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Vertex coloring of graphs via the discharging method

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Vertex coloring of graphs via the discharging method

Abstract : In this thesis, we are interested in various vertex coloring and homomorphism problems of graphs with special emphasis on planar graphs and sparse graphs. We consider proper vertex coloring, acyclic coloring, star coloring, forest coloring, fractional coloring and the list version of most of these concepts.

In Chapter 2, we consider the problem of finding sufficient conditions for a planar graph to be 3-choosable. These conditions are expressed in terms of forbidden subgraphs and our results extend several known results.

The notion of acyclic list coloring of planar graphs was introduced by Borodin, Fon-Der Flaass, Kostochka, Raspaud, and Sopena. They conjectured that every planar graph is acyclically 5-choosable. In Chapter 3, we obtain some sufficient conditions for planar graphs to be acyclically k-choosable with $k \in \{3, 4, 5\}$.

In Chapter 4, we prove that every subcubic graph is 6-star-colorable. On the other hand, Fertin, Raspaud and Reed showed that the Wagner graph cannot be 5-star-colorable. This fact implies that our result is best possible. Moreover, we obtain new upper bounds on star choosability of planar subcubic graphs with given girth.

A k-forest-coloring of a graph G is a mapping π from $V(G)$ to the set $\{1, \cdots, k\}$ such that each color class induces a forest. The vertex-arboricity of G is the smallest integer k such that G has a k-forest-coloring. In Chapter 5, we prove a conjecture of Raspaud and Wang asserting that every planar graph without intersecting triangles has vertex-arboricity at most 2.

Finally, in Chapter 6, we focus on the homomorphism problems of sparse graphs to the Petersen graph. More precisely, we prove that every triangle-free graph with maximum average degree less than 5/2 admits a homomorphism to the Petersen graph. Moreover, we show that the bound on the maximum average degree in our result is best possible.

Keywords : planar graph, acyclic coloring, star coloring, vertex-arboricity, homomorphism, maximum average degree, Petersen graph, cycle.

Discipline : Computer Science

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Coloration des sommets des graphes par la méthode de déchargement

Résumé : Dans cette thèse, nous nous intéressons à differentes colorations des sommets d'un graphe et aux homomorphismes de graphes. Nous nous intéressons plus spécialement aux graphes planaires et aux graphes peu denses. Nous considérons la coloration propre des sommets, la coloration acyclique, la coloration étoilée, la k-forêt-coloration, la coloration fractionnaire et la version par liste de la plupart de ces concepts.

Dans le Chapitre 2, nous cherchons des conditions suffisantes de 3-liste colorabilité des graphes planaires. Ces conditions sont exprimées en termes de sous-graphes interdits et nos résultats impliquent plusieurs résultats connus.

La notion de la coloration acyclique par liste des graphes planaires a été introduite par Borodin, Fon-Der Flaass, Kostochka, Raspaud, et Sopena. Ils ont conjecturé que tout graphe planaire est acycliquement 5-liste coloriable. Dans le Chapitre 3, on obtient des conditions suffisantes pour qu'un graphe planaire admette une k-coloration acyclique par liste avec $k \in \{3, 4, 5\}$.

Dans le Chapitre 4, nous montrons que tout graphe subcubique est 6-étoilé-coloriable. D'autre part, Fertin, Raspaud et Reed ont montré que le graphe de Wagner ne peut pas être 5-étoilé-coloriable. Ce fait implique que notre résultat est optimal. De plus, nous obtenons des nouvelles bornes supérieures sur la choisissabilité étoilé d'un graphe planaire subcubique de maille donnée.

Une k-forêt-coloration d'un graphe G est une application π de l'ensemble des sommets $V(G)$ de G dans l'ensemble de couleurs $1, 2, \cdots, k$ telle que chaque classe de couleur induit une forêt. Le sommet-arboricité de G est le plus petit entier k tel que G a k-forêt-coloration. Dans le Chapitre 5, nous prouvons une conjecture de Raspaud et Wang affirmant que tout graphe planaire sans triangles intersectants admet une sommet-arboricité au plus 2.

Enfin, au Chapitre 6, nous nous concentrons sur le problème d'homomorphisme des graphes peu denses dans le graphe de Petersen. Plus précisément, nous prouvons que tout graphe sans triangles ayant un degré moyen maximum moins de $5/2$ admet un homomorphisme dans le graphe de Petersen. En outre, nous montrons que la borne sur le degré moyen maximum est la meilleure possible.

Mots clefs : graphe planaire, coloration acyclique, coloration étoilée, sommets-arboricité, homomorphisme, degré moyen maximum, graphe de Petersen, cycle.

Discipline : Informatique

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Introduction and Preliminaries

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Many real world situations can conveniently be described by means of a diagram consisting of a set of points together with lines joining certain pairs of these points. For example, the points could represent people, with lines joining pairs of friends; or the points might be communication centers, with lines representing communication links. Notice that in such diagrams one is mainly interested in whether two given points are joined by a line; the manner in which they are joined is immaterial. A mathematical abstraction of situations of this type gives rise to the concept of a graph. The basic concepts of graph theory are simple and can be used to express problems from many different subjects.

Graph coloring is an important field in graph theory. It has a central position in discrete mathematics and is of interest for its applications. Graph coloring deals with the fundamental problem of partitioning a set of objects into classes, according to certain rules. Time tabling, sequencing, and scheduling problems, in their many forms, are basically of this nature.

Most of graph coloring problems come from the famous Four Color Problem which states that any map in a plane can be colored using four colors in such a way that regions sharing a common boundary (other than a single point) have distinct colors. This was a question which Francis Guthrie asked his brother Frederick Guthrie, who was a student of De Morgan in mathematics. In 1976, the Four Color Problem was proved by Appel and Haken [AH76] using computer. So, the Four Color Problem has been changed into the Four Color Theorem ever since.

In this thesis, we mainly study the vertex coloring of graphs via the Discharging Method, which was used to solve the Four Color Problem (and which we extensively use in this thesis). So, in Section 1.3, we start with an introduction of what Discharging Method is and how it works. Before that, we need to define some basic notation used throughout the thesis in Section 1.1 and Section 1.2, followed by an overview of all results (in Section 1.4) of the thesis.

1.1 Definition

1.1.1 Definition of graphs

A graph G is an ordered pair (V, E), where V stands for a finite set whose elements are called *vertices* and E is a set of 2-subsets of V whose elements are called *edges*. We note that with our definition a graph is finite and simple (i.e., no loops and multiple edges). The *order* of G is the number of vertices in G, written as $|G|$. The *size* of G is the number of edges in G, denoted by $||G||$. Sometimes, we use $|E|$ instead of $||G||$.

An *edge* $\{x, y\}$ is said to join the vertices x and y and is denoted by xy. The vertices x and y are the *endvertices* of the edge xy. If $xy \in E(G)$, then we say that x and y are *adjacent vertices* of G, and the vertices x and y are *incident* with the edge xy. We say two edges e and e' are *adjacent* if they have exactly one common endvertex.

The graph with no vertices (and hence no edges) is the *null graph* and the graph with just one vertex is the *trivial graph*. All other graphs are *nontrivial*. An *empty graph* is a graph in which no two vertices are adjacent; that is, its edge set is empty.

1.1.2 Vertex degrees

The *degree* of a vertex v in a graph G, denoted by $d_G(v)$ or $d(v)$, is the number of edges of G incident with v. A vertex of degree zero is called an *isolated vertex*. A vertex of degree k is called a *k-vertex*. A k^+-vertex (or k^--vertex) is a vertex of degree at least (or at most) k. Sometimes, a k-vertex v is said to be a *$k(d)$-vertex* if v is adjacent to d 2-vertices. An edge uv is said a *(b_1, b_2)-edge* if $d(u) = b_1$ and $d(v) = b_2$. We call $N_S(v) = \{u|u \in S, uv \in E(S)\}$ the *neighborhood* of v in S. In particular, if $S = V(G)$, we write $N(v)$ instead of $N_{V(G)}(v)$. Observe that if G is a simple graph, $d(v)$ is the number of neighbors of v in G and thus $d(v) = |N(v)|$. We define $N_S^*(v) = N_S(v) \cup \{v\}$.

The number $\delta(G) = \min\{d(v), v \in V(G)\}$ is the *minimum degree* of G. The number $\Delta(G) = \max\{d(v), v \in V(G)\}$ is the *maximum degree* of G. If all the vertices in G are of degree k, then we call G a *k-regular graph*. In particular, a 3-regular graph is called *cubic* and a graph G with $\Delta(G) \leqslant 3$ is called *subcubic*.

The well known Handshake Lemma establishes a fundamental identity relating the degrees of the vertices of a graph and the number of its edges.

Lemma 1.1.1 *If G is a plane graph, then*

$$\sum_{v \in V(G)} d(v) = 2|E(G)|.$$

This seemingly simple fact plays a prominent role in graph theory, especially in the discharging argument, which will be introduced in Section 1.3. The number $\text{ad}(G) = \frac{\sum_{v \in V(G)} d(v)}{|V(G)|} = \frac{2|E(G)|}{|V(G)|}$ is called an *average degree* of G. The *maximum average degree* of G, denoted by $\text{Mad}(G)$, is the maximum average degree over all induced subgraphs of G, i.e., $\text{Mad}(G) = \max\{\frac{2|E(H)|}{|V(H)|} : H \subseteq G\}$.

1.1.3 Paths, threads and cycles

A *walk* in a graph G is a non-empty alternating sequence of vertices and edges denoted by $W = v_0e_1v_1e_2 \cdots e_kv_k$, where $e_i = v_{i-1}v_i$ for each $1 \leqslant i \leqslant k$. v_0 and v_k are both called *endvertices* of W. The *length* of W is the number of its edges, i.e., k. This walk W is called a *trail* of G if all its edges are distinct and is called a *path* of G if all vertices are distinct. Notice that each path is a trail but the converse is not true.

Suppose $P = v_0v_1 \cdots v_{k-1}v_k$ is a path. We say that v_0, v_k are the two *endvertices* of P and $v_1, \cdots, v_{k-1}$ are the *internal* vertices of P. Moreover, we call such path P a *(v_0,v_k)-path*. A *thread* of G is a path whose all internal vertices are of degree 2 in G. We use *k-thread* to denote a thread with exactly k internal 2-vertices (and length $j+1$). A *maximal thread* is a thread whose two endpoints are both 3-vertices. A k-vertex v is a *$(j_1, j_2, \cdots, j_k)$-vertex* if there are maximal threads starting from v which have $j_1, j_2, \cdots, j_k$ internal vertices, respectively.

A *closed walk* is a walk whose endvertices coincide. A closed path with length at least 3 is called a *cycle*. A cycles with length k is called a *k-cycle*. A k^+-cycle (or k^--cycles) is a cycle with length at least (or at most) k. A *triangle* is synonymous with a 3-cycle. The *girth* of a graph G is the length of its shortest cycle. Similarly, the *odd girth* of a graph G is the length of a shortest odd cycle in G (∞ if G is bipartite).

For $u, v \in V(G)$, the *distance* between u and v, denoted $\text{dist}(u, v)$, is the number of edges in a shortest path connecting them. The distance between two triangles T and T' is defined as the value $\min\{\text{dist}(x, y)|x \in V(T) \text{ and } y \in V(T')\}$. In particular, two triangles are said to be *intersecting* if they have distance 0. The *diameter* of G is the greatest distance between any two vertices in G.

1.1.4 Subgraphs and operations

Let $G = (V, E)$ and $G' = (V', E')$ denote two graphs. We say that G' is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. For convenience, we write $G' \subseteq G$. If $V' = V$ and $G' \subseteq G$, then G' is said to be a *spanning subgraph* of G. If G' contains all edges of G that join two vertices in V', then G' is said to be the subgraph *induced* by V' and is denoted by $G[V']$.

We shall often construct new graphs from old ones by deleting or adding some vertices and edges. Suppose G is a graph. If $S \subset V(G)$, then $G - S = G[V \setminus S]$ is the subgraph of G obtained by deleting the vertices in S and all edges incident with them. Similarly, if $E' \subseteq E(G)$, then $G - E' = (V(G), E(G) \setminus E')$. If $S = \{s\}$ and $E' = \{xy\}$, then this notation is simplified to $G - s$ and $G - xy$. Similarly, if x and y are non-adjacent vertices of G, then $G + xy$ is obtained from G by joining x to y.

1.1.5 Connectivity

Let G be a non-empty graph. We say that G is *connected* if, for each pair $u, v \in V(G)$, there always exists a path connecting them; otherwise G is *disconnected*. A *maximal connected subgraph* of G is a subgraph that is connected and is not properly contained in any other connected subgraph of G. The *components* of a graph G are its maximal connected subgraphs. A *cut-edge* or *cut-vertex* of G is an edge or vertex whose deletion increases the number of components. The *connectivity* of a connected graph G, written as $\kappa(G)$, is the minimum size of a vertex set S such that $G - S$ is disconnected or has only one vertex. A graph G is called k*-connected* if $\kappa(G) \geqslant k$.

1.1.6 Special families of graphs

A *complete graph* is a simple graph in which any two vertices are adjacent. If it has n vertices, we denote it by K_n. Obviously, a complete graph with n vertices is an $(n-1)$-regular graph. A graph is *bipartite* if its vertex set can be partitioned into two subsets X and Y so that every edge has one endvertex in X and one endvertex in Y; such a partition (X, Y) is called a *bipartition* of the graph, and X and Y its *parts*. We denote a bipartite graph G with bipartition (X, Y) by $G[X, Y]$. If $G[X, Y]$ is simple and every vertex in X is joined to every vertex in Y, then G is called a *complete bipartite graph* and denoted by $K_{n,m}$, where $|X| = n$ and $|Y| = m$.

A graph without any cycles is a *forest*, or an *acyclic graph*. A *tree* is a connected forest. The relation of a tree to a forest sounds less absurd if we note that a forest is a disjoint union of trees. In other words, a forest is a graph whose every component is a tree.

1.1.7 Oriented graphs

An *oriented graph* is graph G which consists of a *vertex set* $V(G)$ and an *edge set* $E(G)$ whose elements are ordered pairs of vertices. An *orientation* of a graph G is obtained by choosing an orientation $x \to y$ or $y \to x$ for each edge $xy \in E(G)$.

An *arc* $e = (x, y)$ is considered to be directed from x to y; y is called the *head* and x is called the *tail* of the arc. For a vertex v in G, the number of tail endvertices adjacent to v is called the *indegree* of v, denoted by $d^-(v)$, and the number of head endvertices is its *outdegree*, denoted by $d^+(v)$.

1.1.8 Planar graphs

Let us first read the following brain teaser introduced in [Wes02], which was appeared as early as in [Dud17].

Gas-Water-Electricity Problem Three sworn enemies A, B, C live in houses in the woods. They make paths so that each has a path to each of three utilities, which by tradition are gas, water, and electricity. In order to avoid confrontations, they do not want any of the paths to cross. Can this be done?

This question can be asked in terms of graph theory that whether $K_{3,3}$ can be drawn in the plane without edge crossings. To answer this question, we start with the following definitions.

A graph G is *planar* if it has a drawing without crossings in the plane. Such a drawing is a *planar embedding* of G. A *plane graph* is a particular embedding of a planar graph. A planar embedding of a graph G cuts the plane into a number of arcwise-connected open sets. These sets are called the *faces* of G, denoted by $F(G)$. Each plane graph has exactly one unbounded face, called the *outer face* and others called *internal faces*. A face is said to be *incident* with the vertices and edges in its boundary. The *dual graph* G^* of a plane graph G is a plane graph whose vertices correspond to the faces of G. Two vertices in G^* are adjacent if the corresponding faces in G are adjacent.

The next lemma may be regarded as a dual version of Lemma 1.1.1.

Lemma 1.1.2 *If G is a plane graph, then*

$$\sum_{f \in F(G)} d(f) = 2|E(G)|.$$

There is a simple formula relating the numbers of vertices, edges, and faces in a connected plane graph. It was first established for polyhedral graphs by Euler in 1752, and is known as Euler's Formula, which plays a key role in our proofs, and in general, in the proofs of problems on planar graphs that use the Discharging Method.

Theorem 1.1.3 (Euler's Formula)
Let G be a connected plane graph G with n vertices, m edges, and f faces. Then

$$n - m + f = 2.$$

Using the following corollaries of this theorem one can prove that dense graphs are not planar.

Corollary 1.1.4 *If G is a planar graph with girth $g(G)$, then*

$$|E(G)| \leqslant \frac{g(G)}{g(G)-2}(|V(G)|-2).$$

Corollary 1.1.5 *If G is a planar graph with girth $g(G)$, then*

$$\mathrm{Mad}(G) < \tfrac{2g(G)}{g(G)-2}.$$

Now for example we give a negative answer to the Gas-Water-Electricity Problem. On the one hand, we notice that a solution to this problem is equivalent to a planar embedding of $K_{3,3}$. On the other hand, since $g(K_{3,3}) = 4$, this graph does not satisfy the condition of Corollary 1.1.4.

1.1.9 Basic notation

This section is dedicated to some basic notion used throughout the thesis. We will use i^+ to denote a number equal or greater than i. Let G be a plane graph. For a face $f \in F(G)$, we use $b(f)$ to denote the boundary walk of f and write $f = [u_1u_2\cdots u_n]$ if $u_1, u_2, \cdots, u_n$ are the vertices of $b(f)$ appearing in a boundary walk of f. Sometimes, we write simply $V(f) = V(b(f))$. A face f is *simple* if $b(f)$ forms a cycle. The *degree*, denoted by $d(f)$, of a face f is the number of edges in its boundary $b(f)$. Note that each cut-edge is counted twice. A k^+-face (or k^--face) is a face of degree at least (or at most) k. We say that two cycles (or faces) are *adjacent* if they share at least one edge. Moreover, two adjacent cycles (or faces) are said to be *normally adjacent* if they share exactly two vertices.

For $x \in V(G) \cup F(G)$, if there is no special mention, we usually use $t(x)$ to denote the number of 3-faces adjacent/incident to x and use $n_j(x)$ to denote the number of j-vertices adjacent/incident to x, where j is an integer and $j \geqslant 2$. For $f = [u_1u_2\cdots u_n]$, we use $f_{u_iu_{i+1}}$ to denote the face adjacent to f by a common edge u_iu_{i+1}, where i is taken modulo n. A k-face $f = [v_1v_2\cdots v_k]$ is called an $(a_1, a_2, \cdots, a_k)$*-face* if the degree of the vertex v_i is a_i for $i = 1, 2, \cdots, k$. A 3-vertex v is *light* if it is incident to a 3-face. If a vertex u is adjacent to a 3-vertex v such that the edge uv is not incident to any 3-face, then we call v a *pendant* 3-vertex of u. A *pendant light* 3-vertex is a light and pendant 3-vertex. If v is a pendant light 3-vertex which is incident to an (a_1, a_2, a_3)-face, then we call v is a *pendant light* (a_1, a_2, a_3)*-vertex*. Let $p_3(u)$ denote the number of pendant light 3-vertices of a vertex u.

Sometimes, for simplicity, we use $\{c_1, c_2, \cdots, c_k\}$-cycles to denote the cycles of lengths $c_1, c_2, \cdots$, and c_k, where k is a positive integer. For all figures in this thesis, a vertex is represented by a solid point when all of its incident edges are drawn; otherwise it is represented by a hollow point.

1.2 Graph coloring

A *proper* vertex coloring of G is an assignment π of integers (as colors) to the vertices of G such that $\pi(u) \neq \pi(v)$ if the vertices u and v are adjacent in G. A *k-coloring* is a proper vertex coloring using k colors. Each color class forms an *independent* set of vertices; that is, no two of them are joined by an edge. The *chromatic number*, denoted by $\chi(G)$, is the least cardinal k for which G has a proper k-coloring. Let α, β be any 2 colors. An *alternating* (α, β)*-path in* G is a path in G with each vertex colored α or β.

A *homomorphism* of a graph G to a graph H is a mapping $f : V(G) \to V(H)$ such that $f(x)f(y) \in E(H)$ if $xy \in E(G)$. The graph homomorphisms have been studied as extension of graph colorings. Note that a graph G has a k-coloring if and only if G has a homomorphism to the complete graph K_k. Therefore, the chromatic number of a graph G can be equivalently defined to be the minimum number of vertices in a graph H such that G has a homomorphism to H. In general, a homomorphism of G to a graph H is called an H*-coloring* of G.

We say that L is an *assignment* for the graph G if it assigns a list $L(v)$ of possible colors to each vertex v of G. If G has a proper coloring π such that $\pi(v) \in L(v)$ for all vertices v, then we say that G is L*-colorable* or π is an L*-coloring* of G. The graph G is k*-choosable* (or k*-list colorable*) if it is L-colorable for every assignment L satisfying $|L(v)| \geqslant k$ for all vertices v. The *list chromatic number* of G, denoted $\chi^l(G)$, is the smallest integer k such that G is k-choosable.

The concepts of L-list coloring were introduced by both Vizing [Viz76] in 1976 and Erdős, Rubin and Taylor [ERT79] in 1979. We note that $\chi^l(G) \geqslant \chi(G)$ but $\chi^l(G)$ can be arbitrarily larger than $\chi(G)$. For example, the 2-colorable graph $K_{3,3}$ is not L-colorable for L which is given in Figure 1.1.

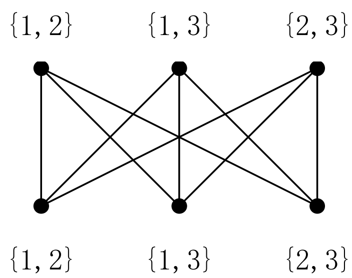

Figure 1.1: The bipartite graph $K_{3,3}$ is 2-colorable but not 2-list colorable.

1.3 Discharging method

As we mentioned before, the Four Color Theorem was proved by Appel and Haken [AH76] in 1976. In fact, its computer-assisted proof used the Discharging Method. This method has increasingly been used to solve problems for graphs with or without assistance of a computer. The method is mostly used for graphs embedded on a surface because of the Euler's formula.

Let $\mathcal{C}$ be the class of planar graphs and suppose we want to prove that every graph in $\mathcal{C}$ has a property P. To do this using the Discharging Method we have 6 main steps below:

Step 1 Suppose $G \in \mathcal{C}$ is a graph which does not satisfy the property P. Most of the time, we choose such a graph G to be minimal.

Step 2 Show that G cannot contain certain subgraphs (this is normally done using minimality of G). Such subgraphs are called *reducible configurations*.

Step 3 Assign *initial weights* to the vertices and the faces of G.

Step 4 Use Euler's formula, $|V(G)|-|E(G)|+|F(G)| = 2$, and Handshake Lemma $\sum_{v\in V(G)} d(v) = \sum_{f\in F(G)} d(f) = 2|E(G)|$ to show that the total sum of initial weights is equal to some constant.

Step 5 [Discharging] Design appropriate discharging rules and redistribute weights accordingly, while preserving the total weights. Once the discharging is finished, a *new weight* for each vertex and face is produced.

Step 6 Using the absence of reducible configurations, we show that the total sum of new weights is now different from the total sum of initial weights. This obvious contradiction demonstrates that such counterexample G does not exist. Therefore, every graph in $\mathcal{C}$ has the property P.

This process may also be called a *discharging argument*. For $x, y \in V(G)\cup F(G)$, we usually use $\tau(x \to y)$ to denote the amount of weights transferred from x to y in the discharging argument of the thesis. Step 5, the discharging part, is a crucial step of this argument. Finding the reducible configurations in Step 2 is also an important part of the proof. However, sometime finding such reducible configurations together with appropriate discharging rules could be extremely difficult just as in the case of the Four Color Theorem.

In most cases of the thesis, we use one standard weight assignment in Step 3. Namely, we assign each vertex v an initial weight $\omega(v) = 2d(v) - 6$ and each face f an initial weight $\omega(f) = d(f) - 6$. The following lemma shows that the total sum of initial weights is equal to -12.

Lemma 1.3.1 *Let G be a connected plane graph with n vertices, m edges and f faces. Then*

$$\sum_{v\in V(G)} (2d(v) - 6) + \sum_{f\in F(G)} (d(f) - 6) = -12. \tag{1.1}$$

Proof. Euler's formula $n - m + f = 2$ yields $(4m - 6n) + (2m - 6f) = -12$. This identity and the Handshake Lemma $\sum_{v\in V(G)} d(v) = \sum_{f\in F(G)} d(f) = 2m$ imply (1.1). This proves Lemma 1.3.1. ∎

1.4 Presentation of results

In this thesis, we are interested in various graph coloring problems, including proper vertex coloring, acyclic coloring, star coloring, forest coloring, fractional coloring and the list version of most of these concepts on planar graphs and sparse graphs. In each of the following parts, we will give a short survey and present our results.

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As we mentioned in Section 1.2, Vizing [Viz76] and Erdős, Rubin and Taylor [ERT79] independently introduced the concepts of L-list coloring and choosability. An easy consequence of Euler's formula is that every planar graph has a 5^--vertex. It implies that every planar graph is 6-choosable. Thomassen improved this result by showing the following:

Theorem 1.4.1 [Tho94] *Every planar graph is* 5*-choosable.*

The bound in Theorem 1.4.1 is best possible, since Voigt [Voi93] and Mirzakhani [Mir96] independently, gave examples to show that there exists a non-4-choosable planar graph. All 2-choosable graphs were characterized completely in [ERT79]. So characterizing planar graphs that are 3- or 4-choosable turned out to be interesting problems in graph coloring. However, Gutner [Gut96] proved that both these problems are NP-complete. Some sufficient conditions for planar graphs to be 4-choosable were established (see more details in **Chapter 5**).

In 1958, Grötzsch [Grö59] proved that planar graphs without 3-cycles are 3-colorable. But not every triangle-free planar graph is 3-choosable. The first example of such graphs was provided by Voigt [Voi95]. Moreover, Thomassen [Tho95] proved that planar graphs with girth at least 5 are 3-choosable. In 1976, Steinberg [JT95b] proposed the following conjecture:

Conjecture 1.4.2 *Every planar graph without* 4*-cycles and* 5*-cycles is* 3*-colorable.*

This challenging conjecture still remains unsolved. In 1990, Erdös suggested the following relaxation of Steinberg's conjecture: What is the smallest integer k such that every planar graph without i-cycles for $4 \leqslant i \leqslant k$ is 3-colorable. The best known upper bound is $k \leqslant 7$ obtained by Borodin, Glebov, Raspaud and Salavatipour [BGRS05]. Recently, Borodin et al. [BGMR09] improved this result by showing that planar graphs without 5-, 7-cycles and adjacent triangles are 3-colorable. It is natural to ask the same question for choosability:

Question 1.4.3 *What is the smallest integer k such that every planar graph without i-cycles for $4 \leqslant i \leqslant k^*$ is* 3*-choosable?*

Borodin [Bor96] proved that $k \leqslant 9$ and Voigt [Voi07] proved that $k \geqslant 6$ by constructing a non-3-choosable planar graph which contains neither 4- nor 5-cycles. Moreover, for a planar graph G, $\chi^l(G) \leqslant 3$ was obtained in the following cases: if G has no $\{4,5,6,9\}$-cycles (Zhang and Wu [ZW05]); or without $\{4,5,7,9\}$-cycles (Zhang and Wu [ZW04]); or without $\{4,6,7,9\}$-cycles (Chen, Lu and Wang [CLW08]); or without $\{4,6,8,9\}$-cycles (Shen and Wang [SW07]); or $\{4,5,8,9\}$-cycles (Wang, Lu and Chen [WLC10]); or $\{4,7,8,9\}$-cycles (Chen, Shen and Wang [CSW10]).

In **Chapter 2**, we will consider the 3-choosability of planar graphs in which each vertex is not incident to some cycles of given lengths, but all vertices can have different restrictions. In other words, we only forbid a certain set of cycles for each vertex and these sets of forbidden cycles are not necessarily the same. More precisely, we will prove that a planar graph G is 3-choosable if it is satisfied one of the following conditions:

- each vertex x is neither incident to cycles of lengths $4, 9, i_x$ with $i_x \in \{5,7,8\}$, nor incident to 6-cycles adjacent to a 3-cycle.
- each vertex x is not incident to cycles of lengths $4, 7, 9, i_x$ with $i_x \in \{5,6,8\}$.

This work is jointly done with Montassier and Raspaud [CMR] and extends five known results in [ZW04, ZW05, SW07, CLW08, CSW10].

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We say a proper vertex coloring of a graph G is *acyclic* if there is no bicolored cycle in G. In other words, every cycle uses at least three colors. The *acyclic chromatic number* of a graph G, denoted by $\chi_a(G)$, is the smallest integer k such that G has an acyclic k-coloring. It is obvious that $\chi(G) \leqslant \chi_a(G)$ for any graph G.

The notion of acyclic coloring of graphs was introduced by Grünbaum [Grü73] in 1973 and studied by Mitchem [Mit74], Albertson and Berman [AB77] and Kostochka [Kos76]. In 1979, Borodin [Bor79] proved Grünbaum's conjecture that every planar graph is acyclically 5-colorable. This bound is best possible. In 1973, Grünbaum [Grü73] gave an example of a 4-regular planar graph which is not acyclically 4-colorable. Furthermore, bipartite planar graphs which are not acyclically 4-colorable were constructed in [KM76], see Figure 1.2.

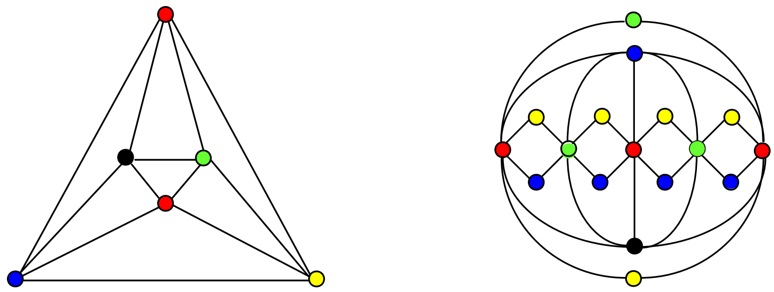

Figure 1.2: Examples of Grünbaum and Kostochka Mel'nikov.

Given an assignment $L = \{L(v)|v \in V(G)\}$ of colors to the vertices of a graph G, we say G is *acyclically L-list colorable* if there is an acyclic coloring π of the vertices such that $\pi(v) \in L(v)$ for every vertex v. The coloring π is called an *acyclic L-coloring* of G. If G is acyclically L-list colorable for any list assignment L with $|L(v)| \geqslant k$ for all $v \in V$, then G is *acyclically k-choosable* or *acyclically k-list colorable*. The *acyclic list chromatic number* or *acyclic choosability* of G, denoted by $\chi_a^l(G)$, is the smallest integer k such that G is acyclically k-choosable.

In 2002, Borodin, Fon-Der Flaass, Kostochka, Raspaud, and Sopena [BFDFK+02] first investigated acyclic list coloring of planar graphs. They proved that every planar graph is acyclically 7-choosable. They also put forward the following challenging conjecture:

Conjecture 1.4.4 [BFDFK+02] *Every planar graph is acyclically 5-choosable.*

This conjecture attracted much attention recently. If Conjecture 1.4.4 were true, then it would strengthen the Borodin's acyclic 5-color theorem [Bor79] and the Thomassen's 5-choosable theorem [Tho94] about planar graphs. However, this challenging conjecture seems to be difficult. As yet, it has been verified only for several restricted classes of planar graphs: those of girth at least 5 (Montassier, Ochem and Raspaud [MOR06]); without 4-cycles and 5-cycles, or without 4-cycles and 6-cycles (Montassier, Raspaud and Wang [MRW07]); without 4-cycles and without triangles at distance less than 3 (Chen and Wang [CW08a]); with neither 4-cycles nor chordal 6-cycles (Zhang and Xu [ZX09]). In particular, in [BI09a], Borodin and Ivanova proved that a planar graph G is acyclically 5-choosable if G does not contain an i-cycle adjacent to a j-cycle where $3 \leqslant j \leqslant 5$ if $i = 3$ and $4 \leqslant j \leqslant 6$ if $i = 4$. This result absorbs most of the previous work in this direction, including [MRW07].

Wang and Chen [WC09] proved that every planar graph without 4-cycles is acyclically 6-choosable. To attack Conjecture 1.4.4, in [CW08a], they proposed the following weak version of Conjecture 1.4.4:

Conjecture 1.4.5 [CW08a] *Every planar graph without 4-cycles is acyclically 5-choosable.*

As far as we know, Conjecture 1.4.5 is still open. In Section 3.2 of **Chapter 3**, we will prove the following result:

Theorem 1.4.6 [CR10d] *Every planar graph with neither 4-cycles nor intersecting triangles is acyclically 5-choosable.*

This result partially confirms the Conjecture 1.4.5 and gives an improvement to the result in [CW08a].

Some sufficient conditions for a planar graph to be acyclically 4-choosable (or colorable) are also obtained. It is proved in [Bor10] that $\chi_a(G) \leqslant 4$ if G contains no $\{4,5\}$-cycles. Moreover, $\chi_a^l(G) \leqslant 4$ was obtained in the following cases: $g(G) \geqslant 5$ (Montassier [Mon07]), which extends two results in [MOR06] and [BKW99]; or if G has no $\{4,5,6\}$-cycles, or without $\{4,5,7\}$-cycles, or without $\{4,5\}$-cycles and intersecting 3-cycles (Montassier, Raspaud and Wang [MRW06a]); or with neither $\{4,5\}$-cycles nor 8-cycles having a triangular chord (Chen and Raspaud [CR10b]); or without $\{4,7,8\}$-cycles (Chen et al. [CRRZ11]); or with neither 4-cycles nor 6-cycles adjacent to a triangle (Borodin, Ivanova, and Raspaud [BIR10]). Moreover, in [MRW06b], Montassier, Raspaud and Wang proposed the following conjecture which is still unsettled.

Conjecture 1.4.7 ("Domaine de la Solitude 2000"Conjecture)
Every planar graph without 4*-cycles is acyclically* 4*-choosable.*

In Section 3.3 of **Chapter 3**, we will prove the following result:

Theorem 1.4.8 [CR10c] *Planar graphs without* 4*-cycles and* 5*-cycle are acyclically* 4*-choosable.*

This result is a new approach to the conjecture 1.4.7 and is best possible in the sense that there are planar graphs without 4- and 5-cycles that are not 3-choosable [Voi07]. Moreover, it extends some results in [Bor10, MRW06a, MRW07, CR09, CR10b]. We remark that the same result is independently obtained by Borodin and Ivanova [BI10] recently.

In the final section of **Chapter 3**, we will consider the acyclic 3-choosability of planar graphs. More generally, we prove the following theorem:

Theorem 1.4.9 [BCIR10] *Every graph* G *with* $\mathrm{Mad}(G) < \frac{14}{5}$ and $g(G) \geqslant 7$ is acyclically 3-choosable.

Since $\mathrm{Mad}(G) < \frac{2g(G)}{g(G)-2}$ for any planar graph G, we deduce from Theorem 1.4.9 that every planar graph with girth at least 7 is acyclically 3-choosable. This is a common strengthening of the facts that such a graph is acyclically 3-colorable (Borodin, Kostochka and Woodall [BKW99]) and that a planar graph of girth at least 8 is acyclically 3-choosable (Montassier, Ochem and Raspaud [MOR06]).

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The condition of no bicolored cycle in the definition of acyclic coloring can be naturally strengthened to the requirement that every pair of colors induces a star forest. A coloring satisfying such condition is called a *star coloring*. Star coloring was also introduced by Grünbaum [Grü73]. The *star chromatic number* $\chi_s(G)$ is defined to be the least number of colors required to obtain a star-coloring of G. The *star list chromatic number* of G, denoted by $\chi_s^l(G)$, is defined analogously.

We notice that every star coloring is an acyclic coloring but a star coloring of a graph may require more colors than that of acyclic coloring. In general, many star coloring questions are not as well understood as their acyclic counterparts. For example, as we mentioned before, Borodin's acyclic 5-color theorem is the best possible. On the other hand, Albertson et al. [ACK+04] proved that every planar graph is 20-star-colorable, and gave an example of a planar graph that requires 10 colors to star color. Albertson et al. [ACK+04] also noted that bounding the acyclic chromatic number bounds the star chromatic number and showed that $\chi_s(G) \leqslant \chi_a(G)(2\chi_a(G) - 1)$ for any graph G. However, determining the minimum star (list) chromatic number of many families of graphs is proved to be a challenging problem. This is indeed the case for families as simple as subcubic graphs.

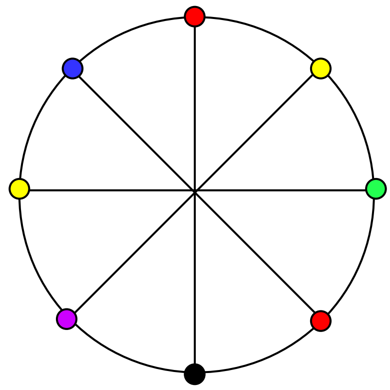

Figure 1.3: The Wagner graph G_W with $\chi_s(G_W) = 6$.

In 2001, Fertin, Raspaud and Reed [FRR01] proved that the Wagner graph is not 5-star-cholorable, see Figure 1.3. In other words, we need to use at least 6 colors to star color some subcubic graphs. On the other hand, Albertson et al. [ACK+04] proved that every subcubic graph is 7-star-choosable. Since $\chi_s(G) \leqslant \chi_s^l(G)$ for every graph G, the above two facts imply the following:

Corollary 1.4.10 *Let $\mathcal{SC}$ denote the family of subcubic graphs. We have that*

$$6 \leqslant \chi_s(\mathcal{SC}) \leqslant \chi_s^l(\mathcal{SC}) \leqslant 7.$$

As far as we know, no any improvement of Corollary 1.4.10 has been done in recent years. So the problem of deciding the star (list) chromatic number of subcubic graphs becomes the main question we are concerned in **Chapter 4**. First, in Section 4.2 of **Chapter 4**, we prove that 6 colors are indeed enough to star color subcubic graphs. More precisely, we prove the following result:

Theorem 1.4.11 [CRW10a] *Every subcubic graph is 6-star-colorable.*

Since the graph we considered in Theorem 1.4.11 is subcubic (not necessarily planar), the proof of Theorem 1.4.11 is relied on a detailed analysis of the structure properties rather than the discharging argument.

As Albertson et al. [ACK+04] investigated the star list chromatic number of subcubic graphs, they also showed that there exists a planar subcubic graph (obtained by adding a pendant vertex to each vertex in a cycle C_n) with arbitrarily high girth that has star chromatic number 4. It means that there does not exist a constant c such that every planar subcubic graph G with $g(G) \geqslant c$ has $\chi_s^l(G) \leqslant 3$.

So in Section 4.3 of **Chapter 4**, our focus is on the problem of finding star list chromatic number of planar subcubic graphs with the girth condition. More precisely, we prove the following result:

Theorem 1.4.12 [CRW10b] *Let G be a planar subcubic graph. Then*

(1) $\chi_s^l(G) \leqslant 6$.

(2) *If* $g(G) \geqslant 8$*, then* $\chi_s^l(G) \leqslant 5$.

(3) *If* $g(G) \geqslant 12$*, then* $\chi_s^l(G) \leqslant 4$.

Notice that the conclusion (1) in Theorem 1.4.12 partially improves the right side of the inequality in Corollary 1.4.10. Moreover, in proving Theorem 1.4.12, we introduce a useful concept *L-in-coloring* which is a good tool to control the star list chromatic number. This concept is an extension of the concept *in-coloring* which was used implicitly by Nešetřil and Ossona de Mendez [NOdM03] and explicitly by Albertson et al. [ACK+04]. More details about L-in-coloring can be found in Section 4.3.1 of **Chapter 4**,

The maximum average degree of graphs is a conventional measure of the sparseness of an arbitrary graph (not necessarily planar). In [KT10], Kündgen and Timmons proved a theorem about the dependence between the maximum average degree of graphs and their star list chromatic number. Their main result is the following:

Theorem 1.4.13 [KT10] *Let G be a graph.*

(1) *If* $\mathrm{Mad}(G) < \frac{8}{3}$*, then* $\chi_s^l(G) \leqslant 6$.

(2) *If* $\mathrm{Mad}(G) < \frac{14}{5}$*, then* $\chi_s^l(G) \leqslant 7$.

(3) *If G is planar and* $g(G) \geqslant 6$*, then* $\chi_s^l(G) \leqslant 8$.

In the final section of **Chapter 4**, we extend the conclusion (3) in Theorem 1.4.13 to a more general result, which avoids the planar constraint. The main result is stated as follows:

Theorem 1.4.14 [CRW09] *Every graph with* $\mathrm{Mad}(G) < 3$ *is* 8*-star-choosable.*

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The *vertex-arboricity* of a graph G is the minimum number $\mathrm{va}(G)$ of subsets into which the vertex set $V(G)$ can be partitioned so that each subset induces a forest. Clearly, $\mathrm{va}(G) \geqslant 1$ for every nonempty graph G and $\mathrm{va}(G) = 1$ if and only if G itself is a forest. There is an equivalent definition to the vertex-arboricity in terms of the coloring version. A *k-forest-coloring* of a graph G is a mapping π from $V(G)$ to the set $\{1, \cdots, k\}$ such that each color class induces a forest. The *vertex-arboricity* $\mathrm{va}(G)$ of G is the smallest integer k such that G has a k-forest-coloring. We should notice that two adjacent vertices can be assigned with the same color in a k-forest-coloring.

The vertex version of arboricity was first introduced by Chartrand, Kronk and Wall [CKW68] in 1968, who named it *point-arboricity*. They proved that $\mathrm{va}(G) \leqslant \lceil \frac{1+\Delta(G)}{2} \rceil$ for any graph G and $\mathrm{va}(G) \leqslant 3$ for any planar graph G. Chartrand and Kronk [CK69] showed this bound is sharp, by giving a planar graph which has vertex-arboricity 3. In fact, this graph was discovered by Tutte, which was used to disprove the conjecture of Tait that every cubic polyhedral graph is hamiltonian (see [Tut46]).

In 1979, Garey and Johnson [GJ79] proved that determining the vertex-arboricity of a graph is NP-hard. Hakimi and Schmeichel [HS89] showed that determining whether $\mathrm{va}(G) \leqslant 2$ is NP-complete for any maximal planar graph G. Recently, Raspaud and Wang [RW08] proved the following theorem:

Theorem 1.4.15 [RW08] *Let G be a planar graph.*

(1) *If G contains no k-cycles for some fixed $k \in \{3, 4, 5, 6\}$, then* $\mathrm{va}(G) \leqslant 2$.

(2) *If G contains no triangles at distance less than 2, then* $\mathrm{va}(G) \leqslant 2$.

Moreover, they proposed the following conjecture:

Conjecture 1.4.16 [RW08] *Every planar graph without intersecting triangles has vertex-arboricity at most* 2.

In **Chapter 5**, we will show that the above conjecture is true.

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In the final chapter, we study homomorphisms of sparse graphs to the Petersen graph. A *homomorphism* of G to H is a mapping $h : V(G) \to V(H)$ such that if $xy \in E(G)$ then $h(x)h(y) \in E(H)$. For more details about homomorphisms see the monograph of Hell and Nešetřil [HN04]. For positive integers k and $n \geqslant 2k$, an (n, k)-*coloring* of a graph G is a mapping c: $V(G) \to \binom{\{1, 2, \cdots, n\}}{k}$ such that for any two adjacent vertices x and y, $c(x)$ and $c(y)$ are disjoint. The concept of (n, k)-coloring is a generalization of the conventional vertex coloring problem. In fact, an $(n, 1)$-coloring is exactly an ordinary proper n-coloring.

The *fractional chromatic number*, denoted $\chi_f(G)$, of a graph G is the infimum of the fractions n/k for which there exists an (n, k)-coloring of G. As is well-known, the fractional chromatic number of a finite graph is always a rational number and the infimum is actually a minimum. The *Kneser graph*, denoted by $K_{n:k}$, is defined to be the graph in which vertices represent subsets of cardinality k taken from $\{1, 2, \cdots, n\}$ and two vertices are adjacent if and only if the corresponding subsets are disjoint. Note that $K_{5:2}$ is the famous Petersen graph. It is easy to observe that a graph G has an (n, k)-coloring if and only if there exists a homomorphism of G to $K_{n:k}$. As a special case, a graph G is $(5, 2)$-colorable if and only if there is homomorphism of G to the Petersen graph. Some background and more details about fractional coloring can be found in the monograph of Scheinerman and Ullman [SU97].

In **Chapter 6**, we will prove that every triangle-free graph with $\mathrm{Mad}(G) < \frac{5}{2}$ is homomorphic to the Petersen graph [CR10a]. In other words, such a graph is $(5, 2)$-colorable. Moreover, we show that the bound on the maximum average degree in our result is best possible. We also propose the following conjecture to conclude the thesis.

Conjecture 1.4.17 *Every graph G with odd girth $2k+1$ and $\mathrm{Mad}(G) < 2 + \frac{1}{k}$ has a fractional $(2k+1, k)$-coloring, where k is a positive integer.*

Chapter 2

3-choosability of planar graphs

Contents

In this chapter, we consider the problem of finding sufficient conditions for a planar graph to be 3-choosable. More specifically, we consider the 3-choosability of planar graphs in which each vertex is not incident to some cycles of given lengths, but all vertices can have different restrictions. This generalizes the approach based on forbidden cycles which corresponds to the case where all vertices have the same restrictions on the incident cycles. We prove that a planar graph G is 3-choosable if it is satisfied one of the following conditions:

- each vertex x is neither incident to cycles of lengths $4, 9, i_x$ with $i_x \in \{5, 7, 8\}$, nor incident to 6-cycles adjacent to a 3-cycle.
- each vertex x is not incident to cycles of lengths $4, 7, 9, i_x$ with $i_x \in \{5, 6, 8\}$.

This work extends five (published) results in [ZW04, ZW05, SW07, CLW08, CSW10].

2.1 Introduction

In 1976, Steinberg conjectured that every planar graph without cycles of lengths 4 and 5 is 3-colorable (see Problem 2.9 [JT95b]). This conjecture remains unsettled. Erdős [Ste93] asked if there exists an integer k such that the absence of cycles with size from 4 to k in a planar graph guarantees its 3-colorability? In [AZ91], Abbott and Zhou showed that such k exists and $k \leqslant 11$. This result was improved to $k \leqslant 9$ by Borodin [Bor96] and, independently, Sanders and Zhao [SZ95], and then it was improved to $k \leqslant 7$ by Borodin et al. [BGRS05]. Today the best known upper bound is $k \leqslant 7$ [BGRS05]. So it is interesting to answer Erdős's question by considering some restricted planar graphs (i.e., without some lengths of cycles).

In a slightly different approach many authors considered coloring planar graphs with 4 forbidden cycles, see [ZW04, ZW05, WC07a, CRW07, LCW07, SW07]. Strengthening results of these types, Wang and Chen [WC07b] proved that every planar graph without 4-, 6- and 8-cycles is 3-colorable; Lu et al. [LWW+09] proved that every planar graph without 4-, 7- and 9-cycles is 3-colorable; Borodin et al. [BGMR09] proved that every planar graph without 5-cycles, 7-cycles and adjacent 3-cycles is 3-colorable, which implies that every planar graph without 4-, 5- and 7-cycles is 3-colorable. Some other results related to 3-colorable planar graphs can be found in [BR03, MRW06b, CW08b, BMR10].

Naturally, we may propose the same question below for choosability:

Question 2.1.1 *What is the smallest integer c such that every planar graph without j-cycles for $4 \leqslant j \leqslant k^*$ is 3-choosable?*

Notice that it is impossible to extend Steinberg's conjecture to list coloring by the example given by Voigt [Voi07] and independently, by Montassier [Mon05b]. Hence $k^* \geqslant 6$. The best known upper bound is $k^* \leqslant 9$ obtained by Borodin [Bor96] in 1996, i.e., every planar graph without $\{4, \cdots, 9\}$-cycles is 3-choosable. These results have also been improved by showing that forbidding 4 cycles of certain size would lead to 3-choosability in planar graphs. We summarize these results in the following theorem:

Theorem 2.1.2 *A planar graph is 3-choosable if it has no*

- (Zhang and Wu [ZW04]) 4-, 5-, 7-, *and* 9-*cycles; or*
- (Zhang and Wu [ZW05]) 4-, 5-, 6-, *and* 9-*cycles; or*
- (Chen, Lu and Wang [CLW08]) 4-, 6-, 7-, *and* 9-*cycles; or*
- (Shen and Wang [SW07]) 4-, 6-, 8-, *and* 9-*cycles; or*
- (Chen, Shen and Wang [CSW10]) 4-, 7-, 8-, *and* 9-*cycles; or*
- (Wang, Lu and Chen [WLC10]) 4-, 5-, 8-, *and* 9-*cycles.*

In this chapter we introduce a new approach to the problem of characterizing planar graphs to be 3-choosable. Instead of forbidding certain cycles in the whole graph, we forbid a certain set of cycles for each vertex and these sets of forbidden cycles not necessarily are the same. More precisely, we prove the following theorems:

Theorem 2.1.3 *Let G be a planar graph in which each vertex x is neither incident to cycles of lengths $4, 9, i_x$ with $i_x \in \{5, 7, 8\}$, nor incident to 6-cycles adjacent to a 3-cycle. Then G is 3-choosable.*

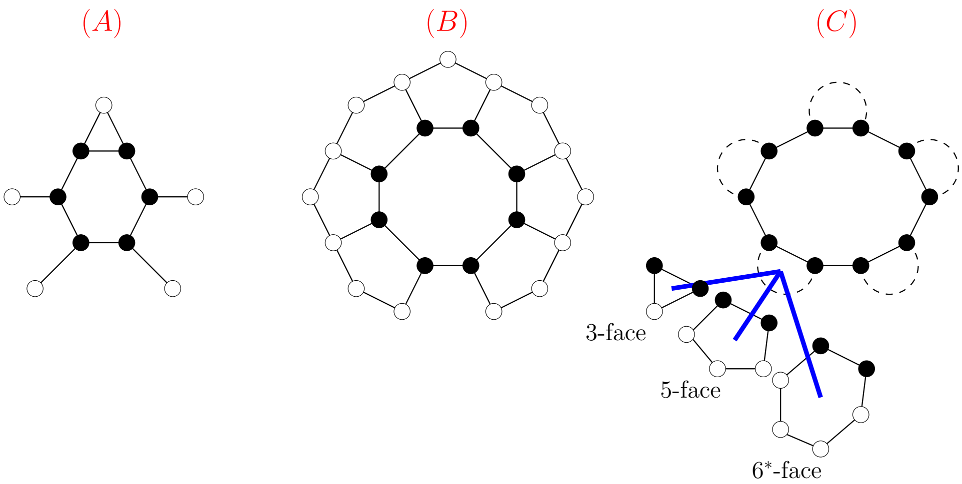


Figure 2.1: (a) Orchid, (b) sunflower, and (c) lotus.

Theorem 2.1.4 *Let G be a planar graph in which every vertex x is not incident to cycles of lengths $4, 7, 9, i_x$ with $i_x \in \{5, 6, 8\}$. Then G is 3-choosable.*

By Theorems 2.1.3 and 2.1.4, it is easy to deduce the following corollary which covers all results in Theorem 2.1.2 except the last conclusion [WLC10].

Corollary 2.1.5 *Every planar graph without $\{4, i, j, 9\}$-cycles with $5 \leqslant i < j \leqslant 8$ and $(i, j) \neq (5, 8)$ is 3-choosable.*

We remark that this is a joint work with Montassier and Raspaud [CMR]. To proceed with the proof of these theorems, we introduce some notation. Let G be a plane graph. A cycle C or a face f is called *triangle-far* if it is not adjacent to any 3-cycles. We call an i-face f an *i^*-face* if there is exactly one 3-face f' adjacent to f, and furthermore f' is adjacent to f normally. Similarly, we call an i-cycle C an *i^*-cycle* if there is exactly one 3-cycle C' adjacent to C, and furthermore C' is adjacent to C normally.

An *orchid* is a simple 6-face incident to six 3-vertices and normally adjacent to a 3-face. A *sunflower* is a simple 8-face incident to eight 3-vertices and adjacent to at least seven 5-faces. A *lotus* is a simple 10-face f incident to ten 3-vertices and adjacent to five clusters that are mutually disjoint with respect to f, where a cluster is either a 3-face, or a 5-face, or a 6^*-face (see Figure 2.1).

2.2 Our main result

To obtain Theorems 2.1.3 and 2.1.4, we prove the following stronger Theorem 2.2.1, whose proof will be postponed to Section 2.3.

Theorem 2.2.1 *Let G be a planar graph with $\delta(G) \geqslant 3$ and G does not contain 4-cycles and 9-cycles. If G further satisfies the following structural properties:*

(C1) *a 5-cycle or a 6-cycle is adjacent to at most one 3-cycle;*

(C2) *a 5^*-cycle is neither normally adjacent to a 5^*-cycle, nor adjacent to an i-cycle with $i \in \{7, 8\}$;*

(C3) *a 6^*-cycle is neither adjacent to a 6-cycle, nor incident to an i-cycle C with $i \in \{3, 5\}$, where C is opposite to such 6^*-cycle by a 4-vertex;*

(C4) *a triangle-far 7-cycle is not adjacent to two 5-cycles which are normally adjacent;*

(C5) *a 7^*-cycle is neither adjacent to a 5-cycle nor a 6^*-cycle.*

Then G contains an orchid or a sunflower or a lotus.

Assuming Theorem 2.2.1, we can easily prove Theorems 2.1.3 and 2.1.4.

Proofs of Theorems 2.1.3 and 2.1.4: Suppose that G_1 or G_2 is a plane presentation of the counterexample to Theorem 2.1.3 and 2.1.4, respectively, with the smallest number of vertices. Thus, G_i is connected ($i = 1, 2$). First, for each $i \in \{1, 2\}$, we observe that $\delta(G_i) \geqslant 3$. Otherwise, let u_i be a vertex of minimum degree in G_i. By the minimality of G_i, $G_i - u_i$ is 3-choosable. Obviously, we can extend any L-coloring such that $\forall x \in V(G) : |L(x)| \geqslant 3$ of $G_i - u_i$ to G_i and ensure that G_i is 3-choosable. Next, in each case, we will show that each G_i contains either an orchid, or a sunflower, or a lotus. Denote N_a, N_b, N_c be the set of black vertices of (a), (b) and (c) in Figure 2.1, respectively. For each $j \in \{a, b, c\}$, one can easily observe that we can extend any L-coloring such that for all $x \in V(G) : |L(x)| \geqslant 3$ of $G_i - N_j$ to N_j and make sure that G_i is 3-choosable. Thus, G_1 and G_2 are both 3-choosable, which are contradictions.

Since G_i does not contain 4-cycles and 9-cycles, for each $i \in \{1, 2\}$, we only need to verify if G_i satisfies all the structural properties (C1) to (C5).

(1) For G_1, since each vertex x is not incident to 6-cycles adjacent to a 3-cycle, we assert that there is neither 5^*-face nor 6^*-face in G_1. Thus, (C1), (C2) and (C3) are satisfied. It remains us to check the properties (C4) and (C5). If (C4) is not satisfied, then there appears a vertex x which is incident to an i_x-cycle with $i_x \in \{5, 7, 8\}$, which contradicts the assumption of G_1. If (C5) is not satisfied, then a vertex y is appeared such that y is incident to an i_y-cycle with $i_y \in \{5, 7, 8\}$, which is a contradiction.

(2) For G_2, because it does not contain 7-cycles, we confirm that there is no 6^*-cycle and 7^*-cycle in G_2. Thus, we only need to check the properties (C1) and (C2). It is easy to establish a 7-cycle or a 4-cycle if a 5-cycle or a 6-cycle is adjacent to at least two 3-cycles. Thus, (C1) is satisfied. Let us check (C2). If there exist two 5^*-cycles that are normally adjacent, then a 9-cycle is produced, which is a contradiction. If a 5^*-cycle is adjacent to an 8-cycle, then there is a vertex incident to a 5-cycle, a 6-cycle and an 8-cycle, which contradicts the assumption of G_2. Therefore, (C2) is satisfied.

This completes the proofs of Theorems 2.1.3 and 2.1.4. ∎

2.3 Proof of Theorem 2.2.1

Let G be a counterexample to Theorem 2.2.1, i.e., an embedded plane graph G with $\delta(G) \geqslant 3$, no cycles of lengths 4 and 9, satisfying the structural properties (C1) to (C5), and containing no orchid, no sunflower, and no lotus (i.e., none of the configurations depicted in Figure 2.1).

2.3.1 Proof of 2-connected case

First, we suppose that G is 2-connected. Thus, every face in G is simple. It means that an m-face is exactly an m-cycle with $m \geqslant 3$. So all cycles mentioned in assumptions (C1) to (C5) can be regarded as faces. We need to discuss some properties of G.

Claim 2.3.1 *For some fixed $i \in \{5, 6, 7, 8\}$, if an i-face is adjacent to a 3-face, then they are normally adjacent.*

Proof. Suppose the claim is false. Let $f_i = [v_1 v_2 \cdots v_i]$ be an i-face and $f_2 = [v_1 v_2 u]$ be a 3-face such that f_1 is adjacent to f_2 and $|V(f_1) \cap V(f_2)| \geqslant 3$. It means that u is equal to some v_j with $j \in \{3, 4, \cdots, i\}$. According to the value of i, one can easily observe that if u is a vertex v_j with $3 \leqslant j \leqslant i$, then G contains either a 2-vertex or a 4-cycle, which is a contradiction. This completes the proof of Claim 2.3.1. ∎

Since G does not contain 9-cycles, we obtain Claims 2.3.2 and 2.3.3 easily by Claim 2.3.1.

Claim 2.3.2 *Each 7-face is adjacent to at most one 3-face.*

Claim 2.3.3 *No 8-face is adjacent to a 3-face.*

Claim 2.3.4 *If two 5-faces are adjacent to each other, then they are normally adjacent.*

Proof. Suppose that there are two adjacent 5-faces $f_1 = [v_1 v_2 \cdots v_5]$ and $f_2 = [v_1 v_2 u v w]$ with $v_1 v_2$ as a common edge. If $|V(f_1) \cap V(f_2)| = 2$, then Claim 2.3.4 follows. Otherwise, by symmetry, we only need to consider the following cases. If $w = v_5$, then $d(v_1) = 2$ which is impossible. If $w = v_4$, then G contains a 4-cycle $v_1 v_2 v_3 v_4 v_1$, which is a contradiction. This implies $u \notin V(f_1)$ and $w \notin V(f_1)$. If $v = v_5$ or $v = v_4$, then a 4-cycle $u v_2 v_1 v_5 u$ or $w v_1 v_5 v_4 w$ can be easily established. This contradiction completes the proof of Claim 2.3.4. ∎

Together with (C2), we have:

Claim 2.3.5 *There is no adjacent two 5^*-faces in G.*

Claim 2.3.6 *A triangle-far 5-face cannot be adjacent to a 5^*-face in G.*

Proof. Suppose to the contrary that a triangle-far 5-face $f_1 = [v_1v_2 \cdots v_5]$ is adjacent to a 5^*-face $f_2 = [v_1v_2u_3u_4u_5]$ by a common edge v_1v_2. By definition, f_1 is not adjacent to any 3-face. By Claim 2.3.4, each u_i cannot be equal to some v_j with $i, j \in \{3, 4, 5\}$. By symmetry, we have to handle the following two cases:

- Assume that v_1u_5u is a 3-face. By Claim 2.3.1, $u \neq v_2, u_3, u_4$. Moreover, $u \neq v_5$ by the choice of f_1. If $u = v_4$ or $u = v_3$, then G contains a 4-cycle, which is impossible. Thus, $u \notin V(f_1) \cup V(f_2)$ and thus G contains a 9-cycle $uv_1v_5v_4v_3v_2u_3u_4u_5u$, which is a contradiction.

- Assume that u_5u_4u is a 3-face. Notice that $u \neq v_1, v_2, u_3$ by Claim 2.3.1. If $u \in \{v_3, v_4, v_5\}$, then a 4-cycle is easily obtained, which is a contradiction. Thus, $u \notin V(f_1) \cup V(f_2)$. Obviously, a 9-cycle $uu_5v_1v_5v_4v_3v_2u_3u_4u$ is established. This contradicts the absence of 9-cycles in G. Therefore, we complete the proof of Claim 2.3.6. ∎

Claim 2.3.7 *No 3-vertex is incident to three 5-faces.*

Proof. Suppose to the contrary that G contains a 3-vertex u adjacent to three vertices v_1, v_2, v_3 and incident to three 5-faces $f_1 = [uv_1x_1x_2v_2]$, $f_2 = [uv_2y_1y_2v_3]$, and $f_3 = [uv_3z_1z_2v_1]$. By Claim 2.3.4, f_i and f_j are normally adjacent for each pair $\{i, j\} \subset \{1, 2, 3\}$. It implies that all vertices in $(V(f_1) \cup V(f_2) \cup V(f_3)) \setminus \{u\}$ are mutually distinct. However, a 9-cycle $v_1x_1x_2v_2y_1y_2v_3z_1z_2v_1$ is established, contradicting the assumption on G. Thus, we complete the proof of Claim 2.3.7. ∎

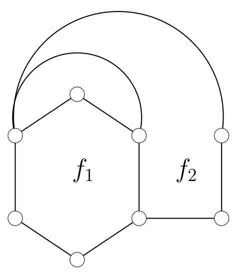


Figure 2.2: A 6-face f_1 is adjacent to a 5-face f_2.

Claim 2.3.8 *Up to isomorphism, a 6-face can be adjacent to a 5-face in an unique way as depicted in Figure 2.2.*

Proof. Assume that a 6-face $f_1 = [v_1v_2 \cdots v_6]$ is adjacent to a 5-face $f_2 = [v_1v_2uvw]$ with v_1v_2 as a common edge. We first suppose that $u, w \notin V(f_1)$. By the absence of 4-cycles in G, we deduce that $v \neq v_3$ and $v \neq v_4$. Otherwise, there is a 4-cycle either $wv_1v_2v_3w$ or $uv_4v_3v_2u$. So by symmetry, we have that $v \notin \{v_5, v_6\}$. However, one can easily check that a 9-cycle $v_2v_3v_4v_5v_6v_1wvuv_2$ is established, which is a contradiction.

Now, w.l.o.g., we may suppose that $w \in V(f_1)$. The following argument is divided into four cases.

- Assume that $w = v_6$. Then v_1 is a 2-vertex, which is a contradiction.

- Assume that $w = v_5$. Obviously, $u \neq v_3$ and $u \neq v_4$. Otherwise, either $d(v_2) = 2$ or a 4-cycle $v_1wuv_2v_1$ is established, which are both contradictions. So we may suppose that $u \notin V(f_1)$. If $v = v_3$, then a 4-cycle wv_1v_2vw is formed. If $v = v_4$, then a 4-cycle vv_3v_2uv is formed. A contradiction is always obtained, which implies that $v \notin V(f_1)$ and thus we are done, see Figure 2.2.

- Assume that $w = v_4$. Then a 4-cycle $v_1v_6v_5wv_1$ is constructed, which is impossible.

- Assume that $w = v_3$. Since G is the plane graph, we see that $u, v \notin V(f_1)$. However, v_2uvwv_2 is a 4-cycle, which is a contradiction.

Therefore, we complete the proof of Claim 2.3.8. ∎

Claim 2.3.9 *No 3-vertex is incident to two 5-faces and one 6-face.*

Proof. Suppose the claim is not true. We assume that there exists a 3-vertex u adjacent to three vertices v_1, v_2, v_3 and incident to two 5-faces $f_1 = [uv_1x_1x_2v_2]$, $f_2 = [uv_2y_1y_2v_3]$, and one 6-face $f_3 = [uv_3z_1z_2z_3v_1]$.

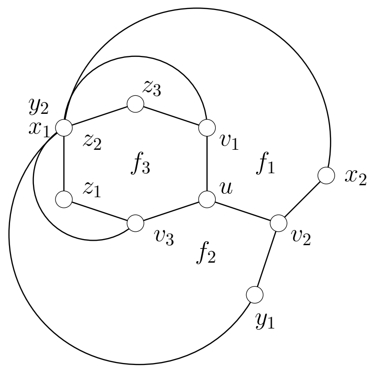


Figure 2.3: A 3-vertex u incident to two 5-faces f_1 and f_2 and a 6-face f_3.

By Claim 2.3.8, $z_2 = y_2 = x_1$, see Figure 2.3. Hence a 4-cycle $z_2v_1uv_3z_2$ exists which is a contradiction. Thus, we complete the proof of Claim 2.3.9. ∎

Claim 2.3.10 *No 3-vertex is incident to one 5-face and two 6-faces.*

Proof. Suppose to the contrary that there exists a 3-vertex u adjacent to three vertices v_1, v_2, v_3 and incident to two 6-faces $f_1 = [uv_3y_1y_2y_3v_1]$, $f_2 = [uv_2z_1z_2z_3v_3]$, and one 5-face $f_3 = [uv_1x_1x_2v_2]$. By Claim 2.3.8, we see that f_1 and f_3 can only be adjacent to each other in an unique way as depicted in Figure 2.2. One can easily observe that $x_1 = y_2$ or $v_2 = y_1$, see Figure 2.4. Next, we will make use of contradictions to show that f_2 cannot exist in G. We have to deal with the following two cases.

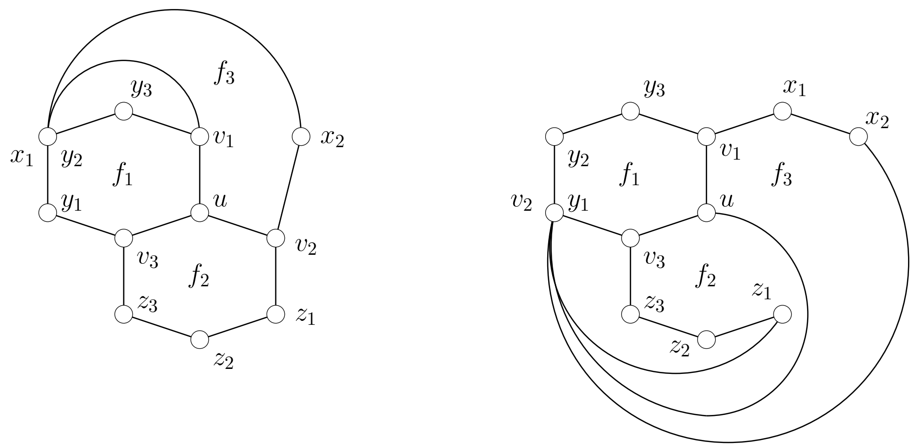


Figure 2.4: A 3-vertex u incident to one 5-face f_3 and two 6-faces f_1 and f_2.

- $x_1 = y_2$. For simplicity, denote $x^* = x_1 = y_2$. By Claim 2.3.8, we see that $x_2 = z_2$. Then a 5-face $x^*v_1uv_2x_2x^*$ adjacent to two 3-cycles $x^*y_3v_1x^*$ and $v_2z_1x_2v_2$ is produced. This contradicts (C1).

