsilon[\mathcal{T},\eta_{e(\mathcal{T})}]$ gives the orientation of the tetrahedron relative to the vertex it is seen from. This sign tells us whether the outwardly pointing normal of the boundary of the four-simplex is future (i.e. $\varepsilon = +1$) or past (i.e. $\varepsilon = -1$) oriented.

For the sake of completeness let us give the missing definitions for the orientation $\varepsilon[\mathcal{T},\eta]$ and the time normals $n^\alpha[\mathcal{T}]$. Let $\mathcal{T}^{\text{target}}$ and $\mathcal{T}^{\text{source}}$ be the same tetrahedron seen from vertices v^{target} and v^{source}, and the intermediate edge $e(\mathcal{T}^{\text{source}}) = e(\mathcal{T}^{\text{target}})$ be oriented from v^{source} towards v^{target}. We define the orientation by setting:

$$\varepsilon[\mathcal{T}^{\text{source}},\pm 1] = \pm 1, \quad \varepsilon[\mathcal{T}^{\text{target}},\pm 1] = \mp 1. \tag{4.48}$$

The time normals are given by equation (3.83) implying:

$$\frac{\mathrm{i}}{\sqrt{2}}\delta^{A\bar{A}} = \big(g_e^{\text{target}}\big)_B{}^A\big(\bar{g}_e^{\text{target}}\big)_{\bar{B}}{}^{\bar{A}} n^{B\bar{B}}[\mathcal{T}^{\text{target}}], \tag{4.49a}$$

$$\frac{\mathrm{i}}{\sqrt{2}}\delta^{A\bar{A}} = \big(g_e^{\text{source}}\big)_B{}^A\big(\bar{g}_e^{\text{source}}\big)_{\bar{B}}{}^{\bar{A}} n^{B\bar{B}}[\mathcal{T}^{\text{source}}]. \tag{4.49b}$$

We do not want to give a detailed analysis of the proposed amplitude (4.47). Let us just make an immediate observation. This concerns causality. The function ε defined in (4.48) assigns to any tetrahedron a local time orientation, and tells us whether the outwardly pointing four-normal of a tetrahedron bounding a four-simplex is future or past oriented—that is, so to say, whether the tetrahedron "enters" or "leaves" the four-simplex. This would distinguish four-simplices corresponding to 3-1 (1-3) moves from those representing 4-1 (1-4) moves, which could eventually introduce a notion of causality for spinfoam gravity.

The main lesson from these considerations is not so much that we can define yet another model, but rather that we are lacking a universal recipe of how to glue the individual amplitudes together. There are many different models floating around [20, 126, 127, 165, 196], and we are lacking the tools to judge which of them are actually valuable and worth to investigate more carefully. Studying the semi-classical limit alone may be too difficult and time consuming. I think, the most powerful guiding principle towards an unambiguous definition of spinfoam gravity, is to go the other way around: To start from a classical discretisation of general relativity, find a Hamiltonian formulation of the discretised theory, and canonically quantise in order to then define a path integral over trajectories in the phase-space of the theory [114].

The results of this thesis achieve this only partially, but yet they clearly support the EPRL model. We have, in fact, only derived the spinfoam face amplitudes: We could show that these amplitudes arise from the canonical quantisation of a version of first-order Regge calculus, with spinors as the elementary configuration variables. What is missing is a principle that could tell us how to glue the individual amplitudes together, and I thik, the only way to find such a principle is to look for a fully covariant phase-space description. This would require first, to lift the dynamics to an even larger phase space, where there are also canonical momenta for the time-normals and Lagrange multipliers appearing in the action (3.46). Defining the canonical integration measure on the reduced phase space of the theory would then lead us to a fully covariant spinfoam model, which would be unique only up to the notorious ordering ambiguities. The techniques needed to study this problem have all been developed in the previous chapters, yet it lies beyond the scope of this thesis to actually complete this task.

4.3 SUMMARY

Let us briefly summarise this chapter. The first part concerned the kinematical structure. We took the classical phase space of twistors on a half link and followed the program of canonical quantistaion. The classical constraint equations turned into quantum operators, that define the physical state space of the theory. Solutions of the first class constraints lie in the kernel of both the Gauß law (4.24) and the "diagonal" simplicity constraint $\hat{D}$ (4.12). The constraints $F_n = 0 = \bar{F}_n$ (as in (3.19)), on the other hand, cannot be simultaneously diagonalised, for they do not Poisson commute among another (3.68, 3.69). Instead, they form a system of second-class constraints. The quantisation proceeds with Gupta and Bleuler. We separated the second class constraints in two parts, one being the Hermitian conjugate of the other. The first half annihilates physical states while the Hermitian conjugate maps them to their orthogonal complement. We have thus imposed the second class constraints weakly—all matrix elements between physical states vanish (4.33) on the physical Hilbertspace of the theory.

The resulting Hilbertspace agrees with the Hilbertspace of a quantised tetrahedron as it appears in the usual loop gravity Hilbertspace. The area matching constraint (3.14) glues these quantised tetrahedra along the bounding triangles, eventually forming a Hilbertspace that is isomorphic to the space of four-valent $SU(2)$ spinnetwork functions.

The last chapter was about the spinfoam amplitude. Here we only have a partial result concerning the dynamics on a spinfoam face. We could derive these amplitudes from the canonical quantisation of a classical action (3.46), which is a version of constrained "BF"-theory written in terms of spinorial variables. To obtain a complete spinfoam model, we have to glue these amplitudes together. We discussed ambiguities in this construction, and argued that only a fully covariant path integral formulation could lead to an unambiguous definition of the transition amplitudes.

5
Conclusion

5.1 DISCUSSION OF THE KEY RESULTS

To canonically quantise gravity it is often thought that one first needs to start from a 3+1 split, study the ADM (Arnowitt–Deser–Miser) formulation in the "right" variables, identify the canonical structure and perform a Schrödinger quantisation. The results of this thesis question this idea. The ADM formulation is very well adapted to a continuous space-time, but in spinfoam gravity we are working with a discretisation of the manifold, hence lacking that assumption. Instead we have simplices glued together and should find a Hamiltonian formulation better adapted to the problem.

After the introductory chapters 1 and 2, we found such a Hamiltonian formalism for the discretised theory. The underlying Hamiltonian generates the time evolution along the edges of the spinfoam. The corresponding time variable parametrises the edges of the discretisation, it is nothing but a coordinate, and does not measure duration as given by a clock.

