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Modèles attractifs en astrophysique et biologie: points critiques et comportement en temps grand des solutions

THÈSE:

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Abstract In this thesis we study the set of solutions of partial differential equations arising from models in astrophysics and biology. We answer the questions of existence but also we try to describe the behavior of some families of solutions when parameters vary. First we study two problems concerned with astrophysics, where we show the existence of particular sets of solutions depending on a parameter using the Lyapunov-Schmidt reduction method. Afterwards a perturbation argument and Banach's Fixed Point Theorem reduce the original problem to a finite-dimensional one, which can be solved, usually, by variational techniques. The rest of the thesis is devoted to the study of the Keller-Segel model, which describes the motion of unicellular amoebae. In its simpler version, the Keller-Segel model is a parabolic-elliptic system which shares with some gravitational models the property that interaction is computed through an attractive Poisson / Newton equation. A major difference is the fact that it is set in a two-dimensional setting, which experimentally makes sense, while gravitational models are ordinarily three-dimensional. For this problem the existence issues are well known, but the behaviour of the solutions during the time evolution is still an active area of research. Here we extend properties already known in particular regimes to a broader range of the mass parameter, and we give a precise estimate of the convergence rate of the solution to a known profile as time goes to infinity. This result is achieved using various tools such as symmetrization techniques and optimal functional inequalities. The last chapters deal with numerical results and formal computations related to the Keller-Segel model.

Résumé Dans cette thèse, nous étudions l'ensemble des solutions d'équations aux dérivées partielles résultant de modèles d'astrophysique et de biologie. Nous répondons aux questions de l'existence, mais aussi nous essayons de décrire le comportement de certaines familles de solutions lorsque les paramètres varient. Tout d'abord, nous étudions deux problèmes issus de l'astrophysique, pour lesquels nous montrons l'existence d'ensembles particuliers de solutions dépendant d'un paramètre à l'aide de la méthode de réduction de Lyapunov-Schmidt. Ensuite un argument de perturbation et le théorème du Point fixe de Banach réduisent le problème original à un problème de dimension finie, et qui peut être résolu, habituellement, par des techniques variationnelles. Le reste de la thèse est consacré à l'étude du modèle Keller-Segel, qui décrit le mouvement d'amibes unicellulaires. Dans sa version plus simple, le modèle de Keller-Segel est un système parabolique-elliptique qui partage avec certains modèles gravitationnels la propriété que l'interaction est calculée au moyen d'une équation de Poisson / Newton attractive. Une différence majeure réside dans le fait que le modèle est défini dans un espace bidimensionnel, qui est expérimentalement consistant, tandis que les modèles de gravitationnels sont ordinairement posés en trois dimensions. Pour ce problème, les questions de l'existence sont bien connues, mais le comportement des solutions au cours de l'évolution dans le temps est encore un domaine actif de recherche. Ici nous étendre les propriétés déjà connues dans des régimes particuliers à un intervalle plus large du paramètre de masse, et nous donneons une estimation précise de la vitesse de convergence de la solution vers un profil donné quand le temps tend vers l'infini. Ce résultat est obtenu à l'aide de divers outils tels que des techniques de symétrisation et des inégalités fonctionnelles optimales. Les derniers chapitres traitent de résultats numériques et de calculs formels liés au modèle Keller-Segel.

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Chapter 1

Introduction (version française)

Cette thèse est consacrée à l'étude de modèles mathématiques issus de divers domaines des sciences, biologie et astrophysique surtout. Ces modèles sont présentés sous la forme *d'équations aux dérivées partielles* qui décrivent comment certaines quantités évoluent au cours du temps ou en fonction de certains paramètres. Toutes les équations étudiées dans ce mémoire impliquent à un certain moment une structure elliptique, éventuellement après une réduction du problème.

L'objectif principal de cette étude est de caractériser les propriétés de l'ensemble des solutions. Nous répondons aux questions fondamentales de l'existence, mais nous essayons aussi de décrire le comportement de certaines familles de solutions lorsque les paramètres varient. Les chapitres 3 et 4 traitent de problèmes liés à l'astrophysique. Dans les deux cas, nous montrons l'existence d'ensembles de solutions particulières dépendant d'un paramètre lorsque ce paramètre est petit, à l'aide de la méthode de réduction de la dimension, ou méthode de Lyapunov-Schmidt. Cette méthode constructive a été largement utilisée dans des problèmes elliptiques non linéaires. Elle repose sur une *bonne* approximation de la solution, communément appelée *ansatz*. Ensuite un argument de perturbation et un point fixe de Banach réduisent le problème original à un problème de dimension finie, qui peut être résolu, par des techniques usuelles du calcul des variations. On obtient ainsi des *tours de bulles* dans le chapitre 3 et des *équilibres relatifs* dans le chapitre 4.

Les autres chapitres sont consacrés à l'étude du modèle Keller-Segel, qui décrit le mouvement d'amibes unicellulaires, *dictyostelium discoideum*. Dans sa version la plus simple, le modèle de Keller-Segel est un système parabolique-elliptique qui partage avec certains modèles gravitationnels la propriété que l'interaction est calculée au travers d'une équation de Poisson attractive, ou équation de Newton. Une différence majeure réside toutefois dans le fait que le modèle Keller-Segel est défini dans un espace bidimensionnel, qui est expérimentalement pertinent, tandis que les modèles de gravitation sont ordinairement en trois dimensions. Pour ce problème, les questions de l'existence sont bien connues, mais le comportement des solutions au cours de l'évolution en temps est encore un domaine actif de recherche. Ici nous étendons les propriétés déjà connues dans le cas particulier du régime de masse petite à tout l'intervalle de masse sous-critique, et nous donnons une estimation précise de la vitesse de convergence de la solution vers un profil limite quand le temps tend vers l'infini. Ce résultat est obtenu à l'aide de divers outils tels que des techniques de symétrisation et des inégalités fonctionnelles optimales. Les derniers chapitres traitent de résultats numériques et de calculs formels liés au modèle Keller-Segel.

1.1 Chapter 3: Le phénomènes des "tours de bulles" dans des équations semi-linéaires elliptiques avec exposants de Sobolev mixtes

Dans le premier chapitre, nous étudions l'équation

$$\left\{ \begin{array}{rcll} \Delta u + u^p + u^q & = & 0 & \text{dans } \mathbb{R}^N \\ u & > & 0 & \text{dans } \mathbb{R}^N \\ \lim_{|x|\to\infty} u(x) & \to & 0 & \end{array} \right. \tag{1.1.1}$$

pour $N \geq 3$, $1 < p < q$, et Δ désigne l'opérateur Laplacien standard. Ce problème est étroitement lié à l'équation d'Emdem-Fowler-Lane

$$\left\{ \begin{array}{rcll} \Delta u + u^p & = & 0 & \text{dans } \mathbb{R}^N \\ u & > & 0 & \text{dans } \mathbb{R}^N \\ \lim_{|x|\to\infty} u(x) & \to & 0 & \end{array} \right. \tag{1.1.2}$$

qui est un modèle de base de la structure interne des étoiles. Il est bien connu que l'exposant critique $p = p^* := \frac{N+2}{N-2}$ a des conséquences cruciales sur la question de l'existence de solutions. Dans [15] les auteurs ont démontré que dans le cas $1 < p < p^*$ il n'y n'a aucune solution positive à (1.1.2). Lorsque $p = p^*$, (voir [1, 17, 3]), toutes les solutions sont explicitement données par

$$u_{\lambda,\xi}(x) \quad = \quad \gamma_N \left(\frac{\lambda}{\lambda^2 + |x-\xi|^2} \right)^{\frac{N-2}{2}} \qquad , \ \gamma_N = (N(N-2))^{\frac{N-2}{4}}$$

où $\lambda > 0$ et $\xi \in \mathbb{R}^N$ sony des paramètres arbitraires. Si $p > p^*$, les solutions sont de la forme $u_\lambda(x) = \lambda^{\frac{2}{p-1}} v(\lambda x)$, avec $\lambda > 0$, et

$$u_\lambda(x) \sim C_{p,N} \, |x|^{\frac{-2}{p-1}}$$

pour une certaine constante $C_{p,N} > 0$.

Dans le cas de l'équation (1.1.1), H. Zou a montré dans [18] que si $p \leq \frac{N}{N-2}$, alors (1.1.1) n'admet aucun état fondamental, et si $q < p^*$, alors il n'y a pas de solution positive. Il a également montré que, si $p > p^*$ alors l'équation admet un nombre infini de solutions qui décroissent comme $|x|^{\frac{-2}{p-1}}$ lorsque $|x|$ tend vers l'infini, et enfin, dans le cas

$$1 < p < p^* < q \, , \tag{1.1.3}$$

il a aussi prouvé que tous les états fondamentaux de (1.1.1) sont radiaux, à translation près.

La question de l'existence de solutions de (1.1.1)sous la restriction (1.1.3) a été partiellement résolue dans le cas légèrement sous/super-critique avec des outils géométriques pour les systèmes dynamiques dans [2]. Le résultat présenté dans le chapitre 3 redémontre les théorèmes d'existence de [2] et donne aussi une approximation asymptotique des solutions avec une méthode plus simple.

Nous considérons en premier lieu le cas légèrement super-critique

$$\left\{ \begin{array}{rcll} \Delta u + u^p + u^{p^*+\varepsilon} & = & 0 & \text{dans } \mathbb{R}^N \\ u & > & 0 & \text{dans } \mathbb{R}^N \\ \lim_{|x|\to\infty} u(x) & \to & 0 & \end{array} \right. \tag{1.1.4}$$

où $\frac{N}{N-2} < p$ est fixé et $\varepsilon > 0$. Dans ce cas, nous sommes capables de montrer que

Théorème 1 *Soit $N \geq 3$ et $\frac{N}{N-2} < p$. Alors pour tout $k \in \mathbb{N}$, il existe, pour un $\varepsilon > 0$ suffisamment petit, une solution u_ε de (1.1.1) de la forme*

$$u_\varepsilon(y) = \gamma_N \sum_{i=1}^{k} \left(\frac{1}{1 + \alpha_i^{\frac{4}{N-2}} \varepsilon^{-(i-1+\frac{1}{p^*-p})\frac{4}{N-2}} |y|^2} \right)^{\frac{N-2}{2}} \alpha_i \, \varepsilon^{-(i-1+\frac{1}{p^*-p})}(1 + o(1))$$

avec $o(1) \to 0$ uniformément dans $\mathbb{R}^N$, quand $\varepsilon \to 0$. Les constantes α_i peuvent être calculées explicitement et ne dépendent que de N et p.

Ce phénomène de concentration est appelé *tour de bulles* dans la littérature.

Dans le cas légèrement sous-critique

$$\begin{cases} \Delta u + u^{p^*-\varepsilon} + u^q = 0 & \text{dans } \mathbb{R}^N \\ u > 0 & \text{dans } \mathbb{R}^N \\ \lim_{|x|\to\infty} u(x) \to 0 \end{cases} \tag{1.1.5}$$

avec $p^* < q$ fixé, pour tout $k \in \mathbb{N}$, nous obtenons un résultat similaire

Théorème 2 *Soit $N \geq 3$ et $p^* < q$ fixés. Alors pour un $k \in \mathbb{N}$ donné, il existe, pour tout $\varepsilon > 0$ assez petit, une solution u_ε du problème (1.1.5) de la forme*

$$u_\varepsilon(y) = \gamma_N \sum_{i=1}^{k} \left(\frac{1}{1 + \beta_i^{\frac{4}{N-2}} \varepsilon^{(i-1+\frac{1}{q-p^*})\frac{4}{N-2}} |y|^2} \right)^{\frac{N-2}{2}} \beta_i \, \varepsilon^{(i-1+\frac{1}{q-p^*})}(1 + o(1))$$

où $o(1) \to 0$ signifie que le reste est uniformément petit sur $\mathbb{R}^N$, et où les constantes β_i ne dépendent que de N et q, et peuvent être calculées explicitement.

Pour prouver ces théorèmes, nous utilisons la transformation dite d'Emden-Fowler, introduite pour la première fois dans [13], qui transforme les dilatations en translations. Le problème qui consiste à chercher une solution sous la forme de k-bulle résolvant (1.1.1) devient équivalent au problème de trouver une solution k-bosses d'une équation du second ordre sur $\mathbb{R}$. Alors une variante de la méthode de Lyapunov-Schimdt permet de réduire le problème à la construction de solutions en dimension finie d'un problème variationnel sur $\mathbb{R}$.

1.2 Chapter 4: Equilibres relatifs en dynamique stellaires des milieux continus

Dans ce chapitre, nous considérons le système de Vlasov-Poisson

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = 0 \\ \\ \phi = -\dfrac{1}{4\pi \, |\cdot|} * \rho \, , \quad \rho := \displaystyle\int_{\mathbb{R}^3} f \, dv \end{cases} \tag{1.2.1}$$

qui modélise la dynamique d'une nuage de particules se déplaçant sous l'effet d'un potentiel gravitationnel à champ moyen ϕ donné par l'équation de Poisson: $\Delta\phi = \rho$. Ici $f = f(t, x, v)$ est une fonction positive ou

nulle dans $L^\infty(\mathbb{R}, L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ dépendant du *temps* $t \in \mathbb{R}$, de la *position* $x \in \mathbb{R}^3$ et de la *vitesse* $v \in \mathbb{R}^3$, tandis que la fonction ρ ne dépend que de t et x.

La première équation de (1.2.1) est l'équation de *Vlasov*, qui est obtenue en écrivant que la masse est transportée par leflot des équations de Newton, alors que le champ gravitationnel est obtenu comme potentiel de champ moyen. Réciproquement, la dynamique des systèmes discrets de particules peut être formellement récupérée en considérant les distributions empiriques, c'est-à-dire les solutions à valeur mesure faites d'une somme de masses de Dirac, en négligeant les termes gravitationnels auto-cohérents liés à l'interaction de chaque masse de Dirac avec elle-même.

Il est également possible de relier (1.2.1) avec des systèmes discrets de la manière suivante. Prenons le cas de N sphères gazeuses, loin l'une de l'autre, de telle manière qu'elles interagissent faiblement entre elles par gravitation. Du point de vue du système (1.2.1), une telle solution est représentée par une fonction de distribution f, dont la densité spatiale ρ a un support compact, formé de plusieurs composants presque sphériques. À grande échelle, l'emplacement de ces sphères est régi au premier ordre par le problème de gravitation à N corps.

Le but de cette étude est de dévoiler ce lien par la construction d'une classe spéciale de solutions: nous allons construire des solutions périodiques en temps, non radialement symétriques, qui généralisent aux équations cinétiques la notion d'*équilibre relatif* pour le problème à N corps discret. Ces solutions ont un mouvement plan rigide de rotation autour d'un axe qui contient le centre de gravité du système, de sorte que la force centrifuge compense l'attraction due à la gravitation.

Par conséquent, nous recherchons des solutions périodiques en temps, qui sont en rotation à vitesse angulaire constante ω. En remplaçant $x = (x', x^3)$ et $v = (v', v^3)$ respectivement par $(e^{i\,\omega\,t}\,x', x^3)$ et $(i\,\omega\,x' + e^{i\,\omega\,t}\,v', v^3)$ et en utilisant des notations complexes de sorte que x', $v' \in \mathbb{R}^2 \approx \mathbb{C}$, Problem (1.2.1) devient

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f - \omega^2\, x' \cdot \nabla_{v'} f + 2\,\omega\, i\, v' \cdot \nabla_{v'} f = 0\,, \\ \phi = -\frac{1}{4\pi\,|\cdot|} * \rho\,, \quad \rho = \displaystyle\int_{\mathbb{R}^3} f\; dv\,, \end{cases} \tag{1.2.2}$$

où nous avons, abusivement, utilisé les mêmes notations pour le potentiel ϕ et la function de distribution f, dans le soucis de ne pas multiplier les notations. Un *équilibre relatif* of (1.2.1) est une solution stationnaire de (1.2.2) et peut être obtenu en considérant des points critiques de la fonctionnelle d'*énergie libre*

$$\mathcal{F}[f] = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \beta(f)\; dx\, dv + \frac{1}{2} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \big(|v|^2 - \omega^2\,|x'|^2\big) f\; dx\, dv - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi|^2\; dx$$

pour une fonction convexe arbitraire β, sous la contrainte de masse $\iint_{\mathbb{R}^3\times\mathbb{R}^3} f\; dx\, dv = M$. Un example typique d'une telle function est donné par

$$\beta(f) = \frac{1}{q}\,\kappa_q^{q-1}\, f^q \tag{1.2.3}$$

pour un certain $q \in (1, \infty)$ et une constant positive κ_q, à fixer ultérieurement. La solution correspondante est connue sous le nom de modèles des *gaz polytropiques*; voir [21, 22, 57, 61].

Il s'ensuit que tout équilibre relatif prend alors la forme $f(x, v) = \gamma\big(\lambda + \frac{1}{2}\,|v|^2 + \phi(x) - \frac{1}{2}\,\omega^2\,|x'|^2\big)$ où $\gamma(s) = \kappa_q^{-1}\,(-s)_+^{1/(q-1)}$ et où λ est constant sur chaque composante du support de f. Le problème est maintenant réduit à une équation de Poisson non-linéaire, à savoir

$$\Delta\phi = g\big(\lambda + \phi(x) - \tfrac{1}{2}\,\omega^2\,|x'|^2\big) \quad \text{if } x \in \mathrm{supp}(\rho)$$

et $\Delta\phi = 0$ dans tous les autres cas, avec $g(\mu) = (-\mu)_+^p$ et $p = \frac{1}{q-1} + \frac{3}{2}$, si κ_q est choisi de manière appropriée. En supposant que la solution a un support composé de N composante disjointes K_i, en

désignant par λ_i la valeur de λ sur K_i et par χ_i la fonction caractéristique de K_i, nous avons fini par réduire le problème à la recherche d'une solution positive $u = -\phi$ de

$$-\Delta u = \sum_{i=1}^{N} \rho_i^\omega \quad \text{in } \mathbb{R}^3\,, \quad \rho_i^\omega = \big(u - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2\big)_+^p\,\chi_i$$

sous la condition de bord asymptotique $\lim_{|x|\to\infty} u(x) = 0$. Nous définissons la masse et le centre de masse associés à chaque composante par $m_i = \int_{\mathbb{R}^3} \rho_i^\omega\,dx$ et $\xi_i^\omega = \frac{1}{m_i}\int_{\mathbb{R}^3} x\,\rho_i^\omega\,dx$ respectivement.

Les fonctionnelles d'*énergie libre* ont été abondamment étudiées durant les dernières années, non seulement pour caractériser des états stationnaires particuliers, mais aussi parce qu'elles fournissent un cadre pour traiter de la *stabilité orbitale,* qui est une question fondamentale dans la mécanique de la gravitation. Dans le contexte de la stabilité orbitale, l'utilisation d'une fonctionnelle d'énergie libre, dont la partie entropique, $\iint_{\mathbb{R}^3\times\mathbb{R}^3} \beta(f)\,dx\,dv$ est parfois appelée fonctionnelle d'énergie *Casimir,* remonte au travail de V.I. Arnold (voir [19, 20, 62]). La caractérisation variationnelle de solutions stationnaires particulières et leur stabilité orbitale ont été étudiées par Y. Guo et G. Rein dans une série d'articles: [33, 34, 35, 36, 53, 54, 55, 56] et aussi par beaucoup d'autres auteurs, voir par exemple [26, 27, 39, 40, 41, 42, 57, 58, 61].

Le principal défaut de telles approches est que les solutions stationnaires sont en un sens triviales: radiales, avec un support composé d'un seule composante connexe. Ici, nous utilisons une approche différente pour construire les solutions, qui remonte à [30] dans le contexte des équations de Schrödinger. Nous ne sommes pas au courant de tentatives d'utilisation de la méthode de la réduction de la dimension dans le cas de non-linéarités en puissances, à l'exception du cas d'une équation de Schrödinger non-linéaire avec non-linéarité en puissance et prenant en compte des forces de Coulomb répulsives (voir([25]), et du cas d'un modèle de Hartree-Fock attractif (voir [38]). Techniquement, nos résultats sont toutefois très proches de ceux de [23, 24].

En comparaison avec les résultats antérieurs sur les systèmes gravitationnels, le principal intérêt de notre approche est de fournir un ensemble bien plus riche de solutions, qui présente un grand intérêt en astrophysique pour la description de systèmes comme les étoiles gazeuses binaires ou d'objets encore plus complexes. La nécessité d'une telle amélioration a été pointée par exemple dans [37]. Une tentative dans cette direction a été faite antérieurement dans le contexte des distances de Wasserstein et de la théorie du transport de masse dans [44]. Le point principal du résultat présenté dans cette thèse est que nous pouvons nous appuyer sur la connaissance de solutions spéciales du problème des N-corps pour construire des solution du problème correspondant en mécanique des milieux continus, qui conservent certaines des propriétés du système discret.

Le principal résultat du chapitre 4 s'énonce ainsi.

Théorème 3 *Soit $N \geq 2$ et $p \in (3/2, 3) \cup (3, 5)$. Pour presque toutes masses m_i, $i = 1, \dots N$, et pour tout $\omega > 0$ suffisamment petit, il exists au moins $[2^{N-1}(N-2)+1]\,(N-2)\,!$ solutions distinctes f_ω de (1.2.2) qui sont telles que*

$$\int_{\mathbb{R}^3} f_\omega\,dv \,=\, \sum_{i=1}^{N} \rho_i^\omega \,+\, o(1)$$

où $o(1)$ signifie que le terme de reste converge uniformément vers 0 quand $\omega \to 0_+$ et s'annulle en dehors de $\cup_{i=1}^N B_R(\xi_i^\omega)$, pour un certain $R > 0$, indépendant de ω.

Avec les notations précédentes, pour tout $i = 1, \dots N$, nous obtenons que

$$\rho_i^\omega(x - \xi_i^\omega) = \lambda_i^p\,\rho_*\big(\lambda_i^{(p-1)/2}\,x\big) \,+\, o(1)$$

où ρ_* est une fonction à support compact, positive ou nulle, à symétrie radiale, croissante, ne dépendant que de p, et λ_i est tel que $m_i = \lambda_i^{(3-p)/2} \int_{\mathbb{R}^3} \rho_*\, dx + o(1)$.

Les points ξ_i^ω sont tels que $\xi_i^\omega = \omega^{-2/3}\,(\zeta_i^\omega, 0)$ où, pour tout $i = 1, \dots N$, $\zeta_i^\omega \in \mathbb{R}^2$ converge quand $\omega \to 0$ vers un point critique de

$$\mathcal{V}(\zeta_1, \dots \zeta_N) \;=\; \frac{1}{8\pi} \sum_{i \neq j = 1}^{N} \frac{m_i\, m_j}{|\zeta_i - \zeta_j|} + \frac{1}{2} \sum_{i=1}^{N} m_i\, |\zeta_i|^2 \,.$$

Ce théorème repose sur use classification des équilibres realties pour le problème à N corps qui a été établie principalement par J.I. Palmore dans [47, 48, 49, 50, 51]. Ici, solutions *distinctes* signifie qu'une solution ne peut pas être déduite d'une autre par un simple changement d'èchelle ou par une rotation. La stratégie consiste à trouver des points critiques de

$$\mathsf{J}\,[u] = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2\, dx - \frac{1}{p+1} \sum_{i=1}^{N} \int_{\mathbb{R}^3} \big(u - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2\big)_+^{p+1}\, \chi_i\, dx \,,$$

en utilisant la solution de

$$-\Delta w_* = (w_* - 1)_+^p =: \rho_* \quad \text{in } \mathbb{R}^3$$

comme "brique élémentaire" sur chacune des composantes connexes. Avec $W_\xi := \sum_{i=1}^N w_i$, $w_i(x) = \lambda_i\, w_*\big(\lambda_i^{(p-1)/2}\,(x - \xi_i)\big)$ et $\xi = (\xi_1, \dots \xi_N)$, nous voulons donc résoudre le problème

$$\Delta\phi + \sum_{i=1}^{N} p\big(W_\xi - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2\big)_+^{p-1} \chi_i\, \phi = -\mathsf{E} - \mathsf{N}[\phi]$$

avec $\lim_{|x|\to\infty} \phi(x) = 0$, où $\mathsf{E} = \Delta W_\xi + \sum_{i=1}^N \big(W_\xi - \lambda_i + \frac{1}{2}\,\omega^2\,|x'|^2\big)_+^p\, \chi_i$ et $\mathsf{N}[\phi]$ est une correction non-linéaire. Un long calcul montre que

$$\mathsf{J}\,[W_\xi] \;= \sum_{i=1}^{N} \lambda_i^{(5-p)/2}\, \mathsf{e}_* \;-\; \omega^{2/3}\, \mathcal{V}(\zeta_1, \dots \zeta_N) \;+\; O(\omega^{4/3})$$

où $\mathsf{e}_* = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla w|^2\, dx - \frac{1}{p+1} \int_{\mathbb{R}^3} (w-1)_+^{p+1}\, dx$ et $\zeta_i = \omega^{2/3}\, \xi_i'$ si les points ξ_i sont tels que, pour tout $\mu > 0$, grand, et pour tout $\omega > 0$, petit, nous avons $|\xi_i| < \mu\, \omega^{-2/3}$ et $|\xi_i - \xi_j| > \mu^{-1}\, \omega^{-2/3}$. Pour localiser chaque K_i dans un voisinage de ξ_i, nous imposons les conditions d'orthogonalité

$$\int_{\mathbb{R}^3} \phi\, \partial_{x_j} w_i\, \chi_i\, dx = 0 \quad \forall\, i = 1,\, 2 \dots N,\; j = 1,\, 2,\, 3 \,, \tag{1.2.4}$$

au prix de multiplicateurs de Lagrange. Des méthodes de point fixe permettent de trouser use solution ϕ sous contrainte. Comme $\xi \mapsto \mathsf{J}\,[W_\xi]$ est une fonction de dimension finis, si $\xi_i = (\zeta_i, 0)$ est tel que $(\zeta_1, \dots \zeta_N)$ est dans un voisinage d'un point critique non dégénéré de $\mathcal{V}$, nous pouvons trouver un point critique ϕ pour lequel les multiplicateurs de Lagrange associés à (1.2.4) sont tous égaux à zéro. Ceci termine la preuve, à un dernier point technique près. Tous les calculs ont été faits à multiplicateurs de Lagrange fixés (multiplicateurs de Lagrange correspondant aux contraintes de masse pour chaque composante K_i). Ces contraintes doivent encore être inversées (afin de permettre de fixer les masses), ce qui introduit une restriction additionnelle, à savoir $p \neq 3$.

Dans cette approche, les équilibres relatifs ont été obtenus dans un régime asymptotique où chaque composante de la fonction de distribution se comporte comme un minimiseur de l'énergie libre quand $\omega = 0$, légèrement perturbé par les autres composantes, et peut être vu à grande échelle comme une *pseudo-particule*. Ces pseudo-particules sont localisées dans un voisinage des points des équilibres relatifs du problème à N corps, qui sont obtenus quand la force centrifuge dans le référentiel en rotation équilibre la force gravitationnelle. Dans le référentiel en rotation, la force centrifuge donne lieu à un potentiel harmonique en x', avec signe négatif, qui entre en compétition avec la non-linéarité. Cette non-linéarité tend en effet quant à elle à agréger la masse en une fonction à symétrie radiale.

Un tel phénomène de *brisure de symétrie* du aux effets de rotation a été étudié dans [26] dans le cas dit *plat,* qui est légèrement plus simple (pas de variable x_3) au prix d'une interaction non-locale. Dans un tel cas, une branche distincte de solutions a été étudiée, qui est issue de la solution radiale correspondant à $\omega = 0$ et se trouve déformée quand $|\omega|$ croît. De telles solutions peuvent être définies comme minimiseurs à condition que leur support soit restreint à une boule bien choisie. Il n'est probablement pas très difficile de trouver des solutions analogues dans le cas de la dimension trois, bien qu'elles soient plus compliquées à calculer numériquement. Il serait alors intéressant de savoir si ces solutions co-existent avec celles qui ont été trouvées dans le Théorème 3 et de les étendre quand ω augmente comme branche de solutions paramètrée par ω. Si les solutions co-existent pour certaines valeurs de ω, et après restriction du support des solutions à une grande boule (ayant toutefois un rayon fini), comparer leur énergie libre avec celle des équilibres relatifs donnerait une nouvelle perspective sur la physique des systèmes gravitationnels. Il s'agit aussi d'un problème très intéressant de brisure de symétrie, pour lequel presque rien n'est connu dans le cas d'un non-linéarité non-locale comme celle du potentiel Newtonien obtenu en résolvant l'équation de Poisson attractive.

1.3 Chapitres 5 et 6: Le système de Keller-Segel en variables auto-similaires

Les chapitres 5 et 6 correspondent à un travail commun avec J. Dolbeault. Ils sont consacrés à l'étude du modèle de Keller-Segel parabolique-elliptique.

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \nabla v) & x \in \mathbb{R}^2 \,, \quad t > 0 \\ v = -\frac{1}{2\pi} \log |\cdot| * u & x \in \mathbb{R}^2 \,, \quad t > 0 \\ u(0,x) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases} \tag{1.3.1}$$

qui décrit le movement d'amibes uni-cellulaires comme *dictyostelium discoideum.* Ici u désigne leur densitée spatiale et il est cohérent de les considérer dans un cadre bi-dimensionnel. Un calcul élémentaire (voir [109, pages 122–124] et [122]) montre que les solutions (avec second moment initial fini) explosent en temps fini si la masse totale est suffisamment grande (plus grande que 8π avec nos conventions), tandis que pour les solutions de masse plus petite, la diffusion domine le comportement asymptotique en temps grand.

Plus précisément, il a été montré dans [100, 72, 66, 88] que si

$$n_0 \in L^1_+ \left(\mathbb{R}^2 \,, (1 + |x|^2)\, dx \right) \,, \quad n_0 \, |\log n_0| \in L^1(\mathbb{R}^2) \quad \text{and} \quad M := \int_{\mathbb{R}^3} n_0 \, dx < 8\,\pi \,, \tag{1.3.2}$$

alors il existe use solution u, au sens des distributions, qui est globale en temps et tells que la masse $M = \int_{\mathbb{R}^3} u(x,t)\, dx$ est conservée au cours de l'évolution. Il n'y a pas de solution stationnaire non-triviale

de (1.3.1) et toute solution converge localement vers zéro quand le temps devient grand. Il est par conséquent approprié d'étudier le comportement asymptotique de u dans des variables auto-similaires, dans lesquelles les échelles de temps et d'espace sont données respectivement par $R(t) := \sqrt{1+2t}$ et $\tau(t) := \log R(t)$, et nous définissons donc les fonctions n et c par

$$u(x,t) := R^{-2}\, n\left(R^{-1}(t)\, x, \tau(t)\right) \quad \text{et} \quad v(x,t) := c\left(R^{-1}(t)\, x, \tau(t)\right) .$$

Dans ces variables, le système peut être réécrit comme

$$\begin{cases} \frac{\partial n}{\partial t} = \Delta n + \nabla\cdot(n\, x) - \nabla\cdot(n\,\nabla c) & x\in\mathbb{R}^2\,, \quad t>0 \\ c = -\frac{1}{2\pi}\log|\cdot| * n & x\in\mathbb{R}^2\,, \quad t>0 \\ n(0,x) = n_0 \geq 0 & x\in\mathbb{R}^2 \end{cases} \tag{1.3.3}$$

L'existence d'une solution stationnaire de (1.3.3) a été démontrée dans [76] par des techniques d'EDO, et dans [107] par des méthodes d'EDP. Cette solution stationnaire est unique d'après [79]. De plus, il a été montré dans [66] que n et ∇c convergent quand $t\to\infty$, respectivement dans $L^1(\mathbb{R}^2)$ et $L^2(\mathbb{R}^2)$ vers cette unique solution stationnaire qui met en jeu des fonctions régulières et à symétrie radiale.

Un simple calcul de l'évolution du second moment montre que les solutions régulières avec masse plus grande que $8\,\pi$ explosent en temps fini; voir par exemple [100]. Le cas $M = 8\,\pi$ a été largement étudié. Nous renvoyons à [79, 80, 81] pour quelques contributions récentes sur ce sujet. Dans les chapitres 5 et 6 nous restreignons notre propos au cas sous-critique $M < 8\,\pi$.

Le taux de convergence vers la solution stationnaire en variables auto-similaires donne, en défaisant le changement de variables, le taux de convergence vers le profil asymptotique des solutions de (1.3.1). Dans [65], il a été montré que si M est inférieure à une certaine masse $M_* \in (0, 8\pi)$, alors la convergence en variables auto-similaires a lieu avec une vitesse exponentielle, qui est principalement gouvernée par la linéarisation du système (1.3.3) autour de la solution stationnaire. Toutefois, l'estimation de M_* montre que sa valeur est significativement plus petite que 8π. Dans le cadre de la symétrie radiale, V. Calvez et J.A. Carrillo ont trouvé dans [67] que le taux de convergence mesuré en distance de Wasserstein ne dépend pas de la masse, dans tout l'intervalle $(0, 8\pi)$. Le but des chapitres 5 et 6 est de montrer une estimation analogue sans condition de symétrie.

Dans le chapitre 5 nous définissons un cadre fonctionnel spécifique et démontrons une inégalité fonctionnelle qui se révèle un outil crucial pour démontrer le principal résultat du chapitre 6. Commençons par décrire brièvement ce cadre fonctionnel. Dans $\mathbb{R}^2$ l'inégalité de Hardy-Littlewood-Sobolev logarithmique a été démontrée avec constantes optimales dans [69] (voir aussi [64]) et peut être écrite comme

$$\int_{\mathbb{R}^3} n\,\log\left(\frac{n}{M}\right)\,dx + \frac{2}{M}\int_{\mathbb{R}^2\times\mathbb{R}^2} n(x)\,n(y)\,\log|x-y|\,dx\,dy + M\,(1+\log\pi) \geq 0 \tag{1.3.4}$$

pour toutes fonction $n\in L^1_+(\mathbb{R}^2)$ avec $M = \int_{\mathbb{R}^3} n\,dx$. De manière équivalente, l'inégalité(1.3.4) peut être écrite comme

$$\int_{\mathbb{R}^3} n\,\log\left(\frac{n}{M\,\mu}\right)\,dx + \frac{2}{M}\iint_{\mathbb{R}^2\times\mathbb{R}^2} (n(x) - M\,\mu(x))\,\log|x-y|\,(n(y) - M\,\mu(y))\,dx\,dy \geq 0$$

où $M = \int_{\mathbb{R}^3} n\,dx$ et $1/\mu(x) = \pi\,(1+|x|^2)^2$ pour tout $x\in\mathbb{R}^2$. De plus, par dualité de Legendre, elle est équivalente à l'inégalité d'Onofri euclidienne.

Pour étudier le système de Keller-Segel écrit en variables auto-similaires, c'est-à-dire l'équation (1.3.3), il se trouve qu'il est commode d'utiliser une forme équivalente de l'inégalité de Hardy-Littlewood-Sobolev logarithmique qui, pourvu que $M < 8\,\pi$, s'exprime sous la forme

$$\int_{\mathbb{R}^3} n\,\log\left(\frac{n}{n_M}\right)\,dx + \frac{1}{4\,\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2}(n(x)-n_M(x))\,\log|x-y|\,(n(y)-n_M(y))\,dx\,dy \geq 0 \qquad (1.3.5)$$

où (n_M, c_M) désigne l'unique solution stationnaire de (1.3.3), donnée par

$$-\Delta c_M = M\,\frac{e^{-\frac{1}{2}|x|^2+c_M}}{\int_{\mathbb{R}^3} e^{-\frac{1}{2}|x|^2+c}\,dx} =: n_M\ ,\quad x\in\mathbb{R}^2\ .$$

En suivant les arguments de [69, 64, 68, 71], on montre par dualité que (1.3.5) correspond à une nouvelle inégalité de type Onofri.

Théorème 4 *Pour tout $M \in (0, 8\,\pi)$, et pour toute fonction régulière ϕ à support compact, on a*

$$\log\left(\int_{\mathbb{R}^2} e^{\phi}\,d\mu_M\right) - \int_{\mathbb{R}^2}\phi\,d\mu_M \leq \frac{1}{2\,M}\int_{\mathbb{R}^3}|\nabla\phi|^2\,dx\ .$$

Ici, $d\mu_M := \frac{1}{M}\,n_M\,dx$ est une mesure de probabilités, et comme dans [70], on trouve un trou spectral en développant la fonctionnelle associée à l'inégalité autour de $\phi \equiv 1$. Par un argument de densité, on montre que l'inégalité s'étend à l'espace fonctionnel des fonctions régulières à support compact, complété au moyen de la norme $\|\phi\|^2 = \int_{\mathbb{R}^3}|\nabla\phi|^2\ dx + (\int_{\mathbb{R}^2}\phi\ d\mu_M)^2$. Ceci constitue le principal résultat du chapitre 5.

L'inégalité obtenue en développant autour de $\phi = 1$ est une inégalité de type Poincaré. Elle a un pendant dans le cadre Hardy-Littlewood-Sobolev, qui peut être écrit comme

$$\mathsf{Q}_1[f] := \int_{\mathbb{R}^2}|f|^2\,d\mu_M + \frac{1}{2\,\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\,\log|x-y|\,f(y)\,d\mu_M(x)\,d\mu_M(y) \geq 0\ .$$

Ceci suggère d'écrire le système de Keller-Segel linéarisé dans l'espace des fonctions de carré intégrable par rapport à la mesure $d\mu_M$, orthogonales au noyau de dimension un de l'opérateur linéarisé $\mathcal{L}$, muni du produit scalaire associé à la forme quadratique Q_1. L'opérateur $\mathcal{L}$ est alors auto-adjoint et nous démontrons une inégalité de trou spectral explicite:

$$\mathsf{Q}_1[f] = \langle f,\,f\rangle \leq \langle f,\,\mathcal{L}\,f\rangle =: \mathsf{Q}_2[f]\ .$$

Le détail des propriétés de $\mathcal{L}$, et en particulier les propriétés de son spectre, est donné plus loin. Avec ces préliminaires en main, on peut étudier le comportement en temps grand des solutions du système de Keller-Segel et faire le lien entre le trou spectral et les taux de convergence.

Au chapitre 6, nous donnons des estimations asymptotiques raffinées sur le comportement en temps grand des solutions du système de Keller-Segel en variables auto-similaires. Il a été démontré dans [65] qu'il existe une masse $M_\star \leq 8\,\pi$ telle que pour toute donnée initiale $n_0 \in L^2(n_M^{-1}\,dx)$ de masse $M < M_\star$ satisfaisant (1.3.2), le système (1.3.3) a une unique solution n vérifiant

$$\int_{\mathbb{R}^2}|n(t,x)-n_M(x)|^2\,\frac{dx}{n_M(x)} \leq C\,e^{-2\,\delta\,t}\quad\forall\ t\geq 0$$

pour deux constantes strictement positives C et δ. De plus δ peut-être prise arbitrairement proche de 1 quand $M \to 0$. Si $M < 8\,\pi$, nous pouvons remarquer que la condition $n_0 \in L^2(n_M^{-1}\,dx)$ est plus forte que (1.3.2). Notre principal résultat est que $M_\star = 8\,\pi$ et $\delta = 1$, au moins pour une large sous-classe de solutions avec n_0 vérifiant la condition technique suivante

$$\exists\, \varepsilon \in (0, 8\,\pi - M) \quad \text{tel que} \quad \int_0^s n_{0,*}(\sigma)\, d\sigma \le \int_{B\left(0,\sqrt{s/\pi}\right)} n_{M+\varepsilon}(x)\, dx \quad \forall\, s \ge 0\,. \tag{1.3.6}$$

Théorème 5 *Supposons que n_0 vérifie la condition technique* (1.3.6)*,*

$$n_0 \in L^2_+(n_M^{-1}\,dx) \quad \textit{et} \quad M := \int_{\mathbb{R}^3} n_0\, dx < 8\,\pi\,.$$

Alors toute solution de (1.3.3) *avec donnée initiale n_0 est telle que*

$$\int_{\mathbb{R}^2} |n(t,x) - n_M(x)|^2\, \frac{dx}{n_M(x)} \le C\, e^{-\,2\,t} \quad \forall\, t \ge 0$$

pour une certaine constante strictement positive C, où n_M est l'unique solution stationnaire de(1.3) *avec masse M.*

Ce résultat s'avère consistant avec les résultats récents de [67] pour le modèle bi-dimensionnel radial et son équivalent unidimensionnel. L'estimation $\delta = 1$ est optimale. Dans un souci d'exhaustivité, mentionnons que des résultats de convergence exponentielle pour les problèmes avec champ moyen ont auparavant été obtenus dans [89, 90], mais seulement dans le cas de potentiels d'interaction basés sur des noyaux bien plus réguliers que G_2.

Pour démontrer le résultat du Théorème 5, nous établissons des estimations uniformes sur $\|n\|_{L^p(\mathbb{R}^2)}$ en appliquant des méthodes de symétrisation analogues à celles de [93, 94], et prouvons ensuite la convergence uniforme de n vers n_M en utilisant la formule de Duhamel. Notre principal outil est une estimation de trou spectral de l'opérateur linéarisé $\mathcal{L}$ et la positivité stricte de l'entropie linéarisée dans un cadre fonctionnel approprié.

1.4 Chapitres 7 et 8: Une étude numérique et une approche par développements asymptotiques formels du modèle de Keller-Segel

Les deux derniers chapitres de cette thèse traitent à nouveau du modèle de Keller-Segel parabolique-elliptique, mais du point de vue numérique et des estimations asymptotiques formelles. Comme cela a été indiqué plus haut, des estimations raffinées du comportement en temps grand des solutions dans le régime de masse sous-critique ont été obtenues dans les chapitres 5 et 6, dans lesquels un cadre fonctionnel adapté à l'opérateur linéarisé a été proprement défini. Dans le chapitre 7, nous retrouvons ces résultats numériquement et donnons des détails sur le spectre de $\mathcal{L}$. Il s'agit d'un travail réalisé avec J. Dolbeault.

Tout d'abord, nous étudions le diagramme de bifurcation des solutions n_M de l'équation (1.3) en fonction du paramètre M. Ensuite, nous nous tournons vers l'opérateur linéarisé en considérant f et g tels que $n\,(1 + f(x,t))$ et $c(x)\,(1 + g(x,t))$ est une solution de (1.3.3). On montre alors que (f, g) résoud le problème non-linéaire

$$\begin{cases} \frac{\partial f}{\partial t} - \mathcal{L}\, f = -\frac{1}{n}\, \nabla \cdot [f\, n\, (\nabla (g\, c))] & x \in \mathbb{R}^2\,, \quad t > 0 \\ -\Delta (c\, g) = f\, n & x \in \mathbb{R}^2\,, \quad t > 0 \end{cases}$$

où $\mathcal{L}$ est l'opérateur linéaire défini par

$$\mathcal{L} f = \frac{1}{n} \nabla \cdot [n \nabla (f - c\, g)]$$

et nous savons que $(f\, n, \nabla(g\, c))(t, \cdot)$ prend au cours du temps ses valeurs dans $L^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$, et tend asymptotiquement vers zéro quand $t \to \infty$. Pour étudier le comportement en temps grand, il est commode de normaliser la solution de manière différente. Ce que nous voulons en fait étudier est le cas où les solutions de (1.3.3) peuvent être écrites sous la forme

$$n(x)\,(1 + \varepsilon\, f(x,t)) \quad \text{and} \quad c(x)\,(1 + \varepsilon\, g(x,t))$$

dans le régime asymptotique correspondant à $\varepsilon \to 0_+$. Formellement, il est alors clair que, à l'ordre ε, le comportement de la solution est donné par $\frac{\partial f}{\partial t} = \mathcal{L} f$. Le noyau de $\mathcal{L}$ a été identifié dans les chapitres précédents, et il a aussi été démontré que l'opérateur $\mathcal{L}$ a un spectre purement discret, et que 1 et 2 sont des valeurs propres.

Dans le chapitre 7, l'objectif est d'identifier les values propres les plus basses et de retrouver que le trou spectral est en fait égal à 1, quelle que soit la masse dans l'intervalle $(0, 8\pi)$. Nous calculons aussi la valeur numérique des autres valeurs propres de $\mathcal{L}$ en bas du spectre, dans la section 7.4, et en tirons quelques conséquences dans la toute dernière section du chapitre: taux de convergence améliorés pour des données initiales centrées et relativement à des solutions auto-similaires optimisées (par scaling: *best matching solutions*). Afin d'établir ces résultats numériques, nous commençons par paramétriser l'ensemble des solutions en fonction du paramètre de masse $M \in (0, 8\pi)$ et considérons les régimes asymptotiques de masse M petite ou proche de 8π. Ensuite nous introduisons l'opérateur linéarisé et étudions son spectre en utilisant diverses méthodes de tir: nous caractérisons ainsi son noyau, son spectre lorsqu'il est restreint aux fonctions radiales et utilisons une décomposition en harmoniques sphériques pour étudier les autres valeurs propres.

Le chapitre 8 est un travail réalisé en collaboration avec Manuel del Pino. Comme cela a été conjecturé par S. Childress et J.K. Percus [133] (voir aussi [147]) on peut avoir un *collapse chemotactique*. Plus précisément, cela signifie que la solution existe globalement en temps, ou bien qu'elle explose en temps fini. Les auteurs sus-mentionnés conjecturent aussi que lorsque la solution explose, l'agrégation se produit par formation d'une masse de Dirac à l'origine, dans le cas des solutions radiales. L'équilibre entre la tendance à diffuser la masse à l'infini et la concentration due au terme de dérive se produit précisément pour la masse critique, 8π.

Dans le cas d'une masse super-critique, c'est-à-dire quand la masse est supérieure à 8π et que la donnée initiale $u_0 \in L^1_+(\mathbb{R}^2, (1 + |x|^2)dx)$ est telle que $u_0 \,|\log u_0| \in L^1(\mathbb{R}^2)$, il est facile en utilisant des estimations de second moment de montrer que les solutions explosent en temps fini. Des contributions-clé sur le phénomène d'explosion ont été apportées en particulier par Herrero, Velázquez dans [140], [139], et par Velázquez dans [152]. Voir aussi [146] pour des mises en évidence numériques du phénomène.

Le cas critique avec masse 8π possède une famille de solutions stationnaires explicites

$$u_\lambda(x) = \frac{8\,\lambda}{(|x|^2 + \lambda)^2}$$

pour tout $\lambda > 0$. Ces solutions stationnaires, qui ont une masse critique mais un second moment infini, jouent un rôle crucial dans l'étude du comportement des solutions qui explosent. En effet, si l'on considère

un domaine borné Ω et le système

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + \nabla \cdot (u \nabla v) & x \in \Omega, \quad t > 0, \\ \Delta v = u - 1 & x \in \Omega, \quad t > 0, \\ u(0,x) = u_0(x) \geq 0 & x \in \Omega. \end{cases} \tag{1.4.1}$$

il a été montré, lorsque la masse est strictement plus grande que 8π qu'il existe formellement une solution qui produit une concentration de masse de $8\,\pi$ et que, le point de concentration étant placé à l'origine, le comportement asymptotique au moment d'explosion est donné par

$$u(x,t) \quad = \quad u_{(T-t)\varepsilon(-\log(T-t))}(|x|) \tag{1.4.2}$$

pour $|x| = O(\sqrt{(T-t)\varepsilon(-\log(T-t))})$ as $t \to T^-$, avec

$$\varepsilon(\tau) \quad = \quad O(e^{\sqrt{2\tau}}) \quad \text{quand } \tau \to \infty. \tag{1.4.3}$$

Ce résultat a été obtenu dans [140] dans le cas radial, et dans [152] dans le cas d'un domaine borné général.

Dans le cas de l'espace euclidien $\mathbb{R}^2$ tout entier, le cas de la masse critique 8π a été considéré dans [79] dans le cas radial. Les auteurs ont montré qu'il existe une solution globale pour une donnée initiale avec second moment fini ou infini. Ils ont aussi montré que la famille u_λ attire les solutions avec donnée initiale de second moment *infini,* définies dans un sens précis, auquel cas il existe une fonctionnelle de Lyapunov. De plus, quand le second moment est fini, Blanchet, Carrillo et Masmoudi ont démontré dans [129] qu'ile existe une solution u^* qui est globale en temps, qui a une *énergie libre* $\mathcal{F}$ finie et telle que

$$u^*(x,t) \quad \to \quad 8\pi\,\delta_0 \quad \text{quand} \quad t \to \infty$$

au sens de la convergence faible-étoile des mesures. Ici, l'énergie libre est définie par

$$\mathcal{F}[u](t) \quad := \quad \int_{\mathbb{R}^2} u(x,t) \log u(x,t) dx - \frac{1}{2} \int_{\mathbb{R}^2} u(x,t)\,(-\Delta)^{-1} u(x,t)\,dx\ .$$

Toutefois la caractérisation du taux d'explosion et la forme du profil limite sont des questions essentiellement ouvertes. Le seul résultat que nous pouvons mentionner dans cette direction est [132]. Avec des méthodes différentes des nôtres, les auteurs obtiennent un développement analogue des solutions. Notre approche est basée sur la méthode des *matched asymptotics* et nous ferons quelques commentaires sur son intérêt par la suite.

Au chapitre 8, nous établissons le comportement d'une solution radiale globale qui explose en temps infini. Comme dans le cas d'une masse super-critique, la famille des solutions stationnaires u_λ approche la solution près de l'origine, mais à grande distance la solution a une forme différente de manière à conserver le second moment fini, et constant dans un certain sens. Nous montrons que quand t tend vers l'infini,

$$u(x,t) \quad \approx \quad u_{\lambda(t)}(|x|) \tag{1.4.4}$$

quand $|x| = O(\sqrt{t})$, et que

$$u(x,t) \quad \approx \quad \kappa_1\,\lambda(t)\,e^{-\frac{|x|^2}{4t}}\,\frac{1}{|x|^4} \tag{1.4.5}$$

dans les autres cas. Le taux d'explosion est donné par

$$\lambda(t)^{-1} \quad = \quad \kappa \Big(\log t + O\big(\log(\log t) \big) \Big) \tag{1.4.6}$$

quand $t \to \infty$, où κ est une constante dépendant du second moment de la donnée initiale u_0.

Dans notre approche, nous introduisons la variable de *masse cumulée* afin de simplifier l'étude de l'équation. Ensuite nous établissons une approximation du comportement de la solution dans une zone intérieure et discutons de la zone de validité de cette approximation. Nous continuons l'étude en établissant une estimation de la solution à grande distance, dans une région qui s'éloigne de l'origine. Enfin, nous obtenons un taux pour l'explosion en combinant les termes asymptotiques correspondant aux deux régions. Dans l'appendice du chapitre 8 nous établissons une brève liste des différences entre ce cas et celui d'un domaine borné pour lequel l'explosion en temps infini se produit aussi ([127], [142], [150]).

Le type de comportement que nous avons trouvé, à savoir la convergence vers une solution *éternelle* de l'équation dans une région intérieure, semble être un phénomène récurrent de la description des *patterns* d'explosion près de la singularité. Par exemple, le même phénomène a été trouvé dans le cas de l'*équation de diffusion rapide logarithmique* dans [144], rigoureusement dans [134] pour le cas radial, et dans [135] pour le cas non-radial. La méthode des *matched expansions* a été utilisée précédemment pour obtenir les profils asymptotiques dans des équations paraboliques avec temps d'explosion infini dans le cas de l'équation de la chaleur semi-linéaire dans [138], [137]. La méthode fonctionne aussi dans le cas de l'explosion en temps fini, par exemple dans le cas du flot de la chaleur harmonique: voir [151], [126]. Même si le développement asymptotique a été établi par Chavanis et Sire dans [132] par des méthodes différentes, par rapport à leur approche, nous pensons que la méthode des *matched expansions* donne une meilleure compréhension du problème linéarisé, qui joue un rôle essentiel pour l'application de la méthode de Lyapunov-Schmidt de réduction de la dimension. La construction de solutions *growing up* par des arguments de perturbation singulière est une ligne de recherche intéressante à développer, avec comme point de départ la solution approchée que nous avons construite.

References

Les références renvoient au chapitre auquel elles correspondent.

Chapter 2

Introduction (english version)

This thesis is devoted to the study of mathematical models arising from various domains of science, mostly astrophysics and biology. These models are presented in the form of *Partial Differential Equations* which describe the behavior of certain quantities as time evolves or some parameters change. All equations studied in this memoir involve at some point an elliptic structure, eventually after a convenient reduction.

The main goal of this study is to characterize the properties of the set of solutions. We answer the basic questions of existence, but we also try to describe the behavior of some families of solutions when parameters vary. Chapters 3 and 4 are concerned with problems related to astrophysics. In both cases we show the existence of particular sets of solutions depending on a parameter when this parameter gets small using the Lyapunov-Schmidt Reduction method. This constructive method has been extensively applied in non-linear elliptic problems. It is based in a *good guess* of the solution, commonly called *ansatz*. Afterwards a perturbation argument and Banach's Point Fixed Theorem reduce the original problem to a finite-dimensional one, which can be solved, usually, by variational techniques. Hence one obtains *bubble towers* solutions in Chapter 3 and *Relative Equilibria* in Chapter 4.

The other the chapters are devoted to the study of the Keller-Segel model, which describes the motion of unicellular amoebae, like *dictyostelium discoideum*. The Keller-Segel model is a parabolic-elliptic system which shares with some gravitational models the property that interaction is computed an attractive Poisson / Newton equation. A major difference is the fact that it is set in a two-dimensional setting, which experimentally makes sense, while gravitational models are ordinarily three-dimensional. For this problem the existence issues are well known, but the behaviour of the solutions during the time evolution is still an active area of research. Here we extend properties already known in particular regimes to a slightly broader range of the mass parameter, and we give a precise estimate of the convergence rate of the solution to a known profile as time goes to infinity. This result is achieved using various tools such as symmetrization techniques and optimal functional inequalities. The final chapters deal with numerical results and formal computations related to the Keller-Segel model.

2.1 Chapter 3: The "Bubble-Tower" phenomenon in a semilinear elliptic equation with mixed Sobolev growth

In the first chapter we study the equation

$$\begin{cases} \Delta u + u^p + u^q &= 0 \quad \text{in } \mathbb{R}^N \\ u &> 0 \quad \text{in } \mathbb{R}^N \\ \lim_{|x|\to\infty} u(x) &\to 0 \end{cases} \tag{2.1.1}$$

for $N \geq 3$, $1 < p < q$, and Δ denotes the standard Laplacian operator. This problem is closely related to the classical Emdem-Fowler-Lane equation

$$\left\{ \begin{array}{rcll} \Delta u + u^p & = & 0 & \text{in } \mathbb{R}^N \\ u & > & 0 & \text{in } \mathbb{R}^N \\ \lim_{|x|\to\infty} u(x) & \to & 0 & \end{array} \right. \tag{2.1.2}$$

which is a basic model of the inner structure of stars. It is well known that the critical exponent $p = p^* := \frac{N+2}{N-2}$ sets a dramatic shift in the existence of solutions. In [15] the authors showed that in the case $1 < p < p^*$ there is no positive solution of (2.1.2). When $p = p^*$, (see [1], [17], [3]), all the solutions are explicitely given by

$$u_{\lambda,\xi}(x) \quad = \quad \gamma_N \left(\frac{\lambda}{\lambda^2 + |x-\xi|^2} \right)^{\frac{N-2}{2}} \qquad , \ \gamma_N = (N(N-2))^{\frac{N-2}{4}}$$

where $\lambda > 0$ and $\xi \in \mathbb{R}^N$ are arbitrary parameters. If $p > p^*$ then the solutions have the form $u_\lambda(x) = \lambda^{\frac{2}{p-1}} v(\lambda x)$, with $\lambda > 0$, and

$$u_\lambda(x) \sim C_{p,N} \, |x|^{\frac{-2}{p-1}}$$

for some known $C_{p,N} > 0$.