- $v_2 = y_1$. Clearly, uv_3y_1u is a 3-cycle which is not a 3-face. For simplicity, let $y^* = v_2 = y_1$. Obviously, $\{z_1, z_2, z_3\} \cap \{y_2, y_3, x_1, x_2\} = \varnothing$ because of the planarity of G. However, a 9-cycle $y^*z_1z_2z_3v_3uv_1x_1x_2y^*$ is easily established, which is impossible. This completes the proof of Claim 2.3.10. ∎

Claim 2.3.11 *No 6^*-face is adjacent to a 5-face in G.*

Proof. Suppose to the contrary that there exists a 6-face $f_1 = [v_1v_2\cdots v_6]$ adjacent to a 5-face $f_2 = [v_1v_2uvw]$ by a common edge v_1v_2. By Claim 2.3.8, w.l.o..g, suppose that $w = v_5$. Note that f_1 is adjacent to a 3-cycle $v_1v_5v_6v_1$ which is not a 3-face. Thus, f_1 cannot be adjacent to any other 3-face by (C1), which means that f_1 cannot be a 6^*-face. This completes the proof of Claim 2.3.11. ∎

The following claim is immediately obtained by Claim 2.3.11.

Claim 2.3.12 *No 6^*-face is adjacent to a 5^*-face in G.*

By (C1) and a similar proof of Claim 2.3.11, we have:

Claim 2.3.13 *No 5^*-face is adjacent to a 6-face in G.*

Furthermore, (C3) implies the following claim:

Claim 2.3.14 *There is no adjacent 6^*-faces in G.*

Since G contains no 4- and 9-faces, it is easy to deduce the following claim by Claim 2.3.6, Claim 2.3.5, Claim 2.3.13 and (C2).

Claim 2.3.15 *No 5^*-face is adjacent to an i-face in G, where $i \in \{4, \cdots, 9\}$.*

Discharging procedure:

We complete the proof with a discharging procedure. We first assign to each vertex v an initial charge $\omega(v)$ such that for all $v \in V(G), \omega(v) = 2d(v) - 6$ and to each face f an initial charge such that for all $f \in F(G), \omega(f) = d(f) - 6$. By Lemma 1.3.1, we see that $\sum_{x \in V(G) \cup F(G)} \omega(x) = -12$.

Before stating the discharging rules, we need to give some notation that will be frequently used in the following argument. For a vertex $v \in V(G)$ and for an integer $i \geqslant 5$, let $m_i(v)$ and $m_{i^*}(v)$ denote the number of triangle-far i-faces and i^*-faces incident to v, respectively. Furthermore, we denote $M_i(v) = m_i(v) + m_{i^*}(v)$ and call a face f a *non-3-face* if $d(f) \neq 3$. Let $f_1 = [xuvy\cdots]$ and $f_2 = [zuvt\cdots]$ denote two adjacent faces by a common edge uv, where f_1 is a 7^+-face while f_2 is a 5- or 5^*- or 6^*-face. If both zu and vt are non-triangular edges of f_2, then we call uv a *good common edge*. We further call such uv a *good common* (b_1, b_2)*-edge* if uv is a (b_1, b_2)-edge.

The discharging rules are defined as follows:

(R1) Each 5^+-face sends 1 to its adjacent 3-face.
(R2) Let v be a 4-vertex.
(R2a) If $t(v) = 2$, then for each non-3-face f, $\tau(v \to f) = 1$.
(R2b) If $t(v) = 1$, then let f_1 denote the incident 3-face and f' be the opposite face of f_1.
(R2b1) If f' is a triangle-far 5-face, then v sends $\frac{2}{3}$ to each incident face different from f_1.
(R2b2) Otherwise, v sends 1 to each incident face which is adjacent to f_1.
(R2c) If $t(v) = 0$, let f_1, f_2, f_3, and f_4 denote the faces of G incident to v in a cyclic order such that the degree of f_1 is the smallest one among all faces incident to v, then we do as follows:
(R2c1) if $M_5(v) = 0$, then v sends $\frac{1}{2}$ to each incident face.
(R2c2) if $M_5(v) = 1$, then v sends $\frac{2}{3}$ to each of f_1, f_2, and f_4 when f_1 is a triangle-far 5-face; or v sends 1 to each of f_2 and f_4 when f_1 is a 5^*-face.
(R2c3) if $M_5(v) = 2$, then
(R2c3.1) v sends $\frac{2}{3}$ to each triangle-far 5-face and $\frac{1}{3}$ to each other incident face when $m_5(v) = 2$.
(R2c3.2) v sends $\frac{2}{3}$ to each incident face of v except the unique 5^*-face when $m_5(v) = 1$ and $m_{5^*}(v) = 1$.
(R2c3.3) v sends 1 to each incident face that is not a 5^*-face when $m_{5^*}(v) = 2$.
(R2c4) if $M_5(v) = 3$, then v gives $\frac{2}{3}$ to each incident triangle-far 5-face.
(R2c5) if $M_5(v) = 4$, then v gives $\frac{1}{2}$ to each incident triangle-far 5-face.
(R3) Let v be a 5-vertex and f be a non-3-face incident to v. Then
(R3a) $\tau(v \to f) = \frac{4}{3}$ if $t(v) = 2$.
(R3b) $\tau(v \to f) = 1$ if $t(v) = 1$.
(R3c) if $t(v) = 0$, v sends 1 to each incident face different from 5^*-faces when $m_{5^*}(v) \geqslant 1$; or sends $\frac{5}{6}$ to each incident 6^*-face and sends $\frac{4 - \frac{5}{6} m_{6^*}(v)}{5 - m_{6^*}(v)}$ to each other incident face when $m_{5^*}(v) = 0$.
(R4) Let f be a 7^+-face. If f' is adjacent to f by a good common edge e, then

(R4a) $\tau(f \to f') = \frac{1}{3}$ if f' is a triangle-far 5-face and e is a $(3,3)$-edge.
(R4b) $\tau(f \to f') = \frac{1}{6}$ if f' is a 6^*-face and e is a $(3,3)$-edge or a $(3,4)$-edge.

(R5) Each 10^+-face sends 1 to each adjacent 5^*-face by a good common $(3^+,3^+)$-edge.

(R6) Each 6^+-vertex sends 1 to each incident face.

In the following, we will prove that the new weight function satisfies $\omega^*(x) \geqslant 0$ for all $x \in V(G) \cup F(G)$, which leads to an obvious following contradiction

$$-12 = \sum_{x \in V(G) \cup F(G)} \omega(x) = \sum_{x \in V(G) \cup F(G)} \omega^*(x) \geqslant 0$$

and hence we complete the proof of 2-connected case of Theorem 2.2.1.

The following observation obviously holds by the absence of 4-cycles in G.

Observation 2.3.1 *For $v \in V(G)$, we have $t(v) \leqslant \left\lfloor \frac{d(v)}{2} \right\rfloor$.*

Since $\delta(G) \geqslant 3$, $d(v) \geqslant 3$ for each vertex $v \in V(G)$. We have to handle the following cases, depending on the value of $d(v)$.

Case 1 $d(v) = 3$.

It is easy to see that $\omega^*(v) = \omega(v) = 2 \times 3 - 6 = 0$ by (R1) to (R6).

Case 2 $d(v) = 4$.

Clearly, $\omega(v) = 2$ and v is incident to at most two 3-faces by Observation 2.3.1. If $t(v) = 2$, then we deduce that $\omega^*(v) = 2 - 2 \times 1 = 0$ by (R2a). If $t(v) = 1$ (v is incident to exactly one 3-face), then depending on the opposite face of such 3-face, v gives either $\frac{2}{3} \times 3 = 2$, or $1 \times 2 = 2$ by (R2b1) or (R2b2). Hence, $\omega^*(v) = 0$. Finally, we only need to consider the case of $t(v) = 0$. We divide the discussion into five subcases in the light of the value of $M_5(v)$.

Subcase 2.1 $M_5(v) = 0$.

This implies that the degree of each face incident to v is at least 6 by the absence of 4-faces. According to (R2c1), $\omega^*(v) \geqslant 2 - \frac{1}{2} \times 4 = 0$.

Subcase 2.2 $M_5(v) = 1$.

It is easy to observe that v sends either $\frac{2}{3} \times 3 = 2$ if $m_5(v) = 1$, or $1 \times 2 = 2$ if $m_{5^*}(v) = 1$ by (R2c2). Thus, v gives totally at most 2 to incident faces. Hence, $\omega^*(v) \geqslant 2 - 2 = 0$.

Subcase 2.3 $M_5(v) = 2$.

If $m_5(v) = 2$, then $\omega^*(v) \geqslant 2 - \frac{2}{3} \times 2 - \frac{1}{3} \times 2 = 0$ by (R2c3.1). If $m_5(v) = m_{5^*}(v) = 1$, then such triangle-far 5-face and 5^*-face cannot be adjacent to each other by Claim 2.3.6. Thus, applying (R2c3.2), $\omega^*(v) \geqslant 2 - \frac{2}{3} \times 3 = 0$. Otherwise, suppose $m_{5^*}(v) = 2$. Notice that v is incident to two 5^*-faces which are opposite to each other by Claim 2.3.5. Thus, $\omega^*(v) \geqslant 2 - 1 \times 2 = 0$ by (R2c3.3).

Subcase 2.4 $M_5(v) = 3$.

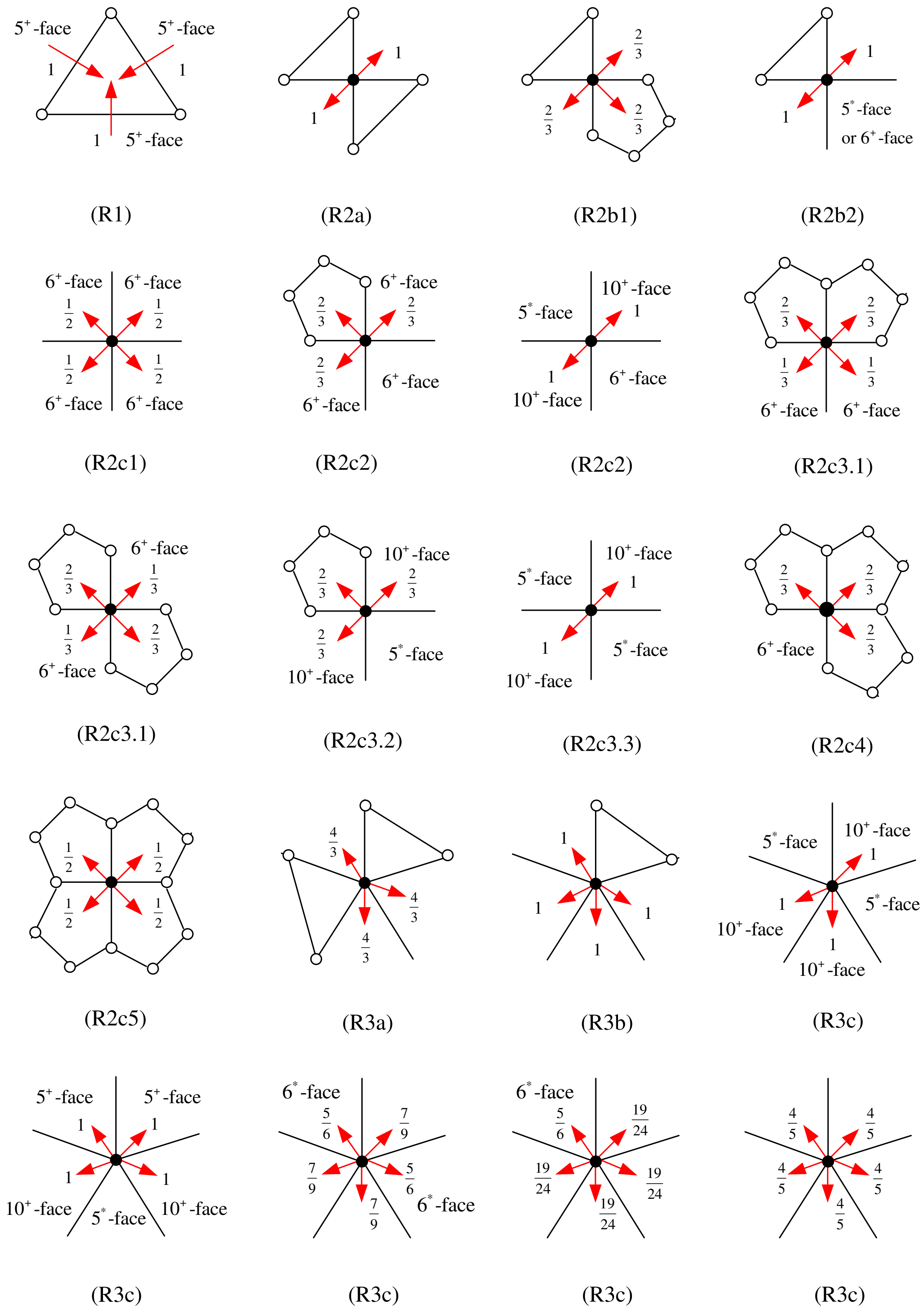


Figure 2.5: Discharging rules (R1) to (R3).

We first notice that $m_{5^*}(v) \neq 3$ since there are no adjacent 5^*-faces in G by Claim 2.3.5. If $1 \leqslant m_{5^*}(v) \leqslant 2$, then there exists at least one triangle-far 5-face adjacent to one 5^*-face, contradicting the Claim 2.3.6. Thus, $m_{5^*}(v) = 0$, which implies that $m_5(v) = 3$. According to (R2c4), we have that $\omega^*(v) \geqslant 2 - \frac{2}{3} \times 3 = 0$.

Subcase 2.5 $M_5(v) = 4$.

One can observe that $m_{5^*}(v) = 0$ by Claim 2.3.6 and Claim 2.3.5. It implies that v is incident to exactly four triangle-far 5-faces. Consequently, we have that $\omega^*(v) \geqslant 2 - \frac{1}{2} \times 4 = 0$ by (R2c5).

Case 3 $d(v) = 5$.

Obviously, $\omega(v) = 4$ and $t(v) \leqslant 2$ by Observation 2.3.1. It is easy to observe that v sends either $\frac{4}{3} \times 3 = 4$ by (R3a) if $t(v) = 2$, or $1 \times 4 = 4$ by (R3b) if $t(v) = 1$. Therefore, $\omega^*(v) \geqslant 4 - 4 = 0$ if $t(v) > 0$. Now we may assume that $t(v) = 0$. This implies that each face incident to v is a 5^+-face combining the fact that G does not contain any 4-faces. By Claim 2.3.5, we have that $m_{5^*}(v) \leqslant 2$. Moreover, the degree of the face adjacent to a 5^*-face is at least 10 by Claim 2.3.15. So by (R3c), $\omega^*(v) \geqslant 4 - 1 \times 4 = 0$ if $m_{5^*}(v) \geqslant 1$; or $\omega^*(v) \geqslant 4 - \frac{5}{6}m_{6^*}(v) - \frac{4-\frac{5}{6}m_{6^*}(v)}{5-m_{6^*}(v)}(5 - m_{6^*}(v)) = 0$ if $m_{5^*}(v) = 0$.

Case 4 $d(v) \geqslant 6$.

According to (R6), we have that $\omega^*(v) \geqslant (2d(v) - 6) - 1 \times d(v) = d(v) - 6 \geqslant 0$.

Let $f \in F(G)$. Then $b(f)$ is a cycle since G is 2-connected. Clearly, $d(f) \neq 4$ and $d(f) \neq 9$ by the absence of 4- and 9-cycles. We write $f = [v_1 v_2 \cdots v_{d(f)}]$ and suppose that f_i is the face of G adjacent to f with $v_i v_{i+1}$ as a common edge, where (and in the following discussion) all indices are taken modulo $d(f)$. Let $m_5(f)$, $m_{5^*}(f)$, and $m_{6^*}(f)$ denote the number of triangle-far 5-faces, 5^*-faces, and 6^*-faces adjacent to f.

Case 5 $d(f) = 3$.

Let f be a 3-face and then $\omega(f) = -3$. Since $\delta(G) \geqslant 3$, f is adjacent to three faces and each adjacent face is neither a 3-face nor a 4-face by the absence of 4-cycles in G. It implies that f gets 3×1 from its adjacent faces by (R1). Thus, $\omega^*(f) \geqslant -3 + 1 \times 3 = 0$.

Case 6 $d(f) = 5$.

Let $f = [v_1 \cdots v_5]$ and then $\omega(f) = -1$. Clearly, f is adjacent to at most one 3-face by (C1).

6.1 First assume that f is a triangle-far 5-face. It follows that there is no 3-face adjacent to f. Thus, f sends nothing to all its adjacent faces. Moreover, each f_i cannot be a 5^*-face by Claim 2.3.6. We need to deal with the following three possibilities, depending on the value of $n_3(f)$.

   a) $n_3(f) = 5$. It means that v_i is a 3-vertex for all $i = 1, \ldots, 5$. If there exists a 6-face adjacent to f, then by Claim 2.3.8 we see that they are adjacent to each other in an unique way as depicted in Figure 2.2. It is easy to see that there is one 4^+-vertex belonging to $V(f)$, which contradicts $n_3(f) = 5$. Thus, each face adjacent to f is either a triangle-far 5-face or a 7^+-face by the absence of 4-faces. Furthermore, we notice that f is adjacent to at most two triangle-far 5-faces which are not adjacent by Claim 2.3.7. So f is adjacent to at least three 7^+-faces such that each 7^+-face is adjacent to f by a good common $(3,3)$-edge. Therefore, applying (R4a), we obtain that $\omega^*(f) \geqslant -1 + 3 \times \frac{1}{3} = 0$.

b) $n_3(f) = 4$. Let v_1 be such a 4^+-vertex and v_j be a 3-vertex for all $j = 2,3,4,5$. Clearly, v_1 gives at least $\frac{1}{2}$ to f by (R2) and (R3). Moreover, f_1 and f_5 cannot be any 6-face by Claim 2.3.8. If $d(f_1) = 5$ and $d(f_5) = 5$, then $d(f_j) \notin \{5,6\}$ with $j \in \{2,4\}$ according to Claim 2.3.7 and Claim 2.3.9. Thus, for $j \in \{2,4\}$, f_j is a 7^+-face by the absence of 4-faces and each f_j is adjacent to f by a good common $(3,3)$-edge. By (R4a), we see that $\tau(f_2 \to f) = \frac{1}{3}$ and $\tau(f_4 \to f) = \frac{1}{3}$. So we obtain that $\omega^*(f) \geqslant -1 + \frac{1}{2} + \frac{1}{3} \times 2 = \frac{1}{6} > 0$.

Now we may suppose that there exists at least one face of f_1 and f_5 which is a 7^+-face, i.e., $d(f_1) \geqslant 7$. Then by (R2), (R3) and (R6), we see that $\tau(v_1 \to f) \geqslant \frac{2}{3}$. Clearly, for each $i \in \{2,3,4\}$, f_i is adjacent to f by a good common $(3,3)$-edge. According to Claim 2.3.7, Claim 2.3.9 and Claim 2.3.10, we see that there exists at least one face of f_2, f_3, f_4 which is a 7^+-face. Hence, $\omega^*(f) \geqslant -1 + \frac{1}{3} + \frac{2}{3} = 0$ by (R4a).

c) $n_3(f) \leqslant 3$. It follows that there are at least two vertices whose degree are both at least 4. By (R2), (R3) and (R6), we derive that $\omega^*(f) \geqslant -1 + \frac{1}{2} \times 2 = 0$.

6.2 Now assume that f is a 5^*-face. It implies that f is adjacent to exactly one 3-face. W.l.o.g., let $f_1 = [vv_1v_2]$ be such a 3-face that it is adjacent to f. By Claim 2.3.1, $v \neq v_i$ for all $i = 3,4,5$. Since $\delta(G) \geqslant 3$, $d(v_i) \geqslant 3$ with $i \in \{1,2,\cdots,5\}$. By Claim 2.3.15, for each $j \in \{2,3,4,5\}$, we see that $d(f_j) \geqslant 10$ and thus both v_3v_4 and v_4v_5 are good common $(3^+,3^+)$-edges. By (R5), $\tau(f_3 \to f) = 1$ and $\tau(f_4 \to f) = 1$. Hence, $\omega^*(f) \geqslant -1 - 1 + 1 \times 2 = 0$ by (R1).

Case 7 $d(f) = 6$.

Let $f = [v_1 \cdots v_6]$ and then $\omega(f) = 0$. If f is a triangle-far 6-face, then $\omega^*(f) = \omega(f) = 0$ by (R1) to (R6). Now, we assume that f is a 6^*-face. W.l.o.g., assume $f_1 = [vv_1v_2]$ is a 3-face adjacent to f. It is obvious that $v \notin V(f)$ by Claim 2.3.1. Furthermore, f is adjacent to at most one 3-face by (C1). So f only need to send 1 to the unique 3-face f_1. Obviously, for each $j \in \{2,\cdots,6\}$, $d(f_j) \notin \{3,4,5,6\}$ by (C1), (C3), Claim 2.3.11 and the absence of 4-cycles in G. Again, by (C1) and the absence of 4-cycles, we assert that $v_3v_5 \notin E(G)$ and $v_3v_6 \notin E(G)$. It implies that each v_i has at least one outgoing neighbor which does not belonging to $V(f)$. Since there is no orchid in G, f is incident to at least one 4^+-vertex. It implies that

$n_3(f) \leqslant 5$. Next, in each case, we will show that the total charge f obtained is at least 1 and thus $\omega^*(f) \geqslant -1+1=0$.

Subcase 7.1 $n_3(f)=5$.

It means that there is exactly one 4^+-vertex incident to f. If $d(v_2) \geqslant 4$, then $\tau(v_2 \to f) \geqslant 1$ by (R2b2), (R3a), (R3b) and (R6) since $d(f_2) \neq 5$. Otherwise, by symmetry, suppose some v_i is a 4^+-vertex, where $i \in \{3,4\}$. Denote v^* be such a 4^+-vertex. First, we observe that each adjacent face different from f_1 is a 7^+-face by the discussion above. If $d(v^*) \geqslant 5$, then $\tau(v^* \to f) \geqslant \frac{5}{6}$ by (R3) and (R6). Since v_5v_6 is a good common $(3,3)$-edge, f_5 sends $\frac{1}{6}$ to f by (R4b). Thus, f gets at least $\frac{5}{6}+\frac{1}{6}=1$ from v^* and f_5. If $d(v^*)=4$, then the opposite face of f, which is incident to f by v^*, cannot be a 3-face or a 5-face by (C3). So v^* is incident to four 6^+-faces and thus v^* gives $\frac{1}{2}$ to f by (R2c1). Consequently, f gets at least $\frac{1}{2}+\frac{1}{6}\times 3=1$ by (R4b).

Subcase 7.2 $0 \leqslant n_3(f) \leqslant 4$.

It implies that there are at least two 4^+-vertices incident to f. It is easy to see that every 5^+-vertex sends at least $\frac{5}{6}$ to f by (R3) and (R6). Moreover, every 4-vertex v_i sends at least $\frac{1}{2}$ to f since the opposite face to f by v_i cannot be any 3-face or 5-face by (C3). Hence, f receives at least $\frac{1}{2}\times 2=1$ from its incident 4^+-vertices.

In what follows, for simplicity, let $p_5(f)$, $p_{5^*}(f)$, and $p_{6^*}(f)$ denote the number of triangle-far 5-face, 5^*-face, and 6^*-face receiving a charge $\frac{1}{3}$, 1, $\frac{1}{6}$ from f, respectively. Clearly, $p_5(f) \leqslant m_5(f)$, $p_{5^*}(f) \leqslant m_{5^*}(f)$ and $p_{6^*}(f) \leqslant m_{6^*}(f)$.

Case 8 $d(f)=7$.

Then $\omega(f)=1$ and Claim 2.3.2 implies that f is adjacent to at most one 3-face.

8.1 First assume that f is a triangle-far 7-face. Noting that $d(f_i) \geqslant 5$ since G contains no 4-faces. By (C2), $m_{5^*}(f)=0$. By (C4), $p_5(f) \leqslant 3$. We will divide the argument into four subcases according to the value of $p_5(f)$.

a) $p_5(f)=3$. Suppose f_1, f_3, f_5 are such three 5-faces that each of them takes a charge $\frac{1}{3}$ from f. By (R4a), we see that all common edges v_1v_2, v_3v_4 and v_5v_6 are good $(3,3)$-edges. This implies that $d(v_i)=3$ with $i \in \{1,\cdots,6\}$. By Claim 2.3.11, one can easily deduce that none of f_2, f_4, f_6, f_7 is a 6^*-face. Thus, $p_{6^*}(f) \leqslant m_{6^*}(f)=0$. Consequently, we deduce that $\omega^*(f) \geqslant 1-\frac{1}{3}\times 3=0$ by (R4a).

b) $p_5(f)=2$. We may suppose that f_i is a 5-face which takes $\frac{1}{3}$ from f. It means that $d(v_i)=d(v_{i+1})=3$ and v_iv_{i+1} is a good common edge. Thus, f_{i-1} and f_{i+1} cannot be any 6^*-face by Claim 2.3.11. It follows immediately that $p_{6^*}(f) \leqslant 7-(2+3)=2$ since $p_5(f)=2$. Consequently, we have that $\omega^*(f) \geqslant 1-\frac{1}{3}\times 2-\frac{1}{6}\times 2=0$ by (R4).

c) $p_5(f)=1$. W.l.o.g., let f_1 be such a triangle-far 5-face that v_1v_2 be a good common $(3,3)$-edge. This implies that neither f_2 nor f_7 is a 6^*-face. Thus, $p_{6^*}(f) \leqslant 7-3=4$. Hence, we have $\omega^*(f) \geqslant 1-\frac{1}{3}-\frac{1}{6}\times 4=0$ by (R4a) and (R4b).

d) $p_5(f) = 0$. If $p_{6^*}(f) = 0$, then according to (R4), we obtain that $\omega^*(f) \geqslant 1 - 0 = 1$. Otherwise, we may let f_1 be a 6^*-face, which takes a charge $\frac{1}{6}$ from f. It is obvious that f_1 must be adjacent to f by a good common $(3,3)$-edge or $(3,4)$-edge, i.e., $d(v_1) = 3$ and $d(v_2) \in \{3,4\}$. It is easy to observe that f_7 cannot be any 6^*-face because of Claim 2.3.14. Thus, $p_{6^*}(f) \leqslant 6$ and $\omega^*(f) \geqslant 1 - \frac{1}{6} \times 6 = 0$ by (R4b).

8.2 Now, w.l.o.g., we assume that f is adjacent to a 3-face $f_1 = [vv_1v_2]$. Then $\tau(f_1 \to f) = 1$. By Claim 2.3.1, we confirm that $v \notin V(f)$. Moreover, for each $j \in \{2, \cdots, 7\}$, we deduce that f_j is neither a 5-face nor a 6^*-face by (C5). It implies that f sends nothing to each f_j with $j \in \{2, \cdots, 7\}$. Applying (R1), we deduce that $\omega^*(f) \geqslant 1 - 1 = 0$.

Case 9 $d(f) = 8$.

Clearly, $\omega(f) = 2$ and f cannot be adjacent to any 3-face by Claim 2.3.3. So we only need to consider the size of $p_5(f)$ and $p_{6^*}(f)$ since they may take charge from f. It is easy to obtain $p_5(f) \leqslant 6$ because there is no sunflower in G. We need to consider the following possibilities by the value of $p_5(f)$.

Subcase 9.1 $p_5(f) = 6$.

It implies that f is incident to at least seven 3-vertices. Thus, the remaining two faces adjacent to f, which are not triangle-far 5-faces, cannot be any 6^*-faces by Claim 2.3.11. So $\omega^*(f) \geqslant 2 - 6 \times \frac{1}{3} = 0$ by (R4).

Subcase 9.2 $p_5(f) = 5$.

Notice that at most one of f_i, with $i \in \{1, 2, \cdots, 8\}$, can be a 6^*-face because no 5-face can be adjacent to a 6^*-face by Claim 2.3.11 again. Therefore, $\omega^*(f) \geqslant 2 - 5 \times \frac{1}{3} - \frac{1}{6} = \frac{1}{6} > 0$.

Subcase 9.3 $0 \leqslant p_5(f) \leqslant 4$.

By (R4), we derive that

$$\begin{aligned}
\omega^*(f) &\geqslant 2 - \frac{1}{3}p_5(f) - \frac{1}{6}p_{6^*}(f) \\
&\geqslant 2 - \frac{1}{3}p_5(f) - \frac{1}{6}(8 - p_5(f)) \\
&= \frac{2}{3} - \frac{1}{6}p_5(f) \\
&\geqslant \frac{2}{3} - \frac{1}{6} \times 4 \\
&= 0.
\end{aligned}$$

Next, we will discuss several cases where $d(f) \geqslant 10$. Let f be a 10^+-face and f' a face adjacent to f. We call f' *special* if it takes charge 1 from f. Let $|F_s(f)|$ denote the number of adjacent special faces. Let S_i be a face adjacent to f by an edge e_i for $i = 1, 2$. If e_1 is not incident to e_2, then we say that S_1 and S_2 are *mutually* *disjoint*. According to (R1) and (R5), we see that only 3-faces and 5^*-faces may take charge 1 from f. It implies that each special face is either a 3-face or a 5^*-face. We first observe the following:

Observation 2.3.2 *If f is adjacent to two special faces by two consecutive edges uv and vw of $b(f)$, then $\tau(v \to f) \geqslant 1$.*

Proof. Let f_{uv} and f_{vw} denote such two special faces adjacent to f by sharing the edges uv and vw, respectively. It suffices to consider the following three cases.

- Assume that f_{uv} and f_{vw} are both 3-faces. By the absence of 4-cycles, we see that $d(v) \geqslant 4$ and thus $\tau(v \to f) \geqslant 1$ by (R2a), (R3a) and (R6).
- Assume that f_{uv} and f_{vw} are both 5^*-faces. By Claim 2.3.5, $d(v) \geqslant 4$. So by (R2c3.3), (R3c) and (R6), we derive that $\tau(v \to f) = 1$.
- Finally, w.l.o.g., we assume that f_{uv} is a 3-face and f_{vw} is a 5^*-face. By (R5), we know that the edge vw is a good common $(3^+, 3^+)$-edge, which implies that $d(v) \geqslant 4$. Applying (R2b2), (R3b) and (R6), we have that $\tau(v \to f) = 1$.

This completes the proof of Observation 2.3.2. ∎

If there exist two special faces which share at least one vertex v that is lied on $b(f)$, i.e., let f_i and f_{i+1} be such two special faces that $v_{i+1} \in V(f_i) \cap V(f_{i+1})$ and $v_{i+1} \in V(f)$, then we see that $\tau(v_{i+1} \to f) \geqslant 1$ by Observation 2.3.2 and f sends at most 2×1 to f_i and f_{i+1}. It means that f takes charge 1 from v_{i+1} and then sends it to f_{i+1}. Thus, we can consider that f_{i+1} takes nothing from f. So in what follows, our main focus is on the special faces adjacent to f that are mutually disjoint. For our convenience, we let $|F_s^*(f)|$ denote the maximal number of special faces adjacent to f which are mutually disjoint. Obviously, $|F_s^*(f)| \leqslant \left\lfloor \frac{d(f)}{2} \right\rfloor$.

Observation 2.3.3 $p_5(f) + p_{6^*}(f) \leqslant d(f) - 2|F_s^*(f)|$.

Proof. W.l.o.g., suppose that f_i is a special face such that neither f_{i-1} nor f_{i+1} is a special face. In order to prove Observation 2.3.3, it suffices to show that f_{i-1} gets nothing from f if it is a 5- or 6^*-face.

First suppose that f_i is a 3-face. If f_{i-1} takes a charge $\frac{1}{3}$ or $\frac{1}{6}$, then by (R4a) and (R4b), we see that $d(v_i) = 4$ and f_{i-1} is a 6^*-face. This contradicts (C3). Now we assume that f_i is a 5^*-face. If $d(v_i) = 3$, then f_{i-1} cannot be any triangle-far 5-face by Claim 2.3.6 and any 6^*-face by Claim 2.3.12 and thus we are done. Now suppose that $d(v_i) \geqslant 4$. Note that if f_{i-1} is a triangle-far 5-face, then f sends nothing to it because $v_{i-1}v_i$ is not a $(3,3)$-edge. If f_{i-1} is a 6^*-face, then we discuss as follows: when v_i is a 5^+-vertex, then $\tau(f \to f_{i-1}) = 0$ since $v_{i-1}v_i$ is neither a $(3,3)$-edge nor a $(3,4)$-edge; when v_i is a 4-vertex, then f_i is the opposite face of f_{i-1} by a 4-vertex v_i, which contradicts (C3). This completes the proof of Observation 2.3.3. ∎

Case 10 $d(f) = 10$.

Then $\omega(f) = 4$ and $|F_s^*(f)| \leqslant 5$. We divide the argument into the following three subcases in light of $|F_s^*(f)|$.

Case 10.1 $|F_s^*(f)| = 5$.

By definition, f is adjacent to five special faces that are mutually disjoint. W.l.o.g., suppose that f_1, f_3, f_5, f_7, f_9 are all these special faces. If f_j is a special face for some fixed $j \in \{2,4,6,8,10\}$, then $\tau(f \to f_j) = 1$, while $\tau(v_j \to f) \geqslant 1$ and $\tau(v_{j+1} \to f) \geqslant 1$ by Observation 2.3.2. Therefore, $\omega^*(f) \geqslant 4 - 1 \times 5 - |F_s^{**}(f)| + 2|F_s^{**}(f)| = -1 + |F_s^{**}(f)| \geqslant 0$, where $|F_s^{**}(f)|$ denotes the number of special faces among $f_2, f_4, f_6, f_8, f_{10}$. In what follows, for each $j \in \{2,4,6,8,10\}$, we suppose that f_j is not a special face. Since G does not contain lotus, there exists at least one 4^+-vertex on $b(f)$, say v_1. If v_1 is a 5^+-vertex, then v_1 sends at least 1 to f by (R3) and (R6). If v_1 is a 4-vertex, then we have two cases: If $d(v_{10}) = 3$, then f_{10} is not a triangle-far 5-face since f_9 is a special face. So $\tau(v_1 \to f) = 1$ by (R2b2), (R2c2) and (R2c3.3); otherwise, $d(v_{10}) \geqslant 4$ and f receives at least $\frac{2}{3} \times 2 = \frac{4}{3}$ from v_1 and v_{10} in total by (R2b1), (R2b2), (R2c2), (R2c3.2) and (R2c3.3). Thus, in each case, we always have that $\omega^*(f) \geqslant 4 - 1 \times 5 + 1 = 0$.

Case 10.2 $|F_s^*(f)| = 4$.

It implies that f is adjacent to exactly four special faces by four common edges which are disjoint each other. Denote S_i be such a special face adjacent to f by a common edge e_i, where $i = 1,2,3,4$. Noting that e_i cannot be incident to e_j for each pair $(i,j) \subset \{1,\cdots,4\}$. Thus, it follows that there exist two vertices v_j, v_k lied on $b(f)$ which are not incident to any common edge e_i with $i \in \{1,\cdots,4\}$. W.l.o.g., assume $j < k$.

First we consider the case that $k = j+1$. Namely, v_jv_k is an edge of $b(f)$. W.l.o.g., we assume that $v_jv_k = v_{10}v_9$ such that f_1, f_3, f_5, f_7 are special faces and f_9 is not. By the proof of Observation 2.3.3, we assert that none of $f_2, f_4, f_6, f_8, f_{10}$ gets charge from f if it is a 5- or 6^*-face. It follows that $p_5(f) + p_{6^*}(f) \leqslant 1$. If $p_5(f) + p_{6^*}(f) = 0$, then we are done since $\omega^*(f) \geqslant 4 - 1 \times 4 = 0$. Otherwise, suppose that f_9 is a triangle-far 5-face or a 6^*-face which gets a charge $\frac{1}{3}$ or $\frac{1}{6}$ from f, respectively. It follows that neither f_8 nor f_{10} is a special face. If f_j is a special face for some $j = 2,4,6$, then similarly we have that $\omega^*(f) \geqslant 4 - 1 \times 4 - \frac{1}{3} - |F_s^{**}(f)| + 2|F_s^{**}(f)| = |F_s^{**}(f)| - \frac{1}{3} \geqslant \frac{2}{3}$, where $|F_s^{**}(f)|$ denotes the number of special faces among f_2, f_4, f_6. So in the following, we assume that f_j is not a special face for each $j = 2,4,6$. By the absence of lotus in G, there exists at least one vertex in $V(f)$ whose degree is at least 4. Let v^* be such a 4^+-vertex. W.l.o.g., we have two subcases below, according to the situation of v^*.

- Assume that $v^* = v_1$. If $d(v^*) \geqslant 5$, then v^* sends at least 1 to f by (R3) and (R6). Otherwise, $d(v^*) = 4$. By (R2b1), (R2b2), and (R2c2), we see that $\tau(v_1 \to f) \geqslant \frac{2}{3}$. Thus, in each case, we always have that $\omega^*(f) \geqslant 4 - 1 \times 4 - \frac{1}{3} + \frac{2}{3} = \frac{1}{3}$.

- Assume that $v^* = v_i$, where $i \in \{2,\cdots,7\}$. Then by a similar discussion as the proof of Case 10.1, we have that $\omega^*(f) \geqslant 4 - 1 \times 4 - \frac{1}{3} + 1 = \frac{2}{3}$.

- Assume that $v^* = v_9$. Namely, $d(v_9) \geqslant 4$. Moreover, we may further assume that f_9 is a 6^*-face (otherwise, f_9 gets nothing from f by (R4a)). If $d(v_9) \geqslant 5$, then $\tau(v_9 \to f) \geqslant \frac{4}{5}$ by (R3) and (R6). If $d(v_9) = 4$, then according to (R2c2), (R2c3.2) and (R2c3.3), it is obvious that v_9 sends at least $\frac{2}{3}$ to f. Thus, we have that $\omega^*(f) \geqslant 4 - 1 \times 4 - \frac{1}{3} + \frac{2}{3} = \frac{1}{3} > 0$.

Now we suppose that $k > j + 1$. It means that $v_k v_j \notin E(f)$. In this case, it is easy to deduce that $p_5(f) + p_{6^*}(f) = 0$ by the proof of Observation 2.3.3. In other words, f only sends charges to its special faces. Therefore, $\omega^*(f) \geqslant 4 - 1 \times 4 = 0$ by (R1) and (R5).

Case 10.3 $0 \leqslant |F_s^*(f)| \leqslant 3$.

If $|F_s^*(f)| = 3$, by a careful inspection, one can easily obtain that $p_5(f) + p_{6^*}(f) \leqslant 10 - (3+4) = 3$. So, $\omega^*(f) \geqslant 4 - 3 \times 1 - \frac{1}{3} \times 3 = 0$ by (R4). If $0 \leqslant |F_s^*(f)| \leqslant 2$, then by Observation 2.3.3, we have that $p_5(f) + p_{6^*}(f) \leqslant 10 - 2|F_s^*(f)|$ and therefore, $\omega^*(f) \geqslant 4 - |F_s^*(f)| - \frac{1}{3}(10 - 2|F_s^*(f)|) = \frac{2}{3} - \frac{1}{3}|F_s^*(f)| \geqslant \frac{2}{3} - \frac{1}{3} \times 2 = 0$.

Case 11 $d(f) = 11$.

Clearly, $\omega(f) = 5$ and $|F_s^*(f)| \leqslant 5$. If $|F_s^*(f)| = 5$, then $p_5(f) + p_{6^*}(f) \leqslant 11 - (5+6) = 0$. So $\omega^*(f) \geqslant 5 - 1 \times 5 = 0$. If $0 \leqslant |F_s^*(f)| \leqslant 4$, then $p_5(f) + p_{6^*}(f) \leqslant 11 - 2|F_s^*(f)|$ by Observation 2.3.3. Then $\omega^*(f) \geqslant 5 - |F_s^*(f)| - \frac{1}{3}(11 - 2|F_s^*(f)|) = \frac{4}{3} - \frac{1}{3}|F_s^*(f)| \geqslant= 0$.

Case 12 $d(f) \geqslant 12$.

By Observation 2.3.3, we have that $p_5(f) + p_{6^*}(f) \leqslant d(f) - 2|F_s^*(f)|$. Moreover, $|F_s^*(f)| \leqslant \lfloor \frac{1}{2} d(f) \rfloor$. Thus, we have that

$$
\begin{aligned}
\omega^*(f) &\geqslant (d(f) - 6) - |F_s^*(f)| - \frac{1}{3}(d(f) - 2|F_s^*(f)|) \\
&= \frac{2}{3}d(f) - 6 - \frac{1}{3}|F_s^*(f)| \\
&\geqslant \frac{2}{3}d(f) - 6 - \frac{1}{3} \times \frac{d(f)}{2} \\
&= \frac{1}{2}d(f) - 6 \\
&\geqslant \frac{1}{2} \times 12 - 6 \\
&= 0.
\end{aligned}
$$

Therefore, we complete the proof of 2-connected case of Theorem 2.2.1. ∎

2.3.2 Proof of non-2-connected case

In what follows, we suppose that G is not a 2-connected plane graph and we will construct a 2-connected plane graph G^* with $\delta(G^*) \geqslant 3$ having neither 4-cycles nor 9-cycles and satisfying all structural properties (C1) to (C5). This obviously contradicts the result just established before.

We remark that the following proof is stimulated by the technique used in [CLW08].

Let B be an end block of G with the unique cut-vertex x. Let f be the outside face of G. Notice that $d_B(x) \geqslant 2$ and $d_B(v) \geqslant 3$ for each $v \in V(B)\backslash\{x\}$. Choosing another vertex y of B such that $y \neq x$ and y lies on the boundary of B. Obviously, x and y are both belonging to $b(f)$. Then we take ten copies of B, i.e., B_k with $k = 1, \cdots, 10$. In each copy B_k, the vertices corresponding to x and y are denoted by x_k and y_k, respectively. Then one can embed B_k, $k = 1, \cdots, 10$, into f in the following way: first, let $B = B_1$. Next, for each $k = 2, \cdots, 10$, consecutively embed B_k into f by identifying x_k with y_{k-1}. Finally, identify y_{10} with a vertex $u \in V(f)\backslash V(B)$. Then the first resulting graph, denoted by G_1.

Obviously, in the processing of constructing G_1, we confirm that there are no new adjacent cycles established. Furthermore, no 4-cycles and 9-cycles are formed. Thus, it is easy to deduce that G_1 satisfies the following structural properties.

(A1) Fewer end blocks than G.

(A2) The minimum degree is at least 3.

(A3) Neither 4-cycles nor 9-cycles.

(A4) A 5-cycle or a 6-cycle is adjacent to at most one 3-cycle.

(A5) A 5^*-cycle is neither adjacent to a 5^*-cycle normally, nor adjacent to an i-cycle with $i \in \{7, 8\}$.

(A6) A 6^*-cycle is not adjacent to a 6-cycle.

(A7) A triangle-far 7-cycle is not adjacent to two 5-cycles which are normally adjacent;

(A8) A 7^*-cycle is neither adjacent to a 5-cycle nor a 6^*-cycle.

Furthermore, we confirm that G_1 also satisfies the following two structural properties:

(P1) G_1 has neither orchid, nor sunflower, nor lotus.

(P2) A 6^*-cycle is not incident to an i-cycle C with $i \in \{3, 5\}$, where C is opposite to such 6^*-cycle by a 4-vertex.

(P1) For some $k \in \{2, \cdots, 10\}$, notice that we just identify some vertex x_k with y_{k-1}. It implies that any new cycle, which is not completely belong to some B_k, must be an 11^+-cycles, i.e., $C^* = x_1 \cdots x_{10} u \cdots x_1$. Thus, any orchid, sunflower, or lotus cannot be established.

(P2) Assume to the contrary that G_1 contains a 6^*-cycle, denoted by C_6^*, which is incident to a 3-cycle C_3 or a 5-cycle C_5 by a 4-vertex v^*. Clearly, v^* must be equal to

u or some vertex x_k with $k \in \{2, \cdots, 10\}$. However, $d_{G_1}(u) = d_{B_{10}}(u) + d_{G\backslash B_1}(u) \geqslant 2 + 3 = 5$ or $d_{G_1}(x_k) = d_{B_{k-1}}(x_k) + d_{B_k}(x_k) \geqslant 3 + 2 = 5$ for all $k \in \{2, \cdots, 10\}$. We always get a contradiction to $d_{G_1}(v^*) = 4$.

Now, if G_1 is 2-connected, then we are done. Otherwise, we may repeat the process described above and finally obtain a desired G^*.

Thus, we complete the proof of Theorem 2.2.1. ∎

2.4 Further research

In 2005, Bordoin, Glebov, Raspaud, and Salavatipour [BGRS05] proved that every planar graph without 4-, 5-, 6- and 7-cycles is 3-colorable. This is a big step to the previous results on a long-standing conjecture of Steinberg. Some authors use a similar way as that of [BGRS05] to obtain some sufficient conditions for planar graphs to be 3-colorable. Among most of them, we are more interested in the results of 3-colorable planar graphs without cycles of three lengths. We summarize them again as follows:

Theorem 2.4.1 *A planar graph is 3-colorable if it has no*

- (Wang and Chen [WC07b]) 4-, 6-, *and* 8-*cycles; or*
- (Lu et al. [LWW+09]) 4-, 7-, *and* 9-*cycles; or*
- (Borodin et al. [BGMR09]) 4-, 5-, *and* 7-*cycles.*

We would like to put forward the following three problems to conclude this chapter.

Problem 2.4.2 *Is every planar graph without* 4-, 5-, *and* 6-*cycles* 3-*colorable?*

Problem 2.4.3 *Is every planar graph without* 4-, 6-, *and* 7-*cycles* 3-*colorable?*

Problem 2.4.4 *Is every planar graph without* 4-, 5-, *and* 8-*cycles* 3-*colorable?*

Chapter 3

Acyclic choosability

Contents

In the previous chapter, we studied the 3-choosability of planar graphs with restrictions. In this chapter, we are interested in a proper L-coloring such that the union of any two color classes induces a forest. Such a coloring is called an *acyclic L-coloring*. In Section 3.1, we give a general introduction and a short survey about acyclic coloring and acyclic L-coloring. In Sections 3.2 to 3.4, we will study, respectively, the acyclic k-choosability of planar graphs for each $k = 5, 4, 3$.

3.1 Introduction

3.1.1 Acyclic coloring

A proper vertex coloring of a graph G is *acyclic* if there is no bicolored cycle in G. Namely, the union of any two color classes induces a forest. The *acyclic chromatic number*, denoted by $\chi_a(G)$, of a graph G is the smallest integer k such that G has an acyclic k-coloring.

The notion of acyclic coloring of graphs was introduced by Grünbaum [Grü73] in 1973 and studied by Mitchem [Mit74], Albertson and Berman [AB77] and Kostochka [Kos76]. In 1979, Borodin [Bor79] confirmed the conjecture of Grünbaum by proving that

Theorem 3.1.1 [Bor79] *Every planar graph is acyclically 5-colorable.*

The bound in Theorem 3.1.1 is sharp. In 1973, Grünbaum [Grü73] gave an example of a 4-regular planar graph which is not acyclically 4-colorable; furthermore, bipartite planar graphs which are not acyclically 4-colorable were constructed in [KM76], see Figure 3.1.

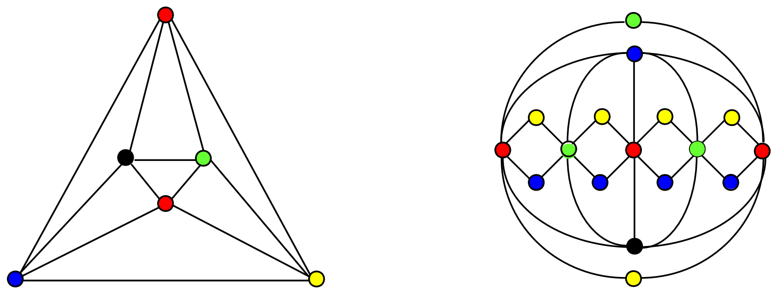


Figure 3.1: Examples of Grünbaum and Kostochka Mel'nikov.

In 1999, Borodin, Kostochka and Woodall improved this bound for planar graphs with large girth.

Theorem 3.1.2 [BKW99]

(1) *If G is planar with girth $g \geqslant 5$, then $\chi_a(G) \leqslant 4$.*

(2) *If G is planar with girth $g \geqslant 7$, then $\chi_a(G) \leqslant 3$.*

3.1.2 Acyclic L-coloring

We say that G is *acyclically L-list colorable* if for a given list assignment $L = \{L(v) : v \in V\}$, there exists a proper acyclic coloring π of G such that $\pi(v) \in L(v)$ for all $v \in V$. If G is acyclically L-list colorable for any list assignment with $|L(v)| \geqslant k$ for all $v \in V$, then G is *acyclically k-choosable* or *acyclic k-list colorable.*

In 2002, Borodin, Fon-Der Flaass, Kostochka, Raspaud, and Sopena [BFDFK+02] first investigated the acyclic list coloring of planar graphs to show the following:

Theorem 3.1.3 [BFDFK+02] *Every planar graph is acyclically 7-choosable.*

Moreover, they proposed the challenging conjecture as follows:

Conjecture 3.1.4 [BFDFK+02] *Every planar graph is acyclically 5-choosable.*

If Conjecture 3.1.4 were true, then it would strengthen both the Borodin's acyclically 5-colorable theorem [Bor79] and the Thomassen's 5-choosable theorem [Tho94] about planar graphs. However, this challenging conjecture seems to be difficult. As yet, it has been verified only for several restricted classes of planar graphs. Wang and Chen [WC09] proved that every planar graph without 4-cycles is acyclically 6-choosable.

Montassier, Ochem and Raspaud [MOR06] studied the acyclic choosability of graphs with bounded maximum average degree.

Theorem 3.1.5 [MOR06]

(1) *Every graph G with* $\mathrm{Mad}(G) < \frac{8}{3}$ *is acyclically 3-choosable.*

(2) *Every graph G with* $\mathrm{Mad}(G) < \frac{19}{6}$ *is acyclically 4-choosable.*

(3) *Every graph G with* $\mathrm{Mad}(G) < \frac{24}{7}$ *is acyclically 5-choosable.*

By the well-known relationship $\mathrm{Mad}(G) < \frac{2g(G)}{g(G)-2}$ for any planar graph G, it is easy to deduce the following:

Corollary 3.1.6 [MOR06]

(1) *Every planar graph G with $g(G) \geqslant 8$ is acyclically 3-choosable.*

(2) *Every planar graph G with $g(G) \geqslant 6$ is acyclically 4-choosable.*

(3) *Every planar graph G with $g(G) \geqslant 5$ is acyclically 5-choosable.*

3.1.3 The relationship between $\chi_a(G)$ and $\chi_a^l(G)$

The notion of acyclic coloring is different from the notion of acyclic list coloring. For any graph G, it is obvious that $\chi_a^l(G) \geqslant \chi_a(G)$. Until now, there is no upper bounds of $\chi_a^l(G)$ in terms of $\chi_a(G)$. Montassier [Mon05a] proved that the list acyclic chromatic number could be strictly greater than the acyclic chromatic number by showing an example (see Figure 3.2) which is acyclically 3-colorable but not acyclically 3-choosable.

3.2 Acyclic 5-choosability

3.2.1 Known results

In this section, we summarize some sufficient conditions for a planar graph to be ayclically 5-choosable. Montassier, Raspaud and Wang [MRW07] proved that every planar graph G without 4-cycles and 5-cycles, or without 4-cycles and 6-cycles is

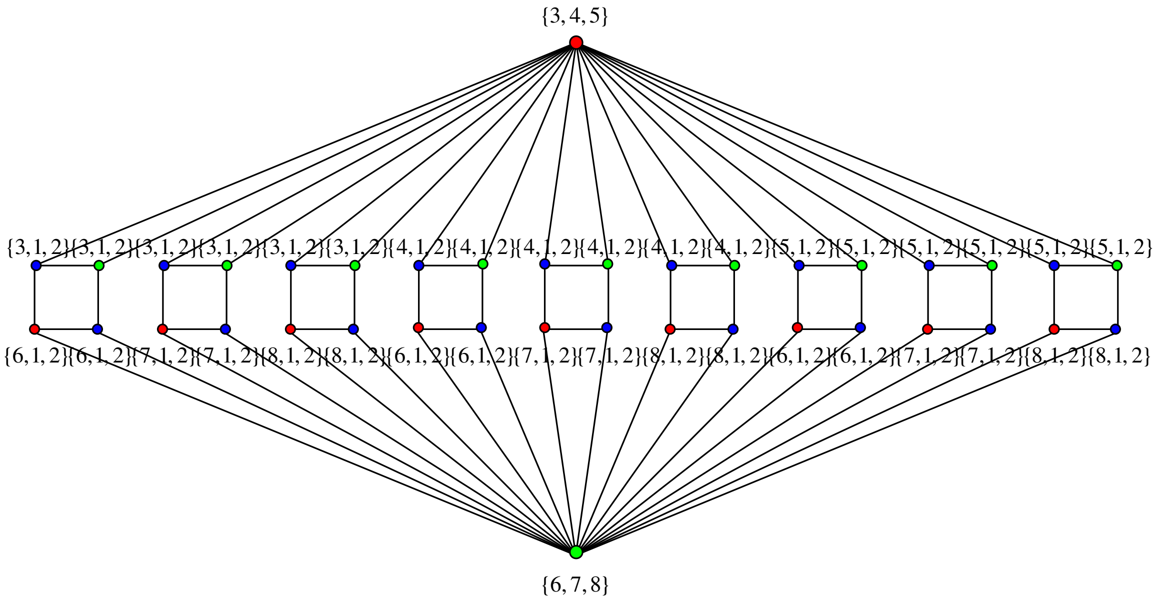


Figure 3.2: The graph G with $\chi_a(G) = 3$ and $\chi_a^l(G) \neq 3$.

acyclically 5-choosable. Chen and Wang [CW08a] studied the 3-cycles at distance d and proved that every planar graph without 4-cycles and without triangles at distance less than 3 is acyclically 5-choosable. Recently, Zhang and Xu [ZX09] proved that every planar graph with neither 4-cycles nor chordal 6-cycles is acyclically 5-choosable. Note that in all these results cycles of length 4 are forbidden. In [BI09a], a common extension of the results in [MRW07] is given: a planar graph is acyclically 5-choosable if it does not contain an i-cycle adjacent to a j-cycle where $3 \leqslant j \leqslant 5$ if $i = 3$ and $4 \leqslant j \leqslant 6$ if $i = 4$.

To attack Conjecture 3.1.4, Chen and Wang [CW08a] proposed a weak version about this conjecture:

Conjecture 3.2.1 *Every planar graph without* 4*-cycles is acyclically* 5*-choosable.*

Conjecture 3.2.1 is still open. In this section, we prove the following result.

Theorem 3.2.2 [CR10d] *Every planar graph with neither* 4*-cycles nor intersecting triangles is acyclically* 5*-choosable.*

Our result is a new approach to Conjecture 3.2.1 and gives an improvement to the result in [CW08a].

3.2.2 Proof of Theorem 3.2.2

The proof of Theorem 3.2.2 is proceeded by a contradiction. We suppose that G is a minimal counterexample (i.e., with the least number of vertices) to the Theorem 3.2.2 which is embedded in the plane. Thus G is connected. We first investigate the structural properties of G in Section 3.2.2.1, then use Euler's formula and discharging argument to derive a contradiction in Section 3.2.2.2.

3.2.2.1 Structural properties

First we have the following Lemmas 3.2.3 to 3.2.6, whose proofs were provided in [MRW07, CW08a, BI09b].

Lemma 3.2.3 [MRW07]
(C1) *There are no* 1*-vertices.*
(C2) *No* 2*-vertex is adjacent to a* 4^-*-vertex.*
(C3) *Let* v *be a* 3*-vertex. Then*
    (C3.1) *If* v *is adjacent to a* 3*-vertex, then* v *is not adjacent to other* 4^-*-vertex;*
    (C3.2) v *is not adjacent to any pendant light 3-vertex.*
(C4) *Let* v *be a* 5*-vertex. Then*
    (C4.1) v *is adjacent to at most one* 2*-vertex;*
    (C4.2) *If* $n_2(v) = 1$*, then* v *is not adjacent to any pendant light* 3*-vertex.*
(C5) *Let* v *be a* 6*-vertex. Then*
    (C5.1) v *is adjacent to at most four* 2*-vertices;*
    (C5.2) *If* $n_2(v) = 4$*, then* v *is not adjacent to any* 3*-vertex.*
(C6) *Each* 7*-vertex is adjacent to at most five* 2*-vertices.*
(C7) *No* 3*-face* $[xyz]$ *with* $d(x) \leqslant d(y) \leqslant d(z)$ *satisfies one of the following:*
    (C7.1) $d(x) = 2$*;*
    (C7.2) $d(x) = d(y) = 3$ *and* $d(z) \leqslant 5$*;*
    (C7.3) $d(x) = 3$ *and* $d(y) = d(z) = 4$*.*

Lemma 3.2.4 [CW08a] *Suppose that* v *is a* 5*-vertex with* $n_2(v) = 1$*. If* v *is incident to a* 3*-face* f*, then* $n_3(f) = 0$*.*

Lemma 3.2.5 [CW08a] *Suppose that* v *is a* 6*-vertex. Then the following hold:*
(A1) *If* $n_2(v) = 2$ *and* v *is incident to a* $(3, 3, 6)$*-face, then* $n_3(v) \leqslant 2$*;*
(A2) *If* $n_2(v) = 3$*, then* $n_3(v) \leqslant 1$*;*
(A3) *If* $n_2(v) = 4$*, then* $t(v) = 0$*.*

Lemma 3.2.6 [CW08a] *Let* v *be a* 7*-vertex. Then*
(B1) *If* $n_2(v) = 4$*, then* $n_3(v) \leqslant 2$*;*
(B2) *If* $n_2(v) = 5$*, then* $n_3(v) = 0$ *and* $t(v) = 0$*.*

Lemma 3.2.7 [BI09a] *If* v *is a pendant light 3-vertex of* v_3*, i.e.,* $f = [vv_1v_2]$ *is a 3-face, then* $d(v_3) \geqslant 5$*.*

In what follows, let L be a list assignment of G with $|L(v)| = 5$ for all $v \in V(G)$. In the following proofs of Lemmas 3.2.8 to 3.2.11, for $v \in V(G)$, we let $v_1, v_2, \cdots, v_{d(v)}$ denote the neighbors of v in clockwise order. If v_i is a 2-vertex, we use u_i to denote the neighbor of v_i different from v. If v_j is a pendant light 3-vertex, we use x_j and y_j to denote the neighbors of v_j different from v such that $[v_jx_jy_j]$ is a 3-face.

Lemma 3.2.8 *Suppose that* v *is a* 5*-vertex with* $n_2(v) = 0$*. Then the following hold:*

(F1) *$p_3(v) \leqslant 3$;*
(F2) *If $t(v) = 1$, then $p_3(v) \leqslant 2$;*
(F3) *If v is incident to a $(5, 3, 4)$-face, then $p_3(v) \leqslant 1$;*
(F4) *If v is incident to a $(5, 3, 5^+)$-face and $p_3(v) = 2$, then $n_3(v) \leqslant 3$.*

Proof. We will make use of contradictions to show (F1)-(F4).

(F1) Suppose to the contrary that $p_3(v) \geqslant 4$. Assume, without loss of generality, that $v_1, \cdots, v_4$ are adjacent pendant light 3-vertices of v. By the minimality of G, $G - \{v, v_1, \cdots, v_4\}$ admits an acyclic L-coloring π. It is obvious that $\pi(x_i) \neq \pi(y_i)$ for all $i = 1, \cdots, 4$. Let $S = \{x_1, \cdots, x_4, y_1, \cdots, y_4\}$. Note that $|L(v) \setminus \{\pi(v_5)\}| \geqslant 4$ and $|S| = 8$. It follows that there exists a color $c \in L(v) \setminus \{\pi(v_5)\}$ appearing at most twice on the set S, say $\pi(x_1) = \pi(x_2) = c$. Then we color v with c, v_1 with a color $a \in L(v_1) \setminus \{c, \pi(v_5), \pi(y_1)\}$, v_2 with a color $b \in L(v_2) \setminus \{a, c, \pi(v_5), \pi(y_2)\}$, and v_i with a color different from $c, \pi(x_i), \pi(y_i)$ for $i = 3, 4$. Since $a \neq b$, we see that any cycle containing edges v_1v and vv_2 is not bicolored. Therefore, the resulting coloring is an acyclic L-coloring because none of x_3, y_3, x_4, y_4 is colored with c.

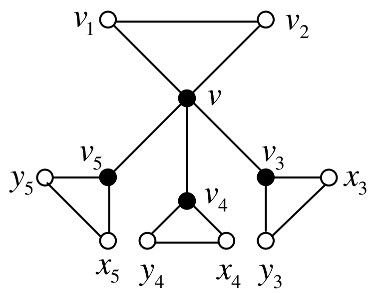


Figure 3.3: A 5-vertex v with $t(v) = 1$ and $p_3(v) = 3$.

(F2) Assume to the contrary that $[vv_1v_2]$ is a 3-face and v_3, v_4, v_5 are adjacent pendant light 3-vertices such that $v_1, v_2, x_3, y_3, x_4, y_4, x_5, y_5$ are in clockwise order, see Figure 3.3. Let $G' = G - \{v, v_3, v_4, v_5\}$. Obviously, G' admits an acyclic L-coloring π by the minimality of G. Moreover, $\pi(v_1) \neq \pi(v_2)$ and $\pi(x_i) \neq \pi(y_i)$ for each $i \in \{3, 4, 5\}$. Denote $S = \{x_3, x_4, x_5, y_3, y_4, y_5\}$. Notice that $|L(v) \setminus \{\pi(v_1), \pi(v_2)\}| \geqslant 3$ and $|S| = 6$. This implies that there exists a color in $L(v) \setminus \{\pi(v_1), \pi(v_2)\}$ appearing at most twice on the set S. We have to consider two cases below.

If there exists a color $c \in L(v) \setminus \{\pi(v_1), \pi(v_2)\}$ which appears at most once on the set S, i.e., $\pi(x_3) = c$, we can color v with c, v_3 with a color different from $c, \pi(v_1), \pi(v_2), \pi(y_3)$, and finally color v_i with a color different from $c, \pi(x_i), \pi(y_i)$ for $i = 4, 5$.

Now we assume that $L(v) = \{1, 2, 3, 4, 5\}$, $\pi(v_1) = 1$, $\pi(v_2) = 2$, and each color in $\{3, 4, 5\}$ appears exactly twice on the set S. W.l.o.g., assume that $\pi(x_3) = \pi(x_5) = 3$, $\pi(y_4) = \pi(y_5) = 4$ and $\pi(y_3) = \pi(x_4) = 5$. If there is no alternating $(5, 1)$-path in G' connecting with y_3 and v_1, then color v with 5, v_4 with $a \in L(v_4) \setminus \{1, 2, 4, 5\}$, v_3 with $b \in L(v_3) \setminus \{2, 3, 5, a\}$, and finally color v_5 with a color distinct to $3, 4, 5$. If there is no alternating $(4, 2)$-path in G' connecting with y_5 and v_2, then color v with 4, v_4 with $c \in L(v_4) \setminus \{1, 2, 4, 5\}$, v_5 with $d \in L(v_5) \setminus \{1, 3, 4, a\}$, and finally color v_3 with a color distinct to $3, 4, 5$. Now, suppose that G' contains an alternating $(5, 1)$-path connecting with y_3 and v_1 and an alternating $(4, 2)$-path connecting with y_5 and v_2. Obviously, it is impossible, since G is an embedded plane graph.

(F3) Assume to the contrary that $[vv_1v_2]$ is a $(5, 3, 4)$-face, i.e., $d(v_1) = 3$ and $d(v_2) = 4$, and v_3, v_4 are adjacent pendant light 3-vertices. Denote $N(v_1) = \{v_1', v_2, v\}$ and $N(v_2) = \{v_1, v_2', v_2'', v\}$. Let $G' = G - \{v, v_1, v_3, v_4\}$. Clearly, G' admits an acyclic L-coloring π by the minimality of G. Moreover, for each $i \in \{3, 4\}$, $\pi(x_i) \neq \pi(y_i)$. Denote $S = \{v_1', x_3, y_3, x_4, y_4\}$. We have to consider following two cases, depending on the colors of v_2 and v_5.

- $\pi(v_2) \neq \pi(v_5)$. Note that $|L(v) \setminus \{\pi(v_2), \pi(v_5)\}| \geqslant 3$ and $|S| = 5$. It implies that there exists a color $c \in L(v) \setminus \{\pi(v_2), \pi(v_5)\}$ appearing at most once on the set S. We first color v with c. If $\pi(v_1') = c$, then assign a color in $L(v_1) \setminus \{\pi(v_2), \pi(v_5), c\}$ to v_1, and then color v_i with a color different from $c, \pi(x_i), \pi(y_i)$ for $i = 3, 4$. Otherwise, assume, w.l.o.g., that $\pi(x_3) = c$. We may color v_3 with a color belonging to $L(v_3) \setminus \{c, \pi(v_2), \pi(v_5), \pi(y_3)\}$ and then color v_4 with a color different from $c, \pi(x_4), \pi(y_4)$. We further color v_1 in the following way: If $\pi(v_1') \neq \pi(v_2)$, we color v_1 with a color distinct to $c, \pi(v_1'), \pi(v_2)$; If $\pi(v_1') = \pi(v_2)$, we color v_1 with a color distinct to $c, \pi(v_2), \pi(v_2'), \pi(v_2'')$. In each case, it is easy to verify that the resulting coloring is acyclic. This contradicts the choice of G.

- $\pi(v_2) = \pi(v_5)$. If $\pi(v_2') = \pi(v_2'')$, then there exists a color in $L(v) \setminus \{\pi(v_2), \pi(v_2')\}$ which appears at most once on the set S. Then the proof can also be given with a similar argument to the previous case. Otherwise, we first recolor v_2 with a color differen from $\pi(v_2), \pi(v_2'), \pi(v_2'')$ and then reduce the proof to the former case.

(F4) Assume to the contrary that $[vv_1v_2]$ is a $(5, 3, 5^+)$-face, i.e., $d(v_1) = 3$ and $d(v_2) \geqslant 5$, v_3, v_4 are adjacent pendant light 3-vertices and v_5 is a 3-vertex. Let $N(v_1) = \{v_1', v_2, v\}$ and $N(v_5) = \{v, v_5', v_5''\}$. Let $G' = G - \{v, v_3, v_4\}$. By the minimality of G, G' admits an acyclic L-coloring π. It is obvious that $\pi(v_1) \neq \pi(v_2)$ and $\pi(x_i) \neq \pi(y_i)$ for each $i \in \{3, 4\}$. Denote $S = \{x_3, y_3, x_4, y_4\}$. Depending on the colors of v_1, v_2, v_5, we need to consider the following three cases.