Our construction started from the topological "BF"-theory (3.1), and took the spinorial framework of loop quantum gravity to parametrise the discretised action. This we did for technical reasons only, spinors do not add anything physically new to the theory. The key idea was then to perform a limiting process that partially brings us back to the continuum. We split every wedge into smaller and smaller parts, until we obtained a continuum action on an edge. Next, we added the simplicity constraints to the action. The equations of motion allowed for a Hamiltonian formulation. We studied the Dirac analysis of the constraint algebra. All constraints are preserved in time (i.e. along the boundary of the spinfoam face) provided the Lagrange multiplier in front of the second-class constraint $F_n = 0$ vanishes.

The classical part concluded with a reflection on the role of torsion in a discrete theory of gravity. We saw, torsion implies the closure of the elementary building blocks of geometry. The Gauß law for each tetrahedron is one of these closure conditions, but there is also an additional four-dimensional closure constraint (3.101) to be fulfilled. This constraint demands that on every four-simplex the outwardly pointing normals of the bounding tetrahedra weighted by their volumes sum up to zero. What happens in three dimensions is also true in four dimensions: The closure constraints guarantee the geometricity of the elementary building blocks through Minkowski's reconstruction theorem, which holds also in Minkowski space. The additional torsional condition is fulfilled only once we go to the solution space of all the equations of motion. In quantum theory the four-dimensional closure constraint therefore holds in the weakest possible way: Only at the saddle point of the spinfoam amplitude [175] we would see the bounding tetrahedra close to form a four-simplex. We argued that this may be yet too weak, and that the four-dimensional closure constraint (3.101) could be imposed more strongly.

In summary, the classical part introduced a canonical formulation of spinfoam gravity adapted to a simplicial discretisation of space-time. This framework should be of general interest, as it provides a solid foundation where different models could fruitfully be compared.

The last section was about quantum theory. With the Hamiltonian formulation of the spinfoam dynamics at hand, canonical quantisation was straight-forward. We used an auxiliary Hilbert space to define the operators. Physical states are in the kernel of the first-class constraints. The second-class constraints act as ladder operators. One of them ($\hat{F}_n$) annihilates physical states, while the other one ($\hat{F}_n^\dagger$) maps them to their orthogonal complement, i.e. into the spurious part of the auxiliary Hilbert space. This is exactly what happens in the Gupta–Bleuler formalism.

Dynamics is determined by the Schrödinger equation. We quantised the classical Hamiltonian and solved the Schrödinger equation that gives the evolution of the quantum states along the boundary of a spinfoam face. This boundary evolution matched the Schrödinger equation introduced by Bianchi in the thermodynamical analysis of spinfoam gravity [113]. Gluing the individual transition amplitudes together, we got the amplitude for a spinfoam face, which was in exact agreement with the EPRL model.

5.2 FUTURE RESEARCH INTERESTS

Generalisation to null hypersurfaces If we replace the time normal appearing in the simplicity constraints (3.18) by a null vector we could define spinfoam models for light-like tetrahedra. The resulting null spinfoams could lead to a better understanding of event-horizons, black holes and the causal structure of the quantum theory. In fact, our spinorial action immediately calls for a generalisation to null hypersurfaces, simply because a spinor ℓ^A defines both a null vector $\ell^\alpha \equiv \mathrm{i}\ell^A\bar{\ell}^{\bar{A}}$ and a null-plane $\Sigma_{AB} \propto \ell_A\ell_B$. First steps towards this generalisation have already been reported by Zhang at the quadrennial Loops conference [197].

Flatness problem and the relation to GR Spinfoam gravity suffers from the so-called flatness problem. The analysis of [171–174] shows that the curvature in a spinfoam face must satisfy an unexpected flatness conditions.

We can see this constraint already at the classical level. The equations of motion for the spinors imply the geometricity of the four-simplex: Each bone bounding a triangle has a unique edge length, and all bones close to form a flat four-simplex in Minkowski space. Each edge is dual to a tetrahedron, and we can ask how the shape of a tetrahedron changes once we move along its dual edge and thus go from one vertex to its neighbouring four-simplex. Looking back at the equations of motion for the spinors (3.71), we can easily show that the Hamiltonian flow preserves the shape of the triangles. The tetrahedron gets boosted, yet the shape of the triangles in the frame of the center of the tetrahedron remains unchanged. This means that we are in a Regge geometry, each bone bounding a triangle has a unique length, from whatever four-simplex we look at it.

Once we are in a Regge geometry, the holonomy $U^A{}_B$ around the spinfoam face must be a pure boost, in the notation of section 3.2.5 this means that:

$$U^A{}_B \stackrel{!}{=} \pm(\pi\omega)^{-1}\big(\mathrm{e}^{-\frac{\Xi}{2}}\omega^A\pi_B - \mathrm{e}^{+\frac{\Xi}{2}}\pi^A\omega_B\big). \tag{5.1}$$

Looking back at equation (3.75) this implies the unexpected [171–174] condition:

$$\beta\Xi \in 2\mathbb{N}_0. \tag{5.2}$$

Notice that this condition follows from the analysis of the classical theory, and has therefore nothing to do with the quantum theory itself. Equation (5.2) raises a problem, because it does not appear in Regge [64, 170] calculus, and even if we would get rid of it, it is far from obvious whether the solutions of the equations of motion for the spinors approximate Ricci flat space times.

That this problem reappears already at the level of the classical action is an important result, for it suggest not to focus on the quantum theory and its semi-classical limit but to better understand the classical theory behind spinfoam gravity. I think, carefully tuning the quantum amplitudes won't fix the trouble. Instead we should look back at the classical theory as defined in chapter 3. We have shown that the quantisation of this theory leads to spinfoam gravity, it suffers from the same problems as the quantum theory, and thus offers an ideal framework to study the issues raised by [171–174]. An immediate possibility would be to abandon equation (3.40), and turn the time normals of the tetrahedra into dynamical variables.

Inclusion of matter To aim at a phenomenology of loop quantum gravity [198–200], strong enough to turn it falsifiable, we need to better understand how matter (our "rulers" and "clocks") couples to the theory. Unfortunately, after decades of research, we still cannot say much about this issue. To overcome this trouble, I can see four roads to attack the problem, three of which I would like to study by myself:

(i) At first, there is what has been always tried in loop quantum gravity when it comes to this problem. Take any standard matter described by some Lagrangian, put in on an irregular lattice corresponding to a spinnetwork state and canonically quantise. Although this approach was tried for all kinds of matter it led to very little physical insight. I think it is time to try different strategies.

(ii) The first idea that comes to my mind originates from an old paper by t' Hooft [201]. I think it is a logical possibility that loop quantum gravity already contains a certain form of matter. If we look at the curvature of our models we find it is concentrated on the two-dimensional surfaces of the spinfoam faces. This curvature has a non-vanishing Ricci part which we can use (employing Einstein's equations) to assign an energy momentum tensor to the spinfoam face. Following this logic one may then be able to reformulate the dynamics of spinfoam gravity as a scattering process of these two-dimensional worldsheets (that now carry energy-momentum) in a locally flat ambient space.