In the case of equation (2.1.1), H. Zou proved in [18] that if $p \leq \frac{N}{N-2}$ then (2.1.1) admits no ground states, and if $q < p^*$ then there is no positive solution. He also showed that if $p > p^*$ then the equation admits infintely many solutions that decay as $|x|^{\frac{-2}{p-1}}$ when $|x|$ goes to infinity, and finally, in the case

$$1 < p \quad < \quad p^* < q \tag{2.1.3}$$

he proved that all the ground states of (2.1.1) are radial around some point.

The question of existence of solutions for (2.1.1) under the restriction (2.1.3) has been partly answered in the slightly sub-supercritical case with geometrical dynamical systems tools in [2]. The result presented in chapter 3 recovers the existence theorems given in [2] and also gives an asymptotic approximation of the solutions with a simpler method.

First we consider the slightly super-critical case

$$\left\{ \begin{array}{rcll} \Delta u + u^p + u^{p^*+\varepsilon} & = & 0 & \text{in } \mathbb{R}^N \\ u & > & 0 & \text{in } \mathbb{R}^N \\ \lim_{|x|\to\infty} u(x) & \to & 0 & \end{array} \right. \tag{2.1.4}$$

where $\frac{N}{N-2} < p$ is fixed and $\varepsilon > 0$. Here we are able to show that

Theorem 1 *Let $N \geq 3$ and $\frac{N}{N-2} < p$. Then for any $k \in \mathbb{N}$ there exists, for all sufficiently small $\varepsilon > 0$, a solution u_ε of (2.1.1) of the form*

$$u_\varepsilon(y) \quad = \quad \gamma_N \sum_{i=1}^{k} \left(\frac{1}{1 + \alpha_i^{\frac{4}{N-2}} \varepsilon^{-(i-1+\frac{1}{p^*-p})\frac{4}{N-2}} \, |y|^2} \right)^{\frac{N-2}{2}} \alpha_i \, \varepsilon^{-(i-1+\frac{1}{p^*-p})}(1 + o(1))$$

with $o(1) \to 0$ uniformly in $\mathbb{R}^N$, as $\varepsilon \to 0$. The constants α_i can be computed explicitly and depend only on N and p.

This concentration phenomenon is called as *bubble-tower* in the literature.

In the slightly sub-critical case

$$\left\{\begin{array}{rcll} \Delta u + u^{p^*-\varepsilon} + u^q & = & 0 & \text{in } \mathbb{R}^N \\ u & > & 0 & \text{in } \mathbb{R}^N \\ \lim_{|x|\to\infty} u(x) & \to & 0 & \end{array}\right. \tag{2.1.5}$$

with $p^* < q$ fixed, for any $k \in \mathbb{N}$, we get a similar result

Theorem 2 *Let $N \geq 3$ and $p^* < q$ fixed. Then given $k \in \mathbb{N}$ there exists, for $\varepsilon > 0$ small enough, a solution u_ε of the problem (2.1.5) of the form*

$$u_\varepsilon(y) = \gamma_N \sum_{i=1}^{k} \left(\frac{1}{1 + \beta_i^{\frac{4}{N-2}} \varepsilon^{(i-1+\frac{1}{q-p^*})\frac{4}{N-2}} |y|^2} \right)^{\frac{N-2}{2}} \beta_i \, \varepsilon^{(i-1+\frac{1}{q-p^*})}(1 + o(1))$$

with $o(1) \to 0$ uniformly on $\mathbb{R}^N$, where the constants β_i depend only on N and q, and can be computed explicitly.

To prove these theorems we use the so-called Emden-Fowler transformation, first introduced in [13], which transforms dilations into translations. The problem of finding a k-bubble solution for (2.1.1) becomes equivalent to the problem of finding a k-bump solution of a second-order equation on $\mathbb{R}$. Then a variation of Lyapunov-Schimdt procedure reduces the construction of the solutions to a finite-dimensional variational problem on $\mathbb{R}$.

2.2 Chapter 4: Relative equilibria in continuous stellar dynamics

In this chapter we consider the Vlasov-Poisson system

$$\left\{\begin{array}{l} \partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = 0 \\ \\ \phi = -\dfrac{1}{4\pi\,|\cdot|} * \rho \,, \quad \rho := \displaystyle\int_{\mathbb{R}^3} f \, dv \end{array}\right. \tag{2.2.1}$$

which models the dynamics of a cloud of particles moving under the action of a mean field gravitational potential ϕ solving the Poisson equation: $\Delta\phi = \rho$. Here $f = f(t, x, v)$ is a nonnegative function in $L^\infty(\mathbb{R}, L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ depending on *time* $t \in \mathbb{R}$, *position* $x \in \mathbb{R}^3$ and *velocity* $v \in \mathbb{R}^3$, while the function ρ depends only on t and x.

The first equation in (2.2.1) is the *Vlasov equation,*, which is obtained by writing that the mass is transported by the flow of Newton's equations, when the gravitational field is computed as a mean field potential. Reciprocally, the dynamics of discrete particle systems can be formally recovered by considering empirical distributions, namely measure valued solutions made of a sum of Dirac masses, and neglecting the self-consistent gravitational terms associated to the interaction of each Dirac mass with itself.

It is also possible to relate (2.2.1) with discrete systems as follows. Consider the case of N gaseous spheres, far away one to each other, in such a way that they weakly interact through gravitation. In terms of system (2.2.1), such a solution is represented by a distribution function f, whose space density ρ is compactly supported, with several nearly spherical components. At large scale, the location of these spheres is governed at leading order by the N-body gravitational problem.

The purpose of this study is to unveil this link by constructing a special class of solutions: we will build time-periodic, non radially symmetric solutions, which generalize to kinetic equations the notion of *relative equilibria* for the discrete N-body problem. Such solutions have a planar solid motion of rotation around an axis which contains the center of gravity of the system, so that the centrifugal force counter-balances the attraction due to gravitation.

Hence we look for time-periodic solutions which are in rotation at constant angular velocity ω. Replacing $x = (x', x^3)$ and $v = (v', v^3)$ respectively by $(e^{i\,\omega\,t}\,x', x^3)$ and $(i\,\omega\,x' + e^{i\,\omega\,t}\,v', v^3)$ and using complex notations so that x', $v' \in \mathbb{R}^2 \approx \mathbb{C}$, Problem (2.2.1) becomes

$$\left\{ \begin{array}{l} \partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f - \omega^2\, x' \cdot \nabla_{v'} f + 2\,\omega\, i\, v' \cdot \nabla_{v'} f = 0\ , \\ \phi = -\frac{1}{4\pi\,|\cdot|} * \rho\ , \quad \rho = \displaystyle\int_{\mathbb{R}^3} f\ dv\ , \end{array} \right. \tag{2.2.2}$$

where we have abusively used the same notations for the potential ϕ and the distribution function f, for sake of simplicity. A *relative equilibrium* of (2.2.1) is a stationary solution of (2.2.2) and can be obtained by considering critical points of the *free energy* functional

$$\mathcal{F}[f] = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \beta(f)\ dx\,dv + \frac{1}{2} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \big(|v|^2 - \omega^2\,|x'|^2\big) f\ dx\,dv - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi|^2\ dx$$

for some arbitrary convex function β, under a mass constraint $\iint_{\mathbb{R}^3\times\mathbb{R}^3} f\ dx\,dv = M$. A typical example of such a function is

$$\beta(f) = \frac{1}{q}\,\kappa_q^{q-1}\,f^q \tag{2.2.3}$$

for some $q \in (1,\infty)$ and some positive constant κ_q, to be fixed later. The corresponding solution is known as the solution of the *polytropic gas model*, see [21, 22, 57, 61].

It follows that any relative equilibrium takes the form $f(x,v) = \gamma\big(\lambda + \frac{1}{2}\,|v|^2 + \phi(x) - \frac{1}{2}\,\omega^2\,|x'|^2\big)$ where $\gamma(s) = \kappa_q^{-1}\,(-s)_+^{1/(q-1)}$ and λ is constant on each component of the support of f. The problem is now reduced to solve a nonlinear Poisson equation, namely

$$\Delta\phi = g\big(\lambda + \phi(x) - \tfrac{1}{2}\,\omega^2\,|x'|^2\big) \quad \text{if } x \in \mathrm{supp}(\rho)$$

and $\Delta\phi = 0$ otherwise, with $g(\mu) = (-\mu)_+^p$ and $p = \frac{1}{q-1} + \frac{3}{2}$, if κ_q is appropriately chosen. Assuming that the solution has N disjoint connected components K_i, denoting by λ_i the value of λ on K_i and by χ_i the characteristic function of K_i, we end up looking for a positive solution $u = -\phi$ of

$$-\Delta u = \sum_{i=1}^N \rho_i^\omega \quad \text{in } \mathbb{R}^3\,, \quad \rho_i^\omega = \big(u - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2\big)_+^p\,\chi_i$$

under the asymptotic boundary condition $\lim_{|x|\to\infty} u(x) = 0$. We define the mass and the center of mass associated to each component by $m_i = \int_{\mathbb{R}^3} \rho_i^\omega\ dx$ and $\xi_i^\omega = \frac{1}{m_i}\int_{\mathbb{R}^3} x\,\rho_i^\omega\ dx$ respectively.

Free energy functionals have been very much studied over the last years, not only to characterize special stationary states, but also because they provide a framework to deal with *orbital stability,* which is a fundamental issue in the mechanics of gravitation. In the context of orbital stability, the use of a free energy functional, whose entropy part, $\iint_{\mathbb{R}^3\times\mathbb{R}^3} \beta(f)\ dx\,dv$ is sometimes also called the *Casimir energy functional,* goes back to the work of V.I. Arnold (see [19, 20, 62]). The variational characterization of special stationary solutions and their orbital stability have been studied by Y. Guo and G. Rein in a series

of papers [33, 34, 35, 36, 53, 54, 55, 56] and by many other authors, see for instance [26, 27, 39, 40, 41, 42, 57, 58, 61].

The main drawback of such approaches is that stationary solutions which are characterized by these techniques are in some sense trivial: radial, with a single simply connected component support. Here we use a different approach to construct the solutions, which goes back to [30] in the context of Schrödinger equations. We are not aware of attempts to use dimensional reduction coupled to power-law non-linearities and Poisson force fields except in the similar case of a nonlinear Schrödinger equation with power law nonlinearity and repulsive Coulomb forces (see [25]), or in the case of an attractive Hartree-Fock model (see [38]). Technically, our results turn out to be closely related to the ones in [23, 24].

Compared to previous results on gravitational systems, the main interest of our approach is to provide a much richer set of solutions, which is definitely of interest in astrophysics for describing complex patterns like binary gaseous stars or even more complex objects. The need of such an improvement was pointed for instance in [37]. An earlier attempt in this direction has been done in the framework of Wasserstein's distance and mass transport theory in [44]. The point of this work is that we can take advantage of the knowledge of special solutions of the N-body problem to produce solutions of the corresponding problem in continuum mechanics, which are still reminiscent of the discrete system.

The main result of chapter 4 goes as follows.

Theorem 3 *Let $N \geq 2$ and $p \in (3/2, 3) \cup (3, 5)$. For almost any masses m_i, $i = 1, \dots N$, and for any sufficiently small $\omega > 0$, there exist at least $[2^{N-1}(N-2)+1]\,(N-2)\,!$ distinct stationary solutions f_ω of (2.2.2) which are such that*

$$\int_{\mathbb{R}^3} f_\omega \, dv \, = \sum_{i=1}^N \rho_i^\omega \, + \, o(1)$$

where $o(1)$ means that the remainder term uniformly converges to 0 as $\omega \to 0_+$ and identically vanishes away from $\cup_{i=1}^N B_R(\xi_i^\omega)$, for some $R > 0$, independent of ω.

With the above notations, for all $i = 1, \dots N$, we have that

$$\rho_i^\omega(x - \xi_i^\omega) = \lambda_i^p \, \rho_*\big(\lambda_i^{(p-1)/2}\, x\big) \, + \, o(1)$$

where ρ_* is non-negative, radially symmetric, non-increasing, compactly supported function, depending only on p, and λ_i is such that $m_i = \lambda_i^{(3-p)/2} \int_{\mathbb{R}^3} \rho_* \, dx + o(1)$.

The points ξ_i^ω are such that $\xi_i^\omega = \omega^{-2/3}\,(\zeta_i^\omega, 0)$ where, for any $i = 1, \dots N$, $\zeta_i^\omega \in \mathbb{R}^2$ converges as $\omega \to 0$ to a critical point of

$$\mathcal{V}(\zeta_1, \dots \zeta_N) \; = \; \frac{1}{8\pi} \sum_{i \neq j = 1}^N \frac{m_i \, m_j}{|\zeta_i - \zeta_j|} + \frac{1}{2} \sum_{i=1}^N m_i \, |\zeta_i|^2 \, .$$

This theorem relies on a classification of relative equilibria for the N-body problems which has been established mostly by J.I. Palmore in [47, 48, 49, 50, 51]. Here *distinct* solutions means that one solution cannot be deduced from another one by a simple scaling or by a rotation. The strategy is to find critical points of

$$\mathsf{J}\,[u] = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx - \frac{1}{p+1} \sum_{i=1}^N \int_{\mathbb{R}^3} \big(u - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2\big)_+^{p+1} \, \chi_i \, dx \, ,$$

by using the solution of

$$-\Delta w_* = (w_* - 1)_+^p =: \rho_* \quad \text{in } \mathbb{R}^3$$

as "building bricks" on each of the connected components. With $W_\xi := \sum_{i=1}^N w_i$, $w_i(x) = \lambda_i\, w_*\big(\lambda_i^{(p-1)/2}\,(x-\xi_i)\big)$ and $\xi = (\xi_1, \dots \xi_N)$, we want to solve the problem

$$\Delta\phi + \sum_{i=1}^N p\big(W_\xi - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2\big)_+^{p-1}\chi_i\,\phi = -\mathsf{E} - \mathsf{N}[\phi]$$

with $\lim_{|x|\to\infty}\phi(x) = 0$, where $\mathsf{E} = \Delta W_\xi + \sum_{i=1}^N \big(W_\xi - \lambda_i + \frac{1}{2}\,\omega^2\,|x'|^2\big)_+^p\,\chi_i$ and $\mathsf{N}[\phi]$ is a nonlinear correction. A lengthy computation shows that

$$\mathsf{J}\,[W_\xi] = \sum_{i=1}^N \lambda_i^{(5-p)/2}\,\mathsf{e}_* \,-\, \omega^{2/3}\,\mathcal{V}(\zeta_1, \dots \zeta_N) \,+\, O(\omega^{4/3})$$

where $\mathsf{e}_* = \frac{1}{2}\int_{\mathbb{R}^3}|\nabla w|^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^3}(w-1)_+^{p+1}\,dx$ and $\zeta_i = \omega^{2/3}\,\xi_i'$ if the points ξ_i are such that, for a large, fixed $\mu > 0$, and all small $\omega > 0$, we have $|\xi_i| < \mu\,\omega^{-2/3}$ and $|\xi_i - \xi_j| > \mu^{-1}\,\omega^{-2/3}$. To localize each K_i in a neighborhood of ξ_i, we impose the orthogonality conditions

$$\int_{\mathbb{R}^3}\phi\,\partial_{x_j} w_i\,\chi_i\,dx = 0 \quad \forall\, i = 1,\, 2 \dots N,\; j = 1,\, 2,\, 3\;, \tag{2.2.4}$$

to the price of Lagrange multipliers. Fixed point methods allow to find a constrained solution ϕ. Since $\xi \mapsto \mathsf{J}\,[W_\xi]$ is a finite dimensional function, if $\xi_i = (\zeta_i, 0)$ is such that $(\zeta_1, \dots \zeta_N)$ is in a neighborhood of a non-degenerate critical point of $\mathcal{V}$, we can find a critical point ϕ for which the Lagrange multipliers associated to (2.2.4) are all equal to zero. This completes the scheme of the proof, up to a last technicality. All above computations have been done in terms of fixed Lagrange multipliers (corresponding to the mass constraints associated to each K_i). These constraints still need to be inverted (in order to fix the masses), thus introducing an additional restriction, namely $p \neq 3$.

In this approach, relative equilibria have been obtained in an asymptotic regime in which each component of the distribution function behaves like a minimizer of the free energy when $\omega = 0$, slightly perturbed by the other components, and can be seen at large scale like *pseudo-particles*. These pseudo-particles are located close to the relative equilibria of the N-body problem which are obtained when the centrifugal force in the rotating frame equilibrates the force of gravitation. In the rotating frame, the centrifugal force gives rise to an harmonic potential in the variable x', with negative sign, which competes with the nonlinearity. The nonlinearity indeed tends to aggregate the mass into spherically symmetric functions.

Such *symmetry breaking* phenomena due to rotation effects have been investigated in [26] in the so-called *flat* case, which is slightly simpler (no x_3 variable) to the price of a nonlocal interaction. In such a case, a different branch of solutions has been considered, which originates from the radial solution corresponding to $\omega = 0$ and gets deformed as $|\omega|$ increases. These solutions can be defined as minimizers, provided their support is restricted to a well chosen ball. It is probably not very difficult to find similar solutions in the full three-dimensional setting, although they will certainly be harder to compute numerically. It would then be of interest to understand if such solutions can co-exists with the ones found in Theorem 3 and to extend them as ω increases as a branch of solutions depending on ω. If solutions co-exist, and after restricting the support of the solutions to a large but finite ball, comparing their energy would definitely provide a new insight into the physics of gravitating systems. This is also a very nice problem of symmetry breaking, for which almost nothing is known in case of a nonlocal nonlinearity such as the one corresponding to the Newtonian potential found by solving the attractive Poisson equation.

2.3 Chapters 5 and 6: The Keller-Segel system in self-similar variables

Chapters 5 and 6 are a joint work with J. Dolbeault. They are devoted to the study of the parabolic-elliptic Keller-Segel model

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \nabla v) & x \in \mathbb{R}^2 , \quad t > 0 \\ v = -\frac{1}{2\pi} \log |\cdot| * u & x \in \mathbb{R}^2 , \quad t > 0 \\ u(0,x) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases} \tag{2.3.1}$$

which describes the motion of unicellular amoebae, like *dictyostelium discoideum*. Here u denotes their spatial density and it makes sense to consider them in a two-dimensional setting. A straightforward computation (see [109, pages 122–124] and [122]) shows that solutions (with second moment initially finite) blow-up in finite time if the total mass is large enough (larger than 8π with our conventions), while, for solutions with smaller masses, the diffusion dominates the large time asymptotics.

More precisely, it has been shown in [100, 72, 66, 88], that if

$$n_0 \in L^1_+ \left(\mathbb{R}^2 , (1 + |x|^2)\, dx \right) , \quad n_0 \, |\log n_0| \in L^1(\mathbb{R}^2) \quad \text{and} \quad M := \int_{\mathbb{R}^3} n_0 \, dx < 8\,\pi , \tag{2.3.2}$$

then there exists a solution u, in the sense of distributions, that is global in time and such that $M = \int_{\mathbb{R}^3} u(x,t)\ dx$ is preserved along the evolution. There is no non-trivial stationary solution to (2.3.1) and any solution converges to zero locally as time gets large. It is therefore convenient to study the asymptotic behavior of u in self-similar variables, where space and time scales are given respectively by $R(t) := \sqrt{1+2t}$ and $\tau(t) := \log R(t)$, and so we define the rescaled functions n and c by

$$u(x,t) := R^{-2}\, n \left(R^{-1}(t)\, x, \tau(t) \right) \quad \text{and} \quad v(x,t) := c \left(R^{-1}(t)\, x, \tau(t) \right) .$$

In this variables the system can be written as

$$\begin{cases} \frac{\partial n}{\partial t} = \Delta n + \nabla \cdot (n\, x) - \nabla \cdot (n \nabla c) & x \in \mathbb{R}^2 , \quad t > 0 \\ c = -\frac{1}{2\pi} \log |\cdot| * n & x \in \mathbb{R}^2 , \quad t > 0 \\ n(0,x) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases} \tag{2.3.3}$$

Existence of a stationary solution to (2.3.3) has been established in [76] by ODE techniques, and in [107] by PDE methods. This stationary solution is unique according to [79]. Moreover it has been shown in [66] that n and ∇c converge as $t \to \infty$, respectively in $L^1(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$ to this unique stationary solution which involves smooth and radially symmetric functions.

A simple computation of the second moment shows that smooth solutions with mass larger than $8\,\pi$ blow-up in finite time; see for instance [100]. The case $M = 8\,\pi$ has been extensively studied. We shall refer to [79, 80, 81] for some recent papers on this topic. The asymptotic regime is of a very different nature in such a critical case. In the chapters 5 and 6 we shall restrict our purpose to the sub-critical case $M < 8\,\pi$.

The rate of convergence towards the stationary solution in self-similar variables gives, after undoing the change of variables, the rate of convergence towards the asymptotic profile for the solutions of (2.3.1). In [65], it has been proved that if M is less than some mass $M_* \in (0, 8\pi)$, then convergence holds at an exponential rate, which is essentially governed by the linearization of System (2.3.3) around the stationary

solution. However, the estimate of the value of M_* was found to be significantly smaller than 8π. In the radially symmetric setting, V. Calvez and J.A. Carrillo have found in [67] that the rate measured with respect to Wasserstein's distance does not depend on the mass, in the whole range $(0, 8\pi)$. The goal of chapters 5 and 6 is to prove a similar estimate with no symmetry assumption.

In chapter 5 we define a proper functional framework and prove a functional inequality which is the crucial tool for obtaining the main result of chapter 6. Let us describe briefly this framework. In $\mathbb{R}^2$ the logarithmic Hardy-Littlewood-Sobolev has been established with optimal constants in [69] (also see [64]) and can be written as

$$\int_{\mathbb{R}^3} n \log\Big(\frac{n}{M}\Big)\,dx + \frac{2}{M}\int_{\mathbb{R}^2\times\mathbb{R}^2} n(x)\,n(y)\,\log|x-y|\,dx\,dy + M\,(1+\log\pi) \ge 0 \tag{2.3.4}$$

for any function $n \in L^1_+(\mathbb{R}^2)$ with $M = \int_{\mathbb{R}^3} n\,dx$. Equivalently, inequality (2.3.4) can be written as

$$\int_{\mathbb{R}^3} n \log\left(\frac{n}{M\,\mu}\right)dx + \frac{2}{M}\iint_{\mathbb{R}^2\times\mathbb{R}^2} (n(x) - M\,\mu(x))\,\log|x-y|\,(n(y) - M\,\mu(y))\,dx\,dy \ge 0$$

where $M = \int_{\mathbb{R}^3} n\,dx$ and $1/\mu(x) = \pi\,(1+|x|^2)^2$ for all $x \in \mathbb{R}^2$. Moreover, by Legendre's duality, it is equivalent to the euclidian Onofri's inequality.

To study the Keller-Segel system written in self-similar variables, equation (2.3.3), it turns out to be convenient to use an equivalent form of the logarithmic Hardy-Littlewood-Sobolev inequality, which, provided that $M < 8\,\pi$, can be written as

$$\int_{\mathbb{R}^3} n \log\left(\frac{n}{n_M}\right)dx + \frac{1}{4\,\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2} (n(x) - n_M(x))\,\log|x-y|\,(n(y) - n_M(y))\,dx\,dy \ge 0 \tag{2.3.5}$$

where (n_M, c_M) denotes the unique stationary solution, of (2.3.3), given by

$$-\Delta c_M = M\,\frac{e^{-\frac{1}{2}|x|^2+c_M}}{\int_{\mathbb{R}^3} e^{-\frac{1}{2}|x|^2+c}\,dx} =: n_M\ , \quad x \in \mathbb{R}^2\ .$$

Following [69, 64, 68, 71], we show by duality that (2.3.5) corresponds to a new Onofri type inequality.

Theorem 4 *For every $M \in (0, 8\,\pi)$, and for all function ϕ smooth and compactly supported, one has*

$$\log\left(\int_{\mathbb{R}^2} e^{\phi}\,d\mu_M\right) - \int_{\mathbb{R}^2}\phi\,d\mu_M \le \frac{1}{2\,M}\int_{\mathbb{R}^3}|\nabla\phi|^2\,dx\ .$$

Here, $d\mu_M := \frac{1}{M}\,n_M\,dx$ is a probability measure, and as in [70], we find a spectral gap by expanding the functional associated with the inequality around $\phi \equiv 1$. By density, it is possible to attain an inequality in the functional space obtained when the set of smooth functions with compact support is completed with respect to the norm $\|\phi\|^2 = \int_{\mathbb{R}^3}|\nabla\phi|^2\,dx + (\int_{\mathbb{R}^2}\phi\,d\mu_M)^2$. This is the main result of chapter 5.

The inequality obtained by expanding around $\phi = 1$ is a Poincaré type inequality. It has a counterpart in the Hardy-Littlewood-Sobolev framework, which can be written as

$$\mathsf{Q}_1[f] := \int_{\mathbb{R}^2}|f|^2\,d\mu_M + \frac{1}{2\,\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\,\log|x-y|\,f(y)\,d\mu_M(x)\,d\mu_M(y) \ge 0\ .$$

This suggests to write the linearized Keller-Segel system on the space of square integrable functions with respect to $d\mu_M$, which are orthogonal to the one-dimensional kernel of the linearized operator $\mathcal{L}$, endowed

with the scalar product associated to the quadratic form Q_1. Then $\mathcal{L}$ is self-adjoint and we prove that it has an explicit spectral gap:

$$\mathsf{Q}_1[f] = \langle f,\, f\rangle \le \langle f,\, \mathcal{L}\, f\rangle =: \mathsf{Q}_2[f]\ .$$

Details on $\mathcal{L}$ and in particular on its spectrum are given later. With these preliminaries in hand, one can then study the large time asymptotics of the solutions to the Keller-Segel system and relate the spectral gap with rates of convergence.

In chapter 6 we provide refined asymptotics concerning the large-time behavior of the solutions of the two-dimensional Keller-Segel system in self-similar variables. It has been shown in [65] that there exists a positive mass $M_\star \le 8\,\pi$ such that for any initial data $n_0 \in L^2(n_M^{-1}\,dx)$ of mass $M < M_\star$ satisfying (2.3.2), System (2.3.3) has a unique solution n such that

$$\int_{\mathbb{R}^2} |n(t,x) - n_M(x)|^2\,\frac{dx}{n_M(x)} \le C\,e^{-\,2\,\delta\,t} \quad \forall\ t \ge 0$$

for some positive constants C and δ. Moreover δ can be taken arbitrarily close to 1 as $M \to 0$. If $M < 8\,\pi$, we may notice that the condition $n_0 \in L^2(n_M^{-1}\,dx)$ is stronger than (2.3.2). Our main result is that $M_\star = 8\,\pi$ and $\delta = 1$, at least for a large subclass of solutions with initial datum n_0 satisfying the following technical assumption

$$\exists\ \varepsilon \in (0, 8\,\pi - M) \quad \text{such that} \quad \int_0^s n_{0,*}(\sigma)\ d\sigma \le \int_{B\left(0,\sqrt{s/\pi}\right)} n_{M+\varepsilon}(x)\ dx \quad \forall\ s \ge 0\ . \tag{2.3.6}$$

Theorem 5 *Assume that n_0 satisfies the technical assumption* (2.3.6)*,*

$$n_0 \in L^2_+(n_M^{-1}\,dx) \quad \textit{and} \quad M := \int_{\mathbb{R}^3} n_0\ dx < 8\,\pi\ .$$

Then any solution of (2.3.3) *with initial datum n_0 is such that*

$$\int_{\mathbb{R}^2} |n(t,x) - n_M(x)|^2\,\frac{dx}{n_M(x)} \le C\,e^{-\,2\,t} \quad \forall\ t \ge 0$$

for some positive constant C, where n_M is the unique stationary solution to (2.3) *with mass M.*

This result turns out to be consistent with the recent results of [67] for the two-dimensional radial model and its one-dimensional counterpart. The estimate $\delta = 1$ is sharp. For completeness, let us mention that results of exponential convergence for problems with mean field have been obtained earlier in [89, 90], but only for interaction potentials involving much smoother kernels than G_2.

To obtain this result, we establish uniform estimates on $\|n\|_{L^p(\mathbb{R}^2)}$ by applying symmetrization techniques as in [93, 94], and then prove the uniform convergence of n to n_M using Duhamel's formula. Our main tool are the spectral gap of the linearized operator $\mathcal{L}$ and the strict positivity of the linearized entropy in the appropriate functional space.

2.4 Chapters 7 and 8: A numerical study and a matched asymptotics analysis of the Keller-Segel model

The last two chapters of this thesis are again concerned with the parabolic-elliptic Keller-Segel model, but they deal with numerical schèmes and formal estimates respectively.

As mentioned previously, refined estimates of the large-time behavior of the solutions in the sub-critical mass case have been obtained in chapters 5 and 6, where the functional setting for the linear operator has also been properly characterized. In chapter 7 we recover these results numerically and give more details on the spectrum of $\mathcal{L}$. This is a joint work with J. Dolbeault.

First we study the bifurcation diagram of the solutions n_M to equation (2.3) in terms of the parameter M. Next, we turn to the linearized operator, consider f and g such that $n\,(1+f(x,t))$ and $c(x)\,(1+g(x,t))$ is a solution to (2.3.3). Then (f,g) solves the nonlinear problem

$$\begin{cases} \frac{\partial f}{\partial t} - \mathcal{L}\,f = -\frac{1}{n}\,\nabla\cdot[f\,n\,(\nabla(g\,c))] & x\in\mathbb{R}^2\,,\quad t>0\\ -\Delta(c\,g) = f\,n & x\in\mathbb{R}^2\,,\quad t>0 \end{cases}$$

where $\mathcal{L}$ is the linear operator defined by

$$\mathcal{L}\,f = \frac{1}{n}\,\nabla\cdot[n\,\nabla(f-c\,g)]$$

and we know that $(f\,n, \nabla(g\,c))(t,\cdot)$ has to evolve in $L^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)$, and asymptotically vanish as $t\to\infty$. To investigate the large time behavior, it is convenient to normalize the solution differently. What we actually want to investigate is the case where solutions of (2.3.3) can be written as

$$n(x)\,(1+\varepsilon\,f(x,t)) \quad\text{and}\quad c(x)\,(1+\varepsilon\,g(x,t))$$

in the asymptotic regime corresponding to $\varepsilon\to 0_+$. Formally, it is then clear that, at order ε, the behavior of the solution is given by $\frac{\partial f}{\partial t} = \mathcal{L}\,f$. The kernel of $\mathcal{L}$ has been identified in the previous chapters, it has also been shown that $\mathcal{L}$ has pure discrete spectrum and that 1 and 2 are eigenvalues.

In chapter 7, the goal is to identify the lowest eigenvalues and recover that the spectral gap is actually equal to 1, whatever the mass is in the range $(0,8\pi)$. We will also establish the numerical value of other eigenvalues of $\mathcal{L}$ at the bottom of its spectrum in Section 7.4, and draw some consequences in the last section of the chapter: improved rates of convergence for centered initial data and faster decay rates for best matching self-similar solutions. In order to obtain this numerical results we first parametrize the set of solutions in terms of the mass parameter $M\in(0,8\pi)$ and consider the asymptotic regimes for M small or M close to 8π. Next we introduce the linearized operator and study its spectrum using various shooting methods: we determine its kernel, the spectrum among radial functions and use a decomposition into spherical harmonics to study the other eigenvalues.

Chapter 8 is a joint work with Manuel del Pino. As conjectured by S. Childress and J.K. Percus [133] (also see [147]) *chemotactic collapse* occurs, meaning that either the solution exists globally in time or blows up in finite time. They also state that the aggregation should occur by formation of a delta Dirac distribution at the origin of the cell density, in case of radial functions. The balance between the tendency to spread mass to infinity by diffusion and the effect of aggregation caused by the drift term happens precisely at the critical mass, 8π.

In the super-critical mass case, that is when the mass is bigger than 8π and the initial datum $u_0\in L^1_+(\mathbb{R}^2,(1+|x|^2)dx)$ is such that $u_0\,|\log u_0|\in L^1(\mathbb{R}^2)$, it is easy to see using second moment estimates, that

solutions blow up in finite time. Key contributions concerning the blow-up phenomena have been made by Herrero, Velázquez in [140], [139], and by Velázquez in [152]. Also see [146] for numerical evidence.

The critical case with mass 8π has an explicit family of stationary solutions

$$u_\lambda(x) = \frac{8\,\lambda}{(|x|^2+\lambda)^2}$$

for any $\lambda > 0$. These stationary solutions, that have critical mass but infinite second moment, play a crucial role on the behavior of blowing up solutions. In fact if we consider a bounded domain Ω and the system

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + \nabla\cdot(u\nabla v) & x\in\Omega, \quad t>0, \\ \Delta v = u - 1 & x\in\Omega, \quad t>0, \\ u(0,x) = u_0(x) \geq 0 & x\in\Omega. \end{cases} \tag{2.4.1}$$

it holds that if the mass is strictly larger than 8π then there exists a formal solution that yields a concentration of mass 8π near the origin chosen to be at thé concentration point), and the asymptotics are given by

$$u(x,t) \;\; = \;\; u_{(T-t)\varepsilon(-\log(T-t))}(|x|) \tag{2.4.2}$$

for $|x| = O(\sqrt{(T-t)\varepsilon(-\log(T-t))})$ as $t\to T^-$, with

$$\varepsilon(\tau) \;\; = \;\; O(e^{\sqrt{2\tau}}) \quad \text{as } \tau\to\infty. \tag{2.4.3}$$

This result was obtained in [140] in the radial case, and in [152] for a general bounded domain.

In the whole euclidean case $\mathbb{R}^2$, the critical mass case 8π was considered in [79] in the radial case. There the authors show that there exists a global solution for initial data with infinite or finite second momentum. They also obtain that the family u_λ attracts solutions with initial data with *infinite* second momentum defined in a precise way, in which case there is a Lyapunov functional. Furthermore when the second momentum is finite Blanchet, Carrillo and Masmoudi proved in [129] that there exists a solution u^* that is global in time, has finite *free energy* $\mathcal{F}$ and such that

$$u^*(x,t) \;\; \to \;\; 8\pi\,\delta_0 \quad \text{as} \quad t\to\infty$$

in the sense of weak-star measures. Here the free energy is defined as

$$\mathcal{F}[u](t) \;\; := \;\; \int_{\mathbb{R}^2} u(x,t)\log u(x,t)dx - \frac{1}{2}\int_{\mathbb{R}^2} u(x,t)\,(-\Delta)^{-1}u(x,t)\,dx\,.$$

However the characterization of the rate of blow-up and the shape of the limiting profile are essentially open questions. The only result one can mention in this direction is [132]. With methods different from ours, the authors arrive at a similar expansion of the solutions. Here we apply the *matched asymptotics* and will comment on its interest below.

In chapter 8, we will formally derive the behavior of a radial global solution with infinite time blow-up. As in the super-critical mass case, the family of stationary solutions u_λ approximate the solution close

to the origin, but far from the origin it has different shape in order to keep the second momentum finite and constant in some sense. We will prove that as t goes to infinity:

$$u(x,t) \approx u_{\lambda(t)}(|x|) \tag{2.4.4}$$

when $|x| = O(\sqrt{t})$, and

$$u(x,t) \approx \kappa_1 \lambda(t) e^{-\frac{|x|^2}{4t}} \frac{1}{|x|^4} \tag{2.4.5}$$

elsewhere. The blow-up rate is given by

$$\lambda(t)^{-1} = \kappa \Big(\log t + O\big(\log(\log t) \big) \Big) \tag{2.4.6}$$

as $t \to \infty$, where κ is a constant depending on the second momentum of the initial data u_0.

In our approach, we have to introduce the *cumulated mass* variable to simplify the study of the equation. Next we derive an approximation of the behavior of the solution in an inner layer, and discuss how far this approximation remains valid. We continue by deriving an estimate of the solution on a remote region moving away from the origin. Finally we obtain the rate of blow-up by matching terms between the asymptotic expansions. In the appendix of chapter 8 we briefly list the differences between this case and the bounded domain case in which infinite-time blow-up also occurs ([127], [142], [150]).

The kind of behavior we have found, namely the convergence to an eternal solution of the equation in an inner layer, appears to be common in describing blow-up patterns around the singularity. For example the same phenomenon has been found for the *logarithmic fast diffusion* equation formally in [144], rigorously in [134] for the radial case, and in [135] in the non-radial case. The matched expansions method has been used to obtain asymptotic profiles in parabolic equations exhibiting infinite time blow-up previously in the case of the semilinear heat equation in [138], [137]. The method also works in the finite time blow-up case, for example in the case of the harmonic map heat flow [151], [126]. Even though this asymptotic expansion has been obtained by Chavanis and Sire in [132] by different methods, compared to their approach, we believe that the matched expansions provides a better understanding of the linearized problem, which plays a crucial role for applying the Lyapunov-Schmidt reduction. The construction of *growing up* solutions via singular perturbations arguments is an interesting line of research to be developed with the approximated solution given here as a starting point.

References

References are attached to each of the chapters.

List of Publications

[1] J. Campos, M. del Pino, and J. Dolbeault, *Relative equilibria in continuous stellar dynamics*, Comm. Math. Phys., 300 (2010), pp. 765–788.

[2] J. F. Campos, *"Bubble-tower" phenomena in a semilinear elliptic equation with mixed Sobolev growth*, Nonlinear Anal., 68 (2008), pp. 1382–1397.

[3] J. F. Campos and M. del Pino, *Matched asymptotics analysis in the Keller-Segel model with critical mass*. Technical report, 2011.

[4] J. Campos Serrano and J. Dolbeault, *Asymptotic estimates for the parabolic-elliptic Keller-Segel model in the plane*, tech. rep., Preprint, http://hal.archives-ouvertes.fr/hal-00706194, 2012.

[5] J. Dolbeault and J. Campos Serrano, *A functional framework for the Keller-Segel system: logarithmic Hardy-Littlewood-Sobolev and related spectral gap inequalities*. Preprint, http://hal.archives-ouvertes.fr/hal-00706249, 2012.

[6] Juan F. Campos and Jean Dolbeault, *A numerical study of linearized Keller-Segel operator in self-similar variables*, tech. rep., Preprint, 2012.

Chapter 3

The "Bubble-Tower" phenomenon in a semilinear elliptic equation with mixed Sobolev growth

In this chapter we consider the following problem

$$\begin{cases} \Delta u + u^p + u^q &= 0 \quad \text{in } \mathbb{R}^N \\ u &> 0 \quad \text{in } \mathbb{R}^N \\ \lim_{|x|\to\infty} u(x) &\to 0 \end{cases} \tag{3.0.1}$$

with $N/(N-2) < p < p^* = (N+2)/(N-2) < q$, $N \geq 3$.
We prove that if p is fixed, and q is close enough to the critical exponent p^*, then there exists a radial solution which behaves like a superposition of *bubbles* of different blow-up orders centered at the origin. Similarly when q is fixed and p is sufficiently close to the critic, we prove the existence of a radial solution which resembles a superposition of *flat bubbles* centered at the origin.

This work has already been published as *"Bubble-Tower" phenomena in a semilinear elliptic equation with mixed Sobolev growth*, in the journal: *Nonlinear Analysis: Theory, Methods and Applications, Volume: 68 Issue: 5 Pages: 1382-1397 Published: MAR 1 2008.*

3.1 Introduction

Let us consider the problem

$$\begin{cases} \Delta u + u^p + u^q &= 0 \quad \text{in } \mathbb{R}^N \\ u &> 0 \quad \text{in } \mathbb{R}^N \\ \lim_{|x|\to\infty} u(x) &\to 0 \end{cases} \tag{3.1.1}$$

for $N \geq 3$, $1 < p < q$, and Δ denotes the standard Laplacian operator. In the case of a single power, namely $1 < p = q$, (3.1.1) is equivalent to the classical Emdem-Fowler-Lane equation

$$\begin{cases} \Delta u + u^p &= 0 \quad \text{in } \mathbb{R}^N \\ u &> 0 \quad \text{in } \mathbb{R}^N \\ \lim_{|x|\to\infty} u(x) &\to 0 \end{cases} \tag{3.1.2}$$

This equation was introduced by Lane in the mid-19th century, as a model of the inner structure of stars. A basic question is that of finding *radial ground states* to this problem, namely a solution $u(x) = u(|x|)$ that is finite up to $r = 0$ with $u'(0) = 0$. It is well known that the critical exponent $p = p^* := \frac{N+2}{N-2}$ sets a dramatic shift in the existence of solutions. In [15] the authors showed that in the case $1 < p < p^*$ there is no positive solution of (3.1.2). When $p = p^*$, (see [1], [17], [3]), all the solutions are constituted by the family

$$u_{\lambda,\xi}(x) \quad = \quad \gamma_N \left(\frac{\lambda}{\lambda^2 + |x-\xi|^2} \right)^{\frac{N-2}{2}} \qquad , \ \gamma_N = (N(N-2))^{\frac{N-2}{4}}$$

where $\lambda > 0, \xi \in \mathbb{R}^N$. In the case $\xi = 0$, this solution is radially symmetric and it has *fast decay*, which means that $u_\lambda(r) = O(r^{-(N-2)})$ as $r \to \infty$. If $p > p^*$ then the solutions have the form $u_\lambda(x) = \lambda^{\frac{2}{p-1}} v(\lambda x)$, with $\lambda > 0$, and

$$u_\lambda(x) \sim C_{p,N} \, |x|^{\frac{-2}{p-1}}$$

where $C_{p,N} = (\frac{2}{p-1}\{\frac{2}{p-1} - (N-2)\})^{\frac{-2}{p-1}}$. This kind of asymptotic behavior is what we call *slow decay*. Let us notice that these solutions still exist when $p = p^*$ but their decay rate is like $r^{-\frac{N-2}{2}}$, which is slower than fast decay.

In the more general case $1 < p < q$, H. Zou proved in [18] that if $p \leq \frac{N}{N-2}$ then (3.1.1) admits no ground states, and if $q < p^*$ then there is no positive solution. He also showed that if $p > p^*$ then (3.1.1) admits infintely many solutions with slow decay, and finally, in the case

$$1 < p \quad < \quad p^* < q \tag{3.1.3}$$

he proved that all ground states of (3.1.1) are radial around some point. The first result of existence of radial ground states for (3.1.1) under the restriction (3.1.3), was given by Lin and Ni in [16]. They found, in the case $q = 2p - 1$, an explicit solution of the form $u(r) = A(B + r^2)^{-1/(p-1)}$ where A, B are positive constants depending on p and N. The question of existence remained open until the work of Bamón, Flores, and del Pino. In [2] the authors proved existence of radial ground states using dynamical systems tools. They proved that for $N/(N-2) < p < p^*$ fixed, given any integer $k \geq 1$, if $q > p^*$ is close enough to p^* then (3.1.1) has at least k radial ground states with fast decay. And if $p^* < q$ is fixed, given any integer $k \geq 1$, if $p < p^*$ is sufficiently close to p^*, then (3.1.1) has at least k radial ground states with fast decay. They also showed that if $q > p^*$ is fixed there exists $\bar{p} > N/(N-2)$ such if $1 < p < \bar{p}$ then there are no radial ground states. Let us notice that this results do not cover the Lin and Ni's solution since it is of slow decay. It can also be shown that slow decay solutions are unique if they exists and, as discussed in [2], [12], their presence is not expected to be generic. It is worthwhile mentioning that in the case $q = 2p - 1$, if the range of p is further restricted to

$$\frac{N + 2\sqrt{N-1}}{N + 2\sqrt{N-1} - 4} \quad < \quad p \tag{3.1.4}$$

then not only Lin and Ni's solution exists, but also infinitely many solutions with fast decay. Moreover if $\frac{N}{N-2} < p < p^* < q$, p satisfy (3.1.4), and there exists a slow decay ground state for (3.1.1), then there are infinitely many ground states with fast decay.

Even though the question of existence of solutions for (3.1.1) under the restriction (3.1.3) has been partly answered in the slightly sub-supercritical case with geometrical dynamical systems tools, the result presented in this chapter recovers the existence theorems given in [2] and also it gives an asymptotic approximation of the solutions with a simpler method.

More precisely we prove the existence of a solution which asymptotically resembles to a superposition of *bubbles* of different blow up orders, centered at the origin. First we consider the case

$$\left\{\begin{array}{rcll} \Delta u + u^p + u^{p^*+\varepsilon} & = & 0 & \text{in } \mathbb{R}^N \\ u & > & 0 & \text{in } \mathbb{R}^N \\ \lim_{|x|\to\infty} u(x) & \to & 0 & \end{array}\right. \tag{3.1.5}$$

where $\frac{N}{N-2} < p$ is fixed and $\varepsilon > 0$. Then we have

Theorem 6 *Let $N \geq 3$ and $\frac{N}{N-2} < p$. Then for any $k \in \mathbb{N}$ there exists, for all sufficiently small $\varepsilon > 0$, a solution u_ε of (3.1.1) of the form*

$$u_\varepsilon(y) = \gamma_N \sum_{i=1}^{k} \left(\frac{1}{1 + \alpha_i^{\frac{4}{N-2}} \varepsilon^{-(i-1+\frac{1}{p^*-p})\frac{4}{N-2}} |y|^2} \right)^{\frac{N-2}{2}} \alpha_i \varepsilon^{-(i-1+\frac{1}{p^*-p})}(1+o(1))$$

with $o(1) \to 0$ uniformly in $\mathbb{R}^N$, as $\varepsilon \to 0$. The constants α_i can be computed explicitly and depend only on N and p.

This kind of concentration phenomena is known as *bubble-tower*, and it has been detected for some semilinear elliptic equations with radial symmetry, see for example [6], [4], [5]. The existence of bubble-tower solutions in the case of a generic domain has been established for the Brezis-Nirenberg problem in [14], see also [10].

In the case

$$\left\{\begin{array}{rcll} \Delta u + u^{p^*-\varepsilon} + u^q & = & 0 & \text{in } \mathbb{R}^N \\ u & > & 0 & \text{in } \mathbb{R}^N \\ \lim_{|x|\to\infty} u(x) & \to & 0 & \end{array}\right. \tag{3.1.6}$$

with $p^* < q$ fixed, for any $k \in \mathbb{N}$, we prove the existence of a solution which behaves like superposition of k *flat bubbles* with a small maximum value which approach to zero uniformily as $\varepsilon \to 0$. In [8] the authors detected this kind of solutions in the problem of finding radially symmetric solutions of the prescribed mean curvature equation.

Theorem 7 *Let $N \geq 3$ and $p^* < q$ fixed. Then given $k \in \mathbb{N}$ exists, for $\varepsilon > 0$ small enough, a solution u_ε of the problem (3.1.6) of the form*

$$u_\varepsilon(y) = \gamma_N \sum_{i=1}^{k} \left(\frac{1}{1 + \beta_i^{\frac{4}{N-2}} \varepsilon^{(i-1+\frac{1}{q-p^*})\frac{4}{N-2}} |y|^2} \right)^{\frac{N-2}{2}} \beta_i \varepsilon^{(i-1+\frac{1}{q-p^*})}(1+o(1))$$

with $o(1) \to 0$ uniformly on $\mathbb{R}^N$, where the constants β_i depend only on N y q, and can be computed explicitly.

To prove these theorems we use the so-called Emden-Fowler transformation, first introduced in [13], which converts dilations into translations, so the problem of finding a k-bubble solution for (3.1.1) becomes equivalent to the problem of finding a k-bump solution of a second-order equation on $\mathbb{R}$. Then a variation of Lyapunov-Schmidt procedure reduces the construction of these solutions to a finite-dimensional

variational problem on $\mathbb{R}$. This kind of reduction, first introduced in [11], has been used to detect bubbling concentration phenomena in [6], [7], and it also can be adapted to certain situations without radial symmetry, for example when symmetry with respect to N axes at a point of the domain is assumed, see for example [9], [7].

The rest of the Chapter is organized as follows, the next three sections are devoted to the proof of the theorem 6: we first state an asymptotical estimate of the energy of the ansatz, which is the key to the method, after that we solve a nonlinear linear problem corresponding to the finite-dimensional reduction, and then solve the finite-dimensional variational problem. The last section is the proof of theorem 7, which is similar to the first one, except for some minor variations.

3.2 The Asymptotic Expansion

We are interested in the problem of finding a solution u of (3.1.5) with fast decay. We can assume that u is radial around the origin, and then (3.1.5) becomes equivalent to

$$\left\{\begin{array}{rcl} u''(r)+\frac{N-1}{r}u'(r)+u^p(r)+u^{p^*+\varepsilon}(r) &=& 0 \quad r\in(0,\infty) \\ u'(0) &=& 0 \\ \lim_{r\to\infty} u(r) &\to& 0 \end{array}\right. \tag{3.2.1}$$

Introducing the change of variable

$$v(x) \;=\; r^{\frac{2}{p^*-1}}u(r)\big|_{r=e^{-\frac{p^*-1}{2}x}} \tag{3.2.2}$$

for $x\in\mathbb{R}$, which is the so-called Emden-Fowler transformation, the problem (3.2.1) becomes

$$\left\{\begin{array}{rcl} v''(x)+\beta[e^{\varepsilon x}v^{p^*+\varepsilon}(x)+e^{-(p^*-p)x}v^p(x)]-v &=& 0 \quad \text{in } \mathbb{R} \\ 0<v(x) &\to& 0 \text{ as } x\to\pm\infty \end{array}\right. \tag{3.2.3}$$

with $\beta=(\frac{2}{N-2})^2$. The functional associated to (3.2.3) is

$$E_\varepsilon(\psi) \;=\; I_\varepsilon(\psi)-\frac{\beta}{p+1}\int_{-\infty}^{\infty} e^{-(p^*-p)x}\,|\psi|^{p+1}\,dx \tag{3.2.4}$$

where

$$I_\varepsilon(\psi) \;=\; \frac{1}{2}\int_{-\infty}^{\infty}\left|\psi'\right|^2dx+\frac{1}{2}\int_{-\infty}^{\infty}|\psi|^2\,dx-\frac{\beta}{p^*+\varepsilon+1}\int_{-\infty}^{\infty}e^{\varepsilon x}\,|\psi|^{p^*+\varepsilon+1}\,dx\,.$$

Let w be the positive radial solution of

$$\Delta w+w^{p^*} \;=\; 0 \quad \text{in } \mathbb{R}^N$$

with $w(0)=\gamma_N$, given by $u_{1,0}$. Now let U be the transformation of w via (3.2.2) given by

$$U(x) \;=\; \gamma_N e^{-x}(1+e^{-(p^*-1)x})^{-\frac{N-2}{2}} \tag{3.2.5}$$

Then U satisfy

$$\left\{\begin{array}{rcl} U''-U+\beta U^{p^*} &=& 0 \quad \text{in } \mathbb{R} \\ 0<U(x) &\to& 0 \text{ as } |x|\to\pm\infty \end{array}\right. \tag{3.2.6}$$

and

$$\gamma_N e^{-|x|} 2^{-\frac{N-2}{2}} \leq U(x) \leq \gamma_N e^{-|x|}$$

therefore $U(x) = O(e^{-|x|})$.
Let us consider $0 < \xi_1 < \xi_2 < \cdots < \xi_k$. We look for a solution of (3.2.3) of the form

$$v(x) \quad = \quad \sum_{i=1}^{k} U(x - \xi_i) + \phi$$

with ϕ small.
We define

$$U_i(x) = U(x - \xi_i) \quad , \quad V(x) = \sum_{i=1}^{k} U_i(x) \tag{3.2.7}$$

with the following choices for the points ξ_i:

$$\begin{aligned} \xi_1 &= -\frac{1}{p^* - p} \log \varepsilon - \log \Lambda_1 \\ \xi_{i+1} - \xi_i &= -\log \varepsilon - \log \Lambda_{i+1} \qquad i = 1, \ldots, k-1 \end{aligned} \tag{3.2.8}$$

where the numbers Λ_i are positive parameters. This choice of the ξ_i's turns out to be convenient in the proof of the following asymptotic expansion of $E_\varepsilon(V)$.

Lemma 1 *Let $N \geq 3$, $p > \frac{N}{N-2}$, $k \in \mathbb{N}$ and $\delta > 0$ fixed. Assume that*

$$\delta < \Lambda_i < \delta^{-1} \qquad \forall i = 1, \ldots, k \tag{3.2.9}$$

Then for $V(x)$ given by (3.2.7), and for the choice (3.2.8) of the points ξ_i, there are positives numbers $a_1, \ldots, a_5$, depending only on N and p, such that

$$\begin{aligned} E_\varepsilon(V) &= k a_1 + \varepsilon \Psi_k(\Lambda) + k a_4 \varepsilon + \varepsilon \Theta_\varepsilon(\Lambda) \\ &\quad - \frac{a_3 k}{2(p^* - p)} ((1-k)(p^* - p) - 2) \varepsilon \log \varepsilon \end{aligned} \tag{3.2.10}$$

where

$$\Psi_k(\Lambda) \quad = \quad a_3 k \log \Lambda_1 - a_5 \Lambda_1^{(p^* - p)} + \sum_{i=2}^{k} [(k - i + 1) a_3 \log \Lambda_i - a_2 \Lambda_i] \tag{3.2.11}$$

with $\Theta_\varepsilon(\Lambda) \to 0$ as $\varepsilon \to 0$, uniformly in the C^1-sense on the set of Λ_i's that satisfy (3.2.9).