(1) Assume that v_1, v_2, v_5 have pairwise distinct colors. W.l.o.g, suppose that $\pi(v_1) = 1, \pi(v_2) = 2$ and $\pi(v_5) = 3$. If there exists a color $c \in L(v) \setminus \{1, 2, 3\}$ which appears at most once on the set S, say $\pi(x_3) = c$, we first color v with c and v_4 with a color different from $c, \pi(x_4), \pi(y_4)$. Then we color v_3 with a color α different from $2, 3, c, \pi(y_3)$. If such coloring is not acyclic, there is only one possible case that $\alpha = 1$ and $\pi(v_1') = c$. So we need to further recolor v_1 with a color different from $1, 2, 3, c$.

Now assume, w.l.o.g., that $L(v) = \{1, 2, 3, 4, 5\}$ and $\pi(x_3) = \pi(x_4) = 4$ and $\pi(y_3) = \pi(y_4) = 5$. If $\pi(v_1') \neq 2$, then recolor v_1 with a color $a \in L(v_1)\setminus\{1, 2, 3, \pi(v_1')\}$, color v with 1 and finally color v_i with a color distinct to $1, 4, 5$ for each $i \in \{3, 4\}$. Otherwise, suppose that $\pi(v_1') = 2$. If $\pi(v_5') = \pi(v_5'')$, then color v with a color $b \in \{4, 5\} \setminus \{\pi(v_5')\}$, v_3 with a color $c \in L(v_3) \setminus \{2, 4, 5\}$ and

v_4 with a color $d \in L(v_4) \setminus \{2,4,5,c\}$. If $\pi(v_5') \neq \pi(v_5'')$, we first recolor v_5 with a color different from $2, 3, \pi(v_5'), \pi(v_5'')$, then color v with 3 and finally give a proper coloring for v_3 and v_4.

(2) Assume that $\pi(v_5) = \pi(v_1) = 1$ and $\pi(v_2) = 2$. If $\pi(v_1') \neq 2$, recolor v_1 with a color different from $1, 2, \pi(v_1')$ and then go back to the previous Case (1). Now suppose that $\pi(v_1') = 2$. It is easy to observe that there exists a color c belonging to $L(v) \setminus \{1,2\}$ which appears at most once on the set S, w.l.o.g., say $\pi(x_3) = c$. We can color v with c, v_3 with a color in $L(v_3) \setminus \{1,2,c,\pi(y_3)\}$, and finally color v_4 with a color different from $c, \pi(x_4), \pi(y_4)$.

(3) Assume that $\pi(v_5) = \pi(v_2) = 1$ and $\pi(v_1) = 2$. If $\pi(v_5') \neq \pi(v_5'')$, then recolor v_5 with a color different from $1, 2, \pi(v_5'), \pi(v_5'')$ and then reduce to the previous Case (1). Now suppose that $\pi(v_5') = \pi(v_5'')$. If there exists a color $c \in L(v) \setminus \{1, 2, \pi(v_5')\}$ appearing at most once on the set S, say $\pi(x_3) = c$, then first color v with c, v_3 with a color distinct to $1, 2, c, \pi(y_3)\}$, and finally color v_4 with a color different from $c, \pi(x_4), \pi(y_4)$. Otherwise, w.l.o.g., assume that $L(v) = \{1, 2, \pi(v_5'), 4, 5\}$ and $\pi(x_3) = \pi(x_4) = 4$ and $\pi(y_3) = \pi(y_4) = 5$. If $\pi(v_1') \neq 1$, we recolor v_1 with a color $a \in L(v_1) \setminus \{1, 2, \pi(v_1')\}$ and then reduce the proof to the previous case (1). Otherwise, we may color v with 4, v_3 with a color $b \in L(v_3) \setminus \{1,4,5\}$ and v_4 with a color in $L(v_4) \setminus \{1,4,5,b\}$. ∎

Lemma 3.2.9 *Suppose that v is a 6-vertex. Then the following hold:*
(Q1) *If $n_2(v) = 3$ and $t(v) = 1$, then $p_3(v) = 0$;*
(Q2) *If $n_2(v) = 2$, then $p_3(v) \leqslant 2$;*
(Q3) *If $n_2(v) = 2$ and $t(v) = 1$, then $p_3(v) \leqslant 1$;*
(Q4) *If $n_2(v) = 1$, then $p_3(v) \leqslant 4$;*
(Q5) *If $n_2(v) = 1$ and v is incident to a $(3,3,6)$-face, then $p_3(v) \leqslant 1$;*
(Q6) *If $n_2(v) = 0$ and v is incident to a $(3,3,6)$-face, then $p_3(v) \leqslant 2$;*
(Q7) *If v is incident to a $(3,4,6)$-face, then*
  (Q7.1) *$n_2(v) \leqslant 2$;*
  (Q7.2) *If $n_2(v) = 1$, then $p_3(v) \leqslant 2$.*

Proof. **(Q1)** Assume to the contrary that $[vv_1v_2]$ is a incident 3-face, v_3, v_4, v_5 are 2-vertices and v_6 is an adjacent pendant light 3-vertex. By the minimality of G, $G - \{v, v_3, v_4, v_5, v_6\}$ admits an acyclic L-coloring π. Obviously, $\pi(v_1) \neq \pi(v_2)$. Let $S = \{u_3, u_4, u_5, x_6, y_6\}$. Since $|L(v) \setminus \{\pi(v_1), \pi(v_2)\}| \geqslant 3$ and $|S| = 5$, there exists a color $c \in L(v) \setminus \{\pi(v_1), \pi(v_2)\}$ appearing at most once on the set S. So we may first color v with c. In order to color the remanent uncolored vertices, w.l.o.g., we have to consider following two cases.

- If $\pi(u_3) = c$, then color v_i with a color different from $c, \pi(v_1), \pi(v_2), \pi(u_i)$ for $i = 3, 4, 5$, and v_6 with a color different from $c, \pi(x_6), \pi(y_6)$ successfully.

- If $\pi(x_6) = c$, then color v_i with a color different from $c, \pi(u_i)$ for $i = 3, 4, 5$, and v_6 with a color different from $c, \pi(v_1), \pi(v_2), \pi(y_6)$ successfully.

(Q2) Suppose to the contrary that v_1, v_2 are 2-vertices and v_3, v_4, v_5 are adjacent pendant light 3-vertices. By the minimality of G, $G - \{v, v_1, v_2, \cdots, v_5\}$ has an acyclic L-coloring π. It is obvious that $\pi(x_i) \neq \pi(y_i)$ for all $i = 3, 4, 5$. Let $S = \{u_1, u_2, x_3, y_3, x_4, y_4, x_5, y_5\}$. Since $|L(v) \setminus \{\pi(v_6)\}| \geqslant 4$ and $|S| = 8$, there exists a color belonging to $L(v) \setminus \{\pi(v_6)\}$ appearing at most twice on the set S.

First assume that there exists a color $c \in L(v) \setminus \{\pi(v_6)\}$ which appears at most once on the set S. We color v with c, v_i with a color different from $c, \pi(v_6), \pi(u_i)$ for $i = 1, 2$, and v_j with a color different from $c, \pi(v_6), \pi(x_j), \pi(y_j)$ for $j = 3, 4, 5$.

Now assume, w.l.o.g., that $L(v) = \{1, 2, 3, 4, 5\}$, $\pi(v_6) = 1$, and each color belonging to $\{2, 3, 4, 5\}$ appears exactly twice on the set S. One can easily observe that there exist two vertices x and y, where $x, y \in S \setminus \{u_1, u_2\}$, such that $\pi(x) = \pi(y)$. W.l.o.g., assume that $\pi(x_3) = \pi(x_4) = 2$. We color v with 2, v_3 with a color $a \in L(v_3) \setminus \{1, 2, \pi(y_3)\}$, v_4 with a color $b \in L(v_4) \setminus \{1, 2, a, \pi(y_4)\}$, v_i with a color different from $2, \pi(u_i)$ for $i = 1, 2$, and finally color v_6 with a color different from $2, \pi(x_6), \pi(y_6)$.

(Q3) Assume to the contrary that $[vv_1v_2]$ is a incident 3-face, v_3, v_4 are 2-vertices and v_5, v_6 are adjacent pendant light 3-vertices. By the minimality of G, $G - \{v, v_3, v_4, v_5, v_6\}$ admits an acyclic L-coloring π. Notice that $\pi(v_1) \neq \pi(v_2)$ and $\pi(x_i) \neq \pi(y_i)$ for each $i \in \{5, 6\}$. Let $S = \{u_3, u_4, x_5, y_5, x_6, y_6\}$. It is easy to observe that $|L(v) \setminus \{\pi(v_1), \pi(v_2)\}| \geqslant 3$ and $|S| = 6$. Basing on this fact, we assert that there exists a color belonging to $L(v) \setminus \{\pi(v_1), \pi(v_2)\}$ appearing at most twice on the set S,

First assume that there exists a color $c \in L(v) \setminus \{\pi(v_1), \pi(v_2)\}$ appearing at most once on the set S. By symmetry, we may color v with c. Then we color the remanent uncolored vertices in the following way: If $\pi(u_3) = c$, color v_3 with a color different from $c, \pi(v_1), \pi(v_2)$, and then assign v_i with a color different from that of its neighbors for $i = 4, 5, 6$. If $\pi(x_5) = c$, color v_5 with a color different from $c, \pi(v_1), \pi(v_2), \pi(y_5)$, and then assign v_j with a color different from that of its neighbors for $j = 3, 4, 6$.

Now, assume that $L(v) = \{1, 2, 3, 4, 5\}$, $\pi(v_1) = 1$, $\pi(v_2) = 2$ and each color in $\{3, 4, 5\}$ appears exactly twice on the set S. If $\pi(u_3) = \pi(u_4)$, say $\pi(u_3) = \pi(u_4) = 3$, then color v with 3, v_3 with a color $a \in L(v_3) \setminus \{1, 2, 3\}$, v_4 with a color $b \in L(v_4) \setminus \{1, 2, 3, a\}$, and finally color v_i with a color distinct to $3, \pi(x_i), \pi(y_i)$ for $i = 5, 6$. Otherwise, w.l.o.g., suppose that $\pi(u_3) = \pi(x_5) = 3$. Then color v with 3, v_5 with $c \in L(v_5) \setminus \{1, 2, 3, \pi(y_5)\}$, v_3 with $d \in L(v_3) \setminus \{1, 2, 3, c\}$, and finally assign a proper coloring for v_4 and v_6 easily.

(Q4) Assume to the contrary that v_1 is a 2-vertex and $v_2, v_3, \cdots, v_6$ are adjacent pendant light 3-vertices. Let $G' = G - \{v, v_1, v_2, \cdots, v_6\}$. By the minimality of G, G' admits an acyclic L-coloring π. Moreover, $\pi(x_i) \neq \pi(y_i)$ for all $i \geqslant 2$ since x_i is adjacent to y_i in G'. Let $S = \{u_1, x_2, y_2, \cdots, x_6, y_6\}$. Note that $|L(v)| = 5$ and $|S| = 11$. Thus, there exists a color $c \in L(v)$ which appears at most twice on the set S. We color v with c. If $\pi(u_1) = \pi(x_2) = c$, color v_1 with a color a different from $c, \pi(u_1)$, v_2 with a color different from $c, a, \pi(y_2)$, and v_i with a color different from $c, \pi(x_i), \pi(y_i)$ for $i = 3, \cdots, 6$. If $\pi(x_2) = \pi(x_3) = c$, color v_2 with a color b different from $c, \pi(y_2)$, v_3 with a color different from $c, b, \pi(y_3)$, and v_i with a color different from its neighbors for $i = 1, 4, 5, 6$.

(Q5) Assume to the contrary that $[vv_1v_2]$ is $(6,3,3)$-face, i.e., $d(v_1) = d(v_2) = 3$, v_3 is a 2-vertex, and v_4, v_5 are adjacent pendant light 3-vertices. Let $N(v_1) = \{v_1', v_2, v\}$ and $N(v_2) = \{v_2', v_1, v\}$. By the minimality of G, $G - \{v, v_1, v_2, \cdots, v_5\}$ admits an acyclic L-coloring π. Obviously, $\pi(x_i) \neq \pi(y_i)$ for each $i \in \{4, 5\}$. Let $S = \{v_1', v_2', u_3, x_4, y_4, x_5, y_5\}$. Since $|L(v) \setminus \{\pi(v_6)\}| \geqslant 4$ and $|S| = 7$, there exists a color $c \in L(v) \setminus \{\pi(v_6)\}$ which appears at most once on the set S. Then we color v with c, v_1 with a color $a \in L(v_1) \setminus \{c, \pi(v_6), \pi(v_1'), \pi(v_2')\}$, v_2 with a color different from $a, c, \pi(v_6), \pi(v_2')$, v_3 with a color different from $\pi(v_6), c, \pi(u_3)$, and finally color v_i with a color different from $c, \pi(v_6), \pi(x_i), \pi(y_i)$ for $i = 4, 5$ successfully.

(Q6) Assume to the contrary that $[vv_1v_2]$ is $(6,3,3)$-face, i.e., $d(v_1) = d(v_2) = 3$, and v_3, v_4, v_5 are adjacent pendant light 3-vertices. Let $N(v_1) = \{v_1', v_2, v\}$ and $N(v_2) = \{v_2', v_1, v\}$. By the minimality of G, $G - \{v, v_1, v_2, \cdots, v_5\}$ has an acyclic L-coloring π. Notice that $\pi(x_i) \neq \pi(y_i)$ for each $i \in \{3, 4, 5\}$. Let $S = \{v_1', v_2', x_3, y_3, x_4, y_4, x_5, y_5\}$. Since $|L(v) \setminus \{\pi(v_6)\}| \geqslant 4$ and $|S| = 8$, there exists a color belonging to $L(v) \setminus \{\pi(v_6)\}$ appearing at most twice on the set S. If there exists a color in $L(v) \setminus \{\pi(v_6)\}$ appearing at most once on S, the proof can also be given with a similar argument to the (Q5). Now assume, w.l.o.g., that $L(v) = \{1, 2, 3, 4, 5\}$, $\pi(v_6) = 1$, and each color in $\{2, 3, 4, 5\}$ appears exactly twice on the set S. It is easy to see that there exist two vertices $x, y \in \{x_3, y_3, x_4, y_4, x_5, y_5\}$ having the same color, set $\pi(x_3) = \pi(x_4) = 2$. We can color v with 2, v_1 with a color a different from $2, \pi(v_1'), \pi(v_2')$, v_2 with a color different from $a, 2, \pi(v_2')$, v_3 with a color $b \in L(v_3) \setminus \{1, 2, \pi(y_3)\}$, v_4 with a color $c \in L(v_4) \setminus \{1, 2, b, \pi(y_4)\}$, and finally assign a proper coloring for v_5.

(Q7) Suppose that $[vv_1v_2]$ is $(6,3,4)$-face such that $d(v_1) = 3$ and $d(v_2) = 4$. Let $N(v_1) = \{v_1', v_2, v\}$ and $N(v_2) = \{v_2', v_2'', v_1, v\}$. We need to consider two cases as follows.

(7.1) Assume to the contrary that v_3, v_4, v_5 are 2-vertices. By the minimality of G, $G - \{v, v_3, v_4, v_5\}$ admits an acyclic L-coloring π. Obviously, $\pi(v_1) \neq \pi(v_2)$. First suppose that v_1, v_2, v_6 are colored mutually distinct. We confirm that there exists a color c belonging to $L(v) \setminus \{\pi(v_1), \pi(v_2), \pi(v_6)\}$ which appears at most once on the set $\{u_3, u_4, u_5\}$, i.e., $\pi(u_3) = c$. So we color v with c, v_3 with a color different from $c, \pi(v_1), \pi(v_2), \pi(v_6)$, and then color v_i with a color distinct to $c, \pi(u_i)$ for $i = 4, 5$. Next, suppose that $\pi(v_6) = \pi(v_1)$. If $\pi(v_1') = \pi(v_2)$, then recolor v_1 with a color different from $\pi(v_1), \pi(v_2), \pi(v_2'), \pi(v_2'')$ and then go back to the former case. Otherwise, we also can recolor v_1 with a color different from $\pi(v_1), \pi(v_2), \pi(v_1')$ and then reduce the argument to the previous case. Finally, suppose that $\pi(v_2) = \pi(v_6)$. If $\pi(v_2') = \pi(v_2'')$, there exists a color $c' \in L(v) \setminus \{\pi(v_1), \pi(v_2), \pi(v_2')\}$ appearing at most once on the set $\{u_3, u_4, u_5\}$ and then the proof can also be given with a similar argument to the previous case. Now we assume that $\pi(v_2') \neq \pi(v_2'')$. If $\pi(v_1) \in \{\pi(v_2'), \pi(v_2'')\}$, we first recolor v_2 with a color distinct to $\pi(v_1'), \pi(v_2), \pi(v_2'), \pi(v_2'')$ and then reduce to the previous case. Otherwise, v_1, v_2', v_2'' have pairwise distinct colors. We may first recolor v_2 with a color distinct to $\pi(v_1), \pi(v_2), \pi(v_2'), \pi(v_2'')$ and reduce the argument to the previous case.

(7.2) Assume to the contrary that v_3 is a 2-vertex and v_4, v_5, v_6 are adjacent pendant light 3-vertices. By the minimality of G, $G - \{v, v_1, v_3, v_4, v_5, v_6\}$ admits an acyclic L-coloring π. Let $S = \{v_1', u_3, x_4, y_4, x_5, y_5, x_6, y_6\}$. It is easy to see that there exists a color belonging to $L(v) \setminus \{\pi(v_2)\}$ appearing at most twice on the set S, since $|L(v) \setminus \{\pi(v_2)\}| \geqslant 4$ and $|S| = 8$. We will discuss the following two cases.

First assume that there exists a color $c \in L(v) \setminus \{\pi(v_2)\}$ which appears at most once on the set S. We color v with c firstly, then color v_3 with a color different from $c, \pi(u_3), \pi(v_2)$, and v_i with a color different from $c, \pi(v_2), \pi(x_i), \pi(y_i)$ for $i = 4, 5, 6$. We further color v_1 in the following way: If $\pi(v_1') = \pi(v_2)$, then assign v_1 with a color in $L(v_1)\setminus\{c, \pi(v_2), \pi(v_2'), \pi(v_2'')\}$. Otherwise, assign a color in $L(v_1)\setminus\{c, \pi(v_2), \pi(v_1')\}$ to v_1.

Now assume, w.l.o.g., that $L(v) = \{1, 2, 3, 4, 5\}$, $\pi(v_2) = 1$, and each color in $\{2, 3, 4, 5\}$ appears exactly twice on the set S. It follows easily that there exist two vertices x and y belonging to $\{x_4, y_4, x_5, y_5, x_6, y_6\}$ having the same color. W.l.o.g., assume that $\pi(x_4) = \pi(x_5) = 2$. We may first color v with 2, v_3 with a color different from $2, \pi(u_3)$, v_4 with a color $a \in L(v_4) \setminus \{1, 2, \pi(y_4)\}$, v_5 with a color $b \in L(v_5) \setminus \{1, 2, a, \pi(y_5)\}$, v_6 with a color different from $2, \pi(x_6), \pi(y_6)$, and finally color v_1 in the following way: If $\pi(v_1') = \pi(v_2) = 1$, then assign v_1 with a color in $L(v_1) \setminus \{1, 2, \pi(v_2'), \pi(v_2'')\}$. Otherwise, assign a color in $L(v_1) \setminus \{1, 2, \pi(v_1')\}$ to v_1. ∎

Lemma 3.2.10 *Suppose that v is a 7-vertex. Then the following hold:*
(P1) *If $n_2(v) = 4$ and $t(v) = 1$, then $p_3(v) = 0$;*
(P2) *If $n_2(v) = 3$ and v is incident to a $(7, 3, 3)$-face, then $p_3(v) \leqslant 1$;*

Proof. **(P1)** Suppose to the contrary that $[vv_1v_2]$ is a 3-face, v_3, v_4, v_5, v_6 are 2-vertices and v_7 is an adjacent pendant light 3-vertex. By the minimality of G, $G - \{v, v_3, v_4, \cdots, v_7\}$ admits an acyclic L-coloring π. Let $S = \{u_3, u_4, u_5, u_6, x_7, y_7\}$. Obviously, $|L(v)\setminus\{\pi(v_1), \pi(v_2)\}| \geqslant 3$ and $|S| = 6$. This fact implies that there exists a color belonging to $L(v) \setminus \{\pi(v_1), \pi(v_2)\}$ appearing at most twice on the set S. If there is a color $c \in L(v) \setminus \{\pi(v_1), \pi(v_2)\}$ appearing at most once on the set S, then proof can also be given with a similar argument to the previous case (Q1). In what follows, suppose that $L(v) = \{1, 2, 3, 4, 5\}$, $\pi(v_1) = 1$, $\pi(v_2) = 2$ and each color belonging to $\{3, 4, 5\}$ appears exactly twice on the set S. Moreover, there are two vertices $u, w \in \{u_3, u_4, u_5, u_6\}$ given the same color, say $\pi(u_3) = \pi(u_4) = 3$. We may color v with 3. Then color v_3 with $a \in L(v_3)\setminus\{1, 2, 3\}$, v_4 with $b \in L(v_4)\setminus\{1, 2, 3, a\}$, v_i with a color different from $3, \pi(u_i)$ for $i = 5, 6$ and v_7 with a color different from $3, \pi(x_7), \pi(y_7)$.

(P2) Suppose to the contrary that $[vv_1v_2]$ is a $(7, 3, 3)$-face such that v_1 and v_2 are both 3-vertices, v_3, v_4, v_5 are 2-vertices and v_6, v_7 are adjacent pendant light 3-vertices. By the minimality of G, $G - \{v, v_1, v_2, \cdots, v_7\}$ admits an acyclic L-coloring π. Let $N(v_1) = \{v_1', v_2, v\}$ and $N(v_2) = \{v_1, v_2', v\}$. Let $S = \{v_1', v_2', u_3, u_4, u_5, x_6, y_6, x_7, y_7\}$. Since $|L(v)| = 5$ and $|S| = 9$, there exists a color $c \in L(v)$ appearing at most once on the set S. We can extend π to G in the following way: color v with c, v_1 with a color a different from $c, \pi(v_1'), \pi(v_2')$, v_2 with a color different from $a, c, \pi(v_1'), \pi(v_2')$, v_i with a color different from $c, \pi(u_i)$ for $i = 3, 4, 5$ and v_j with a color different from $c, \pi(x_j), \pi(y_j)$ for each $j \in \{6, 7\}$. ∎

Lemma 3.2.11 *Suppose that v is an 8-vertex. Then the following hold:*
(S1) $n_2(v) \leqslant 6$;
(S2) *If $t(v) = 1$, then $n_2(v) \leqslant 5$.*

Proof. **(S1)** The proof is similar to (C6) in Lemma 3.2.3.

(S2) Assume to the contrary that $v_1, v_2, \cdots, v_6$ are 2-vertices and $[vv_7v_8]$ is a 3-face. Let π be an acyclic L-coloring of $G - \{v, v_1, v_2, \cdots, v_6\}$. Obviously, $\pi(v_7) \neq \pi(v_8)$. Let $S = \{u_1, u_2, u_3, u_4, u_5, u_6\}$. Then there exists a color $c \in L(v) \setminus \{\pi(v_7), \pi(v_8)\}$ appearing at most twice on the set S, say $\pi(u_1) = \pi(u_2) = c$. Then color v with c, v_1 with a color a different from $\pi(v_7), \pi(v_8), c$, v_2 with a color different from $a, c, \pi(v_7), \pi(v_8)$, and finally color v_i with a color different from $c, \pi(u_i)$ for $i = 3, 4, 5, 6$. ∎

3.2.2.2 Discharging argument

We complete the proof with a discharging procedure. We first assign to each vertex v an initial charge $\omega(v)$ such that for all $v \in V(G), \omega(v) = 2d(v) - 6$ and to each face f an initial charge such that for all $f \in F(G), \omega(f) = d(f) - 6$. Suppose that $f = [v_1v_2v_3]$ is a 3-face with $d(v_1) \leqslant d(v_2) \leqslant d(v_3)$. We use $(d(v_1), d(v_2), d(v_3)) \to (c_1, c_2, c_3)$ to denote that the vertex v_i gives f the amount of weight c_i for $i = 1, 2, 3$.

Our discharging rules are as follows:

(R1) Every 5^+-vertex sends 1 to each adjacent 2-vertex, and $\frac{1}{2}$ to each adjacent pendant light 3-vertex.

(R2) Let $f = [v_1v_2v_3]$ be a 3-face with $d(v_1) \leqslant d(v_2) \leqslant d(v_3)$. We set

- $(3, 3, 6^+) \to (\frac{1}{2}, \frac{1}{2}, 2)$;
- $(3, 4, 5^+) \to (\frac{1}{2}, 1, \frac{3}{2})$;
- $(3, 5^+, 5^+) \to (\frac{1}{2}, \frac{5}{4}, \frac{5}{4})$;
- $(4^+, 4^+, 4^+) \to (1, 1, 1)$.

(R3) Every 4^+-vertex v gives $\frac{1}{5 - n_2(f) - n_3(f)}$ to each incident 5-face f.

Claim 3.2.1 *Suppose that $f = [v_1v_2 \cdots v_5]$ is a 5-face. Let $i \in \{1, 2, \cdots, 5\}$.*
(1) *If $d(v_i), d(v_{i+1}) \geqslant 4$, where i is taken modulo 5, then $\tau(v_i \to f) \leqslant \frac{1}{3}$ and $\tau(v_{i+1} \to f) \leqslant \frac{1}{3}$;*
(2) *If $d(v_i) = 4$, then $\tau(v_i \to f) \leqslant \frac{1}{3}$;*
(3) *If $d(v_i) \geqslant 5$, then $\tau(v_i \to f) \leqslant \frac{1}{2}$.*

Proof. (1) Assume, w.l.o.g., that $i = 1$. Namely, $d(v_1), d(v_2) \geqslant 4$. It follows directly from (C2) and (C3.1) that there are at most two 3^--vertices among the vertices of v_3, v_4, v_5. This implies that $n_2(f) + n_3(f) \leqslant 2$ and therefore (1) holds by (R3).

(2) Assume, w.l.o.g., that v_1 is a 4-vertex. If either v_2 or v_5 is a 4^+-vertex, then we are done by (1). Otherwise, suppose that $d(v_2) = d(v_5) = 3$ by (C2). For each $i \in \{3, 4\}$, $d(v_i) \neq 2$ by (C2) and $d(v_i) \neq 3$ by (C3.1). This means that both v_3 and v_4 are 4^+-vertices and thus (2) holds by (R3).

(3) It follows immediately from (C2) and (C3.1) that there are at most three 3^--vertices incident to f. Hence, (3) holds by (R3). ∎

Similarly, to complete the proof of Theorem 3.2.2, we only need show that the new weight function satisfies $\omega^*(x) \geqslant 0$ for all $x \in V(G) \cup F(G)$.

Lemma 3.2.12 *For every face f, $\omega^*(f) \geqslant 0$.*

Proof. Since G does not contain 4-cycles, there is no 4-faces. Depending on the degree of f, we divide the proof into three cases.

Case 1 $d(f) = 3$.

The initial charge is $\omega(f) = -3$. Let $f = [xyz]$ such that $d(x) \leqslant d(y) \leqslant d(z)$. By (C7.1), $d(x) \geqslant 3$. By (C7.2) and (C7.3), f is either a $(3, 3, 6^+)$-face, or a $(3, 4, 5^+)$-face, or a $(3, 5^+, 5^+)$-face, or a $(4^+, 4^+, 4^+)$-face. In each case, by (R2), we have $\omega^*(f) \geqslant -3+\frac{1}{2}\times 2+2 = 0$, or $\omega^*(f) \geqslant -3+\frac{1}{2}+1+\frac{3}{2} = 0$, or $\omega^*(f) \geqslant -3+\frac{1}{2}+\frac{5}{4}\times 2 = 0$, or $\omega^*(f) \geqslant -3 + 1 \times 3 = 0$.

Case 2 $d(f) = 5$.

Obviously, the initial charge of f is $\omega(f) = -1$. It is easy to see by (C2) and (C3.1) that $5 - n_2(f) - n_3(f) \geqslant 2$. Thus $\omega^*(f) \geqslant -1 + \frac{1}{5-n_2(f)-n_3(f)} \times (5 - n_2(f) - n_3(f)) = 0$ by (R3).

Case 3 $d(f) \geqslant 6$.

It is trivial that $\omega^*(f) = \omega(f) = d(f) - 6 \geqslant 0$.

This completes the proof of Lemma 3.2.12. ∎

It remains to show that for each vertex v, $\omega^*(v) \geqslant 0$. Let $v \in V(G)$. By (C1), $d(v) \geqslant 2$. In the following, let $v_1, v_2, \cdots, v_{d(v)}$ denote the neighbors of v in a cyclic order, and let f_i denote the incident face of v with vv_i and vv_{i+1} as two boundary edges for $i = 1, 2, \cdots, d(v)$, where indices are taken modulo $d(v)$.

If $d(v) = 2$, then the initial charge is $\omega(v) = -2$. By (C2), v is adjacent to two 5^+-vertices. Therefore, $\omega^*(v) \geqslant -2+1\times 2 = 0$ by (R1). If $d(v) = 3$, then the initial charge is $\omega(v) = 0$ and $t(v) \leqslant 1$ by the absence of intersecting triangles. If $t(v) = 0$, then no charge is sent out. So the final charge is also 0. Otherwise, assume that v is incident to a 3-face $[vv_1v_2]$. Lemma 3.2.7 confirms that v_3 is a vertex of degree at least 5. It follows from (R1) and (R2) that $\tau(v_3 \to v) = \frac{1}{2}$ and v sends $\frac{1}{2}$ to $[vv_1v_2]$. Thus, $\omega^*(v) \geqslant 0-\frac{1}{2}+\frac{1}{2} = 0$. In the following, we consider the charge of 4^+-vertices. Let $v \in V(G)$, we use $m_5(v)$ to denote the number of 5-faces incident to v. By the absence of intersecting triangles, $t(v) \leqslant 1$ for $v \in V(G)$. This straightforward fact is tacitly used in the following proofs.

Lemma 3.2.13 *If $d(v) = 4$, then $\omega^*(v) \geqslant 0$.*

Proof. The initial charge is $\omega(v) = 2$. Observe that $n_2(v) = 0$ by (C2) and $p_3(v) = 0$ by Lemma 3.2.7. Therefore, $\omega^*(v) \geqslant 2 - t(v) - \frac{1}{3}(4 - t(v)) = \frac{2}{3} - \frac{2}{3}t(v) \geqslant 0$ by (R2) and (2) of Claim 3.2.1. This completes the proof of Lemma 3.2.13. ∎

A 5-face f incident to v is called *weak* if v gives to f a charge exactly $\frac{1}{2}$. Let $m'_5(v)$ denote the number of weak 5-faces incident to v. First we have the following observation.

Observation 3.2.14 *For any 5^+-vertex v, we have that $n_2(v) + 2t(v) + m'_5(v) \leqslant d(v)$.*

Proof. Suppose that v is a 5^+-vertex. Let

$$\begin{aligned} A &= \{u \in N(v) : d(u) = 2\}, \\ B &= \{u \in N(v) : vu \textit{ is contained in a triangle}\}. \end{aligned}$$

It follows from the definition and (C7.1) that A, B are disjoint and $n_2(v) = |A|$, and $2t(v) = |B|$. Suppose that $f_i = [vv_iw_iw_{i+1}v_{i+1}]$ is a weak 5-face. That is to say that f_i gets $\frac{1}{2}$ from v. Thus both v_i and v_{i+1} are both 3^--vertices by (1) of Claim 3.2.1 and $n_2(f_i) + n_3(f_i) = 3$ by (R3). By symmetry, we have to consider the following three cases, depending on the degree of v_i and v_{i+1}.

- $d(v_i) = d(v_{i+1}) = 2$. Then $d(w_i), d(w_{i+1}) \geqslant 5$ by (C2), which is a contradiction to the fact that $n_2(f_i) + n_3(f_i) = 3$.

- $d(v_i) = 3$ and $d(v_{i+1}) = 2$. By (C2), $d(w_i) \geqslant 3$ and $d(w_{i+1}) \geqslant 5$. Since $n_2(f_i) + n_3(f_i) = 3$, we deduce that $d(w_i) = 3$. Noting that $v_i \notin A$. Moreover, $v_i \notin B$ by Lemma 3.2.7. If either f_{i-1} is not a weak 5-face or f_{i-1} is a weak 5-face but v_{i-1} does not belong to $A \cup B$, then we are done, since $m'_5(v) \leqslant d(v) - |A \cup B| = d(v) - |A| - |B| = d(v) - n_2(v) - 2t(v)$. Otherwise, suppose that $f_{i-1} = [vv_{i-1}u_{i-1}u_iv_i]$ is a weak 5-face and $v_{i-1} \in A \cup B$. By (1) of Claim 3.2.1, $d(v_{i-1}) \leqslant 3$. If $v_{i-1} \in A$, i.e., $d(v_{i-1}) = 2$, then $d(u_{i-1}) \geqslant 5$ by (C2). If $v_{i-1} \in B$, i.e., $f_{i-2} = [vv_{i-2}v_{i-1}]$ is a 3-face, then $d(u_{i-1}) \geqslant 5$ by Lemma 3.2.7. So, in each case, we always have that $d(u_{i-1}) \geqslant 5$. On the other hand, $d(u_i) \geqslant 5$ by (C3.1) because $d(w_i) = 3$. So $n_2(f_{i-1}) + n_3(f_{i-1}) = 3$ and thus v sends at most $\frac{1}{3}$ to f_{i-1} by (R3). This contradicts the assumption of f_{i-1}.

- $d(v_i) = d(v_{i+1}) = 3$. By (C2), w_i and w_{i+1} are vertices of degree at least 3. Since $n_2(f_i) + n_3(f_i) = 3$, w.l.o.g., set $d(w_i) = 3$ and $d(w_{i+1}) \geqslant 4$. It follows that $v_i \notin A$ and $v_i \notin B$ by Lemma 3.2.7. By using a similar discussion as above paragraph, we derive that f_{i-1} cannot be a weak 5-face such that $v_{i-1} \in A \cup B$ and thus $m'_5(v) \leqslant d(v) - n_2(v) - 2t(v)$.

This completes the proof of Observation 3.2.14. ∎

In the following argument, let m be the charge transferring from v to its incident 3-face (if exists). To estimate the total amount of charge sent from a 5^+-vertex v to its incident 3-face, 5-faces, and adjacent 2-vertices and pendant light 3-vertices, by

Observation 3.2.14, we make a rough calculation for v according to (R1) to (R3) as follows:

$$\begin{aligned}
\omega^*(v) &\geqslant 2d(v) - 6 - m - n_2(v) - \frac{1}{2}p_3(v) - \frac{1}{2}m'_5(v) - \frac{1}{3}(m_5(v) - m'_5(v)) \\
&= 2d(v) - 6 - m - n_2(v) - \frac{1}{2}p_3(v) - \frac{1}{6}m'_5(v) - \frac{1}{3}m_5(v) \\
&\geqslant 2d(v) - 6 - m - n_2(v) - \frac{1}{2}p_3(v) - \frac{1}{6}(d(v) - n_2(v) - 2t(v)) - \frac{1}{3}(d(v) - t(v)) \\
&= \frac{3}{2}d(v) - 6 - m - \frac{5}{6}n_2(v) - \frac{1}{2}p_3(v) + \frac{2}{3}t(v) \equiv \sigma(v) \qquad (*).
\end{aligned}$$

Lemma 3.2.15 *If $d(v) = 5$, then $\omega^*(v) \geqslant 0$.*

Proof. The initial charge is $\omega(v) = 4$. By (C4.1), $n_2(v) \leqslant 1$. Moreover, $t(v) \leqslant 1$. According to the value of $t(v)$, the following proof is divided into two cases.

Case 1 $t(v) = 0$.

It follows that $m = 0$. By $(*)$, we have that $\sigma(v) = \frac{3}{2} \times 5 - 6 - \frac{5}{6}n_2(v) - \frac{1}{2}p_3(v) = \frac{3}{2} - \frac{5}{6}n_2(v) - \frac{1}{2}p_3(v) \equiv \sigma^*(v)$. If $n_2(v) = 1$, then $p_3(v) = 0$ by (C4.2) and thus $\sigma^*(v) = \frac{3}{2} - \frac{5}{6} = \frac{2}{3}$. If $n_2(v) = 0$, then $p_3(v) \leqslant 3$ by (F1) and thus $\sigma^*(v) = \frac{3}{2} - \frac{1}{2} \times 3 = 0$.

Case 2 $t(v) = 1$.

Let $f_1 = [vv_1v_2]$ be the 3-face incident to v. By $(*)$, we have

$$\sigma(v) = \frac{3}{2} \times 5 - 6 - m - \frac{5}{6}n_2(v) - \frac{1}{2}p_3(v) + \frac{2}{3} = \frac{13}{6} - m - \frac{5}{6}n_2(v) - \frac{1}{2}p_3(v) \equiv \sigma^*(v).$$

If $n_2(v) = 1$, then $p_3(v) = 0$ by (C4.2) and f_1 must be a $(5, 4^+, 4^+)$-face by Lemma 3.2.4. By (R2), $\tau(v \to f_1) = 1$. Thus, $\sigma^*(v) = \frac{13}{6} - 1 - \frac{5}{6} = \frac{1}{3}$. Now, assume that $n_2(v) = 0$. By (C7), we see that f_1 is either a $(5, 3, 4^+)$-face or a $(5, 4^+, 4^+)$-face. We only need to consider the following three cases, according to the situation of f_1.

- Assume that f_1 is a $(5, 3, 4)$-face. It follows from (F3) that $p_3(v) \leqslant 1$. Moreover, v sends $\frac{3}{2}$ to f_1 by (R2). Thus, $\sigma^*(v) = \frac{13}{6} - \frac{3}{2} - \frac{1}{2} = \frac{1}{6}$.
- Assume that f_1 is a $(5, 3, 5^+)$-face. Namely, $d(v_1) = 3$ and $d(v_2) \geqslant 5$. Let v'_1 be the other neighbor of v_1 not on f_1. Then $\tau(v \to f_1) = \frac{5}{4}$ by (R2). By (F2), $p_3(v) \leqslant 2$. If $p_3(v) \leqslant 1$, then $\sigma^*(v) = \frac{13}{6} - \frac{5}{4} - \frac{1}{2} = \frac{5}{12}$. Now assume that $p_3(v) = 2$. If $m_5(v) \leqslant 3$, then $\omega^*(v) \geqslant 4 - \frac{5}{4} - \frac{1}{2} \times 2 - \frac{1}{2} \times 3 = \frac{1}{4}$ by (R1) and (3) of Claim 3.2.1. Otherwise, suppose that f_i is a 5-face for all $i = 2, 3, 4, 5$. By (F4), $n_3(v) \leqslant 3$. It implies that the vertex v_i with $i \in \{3, 4, 5\}$ which is not a pendant light 3-vertex must be a 4^+-vertex. By (1) of Claim 3.2.1, each of f_{i-1} and f_i gets at most $\frac{1}{3}$ from v, respectively. Therefore, $\omega^*(v) \geqslant 4 - \frac{5}{4} - \frac{1}{2} \times 2 - \frac{1}{3} \times 2 - \frac{1}{2} \times 2 = \frac{1}{12}$ by (R1) and (3) of Claim 3.2.1.
- Assume that f_1 is a $(5, 4^+, 4^+)$-face. By (R2), $\tau(v \to f_1) = 1$. Moreover, $p_3(v) \leqslant 2$ by (F2). Thus, $\sigma^*(v) = \frac{13}{6} - 1 - \frac{1}{2} \times 2 = \frac{1}{6}$.

This completes the proof of Lemma 3.2.15. ∎

Lemma 3.2.16 *If $d(v) = 6$, then $\omega^*(v) \geqslant 0$.*

Proof. The initial charge is $\omega(v) = 6$. By (C5.1), $n_2(v) \leqslant 4$. Moreover, $t(v) \leqslant 1$. Depending on the value of $t(v)$, the following proof is divided into two cases.

Case 1 $t(v) = 0$.

Then $m = 0$ and by $(*)$ we obtain that

$$\sigma(v) = \frac{3}{2} \times 6 - 6 - \frac{5}{6}n_2(v) - \frac{1}{2}p_3(v) = 3 - \frac{5}{6}n_2(v) - \frac{1}{2}p_3(v) \equiv \sigma^*(v).$$

• $n_2(v) = 4$. Then $n_3(v) = 0$ by (C5.2). It means that $p_3(v) = 0$ and v_i is either a 2-vertex or a 4^+-vertex for all $i = 1, \cdots 6$. If $m'_5(v) = 0$, then $\omega^*(v) \geqslant 6 - 1 \times 4 - \frac{1}{3} \times 6 = 0$ by (R1). Otherwise, assume that $f_i = [vv_iu_iu_{i+1}v_{i+1}]$ is a weak 5-face, where $i \in \{1, \cdots, 6\}$ and i is taken modulo 6. By (1) of Claim 3.2.1, we assert that $d(v_i) = d(v_{i+1}) = 2$. So both u_i and u_{i+1} are 5^+-vertices by (C2). It follows that $\tau(v \to f_i) \leqslant \frac{1}{3}$ by (R3), which contradicts the definition of a weak 5-face.

• $n_2(v) = 3$. Then $p_3(v) \leqslant 1$ by (A2). Thus, $\sigma^*(v) = 3 - \frac{5}{6} \times 3 - \frac{1}{2} = 0$.

• $n_2(v) = 2$. Then $p_3(v) \leqslant 2$ by (Q2) and hence $\sigma^*(v) = 3 - \frac{5}{6} \times 2 - \frac{1}{2} \times 2 = \frac{1}{3}$.

• $n_2(v) = 1$. Then $p_3(v) \leqslant 4$ by (Q4) and hence $\sigma^*(v) = 3 - \frac{5}{6} - \frac{1}{2} \times 4 = \frac{1}{6}$.

• $n_2(v) = 0$. Then $p_3(v) \leqslant 6$. Thus, $\sigma^*(v) = 3 - \frac{1}{2} \times 6 = 0$.

Case 2 $t(v) = 1$.

Let $f_1 = [vv_1v_2]$ be the 3-face incident to v. Obviously, $d(v_1), d(v_2) \geqslant 3$ by (C7.1). First we deduce by $(*)$ that

$$\sigma(v) = \frac{3}{2} \times 6 - 6 - m - \frac{5}{6}n_2(v) - \frac{1}{2}p_3(v) + \frac{2}{3} = \frac{11}{3} - m - \frac{5}{6}n_2(v) - \frac{1}{2}p_3(v) \equiv \sigma^*(v).$$

First assume that f_1 is a $(6, 3, 3)$-face. Then $\tau(v \to f_1) = 2$ by (R2) and $n_2(v) \leqslant 2$ by (C5.2) and (A2). By (A1), (Q5) and (Q6), $n_2(v) + p_3(v) \leqslant 2$. Thus, $\sigma^*(v) \geqslant \frac{11}{3} - 2 - \frac{5}{6}n_2(v) - \frac{1}{2}(2 - n_2(v)) = \frac{2}{3} - \frac{1}{3}n_2(v) \geqslant \frac{2}{3} - \frac{1}{3} \times 2 = 0$.

Next assume that f_1 is a $(6, 3, 4)$-face. It follows from (R2) that $\tau(v \to f_1) = \frac{3}{2}$. By (Q7.1), we see that $n_2(v) \leqslant 2$. If $1 \leqslant n_2(v) \leqslant 2$, then $n_2(v) + p_3(v) \leqslant 3$ by (Q3) and (Q7.2). Thus, $\sigma^*(v) = \frac{11}{3} - \frac{3}{2} - \frac{5}{6}n_2(v) - \frac{1}{2}(3 - n_2(v)) = \frac{2}{3} - \frac{1}{3}n_2(v) \geqslant \frac{2}{3} - \frac{1}{3} \times 2 = 0$. If $n_2(v) = 0$, then $p_3(v) \leqslant 4$ and therefore $\sigma^*(v) \geqslant \frac{11}{3} - \frac{3}{2} - \frac{1}{2} \times 4 = \frac{1}{6}$.

Next assume that f_1 is a $(6, 3, 5^+)$-face, i.e., $d(v_1) = 3$ and $d(v_2) \geqslant 5$. We have that $\tau(v \to f_1) = \frac{5}{4}$ by (R2). Moreover, $n_2(v) \leqslant 3$ by (A3).

• $n_2(v) = 3$. Then $n_3(v) \leqslant 1$ by (A2). It means that $p_3(v) = 0$ and v_i is either a 2-vertex or a 4^+-vertex, where $i \in \{3, 4, 5, 6\}$. One can easily check that f_i cannot be a weak 5-face for all $i = 2, \cdots, 5$. Therefore, $\omega^*(v) \geqslant 6 - \frac{5}{4} - 1 \times 3 - \frac{1}{3} \times 5 = \frac{1}{12}$ by (R1).

• $n_2(v) = 2$. Then $p_3(v) \leqslant 1$ by (Q3). It follows that $\sigma^*(v) = \frac{11}{3} - \frac{5}{4} - \frac{5}{6} \times 2 - \frac{1}{2} = \frac{1}{4}$.

• $n_2(v) \leqslant 1$. We conclude that $\sigma^*(v) = \frac{11}{3} - \frac{5}{4} - \frac{5}{6}n_2(v) - \frac{1}{2}(4 - n_2(v)) = \frac{5}{12} - \frac{1}{3}n_2(v) \geqslant \frac{5}{12} - \frac{1}{3} = \frac{1}{12}$.

Finally suppose that f_1 is a $(6, 4^+, 4^+)$-face. By (R2), v sends 1 to f_1. By (A3), $n_2(v) \leqslant 3$. If $n_2(v) = 3$, then $p_3(v) = 0$ by (Q1) and thus $\sigma^*(v) = \frac{11}{3} - 1 - \frac{5}{6} \times 3 = \frac{1}{6}$.

If $n_2(v) \leqslant 2$, then $p_3(v) \leqslant 4-n_2(v)$ and thus $\sigma^*(v) \geqslant \frac{11}{3}-1-\frac{5}{6}n_2(v)-\frac{1}{2}(4-n_2(v)) = \frac{2}{3}-\frac{1}{3}n_2(v) \geqslant \frac{2}{3}-\frac{1}{3}\times 2 = 0$.

This completes the proof of Lemma 3.2.16. ∎

Lemma 3.2.17 *If $d(v) = 7$, then $\omega^*(v) \geqslant 0$.*

Proof. The initial charge is $\omega(v) = 8$. By (C6), $n_2(v) \leqslant 5$. Moreover, $t(v) \leqslant 1$. Depending on the value of $t(v)$, the following proof is divided into two cases.

Case 1 $t(v) = 0$.

It follows that $m = 0$. By ($*$), we have

$$\sigma(v) = \frac{3}{2}\times 7 - 6 - \frac{5}{6}n_2(v) - \frac{1}{2}p_3(v) = \frac{9}{2} - \frac{5}{6}n_2(v) - \frac{1}{2}p_3(v) \equiv \sigma^*(v).$$

- $n_2(v) = 5$. Then $p_3(v) = 0$ by (B2) and thus $\sigma^*(v) = \frac{9}{2}-\frac{5}{6}\times 5 = \frac{1}{3}$.
- $n_2(v) = 4$. Then $p_3(v) \leqslant 2$ by (B1). So $\sigma^*(v) = \frac{9}{2}-\frac{5}{6}\times 4 - \frac{1}{2}\times 2 = \frac{1}{6}$.
- $n_2(v) \leqslant 3$. It is easy to deduce that $\sigma^*(v) \geqslant \frac{9}{2}-\frac{5}{6}n_2(v)-\frac{1}{2}(7-n_2(v)) = 1-\frac{1}{3}n_2(v) \geqslant 1-\frac{1}{3}\times 3 = 0$.

Case 2 $t(v) = 1$.

Let $f_1 = [vv_1v_2]$ be the 3-face incident to v. By ($*$), we derive that

$$\sigma(v) = \frac{3}{2}\times 7 - 6 - m - \frac{5}{6}n_2(v) - \frac{1}{2}p_3(v) + \frac{2}{3} = \frac{31}{6} - m - \frac{5}{6}n_2(v) - \frac{1}{2}p_3(v) \equiv \sigma^*(v).$$

First assume that f_1 is a $(7,3,3)$-face, i.e., $d(v_1) = d(v_2) = 3$. It follows from (B1) that v_i is either a 2-vertex or a 4^+-vertex for each $i \in \{3,\cdots,7\}$. By (C2) and (1) of Claim 3.2.1, one can easily check that v cannot be incident to any weak 5-face, which implies that $\tau(v \to f_i) \leqslant \frac{1}{3}$ for all $i = 2,\cdots,7$. Therefore, $\omega^*(v) \geqslant 8-2-1\times 4-\frac{1}{3}\times 6 = 0$ by (R1) and (R2).

Next, suppose that $n_2(v) = 3$. Then $p_3(v) \leqslant 1$ by (P2) and thus $\sigma^*(v) = \frac{31}{6}-2-\frac{5}{6}\times 3-\frac{1}{2} = \frac{1}{6}$. Finally, suppose that $n_2(v) \leqslant 2$. It follows immediately that $p_3(v) \leqslant 5-n_2(v)$ and therefore $\sigma^*(v) = \frac{31}{6}-2-\frac{5}{6}n_2(v)-\frac{1}{2}(5-n_2(v)) = \frac{2}{3}-\frac{1}{3}n_2(v) \geqslant \frac{2}{3}-\frac{1}{3}\times 2 = 0$.

Now assume that f_1 is a $(7,3,4)$-face. According to (R2), $\tau(v \to f_1) \leqslant \frac{3}{2}$. If $n_2(v) = 4$, then $p_3(v) = 0$ by (P1). So $\sigma^*(v) = \frac{31}{6}-\frac{3}{2}-\frac{5}{6}\times 4 = \frac{1}{3}$. If $n_2(v) \leqslant 3$, then $p_3(v) \leqslant 5-n_2(v)$ and therefore $\sigma^*(v) \geqslant \frac{31}{6}-\frac{3}{2}-\frac{5}{6}n_2(v)-\frac{1}{2}(5-n_2(v)) = \frac{7}{6}-\frac{1}{3}n_2(v) \geqslant \frac{7}{6}-\frac{1}{3}\times 3 = \frac{1}{6}$.

Finally assume that f_1 is a $(7,3^+,5^+)$-face. By (R2), $\tau(v \to f_1) \leqslant \frac{5}{4}$. Moreover, $n_2(v) \leqslant 4$ by (B2). Therefore, $\sigma^*(v) \geqslant \frac{31}{6}-\frac{5}{4}-\frac{5}{6}n_2(v)-\frac{1}{2}(5-n_2(v)) = \frac{17}{12}-\frac{1}{3}n_2(v) \geqslant \frac{17}{12}-\frac{1}{3}\times 4 = \frac{1}{12}$.

This completes the proof of Lemma 3.2.17. ∎

Lemma 3.2.18 *If $d(v) \geqslant 8$, then $\omega^*(v) \geqslant 0$.*

Proof. We recall that $\sigma(v) = \frac{3}{2}d(v) - 6 - m - \frac{5}{6}n_2(v) - \frac{1}{2}p_3(v) + \frac{2}{3}t(v)$.

First assume that $t(v) = 0$. Then $m = 0$. If $d(v) \geqslant 9$, then $\sigma(v) \geqslant \frac{3}{2}d(v) - 6 - \frac{5}{6}n_2(v) - \frac{1}{2}(d(v) - n_2(v)) = d(v) - 6 - \frac{1}{3}n_2(v) \geqslant d(v) - 6 - \frac{1}{3} \times d(v) = \frac{2}{3}d(v) - 6 \geqslant 0$. If $d(v) = 8$, then $n_2(v) \leqslant 6$ by (S1) and thus $\sigma(v) \geqslant \frac{3}{2} \times 8 - 6 - \frac{5}{6}n_2(v) - \frac{1}{2}(8 - n_2(v)) = 2 - \frac{1}{3}n_2(v) \geqslant 2 - \frac{1}{3} \times 6 = 0$.

Now assume that $t(v) = 1$. If $d(v) \geqslant 9$, then $m \leqslant 2$ by (R2) and thus $\sigma(v) \geqslant \frac{3}{2}d(v) - 6 - 2 - \frac{5}{6}n_2(v) - \frac{1}{2}(d(v) - n_2(v) - 2) + \frac{2}{3} = d(v) - \frac{19}{3} - \frac{1}{3}n_2(v) \geqslant d(v) - \frac{19}{3} - \frac{1}{3}(d(v) - 2) = \frac{2}{3}d(v) - \frac{17}{3} \geqslant \frac{1}{3}$. If $d(v) = 8$, then $n_2(v) \leqslant 5$ by (S2) and thus $\sigma(v) \geqslant \frac{3}{2} \times 8 - 6 - 2 - \frac{5}{6}n_2(v) - \frac{1}{2}(8 - n_2(v) - 2) + \frac{2}{3} = \frac{5}{3} - \frac{1}{3}n_2(v) \geqslant 0$.

This completes the proof of Lemma 3.2.18. ∎

3.3 Acyclic 4-choosability

3.3.1 Known results

In this section, we study the acyclic 4-choosability of planar graphs. In [Mon07], Montassier considered the planar graphs with girth at least 5 and improved the result on acyclically 4-colorable [BKW99] to acyclically 4-choosable. In [MRW06a], Montassier, Raspaud and Wang proved that every planar graph G without 4-, 5-, and 6-cycles, or without 4-, 5-, and 7-cycles, or without 4-cycles, 5-cycles and intersecting 3-cycles is acyclically 4-choosable. Moreover, they proposed the following conjecture:

Conjecture 3.3.1 ("Domaine de la Solitude 2000"Conjecture)
Every planar graph without 4-cycles is acyclically 4-choosable.

This conjecture is stronger than Conjecture 3.2.1, which is still unsettled. It seems to be much more difficult. Some sufficient conditions for planar graphs without specific short cycles to be acyclically 4-choosable were established. It is proved in [CRRZ11] that every planar graph without 4-, 7-, and 8-cycles is acyclically 4-choosable. Chen and Raspaud [CR09] proved that every planar graph without 4-, 5-, and 8-cycles is acyclically 4-choosable. Later, they showed in [CR10b] that a planar graph G is still acyclically 4-choosable if G contains no 4-cycles, 5-cycles and an 8-cycle having a triangular chord. Recently, Borodin, Ivanova and Raspaud [BIR10] showed that every planar graph with neither 4-cycles nor triangular 6-cycles is acyclically 4-choosable, which implies that every planar graph without 4- and 6-cycles is acyclically 4-choosable. Note that in all these results cycles of length 4 are forbidden. In this section, we prove the following:

Theorem 3.3.2 [CR10c] *Planar graphs without 4-cycles and 5-cycle are acyclically 4-choosable.*

Our result is a new approach to the conjecture 3.3.1 and is best possible in the sense that there are planar graphs without 4- and 5-cycles that are not 3-choosable [Voi07]. Moreover, it extends some results in [Bor10, MRW06a, MRW07, CR09, CR10b]. We remark that the same result is independently obtained by Borodin and Ivanova [BI10] recently.

3.3.2 Proof of Theorem 3.3.2

Suppose to the contrary that Theorem 3.3.2 is not true. Let G be a counterexample to Theorem 3.3.2 with the least number of vertices. Thus G is connected. We first investigate the structural properties of G in Section 3.3.2.1, then use Euler's formula and discharging technique to derive a contradiction in Section 3.3.2.2.

3.3.2.1 Structural properties

First, we give the following Lemmas 3.3.3 to 3.3.5, whose proof were provided in [MRW06a] and [CR10b], respectively.

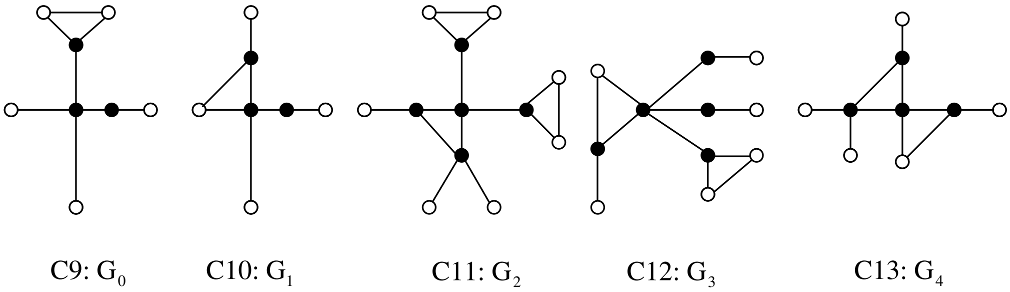


Figure 3.4: Some of reducible configurations in Lemma 3.3.3.

Lemma 3.3.3 [MRW06b] (C1) *There are no* 1*-vertices.*
(C2) *A* 2*-vertex is not incident to a* 3*-face.*
(C3) *A* 2*-vertex is not adjacent to a vertex of degree at most* 3*.*
(C4) *A* 3*-vertex is adjacent to at most one* 3*-vertex.*
(C5) *A* 4*-vertex is adjacent to at most one* 2*-vertex.*
(C6) *There is no* 3*-face incident to two* 3*-vertices and one* 4*-vertex.*
(C7) *A* 5*-vertex is adjacent to at most three* 2*-vertices.*
(C8) *There is no* 5*-vertex incident to a* 3*-face, adjacent to three* 2*-vertices.*
(C9) *G does not contain G_0 as a subgraph.*
(C10) *G does not contain G_1 as a subgraph.*
(C11) *G does not contain G_2 as a subgraph.*
(C12) *G does not contain G_3 as a subgraph.*
(C13) *G does not contain G_4 as a subgraph.*

It is worthy of being mentioned that (C4) was proved independently by Borodin, Kostochka and Woodall in [BKW99] for the acyclic 4-colorings. However, the proof in [BIR10] also works for the acyclic 4-choosability with almost no changes.

Lemma 3.3.4 [CR10b] *If v is a pendant light* 3*-vertex of v_3, i.e., $f = [vv_1v_2]$ is a* 3*-face, then $d(v_3) \geqslant 4$.*

We remark that the proof of Lemma 3.3.4 was inspired from Lemma 1 in [BIR10].

Lemma 3.3.5 [CR10b] *Let v be a 5-vertex with $t(v) = 2$. If one incident 3-face of v is a $(3, 3, 5)$-face, then $n_2(v) = 0$.*

In what follows, let L be a list assignment of G with $|L(v)| = 4$ for all $v \in V(G)$.

Lemma 3.3.6 *If v is a 5-vertex incident to a $(5, 3, 3)$-face and a $(5, 3, 3^+)$-face, then $p_3(v) = 0$.*

Proof. Suppose to the contrary that $f = [vv_1v_2]$ is a $(5, 3, 3)$-face, $f' = [vv_4v_5]$ is a $(5, 3, 3^+)$-face and v_3 is a pendant light 3-vertex such that $[v_3x_3y_3]$ is a 3-face. By definition, we see that v_1, v_2, v_4 are all 3-vertices. For each $i \in \{1, 2, 4\}$, let w_i denote the another neighbor of v_i not on its incident 3-face.

By the minimality of G, $G - \{v, v_1, v_2, v_3\}$ has an acyclic L-coloring π. Notice that $\pi(x_3) \neq \pi(y_3)$ and $\pi(v_4) \neq \pi(v_5)$. Denote $S = \{w_1, w_2, x_3, y_3\}$. Since $|L(v) \setminus \{\pi(v_4), \pi(v_5)\}| \geqslant 2$ and $|S| = 4$, there exists a color $c \in L(v) \setminus \{\pi(v_4), \pi(v_5)\}$ appearing at most twice on the set S. We first suppose that such color c appears at most once on the set S. If one of w_1, w_2 is colored with c, say w_1, then it is easy to extend π to G by coloring v with c, v_1 with a color a different from $c, \pi(v_4), \pi(v_5)$, v_2 with a color different from $a, c, \pi(w_2)$, and v_3 with a color different from $c, \pi(x_3), \pi(y_3)$. Otherwise, w.l.o.g., suppose that $\pi(x_3) = c$. We may color v with c, v_1 with a color a different from $c, \pi(w_1), \pi(w_2)$, v_2 with a color different from $a, c, \pi(w_2)$, and v_3 with a color b different from $c, \pi(y_3), \pi(v_5)$. If the resulting coloring is not acyclic, we deduce that $\pi(w_4) = c$ and $\pi(v_4) = b$. We only need to recolor v_4 with a color in $L(v_4) \setminus \{b, c, \pi(v_5)\}$.

So, in what follows, w.l.o.g., we suppose that $L(v) = \{1, 2, 3, 4\}$, $\pi(v_4) = 1$, $\pi(v_5) = 2$, $\pi(w_1) = \pi(x_3) = 3$ and $\pi(w_2) = \pi(y_3) = 4$. Obviously, there is a color $a \in \{3, 4\} \setminus \{\pi(w_4)\}$. Without loss of generality, assume $a = 3$. We first color v with 3, then color v_3 with a color b different from $2, 3, 4$, v_1 with a color d different from $2, 3, b$, and finally color v_2 with a color different from $3, 4, d$. ∎

Let v be a 4-vertex. If v is incident to exactly two non-adjacent 3-faces, then we call v a 4^*-*vertex*. If a 3-face $f = [v_1v_2v_3]$ is incident to a 4^*-vertex, say v_1, then we call f a $(4^*, d(v_2), d(v_3))$-face.

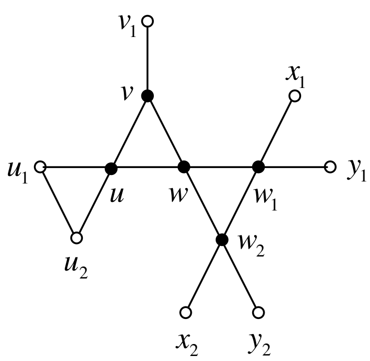


Figure 3.5: A $(3, 4^*, 4^*)$-face $[vwu]$.

Lemma 3.3.7 *Suppose G contains a $(3, 4^*, 4^*)$-face $[vwu]$, depicted in Figure 3.5. Let π be an acyclic L-coloring of $G - v$ and $L(v) = \{1, 2, 3, 4\}$. Then*

(i) *w.l.o.g., $\pi(v_1) = \pi(w) = 1$ and $\pi(u) = 2$;*

(iii) *$L(w) = L(u) = \{1, 2, 3, 4\}$:*

(iii) *$\{\pi(w_1), \pi(w_2)\} = \{\pi(u_1), \pi(u_2)\} = \{3, 4\}$;*

(iv) *$1 \in N(w_i) \setminus \{w\}$ and $1 \in N(u_i) \setminus \{u\}$.*

Proof. Obviously, $\pi(u) \neq \pi(w)$. If $\pi(v_1), \pi(u), \pi(w)$ are mutually distinct, we are easily done by coloring v with a color in $L(v) \setminus \{\pi(v_1), \pi(u), \pi(w)\}$. Otherwise, by symmetry, we may suppose that $\pi(v_1) = \pi(w)$. If there exists a color $c \in L(v) \setminus \{\pi(v_1), \pi(u), \pi(w_1), \pi(w_2)\}$, then it suffices to color v with c. Otherwise, if v cannot be acyclically colored, we may suppose, w.l.o.g., that $\pi(v_1) = \pi(w) = 1$, $\pi(u) = 2$, $\{\pi(w_1), \pi(w_2)\} = \{3, 4\}$, and $1 \in N(w_i) \setminus \{w\}$ for each $i \in \{1, 2\}$. If $L(w) \neq \{1, 2, 3, 4\}$, then we recolor w by a color in $L(w) \setminus \{1, 2, 3, 4\}$ and then go back to the previous case. So, in the following, assume that $L(w) = \{1, 2, 3, 4\}$. Now we erase the color of u and recolor w with 2. Notice that neither u_1 nor u_2 is colored with 2. If $L(u) \neq \{1, 2, \pi(u_1), \pi(u_2)\}$, then color u with a color in $L(u) \setminus \{1, 2, \pi(u_1), \pi(u_2)\}$ and then reduce the proof to the previous case. Otherwise, assume that $L(u) = \{1, 2, \pi(u_1), \pi(u_2)\}$ and assign 1 to u. If v cannot be further given the color 3 or the color 4, then it follows immediately that $\{\pi(u_1), \pi(u_2)\} = \{3, 4\}$ and $1 \in N(u_i) \setminus \{u\}$ for each $i \in \{1, 2\}$ and thus $L(u) = \{1, 2, 3, 4\}$. ∎

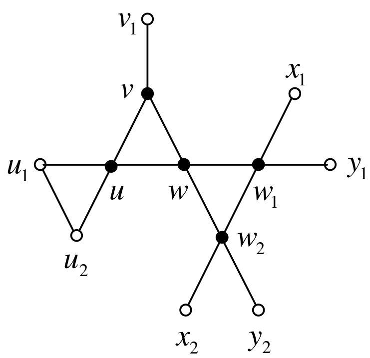


Figure 3.6: The configuration (F).