(iii) Loop quantum gravity is a theory of quantised *area-angle-variables*. I think this suggests not to start from the standard model that couples matter to tetrad (i.e. *length-angle*) variables. Instead we should take the fundamental discreteness of loop quantum gravity seriously, and try to add matter fields to the natural geometrical structures appearing, e.g. the two-dimensional spinfoam faces. In fact, when looking at the kinetic term of the action (3.46) a candidate immediately appears. We could just replace the commuting (π, ω) spinors by anti-commuting Weyl (Majorana) spinors, yielding a simple coupling of uncharged spin $1/2$ particles to a spinfoam.

(iv) The recent understanding of loop quantum gravity in terms of twistors is mirrored [202–206]by similar developments in the study of scattering amplitudes of e.g. $\mathcal{N} = 4$ super Yang–Mills theory . It is tempting to say these results all point towards the same direction eventually yielding a twistorial framework for all interactions. [].

A
Appendices

A.1 THE LORENTZ GROUP

We start by reviewing some basic facts [29, 207] of the Lorentz group L and its corresponding Lie algebra. A linear transformation $X^\mu \mapsto \Lambda^\mu{}_\nu X^\nu$ of the inertial coordiantes in four-dimensional Minkowski space is said to be a Lorentz transformation if it leaves the metric unchanged, that is:

$$\Lambda \in L \Leftrightarrow \eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta}. \tag{A.1}$$

This group falls into four disconnected parts, one of which is the subgroup $L^\uparrow_+$ of special (i.e. $\det(\Lambda) > 0$) orthochronous transformations (i.e. $\Lambda^0{}_0 > 0$). All elements of this subgroub are continuously connected to the identity and can be reached by the exponential map:

$$\Lambda^\alpha{}_\beta = \exp(\omega)^\alpha{}_\beta = \delta^\alpha_\beta + \omega^\alpha{}_\beta + \frac{1}{2}\omega^\alpha{}_\mu \omega^\mu{}_\beta + \ldots. \tag{A.2}$$

Time reversal $T : (X^0,\, X^1,\, X^2,\, X^3) \mapsto (-X^0,\, X^1,\, X^2,\, X^3)$ and parity $P : (X^0,\, X^1,\, X^2,\, X^3) \mapsto (X^0,\, -X^1,\, -X^2,\, -X^3)$ relate the remaining, mutually disconnected parts $L^\uparrow_-$, $L^\downarrow_+$ and $L^\downarrow_-$ of the Lorentz group among one another. We have $L^\uparrow_- = PL^\uparrow_+$, $L^\downarrow_- = TL^\uparrow_+$, and $L^\uparrow_\pm = PTL^\downarrow_\pm$.

To study the Lie algebra we look at tangent vectors at the identity. If Λ_ε is a smooth one parameter family of Lorentz transformations passing through the identity at $\varepsilon = 0$, equation (A.1) implies that the tangent vector

$$\omega^\mu{}_\nu = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} (\Lambda_\varepsilon)^\mu{}_\nu \tag{A.3}$$

is antisymmetric in its lowered indices:

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0. \tag{A.4}$$

We so have that the Lie algebra of the Lorentz group is nothing but:

$$\mathrm{Lie}(L^\uparrow_+) = \mathfrak{so}(1,3) = \left\{\omega^\mu{}_\nu \in \mathbb{R}^4 \otimes (\mathbb{R}^4)^* \middle| \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0\right\}. \tag{A.5}$$

Let us now choose a basis in the Lie algebra and determine both commutation relations and structure constants. Consider the matrices $M_{\alpha\beta}$ defined by explicitly stating their row (i.e. μ) and column (i.e. ν) entries:

$$[M_{\alpha\beta}]^\mu{}_\nu : [M_{\alpha\beta}]^{\mu\nu} = 2\delta^{[\mu}_\alpha \delta^{\nu]}_\beta. \tag{A.6}$$

These matrices form a basis, since for any $\omega \in \mathfrak{so}(1,3)$ we can trivially write $\omega = \frac{1}{2}M_{\alpha\beta}\omega^{\alpha\beta}$. Suppressing the row and column indices of $[M_{\alpha\beta}]^{\mu}{}_{\nu}$ we compute the commutator and find:

$$\big[M_{\alpha\beta}, M_{\mu\nu}\big] = 4\delta_{\alpha}^{[\alpha'}\delta_{\beta}^{\beta']}\eta_{\beta'\mu'}\delta_{\mu}^{[\mu'}\delta_{\nu}^{\nu']}M_{\alpha'\nu'}. \tag{A.7}$$

We can now define the generators of boost K_i and rotations L_i:

$$L_i := \frac{\mathrm{i}}{2}\epsilon^{l}{}_{i}{}^{m}M_{lm}, \quad K_i := \mathrm{i}M_{i0}. \tag{A.8}$$

The commutations relations (A.7) imply for both K_i and L_i that:

$$[L_i, L_j] = \mathrm{i}\epsilon_{ij}{}^{l}L_l, \quad [L_i, K_j] = \mathrm{i}\epsilon_{ij}{}^{l}K_l \quad [K_i, K_j] = -\mathrm{i}\epsilon_{ij}{}^{l}L_l. \tag{A.9}$$

We see, L_i is the generator of rotations leaving invariant the X^0 coordinate, K_i transforms as a vector under rotations, while the commutator of two infinitesimal boosts gives an infinitesimal rotation. We can diagonalise this algebra by introducing the complex generators:

$$\Pi_i := \frac{1}{2}\big(L_i + \mathrm{i}K_i\big), \quad \bar{\Pi}_i := \frac{1}{2}\big(L_i - \mathrm{i}K_i\big). \tag{A.10}$$

That obey the commutation relations of two copies of $\mathfrak{su}(2)$:

$$[\Pi_i, \Pi_j] = \mathrm{i}\epsilon_{ij}{}^{l}\Pi_l, \quad [\bar{\Pi}_i, \bar{\Pi}_j] = \mathrm{i}\epsilon_{ij}{}^{l}\bar{\Pi}_l, \quad [\Pi_i, \bar{\Pi}_j] = 0. \tag{A.11}$$

Introducing this basis does however *not* prove that $\mathfrak{so}(1,3)$ be equal two copies of $\mathfrak{su}(2)$. This becomes explicit when studying how a generic Lie algebra element decomposes into these complex generators. We have

$$\mathfrak{so}(1,3) \ni \omega = \frac{1}{2}M_{\alpha\beta}\omega^{\alpha\beta} = -\mathrm{i}\Pi_i\omega^i - \mathrm{i}\bar{\Pi}_i\bar{\omega}^i. \tag{A.12}$$