Proof. Let us estimate $I_\varepsilon(V)$. First we may write

$$I_\varepsilon(V) \quad = \quad I_0(V) - \frac{\beta}{p^* + 1} \int_{-\infty}^{\infty} (e^{\varepsilon x} - 1) \, |V|^{p^* + \varepsilon + 1} \, dx + A_\varepsilon$$

where

$$\begin{aligned} A_\varepsilon &= \beta \Big(\frac{1}{p^* + 1} - \frac{1}{p^* + \varepsilon + 1} \Big) \int_{-\infty}^{\infty} e^{\varepsilon x} \, |V|^{p^* + \varepsilon + 1} \, dx \\ &\quad + \frac{\beta}{p^* + 1} \int_{-\infty}^{\infty} \big(|V|^{p^* + 1} - |V|^{p^* + \varepsilon + 1} \big) dx \end{aligned}$$

We can prove that

$$A_\varepsilon = k\varepsilon\beta\left(\frac{1}{(1+p^*)^2}\int_{-\infty}^{\infty}|U|^{p^*+1}\,dx - \frac{1}{(1+p^*)}\int_{-\infty}^{\infty}|U|^{p^*+1}\log U dx\right)+o(\varepsilon) \qquad (3.2.12)$$

In a similar way we find

$$\begin{aligned}\int_{-\infty}^{\infty}(e^{\varepsilon x}-1)\,|V|^{p^*+\varepsilon+1}\,dx &= \sum_{l=1}^{k}\int_{\mu_{l-1}}^{\mu_l} x\,|V|^{p^*+1}\,dx \\ &= k\varepsilon\left(\sum_{l=1}^{k}\xi_l\right)\int_{-\infty}^{\infty}U^{p^*+1}dy+o(\varepsilon) \qquad (3.2.13)\end{aligned}$$

Now we define

$$B = \frac{p^*+1}{\beta}\left(I_0(V)-\sum_{i=1}^{k}I_0(U_i)\right)\,.$$

It is not hard to check that

$$B = \int_{-\infty}^{\infty}\left(\sum_{i=1}^{k}U_i^{p^*+1}-\left(\sum_{i=1}^{k}U_i\right)^{p^*+1}\right)+(p^*+1)\int_{-\infty}^{\infty}\sum_{i<j}\left(U_i^{p^*}U_j\right)\,.$$

Let us consider

$$\mu_0=-\infty \quad , \quad \mu_i=\frac{\xi_i+\xi_{i+1}}{2} \quad i=1,\dots,k-1, \quad \mu_k=\infty \qquad (3.2.14)$$

and decompose B as

$$B = \sum_{l=1}^{k}\left(C_1^l-C_0^l+C_2^l\right)$$

where

$$\begin{aligned}C_0^l &= (p^*+1)\int_{\mu_{l-1}}^{\mu_l}U_l^{p^*}\sum_{j<l}^{k}U_j\,,\\ C_1^l &= \int_{\mu_{l-1}}^{\mu_l}\left[U_l^{p^*+1}-\left(\sum_{i=1}^{k}U_i\right)^{p^*+1}+(p^*+1)U_l^{p^*}\sum_{j\neq l}^{k}U_j\right]\,,\\ C_2^l &= \int_{\mu_{l-1}}^{\mu_l}\left(\sum_{i\neq l}^{k}U_i^{p^*+1}+(p^*+1)\sum_{i\neq l}\sum_{i<j}U_i^{p^*}U_j\right)\,.\end{aligned}$$

Now, let us estimate C_1^l. From the mean value theorem we get

$$\left|C_1^l\right| \leq C\int_{\mu_{l-1}}^{\mu_l}(\sum_{i\neq l}^{k}U_i)^2\left(\sum_{i=1}^{k}U_i\right)^{p^*-1}dx\,.$$

If $l \in \{2, \dots, k-1\}$, setting $\rho = -\log \varepsilon$ and using the fact that $U(x) = O(e^{-|x|})$, we get, using (3.2.14)

$$\begin{aligned} \left|C_1^l\right| &\leq C \int_0^{\frac{\rho}{2}+K} e^{-2|\rho-y|} e^{y(p^*-1)} dy \\ &\leq Ce^{-2\rho} \int_0^{\frac{\rho}{2}+K} e^{-(p^*-3)y} dy = O(e^{-\frac{p^*+1}{2}\rho}) = o(\varepsilon) \end{aligned}$$

where K depends only on δ. If $l \in \{1, k\}$, we easily check that $C_1^l = o(\varepsilon)$. Similar arguments yields $C_2^l = o(\varepsilon)$. To estimate C_0^l, we notice that

$$C_0^l = (p^*+1) \int_{\mu_{l-1}}^{\mu_l} U_l^{p^*} U_{l-1} dx + o(\varepsilon) \, .$$

According to (3.2.5), we have $U(x) = C_N \cosh\left(\frac{2x}{N-2}\right)^{-\frac{N-2}{2}}$, with $C_N = \gamma_N 2^{-\frac{N-2}{2}}$. Then

$$\left|U(x+\xi) - C_N e^{-|x+\xi|}\right| = O(e^{-p^*|x+\xi|})$$

when $\xi \to \infty$. Therefore we obtain

$$C_0^l = (p^*+1) C_N e^{\xi_l - \xi_{l-1}} \int_{-\infty}^{\infty} U^{p^*}(x) e^x dx + o(\varepsilon) \, .$$

From these estimates we conclude

$$I_0(V) = kI_0(U) - \beta C_N \int_{-\infty}^{\infty} U^{p^*}(x) e^x dx \left(\sum_{l=2}^{k} e^{\xi_l - \xi_{l-1}} \right) + o(\varepsilon) \, . \tag{3.2.15}$$

Finally, we easily check

$$\int_{-\infty}^{\infty} e^{-(p^*-p)x} V^{p+1}(x) dx = e^{-(p^*-p)\xi_1} \int_{-\infty}^{\infty} e^{-(p^*-p)x} U^{p+1}(x) + o(\varepsilon) \, . \tag{3.2.16}$$

Now we define

$$\left\{ \begin{array}{lcl} a_1 &=& \frac{1}{2}\int_{-\infty}^{\infty} |U'(x)|^2 dx + \frac{1}{2}\int_{-\infty}^{\infty} U^2(x) dx - \frac{\beta}{p^*+1}\int_{-\infty}^{\infty} U^{p^*+1}(x) dx \\ a_2 &=& \beta C_N \int_{-\infty}^{\infty} e^x U^{p^*}(x) dx \\ a_3 &=& \frac{\beta}{p^*+1}\int_{-\infty}^{\infty} U^{p^*+1}(x) dx \\ a_4 &=& \frac{1}{(p^*+1)^2}\int_{-\infty}^{\infty} U^{p^*+1}(x) dx - \frac{1}{p^*+1}\int_{-\infty}^{\infty} U^{p^*+1}(x) \log U(x) dx \\ a_5 &=& \frac{\beta}{p+1}\int_{-\infty}^{\infty} e^{-(p^*-p)x} U^{p+1}(x) dx \end{array} \right. \tag{3.2.17}$$

Collecting the estimates (3.2.12)-(3.2.16), we get the validity of the following expansion

$$E_\varepsilon(V) = ka_1 - a_2 \sum_{l=2}^{k} e^{-(\xi_l - \xi_{l-1})} - \varepsilon a_3 \left(\sum_{i=1}^{k} \xi_i \right) + k\varepsilon a_4 - a_5 e^{-(p^*-p)\xi_1} + o(\varepsilon) \, .$$

Using (3.2.8), this decomposition reads

$$E_\varepsilon(V) = ka_1 + \varepsilon \Psi_k(\Lambda) - \frac{a_3 k}{2(p^*-p)}((1-k)(p^*-p)-2)\varepsilon \log \varepsilon + ka_4\varepsilon + o(\varepsilon)$$

with Ψ_k given by (3.2.11). Moreover, the term $o(\varepsilon)$ is uniform in the set of the Λ_i's that satisfy (3.2.9). The fact that differentiation with respect to Λ leaves $o(\varepsilon)$ of the same order follows from very similar computations, so we omit them. $\square$

Let us notice that the only critical point of Ψ_k is given by

$$\Lambda^* = \left(\left[\frac{a_3 k}{a_5(p^*-p)} \right]^{\frac{1}{p^*-p}}, \frac{(k-1)a_3}{a_2}, \frac{(k-2)a_3}{a_2}, \ldots, \frac{a_3}{a_2} \right)$$

and is nondegenerate. This result will be useful since, if $V+\phi$ is a solution of (3.2.3), with ϕ small, it is natural to expect that this only occurs if Λ corresponds to a critical point of Ψ_k. This is actually true, as we show in the following sections.

3.3 The finite-dimensional reduction

In this section we consider $p > \frac{N}{N-2}$, points $0 < \xi_1 < \cdots < \xi_k$, which are for now arbitrary, and we keep the notation V and U_i as in the previous section. We define

$$Z_i(x) = U_i'(x) \quad , \quad i = 1, \ldots, k \, .$$

Consider the problem of finding a function ϕ for which there are constants c_i, $i = 1, \ldots, k$ such that

$$\left\{ \begin{array}{rcl} \sum_{i=1}^k c_i Z_i & = & -(V+\phi)'' + (V+\phi) \\ & & -\beta \left[e^{\varepsilon x} (V+\phi)_+^{p^*+\varepsilon} + e^{-(p^*-p)x} (V+\phi)_+^p \right] \\ \int_{-\infty}^{\infty} Z_i \phi & = & 0 \quad \forall \; i = 1, \ldots, k \; , \quad \lim_{x \to \pm\infty} \phi(x) = 0 \end{array} \right. \tag{3.3.1}$$

Let us consider the operator

$$\mathcal{L}_\varepsilon \phi = -\phi'' + \phi - \beta \left[(p^*+\varepsilon) e^{\varepsilon x} V^{p^*+\varepsilon-1} + p e^{-(p^*-p)x} V^{p-1} \right] \phi \, .$$

The problem (3.3.1) gets rewritten as

$$\left\{ \begin{array}{ll} \mathcal{L}_\varepsilon(\phi) = N_\varepsilon(\phi) + R_\varepsilon + \sum_{i=1}^k c_i Z_i & \text{en } \mathbb{R} \\ \int_{-\infty}^{\infty} Z_i \phi = 0 \quad \forall \; i = 1, \ldots, k \; , & \lim_{x \to \pm\infty} \phi(x) = 0 \end{array} \right. \tag{3.3.2}$$

where

$$\begin{array}{rcl} N_\varepsilon(\phi) & = & \beta e^{\varepsilon x} \left((V+\phi)_+^{p^*+\varepsilon} - V^{p^*+\varepsilon} - (p^*+\varepsilon) V^{p^*+\varepsilon-1} \phi \right) \\ & & + \beta e^{-(p^*-p)x} \left((V+\phi)_+^p - V^p - p V^{p-1} \phi \right) \; , \\ R_\varepsilon & = & \beta \left(e^{\varepsilon x} V^{p^*+\varepsilon} + e^{-(p^*-p)x} V^p - \sum_{i=1}^k U_i^{p^*}(x) \right) \; . \end{array}$$

Next we introduce next a convenient functional setting to analyze the invertibility of the operator $\mathcal{L}_\varepsilon$ under the conditions of orthogonality. For a small $\sigma > 0$, to be fixed, and a function $\psi : \mathbb{R} \to \mathbb{R}$, we define the norm

$$\|\psi\|_* = \sup_{x \in \mathbb{R}} \left(\sum_{i=1}^k e^{-\sigma|x-\xi_i|} \right)^{-1} |\psi(x)| \; .$$

In order to solve (3.3.1), it is important to understand first its linear part, we consider the problem of, given a function h, finding ϕ such that

$$\left\{ \begin{array}{cc} \mathcal{L}_\varepsilon(\phi) = h + \sum_{i=1}^k c_i Z_i & \text{en } \mathbb{R} \\ \int_{-\infty}^{\infty} Z_i \phi = 0 \quad \forall\ i = 1, \dots, k\ , & \lim_{x\to\pm\infty} \phi(x) = 0 \end{array} \right. \tag{3.3.3}$$

for certain constants c_i. The following result holds

Proposition 1 *There exists positive numbers $\varepsilon_0, \delta_0, R_0$ such that, if*

$$R_0 < \xi_1, \qquad R_0 < \min_{i=1,\dots,k} (\xi_{i+1} - \xi_i), \qquad \xi_k < \frac{\delta_0}{\varepsilon} \tag{3.3.4}$$

then for all $0 < \varepsilon < \varepsilon_0$ and $\forall\ h \in C(\mathbb{R})$ with $\|h\|_* < \infty$, the problem (3.3.3) admits a unique solution $\phi = T_\varepsilon(h)$. Besides, there exists $C > 0$ such that

$$\|T_\varepsilon(h)\|_* \le C\|h\|_* \quad , \quad |c_i| \le C\|h\|_*\ .$$

For the proof we need the following

Lemma 2 *Assume that there are sequences $\varepsilon_n \to 0$ and $0 < \xi_1^n < \cdots < \xi_k^n$ with*

$$\xi_1^n \to \infty, \qquad \min_{i=1,\dots,k} (\xi_{i+1}^n - \xi_i^n) \to \infty, \qquad \xi_k^n = o(\varepsilon_n^{-1})$$

such that for certain functions ϕ_n, h_n with $\|h_n\|_* \to 0$, and scalars c_i^n one has

$$\left\{ \begin{array}{cc} \mathcal{L}_{\varepsilon_n}(\phi_n) = h_n + \sum_{i=1}^k c_i^n Z_i^n & \text{in } \mathbb{R} \\ \int_{-\infty}^{\infty} Z_i^n \phi_n = 0 \quad \forall\ i = 1, \dots, k\ , & \lim_{x\to\pm\infty} \phi_n(x) = 0 \end{array} \right. \tag{3.3.5}$$

with $Z_i^n(x) = U'(x - \xi_i^n)$. Then

$$\lim_{n\to\infty} \|\phi_n\|_* = 0\ .$$

Proof. We will establish first the weaker assertion that

$$\lim_{n\to\infty} \|\phi_n\|_\infty = 0\ .$$

By contradiction, we may assume that $\|\phi_n\|_\infty = 1$. Testing (3.3.5) against Z_l^n and integrating by parts we get

$$\sum_{i=1}^k c_i^n \int_{-\infty}^{\infty} Z_i^n Z_l^n dx \quad = \quad \int_{-\infty}^{\infty} \mathcal{L}_{\varepsilon_n}(Z_l^n)\phi_n dx - \int_{-\infty}^{\infty} h_n Z_l^n dx\ .$$

This defines an "almost diagonal" system in the c_i^n's as $n \to \infty$. Moreover, the facts that $Z_l^n(x) = O(e^{-|x-\xi_l^n|})$, $p > \frac{N}{N-2}$, and that Z_l^n solves

$$-Z'' + (1 - p^*\beta U_l^{p^*-1})Z = 0$$

yields, after an application of dominated convergence, that $\lim_{n\to\infty} c_i^n = 0$. If we set $x_n \in \mathbb{R}^N$ such that $\phi_n(x_n) = 1$, we can assume that $\exists\ i \in \{1, \dots, k\}$ such that for n large enough

$$\exists\ R > 0 \quad \text{such that} \quad |x_n - \xi_i^n| < R \tag{3.3.6}$$

Let us fix an index i such that (3.3.6) holds. We set

$$\tilde{\phi}_n(x) = \phi_n(x + \xi_i^n) \, .$$

From (3.3.5), (3.3.6) and elliptic estimates, we see that passing to a suitable subsequence $\tilde{\phi}_n(x)$ converges uniformly over compacts to a nontrivial bounded solution ϕ of

$$-\phi'' + \phi - \beta p^* U^{p^*} \phi \;\; = \;\; 0 \qquad \text{in } \mathbb{R} \, .$$

Hence $\phi = cU'$, for some $c \neq 0$. However

$$0 = \int_{-\infty}^{\infty} Z_l^n \phi_n \;\; \to \;\; c \int_{-\infty}^{\infty} [U'(x)]^2$$

which is a contradiction. Then necessarily $\|\phi_n\|_{\infty} \to 0$.

Let us observe that (3.3.5) takes the form

$$-\phi_n'' + \phi_n = g_n(x) \tag{3.3.7}$$

with

$$g_n(x) = h_n(x) + \sum_{i=1}^{k} Z_i^n + \beta[(p^* + \varepsilon_n) e^{\varepsilon_n x} V^{p^* + \varepsilon_n - 1} + p e^{-(p^* - p)x} V^{p-1}] \phi_n \, .$$

If $0 < \sigma < \min\{p^* - 1, 1, 2p - 1 - p^*\}$, we have

$$|g_n(x)| \leq \theta_n \sum_{i=1}^{k} e^{-\sigma |x - \xi_i^n|}$$

with $\theta_n \to 0$. Choosing $\bar{c} > 0$ large enough we get that

$$\varphi_n(x) = \bar{c} \; \theta_n \sum_{i=1}^{k} e^{-\sigma |x - \xi_i^n|}$$

is a supersolution of (3.3.7), and $-\varphi_n(x)$ will be a subsolution of (3.3.7). Then

$$|\phi_n| \leq \theta_n \sum_{i=1}^{k} e^{-\sigma |x - \xi_i^n|}$$

and the proof of the lemma is concluded. □

Proof of Proposition 1. Consider

$$H = \left\{ \phi \in H^1(\mathbb{R}) \; : \; \int_{-\infty}^{\infty} Z_i \phi = 0 \; \forall i \in \{1, \ldots, k\} \right\}$$

endowed with the inner product $[\phi, \psi] = \int_{-\infty}^{\infty} (\phi' \psi' + \phi \psi)$. Then the problem (3.3.3) expressed in weak form is equivalent to that of finding $\phi \in H$ such that $\forall \; \psi \in H$

$$[\phi, \psi] = \beta \int_{-\infty}^{\infty} \left[(p^* + \varepsilon) e^{\varepsilon x} V^{p^* + \varepsilon - 1} + p e^{-(p^* - p)x} V^{p-1} \right] \phi \psi + \int_{-\infty}^{\infty} h \psi$$

With the aid of Riesz' representation theorem, this equation gets rewritten in the operational form

$$[\phi, \psi] = [K_\varepsilon(\phi) + \tilde{h}, \psi]$$

where $\tilde{h}$ depends linearly on h, and $K_\varepsilon(\phi)$ is compact. Fredholm's alternative guarantees unique solvability for any $h \in H$, provided that the equation $\phi = K_\varepsilon(\phi)$ has only the trivial solution in H. This latter statement holds for $R_0, \varepsilon_0, \delta_0$ chosen properly, assuming the opposite would lead us to a contradiction with the previous lemma. Continuity can be deduced in a similar way. □

Now we study some differentiability properties of T_ε on the variables ξ_i, that will be important for later purposes. We shall use the notation $\xi = (\xi_1, \ldots, \xi_k)$, and consider the Banach space

$$C_* = \{f \in C(\mathbb{R}) \mid \|f\|_* < \infty\}$$

endowed with the $\|\cdot\|_*$ norm. We also consider the space $\mathcal{L}(C_*)$ of linear operators of C_*.
The following result holds

Proposition 2 *Under the assumptions of the proposition 1, the map $\xi \to T_\varepsilon$ with values in $\mathcal{L}(C_*)$ is of class C^1. Moreover, there is a constant $C > 0$ such that*

$$\|D_\xi T_\varepsilon\|_{\mathcal{L}(C^*)} \leq C$$

uniformly on the vectors ξ that satisfy (3.3.4).

Proof. Fix $h \in C_*$, and let $\phi = T_\varepsilon(h)$ for $\varepsilon < \varepsilon_0$. Consider differentiation with respect to ξ_l. Let us recall that ϕ satisfies

$$\mathcal{L}_\varepsilon(\phi) \;=\; h + \sum_{i=1}^{k} c_i\, Z_i \qquad \text{in } \mathbb{R}$$

and orthogonality conditions, for some constants c_i (uniquely determined). For $j \in \{1, \ldots, k\}$ we define the constants α_j as the solution of

$$\begin{aligned}
\sum_{j=1}^{k} \alpha_j \int_{-\infty}^{\infty} Z_j Z_i &= 0 \quad \forall\, i \neq l\,, \\
\sum_{j=1}^{k} \alpha_j \int_{-\infty}^{\infty} Z_j Z_l &= -\int_{-\infty}^{\infty} \phi \partial_{\xi_l} Z_l\,.
\end{aligned}$$

Again this is an almost diagonal system. We define also the function

$$f(x) = \beta \partial_{\xi_l} F_\varepsilon(x)\phi + c_l \partial_{\xi_l} Z_l - \sum_{j=1}^{k} \alpha_j \mathcal{L}_\varepsilon(Z_j)$$

where

$$F_\varepsilon(x) = \beta \left[(p^* + \varepsilon) e^{\varepsilon x} V^{p^* + \varepsilon - 1} + p e^{-(p^* - p)x} V^{p-1} \right]\,.$$

Hence $\partial_{\xi_l}\phi$ satisfy

$$\partial_{\xi_l}\phi - \sum_{j=1}^{k} \alpha_j Z_j = T_\varepsilon(f) \ .$$

Moreover $|\alpha_i| \leq C\|\phi\|_*$, $|c_i| \leq C\|h\|_*$, $\|\phi\|_* \leq C\|h\|_*$, so that also $\|\partial_{\xi_l}\phi\|_* \leq C\|h\|_*$ Besides $\partial_{\xi_l}\phi$ depends continuously on ξ for this norm, and the validity of the result is proved. $\square$

In what follows we assume, for $M > 0$ large and fixed, the validity of the constraints

$$\begin{aligned} &\frac{1}{p^* - p}\log(M\varepsilon)^{-1} < \xi_1 \quad , \quad \log(M\varepsilon)^{-1} < \min_{i=2,\dots,k}(\xi_i - \xi_{i-1}) \ , \\ &\qquad \xi_k < k\log(M\varepsilon)^{-1} \ . \end{aligned} \tag{3.3.8}$$

For the next purposes it is useful to point out that for σ fixed and small enough, and for $\|\phi\|_\sigma \leq \theta$, where θ is a small and fixed positive number, one can check that

$$\|N(\phi)\|_* \ \leq \ C(\|\phi\|_*^{\min\{p^*,2\}} + \|\phi\|_*^{\min\{2p-p^*,2\}}) \tag{3.3.9}$$

$$\left\| \frac{\partial N_\varepsilon}{\partial \phi} \right\|_* \ \leq \ C\left(\|\phi\|_*^{\min\{p^*-1,2\}} + \|\phi\|_*^{\min\{2p-p^*-1,2\}} \right) \tag{3.3.10}$$

and

$$\|R_\varepsilon\|_* \leq C\varepsilon^\alpha \quad , \quad \|\partial R_\varepsilon\|_* \leq C\varepsilon^\alpha \tag{3.3.11}$$

with $\alpha = \frac{1+\lambda}{2}$, for some $\lambda > 0$ small enough.

Proposition 3 *Assume that conditions (3.3.8) hold. Then there exists a positive C such that for any $\varepsilon > 0$, small enough, there exists a unique solution $\phi = \phi(\xi)$ to the problem (3.3.1), which besides satisfies*

$$\|\phi\|_* \leq C\varepsilon^\alpha \ .$$

Moreover, the map $\xi \to \phi(\xi)$ is of class C^1 for the $\|\cdot\|_*$-norm, and

$$\|D_\xi\phi\|_* \leq C\varepsilon^\alpha \ .$$

Proof. We will only prove that the existence statement is equivalent to solving a fixed point problem. If we define

$$A_\varepsilon(\phi) := T_\varepsilon(N_\varepsilon(\phi) + R_\varepsilon) \ ,$$

then (3.3.2) is equivalent to the fixed point problem $\phi = A_\varepsilon(\phi)$. We will show that A_ε is a contraction in a proper region. Let

$$\mathcal{F}_r = \left\{ \phi \in C_* \ : \ \|\phi\|_* \leq r\varepsilon^\alpha \right\}$$

where $r > 0$ will be fixed later. We have that

$$\begin{aligned} \|A_\varepsilon(\phi)\|_* \ &\leq \ \|T_\varepsilon\left(N_\varepsilon(\phi) + R_\varepsilon\right)\|_* \\ &\leq \ C\|N_\varepsilon(\phi) + R_\varepsilon\|_* \\ &\leq \ C_0\left((r\varepsilon^\alpha)^{\min\{p^*,2\}} + (r\varepsilon^\alpha)^{\min\{2p-p^*,2\}} + \varepsilon^\alpha \right) \ . \end{aligned}$$

Besides we have

$$|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)| \leq C\left((r\varepsilon^\alpha)^{\min\{p^*-1,2\}} + (r\varepsilon^\alpha)^{\min\{2p-p^*-1,2\}}\right)|\phi_1 - \phi_2|$$

and consequently we get

$$\|A(\phi_1) - A(\phi_2)\|_* \leq C_1\left((r\varepsilon^\alpha)^{\min\{p^*-1,2\}} + (r\varepsilon^\alpha)^{\min\{2p-p^*-1,2\}}\right)\|\phi_1 - \phi_2\|_* .$$

If we choose $r \geq 3C_0$, then for ε small enough

$$\begin{aligned} C_0\left((r\varepsilon^\alpha)^{\min\{p^*,2\}} + (r\varepsilon^\alpha)^{\min\{2p-p^*,2\}} + \varepsilon^\alpha\right) &\leq r\varepsilon^\alpha \\ C_1\left((r\varepsilon^\alpha)^{\min\{p^*-1,2\}} + (r\varepsilon^\alpha)^{\min\{2p-p^*-1,2\}}\right) &< 1 \end{aligned}$$

and so there is a unique fixed point of A in $\mathcal{F}_r$.

Concerning now the differentiability of $\xi \to \phi(\xi)$, let

$$B(\xi, \phi) = \phi - T_\varepsilon(N_\varepsilon(\phi) + R_\varepsilon) .$$

Of course we have $B(\xi, \phi(\xi)) = 0$. Now let us write

$$D_\phi B(\xi, \phi)[\theta] = \theta - T_\varepsilon(\theta D_\phi N_\varepsilon(\phi)) = \theta + M(\theta)$$

where

$$M(\theta) = -T_\varepsilon(\theta D_\phi N_\varepsilon(\phi))$$

From (3.3.10) and using the fact that $\phi \in F_r$, we obtain

$$\|M(\theta)\|_* \leq C\left(\varepsilon^{\alpha\min\{p^*-1,2\}} + \varepsilon^{\alpha\min\{2p-p^*-1,2\}}\right)\|\theta\|_* .$$

It follows that for a small ε, the operator $D_\phi B(\varepsilon, \phi)$ is invertible, with uniformly bounded inverse. It also depends continuously on its parameters. Let us differentiate with respect to ξ, we have

$$D_\xi B(\xi, \phi) = -D_\xi T_\varepsilon[N_\varepsilon(\phi) + R_\varepsilon] - T_\varepsilon[D_\xi N_\varepsilon(\xi, \phi) + D_\xi R_\varepsilon]$$

where all these expressions depend continuously on their parameters. Now, the implicit function theorem yields that $\phi(\xi)$ is of class C^1 and

$$D_\xi\phi = -\left(D_\phi B(\xi, \phi)\right)^{-1}\left[D_\xi B(\xi, \phi)\right]$$

so that

$$\|D_\xi\phi\|_* \leq C\left(\|N_\varepsilon(\phi) + R_\varepsilon\|_* + \|D_\xi N_\varepsilon(\xi, \phi)\|_* + \|D_\xi R_\varepsilon\|_*\right) \leq C\varepsilon^\alpha .$$

This concludes the proof. □

3.4 The Finite-Dimensional Variational Problem

In this section we fix $M > 0$ large and assume that conditions (3.3.8) hold for $\xi = (\xi_1, \dots, \xi_k)$. According to the previous sections, our original problem has been reduced to that of finding ξ such that the c_i that appear in (3.3.1), given by proposition 3, are all zero. Thus we need to solve

$$c_i(\xi) = 0 \qquad \forall\, i \in \{1, \dots, k\}\ . \tag{3.4.1}$$

This problem is equivalent to a variational problem. We define

$$J_\varepsilon(\xi) = E_\varepsilon(V + \phi(\xi))\ .$$

Lemma 3 *The function $V + \phi$ is a solution of (3.2.3) $\Leftrightarrow$ ξ is a critic point of J_ε, where $\phi = \phi(\xi)$ is given by proposition 3.*

Proof. Assume that $V + \phi$ solves (3.2.3), integrating (3.2.3) against $\partial_{\xi_l}(V + \phi)$ we get

$$DE_\varepsilon(V + \phi)\partial_{\xi_l}(V + \phi) = 0\ .$$

Now if ξ is a critic point of J_ε, we have

$$\begin{aligned} D_{\xi_l} J_\varepsilon(\xi) = 0 \quad &\Leftrightarrow \quad DE_\varepsilon(V + \phi)\partial_{\xi_l}(V + \phi) = 0 \\ &\Leftrightarrow \quad \sum_{i=1}^{k} c_i \int_{-\infty}^{\infty} Z_i \partial_{\xi_l}(V + \phi) = 0\ . \end{aligned}$$

But $\partial_{\xi_l}(V + \phi) = Z_l + o(1)$ where $o(1) \to 0$ as $\varepsilon \to 0$, uniformly for the $\|\cdot\|_*$-norm. Therefore

$$D_{\xi_l} J_\varepsilon(\xi) = 0 \quad \forall\, l = 1, \dots, k \quad \Leftrightarrow \quad \sum_{i=1}^{k} c_i \int_{-\infty}^{\infty} Z_i[Z_l + o(1)] = 0 \quad \forall\, l = 1, \dots, k$$

which defines an almost diagonal linear system on c_i, and the conclusion follows. □

The next lemma is crucial to find the critical points of J_ε.

Lemma 4 *The following expansion holds*

$$J_\varepsilon(\xi) = E_\varepsilon(V) + o(\varepsilon)$$

where $o(\varepsilon)$ is uniform in the C^1-sense on the vectors ξ which satisfy (3.3.8).

Proof. Using the fact that $DE_\varepsilon(V + \phi)[\phi] = 0$, a Taylor expansion gives

$$\begin{aligned} E_\varepsilon(V + \phi) - E_\varepsilon(V) &= \int_0^1 D^2 E_\varepsilon(V + t\phi)[\phi^2] t dt \\ &= \int_0^1 \int_{-\infty}^{\infty} (N_\varepsilon(\phi) + R_\varepsilon)\phi t dt \\ &\quad + \beta(p^* + \varepsilon) \int_0^1 \int_{-\infty}^{\infty} e^{\varepsilon x} \left[V^{p^* + \varepsilon - 1} - (V + t\phi)^{p^* + \varepsilon - 1} \right] \phi^2 t dt \\ &\quad + \beta p \int_0^1 \int_{-\infty}^{\infty} e^{-(p^* - p)x} \left[V^{p-1} - (V + t\phi)^{p-1} \right] \phi^2 t dt \end{aligned}$$

and since $\|\phi\|_* \leq C\varepsilon^\alpha$, with $\alpha = \frac{1+\lambda}{2}$, we get

$$J_\varepsilon(\xi) - E_\varepsilon(V) = O(\varepsilon^{1+\lambda})$$

uniformly on the points satisfying (3.3.8). Differentiating now with respect to the ξ variables, we obtain

$$\begin{aligned}\partial_{\xi_l}\left(J_\varepsilon(\xi) - E_\varepsilon(V)\right) &= \int_0^1\int_{-\infty}^{\infty}\partial_{\xi_l}[(N_\varepsilon(\phi)+R_\varepsilon)\phi]tdt \\ &+\beta(p^*+\varepsilon)\int_0^1\int_{-\infty}^{\infty}e^{\varepsilon x}\partial_{\xi_l}\left(\left[V^{p^*+\varepsilon-1}-(V+t\phi)^{p^*+\varepsilon-1}\right]\phi^2\right)tdt \\ &+\beta p\int_0^1\int_{-\infty}^{\infty}e^{-(p^*-p)x}\partial_{\xi_l}\left(\left[V^{p-1}-(V+t\phi)^{p-1}\right]\phi^2\right)tdt\,.\end{aligned}$$

From the computations made in the previous propositions we deduce

$$\partial_{\xi_l}\left(J_\varepsilon(\xi) - E_\varepsilon(V)\right) = O(\varepsilon^{1+\lambda})$$

which concludes the proof. □

Proof of Theorem 6. We consider the change of variable

$$\begin{aligned}\xi_1 &= -\frac{1}{p^*-p}\log\varepsilon - \log\Lambda_1 \\ \xi_{i+1}-\xi_i &= -\log\varepsilon - \log\Lambda_i \quad \forall\ i\geq 2\end{aligned}$$

where the Λ_i's are positive parameters. For notational convenience, we set $\Lambda = (\Lambda_1,\ldots,\Lambda_k)$. Hence it suffices to find critical points of

$$\Phi_\varepsilon(\Lambda) = \varepsilon^{-1}J_\varepsilon(\xi(\Lambda))\,.$$

From the previous lemma and the expansion given in lemma 1, we get

$$\nabla\Phi_\varepsilon(\Lambda) = \nabla\Psi_k(\Lambda) + o(1)$$

where $o(1)\to 0$ uniformly on the vectors Λ satisfying $M^{-1} < \Lambda_i < M$ for any fixed large M. As we pointed out before, Ψ_k has only one critical point that we denote Λ^*. Since this critical point is nondegenerate, it follows that the local degree $\deg(\nabla\Phi_\varepsilon, \mathcal{U}, 0)$ is well defined and is nonzero. Here $\mathcal{U}$ denotes an arbitrarily small neighborhood of Λ^*. Hence for a sufficiently small ε

$$\deg(J_\varepsilon, \mathcal{U}, 0) \neq 0\,.$$

We conclude that exists a critical point Λ_ε^* of Φ_ε such that

$$\Lambda_\varepsilon^* = \Lambda^* + o(1)\,.$$

Then for $\xi^* = \xi(\Lambda^*)$ we obtain that

$$v^* = \sum_{i=1}^k U(x-\xi_i^*) + \phi(\xi^*) = \sum_{i=1}^k U(x-\xi_i^*)(1+o(1))$$

is a solution of (3.2.3), and going back in the transformation (3.2.2) we obtain that

$$u_\varepsilon^*(r) = \gamma_N \sum_{i=1}^{k} e^{\xi_i^*} \left(\frac{1}{1 + e^{(p^*-1)\xi_i^*} r^2} \right)^{\frac{N-2}{2}} (1 + o(1))$$

is a solution of (3.1.5), where

$$e^{\xi_i^*} \quad = \quad \varepsilon^{-(i-1)-\frac{1}{p^*-p}} \Pi_{j=1}^{i} (\Lambda_i^*)^{-1} \ .$$

If we set $\alpha_i = \Pi_{j=1}^{i}(\Lambda_i^*)^{-1}$, then we find that

$$\alpha_i = \left[\frac{a_5(p^*-p)}{a_3 k} \right]^{\frac{1}{p^*-p}} \left(\frac{a_2}{a_3} \right)^{i-1} \frac{(k-i)!}{(k-1)!}$$

where the constants a_2, a_3, a_5, are given by (3.2.17). □

3.5 Proof of Theorem 7

In this section we consider the transformation

$$v(x) = r^{\frac{2}{p^*-1}} u(r)\big|_{r=e^{\frac{p^*-1}{2}x}} \tag{3.5.1}$$

and then problem (3.1.6) turns out to be equivalent to

$$\begin{cases} v''(x) + \beta[e^{\varepsilon x} v^{p^*-\varepsilon}(x) + e^{-(q-p^*)x} v^q(x)] - v \quad = \quad 0 \quad \text{in } \mathbb{R} \ , \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 < v(x) \quad \to \quad 0 \quad \text{as } x \to \pm\infty \ . \end{cases} \tag{3.5.2}$$

The associated functional reads

$$\begin{aligned} \hat{E}_\varepsilon(\psi) \quad = \quad & \frac{1}{2}\int_{-\infty}^{\infty} \left|\psi'\right|^2 dx + \frac{1}{2}\int_{-\infty}^{\infty} |\psi|^2 \, dx - \frac{\beta}{p^*-\varepsilon+1}\int_{-\infty}^{\infty} e^{\varepsilon x} \, |\psi|^{p^*-\varepsilon+1} \, dx \\ & - \frac{\beta}{q+1}\int_{-\infty}^{\infty} e^{-(q-p^*)x} \, |\psi|^{q+1} \, dx \ . \end{aligned} \tag{3.5.3}$$

We define $\hat{U}$ as the transformation via (3.5.1) of w, and for small $\varepsilon > 0$ we define

$$\begin{aligned} \hat{\xi}_1 \quad &= \quad -\frac{1}{q-p^*} \log \varepsilon - \log \hat{\Lambda}_1 \\ \hat{\xi}_{i+1} - \hat{\xi}_i \quad &= \quad -\log \varepsilon - \log \hat{\Lambda}_{i+1} \qquad i = 1, \dots, k-1 \end{aligned} \tag{3.5.4}$$

where the points $\hat{\Lambda}_i$ are positive parameters. We look for a solution of (3.5.2) of the form

$$v(x) \quad = \quad \sum_{i=1}^{k} \hat{U}(x - \hat{\xi}_i) + \phi$$

where ϕ is small. In a similar way to the proof of lemma 1, we can prove that for $N \geq 3$, $k \in \mathbb{N}$, $q > p^*$ and $\delta > 0$ fixed, if

$$\delta < \hat{\Lambda}_i < \delta^{-1} \qquad \forall i = 1, \dots, k \tag{3.5.5}$$

then there are constants $b_1, \dots, b_n$ depending only on N and q, such that

$$\hat{E}_\varepsilon(\hat{V}) = kb_1 + \varepsilon\hat{\Psi}_k(\hat{\Lambda}) - \frac{b_3 k}{2(q-p^*)}((1-k)(q-p^*)-2)\varepsilon\log\varepsilon - kb_4\varepsilon + \varepsilon\hat{\Theta}_\varepsilon(\hat{\Lambda}) \tag{3.5.6}$$

where $\hat{\Lambda} = (\hat{\Lambda}_1, \dots, \hat{\Lambda}_k)$ and

$$\hat{\Psi}_k(\hat{\Lambda}) \;=\; b_3 k\log\hat{\Lambda}_1 - b_5\hat{\Lambda}_1^{(q-p^*)} + \sum_{i=2}^{k}[(k-i+1)b_3\log\hat{\Lambda}_i - b_2\hat{\Lambda}_i] \tag{3.5.7}$$

with $\Theta_\varepsilon(\Lambda) \to 0$ as $\varepsilon \to 0$, uniformly in the C^1-sense on the points $\hat{\Lambda}_i$ that satisfy (3.5.5). Besides the constants are given by

$$\left\{\begin{array}{rcl} b_1 &=& \frac{1}{2}\int_{-\infty}^{\infty}\left|\hat{U}'(x)\right|^2 dx + \frac{1}{2}\int_{-\infty}^{\infty}\hat{U}^2(x)dx - \frac{\beta}{p^*+1}\int_{-\infty}^{\infty}\hat{U}^{p^*+1}(x)dx \\ b_2 &=& \beta C_N\int_{-\infty}^{\infty}e^x\hat{U}^{p^*}(x)dx \\ b_3 &=& \frac{\beta}{p^*+1}\int_{-\infty}^{\infty}\hat{U}^{p^*+1}(x)dx \\ b_4 &=& \frac{1}{(p^*+1)^2}\int_{-\infty}^{\infty}\hat{U}^{p^*+1}(x)dx - \frac{1}{p^*+1}\int_{-\infty}^{\infty}\hat{U}^{p^*+1}(x)\log\hat{U}(x)dx \\ b_5 &=& \frac{\beta}{q+1}\int_{-\infty}^{\infty}e^{-(q-p^*)x}\hat{U}^{q+1}(x)dx \end{array}\right. \tag{3.5.8}$$

It follows that the only critical point of $\hat{\Psi}_k$ is nondegenerate and given by

$$\hat{\Lambda}^* \;=\; \left(\left[\frac{b_3 k}{b_5(q-p^*)}\right]^{\frac{1}{q-p^*}}, \frac{(k-1)b_3}{b_2}, \frac{(k-2)b_3}{b_2}, \dots, \frac{b_3}{b_2}\right) .$$

The finite-dimensional reduction can be worked in a way similar to the section 3.3, except for (3.3.9), (3.3.10), that get replaced by

$$\|N(\phi)\|_* \;\le\; C(\|\phi\|_*^{\min\{p^*,2\}} + \|\phi\|_*^{\min\{\frac{p^*+1}{2},\frac{3}{2}\}}) , \tag{3.5.9}$$

$$\left\|\frac{\partial N_\varepsilon}{\partial\phi}\right\|_* \;\le\; C\left(\|\phi\|_*^{\min\{p^*-1,2\}} + \|\phi\|_*^{\min\{\frac{p^*-1}{2},2\}}\right) . \tag{3.5.10}$$

The finite-dimensional variational problem and the conclusion of the theorem can be derived in a way analogous to the one of section 3.4. □

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Chapter 4

Relative equilibria in continuous stellar dynamics

This chapter is devoted to the study of a three dimensional continuous model of gravitating matter rotating at constant angular velocity. In the rotating reference frame, by a finite dimensional reduction, we prove the existence of non radial stationary solutions whose supports are made of an arbitrarily large number of disjoint compact sets, in the low angular velocity and large scale limit. At first order, the solutions behave like point particles, thus making the link with the *relative equilibria* in N-body dynamics.

This work is a joint work with M. del Pino and J. Dolbeault, which has already been published as *Relative Equilibria in Continuous Stellar Dynamics*, in the journal *Communications in Mathematical Physics, December 2010, Volume 300, Issue 3, pp. 765-788.*

4.1 Introduction and statement of the main results

We consider the Vlasov-Poisson system

$$\begin{cases} \partial_t f + v\cdot\nabla_x f - \nabla_x\phi\cdot\nabla_v f = 0 \\ \\ \phi = -\dfrac{1}{4\pi\,|\,\cdot\,|} * \rho\,, \quad \rho := \displaystyle\int_{\mathbb{R}^3} f\,dv \end{cases} \tag{4.1.1}$$

which models the dynamics of a cloud of particles moving under the action of a mean field gravitational potential ϕ solving the Poisson equation: $\Delta\phi = \rho$. Kinetic models like system (4.1.1) are typically used to describe gaseous stars or globular clusters. Here $f = f(t,x,v)$ is the so-called *distribution function,* a nonnegative function in $L^\infty(\mathbb{R}, L^1(\mathbb{R}^3\times\mathbb{R}^3))$ depending on *time* $t\in\mathbb{R}$, *position* $x\in\mathbb{R}^3$ and *velocity* $v\in\mathbb{R}^3$, which represents a density of particles in the *phase space,* $\mathbb{R}^3\times\mathbb{R}^3$. The function ρ is the *spatial density* function and depends only on t and x. The total mass is conserved and hence

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} f(t,x,v)\,dx\,dv = \int_{\mathbb{R}^3}\rho(t,x)\,dx = M$$

does not depend on t.

The first equation in (4.1.1) is the *Vlasov equation,* also known as the *collisionless Boltzmann equation* in the astrophysical literature; see [22]. It is obtained by writing that the mass is transported by the flow of Newton's equations, when the gravitational field is computed as a mean field potential. Reciprocally, the

dynamics of discrete particle systems can be formally recovered by considering empirical distributions, namely measure valued solutions made of a sum of Dirac masses, and neglecting the self-consistent gravitational terms associated to the interaction of each Dirac mass with itself.

It is also possible to relate (4.1.1) with discrete systems as follows. Consider the case of N gaseous spheres, far away one to each other, in such a way that they weakly interact through gravitation. In terms of system (4.1.1), such a solution should be represented by a distribution function f, whose space density ρ is compactly supported, with several nearly spherical components. At large scale, the location of these spheres is governed at leading order by the N-body gravitational problem.

The purpose of this study is to unveil this link by constructing a special class of solutions: we will build time-periodic, non radially symmetric solutions, which generalize to kinetic equations the notion of *relative equilibria* for the discrete N-body problem. Such solutions have a planar solid motion of rotation around an axis which contains the center of gravity of the system, so that the centrifugal force counter-balances the attraction due to gravitation. Let us give some details.

Consider N point particles with masses m_j, located at points $x_j(t) \in \mathbb{R}^3$ and assume that their dynamics is governed by Newton's gravitational equations

$$m_j \frac{d^2 x_j}{dt^2} = \sum_{j \neq k=1}^{N} \frac{m_j\, m_k}{4\pi} \frac{x_k - x_j}{|x_k - x_j|^3} , \quad j = 1, \dots N . \tag{4.1.2}$$

Let us write $x \in \mathbb{R}^3$ as $x = (x', x^3) \in \mathbb{R}^2 \times \mathbb{R} \approx \mathbb{C} \times \mathbb{R}$ where, using complex notations, $x' = (x^1, x^2) \approx x^1 + i\, x^2$ and rewrite system (4.1.2) in coordinates relative to a reference frame rotating at a constant velocity $\omega > 0$ around the x^3-axis. This amounts to carry out the change of variables

$$x = (e^{i\,\omega\, t}\, z',\, z^3), \quad z' = z^1 + i\, z^2 .$$

In terms of the coordinates (z', z^3), system (4.1.2) then reads

$$\frac{d^2 z_j}{dt^2} = \sum_{j \neq k=1}^{N} \frac{m_k}{4\pi} \frac{z_k - z_j}{|z_k - z_j|^3} + \omega^2\, (z'_j, 0) + 2\, \omega\, (i\, \tfrac{dz'_j}{dt}, 0) , \quad j = 1, \dots N . \tag{4.1.3}$$

We consider solutions which are stationary in the rotating frame, namely constant solutions $(z_1, \dots z_N)$ of system (4.1.3). Clearly all z_j's have their third component with the same value, which we assume zero. Hence, we have that

$$z_k = (\xi_k, 0) , \quad \xi_k \in \mathbb{C} ,$$

where the ξ_k's are constants and satisfy the system of equations

$$\sum_{k \neq j=1}^{N} \frac{m_k}{4\pi} \frac{\xi_k - \xi_j}{|\xi_k - \xi_j|^3} + \omega^2\, \xi_j = 0 , \quad j = 1, \dots N . \tag{4.1.4}$$

In the original reference frame, the solution of (4.1.2) obeys to a rigid motion of rotation around the center of mass, with constant angular velocity ω. This solution is known as a *relative equilibrium*, thus taking the form

$$x_j^{\omega}(t) = (e^{i\,\omega\, t}\, \xi_j, 0) , \quad \xi_j \in \mathbb{C} , \quad j = 1, \dots N .$$

System (4.1.4) has a variational formulation. In fact a vector $(\xi_1, \dots \xi_N)$ solves (4.1.4) if and only if it is a critical point of the function

$$\mathcal{V}_m^{\omega}(\xi_1, \dots \xi_N) \;:=\; \frac{1}{8\pi} \sum_{j \neq k=1}^{N} \frac{m_j\, m_k}{|\xi_k - \xi_j|} + \frac{\omega^2}{2} \sum_{j=1}^{N} m_j\, |\xi_j|^2 .$$

Here m denotes $(m_j)_{j=1}^N$. A further simplification is achieved by considering the scaling

$$\xi_j = \omega^{-2/3}\,\zeta_j\ , \quad \mathcal{V}_m^\omega(\xi_1,\dots\xi_N) = \omega^{2/3}\,\mathcal{V}_m(\zeta_1,\dots\zeta_N) \tag{4.1.5}$$

where

$$\mathcal{V}_m(\zeta_1,\dots\zeta_N)\ :=\ \frac{1}{8\pi}\sum_{j\neq k=1}^N \frac{m_j\,m_k}{|\zeta_k-\zeta_j|} + \frac{1}{2}\sum_{j=1}^N m_j\,|\zeta_j|^2\,.$$

This function has in general many critical points, which are all *relative equilibria.* For instance, $\mathcal{V}_m$ clearly has a global minimum point.

Our aim is to construct solutions of gravitational models in continuum mechanics based on the theory of relative equilibria. We have the following result.

Theorem 1 *Given masses m_j, $j=1,\dots N$, and any sufficiently small $\omega>0$, there exists a solution $f_\omega(t,x,v)$ of equation (4.1.1) which is $\frac{2\pi}{\omega}$-periodic in time and whose spatial density takes the form*

$$\rho(t,x) := \int_{\mathbb{R}^3} f_\omega\,dv\ = \sum_{i=1}^N \rho_j(x-x_j^\omega(t))\ +\ o(1)\ .$$

Here $o(1)$ means that the remainder term uniformly converges to 0 as $\omega\to 0_+$ and identically vanishes away from $\cup_{j=1}^N B_R(x_j^\omega(t))$, for some $R>0$, independent of ω. The functions $\rho_j(y)$ are non-negative, radially symmetric, non-increasing, compactly supported functions, independent of ω, with $\int_{\mathbb{R}^3}\rho_j(y)\,dy = m_j$ and the points $x_j^\omega(t)$ are such that

$$x_j^\omega(t) = \omega^{-2/3}\,(e^{i\,\omega\,t}\,\zeta_j^\omega,0)\ , \quad \zeta_j^\omega\in\mathbb{C}\ , \quad j=1,\dots N$$

and

$$\lim_{\omega\to 0_+}\mathcal{V}_m(\zeta_1^\omega,\dots\zeta_N^\omega) = \min_{\mathbb{C}^N}\mathcal{V}_m\ , \quad \lim_{\omega\to 0_+}\nabla\mathcal{V}_m(\zeta_1^\omega,\dots\zeta_N^\omega) = 0\ .$$

The solution of Theorem 1 has a spatial density which is nearly spherically symmetric on each component of its support and these ball-like components rotate at constant, very small, angular velocity around the x^3-axis. The radii of these balls are very small compared with their distance to the axis. We shall call such a solution a *relative equilibrium* of (4.1.1), by extension of the discrete notion. The construction provides much more accurate informations on the solution. In particular, the building blocks ρ_j are obtained as minimizers of an explicit reduced free energy functional, under suitable mass constraints.

It is also natural to consider other discrete relative equilibria, namely critical points of the energy $\mathcal{V}_m$ that may or may not be globally minimizing, and ask whether associated relative equilibria of system (4.1.1) exist. There are plenty of relative equilibria of the N-body problem. For instance, if all masses m_j are equal to some $m_*>0$, a critical point is found by locating the ζ_j's at the vertices of a regular polygon:

$$\zeta_j = r\,e^{2\,i\,\pi(j-1)/N}\ , \quad j=1,\dots N\ , \tag{4.1.6}$$

where r is such that

$$\frac{d}{dr}\Big[\frac{a_N}{4\pi}\,\frac{m_*}{r}+\frac{1}{2}\,r^2\Big]=0 \quad\text{with}\quad a_N := \frac{1}{\sqrt{2}}\sum_{j=1}^{N-1}\frac{1}{\sqrt{1-\cos(2\pi j/N)}}\ ,$$

i.e. $r=(a_N\,m_*/(4\pi))^{1/3}$. This configuration is called the *Lagrange solution,* see [52]. The counterpart in terms of continuum mechanics goes as follows.

Theorem 2 *Let $(\zeta_1, \dots \zeta_N)$ be a regular polygon, namely with ζ_j given by (4.1.6), and assume that all masses are equal. Then there exists a solution f_ω exactly as in Theorem 1, but with $\lim_{\omega\to 0_+}(\zeta_1^\omega, \dots \zeta_N^\omega) = (\zeta_1, \dots \zeta_N)$.*

Further examples of *relative equilibria* in the N-body problem can be obtained for instance by setting $N-1$ point particles of the same mass at the vertices of a regular polygon centered at the origin, then adding one more point particle at the center (not necessarily with the same mass), and finally adjusting the radius. Another family of solutions, known as the *Euler–Moulton solutions* is constituted by arrays of aligned points.

Critical points of the functional $\mathcal{V}_m$ are always degenerate because of their invariance under rotations: for any $\alpha \in \mathbb{R}$ we have

$$\mathcal{V}_m(\zeta_1, \dots \zeta_N) = \mathcal{V}_m(e^{i\,\alpha}\,\zeta_1, \dots e^{i\,\alpha}\,\zeta_N)\ .$$

Let $\bar\zeta = (\bar\zeta_1, \dots \bar\zeta_N)$ be a critical point of $\mathcal{V}_m$ with $\bar\zeta_\ell \neq 0$. After a uniquely defined rotation, we may assume that $\bar\zeta_{\ell 2} = 0$. Moreover, we have a critical point of the function of $2N-1$ real variables

$$\tilde{\mathcal{V}}_m(\zeta_1, \dots \zeta_{\ell 1}, \dots \zeta_N) := \mathcal{V}_m(\zeta_1, \dots (\zeta_{\ell 1}, 0), \dots \zeta_N)\ .$$

We shall say that a critical point of $\mathcal{V}_m$ is *non-degenerate up to rotations* if the matrix $D^2\tilde{\mathcal{V}}_m(\bar\zeta_1, \dots \bar\zeta_{\ell 1}, \dots \bar\zeta_N)$ is non-singular. This property is clearly independent of the choice of ℓ.

Palmore in [47, 48, 49, 50, 51] has obtained classification results for the relative equilibria. In particular, it turns out that for *almost every choice of masses m_j,* all critical points of the functional $\mathcal{V}_m$ are non-degenerate up to rotations. Moreover, in such a case there exist at least $[2^{N-1}(N-2)+1]\,(N-2)\,!$ such distinct critical points. Many other results on *relative equilibria* are available in the literature. We have collected some of them in 4.8 with a list of relevant references. These results have a counterpart in terms of relative equilibria of system (4.1.1).

Theorem 3 *Let $(\zeta_1, \dots \zeta_N)$ be a non-degenerate critical point of $\mathcal{V}_m$ up to rotations. Then there exists a solution f_ω as in Theorem 1, which satisfies, as in Theorem 2, $\lim_{\omega\to 0_+}(\zeta_1^\omega, \dots \zeta_N^\omega) = (\zeta_1, \dots \zeta_N)$.*

This chapter is organized as follows. In the next section, we explain how the search for relative equilibria for the Vlasov-Poisson system can be reduced to the study of critical points of a functional acting on the gravitational potential. The construction of these critical points is detailed in Section 4.3. Sections 4.4 and 4.5 are respectively devoted to the linearization of the problem around a superposition of solutions of the problem with zero angular velocity, and to the existence of a solution of a nonlinear problem with appropriate orthogonality constraints depending on parameters $(\xi_j)_{j=1}^N$ related to the location of the N components of the support of the spatial density. Solving the original problem amounts to make all corresponding Lagrange multipliers equal to zero, which is equivalent to find a critical point of a function depending on $(\xi_j)_{j=1}^N$: this is the variational reduction described in Section 4.6. The proof of Theorems 1, 2 and 3 is given in Section 4.7 while known results on relative equilibria for the N-body, discrete problem are summarized in 4.8.

4.2 The setup

Guided by the representation (4.1.3) of the N-body problem in a rotating frame, we change variables in equation (4.1.1), replacing $x = (x', x^3)$ and $v = (v', v^3)$ respectively by

$$(e^{i\,\omega\,t}\,x', x^3) \quad \text{and} \quad (i\,\omega\,x' + e^{i\,\omega\,t}\,v', v^3)\ .$$

Written in these new coordinates, Problem (4.1.1) becomes

$$\begin{cases} \frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f - \omega^2 \, x' \cdot \nabla_{v'} f + 2 \, \omega \, i \, v' \cdot \nabla_{v'} f = 0 \, , \\ U = -\dfrac{1}{4\pi \, |\cdot|} * \rho \, , \quad \rho := \displaystyle\int_{\mathbb{R}^3} f \, dv \, . \end{cases} \tag{4.2.1}$$

The last two terms in the equation take into account the centrifugal and Coriolis force effects. System (4.2.1) can be regarded as the continuous version of problem (4.1.3). Accordingly, a *relative equilibrium* of System (4.1.1) will simply correspond to a stationary state of (4.2.1).

Such stationary solutions of (4.2.1) can be found by considering for instance critical points of the *free energy* functional

$$\mathcal{F}[f] := \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \beta(f) \, dx \, dv + \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (|v|^2 - \omega^2 \, |x'|^2) f \, dx \, dv - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla U|^2 \, dx$$

for some arbitrary convex function β, under the mass constraint

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} f \, dx \, dv = M \, .$$

A typical example of such a function is

$$\beta(f) = \frac{1}{q} \, \kappa_q^{q-1} \, f^q \tag{4.2.2}$$

for some $q \in (1, \infty)$ and some positive constant κ_q, to be fixed later. An additional restriction, $q > 9/7$, will come from the variational setting. The corresponding solution is known as the solution of the *polytropic gas model*, see [21, 22, 57, 61].

When dealing with stationary solutions, it is not very difficult to rewrite the problem in terms of the potential. A critical point of $\mathcal{F}$ under the mass constraint $\iint_{\mathbb{R}^3 \times \mathbb{R}^3} f \, dx \, dv = M$ is indeed given in terms of U by

$$f(x, v) = \gamma(\lambda + \tfrac{1}{2} \, |v|^2 + U(x) - \tfrac{1}{2} \, \omega^2 \, |x'|^2) \tag{4.2.3}$$

where γ is, up to a sign, an appropriate generalized inverse of β'. In case (4.2.2), $\gamma(s) = \kappa_q^{-1} \, (-s)_+^{1/(q-1)}$, where $s_+ = (s + |s|)/2$ denotes the positive part of s. The parameter λ stands for the Lagrange multiplier associated to the mass constraint, at least if f has a single connected component. At this point, one should mention that the analysis is not exactly as simple as written above. Identity (4.2.3) indeed holds only component by component of the support of the solution, if this support has more than one connected component, and the Lagrange multipliers have to be defined for each component. The fact that

$$U(x) \underset{|x| \to \infty}{\sim} -\frac{M}{4\pi \, |x|}$$

is dominated by $-\frac{1}{2} \, \omega^2 \, |x'|^2$ as $|x'| \to \infty$ is also a serious cause of trouble, which clearly discards the possibility that the free energy functional can be bounded from below if $\omega \neq 0$. This issue has been studied in [26], in the case of the so-called *flat systems*.