Lemma 3.3.8 *Suppose that G contains the configuration **(F)**, depicted in Figure 3.6. Let π be an acyclic L-coloring of $G - v$ and $L(v) = \{1, 2, 3, 4\}$. If $\pi(x_i) \neq \pi(y_i)$ for some fixed $i \in \{1, 2\}$, then $L(w_i) = \{\pi(w_1), \pi(w_2), \pi(x_i), \pi(y_i)\}$.*

Proof. By symmetry, suppose that $\pi(x_1) \neq \pi(y_1)$. By (i) of Lemma 3.3.7, we may assume that $\pi(v_1) = \pi(w) = 1$ and $\pi(u) = 2$ (otherwise, we swap the colors of w and u). By (ii) to (iv) of Lemma 3.3.7, we further suppose, w.l.o.g., that $\pi(x_1) = \pi(x_2) = 1$, $\pi(w_1) = \pi(u_1) = 3$, $\pi(w_2) = \pi(u_2) = 4$, and $L(w) = L(u) = \{1, 2, 3, 4\}$. First, suppose that $\pi(y_1) = 4$. We may first recolor w_1 with a in

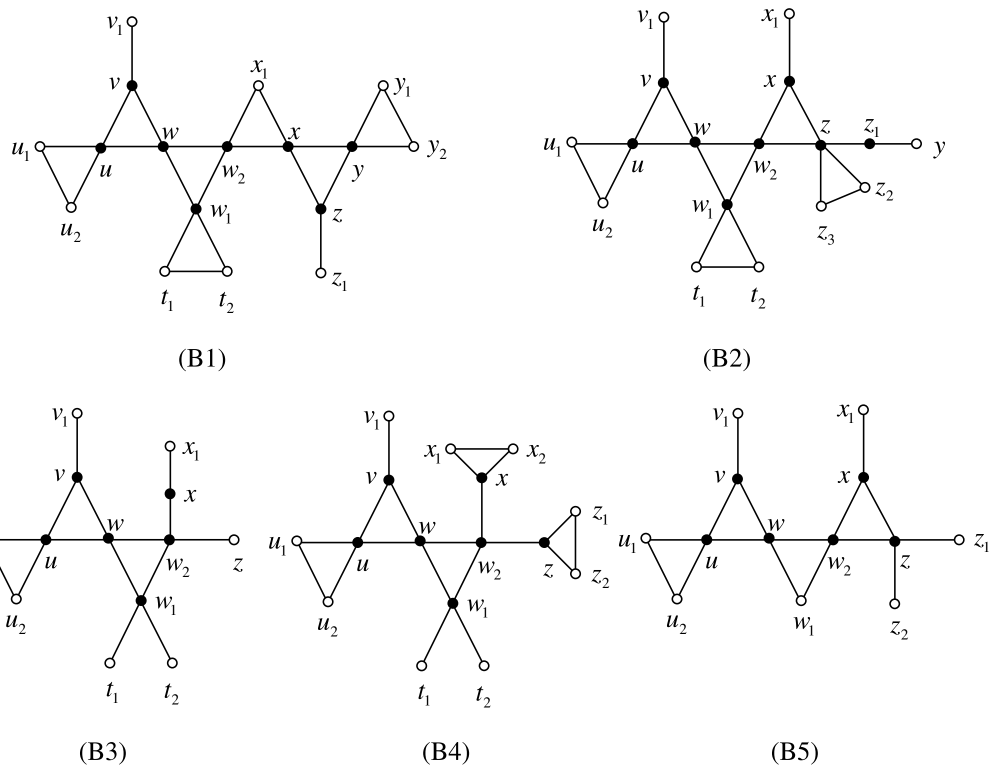


Figure 3.7: The configurations (B1) to (B5).

$L(w_1)$ different from 1, 3, 4. If $\pi(y_2) \neq a$, we are done by coloring v with 3. Otherwise, suppose $\pi(y_2) = a$. We can recolor w_2 with a color $b \in L(w_2) \setminus \{1, 4, a\}$ and finally color v with 4 successfully. Now assume that $\pi(y_1) \neq 4$. If $L(w_1) \neq \{\pi(w_1), \pi(w_2), \pi(x_1), \pi(y_1)\} = \{1, 3, 4, \pi(y_1)\}$, we can extend π to G by recoloring w_1 with a color in $L(w_1) \setminus \{1, 3, 4, \pi(y_1)\}$ and coloring v with 3. This contradicts the assumption of G and thus we complete the proof of Lemma 3.3.8. ∎

Lemma 3.3.9 *G does not contain the configurations* ***(B1)*** *to* ***(B5)*** *depicted in Figure 3.7.*

Proof. In each of the following Case i with $i \in \{1, \cdots, 5\}$, we suppose that G contains the configuration **(Bi)** and let π be an acyclic L-coloring of $G - v$ by the minimality of G. W.l.o.g., suppose that $L(v) = \{1, 2, 3, 4\}$. By (i) to (iii) of Lemma 3.3.7, we may suppose, w.l.o.g., that $\pi(v_1) = \pi(w) = 1$, $\pi(u) = 2$, $\pi(w_1) = \pi(u_1) = 3$, $\pi(w_2) = \pi(u_2) = 4$, and $L(w) = L(u) = \{1, 2, 3, 4\}$. Next, in each case, we will make use of contradictions to show that **(Bi)** cannot exist in G.

Case 1 G contains **(B1)**.

W.l.o.g, we have that $\pi(t_1) = 1$ and $1 \in \{\pi(x_1), \pi(x)\}$ by (iv) of Lemma 3.3.7.

1.1 $\pi(x_1) = 1$. For our convenience, we denote $\pi(x) = \alpha^*$ and $\pi(t_2) = \beta^*$. Notice that α^* could be equal to β^*. By Lemma 3.3.8, we have that $L(w_1) = \{1, 3, 4, \beta^*\}$ and $L(w_2) = \{1, 3, 4, \alpha^*\}$. It follows that $\alpha^*, \beta^* \notin \{3, 4\}$. We first erase the color of w_2. If x can be assigned a *feasible* color γ such that the resulting coloring of $G-\{v,w_2\}$ is still acyclic, then we can extend π to G properly in the following way: If $\gamma=3$, then recolor w_1 with 4, and color w_2 with α^* and v with 3; otherwise, we only need to color w_2 with α^* and color v with 4. Next, we will show how to recolor x with such a feasible color.

1.1.1 $1\notin\{\pi(y),\pi(z)\}$. Denote $\pi(y)=\gamma_1$ and $\pi(z)=\gamma_2$. If $L(x)\neq\{1,\alpha^*,\gamma_1,\gamma_2\}$, then a color $c\in L(x)\setminus\{1,\alpha^*,\gamma_1,\gamma_2\}$ is feasible for x of $G-\{v,w_2\}$ and thus we are done. Now suppose $L(x)=\{1,\alpha^*,\gamma_1,\gamma_2\}$ and erase the color of x. First, we assume that $\pi(z_1)\neq\gamma_1$. We recolor z with a color different from $\gamma_1,\gamma_2,\pi(z_1)$ and then color x with a feasible color γ_2. Otherwise, suppose $\pi(z_1)=\gamma_1$. If $\gamma_2\in\{\pi(y_1),\pi(y_2)\}$, we can recolor y with a color different from $\gamma_1,\pi(y_1),\pi(y_2)$ and then color x with a feasible color γ_1. Now suppose that $\gamma_2,\pi(y_1),\pi(y_2)$ are pairwise distinct. In this case, we may first recolor y with a color $c\in L(y)\setminus\{\gamma_1,\pi(y_1),\pi(y_2)\}$. If $c=\gamma_2$, we further recolor z with a color $c'\in L(z)\setminus\{1,\gamma_1,\gamma_2\}$ and then assign the feasible color γ_1 for x. If $c\neq\gamma_2$, it suffices to color x with the feasible color γ_1.

1.1.2 $\pi(y)=1$ and $\pi(z)\neq 1$. We need to consider two possibilities below.

- $\pi(y_1),\pi(y_2),\pi(z)$ are mutually distinct. We first recolor y with c belonging to $L(y)\setminus\{1,\pi(y_1),\pi(y_2)\}$. If $c\notin\{\alpha^*,\pi(z)\}$, then go back to the previous case 1.1.1. If $c=\alpha^*$, then recolor x with a feasible color $d\in L(x)\setminus\{1,\alpha^*,\pi(z)\}$. Now we suppose that $c=\pi(z)$. In this case, we first recolor z with a color c' different from $1,\pi(z_1),\pi(z)$. If $c'\neq\alpha^*$, we may also reduce the proof to the previous case 1.1.1. If $c'=\alpha^*$, we only need to recolor x with a feasible color $d'\in L(x)\setminus\{1,\alpha^*,c\}$.
- $\pi(y_1)=\pi(z)$. Denote $\pi(y_1)=\gamma_1$ and $\pi(y_2)=\gamma_2$. Observe that $L(x)=\{1,\alpha^*,\gamma_1,\gamma_2\}$; otherwise, there exists a feasible color belonging to $L(x)\setminus\{1,\alpha^*,\gamma_1,\gamma_2\}$ for x and thus we are done. If $\pi(z_1)\neq 1$, we recolor z with a color distinct to $1,\gamma_1,\pi(z_1)$ and then assign the feasible color γ_1 to x. If $\pi(z_1)=1$, we may recolor y with a color different from $1,\gamma_1,\gamma_2$ and then assign the feasible color γ_2 to x.

1.1.3 $\pi(z)=1$ and $\pi(y)\neq 1$. Similarly, we observe that $L(x)=\{1,\alpha^*,\pi(y),\pi(z_1)\}$; otherwise, there is a feasible color in $L(x)\setminus\{1,\alpha^*,\pi(y),\pi(z_1)\}$ for x and thus we are done. This observation implies that $\pi(y)\neq\pi(z_1)$. If there is a color c in $L(z)\setminus\{1,\alpha^*,\pi(y),\pi(z_1)\}$, then recolor z with c and then go back to the previous case 1.1.1. Now assume that $L(z)=\{1,\alpha^*,\pi(y),\pi(z_1)\}$. We only need to recolor z with α^* and then assign the feasible color $\pi(z_1)$ to x.

1.2 $\pi(x)=1$. Similarly, denote $\pi(x_1)=\alpha^*$ and $\pi(t_2)=\beta^*$. By Lemma 3.3.8, we have that $L(w_1)=\{1,3,4,\beta^*\}$ and $L(w_2)=\{1,3,4,\alpha^*\}$. If $3\notin\{\pi(y),\pi(z)\}$, then recolor w_2 with 3, w_1 with 4, and finally color v with 3 successfully. If $4\notin\{\pi(y),\pi(z)\}$, then color v with 4 easily. Now assume that $\{\pi(y),\pi(z)\}=$

$\{3,4\}$. We may further deduce that $\pi(z_1) = 1$ and $1 \in \{\pi(y_1), \pi(y_2)\}$, say $\pi(y_1) = 1$. If there exists a color $c \in L(x) \setminus \{1, 3, 4, \alpha^*\}$, then recolor x with c and color v with 4. Otherwise, assume that $L(x) = \{1, 3, 4, \alpha^*\}$. If $\pi(z) = 3$, then recolor x with 3, z with a color different from 1, 3, 4 and finally color v with 4. If $\pi(z) = 4$, then recolor x with 4, w_2 with 3, w_1 with 4, z with a color different from 1, 3, 4 and finally color v with 3.

Case 2 G contains **(B2)**.

By (iv) of Lemma 3.3.7, w.l.o.g., we assume that $\pi(t_1) = 1$ and $1 \in \{\pi(x), \pi(z)\}$.

2.1 $\pi(x) = 1$. It follows immediately that $\pi(x_1) = 4$. By Lemma 3.3.8, we see that $L(w_2) = \{1, 3, 4, \pi(z)\}$. Observe that $\pi(z) \neq 4$. So we can recolor w_2 with 1, x with a color different from $1, 4, \pi(z)$, w with 4, and finally color v with 3.

2.2 $\pi(z) = 1$. It follows that at least one of z_1, z_2, z_3 is colored with 4. For simplicity, we denote $\pi(x) = \alpha^*$ and $\pi(t_2) = \beta^*$. According to Lemma 3.3.8, we have that $L(w_1) = \{1, 3, 4, \beta^*\}$ and $L(w_2) = \{1, 3, 4, \alpha^*\}$. We have two subcases, depending on the color of x_1.

a) $\pi(x_1) \neq 1$. It is easy to extend π to G by recoloring w_2 with α^*, x with a color different from $1, \alpha^*, \pi(x_1)$, and finally coloring v with 3.

b) $\pi(x_1) = 1$. Observe that at least one of z_1, z_2, z_3 is colored with 3; otherwise, we can recolor w_1 with 4, w_2 with 3 and color v with 3 successfully. This observation reminds us that the color 3 and the color 4 are both appeared on the set $\{z_1, z_2, z_3\}$. Now we first recolor z with a color $c \in L(z) \setminus \{1, \pi(z_2), \pi(z_3)\}$. If $c = 3$, then $\pi(z_1) = 3$ and thus we first recolor z_1 with a color different from $3, \pi(y)$ and then color v with 4. Similarly, if $c = 4$, then $\pi(z_1) = 4$ and hence we can first recolor z_1 with a color different from $4, \pi(y)$, w_2 with 3, w_1 with 4, and then color v with 3. So, in the following, we suppose $c \notin \{3, 4\}$. First assume that $c \neq \alpha^*$. If $c \neq \pi(y)$, it is enough to recolor z_1 with a color distinct to c, $\pi(y)$, and then color v with 4. Otherwise, we recolor z_1 with a color different from $c, \pi(z_2), \pi(z_3)$ and then color v with 4. Now assume that $c = \alpha^*$. We need to first recolor x with a color different from 1, 4, α^* and then go back to the previous case. ∎

Case 3 G contains **(B3)**.

W.l.o.g, we assume that $\pi(t_1) = 1$ and $1 \in \{\pi(z), \pi(x)\}$ by (iv) of Lemma 3.3.7.

3.1 $\pi(z) = 1$. We first recolor w_2 with a in $L(w_2)$ different from 1, 3, 4. If $\pi(x_1) \neq a$, we need to further recolor x with a color different from $a, \pi(x_1)$ and color v with 4. Otherwise, suppose $\pi(x_1) = a$. Then we can recolor x with a color different from $1, 3, a$ and finally color v with 4.

3.2 $\pi(x) = 1$. In this case, we may further suppose that $\pi(x_1) = 4$; otherwise, we can color v with 4 successfully. So $\pi(x) \neq \pi(z)$. By Lemma 3.3.8, we have that $L(w_2) = \{1, 3, 4, \pi(z)\}$. Therefore, we can first recolor w_2 with 1 and w with 4, then recolor x with a color different from 1 and 4, and finally color v with 3.

Case 4 G contains **(B4)**.

By (iv) of Lemma 3.3.7, we may assume, w.l.o.g., that $\pi(t_1) = \pi(x) = 1$. Moreover, at least one of x_1, x_2 is colored with 4; otherwise, we can extend π to G easily by assigning the color 4 to v. By symmetry, assume that $\pi(x_1) = 4$. The following argument is divided into two cases, according to the color of z.

4.1 $\pi(z) = 1$. We first recolor w_2 with $a \in L(w_2) \setminus \{3, 4, \pi(x_2)\}$. If $a \neq 1$, then further color v with 4. Otherwise, suppose $a = 1$. Then recolor x with a color in $L(x) \setminus \{1, 4, \pi(x_2)\}$, z with a color in $L(z) \setminus \{1, \pi(z_1), \pi(z_2)\}$, w with 4, and finally color v with 3.

4.2 $\pi(z) \neq 1$. By Lemma 3.3.8, we see that $L(w_2) = \{1, 3, 4, \pi(z)\}$. We first recolor w_2 with 1, then recolor x with a color distinct to $1, 4, \pi(x_2)$ and recolor w with 4, and finally color v with 3.

Case 5 G contains **(B5)**.

By (iv) of Lemma 3.3.7, we see that either x or z is colored with 1.

5.1 $\pi(x) = 1$. It it easy to obtain that $\pi(x_1) = 4$. Notice that $\pi(z) \neq 4$. If there exists a color c in $L(x) \setminus \{1, 3, 4, \pi(z)\}$, then recolor x with c and color v with 4. Now assume that $L(x) = \{1, 3, 4, \pi(z)\}$. It follows that $\pi(z) \neq 3$. We need to recolor x with 3, w_2 with a color different from 3, 4, $\pi(z)$, w with 4 and finally color v with 3 successfully.

5.2 $\pi(z) = 1$. By symmetry, we may assume that $\pi(z_1) = 4$. First suppose that $\pi(x) = 3$. We first recolor w_2 with a in $L(w_2)$ different from 1, 3, 4. If $\pi(x_1) \neq a$, it is easy to color v with 4. If $\pi(x_1) = a$, we recolor x with a color $b \in L(x) \setminus \{1, 3, a\}$ and then color v with 4. Now suppose that $\pi(x) \neq 3$. If there exists a color $c \in L(w_2) \setminus \{1, 3, 4, \pi(x)\}$, we recolor w_2 with c and then color v with 4. So, in what follows, suppose that $L(w_2) = \{1, 3, 4, \pi(x)\}$. Denote $\pi(x) = \alpha^*$. We have two subcases, depending on the color of x_1.

- $\pi(x_1) = 1$. Firstly, we recolor x with a different from $1, 4, \alpha^*$. If $\pi(z_2) \neq a$, we continue to recolor w_2 with α^*, w with 4 and finally color v with 3. Otherwise, suppose that $\pi(z_2) = a$. It means that z, z_1, z_2 have distinct colors. Then we further recolor z with b different from 1, 4, a. If $b \neq 3$, we only need to color v with 4. If $b = 3$, then we recolor w_2 with 1, w with 4, and finally color v with 3.
- $\pi(x_1) \neq 1$. It is easy to extend π to G by recoloring w_2 with α^*, x with a color different from $1, \alpha^*, \pi(x_1)$, and afterwards coloring v with 4. ∎

Lemma 3.3.10 *G does not contain the configurations **(Q1)** to **(Q4)** depicted in Figure 3.8.*

Proof. In each of the following Case i with $i \in \{1, \cdots, 4\}$, we suppose that G contains the configuration **(Qi)** and let π be an acyclic L-coloring of $G - v$ by the minimality of G. W.l.o.g., suppose that $L(v) = \{1, 2, 3, 4\}$. By (i)-(iv) of Lemma 3.3.7, we may suppose, w.l.o.g., that $\pi(v_1) = \pi(w) = \pi(t_1) = \pi(s_1) = 1$, $\pi(u) = 2$, $\pi(w_1) = 3$, $\pi(w_2) = 4$, $\{\pi(u_1), \pi(u_2)\} = \{3, 4\}$, $1 \in \{\pi(x), \pi(z)\}$, and $L(w) = L(u) = \{1, 2, 3, 4\}$. First we assume, in the following each case, that $\pi(z) = 1$. By Lemma 3.3.8, we obtain that $L(w_1) = \{1, 3, 4, \pi(t_2)\}$ and $L(w_2) = \{1, 3, 4, \pi(x)\}$. Now we swap the colors of w and u. By Lemma 3.3.8 again, we deduce that $\pi(z_1) \notin \{3, 4\}$. So at most one of the colors 3 and 4 is appeared on z_2. If $\pi(z_2) = 4$, we recolor w_2 with 3, w_1 with 4, w with 1, u with 2, and then color v with 3 successfully. Otherwise, we recolor w with 1, u with 2, and color v with 4. We always derive a contradiction. So, in the following, by symmetry, we assume $\pi(x) = \pi(u_3) = 1$. For simplicity, denote $\pi(z) = \alpha^*$ and $\pi(t_2) = \beta^*$. Again $L(w_1) = \{1, 3, 4, \beta^*\}$ and $L(w_2) = \{1, 3, 4, \alpha^*\}$ by Lemma 3.3.8. It means that $\alpha^*, \beta^* \notin \{3, 4\}$. Next, in each case, we will make use of contradictions to show that **(Qi)** cannot exist in G.

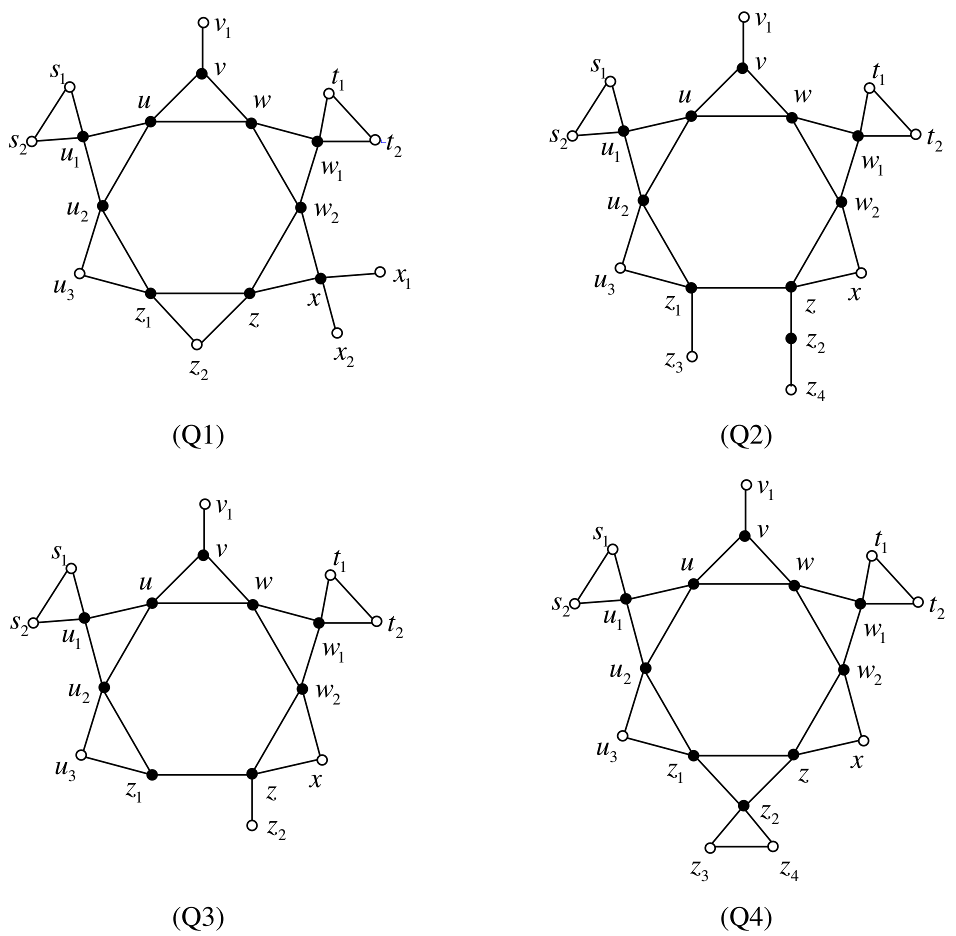


Figure 3.8: The configurations (Q1) to (Q4).

Case 1 G contains **(Q1)**.

By using a similar argument as above, we deduce that $\{\pi(x_1), \pi(x_2)\} = \{3, 4\}$. We first recolor x with $c \in L(x) \setminus \{1, 3, 4\}$. If $c \neq \alpha^*$, then we color v with 4 successfully. Otherwise, we continue to recolor z with d different from $\alpha^*, \pi(z_1), \pi(z_2)$. It is easy to check that the resulting coloring is acyclic. If $d = 1$, then go back to the previous case. If $d = 4$, then recolor w_2 with 3, w_1 with 4, and color v with 3. Otherwise, we color v with 4 easily.

Case 2 G contains **(Q2)**.

Note that $\pi(z_1) \neq 1$. Moreover, $\pi(z_1) \notin \{3, 4\}$ by Lemma 3.3.8. It implies that $\pi(z_1) \neq \pi(u_1)$. We first recolor z with a color $c \in L(z) \setminus \{1, \pi(z_1), \alpha^*\}$, w_2 with α^*, w_1 with $a \in \{3, 4\} \setminus \{c\}$, and finally color v with a color in $\{3, 4\}$ different from a. If the resulting coloring is not acyclic, then we deduce that $\pi(z_4) = c$ and $\pi(z_2) \in \{1, \pi(z_1)\}$. It suffices to further recolor z_2 with a color different from $1, c, \pi(z_1)$.

Case 3 G contains **(Q3)**.

Notice that $\pi(u_3) = 1$. First, we recolor u with 1 and w with 2. By Lemma 3.3.8, $L(u_2) = \{1, 3, 4, \pi(z_1)\}$. It implies that $\pi(z_1) \neq \pi(u_1)$, since $\pi(u_1) \in \{3, 4\}$. So we can extend π to G easily by recoloring z_1 with $c \in L(z_1) \setminus \{1, \alpha^*, \pi(z_1)\}$, u_2 with $\pi(z_1)$, and afterwards coloring v with a color in $\{3, 4\} \setminus \{\pi(u_1)\}$.

Case 4 G contains **(Q4)**.

Denote $\pi(z_1) = \gamma^*$. Similarly, by Lemma 3.3.8, we deduce that $L(u_2) = \{1, 3, 4, \gamma^*\}$. The argument is divided into two subcases, depending on the color of z_2.

4.1 $\pi(z_2) \neq 1$. We first recolor z with c distinct to $1, \alpha^*, \pi(z_2)$. If $c \neq \gamma^*$, we need to recolor w_2 with α^* and then color v with 4 successfully. Otherwise, assume $L(z) = \{1, \alpha^*, \gamma^*, \pi(z_2)\}$. By symmetry, we easily obtain that $L(z_1) = \{1, \alpha^*, \gamma^*, \pi(z_2)\}$. Now we recolor z with γ^*, z_1, w_2 with α^* and afterwards color v with 4. It is easy to check that the resulting coloring is acyclic.

4.2 $\pi(z_2) = 1$. First recolor z_2 with a color $a \in L(z_2) \setminus \{1, \pi(z_3), \pi(z_4)\}$. If $a \notin \{\alpha^*, \gamma^*\}$, then reduce the proof to the previous Case 4.1. Otherwise, by symmetry, suppose that $a = \alpha^*$. We first recolor z with a color c distinct to $1, \alpha^*, \gamma^*$, then recolor w_2 with α^* and w_1 with a color c' in $\{3, 4\} \setminus \{c\}$, and finally color v with a color in $\{3, 4\}$ different from c' successfully. ∎

3.3.2.2 Discharging argument

We complete the proof with a discharging procedure. As usual, we assign to each vertex v an initial charge $\omega(v) = 2d(v) - 6$ and to each face f an initial charge $\omega(f) = d(f) - 6$. Before stating discharging rules, we need to give some notation used in the rest part of this section.

If a 4-vertex v with $t(v) = 2$ is incident to a $(3, 4^*, 4^*)$-face, then we call v a *special 4-vertex*. Suppose that $f = [xyz]$ is a $(3, 4^*, 4^*)$-face such that $d(x) = 3$, $d(y) = d(z) = 4$ and $t(y) = t(z) = 2$. Let f^* denote the face adjacent to f by the common edge yz and let f', f'' denote the opposite face to f by y and z, respectively.

If both f' and f'' are $(4^*,4^*,4^*)$-faces, then we say that f^* is *heavy*. By the absence of 4- and 5-cycles, we observe that $d(f^*)\geqslant 6$. Moreover, by definition, each heavy 6^+-face is adjacent to at least five triangles. For $v\in V(G)$, we denote by $m_6^*(v)$ the number of heavy 6-faces incident to v.

Suppose that $f=[v_1v_2\cdots v_6]$ is a heavy 6-face such that $f_{v_1v_2}$ is a $(3,4^*,4^*)$-face and $f_{v_2v_3},f_{v_6v_1}$ are $(4^*,4^*,4^*)$-faces. If $t(f)=n_4(f)=6$ and $f_{v_3v_4}$ is a $(4,4,5^+)$-face, then we call f a *strong* 6-*face* of the edge v_3v_4.

Our discharging rules are defined as follows:

(R0) Every 3-vertex sends 0.5 to each of its incident 3-face.
(R1) Every 4^+-vertex sends 1 to its adjacent 2-vertex and 0.5 to each of its pendant light 3-vertex.
(R2) Let v be a 4-vertex and f_1,f_2,f_3,f_4 denote the faces of G incident to v in a cyclic order.

(R2a) Assume $t(v)=2$ such that f_1,f_3 are both 3-faces. Then

(R2a1) $\tau(v\to f_1)=1.5$ and $\tau(v\to f_3)=0.5$ if f_1 is a $(3,4^*,4^*)$-face and f_3 is not a $(4^*,4^*,4^*)$-face;

(R2a2) $\tau(v\to f_1)=\tau(v\to f_3)=1$, otherwise.

(R2b) Assume $t(v)=1$ such that f_1 is a 3-face. Then

(R2b1) $\tau(v\to f_1)=1.5$ if f_1 is either a $(3,4,4)$-face or a $(4,4,4)$-face incident to a special 4-vertex;

(R2b2) $\tau(v\to f_1)=1$, otherwise.

(R3) Let v be a 5^+-vertex incident to a 3-face $f=[vxy]$. Then

(R3a) $\tau(v\to f)=2$ if f is a $(5^+,3,3)$-face;

(R3b) $\tau(v\to f)=1.5$ if f is either a $(5^+,3,4)$-face or a $(5^+,4,4)$-face incident to a special 4-vertex;

(R3c) $\tau(v\to f)=1.25$ if f is either a $(5^+,3,5^+)$-face, or a $(5^+,4,5^+)$-face incident to a special 4-vertex, or a $(5^+,4^*,4^*)$-face such that f_{xy} is a strong 6-face of the edge xy;

(R3d) $\tau(v\to f)=1$, otherwise.

(R4) Every heavy 6^+-face sends 0.5 to each of its adjacent $(3,4^*,4^*)$-faces.
(R5) Suppose $f=[v_1v_2\cdots v_6]$ is a heavy 6-face such that $f_{v_1v_2}$ is a $(3,4^*,4^*)$-face.

(R5a) Assume $d(f_{v_4v_5})=3$. Then

(R5a1) $\tau(f_{v_3v_4}\to f)=\tau(f_{v_5v_6}\to f)=0.25$ if $d(v_4)=d(v_5)=4$;

(R5a2) $\tau(v_4\to f)=0.5$ if $d(v_4)\geqslant 5$ and $d(v_5)=4$;

(R5a3) $\tau(v_4\to f)=\tau(v_5\to f)=0.25$, otherwise.

(R5b) Assume $d(f_{v_4v_5})\neq 3$. Then

(R5b1) $\tau(v_4\to f)=0.5$ if $d(v_5)=3$ and $d(v_4)\geqslant 5$;

(R5b2) $\tau(v_4\to f)=0.5$ if $d(v_5)=4$ with $n_2(v_5)=1$ and $d(v_4)\geqslant 5$;

(R5b3) $\tau(v_4\to f)=\tau(v_5\to f)=0.25$, otherwise.

Similarly, to complete the proof of Theorem 3.3.2, it suffices to show that the new weight function satisfies $\omega^*(x)\geqslant 0$ for all $x\in V(G)\cup F(G)$. Obviously, G contains no 4- and 5-faces. We divide the proof into the following several cases:

Case 1 $d(f)=3$.

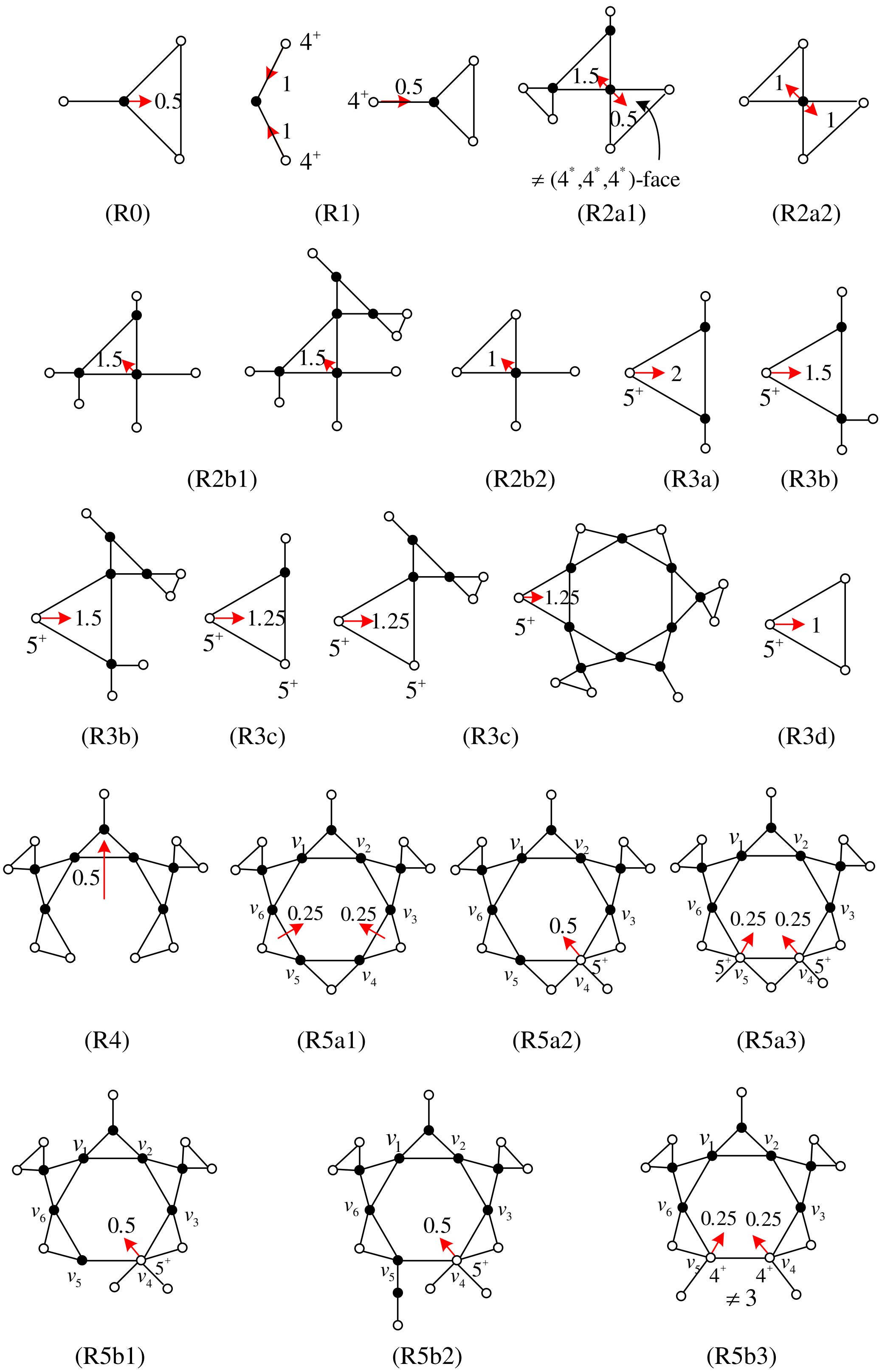


Figure 3.9: Discharging rules (R0) to (R5).

Then $\omega(f) = -3$. Let $f = [v_1v_2v_3]$ such that $d(v_1) \leqslant d(v_2) \leqslant d(v_3)$. By (C2), we derive that $d(v_1) \geqslant 3$. By the absence of 4- and 5-cycles, $d(f_{v_iv_{i+1}}) \geqslant 6$, where $i \in \{1, 2, 3\}$ and i is taken modulo 3. By (C4) and (C6), we see that f is either a $(3, 3, 5^+)$-face, or a $(3, 4^+, 4^+)$-face, or a $(4^+, 4^+, 4^+)$-face.

If f is a $(3, 3, 5^+)$-face, then $\omega^*(f) \geqslant -3 + 0.5 \times 2 + 2 = 0$ by (R0) and (R3a). If f is a $(3, 4, 5^+)$-face, i.e., $d(v_2) = 4$ and $d(v_3) \geqslant 5$, then v_2 cannot be a special 4-vertex by (C13) and thus $\omega^*(f) \geqslant -3+0.5+1+1.5 = 0$ by (R0), (R2) and (R3b). If f is a $(3, 5^+, 5^+)$-face, then $\omega^*(f) \geqslant -3 + 0.5 + 1.25 \times 2 = 0$ by (R0) and (R3c). If f is a $(4, 5^+, 5^+)$-face, then $\omega^*(f) \geqslant -3 + 0.5 + 1.25 \times 2 = 0$ by (R2a1) and (R3c) or $\omega^*(f) \geqslant -3 + 1 \times 3 = 0$ by (R2) and (R3d). If f is a $(5^+, 5^+, 5^+)$-face, then by (R3d), we conclude that $\omega^*(f) \geqslant -3 + 1 \times 3 = 0$.

Now suppose that f is a $(4, 4, 5^+)$-face. Namely $d(v_1) = d(v_2) = 4$ and $d(v_3) \geqslant 5$. If neither v_1 nor v_2 is a special 4-vertex, then $\tau(v_i \to f) \geqslant 1$ by (R2) for each $i = 1, 2$. Then $\omega^*(f) \geqslant -3 + 1 + 1 \times 2 = 0$ by (R3d) or $\omega^*(f) \geqslant -3 + 1 \times 2 + 1.25 - 0.25 = 0$ by (R3c) and (R5a1). Otherwise, by symmetry, assume that v_1 is a special 4-vertex. By the absence of (B5), v_2 cannot be a special 4-vertex. It follows from (R2) and (R3b) that $\tau(v_1 \to f) = 0.5$, $\tau(v_2 \to f) \geqslant 1$ and $\tau(v_3 \to f) = 1.5$. Thus, $\omega^*(f) \geqslant -3 + 0.5 + 1 + 1.5 = 0$.

Next suppose that f is a $(3, 4, 4)$-face. Namely $d(v_1) = 3$ and $d(v_2) = d(v_3) = 4$. Denote f', f'' be respectively, the opposite face to f by v_2 and v_3. If at least one of f' and f'' is a 6^+-face, say f', then v_2 sends 1.5 to f by (R2b1). Moreover, v_3 cannot be a special 4-vertex by (C13). Thus $\omega^*(f) \geqslant -3+0.5+1.5+1 = 0$ by (R0) and (R2). Now, assume that $d(f') = d(f'') = 3$. By definition, v_2, v_3 are special 4-vertices. It is easy to deduce that both f' and f'' are $(4^+, 4^+, 4^+)$-faces by (C13). If at least one of f', f'' is not a $(4^*, 4^*, 4^*)$-face, say f', then by (R2a1), v_2 sends 1.5 to f. Thus, $\omega^*(f) \geqslant -3 + 0.5 + 1.5 + 1 = 0$ by (R0) and (R2). So now assume that f' and f'' are both $(4^*, 4^*, 4^*)$-faces. This implies that $f_{v_2v_3}$ is a heavy 6^+-face, which sends charge 0.5 to f by (R4). Thus, $\omega^*(f) \geqslant -3 + 0.5 + 1 \times 2 + 0.5 = 0$ by (R0) and (R2a).

Finally suppose that f is a $(4, 4, 4)$-face. By (R2), for each $i \in \{1, 2, 3\}$, f gets either 0.5 or at least 1 from v_i. If $\tau(v_i \to f) \geqslant 1$ for all $i = 1, 2, 3$, then $\omega^*(f) \geqslant -3 + 1 \times 3 = 0$. Otherwise, by (R2a1), w.l.o.g., suppose that v_1 is a special 4-vertex and the opposite face to f by v_2 is of degree at least 6. Again by the absence of (B5), v_3 cannot be a special 4-vertex. Therefore, $\omega^*(f) \geqslant -3 + 0.5 + 1.5 + 1 = 0$ by (R2).

Case 2 $d(f) = 6$.

Then $\omega(f) = 0$. By (R4), only heavy 6-faces send charges to its adjacent $(3, 4^*, 4^*)$-faces. Moreover, every 6-face is adjacent to at most one $(3, 4^*, 4^*)$-face by the reducible configurations (B1) and (B5). Suppose $f = [v_1v_2 \cdots v_6]$ is a heavy 6-face such that $f_{v_1v_2}$ is a $(3, 4^*, 4^*)$-face. First assume $f_{v_4v_5}$ is a 3-face. If $d(v_4) = d(v_5) = 4$, then both $f_{v_3v_4}$ and $f_{v_5v_6}$ are $(4, 4, 5^+)$-face by the absence of (Q1). So $\omega^*(f) \geqslant 0 - 0.5 + 0.25 \times 2 = 0$ by (R5a1). Otherwise, f gets either 0.5 by (R5a2) or 0.25×2 by (R5a3) in total from v_4, v_5 and thus $\omega^*(f) \geqslant 0 - 0.5 + 0.5 = 0$. Now assume $d(f_{v_4v_5}) \neq 3$. By the absence of (Q2) and (Q3), we obtain that $\omega^*(f) \geqslant 0 - 0.5 + 0.5 = 0$ or $\omega^*(f) \geqslant 0 - 0.5 + 0.25 \times 2 = 0$ by (R5b).

Case 3 $d(f) \geqslant 7$.

Denote by $t^*(f)$ be the number of $(3, 4^*, 4^*)$-faces adjacent to f. By the absence of (B1) and (B5), we see that any two $(3, 4^*, 4^*)$-faces adjacent to f must be at distance at least 3 on the boundary of f. It follows that $t^*(f) \leqslant \lfloor \frac{d(f)}{4} \rfloor$. By (R4), $\omega^*(f) \geqslant d(f) - 6 - \frac{1}{2}t^*(f) \geqslant d(f) - 6 - \frac{1}{2} \times \frac{d(f)}{4} = \frac{7}{8}d(f) - 6 \geqslant \frac{1}{8}$.

Let $v \in V(G)$. Let $v_1, v_2, \cdots, v_{d(v)}$ denote the neighbors of v in a cyclic order. Let f_i denote the incident face of v with vv_i and vv_{i+1} as two boundary edges for $i = 1, 2, \cdots, d(v)$, where indices are taken modulo $d(v)$. By definition, we first observe the following

Observation 3.3.11 *Every heavy 6-face f is satisfying that $5 \leqslant t(f) \leqslant 6$, $n_4(f) \geqslant 4$, and $n_2(v) = 0$.*

By (C1), $d(v) \geqslant 2$. If $d(v) = 2$ then we easily obtain that $\omega^*(v) \geqslant -2+1\times 2 = 0$ by (C3) and (R1). If $d(v) = 3$, then $\omega^*(v) \geqslant 0 - 0.5 + 0.5 = 0$ by (R0), (R1) and Lemma 3.3.4. So, in what follows, we will show that $\omega^*(v) \geqslant 0$ for each 4^+-vertex v. The proof is divided into three cases according to the value of $d(v)$.

Case 4 $d(v) = 4$.

We have that $\omega(v) = 2$, $t(v) \leqslant 2$ and $n_2(v) \leqslant 1$ by (C5). We have to consider the following three subcases in light of the size of $t(v)$.

(4.1) Assume $t(v) = 2$. Clearly, $n_2(v) = p_3(v) = 0$ by (C2) and the absence of 4-cycles. By (R2a), v sends in total either $1.5 + 0.5 = 2$ or $1 + 1 = 2$ to incident 3-faces. By (R5), v sends nothing to its incident 6^+-faces. Thus, $\omega^*(v) \geqslant 2-2 = 0$.

(4.2) Assume $t(v) = 1$. Let $f_1 = [v_1vv_2]$ be a 3-face. Then $p_3(v) \leqslant 2$. If $n_2(v) = 1$, then $p_3(v) = 0$ by (C9) and f_1 is a $(4, 4^+, 4^+)$-face by (C10). By the absence of (B3), v sends at most 1 to f_1 by (R2b). Moreover, v sends nothing to f_2, f_3, f_3 by (R5) and Observation 3.3.11. Therefore $\omega^*(v) \geqslant 2-1-1 = 0$ by (R1).

Now suppose that $n_2(v) = 0$. By Observation 3.3.11, we see that only f_2 and f_4 could be heavy 6-faces. By (R5b3), we deduce that if f_2 is a heavy 6-face, then $d(v_3) \geqslant 4$. It follows that v_3 cannot be a pendant light 3-vertex. So if $m_6^*(v) = 2$ then $p_3(v) = 0$ and thus $\omega^*(v) \geqslant 2 - 1.5 - 0.25 \times 2 = 0$ by (R2). Next suppose that $m_6^*(v) = 0$. If $p_3(v) = 2$ then f_1 is a $(4^+, 4^+, 4^+)$-face by (C11). Moreover, if f_1 is a $(4, 4, 4)$-face, then neither v_1 nor v_2 is a special 4-vertex by the absence of (B4). Thus $\omega^*(v) \geqslant 2 - 1 - 0.5 \times 2 = 0$ by (R2a2). Otherwise, $\omega^*(v) \geqslant 2 - 1.5 - 0.5 = 0$ by (R2a1). Finally, w.l.o.g., suppose that $f_2 = [vv_2w_1w_2w_3v_3]$ is a heavy 6-face such that $f_{w_1w_2}$ is a $(3, 4^*, 4^*)$-face and both $f_{v_2w_1}$ and $f_{w_2w_3}$ are $(4^*, 4^*, 4^*)$-faces. So v_2 is not a special 4-vertex. If $d(v_1) = 3$, the configuration (B5) is established, which is a contradiction. If v_1 is a special 4-vertex, i.e., the opposite 3-face to f_1 is a $(3, 4^*, 4^*)$-face, then the configuration (B1) is formed, which is also a contradiction. Therefore, by (R2b2), $\tau(v \to f_1) = 1$ and we have that $\omega^*(v) \geqslant 2 - 1 - 0.25 - 0.5 = 0.25$.

(4.3) Assume $t(v) = 0$. By Observation 3.3.11, $m_6^*(v) = 0$. If $n_2(v) = 0$, then $\omega^*(v) \geqslant 2 - 4 \times 0.5 = 0$ by (R1). Otherwise, we suppose $n_2(v) = 1$, which implies that $p_3(v) = 0$ by (C9). By (R1) we conclude that $\omega^*(v) \geqslant 2 - 1 = 1$.

To well estimate the total charge sent out from a 5^+-vertex, we begin with the following claim.

Claim 3.3.1 *Suppose that v is a 5^+-vertex incident to a 3-face f_1 and a heavy 6-face f_2. Then $\tau(v \to f_1) \leqslant 1.5$. In particular, $\tau(v \to f_1) \leqslant 1$ if f_1 is a $(5^+, 4^+, 4^+)$-face and $d(f_3) \neq 3$.*

Proof. Suppose that $f_2 = [vv_2w_1w_2w_3v_3]$ is a heavy 6-face. By definition, we see that $d(v_2) \geqslant 4$ and thus $\tau(v \to f_1) \leqslant 1.5$ by (R3). Now assume that $d(f_3) \neq 3$ and f_1 is a $(5^+, 4^+, 4^+)$-face. There is only one possible case that $f_{w_1w_2}$ is a $(3, 4^*, 4^*)$-face and $f_{v_2w_1}$ and $f_{w_2w_3}$ are both $(4^*, 4^*, 4^*)$-faces. We notice that v_2 is a 4-vertex but not special. If v_1 is a special 4-vertex, i.e., the opposite face to f_1 by v_1 is a $(3, 4^*, 4^*)$-face, then the configuration (B1) is established, which is a contradiction. So, in order to prove $\tau(v \to f_1) \leqslant 1$, by (R3c), we only need show that $f_{v_1v_2}$ (the face adjacent to f_1 by the common edge v_1v_2) is not a strong 6-face of the edge v_1v_2. Let $f_{v_1v_2} = [v_1u_1u_2u_3u_4v_2]$ be a 6-face adjacent to six 3-faces. By definition, either $f_{u_3u_4}$ or $f_{u_1u_2}$ is a $(3, 4^*, 4^*)$-face. If $f_{u_3u_4}$ is a $(3, 4^*, 4^*)$-face, then the configuration (B5) is formed. If $f_{u_1u_2}$ is a $(3, 4^*, 4^*)$-face, then the configuration (Q4) is constructed. We always obtain a contradiction. ∎

Case 5 $d(v) = 5$.

The initial charge is $\omega(v) = 4$. According to (C7), $n_2(v) \leqslant 3$. Moreover, $t(v) \leqslant 2$ by the absence of 4-cycles. We need to handle the following three cases, depending on the value of $t(v)$.

(5.1) Assume $t(v) = 2$. W.l.o.g., let $f_1 = [vv_1v_2]$ and $f_3 = [vv_3v_4]$ be two 3-faces. It follows from (C2) that v_i is neither a 2-vertex nor a pendant light 3-vertex of v for each $i \in \{1, 2, 3, 4\}$. So $n_2(v) + p_3(v) \leqslant 1$. By (R3), $\tau(v \to f_i) \leqslant 2$ for each $i = 1, 2$.

First suppose that at least one of f_1, f_3 taking charge 2 from v. W.l.o.g., suppose f_1 is a $(5, 3, 3)$-face by (R3a). It follows from (A1) that $n_2(v) = 0$. Moreover, by (R5), one can easily check that v sends nothing to f_2 and f_5 since $d(v_1) = d(v_2) = 3$. So $m_6^*(v) \leqslant 1$. If f_3 gets a charge 2 from v, then f_3 is a $(5, 3, 3)$-face by (R3a). Moreover, $p_3(v) = 0$ by Lemma 3.3.6. Similarly, v sends nothing to f_4 since $d(v_4) = 3$. Thus, $\omega^*(v) \geqslant 4 - 2 \times 2 = 0$. Now assume that $\tau(v \to f_3) \leqslant 1.5$. If $p_3(v) + m_6^*(v) \leqslant 1$, then $\omega^*(v) \geqslant 4 - 2 - 1.5 - 0.5 = 0$. Otherwise, assume that v_5 is a pendant light 3-vertex and f_4 is a heavy 6-face. In light of Lemma 3.3.6, f_3 is a $(5, 4^+, 4^+)$-face. It follows immediately from Claim 3.3.1 that $\tau(v \to f_3) \leqslant 1$. Therefore, $\omega^*(v) \geqslant 4 - 2 - 1 - 0.5 - 0.5 = 0$ by (R1).

Next suppose that $\tau(v \to f_i) \leqslant 1.5$ for each $i = 1, 3$. We first assume that $n_2(v) = 1$. By Observation 3.3.11, we see that neither f_4 nor f_5 is a heavy 6-face. If f_2 is not a heavy 6-face, then we are done since $\omega^*(v) \geqslant 4 - 1.5 \times 2 - 1 = 0$. Otherwise, suppose $f_2 = [vv_2u_1u_2u_3v_3]$ is a heavy 6-face such that $f_{u_1u_2}$ is a $(3, 4^*, 4^*)$-face and $f_{v_2u_1}$ and $f_{u_2u_3}$ are both $(4^*, 4^*, 4^*)$-faces. If $d(v_1) = 3$, then the configuration (B2) is established, which is a contradiction. So $d(v_1) \geqslant 4$. By a similar argument as Claim 3.3.1, we obtain that $\tau(v \to f_1) \leqslant 1$ and thus $\omega^*(v) \geqslant 4 - 1.5 - 1 - 0.5 - 1 = 0$. So in the following, we assume that $n_2(v) = 0$ and v_5 is a pendant light 3-vertex. By the absence of (B5), we observe that at most one of f_4, f_5 can be a heavy 6-face. It means that $m_6^*(v) \leqslant 2$. If $m_6^*(v) = 2$, say f_2 and f_5, then $\tau(v \to f_1) \leqslant 1$ by Claim

3.3.1. Hence $\omega^*(v) \geqslant 4 - 1.5 - 1 - 0.5 - 0.5 \times 2 = 0$. Otherwise, we conclude that $\omega^*(v) \geqslant 4 - 1.5 \times 2 - 0.5 - 0.5 = 0$.

(5.2) Assume $t(v) = 1$. W.l.o.g., let $d(f_1) = 3$. By Observation 3.3.11, we see that only f_2 and f_5 can be a heavy 6-face and thus $m_6^*(v) \leqslant 2$. Moreover, $n_2(v) \leqslant 2$ by (C8). We have three cases, depending on $n_2(v)$.

(a) $n_2(v) = 2$. Then $m_6^*(v) \leqslant 1$ since at least one of v_3, v_5 is a 2-vertex. If f_1 is a $(3, 3, 5)$-face, then $p_3(v) = 0$ by (C12) and $m_6^*(v) = 0$. Thus $\omega^*(v) \geqslant 4 - 2 - 1 \times 2 = 0$ by (R1) and (R3). Otherwise, f_1 gets at most 1.5 from v. If $m_6^*(v) + p_3(v) \leqslant 1$, then $\omega^*(v) \geqslant 4 - 1.5 - 1 \times 2 - 0.5 = 0$. Now assume that $p_3(v) = m_6^*(v) = 1$. By (C12) again, $d(v_1), d(v_2) \geqslant 4$. By Claim 3.3.1, $\tau(v \to f_1) \leqslant 1$. Therefore, $\omega^*(v) \geqslant 4 - 1 - 1 \times 2 - 0.5 \times 2 = 0$.

(b) $n_2(v) = 1$. Then $p_3(v) \leqslant 2$ and $m_6^*(v) \leqslant 2$. If $m_6^*(v) + p_3(v) \leqslant 2$, then $\omega^*(v) \geqslant 4 - 2 - 1 - 0.5 \times 2 = 0$. If $m_6^*(v) + p_3(v) = 3$, then f_1 cannot be a $(5, 3, 3)$-face. Thus $\omega^*(v) \geqslant 4 - 1.5 - 1 - 0.5 \times 3 = 0$. So assume that $m_6^*(v) = p_3(v) = 2$. Similarly, by Claim 3.3.1, we affirm that f_1 gets at most 1 from v and therefore $\omega^*(v) \geqslant 4 - 1 - 1 - 0.5 \times 4 = 0$.

(c) $n_2(v) = 0$. Then $p_3(v) \leqslant 3$ and $m_6^*(v) \leqslant 2$. If f_1 is a $(3, 3, 5)$-face, then $m_6^*(v) = 0$ and $\omega^*(v) \geqslant 4 - 2 - 0.5 \times 3 = 0.5$. Otherwise, $\tau(v \to f_1) \leqslant 1.5$ and hence $\omega^*(v) \geqslant 4 - 1.5 - 0.5 \times 5 = 0$.

(5.3) Assume $t(v) = 0$. By (C7), $n_2(v) \leqslant 3$. Clearly, $m_6^*(v) = 0$. Thus, $\omega^*(v) \geqslant 4 - 3 \times 1 - 0.5 \times 2 = 0$.

Case 6 $d(v) \geqslant 6$.

By (R0)-(R5), we notice that the faces getting charge from v are only 3-faces and heavy 6-faces. Suppose f_2 is a heavy 6-face. By (R5), $\tau(v \to f_2) \leqslant 0.5$. By definition, at least one of f_1 and f_3 is a 3-face. If $d(f_1) = d(f_3) = 3$, then we may consider this charge 0.5 (from v to f_2) to be firstly given on average to f_1 and f_3 and then transferred from f_1, f_3 to f_2, respectively. If f_1 is a 3-face and f_3 is not, then we may consider this charge 0.5 (from v to f_2) to be directly given to f_1 and then transferred to f_2. By Claim 3.3.1, it is easy to deduce that v sends a charge at most 2 to each incident 3-faces and nothing to heavy 6-faces. Therefore, $\omega^*(v) \geqslant 2d(v) - 6 - 2t(v) - n_2(v) - 0.5p_3(v) \geqslant 2d(v) - 6 - 2t(v) - (n_2(v) + p_3(v)) \geqslant 2d(v) - 6 - 2t(v) - (d(v) - 2t(v)) = d(v) - 6 \geqslant 0$.

Therefore, we complete the proof of Theorem 3.3.2. ∎

3.4 Acyclic 3-choosability

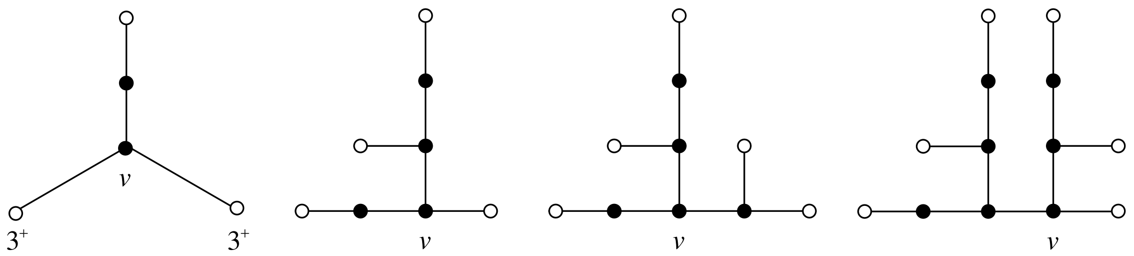
In this section, we prove the following theorem:

Theorem 3.4.1 *Every planar graph with girth 7 is acyclically 3-choosable.*

This is a common strengthening of the facts that such a graph is acyclically 3-colorable (Borodin, Kostochka and Woodall [BKW99]) and that a planar graph with girth 8 is acyclically 3-choosable (Montassier, Ochem and Raspaud [MOR06]). More generally, we prove the following theorem:

Theorem 3.4.2 *Every graph G with* $\mathrm{Mad}(G) < \frac{14}{5}$ and $g(G) \geqslant 7$ is acyclically 3-choosable.

We remark that this work is jointly done with Borodin, Ivanova and Raspaud. It has been published in Discrete Mathematics [BCIR10]. The organizing of this section is as follows: In Section 3.4.1, we will give some useful preliminaries of G. Then, in Section 3.4.2, we will show some reducible configurations. Finally, we use the Discharging argument to derive a contradiction in Section 3.4.3 and thus complete the proof of Theorem 3.4.2. We begin with some notation.



(1) A minor vertex *v*. (2) An ugly 3-vertex *v*. (3) A special ugly 3-vertex *v*. (4) A heavy 3-vertex *v*.

Figure 3.10: Four definitions of a 3-vertex v.

Let G be a plane graph. A 3-vertex v is called *minor* if $n_2(v) = 1$, see Figure 3.10 (1). By definition, we see that each minor vertex is also a $(1^+, 0, 0)$-vertex. A minor vertex v is called *ugly* if v is adjacent to a minor vertex, see Figure 3.10 (2). A *special ugly 3-vertex* is an ugly 3-vertex that is not adjacent to any 4^+-vertex, see Figure 3.10 (3). We call a 3-vertex v *heavy* if $n_2(v) = 0$ and v is adjacent to one minor vertex and one special ugly 3-vertex, see Figure 3.10 (4).

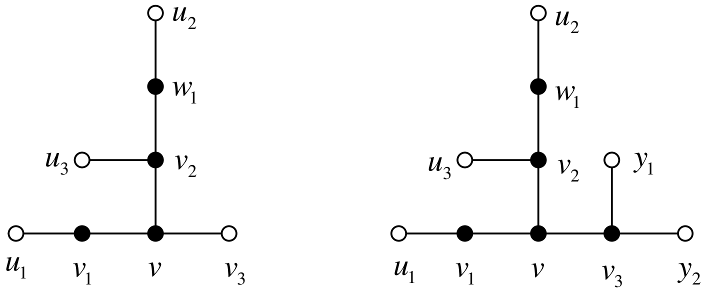
3.4.1 Preliminaries

Suppose to the contrary that Theorem 3.4.2 is false. Let G be a counterexample to Theorem 3.4.2 with the least number of vertices. Thus, G is connected. In what follows, let L be a list assignment of G with $|L(v)| = 3$ for all $v \in V(G)$.

Lemma 3.4.3 *Suppose v is an ugly 3-vertex as depicted in Figure 3.11 (A). Let π be an acyclic L-coloring of $G - v$ and $L(v) = \{1, 2, 3\}$. Then, $\pi(u_1) = \pi(u_3) = i$ and $\pi(v_3) = \pi(u_2) = j$, where $\{i, j\} \subseteq \{1, 2, 3\}$.*

Proof. The proof is divided into the following three cases, depending on the colors of v_1, v_2 and v_3.

Case 1 Assume $\pi(v_1)$, $\pi(v_2)$ and $\pi(v_3)$ are mutually distinct.



(A) An ugly 3-vertex v. (B) A special ugly 3-vertex v.

Figure 3.11: The configurations (A) and (B) in Lemmas 3.4.3-3.4.4.

If there exists a color c belonging to $L(v) \setminus \{\pi(v_1), \pi(v_2), \pi(v_3)\}$, it is easy to extend π to G by coloring v with c. Now, w.l.o.g., suppose that $\pi(v_1) = 1$, $\pi(v_2) = 2$ and $\pi(v_3) = 3$. We only need to recolor v_1 with a color in $L(v_1) \setminus \{1, \pi(u_1)\}$ and then color v with 1.

Case 2 Assume exactly two vertices of v_1, v_2, v_3 are colored with the same color.

(2.1) $\pi(v_1) = \pi(v_2) \neq \pi(v_3)$. If there exists a color c in $L(v) \setminus \{\pi(v_1), \pi(v_3), \pi(u_1)\}$, then we assign v with c properly. Otherwise, w.l.o.g., assume that $\pi(v_1) = \pi(v_2) = 1$, $\pi(v_3) = 2$ and $\pi(u_1) = 3$. If $L(v_1) \neq \{1, 2, 3\}$, we recolor v_1 with a color in $L(v_1) \setminus \{1, 2, 3\}$ and then go back to the previous Case 1. So in the following, we suppose $L(v_1) = \{1, 2, 3\}$. If v cannot be colored with 3, we affirm that there is an alternating $(1, 3)$-path starting from v_1 and ending at v_2. We have two possibilities below:

- $\pi(u_3) = 3$. Namely, $\pi(u_3) = \pi(u_1)$. If $\pi(u_2) = \pi(v_3) = 2$, then we are done. Otherwise, we first erase the color of w_1. Then recolor v_1 with 2, color v with 1, and recolor v_2 with a color c different from $1, 3$. If $\pi(u_2) \neq c$, it suffices to further color w_1 with a color in $L(w_1) \setminus \{c, \pi(u_2)\}$. Otherwise, we may color w_1 with a color distinct to c and 3.
- $\pi(u_3) \neq 3$. It implies that $\pi(w_1) = 3$ and $\pi(u_2) = 1$. Erase the colors of v_2 and w_1. We first recolor v_1 with 2, color v with 1, and then color v_2 with c^* in the following way: If $\pi(u_2) = 1$, set $c^* \in L(v_2) \setminus \{1, 2\}$; otherwise, set $c^* \in L(v_2) \setminus \{1, \pi(u_3)\}$. Finally, color w_1 with a color distinct to c^* and 1.

(2.2) $\pi(v_1) = \pi(v_3) \neq \pi(v_2)$. Similarly, if there exists a color c in $L(v) \setminus \{\pi(v_1), \pi(v_2), \pi(u_1)\}$, then we assign v with c properly. Otherwise, w.l.o.g., assume that $\pi(v_1) = \pi(v_3) = 1$, $\pi(v_2) = 2$ and $\pi(u_1) = 3$. We may first recolor v_1 with a color $c \in L(v_1) \setminus \{1, 3\}$ and then go back to the previous Case 2.1 or Case 1.

(2.3) $\pi(v_2) = \pi(v_3) \neq \pi(v_1)$. First assume $\pi(v_2) \notin \{1, 2, 3\}$. We first color v with a color c different from $\pi(w_1)$ and $\pi(u_3)$. If $c \neq \pi(v_1)$, then the resulting coloring is obviously acyclic, which is a contradiction. Otherwise, we only need to further recolor v_1 with a color distinct to c and $\pi(u_1)$. So, in what follows, w.l.o.g., assume $\pi(v_2) = \pi(v_3) = 1$. If $\pi(u_2) = 1$, we can first recolor v_2 with a color c distinct to 1 and $\pi(u_3)$, then recolor w_1 with a color different from $1, c$, and then go back to the previous Case 1 or Case 2.1 depending on the color c. Now we may suppose that $\pi(u_2) \neq 1$. Firstly, color v with a color c' different from $1, \pi(u_3)$. Similarly, if $c' \neq \pi(v_1)$, then the resulting coloring is obviously acyclic, which is a contradiction. Otherwise, we only need to further recolor v_1 with a color distinct to c' and $\pi(u_1)$.

Case 3 Assume that $\pi(v_1) = \pi(v_2) = \pi(v_3)$.

It is easy to recolor v_1 with a color different from $\pi(v_1)$ and $\pi(u_1)$ and thus we go back to the former Case 2.

In each possible case, we are always able to extend π to G, which contradicts the assumption of G. Therefore, we complete the proof of Lemma 3.4.3. ∎

Lemma 3.4.4 *Suppose v is a special ugly 3-vertex as depicted in Figure 3.11 (B). Let π be an acyclic L-coloring of $G - v$ and $L(v) = \{1, 2, 3\}$. Then*

($P1$) $\{\pi(y_1), \pi(y_2), \pi(v_3)\} = \{1, 2, 3\}$;

($P2$) $L(v_3) = \{1, 2, 3\}$;

($P3$) *There exist an alternating $(\pi(v_3), \pi(y_1))$-path $v_3y_1\cdots$ and an alternating $(\pi(v_3), \pi(y_2))$-path $v_3y_2\cdots$ in $G - v$.*

Proof. By Lemma 3.4.4, w.l.o.g., we assume that $\pi(u_1) = \pi(u_3) = 1$ and $\pi(v_3) = \pi(u_2) = 3$. If $L(v_1) \neq \{1, 2, 3\}$, we first recolor v_1 with $a \in L(v_1) \setminus \{1, 2, 3\}$, v_2 with $b \in L(v_2) \setminus \{1, 3\}$, w_1 with a color distinct to $1, 3$, and then color v like this: If $b \neq a$, color v with 1; otherwise, color v with 2. So, in the following, we suppose that $L(v_1) = \{1, 2, 3\}$. To show (P1) to (P3), we will make use of contradictions.

(P1) Assume to the contrary that $\{\pi(y_1), \pi(y_2)\} \neq \{1, 2\}$. If neither y_1 nor y_2 is colored with 1, then we give the color 1 to v, a color different from $1, \pi(v_2)$ to v_1 and thus an acyclic L-coloring is obtained. Now suppose that neither y_1 nor y_2 is colored with 2. We first erase the color of w_1. Then color v with 2, recolor v_1 with 3, recolor v_2 with a color $c^* \in L(v_2) \setminus \{1, 2\}$, and finally color w_1 with α in the following way: If $c^* = 3$, set $\alpha \in L(w_1) \setminus \{1, 3\}$; otherwise, set $\alpha \in L(w_1) \setminus \{3, c^*\}$.

(P2) Assume to the contrary that $L(v_3) \neq \{1, 2, 3\}$. By (P1), $\{\pi(y_1), \pi(y_2)\} = \{1, 2\}$, since $\pi(v_3) = 3$. We may give a new color belonging to $L(v_3) \setminus \{1, 2, 3\}$ to v_3 to obtain an acyclic L-coloring of $G - v$, which contradicts Lemma 3.4.3.

(P3) By symmetry, assume $\pi(y_1) = 1$ and $\pi(y_2) = 2$ by (P2). If none of paths $v_3y_1\cdots$ is an alternating $(3, 1)$-path, then we can extend π to G by coloring v with 1 and recoloring v_1 with a color different from $1, \pi(v_2)$. If none of paths $v_3y_2\cdots$ is an alternating $(3, 2)$-path, we first erase the color of w_1. Then, color v with 2, recolor v_1 with 3, recolor v_2 with $c^* \in L(v_2) \setminus \{1, 2\}$, and finally color w_1 with α in the following way: If $c^* = 3$, set $\alpha \in L(w_1) \setminus \{1, 3\}$; or else, set $\alpha \in L(w_1) \setminus \{3, c^*\}$. We always obtain a contradiction to the assumption of G. Therefore, (P3) holds. ∎

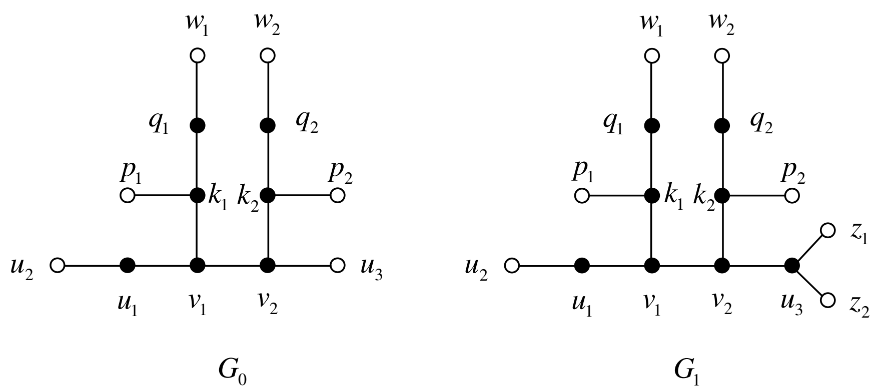


Figure 3.12: Two subgraphs G_0 and G_1 mentioned in Lemmas 3.4.5-3.4.6.