Where there appears the complex component vector:

$$\omega^i = \frac{1}{2}\epsilon_{l}{}^{i}{}_{m}\omega^{lm} + \mathrm{i}\omega^{i}{}_{o}. \tag{A.13}$$

We close this section by giving the Casimirs of the Lorentz group. In our conventions we can write them as:

$$C_1 := \frac{1}{4}\epsilon^{\alpha\beta\mu\nu}M_{\alpha\beta}M_{\mu\nu} = 2L_iK^i = 4\mathfrak{Im}\big(\Pi_i\Pi^i\big), \tag{A.14a}$$

$$C_2 := \frac{1}{2}M_{\alpha\beta}M^{\alpha\beta} = K_iK^i - L_iL^i = -4\mathfrak{Re}\big(\Pi_i\Pi^i\big). \tag{A.14b}$$

A.2 SPINORS AND THE LORENTZ GROUP

The $SL(2,\mathbb{C})$ group is the universal cover of the group $L_{+}^{\uparrow}$ of proper orthochronous Lorentz transformations. We can best understand the intertwining map Λ relating the first with the second when studying the vector space of anti-Hermitian* 2×2 matrices. Given such a matrix we denote its row indices by roman capitals $A, B, C, \dots \in \{0,1\}$, while indices referring to the column should carry a makron, i.e. an "over-bar" such

*That we do not work here with Hermitian matrices but rather use anti-Hermitian elements of $\mathbb{C}^2 \otimes \bar{\mathbb{C}}^2$ is related to the choice of signature $(-,+,+,+)$ that we have agreed on earlier.

that we write $\bar{A}, \bar{B}, \bar{C}, \dots \in \{\bar{0}, \bar{1}\}$. The unmarked indices A, B, C refer to $\mathbb{C}^2$ and we call them *left-handed*, while their brothers $\bar{A}, \bar{B}, \bar{C}, \dots$ belong to the complex conjugate vector space $\bar{\mathbb{C}}^2$, and we call them *right-handed* or *of opposite chirality*. The logic behind this terminology should become clear in a moment. Using this index-notation we can express the antihermiticity of a matrix X by saying:

$$X^{A\bar{B}} \text{ is antihermitian} \Leftrightarrow \overline{X^{A\bar{B}}} \equiv \bar{X}^{\bar{A}B} = -X^{B\bar{A}}. \tag{A.15}$$

The soldering matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.16}$$

form a basis in the vector space of all hermitian 2×2 matrices. We introduce the component vector $X^\mu \in \mathbb{R}^4$ of $X^{A\bar{A}}$ with respect to this basis and set:

$$X^{A\bar{A}} = \frac{\mathrm{i}}{\sqrt{2}} \sigma^{A\bar{A}}{}_\alpha X^\alpha. \tag{A.17}$$

We can write this more abstractly by equating the component-vector with the matrix itself. i.e.:

$$X^{A\bar{A}} = X^\alpha. \tag{A.18}$$

The isomorphism (A.17) allows us to identify Minkowski indices with pairs of spinorial idices, for a generic world tensor $T^{\alpha\beta\dots}$ we can thus write:

$$T^{\alpha\beta\dots} = T^{A\bar{A}B\bar{B}\dots} \tag{A.19}$$

While the individual ordering of ordinary (right-handed) indices $A, B, C, \dots$ and their left-handed brothers $\bar{A}, \bar{B}, \bar{C}$ matters, the relative ordering has no significance. With our choice of signature it turns out to be useful to declare the exchange of two adjacent indices of opposite chirality to result in an overall minus sign, e.g.:

$$X^{A\bar{A}} = -X^{\bar{A}A}. \tag{A.20}$$

A short moment of reflection reveals now that the determinant of the matrix (A.17) turns into the Minkowski inner product of its components, for any $X^\mu \in \mathbb{R}^4$:

$$2\det(X) = -(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 = \eta_{\alpha\beta} X^\alpha X^\beta. \tag{A.21}$$

Let us now understand how Lorentz transformations can act on these spinorial indices. For any element $g \in SL(2, \mathbb{C})$ the definition

$$\forall g \in SL(2, \mathbb{C}) : \big(\Lambda(g)X\big)^{A\bar{A}} = g^A{}_B X^{B\bar{B}} \bar{g}^{\bar{A}}{}_{\bar{B}} \equiv g X g^\dagger \tag{A.22}$$

defines a representation of $SL(2, \mathbb{C})$ on the vector space $\mathbb{R}^4$ formed by the components X^μ. This representation defines a linear map

$$\Lambda(g) : \mathbb{R}^4 \to \mathbb{R}^4, X^\mu \mapsto \big(\Lambda(g)X\big)^\mu = \Lambda(g)^\mu{}_\nu X^\nu. \tag{A.23}$$

The component matrix $\Lambda^\mu{}_\nu$ must be a Lorentz transformation. This can be seen by looking at the Minkowski norm as defined by equation (A.21), and noting that the definition (A.22) cannot affect this norm simply because the determinant of a product of matrices equals the product of the determinant of the individual constituents, that is:

$$\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}X^{\alpha}\Lambda^{\nu}{}_{\beta}X^{\beta} = 2\det(gXg^{\dagger}) = 2\det(g)\det(X)\overline{\det(g)} = \\ = 2\det(X) = X_{\Lambda}X^{\alpha}. \tag{A.24}$$

Where we used that for any $g \in SL(2,\mathbb{C})$ we have, ipso facto, $\det(g) = 1$. Since (A.2) holds for all $X^{\mu} \in \mathbb{R}^4$, we can see $\Lambda(g)$ must be a Lorentz transformation. Since $SL(2,\mathbb{C})$ is simply connected and the map $g \mapsto \Lambda(g)$ is continuous, we can also continuously connect the image $\Lambda^{\mu}{}_{\nu}(g)$ with the identity δ^{μ}_{ν}. Therefore, $\Lambda(g)$ must be a proper orthochronous Lorentz transformation, i.e. an element of $L^{\uparrow}_{+}$. In fact, $\Lambda(g)$ can cover all of $L^{\uparrow}_{+}$. This can be seen by e.g. studying the corresponding Lie algebras $\mathfrak{so}(1,3)$ and $\mathfrak{sl}(2,\mathbb{C})$, establishing the isomorphism between the two, and recognising that for both $L^{\uparrow}_{+}$ and $SL(2,\mathbb{C})$ the exponential map can reach any element of the two respective groups. The homomorphism $\Lambda : SL(2,\mathbb{C}) \to L^{\uparrow}_{+}$ so established is however not invertible; for every $\lambda \in L^{\uparrow}_{+}$ there are exactly two elements $g, g' \in SL(2,\mathbb{C})$, equal up to a sign $g = -g'$, that are both mapped towards the same $\lambda = \Lambda(g) = \Lambda(-g)$. We thus see, $SL(2,\mathbb{C})$ is the universal cover of the group of proper orthochronous Lorentz transformation, and thus plays the same role that $SU(2)$ has for $SO(3)$.