Finding a stationary solution in the rotating frame amounts to solving a non-linear Poisson equation, namely

$$\Delta U = g(\lambda + U(x) - \tfrac{1}{2} \, \omega^2 \, |x'|^2) \quad \text{if } x \in \mathrm{supp}(\rho) \tag{4.2.4}$$

and $\Delta U = 0$ otherwise, where g is defined by

$$g(\mu) := \int_{\mathbb{R}^3} \gamma(\mu + \tfrac{1}{2}\,|v|^2)\; dv\;.$$

Hence, the problem can also be reduced to look for a critical point of the functional

$$\mathcal{J}[U] := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla U|^2\; dx + \int_{\cup_i K_i} G(\lambda + U(x) - \tfrac{1}{2}\,\omega^2\,|x'|^2)\; dx - \int_{\cup_i K_i} \lambda\,\rho\; dx\;,$$

where $\lambda = \lambda[x, U]$ is now a functional which is constant with respect to x, with value λ_i, on each connected component K_i of the support of $\rho(x) = g(\lambda[x,U] + U(x) - \frac{1}{2}\,\omega^2\,|x'|^2)$, $x \in \cup_i K_i$ and implicitly determined by the condition

$$\int_{K_i} g(\lambda_i + U(x) - \tfrac{1}{2}\,\omega^2\,|x'|^2)\, dx = m_i\;.$$

By G, we denote a primitive of g and the total mass is $M = \sum_{i=1}^N m_i$. Hence we can rewrite $\mathcal{J}$ as

$$\mathcal{J}[U] = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla U|^2\; dx + \sum_{i=1}^N \left[\int_{K_i} G(\lambda_i + U(x) - \tfrac{1}{2}\,\omega^2\,|x'|^2)\; dx - m_i\,\lambda_i\right]\;. \tag{4.2.5}$$

We may also observe that critical points of $\mathcal{F}$ correspond to critical points of the *reduced free energy* functional

$$\mathcal{G}[\rho] := \int_{\mathbb{R}^3} \left(h(\rho) - \frac{1}{2}\,\omega^2\,|x'|^2\,\rho\right)\; dx - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla U|^2\; dx$$

acting on the spatial densities if $h(\rho) = \int_0^\rho g^{-1}(-s)\, ds$. Also notice that, using the same function γ as in (4.2.3), to each distribution function f, we can associate a *local equilibrium*, or *local Gibbs state*,

$$G_f(x,v) = \gamma(\mu(\rho(x)) + \tfrac{1}{2}\,|v|^2)$$

where μ is such that $g(\mu) = \rho$. This identity defines $\mu = \mu(\rho) = g^{-1}(\rho)$ as a function of ρ. Furthermore, by convexity, it follows that $\mathcal{F}[f] \geq \mathcal{F}[G_f] = \mathcal{G}[\rho]$ if $\rho(x) = \int_{\mathbb{R}^3} f(x,v)\; dv$, with equality if f is a local Gibbs state. See [28] for more details.

Summarizing, the heuristics are now as follows. The various components K_i of the support of the spatial density ρ of a critical point are assumed to be far away from each other so that the dynamics of their center of mass is described by the N-body point particles system, at first order. On each component K_i, the solution is a perturbation of an isolated minimizer of the free energy functional $\mathcal{F}$ (without angular rotation) under the constraint that the mass is equal to m_i. In the spatial density picture, on K_i, the solution is a perturbation of a minimizer of the reduced free energy functional $\mathcal{G}$.

To further simplify the presentation of our results, we shall focus on the model of *polytropic gases* corresponding to (4.2.2). In such a case, with $p := \frac{1}{q-1} + \frac{3}{2}$, g is given by

$$g(\mu) = (-\mu)_+^p$$

if the constant κ_q is fixed so that $\kappa_q = 4\pi\sqrt{2}\int_0^{+\infty} \sqrt{t}\,(1+t)^{\frac{1}{q-1}}\; dt = (2\pi)^{\frac{3}{2}}\frac{\Gamma(\frac{q}{q-1})}{\Gamma(\frac{3}{2}+\frac{q}{q-1})}$. For compactness reasons, we shall further restrict p to be subcritical, so that the range covered by our aproach is $p \in 3/2, 5)$.

Free energy functionals have been very much studied over the last years, not only to characterize special stationary states, but also because they provide a framework to deal with *orbital stability,* which

is a fundamental issue in the mechanics of gravitation. The use of a free energy functional, whose entropy part, $\iint_{\mathbb{R}^3\times\mathbb{R}^3} \beta(f)\, dx\, dv$ is sometimes also called the *Casimir energy functional,* goes back to the work of V.I. Arnold (see ï¿½[19, 20, 62]). The variational characterization of special stationary solutions and their orbital stability have been studied by Y. Guo and G. Rein in a series of papers [33, 34, 35, 36, 53, 54, 55, 56] and by many other authors, see for instance [26, 27, 39, 40, 41, 42, 57, 58, 61].

The main drawback of such approaches is that stationary solutions which are characterized by these techniques are in some sense trivial: radial, with a single simply connected component support. Here we use a different approach to construct the solutions, which goes back to [30] in the context of Schrödinger equations. We are not aware of attempts to use dimensional reduction coupled to power-law non-linearities and Poisson force fields except in the similar case of a nonlinear Schrödinger equation with power law nonlinearity and repulsive Coulomb forces (see [25]), or in the case of an attractive Hartree-Fock model (see [38]). Technically, our results turn out to be closely related to the ones in [23, 24].

Compared to previous results on gravitational systems, the main interest of our approach is to provide a much richer set of solutions, which is definitely of interest in astrophysics for describing complex patterns like binary gaseous stars or even more complex objects. The need of such an improvement was pointed for instance in [37]. An earlier attempt in this direction has been done in the framework of Wasserstein's distance and mass transport theory in [44]. The point of this work is that we can take advantage of the knowledge of special solutions of the N-body problem to produce solutions of the corresponding problem in continuum mechanics, which are still reminiscent of the discrete system.

4.3 Construction of relative equilibria

4.3.1 Some notations

We denote by $x = (x', x^3) \in \mathbb{R}^2 \times \mathbb{R}$ a generic point in $\mathbb{R}^3$. We may reformulate Problem (4.2.4) in terms of the potential $u = -U$ as follows. Given N positive numbers $\lambda_1, \dots \lambda_N$ and a small positive parameter ω, we consider the problem of finding N non-empty, compact, disjoint, connected subsets K_i of $\mathbb{R}^3$, $i = 1,\, 2 \dots N$, and a positive solution u of the problem

$$-\Delta u = \sum_{i=1}^{N} \rho_i \quad \text{in } \mathbb{R}^3\,, \quad \rho_i := (u - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2)_+^p\,\chi_i\,, \tag{4.3.1}$$

$$\lim_{|x|\to\infty} u(x) = 0\,, \tag{4.3.2}$$

where χ_i denotes the characteristic function of K_i. We define the mass and the center of mass associated to each component by

$$m_i^\omega := \int_{\mathbb{R}^3} \rho_i\, dx \quad \text{and} \quad x_i^\omega := \frac{1}{m_i}\int_{\mathbb{R}^3} x\,\rho_i\, dx\,.$$

In our construction, when $\omega \to 0$, the sets K_i are asymptotically balls centered around x_i. It is crucial to localize the support of ρ_i since $u - \lambda_i + \frac{1}{2}\,\omega^2\,|x'|^2$ is always positive for large values of $|x'|$. We shall find a solution of (4.3.1) as a critical point u of the functional

$$\mathsf{J}\,[u] = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2\, dx - \frac{1}{p+1}\sum_{i=1}^{N}\int_{\mathbb{R}^3} \big(u - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2\big)_+^{p+1}\,\chi_i\, dx\,, \tag{4.3.3}$$

so that $-u$ is a critical point of $\mathcal{J}$ in (4.2.5) in the case (4.2.2), namely $G(-s) = \frac{1}{p+1}\,s_+^{p+1}$.

Heuristically, our method goes as follows. We first consider the so-called *basic cell* problem: we characterize the solution with a single component support, when $\omega = 0$ and then build an *ansatz* by considering approximate solutions made of the superposition of basic cell solutions located close to relative equilibrium points, when they are far apart from each other. This can be done using the scaling invariance, in the low angular velocity limit $\omega \to 0_+$. The proof of our main results will be given in Sections 4.4–4.7. It relies on a dimensional reduction of the variational problem: we shall prove that for a well chosen u, $\mathsf{J}[u] = \sum_{i=1}^N \lambda_i^{5-p}\,\mathsf{e}_* - \mathcal{V}_m^\omega(\xi_1,\dots\xi_N) + o(1)$ for some constant e_*, up to $o(1)$ terms, which are uniform in $\omega > 0$, small. Hence finding a critical point for J will be reduced to look for a critical point of $\mathcal{V}_m^\omega$ as a function of $(\xi_1,\dots\xi_N)$.

4.3.2 The basic cell problem

Let us consider the following problem

$$-\Delta w = (w-1)_+^p \quad \text{in } \mathbb{R}^3\,. \tag{4.3.4}$$

Lemma 1 *Under the condition* $\lim_{|x|\to\infty} w(x) = 0$*, Equation* (4.3.4) *has a unique solution, up to translations, which is positive and radially symmetric if* $p \in (1,5)$*.*

Proof 1 *Since* p *is subcritical, it is well known that the problem*

$$-\Delta Z = Z^p \quad \textit{in } B_1(0)$$

with homogeneous Dirichlet boundary conditions, $Z = 0$*, on* $\partial B_1(0)$*, has a unique positive solution, which is also radially symmetric (see [32]). For any* $R > 0$*, the function* $Z_R(x) := R^{-2/(p-1)}\,Z(x/R)$ *is the unique radial, positive solution of*

$$-\Delta Z_R = Z_R^p \quad \textit{in } B_R(0)$$

with homogeneous Dirichlet boundary conditions on $\partial B_R(0)$*. According to [31, 32], any positive solution of* (4.3.4) *is radially symmetric, up to translations. Finding such a solution* w *of* (4.3.4) *is equivalent to finding numbers* $R > 0$ *and* $m_* > 0$ *such that the function, defined by pieces as* $w = Z_R + 1$ *in* B_R *and* $w(x) = m_*/(4\pi\,|x|)$ *for any* $x \in \mathbb{R}^3$ *such that* $|x| > R$*, is of class* C^1*. These numbers are therefore uniquely determined by*

$$w(R^-) = 1 = \frac{m_*}{4\pi\,R} = w(R^+)\,,\ w'(R^-) = R^{-\frac{2}{p-1}-1}\,Z'(1) = -\frac{m_*}{4\pi\,R^2} = w'(R^+)\,,$$

which uniquely determines the solution of (4.3.4)*.* □

Now let us consider the slightly more general problem

$$-\Delta w^\lambda = (w^\lambda - \lambda)_+^p \quad \text{in } \mathbb{R}^3$$

with $\lim_{|x|\to\infty} w^\lambda(x) = 0$. For any $\lambda > 0$, it is straightforward to check that it has a unique radial solution given by

$$w^\lambda(x) = \lambda\, w\big(\lambda^{(p-1)/2}\,x\big) \quad \forall\, x \in \mathbb{R}^3\,.$$

Let us observe, for later reference, that

$$\int_{\mathbb{R}^3} (w^\lambda - \lambda)_+^p\,dx = \lambda^{(3-p)/2} \int_{\mathbb{R}^3} (w-1)_+^p\,dx =: \lambda^{(3-p)/2}\,m_*\,. \tag{4.3.5}$$

Moreover, w^λ is given by

$$w^\lambda(x) = \frac{m_*}{4\pi\,|x|}\,\lambda^{(3-p)/2} \quad \forall\, x \in \mathbb{R}^3 \text{ such that } |x| > R\,\lambda^{-(p-1)/2}\,.$$

4.3.3 The ansatz

We consider now a first approximation of a solution of (4.3.1)-(4.3.2), built as a superposition of the radially symmetric functions w^{λ_i} translated to points ξ_i, $i = 1, \dots N$ in $\mathbb{R}^2 \times \{0\}$, far away from each other:

$$w_i(x) := w^{\lambda_i}(x - \xi_i) \ , \quad W_\xi := \sum_{i=1}^N w_i \ .$$

Recall that we are given the masses $m_1, \dots m_N$. We choose, according to formula (4.3.5), the positive numbers λ_i so that

$$\int_{\mathbb{R}^3} (w_i - \lambda_i)_+^p \, dx \ = \ m_i \quad \text{for all } i = 1, \dots N \ .$$

By ξ we denote the array $(\xi_1, \xi_2, \dots \xi_N)$.

We shall assume in what follows that the points ξ_i are such that for a large, fixed $\mu > 0$, and all small $\omega > 0$ we have

$$|\xi_i| < \mu \, \omega^{-2/3} \, , \quad |\xi_i - \xi_j| > \mu^{-1} \, \omega^{-2/3} \ . \tag{4.3.6}$$

Equivalently,

$$|\zeta_i| < \mu \ , \quad |\zeta_i - \zeta_j| > \mu^{-1} \quad \text{where} \quad \zeta_i := \omega^{-2/3} \, \xi_i \ . \tag{4.3.7}$$

We look for a solution of (4.3.1) of the form

$$u = W_\xi + \phi$$

for a convenient choice of the points ξ_i, where ϕ is globally uniformly small when compared with W_ξ. For this purpose, we consider a fixed number $R > 1$ such that

$$\text{supp}\,(w^{\lambda_i} - \lambda_i)_+ \subset B_{R-1}(0) \quad \forall \ i = 1, \, 2 \dots N$$

and define the functions

$$\chi(x) = \begin{cases} 1 & \text{if } |x| < R \\ \\ 0 & \text{if } |x| \geq R \end{cases} \quad \text{and} \quad \chi_i(x) = \chi(x - \xi_i) \ .$$

Thus we want to find a solution to the problem

$$\Delta(W_\xi + \phi) + \sum_{i=1}^N (W_\xi - \lambda_i + \phi + \tfrac{1}{2} \, \omega^2 \, |x'|^2)_+^p \, \chi_i = 0 \quad \text{in } \mathbb{R}^3$$

with $\lim_{|x| \to \infty} \phi(x) = 0$, that is we want to solve the problem

$$\Delta \phi + \sum_{i=1}^N p (W_\xi - \lambda_i + \tfrac{1}{2} \, \omega^2 \, |x'|^2)_+^{p-1} \chi_i \, \phi = -\mathsf{E} - \mathsf{N}[\phi] \tag{4.3.8}$$

where

$$\mathsf{E} := \Delta W_\xi + \sum_{i=1}^N (W_\xi - \lambda_i + \tfrac{1}{2} \, \omega^2 \, |x'|^2)_+^p \, \chi_i \ ,$$

$$\mathsf{N}[\phi] := \sum_{i=1}^N \Big[(W_\xi - \lambda_i + \tfrac{1}{2} \, \omega^2 \, |x'|^2 + \phi)_+^p - (W_\xi - \lambda_i + \tfrac{1}{2} \, \omega^2 \, |x'|^2)_+^p$$
$$- p (W_\xi - \lambda_i + \tfrac{1}{2} \, \omega^2 \, |x'|^2)_+^{p-1} \phi \Big] \, \chi_i \ .$$

4.4 A linear theory

The purpose of this section is to develop a solvability theory for the operator

$$\mathsf{L}[\phi] = \Delta\phi + \sum_{i=1}^{N} p(W_\xi - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2)_+^{p-1}\chi_i\,\phi\ .$$

To this end we introduce the norms

$$\|\phi\|_* = \sup_{x\in\mathbb{R}^3}\left(\sum_{i=1}^{N}|x-\xi_i|+1\right)|\phi(x)|\ ,\quad \|h\|_{**} = \sup_{x\in\mathbb{R}^3}\left(\sum_{i=1}^{N}|x-\xi_i|^4+1\right)|h(x)|\ .$$

We want to solve problems of the form $\mathsf{L}[\phi]=h$ with h and ϕ having the above norms finite. Rather than solving this problem directly, we consider a *projected problem* of the form

$$\mathsf{L}[\phi] = h + \sum_{i=1}^{N}\sum_{j=1}^{3} c_{ij}\,Z_{ij}\,\chi_i\ , \tag{4.4.1}$$

$$\lim_{|x|\to\infty}\phi(x) = 0\ , \tag{4.4.2}$$

where $Z_{ij} := \partial_{x_j} w_i$, subject to orthogonality conditions

$$\int_{\mathbb{R}^3}\phi\,Z_{ij}\,\chi_i\,dx = 0 \quad \forall\, i=1,\,2\ldots N,\ j=1,\,2,\,3\ . \tag{4.4.3}$$

Equation (4.4.1) involves the coefficients c_{ij} as Lagrange multipliers associated to the constraints (4.4.3). If we can solve the equations $\mathsf{L}[\psi]=h$ and $\mathsf{L}[Y_{ij}]=Z_{ij}$, and if we define c_{ij} such that $\int_{\mathbb{R}^3}\psi\,Z_{ij}\,dx + c_{ij}\int_{\mathbb{R}^3}Y_{ij}\,Z_{ij}\,dx = 0$, then we observe that $\phi=\psi+\sum_{i,\,j}c_{ij}\,Y_{ij}$ solves (4.4.1) and satisfies (4.4.3). However, for existence, we will rather reformulate the question as a constrained variational problem; see Equation (4.4.6) below.

Lemma 2 *Assume that* (4.3.6) *holds. Given h with $\|h\|_{**}<+\infty$, Problem* (4.4.1)-(4.4.3) *has a unique solution $\phi =: \mathsf{T}[h]$ and there exists a positive constant C, which is independent of ξ such that, for $\omega>0$ small enough,*

$$\|\phi\|_* \le C\,\|h\|_{**}\ . \tag{4.4.4}$$

Proof 2 *In order to solve* (4.4.1)-(4.4.3)*, we first establish* (4.4.4) *as an* a priori *estimate. Assume by contradiction the existence of sequences $\omega_n\to 0$, ξ_i^n satisfying* (4.3.6) *for $\omega=\omega_n$, of functions ϕ_n, h_n and of constants c_{ij}^n for which*

$$\|\phi_n\|_* = 1\ ,\quad \lim_{n\to\infty}\|h_n\|_{**} = 0\ ,$$
$$\int_{\mathbb{R}^3}\phi_n\,Z_{ij}\,\chi_i\,dx = 0 \quad \forall\, i\,,\ j \quad \textit{and} \quad L[\phi_n] = h_n + \sum_{i=1}^{N}\sum_{j=1}^{3} c_{ij}^n\,Z_{ij}\,\chi_i\ .$$

Testing the equation against $Z_{k\ell}$, we obtain, after an integration by parts,

$$\int_{\mathbb{R}^3} p\left[\sum_{i=1}^{N}(W_{\xi^n}-\lambda_i+\tfrac{1}{2}\,\omega_n^2\,|x'|^2)_+^{p-1}\chi_i-(w_k-\lambda_k)_+^{p-1}\right]\phi_n\,Z_{k\ell}\,dx$$
$$=\int_{\mathbb{R}^3} h_n\,Z_{k\ell}\,dx+\sum_{i=1}^{N}\sum_{j=1}^{3}c_{ij}^n\int_{\mathbb{R}^3}Z_{k\ell}\,Z_{ij}\,\chi_i\,dx\,. \quad (4.4.5)$$

Here we have used $\Delta Z_{k\ell}=p\,(w_k-\lambda_k)_+^{p-1}\,Z_{k\ell}$. *The integrals in the sum can be estimated as follows:*

$$\int_{\mathbb{R}^3}|Z_{k\ell}\,Z_{ij}\,\chi_i|\,dx=\int_{B(0,R)}|\partial_{x_\ell}w^{\lambda_k}(x)\,\partial_{x_j}w^{\lambda_i}(x+\xi_i^n-\xi_k^n)|\,dx$$
$$\le\int_{B(0,R)}|\partial_{x_\ell}\,w^{\lambda_k}(x)|\,\frac{C}{|x+\xi_i^n-\xi_k^n|^2}\,dx=O(\omega_n^{4/3})$$

for some generic constant $C>0$ *which will change from line to line. Now we turn our attention to the left-hand side of* (4.4.5). *Since* $p-1>0$, *we first notice that*

$$\int_{\mathbb{R}^3}\Big|\sum_{i=1}^{N}(W_{\xi^n}-\lambda_i+\tfrac{1}{2}\,\omega_n^2\,|x'|^2)_+^{p-1}\chi_i-(w_k-\lambda_k)_+^{p-1}\Big|\,|\phi_n\,Z_{k\ell}|\,dx$$
$$\le\int_{\mathbb{R}^3}\sum_{i=1,i\neq k}^{N}(W_{\xi^n}-\lambda_i+\tfrac{1}{2}\,\omega_n^2\,|x'|^2)_+^{p-1}\chi_i\,|\phi_n\,Z_{k\ell}|\,dx$$
$$+C\int_{\mathbb{R}^3}\Big(\tfrac{1}{2}\,\omega_n^2\,|x'|^2+\sum_{i=1,i\neq k}^{N}w_i\Big)\,\chi_k\,|\phi_n\,Z_{k\ell}|\,dx\,.$$

It is not hard to check that

$$\int_{\mathbb{R}^3}\sum_{i=1,i\neq k}^{N}(W_{\xi^n}-\lambda_i+\tfrac{1}{2}\,\omega_n^2\,|x'|^2)_+^{p-1}\chi_i\,|\phi_n\,Z_{k\ell}|\,dx$$
$$\le C\,\|\phi_n\|_*\left(\sum_{i=1,i\neq k}^{N}\int_{B(0,R)}|\partial_{x_\ell}w^{\lambda_k}(x+\xi_i-\xi_k)|\,dx\right)=O(\omega_n^{4/3})$$

and

$$\int_{\mathbb{R}^3}\Big(\tfrac{1}{2}\,\omega_n^2\,|x'|^2+\sum_{i=1,i\neq k}^{N}w_i\Big)\,\chi_k\,|\phi_n\,Z_{k\ell}|\,dx$$
$$\le C\,\|\phi_n\|_*\left(\omega_n^2+\sum_{i=1,i\neq k}^{N}\int_{B(0,R)}\frac{|\partial_{x_\ell}w^{\lambda_k}|}{|x+\xi_k-\xi_i|}\right)dx=O(\omega_n^{2/3})\,.$$

Summarizing, we have found that, for each $k=1,\,2\ldots N$

$$O(\omega_n^{2/3})\,\|\phi_n\|_*=\int_{\mathbb{R}^3}h_n\,Z_{k\ell}\,dx+c_{k\ell}^n\int_{\mathbb{R}^3}|Z_{k\ell}|^2\,\chi_k\,dx+O(\omega_n^{4/3})\sum_{(i,j)\neq(k,\,\ell)}|c_{ij}^n|\,,$$

from which we deduce that $c^n_{k\ell} = O(\omega_n^{2/3}) + O(\|h_n\|_{**}) \to 0$ *for all* k, ℓ. *We may indeed notice that for* ω_n *small enough, the above equations define an almost diagonal system, so that the coefficients* c_{ij} *are uniquely determined.*

Let us prove that $\lim_{n\to\infty} \|\phi_n\|_{L^\infty(\mathbb{R}^3)} = 0$. *If not, since* $\|\phi_n\|_* = 1$, *we may assume that there is an index* i *and a sufficiently large number* $R > 0$ *for which*

$$\liminf_{n\to\infty} \|\phi_n\|_{L^\infty(B_R(\xi_i))} > 0 \, .$$

Using elliptic estimates, and defining $\psi_n(x) = \phi_n(\xi_i^n + x)$, *we may assume that* ψ_n *uniformly converges in* C^1 *sense over compact subsets of* $\mathbb{R}^3$ *to a bounded, non-trivial solution* ψ *of the equation*

$$\begin{aligned} \Delta\psi + p\left(w^{\lambda_i} - \lambda_i\right)_+^{p-1}\psi = 0 \, , \\ \int_{\mathbb{R}^3} \psi\, \partial_{x_j} w^{\lambda_i}\, \chi \, dx = 0 \quad \forall\, j = 1,\, 2,\, 3 \, . \end{aligned}$$

According to [31, Lemma 5], ψ *must be a linear combination of the functions* $\partial_{x_j} w^{\lambda_i}$, $j = 1, 2, 3$. *The latter orthogonality conditions yield* $\psi \equiv 0$. *This is a contradiction and the claim is proven. Finally, let*

$$\tilde{h}_n := h_n + \sum_{i=1}^{N}\sum_{j=1}^{3} c^n_{ij}\, Z_{ij}\, \chi_i \, .$$

Then we have that

$$|\tilde{h}_n(x)| \le \Big(O(\omega_n^{2/3}) + O(\|h_n\|_{**})\Big) \sum_{i=1}^{k} \frac{1}{1 + |x - \xi_i^n|^4}$$

and hence $\tilde{\phi}_n$, *the unique solution in* $\mathbb{R}^3$ *of*

$$\Delta\tilde{\phi}_n = \tilde{h}_n \, , \quad \lim_{|x|\to\infty} \tilde{\phi}_n(x) = 0 \, ,$$

satisfies

$$|\tilde{\phi}_n(x)| \le \Big(O(\omega_n^{2/3}) + O(\|h_n\|_{**})\Big) \sum_{i=1}^{k} \frac{1}{|x - \xi_i^n|} \, .$$

Now, since $\phi_n - \tilde{\phi}_n$ *is harmonic in* $\mathbb{R}^3 \setminus \cup_i B_R(\xi_i^n)$, *it tends to zero as* $|x| \to \infty$ *and gets uniformly small on the boundary of this set. By the maximum principle, we get the estimate*

$$|\phi_n(x)| \le \Big(O(\omega_n^{2/3}) + O(\|h_n\|_{**})\Big) \sum_{i=1}^{k} \frac{1}{|x - \xi_i^n|} \quad \text{on } \mathbb{R}^3 \setminus \cup_i B_R(\xi_i^n) \, .$$

This shows that $\lim_{n\to\infty} \|\phi_n\|_* = 0$, *a contradiction with* $\|\phi_n\|_* = 1$, *and* (4.4.4) *follows.*

Now, for existence issues, we observe that problem (4.4.1)-(4.4.3) *can be set up in variational form in the Hilbert space*

$$\mathcal{H} = \Big\{ \phi \in \mathcal{D}^{1,2}(\mathbb{R}^3) \ : \ \int_{\mathbb{R}^3} \phi\, Z_{ij}\, \chi_i \, dx = 0 \quad \forall\, i = 1,\, 2 \ldots N,\ j = 1,\, 2,\, 3 \Big\}$$

endowed with the inner product $\langle \phi, \psi \rangle = \int_{\mathbb{R}^3} \nabla\psi \cdot \nabla\phi \, dx$, *as*

$$\int_{\mathbb{R}^3} \nabla\phi \cdot \nabla\psi \, dx - \int_{\mathbb{R}^3} \sum_{i=1}^{N} p(W_\xi - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2)_+^{p-1}\,\chi_i\,\phi\,\psi \, dx + \int_{\mathbb{R}^3} \psi\, h \, dx = 0 \tag{4.4.6}$$

for all $\psi \in \mathcal{H}$. *Since the potential defined by the second term of the above equality is compactly supported and* h *decays sufficiently fast, this equation takes the form* $\phi + \mathsf{K}[\phi] = \tilde{h}$ *where* K *is a compact linear operator of* $\mathcal{H}$. *The equation for* $\tilde{h} = 0$ *has just the trivial solution in view of estimate* (4.4.4). *Fredholm's alternative thus applies to yield existence. This concludes the proof of Lemma 2.* □

Notice that the convergence in the norm $\|\cdot\|_{**}$-norm guarantees that there is no issue with the localization of the support of the components of the spatial density.

We conclude this section with some considerations on the differentiability of the solution with respect to the parameter ξ. Let us assume that $h = h(\cdot, \xi)$ defines a continuous operator into the space of functions with finite $\|\cdot\|_{**}$-norm. We also assume that $\|\partial_\xi h(\cdot, \xi)\|_{**} < +\infty$. Let $\phi = \phi(\cdot, \xi)$ be the unique solution of Problem (4.4.1)-(4.4.3) for that right hand side, with corresponding constants $c_{ij}(\xi)$. Then ϕ is differentiable in ξ. Moreover $\partial_\xi\phi$ can be decomposed as

$$\partial_\xi\phi = \psi + \sum_{ij} d_{ij}\, Z_{ij}\, \chi_j$$

where ψ solves

$$\begin{aligned}
\mathsf{L}[\psi] = \partial_\xi h \; - \sum_{i=1}^{N} p\,\partial_\xi \Big[(W_\xi - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2)_+^{p-1} \chi_i \Big]\, \phi \\
+ \sum_{i=1}^{N} \sum_{j=1}^{3} \Big[c_{ij}\, \partial_\xi (Z_{ij}\,\chi_i) + b_{ij}\, Z_{ij}\,\chi_i \Big] , \\
\lim_{|x|\to\infty} \psi(x) = 0 \,, \quad \int_{\mathbb{R}^3} \psi\, Z_{ij}\,\chi_i \, dx = 0 \quad \forall\, i = 1,\, 2 \ldots N,\; j = 1,\, 2,\, 3 \,,
\end{aligned}$$

and the constants d_{ij} are chosen so that $\eta := \sum_{i=1}^{N} \sum_{j=1}^{3} d_{ij}\, Z_{ij}$ satisfies

$$\int_{\mathbb{R}^3} \chi_{ij}\, Z_{ij}\, \eta \, dx = -\int_{\mathbb{R}^3} \partial_\xi(\chi_{ij}\, Z_{ij})\phi \, dx \quad \forall\, i = 1,\, 2 \ldots N,\; j = 1,\, 2,\, 3 \,.$$

Lemma 3 *With the same notations and conditions as in Lemma 2, we have*

$$\|\partial_\xi\phi(\cdot, \xi)\|_* \;\le\; C \left(\|h(\cdot, \xi)\|_{**} + \|\partial_\xi h(\cdot, \xi)\|_{**} \right) .$$

$\ddot{i}\dot{\partial}\frac{1}{2}$

4.5 The projected nonlinear problem

Next we want to solve a *projected version* of the nonlinear problem (4.3.8) using Lemma 2. Thus we consider the problem of finding ϕ with $\|\phi\|_* < +\infty$, solution of

$$\mathsf{L}[\phi] = -\mathsf{E} - \mathsf{N}[\phi] + \sum_{i=1}^{N} \sum_{j=1}^{3} c_{ij}\, Z_{ij}\, \chi_i \tag{4.5.1}$$

$$\lim_{|x|\to+\infty} \phi(x) = 0 \tag{4.5.2}$$

where the coefficients c_{ij} are Lagrange multipliers associated to the orthogonality conditions

$$\int_{\mathbb{R}^3} \phi\, Z_{ij}\, \chi_i\, dx = 0 \quad \forall\, i = 1,\, 2 \ldots N,\ j = 1,\, 2,\, 3\,. \tag{4.5.3}$$

In other words, we look for a critical point of the functional J defined by (4.3.3) under the constraints (4.5.3).

For this purpose, we first have to measure the error E. We recall that

$$\begin{aligned}
\mathsf{E} &= \Delta W_\xi + \sum_{i=1}^N (W_\xi - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2)_+^p\,\chi_i \\
&= \sum_{i=1}^N \Big[\Big(w_i + \sum_{j\neq i} w_j - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2 \Big)_+^p - (w_i - \lambda_i)_+^p \Big]\,\chi_i \\
&= \sum_{i=1}^N p\,\Big[w_i - \lambda_i + t\,\Big(\sum_{j\neq i} w_j + \tfrac{1}{2}\,\omega^2\,|x'|^2 \Big) \Big]_+^{p-1} \Big(\sum_{j\neq i} w_j + \tfrac{1}{2}\,\omega^2\,|x'|^2 \Big)\,\chi_i
\end{aligned}$$

for some function t taking values in $(0,1)$. It follows that

$$|\mathsf{E}| \le C \sum_{i=1}^N \Big[\sum_{j\neq i} \frac{1}{|\xi_i - \xi_j|} + \frac{1}{2}\,\omega^2\,|\xi_i|^2 \Big]\,\chi_i \ \le C\,\omega^{2/3} \sum_{i=1}^N \chi_i\,,$$

from which we deduce the estimate

$$\|\mathsf{E}\|_{**} \le C\,\omega^{2/3}\,.$$

As for the operator $\mathsf{N}[\phi]$, we easily check that for $\|\phi\|_* \le 1$,

$$\big|\,\mathsf{N}[\phi]\,\big| \le C \sum_{i=1}^N |\phi|^\gamma\,\chi_i \quad \text{with } \gamma := \min\{p, 2\}\,,$$

which implies

$$\big\|\,\mathsf{N}[\phi]\,\big\|_{**} \le C\,\|\phi\|_*^\gamma\,.$$

Let T be the linear operator defined in Lemma 2. Equation (4.5.1) can be rewritten as

$$\phi = \mathsf{A}[\phi] := -\mathsf{T}\big[\mathsf{E} + \mathsf{N}[\phi]\big]\,.$$

Clearly the operator A maps the region

$$\mathcal{B} = \Big\{ \phi \in L^\infty(\mathbb{R}^3)\ :\ \|\phi\|_* \le K\,\omega^{2/3} \Big\}$$

into itself if the constant K is fixed, large enough. It is straightforward to check that $\mathsf{N}[\phi]$ satisfies in this region a Lipschitz property of the form

$$\big\|\,\mathsf{N}[\phi_1] - \mathsf{N}[\phi_2]\,\big\|_{**} \le \kappa_\omega\,\|\phi_1 - \phi_2\|_*$$

for some positive κ_ω such that $\lim_{\omega\to 0}\kappa_\omega = 0$, and hence existence of a unique fixed point ϕ of A in $\mathcal{B}$ immediately follows for ω small enough. We have then solved the projected nonlinear problem.

Since the error E is even with respect to the variable x^3, uniqueness of the solution of (4.5.1)-(4.5.3) implies that this symmetry is also valid for ϕ itself, and besides, the numbers c_{i3} are automatically all zero. Summarizing, we have proven the following result.

Lemma 4 *Assume that $\xi = (\xi_1, \xi_2, \dots \xi_N) \in \mathbb{R}^{2N}$ is given and satisfies* (4.3.6). *Then Problem* (4.5.1)-(4.5.3) *has a unique solution ϕ_ξ which depends continuously on ξ and ω for the $\|\ \|_*$-norm and satisfies $\|\phi_\xi\|_* \le C\,\omega^{2/3}$ for some positive C, which is independent of ω, small enough. Besides, the numbers c_{i3} are all equal to zero for $i = 1, 2 \dots N$.*

It is important to mention that ϕ_ξ also defines a continuously differentiable operator in its parameter. Indeed, combining its fixed point characterization with the implicit function theorem and the result of Lemma 3, we find in fact that

$$\|\partial_\xi \phi_\xi\|_* \le C\,\omega^{2/3}\,.$$

We leave the details to the reader.

With the complex notation of Section 4.1, let us consider the rotation $e^{i\,\alpha}$ of an angle α around the x^3-axis and let $e^{i\,\alpha}\,\xi = (e^{i\,\alpha}\,\xi_1, \dots e^{i\,\alpha}\,\xi_N)$. By construction, there is a rotational symmetry around the x^3-axis, which is reflected at the level of Problem (4.5.1)-(4.5.3) as follows.

Lemma 5 *Consider the solution ϕ found in Lemma 4. For any $\alpha \in \mathbb{R}$ and any $(x', x^3) \in \mathbb{C}\times\mathbb{R}$, we have that*

$$\phi_{e^{i\,\alpha}\xi}(x', x^3) = \phi_\xi(e^{-i\,\alpha}\,x', x^3)\,.$$

The proof is a direct consequence of uniqueness and rotation invariance of the equation satisfied by ϕ_ξ.

4.6 The variational reduction

We consider the functional J defined in (4.3.3). Our goal is to find a critical point satisfying (4.5.3), of the form $u = W_\xi + \phi_\xi$. We estimate $\mathsf{J}\,[W_\xi]$ by computing first

$$\int_{\mathbb{R}^3} |\nabla W_\xi|^2\,dx = \int_{\mathbb{R}^3} \Big|\sum_{i=1}^N \nabla w_i\Big|^2\,dx = \sum_{i=1}^N \int_{\mathbb{R}^3} |\nabla w_i|^2\,dx + \sum_{i\neq j}\int_{\mathbb{R}^3} \nabla w_i\cdot\nabla w_j\,dx\,.$$

The last term of the right hand side can be estimated by

$$\begin{aligned}\int_{\mathbb{R}^3} \nabla w_i\cdot\nabla w_j\,dx &= -\int_{\mathbb{R}^3} \Delta w_i\,w_j\,dx = \int_{\mathbb{R}^3} (w_i-\lambda_i)_+^p\,w_j\,dx\\ &= \int_{\mathbb{R}^3} \big(w^{\lambda_i}-\lambda_i\big)_+^p\,w^{\lambda_j}(x+\xi_i-\xi_j)\,dx\\ &= \int_{\mathbb{R}^3} \big(w^{\lambda_i}-\lambda_i\big)_+^p\,\frac{m_j}{4\pi\,|x+\xi_i-\xi_j|}\,dx\,.\end{aligned}$$

If we Taylor expand $x \mapsto |x+\xi_i-\xi_j|^{-1}$ around $x=0$, we obtain by (4.3.7)

$$\int_{\mathbb{R}^3} \left(w^{\lambda_i}-\lambda_i\right)_+^p \frac{m_j}{4\pi\,|x+\xi_i-\xi_j|}\,dx$$
$$= \int_{B(0,R)} \left(w^{\lambda_i}-\lambda_i\right)_+^p \frac{m_j}{4\pi}\left(\tfrac{1}{|\xi_i-\xi_j|}-\tfrac{(\xi_i-\xi_j)\cdot x}{|\xi_i-\xi_j|^3}+O(\omega^2\,|x|^2)\right)dx$$
$$= \frac{m_i\,m_j}{4\pi\,|\xi_i-\xi_j|}+O(\omega^{4/3})$$

where $m_i=\int_{\mathbb{R}^3}\left(w^{\lambda_i}-\lambda_i\right)_+^p dx = m_*\,\lambda_i^{(3-p)/2}$. Next we find that

$$\int_{\mathbb{R}^3}\Big(w_i+\sum_{j\neq i}w_j-\lambda_i+\tfrac12\,\omega^2\,|x'|^2\Big)_+^{p+1}\chi_i\,dx$$
$$= \int_{\mathbb{R}^3}(w_i-\lambda_i)_+^{p+1}\,dx$$
$$+(p+1)\int_{\mathbb{R}^3}(w_i-\lambda_i)_+^p\Big(\sum_{j\neq i}w_j+\tfrac12\,\omega^2\,|x'|^2\Big)\,dx\;+\;O(\omega^{4/3})$$
$$= \int_{\mathbb{R}^3}(w_i-\lambda_i)_+^{p+1}\,dx+(p+1)\Big(\sum_{j\neq i}\frac{m_i\,m_j}{4\pi\,|\xi_i-\xi_j|}+\tfrac12\,\omega^2 m_i\,|\xi_i|^2\Big)\;+\;O(\omega^{4/3})\,.$$

Let us define

$$\mathsf{e}_*:=\frac12\int_{\mathbb{R}^3}|\nabla w|^2\,dx-\frac{1}{p+1}\int_{\mathbb{R}^3}(w-1)_+^{p+1}\,dx\,.$$

Combining the above estimates, we obtain that

$$\mathsf{J}\,[W_\xi]\;=\sum_{i=1}^N\lambda_i^{5-p}\,\mathsf{e}_*\;-\;\mathcal{V}_m^\omega(\xi_1,\dots\xi_N)\;+\;O(\omega^{4/3})\,, \tag{4.6.1}$$

where $\mathcal{V}_m^\omega(\xi)=\sum_{i\neq j}\frac{m_i\,m_j}{8\pi\,|\xi_i-\xi_j|}+\frac12\,\omega^2\sum_{i=1}^N m_i\,|\xi_i|^2$ has been introduced in Section 4.1. Here the $O(\omega^{4/3})$ term is uniform as $\omega\to0$ on the set of ξ satisfying the constraints (4.3.6). This approximation is also uniform in the C^1 sense. Indeed, we directly check that

$$\nabla_\xi\mathsf{J}\,[W_\xi]=-\nabla_\xi\mathcal{V}_m^\omega(\xi)+O(\omega^{4/3})\,. \tag{4.6.2}$$

According to (4.1.5), we have $\mathcal{V}_m^\omega(\xi)=\omega^{2/3}\,\mathcal{V}_m(\zeta)$ for $\zeta=\omega^{2/3}\,\xi$. We get a solution of Problem (4.3.1)-(4.3.2) as soon as all constants c_{ij} are equal to zero in (4.5.1).

Lemma 6 *With the above notations, $c_{ij}=0$ for all $i=1,\,2\dots N$, $j=1,\,2,\,3$ if and only if ξ is a critical point of the functional $\xi\mapsto\Lambda(\xi):=\mathsf{J}\,[W_\xi+\phi_\xi]$.*

Proof 3 *We have already noticed in Lemma 4 that the numbers c_{i3} are all equal to zero. On the other hand, we have that*

$$\partial_{\xi_{ij}}\Lambda=D\mathsf{J}\,[W_\xi+\phi_\xi]\cdot\partial_{\xi_{ij}}(W_\xi+\phi_\xi)=\sum_{k,\,\ell}c_{k\ell}\int_{\mathbb{R}^3}\partial_{\xi_{ij}}(W_\xi+\phi_\xi)\,Z_{k\ell}\,\chi_\ell\,dx$$
$$=c_{ij}\,\Big(\int_{\mathbb{R}^3}|Z_{ij}|^2\,\chi_i\,dx\Big)\;+\;\Big(\sum_{(k,\,\ell)\neq(i,j)}c_{k\ell}\Big)\,O(\omega^{2/3})\,. \tag{4.6.3}$$

From here the assertion of the lemma readily follows, provided that ω is sufficiently small. □

Remark 1 *An important observation that follows from the rotation invariance of the equation is the following. Assume that the point ξ is such that $\xi_\ell = (\xi_{\ell 1}, 0) \neq (0,0)$ for some $\ell \in \{1, \dots N\}$. Then if*

$$\partial_{\xi_{kj}} \Lambda(\xi) = 0 \text{ for all } k = 1, \dots N\,,\ j = 1, 2, \quad (k,j) \neq (\ell, 2)\,,$$

it follows that actually ξ is a critical point of Λ. Indeed, differentiating in α the relation $\Lambda(e^{i\,\alpha}\,\xi) = \Lambda(\xi)$ we get

$$0 = \sum_{k=1}^{N} \partial_{\xi_k} \Lambda(\xi) \cdot i\,\xi_k \;=\; \partial_{\xi_\ell} \Lambda(\xi) \cdot i\,\xi_\ell \;=\; -\,\xi_{\ell 1}\,\partial_{\xi_{\ell 2}} \Lambda(\xi)\,,$$

and the result follows.

4.7 Proofs of Theorems 1-3

Let us consider the solution ϕ_ξ of (4.5.1)-(4.5.3), *i.e.* of the problem

$$\mathsf{L}[\phi_\xi] = -\mathsf{E} - \mathsf{N}[\phi_\xi] + \sum_{i=1}^{N} \sum_{j=1}^{3} c_{ij}(\xi)\, Z_{ij}\, \chi_i$$

$$\lim_{|x| \to +\infty} \phi_\xi(x) = 0$$

given by Lemma 4. We will then get a solution of Problem (4.3.1)-(4.3.2), of the desired form $u = W_\xi + \phi_\xi$, inducing the ones for Theorems 1 and 3, if we can adjust ξ in such a way that

$$c_{ij}(\xi) = 0 \quad \text{for all } \ i = 1,\, 2 \dots N,\ j = 1,\, 2,\, 3\,.$$

According to Lemma 6, this is equivalent to finding a critical point of the functional

$$\Lambda(\xi) \;:=\; \mathsf{J}\,[W_\xi + \phi_\xi]\,.$$

We expand this functional as follows:

$$\mathsf{J}\,[W_\xi] = \mathsf{J}\,[W_\xi + \phi_\xi] - D\mathsf{J}\,[W_\xi + \phi_\xi] \cdot \phi_\xi + \frac{1}{2} \int_0^1 D^2 \mathsf{J}\,[W_\xi + (1-t)\,\phi_\xi] \cdot (\phi_\xi\,,\, \phi_\xi)\; dt\,.$$

By definition of ϕ_ξ we have that $D\mathsf{J}\,[W_\xi + \phi_\xi] \cdot \phi_\xi = 0$. On the other hand, using Lemma 4, we check directly, out of the definition of ϕ_ξ, that

$$D^2 \mathsf{J}\,[W_\xi + (1-t)\,\phi_\xi] \cdot (\phi_\xi\,,\, \phi_\xi) = O(\omega^{4/3})$$

uniformly on points ξ_i satisfying constraints (4.3.6). Hence, from expansion (4.6.1) we obtain that

$$\Lambda(\xi) = \sum_{i=1}^{N} \lambda_i^{5-p}\, \mathsf{e}_* \;-\; \mathcal{V}_m^\omega(\xi) \;+\; O(\omega^{4/3})\,.$$

We claim that this expansion also holds in the C^1 sense. Let us first observe that

$$\int_{\mathbb{R}^3} \mathsf{E}\, \partial_\xi W_\xi\; dx = \nabla_\xi \mathsf{J}\,[W_\xi] \quad \text{and} \quad \partial_{\xi_{ij}} W_\xi = Z_{ij}\,.$$

Then, testing equation (4.5.1) against Z_{ij}, we see that

$$\int_{\mathbb{R}^3} \big(\mathsf{N}[\phi]\, Z_{ij} + \mathsf{L}[Z_{ij}]\big)\, \phi\, dx = -\int_{\mathbb{R}^3} \mathsf{E}\, Z_{ij}\, dx + \sum_{k\ell} c_{k\ell} \int_{\mathbb{R}^3} Z_{ij}\, Z_{k\ell}\, \chi_i\, dx\ .$$

Next we observe that

$$\big\|\, \mathsf{L}[Z_{ij}]\, \big\|_{**} = O(\omega^{2/3}) \quad \text{and} \quad \int_{\mathbb{R}^3} Z_{ij}\, Z_{k\ell}\, \chi_i\, dx = O(\omega^{2/3}) \quad \text{if } (i,j) \neq (k,\ell)\ .$$

By Lemma 4, $\|\phi_\xi\|_* = O(\omega^{2/3})$, and so we get

$$\big(\, 1 + O(\omega^{2/3})\,\big)\, c_{ij} \;=\; O(\omega^{4/3}) \;+\; \partial_{\xi_{ij}} \mathsf{J}\,[W_\xi]\ .$$

Hence, according to relation (4.6.3), we obtain

$$(\, \mathrm{I}_{3N} + O(\omega^{2/3})\,)\, \nabla_\xi \Lambda(\xi) \;=\; \nabla_\xi \mathsf{J}\,[W_\xi] \;+\; O(\omega^{4/3})$$

where $\nabla_\xi \mathsf{J}\,[W_\xi]$ has been computed in (4.6.2). Summarizing, we have found that

$$\nabla_\xi \Lambda(\xi) = -\nabla_\xi \mathcal{V}_m^\omega(\xi_1, \ldots \xi_N) \;+\; O(\omega^{2/3})\ .$$

Therefore, setting $\xi = \omega^{2/3}\, \zeta$ with $\zeta = (\zeta_1, \ldots \zeta_N)$ and defining $\Gamma(\zeta) := \Lambda(\xi)$ on $\mathfrak{B}_\mu := \big\{\zeta \in \mathbb{R}^{2N} \,:\, (4.3.7) \text{ holds}\big\}$, we have shown the following result.

Proposition 1 *With the above notations, we have that*

$$\Gamma(\zeta) = \sum_{i=1}^{N} \lambda_i^{5-p}\, \mathsf{e}_* \;-\; \omega^{2/3}\, \mathcal{V}_m(\zeta) \;+\; O(\omega^{4/3})$$
$$\nabla\Gamma(\zeta) = \;-\nabla \mathcal{V}_m(\zeta) \;+\; O(\omega^{2/3})$$

uniformly on ζ satisfying (4.3.7). *Here the terms $O(\cdot)$ are continuous functions of ζ defined on $\mathfrak{B}_\mu$.*

4.7.1 Proof of Theorem 1

If $\mu > 0$ is fixed large enough, we have that

$$\inf_{\mathfrak{B}_\mu} \mathcal{V}_m \;<\; \inf_{\partial\mathfrak{B}_\mu} \mathcal{V}_m\ .$$

Fixing such a μ, we get from Proposition 1 that, for all sufficiently small ω,

$$\sup_{\mathfrak{B}_\mu} \Gamma \;>\; \sup_{\partial\mathfrak{B}_\mu} \Gamma\ .$$

so that the functional Λ has a maximum value somewhere in $\omega^{2/3}\, \mathfrak{B}_\mu$, which is close to a maximum value of $\mathcal{V}_m^\omega$. This value is achieved at critical point of Λ, and hence a solution with the desired features exists. The construction is concluded.

□

4.7.2 Proof of Theorem 2

When $(\zeta_1, \dots \zeta_N)$ is a regular polygon with ζ_j given by (4.1.6) and all masses are equal, the system is invariant under the rotation defined by

$$x = \underbrace{(x^1, x^2, x^3)}_{\in \mathbb{R}^3} \approx \underbrace{((x^1 + i\, x^2),\, x^3)}_{\in \mathbb{C}\times\mathbb{R}} \mapsto (e^{2\, i\, \pi/N}(x^1 + i\, x^2),\, x^3) =: \mathcal{R}_N\, x\ .$$

We can therefore pass to the quotient with respect to this group of invariance and look for solutions u which are invariant under then action of $\mathcal{R}_N$ and moreover symmetric with respect to the reflections $(x^1, x^2, x^3) \mapsto (x^1, -x^2, x^3)$ and $(x^1, x^2, x^3) \mapsto (x^1, x^2, -x^3)$. Here we assume that $(\zeta_1, \dots \zeta_N)$ is contained in the plane $\{x^3 = 0\}$ and $\zeta_1 = (r, 0, 0)$. Altogether this amounts to look for critical points of the functional

$$\mathsf{J}_1[u] = \frac{1}{2}\int_{\Omega_1} |\nabla u|^2\, dx - \frac{1}{p+1}\int_{\Omega_1} \big(u - \lambda_1 + \tfrac{1}{2}\,\omega^2\,|x'|^2\big)_+^{p+1}\,\chi_1\, dx$$

where $\Omega_1 = \{x = (x', x^3) \in \mathbb{R}^3 \approx \mathbb{C}\times\mathbb{R} \,:\, x' = r\, e^{i\,\theta} \text{ s.t. } -\frac{\pi}{N} < \theta < \frac{\pi}{N}\}$ and $u \in H^1(\Omega)$ is invariant under the two above reflections and such that $\nabla u \cdot n = 0$ on $\partial\Omega \setminus \{0\}\times\mathbb{R}$ and χ_1 is the characteristic function of the support of $\rho_1 = \big(u - \lambda_1 + \frac{1}{2}\,\omega^2\,|x'|^2\big)_+^{p+1}\,\chi_1$ in Ω_1. Here $n = n(x)$ denotes the unit outgoing normal vector at $x \in \partial\Omega_1$. With J defined by (4.3.3), it is straightforward to see that $\mathsf{J}\,[u] = N\,\mathsf{J}_1[u]$ if u is extended to $\mathbb{R}^3$ by assuming that $u(\mathcal{R}_N\, x) = u(x)$. With these notations, we find that

$$\mathcal{V}_m(\zeta_1, \dots \zeta_N) = N\, m_*\,(\frac{a_N}{4\pi}\,\frac{m_*}{r} + \frac{1}{2}\,r^2)\ .$$

The proof goes as for Theorem 1. Because of the symmetry assumptions, $c_{1j} = 0$ if $j = 2$ or 3. Details are left to the reader. □

4.7.3 Proof of Theorem 3

We look for a critical point of the functional Γ of Proposition 1 in a neighborhood of a critical point ζ of $\mathcal{V}_m$, which is nondegenerate up to rotations. With no loss of generality, we may assume that $\zeta_1 \neq 0$, $\zeta_{12} = 0$ and denote by $\tilde{\mathcal{V}}_m$ the restriction of $\mathcal{V}_m$ to $(\mathbb{R}\times\{0\})\times(\mathbb{R}^2)^{N-1} \ni \zeta$. Similarly, we denote by $\tilde{\Gamma}$ the restriction of Γ to $(\mathbb{R}\times\{0\})\times(\mathbb{R}^2)^{N-1}$.

By assumption, ζ is a non-degenerate critical point of $\tilde{\mathcal{V}}_m$, *i.e.* an isolated zero of $\nabla\tilde{\mathcal{V}}_m$. Besides, its local degree is non-zero. It follows that on an arbitrarily small neighborhood of that point, the degree for $\nabla\tilde{\Gamma}$ is non-zero for all sufficiently small ω. Hence there exists a zero $\zeta^\omega \in (\mathbb{R}\times\{0\})\times(\mathbb{R}^2)^{N-1}$ of $\nabla\tilde{\Gamma}$ as close to ï¿½ζ as we wish. From the rotation invariance, it follows that ζ^ω is also a critical point of Γ. The proof of Theorem 3 is concluded. □

4.8 Facts on Relative Equilibria

In this Appendix we have collected some results on the N-body problem introduced in Section 4.1 which are of interest for the proofs of Theorems 1-3, with a list of relevant references.

Non-degeneracy of relative equilibria in a standard form

Relative equilibria are by definition critical points of the function $\mathcal{V}_m : \mathbb{R}^{2N} \to \mathbb{R}$ defined by

$$\mathcal{V}_m(\zeta_1, \zeta_2, \dots \zeta_N) = \frac{1}{8\pi}\sum_{i\neq j=1}^{N}\frac{m_i\, m_j}{|\zeta_j - \zeta_i|} + \frac{1}{2}\sum_{i=1}^{N} m_i\,|\zeta_i|^2\,.$$

Here we assume that $N \geq 2$, and $m_i > 0$, $i = 1, \dots N$ are given parameters.

Following Smale in [59], we can rewrite this problem as follows. Let us consider the $(2N-3)$-dimensional manifold

$$S_m := \Big\{ q = (q_1, \dots q_N) \in \mathbb{R}^{2N} \, : \, \sum_{i=1}^N m_i \, (q_i, \tfrac{1}{2} \, |q_i|^2) = (0,1) \, , \; q_i \neq q_j \text{ if } i \neq j \Big\} \, .$$

The problem of finding critical points of the functional

$$U_m(q_1, \dots q_N) = \frac{1}{8\pi} \sum_{i \neq j = 1}^N \frac{m_i \, m_j}{|q_j - q_i|}$$

on S_m is equivalent to that of relative equilibria; see for instance [29]. Let us give some details. Let $\bar{q}$ be a critical point of U_m on S_m. Then by definition, there are Lagrange multipliers $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}^2$ for which

$$-\frac{1}{8\pi} \sum_{i \neq j}^N \frac{m_i \, m_j}{|\bar{q}_i - \bar{q}_j|^3} \, (\bar{q}_i - \bar{q}_j) \, = \lambda \, m_j \, \bar{q}_j + m_j \, \mu \quad \forall \, j = 1, \dots N \, .$$

First, adding in j the above relations and using that $M = \sum_{j=1}^N m_j > 0$ we obtain that $\mu = 0$. Second, taking the scalar product of $\mathbb{R}^2$ against $\bar{q}_j$ and then adding in j, we easily obtain that $U_m(\bar{q}) = \lambda$. From here it follows that the point $\bar{\zeta} = \lambda^{1/3} \, \bar{q}$ is a critical point of the functional $\mathcal{V}_m$, hence a relative equilibrium.