Lemma 3.4.5 *Suppose G contains the subgraph G_0 depicted in Figure 3.12. Let π be an acyclic L-coloring of $G - v_1$ and $L(v_1) = \{1, 2, 3\}$. Then $\pi(p_2) = \pi(v_2) = \pi(w_1)$.*

Proof. By Lemma 3.4.3, w.l.o.g., we assume that $\pi(u_2) = \pi(p_1) = 1$ and $\pi(v_2) = \pi(w_1) = 3$. Moreover, $L(v_2) = \{1, 2, 3\}$ and $\{\pi(k_2), \pi(u_3)\} = \{1, 2\}$ by (P1)-(P2) of Lemma 3.4.4.

Assume to the contrary that $\pi(p_2) \neq \pi(v_2)$, namely, $\pi(p_2) \neq 3$. For our convenience, denote $\pi(k_2) = a$ and $\pi(u_3) = b$. Recall that $L(v_2) = \{3, a, b\}$. Since $\pi(p_2) \neq 3$, we affirm that $\pi(q_2) = 3$ and $\pi(w_2) = a$ by (P3) of Lemma 3.4.4. Erase the color of k_2. We first recolor v_2 by a, then color k_2 with $c^* \in L(k_2) \setminus \{a, \pi(p_2)\}$, and finally recolor q_2 with a color different from a and c^*. It is easy to check that the resulting coloring of $G - v_1$ is still acyclic. However, now the colors of w_1 and v_2 are different. This contradicts Lemma 3.4.3 and thus we complete the proof of Lemma 3.4.5. ∎

Lemma 3.4.6 *Suppose G contains a subgraph G_1 depicted in Figure 3.12. Let π be an acyclic L-coloring of $G - v_1$ and $L(v_1) = \{1, 2, 3\}$. Then*

(i) $\pi(w_2) = \pi(u_3)$;

(ii) $\{\pi(u_3), \pi(z_1), \pi(z_2)\} = \{1, 2, 3\}$;

(iii) $L(u_3) = \{1, 2, 3\}$.

Proof. By Lemma 3.4.3, w.l.o.g., we assume that $\pi(u_2) = \pi(p_1) = 1$ and $\pi(v_2) = \pi(w_1) = 3$. It implies that $\pi(p_2) = 3$ by Lemma 3.4.5. Moreover, by (P1)-(P2) of Lemma 3.4.4, we have that $L(v_2) = \{1, 2, 3\}$ and $\{\pi(k_2), \pi(u_3)\} = \{1, 2\}$. By symmetry, let $\pi(k_2) = 1$ and $\pi(u_3) = 2$.

(i) Assume to the contrary that $\pi(w_2) \neq \pi(u_3)$, namely $\pi(w_2) \neq 2$. Erase the colors of k_2 and q_2. We first recolor v_2 with 1, color k_2 with $\alpha \in L(k_2) \setminus \{1, 3\}$, and then do as follows: If $\alpha = 2$, we further color q_2 with a color different from 2 and $\pi(w_2)$; If $\alpha = \pi(w_2)$, we further choose a color in $L(q_2) \setminus \{\alpha, 3\}$ for q_2; if $\alpha \notin \{2, \pi(w_2)\}$, we further color q_2 with a color distinct to α and $\pi(w_2)$. In each case, one can easily check that the resulting coloring of $G - v_1$ is still acyclic. However, the (new) colors of w_1 and v_2 are different. This contradicts Lemma 3.4.3.

(ii) Obviously, at least one of z_1, z_2 is colored with 3 by (P3) of Lemma 3.4.4. By symmetry, assume $\pi(z_1) = 3$. In order to show (ii), we only need to show that $\pi(z_2) = 1$. Now we suppose to the contrary that $\pi(z_2) \neq 1$. We may first recolor v_2 with 1, k_2 with $\alpha \in L(k_2) \setminus \{1, 3\}$, and further recolor q_2 with β in the following way: If $\alpha = 2$, set $\gamma \in L(q_2) \setminus \{2, 3\}$, since $\pi(w_2) = \pi(u_3) = 2$ by (i); otherwise, set $\gamma \in L(q_2) \setminus \{2, \alpha\}$. It is easy to verify that in each case the resulting coloring of $G - v_1$ is acyclic. But the (new) colors of w_1 and v_2 are different. This contradicts Lemma 3.4.3.

(iii) If $L(u_3) \neq \{1, 2, 3\}$, we may recolor u_3 with a color in $L(u_3) \setminus \{1, 2, 3\}$ and thus obtain a contradiction to (P1) of Lemma 3.4.4. ∎

3.4.2 Reducible configurations

In this section, we show several configurations which cannot exist in G.

Claim 3.4.1 (F1) *There is no* 1*-vertex.*
(F2) *There is no i-vertex with $2 \leqslant i \leqslant 4$ adjacent to $i - 1$ 2-vertices.*
(F3) *A* 3*-vertex is not adjacent to one* 2*-vertex and two minor vertices.*
(F4) *A* 3*-vertex is not adjacent to three minor vertices.*

Proof. In each of following cases, we will show how to derive an acyclic L-coloring of G, which contradicts the choice of G.

(F1) Obvious.

(F2) Suppose to the contrary that G contains an i-vertex v adjacent to $i - 1$ 2-vertices $v_1, v_2, \cdots, v_{i-1}$ such that $d(v_1) = d(v_2) = \cdots = d(v_{i-1}) = 2$. For each $j \in \{1, \cdots, i-1\}$, let v'_j denote the other neighbor of v_j different from v. Clearly, $G - \{v, v_1, \cdots, v_{i-1}\}$ admits an acyclic L-coloring π by the minimality of G. It is easy to deduce that there is a color c belonging to $L(v) \setminus \pi(v_i)$ appeared at most once on the set $\{v'_1, \cdots, v'_{i-1}\}$. By symmetry, assume $\pi(v'_1) = c$. We first color v with c, then color v_1 with a color different from c and $\pi(v_4)$, and finally color v_j with a color different from its neighbors for each $j = 2, \cdots, i-1$.

(F3) Suppose to the contrary that G contains a 3-vertex v adjacent to a 2-vertex v_1, and two minor vertices v_2 and v_3, depicted in Figure 3.13. Obviously, $G - v$ has an acyclic L-coloring π by the minimality of G. Let $L(v) = \{1, 2, 3\}$. By Lemma 3.4.3, w.l.o.g., suppose that $\pi(u_1) = \pi(k_1) = 1$ and $\pi(v_3) = \pi(u_2) = 3$. According to (P1)-(P2) of Lemma 3.4.4, we have that $L(v_3) = \{1, 2, 3\}$ and $\{\pi(k_2), \pi(w_2)\} = \{1, 2\}$. Moreover, we definitely assert that $\pi(u_3) = 3$ by (P3) of Lemma 3.4.4. Now we first recolor v_3 with $\pi(w_2)$, w_2 with a color different from $3, \pi(w_2)$, and then obtain a contradiction to Lemma 3.4.3, since the (new) colors of u_2 and v_3 are distinct.

(F4) Suppose to the contrary that G contains a 3-vertex v adjacent to three minor vertices as depicted in Figure 3.13. Obviously, $G - w_1$ has an acyclic L-coloring π by the minimality of G. Let $L(w_1) = \{1, 2, 3\}$. If $\pi(u_1) \neq \pi(v_1)$, it is

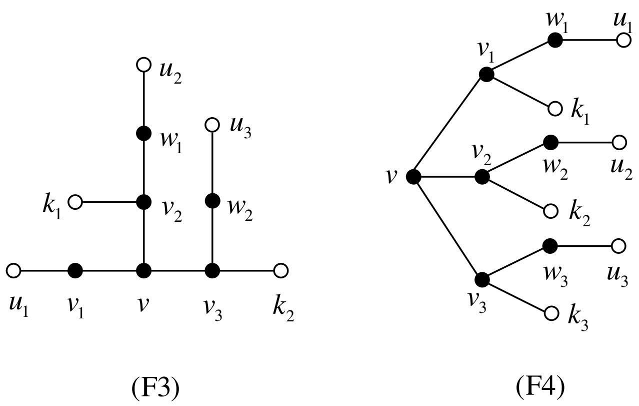


Figure 3.13: Two reducible configurations (F3) and (F4) in Claim 3.4.1.

easy to color w_1. Now, we suppose that $\pi(u_1) = \pi(v_1)$. If w_1 cannot be colored properly, w.l.o.g., assume that $\pi(u_1) = \pi(v_1) = 1$, $\pi(k_1) = 2$, $\pi(v) = 3$ and $1 \in \{\pi(v_2), \pi(v_3)\}$. If $L(v_1) \neq L(w_1)$, namely, $L(v_1) \neq \{1,2,3\}$, then recolor v_1 with a color in $L(v_1) \neq \{1,2,3\}$ and then go back to the previous case. In what follows, we suppose $L(v_1) = \{1,2,3\}$. The following proof is divided into two case, depending on the colors of v_2 and v_3.

Case 1 $\pi(v_2) = \pi(v_3) = 1$.

If none of w_2, k_2, w_3, k_3 is colored with 3, then it is easy to assign 3 to w_1. Otherwise, by symmetry, we have to consider the following two possibilities.

(1.1) Assume that $3 \notin \{\pi(k_2), \pi(k_3)\}$. Then, w.l.o.g., we may suppose that $\pi(w_2) = 3$ and $\pi(u_2) = 1$.

- $\pi(k_2) \neq 2$. We recolor v with $c \in L(v) \setminus \{1,3\}$, v_1 with 3, and color w_1 with 2. If the obtained coloring is not acyclic, we deduce that $\pi(k_2) = c$ and $c \in \{\pi(w_3), \pi(k_3)\}$. So only we need to further recolor v_2 with $c^* \in L(v_2) \setminus \{c, 1\}$ and w_2 with a color distinct to c^* and 1.
- $\pi(k_2) = 2$. If $L(v) \neq \{1,2,3\}$, then recolor v with $c \in L(v) \setminus \{1,2,3\}$ and color w_1 with 3 successfully. Now suppose $L(v) = \{1,2,3\}$. We first recolor v with 2, v_1 with 3 and then color w_1 by 2. If the resulting coloring is not acyclic, we deduce that $2 \in \{\pi(w_3), \pi(k_3)\}$. If $L(v_2) \neq \{1,2,3\}$, it is easy to obtain an acyclic L-coloring of G by further recoloring v_2 with a color not in $\{1,2,3\}$. Now suppose $L(v_2) = \{1,2,3\}$. We can first reassign v_2 with 3, w_2 with a color different from 1 and 3, v with 1, v_3 with a color $a \in L(v_3) \setminus \{1, \pi(k_3)\}$, and then reassign w_3 with γ in the following way: If $\pi(u_3) = a$, set $\gamma \in L(w_3) \setminus \{a, \pi(k_3)\}$; otherwise, set $\gamma \in L(w_3) \setminus \{a, \pi(u_3)\}$.

(1.2) Assume that $3 \in \{\pi(k_2), \pi(k_3)\}$. W.l.o.g., suppose $\pi(k_2) = 3$. We first recolor v with $c \in L(v) \setminus \{1,3\}$, v_1 with 3, and then color w_1 with 2. If such coloring is not acyclic, we derive that $\pi(w_2) = c$, $\pi(u_2) = 1$ and $c \in \{\pi(w_3), \pi(k_3)\}$. If

$L(v_2) \neq \{1,3,c\}$, we further recolor v_2 with a color in $L(v_2) \setminus \{1,3,c\}$ to obtain an acyclic L-coloring of G. If $L(v) \neq \{1,3,c\}$, we also can further recolor v with a color in $L(v) \setminus \{1,3,c\}$ to obtain an acyclic L-coloring of G. These contradictions mean that $L(v) = L(v_2) = \{1,3,c\}$.

- $\pi(k_3) \neq c$. Then $\pi(w_3) = c$ and $\pi(u_3) = 1$. So we can continue to recolor v_2 with c, w_2 with a color different from $1, c$, v with 1, v_3 with c' different from $1, \pi(k_3)$, and finally recolor w_3 with a color distinct to c' and 1. By a careful inspection, one can deduce that the resulting coloring is acyclic, which is a contradiction.
- $\pi(k_3) = c$. We continue to recolor v_2 with c, w_2 with a color different from $1, c$, v with 1, v_3 with a color c^* belonging to $L(v_3) \setminus \{1, c\}$, and further recolor w_3 with γ as follows: If $\pi(u_3) = c^*$, set $\gamma \in L(u_3) \setminus \{c^*, c\}$; otherwise, set $\gamma \in L(u_3) \setminus \{c^*, \pi(u_3)\}$.

Case 2 $\pi(v_2) = 1$ and $\pi(v_3) \neq 1$.

If there exits a color $c \in L(v) \setminus \{1, 3, \pi(v_3)\}$, recolor v with c, v_1 with 3 and then color w_1 with 2 successfully. Now suppose $L(v) = \{1, 3, \pi(v_3)\}$. If $3 \notin \{\pi(w_2), \pi(k_2)\}$, then it is easy to color w_1 with 3 to extend π to G successfully. Otherwise, we need to deal with the following two subcases.

(2.1) $\pi(k_2) \neq 3$. It follows immediately that $\pi(w_2) = 3$ and $\pi(u_2) = 1$. Thus, we can first recolor v_2 with $c \in L(v_2) \setminus \{1, \pi(k_2)\}$, w_2 with a color different from $1, c$, and then recolor v with 1, v_1 with 3, and finally color w_1 with 2 successfully.

(2.2) $\pi(k_2) = 3$. If $\pi(w_2) = 3$, then reduce to the previous case. So assume that $\pi(w_2) \neq 3$. We first recolor v_2 with $c^* \in L(v_2) \setminus \{1, 3\}$, v with 1, v_1 to 3, then color w_1 with 2, and finally recolor w_2 in this way: If $\pi(u_2) \neq c^*$, recolor w_2 with a color distinct to $\pi(u_2), c^*$; otherwise, keep the color of w_2 as before. ∎

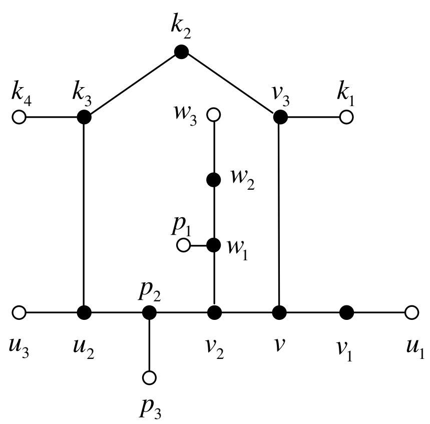


Figure 3.14: The subgraph G_2 in Claim 3.4.2.

Claim 3.4.2 *G does not contain the subgraph G_2 as shown in Figure 3.14.*

Proof. Suppose to the contrary that G contains such a subgraph G_2. By definition, we observe that v is an special ugly 3-vertex. Obviously, $G - v$ admits an acyclic L-coloring π by the choice of G. Let $L(v) = \{1, 2, 3\}$. By Lemma 3.4.3, w.l.o.g., assume that $\pi(u_1) = \pi(k_1) = 1$ and $\pi(v_2) = \pi(k_3) = 3$. According to (P1)-(P2) of Lemma 3.4.4, we have that $L(v_2) = \{1, 2, 3\}$ and $\{\pi(w_1), \pi(p_2)\} = \{1, 2\}$.

Case 1 Assume $\pi(w_1) = 1$ and $\pi(p_2) = 2$.

By Lemma 3.4.5 and (i) of Lemma 3.4.6, we have that $\pi(p_1) = \pi(v_2) = 3$ and $\pi(w_3) = \pi(p_2) = 2$. Moreover, by (ii)-(iii) of Lemma 3.4.6, we know that $L(p_2) = \{1, 2, 3\}$ and $\{\pi(u_2), \pi(p_3)\} = \{1, 3\}$. More specifically, $\pi(u_2) = 1$ and $\pi(p_3) = 3$, since $\pi(k_3) = 3$. The following proof is divided into two cases:

(1.1) $\pi(u_3) \neq 2$. We first recolor v_2 with 1, w_1 with a color c different from $1, 3$, and then recolor w_2 in this way: If $c \neq 2$, recolor w_2 easily; otherwise, recolor w_2 with a color different from 2 and 3. It is easy to verify that the obtained coloring of $G - v$ is acyclic and thus a contradiction to Lemma 3.4.3 is produced, since $\pi(v_2) = 1 \neq 3 = \pi(k_3)$.

(1.2) $\pi(u_3) = 2$. We first color v with 2, recolor v_1 with a color different from 1, 2, and v_3 with a color c different from 1, 2. If $c \neq 3$, we continue to color k_2 with a color different from 3 and c. By a careful inspection, the resulting coloring is acyclic. If $c = 3$, we continue to recolor k_2 with a color c' distinct to $1, 3$. We note that if the resulting coloring is not acyclic, then $c' = 2$ and $\pi(k_4) = 2$ such that one of paths $\cdots p_3 p_2 v_2 v v_3 k_2 k_3 k_4 \cdots$ is an alternating $(3, 2)$-path. So, we need to destroy such danger path by recoloring p_2 with 1, u_2 with $\alpha \in L(u_2) \setminus \{1, 2\}$, k_3 with a color different from $2, 3$ if $\alpha = 3$, then recolor w_1 with a color $\beta \in L(w_1) \setminus \{1, 3\}$, and finally recolor w_2 with γ in the following way: If $\beta = 2$, set $\gamma \in L(w_2) \setminus \{2, 3\}$; otherwise, set $\gamma \in L(w_2) \setminus \{2, \beta\}$.

Case 2 Assume $\pi(w_1) = 2$ and $\pi(p_2) = 1$.

Though the argument is very similar to the above Case 1, we like to write, for completeness, its details.

By Lemma 3.4.5 and (i) of Lemma 3.4.6, we have that $\pi(p_1) = \pi(v_2) = 3$ and $\pi(w_3) = \pi(p_2) = 1$. Moreover, by (ii)-(iii) of Lemma 3.4.6, we know that $L(p_2) = \{1, 2, 3\}$ and $\{\pi(u_2), \pi(p_3)\} = \{2, 3\}$. More specifically, $\pi(u_2) = 2$ and $\pi(p_3) = 3$, since $\pi(k_3) = 3$. The following proof is divided into two cases:

(2.1) $\pi(u_3) \neq 1$. We first recolor v_2 with 2, w_1 with a color c different from $2, 3$, and then recolor w_2 in this way: If $c \neq 1$, recolor w_2 easily; otherwise, recolor w_2 with a color different from 1 and 3. It is easy to verify that the obtained coloring of $G - v$ is acyclic and thus a contradiction to Lemma 3.4.3 is produced, since $\pi(v_2) = 2 \neq 3 = \pi(k_3)$.

(2.2) $\pi(u_3) = 1$. We first color v with 2, recolor v_1 with a color different from 1, 2, and v_3 with a color c different from 1, 2. If $c \neq 3$, we continue to color k_2 with a color different from 3 and c. By a careful inspection, the resulting coloring is acyclic. If $c = 3$, we continue to recolor k_2 with a color c' distinct to $1, 3$.

We note that if the resulting coloring is not acyclic, then $c' = 2$ and $\pi(k_4) = 2$ such that one of paths $\cdots p_1 w_1 v_2 v v_3 k_2 k_3 k_4 \cdots$ is an alternating $(3,2)$-path. So, we need to destroy such danger path by recoloring v_2 with 1, p_2 with 2, u_2 with $\alpha \in L(u_2) \setminus \{1,2\}$, and further recoloring k_3 with a color different from $2,3$ if $\alpha = 3$. ∎

An m-cycle $C = v_1 v_2 \cdots v_m v_1$ is called an $(a_1, a_2, \cdots, a_m, a_1)$-*cycle* if the degree of the vertex v_i is a_i for $i = 1, 2, \cdots, m$. Note that the 7-cycle $\mathcal{C} = k_2 v_3 v v_2 p_2 u_2 k_3 k_2$ depicted in G_2 is a $(2,3,3,3,3,3,3,2)$-cycle which is incident to a 2-vertex k_2, a special ugly 3-vertex v, a heavy 3-vertex v_2 and other four 3-vertices v_3, p_2, u_2, k_3. For convenience, we call $\mathcal{C}$ a *bizarre* $(2,3,3^{SU},3^H,3,3,3,2)$-*cycle*, where 3^{SU} and 3^H denote that the corresponding vertex in $\mathcal{C}$ is a special ugly 3-vertex and a heavy 3-vertex, respectively.

In what follows, let $\mathcal{B}$ denote the set of black vertices in Figure 3.15 to Figure 3.17.

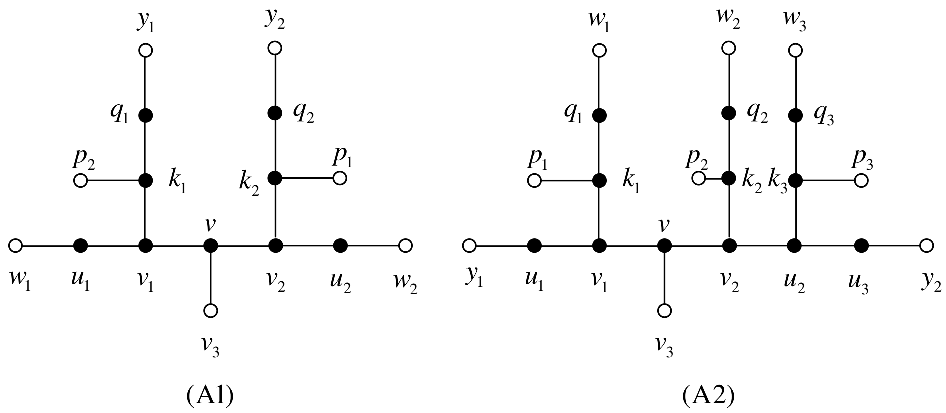


Figure 3.15: Two reducible configurations (A1) and (A2) in Claim 3.4.3.

Claim 3.4.3 (A1) *A* 3*-vertex is not adjacent to two special ugly* 3*-vertices.*

(A2) *A* 3*-vertex is not adjacent to a special ugly* 3*-vertex and a heavy* 3*-vertex.*

Proof. **(A1)** Suppose to the contrary that G contains a 3-vertex v adjacent to two special ugly 3-vertices v_1 and v_2, depicted in Figure 3.15 (A1). By (F2) and the assumption of $g(G) \geqslant 7$, we assert that there is no cycle induced by the vertices of $\mathcal{B}$. By the minimality of G, $G - v_1$ has an acyclic L-coloring π. Without loss of generality, let $L(v_1) = \{1,2,3\}$. According to Lemma 3.4.3, we may assume, w.l.o.g., that $\pi(w_1) = \pi(p_2) = 1$ and $\pi(v) = \pi(y_1) = 3$. Moreover, $L(v) = \{1,2,3\}$ and $\{\pi(v_2), \pi(v_3)\} = \{1,2\}$ by (P1)-(P2) of Lemma 3.4.4, respectively. The following proof is divided into two cases below:

- Assume $\pi(v_3) = 1$ and $\pi(v_2) = 2$. By (P3) of Lemma 3.4.4, $G - v_1$ contains at least one alternating $(3,2)$-path starting from the edge vv_2. It follows that at least one of q_2, p_1, w_2 is colored with 2. Now we recolor v with 2 and erase the color of v_2. Then color v_1 with 3, recolor u_1 and k_1 with a color different from

1 and 3, respectively, and recolor q_1 with a color different from its neighbors. Obviously, the resulting coloring of $G - v_2$ is acyclic. Denote $L(v_2) = \{2, a, b\}$. By Lemma 3.4.3, $\pi(y_2) = \pi(v) = 2$ and $\pi(w_2) = \pi(p_1) = a \neq 2$, which is a contradiction.

- Assume $\pi(v_3) = 2$ and $\pi(v_2) = 1$. We recolor v with 1 and erase the color of v_2. The following argument is similar to the above case.

(A2) Suppose to the contrary that G contains a 3-vertex v adjacent to a special ugly 3-vertex v_1 and a heavy 3-vertex v_2, depicted in Figure 3.15 (A2). By (F2) and the assumption of $g(G) \geqslant 7$, it is easy to deduce that if there is a cycle induced by the vertices of $\mathcal{B}$ then $q_1 = q_3$. However, if $q_1 = q_3$, then the configuration G_2 is produced, which is contradiction to Claim 3.4.2. So in the following, we claim that there is no cycle induced by the vertices of $\mathcal{B}$.

Obviously, $G - u_2$ admits an acyclic L-coloring π by the choice of G. W.l.o.g., let $L(u_2) = \{1, 2, 3\}$. According to Lemma 3.4.3, we may assume, w.l.o.g., that $\pi(p_3) = \pi(y_2) = 1$ and $\pi(v_2) = \pi(w_3) = 3$. Moreover, $L(v_2) = \{1, 2, 3\}$ and $\{\pi(v), \pi(k_2)\} = \{1, 2\}$ by (P1)-(P2) of Lemma 3.4.4, respectively.

Case 1 Assume that $\pi(v) = 2$ and $\pi(k_2) = 1$.

By Lemma 3.4.5 and (i) of Lemma 3.4.6, we have that $\pi(p_2) = \pi(v_2) = 3$ and $\pi(w_2) = \pi(v) = 2$. Moreover, by (ii)-(iii) of Lemma 3.4.6, we know that $L(v) = \{1, 2, 3\}$ and $\{\pi(v_1), \pi(v_3)\} = \{1, 3\}$. We have two possibilities as follows.

(1.1) Assume $\pi(v_1) = 1$ and $\pi(v_3) = 3$. Erase the color of v_1. We first color u_2 with 3, then recolor v_2 with 2, v with 1, u_3, k_3 with a color different from 1 and 3, respectively, and finally recolor q_3 with a color different from its neighbors. By a careful inspection, it is easy to see that the resulting coloring of $G - v_1$ is acyclic. By (P2) of Lemma 3.4.4, we deduce that $L(v_1) = \{1, 2, 3\}$, since $L(v) = \{1, 2, 3\}$. However, none of paths $vv_2 \cdots$ in $G - v_1$ could be an alternating $(1, 2)$-path, which contradicts (P3) of Lemma 3.4.4.

(1.2) Assume $\pi(v_1) = 3$ and $\pi(v_3) = 1$. The argument is very similar to the above Case 1.1.

Case 2 Assume that $\pi(v) = 1$ and $\pi(k_2) = 3$.

The proof is very similar to that of Case 1. ∎

Claim 3.4.4 (B1) *A* 3*-vertex is not adjacent to one heavy* 3*-vertex and two minor vertices.*

(B2) *A* 3*-vertex is not adjacent to two heavy* 3*-vertices.*

Proof. **(B1)** Suppose to the contrary that G contains a 3-vertex v adjacent to a heavy 3-vertex v_1 and two minor vertices v_2 and v_3, as depicted in Figure 3.16. Since $g(G) \geqslant 7$ and G contains no adjacent 2-vertices by (F2), we affirm that there is no cycle induced by the vertices of $\mathcal{B}$.

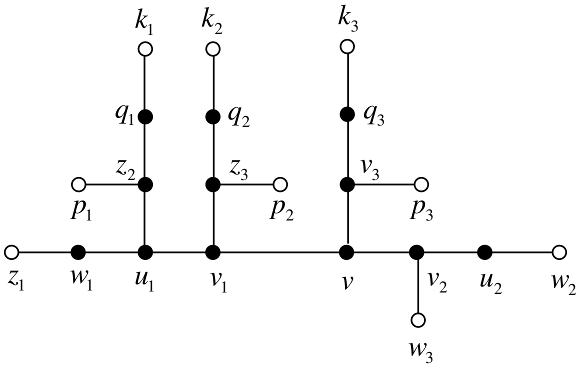


Figure 3.16: Reducible configuration (B1) in Claim 3.4.4.

Obviously, $G - u_1$ has an acyclic L-coloring π. W.l.o.g., let $L(u_1) = \{1, 2, 3\}$. By Lemma 3.4.3, w.l.o.g., we may suppose $\pi(z_1) = \pi(p_1) = 1$ and $\pi(v_1) = \pi(k_1) = 3$. By (P1)-(P2) of Lemma 3.4.4, $L(v_1) = \{1, 2, 3\}$ and $\{\pi(v), \pi(z_3)\} = \{1, 2\}$. The following proof is divided into two cases, according to the colors of v and z_3.

Case 1 Assume $\pi(z_3) = 1$ and $\pi(v) = 2$.

By Lemma 3.4.5 and (i) of Lemma 3.4.6, we see that $\pi(p_2) = \pi(v_1) = 3$ and $\pi(k_2) = \pi(v) = 2$. By (ii)-(iii) of Lemma 3.4.6, we know that $L(v) = \{1, 2, 3\}$ and $\{\pi(v_2), \pi(v_3)\} = \{1, 3\}$. By symmetry, let $\pi(v_2) = 1$ and $\pi(v_3) = 2$. Moreover, either $v_1vv_3q_3k_3\cdots$ or $v_1vv_3p_3\cdots$ is an alternating $(3, 2)$-path according to (P3) of Lemma 3.4.4. We have to discuss two possibilities below.

(1.1) Assume $v_1vv_3q_3k_3\cdots$ is an alternating $(3, 2)$-path. It implies that $\pi(q_3) = 2$ and $\pi(v_3) = \pi(k_3) = 3$. We can recolor v with 3, v_1 with 2, v_3 with a color c different from $3, \pi(p_3)$, and q_3 with a color different from $3, c$. One can easily check that the obtained coloring is acyclic. However, a contradiction to Lemma 3.4.3 is obtained, since $\pi(v_1) = 2 \neq 3 = \pi(k_1)$.

(1.2) Assume $v_1vv_3p_3\cdots$ is an alternating $(3, 2)$-path. It follows that $\pi(v_3) = 3$ and $\pi(p_3) = 2$. We first recolor v with 3, v_1 with 2, and v_3 with a color c different from $2, 3$. If $\pi(k_3) \neq c$, we continue to recolor q_3 with a color different from $c, \pi(k_3)$ and thus obtain an acyclic L-coloring of $G - u_1$ such that $\pi(v_1) = 2 \neq 3 = \pi(k_1)$. This contradicts Lemma 3.4.3. So now we suppose that $\pi(k_3) = c$. We continue to recolor q_3 with c' distinct to $c, 2$. If the resulting coloring of $G - u_1$ is not acyclic, we assert that $c = 1$, $c' = 3$, and either $v_2u_2w_2\cdots$ or $v_2w_3\cdots$ is an alternating $(1, 3)$-path.

- Assume $v_2u_2w_2\cdots$ is an alternating $(1, 3)$-path. Then $\pi(u_2) = 3$ and $\pi(w_2) = 1$. If $\pi(w_3) \neq 2$, we may recolor v with 2 and v_1 with 3 to derive an acyclic L-coloring of $G - u_1$ such that $\{\pi(v), \pi(v_2), \pi(v_3)\} = \{1, 2\} \neq \{1, 2, 3\}$, which is a contradiction to (ii) of Lemma 3.4.6. So now we suppose that $\pi(w_3) = 2$. We first recolor u_2 with a color $\alpha \in L(u_2) \setminus \{1, 3\}$. If $\alpha \neq 2$, then such coloring is an acyclic L-coloring of $G - u_1$ with $\pi(v_1) = 2 \neq 3 = \pi(k_1)$, which contradicts Lemma 3.4.3.

So suppose $\alpha = 2$. In this case, we continue to recolor v_2 with a color $\beta \in L(v_2) \setminus \{1, 2\}$. If $\beta \neq 3$, then we are done by similar reason as above. If $\beta = 3$, we further recolor v with 1, v_3 with 3, and q_3 with a color distinct to $1, 3$. By careful inspection, the obtained coloring is acyclic. However, $\pi(v_1) = 2 \neq 3 = \pi(k_1)$, which contradicts Lemma 3.4.3.

- Assume $v_2 w_3 \cdots$ is an alternating $(1, 3)$-path. Similarly, we deduce that $\pi(u_2) = 2$ and $\pi(w_2) = 1$. This case seems to be easy to discuss. We first recolor u_2 with a color $\alpha \in L(u_2) \setminus \{1, 2\}$. If $\alpha \neq 3$, then further recolor v with 2 and v_1 with 3 to obtain an acyclic L-coloring of $G - u_1$ such that $\{\pi(v_2), \pi(v_3), \pi(v)\} = \{1, 2\} \neq \{1, 2, 3\}$. This contradicts (ii) of Lemma 3.4.6. If $\alpha = 3$, we continue to recolor v_2 with a color different from $1, 3$, and thus similarly derive a contradiction to Lemma 3.4.3.

Case 2 Assume $\pi(z_3) = 2$ and $\pi(v) = 1$.

The proof is very similar to that of Case 1.

(B2) Suppose to the contrary that G contains a 3-vertex v adjacent to two heavy 3-vertices v_1 and v_2 depicted in Figure 3.17 (1). We have to consider the following two cases depending on the cycles formed by the vertices of $\mathcal{B}$.

Case 1 There is no cycle induced by the vertices of $\mathcal{B}$.

As Figure 3.17 (1) shown, it is obvious that $G - u_1$ has an acyclic L-coloring π by the choice of G. W.l.o.g., let $L(u_1) = \{1, 2, 3\}$. By Lemma 3.4.3, w.l.o.g., we may suppose $\pi(k_1) = \pi(y_1) = 1$ and $\pi(v_1) = \pi(p_1) = 3$. By (P1)-(P2) of Lemma 3.4.4, we have that $L(v_1) = \{1, 2, 3\}$ and $\{\pi(v), \pi(z_2)\} = \{1, 2\}$.

Case 1.1 Assume $\pi(z_2) = 1$ and $\pi(v) = 2$.

By Lemma 3.4.5 and (i) of Lemma 3.4.6, $\pi(y_2) = \pi(v_1) = 3$ and $\pi(p_2) = \pi(v) = 2$. By (ii)-(iii) of Lemma 3.4.6, we know that $L(v) = \{1, 2, 3\}$ and $\{\pi(v_2), \pi(v_3)\} = \{1, 3\}$. We have two possibilities below.

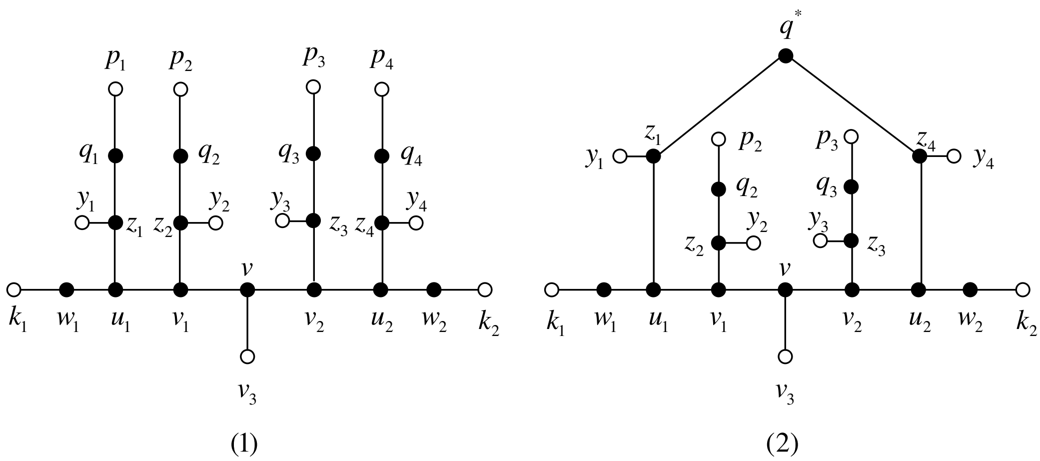


Figure 3.17: Reducible configuration (B2) in Claim 3.4.4.

(1.1.1) $\pi(v_2) = 3$ and $\pi(v_3) = 1$. We first color u_1 with 3, then recolor v with 3, v_1 with 2, w_1 with a color distinct to $1, 3$, z_1 with a color c distinct to $1, 3$, and q_1 with a color different from the 3 and c. Now erase all the colors of v_2, u_2, z_3, q_3. We continue to color v_2 with a color a distinct to 1 and 3, z_3 with a color b different from a and $\pi(y_3)$, and then color q_3 in this way: If $b = \pi(p_3)$, color q_3 with a color different from b and $\pi(y_3)$; otherwise, color q_3 with a color different from b and $\pi(p_3)$. By a careful inspection, we observe that the resulting coloring of $G - u_2$ is acyclic. Moreover, by definition, u_2 is a special ugly 3-vertex. So by (P1) of Lemma 3.4.4, we deduce that $L(u_2) = \{3, a, b\}$. Moreover, $\pi(p_4) = \pi(v_2) = a$ and $\pi(y_4) = \pi(k_2) \in \{3, b\}$ by Lemma 3.4.3. Thus, a contradiction to (P3) of Lemma 3.4.4 is easily obtained, since there is no alternating $(a, 3)$-path $v_2vv_1\cdots$ in $G - u_2$.

(1.2) $\pi(v_2) = 1$ and $\pi(v_3) = 3$. We first color u_1 with 3, then recolor v with 1, v_1 with 2, w_1, z_1 with a color distinct to $1, 3$, respectively, and q_1 with a color different from the colors of p_1 and z_1. Now erase all the colors of v_2, u_2, z_3, q_3. The following argument is similar to the proof of Case 1.1.1.

Case 1.2 Assume $\pi(z_2) = 2$ and $\pi(v) = 1$.

The proof is very similar to that of Case 1.1.

Case 2 There exists a 7^+-cycle induced by the vertices of $\mathcal{B}$.

It follows that at least two of black vertices coincide. Denote $B_1 = \{w_1, w_2, q_1, q_2, q_3, q_4\}$ be the set of black 2-vertices and $B_2 = \{z_1, z_2, z_3, z_4, u_1, u_2, v_1, v_2, v\}$ be the set of black 3-vertices, respectively. It is obvious that x cannot coincide y if $x \in B_1$ and $y \in B_2$. Furthermore, any two vertices in B_2 cannot coincide because $g(G) \geqslant 7$. All these facts imply that two vertices in B_1 coincide. By symmetry, we have to deal with the following three subcases:

(2.1) Assume $w_1 = q_4$. Denote $w^* = w_1 = q_4$. Obviously, $\mathcal{C} = w^*z_4u_2v_2vv_1u_1w^*$ is a 7-cycles. More specifically, $\mathcal{C}$ is a bizarre $(2, 3, 3^{SU}, 3^H, 3, 3, 3, 2)$-cycle, which is a contradiction to Claim 3.4.2.

(2.2) Assume $q_1 = q_3$. Denote $q^* = q_1 = q_3$. It is easy to see that $\mathcal{C} = q_1z_1u_1v_1vv_2z_3q_1$ is a bizarre $(2, 3, 3^{SU}, 3^H, 3, 3, 3, 1)$-cycle, which contradicts Claim 3.4.2.

(2.3) Assume $q_1 = q_4$. As Figure 3.17 (2) shown, we denote $q^* = q_1 = q_4$. Noting that $C^* = u_1z_1q^*z_4u_2v_2vv_1u_1$ is an 8-cycle. Moreover, this is the unique cycle induced by the vertices of $\mathcal{B}$. Since G is the minimal counterexample, $G - u_1$ has an acyclic L-coloring π. W.l.o.g., let $L(u_1) = \{1, 2, 3\}$. By Lemma 3.4.3, w.l.o.g., we may suppose $\pi(k_1) = \pi(y_1) = 1$ and $\pi(v_1) = \pi(z_4) = 3$. By (P1)-(P2) of Lemma 3.4.4, $L(v_1) = \{1, 2, 3\}$ and $\{\pi(v), \pi(z_2)\} = \{1, 2\}$. In this thesis, we only show the proof of the case that $\pi(z_2) = 1$ and $\pi(v) = 2$, since the other case is very similar.

By Lemma 3.4.5 and (i) of Lemma 3.4.6, $\pi(y_2) = \pi(v_1) = 3$ and $\pi(p_2) = \pi(v) = 2$. By (ii)-(iii) of Lemma 3.4.6, we know that $L(v) = \{1, 2, 3\}$ and

$\{\pi(v_2), \pi(v_3)\} = \{1, 3\}$. Though the following discussion is very similar to Case 1, we would like to complete its details.

- $\pi(v_2) = 3$ and $\pi(v_3) = 1$. We first color u_1 with 3, then recolor v with 3, v_1 with 2, w_1 with a color distinct to $1, 3$, z_1 with a color c distinct to $1, 3$, and q^* with a color different from 3 and c. Now erase all the colors of v_2, u_2, z_3, q_3. We continue to color v_2 with a color a distinct to 1 and 3, z_3 with a color b different from a and $\pi(y_3)$, and then color q_3 in this way: If $b = \pi(p_3)$, color q_3 with a color different from b and $\pi(y_3)$; otherwise, color q_3 with a color different from b and $\pi(p_3)$. By a careful inspection, one can easily check that the resulting coloring of $G - u_2$ is acyclic. By definition, u_2 is a special ugly 3-vertex. So by (P1) of Lemma 3.4.4, we have that $L(u_2) = \{3, a, b\}$. Thus, a contradiction to (P3) of Lemma 3.4.4 is easily obtained, since there is no alternating $(a, 3)$-path $v_2 v v_1 \cdots$ in $G - u_2$.
- $\pi(v_2) = 1$ and $\pi(v_3) = 3$. We first color u_1 with 3, then recolor v with 1, v_1 with 2, w_1 with a color distinct to $1, 3$, z_1 with a color c distinct to $1, 3$, and q^* with a color different from 3 and c. Now erase all the colors of v_2, u_2, z_3, q_3. The following argument is similar to the above case. ∎

Since every ugly 3-vertex is a minor vertex, it is easy to deduce the following Claim 3.4.5 by (B1) of Claim 3.4.4.

Claim 3.4.5 *A 3-vertex is not adjacent to an ugly 3-vertex, a minor vertex and a heavy 3-vertex.*

3.4.3 Proof of Theorem 3.4.2

Now we use a discharging argument with initial charge $\omega(v) = d(v)$ at each vertex v and with the following discharging rules (R1)-(R4). We write ω^* to denote the charge at each vertex v after we apply the discharging rules. Note that the discharging rules do not change the sum of the charges. To complete the proof, we show that $\omega^*(v) \geqslant \frac{14}{5}$ for all $v \in V(G)$. This leads to the following obvious contradiction:

$$\frac{14}{5} \leqslant \frac{\sum_{v \in V(G)} \omega^*(v)}{|V(G)|} = \frac{\sum_{v \in V(G)} \omega(v)}{|V(G)|} = \frac{2|E(G)|}{|V(G)|} \leqslant \text{Mad}(G) < \frac{14}{5}.$$

Hence no counterexample can exist.

Our discharging rules are defined as follows:

(R1) Every 2-vertex gets a charge equal to $\frac{2}{5}$ from each of its adjacent 3^+-vertex.

(R2) Let v be a 3-vertex.

(R2.1) If v is an ugly 3-vertex, then v gets a charge equal to $\frac{1}{5}$ from its neighbor that is neither a 2-vertex nor a minor vertex.

(R2.2) If v is a minor vertex that is not ugly, then v gets a charge equal to $\frac{1}{10}$ from each of its neighbors of degree at least 3.

(R2.3) If v is a heavy 3-vertex, then v gets a charge equal to $\frac{1}{10}$ from its neighbor that is neither a minor vertex nor an ugly 3-vertex.

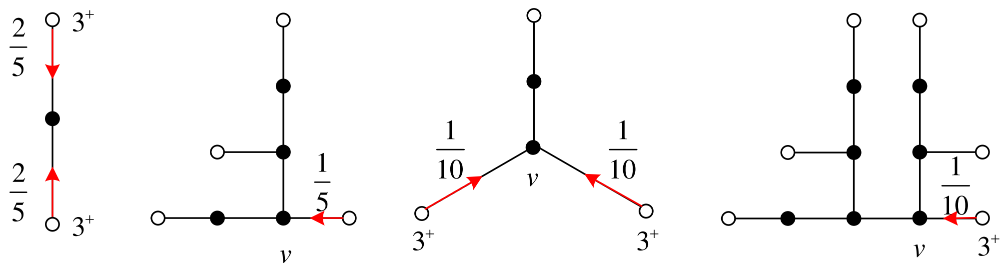


Figure 3.18: Discharging rules (R1) to (R2).

Let $v \in V(G)$. Denote $v_1, v_2, \cdots, v_{d(v)}$ be the neighbors of v in a cyclic order. The proof is divided into four cases according to the value of $d(v)$.

Case 1 $d(v) = 2$.

Then $\omega(v) = 2$, $d(v_1), d(v_2) \geqslant 3$ by (F2). Thus, $\omega^*(v) \geqslant 2 + 2 \times \frac{2}{5} = \frac{14}{5}$ by (R1).

Case 2 $d(v) = 3$.

Then $\omega(v) = 3$. Clearly, $n_2(v) \leqslant 1$ by (F2). It suffices to consider the following two cases, depending on the value of $n_2(v)$.

(2.1) Assume $n_2(v) = 1$. W.l.o.g., suppose that v_1 is a 2-vertex. Namely, v is a $(1, 0, 0)$-vertex. By (R1), $\tau(v \to v_1) = \frac{2}{5}$. According to (F2), we assert that there is at most one of v_2 and v_3 that is a minor vertex. By symmetry, we have two possibilities below:

- Assume that v_2 is a minor vertex. By definition, v is an ugly 3-vertex. Since v_3 is neither a 2-vertex nor a minor vertex, v gets $\frac{1}{5}$ from v_3 by (R2.1). Moreover, v_3 cannot be an ugly 3-vertex by (F2) again. If v_3 is a heavy 3-vertex, by definition, we may suppose that v_3 is adjacent to a minor vertex u_1 and an ugly 3-vertex u_2. It is easy to see that v_3 is adjacent to two ugly 3-vertices v and u_2, which is a contradiction to (A1). Thus, v sends nothing to v_3 and we have that $\omega^*(v) \geqslant 3 - \frac{2}{5} + \frac{1}{5} = \frac{14}{5}$ by (R1) and (R2).
- Now assume that neither v_2 nor v_3 is a minor vertex. This implies that none of v, v_2 and v_3 is an ugly 3-vertex. If neither v_2 nor v_3 is a heavy 3-vertex, then we are done, since $\omega^*(v) \geqslant 3 - \frac{2}{5} + 2 \times \frac{1}{10} = \frac{14}{5}$ by (R1) and (R2.2). Otherwise, by symmetry, assume v_2 is a heavy 3-vertex. Denote $N(v_2) = \{v, u_1, u_2\}$ such that u_1 is a minor vertex and u_2 is an ugly 3-vertex. One can easily observe that u_1, u_2 are both minor vertices. This

fact implies that v_2 is adjacent to three minor vertices u_1, u_2, v, which contradicts (F4).

(2.2) Assume $n_2(v) = 0$. Then v_i is a 3^+-vertex for each $i \in \{1, 2, 3\}$. By (F4), there are at most two minor vertices among v_1, v_2 and v_3. By symmetry, we have to handle the following three cases:

- First assume that v_1 and v_2 are minor vertices and v_3 is not.
  - If v_1 and v_2 are both ugly 3-vertices, then it contradicts (A1).
  - If v_1 is an ugly 3-vertex and v_2 is not, then v is a heavy 3-vertex by definition. So $\tau(v \to v_1) = \frac{1}{5}$ by (R2.1) and $\tau(v \to v_2) = \frac{1}{10}$ by (R2.2). On the other hand, v gets $\frac{1}{10}$ from v_3 by (R2.3), since v_3 is neither a minor vertex nor a ugly 3-vertex. Furthermore, we notice that v_3 cannot be a heavy 3-vertex by Claim 3.4.5. Therefore, $\omega^*(v) \geqslant 3 - \frac{1}{5} - \frac{1}{10} + \frac{1}{10} = \frac{14}{5}$.
  - Now we assume that neither v_1 nor v_2 is ugly. According to (R2.2), each of v_1, v_2 gets $\frac{1}{10}$ from v. It is easy to observe that v_3 is not ugly, since it is not a minor vertex. Moreover, by (B1), we assert that v_3 is not a heavy 3-vertex. All these facts ensure that v sends nothing to v_3. Therefore, $\omega^*(v) \geqslant 3 - 2 \times \frac{1}{10} = \frac{14}{5}$.
- Next assume that v_1 is a minor vertex and v_2, v_3 are not.

  It means that neither v_2 nor v_3 is an ugly 3-vertex. If v_1 is not ugly, then v sends at most $\frac{1}{10}$ to v_1 and there is at most one heavy 3-vertex of v_2 and v_3 by (B2). Hence, $\omega^*(v) \geqslant 3 - \frac{1}{10} - \frac{1}{10} = \frac{14}{5}$ by (R2.2) and (R2.3). Otherwise, we suppose v_1 is an ugly 3-vertex. It follows from (R2.1) that $\tau(v \to v_1) = \frac{1}{5}$. By (A2), neither v_2 nor v_3 is a heavy 3-vertex. So v sends nothing to v_2 and v_3. Hence, we conclude that $\omega^*(v) \geqslant 3 - \frac{1}{5} = \frac{14}{5}$.
- Finally assume that none of v_1, v_2 and v_3 is a minor vertex.

  It implies that none of v_1, v_2 and v_3 is an ugly 3-vertex. According to (B2) again, we defer that at most two vertices of v_1, v_2, v_3 are heavy 3-vertices. Thus, by (R2.3), $\omega^*(v) \geqslant 3 - 2 \times \frac{1}{10} = \frac{14}{5}$.

Case 3 $d(v) = 4$.

Obviously, the initial charge is $\omega(v) = 4$ and $n_2(v) \leqslant 2$ by (F2). Thus $\omega^*(v) \geqslant 4 - 2 \times \frac{2}{5} - 2 \times \frac{1}{5} = \frac{14}{5}$ by (R1) and (R2).

Case 4 $d(v) \geqslant 5$.

By (R1) and (R2), v sends a charge at most $\frac{2}{5}$ to each of its neighbors. Thus, $\omega^*(v) \geqslant d(v) - \frac{2}{5}d(v) = \frac{3}{5}d(v) \geqslant 3 > \frac{14}{5}$.

Therefore, we complete the proof of Theorem 3.4.2. ∎

3.5 Concluding remarks

An *oriented k-coloring* of an oriented graph $G = (V, A)$ is a mapping φ from $V(G)$ to a set of k colors such that (1) $\varphi(u) \neq \varphi(v)$ whenever $\overrightarrow{uv} \in A$, and (2) $\varphi(u) \neq \varphi(y)$ whenever $\overrightarrow{uv}, \overrightarrow{xy} \in A$ and $\varphi(v) = \varphi(x)$. In other words, an oriented k-coloring of an oriented graph $\overrightarrow{G}$ is a partition of vertex set into k color classes such that no two adjacent vertices belong to the same color class and all the arcs linking two color classes have the same direction. The *oriented chromatic number* of an oriented graph $\overrightarrow{G}$, denoted by $\chi_o(\overrightarrow{G})$, is defined as the least integer k such that $\overrightarrow{G}$ admits an oriented k-coloring. The oriented chromatic number of an undirected graph G, denoted by $\chi_o(G)$, is defined as the maximum oriented chromatic number of its orientations.

In 1994, Raspaud and Sopena [RS94] established an interesting relation between the oriented chromatic number and the acyclic chromatic number of a graph G:

Theorem 3.5.1 [RS94] *If $\chi_a(G) = k$, then $\chi_o(G) \leqslant k \cdot 2^{k-1}$.*

By Borodin's acyclic 5-color theorem [Bor79], it follows immediately from Theorem 3.5.1 that the oriented chromatic number of a planar graph is at most 80. Since, for any graph G, $\chi_a(G) \leqslant \chi_a^l(G)$, our Theorem 3.3.2 implies clearly the following result concerning the oriented chromatic number of planar graphs.

Theorem 3.5.2 *If G is a planar graph without 4- and 5-cycles, then $\chi_o(G) \leqslant 32$.*

Voigt [Voi95] constructed a planar triangle-free graph which is not 3-choosable, and Thomassen [Tho95] proved that each planar graph with girth at least 5 is 3-choosable. Combining our Theorem 3.4.2, we would like to propose the following conjecture:

Conjecture 3.5.3 *Every planar graph with girth at least 5 is acyclically 3-choosable.*

Chapter 4

Star coloring and star list coloring

Contents

The results presented in this chapter are joint work [CRW10b, CRW10a, CRW09] with Raspaud and Wang. In this chapter, we investigate the star (list) coloring of graphs. First, in Section 4.1, we give a chief survey in this direction. Then, in Section 4.2, we obtain a tight upper bound on star chromatic number of subcubic graphs. Finally, in Sections 4.3 to 4.4, we study, respectively, the L-star-coloring of planar subcubic graphs and sparse graphs.

4.1 Introduction

When Grünbaum [Grü73] introduced the acyclic notion, he also noted that the condition that the union of any two color classes inducing a forest can be generalized to other bipartite graphs. Among other problems, he suggested requiring that the union of any pair of color classes induces a star forest, namely, a proper coloring avoiding 2-colored paths with four vertices. A proper coloring of the vertices of a graph G is called a *star-coloring* if the union of every two color classes induces a star forest. In other words, no path on four vertices is 2-colored. The *star chromatic number* of G, denoted by $\chi_s(G)$, is the smallest integer k for which G admits a star coloring with k colors.

Grünbaum noted (without proof) that bounding the acyclic chromatic number bounds the star chromatic number. We state the result, a proof of which was given by Fertin, Raspaud and Reed [FRR01].

Theorem 4.1.1 [FRR01] *If $\chi_a(G) = k$, then $\chi_s(G) \leqslant k \cdot 2^{k-1}$.*

Let $\mathcal{P}$ denote the family of planar graphs. By Borodin's acyclically 5-colorable theorem and Theorem 4.1.1, it is easy to obtain that $\chi_s(\mathcal{P}) \leqslant 80$. In 2003, Nešetřil and Ossona de Mendez [NOdM03] made a big step by showing that $\chi_s(\mathcal{P}) \leqslant 30$. This result also implies that every triangle-free planar graph can be star colored using 18 colors, whereas Kierstead, Kündgen, and Timmons [KKT09] gave an example of a bipartite planar graph that requires 8 colors to star color. One year later, Albertson et al. [ACK+04] further decreased the upper bound 30 to 20 and gave a lower bound by showing an example of a planar graph H using at least 10 colors to star color. It follows that $10 \leqslant \chi_s(\mathcal{P}) \leqslant 20$. Moreover, they made an improvement of Theorem 4.1.1 by showing the following:

Theorem 4.1.2 [ACK+04] *If $\chi_a(G) = k$, then $\chi_s(G) \leqslant k(2k-1)$.*

In [BCM+09], Bu et al. studied the star chromatic number of graphs with given maximum average degree by showing that:

Theorem 4.1.3 [BCM+09] *Let G be a graph.*

(1) *If* $\mathrm{Mad}(G) < \frac{26}{11}$, *then* $\chi_s(G) \leqslant 4$.

(2) *If* $\mathrm{Mad}(G) < \frac{18}{7}$ *and* $g(G) \geqslant 6$, *then* $\chi_s(G) \leqslant 5$.

(3) *If* $\mathrm{Mad}(G) < \frac{8}{3}$ *and* $g(G) \geqslant 6$, *then* $\chi_s(G) \leqslant 6$.

By the well-known inequality $\mathrm{Mad}(G) < \frac{2g(G)}{g(G)-2}$, it is easy to deduce from Theorem 4.1.3 that for a planar graph G, $\chi_s(G) \leqslant 4$ if $g(G) \geqslant 13$, $\chi_s(G) \leqslant 5$ if $g(G) \geqslant 9$, and $\chi_s(G) \leqslant 6$ if $g(G) \geqslant 8$.

Other star-coloring results related to planar graphs are provided in Timmons's master's thesis [Tim07].

We say that G is *L-star-colorable* if for a given list assignment L there is a star-coloring c such that $c(v) \in L(v)$. If G is L-star-colorable for any list assignment L with $|L(v)| \geqslant k$ for all $v \in V(G)$, then G is *k-star-choosable*. The *star list chromatic number*, or *star choice number*, denoted by $\chi_s^l(G)$, of G is the smallest integer k such that G is k-star-choosable.

L-star-coloring has been recently investigated by many authors. Kierstead, Kündgen and Timmons [KKT09] showed that bipartite planar graphs are 14-star-choosable. In [KT10], Kündgen and Timmons proved a theorem about the dependence between the maximum average degree of graphs and their star list chromatic number. Their main result is the following:

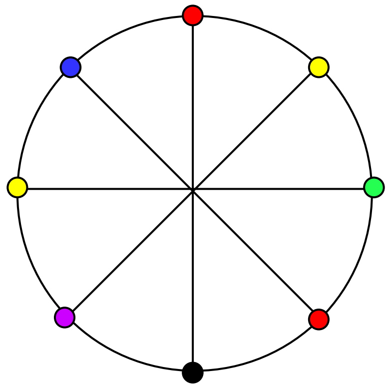


Figure 4.1: A cubic graph G_s with $\chi_s(G_s) = 6$.

Theorem 4.1.4 [KT10] *Let G be a graph.*

(1) *If* $\mathrm{Mad}(G) < \frac{8}{3}$, *then* $\chi_s^l(G) \leqslant 6$.

(2) *If* $\mathrm{Mad}(G) < \frac{14}{5}$, *then* $\chi_s^l(G) \leqslant 7$.

(3) *If G is planar and $g(G) \geqslant 6$, then $\chi_s^l(G) \leqslant 8$.*

We have to notice that the conclusion (1) in Theorem 4.1.4 is stronger than the third conclusion in Theorem 4.1.3. By the relationship mentioned before and (1) and (2) in Theorem 4.1.4, we immediately derive that for a planar graph G, we have $\chi_s^l(G) \leqslant 6$ if $g(G) \geqslant 8$, and $\chi_s^l(G) \leqslant 7$ if $g(G) \geqslant 7$.

By definition, we see that every star coloring is an acyclic coloring but star coloring a graph typically requires more colors than acyclically coloring the same graph. Moreover, determining the minimum (list) chromatic number of many families of graphs is proved to be a challenging problem. This is indeed the case for families as simple as subcubic graphs.

Basing on this point, in this chapter, we mainly work on subcubic graphs. More specifically, in Section 4.2, we shall give an upper bound on star chromatic number of subcubic graphs and show this bound is sharp and in Section 4.3, we obtain some new upper bounds on star choosability of planar subcubic graphs with girth condition. Finally, in Section 4.4, we extend the conclusion (3) in Theorem 4.1.4 to a more general result, which avoids the planar constraint.

4.2 Subcubic graphs are 6-star-colorable

In this section, we prove the following theorem, which is best possible based on the example showed by Fertin, Raspaud and Reed in [FRR01], see Figure 4.1.

Theorem 4.2.1 [CRW10a] *Every subcubic graph is 6-star-colorable.*

Proof. Let $C = \{1, 2, \cdots, 6\}$ denote a set of six colors. Suppose to the contrary that the theorem is not true. Let G be a counterexample with the least number of vertices, i.e., a subcubic graph without any 6-star-coloring by using color set C, but for any subgraph G' with $|G'| < |G|$ admits a 6-star-coloring using C. Therefore, G is connected. We need to discuss some properties of G.

Claim 4.2.1 *G does not contain 1-vertices.*

Proof. Suppose that x is a 1-vertex of G and y is the neighbor of x. Since $\Delta(G) \leqslant 3$, there are at most two neighbors of y different from x. By the minimality of G, $G - x$ has a 6-star-coloring π by using C. Obviously, we can assign a color in C to x, different from the colors of y and the neighbors of y. It follows that G is 6-star-colorable, which is a contradiction. ∎

Claim 4.2.2 *G does not contain 2-vertices.*

Proof. Suppose to the contrary that there is a 2-vertex x adjacent to u and v. Then, $G' = G - x$ has a 6-star-coloring π using C by the minimality of G. We will extend π to x to derive a contradiction.

If $|N_{G'}(u) \cup N_{G'}(v) \cup \{u, v\}| \leqslant 5$, we can color x with a color in C different from the colors of all vertices in $N_{G'}(u) \cup N_{G'}(v) \cup \{u, v\}$ because $|C| = 6$. Otherwise, we may assume that $|N_{G'}(u) \cup N_{G'}(v) \cup \{u, v\}| = 6$, where $N_{G'}(u) = \{u_1, u_2\}$, $N_{G'}(v) = \{v_1, v_2\}$, and u, v, u_1, u_2, v_1, v_2 are mutually distinct. If there is $c \in C\backslash\{\pi(u), \pi(v), \pi(u_1), \pi(u_2), \pi(v_1), \pi(v_2)\}$, we color x with c. Otherwise, we may assume that $\pi(u) = 1$, $\pi(v) = 2$, $\pi(u_1) = 3$, $\pi(u_2) = 4$, $\pi(v_1) = 5$, and $\pi(v_2) = 6$. If the color 1 did not appear in $N_G(u_1)\backslash\{u\}$ or not in $N_G(u_2)\backslash\{u\}$, we color x with 3 or 4. If 1 appeared in both $N_G(u_1)\backslash\{u\}$ and $N_G(u_2)\backslash\{u\}$, we color x with 1, then recolor u with a color $c \in C$ different from 3, 4 and those colors used in $(N_G(u_1) \cup N_G(u_2)) \setminus \{u\}$. Since the total number of colors used on $(N_G(u_1) \cup N_G(u_2)) \setminus \{u\}$ is at most 3, such color c exists. ∎

Claim 4.2.1 and Claim 4.2.2 imply that G is 3-regular, i.e., every vertex in G has exactly three neighbors.

Claim 4.2.3 *G has no 3-cycle.*

Proof. Suppose that $T = v_1v_2v_3v_1$ is a 3-cycle of G. For each $i = 1, 2, 3$, let u_i be the neighbor of v_i not in $V(T)$, and x_i, y_i be two other neighbors of u_i different from v_i. Let $G' = G - V(T)$. Then G' admits a 6-star-coloring π by the minimality of G. It suffices to color v_1 with $a \in C \setminus \{\pi(u_1), \pi(u_2), \pi(u_3), \pi(x_1), \pi(y_1)\}$, v_2 with $b \in C\backslash\{a, \pi(u_2), \pi(u_3), \pi(x_2), \pi(y_2)\}$, and v_3 with a color in $C\backslash\{a, b, \pi(u_3), \pi(x_3), \pi(y_3)\}$. It is easy to verify that the extended coloring is a 6-star-coloring of G, deriving a contradiction. ∎

Claim 4.2.4 *G has no 4-cycle.*

Proof. Suppose that $C_4 = v_1v_2v_3v_4v_1$ is a 4-cycle of G. For $i \in \{1, 2, 3, 4\}$, let u_i denote the the neighbor of v_i not in C_4, and x_i, y_i be two other neighbors of u_i different from v_i. By Claim 4.2.3, we see that $v_1v_3 \notin E(G)$ and $v_2v_4 \notin E(G)$. By the minimality of G, $G' = G - V(C_4)$ admits a 6-star-coloring π by using the color set C. First, we color v_2 with $a \in C \setminus \{\pi(u_1), \pi(u_2), \pi(u_3), \pi(x_2), \pi(y_2)\}$ and v_4 with $b \in C \setminus \{\pi(u_1), \pi(u_3), \pi(u_4), \pi(x_4), \pi(y_4)\}$. Then we consider the following subcases:

Case 1 $a \neq b$.

For $i = 1, 3$, we color v_i with a color in $C \setminus \{a, b, \pi(u_i), \pi(x_i), \pi(y_i)\}$.

Case 2 $a = b$.

In this case, we assume that $\pi(u_1) = 1$, $\pi(u_3) = 2$, $\pi(u_2) = 3$, $\pi(x_2) = 4$, $\pi(y_2) = 5$, $a = b = 6$, and $\{\pi(u_4), \pi(x_4), \pi(y_4)\} = \{3, 4, 5\}$. It means that u_1, u_2, u_3, u_4 are mutually distinct. We first recolor v_4 by 2, then color v_3 with $c \in C \setminus \{2, 6, \pi(u_4), \pi(x_3), \pi(y_3)\}$. Afterwards, we need to consider two possibilities as follows:

(i) $c = 3$. This implies $\pi(u_4) \neq 3$. We recolor v_2 with 1, then color v_1 with a color in $C \setminus \{1, 2, 3, \pi(x_1), \pi(y_1)\}$.

(ii) $c \neq 3$. First, we assume that $\pi(u_4) = 3$ and $\{\pi(x_4), \pi(y_4)\} = \{4, 5\}$. If $3 \notin \{\pi(x_1), \pi(y_1)\}$, then color v_1 with 3. Otherwise, w.l.o.g., suppose $\pi(x_1) = 3$. Then recolor v_2 with 2, and color v_1 with $d \in C \setminus \{1, 2, 3, c, \pi(y_1)\}$. Next, assume, without loss of generality, that $\pi(u_4) = 4$ and $\{\pi(x_4), \pi(y_4)\} = \{3, 5\}$. We color v_1 with $d \in C \setminus \{1, 2, 6, \pi(x_1), \pi(y_1)\}$. If $d = c$, it follows that $c = d \notin \{1, 2, 3, 4, 6\}$, we need to recolor v_4 with 1. ∎

A *partial coloring* will denote a coloring of $V' \subseteq V(G)$, such that the graph $G[V']$ induced by V' is 6-star-colorable. A color $\alpha \in C$ is *feasible* for a vertex v if assigning color α to v still results in a partial coloring. Let π be a partial coloring of G. For $v \in V(G)$ and $u \in N(v)$, we say that u is a *nice neighbor of* v if there exists $u' \in N(u) \setminus \{v\}$ such that $\pi(v) = \pi(u')$. Otherwise, we say u is a *bad neighbor of* v. We call a feasible color α *safe* for v if at least two colored neighbors of v are bad neighbors of v after coloring v with α.

In the following, we assume that G is a 3-regular graph with girth at least 5 by Claim 4.2.3 and Claim 4.2.4. We begin with the following two claims, which play an important role in proving Claim 4.2.7.