Let us now delve more into the structure of $\mathbb{C}^2$ and its complex conjugate vector space $\bar{\mathbb{C}}^2$. Complex conjugation of the components relates one with the other:

$$\overline{\cdot} : \mathbb{C}^2 \to \bar{\mathbb{C}}^2, \quad \omega^A \mapsto \overline{\omega^A} = \bar{\omega}^{\bar{A}}, \tag{A.25}$$

and analogously for the dual vector spaces $(\mathbb{C}^2)^*$, and $(\bar{\mathbb{C}}^2)^*$, the elements of which we can write as ω_A and $\bar{\omega}_{\bar{A}}$ respectively. The spinor indices transform under the fundamental or defining transformation of $SL(2,\mathbb{C})$, that is just by matrix multiplication:

$$(g\omega)^A = g^A{}_B\omega^B. \tag{A.26}$$

All finite dimensional representations of $SL(2,\mathbb{C})$ are labelled by spins $(j,k) \in \frac{1}{2}\mathbb{N}_0 \times \frac{1}{2}\mathbb{N}_0$ and can be constructed by simply tensoring the fundamental representation.

$$\mathcal{H}_{j,k} := \mathrm{sym}\Big(\bigotimes^{2j}\mathbb{C}^2\Big) \otimes \mathrm{sym}\Big(\bigotimes^{2k}\bar{\mathbb{C}}^2\Big), \tag{A.27}$$

where sym denotes total symmetrisation of the respective tensor product. For an element $\Psi \in \mathcal{H}_{j,k}$ the irreducible group action is simply given by:

$$(g\Psi)^{A_1\dots A_{2j}\bar{A}_1\dots\bar{A}_{2\bar{j}}} = g^{A_1}{}_{B_1}\cdots g^{A_{2j}}{}_{B_{2j}}\bar{g}^{\bar{A}_1}{}_{\bar{B}_1}\cdots\bar{g}^{\bar{A}_{2\bar{j}}}{}_{\bar{B}_{2k}}\Psi^{B_1\dots B_{2j}\bar{B}_1\dots\bar{B}_{2k}} \tag{A.28}$$

Comparing this equation with (A.17) we see Minkwoski vectors belong to the $(\frac{1}{2},\frac{1}{2})$ representation of $SL(2,\mathbb{C})$.

We have seen complex conjugation relates $\mathbb{C}^2$ with $\bar{\mathbb{C}}^2$, what we now need is an object allowing us to move the $A, B, C, \dots$ and $\bar{A}, \bar{B}, \bar{C}$ indices, that is a map from e.g. $\mathbb{C}^2$ to its algebraic dual $(\mathbb{C}^2)^*$. This should be done respecting the symmetry group in question. To rise and lower Minkowski indices, we use the metric $\eta_{\mu\nu}$ and not any other non-degenerate two-index tensor, simply because $\eta_{\mu\nu}$ keeps unchanged if going from one inertial frame to another, while for a generic tensor this would not be not true anymore. We have seen, in e.g. (A.22, A.2) that $SL(2,\mathbb{C})$ linearly acts onto these indices when performing a Lorentz transformation. But there is an invariant object for this $SL(2,\mathbb{C})$ action, easily identified as the anti-symmetric ϵ-tensor $\epsilon_{AB} = \epsilon_{BA}$ by observing:

$$\forall g \in SL(2,\mathbb{C}) : \epsilon_{CD} g^C{}_A g^D{}_B = \det(g)\epsilon_{AB} = \epsilon_{AB}. \tag{A.29}$$

Since, again, for any $g \in SL(2,\mathbb{C})$, $\det(g) = 1$. We define its contravariant version ϵ^{AB} by demanding:

$$\epsilon_{AC}\epsilon^{BC} \equiv \epsilon_A{}^B \stackrel{!}{=} \delta^B_A, \quad \bar{\epsilon}_{\bar{A}\bar{B}} := \overline{\epsilon_{AB}}, \quad \bar{\epsilon}^{\bar{A}\bar{B}} := \overline{\epsilon^{AB}} \tag{A.30}$$

We can now use the ϵ-tensor to establish the natural isomorphism between $\mathbb{C}^2$ and its dual vector space:

$$\mathbb{C}^2 \ni \omega^A \longmapsto \omega_A = \epsilon_{BA}\omega^A \in \big(\mathbb{C}^2\big)^*, \tag{A.31}$$

$$\big(\mathbb{C}^2\big)^* \ni \omega_A \longmapsto \omega^A = \epsilon^{AB}\omega_B \in \mathbb{C}^2. \tag{A.32}$$

and equally for the complex conjugate vector space. Here one has to be careful with index positions, particularly illustrated by the identity:

$$\pi_A\omega^A = -\pi^A\omega_A. \tag{A.33}$$

We finally fix our conventions by choosing the matrix elements of the ϵ-tensor as:

$$\epsilon_{01} = 1 = -\epsilon_{10}, \quad \epsilon_{00} = 0 = \epsilon_{11} \tag{A.34}$$

That the ϵ tensor plays the role of a metric is particularly well ilustrated when writing the Minkowski metric according to (A.19). Equation implies:

$$\eta_{\alpha\beta} = \eta_{A\bar{A}B\bar{B}} = \epsilon_{AB}\bar{\epsilon}_{\bar{A}\bar{B}} \tag{A.35}$$

We close this section by studying the Lie algebra of $SL(2,\mathbb{C})$, and giving the relation to $\mathfrak{so}(1,3)$. A possible basis in $\mathfrak{sl}(2,\mathbb{C})$ is given by the wedge product of the soldering forms:

$$\Sigma^A{}_{B\alpha\beta} = -\frac{1}{2}\sigma^{A\bar{C}}{}_{[\alpha}\bar{\sigma}_{\bar{C}B\beta]} \tag{A.36}$$

where $\bar{\sigma}_{\bar{A}A\alpha}$ is nothing but $\sigma_{A\bar{A}\alpha}$, and the antisymmetrisation has to be taken over the pair $[\alpha,\beta]$ of indices only. These matrices obey the commutation relations of the Lorentz algebra. Surpressing the spinor indices $A, B, \dots$ we have in fact:

$$\big[\Sigma_{\alpha\beta}, \Sigma_{\mu\nu}\big] = 4\delta^{\alpha'}_{[\alpha}\delta^{\beta'}_{\beta]}\eta_{\beta'\mu'}\delta^{\mu'}_{[\mu}\delta^{\nu'}_{\nu]}\Sigma_{\mu'\nu'} \tag{A.37}$$