With the reparametrization of $\mathbb{R}^{2N}$ given by

$$\zeta(\alpha, p, q) = (\zeta_1, \dots \zeta_N) = (\alpha \, q_1 + p, \dots \alpha \, q_N + p) \, , \quad (\alpha, p, q) \in \mathbb{R} \times \mathbb{R}^2 \times S_m \, ,$$

the Hessian matrix of $\mathcal{V}_m$ at the critical point $\bar{\zeta} = \zeta(\lambda^{1/3}, 0, \bar{q})$ found above is represented as the block matrix

$$D^2 \mathcal{V}_m(\bar{\zeta}) = \begin{pmatrix} 2 & & \\ & M \, \mathrm{I}_2 & \\ & & \lambda^{-1/3} \, D^2_{S_m} U_m(\bar{q}) \end{pmatrix}$$

where I_2 is the 2×2 identity matrix and $D^2_{S_m}$ represents the second covariant derivative on S_m. Reciprocally, we check that a critical point $\bar{\zeta} = (\zeta_j)_{j=1}^N$ of $\mathcal{V}_m$ necessarily satisfies $\sum_{j=1}^N m_j \, \zeta_j = 0$. Defining $\bar{q} = \big(\frac{1}{2} \sum_{j=1}^N m_j \, |\zeta_j|^2 \big)^{-1/2} \, \bar{\zeta}$, we readily check that $\bar{q}$ is a critical point of U_m in S_m.

Any rotation $e^{i \, \alpha} \, \bar{q}$ of a critical point $\bar{q}$ of U_m on S_m is also a critical point. We say that two such critical points are *equivalent* in S_m. Let us denote by $\mathcal{S}_m$ the quotient manifold of S_m by this equivalence relation. On $\mathcal{S}_m$, critical points of the potential U_m yield critical points of U_m on S_m and hence equivalence classes of critical points $e^{i \, \alpha} \, \zeta$ for $\mathcal{V}_m$ using the reparametrization.

A critical point $\tilde{q}$ of U_m on $\mathcal{S}_m$ is said to be *non-degenerate* if the second variation of U_m at $\tilde{q}$ is non-singular. Let us assume that $\tilde{q}_\ell \neq 0$, with either $\ell = 1$, or $\ell = 2$ if $\tilde{q}_1 = 0$. Then there is a unique representative $\bar{q}$ of this class of equivalence for which $\bar{q}_{\ell 2} = 0$. It is a routine verification to check that $\bar{q}$ is then a critical point of U_m on the $(2N-4)$-dimensional manifold

$$\mathsf{S}_m := \big\{ q \in S_m \, : \, q_\ell \neq 0 \text{ as above} \, , \; q_{\ell 2} = 0 \big\} \, .$$

Moreover, the second derivative of U_m on $\mathcal{S}_m$ at $\tilde{q}$ is non-degenerate if and only if $D^2_{\mathsf{S}_m} U_m(\bar{q})$ is non-singular. Because of the expression of $D^2 \mathcal{V}_m(\bar{\zeta})$, we see that $\bar{\zeta}$ is non-degenerate as a critical point of $\mathcal{V}_m$ on the space of $\zeta \in \mathbb{R}^{2N}$ with $q_{\ell 2} = 0$, which is the notion of *non-degeneracy up to rotations* of a relative equilibrium that we have used in this chapter. Finally we define the *index* of a non-degenerate relative equilibrium $\bar{\zeta}$ as the number of negative eigenvalues of $D^2_{\mathsf{S}_m} U_m(\bar{q})$.

4.8.1 Some results on classification of relative equilibria

For simplicity, we will assume that masses are all different: for any i, $j = 1, \dots N$, if $m_i = m_j$, then $i = j$. This is the generic case.

The cases $N = 2$, 3 are well known; see for instance [45]. For $N = 2$, the only class of critical points is such that

$$|\zeta_1 - \zeta_2| = (\tfrac{M}{4\pi})^{1/3} \quad \text{and} \quad m_1\,\zeta_1 + m_2\,\zeta_2 = 0 \quad \text{with } M = m_1 + m_2\ .$$

For $N = 3$, there are two types of solutions, the Lagrange and the Euler solutions. The *Lagrange solutions* are such that their center of mass is fixed at the origin, the masses are located at the vertices of an equilateral triangle, and the distance between each point is $(M/(4\pi))^{1/3}$ with $M = m_1+m_2+m_3$. They give rise to two classes of solutions corresponding to the two orientations of the triangle when labeled by the masses. The *Euler solutions* are made of aligned points and provide three classes of critical points, one for each ordering of the masses on the line.

In the case $N \geq 4$, the classes of solutions for which all points are collinear still exist (see [46]) and are known as the *Moulton solutions*. But the configuration of relative equilibria where all particles are located at the vertices of a regular N-polygon exists if and only if all masses are equal; see [43, 60, 52, 29, 63]. Various classification results which have been obtained by Palmore are summarized below.

Theorem 4 ([47, 48, 49, 50, 51]) *We have the following multiplicity results.*

(a) *For $N \geq 3$, the index of a relative equilibrium is always greater or equal than $N - 2$. This bound is achieved by Moulton's solutions.*

(b) *For $N \geq 3$, there are at least $\mu_i(N) := \binom{N}{i}(N - 1 - i)\,(N - 2)\,!$ distinct relative equilibria in $\mathcal{S}_m$ of index $2N - 4 - i$ if U_m is a Morse function. As a consequence, there are at least*

$$\sum_{i=0}^{N-2} \mu_i(N) = [2^{N-1}(N - 2) + 1]\,(N - 2)\,!$$

distinct relative equilibria in $\mathcal{S}_m$ if U_m is a Morse function.

(c) *For every $N \geq 3$ and for almost all masses $m \in \mathbb{R}^N_+$, U_m is a Morse function.*

(d) *There are only finitely many classes of relative equilibria for every $N \geq 3$ and for almost all masses $m = (m_i)_{i=1}^N \in \mathbb{R}^N_+$.*

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Chapter 5

A functional framework for the Keller-Segel system: logarithmic Hardy-Littlewood-Sobolev and related spectral gap inequalities

This chapter is devoted to several inequalities deduced from a special form of the logarithmic Hardy-Littlewood-Sobolev, which is adapted to the characterization of stationary solutions of a Keller-Segel system written in self-similar variables, in case of a subcritical mass. For the corresponding evolution problem, such functional inequalities play an important role for identifying the rate of convergence of the solutions towards the stationary solution with same mass.

This is a joint work with J. Dolbeault.

5.1 Introduction

In $\mathbb{R}^2$, the logarithmic Hardy-Littlewood-Sobolev has been established with optimal constants in [69] (also see [64]) and can be written as

$$\int_{\mathbb{R}^3} n\,\log\Big(\frac{n}{M}\Big)\,dx+\frac{2}{M}\int_{\mathbb{R}^2\times\mathbb{R}^2} n(x)\,n(y)\,\log|x-y|\,dx\,dy+M\,(1+\log\pi)\ge 0 \tag{5.1.1}$$

for any function $n\in L^1_+(\mathbb{R}^2)$ with $M=\int_{\mathbb{R}^3} n\,dx$. As a consequence (see [72]), the *free energy* functional

$$F[n]:=\int_{\mathbb{R}^3} n\,\log n\,dx+\frac12\int_{\mathbb{R}^3}|x|^2\,n\,dx-\frac12\int_{\mathbb{R}^3} n\,c\,dx+K\quad\text{with}\quad c=(-\Delta)^{-1}n:=-\frac{1}{2\,\pi}\,\log|\cdot|*n$$

is bounded from below if $M\in(0,8\,\pi]$. Here $K=K(M)$ is a constant to be fixed later. We may observe that F is not bounded from below if $M>8\,\pi$, for instance by considering $\lambda\mapsto F[n_\lambda]$ where $n_\lambda(x)=\lambda^2\,n(\lambda\,x)$ for some given function n, and by taking the limit $\lambda\to\infty$. See [73] for further details. Equality in (5.1.1) is achieved by

$$\mu(x):=\frac{1}{\pi\,(1+|x|^2)^2}\quad\forall\,x\in\mathbb{R}^2\,,$$

which solves $-\Delta\log\mu=8\,\pi\,\mu$ and can be inverted as $(-\Delta)^{-1}\mu=\frac{1}{8\,\pi}\,\log\mu+\frac{1}{8\,\pi}\,\log\pi$.

Consider the probability measure $d\mu := \mu\,dx$. Written in Euclidean form, Onofri's inequality (see [74] for the equivalent version on the sphere)

$$\log\left(\int_{\mathbb{R}^2} e^{\,\phi}\,d\mu\right) - \int_{\mathbb{R}^2} \phi\,d\mu \le \frac{1}{16\,\pi}\,\int_{\mathbb{R}^3} |\nabla\phi|^2\,dx \tag{5.1.2}$$

plays in dimension $d = 2$ the role of Sobolev's inequality in higher dimensions. The inequality holds for any smooth function with compact support and, by density, for any function ϕ in the space $\mathcal{H}_M$ obtained by completion with respect to the norm given by: $\|\phi\|^2 = \int_{\mathbb{R}^3} |\nabla\phi|^2\,dx + (\int_{\mathbb{R}^2} \phi\,d\mu_M)^2$. Onofri's inequality can be seen as the *dual* inequality of the logarithmic Hardy-Littlewood-Sobolev, *cf* [69, 64, 68, 71].

The rescaled parabolic-elliptic Keller-Segel system reads

$$\frac{\partial n}{\partial t} = \Delta n + \nabla\cdot(n\,x) - \nabla\cdot(n\,\nabla c)\;,\quad c = (-\Delta)^{-1} n\;,\quad x\in\mathbb{R}^2\,,\quad t>0 \tag{5.1.3}$$

Assume that the initial datum is $n(0,\cdot) = n_0$. If $M = \int_{\mathbb{R}^3} n_0\,dx > 8\,\pi$, solutions blow-up in finite time. If $n_0 \in L^1_+\big(\mathbb{R}^2\,,(1+|x|^2)\,dx\big)$, $n_0\,|\log n_0| \in L^1(\mathbb{R}^2)$ and $M < 8\,\pi$, solutions globally exists and it has been shown in [66, Theorem 1.2] that

$$\lim_{t\to\infty} \|n(t,\cdot) - n_M\|_{L^1(\mathbb{R}^2)} = 0 \quad\text{and}\quad \lim_{t\to\infty} \|\nabla c(t,\cdot) - \nabla c_M\|_{L^2(\mathbb{R}^2)} = 0\;,$$

where (n_M, c_M) is the unique, smooth and radially symmetric solution of

$$-\,\Delta c_M = M\,\frac{e^{-\frac12\,|x|^2 + c_M}}{\int_{\mathbb{R}^3} e^{-\frac12\,|x|^2+c}\,dx} =: n_M\;,\quad x\in\mathbb{R}^2\;. \tag{5.1.4}$$

Notice that $n_M = M\,e^{c_M-|x|^2/2}/\int_{\mathbb{R}^3} e^{c_M-|x|^2/2}\,dx$ with $c_M = (-\Delta)^{-1} n_M$. The case $M = 8\,\pi$ has also been extensively studied, and it will be briefly discussed in chapter 8.

Ineq. (5.1.2) and the Moser-Trudinger inequality have been repeatedly used to study the Keller-Segel system in bounded domains. In the whole space case, Ineq. (5.1.1) turns out to be very convenient. Ineq. (5.1.2) and Ineq. (5.1.1) correspond to the $M = 8\,\pi$ case. For $M < 8\,\pi$, we will establish a new inequality of Onofri type, which is our first main result: see Theorem 8.

An important issue in the study of (5.1.3) is to characterize the rate of convergence of n towards n_M. See [65, 67]. For this purpose, it is convenient to linearize the Keller-Segel system (5.1.3) by considering

$$n(t,x) = n_M(x)\,(1+\varepsilon\,f(t,x)) \quad\text{and}\quad c(t,x) = c_M(x)\,(1+\varepsilon\,g(t,x))$$

and formally take the limit as $\varepsilon\to 0$. At order $O(\varepsilon)$, (f,g) solves

$$\frac{\partial f}{\partial t} = \frac{1}{n_M}\nabla\cdot\big[n_M\nabla(f - g\,c_M)\big] =: \mathcal{L}\,f \quad\text{and}\quad g\,c_M = (-\Delta)^{-1}(f\,n_M)\;. \tag{5.1.5}$$

As we shall see in Section 5.3, several spectral gap inequalities are related with (5.1.1) and involve the linear operator $\mathcal{L}$. Detailed proofs and applications to the full Keller-Segel system (5.1.3) will be given in the next chapter, whose main result is that, $\|n(t,\cdot) - n_M\|_{L^1(\mathbb{R}^2)} = O(e^{-t})$ as $t\to\infty$.

5.2 Duality and stationary solutions of the Keller-Segel model in self-similar variables

For any $M \in (0, 8\,\pi)$, the function c_M given by (5.1.4) can be characterized either as a minimizer of

$$F^*[c] := \frac{1}{2}\int_{\mathbb{R}^3} n\,c\,dx - M\,\log\left(\int_{\mathbb{R}^3} e^{-\frac{1}{2}\,|x|^2+c}\,dx\right)$$

where n and c are related through the Poisson equation, $-\Delta c = n$, or in terms of n, seen as a minimizer of the functional $n \mapsto F[n]$. Inspired by [64, 68, 69, 71], we can characterized the corresponding functional inequalities and observe that they are *dual* of each other. Let us give some details.

Consider the functional $n \mapsto F[n] = F_1[n] - F_2[n]$ (for an appropriate choice of the constant K) on the set $\mathcal{X}_M$ of all nonnegative integrable functions with mass $M > 0$, where

$$F_1[n] = \int_{\mathbb{R}^3} n\,\log\left(\frac{n}{n_M}\right)dx \quad\text{and}\quad F_2[n] = \frac{1}{2}\int_{\mathbb{R}^3}(n-n_M)\,(-\Delta)^{-1}(n-n_M)\,dx\,.$$

Since n_M is a minimizer for F and $F[n_M] = 0$, we actually have the functional inequality $F_1[n] \geq F_2[n]$ for any $n \in \mathcal{X}_M$. This inequality can be rewritten as

$$\int_{\mathbb{R}^3} n\,\log\left(\frac{n}{n_M}\right)dx + \frac{1}{4\,\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2}(n(x)-n_M(x))\,\log|x-y|\,(n(y)-n_M(y))\,dx\,dy \geq 0 \tag{5.2.1}$$

for any nonnegative $n \in L^1(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^3} n\,dx = M < 8\,\pi$.

By Legendre's duality, we have: $F_1^*[\phi] \leq F_2^*[\phi]$ where $F_i^*[\phi] := \sup_{n\in\mathcal{X}_M}\left(\int_{\mathbb{R}^3}\phi\,n\,dx - F_i[n]\right)$, $i = 1, 2$, is defined on $L^\infty(\mathbb{R}^2)$. A straightforward computation shows that $F_1^*[\phi] = \int_{\mathbb{R}^3}\phi\,n\,dx - F_1[n]$ if and only if $\log(\frac{n}{n_M}) = \phi - \log\left(\int_{\mathbb{R}^2} e^\phi\,d\mu_M\right) + \log M$, so that

$$F_1^*[\phi] = M\,\log\left(\int_{\mathbb{R}^2} e^\phi\,d\mu_M\right) - M\,\log M\,.$$

Here $d\mu_M$ is the probability measure

$$d\mu_M := \mu_M\,dx\,,\quad\text{with}\quad \mu_M := \frac{1}{M}\,n_M\,.$$

It is clear that we can impose at no cost that $\int_{\mathbb{R}^2}\phi\,d\mu_M = 0$. It is also standard to observe that $F_2^*[\phi] = \int_{\mathbb{R}^3}\phi\,n\,dx - F_2[n]$ if and only if $\phi = (-\Delta)^{-1}(n-n_M)$, so that

$$F_2^*[\phi] = \frac{1}{2}\int_{\mathbb{R}^3}|\nabla\phi|^2\,dx\,.$$

Notice that $\int_{\mathbb{R}^3}|\nabla\phi|^2\,dx$ is well defined as $-\Delta\phi = n - n_M$ is integrable and such that $\int_{\mathbb{R}^3}(n-n_M)\,dx = 0$.

Theorem 8 *For any $M \in (0, 8\,\pi)$, with n_M defined as the unique minimizer of F,* i.e. *the unique solution n_M given by* (5.1.4)*, with $c_M = (-\Delta)^{-1}\,n_M$, we have the following inequality:*

$$\log\left(\int_{\mathbb{R}^2} e^\phi\,d\mu_M\right) - \int_{\mathbb{R}^2}\phi\,d\mu_M \leq \frac{1}{2\,M}\int_{\mathbb{R}^3}|\nabla\phi|^2\,dx \quad \forall\,\phi\in\mathcal{H}_M\,. \tag{5.2.2}$$

As a consequence, if we consider the special case $\phi = 1 + \varepsilon\,\psi$ and consider the limit $\varepsilon \to 0$ in (5.2.2), as in [70], we get an interesting spectral gap inequality.

Corollary 1 *With the above notations, for any $\psi \in \mathcal{H}_M$, the following inequality holds*

$$\int_{\mathbb{R}^2} \left|\psi - \overline{\psi}\right|^2 d\mu_M \le \int_{\mathbb{R}^3} |\nabla\psi|^2\, dx \quad \textit{where} \quad \overline{\psi} = \int_{\mathbb{R}^2} \psi\, d\mu_M\ .$$

5.3 Linearized Keller-Segel model, spectral gap inequalities and consequences

Exactly as for Ineq. (5.2.2), we observe that

$$\mathsf{Q}_1[f] := \int_{\mathbb{R}^2} |f|^2\, d\mu_M + \frac{1}{2\,\pi} \iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\, \log|x-y|\, f(y)\, d\mu_M(x)\, d\mu_M(y) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\, F[n_M(1+\varepsilon\, f)] \ge 0\ .$$

Notice that $\mathsf{Q}_1[f] = \int_{\mathbb{R}^3} |f|^2\, n_M\ dx - \int_{\mathbb{R}^3} |\nabla(g\, c_M)|^2\ dx$ if $\int_{\mathbb{R}^2} f\ d\mu_M = 0$. We also notice that $f_{0,0} := \partial_M \log n_M$ generates the kernel $Ker(\mathcal{L})$ considered as an operator on $L^2(\mathbb{R}^2, d\mu_M)$ and the functions $f_{1,i} := \partial_{x_i} \log n_M$ with $i = 1,\ 2$ and $f_{0,1} := x\cdot\nabla \log n_M$ are eigenfunctions of $\mathcal{L}$ with eigenvalues 1 and 2 respectively; moreover they generate the corresponding eigenspaces (see chapter 6 for details). It is remarkable that $\mathsf{Q}_1[f] = 0$ if and only if $f \in Ker(\mathcal{L})$ and this allows to establish a first spectral gap inequality.

Theorem 9 *There exists $\kappa > 1$ such that*

$$\int_{\mathbb{R}^2} f^2\, d\mu_M \le \kappa\, \mathsf{Q}_1[f] \quad \forall\ f \in L^2(\mathbb{R}^2, f\mu_M) \quad \textit{such that} \int_{\mathbb{R}^2} f\, f_{0,0}\, d\mu_M = 0\ .$$

Since $f_{0,0}$ is the unique solution (up to a normalization) of $-\Delta f_{0,0} = f_{0,0}\, n_M$, we may notice that an optimal function in Corollary 1 solves the equation $-\Delta\psi = (\psi - \bar{\psi})\, \mu_M$ and is therefore given by $\psi = f_{0,0} - \mu_M \int_{\mathbb{R}^2} f_{0,0}\, d\mu_M$, up to a multiplication by a constant.

The proof of Theorem 9 relies on spectral properties of Schrödinger operators, this will be shown in detail in the following chapter. Since $\mathsf{Q}_1[f] = 0$ if and only if $f \in Ker(\mathcal{L})$, that is if f is proportional to $f_{0,0}$, we can define the scalar product $\langle\cdot,\cdot\rangle$ induced by the quadratic form Q_1 on the space $\mathcal{D}_M$ orthogonal of $f_{0,0}$ in $L^2(\mathbb{R}^2, d\mu_M)$. With this definition, we have $\mathsf{Q}_1[f] = \langle f, f\rangle$. On the space $\mathcal{D}_M$ with scalar product $\langle\cdot,\cdot\rangle$, the operator $\mathcal{L}$ is self-adjoint. Let

$$\mathsf{Q}_2[f] := \langle f,\, \mathcal{L}\, f\rangle\ .$$

Then we have a second spectral gap inequality.

Theorem 10 *For any function $f \in \mathcal{D}_M$, we have*

$$\mathsf{Q}_1[f] \le \mathsf{Q}_2[f]\ .$$

Moreover, if f is a radial function, then we have $2\,\mathsf{Q}_1[f] \le \mathsf{Q}_2[f]$. The operator $\mathcal{L}$ has only discrete spectrum as a consequence of Persson's lemma, or as can be shown by direct investigation using the tools of the concentration-compactness method and the Sturm-Liouville theory. By rewriting the spectral

problem for $\mathcal{L}$ in terms of *cumulated densities,* it is possible to prove that the eigenspace corresponding to the lowest non-zero eigenvalue is generated by $f_{1,i}$ with $i = 1, 2$, which completes the proof. Details are given in the following chapter.

As a simple consequence, if f is a solution to (5.1.5), then

$$\frac{d}{dt}\langle f, f\rangle = -\langle f, \mathcal{L}\, f\rangle \le -2\,\langle f, f\rangle \ ,$$

which shows the exponential convergence of f towards 0. The nonlinear Keller-Segel model (5.1.3) can be rewritten in terms of $f := (n - n_M)/n_M$ and $g := (c - c_M)/c_M$ as

$$\frac{\partial f}{\partial t} - \mathcal{L}\, f = -\frac{1}{n_M}\nabla\cdot\left[f\, n_M(\nabla(g\, c_M))\right] \ .$$

Estimates based on the Duhamel formula allow to prove that $t \mapsto \mathsf{Q}_1[f(t,\cdot)]$ is bounded uniformly with respect to $t > 0$ and

$$\frac{d}{dt}\,\mathsf{Q}_1[f(t,\cdot)] \le -\,\mathsf{Q}_1[f(t,\cdot)]\left[2 - \delta(t,\varepsilon)\left(\mathsf{Q}_1[f(t,\cdot)]^{\frac{1-\varepsilon}{2-\varepsilon}} + \mathsf{Q}_1[f(t,\cdot)]^{\frac{1}{2+\varepsilon}}\right)\right] \ .$$

for any $\varepsilon > 0$ small enough, for some continuous δ such that $\lim_{t\to\infty}\delta(t,\varepsilon) = 0$.

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Chapter 6

Asymptotic estimates for the parabolic-elliptic Keller-Segel model in the plane

In this chapter we investigate the large-time behavior of the solutions of the two-dimensional Keller-Segel system in self-similar variables, when the total mass is subcritical, that is less than $8\,\pi$ after a proper adimensionalization. It was known from previous works that all solutions converge to stationary solutions, with exponential rate when the mass is small. Here we remove this restriction and show that the rate of convergence measured in relative entropy is exponential for any mass in the subcritical range, and independent of the mass. The proof relies on symmetrization techniques, which are adapted from a paper of J.I. Díaz, T. Nagai, and J.-M. Rakotoson, and allow us to establish uniform estimates for L^p norms of the solution. Exponential convergence is obtained by the mean of a linearization in a space which is defined consistently with relative entropy estimates and in which the linearized evolution operator is self-adjoint. The core of proof relies on several new spectral gap estimates which are of independent interest.

This is a joint work with J. Dolbeault and it has been submitted to the journal *Communications in Partial Differential Equations*.

6.1 Introduction

Consider the two-dimensional parabolic-elliptic Keller-Segel system

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \nabla\cdot(u\,\nabla v) & x\in\mathbb{R}^2\,, \quad t>0\,,\\ v = G_2 * u & x\in\mathbb{R}^2\,, \quad t>0\,,\\ u(0,x) = n_0 \ge 0 & x\in\mathbb{R}^2\,, \end{cases} \tag{6.1.1}$$

where G_2 denotes the Green function associated to $-\Delta$ on $\mathbb{R}^2$:

$$G_2(x) := -\frac{1}{2\pi}\,\log|x|\,, \quad x\in\mathbb{R}^2\,.$$

The equation for the *mass density* u is parabolic, while the *chemo-attractant density* v solves an (elliptic) Poisson equation: $-\Delta v = u$. The drift term corresponds to an attractive mean-field nonlinearity, which

has attracted lots of attention in mathematical biology in the recent years: see [97, 98, 99, 109, 111, 112] for some recent overviews. According to [100, 96, 84, 88], it is known that if

$$n_0 \in L^1_+\left(\mathbb{R}^2\,,(1+|x|^2)\,dx\right)\;,\quad n_0\,|\log n_0| \in L^1(\mathbb{R}^2)\quad\text{and}\quad M := \int_{\mathbb{R}^3} n_0\,dx < 8\,\pi\;, \tag{6.1.2}$$

then there exists a solution u, in the sense of distributions, that is global in time and such that $M = \int_{\mathbb{R}^3} u(t,x)\,dx$ is conserved along the evolution in the euclidean space $\mathbb{R}^2$. There is no non-trivial stationary solution of (6.1.1) and any solution converges to zero locally as time gets large. In order to study the asymptotic behavior of u, it is convenient to work in self-similar variables. We define $R(t) := \sqrt{1+2\,t}$, $\tau(t) := \log R(t)$, and the rescaled functions n and c by

$$u(t,x) := R^{-2}(t)\,n\left(\tau(t),R^{-1}(t)\,x\right)\quad\text{and}\quad v(t,x) := c\left(\tau(t),R^{-1}(t)\,x\right)\;.$$

This time-dependent rescaling is the one of the heat equation. We observe that the non-linear term is also invariant under such a rescaling. The rescaled systems reads

$$\left\{\begin{array}{ll} \frac{\partial n}{\partial t} = \Delta n + \nabla\cdot(n\,x) - \nabla\cdot(n\,\nabla c) & x\in\mathbb{R}^2\,,\quad t>0\,,\\ c = G_2 * n & x\in\mathbb{R}^2\,,\quad t>0\,,\\ n(0,x) = n_0 \ge 0 & x\in\mathbb{R}^2\,. \end{array}\right. \tag{6.1.3}$$

Under Assumptions (6.1.2), it has been shown in [84, Theorem 1.2] that

$$\lim_{t\to\infty}\|n(t,\cdot)-n_\infty\|_{L^1(\mathbb{R}^2)} = 0\quad\text{and}\quad \lim_{t\to\infty}\|\nabla c(t,\cdot)-\nabla c_\infty\|_{L^2(\mathbb{R}^2)} = 0\;,$$

where (n_∞, c_∞) solves

$$n_\infty = M\,\frac{e^{c_\infty-|x|^2/2}}{\int_{\mathbb{R}^3} e^{c_\infty-|x|^2/2}\,dx}\quad\text{with}\quad c_\infty = G_2 * n_\infty\;. \tag{6.1.4}$$

Moreover, n_∞ is smooth and radially symmetric. Existence of a solution to (6.1.4) has been established in [76] by ordinary differential equation techniques and in [107] by partial differential equation methods. The uniqueness has been shown in [79]. To recall the dependence of n_∞ in M, we will write it as $n_{\infty,M}$ whenever needed.

A simple computation of the second moment shows that smooth solutions with mass larger than $8\,\pi$ blow-up in finite time; see for instance [100]. The case $M = 8\,\pi$ has been extensively studied. We shall refer to [79, 80, 81] for some recent papers on this topic. The asymptotic regime is of a very different nature in such a critical case. In the present chapter, we shall restrict our purpose to the sub-critical case $M < 8\,\pi$.

In [82] it has been proved that there exists a positive mass $M_\star \le 8\,\pi$ such that for any initial data $n_0 \in L^2(n_\infty^{-1}\,dx)$ of mass $M < M_\star$ satisfying (6.1.2), System (6.1.3) has a unique solution n such that

$$\int_{\mathbb{R}^2} |n(t,x)-n_\infty(x)|^2\,\frac{dx}{n_\infty(x)} \le C\,e^{-\delta\,t}\quad\forall\,t\ge 0$$

for some positive constants C and δ. Moreover δ can be taken arbitrarily close to 1 as $M\to 0$. If $M < 8\,\pi$, we may notice that the condition $n_0 \in L^2(n_\infty^{-1}\,dx)$ is stronger than (6.1.2). Our main result is that $M_\star = 8\,\pi$ and $\delta \ge 1$, at least for a large subclass of solutions with initial datum n_0 satisfying the following technical assumption:

$$\exists\,\varepsilon\in(0,8\,\pi-M)\quad\text{such that}\quad \int_0^s n_{0,*}(\sigma)\,d\sigma \le \int_{B\left(0,\sqrt{s/\pi}\right)} n_{\infty,M+\varepsilon}(x)\,dx\quad\forall\,s\ge 0\;. \tag{6.1.5}$$

Here $n_{0,*}(\sigma)$ stands for the symmetrized function associated to n_0. Details will be given in Section 6.2.

Theorem 11 *Assume that n_0 satisfies* (6.1.5)*,*

$$n_0 \in L^2_+(n_\infty^{-1}\,dx) \quad \text{and} \quad M := \int_{\mathbb{R}^3} n_0\,dx < 8\,\pi\,.$$

Then any solution of (6.1.3) *with initial datum n_0 is such that*

$$\int_{\mathbb{R}^2} |n(t,x) - n_\infty(x)|^2\,\frac{dx}{n_\infty(x)} \le C\,e^{-2\,t} \quad \forall\, t \ge 0$$

for some positive constant C, where n_∞ is the unique stationary solution to (6.1.4) *with mass M.*

This result is consistent with the recent results of [85] for the two-dimensional radial model and its one-dimensional counterpart (see Proposition 24 for more comments). For completeness, let us mention that results of exponential convergence for problems with mean field have been obtained earlier in [89, 90], but only for interaction potentials involving much smoother kernels than G_2.

This chapter is organized as follows. In Section 6.2, we will apply symmetrization techniques as in [93, 94] to establish uniform estimates on $\|n\|_{L^p(\mathbb{R}^2)}$. Then we will prove the uniform convergence of n to n_∞ using Duhamel's formula: see Corollary 17 in Section 6.3. Section 6.4 is devoted to the linearization of the problem around n_∞ and to the study of the spectral gap of the linearized operator. A strict positivity result for the linearized entropy is also needed and will be proved in Section 6.5. The proof of Theorem 11 is completed in the last section. It is based on two estimates: Theorems 23 and 25 (also see Corollary 26) that are established in Sections 6.4 and 6.5 respectively. Some of the results of Sections 6.4 and 6.5 (see Theorem 23 and Corollary 26) have been announced without proof in [86] in connection with a new Onofri type inequality, whose linearized form is given in Inequality (6.5.2).

6.2 Symmetrization

In this section, we adapt the results of J.I. Díaz, T. Nagai, and J.-M. Rakotoson in [94] to the setting of self-similar variables. Several key estimates are based on an earlier work of J.I. Díaz and T. Nagai for the bounded domain case: see [93]. We shall therefore only sketch the main steps of the method and emphasize the necessary changes.

To any measurable function $u : \mathbb{R}^2 \mapsto [0,+\infty)$, we associate the distribution function defined by $\mu(t,\tau) := |\{u > \tau\}|$ and its decreasing rearrangement given by

$$u_* : [0,+\infty) \;\to\; [0,+\infty]\,, \quad s \;\mapsto\; u_*(s) = \inf\{\tau \ge 0 \;:\; \mu(t,\tau) \le s\}\,.$$

We adopt the following convention: for any time-dependent function $u : (0,+\infty)\times\mathbb{R}^2 \to [0,+\infty)$, we will also denote by u_* the decreasing rearrangement of u with respect to the spatial variable, that is, $u_*(t,s) = u(t,.)_*(s)$.

Rearrangement techniques are a standard tool in the study of partial differential equations: in the framework of chemotaxis, see for instance [75, 91, 106] in case of bounded domains, and [94] for unbounded domains. Let us briefly recall some properties of the decreasing rearrangement:

(a) For every measurable function $F : \mathbb{R}^+ \mapsto \mathbb{R}^+$, we have

$$\int_{\mathbb{R}^3} F(u)\,dx = \int_0^\infty F(u_*)\,dr\,.$$

In particular, if $u \in L^p(\mathbb{R}^N)$ with $1 \le p \le \infty$, then $\|u\|_{L^p(\mathbb{R}^N)} = \|u_*\|_{L^p(\mathbb{R}^N)}$.

(b) If $u \in W^{1,q}(0,T;L^p(\mathbb{R}^N))$ is a nonnegative function, with $1 \le p < \infty$ and $1 \le q \le \infty$, then $u_* \in W^{1,q}(0,T;L^p(0,\infty))$ and the formula

$$\int_0^{\mu(t,\tau)} \frac{\partial u_*}{\partial t}(t,\sigma)\,d\sigma = \int_{\{u(t,\cdot)>\tau\}} \frac{\partial u}{\partial t}(t,x)\,dx$$

holds for almost every $t \in (0,T)$. Here $\mu(t,\tau)$ denotes $|\{u(t,\cdot) > \tau\}|$. See [94, Theorem 2.2, (ii), p. 167] for a statement and a proof.

As in [94], let us consider a solution (n,c) of (6.1.3) and define

$$k(t,s) := \int_0^s n_*(t,\sigma)\,d\sigma$$

The strategy consists in finding a differential inequality for $k(t,s)$. Then, using a comparison principle, we will obtain an upper bound on the L^p norm of n. In [94], the method was applied to (6.1.1). Here we adapt it to the solution in rescaled variables, that is (6.1.3).

Lemma 12 *If n is a solution of* (6.1.3) *with initial datum n_0 satisfying the assumptions of Theorem 11, then the function $k(t,s)$ satisfies*

$$k \in L^\infty\left([0,+\infty)\times(0,+\infty)\right) \cap H^1\left([0,+\infty);W^{1,p}_{\rm loc}(0,+\infty)\right) \cap L^2\left([0,+\infty);W^{2,p}_{\rm loc}(0,+\infty)\right)$$

and

$$\begin{cases} \frac{\partial k}{\partial t} - 4\,\pi\,s\,\frac{\partial^2 k}{\partial s^2} - (k+2\,s)\,\frac{\partial k}{\partial s} \le 0 & \text{a.e. in } (0,+\infty)\times(0,+\infty)\,, \\ k(t,0) = 0\,, \quad k(t,+\infty) = \int_{\mathbb{R}^3} n_0\,dx & \text{for } t \in (0,+\infty)\,, \\ k(0,s) = \int_0^s (n_0)_*\,d\sigma & \text{for } s \ge 0\,. \end{cases}$$

Proof. The proof follows the method of [94, Proposition 3.1]. We will therefore only sketch the main steps that are needed to adapt the results to the setting of self-similar variables and refer to [94] for all technical details.

For $\tau \in (0, n_*(t,0))$ and $h > 0$, define the truncation function $T_{\tau,h}$ on $(-\infty,+\infty)$ by

$$T_{\tau,h} \;=\; \begin{cases} 0 & \text{if } s \le \tau \\ s-\tau & \text{if } \tau < s \le \tau + h \\ h & \text{if } \tau + h < s \end{cases}$$

and observe that $T_{\tau,h}(n(t,\cdot))$ belongs to $W^{1,p}(\mathbb{R}^2)$ since $n(t,\cdot) \in W^{1,p}(\mathbb{R}^2)$ and $T_{\tau,h}$ is Lipschitz continuous. Now we integrate (6.1.3) against $T_{\tau,h}(n)$ over $\mathbb{R}^2$, and integrate by parts to obtain

$$\int_{\mathbb{R}^3} \frac{\partial n}{\partial t}\,T_{\tau,h}(n)\,dx + \int_{\mathbb{R}^3} \nabla n\,\nabla T_{\tau,h}(n)\,dx = \int_{\mathbb{R}^3} n\,(\nabla c - x)\,\nabla T_{\tau,h}(n)\,dx\,.$$

We have that $|\{n=\tau\}| = 0$ for almost every $\tau \ge 0$. Hence one can prove that

$$\lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}^3} \frac{\partial n}{\partial t}\,T_{\tau,h}(n)\,dx = \int_{\{n>\tau\}} \frac{\partial n}{\partial t}(t,x)\,dx = \frac{\partial k}{\partial t}(t,\mu(t,\tau))\,.$$

Next we observe that

$$\begin{aligned}\lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}^3}\nabla n\,\nabla T_{\tau,h}(n)\,dx &= \lim_{h\to 0}\frac{1}{h}\left(\int_{\{n>\tau\}}|\nabla n|^2\,dx-\int_{\{n>\tau+h\}}|\nabla n|^2\,dx\right)\\ &= \frac{\partial}{\partial\tau}\int_{\{n>\tau\}}|\nabla n|^2\,dx\,.\end{aligned}$$

Consider the function

$$\Phi_{\tau,h}=\int_0^s\sigma\,\frac{\partial T_{\tau,h}}{\partial\sigma}(\sigma)\,d\sigma=\begin{cases}0 & \text{if } s\le\tau\,,\\ \frac{1}{2}\,(s^2-\tau^2) & \text{if } \tau<s\le\tau+h\,,\\ h(\tau+\frac{h}{2}) & \text{if } \tau+h<s\,.\end{cases}$$

Integrating the Poisson equation for c against $\Phi_{\tau,h}(n)$, we get

$$\int_{\mathbb{R}^3}\nabla c\,\nabla\Phi_{\tau,h}(n)\,dx=\int_{\mathbb{R}^3}n\,\nabla c\,\nabla T_{\tau,h}(n)\,dx=\int_{\mathbb{R}^3}n\,\Phi_{\tau,h}(n)\,dx\,,$$

thus proving that

$$\begin{aligned}&\lim_{h\to 0_+}\frac{1}{h}\int_{\mathbb{R}^3}n\,\nabla c\,\nabla T_{\tau,h}(n)\,dx\\ &=\lim_{h\to 0_+}\left(\frac{1}{2\,h}\int_{\{\tau<n\le\tau+h\}}n\,(n^2-\tau^2)\,dx+\int_{\{n>\tau+h\}}n\,\left(\tau+\tfrac{h}{2}\right)\,dx\right)\\ &=\tau\int_{\{n>\tau\}}n\,dx=\frac{\partial k}{\partial s}(t,\mu(t,\tau))\,k(t,\mu(t,\tau))\end{aligned}$$

since $\tau=n_*(t,\mu(t,\tau))=\frac{\partial k}{\partial s}(t,\mu(t,\tau))$ and $\int_{\{n>\tau\}}n\,dx=\int_0^{\mu(t,\tau)}n_*(t,s)\,ds=k(t,\mu(t,\tau))$. On the other hand,

$$\begin{aligned}\lim_{h\to 0_+}\frac{1}{h}\int_{\mathbb{R}^3}n(x)\,x\cdot\nabla T_{\tau,h}(x)\,dx\,dx &= \lim_{h\to 0_+}\frac{1}{h}\int_{\mathbb{R}^3}x\cdot\nabla\Phi_{\tau,h}(n)\,dx=-\lim_{h\to 0_+}\frac{2}{h}\int_{\mathbb{R}^3}\Phi_{\tau,h}(n)\,dx\\ &= -2\,\tau\,|\{n>\tau\}|\,=\,-2\,\frac{\partial k}{\partial s}(t,\mu(t,\tau))\,\mu(t,\tau)\,.\end{aligned}$$

Using the inequality

$$4\,\pi\,\mu(t,\tau)\le\frac{\partial\mu}{\partial\tau}(t,\tau)\,\frac{\partial}{\partial\tau}\int_{\{n>\tau\}}|\nabla n|^2\,dx\,,$$

(see [93, Proof of Lemma 4, p. 669], and also [106, pp. 25-26] or [91, p. 20], and [110] for an earlier reference) we obtain

$$1\le-\frac{\frac{\partial\mu}{\partial\tau}(t,\tau)}{4\,\pi\,\mu(t,\tau)}\,\left(-\frac{\partial k}{\partial t}(t,\mu(t,\tau))+\frac{\partial k}{\partial s}(t,\mu(t,\tau))\big(k(t,\mu(t,\tau))+2\,\mu(t,\tau)\big)\right)$$

for almost every $\tau\in(0,n_*(t,0))$. Integrating over $(\tau_1,\tau_2)\subset(0,n_*(t,0))$, as in [92, Lemma 2], we get

$$\frac{1}{4\,\pi}\int_{\mu(t,\tau_1)}^{\mu(t,\tau_2)}\left(-\frac{\partial k}{\partial t}(t,s)+\frac{\partial k}{\partial s}(t,s)\big(k(t,s)+2\,s\big)\right)\,\frac{ds}{s}\;\le\;\tau_1-\tau_2$$

where

$$\tau_1 - \tau_2 \;=\; \frac{\partial k}{\partial s}(t,\mu(t,\tau_1)) - \frac{\partial k}{\partial s}(t,\mu(t,\tau_2))\;.$$

Hence dividing by $(\mu(t,\tau_2)-\mu(t,\tau_2))$ and then taking the limit completes the proof.

The next result is adapted from [93, Proposition A.1, p. 676] and [94, Proposition 3.2, p. 172]. Although it is unnecessarily general for our purpose, as the function g below is extremely well defined (and independent of t), we keep it as in J.I. Díaz *et al.* and give a sketch of the proof, for completeness.

Proposition 13 *Let f, g be two continuous functions on $Q=\mathbb{R}^+\times(0,+\infty)$ such that*

(i) *f, $g\in L^\infty(Q)\cap L^2(0,+\infty;W^{2,2}_{\rm loc}(0,+\infty))$, $\frac{\partial f}{\partial t}$, $\frac{\partial g}{\partial t}\in L^2(0,+\infty;L^2_{\rm loc}(0,+\infty))$,*

(ii) *$\left|\frac{\partial f}{\partial s}(t,s)\right|\le C(t)$ and $\left|\frac{\partial g}{\partial s}(t,s)\right|\le C(t)\max\{s^{-1/2},1\}$, for some continuous function $t\mapsto C(t)$ on $\mathbb{R}^+$.*

If f and g satisfy

$$\left\{\begin{array}{c} \frac{\partial f}{\partial t}-4\,\pi\,s\,\frac{\partial^2 f}{\partial s^2}-(f+2\,s)\,\frac{\partial f}{\partial s}\le \frac{\partial g}{\partial t}-4\,\pi\,s\,\frac{\partial^2 g}{\partial s^2}-(g+2\,s)\,\frac{\partial g}{\partial s}\ \text{a.e. in } Q\;,\\ f(t,0)=0=g(t,0)\quad\text{and}\quad f(t,+\infty)\le g(t,+\infty)\ \text{for any } t\in(0,+\infty)\;,\\ f(0,s)\le g(0,s)\ \text{for } s\ge 0\;,\ \text{and } g(t,s)\ge 0\ \text{in } Q\;,\end{array}\right.$$

then $f\le g$ on Q.

Proof. Take $w=f-g$. We have

$$\frac{\partial w}{\partial t}-4\,\pi\,s\,\frac{\partial^2 w}{\partial s^2}-\,2\,s\,\frac{\partial w}{\partial s}\le w\,\frac{\partial f}{\partial s}+g\,\frac{\partial w}{\partial s}\,.$$

Multiplying by w_+/s, and integrating over (δ,L) with $0<\delta<1<L$, we obtain

$$\begin{aligned}\frac12\frac{\partial}{\partial t}\int_\delta^L\frac{w_+^2}{s}\,ds+4\,\pi\int_\delta^L\left(\frac{\partial w_+}{\partial s}\right)^2ds-\left[4\,\pi\,\frac{\partial w_+}{\partial s}(t,s)\,w_+(t,s)\right]_{s=\delta}^{s=L}-\int_\delta^L\frac{\partial}{\partial s}\left(w_+^2\right)\,ds\\ \le\int_\delta^L\left(w_+^2\,\frac{\partial f}{\partial s}+w_+\,\frac{\partial w}{\partial s}\,g\right)\,\frac{ds}{s}\end{aligned}$$

thus showing that

$$\frac12\,\frac{\partial}{\partial t}\int_\delta^L\frac{w_+^2}{s}\,ds+4\,\pi\int_\delta^L\left(\frac{\partial w_+}{\partial s}\right)^2ds\le C(t)\int_\delta^L\frac{w_+^2}{s}\,ds+\int_\delta^L\frac{w_+}{s}\,\frac{\partial w}{\partial s}\,g\,ds+G(t,\delta,L)\;,$$

where $G(t,\delta,L)$ uniformly (with respect to $t\ge0$) converges to 0 as $\delta\to0$ and $L\to+\infty$. Now using the fact that $g(t,s)/\sqrt{s}\le C(t)$ we obtain that, for some constant $K>0$,

$$\int_\delta^L\frac{w_+}{s}\,\frac{\partial w}{\partial s}\,g\,ds\le4\,\pi\int_\delta^L\left(\frac{\partial w_+}{\partial s}\right)^2ds+K\,C^2(t)\int_\delta^L\frac{w_+^2}{s}\,ds\;,$$

yielding

$$\frac12\frac{\partial}{\partial t}\int_\delta^L\frac{w_+^2}{s}\,ds\le(1+K\,C(t))\,C(t)\int_\delta^L\frac{w_+^2}{s}\,ds+G(t,\delta,L)\;.$$

From Gronwall's lemma and $w_+(0,s)=0$, with $R(t):=2\int_0^t(1+K\,C(\tau))\,C(\tau)\,d\tau$, it follows that

$$\int_\delta^L \frac{w_+^2}{s}\,ds \le 2\,e^{R(t)}\int_0^t e^{-R(\tau)}\,G(\tau,\delta,L)\,d\tau\;.$$

Taking the limit as $\delta\to 0$ and $L\to+\infty$, we obtain

$$\int_0^\infty \frac{w_+^2}{s}\,ds \le 0\;,$$

which implies $f\le g$ and concludes the proof.

Using Lemma 12 and Proposition 13, we can now establish uniforms bounds on $\|n\|_{L^p(\mathbb{R}^2)}$ and $\|\nabla c\|_{L^\infty(\mathbb{R}^2)}$.

Theorem 14 *Assume that $n_0\in L^2_+(n_\infty^{-1}\,dx)$ satisfies (6.1.5) and $M:=\int_{\mathbb{R}^3} n_0\,dx<8\,\pi$. Then there exist positive constants $C_1=C_1(M,p)$ and $C_2=C_2(M,p)$ such that*

$$\|n\|_{L^p(\mathbb{R}^2)}\le C_1\quad\textit{and}\quad\|\nabla c\|_{L^\infty(\mathbb{R}^2)}\le C_2\;.$$

Proof. The function $M_\varepsilon(s):=\int_{B(0,\sqrt{s/\pi})} n_{\infty,M+\varepsilon}\,dx$ satisfies

$$4\,\pi\,s\,M_\varepsilon''+2\,s\,M_\varepsilon'+M_\varepsilon\,M_\varepsilon'=0\;.$$

By direct application of Proposition 13, we obtain

$$k(t,s)\le M_\varepsilon(s)\quad\forall\,(t,s)\in Q\;.$$

By [75, p. 74] or [91, Lemma 1.33], we deduce

$$\|n_*\|_{L^p(0,\infty)}\le\|M_\varepsilon'\|_{L^p(0,\infty)}\;,$$

which yields the result. More details on M_ε and *cumulated densities* will be given in Section 6.4.5.

6.3 Uniform convergence

Wit the boundedness results of Section 6.2 in hands, we can now prove a result of uniform convergence for n and ∇c, if (n,c) is given as a solution of (6.1.3) satisfying the assumptions of Theorem 11.

Consider the kernel associated to the Fokker-Planck equation

$$K(t,x,y):=\frac{1}{2\pi\,(1-e^{-2\,t})}\,e^{-\frac12\,\frac{|x-e^{-t}y|^2}{1-e^{-2\,t}}}\quad x\in\mathbb{R}^2\,,\quad y\in\mathbb{R}^2\,,\quad t>0\;.$$

This definition deserves some explanations. If n is a solution of

$$\frac{\partial n}{\partial t}=\Delta n+\nabla\cdot(n\,x)$$

with initial datum n_0, then $u(\tau,\xi)=R^{-2}\,n\left(\log R,R^{-1}\,\xi\right)$ with $R=R(\tau)=\sqrt{1+2\,\tau}$ is a solution of the heat equation

$$\frac{\partial u}{\partial\tau}=\Delta u\;,\quad u(\tau=0,\cdot)=n_0\;,$$

whose solution is given by

$$u(\xi,\tau) = \frac{1}{4\,\pi\,\tau}\int_{\mathbb{R}^2} e^{-\frac{|\xi-y|^2}{4\tau}}\, n_0(y)\, dy\ .$$

By undoing the change of variables, we get that the solution of the Fokker-Planck equation is given by

$$n(t,x) = \int_{\mathbb{R}^2} K(t,x,y)\, n_0(y)\, dy\ .$$

Consider now a solution of (6.1.3). We have the following Duhamel formula.

Lemma 15 *Assume that n is a solution of* (6.1.3) *with initial data satisfying* (6.1.2)*. Then for any $t>0$, $x\in\mathbb{R}^2$, we have*

$$n(t,x) = \int_{\mathbb{R}^2} K(t,x,y)\, n_0(y)\, dy + \int_0^t\int_{\mathbb{R}^2} \nabla_x K(t-s,x,y)\cdot n(s,y)\,\nabla c(s,y)\, dy\, ds\ .$$

This is a standard fact whose proof relies on the fact that $(t,x)\mapsto K(t,x,y)$ is a solution of the Fokker-Planck equation with a δ-Dirac function initial value. Details are left to the reader.

Using the semi-group property, the expression for $n(2\,t,x)$ found in Lemma 15 can be written in terms of $n(t,x)$ as

$$n(2\,t,x) = \int_{\mathbb{R}^2} K(t,x,y)\, n(t,y)\, dy + \int_t^{2t}\int_{\mathbb{R}^2} \nabla_x K(2\,t-s,x,y)\cdot n(s,y)\,\nabla c(s,y)\, dy\, ds$$

for any $t\geq 0$. Since n_∞ is a stationary solution, we can also write that

$$n_\infty(x) = \int_{\mathbb{R}^2} K(t,x,y)\, n_\infty(y)\, dy + \int_t^{2t}\int_{\mathbb{R}^2} \nabla_x K(2\,t-s,x,y)\cdot n_\infty(y)\,\nabla c_\infty(y)\, dy\, ds$$

for any $t\geq 0$. By taking the difference of the two expressions written for $n(t,x)$ and n_∞ respectively, we get that

$$\begin{aligned} n(2\,t,x) - n_\infty(x) = {} & \int_{\mathbb{R}^2} K(t,x,y)\,(n(t,y)-n_\infty(y))\, dy\, ds \\ & + \int_t^{2t}\int_{\mathbb{R}^2} \nabla_x K(2\,t-s,x,y)\cdot (n(s,y)\,\nabla c(s,y) - n_\infty(y)\,\nabla c_\infty(y))\, dy\ . \end{aligned}$$

This provides a straightforward estimate, which goes as follows:

$$\begin{aligned} \|n(2\,t,\cdot)-n_\infty\|_{L^\infty(\mathbb{R}^2)} \leq {} & \|K(t,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}\, \|n(t,\cdot)-n_\infty\|_{L^1(\mathbb{R}^2)} \\ & + \int_0^t \|\nabla K(s,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^2_y;L^r(\mathbb{R}^2_x))}\, ds + \mathcal{R}(t) \end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$ with $p\in[1,\infty)$, $q\in[2,\infty)$ and $r\in(1,2)$, and where

$$\mathcal{R}(t) := \sup_{s\in(t,2\,t)} \left(\|n(s,\cdot)\|_{L^p(\mathbb{R}^2)}\, \|\nabla c(s,\cdot)-\nabla c_\infty\|_{L^q(\mathbb{R}^2)} + \|n(s,\cdot)-n_\infty\|_{L^p(\mathbb{R}^2)}\, \|\nabla c_\infty\|_{L^q(\mathbb{R}^2)}\right)$$

converges to 0 by Theorem 14 and the fact that

$$\lim_{t\to\infty} \|n(t,\cdot) - n_\infty\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{t\to\infty} \|\nabla c(t,\cdot) - \nabla c_\infty\|_{L^2(\mathbb{R}^2)} = 0$$

according to [84, Theorem 1.2]. Hence we have shown the *uniform convergence* of n towards n_∞ as $t \to \infty$ and

$$\lim_{t\to\infty} \|n(t,\cdot) - n_\infty\|_{L^p(\mathbb{R}^2)} = 0$$

for any $p \in [1,\infty]$, by Hölder's interpolation. As for the convergence of $\nabla c(t,\cdot)$ towards ∇c_∞ as $t \to \infty$ in $L^q(\mathbb{R}^2)$ for $q \in (2,\infty]$, we need one more interpolation inequality.

Lemma 16 *If $h = (-\Delta)^{-1}\rho$ for some function $\rho \in L^{2-\varepsilon} \cap L^{2+\varepsilon}(\mathbb{R}^2)$, with $\varepsilon \in (0,1)$, then there exists an explicit positive constant $C = C(\varepsilon)$ such that*

$$\|\nabla h\|_{L^\infty(\mathbb{R}^2)} \le C \left(\|\rho\|_{L^{2-\varepsilon}(\mathbb{R}^2)} + \|\rho\|_{L^{2+\varepsilon}(\mathbb{R}^2)} \right) .$$

Proof. This follows by a direct computation. We can estimate $|\nabla h|$ by

$$|\nabla h(x)| = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\rho(y)}{|x-y|}\, dy$$

for any $x \in \mathbb{R}^2$ and split the integral into two pieces corresponding to $|x-y| < 1$ and $|x-y| \ge 1$: by Hölder's inequality, we obtain that

$$\frac{1}{2\pi} \int_{|x-y|<1} \frac{\rho(y)}{|x-y|}\, dy \le C_1(\varepsilon)\, \|\rho\|_{L^{2+\varepsilon}(\mathbb{R}^2)}$$

with $C_1(\varepsilon) = \frac{1}{2\pi} \left(2\pi\,(1+\varepsilon)/\varepsilon\right)^{(1+\varepsilon)/(2+\varepsilon)}$ and

$$\frac{1}{2\pi} \int_{|x-y|\ge 1} \frac{\rho(y)}{|x-y|}\, dy \le C_2(\varepsilon)\, \|\rho\|_{L^{2-\varepsilon}(\mathbb{R}^2)}$$

with $C_2(\varepsilon) = \frac{1}{2\pi} \left(2\pi\,(1-\varepsilon)/\varepsilon\right)^{(1-\varepsilon)/(2-\varepsilon)}$. The conclusion holds with $C = \max_{i=1,2} C_i$.

Hence we have also shown the *uniform convergence* of ∇c towards ∇c_∞ as $t \to \infty$. By Hölder's interpolation, the convergence holds in $L^q(\mathbb{R}^2)$ for any $q \in [2,\infty]$. Summarizing all results of this section, we have shown the following limits.

Corollary 17 *Assume that n is a solution of* (6.1.3) *with initial data satisfying the assumptions of Theorem 11. Then*

$$\lim_{t\to\infty} \|n(t,\cdot) - n_\infty\|_{L^p(\mathbb{R}^2)} = 0 \quad \textit{and} \quad \lim_{t\to\infty} \|\nabla c(t,\cdot) - \nabla c_\infty\|_{L^q(\mathbb{R}^2)} = 0$$

for any $p \in [1,\infty]$ and any $q \in [2,\infty]$.