Claim 4.2.5 *Let π be a partial coloring of G. Suppose x is a colored 3-vertex with two colored neighbors x_1, x_2 and one uncolored neighbor x_3. If both x_1 and x_2 are nice neighbors of x, then there exists a partial coloring π' and a safe color $\alpha \in C$ for x such that both x_1 and x_2 become bad neighbors of x with respect to π'.*

Proof. Denote $N(x_1) = \{x, x_1', x_1''\}$ and $N(x_2) = \{x, x_2', x_2''\}$. By Claim 4.2.4, we see that x_1', x_1'', x_2', x_2'' are mutually distinct. Since x_1, x_2 are both nice neighbors of x, by symmetry, we may suppose that $\pi(x_1') = \pi(x_2') = \pi(x)$. It follows immediately that there exists a feasible color $\alpha \in C \setminus \{\pi(x), \pi(x_1), \pi(x_2), \pi(x_1''), \pi(x_2'')\}$ because $|C| = 6$. Let $\pi'(x) = \alpha$ and $\pi'(u) = \pi(u)$ for colored vertex $u \in V(G) \setminus \{x\}$. It is easy to check that α is safe since x_1 and x_2 become bad neighbors of x. Moreover, π' is still a partial coloring of G. This completes the proof of Claim 4.2.5. ∎

Claim 4.2.6 *Let π be a partial coloring of G. Suppose x is a colored 3-vertex with two colored neighbors x_1, x_2 and one uncolored neighbor x_3. If x_1 is a nice neighbor, then there exists a partial coloring π' and a feasible color $\beta \in C$ for x such that one of the following holds:*

(A1) *$\pi'(x) = \beta$ and $\pi'(u) = \pi(u)$ for colored vertex $u \in V(G)\backslash\{x\}$ such that both x_1 and x_2 become bad neighbors of x with respect to π';*

(A2) *$\pi'(x) = \beta = \pi(x_2)$, $\pi'(x_2) = \beta^*$, where β^* is a safe color for x_2, and $\pi'(u) = \pi(u)$ for colored vertex $u \in V(G)\backslash\{x, x_2\}$ such that both x_1 and x_2 become bad neighbors of x with respect to π';*

(A3) *$\pi'(x) = \beta \in \{\pi(x_2'), \pi(x_2'')\}$ and $\pi'(u) = \pi(u)$ for colored vertex $u \in V(G)\backslash\{x\}$ such that x_1 becomes a bad neighbor of x with respect to π'.*

Proof. Denote $N(x_1) = \{x, x_1', x_1''\}$ and $N(x_2) = \{x, x_2', x_2''\}$. Notice that $x_3 \notin \{x_1', x_1'', x_2', x_2''\}$ by the absence of 3-cycles in G. Furthermore, x_1', x_1'', x_2', x_2'' are mutually distinct by Claim 4.2.4. Since x_1 is a nice neighbor of x, without loss of generality, we suppose that $\pi(x_1') = \pi(x)$.

If there exists a color a different from the colors (if colored) of x, x_1, x_2, x_1'', x_2', and x_2'', then we set $\pi'(x) = \beta$. For other colored vertex $u \in V(G)\backslash\{x\}$, we set $\pi'(u) = \pi(u)$. It is easy to see that β is a feasible color for x and (A1) holds.

Otherwise, we may assume that $\pi(x) = \pi(x_1') = 1$, $\pi(x_1) = 2$, $\pi(x_2) = 3$, $\pi(x_1'') = 4$, $\pi(x_2') = 5$, and $\pi(x_2'') = 6$. This means that x_2 is a bad neighbor of x. We first erase the color x and need to consider the following two cases.

(i) If x_2' and x_2'' are both nice neighbors of x_2, there exists a safe color β^* for x_2 by Claim 4.2.5 and then we can set $\pi'(x_2) = \beta^*$, $\pi'(x) = \pi(x_2) = 3$, and finally set $\pi'(u) = \pi(u)$ for any colored vertex $u \in V(G)\backslash\{x, x_2\}$. Since all vertices x_1', x_1'', x_2', x_2'' keep the same colors as before and none of them was colored with 3, we deduce that π' is proper partial coloring and both x_1 and x_2 become bad neighbors of x with respect to π'. Hence, (A2) holds.

(ii) Now, w.l.o.g., we may suppose that x_2' is a bad neighbor of x_2, i.e., x_2's two other neighbors different from x_2 are not colored with 3. In this case, we can set $\pi'(x) = \pi(x_2') = 5$, and $\pi'(u) = \pi(u)$ for any colored vertex $u \in V(G)\backslash\{x\}$. One can easily check that such coloring still ensures that there is no 2-colored path on four vertices. So π' is also a partial coloring of G. Since $\{\pi'(x_1'), \pi'(x_1'')\} = \{\pi(x_1'), \pi(x_1'')\} = \{1, 4\}$, x_1 becomes a bad neighbor of x with respect to π'. Therefore, we obtain (A3). ∎

Remark 1: Let π be a partial coloring of G. Assume that x is adjacent to two colored vertices x_1, x_2 and one uncolored vertex x_3. We further suppose that x_1 is a nice neighbor of x. For each $i \in \{1, 2\}$, denote x_i', x_i'' be the other two neighbors of x_i distinct to x. If (A2) of Claim 4.2.6 holds, then it follows from the proof of case (i) that $\pi'(x_2') = \pi(x_2') \neq \pi(x_2'') = \pi'(x_2'')$. Moreover, there exists $u \in N(x_2')\backslash\{x_2\}$ and $v \in N(x_2'')\backslash\{x_2\}$ such that $\pi'(u) = \pi'(v) = \pi(x_2) = \pi'(x)$.

We will conclude the proof of Theorem 4.2.1 by showing the following Claim 4.2.7, which is a contradiction to the assumption of G.

Claim 4.2.7 *G contains no 3-vertex.*

Proof. Suppose to the contrary that G contains a 3-vertex v adjacent to x, y and z. We denote by x_1, x_2 (resp. y_1, y_2, z_1, z_2) the other two neighbors of x (resp. y, z) different from v. Let $S = \{x_1, x_2, y_1, y_2, z_1, z_2\}$. For each vertex $u \in S$, let u', u'' denote the other two neighbors of u different from x, y, z, see Figure 4.2. For simplicity, we use $\pi(S)$ to denote the color set $\{\pi(x_1), \pi(x_2), \pi(y_1), \pi(y_2), \pi(z_1), \pi(z_2)\}$. By Claim 4.2.3, $S \cap \{x, y, z\} = \varnothing$. Moreover, $|S| = 6$ by Claim 4.2.4. Let $G' = G - v$. By the minimality of G, G' admits a 6-star-coloring π by using the color set C. By symmetry, the following proof is divided into three lemmas, each of which shows that in any case π can be extended to v successfully. Thus, we always derive a 6-star-coloring of G and thus conclude the proof of Claim 4.2.7.

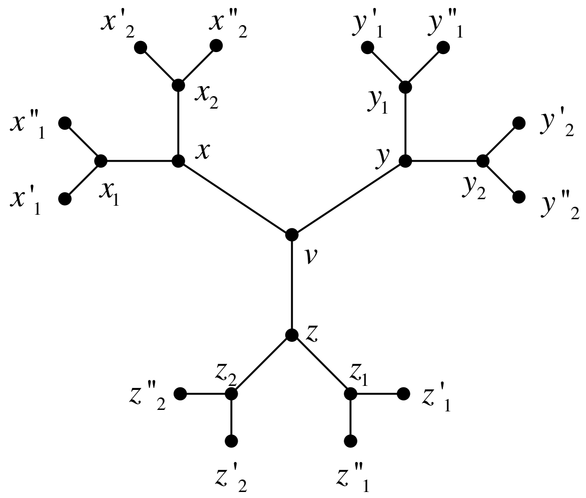


Figure 4.2: A 3-vertex v is adjacent to x, y and z.

Lemma 4.2.2 *If $\pi(x) = \pi(y) = \pi(z)$, then π can be extended to v successfully.*

Proof. Without loss of generality, we suppose that $\pi(x) = \pi(y) = \pi(z) = 1$. If there is $a \in C \backslash (\{1\} \cup \pi(S))$, we extend π to v by assigning v with a. Otherwise, w.l.o.g., assume that $\pi(x_1) = 2, \pi(x_2) = 3, \pi(y_1) = 4, \pi(y_2) = 5, \pi(z_1) = 6$ and $\pi(z_2) \in \{2, \cdots, 6\}$. By symmetry, we further assume that $\pi(z_2) \notin \{4, 5\}$. Denote $S_2 = \{y_1', y_1'', y_2', y_2''\}$. By Claims 4.2.3 and 4.2.4, $S_2 \cap \{y_1, y_2\} = \varnothing$ and $|S_2| = 4$. Moreover, any vertex in S_2 could be coincide with the vertex in $\{x_1, x_2, z_1, z_2\}$. Depending on the situations of y_1 and y_2, we have to handle three cases below.

Case 1 Assume that y_1 and y_2 are both nice neighbors of y.

By Claim 4.2.5, we may first recolor y with a safe color α. Obviously, $\alpha \neq 4$ since $\pi(y_1) = 4$. Then we color v with 4. Since none of vertex in $S \backslash \{y_1\}$ is colored with 4 and $\alpha \notin \{\pi(y_1'), \pi(y_1'')\}$, the resulting coloring is a proper 6-star-coloring.

Case 2 Assume that exactly one of y_1 and y_2 is a nice neighbor of y.

By symmetry, assume that y_1 is a nice neighbor of y and y_2 is not, say $\pi(y_1') = 1$. By Claim 4.2.6, we see that y can be given a feasible color β in three ways. So, we first recolor y with β. If either (A1) or (A3) holds, then in each case, y_1 becomes a bad neighbor of y after recoloring y and y_2 still remains the same color as before. So we finally color v with 4 properly since $\beta \neq 4$ and none of vertex in $S\backslash\{y_1\}$ is colored with 4.

Now, we suppose that (A2) holds. Namely, $\beta = 5$ and y_2 has been already recolored by a safe color, say β^*. In this case, we also color v with 4. It is easy to check that the resulting coloring is a 6-star-coloring since none of x_1, x_2, z_1, z_2 is colored with 4 and both y_1 and y_2 become nice neighbors of y after recoloring y and y_2.

Case 3 Assume that neither y_1 nor y_2 is a nice neighbor of y.

It follows immediately that $1 \notin \{\pi(y_1'), \pi(y_1''), \pi(y_2'), \pi(y_2'')\}$. For our convenience, we write that $\pi(y_1') = a$, $\pi(y_1'') = b$, $\pi(y_2') = c$, $\pi(y_2'') = d$ and notice that two of them can be equal. If there exists a color α belonging to $C\backslash\{1, 4, 5, a, b, c, d\}$, we recolor y with α and then color v with 4. Otherwise, we obtain that $3 \leqslant |\{a, b, c, d\}| \leqslant 4$ and thus $\{2, 3, 6\} \subseteq \{a, b, c, d\} \subseteq \{2, 3, 6, i\}$, where $i \in \{4, 5\}$. Assume, w.l.o.g., that $i = 4$. We have to consider the following two subcases.

Case 3.1 $\{a, b, c, d\} = \{2, 3, 4, 6\}$.

Obviously, $4 \in \{c, d\}$ since $\pi(y_1) = 4$. By symmetry, suppose that $d = 4$. It follows easily that $\{a, b, c\} = \{2, 3, 6\}$. For $u \in \{y_1', y_1''\}$, if u is not a nice neighbor of y_1, we may first recolor y with the color of u and then color v with 5. So now, suppose that both y_1' and y_1'' are nice neighbors of y_1. Then erase the color of y. By Claim 4.2.5, we assign a safe color α to y_1. If $\alpha \neq 1$, then color y with 1 and v with 4 properly since $\alpha \neq 4$ and $\pi(z_2) \neq 4$. Otherwise, color y with a properly since $a \in \{2, 3, 6\}$ and v with 4 successfully.

Case 3.2 $\{a, b, c, d\} = \{2, 3, 6\}$.

By symmetry, we only need to consider the following two possibilities.

- $a = b$. Clearly, $c \neq d$. Similarly, for $u \in \{y_2', y_2''\}$, if u is not a nice neighbor of y_2, we can first recolor y with the color of u and then color v with 4. So now, we suppose that both y_2' and y_2'' are nice neighbors of y_2. Then erase the color of y. According to Claim 4.2.5, we first recolor y_2 with a safe color α. If $\alpha \neq 1$, then color y with 1 and v with 5 properly because $\alpha \neq 5$, $\pi(z_2) \neq 5$, and $1 \notin \{a, c, d\}$. Otherwise, color y with 5 and v with 4 successfully by the fact that $\pi(z_2) \neq 4$ and $a \neq 5$.

- $a = c$. Obviously, $b \neq d$. If y_1'' is a bad neighbor of y_1, we recolor y with b and color v with 5 since $\pi(z_2) \neq 5$ and $b \notin \{a, d\}$. Otherwise, let $N(y_1') = \{y_1, p_1, p_2\}$, $N(y_1'') = \{y_1, p_3, p_4\}$, and by symmetry we suppose $\pi(p_3) = 4$. We erase the color of y. By Claim 4.2.6, we can give a feasible color β to y_1 in three different ways. So, we first recolor y_1 with β. If β satisfies (A1) or (A3), then y_1' is still colored with a and y_1'' is still colored with b. Moreover, after recoloring y_1 with β, y_1'' becomes a bad neighbor of y_1. Afterwards, we color y and v in the following way: If $\beta = 5$, color y with 1 and v with 4; Otherwise, first color y with b. Since $\pi(y_1') = a \neq b$ and $b \neq d$, the color b is feasible for y. Then we assign 4 to v successfully by the fact that $\beta \neq 4$ and $\pi(z_2) \neq 4$.

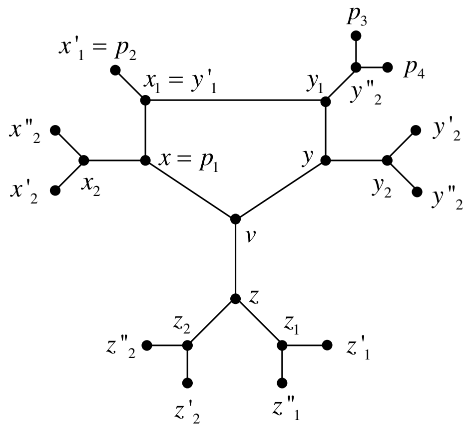


Figure 4.3: (A2) holds and $y_1' = x_1$.

Now, we suppose that (A2) holds. Namely, $\beta = \pi(y_1') = a \in \{2, 3, 6\}$ and y_1' has been recolored by a safe color, say β^*. Moreover, both y_1' and y_1'' become nice neighbors of y_1 after recoloring y_1. If $y_1' \notin \{x_1, x_2, z_1, z_2\}$, then none of x_1, x_2, z_1, z_2 was recolored in the process of recoloring y_1'. Thus, we color y with 4 properly since $4 \notin \{a, d\}$, and finally color v with 5 successfully because $\pi(z_2) \neq 5$. Otherwise, one of the following holds:

- $y_1' = x_1$. W.l.o.g., set $p_1 = x$ and $p_2 = x_1'$, see Figure 4.3. Then there exists a vertex in $N(p_1)\backslash\{x_1\} = \{x_2, v\}$ colored with 2 by Remark 1. This is impossible since x_2 is colored with 3 and v remains uncolored.
- $y_1' = x_2$. The proof if similar to the above case.

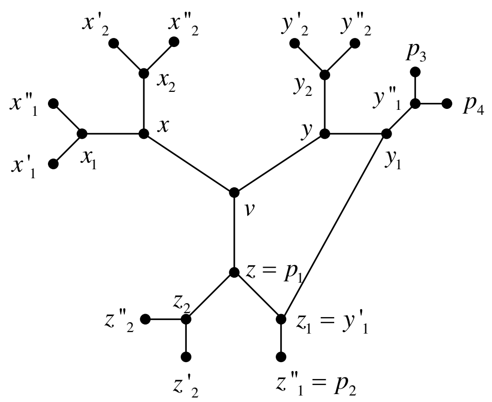


Figure 4.4: (A2) holds and $y_1' = z_1$.

- $y_1' = z_1$. W.l.o.g., set $p_1 = z$ and $p_2 = z_1''$, see Figure 4.4. This implies that $\beta = a = \pi(y_1') = 6$. If z_2 is not colored with 6, then we deduce that this case does not exist by using a similar proof as above. So, in what follows, assume that $\pi(z_2) = 6$. By Remark 1, we derive that $\pi(p_1) \neq \pi(p_2)$ and thus $\pi(p_2) \neq 1$. Hence, we further color y with 4 properly since $4 \notin \{a, d\}$

and $6 \notin \{\pi(p_1), \pi(p_2)\}$, and then color v with 5 easily since $\pi(p_2) \neq 1$ and $\pi(z_2) \neq 5$.

- $y_1' = z_2$. The proof is similar to the above case.

This completes the proof of Lemma 4.2.2. ∎

Lemma 4.2.3 *If $\pi(x) = \pi(y) \neq \pi(z)$, then π can be extended to v successfully.*

Proof. We suppose, w.l.o.g., that $\pi(x) = \pi(y) = 1$ and $\pi(z) = 2$. If there exists $a \in C \backslash (\{1, 2\} \cup \pi(S))$, we color v with a. Otherwise, there must be four vertices in S colored with 3,4,5,6, respectively. For our convenience, we call such four vertices special. Let $S_2 = \{y_1', y_1'', y_2', y_2''\}$. It follows from Claim 4.2.3 that $S_2 \cap \{y_1, y_2\} = \varnothing$. Moreover, $|S_2| = 4$ by Claim 4.2.4. We have to notice that any vertex in S_2 could be coincide with the vertex in $\{x_1, x_2, z_1, z_2\}$. To extend π to v, we have to consider the following four cases, according to the situations of those special vertices.

Case 1 $\{\pi(x_1), \pi(x_2), \pi(y_1), \pi(y_2)\} = \{3, 4, 5, 6\}$.

Without loss of generality, we set that $\pi(x_1) = 3$, $\pi(x_2) = 4$, $\pi(y_1) = 5$ and $\pi(y_2) = 6$. We have to handle the following two subcases, according to the colors of z_1 and z_2.

Case 1.1 Assume either $\{3, 4\} \cap \{\pi(z_1), \pi(z_2)\} = \varnothing$ or $\{5, 6\} \cap \{\pi(z_1), \pi(z_2)\} = \varnothing$.

W.l.o.g., we suppose that $\{5, 6\} \cap \{\pi(z_1), \pi(z_2)\} = \varnothing$. Namely, the colors 5 and 6 do not appear on z_1 and z_2. We discuss the three possibilities below.

(a) Assume that y_1 and y_2 are both nice neighbors of y.

We may assume, w.l.o.g., that $\pi(y_1') = \pi(y_2') = 1$. If $\{\pi(y_1''), \pi(y_2'')\} \neq \{3, 4\}$, we may first recolor y with a safe color $\alpha \neq 2$ by Claim 4.2.5 and then color v with 5 successfully. Now, we may assume, w.l.o.g., that $\pi(y_1'') = 3$, $\pi(y_2'') = 4$ and erase the color of y. If y_1'' is a bad neighbor of y_1, then color y with 3 and v with 6. Otherwise, let $N(y_1') = \{y_1, p_1, p_2\}$, $N(y_1'') = \{y_1, p_3, p_4\}$, and $\pi(p_3) = 5$. By Claim 4.2.6, we may assign y_1 with a feasible color β in three ways.

If either (A1) or (A3) holds, then y_1' and y_1'' were not recolored in the process of recoloring y_1. In other words, both of them keep the same colors as before. Furthermore, y_1'' becomes a bad neighbor of y_1 after recoloring y_1. It is obvious that the color 5 is a safe color for y since $5 \notin \{\pi(y_1'), \pi(y_1''), \pi(y_2'), \pi(y_2'')\} = \{1, 3, 4\}$. So we assign 5 to y. Then, assign 6 to v successfully since $6 \notin \{\pi(z_1), \pi(z_2)\}$.

Next, suppose that (A2) holds. Namely, $\beta = \pi(y_1') = 1$ and y_1' has been already given a safe color β^*. Moreover, after recoloring y_1 and y_1', both y_1' and y_1'' become nice neighbors of y_1. If $y_1' \notin \{x_1, x_2, z_1, z_2\}$, then none of x_1, x_2, z_1, z_2 was recolored in the process of recoloring y_1'. Thus, we can extend π to G by assigning 5 to y and 6 to v. Otherwise, suppose that $y_1' \in \{z_1, z_2\}$ since $\pi(y_1') = 1$. By symmetry, set $y_1' = z_1$, $p_1 = z$ and $p_2 = z_1''$, see Fig. 4. By Remark 1, $\pi(p_2) \neq \pi(z)$. In other words, $\pi(p_2) \neq 2$. Therefore, it is easy to color y with 5 and v with 6 to derive a 6-star-coloring of G.

(b) Assume that exactly one of y_1 and y_2 is a nice neighbor of y.

Without loss of generality, assume that $\pi(y_1') = 1$ and $1 \notin \{\pi(y_2'), \pi(y_2'')\}$. We first erase the color of y.

(b1) First assume that there is a color a belonging to $C \setminus \{1, 5, 6, \pi(y_1''), \pi(y_2'), \pi(y_2'')\}$. If $a \neq 2$, then color y with a and v with 5. Otherwise, we may suppose that $\{3, 4\} \subseteq \{\pi(y_1''), \pi(y_2'), \pi(y_2'')\}$. This implies that $|\{\pi(y_1''), \pi(y_2'), \pi(y_2'')\}| \geqslant 2$. Furthermore, we note that $2 \notin \{\pi(y_1''), \pi(y_2'), \pi(y_2'')\}$.

(b1.1) If y_2' and y_2'' are both nice neighbors of y_2, we first recolor y_2 with a safe color α by Claim 4.2.5. If $\alpha = 1$, color y with a color b in $\{\pi(y_2'), \pi(y_2'')\} \setminus \{5, \pi(y_1'')\}$. Clearly, such color b exists and $b \notin \{1, 2\}$. We further color v with 5 properly since $b \notin \{\pi(y_1'), \pi(y_1''), \pi(z_1), \pi(z_2)\}$. If $\alpha = 5$, then color y with 2 and color v with 6. It is easy to see that the resulting coloring is a 6-star-coloring since $2 \notin \{\pi(y_1'), \pi(y_1''), \pi(y_2'), \pi(y_2'')\}$ and $6 \notin \{\pi(z_1), \pi(z_2)\}$. If $\alpha \notin \{1, 5\}$, we reassign color 1 to y and assign color 6 to v.

(b1.2) If exactly one of y_2' and y_2'' is a nice neighbor of y_2. Assume, w.l.o.g., that y_2' is a nice neighbor of y_2. Let $N(y_2') = \{y_2, q_1, q_2\}$ and $N(y_2'') = \{y_2, q_3, q_4\}$. By symmetry, assume $\pi(q_1) = 6$. By Claim 4.2.6, we first recolor y_2 with a feasible color β which is differen from 6 in the following three ways.

If either (A1) or (A3) holds, then y_2' and y_2'' still remain the same colors as before and y_2' becomes a bad neighbor of y_2 after recoloring y_2. We further color y and v in the following way: If $\beta \in \{1, 5\}$, color y with 2 and v with 6. It is easy to verify that such coloring is proper since the color 2 does not appear on the vertex in $\{y_1', y_1'', y_2', y_2''\}$ and the color 6 does not appear on z_1 and z_2. If $\beta \notin \{1, 5\}$, reassign color 1 to y and assign color 6 to v successfully basing on the fact that $1 \notin \{\pi(y_2'), \pi(y_2'')\}$.

Now, suppose that (A2) holds. It follows that $\beta = \pi(y_2'')$ and y_2'' has been recolored by a safe color, say β^*. Moreover, neither y_2' nor y_2'' is a nice neighbor of y_2 after recoloring y_2. Obviously, $\beta \notin \{1, 2\}$ since the former color of y_2'' is neither 1 nor 2. It means that $\beta \in \{3, 4, 5\}$.

- $y_2'' \notin \{x_1, x_2, z_1, z_2\}$. Then none of x_1, x_2, z_1, z_2 was recolored in the process of recoloring y_2''. Depending on β, we have two coloring ways to extend π to y and v: Suppose $\beta = 5$. This implies that the former color of y_2'' was 5. Thus, $\{\pi(y_1''), \pi(y_2')\} = \{3, 4\}$. If $\beta^* = 2$, color y with 6 and v with 5. If $\beta^* \neq 2$, we color y with 2 and v with 6 properly. Now we suppose that $\beta \in \{3, 4\}$. We reassign color 1 to y and assign color 6 to v properly.

- $y_2'' \in \{x_1, x_2, z_1, z_2\}$. Remark 1 and the fact that $\pi(x_1) \neq \pi(x_2)$ imply that $y_2'' \notin \{x_1, x_2\}$. So, by symmetry, suppose that $y_2'' = z_1$, $q_3 = z$ and $q_4 = z_1''$, see Figure 4.5. It is easy to deduce that $\beta \in \{3, 4\}$ since the former color of y_2'' was neither 1, nor 5, nor 6. Moreover, Remark 1 asserts that $\pi(q_4) \neq \pi(q_3)$ and hence $\pi(q_4) \neq 2$. So we reassign color 1 to y and assign color 6 to v.

(b1.3) If neither y_2' nor y_2'' is a nice neighbor of y_2, then we first color y with a color b in $\{\pi(y_2'), \pi(y_2'')\} \setminus \{5, \pi(y_1'')\}$. Notice that such coloring b exists since $|\{\pi(y_1''), \pi(y_2'), \pi(y_2'')\}| \geqslant 2$ and $\{3, 4\} \subseteq \{\pi(y_1)''), \pi(y_2'), \pi(y_2'')\}$. By a careful inspection, we see that b is feasible for y. Moreover, $b \neq 2$ and y_1 becomes a bad neighbor

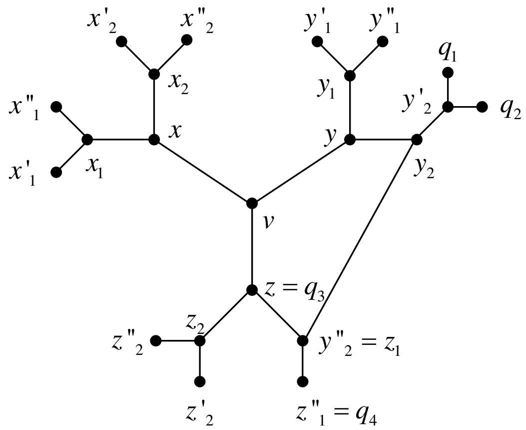


Figure 4.5: (A2) holds and $y_2'' = z_1$.

of y after assigning color b to y. Therefore, we can further color v with 5 to extend π to the whole graph G.

(b2) Now, we suppose that $\{\pi(y_1''), \pi(y_2'), \pi(y_2'')\} = \{2, 3, 4\}$. It follows that $\{\pi(y_1'), \pi(y_1''), \pi(y_2'), \pi(y_2'')\} = \{1, 2, 3, 4\}$.

(b2.1) If y_2' and y_2'' are both nice neighbors of y_2, we recolor y_2 with a safe color α by Claim 4.2.5, color y with 6 properly since $6 \notin \{\pi(y_1'), \pi(y_1''), \pi(y_2'), \pi(y_2'')\}$, and finally color v with 5 successfully by the fact that z_i is not colored with 5 for each $i \in \{1, 2\}$.

(b2.2) If exactly one of y_2' or y_2'' is a nice neighbor of y_2. Assume, w.l.o.g., that y_2' is a nice neighbor of y_2. Let $N(y_2') = \{y_2, q_1, q_2\}$ and $N(y_2'') = \{y_2, q_3, q_4\}$. By symmetry, set $\pi(q_1) = 6$. By Claim 4.2.6, we are able to give a feasible color β to y_2 such that (A_i) holds for some fixed $i \in \{1, 2, 3\}$.

First, suppose that either (A1) or (A3) holds. Notice that y_2' and y_2'' were not recolored in the process of recoloring y_2. Moreover, y_2' becomes a bad neighbor of y_2 after recoloring y_2. Thus, we color y with 6 since $\beta \neq 6$ and $\{\pi(y_1'), \pi(y_1''), \pi(y_2'), \pi(y_2'')\} = \{1, 2, 3, 4\}$, and finally color v with 5.

We now suppose that (A2) holds. Namely, $\beta = \pi(y_2'')$ and y_2'' has been recolored by a safe color β^*. Moreover, (A2) affirms that both y_2' and y_2'' become bad neighbors of y_2 after recoloring y_2 and y_2''. Obviously, $\beta \in \{2, 3, 4\}$ since the former color of y_2'' belongs to $\{2, 3, 4\}$.

- $y_2'' \notin \{x_1, x_2, z_1, z_2\}$. Then none of x_1, x_2, z_1, z_2 was recolored in the process of recoloring y_2''. We first reassign color 1 to y properly since $\beta \neq 5$. Then assign color 6 to v successfully due to the fact that $\beta \neq 6$ and $6 \notin \{\pi(z_1), \pi(z_2)\}$.

- $y_2'' \in \{x_1, x_2, z_1, z_2\}$. Similarly, since $\pi(x_1) \neq \pi(x_2)$ and v is still uncolored, we deduce that $y_2'' \notin \{x_1, x_2\}$ by Remark 1. So, by symmetry, suppose that $y_2'' = z_1$, $q_3 = z$ and $q_4 = z_1''$, see Figure 4.5. Since $\pi(z) = 2$, we have that $\beta \in \{3, 4\}$. Moreover, $\pi(q_4) \neq \pi(z)$ by Remark 1 and thus $\pi(q_4) \neq 2$. So we can extend π to the whole graph G by coloring y with 6 and v with 5.

(b2.3) If neither y_2' nor y_2'' is a nice neighbor of y_2, then color y with a color a in $\{\pi(y_2'), \pi(y_2'')\}\backslash\{2\}$. We recall that $\{\pi(y_1''), \pi(y_2'), \pi(y_2'')\} = \{2, 3, 4\}$. It implies that such coloring a exists. Moreover, y_1 becomes a bad neighbor of y after coloring y with a. Afterwards, we color v with 5 to obtain a proper 6-star-coloring of G.

(c) Assume that neither y_1 nor y_2 is a nice neighbor of y.

It follows directly that none of y_1', y_1'', y_2', y_2'' is colored with 1. We divide the following proof into three subcases according to the situations of x_1 and x_2.

(c1) If x_1 and x_2 are both nice neighbors of x, i.e., $\pi(x_1') = \pi(x_2') = 1$, then x can be given a safe color α different from 1 by Claim 4.2.5 Finally, color v with a color belonging to $\{5, 6\}\backslash\{\alpha\}$ properly since $\{5, 6\} \cap \{\pi(z_1), \pi(z_2)\} = \varnothing$.

(c2) If exactly one of x_1 and x_2 is a nice neighbor of x, say x_1 and $\pi(x_1') = 1$, then we give a feasible color β to x in three ways by Claim 4.2.6. If either (A1) or (A3) holds, then x_1 and x_2 still remain the same colors as before. This means that $\beta \in \{2, 5, 6\}$. Furthermore, x_1 becomes a bad neighbor of x after recoloring x with β. If $\beta = 2$, then color v with 5 properly since neither z_1 nor z_2 is colored with 5. Otherwise, suppose that $\beta \in \{5, 6\}$. We color v with a color in $\{5, 6\}\backslash\{\beta\}$ to obtain a 6-star-coloring of G.

Now, suppose that (A2) holds. Namely, $\beta = \pi(x_2) = 4$ and x_2 has been recolored by a safe color, say β^*. Moreover, both x_1 and x_2 become bad neighbors of x after recoloring x and x_2. In this case, we may further color v with 5 successfully.

(c3) Assume that neither x_1 nor x_2 is a nice neighbor of x. For simplicity, we write that $\pi(x_1') = r_1$, $\pi(x_1'') = r_2$, $\pi(x_2') = r_3$ and $\pi(x_2'') = r_4$. So, $1 \notin \{r_1, r_2, r_3, r_4\}$. If there exists a color a belonging to $C\backslash\{1, 3, 4, r_1, r_2, r_3, r_4\}$, we recolor x with a, and v with a color belonging to $\{5, 6\}\backslash\{a\}$. Otherwise, suppose that $\{2, 5, 6\} \subseteq \{r_1, r_2, r_3, r_4\}$. This implies that $|\{r_1, r_2, r_3, r_4\}| \geqslant 3$. By symmetry, we need to discuss the following two possibilities, depending on the value of r_4.

(c3.1) Assume that $r_4 \in \{3, r_3\}$.

In each case, we always have that $\{r_1, r_2, r_3\} = \{2, 5, 6\}$. Namely, all r_1, r_2, r_3 are mutually different. If $u \in \{x_1', x_1''\}$ is a bad neighbor of x_1, we recolor x with $\pi(u)$, and then color v with a color in $\{5, 6\}\backslash\{\pi(u)\}$. So, in the following discussion, assume that both x_1' and x_1'' are nice neighbors of x_1. We erase the color of x firstly. By Claim 4.2.5, we assign a safe color α to x_1. Next, we will show how to extend π to G.

- $\alpha = 4$. We further color x with 1. Obviously, the color 1 is a feasible color for x since none of x_1', x_1'', x_2', x_2'' is colored with 1. Then, we color v with 3. If such coloring is not feasible for v, then z must have a nice neighbor colored with 3, i.e., $\pi(z_1) = 3$ and $\pi(z_1') = 2$. We erase the color of v. By Claim 4.2.6, z can be given a feasible color β in three different ways.

  If either (A1) or (A3) holds, then z_1 and z_2 still remain the same colors as before. Moreover, z_1 becomes a bad neighbor of z after recoloring z. Thus, we can color v with 2 to derive a 6-star-coloring of the whole graph G since $\pi(z_2) \neq 2$. Now, we suppose that (A2) holds. Namely, $\beta = \pi(z_2)$ and z_2 has been recolored by a safe color β^*. If $\beta = 1$, then reduce to the previous Lemma

4.2.2. Otherwise, we can color v with 2 successfully since neither zz_2z_2' nor zz_2z_2'' is 2-colored after recoloring z and z_2.

- $\alpha \neq 4$. We recall that $\{r_1, r_2, r_3\} = \{2, 5, 6\}$ and do as follows: If $\{r_1, r_2\} = \{5, 6\}$, then $\alpha \notin \{5, 6\}$ and $r_3 = 2$. We color x with 5 and v with 6. Otherwise, w.l.o.g., set $r_1 = 2$. It follows that $\alpha \neq 2$ and $r_3 \in \{5, 6\}$. Since $r_4 \in \{3, r_3\}$, we deduce that $r_4 \neq 2$ and thus $\pi(x_2'') \neq 2$. Hence, we color x with 2 and finally color v with a color in $\{5, 6\}\backslash\{\alpha\}$.

(c3.2) Assume that $r_4 \in \{r_1, r_2\}$.

Without loss of generality, assume that $r_4 = r_1$. It means that $r_2 \neq r_3$ and hence $\{r_1, r_2, r_3\} = \{2, 5, 6\}$. If x_1'' is not a nice neighbor of x_1, then recolor x with r_2 and finally color v with a color in $\{5, 6\}\backslash\{r_2\}$. Otherwise, let $N(x_1') = \{x_1, p_1, p_2\}$, $N(x_1'') = \{x_1, p_3, p_4\}$ and suppose that $\pi(p_3) = 3$. First we erase the color of x. Then, by Claim 4.2.6, we give a feasible color β to x_1 in three ways.

If either (A1) or (A3) holds, then x_1' and x_1'' keep the same colors as before. Furthermore, x_1'' becomes a bad neighbor of x_1 after recoloring x_1. So, we can color x with 3 properly since $3 \notin \{r_1, r_2, r_3\}$, and finally color v with 5 successfully.

Now, suppose that (A2) holds. Namely, $\beta = \pi(x_1') = r_1 \in \{2, 5, 6\}$ and x_1' has been recolored by a safe color β^*. Noting that both x_1' and x_1'' become bad neighbors of x_1 after recoloring x_1 and x_1'. Depending on the situation of x_1', we have to subcases below.

- $x_1' \notin \{y_1, y_2, z_1, z_2\}$. Then none of y_1, y_2, z_1, z_2 was recolored in the process of recoloring x_1 and x_1'. In this case, we may color x with 3 and v with a color in $\{5, 6\}\backslash\{r_1\}$ successfully.

- $x_1' \in \{y_1, y_2, z_1, z_2\}$. Since $\beta = r_1 \in \{2, 5, 6\}$ and $\{\pi(z_i)\} \in \{1, 3, 4\}$ for each $i \in \{1, 2\}$, we derive that $x_1' \notin \{z_1, z_2\}$. So, we suppose that $x_1' \in \{y_1, y_2\}$. However, by Remark 1, it is impossible since $\pi(y_1) \neq \pi(y_2)$.

Case 1.2 Assume that $z_1 \in \{3, 4\}$ and $z_2 \in \{5, 6\}$.

Suppose, w.l.o.g., that $z_1 = 3$ and $z_2 = 5$. We begin with the following Claims 4.2.8 to 4.2.10.

Claim 4.2.8 *x_2 is a bad neighbor of x.*

Proof. Suppose to the contrary that x_2 is a nice neighbor of x. W.l.o.g., assume that $\pi(x_2') = 1$. We first recolor x with a feasible color β by Claim 4.2.6. If either (A1) or (A3) holds, then x_1 and x_2 keep the same colors as before. Namely, the color of x_1 is still 3 and the color of x_2 is still 4. Moreover, x_2 becomes a bad neighbor of x after recoloring x. If $\beta \neq 2$, we can color v with 4 since $\beta \neq 4$. Otherwise, we assign color 6 to v. If such coloring is not feasible for v, we infer that one of y_2' and y_2'' is colored with 1, say y_2'. Now, we erase the color 6 from v. By Claim 4.2.6, we give a feasible color γ to y satisfying (A1), (A2) or (A3). If $\gamma = 2$, then reduce the following proof to Lemma 4.2.2. Otherwise, in each case, we can color v with 1 to derive a 6-star-coloring of G.

Now, suppose that (A2) holds. Namely, $\beta = \pi(x_1) = 3$ and x_1 has been recolored by a safe color β^*. By (A2), we assert that both x_1 and x_2 become bad neighbors of x after recoloring x and x_1. So, it is easy to color v with 4 to derive a 6-star-coloring of G, which is a contradiction. ∎

By a similar proof as Claim 4.2.8, we deduce the following Claim 4.2.9.

Claim 4.2.9 *y_2 is a bad neighbor of y.*

Claim 4.2.10 *z_2 is a bad neighbor of z.*

Proof. Assume to the contrary that z_2 is a nice neighbor of z and suppose that $\pi(z_2') = 2$. Then, we first recolor z with a feasible color β by Claim 4.2.6. If either (A1) or (A3) holds, then z_1 and z_2 keep the same colors as before. Furthermore, z_2 becomes a bad neighbor of z after recoloring z. If $\beta = 1$, then reduce to Lemma 4.2.2. Otherwise, we can color v with 2 successfully. Now, we suppose that (A2) holds. Namely, $\beta = \pi(z_1) = 3$ and z_1 has been recolored by a safe color β^*. By (A2), we see that both z_1 and z_2 become bad neighbors of z after recoloring z with β and z_1 with β^*. Therefore, we may color v with 2 to obtain a 6-star-coloring of G, which contradicts the choice of G. ∎

By Claim 4.2.9, the following proof is divided into two possible cases according to the situation of y_1.

(d1) Assume that y_1 is a nice neighbor of y.
W.l.o.g., set $\pi(y_1') = 1$. Firstly, recolor y with a feasible color β by Claim 4.2.6. If either (A1) or (A3) holds, then y_1 and y_2 keep the same colors as before. Moreover, y_1 becomes a bad neighbor of y after recoloring y. If $\beta = 2$, we color v with 4 properly by Claim 4.2.8. Otherwise, we color v with 5 properly by Claim 4.2.10. Now, we suppose that (A2) holds. Namely, y is given a feasible color β which is equal to $\pi(z_2) = 6$ and y_2 has been recolored by a safe color β^*. The condition (A2) ensures that both y_1 and y_2 become bad neighbors of y after recoloring y with 6 and y_2 with β^*. Therefore, we further color v with 4 successfully by Claim 4.2.8.

(d2) Assume that y_1 is a bad neighbor of y.
It means that none of y_1', y_1'', y_2', y_2'' is colored with 1. For our convenience, we write that $\pi(y_1') = r_1$, $\pi(y_1'') = r_2$, $\pi(y_2') = r_3$ and $\pi(y_2'') = r_4$.

(d2.1) Assume that there exists $a \in C \backslash \{1, 5, 6, r_1, r_2, r_3, r_4\}$. Then we recolor y with a. If $a = 2$, we further color v with 4 properly by Claim 4.2.8. Otherwise, we color v with 6 successfully.

(d2.2) Now, assume that $\{2, 3, 4\} \subseteq \{r_1, r_2, r_3, r_4\}$. First erase the color of y. Obviously, $\{r_1, r_2, r_3, r_4\} \subseteq \{2, 3, 4, i\}$, where $i \in \{5, 6\}$. We have to deal with the following two subcases.

- Assume that $\{r_1, r_2\} \cap \{r_3, r_4\} = \varnothing$. By symmetry, we have two subcases below.
  - $r_4 \in \{5, r_3\}$. It follows immediately that $\{r_1, r_2, r_3\} = \{2, 3, 4\}$. If $u \in \{y_1', y_1''\}$ is a bad neighbor of y_1, we first color y with $\pi(u)$ and then do

as follows: If $\pi(u) = 2$, color v with 4 proper by Claim 4.2.8; Otherwise, we color v with 6 properly since neither y_2' nor y_2'' is colored with $\pi(u)$.

Now, we may assume that both y_1' and y_1'' are nice neighbors of y_1. By Claim 4.2.5, we may recolor y_1 with a safe color α. If $\alpha = 6$, then color y with 1. Since $1 \notin \{r_1, r_2, r_3, r_4\}$, such coloring is feasible for y. Finally, we color v with 5 successfully by Claim 4.2.10. If $\alpha = 4$, then $\{r_1, r_2\} = \{2, 3\}$ and $\pi(r_3) = 4$. We may color y with 3 and v with 6. We easily observe that the resulting coloring is aa 6-star-coloring since $\pi(y_2'') \in \{5, r_3\} = \{5, 4\}$. If $\alpha \notin \{4, 6\}$, then color y with a color in $\{r_1, r_2\} \backslash \{4\}$ and finally color v with 4 successfully by Claim 4.2.8.

– $r_1 \in \{6, r_2\}$. It follows that $\{r_2, r_3, r_4\} = \{2, 3, 4\}$. If $u \in \{y_2', y_2''\}$ is a bad neighbor of y_2, we color y with $\pi(u)$. If $\pi(u) = 2$, we color v with 4 properly by Claim 4.2.8. If $\pi(u) \neq 2$, we color v with 5 successfully according to Claim 4.2.10.

  Now, we may assume that both y_2' and y_2'' are nice neighbors of y_2. By Claim 4.2.5, we recolor y_2 with a safe color α. If $\alpha = 5$, then color y with 1. Because $1 \notin \{r_2, r_3, r_4\}$ and $r_1 \in \{6, r_2\}$, such coloring is feasible for y. Then, we further color v with 6. If $\alpha = 4$, then $\{r_3, r_4\} = \{2, 3\}$ and $\pi(r_2) = 4$. We may color y with 3 since neither y_1' nor y_1'' is colored with 3. Then, color v with 5 properly by Claim 4.2.10. If $\alpha \notin \{4, 5\}$, color y with a color in $\{r_3, r_4\} \backslash \{4\}$ and afterward color v with 4 properly by Claim 4.2.8.

- Assume that $\{r_1, r_2\} \cap \{r_3, r_4\} \neq \varnothing$.

  It is obvious that there is at most one color belonging to $\{r_1, r_2\} \cap \{r_3, r_4\}$. W.l.o.g., assume that $r_1 = r_4$. First, suppose that y_1'' is a bad neighbor of y_1. We color y with r_2. If $r_2 = 2$, then color v with 4 properly due to Claim 4.2.8. Otherwise, we assign 6 to v since neither y_2' nor y_2'' is colored with r_2. So next, we let $N(y_1') = \{y_1, p_1, p_2\}$, $N(y_1'') = \{y_1, p_3, p_4\}$ and, w.l.o.g., suppose that $\pi(p_3) = 5$. By Claim 4.2.6, we may recolor y_1 with a feasible color β in three ways.

  – If either (A1) or (A3) holds, then y_1' and y_1'' were not recolored in the process of recoloring y_1. We further color y with 5 properly since $5 \notin \{r_1, r_2, r_3, r_4\}$ and v with 4 properly by Claim 4.2.8.

  – Now, we suppose that (A2) holds. It implies that $\beta = \pi(y_1') = r_1 = r_4 \in \{2, 3, 4\}$. Moreover, y_1' has been recolored by a safe color β^*. After recoloring y_1 and y_1', both y_1' and y_1'' become bad neighbors of y_1. On the other hand, Remark 1 asserts that there exists $u \in N(p_1) \backslash \{y_1'\}$ and $w \in N(p_2) \backslash \{y_1'\}$ such that u and w are both colored with r_1. This fact guarantees that y_1' cannot be coincided with a vertex in $\{x_1, x_2, z_1, z_2\}$. It means that none of x_1, x_2, z_1, z_2 was recolored in the process of recoloring y_1'. Therefore, we can color y with 5 and v with 6. It is easy to verify that the resulting coloring is a proper 6-star-coloring.

Case 2 $\{\pi(x_1), \pi(x_2), \pi(y_i), \pi(z_j)\} = \{3, 4, 5, 6\}$, where $i, j \in \{1, 2\}$.

We assume, without loss of generality, that $\pi(x_1) = 3$, $\pi(x_2) = 4$, $\pi(y_1) = 5$, and $\pi(z_1) = 6$. It is easy to see that $\pi(y_2) \neq 6$. Otherwise, we are in the previous Case 1 in this lemma. Moreover, $\pi(z_2) \neq 6$. To see that, we do as follows: If $2 \notin \{\pi(z_1'), \pi(z_1''), \pi(z_2'), \pi(z_2'')\}$, color v with 6; Otherwise, assume, w.l.o.g., that $\pi(z_1') = 2$. Then recolor z with a color a different from 2, 6, $\pi(z_1'')$, $\pi(z_2')$ and $\pi(z_2'')$. If $a = 1$, then go back to the previous Lemma 4.2.2. Or else, we assign 6 to v successfully. So next, suppose that $\pi(z_2) \neq 6$ and z_1 is a nice neighbor of z, say $\pi(z_1') = 2$. By Claim 4.2.6, we may first recolor z with a feasible color β in three ways.

If either (A1) or (A3) holds, then z_1 and z_2 keep the same colors as before. If $\beta = 1$, then reduce to Lemma 4.2.2. Otherwise, we assign 6 to v properly since z_1 becomes a bad neighbor of z after recoloring z by (A1) or (A3).

Now, suppose that (A2) holds. Namely, $\beta = \pi(z_2) \neq 6$ and z_2 has been recolored by a safe color β^*. Moreover, after recoloring z with β and z_2 with β^*, both z_1 and z_2 become bad neighbors of z. If $\beta = 1$, the following proof is reduced to Lemma 4.2.2. Otherwise, we again assign 6 to v successfully.

Case 3 $\{\pi(x_1), \pi(x_2), \pi(z_1), \pi(z_2)\} = \{3, 4, 5, 6\}$.

Without loss of generality, we set $\pi(x_1) = 3$, $\pi(x_2) = 4$, $\pi(z_1) = 5$ and $\pi(z_2) = 6$. For $i \in \{1, 2\}$, we have that $\pi(y_i) \notin \{5, 6\}$. Otherwise, we are in the previous Case 2. Moreover, both z_1 and z_2 are nice neighbors of z, since we may extend π to G by coloring v with $\pi(z_i)$ if z_i is a bad neighbor of z with $i \in \{1, 2\}$. Now, by Claim 4.2.5, we first recolor z with a safe color α. If $\alpha = 1$, then go back to Lemma 4.2.2. Otherwise, we assign 6 to v properly.

Case 4 $\{\pi(x_i), \pi(y_j), \pi(z_1), \pi(z_2)\} = \{3, 4, 5, 6\}$, where $i, j \in \{1, 2\}$.

We suppose, w.l.o.g., that $\pi(x_1) = 3$, $\pi(y_1) = 4$, $\pi(z_1) = 5$ and $\pi(z_2) = 6$. If $5 \in \{\pi(x_2), \pi(y_2)\}$, then we go back to the previous Case 2. Similarly, if $6 \in \{\pi(x_2), \pi(y_2)\}$, then we may go back to the previous Case 2. Thus, in what follows, assume that $5, 6 \notin \{\pi(x_2), \pi(y_2)\}$. One can easily observe that both z_1 and z_2 are nice neighbors of z. If not, we may color v with $\pi(z_i)$, where z_i is a bad neighbor of z with $i \in \{1, 2\}$. Now, by Claim 4.2.5, we first recolor z with a safe color α. If $\alpha = 1$, then go back to Lemma 4.2.2. Otherwise, we assign 6 to v.

This completes the proof of Lemma 4.2.3. ∎

Lemma 4.2.4 *If $\pi(x), \pi(y), \pi(z)$ are mutually different, then π can be extended to v successfully.*

Proof. By symmetry, we suppose that $\pi(x) = 1$, $\pi(y) = 2$ and $\pi(z) = 3$. We recall that $S = \{x_1, x_2, y_1, y_2, z_1, z_2\}$. If π cannot be extended to v, there must exist three vertices in S, say u_1, u_2, u_3, such that $\{\pi(u_1), \pi(u_2), \pi(u_3)\} = \{4, 5, 6\}$ and each u_i is a nice neighbor of $u \in \{x, y, z\}$ if $uu_i \in E(G)$. We assume, w.l.o.g., that $\pi(u_1) = 4, \pi(u_2) = 5$ and $\pi(u_3) = 6$. By symmetry, we have to deal with two cases, according to the situations of u_1, u_2, u_3.

Case 1 $u_1 = x_1$, $u_2 = x_2$ and $u_3 = y_1$.

It implies that $\pi(x_1) = 4$, $\pi(x_2) = 5$ and $\pi(y_1) = 6$. Moreover, x_1, x_2 are both nice neighbors of x and y_1 is a nice neighbor of y. Basing on this fact, we first recolor x with a safe color α by Claim 4.2.5. If $\alpha \in \{2,3\}$, then reduce the following proof to Lemma 4.2.3. Otherwise, $\alpha = 6$. If there exists a color $c \in \{1,4,5\} \backslash \{\pi(y_2), \pi(z_1), \pi(z_2)\}$, then color v with c. Otherwise, we suppose that $\{\pi(y_2), \pi(z_1), \pi(z_2)\} = \{1,4,5\}$. This means that y_2, z_1, and z_2 ar colored mutually distinct. If y_2 is not a nice neighbor of y, then we color v with $\pi(y_2)$ successfully. Otherwise, by Claim 4.2.5, we can first recolor y with a safe color β and finally color v with 2 successfully since $2 \notin \{\pi(x_1), \pi(x_2), \pi(y_1), \pi(y_2), \pi(z_1), \pi(z_2)\}$.

Case 2 $u_1 = x_1$, $u_2 = y_1$ and $u_3 = z_1$.

It follows that $\pi(x_1) = 4$, $\pi(y_1) = 5$ and $\pi(z_1) = 6$. Moreover, x_1 (resp. y_1, z_1) is a nice neighbor of x (resp. y, z). Without loss of generality, set $\pi(x'_1) = 1$. If $\pi(y_2) = 4$ and y_2 is a nice neighbor of y, then reduce to the previous Case 1. So, if $\pi(y_2) = 4$ then y_2 must be a bad neighbor of y. Similarly, if $\pi(z_2) = 4$ then z_2 must be a bad neighbor of z.

First, assume that $\pi(x_2) = 4$. Then, we recolor x with a proper color c belonging to $C \backslash \{1, 4, \pi(x''_1), \pi(x'_2), \pi(x''_2)\}$. If $c \in \{2,3\}$, then reduce the proof to Lemma 4.2.3. Otherwise, we assign color 4 to v successfully.

Now, suppose that $\pi(x_2) \neq 4$. It is easy to deduce that $\pi(x_2) \in \{2,3,5,6\}$. Moreover, if $\pi(x_2) \in \{5,6\}$ then x_2 must be a bad neighbor of x since otherwise we are in the previous Case 1. By Claim 4.2.6, we can recolor x with a feasible color β in three ways. Notice that $\beta \neq 4$ since we did not recolor the vertex x_1 in the process of recoloring x.

- If either (A1) or (A3) holds, then after recoloring x, x_1, x_2 keep the same colors as before and x_1 becomes a bad neighbor of x. If $\beta \in \{2,3\}$, the following argument is reduced to Lemma 4.2.3. Otherwise, $\beta \in \{5,6\}$. We further color v with 4 successfully by the fact that none of $vyy_2y'_2$, $vyy_2y''_2$, $vzz_2z'_2$ and $vzz_2z''_2$ is a 2-colored path.

- Now, suppose that (A2) holds. It means that $\beta = \pi(x_2) \neq 4$ and x_2 has been recolored with a safe color β^*. By (A2), we see that x_1 and x_2 become both bad neighbors of x after recoloring x and x_2. If $\beta \in \{2,3\}$, the proof is reduced to Lemma 4.2.3. Otherwise, we color v with 4 to derive a proper 6-star-coloring of G.

This completes the proof of Lemma 4.2.4. ∎

4.3 Star choosability of planar subcubic graphs

In this section, our main result is stated as follows:

Theorem 4.3.1 [CRW10b] *Let G be a planar subcubic graph.*

(1) $\chi^l_s(G) \leqslant 6$.

(2) *If* $g(G) \geqslant 8$*, then* $\chi_s^l(G) \leqslant 5$.

(3) *If* $g(G) \geqslant 12$*, then* $\chi_s^l(G) \leqslant 4$.

We need to point out that the conclusion (1) in Theorem 4.3.1 partially improves one result in [ACK+04], which says that every subcubic graph is 7-star-choosable.

Let G_n denote the graph obtained by adding a pendant vertex to each vertex in a cycle, C_n, of length n. Albertson et al. [ACK+04] observed that $\chi_s(G_n) = 4$ for any $n \geqslant 4$ and $n \not\equiv 0 \pmod 3$. In other words, there exists a planar subcubic graph with arbitrary high girth has star chromatic number 4. This example shows that Theorem 4.3.1 is best possible in the sense that there does not exist a constant c such that every planar subcubic graph G with $g(G) \geqslant c$ has $\chi_s^l(G) \leqslant 3$.

In Section 4.3.1, we give some preliminary notation and facts, which are used in the following sections. The proof of Theorem 4.3.1 is divided into three parts, which are arranged in Section 4.3.2, Section 4.3.3, and Section 4.3.4, separately. Recall that $N_H^*(v) = N_H(v) \cup \{v\}$ for any $v \in V(H)$. For simplicity, in the sequel, we write $N^*(v)$ instead of $N_H^*(v)$ if there is no confusion about the context.

4.3.1 Preliminaries

In order to study the star chromatic number of graphs, we first introduce the following useful concept, which was used implicitly by Nešetřil and Ossona de Mendez [NOdM03], and explicitly by Albertson et al. [ACK+04], who formalized the connection to star coloring.

A proper coloring of an oriented graph G is called an *in-coloring* if for every 2-colored P_3 on three vertices in G, the edges are directed towards the middle vertex. A coloring of G is an in-coloring if it is an in-coloring of some orientation of G. An *L-in-coloring* of G is an in-coloring of G such that the colors are chosen from the lists assigned to each vertex.

Though the proof of the following Lemma 4.3.2 is very similar to that of Lemma 3.2 in [ACK+04], we like to write, for completeness, its details.

Lemma 4.3.2 *An L-coloring of a graph G is an L-star-coloring if and only if it is an L-in-coloring of some orientation of G.*

Proof. Given an L-star-coloring, we can construct an orientation by directing the edges towards the center of the star in each star-forest corresponding to the union of two color classes.

Conversely, consider an L-in-coloring of $\overrightarrow{G}$, an orientation of G. Let $P_3 = uvwz$ be any path on four vertices in G. We may assume the edge vw is directed towards w in $\overrightarrow{G}$. For the given coloring to be an L-in-coloring at v, we must have three different colors on u, v and w. ∎

4.3.2 General planar subcubic graphs

In this section, we prove the conclusion (1) in Theorem 4.3.1. That is, we have the following:

Theorem 4.3.3 *If G is a planar subcubic graph, then $\chi_s^l(G) \leqslant 6$.*

Proof. Suppose to the contrary that the theorem is not true. Let G be a counterexample with the least vertex number, i.e., a plane subcubic graph without L-star-coloring for some list assignment L such that $|L(v)| = 6$ for all vertices $v \in V(G)$.

By a careful inspection, one may observe that the Claims 4.2.1-4.2.4 also work in the 6-star-choosability. So in what follows, we may suppose that G is a 3-regular plane graph with $g(G) \geqslant 5$. It follows that no k-cycle with $5 \leqslant k \leqslant 6$ has a chord. These facts immediately implies the following Claim 4.3.1 and Claim 4.3.2.

Claim 4.3.1 *If f is a k-face with $5 \leqslant k \leqslant 6$, then $b(f)$ is a cycle.*

Claim 4.3.2 *If a k-face f, with $5 \leqslant k \leqslant 6$, is adjacent to a 5-face f', then f and f' are normally adjacent.*

Claim 4.3.3 *G contains no two 5-faces that are happily adjacent.*

Proof. Assume that $f = [v_1v_2v_3v_7v_8]$ and $f' = [v_3v_4v_5v_6v_7]$ are adjacent 5-faces with v_3v_7 as a common edge. By Claim 4.3.2, f and f' are normally adjacent, i.e., $v_i \neq v_j$ for each pair $\{i, j\} \subseteq \{1, 2, \cdots, 8\}$. For each $i \in \{1, 2, 4, 5, 6, 8\}$, let u_i denote the another neighbor of v_i different from v_{i-1} and v_{i+1}, where i is taken modulo 8, and let x_i, y_i denote the other neighbors of u_i different from v_i. Now suppose to the contrary that f and f' are happily adjacent. By definition, we see that each u_i does not belong to $V(f) \cup V(f')$. But we should notice that u_i could be equal to u_j, where $\{i, j\} \subseteq \{1, 2, 4, 5, 6, 8\}$. If this indeed is the case, we still say that u_i is the another neighbor of v_i and u_j is the another neighbor of v_j. Let $G' = G - V(f \cup f')$. By the minimality of G, G' admits an L-star-coloring π. In the following, for each $i \in \{1, 2, 4, 5, 6, 8\}$, we set

$$L^*(v_i) = L(v_i) \setminus \{\pi(u_i), \pi(x_i), \pi(y_i)\}.$$

Obviously, $|L^*(v_i)| \geqslant 3$. We first color v_1 with $a \in L^*(v_1) \setminus \{\pi(u_2), \pi(u_8)\}$. Then color v_2 with $b \in L^*(v_2) \setminus \{a, \pi(u_1)\}$ and v_8 with $c \in L^*(v_8) \setminus \{a, \pi(u_1)\}$. To extend π to G, we need to consider the following two cases, depending on the colors of b and c.

Case 1 Assume that $b \neq c$.

It implies that $a, b, c, \pi(u_1)$ are mutually distinct colors. We first color v_4 with $d_1 \in L^*(v_4) \setminus \{b, \pi(u_5)\}$ and v_6 with $d_2 \in L^*(v_6) \setminus \{c, \pi(u_5)\}$. Then, it remains to handle two possibilities as follows:

• Assume that $d_1 \neq d_2$. We may assign a color $d \in L^*(v_5) \setminus \{d_1, d_2\}$ to v_5 firstly. Then color v_3 with α_1 different from b, c, d_1, d_2 and v_7 with α_2 different from b, c, d_1, d_2, α_1. By a careful inspection, the resulting coloring of G is an L-star-coloring, a contradiction.

• Assume that $d_1 = d_2$. It follows that $L(v_4) = \{b, d_1, \pi(u_5), \pi(u_4), \pi(x_4), \pi(y_4)\}$ and $L(v_6) = \{c, d_1, \pi(u_5), \pi(u_6), \pi(x_6), \pi(y_6)\}$. So $b, c, d_1, \pi(u_5)$ are pairwise different. Denote $\pi(u_5) = \alpha$. First assign the color α to v_4. Then color v_5 with d

in $L(v_5) \setminus \{d_1, \alpha, \pi(u_4), \pi(x_5), \pi(y_5)\}$. Finally, we color v_3 with α_1 different from b, c, d_1, α, d and v_7 with α_2 different from $b, c, d_1, \alpha, \alpha_1$. It is easy to check that the resulting coloring of G is an L-star-coloring, a contradiction.

Case 2 Assume that $b = c$.

It follows that $L(v_i) = \{a, b, \pi(u_1), \pi(u_i), \pi(x_i), \pi(y_i)\}$ for each $i \in \{2, 8\}$. Then we color v_5 with $c \in L^*(v_5) \setminus \{\pi(u_4), \pi(u_6)\}$, v_4 with $d_1 \in L^*(v_4) \setminus \{c, \pi(u_5)\}$ and v_6 with $d_2 \in L^*(v_6) \setminus \{c, \pi(u_5)\}$. If $d_1 \neq d_2$, then reduce to the previous Case 1. Otherwise, suppose that $d_1 = d_2$. It follows that $L(v_i) = \{d_1, c, \pi(u_5), \pi(u_i), \pi(x_i), \pi(y_i)\}$ for each $i \in \{4, 6\}$. If $b \neq d_1$, then color v_3 with γ different from a, b, c, d_1 and v_7 with γ' different from γ, a, b, c, d_1. Otherwise, suppose that $b = d_1$. Namely, v_2, v_4, v_6, v_8 have the same color b. Denote $\pi(u_5) = \alpha$. Then we first recolor v_4 with α, v_6 with c, v_5 with $\beta \in L(v_5) \setminus \{c, \alpha, \pi(u_4), \pi(x_5), \pi(y_5)\}$, and then color v_3 with $\gamma_1 \in L(v_3) \setminus \{\alpha, \beta, a, b, c\}$ and v_7 with $\gamma_2 \in L(v_7) \setminus \{\gamma_1, \alpha, a, b, c\}$. In each case, one can easily check that the extending coloring is an L-star-coloring. This contradicts the choice of G and thus we complete the proof of Claim 4.3.3. ∎

In each proof of Claim 4.3.4 and Claim 4.3.5, we use $\mathcal{B}$ to denote the set of all solid vertices, depicted in Figures 4.6-4.7. Let $G' = G - \mathcal{B}$. By the minimality of G, G' has an L-star-coloring π. By Lemma 4.3.2, G' admits an L-in-coloring c for some orientation $\overrightarrow{G'}$ of G'. We give an orientation of the edge set $E(G[\mathcal{B}])$ and those edges between $V(G')$ and $\mathcal{B}$, then extend c to $\mathcal{B}$ to obtain an L-in-coloring of $\overrightarrow{G}$. By Lemma 4.3.2, G has an L-star-coloring, which contradicts the choice of G. For $v \in \mathcal{B}$, we use $S(v)$ to denote the set of vertices forbidden on v by the definition of L-in-coloring when we are about to color v.

We remark that the followng proofs of Claim 4.3.4 and Claim 4.3.5 seem to be easy but constructing proper orientations in each case is indeed very difficult; especially the Case 1 of Claim 4.3.5.

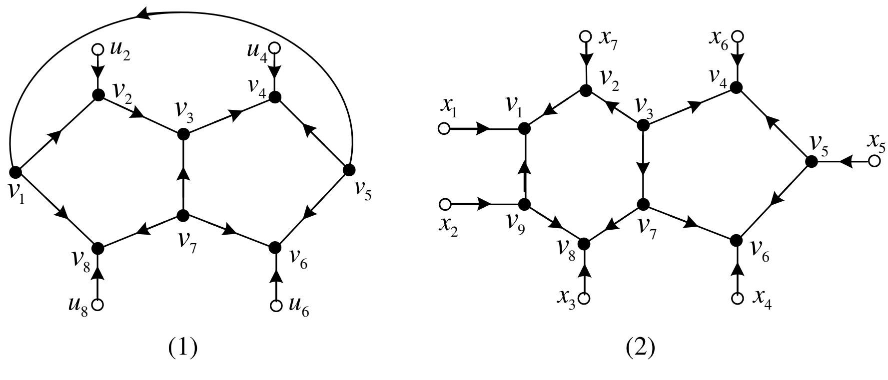


Figure 4.6: Two reducible configurations in Claim 4.3.3 and Claim 4.3.4.

Claim 4.3.4 *G contains no adjacent 5-faces.*

Proof. Suppose to the contrary that $f = [v_1v_2v_3v_7v_8]$ and $f' = [v_3v_4v_5v_6v_7]$ are adjacent 5-faces with the common edge v_3v_5. Again, f and f' are normally adjacent by Claim 4.3.2. For each $i \in \{1,2,4,5,6,8\}$, let u_i denote the another neighbor of v_i different from v_{i-1} and v_{i+1}, where i is taken modulo 8. By Claim 4.3.4, we may further suppose that f and f' are not happily adjacent. Then only obstacle is that $v_1v_5 \in E(G)$ by the absence of 3- and 4-cycles in G. Let $\mathcal{B} = V(f) \cup V(f')$ and $G' = G - \mathcal{B}$. By the minimality of G, G' has an L-in-coloring c for its some orientation $\overrightarrow{G'}$. We define orientations for $E(G[\mathcal{B}])$ and those edges between $V(G')$ and $\mathcal{B}$, as shown in Figure 4.6 (1). Based on c, we can color $v_8, v_6, v_4, v_2, v_3, v_1, v_5, v_7$, successively, since

$S(v_8) = N^*(u_8)$, $S(v_6) = N^*(u_6) \cup \{v_8\}$,
$S(v_4) = N^*(u_4) \cup \{v_6\}$, $S(v_2) = N^*(u_2) \cup \{v_4, v_8\}$,
$S(v_3) = \{u_2, v_2, v_4, v_6, v_8\}$, $S(v_1) = \{v_2, v_3, v_4, v_6, v_8\}$,
$S(v_5) = \{v_1, v_2, v_4, v_6, v_8\}$, $S(v_7) = \{v_3, v_4, v_6, v_8\}$.

Noting that $|S(v)| \leqslant 5$ for each vertex $v \in \mathcal{B}$ and by a careful inspection, we can show that the resulting coloring is an L-in-coloring of G. By Lemma 4.3.2, G has an L-star-coloring, which is a contradiction. ∎

Claim 4.3.5 *There is no* 5*-face adjacent to a* 6*-face.*

Proof. Suppose that there is a 5-face $f = [v_3v_4v_5v_6v_7]$ adjacent to a 6-face $f' = [v_3v_7v_8v_9v_1v_2]$ with v_3v_7 as their common edge. By Claim 4.3.2, f and f' are normally adjacent. The proof is divided into two cases as follows:

Case 1 There is no strange edge e'.