Using this basis we can write the isomorphism induced by (A.2) between $\mathfrak{so}(1,3)$ and $\mathfrak{sl}(2,\mathbb{C})$ by stating:

$$\Lambda_* : \mathfrak{sl}(2,\mathbb{C}) \ni \frac{1}{2}\Sigma_{\alpha\beta}\omega^{\alpha\beta} \longmapsto \omega^\alpha{}_\beta \in \mathfrak{so}(1,3). \tag{A.38}$$

These generators correspond to the selfdual sector of the Lorentz algebra, e.g.

$$\Sigma_{\alpha\beta}P^{\alpha\beta}{}_{\mu\nu} = \Sigma_{\mu\nu}. \tag{A.39}$$

Where we have introduced the selfdual projector

$$P^{\alpha\beta}{}_{\mu\nu} = \frac{1}{2}\big(\delta^{[\alpha}_\mu\delta^{\beta]}_\nu - \frac{\mathrm{i}}{2}\epsilon^{\alpha\beta}{}_{\mu\nu}\big). \tag{A.40}$$

Furthermore for any $\omega \in \mathfrak{so}(1,3)$ we find that:

$$\frac{1}{2}\Sigma_{\alpha\beta}\omega^{\alpha\beta} = \tau_i\Big(\frac{1}{2}\epsilon_m{}^{in}\omega^m{}_n + \mathrm{i}\omega^i{}_o\Big) =: \tau_i\omega^i, \tag{A.41}$$

where $\sigma_i = 2\mathrm{i}\tau_i$ are the Pauli spin matrices, and for any $\omega \in \mathfrak{so}(1,3)$

$$\omega^i = \frac{1}{2}\epsilon_m{}^{in}\omega^m{}_n + \mathrm{i}\omega^i{}_o \tag{A.42}$$

denote its selfdual components (A.13). Choosing these complex coordinates on $\mathfrak{sl}(2,\mathbb{C})$, we can simplify calculations, in fact the commutation relations between the selfdual generators are nothing but

$$\big[\tau_i, \tau_j\big] = \epsilon_{ij}{}^m\tau_m. \tag{A.43}$$

The differential map, that is the push forward $\Lambda_* : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{so}(1,3)$, induces an isomorphism between the two respective Lie algebras.

$$\Lambda_*\tau_i = -\mathrm{i}\Pi_i, \quad \Lambda_*\bar{\tau}_i = -\mathrm{i}\bar{\Pi}_i, \tag{A.44}$$

where Π_i are the selfdual generators as defined by (A.10).

A.3 SU(2) SPINORS AND THE BRA-KET NOTATION

If we fix a future oriented timelike normal n^μ we can introduce some important $SU(2)$ structures. First of all we define a Hermitian metric

$$\delta_{A\bar{A}} = \sigma_{A\bar{A}\mu}n^\mu. \tag{A.45}$$

We can use this metric to introduce the Hermitian conjugate of a contravariant spinor, that is a map from $\mathbb{C}^2$ towards its dual vector space, we define:

$$\omega^\dagger_A := \delta_{A\bar{A}}\bar{\omega}^{\bar{A}}. \tag{A.46}$$

We can now also define the Pauli matrices in the frame of the normal:

$$\sigma^A{}_{B\mu} := -2\Sigma^A{}_{B\mu\nu}n^\nu. \tag{A.47}$$

These are purely spatial, i.e.:

$$\sigma^A{}_{B\mu}n^\mu = 0. \tag{A.48}$$

We introduce the induced spatial metric, together with the spatial Levi-Civita tensor:

$$h_{\mu\nu} = n_\mu n_\nu + \eta_{\mu\nu}, \quad \varepsilon_{\alpha\beta\rho} = \varepsilon_{\mu\alpha\beta\rho}n^\mu, \tag{A.49}$$

and recover the Pauli identity

$$\sigma^A{}_{C\mu}\sigma^C{}_{B\mu} = \delta^A_B h_{\mu\nu} + \mathrm{i}\varepsilon_{\mu\nu}{}^\alpha\sigma^A{}_{B\alpha}. \tag{A.50}$$

Important relations are also

$$h^\alpha{}_\mu h^\beta{}_\nu\Sigma^A{}_B{}^{\mu\nu} = \frac{1}{2\mathrm{i}}\varepsilon^{\alpha\mu\beta}\sigma^A{}_{B\mu}, \tag{A.51}$$

and

$$\sigma^A{}_{B\alpha} = h_\alpha{}^\beta\sigma^{A\bar{B}}{}_\beta\delta_{B\bar{B}}, \quad \sigma^A{}_{B\alpha} = -h_\alpha{}^\beta\bar{\sigma}_{\bar{A}B}{}^\beta\delta^{A\bar{A}}. \tag{A.52}$$

The last two equations imply the Hermiticity of the Pauli matrices:

$$\sigma^{A}{}_{Ba} = \delta^{A\bar{A}}\overline{\sigma^{A}{}_{Ba}}\delta_{B\bar{B}} \tag{A.53}$$

Finally, let us also note, that sometimes a bra-ket notation can also be very useful, a dictionary is given by the relations:

$$|\omega\rangle \equiv \omega^{A}, \quad \langle\omega| \equiv \omega^{\dagger}_{A}, \quad [\omega| \equiv \omega_{A}, \quad |\omega] = \delta^{A\bar{A}}\bar{\omega}_{\bar{A}}, \tag{A.54}$$

together with

$$\|\omega\| = \sqrt{\delta_{A\bar{A}}\bar{\omega}^{A}\omega^{A}} = \langle\omega|\omega\rangle. \tag{A.55}$$

A.4 UNITARY REPRESENTATIONS

labelunitaryappdxWe will give here a short overview over the unitary irreducible representations of the Lorentz group, key for understanding the EPRL model. Further reading may be found in [193, 194, 208].

The representation

$$SL(2,\mathbb{C}) \ni g : \omega \in \mathbb{C}^2 \mapsto g\omega \equiv g^{A}{}_{B}\omega^{B}. \tag{A.56}$$

of $SL(2,\mathbb{C})$ on $\mathbb{C}^2$ is already irreducible, but not unitary. The induced representations on functions $f : \mathbb{C}^2 \to \mathbb{C}$, with the natural $L^2(\mathbb{C}^2, d^4\omega)$ inner product is unitary though reducible. This immediately follows from the homogeneity and unimodularity of the transformation. Irreducible unitary representations are then built just from homogenous functions on $\mathbb{C}^2$.