6.4 Spectral gap of the linearized operator $\mathcal{L}$

Assume that n is a solution of (6.1.3) and consider f and g defined for any $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^2$ by

$$n(t,x) = n_\infty(x)\,(1+f(t,x)) \quad \text{and} \quad c(t,x) = c_\infty(x)\,(1+g(t,x))\;.$$

Then (f,g) is a solution of the nonlinear problem

$$\left\{ \begin{array}{ll} \frac{\partial f}{\partial t} - \mathcal{L}\,f = -\frac{1}{n_\infty}\nabla\cdot[f\,n_\infty\,\nabla(g\,c_\infty)] & x\in\mathbb{R}^2\,,\quad t>0\,,\\ -\Delta(g\,c_\infty) = f\,n_\infty & x\in\mathbb{R}^2\,,\quad t>0\,, \end{array}\right. \tag{6.4.1}$$

where $\mathcal{L}$ is the linear operator

$$\mathcal{L}\,f = \frac{1}{n_\infty}\nabla\cdot[n_\infty\nabla(f-g\,c_\infty)]\,.$$

The goal of this section is to establish that $\mathcal{L}$ has a spectral gap in an appropriate functional setting. To characterize the spectrum of $\mathcal{L}$, it is indeed necessary to specify the domain of the operator $\mathcal{L}$. Heuristically, it is simpler to identify the eigenfunctions corresponding to the lowest eigenvalues and define only afterwards the norm for which $\mathcal{L}$ turns out to be self-adjoint. We will
- identify some eigenfunctions of the linearized Keller-Segel operator $\mathcal{L}$ in Section 6.4.1,
- characterize the kernel of $\mathcal{L}$ in Section 6.4.2,
- determine an adapted functional setting for $\mathcal{L}$ and related operators in Section 6.4.3,
- show that the spectrum of $\mathcal{L}$ is discrete in Section 6.4.4,
- and finally establish a spectral gap inequality in Section 6.4.5.

6.4.1 Some eigenfunctions of the linearized Keller-Segel operator $\mathcal{L}$

Using the fact that n_∞ depends on $x=(x_1,x_2)\in\mathbb{R}^2$ and on the mass parameter M, we observe that the functions

$$\begin{aligned} f_{0,0} &= \partial_M \log n_{\infty,M}\;,\\ f_{1,i} &= \partial_{x_i}\log n_{\infty,M}\;,\quad i=1\,,\,2\,,\\ f_{0,1} &= x\cdot\nabla\log n_{\infty,M}\;, \end{aligned}$$

are eigenfunctions of $\mathcal{L}$. Here $\partial_M n_{\infty,M}$ denotes the derivative of the function $n_\infty = n_{\infty,M}$ with respect to the mass parameter M, while ∂_{x_i} stands for ∂/∂_{x_i}. We shall use two indices for the numbering of the eigenfunctions because of a spherical harmonics decomposition that will be studied in Section 6.4.5. A precise statements goes as follows.

Lemma 18 *With the above notations, we have*

$$\begin{aligned} \mathcal{L}\,f_{0,0} &= 0\,,\\ \mathcal{L}\,f_{1,i} &= -f_{1,i}\;,\\ \mathcal{L}\,f_{0,1} &= -2\,f_{0,1}\,. \end{aligned}$$

Proof. Assume that $M \in (0, 8\,\pi)$ and consider the unique solution n_∞ of (6.1.4), which is also the unique stationary solution of (6.1.1) such that (6.1.2) holds. For brevity, we shall omit to mention the dependence of $n_\infty = n_{\infty,M}$ in M.

Let us differentiate with respect to M each term of $\Delta\, n_\infty + \nabla \cdot (n_\infty\, x) - \nabla \cdot (n_\infty \nabla c_\infty) = 0$, where $c_\infty = G_2 * n_\infty$. It is straightforward to check that $g_{0,0} := \partial_M \log c_\infty$ is such that $g_{0,0}\, c_\infty = G_2 * (f_{0,0}\, n_\infty)$ and $\mathcal{L}\, f_{0,0} = 0$. Since

$$-\Delta\, c_\infty = M\, \frac{e^{c_\infty - \frac{1}{2}\,|x|^2}}{\int_{\mathbb{R}^3} e^{c_\infty - \frac{1}{2}\,|x|^2}\, dx} = n_\infty\ ,$$

it is clear that $g_{0,0}$ is non-trivial, and therefore $f_{0,0} = \partial_M \log n_\infty$ is a non-trivial solution to $\mathcal{L}\, f = 0$.

By computing

$$0 = \frac{\partial}{\partial x_1}\Big(\Delta n_\infty + \nabla \cdot (x\, n_\infty) - \nabla \cdot (n_\infty\, \nabla c_\infty)\Big) \quad \text{with} \quad -\,\Delta\left(\frac{\partial c_\infty}{\partial x_1}\right) = \frac{\partial n_\infty}{\partial x_1}$$

and observing that

$$\frac{\partial}{\partial x_1} \nabla \cdot (x\, n_\infty) = \frac{\partial n_\infty}{\partial x_1} + \nabla \cdot \left(x\, \frac{\partial n_\infty}{\partial x_1}\right)\ ,$$

we obtain that $f_{1,1} := \partial_{x_1} \log n_\infty$ associated with $g_{1,1} = \frac{1}{c_\infty}\, \partial_{x_1} c_\infty$ is an eigenfunction of $\mathcal{L}$, such that $-\mathcal{L}\, f_{1,1} = f_{1,1}$. The same observation holds if we differentiate with respect to x_i, $i = 2$.

Next consider the dilation operator $D := x \cdot \nabla$. If a is a vector valued function, an elementary computation shows that

$$D\,(\nabla \cdot a) = \nabla \cdot (D\, a) - \nabla \cdot a\ .$$

Since $a = \nabla n_\infty + x\, n_\infty - n_\infty\, \nabla c_\infty$ is such that $\nabla \cdot a = 0$, we get $D\,(\nabla \cdot a) = \nabla \cdot (D\, a)$ and hence

$$0 = D\Big(\Delta n_\infty + \nabla \cdot (x\, n_\infty) - \nabla \cdot (n_\infty\, \nabla c_\infty)\Big) = \nabla \cdot D\left(\nabla n_\infty + x\, n_\infty - n_\infty\, \nabla c_\infty\right).$$

Next, we observe that

$$D\,(\nabla n_\infty) = \nabla\,(D\, n_\infty) - \nabla\, n_\infty$$

so that

$$\nabla \cdot D\,(\nabla n_\infty) = \Delta(D\, n_\infty) - \Delta\, n_\infty\ .$$

It is also straightforward to observe that

$$D\,(x\, n_\infty) = x\, n_\infty + x\, D\, n_\infty \quad \text{and} \quad D\,(\nabla c_\infty) = \nabla\,(D\, c_\infty) - \nabla\, c_\infty\ .$$

Let $f_{0,1} = 1 + \frac{1}{2}\, D\, \log n_\infty = 1 + \frac{1}{2\, n_\infty}\, D\, n_\infty$. By writing $D\,(\Delta c_\infty + n_\infty) = 0$, we get

$$-\Delta\,(D\, c_\infty) + 2\, \Delta c_\infty = D\, n_\infty = 2\,(f_{0,1} - 1)\, n_\infty\ ,$$

since

$$D\,(\Delta c) = \Delta(D\, c) - \,2\, \Delta c\ .$$

Hence, using the fact that $2\, \Delta c_\infty = -\,2\, n_\infty$, the function $g_{0,1} := \frac{1}{c_\infty}\, (-\Delta)^{-1}(n_\infty\, f_{0,1})$ is given by

$$c_\infty\, g_{0,1} = \frac{1}{2}\, D\, c_\infty\ .$$

Collecting these identities, we have found that

$$2\,n_\infty\,\mathcal{L}\,(D\,\log n_\infty) - \nabla\cdot\Big[\nabla n_\infty - x\,n_\infty - \,2\,n_\infty\Big(\nabla(c_\infty\,g_{0,1}) - \nabla c_\infty\Big) + n_\infty\,D\,(\nabla c_\infty)\Big] = 0\;.$$

Using

$$2\Big(\nabla(c_\infty\,g_{0,1}) - \nabla c_\infty\Big) = \nabla(D\,c_\infty) - \,2\,\nabla c_\infty = D\,(\nabla c_\infty) - \nabla c_\infty\;,$$

this gives

$$n_\infty\,\mathcal{L}\,(D\,\log n_\infty) - \Delta n_\infty + \nabla\cdot(x\,n_\infty - n_\infty\,\nabla c_\infty) = 0\;.$$

Hence, owing to the fact that $D\,\log n_\infty = 2\,(f_{0,1} - 1)$ and

$$n_\infty\,\mathcal{L}\,(D\,\log n_\infty) = 2\,n_\infty\,\mathcal{L} f_{0,1} + 2\,\nabla\cdot(n_\infty\nabla c_\infty)\;,$$

we get

$$\begin{aligned}-\,2\,n_\infty\,\mathcal{L}\,f_{0,1} &= -\Delta n_\infty + \nabla\cdot(x\,n_\infty + n_\infty\,\nabla c_\infty)\\ &= 2\,\nabla\cdot(x\,n_\infty) = 4\,n_\infty\,\left(1 + \frac{D\,n_\infty}{2\,n_\infty}\right) = 4\,n_\infty\,f_{0,1}\;.\end{aligned}$$

We have finally found that $-\mathcal{L}\,f_{0,1} = 2\,f_{0,1}$, which completes the proof.

Remark 1 *The fact that* 1 *and* 2 *are eigenvalues of* $\mathcal{L}$ *was known in the limit* $M \to 0_+$*: see [82]. It is remarkable that these two eigenvalues are independent of* M *but this can be explained by noticing that the corresponding eigenfunctions are associated with invariances of the problem before rescaling.*

The functions $\partial_{x_i}\log n_\infty$*,* $i = 1,\,2$ *correspond to the invariance under translation in the directions* x_i*. A decentered self-similar solution would converge in self-similar variables to the stationary solution, in relative entropy, exactly at a rate* e^{-t}*, thus showing that* $\lambda_{1,1} = \lambda_{1,2} = 1$ *are eigenvalues by considering the asymptotic regime.*

The function $D\,\log n_\infty$ *is associated with the scaling invariance. In original variables, a scaling factor corresponds to a translation in time at the level of the self-similar solution and it can easily be checked that, in self-similar variables, a solution corresponding to the stationary solution rescaled by a factor different from* 1 *converges, in relative entropy, exactly at a rate* $e^{-\,2\,t}$*, thus showing that* $\lambda_{0,1} = 2$ *is also an eigenvalue by considering the asymptotic regime.*

6.4.2 The kernel of the linearized Keller-Segel operator $\mathcal{L}$

By definition of n_∞, we know that $\log n_\infty = \mu_0(M) + c_\infty - \frac{1}{2}\,|x|^2$, so that $f_{0,0} = \mu_0'(M) + g_{0,0}\,c_\infty$ where $g_{0,0} = \partial_M \log c_\infty$ is such that $-\Delta(g_{0,0}\,c_\infty) = -\Delta f_{0,0} = f_{0,0}\,n_\infty$. The normalization constant μ_0 is determined by the condition that $M = \int_{\mathbb{R}^3} n_\infty\;dx$, that is $\mu_0 = \log M - \log\big(\int_{\mathbb{R}^3} e^{c_\infty - |x|^2/2}\;dx\big)$. By differentiating with respect to M, we also get that

$$\mu_0'(M) = \frac{1}{M}\left[1 - \int_{\mathbb{R}^3} g_{0,0}\,n_\infty\,c_\infty\;dx\right]\;.$$

The function $f = f_{0,0}$ solves $\mathcal{L}\,f = 0$ and is such that the equation for $g = f/c_\infty$ reads

$$-\Delta\,f = n_\infty\,f\;.$$

It is not *a priori* granted that such an equation has at most one solution, up to a multiplication by a constant. The uniqueness issue is the purpose of our next result.

Proposition 4 *The kernel $Ker(\mathcal{L})$ is generated by $f_{0,0} = \partial_M \log n_\infty$, which is the unique solution in $L^2(\mathbb{R}^2, n_\infty\, dx)$, up to a multiplication by a constant, to*

$$-\Delta f_{0,0} = f_{0,0}\, n_\infty \ .$$

Proof. We have already seen that $f_{0,0} \in Ker(\mathcal{L})$. It remains to prove that $Ker(\mathcal{L})$ is one-dimensional. Let f be such that $\mathcal{L}\, f = 0$ and $g = c_\infty^{-1}\, G_2 * (f\, n_\infty)$. An elementary computation shows that

$$0 = \int_{\mathbb{R}^3} \mathcal{L}\, f\, (f - g\, c_\infty)\, n_\infty\, dx = \int_{\mathbb{R}^3} |\nabla (f - g\, c_\infty)|^2\, n_\infty\, dx \ ,$$

thus proving that $f = g\, c_\infty + \mu_0'$ for some real constant μ_0' (depending eventually on M, with the same notations as above). Hence any solution of $\mathcal{L}\, f = 0$ has to solve

$$\mathcal{H}\, f = 0$$

where $\mathcal{H} := -\Delta - n_\infty$ is a Schrödinger operator with potential n_∞, at least if one assumes that $\nabla(f - G_2 * (f\, n_\infty))$ belongs to $L^2(n_\infty\, dx)$. As we shall see later in the discussion of the domain of definition of $\mathcal{L}$, this is indeed a natural assumption. Altogether, we are interested in characterizing the ground state of the Schrödinger operator $\mathcal{H}$ (with energy level 0) and prove that it is uniquely determined, up to a multiplication by a constant. It is clear that $\mathcal{H}$ has no negative eigenvalue, otherwise the *free energy* functional

$$F[n] := \int_{\mathbb{R}^3} n\, \log\left(\frac{n}{n_\infty}\right) dx + \frac{1}{4\,\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (n(x) - n_\infty(x))\, \log|x-y|\, (n(y) - n_\infty(y))\, dx\, dy$$

would not achieve its minimum for $n = n_\infty$ (see [84] for a proof).

Since n_∞ is radially symmetric (see for instance [84] for a summary of known results), Schwarz' symmetrization applied to $\mathcal{H}$ shows that the ground state is radially symmetric. The function n_∞ seen as a potential, is smooth. By standard elliptic theory, the ground state is smooth as well. Hence, if $f \in H^1(\mathbb{R}^2)$ solves $\mathcal{H}\, f = 0$, it is uniquely determined as a solution of an ordinary differential equation by the Cauchy-Lipschitz theorem, up to a standard analysis at the origin. Indeed, by considering abusively n_∞ and f as functions of $r = |x|$, we find that f is given by

$$\begin{aligned} &f'' + \tfrac{1}{r}\, f' + n_\infty\, f = 0 \\ &f(0) = 1 \ , \quad f'(0) = 0 \end{aligned}$$

(up to a multiplication by an arbitrary constant). This concludes the proof.

6.4.3 Functional setting and operators

In order to go further in the spectral analysis, to define correctly the domain of the operator $\mathcal{L}$, to justify the assumption that $\nabla(f - G_2 * (f\, n_\infty))$ belongs to $L^2(n_\infty\, dx)$ and to establish spectral gap estimates which are crucial for our analysis, some considerations on the functional setting are in order.

Lemma 19 *Assume that $M \in (0, 8\,\pi)$ and consider n_∞ defined by* (6.1.4)*. Then*

$$-\frac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x)\, n_\infty(x)\, \log|x-y|\, f(y)\, n_\infty(y)\, dx\, dy = \int_{\mathbb{R}^3} f\, n_\infty\, g\, c_\infty\, dx \le \int_{\mathbb{R}^3} |f|^2\, n_\infty\, dx \qquad (6.4.2)$$

for any $f \in L^2(\mathbb{R}^2, n_\infty\,dx)$*, where* $g\,c_\infty = G_2 * (f\,n_\infty)$*. Moreover, if*

$$\int_{\mathbb{R}^3} f\,f_{0,0}\,n_\infty\,dx = 0\;, \tag{6.4.3}$$

then equality holds in the above inequality if and only if $f = 0$*.*

Notice that, if $f \in L^2(\mathbb{R}^2, n_\infty\,dx)$ is such that

$$\int_{\mathbb{R}^3} f\,n_\infty\,dx = 0\;, \tag{6.4.4}$$

then (6.4.2) can be written as

$$\int_{\mathbb{R}^3} |\nabla(g\,c_\infty)|^2\,dx \le \int_{\mathbb{R}^3} |f|^2\,n_\infty\,dx\;. \tag{6.4.5}$$

It is indeed well known that $\nabla(g\,c_\infty)$ is in $L^2(\mathbb{R}^2, dx)$ as a solution of $-\Delta(g\,c_\infty) = f\,n_\infty$ only if (6.4.4) holds. Lemma 19 will be improved in Section 6.5 (see Corollary 26); the proof of such a result is independent of the remainder of this section.

Proof. To prove the inequality, we recall that the free energy $n \mapsto F[n]$ achieves its minimum for $n = n_\infty$ according to the logarithmic Hardy-Littlewood-Sobolev inequality (see [95] for detailed considerations on this formulation of the inequality), and observe that

$$\mathsf{Q}_1[f] := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\,F[n_\infty(1+\varepsilon\,f)] \ge 0$$

for any smooth function f with compact support satisfying (6.4.3). The inequality then holds for any $f \in L^2(\mathbb{R}^2, n_\infty\,dx)$ by density of smooth functions with compact support in $L^2(\mathbb{R}^2, n_\infty\,dx)$.

If equality holds in (6.4.2), then the Euler-Lagrange equation amounts to $-\Delta f = f\,n_\infty$, which characterizes the kernel $Ker(\mathcal{L})$ according to Proposition 4.

Consider the quadratic form Q_1 on $L^2(\mathbb{R}^2, n_\infty\,dx)$, which takes the form

$$\mathsf{Q}_1[f] = \int_{\mathbb{R}^3} |f|^2\,n_\infty\,dx + \frac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\,n_\infty(x)\,\log|x-y|\,f(y)\,n_\infty(y)\,dx\,dy\;.$$

By (6.4.2), it is nonnegative, and positive semi-definite on the orthogonal of the kernel of $\mathcal{L}$, for the natural scalar product on $L^2(n_\infty\,dx)$, *i.e.* for any $f \in L^2(n_\infty\,dx)$ such that (6.4.3) holds. Using previous notations, we may write

$$\mathsf{Q}_1[f] = \int_{\mathbb{R}^3} f\,(f - g\,c_\infty)\,n_\infty\,dx \quad \text{with} \quad g\,c_\infty = G_2 * (f\,n_\infty)\;.$$

If (6.4.4) holds, we can also observe that

$$\mathsf{Q}_1[f] = \int_{\mathbb{R}^3} |f|^2\,n_\infty\,dx - \int_{\mathbb{R}^3} |\nabla(g\,c_\infty)|^2\,dx\;.$$

To Q_1 we associate its polar form L_1 defined on smooth functions with compact support such that (6.4.3) holds and define its Friedrich's extension on $L^2(n_\infty\,dx)$, that we still denote by L_1. By construction,

L_1 is a positive self-adjoint operator with domain $\mathcal{D}(\mathsf{L}_1) \subset L^2(n_\infty\,dx)$. On $\mathcal{D}(\mathsf{L}_1)$, we shall denote by $\langle\cdot,\cdot\rangle$ the scalar product induced by L_1. Explicitly, this means that

$$\langle f, \tilde f\rangle = \int_{\mathbb{R}^3} f\,\tilde f\,n_\infty\,dx + \frac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\,n_\infty(x)\,\log|x-y|\,\tilde f(y)\,n_\infty(y)\,dx\,dy\;.$$

The scalar product $\langle\cdot,\cdot\rangle$ induced by L_1 is defined on the orthogonal of $f_{0,0}$, but can be extended as a bilinear form to $L^2(n_\infty\,dx)$. If $f \in L^2(n_\infty\,dx)$ is such that (6.4.4) holds, then we notice that

$$\langle f, f_{0,0}\rangle = \int_{\mathbb{R}^3} f\,(f_{0,0} - G_2 * f_{0,0})\,n_\infty\,dx = 0$$

because $f_{0,0} = G_2 * (f_{0,0}\,n_\infty) + \mu_0'$. With these notations, notice that we have

$$\langle f, f\rangle = \mathsf{Q}_1[f] \ge 0$$

for any $f \in \mathcal{D}(\mathsf{L}_1)$, with equality if and only if $f = 0$.

We can also define the quadratic form Q_2 as

$$\mathsf{Q}_2[f] := \int_{\mathbb{R}^3} |\nabla(f - g\,c_\infty)|^2\,n_\infty\,dx \quad\text{with}\quad g = \frac{1}{c_\infty}\,G_2 * (f\,n_\infty)\;.$$

As for Q_1, we define Q_2 on the set of smooth functions such that (6.4.3) holds and extend it. The associated self-adjoint nonnegative operator is denoted by L_2 and it is again a self-adjoint operator, with domain $\mathcal{D}(\mathsf{L}_2) \subset L^2(n_\infty\,dx)$.

Proposition 5 *With the above notations, the restriction of $\mathcal{L}$ to $\mathcal{D}(\mathsf{L}_1)$ is a self-adjoint operator for the scalar product $\langle\cdot,\cdot\rangle$ with domain $\mathcal{D}(\mathsf{L}_2)$, such that*

$$\langle f, \mathcal{L}\,f\rangle = -\,\mathsf{Q}_2[f] \quad \forall\, f \in \mathcal{D}(\mathsf{L}_2)$$

and $\mathrm{Ker}(\mathcal{L}) \cap \mathcal{D}(\mathsf{L}_2) = \{0\}$.

Remark 2 *The function $f_{0,0}$ is an eigenfunction of $\mathcal{L}$ but this is not the case of $f \equiv 1$. With the notations of Section 6.4.1, the functions $f_{1,i}$ are orthogonal to $f \equiv 1$ in $L^2(\mathbb{R}^2, n_\infty\,dx)$ for $i = 1,\,2$, but this is the case neither for $f_{0,0}$ nor for $f_{0,1}$.*

6.4.4 The spectrum of $\mathcal{L}$ is discrete

We define

$$\Lambda_1 := \inf_{f\in\mathcal{D}(\mathsf{L}_2)\setminus\{0\}} \frac{\mathsf{Q}_2[f]}{\mathsf{Q}_1[f]} \quad\text{and}\quad \Lambda_\infty := \lim_{R\to\infty} \inf_{\substack{f\,\in\,\mathcal{D}(\mathsf{L}_2)\,\setminus\,\{0\}\\ \mathrm{supp}(f)\,\subset\,\mathbb{R}^2\,\setminus\,B(0,R)}} \frac{\mathsf{Q}_2[f]}{\mathsf{Q}_1[f]}\;.$$

First, let us give a heuristic approach of the problem. As an application of Persson's method (see [108]), the bottom of the essential spectrum of $\mathcal{L}$ can be characterized as

$$\inf\sigma_{\mathrm{ess}}(\mathcal{L}) = \Lambda_\infty\;.$$

To prove that $\mathcal{L}$ has a spectral gap on $\mathcal{D}(\mathsf{L}_2)$, it is enough to show that Λ_∞ is positive: either $\Lambda_1 = \Lambda_\infty$, or $\Lambda_1 < \Lambda_\infty$ is a nonnegative eigenvalue, which cannot be equal to 0. This is summarized in the following statement.

Proposition 20 *With the above notations, Λ_1 is positive and*

$$\Lambda_1\,\mathsf{Q}_1[f]\le\mathsf{Q}_2[f]\quad\forall\,f\in\mathcal{D}(\mathsf{L}_2)\;. \tag{6.4.6}$$

For any $f\in\mathcal{D}(\mathsf{L}_2)$, if (6.4.4) holds, then Inequality (6.4.6) can be reformulated as

$$\Lambda_1\int_{\mathbb{R}^3}|f|^2\,n_\infty\,dx\le\int_{\mathbb{R}^3}|\nabla(f-g\,c_\infty)|^2\,n_\infty\,dx+\Lambda_1\int_{\mathbb{R}^3}|\nabla(g\,c_\infty)|^2\,dx\;.$$

The proof of Proposition 20 can be done by considering $\mathcal{L}$ as a perturbation of the operator $f\mapsto n_\infty^{-1}\,\nabla\cdot(n_\infty\,\nabla f)$ defined on $L^2(\mathbb{R}^2,n_\infty\,dx)$. This was the method of [82]. However on such a space $\mathcal{L}$ is not self-adjoint and justifications are delicate because of the logarithmic kernel, away from the small mass regime.

In practice, Persson's method is not well designed either to handle convolution operators, although it can probably be adapted with little effort. This may even have been done, but we are not aware of such a result. Moreover, as we shall see below, we have: $\Lambda_\infty=\infty$, which further simplifies the proof. For these reasons, we will therefore give a direct proof, based on some of the tools of the concentration-compactness method (see [101, 102, 103, 104, 105]) and adapted to the case of a bounded measure, $n_\infty\,dx$, as in [83]. In that framework, Λ_∞ corresponds to the *problem at infinity.* For simplicity, let us split the proof into Lemmas 21 and 22.

Lemma 21 *With the above notations, $\Lambda_\infty=\infty$.*

Proof. Recall that

$$n_\infty=M\,\frac{e^{c_\infty-\frac12\,|x|^2}}{\int_{\mathbb{R}^3}e^{c_\infty-\frac12\,|x|^2}\,dx}$$

where $c_\infty=(-\Delta)^{-1}\,n_\infty$ is such that

$$\limsup_{|x|\to\infty}\Big|\,c_\infty(x)+\frac{M}{2\pi}\,\log|x|\,\Big|<\infty\;.$$

As a consequence, we know that

$$n_\infty(x)\sim|x|^{-\alpha}\,e^{-\frac12\,|x|^2}\quad\text{as}\quad|x|\to+\infty\;,\quad\text{with}\quad\alpha=\frac{M}{2\pi}\;.$$

We can expand the square $|\nabla(f-g\,c_\infty)|^2$ and get

$$\begin{aligned}\mathsf{Q}_2[f]=\int_{\mathbb{R}^3}|\nabla(f-g\,c_\infty)|^2\,n_\infty\,dx\\ =\int_{\mathbb{R}^3}|\nabla f|^2\,n_\infty\,dx+\int_{\mathbb{R}^3}|\nabla(g\,c_\infty)|^2\,n_\infty\,dx\\ +2\int_{\mathbb{R}^3}f\,\nabla(g\,c_\infty)\cdot\nabla n_\infty\,dx-2\int_{\mathbb{R}^3}f\,\big(-\Delta(g\,c_\infty)\big)\,n_\infty\,dx\;.\end{aligned}$$

Assume that f is supported in $\mathbb{R}^2\setminus B(0,R)$, for $R>0$, large. Then

$$\begin{aligned}\int_{\mathbb{R}^3}f\,\big(-\Delta(g\,c_\infty)\big)\,n_\infty\,dx=\int_{\mathbb{R}^3}|f|^2\,n_\infty^2\,dx\\ \le\sup_{|x|>R}n_\infty(x)\int_{\mathbb{R}^3}|f|^2\,n_\infty\,dx\sim R^{-\alpha}\,e^{-\frac12\,R^2}\int_{\mathbb{R}^3}|f|^2\,n_\infty\,dx\end{aligned}$$

on the one hand, and we know from Persson's method that

$$\lim_{R\to\infty} \inf_{\substack{f\in\mathcal{D}(\mathsf{L}_2)\setminus\{0\}\\ \mathrm{supp}(f)\subset\mathbb{R}^2\setminus B(0,R)}} \frac{\int_{\mathbb{R}^3}|\nabla f|^2\,n_\infty\,dx}{\int_{\mathbb{R}^3}|f|^2\,n_\infty\,dx} = +\infty$$

on the other hand, so that, for any $\varepsilon>0$, there exists $R>0$ large enough for which

$$\int_{\mathbb{R}^3}|f|^2\,n_\infty\,dx \le \varepsilon\int_{\mathbb{R}^3}|\nabla f|^2\,n_\infty\,dx$$

for any function $f\in H^1(\mathbb{R}^2,n_\infty\,dx)$. Equivalently, we can write that there exists a positive function $R\mapsto\varepsilon(R)$ such that $\lim_{R\to+\infty}\varepsilon(R)=0$ and

$$0\le\int_{\mathbb{R}^3} f\,\big(-\Delta(g\,c_\infty)\big)\,n_\infty\,dx \le R^{-\alpha}\,e^{-\frac12\,R^2}\,\varepsilon(R)$$

uniformly with respect to f as soon as it is supported in $\mathbb{R}^2\setminus B(0,R)$.

Assume first that Condition (6.4.4) is satisfied. We notice that

$$\left|\int_{\mathbb{R}^3} f\,\nabla(g\,c_\infty)\cdot\nabla n_\infty\,dx\right| \le 2\int_{\mathbb{R}^3}|f\,\sqrt{n_\infty}|\;|\nabla(g\,c_\infty)|\;|(\nabla c_\infty-x)\,\sqrt{n_\infty}|\,dx$$

can be estimated by

$$\left|\int_{\mathbb{R}^3} f\,\nabla(g\,c_\infty)\cdot\nabla n_\infty\,dx\right| \le 2\sup_{|x|>R}\left|(\nabla c_\infty-x)\,\sqrt{n}_\infty\right|\left(\int_{\mathbb{R}^3}|f|^2\,n_\infty\,dx\int_{\mathbb{R}^3}|\nabla(g\,c_\infty)|^2\,dx\right)^{1/2}.$$

By Lemma 19, we know that $\mathsf{Q}_1[f]\ge0$ is equivalent to (6.4.5) and find that

$$\begin{aligned}\left|\int_{\mathbb{R}^3} f\,\nabla(g\,c_\infty)\cdot\nabla n_\infty\,dx\right| &\le 2\sup_{|x|>R}\left|(\nabla c_\infty-x)\,\sqrt{n}_\infty\right|\int_{\mathbb{R}^3}|f|^2\,n_\infty\,dx\\ &\le 2\sup_{|x|>R}\left|(\nabla c_\infty-x)\,\sqrt{n}_\infty\right|\,\varepsilon(R)\int_{\mathbb{R}^3}|\nabla f|^2\,n_\infty\,dx\,.\end{aligned}$$

On the other hand, since Condition (6.4.4) is satisfied, we know for free that

$$\mathsf{Q}_1[f]=\int_{\mathbb{R}^3}|f|^2\,n_\infty\,dx-\int_{\mathbb{R}^3}|\nabla(g\,c_\infty)|^2\,dx\le\int_{\mathbb{R}^3}|f|^2\,n_\infty\,dx\,.$$

As a consequence, we have obtained that

$$\lim_{R\to\infty}\inf_{\substack{f\in\mathcal{D}(\mathsf{L}_2)\setminus\{0\}\\ \mathrm{supp}(f)\subset\mathbb{R}^2\setminus B(0,R)}}\frac{\mathsf{Q}_2[f]}{\mathsf{Q}_1[f]}=\lim_{R\to\infty}\inf_{\substack{f\in\mathcal{D}(\mathsf{L}_2)\setminus\{0\}\\ \mathrm{supp}(f)\subset\mathbb{R}^2\setminus B(0,R)}}\frac{\int_{\mathbb{R}^3}|\nabla f|^2\,n_\infty\,dx}{\int_{\mathbb{R}^3}|f|^2\,n_\infty\,dx}=+\infty\,,$$

which proves our claim.

If Condition (6.4.4) is not satisfied, the proof is more complicated. By homogeneity, there is no restriction to assume that $\frac1M\int_{\mathbb{R}^3}f^2\,n_\infty\,dx=1$. Let $\theta:=\frac1M\int_{\mathbb{R}^3}f\,n_\infty\,dx$ and $\tilde f:=f-\theta$, $\tilde g:=g-\theta$.

Then $\int_{\mathbb{R}^3} \tilde{f}\, n_\infty\, dx = 0$. Notice that, by the Cauchy-Schwarz inequality, we have: $\theta \in [-1,1]$. Moreover, if $\theta \neq 0$, then $B(0,R)$ is contained in $\mathrm{supp}(\tilde{f})$.

With these notations, we first have to estimate

$$\int_{\mathbb{R}^3} f\, \nabla(g\, c_\infty)\cdot\nabla n_\infty\, dx = 2\,\theta \int_{\mathbb{R}^3} f\,\sqrt{n}_\infty\, \nabla c_\infty\cdot\nabla\sqrt{n}_\infty\, dx + 2\int_{\mathbb{R}^3} f\,\sqrt{n}_\infty\,\nabla(\tilde{g}\, c_\infty)\cdot\nabla\sqrt{n}_\infty\, dx\;.$$

By the Cauchy-Schwarz inequality, we have

$$\left|\int_{\mathbb{R}^3} f\,\nabla c_\infty\cdot\nabla\sqrt{n}_\infty\, dx\right|^2 \le \int_{\mathbb{R}^3} f^2\, n_\infty\, dx \int_{\mathbb{R}^2\setminus B(0,R)} |\nabla c_\infty|^2\,|\nabla\sqrt{n}_\infty|^2\, dx\;,$$

and it is simple to check that the last integral in the right hand side converges to 0 as $R\to\infty$. The second integral can be estimated as before by writing

$$\left|\int_{\mathbb{R}^3} f\,\sqrt{n}_\infty\,\nabla(\tilde{g}\, c_\infty)\cdot\nabla\sqrt{n}_\infty\, dx\right|^2 \le \sup_{|x|>R}\left|(\nabla c_\infty - x)\,\sqrt{n}_\infty\right|^2 \int_{\mathbb{R}^3} |f|^2\, n_\infty\, dx \int_{\mathbb{R}^3} |\nabla(\tilde{g}\, c_\infty)|^2\, dx$$

and by recalling that $\int_{\mathbb{R}^3} |\tilde{f}|^2\, n_\infty\, dx = M$. From these estimates, we conclude that

$$\lim_{R\to\infty} \inf_{\substack{f\,\in\,\mathcal{D}(\mathsf{L}_2)\setminus\{0\}\\ \mathrm{supp}(f)\,\subset\,\mathbb{R}^2\setminus B(0,R)}} \mathsf{Q}_2[f] = \infty\;.$$

We also need to estimate $\mathsf{Q}_1[f]$ and this can be done by showing that

$$\mathsf{Q}_1[f] = M + \frac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2} (f\, n_\infty)(x)\,\log|x-y|\,(f\, n_\infty)(y)\, dx\, dy$$

is bounded from above if we still impose that $\int_{\mathbb{R}^3} |f|^2\, n_\infty\, dx = M$. Using the crude estimate

$$2\,\log|x-y| \le |x-y|^2 \le 2\,(|x|^2+|y|^2)\quad \forall\,(x,y)\in\mathbb{R}^2\times\mathbb{R}^2$$

and, as a consequence,

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} (f\, n_\infty)(x)\,\log|x-y|\,(f\, n_\infty)(y)\, dx\, dy \le 2\int_{\mathbb{R}^3} f\, n_\infty\, dx \int_{\mathbb{R}^2} |y|^2\,(f\, n_\infty)(y)\, dy\;,$$

we conclude by observing that

$$\int_{\mathbb{R}^3} f\, n_\infty\, dx \le \sqrt{M}\left(\int_{\mathbb{R}^2\setminus B(0,R)} n_\infty\, dx\right)^{1/2}$$

$$\text{and}\quad \int_{\mathbb{R}^2} |y|^2\,(f\, n_\infty)(y)\, dy \le \sqrt{M}\left(\int_{\mathbb{R}^2\setminus B(0,R)} |y|^4\, n_\infty\, dy\right)^{1/2}$$

both converge to 0 as $R\to\infty$.

Lemma 22 *With the above notations, $\Lambda_1 > 0$.*

Proof. Tools for the proof of this lemma are to a large extent standard in concentration-compactness methods or when applied to models of quantum chemistry, so we shall only sketch the main steps and omit as much as possible the technicalities of such an approach. We will actually prove a result that is stronger than the one of Lemma 22: Λ_1 is achieved by some function $f \in \mathcal{D}(\mathsf{L}_2)$.

Consider a minimizing sequence $(f_n)_{n\in\mathbb{N}}$ for the functional $f \mapsto \mathsf{Q}_2[f]/\mathsf{Q}_1[f]$ defined on $\mathcal{D}(\mathsf{L}_2) \setminus \{0\}$. By homogeneity, we may assume that $\int_{\mathbb{R}^3} f^2\, n_\infty\, dx = 1$ for any $n \in \mathbb{N}$, with no restriction, while $\mathsf{Q}_2[f_n]$ is bounded uniformly in $n \in \mathbb{N}$. Let $F_n := f_n \sqrt{n_\infty}$. In the framework of concentration-compactness methods, for any given $\varepsilon > 0$, it is a standard result that one can decompose F_n as

$$F_n = F_n^{(1)} + F_n^{(2)} + \widetilde{F}_n$$

for any $n \in \mathbb{N}$, with

$$\int_{\mathbb{R}^3} |F_n^{(1)}|^2\, dx + \int_{\mathbb{R}^3} |F_n^{(2)}|^2\, dx + \int_{\mathbb{R}^3} |\widetilde{F}_n|^2\, dx = 1$$

where $F_n^{(1)}$, $F_n^{(2)}$ and $\widetilde{F}_n$ are supported respectively in $B(0, 2\,R)$, $\mathbb{R}^2 \setminus B(0, R_n)$ and $B(0, 2\,R_n) \setminus B(0, R)$, $R_n > R > 1$, $\lim_{n\to\infty} R_n = \infty$,

$$\int_{\mathbb{R}^3} |F_n^{(1)}|^2\, dx \ge \theta - \varepsilon \quad \text{and} \quad \int_{\mathbb{R}^3} |F_n^{(2)}|^2\, dx \ge 1 - \theta - \varepsilon$$

for some $\theta \in [0, 1]$. As a consequence, we also have that $\int_{\mathbb{R}^3} |\widetilde{F}_n|^2\, dx \le 2\,\varepsilon$. A standard method to obtain such a decomposition is based on the IMS truncation method, which goes as follows. Take a smooth truncation function χ with the following properties: $0 \le \chi \le 1$, $\chi(x) = 1$ for any $x \in B(0, 1)$, $\chi(x) = 0$ for any $x \in \mathbb{R}^2 \setminus B(0, 2)$, and define $\chi_R(x) := \chi(x/R)$ for any $x \in \mathbb{R}^2$. Then for an appropriate choice of R and $(R_n)_{n\in\mathbb{N}}$, we can choose

$$F_n^{(1)} = \chi_R\, F_n \quad \text{and} \quad F_n^{(2)} = \sqrt{1 - \chi_{R_n}^2}\, F_n\ .$$

If $\theta = 1$, then $(F_n^{(1)})_{n\in\mathbb{N}}$ strongly converges in $L^2_{\rm loc}(\mathbb{R}^2, dx)$ to some limit F and we have

$$\liminf_{n\to\infty} \int_{\mathbb{R}^3} |\nabla F_n|^2\, dx \ge \int_{\mathbb{R}^3} |\nabla F|^2\, dx \quad \text{and} \quad \int_{\mathbb{R}^3} |F|^2\, dx \ge 1 - \varepsilon\ .$$

Now we repeat the argument as $\varepsilon = \varepsilon_n \to 0_+$, take a diagonal subsequence that we still denote by $(F_n)_{n\in\mathbb{N}}$, define

$$F_n^{(1)} = \chi_{R_n^{(1)}}\, F_n\ , \quad F_n^{(2)} = \sqrt{1 - \chi_{R_n^{(2)}}^2}\, F_n \quad \text{and} \quad \widetilde{F}_n = F_n - F_n^{(1)} - F_n^{(2)}\ ,$$

where $\lim_{n\to\infty} R_n^{(1)} = +\infty$ and $R_n^{(2)} \ge 2\, R_n^{(1)}$ for any $n \in \mathbb{N}$. Since limits obtained above by taking a diagonal subsequence coincide on larger and larger centered balls (that is when R increases), we find a nontrivial minimizer $f = F/\sqrt{n_\infty}$ such that $\int_{\mathbb{R}^3} F^2\ dx = \int_{\mathbb{R}^3} f^2\, n_\infty\ dx = 1$, since all other terms are relatively compact. Notice that $\mathsf{Q}_1[f] > 0$ because the condition (6.4.3) is preserved by passing to the limit. Hence Λ_1 is achieved and we know that Λ_1 is positive because Q_1 is positive semi-definite.

Assume now that $\theta < 1$. We know that $\int_{\mathbb{R}^3} |\widetilde{F}_n|^2\, dx \le 2\,\varepsilon_n$ and hence

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} |\widetilde{F}_n|^2\, dx = 0\ .$$

It is not difficult to see that cross terms do not play any role in the integrals involving convolution kernels, as it is standard for Hartree type (or Schrödinger-Poisson) models. As a consequence, we can write that

$$\lim_{n\to\infty} \mathsf{Q}_1[f_n] = \lim_{n\to\infty} \mathsf{Q}_1[f_n^{(1)}] + \lim_{n\to\infty} \mathsf{Q}_1[f_n^{(2)}]$$

where $f_n^{(i)} := F_n^{(i)}/\sqrt{n}_\infty$, $i = 1, 2$. Proceeding as above, we may find a limit f of $(f_n^{(1)})_{n\in\mathbb{N}}$, in $L^2(n_\infty\,dx)$. It is then straightforward to observe that

$$\Lambda_1 = \lim_{n\to\infty} \frac{\mathsf{Q}_2[f_n]}{\mathsf{Q}_1[f_n]} \geq \lim_{n\to\infty} \frac{\mathsf{Q}_2[f] + \mathsf{Q}_2[f_n^{(2)}]}{\mathsf{Q}_1[f] + \mathsf{Q}_1[f_n^{(2)}]} \,.$$

If $\theta > 0$, we know that $\mathsf{Q}_2[f] \geq \Lambda_1\, \mathsf{Q}_1[f]$ and $\lim_{n\to\infty} \mathsf{Q}_2[f_n^{(2)}]/\mathsf{Q}_1[f_n^{(2)}] > \Lambda_1$ by Lemma 21, so that $\lim_{n\to\infty} \mathsf{Q}_2[f_n^{(2)}] = 0$ and f is a nontrivial minimizer: we are back to the case $\theta = 1$, but with a different normalization of f. If $\theta = 0$, it is clear that $\mathsf{Q}_1[f] = 0$ and we get

$$\Lambda_1 \geq \lim_{n\to\infty} \frac{\mathsf{Q}_2[f_n^{(2)}]}{\mathsf{Q}_1[f_n^{(2)}]} = \infty \,,$$

again by Lemma 21, a contradiction with the fact that $\mathsf{Q}_2[f]/\mathsf{Q}_1[f]$ takes finite values for arbitrary test functions in $\mathcal{D}(\mathsf{L}_2) \setminus \{0\}$. This concludes our proof.

Remark 3 *For any $k \in \mathbb{N}$, $k \geq 1$, define the Raleigh quotient*

$$\Lambda_k := \inf_{\substack{f \in \mathcal{D}(\mathsf{L}_2) \setminus \{0\} \\ \langle f_j, f\rangle = 0\,,\ j = 0,\, 1, \ldots k-1}} \frac{\mathsf{Q}_2[f]}{\mathsf{Q}_1[f]}$$

where f_j denotes a critical point associated to Λ_j. Critical points are counted with multiplicity. Since the orthogonality condition $\langle f_j, f\rangle = 0$ is preserved by taking the limit along the weak topology of $L^2(n_\infty\,dx)$, building a minimizing sequence for $k \geq 1$ goes as in the case $k = 1$. It is easy to check that Λ_k is then an eigenvalue of $\mathcal{L}$ considered as an operator on $\mathcal{D}(\mathsf{L}_2)$ with scalar product $\langle\cdot,\cdot\rangle$, for any $k \geq 1$and $\lim_{k\to\infty} \Lambda_k = \infty$.

6.4.5 A spectral gap inequality

We are now going to prove that Ineq. (6.4.6) holds with $\Lambda_1 = \lambda_{1,i} = 1$, $i = 1, 2$. This is our first main estimate.

Theorem 23 *For any function $f \in \mathcal{D}(\mathsf{L}_2)$, we have*

$$\mathsf{Q}_1[f] \leq \mathsf{Q}_2[f] \,.$$

Recall that, with the notations of Section 6.4, $\mathsf{Q}_1[f] = \langle f, f\rangle$ and $\mathsf{Q}_2[f] = \langle f,\, \mathcal{L}\, f\rangle$.

Proof. We have to compute the lowest positive eigenvalue of $\mathcal{L}$. After a reformulation in terms of cumulated densities for the solution of (6.1.3) and for the eigenvalue problem for $\mathcal{L}$, we will identify the lowest eigenvalue $\lambda_{0,1} = 2$ when $\mathcal{L}$ is restricted to radial functions, and the lowest ones, $\lambda_{1,1} = \lambda_{1,2} = 1$, when $\mathcal{L}$ is restricted to functions corresponding to the $k = 1$ component in the decomposition into spherical harmonics.

Step 1. Reformulation in terms of cumulated densities.

Among spherically symmetric functions, it is possible to reduce the problem to a single ordinary differential equation.

Consider first a stationary solution (n_∞, c_∞) of (6.1.3) and as in [78] or [77] (also see references therein), let us rewrite the system in terms of the *cumulated densities* Φ and Ψ defined by

$$\Phi(s) := \frac{1}{2\pi} \int_{B(0,\sqrt{s})} n_\infty(x)\, dx \ ,$$
$$\Psi(s) := \frac{1}{2\pi} \int_{B(0,\sqrt{s})} c_\infty(x)\, dx \ .$$

Notice that $\Phi(s) = \frac{1}{2\,\pi}\, M_\varepsilon(\pi\, s)$ for $\varepsilon = 0$, with the notations of the proof of Theorem 14. The motivation for such a reformulation is that the system can be rewritten in terms of a nonlinear, local, ordinary differential equation for Φ using the fact that n_∞ is radial. With a slight abuse of notations, we can consider n_∞ and c_∞ as functions of $r = |x|$. Elementary computations show that

$$n_\infty(\sqrt{s}) = 2\,\Phi'(s) \quad \text{and} \quad n'_\infty(\sqrt{s}) = 4\,\sqrt{s}\,\Phi''(s) \ ,$$
$$c_\infty(\sqrt{s}) = 2\,\Psi'(s) \quad \text{and} \quad c'_\infty(\sqrt{s}) = 4\,\sqrt{s}\,\Psi''(s) \ .$$

After one integration with respect to $r = \sqrt{s}$, the Poisson equation $-\Delta c_\infty = n_\infty$ can be rewritten as

$$-\sqrt{s}\, c'_\infty(\sqrt{s}) = \Phi(s)$$

while the equation for n_∞, after an integration on $(0, r)$, is

$$n'_\infty(\sqrt{s}) + \sqrt{s}\, n_\infty(\sqrt{s}) - n_\infty(\sqrt{s})\, c'_\infty(\sqrt{s}) = 0 \ .$$

These two equations written in terms of Φ and Ψ are

$$-\,4\,s\,\Psi'' = \Phi$$

and

$$\Phi'' + \frac{1}{2}\,\Phi' -\, 2\,\Phi'\,\Psi'' \,.$$

After eliminating Ψ'', we find that Φ is the solution of the ordinary differential equation

$$\Phi'' + \frac{1}{2}\,\Phi' + \frac{1}{2\,s}\,\Phi\,\Phi' = 0 \tag{6.4.7}$$

with initial conditions $\Phi(0) = 0$ and $\Phi'(0) = \frac{1}{2}\,n(0) =: a$, so that all solutions can be parametrized in terms of $a > 0$.

Consider next the functions f and g involved in the linearized Keller-Segel system (6.1.3) and define the corresponding *cumulated densities* given by

$$\phi(s) := \frac{1}{2\pi} \int_{B(0,\sqrt{s})} (f\, n_\infty)(x)\, dx \ ,$$
$$\psi(s) := \frac{1}{2\pi} \int_{B(0,\sqrt{s})} (g\, c_\infty)(x)\, dx \ .$$

If $g\,c_\infty = (-\Delta)^{-1}(f\,n_\infty)$ and f is a solution of the eigenvalue problem

$$-\,\mathcal{L}\,f = \lambda\,f\ ,$$

then we can make a computation similar to the above one and get

$$(n_\infty\,f)(\sqrt{s}) = 2\,\phi'(s)\ ,\quad (n_\infty\,f')(\sqrt{s}) = 4\,\sqrt{s}\,\phi''(s) - \,2\,\frac{n'_\infty}{n_\infty}(\sqrt{s})\,\phi'(s)\ ,$$

$$(g\,c_\infty)(\sqrt{s}) = 2\,\psi'(s)\quad\text{and}\quad (g\,c_\infty)'(\sqrt{s}) = 4\,\sqrt{s}\,\psi''(s)\ .$$

The equations satisfied by f and g are

$$-\sqrt{s}\,(g\,c_\infty)'(\sqrt{s}) = \phi(s)$$

and

$$\sqrt{s}\,\Big((n_\infty\,f')(\sqrt{s}) - n_\infty\,(g\,c_\infty)'(\sqrt{s})\Big) + \lambda\,\phi(s) = 0\ .$$

These two equations written in terms of ϕ and ψ become

$$-\,4\,s\,\psi'' = \phi$$

and

$$4\,s\left(\phi'' - \frac{\Phi''}{\Phi'}\,\phi' - \,2\,\Phi'\,\psi''\right) + \lambda\,\phi = 0\ .$$

After eliminating ψ'', we find that ϕ is the solution of the ordinary differential equation

$$\phi'' - \frac{\Phi''}{\Phi'}\,\phi' + \frac{\lambda + 2\,\Phi'}{4\,s}\,\phi = 0\ .$$

Taking into account the equation for Φ, that is

$$-\frac{\Phi''}{\Phi'} = \frac{1}{2} + \frac{\Phi}{2\,s}\ ,$$

we can also write that ϕ solves

$$\phi'' + \frac{s+\Phi}{2\,s}\,\phi' + \frac{\lambda + 2\,\Phi'}{4\,s}\,\phi = 0\ . \tag{6.4.8}$$

Recall that the set of solutions to (6.4.7) is parametrized by $a = \Phi'(0)$. It is straightforward to remark that $\phi = \frac{d}{da}\Phi$ solves (6.4.8) with $\lambda = 0$. The reader is invited to check that $s \mapsto s\,\Phi'(s)$ provides a nonnegative solution of (6.4.8) with $\lambda = 2$.

Step 2. Characterization of the radial ground state.

It is possible to rewrite (6.4.8) as

$$\frac{d}{ds}\left(e^{\alpha(s)}\,\frac{d\phi}{ds}\right) + \frac{\lambda + 2\,\Phi'}{4\,s}\,e^{\alpha(s)}\,\phi = 0\quad\text{with}\quad \alpha(s) := \frac{s}{2} + \frac{1}{2}\int_0^s \frac{\phi(\sigma)}{\sigma}\,d\sigma\ .$$

The equation holds on $(0,\infty)$ and boundary conditions are $\phi(0) = 0$ and $\lim_{s\to\infty}\phi(s) = 0$. By the Sturm-Liouville theory, we know that $\lambda = 2 = \lambda_{0,1}$ is then the lowest positive eigenvalue such that ϕ is nonnegative and satisfies the above boundary conditions.

In other words, we have shown that the function $f_{0,1}$ found in Section 6.4.1 generates the eigenspace corresponding to the lowest positive eigenvalue of $\mathcal{L}$ restricted to radial functions.

Step 3. Spherical harmonics decomposition.

We have to deal with non-radial modes of $\mathcal{L}$. Since n_∞ and c_∞ are both radial, we can use a spherical harmonics decomposition for that purpose. As in [87], the eigenvalue problem for the operator $\mathcal{L}$ amounts to solve among radial functions f and g the system

$$-f'' - \frac{1}{r}\,f' + \frac{k^2}{r^2}\,f + (r - c_\infty')\,(f' - (g\,c_\infty)') - n_\infty\,f = \lambda\,f\ ,$$
$$-(g\,c_\infty)'' - \frac{1}{r}\,(g\,c_\infty)' + \frac{k^2}{r^2}\,(g\,c_\infty) = n_\infty\,f\ ,$$

for some $k \in \mathbb{N}$, $k \geq 1$. Here as above, we make the standard abuse of notations that amounts to write n_∞ and c_∞ as a function of $r = |x|$. It is straightforward to see that $k = 1$ realizes the infimum of the spectrum of $\mathcal{L}$ among non-radial functions. The function $f = -n_\infty'$ provides a nonnegative solution for $k = 1$ and $\lambda = 1$. It is then possible to conclude using the following observation: f is a radial C^2 solution if and only if $r \mapsto r\,f =: \tilde{f}(r)$ solves $-\mathcal{L}\,\tilde{f} = (\lambda + 1)\,\tilde{f}$ among radial functions, and we are back to the problem studied in Step 2. The value we look for is therefore $\lambda = 1 = \lambda_{1,1} = \lambda_{1,2}$.

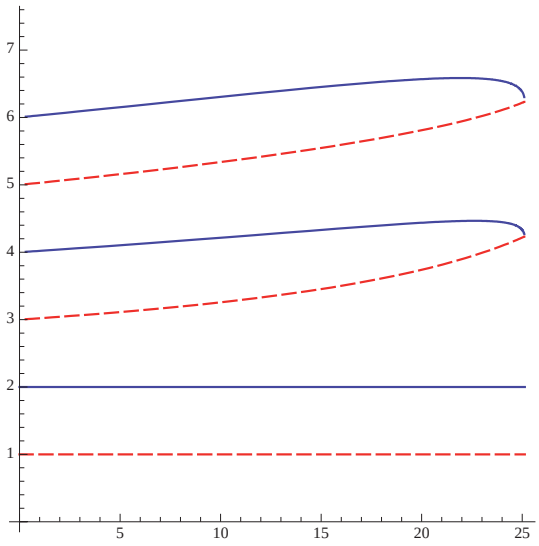

Figure 6.1: *Using a shooting method, one can numerically compute the lowest eigenvalues of $-\mathcal{L}$ for $k = 0$ (radial functions) and for the $k = 1$ component of the spherical harmonics decomposition (dashed curve). The plot shows that 1 and 2 are the lowest eigenvalues, when mass varies between 0 and $8\,\pi \approx 25.1327$.*

In other words, we have shown that the functions $f_{1,1}$ and $f_{1,2}$ found in Section 6.4.1 generate the eigenspace corresponding to the lowest positive eigenvalue of $\mathcal{L}$ corresponding to $k = 1$. We are now in position to conclude the proof of Theorem 23.

Either the spectral gap is achieved among radial functions and $\Lambda_1 = 2$, or it is achieved among functions in one of the non-radial components corresponding to the spherical harmonics decomposition: the one given by $k = 1$ minimizes the gap and hence we obtain $\Lambda_1 = 1$. See Fig. 5.1 for an illustration.

As a consequence of Step 2 in the proof of Theorem 23, we find that the following inequality holds.

Proposition 24 *For any radial function $f \in \mathcal{D}(\mathsf{L}_2)$, we have*

$$2\,\mathsf{Q}_1[f] \le \mathsf{Q}_2[f]\,.$$

where, with the notations of Section 6.4, $\mathsf{Q}_1[f] = \langle f, f\rangle$ and $\mathsf{Q}_2[f] = \langle f,\, \mathcal{L}\, f\rangle$.

This observation has to be related with recent results of V. Calvez and J.A. Carrillo. As a consequence, the rate $e^{-2\,t}$ in Theorem 11 can be replaced by $e^{-4\,t}$ when solutions are radially symmetric, consistently with [85, Theorem 1.2]. The necessary adaptations (see Section 6.6) are straightforward.

6.5 A strict positivity result for the linearized entropy

Lemma 19 can be improved and this is our second main estimate.

Theorem 25 *There exists $\Lambda > 1$ such that*

$$\Lambda \int_{\mathbb{R}^3} f\, n_\infty\, (-\Delta)^{-1}\, (f\, n_\infty)\, dx \le \int_{\mathbb{R}^3} |f|^2\, n_\infty\, dx \tag{6.5.1}$$

for any $f \in L^2(\mathbb{R}^2, n_\infty\, dx)$ such that (6.4.3) holds.

Proof. Let us give an elementary proof based on two main observations: the equivalence with a Poincaré type inequality using Legendre's transform, and the application of a concentration-compactness method for proving the Poincaré inequality. Recall that by Lemma 19 we already know that (6.5.1) holds with $\Lambda = 1$.