It implies that there is no strange edge joining a vertex in $\{v_1, v_2, v_8, v_9\}$ to a vertex in $\{v_4, v_5, v_6\}$. Both $b(f)$ and $b(f')$ are cycles without a chord by Claim 4.3.1 and Claim 4.3.2. Let $\mathcal{B} = V(f) \cup V(f')$ and $G' = G - \mathcal{B}$. We define orientations for $E(G[\mathcal{B}])$ and those edges between $V(G')$ and $\mathcal{B}$, as shown in Figure 4.6 (2). To extend c to $\mathcal{B}$, we can color $v_1, v_8, v_9, v_6, v_4, v_2, v_5, v_3, v_7$, successively, because

$S(v_1) = N^*(x_1) \cup \{x_2, x_7\}$, $S(v_8) = N^*(x_3) \cup \{x_2, v_1\}$,
$S(v_9) = N^*(x_2) \cup \{v_1, v_8\}$, $S(v_6) = N^*(x_4) \cup \{v_8, x_5\}$,
$S(v_4) = N^*(x_6) \cup \{x_5, v_6\}$, $S(v_2) = N^*(x_7) \cup \{v_1, v_4\}$,
$S(v_5) = N^*(x_5) \cup \{v_4, v_6\}$, $S(v_3) = \{v_1, v_2, v_4, v_6, v_8\}$,
$S(v_7) = \{v_2, v_3, v_4, v_6, v_8\}$.

Since $|S(v)| \leqslant 5$ for each vertex $v \in \mathcal{B}$, it is easy to show that the resulting coloring is an L-in-coloring of G. This is impossible.

Case 2 There exists a strange edge.

This means that there is at least one strange edge v_jv_k, where $j \in \{1,2,8,9\}$ and $k \in \{4,5,6\}$. In view of the previous analysis, all possible strange edges must belong to the set $\{v_4v_9, v_6v_1, v_1v_5, v_1v_9\}$ and, because of the plane embedding of G, at most one of these edges occurs. By the symmetry, we only need to discuss the following two subcases:

Case 2.1 $v_4v_9 \in E(G)$.

Let $\mathcal{B} = V(f) \cup V(f')$ and $G' = G - \mathcal{B}$. We define orientations for $E(G[\mathcal{B}])$ and those edges between $V(G')$ and $\mathcal{B}$, as shown in Figure 4.7(1). We color $v_5, v_6, v_1, v_8, v_3, v_7, v_2, v_4, v_9$, successively, such that

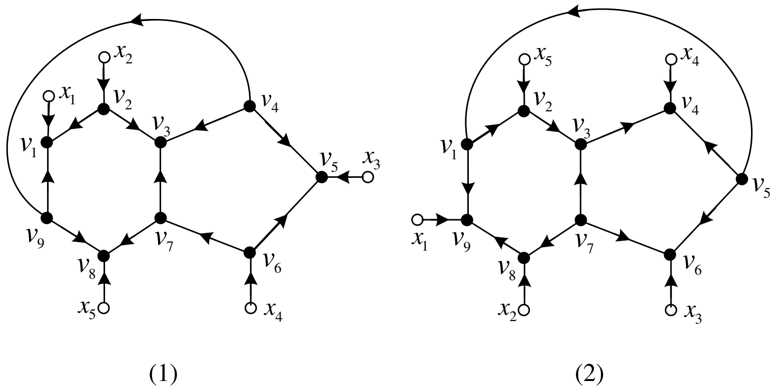


Figure 4.7: Two reducible configurations of Case 2 in Claim 4.3.4.

$S(v_5) = N^*(x_3) \cup \{x_4\}$, $S(v_6) = N^*(x_4) \cup \{v_5\}$,
$S(v_1) = N^*(x_1) \cup \{x_2\}$, $S(v_8) = N^*(x_5) \cup \{v_1, v_6\}$,
$S(v_3) = \{x_2, v_1, v_5, v_6, v_8\}$, $S(v_7) = \{x_4, v_3, v_5, v_6, v_8\}$,
$S(v_2) = N^*(x_2) \cup \{v_1, v_3\}$, $S(v_4) = \{v_1, v_3, v_5, v_8\}$,
$S(v_9) = \{v_1, v_3, v_4, v_5, v_8\}$.

Case 2.2 $v_1v_5 \in E(G)$.

Let $\mathcal{B} = V(f) \cup V(f')$ and $G' = G - \mathcal{B}$. We define orientations for $E(G[\mathcal{B}])$ and those edges between $V(G')$ and $\mathcal{B}$, as shown in Figure 4.7(2). We color $v_9, v_8, v_6, v_4, v_2, v_3, v_1, v_5, v_7$, successively, such that
$S(v_9) = N^*(x_1) \cup \{x_2\}$, $S(v_8) = N^*(x_2) \cup \{v_9\}$,
$S(v_6) = N^*(x_3) \cup \{v_8\}$, $S(v_4) = N^*(x_4) \cup \{v_6\}$,
$S(v_2) = N^*(x_5) \cup \{v_4, v_9\}$, $S(v_3) = \{x_5, v_2, v_4, v_6, v_8\}$,
$S(v_1) = \{v_2, v_3, v_4, v_6, v_9\}$, $S(v_5) = \{v_1, v_2, v_4, v_6, v_9\}$,
$S(v_7) = \{v_3, v_4, v_6, v_8, v_9\}$. ∎

We complete the proof with a discharging procedure. As usual, we define a weight function ω on the vertices and faces of G by letting $\omega(v) = 2d(v) - 6$ if $v \in V(G)$ and $\omega(f) = d(f) - 6$ if $f \in F(G)$. Our discharging rules are defined as follows:

(R1) Every 5-face gives $\frac{1}{5}$ to each incident vertex.

(R2) Let v be a 3-vertex incident to the faces f_1, f_2, f_3 with $d(f_1) \leqslant d(f_2) \leqslant d(f_3)$.

(R2a) If $d(f_1) = 5$ and $d(f_2) \geqslant 7$, then f_i gives $\frac{2}{5}$ to v for each $i = 2, 3$.

(R2b) If $d(f_1) \geqslant 6$, then f_i gives $\frac{1}{3}$ to v for each $i = 1, 2, 3$.

We only need to show that $\omega^*(x) \geqslant 0$ for all $x \in V(G) \cup F(G)$.

Let $v \in V(G)$. Then $d(v) = 3$ and $\omega(v) = -1$. Let f_1, f_2, f_3 denote the faces incident to v in G with $d(f_1) \leqslant d(f_2) \leqslant d(f_3)$. Thus, $d(f_i) \geqslant 5$ for all $i = 1, 2, 3$ because G contains no 3-faces and 4-faces.

First, assume $d(f_1) = 5$. Then $d(f_2) \geqslant 7$ by Claim 4.3.4 and Claim 4.3.5. Thus $\omega^*(v) \geqslant -1 + \frac{1}{5} + \frac{2}{5} \times 2 = 0$ by (R1) and (R2a). Otherwise, $d(f_1) \geqslant 6$. By (R2b), we derive immediately that $\omega^*(v) \geqslant -1 + \frac{1}{3} \times 3 = 0$.

Let $f \in F(G)$. Then $d(f) \geqslant 5$. We consider some cases, depending on the size of $d(f)$.

- $d(f) = 5$. Then $\omega(f) = 1$. It is obvious that $\omega^*(f) \geqslant 1 - \frac{1}{5} \times 5 = 0$ by (R1).
- $d(f) = 6$. By Claim 4.3.5, f is not incident to any 5-face. Every boundary vertex of f must be incident to three 6^+-faces. Thus, by (R2b), $\omega^*(f) \geqslant 2 - \frac{1}{3} \times 6 = 0$.
- $d(f) \geqslant 7$. By Claim 4.3.4, every boundary vertex of f is incident to at most one 5-face. Thus, f gives at most $\frac{2}{5}$ to each boundary vertex by (R2), so that $\omega^*(f) \geqslant d(f) - 4 - \frac{2}{5}d(f) = \frac{3}{5}d(f) - 4 \geqslant \frac{3}{5} \times 7 - 4 = \frac{1}{5}$.

This completes the proof of Theorem 4.2. ∎

4.3.3 Planar subcubic graphs of girth at least 8

We start with some definitions. A 3-vertex v is said to be *Type* 1 if it is a $(1,1,0)$-vertex; *Type* 2 if it is a $(0,0,0)$-vertex and is adjacent to exactly two Type 1 vertices; and *Type* 3 if it is a $(1,0,0)$-vertex and is adjacent to exactly one Type 1 vertex. Let T_i denote the set of Type i vertices in G for each $i = 1,2,3$.

Lemma 4.3.4 *A planar subcubic graph with $g(G) \geqslant 8$ contains one of the following eleven configurations:*
(C1) *A 1^--vertex.*
(C2) *Two adjacent 2-vertices.*
(C3) *A $(1,1,1)$-vertex.*
(C4) *Two adjacent $(1,1,0)$-vertices.*
(C5) *A $(1,0,0)$-vertex v is adjacent to one 2-vertex, one $(1,1,0)$-vertex and one $(1,0,0)$-vertex.*
(C6) *A $(0,0,0)$-vertex is adjacent to two $(1,1,0)$-vertices and one $(1,0,0)$-vertex.*
(C7) *A $(0,0,0)$-vertex is adjacent to two Type 2 vertices.*
(C8) *A $(0,0,0)$-vertex is adjacent to two Type 3 vertices.*
(C9) *A $(0,0,0)$-vertex is adjacent to one Type 1 vertex and one Type 2 vertex.*
(C10) *A $(0,0,0)$-vertex is adjacent to one Type 1 vertex and one Type 3 vertex.*
(C11) *A $(0,0,0)$-vertex is adjacent to one Type 2 vertex and one Type 3 vertex.*

Proof. Suppose that G is a counterexample of the lemma, i.e., a plane subcubic graph with $g(G) \geqslant 8$ and containing none of the configurations (C1)-(C11), as depicted in Figure 4.8.

Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ can be written as the following new form:

$$\sum_{v \in V(G)} (3d(v) - 8) + \sum_{f \in F(G)} (d(f) - 8) = -16. \tag{4.1}$$

We define an initial charge $\omega(v) = 3d(v) - 8$ for each $v \in V(G)$, and $\omega(f) = d(f) - 8$ for each $f \in F(G)$. Then, we design the following discharging rules:

(R1) A $(0,0)$-vertex gets 1 from each of its neighbors.

(R2) A Type 1 vertex gets 1 from its neighbor of degree 3.

(R3) A Type 2 vertex gets 1 from its neighbor of degree 3 that is not Type 1.

(R4) A Type 3 vertex gets 1 from its neighbor of degree 3 that is not Type 1.

Let $\omega^*(x)$ denote the new weight function after the discharging process is complete. Similar to the proof of Theorem 4.3.3, it suffices to show that $\omega^*(x) \geqslant 0$ for all $x \in V(G) \cup F(G)$.

Let $f \in F(G)$. Since $g(G) \geqslant 8$, $d(f) \geqslant 8$. Thus, $\omega^*(f) = d(f) - 8 \geqslant 0$. Let $v \in V(G)$. Since (C1) is excluded from G, we see that $2 \leqslant d(v) \leqslant 3$.

Case 1 $d(v) = 2$.

Then $\omega(v) = 3 \times 2 - 8 = -2$. Since G contains no (C2), v is adjacent to two 3-vertices, i.e., v is a $(0,0)$-vertex. By (R1), $\omega^*(v) \geqslant -2 + 1 \times 2 = 0$.

Case 2 $d(v) = 3$.

Then $\omega(v) = 3\times3-8 = 1$. Since G contains no (C2), v is not an initial vertex of k-threads for any $k \geqslant 2$. Let v_1, v_2, v_3 be the neighbors of v with $d(v_1) \leqslant d(v_2) \leqslant d(v_3)$. Obviously, at most two of v_1, v_2, v_3 are of degree 2 in G by the absence of (C3), i.e., $d(v_3) = 3$. We need to consider the following subcases:

Case 2.1 v is a $(1,1,0)$-vertex.

Namely, v is a type 1 vertex such that $d(v_1) = d(v_2) = 2$ and $d(v_3) = 3$. By (R1), v gives 1 to each of v_1 and v_2. If v sends nothing to v_3, then $\omega^*(v) \geqslant 1-2\times1+1 = 0$ by (R2). Otherwise, by (R1)-(R4), we deduce that $v_3 \in T_1$. Then we take v to be v, and v_3 to be t in Figure 4.8 and thus (C4) is established, which is a contradiction.

Case 2.2 v is a $(1,0,0)$-vertex.

We see that $d(v_1) = 2$ and $d(v_2) = d(v_3) = 3$. Since G contains no (C2), v_1 is a $(0,0)$-vertex. By (R1), v needs to give 1 to v_1. For each $i \in \{2,3\}$, let x_i, y_i denote the other two neighbors of v_i different from v. If v gives nothing to v_2 and v_3, then $\omega^*(v) \geqslant 1 - 1 = 0$. Otherwise, we only need to consider the following three cases.

• Assume $v_2 \in T_2$. By definition, both x_2 and y_2 are of type 1, i.e., $(1,1,0)$-vertices. Then we take v_2 to be v, x_2 to be w, y_2 to be t, and v to be s in Figure 4.8 and thus (C6) is formed, which is a contradiction.

• Assume $v_2 \in T_3$. By definition, w.l.o.g., suppose that x_2 is a 2-vertex and y_2 is a $(1,1,0)$-vertex. Then we take v_2 to be v, x_2 to be w_1, y_2 to be u, and v to be t in Figure 4.8 and thus (C5) is formed, which is a contradiction.

• Finally assume that $v_2 \in T_1$ and $v_3 \notin T_2 \cup T_3$. Since G contains no (C5), we see that $v_3 \notin T_1$. So by definition, v is a type 3 vertex, which gets 1 from v_3 by (R4). Therefore, $\omega^*(v) \geqslant 1 - 1 \times 2 + 1 = 0$ by (R1) and (R2).

Case 2.3 v is a $(0, 0, 0)$-vertex.

We see that $d(v_i) = 3$ for all $i = 1, 2, 3$. If at most one of v_1, v_2, v_3 gets 1 from v, then $\omega^*(v) \geqslant 1 - 1 = 0$. Otherwise, we may assume that v gives 1 to each of v_1 and v_2, respectively. By (R2)-(R4) and symmetry, there are some subcases to be argued as follows.

• If $v_1, v_2 \in T_2$, then we take v to be v, v_1 to be u, v_2 to be w, and v_3 to be t in Figure 4.8 and hence (C7) is established.

• If $v_1, v_2 \in T_3$, then we take v to be v, v_1 to be u, v_2 to be w, and v_3 to be t in Figure 4.8 and hence (C8) is established.

• If $v_1 \in T_1$ and $v_2 \in T_2$, then we take v to be v, v_1 to be u, v_2 to be t, and v_3 to be s in Figure 4.8 and hence (C9) is established.

• If $v_1 \in T_1$ and $v_2 \in T_3$, then we take v to be v, v_1 to be u, v_2 to be t, and v_3 to be s in Figure 4.8 and hence (C10) is established.

• If $v_1 \in T_2$ and $v_2 \in T_3$, then we take v to be v, v_1 to be u, v_2 to be t, and v_3 to be s in Figure 4.8 and hence (C11) is established.

• Assume that $v_1, v_2 \in T_1$. By the absence of (C6), $v_3 \notin T_1 \cup T_3$. It means that v is a Type 2 vertex. Moreover, $v_3 \notin T_2$ by a similar discussion as above. Thus, v_3 gives 1 to v by (R3) and we have that $\omega^*(v) \geqslant 1 - 2 \times 1 + 1 = 0$. ∎

Theorem 4.3.5 *If G is a planar subcubic graph with $g(G) \geqslant 8$, then $\chi_s^l(G) \leqslant 5$.*

Proof. We prove the theorem by induction on the vertex number of G. If $|G| \leqslant 3$, the result holds obviously. Let G be a planar subcubic graph with $|G| \geqslant 4$ and $g(G) \geqslant 8$. Let L be an assignment for G such that $|L(v)| = 5$ for all $v \in V(G)$. By Lemma 4.3.4, G contains one of the configurations (C1)-(C11). For each case, we use $\mathcal{B}$ to denote the set of all solid vertices and set $G' = G - \mathcal{B}$. Note that G' is a planar subcubic graph with $g(G') \geqslant g(G) \geqslant 8$ and $|G'| < |G|$. By the induction hypothesis, G' has an L-in-coloring c for its orientation $\overrightarrow{G'}$. To extend c to the whole graph G, we need to handle, separately, Cases (C1)-(C11). Again, for $v \in \mathcal{B}$, we use $S(v)$ to denote the set of vertices forbidden on v when we are about to color v.

(C1) There is a 1-vertex v adjacent to a vertex u.

Let $\mathcal{B} = \{v\}$ and $G' = G - \mathcal{B}$. We define an orientation for the edge uv, as shown in Figure 4.8 (C1). We can color v with a color in $L(v) \backslash S(v)$ because $S(v) = N^*(u)$ and $|N^*(u)| \leqslant 3$.

(C2) There are two adjacent 2-vertices u and v.

Let $\mathcal{B} = \{u, v\}$ and $G' = G - \mathcal{B}$. We define orientations for edges wu, uv, vt, as shown in Figure 4.8 (C2). We can color u and v in such order, since $S(u) = N^*(w) \cup \{t\}$ and $S(v) = N^*(t) \cup \{u\}$.

(C3) There is a $(1, 1, 1)$-vertex v.

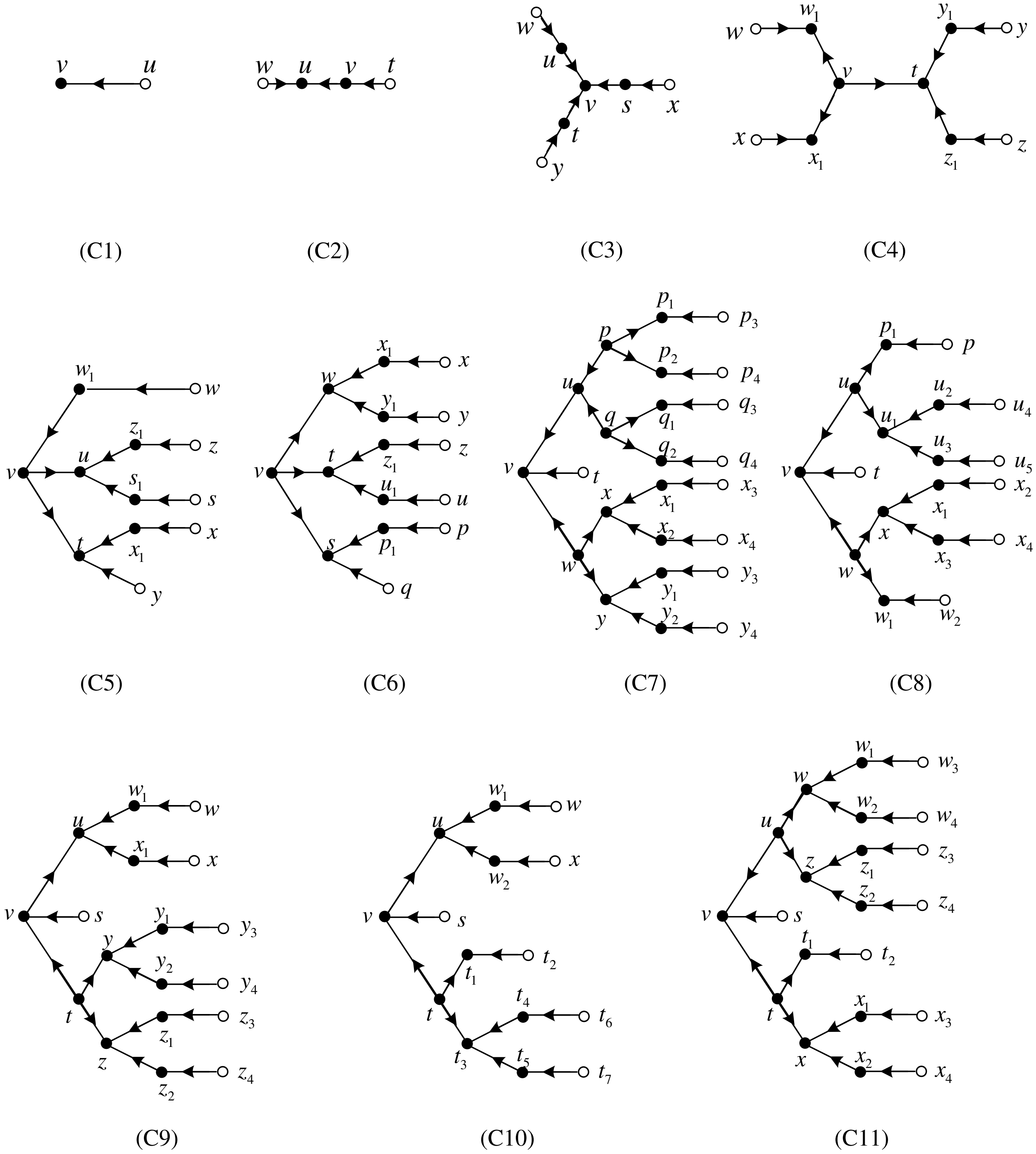


Figure 4.8: Eleven key configurations in Lemma 4.3.4.

Let $\mathcal{B} = \{v, u, s, t\}$ and $G' = G - \mathcal{B}$. We define orientations for edges vu, vs, vt, uw, sx, ty, as shown in Figure 4.8 (C3). We color u, v, s, t, successively, such that

$S(u) = N^*(w)$, $S(v) = \{u, w, x, y\}$,
$S(s) = N^*(x) \cup \{v\}$, $S(t) = N^*(y) \cup \{v\}$.

(C4) There are two adjacent $(1, 1, 0)$-vertices v and t.

Let $\mathcal{B} = \{v, t, x_1, y_1, z_1, w_1\}$ and $G' = G - \mathcal{B}$. We define orientations for $E(G[\mathcal{B}])$ and those edges between $V(G')$ and $\mathcal{B}$, as shown in Figure 4.8 (C4). We color w_1, x_1, t, v, y_1, z_1, successively, such that

$S(w_1) = N^*(w)$, $S(x_1) = N^*(x) \cup \{w_1\}$, $S(t) = \{y, z, w_1, x_1\}$,
$S(v) = \{w_1, x_1, t\}$, $S(y_1) = N^*(y) \cup \{t\}$, $S(z_1) = N^*(z) \cup \{t\}$.

(C5) There is a $(1, 0, 0)$-vertex v adjacent to a 2-vertex w_1, a $(1, 1, 0)$-vertex u and a $(1, 0, 0)$-vertex t.

Let $\mathcal{B} = \{v, u, t, w_1, s_1, x_1, z_1\}$ and $G' = G - \mathcal{B}$. We define orientations for $E(G[\mathcal{B}])$ and those edges between $V(G')$ and $\mathcal{B}$, as shown in Figure 4.8 (C5). We color $t, x_1, w_1, u, v, z_1, s_1$, successively, such that

$S(t) = N^*(y) \cup \{x\}$, $S(x_1) = N^*(x) \cup \{t\}$, $S(w_1) = N^*(w) \cup \{t\}$,
$S(u) = \{w_1, t, s, z\}$, $S(v) = \{t, w_1, w, u\}$, $S(z_1) = N^*(z) \cup \{u\}$,
$S(s_1) = N^*(s) \cup \{u\}$.

(C6) There is a $(0, 0, 0)$-vertex v adjacent to two $(1, 1, 0)$-vertices w, t and a $(1, 0, 0)$-vertex s.

Let $\mathcal{B} = \{v, w, s, t, x_1, y_1, z_1, u_1, p_1\}$ and $G' = G - \mathcal{B}$. We define orientations for $E(G[\mathcal{B}])$ and those edges between $V(G')$ and $\mathcal{B}$, as shown in Figure 4.8 (C6). We color $s, p_1, w, t, v, x_1, y_1, z_1, u_1$, successively, such that

$S(s) = N^*(q) \cup \{p\}$, $S(p_1) = N^*(p) \cup \{s\}$, $S(w) = \{x, y, s\}$,
$S(t) = \{z, u, w, s\}$, $S(v) = \{t, w, s\}$, $S(x_1) = N^*(x) \cup \{w\}$,
$S(y_1) = N^*(y) \cup \{w\}$, $S(z_1) = N^*(z) \cup \{t\}$, $S(u_1) = N^*(u) \cup \{t\}$.

(C7) There is a $(0, 0, 0)$-vertex v adjacent to two Type 2 vertices u and w.

Let $\mathcal{B} = \{v, u, w, p, q, x, y, p_1, p_2, q_1, q_2, x_1, x_2, y_1, y_2\}$ and $G' = G - \mathcal{B}$. We define orientations for $E(G[\mathcal{B}])$ and those edges between $V(G')$ and $\mathcal{B}$, as shown in Figure 4.8 (C7). We color $p_1, p_2, q_1, q_2, u, v, p, q, x, y, w, x_1, x_2, y_1, y_2$, successively, such that

$S(p_1) = N^*(p_3)$, $S(p_2) = N^*(p_4) \cup \{p_1\}$, $S(q_1) = N^*(q_3)$,
$S(q_2) = N^*(q_4) \cup \{q_1\}$, $S(u) = \{p_1, p_2, q_1, q_2\}$, $S(v) = N^*(t) \cup \{u\}$
$S(p) = \{p_1, p_2, u, v\}$, $S(q) = \{q_1, q_2, u, v\}$, $S(x) = \{x_3, x_4, v\}$
$S(y) = \{y_3, y_4, x, v\}$, $S(w) = \{v, x, y\}$, $S(x_1) = N^*(x_3) \cup \{x\}$
$S(x_2) = N^*(x_4) \cup \{x\}$, $S(y_1) = N^*(y_3) \cup \{y\}$, $S(y_2) = N^*(y_4) \cup \{y\}$.

(C8) There is a $(0, 0, 0)$-vertex v adjacent to two Type 3 vertices u and w.

Let $\mathcal{B} = \{v, u, w, p_1, u_1, u_2, u_3, x, w_1, x_1, x_3\}$ and $G' = G - \mathcal{B}$. We define orientations for $E(G[\mathcal{B}])$ and those edges between $V(G')$ and $\mathcal{B}$, as shown in Figure 4.8 (C8). We color $p_1, v, u_1, u, u_2, u_3, w_1, x, w, x_1, x_3$, successively, such that

$S(p_1) = N^*(p)$, $S(v) = N^*(t) \cup \{p_1\}$, $S(u_1) = \{u_4, u_5, v, p_1\}$,

$S(u) = \{p_1, u_1, v\}$, $S(u_2) = N^*(u_4) \cup \{u_1\}$, $S(u_3) = N^*(u_5) \cup \{u_1\}$,
$S(w_1) = N^*(w_2) \cup \{v\}$, $S(x) = \{x_2, x_4, w_1, v\}$, $S(w) = \{w_1, x, v\}$,
$S(x_1) = N^*(x_2) \cup \{x\}$, $S(x_3) = N^*(x_4) \cup \{x\}$.

(C9) There is a $(0,0,0)$-vertex v adjacent to a Type 1 vertex u and a Type 2 vertex t.

Let $\mathcal{B} = \{v, u, t, y, z, w_1, x_1, y_1, y_2, z_1, z_2\}$ and $G' = G - \mathcal{B}$. We define orientations for $E(G[\mathcal{B}])$ and those edges between $V(G')$ and $\mathcal{B}$, as shown in Figure 4.8 (C9). We color $v, u, w_1, x_1, y, z, t, y_1, y_2, z_1, z_2$, successively, such that
$S(v) = N^*(s)$, $S(u) = \{w, x, v, s\}$, $S(w_1) = N^*(w) \cup \{u\}$,
$S(x_1) = N^*(x) \cup \{u\}$, $S(y) = \{y_3, y_4, v\}$, $S(z) = \{z_3, z_4, y, v\}$,
$S(t) = \{y, z, v, u\}$, $S(y_1) = N^*(y_3) \cup \{y\}$, $S(y_2) = N^*(y_4) \cup \{y\}$,
$S(z_1) = N^*(z_3) \cup \{z\}$, $S(z_2) = N^*(z_4) \cup \{z\}$.

(C10) There is a $(0,0,0)$-vertex v adjacent to a Type 1 vertex u and a Type 3 vertex t.

Let $\mathcal{B} = \{v, u, t, w_1, w_2, t_1, t_3, t_4, t_5\}$ and $G' = G - \mathcal{B}$. We define orientations for $E(G[\mathcal{B}])$ and those edges between $V(G')$ and $\mathcal{B}$, as shown in Figure 4.8 (C10). We color $v, u, w_1, w_2, t_1, t_3, t, t_4, t_5$, successively, such that
$S(v) = N^*(s)$, $S(u) = \{w, x, v, s\}$, $S(w_1) = N^*(w) \cup \{u\}$,
$S(w_2) = N^*(x) \cup \{u\}$, $S(t_1) = N^*(t_2) \cup \{v\}$, $S(t_3) = \{t_1, t_6, t_7, v\}$,
$S(t) = \{t_1, t_3, v, u\}$, $S(t_4) = N^*(t_6) \cup \{t_3\}$, $S(t_5) = N^*(t_7) \cup \{t_3\}$.

(C11) There is a $(0,0,0)$-vertex v adjacent to a Type 2 vertex u and a Type 3 vertex t.

Let $\mathcal{B} = \{v, u, t, w, z, x, t_1, w_1, w_2, z_1, z_2, x_1, x_2\}$ and $G' = G - \mathcal{B}$. We define orientations for $E(G[\mathcal{B}])$ and those edges between $V(G')$ and $\mathcal{B}$, as shown in Figure 4.8 (C11). We color $v, w, z, u, w_1, w_2, z_1, z_2, t_1, x, t, x_1, x_2$, successively, such that
$S(v) = N^*(s)$, $S(w) = \{v, w_3, w_4\}$, $S(z) = \{z_3, z_4, v, w\}$,
$S(u) = \{w, z, v\}$, $S(w_1) = N^*(w_3) \cup \{w\}$, $S(w_2) = N^*(w_4) \cup \{w\}$,
$S(z_1) = N^*(z_3) \cup \{z\}$, $S(z_2) = N^*(z_4) \cup \{z\}$, $S(t_1) = N^*(t_2) \cup \{v\}$,
$S(x) = \{x_3, x_4, v, t_1\}$, $S(t) = \{t_1, x, v\}$, $S(x_1) = N^*(x_3) \cup \{x\}$,
$S(x_2) = N^*(x_4) \cup \{x\}$. ∎

4.3.4 Planar subcubic graphs of girth at least 12

In this section, we prove the conclusion (3) in Theorem 4.3.1. That is, we have the following:

Theorem 4.3.6 *If G is a planar subcubic graph with $g(G) \geqslant 12$, then $\chi_s^l(G) \leqslant 4$.*

Proof. Suppose to the contrary that G is a counterexample with the least number of vertices, i.e., a plane subcubic graph with $g(G) \geqslant 12$, without L-star-coloring for some assignment L with $|L(v)| = 4$ for all $v \in V(G)$, but its any subgraph G' with $|G'| < |G|$ admits an L-star-coloring. Clearly, G is connected. Similar to the proof of Claim 4.2.1 in Theorem 4.2.1, we can conclude that G does not contain 1-vertices.

Claim 4.3.6 *There is no 2-vertex adjacent to two 2-vertices.*

Proof. Assume that there is a 2-vertex x adjacent to two 2-vertices x_1 and x_2. For $i = 1, 2$, let $y_i \neq x$ be the second neighbor of x_i. Let $G' = G - \{x, x_1, x_2\}$. By the minimality of G, G' has an L-star-coloring ϕ. We color x_1 with $a \in L(x_1)\backslash N^*(y_1)$ and color x_2 with $b \in L(x_2)\backslash N^*(y_2)$. If $a \neq b$, we color x with a color in $L(x)\backslash\{a, b\}$. Otherwise, we color x with a color in $L(x)\backslash\{a, \phi(y_1), \phi(y_2)\}$. Since $|N^*(y_1)| \leqslant 3$ and $|N^*(y_2)| \leqslant 3$, the constructed coloring is an L-star-coloring of G, a contradiction. ∎

Claim 4.3.7 *If v is a $(2,1,1)$-vertex with three maximal threads $vv_1u_1w_1$, vv_2w_2 and vv_3w_3, then the following statements hold:*
(1) w_1 is neither $(2,2,0^+)$-vertex nor $(2,1^+,1^+)$-vertex.
(2) For each $i = 2, 3$, w_i is neither $(2,1^+,0^+)$-vertex nor $(1^+,1^+,1^+)$-vertex.

Proof. Assume that the claim is not true. By the minimality of G, $G' = G - \{v, v_1, v_2, v_3, u_1\}$ has an L-star-coloring ϕ. We color u_1 with $a \in L(u_1)\backslash N^*(w_1)$, v_2 with $b \in L(v_2)\backslash N^*(w_2)$, and v_3 with $c \in L(v_3)\backslash N^*(w_3)$. Let $A = \{b, c, \phi(w_2), \phi(w_3)\}$. Then $2 \leqslant |A| \leqslant 4$. We need to handle the following three possibilities according to the value of $|A|$.

• $|A| = 2$.

If $b = c$ and $\phi(w_2) = \phi(w_3)$, we color v with d belonging to $L(v)\backslash\{a, b, \phi(w_2)\}$ and v_1 with a color in $L(v_1)\backslash\{a, d, \phi(w_1)\}$. Otherwise, we may set $b = \phi(w_3)$ and $c = \phi(w_2)$. We color v with $d \in L(v)\backslash\{a, b, c\}$ and v_1 with a color in $L(v_1)\backslash\{a, d, \phi(w_1)\}$. Thus, ϕ is extended to the whole graph G, a contradiction.

• $|A| = 4$.

Without loss of generality, we may assume that $b = 1$, $c = 2$, $\phi(w_2) = 3$, and $\phi(w_3) = 4$.

First, assume that $a \notin \{3, 4\}$. We color v with $d \in L(v)\backslash\{1, 2, a\}$. If $d \notin \{3, 4\}$, we color v_1 with a color in $L(v_1)\backslash\{a, d, \phi(w_1)\}$. Otherwise, say $d = 3$, we color v_1 with a color in $L(v_1)\backslash\{1, 3, a\}$.

Next, assume that $a \in \{3, 4\}$, say $a = 3$. We color v with $d \in L(v)\backslash\{1, 2, 3\}$ and v_1 with a color in $L(v_1)\backslash\{2, 3, d\}$.

• $|A| = 3$.

(1) Assume that $b = c$, say $b = c = 1$, $\phi(w_2) = 2$, and $\phi(w_3) = 3$. If there is $d \in L(v)\backslash\{1, 2, 3, a\}$, we color v with d, then color v_1 with a color in $L(v_1)\backslash\{a, d, \phi(w_1)\}$. Otherwise, we may assume that $L(v) = \{1, 2, 3, 4\}$ and $a = 4$. We color v with 4 and v_1 with a color in $L(v_1)\backslash\{1, 4, \phi(w_1)\}$.

(2) Assume that $b \neq c$, say $b = 1$ and $c = 2$. We have to consider two cases by symmetry:

(2.1) $\phi(w_2) = 3$ and $\phi(w_3) = 1$.

If $a = 3$, we color v with $d \in L(v)\backslash\{1, 2, 3\}$ and v_1 with a color in $L(v_1)\backslash\{3, d, \phi(w_1)\}$. Otherwise, we can color v with $d \in L(v)\backslash\{a, 1, 2\}$ and v_1 with a color in $L(v_1)\backslash\{1, a, d\}$.

(2.2) $\phi(w_2) = \phi(w_3) = 3$.

If $a \in \{1, 2, 3\}$, then we color v with $d \in L(v)\backslash\{1, 2, 3\}$ and v_1 with a color in $L(v_1)\backslash\{a, d, \phi(w_1)\}$. Otherwise, we may assume that $a = 4$. If $L(v) \neq \{1, 2, 3, 4\}$, then we color v with $d \in L(v)\backslash\{1,2,3,4\}$ and v_1 with a color in $L(v_1)\backslash\{4,d,\phi(w_1)\}$. So suppose that $L(v)=\{1,2,3,4\}$. If $\phi(w_1)\neq 3$, then we color v with 4 and v_1 with a color in $L(v_1)\backslash\{1,2,4\}$. Otherwise, we assume that $\phi(w_1)=3$. Let p_1 and p_2 be the neighbors of w_1 different from u_1. If there exists $a' \in L(u_1)\backslash\{3,4,\phi(p_1),\phi(p_2)\}$, then we recolor u_1 with a', color v with 4 and v_1 with a color in $L(v_2)\backslash\{3,4,a'\}$. Thus, we may suppose that $\phi(p_1)=\alpha$, $\phi(p_2)=\beta$ and $L(u_1)=\{3,4,\alpha,\beta\}$.

In order to derive a contradiction, it is enough to handle the following four cases by symmetry:

(2.2.1) w_1 is a $(2,1^+,1^+)$-vertex.

Let $w_1p_1q_1$ and $w_1p_2q_2$ be the two 1^+-threads starting from w_1. If either q_1 or q_2 is not colored with 3, say $\phi(q_1)\neq 3$, then we can recolor u_1 with α, color v with 4 and v_1 with a color in $L(v_1)\backslash\{3,4,\alpha\}$. If $\phi(q_1)=\phi(q_2)=3$, we recolor w_1 with $\gamma \in L(w_1)\backslash\{3,\alpha,\beta\}$ and u_1 with a color in $L(u_1)\backslash\{\alpha,\beta,\gamma\}$, and then reduce to the previous case where $a\neq 4$.

(2.2.2) w_1 is a $(2,2,0^+)$-vertex.

Let $w_1p_1q_1p$ be the other 2-thread starting from w_1, different from $w_1u_1v_1v$. Let u_1,u_2 be the neighbors of p_2 different from w_1. If $\phi(q_1)\neq 3$, we recolor u_1 with α and then reduce to the previous case where $a\neq 4$. If $3\notin\{\phi(u_1),\phi(u_2)\}$, we have a similar handling. Thus, we may suppose that $\phi(q_1)=\phi(u_1)=3$. If $L(w_1)\neq\{3,\alpha,\beta,\phi(u_2)\}$, then we recolor w_1 with a color in $L(w_1)\backslash\{3,\alpha,\beta,\phi(u_2)\}$ and u_1 with 3, then reduce to the previous case. So assume that $L(w_1)=\{3,\alpha,\beta,\phi(u_2)\}$. We recolor w_1 with α, u_1 with 3, p_1 with a color in $L(p_1)\backslash\{3,\alpha,\phi(p)\}$, and then reduce to the previous case.

(2.2.3) w_2 is a $(1^+,1^+,1^+)$-vertex.

Let $w_2x_1y_1$ and $w_2x_2y_2$ be the two 1^+-threads starting from w_2, different from w_2v_2v. Let $\phi(x_1)=c_1$ and $\phi(x_2)=c_2$. If $L(v_2)\neq\{1,3,c_1,c_2\}$, then we recolor v_2 with a color in $L(v_2)\backslash\{1,3,c_1,c_2\}$, and color v with 1 and v_2 with a color in $L(v_2)\backslash\{1,3,4\}$. Thus, we may suppose that $c_1\neq c_2$ and $L(v_2)=\{1,3,c_1,c_2\}$. If at most one of y_1,y_2 is colored with 3, say $\phi(y_2)\neq 3$, then we recolor v_2 with c_2, color v with 1 and v_1 with a color in $L(v_1)\backslash\{1,3,4\}$. If $\phi(y_1)=\phi(y_2)=3$, we recolor w_2 with a color in $L(w_2)\backslash\{3,c_1,c_2\}$, v_2 with 3, then go back to the former case.

(2.2.4) w_2 is a $(2,1^+,0^+)$-vertex.

Let $w_2s_1s_2s$ be the 2-thread starting from w_2. Let $t\in N_G(w_2)\backslash\{s_1,v_2\}$ and $t_1,t_2\in N_G(t)\backslash\{w_2\}$. Let $\phi(s_1)=c_1$ and $\phi(t)=c_2$. If $L(v_2)\neq\{1,3,c_1,c_2\}$, then we recolor v_2 with a color in $L(v_2)\backslash\{1,3,c_1,c_2\}$ and v with 1, then color v_2 with a color in $L(v_2)\backslash\{1,3,4\}$. Thus, we may suppose that $c_1\neq c_2$ and $L(v_2)=\{1,3,c_1,c_2\}$. If $\phi(s_2)\neq 3$, we recolor v_2 with c_1, color v with 1 and v_1 with a color in $L(v_1)\backslash\{1,3,4\}$. If $3\notin\{\phi(t_1),\phi(t_2)\}$, we have a similar proof. So assume that $\phi(s_2)=3$ and $\phi(t_1)=3$. If $L(w_2)\neq\{3,c_1,c_2,\phi(t_2)\}$, then we recolor w_2 with a color in $L(w_2)\backslash\{3,c_1,c_2,\phi(t_2)\}$ and v_2 with 3, then go back to the previous case. If $L(w_2)=\{3,c_1,c_2,\phi(t_2)\}$, we recolor s_1 with a color in $L(s_1)\backslash\{3,c_1,\phi(s)\}$, w_2 with c_1, v_2 with 3, and then reduce to the former case. ∎

By Lemma 4.3.2, any subgraph G' with $|G'|<|G|$ has an L-in-coloring c for its some orientation $\overrightarrow{G'}$. In the proofs of Claims 4.3.8-4.3.12, we first remove a proper subset $\mathcal{B}$ of $V(G)$ and let $G'=G-\mathcal{B}$, then establish an orientation for $E(G[\mathcal{B}])$ and

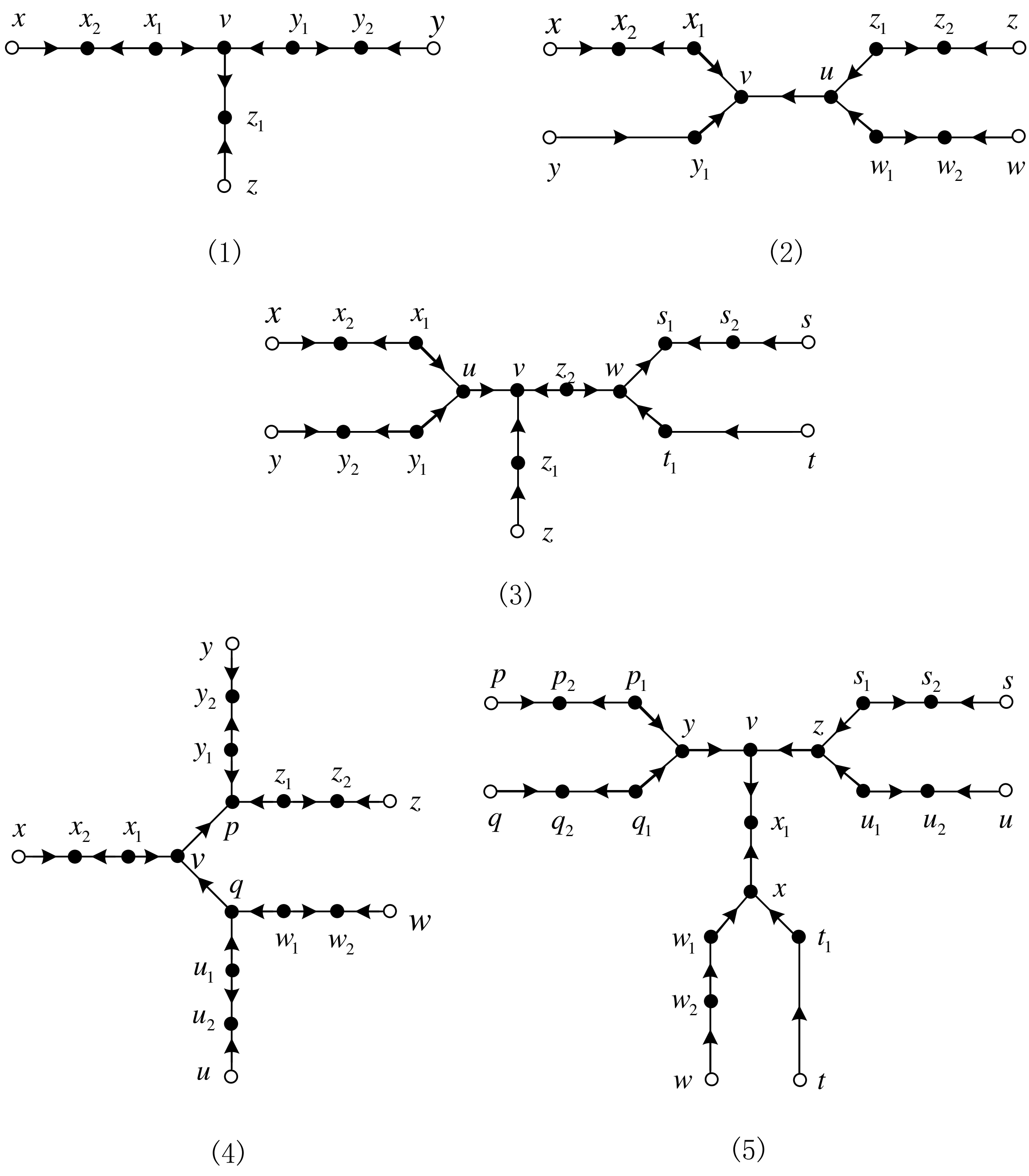


Figure 4.9: Five reducible configurations in Claims 4.3.8- 4.3.12.

those edges between $V(G')$ and $\mathcal{B}$, finally extend an L-in-coloring of $\overrightarrow{G'}$ to the whole graph $\overrightarrow{G}$. Again, by Lemma 4.3.2, we get an L-star-coloring of G, contradicting the choice of G.

Claim 4.3.8 *G contains no $(2,2,1^+)$-vertex.*

Proof. Suppose to the contrary that G contains a $(2,2,1^+)$-vertex v such that vx_1x_2x, vy_1y_2y, and vz_1z are three threads starting from v. Let $\mathcal{B} = \{v, x_1, x_2, y_1, y_2, z_1\}$ and $G' = G - \mathcal{B}$. We define orientations for $E(G[\mathcal{B}])$ and those edges between $V(G')$ and $\mathcal{B}$, as shown in Figure 4.9 (1). We color $x_2, y_2, z_1, v, x_1, y_1$, successively, such that

$S(x_2) = N^*(x)$, $S(y_2) = N^*(y)$, $S(z_1) = N^*(z)$,
$S(v) = \{x_2, y_2, z_1\}$, $S(x_1) = \{x_2, z_1, v\}$, $S(y_1) = \{v, z_1, y_2\}$. ∎

Claim 4.3.9 *There is no $(2,1^+,0^+)$-vertex adjacent to a $(2,2,0)$-vertex.*

Proof. Suppose to contrary that there is a $(2,1^+,0^+)$-vertex v adjacent to $(2,2,0)$-vertex u. Let $\mathcal{B} = \{u, v, x_1, x_2, y_1, z_1, z_2, w_1, w_2\}$ and $G' = G - \mathcal{B}$. We define orientations for $E(G[\mathcal{B}])$ and those edges between $V(G')$ and $\mathcal{B}$, as shown in Figure 4.9 (2). We color $x_2, y_1, z_2, w_2, v, u, x_1, z_1, w_1$, successively, such that

$S(x_2) = N^*(x)$, $S(y_1) = N^*(y)$, $S(z_2) = N^*(z)$,
$S(w_2) = N^*(w)$, $S(v) = \{x_2, y_1, y\}$, $S(u) = \{z_2, w_2, v\}$,
$S(x_1) = \{x_2, v\}$, $S(z_1) = \{z_2, u, v\}$, $S(w_1) = \{w_2, u, v\}$. ∎

Claim 4.3.10 *Suppose that v is a 3-vertex adjacent to u, z_1, z_2 such that $d(u) = 3$ and $d(z_1) = d(z_2) = 2$. Let $z \neq v$ be the second neighbor of z_1, and $w \neq v$ be the second neighbor of z_2. If u is a $(2,2,0)$-vertex, then neither z nor w is a $(2,1,1)$-vertex.*

Proof. Suppose to the contrary that w is a $(2,1,1)$-vertex. Let $\mathcal{B} = \{u, v, w, x_1, x_2, y_1, y_2, z_1, z_2, s_1, s_2, t_1\}$ and $G' = G - \mathcal{B}$. We define orientations for $E(G[\mathcal{B}])$ and those edges between $V(G')$ and $\mathcal{B}$, as shown in Figure 4.9 (3). We color $x_2, y_2, z_1, s_2, t_1, s_1, w, v, u, x_1, y_1, z_2$, successively, such that

$S(x_2) = N^*(x)$, $S(y_2) = N^*(y)$, $S(z_1) = N^*(z)$,
$S(s_2) = N^*(s)$, $S(t_1) = N^*(t)$, $S(s_1) = \{s, s_2, t_1\}$,
$S(w) = \{t, s_1, t_1\}$, $S(v) = \{z, w, z_1\}$, $S(u) = \{v, x_2, y_2\}$,
$S(x_1) = \{u, v, x_2\}$, $S(y_1) = \{u, v, y_2\}$, $S(z_2) = \{v, w, s_1\}$. ∎

Claim 4.3.11 *There is no $(2,0,0)$-vertex adjacent to exactly two $(2,2,0)$-vertices.*

Proof. Suppose to contrary that there is a $(2,2,0)$-vertex v adjacent to two $(2,2,0)$-vertices p and q. Let $\mathcal{B} = \{v, p, q, x_1, x_2, y_1, y_2, u_1, u_2, z_1, z_2, w_1, w_2\}$ and $G' = G - \mathcal{B}$. We define orientations for $E(G[\mathcal{B}])$ and those edges between $V(G')$ and $\mathcal{B}$, as shown in Figure 4.9 (4). We color $x_2, y_2, z_2, w_2, u_2, p, q, v, x_1, y_1, z_1, w_1, u_1$, successively, such that

$S(x_2) = N^*(x)$, $S(y_2) = N^*(y)$, $S(z_2) = N^*(z)$,

$S(w_2) = N^*(w)$, $S(u_2) = N^*(u)$, $S(p) = \{y_2, z_2\}$,
$S(q) = \{w_2, u_2, p\}$, $S(v) = \{x_2, p, q\}$, $S(x_1) = \{x_2, v, p\}$,
$S(y_1) = \{y_2, p\}$, $S(z_1) = \{z_2, p\}$, $S(w_1) = \{w_2, q, v\}$,
$S(u_1) = \{u_2, q, v\}$. ∎

Claim 4.3.12 *Suppose that v is a 3-vertex adjacent to y, z, x_1 such that $d(y) = d(z) = 3$ and $d(x_1) = 2$. Let $x \neq v$ be the second neighbor of x_1. If both y and z are $(2, 2, 0)$-vertices, then x is not a $(2, 1, 1)$-vertex.*

Proof. Suppose to the contrary that x is a $(2, 1, 1)$-vertex. Let $\mathcal{B} = \{v, x, y, z, x_1, t_1, w_1, w_2, u_1, u_2, s_1, s_2, p_1, p_2, q_1, q_2\}$ and $G' = G - \mathcal{B}$. We define orientations for $E(G[\mathcal{B}])$ and those edges between $V(G')$ and $\mathcal{B}$, as shown in Figure 4.9 (5). We color $w_2, t_1, p_2, q_2, s_2, u_2, x, w_1, x_1, z, y, v, s_1, u_1, p_1, q_1$, successively, such that
$S(w_2) = N^*(w)$, $S(t_1) = N^*(t)$, $S(p_2) = N^*(p)$,
$S(q_2) = N^*(q)$, $S(s_2) = N^*(s)$, $S(u_2) = N^*(u)$,
$S(x) = \{w_2, t, t_1\}$, $S(w_1) = \{w_2, w, x\}$, $S(x_1) = \{w_1, x, t_1\}$,
$S(z) = \{s_2, u_2, x_1\}$, $S(y) = \{p_2, q_2, x_1\}$, $S(v) = \{x_1, y, z\}$,
$S(s_1) = \{s_2, v, z\}$, $S(u_1) = \{u_2, z, v\}$, $S(p_1) = \{p_2, y, v\}$,
$S(q_1) = \{q_2, y, v\}$. ∎

This time, we use the following rewritten Euler's formula:

$$\sum_{v \in V(G)} (5d(v) - 12) + \sum_{f \in F(G)} (d(f) - 12) = -24. \tag{4.2}$$

We define $\omega(v) = 5d(v) - 12$ for each $v \in V(G)$ and $\omega(f) = d(f) - 12$ for each $f \in F(G)$. New discharging rules are designed as follows:

(R1) Each $(1, 0)$-vertex gets 2 from its neighbor of degree 3.

(R2) Each $(0, 0)$-vertex gets 1 from each of its neighbors.

(R3) Suppose that v is a $(2, 1, 1)$-vertex with two 1-threads vv_1u_1 and vx_2u_2. Then v gets 0.5 from each of u_1 and u_2.

(R4) Each $(2, 2, 0)$-vertex gets 1 from its neighbor of degree 3.

Let $\omega^*(x)$ denote the new charge function after the discharging process is complete. It suffices to verify that $\omega^*(x) \geqslant 0$ for all $x \in V(G) \cup F(G)$.

Let $f \in F(G)$. Since $g(G) \geqslant 12$, $d(f) \geqslant 12$. So, $\omega^*(f) = d(f) - 12 \geqslant 0$.

Let $v \in V(G)$. Then $2 \leqslant d(v) \leqslant 3$. We need to consider two cases:

Case 1 $d(v) = 2$.

We see that $\omega(v) = 5 \times 2 - 12 = -2$. By Claim 4.3.6, v is not a $(1, 1)$-vertex. If v is a $(1, 0)$-vertex, then $\omega^*(v) \geqslant -2 + 2 = 0$ by (R1). If v is a $(0, 0)$-vertex, then $\omega^*(v) \geqslant -2 + 1 + 1 = 0$ by (R2).

Case 2 $d(v) = 3$.

We see that $\omega(v) = 5 \times 3 - 12 = 3$. By Claim 4.3.8, v is not a $(2, 2, 1^+)$-vertex. So we have to consider several subcases as follows:

• v is a $(2, 2, 0)$-vertex.

Let z denote the neighbor of v with $d(z) = 3$. By Claim 4.3.9, v is not a $(2, 2, 0)$-vertex. By (R4), v gets 1 from z. Hence, $\omega^*(v) \geqslant 3-2\times 2+1 = 0$ by (R1).

• v is a $(2, 1, 1)$-vertex.

Let vy_1y and vz_1z be two 1-threads starting from v. By Claim 4.3.7, both y and z are not a $(2, 1, 1)$-vertex. So v gets 0.5 from each of y and z by (R3). Consequently, $\omega^*(v) \geqslant 3 - 2 - 1 \times 2 + 0.5 \times 2 = 0$ by (R1) and (R2).

• v is a $(2, 1, 0)$-vertex.

Let vy_1y denote the 1-thread starting from v, and z the neighbor of degree 3 of v. By Claim 4.3.9, z is not a $(2, 2, 0)$-vertex, hence gets nothing from v. By Claim 4.3.7, y is not a $(2, 1, 1)$-vertex, hence gets nothing from v. By (R1) and (R2), $\omega^*(v) \geqslant 3 - 2 - 1 = 0$.

• v is a $(2, 0, 0)$-vertex.

Let y and z be the neighbors of degree 3 of v. By Claim 4.3.11, at most one of y and z is a $(2, 2, 0)$-vertex, and hence v sends 1 to at most one of y and z. By (R1) and (R4), $\omega^*(v) \geqslant 3 - 2 - 1 = 0$.

• v is a $(1, 1, 1)$-vertex.

Let vx_1x, vy_1y and vz_1z be three 1-threads starting from v. By Claim 4.3.7, each of x, y, z is not a $(2, 1, 1)$-vertex, hence gets nothing from v. Since v gives exactly 1 to each of x_1, y_1, z_1 by (R2), $\omega^*(v) \geqslant 3 - 1 \times 3 = 0$.

• v is a $(1, 1, 0)$-vertex.

Let vx_1x and vy_1y be two 1-threads starting from v, and z be the neighbor of degree 3 of v. If z is not a $(2, 2, 0)$-vertex, then it is easy to deduce that $\omega^*(v) \geqslant 3 - 1 \times 2 - 0.5 \times 2 = 0$ by (R2) and (R3). If z is a $(2, 2, 0)$-vertex, then neither x nor y is a $(2, 1, 1)$-vertex by Claim 4.3.10. Thus, $\omega^*(v) \geqslant 3 - 1 \times 3 = 0$ by (R2) and (R4).

• v is a $(1, 0, 0)$-vertex.

Let vx_1x be the 1-thread starting from v. Let y and z be the neighbors of degree 3 of v. If at most one of y and z is a $(2, 2, 0)$-vertex, then $\omega^*(v) \geqslant 3-1-1-0.5 = 0.5$ by (R2), (R3) and (R4). Otherwise, assume that y and z are both $(2, 2, 0)$-vertices. By Claim 4.3.12, x cannot be a $(2, 1, 1)$-vertex. It follows that v sends at most 1 to each of its neighbors by (R2) and (R4). Therefore, $\omega^*(v) \geqslant 3 - 1 \times 3 = 0$.

• v is a $(0, 0, 0)$-vertex.

It is easy to deriver that $\omega^*(v) \geqslant 3 - 1 \times 3 = 0$ by (R4). ∎

4.4 Star choosability of sparse graphs

In this section, we extend the conclusion (3) in Theorem 4.1.4 to a more general result, which avoids the planar constraint. More precisely, we prove the following:

Theorem 4.4.1 [CRW09] *Every graph G with* $\text{Mad}(G) < 3$ *is 8-star-choosable.*

Suppose $\overrightarrow{G}$ is an oriented graph. For $v \in V(\overrightarrow{G})$, we define the *outdegree vertices set* of v by $D^+_{\overrightarrow{G}}(v) = \{u | u \in N_{\overrightarrow{G}}(v) \text{ and } v \to u\}$. A *special* orientation $\overrightarrow{G}$ of G is an orientation in which each vertex v satisfies $|D^+_{\overrightarrow{G}}(v)| \leqslant 2$.

So, in order to control the number of colors used in an in-coloring, it is useful to bound the maximum outdegree of the orientation $\overrightarrow{G}$. In 1981, Taris [Tar81] observed a fact that a graph has an orientation with maximum outdegree at most d if and only if $\text{Mad}(G) \leqslant 2d$. This implies that every graph with $\text{Mad}(G) < 3$ has an orientation with maximum outdegree at most 2. Therefore, to obtain our Theorem 4.4.1, we only need to prove the following Theorem 4.4.2 by Lemma 4.3.2.

Theorem 4.4.2 *Every graph G with* $\text{Mad}(G) < 3$ has an orientation of maximum outdegree at most 2 which admits an 8-in-coloring.

4.4.1 Proof of Theorem 4.4.2

Suppose to the contrary that G is a counterexample with the least number of vertices to Theorem 4.4.2. Thus G is connected. Moreover, for any subgraph H with $|H| < |G|$ admits an 8-in-coloring of some special orientation $\overrightarrow{H}$. We first discuss some properties of G, then use discharging technique to derive a contradiction.

In what follows, let L be a list assignment of G with $|L(v)| = 8$ for all $v \in V(G)$. By the definition of maximum average degree and Tarsi's observation, we first note the following statement.

Observation 4.4.3 *Every subgraph $H \subseteq G$ admits a special orientation.*

So, in the following argument, we always admit a special orientation $\overrightarrow{H}$ of H. Moreover, for $v \in V(\overrightarrow{H})$, define $N^*_{\overrightarrow{H}}(v) = D^+_{\overrightarrow{H}}(v) \cup \{v\}$. It is obvious that $|N^*_{\overrightarrow{G}}(v)| \leqslant 3$. For simplicity, we write $N^*(v)$ for $N^*_{\overrightarrow{H}}(v)$. We further use $S(v)$ to denote the set of vertices forbidden on v by the definition of L-in-coloring when we are about to color v.

Claim 4.4.1 *G contains no 1-vertex.*

Proof. Suppose that x is a 1-vertex of G and y is the neighbor of x. Let $H = G - \{x\}$. By the minimality of G, H admits an L-in-coloring c of some special orientation $\overrightarrow{H}$. We orient x to y to establish an orientation $\overrightarrow{G}$ of G. Clearly, the resulting orientation $\overrightarrow{G}$ is special. Now, we assign a color to x in $L(x)$, different from the colors of the vertices in $N^*(y)$. It is easy to see that the color for x is reasonable and thus we extend c to G. A contradiction. ∎

In the proof of Claims 4.4.2 to 4.4.8, we use $\mathcal{B}$ to denote the set of all solid vertices, depicted in Figure 4.10 to Figure 4.17. Let $H = G - \mathcal{B}$. By the minimality of G, H admits an L-in-coloring c of some special orientation $\overrightarrow{H}$. We give an orientation of $G[\mathcal{B}]$ and those edges between $V(H)$ and $\mathcal{B}$ such that the resulting orientation $\overrightarrow{G}$ is special. Then we extend c to $\mathcal{B}$ to obtain an L-in-coloring of $\overrightarrow{G}$, which contradicts the choice of G.

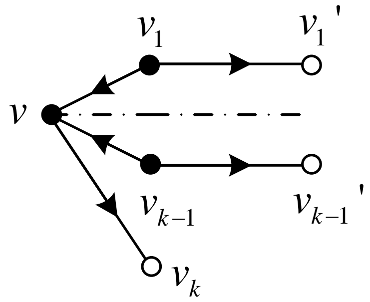


Figure 4.10: v is a $k(k-1)$-vertex.

Claim 4.4.2 *G contains no $k(k-1)$-vertex for any $2 \leqslant k \leqslant 5$.*

Proof. Assume to the contrary that v is a $k(k-1)$-vertex with $2 \leqslant k \leqslant 5$. Denote $v_1, \cdots, v_k$ be the neighbors of v. Without loss of generality, assume that $d(v_i) = 2$ for all $i \in \{1, \cdots, k-1\}$ and $d(v_k) \geqslant 2$. For each $i \in \{1, \cdots, k-1\}$, let v_i' be the other neighbor of v_i different from v.

Let $\mathcal{B} = \{v, v_1, \cdots, v_{k-1}\}$ and $H = G - \mathcal{B}$. By the minimality of G, H has an L-in-coloring c of some special orientation $\overrightarrow{H}$. We construct an orientation for the edge set $E(G[\mathcal{B}])$ and those edges between $V(H)$ and $\mathcal{B}$, as shown in Figure 4.10. One can easily check that the resulting orientation $\overrightarrow{G}$ is also special. Notice that $|N^*(u)| \leqslant 3$ for each $u \in \{v_1', \cdots, v_{k-1}', v_k\}$. Based on c, we can color $v, v_1, \cdots, v_{k-1}$, successively, because

- $S(v) = N^*(v_k) \cup \{v_1', \cdots, v_{k-1}'\}$;
- $S(v_i) = N^*(v_i') \cup \{v, v_k\}$, for each $i \in \{1, \cdots, k-1\}$.

Obviously, for each vertex $x \in \mathcal{B}$ we have $|S(x)| \leqslant 3 + (k-1) = k+2 \leqslant 7$ because $2 \leqslant k \leqslant 5$. By a careful inspection, the resulting coloring is an L-in-coloring of $\overrightarrow{G}$. A contradiction. ∎

Assume that $P = v_1 v_2 \cdots v_n$ is an induced path with $n \geqslant 3$ and all internal vertices are 3-vertices. If $d(v_1) = d(v_n) = 2$ then P is called a *good path.* If $d(v_1) = 2$ and $d(v_n) \geqslant 4$ then P is called a *bad path.* If $d(v_1) = 2$ and $d(v_n) = 3$ then P is called a *terrible path.* For simplicity, we use $P(v_1 \to v_n)$ to denote an orientation for the edge set $E(P)$ in such a way that $v_i \to v_{i+1}$ for each $i \in \{1, \cdots, n-1\}$.

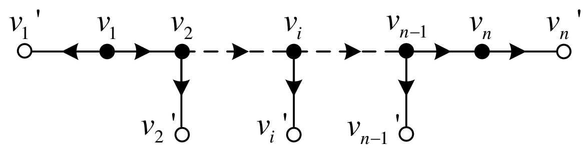


Figure 4.11: A good path $P = v_1 v_2 \cdots v_n$.

Claim 4.4.3 *There is no good path in G.*

Proof. Assume to the contrary that there exists a good path $P = v_1 v_2 \cdots v_n$ with $n \geqslant 3$ in G. It implies that v_1, v_n are both 2-vertices and the remaining vertices are all 3-vertices. Since P is an induced path, for each vertex $v_i \in V(P)$, we may let v_i' be the other neighbor of v_i which is not on P.

Let $\mathcal{B} = \{v_1, \cdots, v_n\}$ and $H = G - \mathcal{B}$. By the minimality of G, H has an L-in-coloring c of some special orientation $\overrightarrow{H}$. We define an orientation for the edge set $E(G[\mathcal{B}]) \cup \{v_1v_1', \cdots, v_iv_i', \cdots, v_nv_n'\}$ in the following way: $P(v_1 \to v_n)$ and $v_j \to v_j'$ for each $j \in \{1, \cdots, n\}$, as depicted in Figure 4.11. It is easy to check that the resulting orientation $\overrightarrow{G}$ is special. We can color $v_1, v_2, \cdots, v_n$, successively, such that

- $S(v_1) = N^*(v_1') \cup \{v_2'\}$;
- $S(v_2) = N^*(v_2') \cup \{v_1, v_1', v_3'\}$;
- $S(v_i) = N^*(v_i') \cup \{v_{i-1}, v_{i-2}, v_{i-1}', v_{i+1}'\}$, for each $i \in \{3, \cdots n-1\}$;
- $S(v_n) = N^*(v_n') \cup \{v_{n-1}, v_{n-2}, v_{n-1}'\}$.

Since $|S(v)| \leqslant 7$ for each vertex $v \in \mathcal{B}$, the resultant coloring is an L-in-coloring of G. A contradiction. ∎

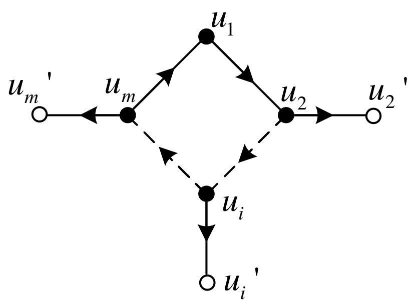


Figure 4.12: A good cycle $C = u_1u_2 \cdots u_mu_1$.

A cycle C is called *good* if C is formed from a good path $P = v_1v_2 \cdots v_n$ by identifying 2-vertices v_1 and v_n.

Claim 4.4.4 *There is no good cycle in G.*

Proof. Suppose to the contrary that $C = u_1u_2 \cdots u_mu_1$ is a good cycle such that $d(u_1) = 2$ and $d(u_i) = 3$ for all $i \in \{2, \cdots, m\}$. Notice that $m \geqslant 3$. Since C is formed from a good path which is also an induced path, we may let u_i' be the third neighbor of u_i that is not on C, for each $i \in \{2, \cdots, m\}$.