For the principle series, the weights of homogeneity are parametrised by a half integer $2k \in \mathbb{Z}$ and some $\rho \in \mathbb{R}$. That is we are dealing with functions

$$\forall\lambda \neq 0,\, \omega^{A} \in \mathbb{C}^2 - \{0\} : f(\lambda\omega^{A}) = \lambda^{-k-1+\mathrm{i}\rho}\bar{\lambda}^{+k-1+\mathrm{i}\rho} f(\omega^{A}). \tag{A.57}$$

From this formula we can easily see that if the pair (ρ, k) labels an irreducible unitary representation, its complex conjugate representation is labelled by $(-\rho, -k)$. A canonical basis in this infinite-dimensional space is given by the following functions,

$$f^{(\rho,k)}_{j,m} = \sqrt{\frac{2j+1}{\pi}}\|\omega\|^{2(\mathrm{i}\rho-1)} R^{(j)}(U^{-1}(\omega))^{k}{}_{m}, \tag{A.58}$$

where $j \geq k$ and $m = -j, \ldots, j$, and

$$R^{j}(U)^{m}{}_{n} = \langle j, m|R^{(j)}(U)|j, n\rangle, \quad \text{for} \quad U(\omega) = \frac{1}{\|\omega\|}\begin{pmatrix} \omega^0 & -\bar{\omega}^{\bar{1}} \\ \omega^1 & \bar{\omega}^{\bar{0}} \end{pmatrix} \in SU(2), \tag{A.59}$$

are the entries of the spin-j Wigner matrix for the $SU(2)$ element $U(\omega)$ constructed from the spinor. The basis elements (A.58) diagonalise a complete set of commuting operators:

$$\big(\widehat{L}^2 - \widehat{K}^2\big) f^{(\rho,k)}_{j,m} = (k^2 - \rho^2 - 1) f^{(\rho,k)}_{j,m}, \qquad \widehat{L}_i\widehat{K}^i f^{(\rho,k)}_{j,m} = -k\rho f^{(\rho,k)}_{j,m} \tag{A.60a}$$

$$\widehat{L}^2 f^{(\rho,k)}_{j,m} = j(j+1) f^{(\rho,k)}_{j,m}, \qquad \widehat{L}_3 f^{(\rho,k)}_{j,m} = m f^{(\rho,k)}_{j,m} \tag{A.60b}$$

where $\widehat{L}_i$ and $\widehat{K}_i$ are the quantisation of the generators introduced in (A.8).

It is quite convenient to introduce a multi-index notation to group the pair (j, m) into a single index μ. We will also use the notation $\bar{\mu}$ to keep track of the complex conjugate representation, and use Einstein's summation convention for the μ indices. With our choices, the matrix representation of the group is the right action, defined according to:

$$\begin{aligned}\big(D(g)f_\mu^{(\rho,k)}\big)(\omega^A) :&= f_\mu^{(\rho,k)}\big((g^{-1})^A{}_B\omega^B\big) = f_\mu^{(\rho,k)}(-\omega^B g_B{}^A) = \\ &= D^{(\rho,k)}(g)^\nu{}_\mu f_\nu^{(\rho,k)}(\omega^A).\end{aligned} \tag{A.61}$$

Since the representation is unitary, it admits an $SL(2,\mathbb{C})$-invariant Hermitian inner product. This is defined as a surface integral on $\mathrm{P}\mathbb{C}^2 \subset \mathbb{C}^2$,*

$$\langle f_\mu^{(\rho,k)} | f_\nu^{(\rho,k)}\rangle = \frac{\mathrm{i}}{2}\int_{\mathrm{P}\mathbb{C}^2} \omega_A \mathrm{d}\omega^A \wedge \bar{\omega}_{\bar{A}}\mathrm{d}\bar{\omega}^{\bar{A}}\,\overline{f_\mu^{(\rho,k)}(\omega^A)}f_\nu^{(\rho,k)}(\omega^A) = \delta_{\bar{\mu}\nu}, \tag{A.62}$$

its value being independent of the way $\mathrm{P}\mathbb{C}^2$ is embedded into $\mathbb{C}^2$ thanks to the homogeneity of the integrand.

The $SL(2,\mathbb{C})$ group locally represents the group of special orthochronous transformations. To recover the full Lorentz group we also need parity

$$\big(Pf_\mu^{(\rho,k)}\big)(\omega^A) = f_\mu^{(\rho,k)}(\delta^{A\bar{A}}\bar{\omega}_{\bar{A}}), \tag{A.63}$$

and time reversal

$$\big(Tf_\mu^{(\rho,k)}\big)(\omega^A) = \epsilon_{\alpha\mu}\delta^{\alpha\bar{\nu}}\,\overline{f_\nu^{(\rho,k)}(\delta^{A\bar{A}}\bar{\omega}_{\bar{A}})}, \tag{A.64}$$

both of which have recently gained [209] some interest in LQG. From (A.57) we can realise parity and time reversal map the irreducible unitary representation of labels (ρ, k) to those of $(\rho, -k)$ and $(-\rho, k)$ respectively.

In each representation space there are two invariants, the first one is the Hermitian inner product (A.62) introduced in above, the second one is the ϵ-invariant

$$\begin{aligned}\big[f_\mu^{(\rho,k)}\big| f_\nu^{(\rho,k)}\big\rangle :=& \frac{k-\mathrm{i}\rho}{4\pi}\int_{P\mathbb{C}^2\times P\mathbb{C}^2} \omega_A\mathrm{d}\omega^A\wedge\bar{\omega}_{\bar{A}}\mathrm{d}\bar{\omega}^{\bar{A}}\wedge\pi_A\mathrm{d}\pi^A\wedge\bar{\pi}_{\bar{A}}\mathrm{d}\bar{\pi}^{\bar{A}}(\pi_A\omega^A)^{k-1-\mathrm{i}\rho}(\bar{\pi}_{\bar{A}}\bar{\omega}^{\bar{A}})^{-k-1-\mathrm{i}\rho} \\ &\cdot f_\mu^{(\rho,k)}(\pi^A)f_\nu^{(\rho,k)}(\omega^A) = \epsilon_{\mu\nu}.\end{aligned} \tag{A.65}$$

Its matrix elements are

$$\epsilon_{(j,m)(j',m')} = (-1)^{k-m}\delta_{j,j'}\delta_{m,-m'}\frac{\Gamma(k+1-\mathrm{i}\rho)}{\Gamma(k+1+\mathrm{i}\rho)}\frac{\Gamma(j+1+\mathrm{i}\rho)}{\Gamma(j+1-\mathrm{i}\rho)}, \tag{A.66}$$

where Euler's Γ function appears. Though infinite dimensional, each of the invariants comes with an inverse, and

$$\left.\begin{aligned}\delta^{\mu\bar{\alpha}}\delta_{\nu\bar{\alpha}} = \delta^\mu_\nu = \epsilon^{\mu\alpha}\epsilon_{\nu\alpha}, \quad \delta^{\mu\bar{\nu}} = \overline{\delta^{\nu\bar{\mu}}}, \\ \epsilon^{\mu\nu} = \delta^{\mu\bar{\alpha}}\delta^{\nu\bar{\beta}}\bar{\epsilon}_{\bar{\alpha}\bar{\beta}}, \quad \epsilon^{\mu\nu} = (-1)^{2k}\epsilon^{\nu\mu}.\end{aligned}\right\} \tag{A.67}$$

Thanks to the completeness of the basis, (A.65) and (A.62) imply for each irreducible subspace (ρ, k) a relation between the ket and its dual,

$$[f_\mu| = \frac{\pi}{\mathrm{i}\rho - k}\epsilon_{\mu\nu}\delta^{\nu\bar{\alpha}}\langle f_\alpha|. \tag{A.68}$$

*Because of the homogeneity, integrating over all of $\mathbb{C}^2$ would lead to divergences.