Step 1. We claim that Inequality (6.5.1) is equivalent to

$$\Lambda \int_{\mathbb{R}^3} |h|^2\, n_\infty\, dx \le \int_{\mathbb{R}^3} |\nabla h|^2\, dx \tag{6.5.2}$$

for any $h \in L^2(\mathbb{R}^2, n_\infty\, dx)$ such that the condition $\int_{\mathbb{R}^3} h\, f_{0,0}\, n_\infty\, dx = 0$ holds, *i.e.* such that h satisfies (6.4.3). Let us prove this claim.

Assume first that (6.5.2) holds and take Legendre's transform of both sides with respect to the natural scalar product in $L^2(n_\infty\, dx)$: for any $f \in L^2(\mathbb{R}^2, n_\infty\, dx)$ such that (6.4.3) holds,

$$\sup_h \left(\int_{\mathbb{R}^3} f\, h\, n_\infty\, dx - \frac{1}{2} \int_{\mathbb{R}^3} h^2\, n_\infty\, dx \right) \ge \sup_h \left(\int_{\mathbb{R}^3} f\, h\, n_\infty\, dx - \frac{1}{2\Lambda} \int_{\mathbb{R}^3} |\nabla h|^2\, dx \right)$$

where the supremum is taken on both sides on all functions h in $L^2(\mathbb{R}^2, n_\infty\, dx)$ such that h satisfies (6.4.3). Since semi-definite positive quadratic forms are involved, the suprema are achieved by convexity. For the left hand side, we find that the optimal function satisfies

$$f = h + \mu\, f_{0,0}$$

for some Lagrange multiplier $\mu \in \mathbb{R}$. However, if we multiply by $f_{0,0}\, n_\infty$, we get that $\mu = 0$, so that the left hand side of the inequality is simply $\frac{1}{2} \int_{\mathbb{R}^3} f^2\, n_\infty\, dx$. As for the right hand side, we find that the optimal function f is such that

$$f\, n_\infty = -\frac{1}{\Lambda}\, \Delta h + \mu\, f_{0,0}\, n_\infty$$

for some Lagrange multiplier $\mu \in \mathbb{R}$. In that case, if we multiply by $(-\Delta)^{-1}(f_{0,0}\, n_\infty) = f_{0,0}$, we get that

$$\mu \int_{\mathbb{R}^3} f_{0,0}^2\, n_\infty\, dx = \frac{1}{\Lambda} \int_{\mathbb{R}^3} \Delta h\, (-\Delta)^{-1}(f_{0,0}\, n_\infty)\, dx = -\frac{1}{\Lambda} \int_{\mathbb{R}^3} h\, f_{0,0}\, n_\infty\, dx = 0$$

thus proving that $\mu = 0$ as well. Hence the right hand side of the inequality is simply

$$\frac{\Lambda}{2} \int_{\mathbb{R}^3} f\, n_\infty\, (-\Delta)^{-1}\, (f\, n_\infty)\, dx\ ,$$

which establishes (6.5.1). It is left to the reader to check that Inequality (6.5.2) can also be deduced from (6.5.1) by a similar argument.

Step 2. Let us prove that (6.5.2) holds for some $\Lambda > 1$. Consider an optimizing sequence of functions $(h_n)_{n\ge1}$ such that $\int_{\mathbb{R}^3} h_n^2\, n_\infty\, dx = 1$ and $\int_{\mathbb{R}^3} h_n\, f_{0,0}\, n_\infty\, dx = 0$ for any $n \ge 1$, and

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla h_n|^2\, n_\infty\, dx = \Lambda\ .$$

As in the proof of Lemma 22, we are going to use the IMS truncation method. Consider a smooth function χ with the following properties: $0 \le \chi \le 1$, $\chi(x) = 1$ for any $x \in B(0,1)$, $\chi(x) = 0$ for any $x \in \mathbb{R}^2 \setminus B(0,2)$, and define $\chi_R(x) := \chi(x/R)$ for any $x \in \mathbb{R}^2$. It is standard in concentration-compactness methods that for any $\varepsilon > 0$, one can find a sequence of positive numbers $(R_n)_{n\ge1}$ such that

$$h_n^{(1)} = \chi_{R_n}\, h_n \quad \text{and} \quad h_n^{(2)} = \sqrt{1 - \chi_{R_n}^2}\, h_n\ ,$$

and, up to the extraction of a subsequence, there exists a function h such that

$$\int_{\mathbb{R}^3} |\nabla h_n^{(1)}|^2\, n_\infty\, dx \ge \eta\, \Lambda - \varepsilon \quad \text{and} \quad \int_{\mathbb{R}^3} |\nabla h_n^{(2)}|^2\, n_\infty\, dx \ge (1-\eta)\, \Lambda - \varepsilon$$

for some $\eta \in [0,1]$, where the sequence $(\nabla h_n^{(1)})_{n\ge1}$ strongly converges to ∇h and

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} |h_n^{(1)}|^2\, n_\infty\, dx = \int_{\mathbb{R}^3} h^2\, n_\infty\, dx =: \theta$$

(this implies the strong convergence of $(h_n^{(1)})_{n\ge1}$ towards h in $L^2(n_\infty\, dx)$) because

$$\int_{\mathbb{R}^2\setminus B(0,R)} |h_n^{(1)}|^2\, n_\infty\, dx \le \left(\int_{\mathbb{R}^3} |h_n^{(1)}|^{\frac{2d}{d-2}}\, n_\infty\, dx \right)^{\frac{d-2}{d}} \left(\int_{\mathbb{R}^2\setminus B(0,R)} n_\infty^{\frac{d}{2}}\, dx \right)^{\frac{2}{d}}$$

is uniformly small as $R \to \infty$ by Sobolev's inequality and because the last term of the right hand side is such that $\lim_{R\to\infty} \int_{\mathbb{R}^2\setminus B(0,R)} n_\infty^{d/2}\, dx = 0$. Of course, we know that

$$\eta\, \Lambda \ge \int_{\mathbb{R}^3} |\nabla h|^2\, dx \ge \Lambda\, \theta$$

by definition of Λ. The above estimate also guarantees that

$$\int_{\mathbb{R}^2\setminus B(0,R)} |h_n^{(2)}|^2\, n_\infty\, dx =: \varepsilon_n \to 0 \quad \text{as} \quad n \to \infty\ .$$

By construction of $(h_n^{(1)})_{n\ge1}$ and $(h_n^{(2)})_{n\ge1}$, we know that $|h_n^{(1)}|^2+|h_n^{(2)}|^2=|h_n|^2$ and hence $\theta=1$. This also means that $\eta=1$ and hence h is a minimizer, since the constraint passes to the limit:

$$\int_{\mathbb{R}^3} h\, f_{0,0}\, n_\infty\, dx = \lim_{n\to\infty} \int_{\mathbb{R}^3} h_n\, f_{0,0}\, n_\infty\, dx = 0\ .$$

The function h is a solution of the Euler-Lagrange equation:

$$-\Delta h = \Lambda\, h\, n_\infty\ .$$

By Proposition 4, if $\Lambda=1$, then h and $f_{0,0}$ are collinear, which is a contradiction with the constraint. This proves that $\Lambda>1$.

Notice that the functions n_∞ and c_∞ being radial symmetric, we know that a decomposition into spherical harmonics allows to reduce the problem of computing all eigenvalues to radially symmetric eigenvalue problems. This provides a method to compute the explicit value of Λ, at least numerically.

Remark 4 *Inequality* (6.5.2) *is a Poincaré inequality, which has already been established in [86] as a linearized version of an Onofri type inequality. This Onofri inequality is dual of the logarithmic Hardy-Littlewood-Sobolev type inequality that has been established in [84] and according to which the free energy functional $F[n]$ is nonnegative.*

A straightforward consequence of Theorem 25 is that we can estimate $\int_{\mathbb{R}^3} f^2\, n_\infty\ dx$ in terms of $\mathsf{Q}_1[f]=\int_{\mathbb{R}^3} f\,(f-g\,c_\infty)\, n_\infty\ dx$.

Corollary 26 *For the same value of $\Lambda>1$ as in Theorem 25, we have*

$$\int_{\mathbb{R}^3} f^2\, n_\infty\ dx \le \frac{\Lambda}{\Lambda-1}\,\mathsf{Q}_1[f]$$

for any $f\in L^2(\mathbb{R}^2, n_\infty\, dx)$ such that (6.4.3) *holds.*

Proof. We may indeed write

$$\mathsf{Q}_1[f]=\frac{\Lambda-1}{\Lambda}\int_{\mathbb{R}^3} f^2\, n_\infty\ dx+\frac{1}{\Lambda}\left(\int_{\mathbb{R}^3}|f|^2\, n_\infty\ dx-\Lambda\int_{\mathbb{R}^3} f\, n_\infty\,(-\Delta)^{-1}\,(f\, n_\infty)\ dx\right)$$

and use the fact that the last term of the right hand side is nonnegative.

6.6 The large time behavior

This section is devoted to the proof of Theorem 11. Our approach is guided by the analysis of the evolution equation corresponding to the linearization of the Keller-Segel system: see Section 6.6.1. The key estimates for the nonlinear evolution problem have been stated in Theorem 23, Theorem 25, and Corollary 26. Nonlinear terms are estimated using Corollary 17.

6.6.1 A linearized evolution problem

We recall that the restriction of $\mathcal{L}$ to $\mathcal{D}(\mathsf{L}_1)$ is a self-adjoint operator with domain $\mathcal{D}(\mathsf{L}_2)$, such that

$$\langle f, \mathcal{L}\, f\rangle = -\,\mathsf{Q}_2[f] \quad \forall\, f \in \mathcal{D}(\mathsf{L}_2)\;.$$

and $\mathrm{Ker}(\mathcal{L}) \cap \mathcal{D}(\mathsf{L}_2) = \{0\}$

By Proposition 5, any solution $(t,x) \mapsto f(t,x)$ of the linearized Keller-Segel model

$$\left\{\begin{array}{ll} \frac{\partial f}{\partial t} = \mathcal{L}\, f & x \in \mathbb{R}^2\,,\quad t > 0\,,\\ -\Delta(g\, c_\infty) = f\, n_\infty & x \in \mathbb{R}^2\,,\quad t > 0\,,\end{array}\right.$$

has an exponential decay, since we know that

$$\frac{d}{dt}\langle f(t,\cdot), f(t,\cdot)\rangle = 2\,\langle f(t,\cdot),\, \mathcal{L}\, f(t,\cdot)\rangle\;,$$

that is

$$\frac{d}{dt}\,\mathsf{Q}_1[f(t,\cdot)] = -\,2\,\mathsf{Q}_2[f(t,\cdot)] \le -\,2\,\mathsf{Q}_1[f(t,\cdot)]$$

by Theorem 23. Hence we obtain

$$\mathsf{Q}_1[f(t,\cdot)] \le \mathsf{Q}_1[f(0,\cdot)]\, e^{-\,2\,t} \quad \forall\, t \in \mathbb{R}^+\,.$$

Here we adopt the usual convention that $\mathsf{Q}_2[f] = +\infty$ for any $f \in \mathcal{D}(\mathsf{L}_1) \setminus \mathcal{D}(\mathsf{L}_2)$.

6.6.2 Proof of Theorem 11

As in [82], Eq. (6.1.3) can be rewritten in terms of $f = (n - n_\infty)/n_\infty$ and $g = (c - c_\infty)/c_\infty$ in the form of (6.4.1), that is

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f - \frac{1}{n_\infty}\,\nabla\left[n_\infty\, f\, \nabla(g\, c_\infty)\right]\;.$$

The computation for the linearized problem established in Section 6.6.1 can be adapted to the nonlinear case and gives

$$\frac{d}{dt}\,\mathsf{Q}_1[f(t,\cdot)] = -\,2\,\mathsf{Q}_2[f(t,\cdot)] + \int_{\mathbb{R}^3} \nabla(f - g\, c_\infty)\, f\, n_\infty \cdot \nabla(g\, c_\infty)\; dx\;,$$

with $-\,2\,\mathsf{Q}_2[f(t,\cdot)] \le -\,2\,\mathsf{Q}_1[f(t,\cdot)]$ according to Theorem 23. To get an estimate on the asymptotic behaviour, we have to establish an estimate of the last term of the right hand side, which is cubic in terms of f. For this purpose, we apply Hölder's inequality and Lemma 16 to get

$$\left(\int_{\mathbb{R}^3} \nabla(f - g\, c_\infty)\, f\, n_\infty \cdot \nabla(g\, c_\infty)\; dx\right)^2 \le \mathsf{Q}_2[f] \int_{\mathbb{R}^3} f^2\, n_\infty\; dx\, \|\nabla(g\, c_\infty)\|^2_{L^\infty(\mathbb{R}^2)}\;.$$

Using Lemma 16 and Corollary 26, we find that the right hand side can be bounded by

$$C(\varepsilon)\,\frac{\Lambda}{\Lambda - 1}\,\mathsf{Q}_1[f]\,\mathsf{Q}_2[f]\,\left(\|f\, n_\infty\|_{L^{2-\varepsilon}(\mathbb{R}^2)} + \|f\, n_\infty\|_{L^{2+\varepsilon}(\mathbb{R}^2)}\right)\;.$$

From [84, Theorem 1.2] and Corollary 17 we know that $\lim_{t\to\infty} \|f(t,\cdot)\, n_\infty\|_{L^1(\mathbb{R}^2)} = 0$ and $\lim_{t\to\infty} \|f(t,\cdot)\, n_\infty\|_{L^\infty(\mathbb{R}^2)} = 0$ and therefore, for any given $\varepsilon \in (0,1)$, there exists a continuous function $t \mapsto \delta(t,\varepsilon)$ with $\lim_{t\to\infty} \delta(t,\varepsilon) = 0$ such that

$$\|f(t,\cdot)\, n_\infty\|_{L^{2-\varepsilon}(\mathbb{R}^2)} + \|f(t,\cdot)\, n_\infty\|_{L^{2+\varepsilon}(\mathbb{R}^2)} \le \delta(t,\varepsilon)\ .$$

As a consequence, we know that

$$\frac{d}{dt}\,\mathsf{Q}_1[f(t,\cdot)] \le -\,2\,\mathsf{Q}_2[f(t,\cdot)] + \delta(t,\varepsilon)\,\sqrt{\mathsf{Q}_1[f(t,\cdot)]\,\mathsf{Q}_2[f(t,\cdot)]} \le (\delta(t,\varepsilon) - 2)\,\mathsf{Q}_2[f(t,\cdot)]\ ,$$

where the last inequality is a consequence of Theorem 23. This proves that $\mathsf{Q}_1[f(t,\cdot)]$ is uniformly bounded with respect to t as $t\to\infty$: there exists a constant $\mathcal{Q} > 0$ such that

$$\mathsf{Q}_1[f(t,\cdot)] \le \mathcal{Q} \quad \forall\, t \ge 0\ .$$

Now we can give a more detailed estimate. Using again Hölder's inequality, we find that

$$\int_{\mathbb{R}^3} (|f|\, n_\infty)^{2+\varepsilon}\, dx \le \mathsf{Q}_1[f]\, \|f\, n_\infty\|^{\varepsilon}_{L^\infty(\mathbb{R}^2)}\, \|n_\infty\|_{L^\infty(\mathbb{R}^2)}\ ,$$

that is

$$\|f\, n_\infty\|_{L^{2+\varepsilon}(\mathbb{R}^2)} \le (\mathsf{Q}_1[f])^{\frac{1}{2+\varepsilon}}\ \|f\, n_\infty\|^{\frac{\varepsilon}{2+\varepsilon}}_{L^\infty(\mathbb{R}^2)}\, \|n_\infty\|^{\frac{1}{2+\varepsilon}}_{L^\infty(\mathbb{R}^2)}$$

and

$$\|f\, n_\infty\|_{L^{2-\varepsilon}(\mathbb{R}^2)} \le \|f\, n_\infty\|^{\frac{\varepsilon}{2-\varepsilon}}_{L^1(\mathbb{R}^2)}\, \|f\, n_\infty\|^{\frac{2\,(1-\varepsilon)}{2-\varepsilon}}_{L^2(\mathbb{R}^2)} \le \|f\, n_\infty\|^{\frac{\varepsilon}{2-\varepsilon}}_{L^1(\mathbb{R}^2)}\ (\mathsf{Q}_1[f])^{\frac{1-\varepsilon}{2-\varepsilon}}\ \|n_\infty\|^{\frac{1-\varepsilon}{2-\varepsilon}}_{L^\infty(\mathbb{R}^2)}\ .$$

Recall that $\lim_{t\to\infty} \|f(t,\cdot)\, n_\infty\|_{L^1(\mathbb{R}^2)} = 0$ and $\lim_{t\to\infty} \|f(t,\cdot)\, n_\infty\|_{L^\infty(\mathbb{R}^2)} = 0$ according to [84, Theorem 1.2] and Corollary 17 respectively. For any given $\varepsilon \in (0,1)$, there exists a continuous function (that we again denote by δ): $t \mapsto \delta(t,\varepsilon)$, with $\lim_{t\to\infty} \delta(t,\varepsilon) = 0$, such that

$$\frac{d}{dt}\,\mathsf{Q}_1[f] \le -\,2\,\mathsf{Q}_2[f] + \delta(t,\varepsilon)\,(\mathsf{Q}_2[f])^{\frac12}\,(\mathsf{Q}_1[f])^{\frac12}\,\Big((\mathsf{Q}_1[f])^{\frac{1-\varepsilon}{2-\varepsilon}} + \mathsf{Q}_1[f])^{\frac{1}{2+\varepsilon}}\Big)\ .$$

Since $\mathsf{Q}_1[f] \le \mathsf{Q}_2[f]$, this provides the estimate

$$\frac{d}{dt}\,\mathsf{Q}_1[f] \le -\sqrt{\mathsf{Q}_2[f]}\,\Big(2\,\sqrt{\mathsf{Q}_2[f]} - \delta(t,\varepsilon)\,(\mathsf{Q}_1[f])^{\frac{4-3\,\varepsilon}{2\,(2-\varepsilon)}} - \delta(t,\varepsilon)\,(\mathsf{Q}_1[f])^{\frac{4+\varepsilon}{2\,(2+\varepsilon)}}\Big)\ .$$

Altogether, this proves that

$$\frac{d}{dt}\,\mathsf{Q}_1[f] \le -\,\mathsf{Q}_2[f]\,\Big[2 - \delta(t,\varepsilon)\,\Big(\mathcal{Q}^{\frac{1-\varepsilon}{2-\varepsilon}} + \mathcal{Q}^{\frac{1}{2+\varepsilon}}\Big)\Big]\ .$$

For $t > 0$ large enough, the right hand side becomes negative and we have found that

$$\frac{d}{dt}\,\mathsf{Q}_1[f(t,\cdot)] \le -\,\mathsf{Q}_1[f(t,\cdot)]\,\Big[2 - \delta(t,\varepsilon)\,\Big(\mathcal{Q}^{\frac{1-\varepsilon}{2-\varepsilon}} + \mathcal{Q}^{\frac{1}{2+\varepsilon}}\Big)\Big]\ ,$$

thus showing that

$$\limsup_{t\to\infty} e^{(2-\eta)\,t}\,\mathsf{Q}_1[f(t,\cdot)] < \infty$$

for any $\eta \in (0,2)$. Actually we know from the above estimates that

$$2\,\sqrt{\mathsf{Q}_2[f(t,\cdot)]} - \delta(t,\varepsilon)\,(\mathsf{Q}_1[f(t,\cdot)])^{\frac{4-3\,\varepsilon}{2\,(2-\varepsilon)}} - \delta(t,\varepsilon)\,(\mathsf{Q}_1[f(t,\cdot)])^{\frac{4+\varepsilon}{2\,(2+\varepsilon)}}$$

is positive for $t > 0$, large enough, thus showing that

$$\frac{d}{dt}\,\mathsf{Q}_1[f(t,\cdot)] \le -\,\mathsf{Q}_1[f(t,\cdot)]\left[2 -\,\delta(t,\varepsilon)\left(\mathsf{Q}_1[f(t,\cdot)]^{\frac{1-\varepsilon}{2-\varepsilon}} + \mathsf{Q}_1[f(t,\cdot)]^{\frac{1}{2+\varepsilon}}\right)\right] .$$

As a consequence, we finally get that

$$\limsup_{t\to\infty} e^{2\,t}\,\mathsf{Q}_1[f(t,\cdot)] < \infty ,$$

which completes the proof of Theorem 11. □

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Chapter 7

A numerical study of the linearized Keller-Segel operator in self-similar variables

This chapter is devoted to a numerical study of the Keller-Segel model in self-similar variables. We first parametrize the set of solutions in terms of the mass parameter $M \in (0, 8\pi)$ and consider the asymptotic regimes for M small or M close to 8π. Next we introduce the linearized operator and study its spectrum using various shooting methods: we determine its kernel, the spectrum among radial functions and use a decomposition into spherical harmonics to study the other eigenvalues. As a result, we numerically observe that the spectral gap of the linearized operator is independent of M and equal to 1, which is compatible with known results in the limiting regime corresponding to $M \to 0_+$, and with the theoretical results obtained in the previous chapters. We also compute other eigenvalues, which allows to state several claims on various refined asymptotic expansions of the solutions in the large time regime.

This is a joint work with J. Dolbeault and it has been publisehd as a technical report of the CEREMADE.

7.1 Introduction

In its simplest version, the parabolic-elliptic Keller-Segel model (also known as the Patlak-Keller-Segel model, see [124, 121])

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \, \nabla v) & x \in \mathbb{R}^2 \,, \quad t > 0 \\ v = -\frac{1}{2\pi} \log |\cdot| * u & x \in \mathbb{R}^2 \,, \quad t > 0 \\ u(0, x) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases} \tag{7.1.1}$$

describes the motion of unicellular amoebae, like *dictyostelium discoideum,* which move freely and diffuse. Here u denotes their spatial density and it makes sense to consider them in a two-dimensional setting like the one of a Petri dish. Under certain circumstances, they emit a *chemo-attractant* and eventually start to aggregate by moving in the direction of the largest concentration of the chemo-attractant. This is modeled in the above equations by the drift term $\nabla \cdot (u \, \nabla v)$. The life cycle of *dictyostelium discoideum* has attracted lots of attention in the community of biologists. Trying to understand the competition

between the diffusion and the drift is a key issue in the aggregation process, which has also motivated quite a few studies among mathematicians interested in applications of PDEs to biology. See [125] for a recent overview on the topic.

An easy computation (see [125, pages 122–124] and [122]) shows that solutions (with second moment initially finite) blow-up in finite time if the total mass is large enough (larger than 8π with our conventions), that is they describe an aggregate, while, for solutions with smaller masses, the diffusion dominates the large time asymptotics.

More precisely, it has been shown in [120, 118, 116] that, for initial data $n_0 \in L^1_+\big(\mathbb{R}^2\,,(1+|x|^2)\,dx\big)$ such that $n_0\,|\log n_0| \in L^1(\mathbb{R}^2)$ and $M := \int_{\mathbb{R}^3} n_0\ dx < 8\pi$, there exists a solution u, in the sense of distributions, that is global in time and such that $M = \int_{\mathbb{R}^3} u(x,t)\ dx$ is preserved along the evolution. There is no non-trivial stationary solution to (7.1.1) and any solution converges to zero locally as time gets large. In order to study the asymptotic behavior of u, it is therefore convenient to work in self-similar variables. In the space and time scales given respectively by $R(t) := \sqrt{1+2t}$ and $\tau(t) := \log R(t)$, we define the rescaled functions n and c by

$$u(x,t) := R^{-2}\,n\left(R^{-1}(t)\,x,\tau(t)\right) \quad \text{and} \quad v(x,t) := c\left(R^{-1}(t)\,x,\tau(t)\right)\ .$$

This time-dependent rescaling is the one of the heat equation. Since the nonlinear term is invariant under this rescaling, it is also present in the rescaled system without time-dependent coefficient. This system can be written as

$$\left\{\begin{array}{ll} \frac{\partial n}{\partial t} = \Delta n + \nabla\cdot(n\,x) - \nabla\cdot(n\,\nabla c) & x\in\mathbb{R}^2\,,\quad t>0 \\ c = -\frac{1}{2\pi}\log|\cdot| * n & x\in\mathbb{R}^2\,,\quad t>0 \\ n(0,x) = n_0 \ge 0 & x\in\mathbb{R}^2 \end{array}\right. \tag{7.1.2}$$

and it has been shown in [116] that n and ∇c converge as $t\to\infty$, respectively in $L^1(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$ to a unique stationary solution given by smooth and radially symmetric functions.

We are interested in estimating the rate of convergence towards the stationary solution in self-similar variables. After undoing the change of variables, this gives the rate of convergence towards the asymptotic profile for the solutions of (7.1.1). Existence of a stationary solution to (7.1.2) has been established in [113] by ODE techniques, and in [123] by PDE methods. The uniqueness has been shown in [114]. In [115], it has been proved that if M is less than some mass $M_*\in(0,8\pi)$, then convergence holds at an exponential rate, which is essentially governed by the linearization of System (7.1.2) around the stationary solution. However, the estimate of the value of M_* was found to be significantly smaller than 8π. In the radially symmetric setting, V. Calvez and J.A. Carrillo have found in [117] that the rate measured with respect to Wasserstein's distance does not depend on the mass, in the whole range $(0,8\pi)$. To establish refined estimates a proper the functional setting for the linear operator must be characterized, this is carried out in detail in chapters 5 and 6. Here we will recoverall these results numerically and give more detailed estimates on the asymptotic behavior of the solutions.

Consider the unique stationary solution to (7.1.2), which is characterized as the solution to

$$-\Delta c = n = M\,\frac{e^{-\frac12|x|^2+c}}{\int_{\mathbb{R}^3} e^{-\frac12|x|^2+c}\,dx}\ , \quad x\in\mathbb{R}^2 \tag{7.1.3}$$

for any given mass $M\in(0,8\pi)$. The bifurcation diagram of the solutions in terms of the parameter M will be considered in Section 7.2.

Next, consider f and g such that $n\,(1+f(x,t))$ and $c(x)\,(1+g(x,t))$ is a solution to (7.1.2). Then (f,g) solves the nonlinear problem

$$\begin{cases} \frac{\partial f}{\partial t} - \mathcal{L}\,f = -\frac{1}{n}\,\nabla\cdot[f\,n\,(\nabla(g\,c))] & x\in\mathbb{R}^2\,,\quad t>0 \\ -\Delta(c\,g) = f\,n & x\in\mathbb{R}^2\,,\quad t>0 \end{cases}$$

where $\mathcal{L}$ is the linear operator defined by

$$\mathcal{L}\,f = \frac{1}{n}\,\nabla\cdot[n\,\nabla(f-c\,g)]$$

and we know that $(f\,n,\nabla(g\,c))(t,\cdot)$ has to evolve in $L^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)$, and asymptotically vanish as $t\to\infty$. To investigate the large time behavior, it is convenient to normalize the solution differently. What we actually want to investigate is the case where solutions of (7.1.2) can be written as

$$n(x)\,(1+\varepsilon\,f(x,t)) \quad\text{and}\quad c(x)\,(1+\varepsilon\,g(x,t))$$

in the asymptotic regime corresponding to $\varepsilon\to 0_+$. Formally, it is then clear that, at order ε, the behavior of the solution is given by $\frac{\partial f}{\partial t} = \mathcal{L}\,f$. The kernel of $\mathcal{L}$ has been identified in chapter 6. It has also been shown that $\mathcal{L}$ has pure discrete spectrum and that 1 and 2 are eigenvalues. In this report, our goal is to identify the lowest eigenvalues and recover that the spectral gap is actually equal to 1, whatever the mass is in the range $M_*\in(0,8\pi)$. We will also establish the numerical value of other eigenvalues of $\mathcal{L}$ at the bottom of its spectrum in Section 7.4, and draw some consequences in the last section of this report: improved rates of convergence for centered initial data and faster decay rates for best matching self-similar solutions.

7.2 Bifurcation diagram and qualitative properties of the branch of solutions corresponding to $M\in(0,8\pi)$

We can numerically solve (7.1.3) among radial solutions as follows. Let

$$\phi(r) = b + c(x)$$

for some $b\in\mathbb{R}$, $r=|x|$, for any $x\in\mathbb{R}^2$, such that

$$M\,\frac{e^{-b}}{\int_{\mathbb{R}^3} e^{-\frac{1}{2}\,|x|^2+c}\,dx} = 1 \quad\Longleftrightarrow\quad b = \log M - \log\left(\int_{\mathbb{R}^3} e^{-\frac{1}{2}\,|x|^2+c}\,dx\right).$$

Then the function $r\mapsto\phi(r)$ solves

$$-\phi'' - \frac{1}{r}\,\phi' = e^{-\frac{1}{2}\,r^2+\phi}\,,\quad r>0$$

with initial conditions $\phi(0)=a$, $\phi'(0)=0$. To emphasize the dependence in $a\in\mathbb{R}$, we will denote the solution by ϕ_a. Since

$$1 = M\,\frac{1}{\int_{\mathbb{R}^3} e^{-\frac{1}{2}\,|x|^2+b+c}\,dx} = \frac{M}{2\pi\int_0^\infty r\,e^{-\frac{1}{2}\,r^2+\phi}\,dr}\,,$$

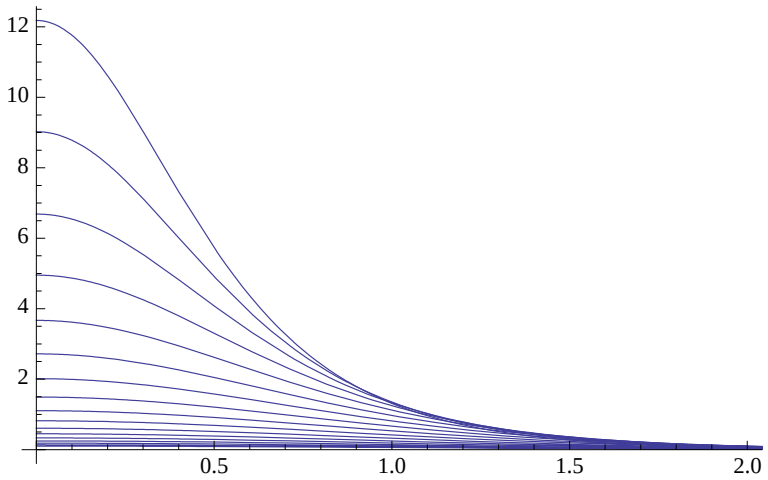

Figure 7.1: *The density n_a as a function of $r = |x|$ for $a = -2.3$, -1.9,...-0.2, 0.1, 0.4, 0.7... 2.5.*

all radial solutions of (7.1.3) can therefore be parametrized by $a \in \mathbb{R}$, using $M = M(a)$ with

$$M(a) := 2\pi \int_0^\infty e^{-\frac{1}{2}r^2+\phi_a}\, dr\ .$$

The density

$$n_a(x) := M(a)\, \frac{e^{-\frac{1}{2}|x|^2+c_a}}{\int_{\mathbb{R}^3} e^{-\frac{1}{2}|x|^2+c_a}\, dx}\ , \quad x \in \mathbb{R}^2$$

can be directly computed as

$$n_a(x) = M(a)\, \frac{e^{-\frac{1}{2}r^2+\phi_a(r)}}{2\pi \int_0^\infty r\, e^{-\frac{1}{2}r^2+\phi_a}\, dr} = e^{-\frac{1}{2}r^2+\phi_a(r)}$$

with $r = |x|$ (see Fig. 6.1).

Moreover it is clear that c in (7.1.3) is determined only up to the addition of a constant. This constant can be fixed by assuming that

$$\lim_{|x|\to\infty} \left(c(x) + \frac{M}{2\pi}\, \log|x| \right) = 0\ ,$$

and we will denote by c_a the corresponding solution. Hence, with

$$b(a) := \lim_{r\to\infty} \left(\phi_a(r) + \frac{M(a)}{2\pi}\, \log r \right)$$

we finally recover that

$$c_a(x) = \phi_a(|x|) - b(a)$$

(see Fig. 6.2).

The above considerations allow to parametrize by a the bifurcation diagram of the solutions of (7.1.3) in $L^\infty(\mathbb{R}^2)$ in terms of the mass M: see Figs. 6.3 and 6.4.

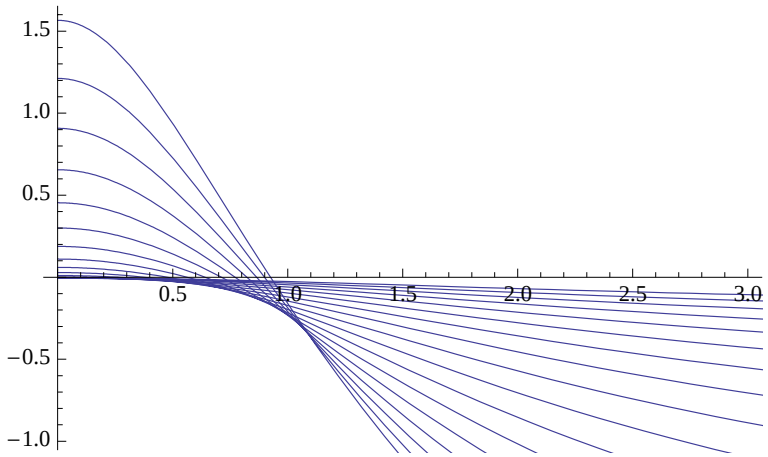

Figure 7.2: *The function c_a as a function of $r = |x|$ for $a = -2.3,\ -1.9,\ldots\ 2.5$.*

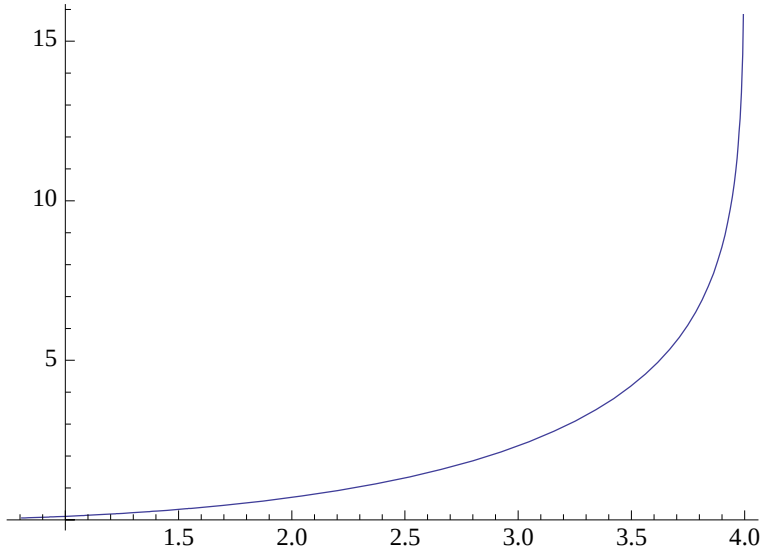

Figure 7.3: *The bifurcation diagram associated to solutions of* (7.1.3) *can be parametrized by $a \mapsto (\frac{1}{2\pi} M(a), \|c_a\|_{L^\infty(\mathbb{R}^2)})$. Here $\|c_a\|_{L^\infty(\mathbb{R}^2)} = c_a(0) = a - b(a)$. Such a diagram is qualitatively very similar to the one of the Keller-Segel system in a ball with no flux boundary conditions.*

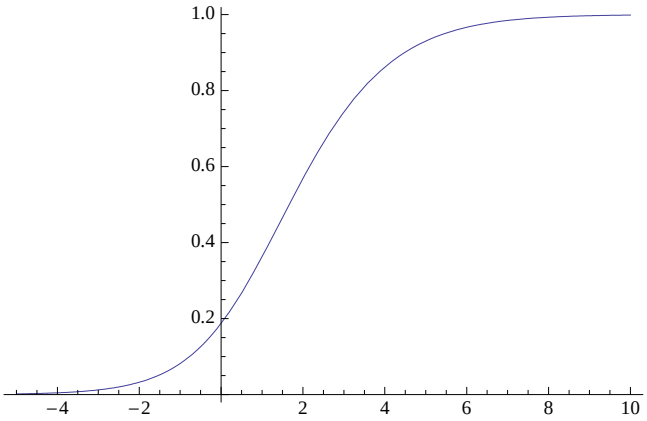

Figure 7.4: *The mass can be computed as* $M(a) = 2\pi \int_0^\infty n_a(r)\, r\, dr$. *Plot of* $a \mapsto \frac{1}{8\pi} M(a)$.

7.3 Asymptotic regimes

Before studying the eigenvalue problem associated to $\mathcal{L}$, it makes sense to investigate the limiting behaviors of the solutions of (7.1.3) as $a \to +\infty$ and $a \to 0$, in order to check the accuracy of our numerical approach. The regime $a \to +\infty$ is by itself interesting. Roughly speaking, concentration, which is numerically observed as the mass M approaches 8π, suggests that for $M = 8\pi$ the limiting problem is governed by the stationary solutions of (7.1.1). This is indeed what occurs and is confirmed by a simple asymptotic expansion.

7.3.1 The large, positive a regime

It can be numerically observed in Fig. 6.3 that

$$\lim_{a \to +\infty} M(a) = 8\pi \, .$$

With $\lambda(a) = 2\sqrt{2}\, e^{-a/2}$, we moreover observe that $\lambda(a)^2\, n_a(\lambda(a)\, x)$ converges as $a \to +\infty$ to

$$n_\star(x) := \frac{8}{(1+|x|^2)^2} \, , \quad x \in \mathbb{R}^2$$

which is the well known solution to the unscaled Keller-Segel model with mass $8\pi = \int_{\mathbb{R}^3} n_\star \, dx$. See Fig. 6.5.

The following asymptotics are not very difficult to recover heuristically. Let

$$c_\star(x) := -2 \log(1+|x|^2) \, , \quad x \in \mathbb{R}^2$$

and observe that $c_\star$ solves

$$-\Delta c_\star = 8\pi \, \frac{e^{c_\star}}{\int_{\mathbb{R}^3} e^{c_\star} \, dx} = n_\star$$

with $8\pi\, e^{c_\star} / \int_{\mathbb{R}^3} e^{c_\star} \, dx = n_\star$. Actually all radial solutions of the above equation are of the form $x \mapsto (\lambda^2\, n_\star(\lambda\, x), c_\star(\lambda\, x) + \mu)$ for any $\lambda > 0$ and $\mu \in \mathbb{R}$. Now, for our special choice of $c_\star$, we have $\int_{\mathbb{R}^3} e^{c_\star} \, dx = \pi$, and hence $c_\star$ is the unique solution to

$$-\Delta c_\star = 8\, e^{c_\star}$$

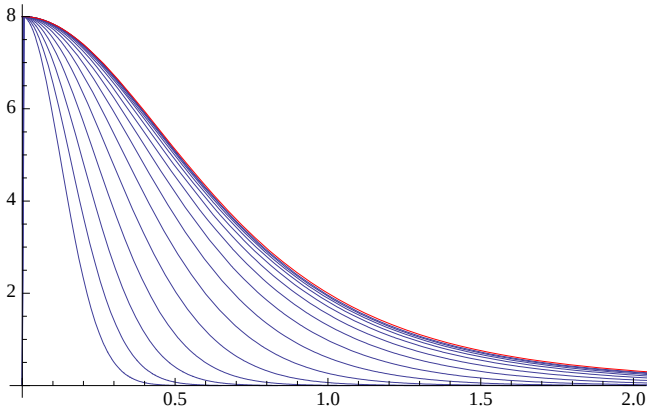

Figure 7.5: *The $a \to +\infty$ case. As a function of $r = |x|$, $\lambda(a)^2\, n_a(\lambda(a)\, x)$ is plotted in blue for $a = -2$, -1.5,... 5, while the limiting profile $r \mapsto n_\star(r)$ is shown in red.*

such that $c_\star(0) = 0$. Let $\psi_a(r) := \phi_a(\lambda\, r) - a$ and observe that

$$-\psi_a'' - \frac{1}{r}\,\psi_a' = e^{\psi_a + a + 2\log\lambda + \frac{\lambda^2}{2}\, r^2} = 8\, e^{\psi_a + \frac{\lambda^2}{2}\, r^2}$$

if $\lambda = \lambda(a) = 2\,\sqrt{2}\, e^{-a/2}$. Hence, since $\psi_a(0) = 0$ and $\lim_{a\to\infty} \lambda(a) = 0$, it is clear that ψ_a converges to $c_\star$. This justifies the fact that $\lambda(a)^2\, n_a(\lambda(a)\,\cdot)$ converges to $n_\star$ as $a \to +\infty$.

7.3.2 The large, negative a regime

When $a \to -\infty$, it is elementary to observe that $a - \phi_a(r) \sim e^a\, \psi(r)$ where ψ solves

$$-\psi'' - \frac{1}{r}\,\psi' = e^{-\frac{1}{2}\, r^2}$$

with $\psi(0) = \psi'(0) = 0$. Integrating this equation, we find that

$$2\,\psi(r) := \int_0^{r^2/2} \left(1 - e^{-s}\right)\, \frac{ds}{s} = \gamma + \Gamma(0, \tfrac{1}{2}\, r^2) - \log 2 + 2\, \log r\ .$$

Here $\Gamma(x, y) = \int_y^{+\infty} t^{x-1}\, e^{-t}\, dt$ is the *Incomplete Gamma Function* and $\gamma \approx 0.577216$ is *Euler's constant.* See Fig. 6.6.

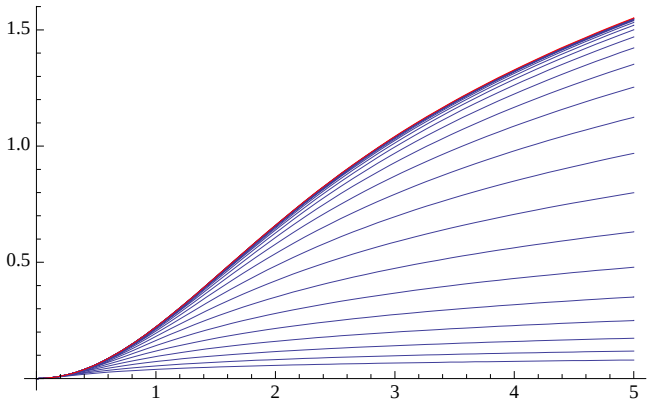

Figure 7.6: *The $a \to -\infty$ case. The function $e^{-a}\,(a-\phi_a)$ as a function of $r=|x|$ for $a=-5,\,-4.5,\ldots\,5$ and the limiting profile $r \mapsto \psi(r)$ (in red).*

7.4 Linearization and spectral gap

Consider the linearized Keller-Segel operator $\mathcal{L}$ introduced at the beginning of this report. Since the function n_a is involved in the linearization, to emphasize the dependence in the parameter a, we shall use the notation $\mathcal{L}_a$. Recall that this operator is defined by

$$\mathcal{L}_a\, f := \frac{1}{n_a} \nabla \cdot [n_a\, \nabla (f - \varphi_f)] \ , \quad x \in \mathbb{R}^2$$

where

$$-\Delta\, \varphi_f = n_a\, f \ .$$

7.4.1 Kernel of $\mathcal{L}_a$

A derivation of ϕ_a with respect to a provides a solution to $\mathcal{L}_a\, f = 0$, $f(0) = 1$, $f'(0) = 0$. By the Cauchy-Lipschitz theorem (and an appropriate analysis at $r = 0$), this solution is unique. See chapter 6 for more details. Hence we have found a solution f_a of

$$-f_a'' - \frac{1}{r}\, f_a' = e^{-\frac{1}{2}\, r^2 + \phi_{f_a}}\, f_a \ , \quad r > 0$$

with initial conditions $f_a(0) = 1$ and $f_a'(0) = 0$, which generates $\mathrm{Ker}(\mathcal{L}_a)$. See Figs. 6.7 and 6.8 for some plots of the solution for various values of a.

7.4.2 Non-zero eigenvalues of $\mathcal{L}_a$

According to chapter 6, $\mathcal{L}_a$ has no continuous spectrum. All non-zero eigenvalues of $-\mathcal{L}_a$ are positive and hence there is a positive spectral gap that can be fully determined using a decomposition in spherical harmonics: for a given $a \in \mathbb{R}$, the spectrum is obtained by solving the radial eigenvalue problems

$$-\mathcal{L}_a^{(k)}\, f_{k,\ell} = \lambda_{k,\ell}\, f_{k,\ell} \ , \quad \ell \in \mathbb{N}$$

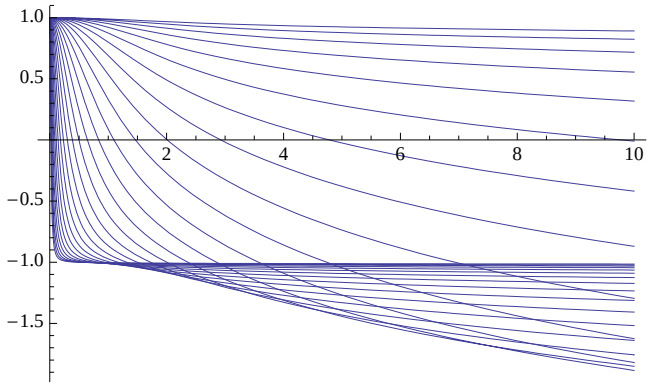

Figure 7.7: *The function f_a as a function of $r = |x|$ for $a = -3, -2, \ldots 10$.*

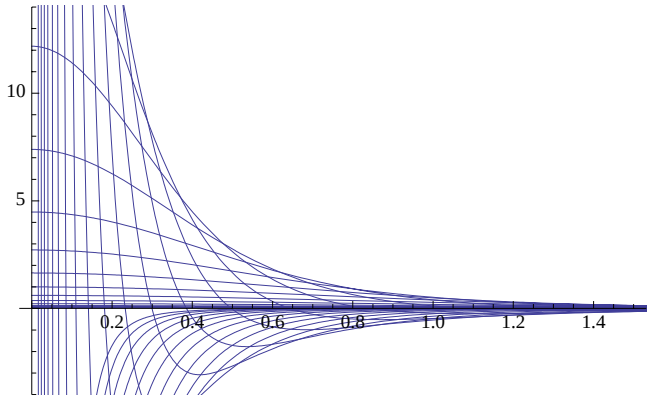

Figure 7.8: *The density $f_a\, n_a$ as a function of $r = |x|$ for $a = -3, -2, \ldots 10$.*

where, for any $k \in \mathbb{N}$,

$$-\mathcal{L}_a^{(k)} f = -f'' - \frac{1}{r} f' + \frac{k^2}{r^2} f + (r - c_a')\,(f' - \psi') - n_a\, f$$

and, with previous notations, $\psi = c_a\, g$ is obtained as the solution to

$$-\psi'' - \frac{1}{r}\,\psi' + \frac{k^2}{r^2}\,\psi = n_a\, f\ .$$

Here we draw the attention of the reader about the numbering of the eigenvalues, which differs from the one adopted in chapter 6.

7.4.3 Spectrum of $\mathcal{L}_a$ restricted to radial functions

To determine the spectrum of $\mathcal{L}_a^{(0)}$, we can use a simple shooting method that goes as follows. Owing to the fact that if λ is an eigenvalue, then $\lim_{r\to\infty} f(r) = 0$, we solve the equation

$$\mathcal{L}_a^{(0)} f + \lambda\, f = 0$$

with initial conditions $f(0) = 1$ and $f'(0) = 0$. We numerically recover that $\lambda = 0$ is an eigenvalue and find that the lowest non-zero eigenvalue of $-\mathcal{L}_a^{(0)}$ is exactly 2. See Figs. 6.9–13. This is consistent with the results of [117].

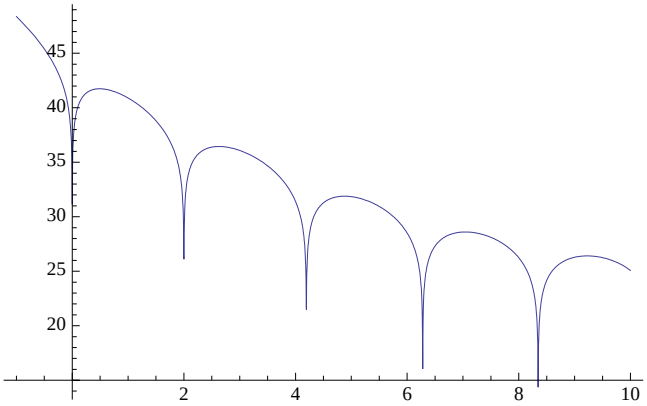

Figure 7.9: *Plot of* $\lambda \mapsto \log\left(1 + f(\lambda, R)^2\right)$ *with* $R = 7$, $a = 1$, *where* $r \mapsto f(\lambda, r)$ *is the solution to* $\mathcal{L}_a^{(0)} f + \lambda f = 0$ *such that* $f(0) = 1$ *and* $f'(0) = 0$. *Each local minimum corresponds to an eigenvalue in the limit* $R \to \infty$. *First minima (from the left) are located exactly at* $\lambda = 0$ *and* $\lambda = 2$.

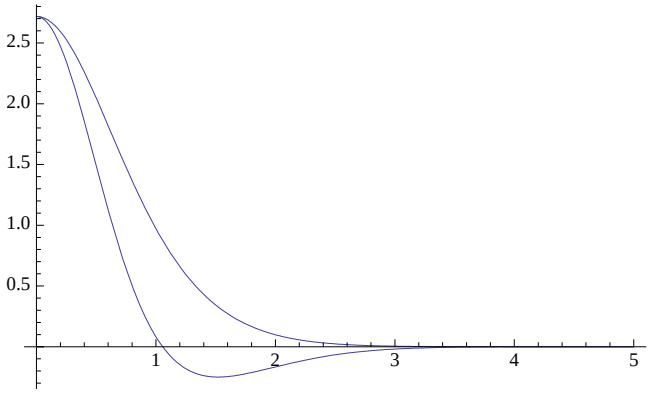

Figure 7.10: *Plot of* $r \mapsto n_a(r)$ *and* $r \mapsto n_a(r)\, f(r)$ *for* $a = 1$, $\lambda = 2$. *The eigenfunction* f *changes sign once. The total mass for* $a = 1$ *is* $M(a) = 2\pi \int_0^\infty n_a(r)\, r\, dr \approx 9.10875$.

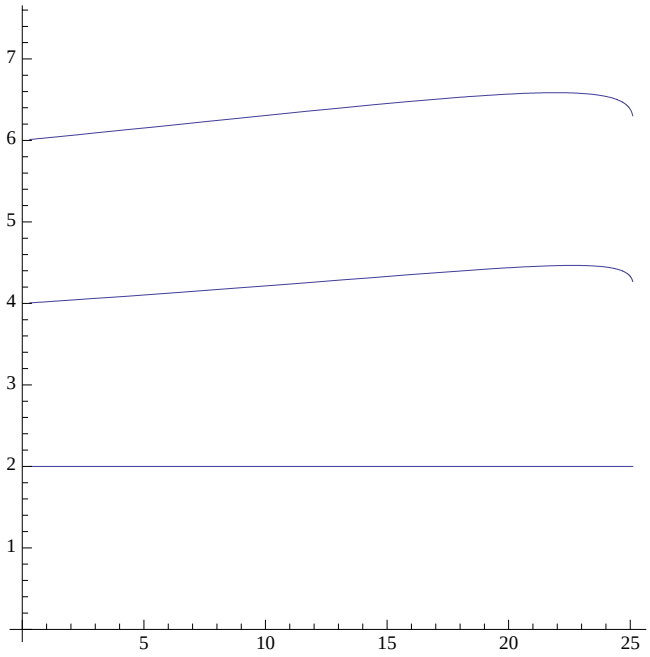

Figure 7.11: *Lowest eigenvalues in the spectrum in* $\mathcal{L}_a^{(0)}$*, as a function of* $M = M(a)$*.*

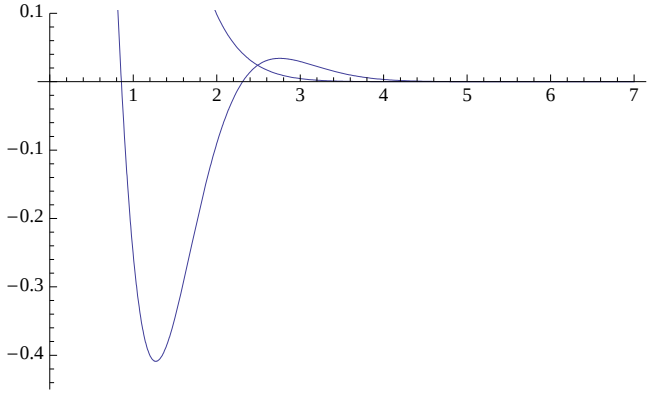

Figure 7.12: *Detail of the plots of* $r \mapsto n_a(r)$ *and* $r \mapsto n_a(r)\, f(r)$ *for* $a = 1$*,* $\lambda \approx 4.1944$*. The eigenfunction* f *changes sign twice.*

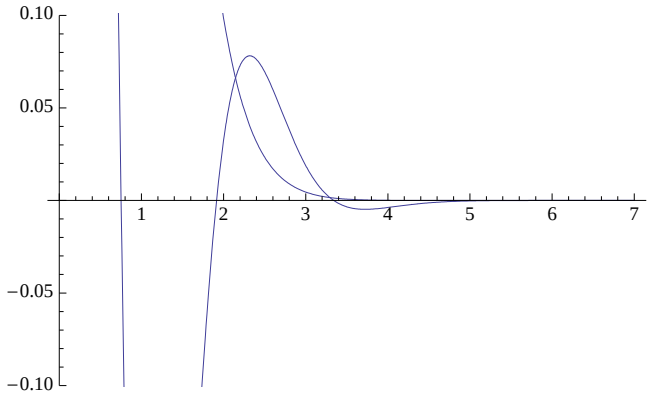

Figure 7.13: *Detail of the plots of* $r \mapsto n_a(r)$ *and* $r \mapsto n_a(r)\, f(r)$ *for* $a = 1$, $\lambda \approx 6.27881$. *The eigenfunction* f *changes sign three times.*

7.4.4 Spectrum of $\mathcal{L}_a^{(1)}$: the $k = 1$ component of the spectrum

By construction, we know that the spectrum $\mathrm{sp}(\mathcal{L}_a)$ of $\mathcal{L}_a$ can be decomposed using the spherical harmonics decomposition as

$$\mathrm{sp}(\mathcal{L}_a) = \bigcup_{k \in \mathbb{N}} \mathrm{sp}\big(\mathcal{L}_a^{(k)}\big)$$

where

$$\mathrm{sp}\big(\mathcal{L}_a^{(k)}\big) = (\lambda_{k,\ell})_{\ell \in \mathbb{N}}$$

for any $k \in \mathbb{N}$. Recall that $\{\lambda_{0,0}\}$ corresponds to the kernel of $\mathcal{L}_a$:

$$\lambda_{0,0} = 0 \, .$$

Moreover, we know that these spectra are ordered, in the sense that

$$\lambda_{k_1,\ell} \leq \lambda_{k_2,\ell} \quad \text{if} \quad k_1 \leq k_2 \, .$$

As a consequence, to determine the spectral gap, we only need to find the minimum of $\lambda_{0,1}$ and $\lambda_{1,0}$.

Numerically, we have observed that $\lambda_{0,1} = 2$ is an eigenvalue. This mode associated is to dilations. The mode associated to translations is in the component $k = 1$ and corresponds to an eigenvalue $\lambda_{1,\ell} = 1$, for some ℓ to be determined. See chapter 6 for the justification of the role of dilations and translations, and Section 7.5 for more detailed comments. Let us check numerically that $\ell = 0$ (*i.e.* that there is no other mode in the component $k = 1$ corresponding to an eigenvalue in $(0, 1)$), so that the spectral gap is

$$\lambda_{1,0} - \lambda_{0,0} = 1$$

and that this holds true for any value of $a \in \mathbb{R}$.