Since both $\delta^{\nu\bar{\nu}}$ and $\epsilon_{\mu\nu}$ are invariant this map commutes with the group action, and implicitly shows the representation labelled by (ρ, k) is unitarily equivalent to its complex conjugate, that is the $(-\rho, -k)$ representation.

The map (A.68) allows us to relate the bilinear invariant (A.65) to the Hermitian inner product (A.62). The dual vector can be obtained also by Fourier transform, up to a phase. In fact, we have

$$\begin{aligned} f^{(\rho,k)}_{\bar{\mu}}(\pi_A) :&= \frac{1}{\pi^2}\int_{\mathbb{C}^2} d^4\omega \mathrm{e}^{\mathrm{i}\pi_A\omega^A - \mathrm{cc.}}\overline{f^{(\rho,k)}_{\mu}(\omega^A)} = \\ &= \mathrm{e}^{-\mathrm{i}\pi k}\frac{\Gamma(k+1-\mathrm{i}\rho)}{\Gamma(k+1+\mathrm{i}\rho)}\delta_{\mu\bar{\mu}}\epsilon^{\mu\nu} f^{(\rho,k)}_{\nu}(\pi^A), \end{aligned} \tag{A.69}$$

and defines an antilinear map from the (ρ, k) representation onto itself, whereas complex conjugation maps the (ρ, k) towards the $(-\rho, -k)$ representation, implicitly showing that (ρ, k) and $(-\rho, -k)$ are unitarily equivalent. To proof this formula one proceeds as follows: First, thanks to the $SL(2, \mathbb{C})$ invariance of the integral, one can realise the left hand side equals the right hand side up to a constant. This constant can only depend on the labels ρ and k. Next, one shows, this constant has unit norm. Calculating the integral for the states of spin labels $k = j = m$, eventually gives the phase appearing in (A.69).

A.5 SYMBOLS AND CONVENTIONS

Index conventions

- $a, b, c, \ldots$ abstract indices in four(three)-dimensional (co)tangent-space
- $\alpha, \beta, \gamma, \ldots$ abstract indices in four dimensional (internal) Minkowski space
- $i, j, k, \ldots$ abstract indices in three dimensional (internal) Euclidean space
- $A, B, C, \ldots$ spinor indices transforming under the fundamental representation of $SL(2, \mathbb{C})$
- $\bar{A}, \bar{B}, \bar{C}, \ldots$ spinor indices transforming under the complex conjugate representation of $SL(2, \mathbb{C})$

Symmetrisations

- $\omega_{[a_1 \ldots a_N]} \ldots$ antisymmetrisation defined as $\omega_{[a_1\ldots a_N]} = \frac{1}{N!}\sum_{\pi\in S_N}\mathrm{sign}(\pi)\omega_{a_{\pi(1)}\cdots a_{\pi(N)}}$
- $\omega_{(a_1 \ldots a_N)} \ldots$ symmetrisation defined as $\omega_{(a_1\ldots a_N)} = \frac{1}{N!}\sum_{\pi\in S_N}\omega_{a_{\pi(1)}\cdots a_{\pi(N)}}$

Constants

- $c \ldots$ speed of light
- $G \ldots$ Newton's constant
- $\Lambda \ldots$ cosmological constant
- $\hbar \ldots$ reduced Planck's constant
- $\ell_{\mathrm{P}} \ldots$ reduced Planck lenght $\ell_{\mathrm{P}} = \sqrt{\frac{8\pi\hbar G}{c^3}}$
- $\beta \ldots$ Barbero–Immizi parameter

Special tensors

- $\eta_{\alpha\beta} \ldots$ internal Minkowski metric, $\eta = \mathrm{diag}(-1, 1, 1, 1)$
- $\varepsilon_{\alpha\beta\mu\nu} \ldots$ internal Levi-Civita tensor, $\epsilon_{0123} = 1$

$(\underset{\sim}{\eta}_{a_1 \dots a_N})$, $\tilde{\eta}^{a_1 \dots a_N}$ ...	(inverse) Levi-Civita density
g_{ab} ...	space-time metric
h_{ab} ...	induced metric on a spatial slice Σ_t
n_a ...	future oriented normal to Σ_t
$(\eta_\alpha{}^a)$, $\eta^\alpha{}_a$ ...	(co)tetrad
$(e_i{}^a)$, $e^i{}_a$ ...	(co)triad

Relations

$\approx$ ...	"approximately", or "equal up to terms constrained to vanish"
$\equiv$ ...	"to be identified term by term", "difference only in notation", "isomorphic"
$\overset{?}{=}$ ...	"to be checked"
$\overset{!}{=}$ ...	"required"

Differential calculus

d ...	exterior derivative: $\mathrm{d}\omega = \frac{1}{N!}\partial_{[\alpha_1}\omega_{\alpha_2 \dots \alpha_{N+1}]}\mathrm{d}x^{\alpha_1} \wedge \dots \wedge \mathrm{d}x^{\alpha_{N+1}}$
$\mathcal{D}$ ...	covariant exterior $SO(1,3)$ derivative in four dimensions: $\mathcal{D}V^\alpha = \mathrm{d}V^\alpha + \omega^\alpha{}_\beta \wedge V^\beta$ ($\omega^\alpha{}_\beta$ is the connection)
∇ ...	metric compatibel Levi-Civita derivative in four dimensions
$\mathfrak{D}$ ...	metric compatibel Levi-Civita derivative in three dimensions
D ...	projection of $\mathcal{D}$ onto a spatial slice

Abbreviations

ADM ...	referring to the 3+1 split of general relativity due to Arnowitt, Deser and Misner
BC ...	boundary conditions
GR ...	general relativity
EH ...	Einstein Hilbert
EOM ...	equations of motion
LQG ...	loop quantum gravity
QG ...	quantum gravity
cc. ...	the complex conjugate of everything preceding
hc. ...	the Hermitian conjugate of everything preceding
$(a \leftrightarrow b)$ ...	the preceding term with just a and b exchanged

B
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