For this purpose, we determine $(\lambda_{k,\ell})_{\ell \in \mathbb{N}}$ by solving the system of ODEs

$$\begin{aligned} -f'' - \frac{1}{r}\, f' + \frac{k^2}{r^2}\, f + (r - c_a')\, (f' - \psi') - n_a\, f &= \lambda\, f \, , \\ -\psi'' - \frac{1}{r}\, \psi' + \frac{k^2}{r^2}\, \psi &= n_a\, f \, . \end{aligned}$$

Boundary conditions have to be determined appropriately. Let us focus on the case $k = 1$. We may fix $f(0) = 0$, $f'(0) = 1$ and $\psi(0) = 0$ without restriction. However, $p = -\psi'(0)$ has to be determined, and this cannot be done by a simple Taylor expansion around $r = 0_+$, as can be checked to the price of a painful computation, that we shall omit here. A numerical scheme has therefore to be invoked.

Before doing so, let us make an ansatz, which turns out to be very convenient. For the special choice of $\psi'(0) = -\frac{e^a}{e^a+2}$, we can plot $\lambda \mapsto \log\big(1 + f(\lambda, R)^2\big)$ as for the case $k = 0$. See Fig. 6.14. An explanation for this ansatz will be given below. The lowest eigenvalue found in the framework of this ansatz has the value 1 and corresponds to

$$f(r) = v_a'(r) - r\ , \quad \psi(r) = c_a'(r) \quad \forall\, r > 0$$

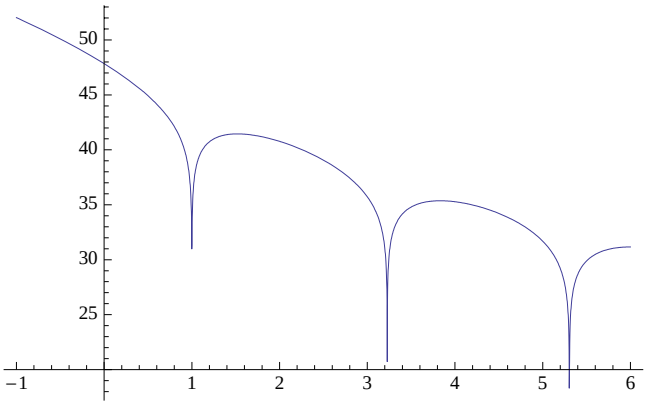

Figure 7.14: *Plot of* $\lambda \mapsto \log\big(1 + f(\lambda, R)^2\big)$ *with* $R = 7$, $a = 1$, *where* $r \mapsto f(\lambda, r)$ *is the solution to* $\mathcal{L}_a^{(1)} f + \lambda f = 0$ *such that* $f(0) = 1$, $f'(0) = 0$, $\psi(0) = 0$ *and* $\psi'(0) = -\frac{e^a}{e^a+2}$. *In the limit* $R \to \infty$, *the first minimum (from the left) is located exactly at* $\lambda = 1$. *However, because of the ansatz, we have no guarantee that there is no other eigenvalues, or even that the other minima are actually eigenvalues. With* $R = 7$, *the second minimum (from the left) is achieved for* $\lambda \approx 3.22762$. *See Fig. 6.15.*

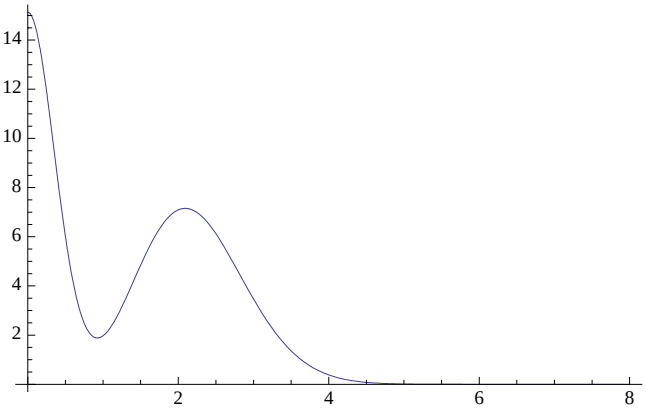

Figure 7.15: *Case* $R = 7$, $a = 1$ *and* $\lambda \approx 3.23$. *One has to test if the solution to the ODE system with same ansatz as in Fig. 6.14 is in the space* $H^1(0, \infty; n_a\, r\, dr)$. *The plot of* $r \mapsto (|f'|^2 + [f|^2)\, n_a$ *is shown above. Clearly the solution found numerically is admissible.*

Now let us come back to the general case. For a given a and λ, we can consider the function which associates to a given $p > 0$ the value of $h(a, \lambda, p, R) := \int_0^R (|f'|^2 + |f|^2)\, n_a\, r\, dr$, for R large enough. See Fig. 6.16.

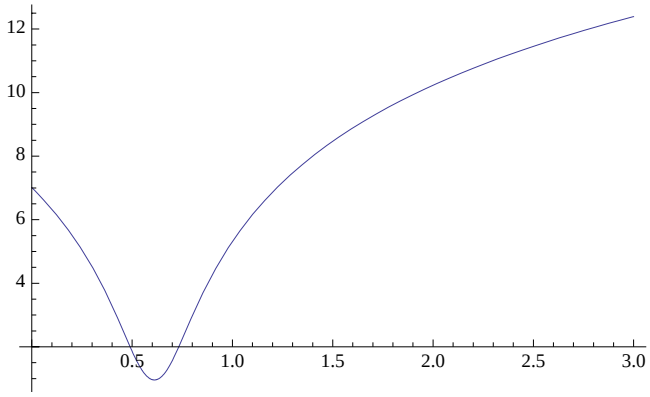

Figure 7.16: *Plot of $p \mapsto \log(1 + h(a, \lambda, p, R)^2)$ for $a = 1$, $\lambda = 0.5$. The difficulty comes from the instability of the solutions with respect to the parameters λ and p. If the function f is not in the space $H^1(0, \infty; n_a\, r\, dr)$, then $h(a, \lambda, p, R)^2$ uniformly diverges as $R \to +\infty$. A possible method is therefore to find the value of p that realizes the minimal value of $h(a, \lambda, p, R)^2$ for a given $R > 0$, and then select for which value of λ this quantity converges to a finite value as $R \to +\infty$. In practice, only rather small values of R can be taken into account, which makes the method inaccurate. Here $R = 3$.*

The main advantage in the approach used for plotting Fig. 6.14 is that the expression of $\psi'(0)$ was explicitly known in terms of a, at least for one solution. This, however, suggests a new shooting criterion, which goes as follows.

Solutions corresponding to $k = 1$ have to solve the Poisson equation

$$-\Delta\left(\psi(r)\,\frac{x_1}{r}\right) = n_a(r)\, f(r)\,\frac{x_1}{r}\,, \quad r = |x|\,, \quad x = (x_1, x_2) \in \mathbb{R}^2$$

with $i = 1, 2$, and can be expressed as

$$\Psi(x) := \psi(r)\,\frac{x_1}{r} = -\frac{1}{2\pi}\int_{\mathbb{R}^3} \log|x - y| * n_a(|y|)\, f(|y|)\,\frac{y_1}{|y|}\, dx\,.$$

Consider the case $i = 1$ and let $\theta \in [0, 2\pi)$ be such that $\frac{x_1}{r} = \cos\theta$.

$$(1, 0)\cdot\nabla\Psi(0) = \psi'(0) = -\frac{1}{2\pi}\int_0^{2\pi}\cos^2\theta\, d\theta \int_0^\infty n_a(r)\, f(r)\, dr$$
$$= -\frac{1}{2}\int_0^\infty n_a(r)\, f(r)\, dr\,.$$

Notice that

$$\int_0^\infty n_a(r)\, f(r)\, dr = \int_{\mathbb{R}^3}\frac{n_a(|x|)\, f(|x|)}{2\pi\,|x|}\, dx\,.$$

This observation provides a new shooting criterion (see Figs. 6.17 and 6.18): any solution has to satisfy the condition

$$s(a, \lambda, p) = 0 \quad \text{where} \quad s(a, \lambda, p) := \left(2\,p + \int_0^\infty n_a(r)\, f(r)\, dr\right)^2\,,$$

where f and ψ are solutions with $f(0) = 0$, $f'(0) = 1$, $\psi(0) = 0$ and $p = -\psi'(0)$. Notice that we recover that $\psi'(0) = \frac{1}{2}\, e^a$ if $f(r) = \phi'(r) - r$, with $f(0) = 0$ and $f'(0) = \phi''(0) - 1 = -(1 + \frac{1}{2}\, e^a)$, or, if we impose $f'(0) = 1$ (which can always be done because we solve a linear problem), $\psi'(0) = -\frac{e^a}{e^a+2}$.

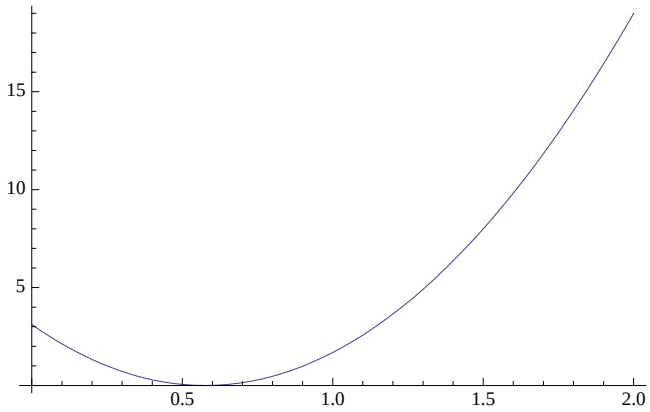

Figure 7.17: *Plot of $p \mapsto s(a, \lambda, p)$ for $a = 1$ and $\lambda = 1$. We numerically recover the fact that $p = \frac{e^a}{e^a+2} \approx 0.576117$.*

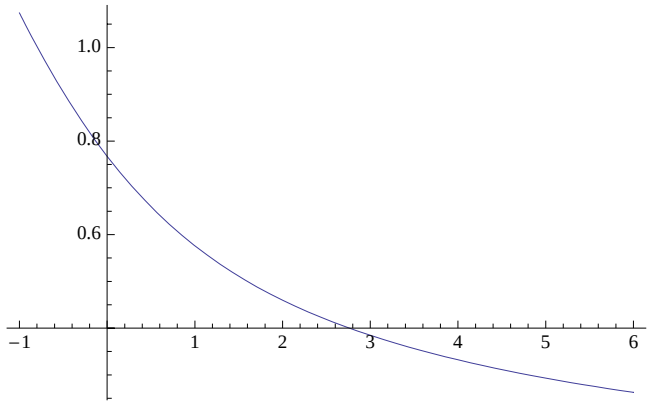

Figure 7.18: *Solving $s(a, \lambda, p) = 0$ determines $p = p(a, \lambda)$. Here is shown the plot of $\lambda \mapsto p(a, \lambda)$ for $a = 1$.*

By considering a shooting criterion similar to the one for $k = 0$, we obtain the spectrum of $-\mathcal{L}_a^{(1)}$. See Fig. 6.19. This completes the study of the spectrum corresponding to $k = 1$. Plotting the spectrum of $-\mathcal{L}_a^{(1)}$ as a function of a, or equivalently as a function of the mass M can now be done: see Figs. 6.19 and 6.20.

Wit these results in hands, it is easy to check that $\lambda_{1,0} = 1$ for any $M \in (0, 8\pi)$. As a consequence, the spectral gap of $-\mathcal{L}_a$ is $\lambda_{1,0} - \lambda_{0,0} = 1$. See Fig. 6.21.

For higher values of k, that is $k \geq 2$, the same numerical methods than for $k = 1$ holds. Numerically we observe (see Fig. 6.22) that the lowest eigenvalue for $k = 2$ is $\lambda_{2,0}$ which takes values larger than 4. The branch originates from $\lambda = 4$ when $M \to 0_+$.

As a consequence, the lowest eigenvalues of $\mathcal{L}$ are

$$\lambda_{0,0} = 0 < \lambda_{1,0} = 1 < \lambda_{0,1} = 2 < 3 < \lambda_{1,1} < \lambda_{0,2} < \lambda_{2,0}$$

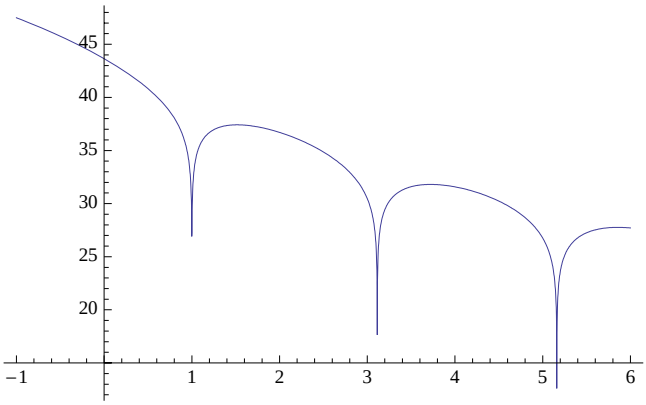

Figure 7.19: *Plot of* $\lambda \mapsto \log\left(1 + f(\lambda, R)^2\right)$ *with* $R = 7$, $a = 1$, *where* $r \mapsto f(\lambda, r)$ *is the solution to* $\mathcal{L}_a^{(1)} f + \lambda f = 0$ *such that* $f(0) = 1$ *and* $f'(0) = 0$, $\psi(0) = 0$ *and* $\psi'(0) = -p(a, \lambda)$. *In the limit* $R \to \infty$, *the first minimum (from the left) is located exactly at* $\lambda = 1$. *Each minimum determines an eigenvalue of* $-\mathcal{L}_a^{(1)}$.

but $\lambda_{1,1}$ is not constant as M varies in $(0, 8\pi)$.

7.5 Concluding remarks

From a physics viewpoint, understanding why 0, 1 and 2 are eigenvalues is not very difficult.

(a) The stationary solution depends on the mass. Differentiating the equation with respect to the mass parameter immediately provides an element of the kernel, which turns out to be one-dimensional as can be shown by elementary considerations (uniqueness of the solution to an ODE by the Cauchy-Lipschitz theorem). As far as we are interested in the long time asymptotics of the solutions to (7.1.2), such a degree of freedom is not relevant for the evolution problem because the conservation of mass uniquely determines the limiting stationary solution.

(b) The Keller-Segel model before rescaling is an autonomous system: it does not depend explicitly on x. Any translation of the initial datum gives rise to a solution to the evolution problem translated by the same quantity, and it is straightforward to realize that the position of the center of mass is preserved along the evolution. In the rescaled variables, it is clear that a solution corresponding to an initial datum made of a decentered self-similar profile exponentially converges towards the same self-similar profile, but centered. When linearizing, this provides an eigenmode (that can be computed by applying the operators which infinitesimally generate the translations, $\partial/\partial x_1$ or $\partial/\partial x_2$) and a direct computation shows that the corresponding eigenvalue is 1.

(c) The reason why $x \cdot \nabla$ also generates an eigenmode is slightly more subtle. In the original variables, the self-similar solutions explicitly depend on t, and a shift in t amounts to a scaling of the self-similar solutions. Notice indeed that a solution translated in t is still a solution. Once the self-similar change of variables has been done, any shift with respect to t amounts to a scaling on the solution and thus explains why 2 is an eigenvalue.

More details on mathematical aspects of these observations can be found in chapter 6. Hence it is easy to understand why 0, 1 and 2 are eigenvalues, independently of the mass M. We have moreover identified

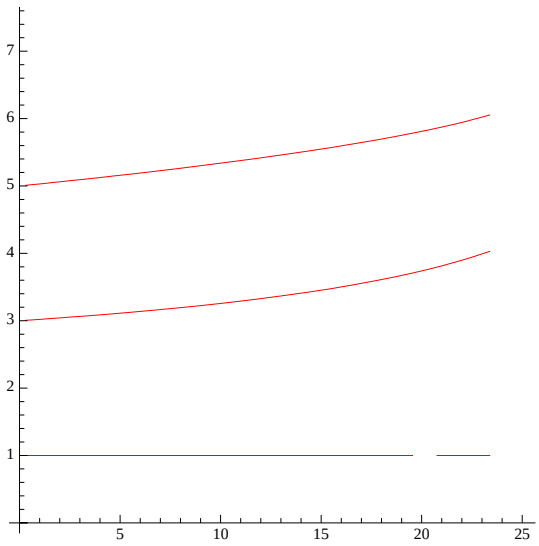

Figure 7.20: *Lowest eigenvalues of* $-\mathcal{L}_a^{(1)}$ *as a function of* $M(a)$*. Missing values corresponding to* $\lambda = 1$ *are due to numerical errors.*

the invariances that explain such facts. In the limit $M \to 0_+$, it has been observed in [115] that the spectrum of $\mathcal{L}$ is the same as the Fokker-Planck operator.

It has been stablished in chapter 6 that the spectrum of $\mathcal{L}$ governs the rate of convergence of the solutions to (7.1.2): *for any* $M \in (0, 8\,\pi)$*, if* $n_0 \in L^2_+(n^{-1}\,dx)$ *and* $M := \int_{\mathbb{R}^3} n_0\ dx < 8\,\pi$*, then any solution to* (7.1.2) *with initial datum* n_0 *satisfies*

$$\int_{\mathbb{R}^2} |n(t,x) - n_\infty(x)|^2\,\frac{dx}{n_\infty(x)} \le C\,e^{-2\,\lambda\,t} \quad \forall\ t \ge 0$$

for some positive constant C*, where* n_∞ *is the unique stationary solution to* (7.1.2) *with mass* M *and*

$$\lambda = \lambda_{1,0} = 1\ ,$$

provided the following technical condition is satisfied

$$\exists\ \varepsilon \in (0, 8\,\pi - M) \quad \textit{such that} \quad \int_0^s u_{0,*}(\sigma)\ d\sigma \le \int_{B\left(0,\sqrt{s/\pi}\right)} n_{\infty,M+\varepsilon}(x)\ dx$$

for any $s \ge 0$*. Here* $u_{0,*}(\sigma)$ *stands for the symmetrized function associated to* n_0*.*

If additionally the initial datum satisfies $\int_{\mathbb{R}^3} x\,n_0\ dx = 0$, then

$$\lambda = \lambda_{0,1} = 2\ .$$

Based on a similar approach that has been developed in the framework of the fast diffusion equation in [119], we can even define the *best matching asymptotic profile* as the function $\tilde{n}_\infty(t,x) = n_{\infty,\sigma(t)}(x)$ where $n_{\infty,\sigma} := \sigma^2\,n_\infty(\sigma\cdot)$ and $\sigma = \sigma(t)$ realizes the infimum

$$\mu \mapsto \int_{\mathbb{R}^2} |n(t,x) - n_{\infty,\mu}(x)|^2\,\frac{dx}{n_{\infty,\mu}(x)}\ .$$

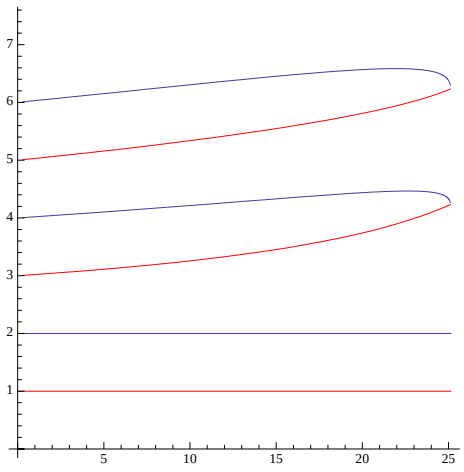

Figure 7.21: *Results of Figs. 6.11 and 6.20 are shown on a single picture. The lowest eigenvalues of* $-\mathcal{L}_a$ *are therefore* 0, 1 *and* 2, *thus establishing that the spectral gap of* $-\mathcal{L}_a$ *is* 1.

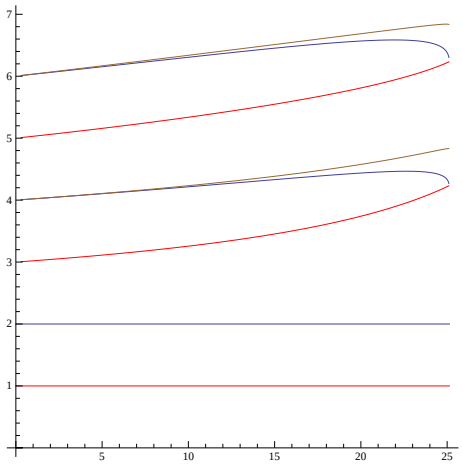

Figure 7.22: *Lowest eigenvalues, for* $k = 0$ *(blue),* $k = 1$ *(red) and* $k = 2$ *(brown).*

If $\int_{\mathbb{R}^3} x\, n_0\, dx = 0$, then it follows from an analysis similar to the one of [119] that

$$\int_{\mathbb{R}^2} |n(t,x) - \tilde{n}_\infty(x)|^2 \, \frac{dx}{\tilde{n}_\infty(x)} \leq C\, e^{-2\,\lambda_{1,1}\,t} \quad \forall\, t \geq 0$$

and our numerical results show that $\lambda_{1,1} \geq 3$.

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Chapter 8

Matched Asymptotics Analysis in the Keller-Segel model with critical Mass

In this last chapter we consider the Keller-Segel model in the plane in the critical mass case. It is known that infinite time aggregation occurs, which means that there exists a solution that blows-up in infinite time. The limiting profile is a delta Dirac measure located at the center of mass, of total mass 8π. Here by a formal method we derive a refined asymptotic behavior for radial blowing up solutions. The space turns out to be divided in three regions, the inner region expanding from the origin at rate $O(\sqrt{t})$, a remote region and an intermediate one.

This is a joint work with M. del Pino.

8.1 Introduction

The Keller-Segel system describes the collective motion of cells that are attracted by a self-emitted chemical substance. There are numerous versions of this model for chemotaxis, we refer the reader to the very nice review papers [29, 30] and references therein. Historically the key papers are the contribution of E.F. Keller and L.A. Segel [143] in the 1970, and Patlak [148] in 1953. Here we consider the parabolic-elliptic Keller-Segel system in the whole plane

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + \nabla \cdot (u \nabla v) & x \in \mathbb{R}^2, \quad t > 0, \\ \Delta v = u & x \in \mathbb{R}^2, \quad t > 0, \\ u(0,x) = u_0(x) \geq 0 & x \in \mathbb{R}^2. \end{cases} \tag{8.1.1}$$

It is usual to include a parameter $\chi > 0$ in front of the nonlinear term $\nabla \cdot (u \nabla v)$, which measures the *sensitivity* of the bacteriae to the chemoattractant. Such a parameter can be removed by scaling which, however, does not preserve the total mass of the system

$$m := \int_{\mathbb{R}^2} u_0(x)\, dx = \int_{\mathbb{R}^2} u(x,t)\, dx.$$

Assuming the existence of a smooth, fast decaying and nonnegative solution in $L^1([0,T] \times \mathbb{R}^2)$ for all $T > 0$ then formally it holds that m remains constant in time, and also its center of mass and its second

momentum

$$\begin{aligned} m_1 &:= \int_{\mathbb{R}^2} x\, u_0(x)\, dx = \int_{\mathbb{R}^2} x\, u(x,t)\, dx, \\ m_2 &:= \int_{\mathbb{R}^2} |x|^2\, u_0(x)\, dx = \int_{\mathbb{R}^2} |x|^2\, u(x)\, dx. \end{aligned}$$

Due to translational invariance we will assume $m_1 = 0$.

It is usual to impose no-flux boundary conditions when the system is posed in a bounded domain. Since here we are not interested in boundary effects we state the problem in the whole space without boundary conditions. The dimension 2 is critical for the L^1 norm.

The Poisson equation $\Delta v = u$ determines v up to a harmonic function. We will define the concentration directly by

$$v = \frac{1}{2\pi} \log|\cdot| * u \qquad x \in \mathbb{R}^2, t > 0. \tag{8.1.2}$$

As conjectured by S. Childress and J.K. Percus [133] (also see [147]) *chemotactic collapse* occurs, meaning that either the solution exists globally in time or it blows up in finite time. They also state that the aggregation should be proceeded by the formation of a delta Dirac measure located at the center of mass of the cell density. The balance between the tendency to spread mass to infinity by diffusion and the effect of aggregation caused by the drift term happens precisely at the critical mass $m = 8\pi$.

In fact, in [136], [131], it has been shown that if $m < 8\pi$ and under the hypothesis

$$u_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2)dx), \qquad \text{and} \qquad u_0 \,|\log u_0| \in L^1(\mathbb{R}^2) \tag{8.1.3}$$

there exists a solution u global in time, in a distribution sense. This result completes the picture given in [141]. Furthermore the asymptotic behavior of the solutions, is given uniquely by *self-similar* profiles. Refined estimates of the distance between these profiles and the solutions are given in [130]. Concerning the radially symmetric solutions, self similar behavior has been obtained in [128].

When the mass m is bigger than 8π it is easy to see, under (8.1.3), using second moment estimates, that global solutions cannot exists and they blow up in finite time. Key contributions concerning the blow-up phenomena have been made by Herrero, Velázquez in [140], [139], and by Velázquez in [152]. See also [146] for numerical evidence.

The critical case $m = 8\pi$ has an explicit family of stationary solutions

$$u_\lambda(x) = \frac{8\,\lambda}{(|x|^2 + \lambda)^2}$$

for $\lambda > 0$. These stationary solutions, that have critical mass but infinite second moment, play a crucial role on the behavior of blowing up solutions. In fact if we consider a bounded domain Ω and the system

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + \nabla \cdot (u \nabla v) & x \in \Omega, \quad t > 0, \\ \Delta v = u - 1 & x \in \Omega, \quad t > 0, \\ u(0,x) = u_0(x) \geq 0 & x \in \Omega, \end{cases} \tag{8.1.4}$$

which is similar to (8.1.1), it holds that if $m > 8\pi$ then there exists a formal solution that yields a concentration of mass 8π near the origin, and the asymptotics are given by

$$u(x,t) \quad = \quad u_{(T-t)\varepsilon(-\log(T-t))}(|x|) \tag{8.1.5}$$

for $|x| = O(\sqrt{(T-t)\varepsilon(-\log(T-t))})$ as $t \to T^-$, with

$$\varepsilon(\tau) \quad = \quad O(e^{\sqrt{2\tau}}) \quad \text{ as } \tau \to \infty. \tag{8.1.6}$$

This result was obtained in [140] in the radial case, and in [140] for a general bounded domain.

The critical mass case $m = 8\pi$ was considered in [128] for the radial case where the authors show that there exists a global solution for initial data with infinite or finite second momentum. They also obtain that the family u_λ attract solutions with initial data with *infinite* second momentum defined in a precise way, in which case there is a Lyapunov functional. Furthermore when the second momentum is finite Blanchet, Carrillo and Masmoudi proved in [129] that there exists a solution u^* that is global in time, has finite *free energy* $\mathcal{F}$ given by

$$\mathcal{F}[u](t) \quad := \quad \int_{\mathbb{R}^2} u(x,t)\log u(x,t)dx - \frac{1}{2}\int_{\mathbb{R}^2} u(x,t)v(x,t)dx,$$

and moreover

$$u^*(x,t) \quad \to \quad 8\pi\delta_0 \quad \text{as} t \to \infty$$

in the sense of weak-star measures.

However the question regarding the rate of the blow-up and the shape of the limiting profile has been largely left open. In this chapter we formally derive the behavior of a radial global solution with infinite time blow-up using the *matching asymptotics* method and find that, like in the super-critical mass case, the family of stationary solutions u_λ approximate the solution close to the origin, but far from the origin it has different shape in order to keep the second momentum finite and constant in some sense.

We will prove that as t goes to infinity:

$$u(x,t) \quad \approx \quad u_{\lambda(t)}(|x|) \tag{8.1.7}$$

when $|x| = O(\sqrt{t})$, and

$$u(x,t) \quad \approx \quad \kappa_1\,\lambda(t)\,e^{-\frac{|x|^2}{4t}}\,\frac{1}{|x|^4} \tag{8.1.8}$$

elsewhere. The blow-up rate is given by

$$\lambda(t)^{-1} \quad = \quad \kappa\Big(\log t + O\big(\log(\log t)\big)\Big) \tag{8.1.9}$$

as $t \to \infty$, where κ is a constant depending on the second momentum of the initial data u_0.

This kind of behavior, namely the convergence to an eternal solution of the equation in an inner layer, appears to be common in describing blow-up patterns around the singularity. For example the same phenomenon has been found for the *logarithmic fast diffusion* formally in [144], rigorously in [134] for the radial case, and in [135] in the non-radial case.

The matched expansions method has been used to obtain asymptotic profiles in parabolic equations exhibiting infinite time blow-up previously in the semilinear heat equation in [138], [137]. The method also works in the finite time blow-up case, for example for the harmonic map heat flow [151], [126].

Our asymptotic expansion has been obtained independently by Chavanis and Sire in [132], by different methods. However the matched expansions approach provides a better understanding of the linearized problem, which plays a crucial role for applying the Lyapunov-Schmidt reduction. The construction of *growing up* solutions via singular perturbations arguments is an interesting line of research to be developed with the approximated solution given here as a starting point.

This chapter is organized as follows: in the next section we will describe the setting of the problem and the notation, and we will introduce the *cumulated mass* variable to simplify the study of the equation. Next we derive an approximation of the behavior of the solution in an inner layer, and in section 4 we will see how far this approximation should remain valid. We continue by deriving an estimate of the solution on a remote region moving away from the origin. Finally we obtain the rate of blow-up by matching terms between the asymptotic expansions. In the appendix we briefly mention the differences between this case and the bounded domain case in which also infinite time blow-up occurs (see [127, 142, 150]).

8.2 Setting of the problem

We will assume that the initial data u_0 is radially symmetric and decaying, which leads to radial symmetry of u and the same monotonicity property, see [149]. Let us consider the *cumulated mass* function of u,

$$M(r,t) = 8\pi - \int_{B(0,\sqrt{r})} u(x,t)dx,$$

which solves the 1-d parabolic equation

$$\partial_t M(r,t) = 4r\partial_{rr}M(r,t) + 8\partial_r M(r,t) - \frac{1}{\pi}M(r,t)\partial_r M(r,t) \tag{8.2.1}$$

on $(0,\infty)\times(0,\infty)$, together with the boundary conditions

$$M(0,t) = 8\pi, \quad M(\infty,t) \equiv \lim_{r\to\infty} M(r,t) = 0, \tag{8.2.2}$$

$$M(r,0) = M_0(r) := \int_{B(0,\sqrt{r})^c} u_0(x)dx. \tag{8.2.3}$$

It is usual to use this transformation to study the Keller-Segel in the radial case, (see for instance [128, 142, 127, 149]), because it is simpler than the original one since this is a local equation. Clearly if M solves (8.2.1) then

$$u(x,t) = -\frac{1}{\pi}\partial_r M(|x|^2,t) \tag{8.2.4}$$

solves (8.1.1).

Formally it is easy to see that

$$\|M(\cdot,t)\|_{L^1(0,\infty)} = \|M_0(\cdot)\|_{L^1(0,\infty)}, \tag{8.2.5}$$

which is actually the analogue of the conservation of the second momentum for the function u (for a rigorous proof see [128]). The problem (8.2.1), (8.2.2), (8.2.3) has a family of steady solutions given by

$$S_\lambda(r,t) = 8\pi\frac{\lambda}{r+\lambda}. \tag{8.2.6}$$

But since $S_\lambda \notin L^1(0,\infty)$ we cannot expect that M gets closer to an element of this family as t goes to infinity. However the family $(S_\lambda)_{\lambda>0}$ is useful to provide an approximation of M close to origin.

In what will follows we will consider a global solution $u(x,t) \to 8\pi\,\delta_0$ which is radially symmetric, and denote $M(r,t)$ its associated cumulated mass function. Since u is radially decaying, $\forall\ t>0$

$$u(0,t) = \sup_{x\in\mathbb{R}^2} u(x,t) > 0.$$

Define

$$\alpha(t) = \frac{1}{u(0,t)}.$$

Since $\alpha(t) \to 0$ as $t\to\infty$, our goal is to estimate the behavior of $\alpha(t)$ for large values of t. Since u is regular and attains its maximum at the origin, we see that

$$\begin{aligned} \partial_t u(0,t) &= \Delta u(0,t) - \nabla\cdot(u(0,t)\nabla v(0,t)) \\ -\frac{\partial_t u(0,t)}{u^2(0,t)} &= -\frac{\Delta u(0,t)}{u^2(0,t)} - 1 \\ \alpha'(t) &\geq -1. \end{aligned}$$

We will assume that there exists an upper bound for $\alpha'(t)$ and that $|\alpha''(t)|$ is also bounded.

8.3 Inner Region Convergence

Let us consider now

$$V_\alpha(\xi,\tau) = M(\alpha(t(\tau))\xi, t(\tau)). \tag{8.3.1}$$

where

$$\tau(t) = \int_0^t \frac{1}{\alpha(r)}\,dr.$$

It is easy to see that τ is a diffeomorphism between $(0,\infty)$ and $(0,\infty)$. This scaled function solves

$$\partial_\tau V_\alpha = 4\xi\partial_{\xi\xi}V_\alpha + 8\partial_\xi V_\alpha - \frac{1}{\pi}V_\alpha\partial_\xi V_\alpha - g(\tau)\xi\partial_\xi V_\alpha \quad \text{on } (0,\infty)\times(0,\infty) \tag{8.3.2}$$

where $g(\tau) = -\alpha'(t(\tau))$. It is positive for any τ large enough. This change of variables is made to prevent the blow-up of $\partial_\xi V_\alpha$ at the origin, since it holds

$$\partial_\xi V_\alpha(0,\tau) = -\pi. \tag{8.3.3}$$

Based on the assumptions on α' we deduce that $g(\tau+\tau_k)\to 0$ for any sequence $\tau_k\to\infty$, and then, from classical regularity theory of parabolic equations ([145]) and from (8.3.3), we deduce that $\partial V(\xi,\tau+\tau_k)$ is bounded on compacts subsets of $[0,\infty)$. It follows from Arzelà-Ascoli's theorem that there exists a subsequence τ_{k_j}, for which $V_\alpha(\xi,\tau+\tau_{k_j})$ converges uniformly over compacts sets of $[0,\infty)\times[-\infty,\infty)$ to some function$\bar{V}$, where $\bar{V}$ is an eternal solution of

$$\partial_\tau \bar{V} = 4\xi\partial_{\xi\xi}\bar{V} + \partial_\xi \bar{V}_\alpha - \frac{1}{\pi}\bar{V}\partial_\xi\bar{V} \quad \text{on } (0,\infty). \tag{8.3.4}$$

Furthermore, we have

$$\begin{aligned} \bar{V}(0,\tau) &= 0, && (8.3.5)\\ \bar{V}(\xi,\tau) &\in [0,8\pi], \quad \forall\, \xi \geq 0, \tau \in \mathbb{R}, && (8.3.6)\\ \partial_\xi \bar{V}(0,\tau) &= -\pi, && (8.3.7)\\ \partial_\xi \bar{V}(\xi,\tau) &\geq 0, \quad \forall\, \xi \geq 0, \tau \in \mathbb{R}. && (8.3.8) \end{aligned}$$

We will suppose that $\bar{V}$ is actually a stationary solution

$$\bar{V}(\xi,\tau) = \bar{V}(\xi).$$

Moreover, it holds $\lim_{\xi\to\infty} \bar{V}(\xi) = 8\pi$, because if $\hat{V} < 8\pi$ then necessarily $V \equiv 0$ since there are no non-trivial solutions to problem (8.3.4) with $\lim_{\xi\to\infty} \bar{V}(\xi) < 8\pi$, which contradicts (8.3.7). When $\hat{V} = 8\pi$ all solutions are given by the family

$$S_\lambda = 8\pi \frac{\lambda}{\xi+\lambda}, \qquad \lambda \in \mathbb{R}.$$

Then using (8.3.7), we get

$$V_\alpha(\xi,\tau) \;\to\; S_8(\xi)$$

uniformly over compact sets. Thus it holds

$$M(s,t) \to S_8\left(\frac{s}{\alpha(t)}\right) = 64\pi \frac{\alpha(t)}{s+8\alpha(t)} \qquad \text{as } t\to\infty \tag{8.3.9}$$

for all $s \leq K\alpha(t)$, and constant $K > 0$.

8.4 Refined Inner Approximation

Let us consider now the change of variables

$$T(l,\tau) = 8\pi - V_\alpha(e^{-2l},\tau).$$

Then T solves

$$\partial_\tau T = e^{2l}\left(\partial_{ll}T + 2\partial_l T - \frac{1}{2\pi} T\partial_l T\right) + \frac{1}{2} g(\tau)\partial_l T.$$

From the previous section we know that formally for every sequence $\tau_k \to \infty$ there exist a subsequence τ_{k_j} such that

$$T(l,\tau+\tau_{k_j}) \to T_0(l) := 8\pi - S_8(e^{-2l}) = 8\pi \frac{e^{-2l}}{e^{-2l}+8} \quad \text{as } j\to\infty$$

uniformly over compact sets of $(-\infty,\infty)\times(0,\infty)$. Since $g(\tau)$ becomes smaller as $\tau\to\infty$ we suppose an expansion of T of the form

$$T(l,\tau) = T_0(l) + g(\tau)R(l,\tau),$$

where R solves

$$\begin{aligned}\partial_\tau R &= e^{2s}\left(\partial_{ss}R + 2\partial_s R - \frac{1}{2\pi}\left[T_0' R + T_0\partial_s R + g(\tau)R\partial_s R\right]\right)\\ &\quad \frac{1}{2}T_0' + \frac{1}{2}g(\tau)\partial_l R - \frac{g'(\tau)}{g(\tau)}R, \end{aligned} \tag{8.4.1}$$

and

$$T_0'' + 2T_0' - \frac{1}{2\pi}T_0T_0' = 0.$$

We will further assume that $g'g^{-1}(\tau) \to 0$ as τ becomes bigger, and we will also suppose that $R(l,\tau) \to T_1$ as $\tau \to \infty$, where T_1 is a stationary solution of

$$\mathcal{L}(T_1) := T_1'' + 2T_1' - \frac{1}{2\pi}\left(T_0'T_1 + T_0T_1'\right) = -\frac{1}{2}e^{-2l}T_0', \quad \forall\, l \in \mathbb{R}.$$

Since

$$\begin{aligned}\lim_{l\to\infty} T(l,\tau) &= 0 = \lim_{l\to\infty} T_0(l),\\ \lim_{l\to\infty} -\frac{1}{2}e^{2s}\partial_l T &= \pi = \lim_{l\to\infty} -\frac{1}{2}e^{2s}T_0'(l),\end{aligned}$$

we will impose consistent boundary conditions on T_1, namely

$$\lim_{l\to\infty} T_1(l) = 0, \qquad \lim_{l\to\infty} -\frac{1}{2}e^{2s}T_1'(l) = 0. \tag{8.4.2}$$

It is easy to see that

$$-\frac{1}{128\pi}T_0'(l) = \frac{1}{(e^{-l}+8e^l)^2}$$

is in the Kernel of $\mathcal{L}$, then using the reduction of order method we conclude that $\mathrm{Ker}\mathcal{L}$ is generated by

$$\left\{\frac{1}{(e^{-l}+8e^l)^2},\ \frac{1}{(e^{-l}+8e^l)^2}\left(-\frac{1}{2}e^{-2l} + 16l + 32e^{2l}\right)\right\}.$$

Therefore, the variation of parameters and the conditions (8.4.2) gives us

$$\begin{aligned}T_1(l) &= \frac{64\pi}{(e^{-l}+8e^l)^2}\Big\{\int_l^\infty (-\frac{1}{2}e^{-2\theta} + 16\theta + 32e^{2\theta})\frac{e^{-2\theta}}{(e^{-\theta}+8e^\theta)^2}d\theta\\ &\quad -(-\frac{1}{2}e^{-2l} + 16l + 32e^{2l})\int_l^\infty \frac{e^{-2\theta}}{(e^{-\theta}+8e^\theta)^2}d\theta\Big\}\\ &= \frac{64\pi}{(e^{-l}+8e^l)^2}\Bigg\{\left(\frac{1}{2}e^{-2l} - 16l - 32e^{2l}\right)\frac{1}{2}\left(\log(\frac{1}{8}e^{-2l}+1) - \frac{e^{-2l}}{e^{-2l}+8}\right)\\ &\quad -\left(\frac{1}{4} + \frac{2l}{e^{-2l}+8}\right)e^{-2l} - \int_0^{e^{-2l}}\frac{\log\theta}{\theta+8}d\theta\Bigg\}.\end{aligned}$$

Then we get that for $l \ll -1$ and large values of τ

$$T(l,\tau) \approx 8\pi - \frac{64\pi}{e^{-2l}+8} + 16\pi g(\tau)\Big(-2l - \log(8) - 2\Big), \tag{8.4.3}$$

which in terms of the function V_α it reads

$$V_\alpha(\xi,\tau) \approx \frac{64\pi}{\xi+8} - 16\pi g(\tau)\Big(\log(\xi)-\log(8)-2\Big), \tag{8.4.4}$$

and gives us the refined asymptotic inner expansion of M:

$$M_\infty(s,t) := \frac{64\pi\alpha(t)}{s+8\alpha(t)} + 16\pi\alpha'(t)\Big(\log(s)-\log(8\alpha(t))-2\Big). \tag{8.4.5}$$

Then it holds

$$M(s,t) \approx M_\infty(s,t)$$

as $t\to\infty$, for $s=O(\alpha(t))$. Let us see how far this expression is still a good approximation.

8.5 Outer Region Behavior

Now we want to estimate the behavior of M far from zero. Although the system has a self-similar symmetry, it seems unlikely to have that M approaches zero in a self-similar way, since the mass for the system (8.1.1) is conserved and u converges to Dirac's delta measure of mass 8π. Thus u cannot be sending mass to infinity as $t\to\infty$. Let us suppose that it is the case, namely that for large values of t

$$M(s,t)\approx\phi\left(\frac{s}{t}\right)$$

with ϕ such that

$$\phi''(r)+\left(\frac{2}{r}+\frac{1}{4}\right)\phi'(r) = 0.$$

Hence

$$\phi'(r)=\kappa r^{-2}e^{\frac{1}{4}r}$$

for some $\kappa\in\mathbb{R}$. Then $u\approx\frac{\pi}{t}\phi'(\frac{s^2}{t})$, but we know that for all $\varepsilon>0$ there exist $T,R>0$ such that, for all $t>T$

$$\begin{aligned}\varepsilon &> \int_R^\infty \frac{\pi}{t}\phi'\left(\frac{s^2}{t}\right)sds\\ &= \int_R^\infty \frac{\pi}{t}\phi'\left(\frac{s^2}{t}\right)sds\\ &= \frac{\pi}{2}\phi\left(\frac{s^2}{t}\right)|_R^\infty,\end{aligned}$$

which provides us with a contradiction for large values of $t>0$.

Now let us write

$$h(s,t):=\frac{1}{\alpha(t)}M(s,t).$$

Then h solves the equation

$$\partial_t h = 4s\partial_{ss}h+8\partial_s h-\frac{\alpha(t)}{\pi}h\partial_s h+\frac{\alpha'(t)}{\alpha(t)}h. \tag{8.5.1}$$

Assuming that, as $t \to \infty$, h stabilizes to a steady solution h_∞ of

$$\partial_t h_\infty = 4s\partial_{ss} h_\infty + 8\partial_s h_\infty$$

would gives us

$$h(s,t) \approx h_\infty(s) = \frac{c_1}{s} + c_0 \tag{8.5.2}$$

for some constants $c_1, c_2 \in \mathbb{R}$. From the previous section we already know that for small s

$$\begin{aligned} h(s,t) &= \frac{1}{\alpha(t)} V_\alpha\left(\frac{s}{\alpha(t)}, t\right) \\ &\approx \frac{64\pi}{s+8\alpha(t)} + 16\pi\frac{\alpha'(t)}{\alpha(t)}\Big(\log(s) - \log(\alpha(t)) - \log(8) - 2\Big). \end{aligned} \tag{8.5.3}$$

Now by matching the terms $O(s^{-1})$ from equations (8.5.3) with (8.5.2), we get $c_1 = 64\pi$. Since $h \geq 0$ we get $c_0 \geq 0$; if we suppose $c_0 > 0$ then, after matching the constant terms in equations (8.5.3) and (8.5.2), we would obtain

$$\begin{aligned} 16\pi\frac{\alpha'(t)}{\alpha(t)}\left(\log(\alpha(t)) + \log 8 + 2\right) &\sim -c_0 \\ \partial_t\left(\log\alpha(t)^2\right) + 2(\log 8 + 2)\partial_t(\log\alpha(t)) + \frac{c_0}{8\pi} &\sim 0 \\ \left(\log\alpha(t)^2\right) + 2(\log 8 + 2)(\log\alpha(t)) + \frac{c_0}{8\pi}t &\sim 0 \end{aligned}$$

which leads to

$$\log\alpha(t) \sim \frac{-2(\log 8 + 2) \pm \sqrt{4(\log 8 + 2)^2 - 4c_0 t(8\pi)^{-1}}}{2},$$

thus giving a contradiction for large values of t. Hence $c_0 = 0$ and the behavior of M over compact sets far from the origin looks like

$$M(s,t) \sim \alpha(t)\frac{64\pi}{s}. \tag{8.5.4}$$

This behavior is asymptotically the same as in the shrinking region $O(\alpha(t))$ given at main order by

$$S_{8\alpha}(r) = 64\pi\frac{\alpha}{r+8\alpha}.$$

If we define J as the evaluation of $\tilde{M}$ on the equation (8.2.1):

$$J(\tilde{M})(r,t) := \partial_t\tilde{M}(r,t) - 4r\partial_{rr}\tilde{M}(r,t) - 8\partial_r\tilde{M}(r,t) + \frac{1}{\pi}\tilde{M}(r,t)\partial_r\tilde{M}(r,t),$$

it turns out that

$$\left|J\left(S_{8\alpha(t)}(r)\right)\right| = \left|\frac{64\pi\alpha'}{r+8\alpha}\left(1 - \frac{8\alpha}{r+8\alpha}\right)\right|$$

is smaller than

$$\left|J(64\pi\alpha s^{-1})\right| \;=\; \left|2^6\pi\frac{\alpha'}{s}-2^{11}\pi\frac{\alpha^2}{s^3}\right|$$

under the assumption $|\alpha'(t)| \ll |\alpha(t)|$. For this reason we expect $S_{8\alpha}(r) = 64\pi\alpha(r+8\alpha)^{-1}$ to be a better approximation of M than $64\pi\alpha s^{-1}$ over bounded sets. Furthermore, we use the refined estimate (8.4.5) instead of (8.5.4) as the approximated solution, not only in the shrinking region but over compact sets too.

8.6 Remote Region Behavior

In order to estimate the blow-up rate we need yet to go further and use the fact that $\int_0^\infty M(s,t)ds$ remains constant. For this we have to estimate the behavior in a certain region moving far from the origin as t goes to ∞. Let us define

$$F(\eta,\omega) = \frac{t}{\alpha(t)}M\left(t\eta,t\right).$$

with $\omega = \log t$. This scaling is chosen so that for small η it holds $F \sim O(s^{-1})$, and also to use the self-similar symmetry of the equation. Then F satisfies

$$\partial_\omega F(\eta,\omega) \;=\; 4\eta\partial_{\eta\eta}F + 8\partial_\eta F + \eta\partial_\eta F + \frac{\alpha(t)}{t\pi}F\partial_\eta F + \left(1-\frac{t\alpha'(t)}{\alpha(t)}\right)F. \tag{8.6.1}$$

Assume that

$$\frac{t\alpha'(t)}{\alpha(t)} \to -q \le 0, \qquad \text{as } t\to\infty$$

and also that for every $\omega_n \to \infty$ there exists a subsequence of $F(s,\omega+\omega_n)$ which converges uniformly over compact sets of $(0,\infty)\times(0,\infty)$ to a solution F_∞ of the steady equation

$$4\eta F_\infty'' + (8+\eta)\,F_\infty' + (1+q)F_\infty \;=\; 0. \tag{8.6.2}$$

With

$$G(\eta) := \eta^{1+q}F_\infty(\eta),$$

we obtain

$$G'' + \left(\frac{1}{4}-\frac{2q}{\eta}\right)G' + \frac{q(q+1)}{4\eta^2}G \;=\; 0.$$

Then for large values of η, we obtain

$$G(\eta) \approx k_1\eta^{2q}e^{-\frac{1}{4}\eta} + k_0,$$

which provides us with

$$F_\infty(\eta) \;\approx\; \frac{k_1}{\eta^{1-q}}e^{-\frac{1}{4}\eta} + \frac{k_0}{\eta^{1+q}}.$$

From here we see that necessarily $q=0$, otherwise there would be no matching terms. Though one might think that $k_0 = c_1 = 64\pi$ because of the matching, this does not seem to be the case since we expect the integral of the solution to be finite in this remote region. We will then suppose $k_0 = 0$ and the constant $k_1 > 0$ will be fixed by the matching procedure.

8.7 Matching

Expanding F_∞ for small η gives

$$F_\infty(\eta,t) \quad = \quad k_1\left(\frac{1}{\eta}-\frac{1}{4}+\frac{\eta}{4^2}+\dots\right). \tag{8.7.1}$$

On the other hand we have

$$\frac{t}{\alpha(t)}M_\infty\left(t\eta,t\right) \quad = \quad \frac{64\pi}{\eta+8\alpha(t)t^{-1}}+16\pi\frac{\alpha'(t)t}{\alpha(t)}\Big(\log(\eta)-\log(8\alpha(t)t^{-1})-2\Big) \tag{8.7.2}$$

Now by matching the terms of order s^{-1} between expressions (8.7.1) and (8.7.2) we get $k_1=64\pi$, whereas the constant term in (8.7.2) matches (8.7.2) so that

$$\frac{\alpha'(t)t}{\alpha(t)}\Big(\log(8\alpha(t)t^{-1})-2\Big) \quad \sim \quad 1.$$

Since $\alpha'\alpha^{-1}t\to 0$ it seems natural to expect

$$|\log\alpha(t)| \quad \ll \quad |\log t|$$

as $t\to 0$. Hence if we write only the leading term in the asymptotic equation we get

$$-\frac{\alpha'(t)t}{\alpha(t)}\log t \quad \sim \quad 1.$$

Let us recall that the blow-up rate is given by $u(0,t)=\alpha(t)^{-1}$. Hence we have

$$\frac{\partial_t u(0,t)t}{u(0,t)}\log(t) \quad \sim \quad 1.$$

We deduce that

$$u(0,t) \quad \sim \quad \kappa\log t, \tag{8.7.3}$$

for some $\kappa>0$. To fix this constant we use that $\int_{\mathbb{R}^+}M(s,t)ds$ remains constant. At main order, we have

$$\int_0^t\frac{64\pi\alpha(t)}{s+8\alpha(t)}ds+\int_t^\infty k_1\frac{\alpha(t)}{s}e^{-\frac{1}{4}\frac{s}{t}}ds \quad \sim \quad \int_0^\infty M_0(s)ds.$$

Hence

$$\begin{aligned}64\pi\alpha(t)\left[\log(t+8\alpha(t))-\log(8\alpha(t))\right]+k_1\alpha(t)\int_1^\infty\frac{1}{s}e^{-\frac{1}{4}s}ds \quad &\sim \quad \int_0^\infty M_0(s)ds,\\ 64\pi\alpha(t)\left[\log(t)-\log(\alpha(t))+K_0\right] \quad &\sim \quad \int_0^\infty M_0(s)ds,\end{aligned}$$

from where we conclude that as $t\to\infty$

$$u(0,t) \quad \sim \quad \kappa\left(\log t+O\left(\log(\log t)\right)\right), \tag{8.7.4}$$

with κ given by

$$\kappa \quad = \quad \frac{64\pi}{\int_0^\infty M_0(s)ds}. \tag{8.7.5}$$

We point out the fact that the asymptotic behavior of the blow-up rate is consistent with the assumptions previously made on $\alpha(t)$.

8.8 The Case of the Bounded Domain

For the bounded radially symmetric domain case the infinite time blow-up was detected in [127], the blow-up rate

$$u(0,t) = O(e^{2\sqrt{2t}})$$

was obtained formally in [150] and rigorously in [142]. As mentioned in [127] the bounded domain case and the whole space are completely different cases. In particular the blow-up rate obviously differs. We will discuss briefly how the matched asymptotics expansion method can be adapted to the bounded domain case.

Let us consider now the chemotaxis problem posed in a ball

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + \nabla \cdot (u \nabla v) & x \in B(0,1), \quad t > 0, \\ \Delta v = u & x \in B(0,1), \quad t > 0, \\ u(0,x) = u_0(x) \geq 0 & x \in B(0,1), \end{cases} \tag{8.8.1}$$

supplemented with boundary conditions

$$\frac{\partial u}{\partial \nu} + u \frac{\partial c}{\partial \nu} = 0, \quad x \in \partial B(0,1), \quad t > 0, \tag{8.8.2}$$

$$v = 0, \quad x \in \partial B(0,1), \quad t > 0. \tag{8.8.3}$$

Again we define the cumulated mass function

$$U(r,t) = 8\pi - \int_{B(0,\sqrt{r})} u(x,t)dx,$$

which solves the one-dimensional parabolic equation

$$\partial_t U(r,t) \quad = \quad 4r\partial_{rr} U(r,t) + 8\partial_r U(r,t) - \frac{1}{\pi} U(r,t)\partial_r U(r,t) \tag{8.8.4}$$

on $(0,\infty) \times (0,\infty)$, but now with the boundary conditions

$$U(0,t) \quad = \quad 8\pi, \quad U(1,t) = 0, \tag{8.8.5}$$

$$U(r,0) \quad = \quad U_0(r) := 8\pi - \int_{B(0,\sqrt{r})} u_0(x)dx. \tag{8.8.6}$$

Similarly we put

$$\alpha(t) = u(0,t)^{-1},$$

and study $U(\alpha(t)r, t)$. It is easy to see that the inner expansion holds in the same way as the case of the whole space domain, giving

$$U(r,t) \quad \approx \quad M_\infty(r,t),$$

as $t \to \infty$, for $r = O(\alpha(t))$, with M_∞ as in (8.4.5). Similarly as $t \to \infty$

$$\frac{1}{\alpha(t)} U(r,t) \quad \approx \quad \frac{c_1}{s} - c_0.$$

On the other hand, from the inner region behavior we get

$$\frac{1}{\alpha(t)}U(r,t) \approx \frac{64\pi}{s+8\alpha(t)} + 16\pi\frac{\alpha'(t)}{\alpha(t)}\Big(\log(s) - \log(\alpha(t)) - \log(8) - 2\Big).$$

By matching the terms of order $O(s^{-1})$ between the previous expansions we obtain $c_1 = 64\pi$. Imposing the boundary condition (8.8.5) we get $c_0 = c_1 = 64\pi$, then if we match the terms with no dependence in s we obtain

$$\begin{aligned} 16\pi\frac{\alpha'(t)}{\alpha(t)}\left(\log(\alpha(t)) + \log 8 + 2\right) &\sim 64\pi \\ \partial_t\left(\log\alpha(t)^2\right) + 2(\log 8 + 2)\partial_t(\log\alpha(t)) - 8 &\sim 0, \end{aligned}$$

which leads to

$$u(0,t) = \alpha(t)^{-1} = O(e^{2\sqrt{2t}}), \tag{8.8.7}$$

and allows us to recover the asymptotic behavior already obtained in [142] and [150].

As we can see the boundary condition (8.8.5) has a strong effect on the asymptotic behavior of the solution U, that rules out the persistence of M_∞ as an approximation, which is in contrast with the estimates obtained for the whole space case.

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