Let $\mathcal{B} = \{u_1, \cdots, u_m\}$ and $H = G - \mathcal{B}$. By the choice of G, H admits an L-in-coloring c of a special orientation $\overrightarrow{H}$. We define an orientation for the edge set $E(G[\mathcal{B}]) \cup \{u_2u_2', \cdots, u_iu_i', \cdots, u_mu_m'\}$ in the following way: for each $j \in \{2, \cdots, m-1\}$, set $u_j \to u_{j+1}$, $u_j \to u_j'$; we further set $u_1 \to u_2$, $u_m \to u_1$ and $u_m \to u_m'$, see Figure 4.12. We notice that the resulting orientation $\overrightarrow{G}$ is also special. Based on c, we can color $u_2, u_3, \cdots, u_m, u_1$, successively, such that

- $S(u_2) = N^*(u_2') \cup \{u_3'\}$;
- $S(u_i) = N^*(u_i') \cup \{u_{i-1}, u_{i-2}, u_{i-1}', u_{i+1}'\}$, for each $i \in \{3, \cdots, m-1\}$;
- $S(u_m) = N^*(u_m') \cup \{u_2, u_{m-1}, u_{m-2}, u_{m-1}'\}$;
- $S(u_1) = \{u_2, u_2', u_3, u_m, u_m', u_{m-1}\}$.

Since $|S(v)| \leqslant 7$ for each vertex $v \in \mathcal{B}$, the resultant coloring is an L-in-coloring of G. ∎

A cycle C is called *light* if every vertex is of degree 3. C is called *simple* if it has no chords. A *simple light cycle* is a light cycle that is simple. Suppose that $C = v_1v_2 \cdots v_nv_1$ is a simple light cycle. If there exists a terrible path P connecting one vertex in C, say v_1, such that $V(P) \cap V(C) = \{v_1\}$, then C is called a *removable cycle*, where v_1 is called a *heavy* 3-vertex of C.

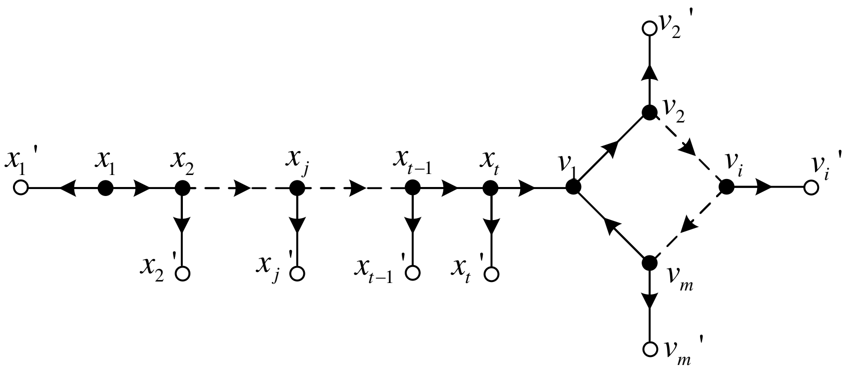


Figure 4.13: A removable cycle $C = v_1v_2 \cdots v_mv_1$ with a heavy 3-vertex v_1.

Claim 4.4.5 *There is no removable cycle in G.*

Proof. Suppose to the contrary that there exists a removable cycle $C = v_1v_2 \cdots v_mv_1$ with a heavy 3-vertex v_1 such that $P = x_1 \cdots x_tv_1$ is a terrible path. Namely, x_1 is a 2-vertex and the remaining other vertices of P are 3-vertices such that $V(P) \cap V(C) = \{v_1\}$. For each $i \in \{2, \cdots, m\}$, let v_i' be the another neighbor of v_i not on C. Since P is an induced path, we further let x_j' be the another neighbor of x_j not on P for each $j \in \{1, \cdots, t\}$. In the following, denote $A_1 = \{x_1, \cdots, x_t\}$ and $A_2 = \{v_2, \cdots, v_m\}$. We have to consider the following two cases.

Case 1 $wz \notin E(G)$ for all $w \in A_1$ and $z \in A_2$.

It means that the third neighbor of x_j is not in C. Let $\mathcal{B} = V(C) \cup V(P)$ and $H = G - \mathcal{B}$. By the choice of G, H admits an L-in-coloring c of a special orientation $\overrightarrow{H}$. We define an orientation for the edge set $E(G[\mathcal{B}])$ and those edges between $V(H)$ and $\mathcal{B}$, as shown in Figure 4.13. By a careful inspection, $\overrightarrow{G}$ is a special orientation. So we can color $v_2, \cdots, v_m, v_1, x_t, \cdots, x_1$, successively, such that

- $S(v_2) = N^*(v_2') \cup \{v_3'\}$;
- $S(v_i) = N^*(v_i') \cup \{v_{i-1}, v_{i-2}, v_{i-1}', v_{i+1}'\}$, for each $i \in \{3, \cdots, m-1\}$;
- $S(v_m) = N^*(v_m') \cup \{v_{m-1}, v_{m-2}, v_{m-1}', v_2\}$;
- $S(v_1) = \{v_2, v_2', v_3, v_m, v_m', v_{m-1}\}$;
- $S(x_t) = N^*(x_t') \cup \{x_{t-1}', v_1, v_2\}$;
- $S(x_{t-1}) = N^*(x_{t-1}') \cup \{x_{t-2}', x_t, x_t', v_1\}$;
- $S(x_j) = N^*(x_j') \cup \{x_{j-1}', x_{j+1}, x_{j+2}, x_{j+1}'\}$, for each $j \in \{t-2, \cdots, 2\}$;
- $S(x_1) = N^*(x_1') \cup \{x_2, x_2', x_3\}$.

Case 2 $wz \in E(G)$, where $w \in A_1$ and $z \in A_2$.

Case 2.1 $w = x_1$ and $z \in A_2$.

This means that $x_1v_s \in E(G)$, where $s \in \{2, 3, \cdots, m\}$. If none of $v_{s+1}, \cdots, v_m$ is adjacent to x_j for some fixed $j \in \{2, \cdots, t\}$, then $x_1x_2 \cdots x_tv_1v_m \cdots v_sx_1$ is a good cycle, which contradicts Claim 4.4.4. Otherwise, we may suppose that $x_jv_k \in E(G)$ for some fixed $k \in \{s+1, \cdots, m\}$ such that there is no edge between $\{x_2, x_3 \cdots, x_{j-1}\}$ and $\{v_{s+1}, v_{s+2}, \cdots, v_{k-1}\}$. However, a good cycle $x_1x_2 \cdots x_jv_kv_{k-1} \cdots v_sx_1$ is established, contradicting Claim 4.4.4.

Case 2.2 $w \in \{x_2, x_3, \cdots, x_t\}$ and $z \in A_2$.

We may suppose that $x_jv_s \in E(G)$ for some fixed $s \in \{2, \cdots, m\}$ such that there is no edge between $\{x_2, x_3 \cdots, x_{j-1}\}$ and $V(C) - \{v_1\}$. If $x_lv_q \notin E(G)$ for all $l \in \{j+1, j+2, \cdots, t\}$ and $q \in \{s+1, s+2, \cdots, m\}$, then a removable cycle $x_jx_{j+1} \cdots x_tv_1v_m \cdots v_sx_j$ with a heavy 3-vertex x_j is formed and then the proof is reduced to the former Case 1. Otherwise, we may suppose that $x_kv_q \in E(G)$ for some fixed $q \in \{s+1, s+2, \cdots, m\}$ such that there is no edge between $\{x_{j+1}, x_{j+2}, \cdots, x_{k-1}\}$ and $\{v_{s+1}, v_{s+2}, \cdots, v_{q-1}\}$. However, a removable cycle $x_jx_{j+1} \cdots x_kv_qv_{q-1} \cdots v_sx_j$ with a heavy 3-vertex x_j is constructed and then go back to the previous Case 1. ∎

Suppose that $P = v_1v_2 \cdots v_n$ is a bad path such that $d(v_1) = 2$, $d(v_n) \geqslant 4$, and $d(v_i) = 3$ for all $i \in \{2, \cdots, n-1\}$. We say that v_n is a *sponsor* of v_2 and v_2 is a *target* of v_n. Moreover, let $\mathcal{T}(v_n)$ denote the set of targets of v_n and let $\mathcal{S}(v_2)$ denote the set of sponsors of v_2.

Claim 4.4.6 *For each 4^+-vertex v, we have $|\mathcal{T}(v)| \leqslant d(v) - n_2(v)$.*

Proof. Let x_1 be a 3^+-vertex adjacent to v. It suffices to show that there is at most one bad path starting from edge vx_1. If $d(x_1) \geqslant 4$, then vx_1 is not a bad path and thus we are done. Otherwise, we may suppose that $P = vx_1 \cdots x_m$ is a bad path with a target x_{m-1} such that $d(x_m) = 2$ and $d(x_i) = 3$ for all $i = 1, \cdots, m-1$. Next, we are going to show that there is no other bad path starting from edge vx_1 and thus conclude the proof of Claim 4.4.6.

Without loss of generality, assume that $P \neq P' = vx_1 \cdots x_ix'_{i+1} \cdots x'_{s-1}x'_s$ is a bad path with a target x'_{s-1} of v. So $d(x'_s) = 2$ and $d(x'_k) = 3$ for all $k \in \{i+1, \cdots, s-1\}$. Let $B_1 = \{x'_{i+1}, x'_{i+2}, \cdots, x'_s\}$ and $B_2 = \{x_{i+1}, x_{i+2}, \cdots, x_m\}$. The proof is divided into the two cases below.

Case 1 $wz \notin E(G)$ for all $w \in B_1$ and $z \in B_2$.

This implies that $B_1 \cap B_2 = \varnothing$. It is easy to observe that a good path $x_mx_{m-1} \cdots x_{i+1}x_ix'_{i+1}\ x'_{i+2} \cdots x'_s$ is established. This contradicts Claim 4.4.3.

Case 2 $wz \in E(G)$, where $w \in B_1$ and $z \in B_2$.

By symmetry, we only need to consider the following two possibilities.

Case 2.1 $w \in \{x'_{i+1}, x'_{i+2}, \cdots, x'_{s-1}\}$ and $z \in \{x_{i+1}, x_{i+2}, \cdots, x_{m-1}\}$.

Denote $z = x_k$ for some fixed $k \in \{i+1, i+2, \cdots, m-1\}$. We may assume $x'_j = w$ such that $x'_j x_k \in E(G)$ and x'_j is the nearest 3-vertex to x_i on P'. In other words, there is no edges between $\{x'_{i+1}, x'_{i+2}, \cdots, x_{j-1}\}$ and $\{x_{i+1}, x_{i+2}, \cdots, x_{k-1}\}$. It is obvious that $x_i x_{i+1} \cdots x_k x'_j x'_{j-1} \cdots x'_{i+1} x_i$ is a simple light cycle with a heavy 3-vertex x_k. So such kind of cycle is removable, a contradiction to Claim 4.4.5.

Case 2.2 $w \in \{x'_{i+1}, x'_{i+2}, \cdots, x'_{s-1}\}$ and $z = x_m$.

Denote $w = x_j$, where $j \in \{i+1, \cdots, s-1\}$. Clearly, $x_m x_{m-1} \cdots x_{i+1} x_i x'_{i+1} \cdots x'_j x_m$ is a good cycle, which is impossible by Claim 4.4.4. ∎

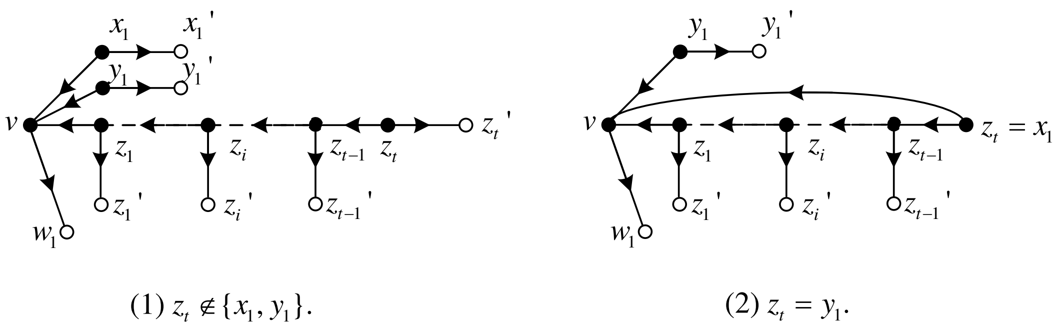


Figure 4.14: v is a 4(2)-vertex with a target z_{t-1}.

Claim 4.4.7 *If v is a 4(2)-vertex, then* $|\mathcal{T}(v)| = 0$.

Proof. Let v be a 4(2)-vertex with four neighbors x_1, y_1, w_1, z_1 such that $d(x_1) = d(y_1) = 2$ and $d(z_1), d(w_1) \geqslant 3$. Suppose to the contrary that $|\mathcal{T}(v)| \geqslant 1$. We further suppose that $P = vz_1 \cdots z_t$ is a bad path connecting v and v's target z_{t-1}. Let $N_G(x_1) = \{v, x'_1\}$ and $N_G(y_1) = \{v, y'_1\}$. For each $k \in \{1, \cdots, t\}$, let z'_k be the another neighbor of z_k that is not on P. Obviously, $x_1 \neq y_1$. Let $\mathcal{B} = V(P) \cup \{x_1, y_1\}$ and $H = G - \mathcal{B}$. Let c denote an L-in-coloring of H for its special orientation $\overrightarrow{H}$. By symmetry of G, we only need to consider two cases below.

Case 1 $z_t \notin \{x_1, y_1\}$.

We define an orientation for the edge set $E(G[\mathcal{B}])$ and those edges between $V(H)$ and $\mathcal{B}$, as depicted in Figure 4.14 (1). It is easy to inspect that the resulting orientation of $\overrightarrow{G}$ is a special orientation. Basing on c, we can color $v, x_1, y_1, z_1, \cdots, z_t$, successively, such that

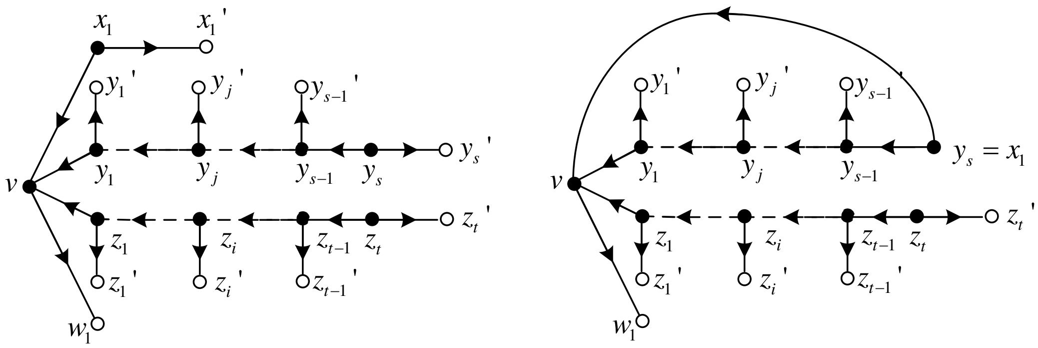
- $S(v) = N^*(w_1) \cup \{x'_1, y'_1, z'_1\}$;
- $S(x_1) = N^*(x'_1) \cup \{v, w_1\}$;
- $S(y_1) = N^*(y'_1) \cup \{v, w_1\}$;
- $S(z_1) = N^*(z'_1) \cup \{v, w_1, z'_2\}$;
- $S(z_2) = N^*(z'_2) \cup \{z_1, z'_1, v, z'_3\}$;
- $S(z_i) = N^*(z'_i) \cup \{z_{i-1}, z_{i-2}, z'_{i-1}, z'_{i+1}\}$, for each $i \in \{3, \cdots, t-1\}$;
- $S(z_t) = N^*(z'_t) \cup \{z_{t-1}, z'_{t-1}, z_{t-2}\}$.

Case 2 $z_t = y_1$.

We define an orientation for the edge set $E(G[\mathcal{B}])$ and those edges between $V(H)$ and $\mathcal{B}$, as shown in Figure 4.14 (2). We observe that the resulting orientation of $\overrightarrow{G}$ is special. Basing on c, we may color $v, x_1, z_1, \cdots, z_t$, successively, such that

- $S(v) = N^*(w_1) \cup \{x_1', z_1'\}$;
- $S(x_1) = N^*(x_1') \cup \{v, w_1\}$;
- $S(z_1) = N^*(z_1') \cup \{v, w_1, z_2'\}$;
- $S(z_2) = N^*(z_2') \cup \{z_1, z_1', v, z_3'\}$;
- $S(z_i) = N^*(z_i') \cup \{z_{i-1}, z_{i-2}, z_{i-1}', z_{i+1}'\}$, for each $i \in \{3, \cdots, t-1\}$;
- $S(z_t) = \{v, w_1, z_{t-1}, z_{t-1}', z_{t-2}\}$.

Therefore, we complete the proof of Claim 4.4.7. ∎



(1) $x_1 \neq y_s \neq z_t \neq x_1$.   (2) $x_1 = y_s \neq z_t$.

Figure 4.15: The Case 1 in Claim 4.4.8.

Claim 4.4.8 *If v is a $4(1)$-vertex, then $|\mathcal{T}(v)| \leqslant 1$.*

Proof. Let v be a $4(1)$-vertex with four neighbors x_1, y_1, w_1, z_1 such that x_1 is a 2-vertex and y_1, z_1, w_1 are all 3^+-vertices. Suppose to the contrary that $|\mathcal{T}(v)| \geqslant 2$. Now, assume that there exist two bad paths P, P', respectively, starting from vy_1, vz_1. We denote by $P = vy_1 \cdots y_s$ and $P' = vz_1 \cdots z_t$. Obviously, $d(y_s) = d(z_t) = 2$ and the remaining internal vertices of P and P' are all 3-vertices. Denote x_1' be the other neighbor of x_1 distinct from v. Let y_j' be the third neighbor of y_j that is not on P. Similarly, let z_k' be the third neighbor of z_k that is not on P'. For our convenience, we denote $C_1 = \{y_1, \cdots, y_s\}$ and $C_2 = \{z_1, \cdots, z_t\}$. We only need to consider the two cases as follows.

Case 1 $yz \notin E(G)$ for all $y \in C_1$ and $z \in C_2$

This implies that $C_1 \cap C_2 = \varnothing$. Let $\mathcal{B} = V(P) \cup V(P') \cup \{x_1\}$ and $H = G - \mathcal{B}$. Let c denote an L-in-coloring of H for its special orientation $\overrightarrow{H}$. To complete the proof of Case 1, we have to discuss the following two possibilities, depending on the situations of x_1, y_s and z_t.

Case 1.1 $x_1 \neq y_s \neq z_t \neq x_1$.

We define an orientation for the edge set $E(G[\mathcal{B}])$ and those edges between $V(H)$ and $\mathcal{B}$, as shown in Figure 4.15 (1). One can easily check that the resulting orientation of $\overrightarrow{G}$ is also special. We color $v, x_1, y_1, \cdots, y_s, z_1, \cdots, z_t$, successively, such that

- $S(v) = N^*(w_1) \cup \{x_1', y_1', z_1'\}$;
- $S(x_1) = N^*(x_1') \cup \{v, w_1\}$;
- $S(y_1) = N^*(y_1') \cup \{v, w_1, y_2'\}$;
- $S(y_2) = N^*(y_2') \cup \{y_1, y_1', v, y_3'\}$;
- $S(y_j) = N^*(y_j') \cup \{y_{j-1}, y_{j-2}, y_{j-1}', y_{j+1}'\}$, for each $j \in \{3, \cdots, s-1\}$;
- $S(y_s) = N^*(y_s') \cup \{y_{s-1}, y_{s-1}', y_{s-2}\}$;
- $S(z_1) = N^*(z_1') \cup \{v, w_1, z_2'\}$;
- $S(z_2) = N^*(z_2') \cup \{z_1, z_1', v, z_3'\}$;
- $S(z_i) = N^*(z_i') \cup \{z_{i-1}, z_{i-2}, z_{i-1}', z_{i+1}'\}$, for each $i \in \{3, \cdots, t-1\}$;
- $S(z_t) = N^*(z_t') \cup \{z_{t-1}, z_{t-1}', z_{t-2}\}$.

Case 1.2 $x_1 = y_s \neq z_t$.

We define an orientation for the edge set $E(G[\mathcal{B}])$ and those edges between $V(H)$ and $\mathcal{B}$, as depicted in Figure 4.15 (2). It is easy to observe that the resulting orientation of $\overrightarrow{G}$ is special. We color $v, y_1, \cdots, y_s, z_1, \cdots, z_t$, successively, such that

- $S(v) = N^*(w_1) \cup \{y_1', z_1'\}$;
- $S(y_1) = N^*(y_1') \cup \{v, w_1, y_2'\}$;
- $S(y_2) = N^*(y_2') \cup \{y_1, y_1', v, y_3'\}$;
- $S(y_j) = N^*(y_j') \cup \{y_{j-1}, y_{j-2}, y_{j-1}', y_{j+1}'\}$, for each $j \in \{3, \cdots, s-2\}$;
- $S(y_{s-1}) = N^*(y_{s-1}') \cup \{y_{s-2}, y_{s-2}', y_{s-3}, v\}$;
- $S(y_s) = \{v, w_1, y_{s-1}, y_{s-1}', y_{s-2}\}$;
- $S(z_1) = N^*(z_1') \cup \{v, w_1, z_2'\}$;
- $S(z_2) = N^*(z_2') \cup \{z_1, z_1', v, z_3'\}$;
- $S(z_i) = N^*(z_i') \cup \{z_{i-1}, z_{i-2}, z_{i-1}', z_{i+1}'\}$, for each $i \in \{3, \cdots, t-1\}$;
- $S(z_t) = N^*(z_t') \cup \{z_{t-1}, z_{t-1}', z_{t-2}\}$.

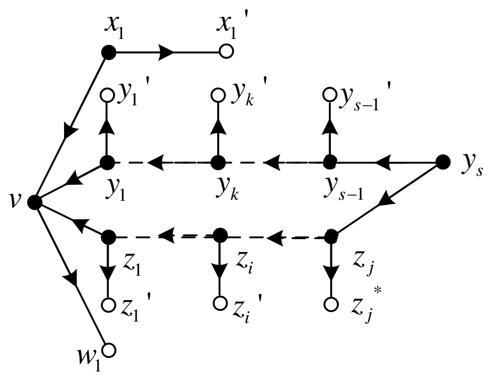


Figure 4.16: The Case 2.1.1 in Claim 4.4.8.

Case 2 $yz \in E(G)$, where $y \in C_1$ and $z \in C_2$.

We need to consider the following two subcases, according to the situation of z.

Case 2.1 $z \in \{z_1, \cdots, z_{t-1}\}$.

We may denote $z = z_j$ such that z_j is the nearest 3-vertex to v on P'. The proof is divided into two possibilities.

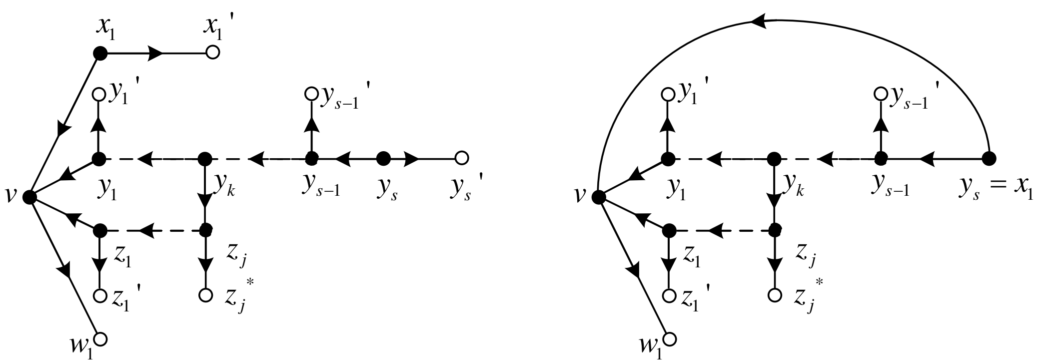
Case 2.1.1 $y = y_s$.

This means that $z_j y_s \in E(G)$ for some fixed $j \in \{1, \cdots, t\}$. Let $\mathcal{B} = V(P) \cup \{x_1, z_1, \cdots, z_j\}$ and $H = G - \mathcal{B}$. Let c denote an L-in-coloring of H for its special orientation $\overrightarrow{H}$. We define an orientation for the edge set $E(G[\mathcal{B}])$ and those edges between $V(H)$ and $\mathcal{B}$, as shown in Figure 4.16 (3). Obviously, the resulting orientation $\overrightarrow{G}$ is special. We color $v, x_1, y_1, \cdots, y_{s-1}, z_1, \cdots, z_j, y_s$, successively, such that

- $S(v) = N^*(w_1) \cup \{x'_1, y'_1, z'_1\}$;
- $S(x_1) = N^*(x'_1) \cup \{v, w_1\}$;
- $S(y_1) = N^*(y'_1) \cup \{v, w_1, y'_2\}$;
- $S(y_2) = N^*(y'_2) \cup \{y_1, y'_1, v, y'_3\}$;
- $S(y_k) = N^*(y'_k) \cup \{y_{k-1}, y_{k-2}, y'_{k-1}, y'_{k+1}\}$, for each $k \in \{3, \cdots, s-2\}$;
- $S(y_{s-1}) = N^*(y'_{s-1}) \cup \{y_{s-2}, y'_{s-2}, y_{s-3}\}$;
- $S(z_1) = N^*(z'_1) \cup \{v, w_1, z'_2\}$;
- $S(z_2) = N^*(z'_2) \cup \{z_1, z'_1, v, z'_3\}$;
- $S(z_i) = N^*(z'_i) \cup \{z_{i-1}, z_{i-2}, z'_{i-1}, z'_{i+1}\}$, for each $i \in \{3, \cdots, j-2\}$;
- $S(z_{j-1}) = N^*(z'_{j-1}) \cup \{z_{j-2}, z'_{j-2}, z_{j-3}, z^*_j\}$;
- $S(z_j) = N^*(z^*_j) \cup \{z_{j-1}, z'_{j-1}, z_{j-2}, y_{s-1}\}$;
- $S(y_s) = \{y_{s-1}, y'_{s-1}, y_{s-2}, z_j, z^*_j, z_{j-1}\}$,

Case 2.1.2 $y \in \{y_1, \cdots, y_{s-1}\}$.

Without loss of generality, we may let $y = y_k$. If $z_j y_s \in E(G)$, then a good cycle $y_s y_{s-1} \cdots y_k z_j y_s$ is formed, contradicting Claim 4.4.4. If $z_j y_l \in E(G)$ for some fixed $l \in \{k+1, k+2, \cdots, s-1\}$, then a removable cycle $y_l y_{l-1} \cdots y_k z_j y_l$ with a heavy 3-vertex y_l is formed, contradicting Claim 4.4.5. So, in what follows, we suppose that there is no edge connecting z_j and one vertex belonging to $\{y_{k+1}, \cdots, y_s\}$. On the other hand, we recall that z_q with $q \in \{1, \cdots, j-1\}$ is not adjacent to any vertex of $y_{k+1}, \cdots, y_s$ since z_j is the nearest vertex to v on P'.



(1) $z_j y_k \in E(G)$ and $x_1 \neq y_s$. (2) $z_j y_k \in E(G)$ and $x_1 = y_s$.

Figure 4.17: The Case 2.1.2 in Claim 4.4.8.

Let $\mathcal{B} = V(P) \cup \{x_1, z_1, \cdots, z_j\}$ and $H = G - \mathcal{B}$. Let c denote an L-in-coloring of H for its special orientation $\overrightarrow{H}$. We need to deal with the following two possibilities, according to the situations of x_1 and y_s.

(i) $x_1 \neq y_s$.

We define an orientation for the edge set $E(G[\mathcal{B}])$ and those edges between $V(H)$ and $\mathcal{B}$, as depicted in Figure 4.17 (1). Clearly, the resulting orientation of $\overrightarrow{G}$ is special. We color $v, x_1, z_1, \cdots, z_j, y_1, \cdots, y_s$, successively, such that

- $S(v) = N^*(w_1) \cup \{x'_1, y'_1, z'_1\}$;
- $S(x_1) = N^*(x'_1) \cup \{v, w_1\}$;
- $S(z_1) = N^*(z'_1) \cup \{v, w_1, z'_2\}$;
- $S(z_i) = N^*(z'_i) \cup \{z_{i-1}, z_{i-2}, z'_{i-1}, z'_{i+1}\}$, for each $i \in \{2, \cdots, j-2\}$;
- $S(z_{j-1}) = N^*(z'_{j-1}) \cup \{z_{j-2}, z'_{j-2}, z_{j-3}, z^*_j\}$;
- $S(z_j) = N^*(z^*_j) \cup \{z_{j-1}, z_{j-2}, z'_{j-1}\}$;
- $S(y_1) = N^*(y'_1) \cup \{v, w_1, y'_2\}$;
- $S(y_l) = N^*(y'_l) \cup \{y_{l-1}, y_{l-2}, y'_{l-1}, y'_{l+1}\}$, for each $l \in \{2, \cdots, k-2\}$;
- $S(y_{k-1}) = N^*(y'_{k-1}) \cup \{y_{k-2}, y_{k-3}, y'_{k-2}, z_j\}$;
- $S(y_k) = \{y_{k-1}, y_{k-2}, y'_{k-1}, z_j, z_{j-1}, z^*_j, y'_{k+1}\}$;
- $S(y_{k+1}) = N^*(y_{k+1}) \cup \{y_k, y_{k-1}, z_j, y'_{k+2}\}$;
- $S(y_p) = N^*(y'_p) \cup \{y_{p-1}, y_{p-2}, y'_{p-1}, y'_{p+1}\}$ for each $p \in \{k+2, \cdots, s-1\}$;
- $S(y_s) = N^*(y'_s) \cup \{y_{s-1}, y'_{s-1}, y_{s-2}\}$.

(ii) $x_1 = y_s$.

We define an orientation for the edge set $E(G[\mathcal{B}])$ and those edges in $V(H)$ and $\mathcal{B}$, as shown in Figure 4.17 (2). Noting that the resulting orientation of $\overrightarrow{G}$ is special. We color $v, z_1, \cdots, z_j, y_1, \cdots, y_s$, successively, such that

- $S(v) = N^*(w_1) \cup \{y'_1, z'_1\}$;
- $S(z_1) = N^*(z'_1) \cup \{v, w_1, z'_2\}$;
- $S(z_i) = N^*(z'_i) \cup \{z_{i-1}, z_{i-2}, z'_{i-1}, z'_{i+1}\}$, for each $i \in \{2, \cdots, j-2\}$;
- $S(z_{j-1}) = N^*(z'_{j-1}) \cup \{z_{j-2}, z'_{j-2}, z_{j-3}, z^*_j\}$;
- $S(z_j) = N^*(z^*_j) \cup \{z_{j-1}, z_{j-2}, z'_{j-1}\}$;
- $S(y_1) = N^*(y'_1) \cup \{v, w_1, y'_2\}$;
- $S(y_l) = N^*(y'_l) \cup \{y_{l-1}, y_{l-2}, y'_{l-1}, y'_{l+1}\}$, for each $l \in \{2, \cdots, k-2\}$;
- $S(y_{k-1}) = N^*(y'_{k-1}) \cup \{y_{k-2}, y_{k-3}, y'_{k-2}, z_j\}$;
- $S(y_k) = \{y_{k-1}, y_{k-2}, y'_{k-1}, z_j, z_{j-1}, z^*_j, y'_{k+1}\}$;
- $S(y_{k+1}) = N^*(y_{k+1}) \cup \{y_k, y_{k-1}, z_j, y'_{k+2}\}$;
- $S(y_p) = N^*(y'_p) \cup \{y_{p-1}, y_{p-2}, y'_{p-1}, y'_{p+1}\}$ for each $p \in \{k+2, \cdots, s-2\}$;
- $S(y_{s-1}) = N^*(y'_{s-1}) \cup \{y_{s-2}, y'_{s-2}, y_{s-3}, v\}$;
- $S(y_s) = \{y_{s-1}, y'_{s-1}, y_{s-2}, v, w_1\}$.

Case 2.2 $z = z_t$.

The proof can be reduced to the previous Case 2.1.1.

Therefore, we complete the proof of Claim 4.4.8 ∎

Now we use a discharging argument with initial charge $\omega(v) = d(v)$ at each vertex v and with the following two discharging rules (R1) and (R2). We write

ω^* to denote the charge at each vertex v after we apply the discharging rules. To complete the proof, we show that $\omega^*(v) \geqslant 3$ for all $v \in V(G)$. This leads to the following obvious contradiction:

$$3 \leqslant \frac{\sum_{v\in V(G)} \omega^*(v)}{|V(G)|} = \frac{\sum_{v\in V(G)} \omega(v)}{|V(G)|} = \frac{2|E(G)|}{|V(G)|} \leqslant \text{Mad}(G) < 3.$$

Hence no counterexample can exist.

Our discharging rules are defined as follows:

(R1) Each 2-vertex gets $\frac{1}{2}$ from each of its neighbors.

(R2) Each 3(1)-vertex gets $\frac{1}{4}$ from each of its sponsors.

Let us check that $\omega^*(v) \geqslant 3$ for each $v \in V(G)$. By Claim 4.4.1, we derive that $\delta(G) \geqslant 2$. In the following argument, we let $v_1, v_2, \cdots, v_{d(v)}$ denote all neighbors of v in a cyclic order. The following discussion is divided into five cases.

Case 1 $d(v) = 2$.

Then $\omega(v) = 2$. By Claim 4.4.2, there is no 2(1)-vertex. It means that v_1, v_2 are both 3^+-vertices. Therefore, $\omega^*(v) \geqslant 2 + \frac{1}{2} \times 2 = 3$ by (R1).

Case 2 $d(v) = 3$.

Obviously, $\omega(v) = 3$. We begin with the following claim.

Claim 4.4.9 *If v is a* 3(1)*-vertex, then* $|\mathcal{S}(v)| \geqslant 2$.

Proof. Without loss of generality, suppose that v_1 is a 2-vertex and v_2, v_3 are both 3^+-vertices. By Claim 4.4.3, it is easy to deduce that there exist at least two bad paths, respectively, starting from vv_2 and vv_3. It follows immediately that $|\mathcal{S}(v)| \geqslant 2$. ∎

According to Claim 4.4.2, we infer that v is neither a 3(2)-vertex nor a 3(3)-vertex. So, it suffices to consider the following two subcases.

• If v is a 3(0)-vertex, then v sends nothing to each v_i by (R1) and (R2) and thus $\omega^*(v) = 3$.

• Now we suppose that v is a 3(1)-vertex. Without loss of generality, assume $d(v_1) = 2$ and $d(v_2), d(v_3) \geqslant 3$. By (R1), v sends a charge $\frac{1}{2}$ to v_1. On the other hand, by Claim 4.4.9, we observe that v has at least two sponsors, each of which sends a charge $\frac{1}{4}$ to v by (R2). Therefore, $\omega^*(v) \geqslant 3 - \frac{1}{2} + \frac{1}{4} \times 2 = 3$.

Case 3 $d(v) = 4$.

This implies that $\omega(v) = 4$. By using Claim 4.4.2, we derive that v is neither a 4(4)-vertex nor a 4(3)-vertex. If v is a 4(2)-vertex, then $|\mathcal{T}(v)| = 0$ by Claim 4.4.7. So, $\omega^*(v) \geqslant 4 - \frac{1}{2} \times 2 = 3$ by (R2). If v is a 4(1)-vertex, then $|\mathcal{T}(v)| \leqslant 1$ by Claim 4.4.8 and therefore $\omega^*(v) \geqslant 4 - \frac{1}{2} - \frac{1}{4} \times 1 = 3\frac{1}{4} > 3$ by (R2). Finally, we suppose that v is a 4(0)-vertex. It means that $n_2(v) = 0$. Then $|\mathcal{T}(v)| \leqslant 4$ by Claim 4.4.6 and we conclude that $\omega^*(v) \geqslant 4 - \frac{1}{4} \times 4 = 3$ by (R2).

Girth	Best Known Bounds	
g	Lower bound	Upper bound
3	10 [ACK+04]	20 [ACK+04]
4	8 [KKT09]	18 [NOdM03]
5	6 [Tim07]	16 [ACK+04]
6	5 [Tim07]	8 [KT10]
7	5[Tim08]	7 [Tim07]
8	4 [ACK+04]	6 [Tim07] [BCM+09]
9 – 13	4 [ACK+04]	5 [Tim08]
14+	4 [ACK+04]	4 [Tim08]

Table 4.1: Best known bounds

Case 4 $d(v) = 5$.

Obviously, $\omega(v) = 5$. By Claim 4.4.2, we deduce that $n_2(v) \leqslant 3$. Moreover, it follows immediately from Claim 4.4.6 that $|\mathcal{T}(v)| \leqslant 5 - n_2(v)$. So, by applying (R1) and (R2), we obtain that $\omega^*(v) \geqslant 5 - \frac{1}{2}n_2(v) - \frac{1}{4}|\mathcal{T}(v)| \geqslant 5 - \frac{1}{2}n_2(v) - \frac{1}{4}(5 - n_2(v)) = 3\frac{3}{4} - \frac{1}{4}n_2(v) \geqslant 3\frac{3}{4} - \frac{1}{4} \times 3 = 3$.

Case 5 $d(v) \geqslant 6$.

It follows directly from Claim 4.4.6 that $\omega^*(v) \geqslant d(v) - \frac{1}{2}n_2(v) - \frac{1}{4}|\mathcal{T}(v)| \geqslant d(v) - \frac{1}{2}n_2(v) - \frac{1}{4}(d(v) - n_2(v)) = \frac{3}{4}d(v) - \frac{1}{4}n_2(v) \geqslant \frac{3}{4}d(v) - \frac{1}{4}d(v) = \frac{1}{2}d(v) \geqslant 3$. ∎

4.5 Known bounds and open problems

The Table 4.1 shows the current best known bounds for the star chromatic number of planar graphs with girth g. The best known bound is given along with the corresponding reference.

By Table 4.1, we see that planar graphs with girth 4 are 18-star-colorable. Recently, Kierstead, Kündgen and Timmons [KKT09] showed that bipartite planar graphs are 14-star-choosable. Since the girth of bipartite planar graphs is also 4, it seems to be interesting to study the following problem.

Problem 4.5.1 *Can the upper bound* 18 *on star chromatic number of planar graphs with girth* 4 *be improved to* 14*?*

Besides, Table 4.1 also shows that planar graphs of girth at least 14 can be star colored with 4 colors and there is a planar graph with girth 7 that requires 5 colors to star color. Finally, we like to conclude this chapter by the following problem:

Problem 4.5.2 *What is the smallest girth g such that planar graphs with girth g is* 4*-star-colorable.*

Chapter 5

Vertex arboricity

Contents

In this chapter, we study the vertex-arboricity of graphs, which has significant applications in various problems of colorings and partitions of graphs. In Section 5.1, we will give a brief introduction. And then in Section 5.2, we prove the conjecture of Raspaud and Wang in [RW08] asserting that every planar graph without intersecting triangles has vertex-arboricity at most 2.

5.1 Introduction

The *vertex-arboricity* $\mathrm{va}(G)$ of a graph G is the minimum number of subsets into which vertex set $V(G)$ can be partitioned so that each subset induces a forest; such a partition is called an *acyclic partition* of $V(G)$. Clearly, $\mathrm{va}(G) \geqslant 1$ for every nonempty graph G and $\mathrm{va}(G) = 1$ if and only if G itself is a forest.

This vertex version of arboricity was first introduced by Chartrand, Kronk, and Wall [CKW68] in 1968, who named it point-arboricity. They proved that $\mathrm{va}(G) \leqslant \lceil \frac{1+\Delta(G)}{2} \rceil$ for any graph G and $\mathrm{va}(G) \leqslant 3$ for any planar graph G. Chartrand and Kronk [CK69] showed this bound is sharp, by giving a planar graph which has vertex-arboricity 3. In fact, this graph was discovered by Professor W. T. Tutte, which was used to disprove the conjecture of P. G. Tait that the graph of every cubic convex polyhedron is hamiltonian (see [Tut46]).

The upper bound 3 for $\mathrm{va}(G)$ on planar graphs has also been studied by Chartrand and Kronk [CK69], Grünbaum [Grü73], Goddard [God91], and Poh[Poh90].

Among them, Goddard [God91] and Poh [Poh90], independently, proved a stronger result that the vertex set of any planar graph can be partitioned into three sets such that each set induces a linear forest. The path version of vertex-arboricity, called *linear vertex-arboricity*, has also been studied extensively in [Poh90, AGLW91, ALW94, Mat90].

In 1979, Garey and Johnson [GJ79] proved that determining the vertex-arboricity of a graph is NP-hard. Hakimi and Schmeichel [HS89] showed that determining whether $\text{va}(G) \leqslant 2$ is NP-complete for maximal planar graphs G. Stein [Ste71] characterizes completely maximal planar graph G with at least 4 vertices by proving that $\text{va}(G) = 2$ if and only if its dual graph G^* is Hamiltonian. This result was further strengthened by Hakimi and Schmeichel [HS89] by showing that a plane graph G has $\text{va}(G) = 2$ if and only if its dual graph G^* contains a connected Eulerian spanning subgraph. The reader is referred to [Bur86, CCC04, CH96, Coo74, Wan88, Škr02] for other results about the vertex-arboricity of graphs.

Now we introduce an equivalent definition to the vertex-arboricity in terms of the coloring version. A *k-forest-coloring* of a graph G is a mapping π from $V(G)$ to the set $\{1, \cdots, k\}$ such that each color class induces a forest. The *vertex-arboricity* $\text{va}(G)$ of G is the smallest integer k such that G has a k-forest-coloring. We should notice that two adjacent vertices can be assigned with the same color in a k-forest-coloring.

Raspaud and Wang [RW08] gave some sufficient conditions on a planar graph to have vertex-arboricity at most 2. Their main results are stated as follows:

Theorem 5.1.1 [RW08] *Let G be a planar graph.*

(1) *If G contains no k-cycles for some fixed $k \in \{3, 4, 5, 6\}$, then $\text{va}(G) \leqslant 2$.*

(2) *If G contains no triangles at distance less than 2, then $\text{va}(G) \leqslant 2$.*

In 2000, Borodin, Kostochka and Toft [BKT00] first introduced the list vertex-arboricity. In terms of the list coloring version, we say G is *L-forest-colorable* if for any sets $L(v)$ of cardinality at least k at its vertices, one can choose an element (color) for each vertex v from its list $L(v)$ so that the subgraph induced by every color class is a forest (an acyclic graph). In [BI08a], Borodin and Ivanova improved the conclusion (2) in Theorem 5.1.1 to the list vertex-arboricity.

Together with Raspaud and Wang, we give a positive answer to one of their conjectures in [RW08]. More specifically, we prove the following:

Theorem 5.1.2 [CRW10c] *Every planar graph G without intersecting triangles has vertex-arboricity at most 2.*

5.2 Proof of Theorem 5.1.2

Suppose to the contrary that the theorem is not true. Let G be a counterexample with the least number of vertices. Thus, G is connected. Since G contains no intersecting triangles, every subgraph of G also contains no intersecting triangles.

This straightforward fact is tacitly used in the following proofs. In the following, let $C = \{a, b\}$ denote the color set. We first investigate the structural properties of G in Section 5.2.1, then use Euler's formula and discharging argument to derive a contradiction in Section 5.2.2.

5.2.1 Structural properties

Claim 5.2.1 *The minimum degree $\delta(G) \geqslant 4$.*

Proof. Assume to the contrary that G contains a 3^--vertex v. By the minimality of G, $G - \{v\}$ is 2-forest-colorable and thus it has an acyclic partition (V_1, V_2). Obviously, there is some V_i, say V_1, such that v is adjacent to at most one vertex in V_1. So $(V_1 \cup \{v\}, V_2)$ is an acyclic partition of G, which is a contradiction. This completes the proof of Claim 5.2.1. ∎

We begin with some basic definitions which are used throughout this section. A k-face $f = [u_1 u_2 \cdots u_k]$ of G is called *light* if $d(u_i) = 4$ for all $i = 1, \cdots, k$. Let $f = [v_1 v_2 \cdots v_5]$ be a 5-face in G. If $d(v_1) = 5$, $d(v_i) = 4$ for all $i = 2, 3, 4, 5$, and f is adjacent to exactly two light 4-faces by sharing edges $v_2 v_3$ and $v_4 v_5$, respectively, then we call f *bad*. Otherwise, we call f *good*. For $x \in V(G) \cup F(G)$ and an integer $i \geqslant 3$, we use $m_i(x)$ denote the number of i-faces incident or adjacent to x and $l(x)$ to denote the number of light 4-faces incident or adjacent to x. Obviously, $l(x) \leqslant m_4(x)$.

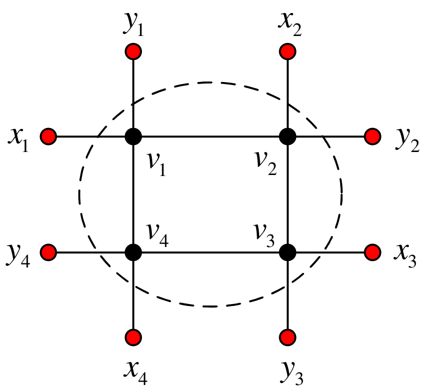


Figure 5.1: $x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4$ are all colored with a, i.e., red.

Lemma 5.2.1 *Let $f = [v_1 v_2 v_3 v_4]$ be a light 4-face and $H = G - V(f)$. If a 2-forest-coloring π of $G - V(f)$ cannot be extended to G, then the following conditions hold.*

(1) *All vertices in $\bigcup_{i=1}^{i=4} N_H(v_i)$ are assigned with the same color, say a, see Figure 5.1.*

(2) *f is adjacent to at least one 5^+-face.*

Proof. For $i \in \{1, 2, 3, 4\}$, let x_i, y_i be the other two neighbors of v_i not on f. Suppose π is a 2-forest-coloring of $G - V(f)$ which cannot be extended to G. Let

f_i be the face adjacent to f by the common edge v_iv_{i+1}, where i is taken modulo 4. Let $S(a)$ denote the subset of $\{\{x_1, y_1\}, \{x_2, y_2\}, \{x_3, y_3\}, \{x_4, y_4\}\}$ which satisfies that all vertices in $S(a)$ get the same color a in the coloring π. Thus $0 \leqslant |S(a)| \leqslant 4$. We will make contradiction to show (1) and (2).

(1) Suppose to the contrary that $|S(a)| \neq 4$. It implies that $0 \leqslant |S(a)| \leqslant 3$. Since G contains no adjacent triangles, $v_1v_3 \notin E(G)$ and $v_2v_4 \notin E(G)$. We have to consider the following four cases, depending on the value of $|S(a)|$.

- $|S(a)| = 3$. Without loss of generality, assume that $\pi(x_i) = \pi(y_i) = a$ for all $i = 1, 2, 3$ and one of x_4 and y_4 is colored with b. We can color v_1, v_2, v_3 with b, and v_4 with a.
- $|S(a)| = 2$. First assume, without loss of generality, that $\pi(x_1) = \pi(y_1) = \pi(x_2) = \pi(y_2) = a$ and $\pi(x_3) = \pi(x_4) = b$. If both y_3 and y_4 are colored with b, we color v_1, v_2 with b and v_3, v_4 with a. Otherwise, w.l.o.g., assume that $\pi(y_3) = a$. We color v_1, v_2, v_3 with b and v_4 with a. Now assume, w.l.o.g., that $\pi(x_1) = \pi(y_1) = \pi(x_3) = \pi(y_3) = a$ and $\pi(x_2) = \pi(x_4) = b$. If $\pi(y_2) = \pi(y_4) = b$, then color v_1, v_3 with b and v_2, v_4 with a. Otherwise, at least one of y_2 and y_4 is colored with b, say y_2. Thus color v_1, v_2, v_3 with b and v_4 with a.
- $|S(a)| = 1$. Without loss of generality, assume that $\pi(x_1) = \pi(y_1) = a$ and $\pi(x_2) = \pi(x_3) = \pi(x_4) = b$. If at least two of y_2, y_3, y_4 are colored with b, then reduce the proof to the former case. If none of y_2, y_3, y_4 is colored with b, i.e., $\pi(y_2) = \pi(y_3) = \pi(y_4) = a$, then we color v_1, v_3 with b and v_2, v_4 with a. Now, suppose that exactly one of y_2, y_3, y_4 is colored with b. If $\pi(y_2) = b$, then $\pi(y_3) = \pi(y_4) = a$ and thus we may color v_1, v_3 with b and v_2, v_4 with a. If $\pi(y_3) = b$, then $\pi(y_2) = \pi(y_4) = a$ and therefore we color v_1, v_4 with b and v_2, v_3 with a.
- $|S(a)| = 0$. It implies that $\{\pi(x_i), \pi(y_i)\} = \{a, b\}$ for all $i = 1, 2, 3, 4$. Hence, it suffices to color v_1, v_3 with a and v_2, v_4 with b.

It is easy to verify that in each possible case the extended coloring is a 2-forest-coloring of G, driving a contradiction.

(2) Assume to the contrary that $3 \leqslant d(f_i) \leqslant 4$ for all $i = 1, 2, 3, 4$. It means that either $y_i = x_{i+1}$ or $y_ix_{i+1} \in E(G)$ for each $i \in \{1, 2, 3, 4\}$ and i is taken modulo 4. Since π cannot be extended to $V(f)$, we may assume that $\pi(x_i) = \pi(y_i) = a$ for all $i = 1, 2, 3, 4$ by (1). If there exists a vertex v_i which can be given the color a without arising any monochromatic cycle, then we color the remaining vertices with b to obtain a 2-forest-coloring of G, a contradiction. Otherwise, suppose that for each $i \in \{1, 2, 3, 4\}$ there exists a path P_i connecting x_i and y_i in H such that all vertices in P_i are colored with a. Therefore, a monochromatic cycle C formed by $\cup_{i=1}^{i=4} P_i$ and some edges y_1x_2, y_2x_3, y_3x_4 and y_4x_1 (if exist) is established in H. This contradicts the choice of H.

Therefore, we complete the proof of Lemma 5.2.1. ∎

Claim 5.2.2 *There are no adjacent light 4-faces in G.*

Proof. Suppose to the contrary that there are 4-faces $f_1 = [v_1v_2v_5v_6]$ and $f_2 = [v_2v_3v_4v_5]$ adjacent by sharing one common edge v_2v_5 such that $d(v_i) = 4$ for all $i = 1, 2, 3, 4, 5, 6$, see Figure 5.2. Since G does not contain adjacent triangles, $v_1, \cdots, v_6$ are mutually distinct. Let $H = G - V(f_1)$. Then H admits a 2-forest-coloring π

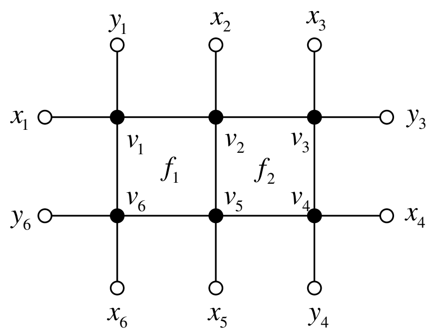


Figure 5.2: Adjacent light 4-faces f_1 and f_2.

by the minimality of G. If π can be extended to G, then we are done. Otherwise, by Lemma 5.2.1, we suppose that $x_1, y_1, x_2, v_3, v_4, x_5, x_6, y_6$ are all colored with the same color a. If at least one vertex in $\{x_3, y_3, x_4, y_4\}$ is colored with a, i.e., $\pi(x_3) = a$, then recolor v_3 with b, color v_1, v_5, v_6 with b and v_2 with a. Otherwise, it suffices to color v_1, v_5, v_6 with b and v_2 with a. It is easy to see that π is extended to the whole graph G in each possible case. This completes the proof of Claim 5.2.2. ∎

The following claim was proved by Raspaud and Wang in [RW08].

Claim 5.2.3 *G contains no 5-cycle $C = v_1v_2 \cdots v_5v_1$ with a chord v_2v_5 such that $d(v_i) = 4$ for all $i = 1, 2, \cdots, 5$.*

Claim 5.2.4 *A light 4-face cannot be adjacent to a light 5-face.*

Proof. Suppose to the contrary that $f = [v_1v_2v_3v_4]$ is a $(4, 4, 4, 4)$-face adjacent to a $(4, 4, 4, 4, 4)$-face $f' = [v_2v_3u_1u_2u_3]$ by sharing a common edge v_2v_3, see Figure 5.3. By definition, it is easy to know that $d(v_i) = 4$ for all $i = 1, \cdots 4$ and $d(u_j) = 4$ for all $j = 1, 2, 3$. Moreover, $u_1, u_3 \notin V(f)$ by the absence of adjacent triangles in G. If $u_2 = v_1$, then $C = u_3v_1v_4v_3v_2u_3$ is a 5-cycle with a chord v_2v_4 such that all vertices in C are of degree 4. This contradicts Claim 5.2.3. Thus, $V(f) \cap V(f') = \{v_2, v_3\}$. By the minimality of G, $G - V(f)$ admits a 2-forest-coloring π. If π can be extended to G, then we are done. Otherwise, by Lemma 5.2.1, we suppose that $x_1, y_1, x_2, u_3, u_1, x_3, x_4, y_4$ are all assigned with the same color a. The following discussion is divided into two cases, according to the color of u_2.

- $\pi(u_2) = a$. If at most one of s_1 and t_1 is colored with b, we recolor u_1 with b and then color v_1, v_2, v_4 with b and v_3 with a. So assume $\pi(s_1) = \pi(t_1) = b$. By symmetry, we also assume $\pi(s_3) = \pi(t_3) = b$. Then, we color v_1, v_2, v_4 with b and v_3 with a. If the resulting coloring is not a 2-forest-coloring, there is only one possible case that one of s_2 and t_2 is colored with a, say s_2. Therefore, we may further recolor u_2 with b to extend π to G successfully.
- $\pi(u_2) = b$. If neither s_1 nor t_1 is colored with a, then color v_1, v_2, v_4 with b and v_3 with a. So assume $\pi(s_1) = a$. Similarly, we assume that $\pi(s_3) = a$. If $\pi(s_2) = \pi(t_2) = a$, then recolor u_1 with b, color v_1, v_2, v_4 with b and v_3 with a. Otherwise, recolor u_1, u_3 with b, u_2 with a, color v_1, v_2, v_4 with b and v_3 with a.

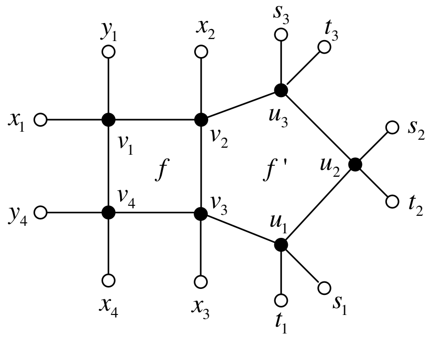


Figure 5.3: A light 4-face f is adjacent to a light 5-face f'.

It is easy to check that the resulting coloring in each possible case does not produce a monochromatic cycle, thus π is extended to a 2-forest-coloring of G, a contradiction. Therefore, we complete the proof of Claim 5.2.4. ∎

Claim 5.2.5 *If a* $(5,4,4,4,4)$*-face is adjacent to a light* 4*-face, then they are normally adjacent.*

Proof. Suppose that $f^* = [v_1v_2\cdots v_5]$ is a $(5,4,4,4,4)$-face adjacent to a (4,4,4,4)-face f. Obviously, $|V(f^*)\cap V(f)| \neq 4$. If $|V(f^*)\cap V(f)| = 2$, then we are done. So, in what follows, we assume that $|V(f^*)\cap V(f)| = 3$. By symmetry, we only need to consider the following two cases.

Case 1 $V(f^*)\cap V(f) = \{v_2, v_3, v_4\}$.

We first assume that $f = [v_2v_3wv_4]$. Clearly, $w \notin \{v_4, v_5\}$. Then two adjacent triangles $v_2v_3v_4v_2$ and $v_3v_4wv_3$ are formed, a contradiction. Now assume that $f = [v_4v_3wv_2]$. Similarly, $w \notin \{v_1, v_5\}$. It is easy to observe that a 3-cycle $v_2v_3v_4v_2$ is adjacent to a 3-cycle $v_2v_3wv_2$, a contradiction.

Case 2 $V(f^*)\cap V(f) = \{v_2, v_3, v_5\}$.

We first assume that $f = [v_2v_3v_5w]$. Clearly, $w \notin \{v_1, v_4\}$. It is easy to see that $C = v_4v_3v_2wv_5v_4$ is a 5-cycle with a chord v_3v_5 such that all vertices in C are of degree 4. This contradicts Claim 5.2.3.

Now, assume that $f = [v_2v_3wv_5]$. Notice that $w \notin \{v_1, v_4\}$. Let w_1, w_2 be the neighbors of w different from v_3 and v_5. Let x_4, y_4 be the neighbors of v_4 different from v_3 and v_5. Let x_2 be the neighbor of v_2 different from v_1, v_3 and v_5. Let x_3 be the neighbor of v_3 different from v_2, v_4 and w. By the minimality of G, $G - V(f)$ admits a 2-forest-coloring π. If π can be extended to G, then it contradicts the choice of G. Otherwise, by Lemma 5.2.1, we suppose that $v_1, x_2, x_3, v_4, w_1, w_2$ are all colored with the same color a. If neither x_4 nor y_4 is colored with a, then color v_3 with a and v_2, v_5, w with b. Otherwise, we first recolor v_4 with b, and then color v_5 with a and v_2, v_3, w with b. In each case, we extend π to G successfully, a contradiction.

Therefore, we complete the proof of Claim 5.2.5. ∎

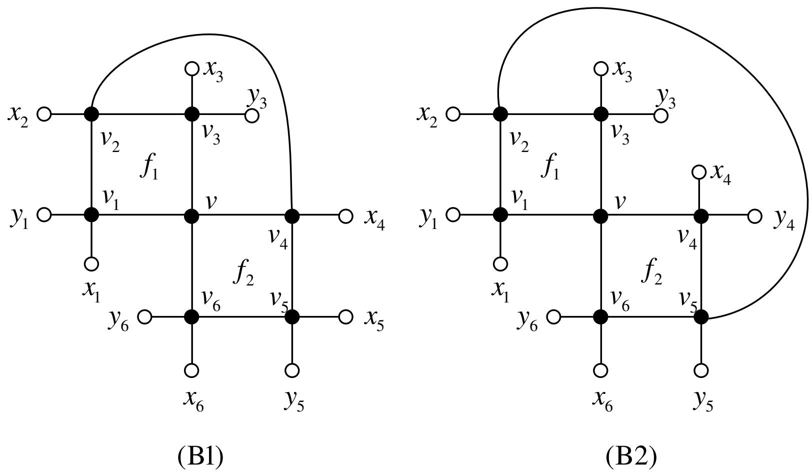


Figure 5.4: The reducible configurations (B1) and (B2) in Claim 5.2.6.

Claim 5.2.6 *Suppose that* $f_1 = [vv_1v_2v_3]$ *and* $f_2 = [vv_4v_5v_6]$ *are two light* 4*-faces which intersect at the unique vertex* v*. Then* G *does not contain the configuration* ***(B1)*** *and* ***(B2)*** *as shown in Figure 5.4.*

Proof. In each case, let $H = G - \{v, v_1, v_2, v_3\}$. By the minimality of G, H admits a 2-forest-coloring π. Next, we will show that π can be extended to G and thus arrive at a contradiction.

(1) Assume G contains (B1). If π cannot be extended to $\{v, v_1, v_2, v_3\}$, by Lemma 5.2.1, we suppose that $x_1, y_1, x_2, v_4, x_3, y_3, v_6$ are all colored with a. In this case, we color v with a and v_1, v_2, v_3 with b. If the resulting coloring is not a 2-forest-coloring, one of x_4 and v_5 must be colored with a. Then, we further recolor v_4 with b.

(2) Assume G contains (B2). Similarly, if π cannot be extended to $\{v, v_1, v_2, v_3\}$, by Lemma 5.2.1, we suppose that $x_1, y_1, x_2, v_5, x_3, y_3, v_4, v_6$ are all colored with a. In this case, we first recolor v_5 with b and then extend π to the remaining uncolored vertices easily by (1) of Lemma 5.2.1.

Thus, we complete the proof of Lemma 5.2.6. ∎

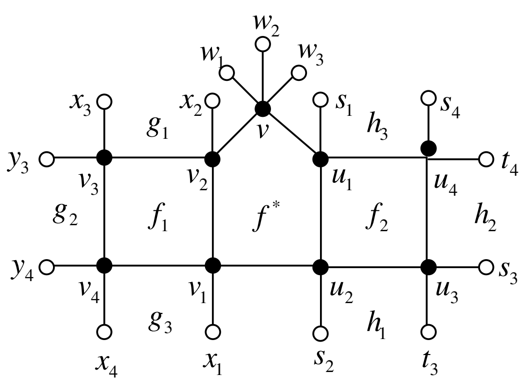


Figure 5.5: The configuration in Lemma 5.2.2.

Lemma 5.2.2 *Suppose that $f^* = [vu_1u_2v_1v_2]$ is a $(5, 4, 4, 4, 4)$-face adjacent to two light 4-faces $f_1 = [v_1v_2v_3v_4]$ and $f_2 = [u_1u_2u_3u_4]$ by the common edge v_1v_2 and u_1u_2, respectively, see Figure 5.5. Let $H = G-V(f_1)$. If a 2-forest-coloring π of $G-V(f_1)$ cannot be extended to G, then either f_1 or f_2 is adjacent to at least two 5^+-faces.*

Proof. By Claim 5.2.5, we see that $\{v_3, v_4\} \cap \{v, u_1, u_2\} = \varnothing$ and $\{u_3, u_4\} \cap \{v, v_1, v_2\} = \varnothing$. If $u_3 = v_4$, then $C = u_2v_4v_3v_2v_1u_2$ is a 5-cycle with a chord v_1v_4 such that all vertices in C have degree 4. This contradicts Claim 5.2.3. If $u_3 = v_3$, then f_1 intersects f_2 at v_3 such that v_1 is adjacent to u_2, contradicting to (B1). So, suppose that $u_3 \notin \{v_3, v_4\}$. If $u_4 = v_4$, then f_1 intersects f_2 at v_4 such that $v_1u_2 \in E(G)$, contradicting to (B1). If $u_4 = v_3$, then f_1 and f_2 intersect at v_3 such that $v_1u_2 \in E(G)$, which is a contradiction to (B2). Thus, in the following argument, we suppose that $\{u_3, u_4\} \cap \{v_3, v_4\} = \varnothing$. Let g_{i-1} denote the face adjacent to f_1 by the common edge v_iv_{i+1}, where $i \in \{2, 3, 4\}$ and i is taken modulo 4. Let h_{j-1} denote the face adjacent to f_2 by the common edge u_ju_{j+1}, where $j \in \{2, 3, 4\}$ and j is taken modulo 4, see Figure 5.5.

Assume to the contrary that $3 \leqslant d(g_i) \leqslant 4$ and $3 \leqslant d(h_j) \leqslant 4$ for all $i, j = 1, 2, 3$. Denote $H = G - V(f_1)$. By the minimality of G, H has a 2-forest-coloring π. If π can be extended to G, then we arrive at a contradiction to the assumption on G. Otherwise, assume w.l.o.g., that $u_2, x_1, x_2, v, x_3, y_3, x_4, y_4$ are all colored with a by Lemma 5.2.1. We have to deal with the following five cases.

Case 1 Assume that at most one of u_1, u_3, s_2 is colored with b.

Then recolor u_2 with b, color v_1 with a and v_2, v_3, v_4 with b.

Case 2 Assume that all u_1, u_3, s_2 are colored with b.

Then color v_1 with a and v_2, v_3, v_4 with b.

Case 3 Assume that $\pi(u_1) = a$ and $\pi(u_3) = \pi(s_2) = b$.

If there is no monochromatic cycle arising after recoloring u_1 with b, then recolor u_1 with b firstly and then go back to the previous Case 2. Otherwise, suppose that $\pi(s_1) = \pi(u_4) = b$. If one of s_3 and t_3 is colored with b, then recolor u_3 with a, u_2 with b and then color v_1 with a and v_2, v_3, v_4 with b. So assume that neither s_3 nor t_3 is colored with b. If at least one of s_4 and t_4 is colored with b, then recolor u_4 with a, u_1 with b and then reduce the proof to the former Case 2. Now, assume that $b \notin \{\pi(s_4), \pi(t_4)\}$. Therefore, we firs recolor u_2 with b, and then extend π to G by coloring v_1 with a and v_2, v_3, v_4 with b.

Case 4 Assume that $\pi(u_3) = a$ and $\pi(u_1) = \pi(s_2) = b$.

If the color b did not appear on s_1 and u_4, then recolor u_2 with b, and color v_1 with a and v_2, v_3, v_4 with b. If the color a did not appear on s_1 and u_4, then switch the colors of u_1 and u_2, then color v_1 with a and finally color v_2, v_3, v_4 with b. Otherwise, suppose that $\{\pi(s_1), \pi(u_4)\} = \{a, b\}$. We have two possibilities below.

- $\pi(s_1) = b$ and $\pi(u_4) = a$. If at most one of s_4 and t_4 is colored with b, then recolor u_2, u_4 with b, u_1 with a, and color v_1 with a and v_2, v_3, v_4 with b. Hence, assume $\pi(s_4) = \pi(t_4) = b$. If at most one of s_3 and t_3 is colored with b, then recolor u_3 with b and then go back to the previous Case 2. Otherwise, set $\pi(s_3) = \pi(t_3) = b$. In this case, we may first switch the colors of u_1 and u_2 and then color v_1 with a and v_2, v_3, v_4 with b successfully.

- $\pi(s_1) = a$ and $\pi(u_4) = b$. If $b \notin \{\pi(s_4), \pi(t_4)\}$, then recolor u_2 with b and color v_1 with a and v_2, v_3, v_4 with b successfully. If $a \notin \{\pi(s_3), \pi(t_3)\}$, then color v_1 with a and finally color v_2, v_3, v_4 with b. So, w.l.o.g., assume that $\pi(s_3) = a$ and $\pi(s_4) = b$. In this case, we can first switch the colors of u_3 and u_4 and then reduce the proof to the former Case 2.

Case 5 Assume that $\pi(s_2) = a$ and $\pi(u_1) = \pi(u_3) = b$.

First we consider the case that $\pi(u_4) = a$. If either $\pi(s_1) \neq b$ or $b \notin \{\pi(s_3), \pi(t_3)\}$, then recolor u_2 with b, color v_1 with a and v_2, v_3, v_4 with b. So, w.l.o.g., assume that $\pi(s_1) = b$ and $\pi(s_3) = b$. We first switch the colors of u_1 and u_2, then color v_1 with a and finally color v_2, v_3, v_4 with b. If the resulting coloring is not a 2-forest-coloring, at least one of s_4 and t_4 is colored with a. Thus, we further recolor u_3 with a and u_4 with b.

Now we consider the case that $\pi(u_4) = b$. If at most one of s_3, t_3 is colored with a, then first switch the colors of u_2 and u_3, then color v_1 with a and finally color v_2, v_3, v_4 with b. So assume that $\pi(s_3) = \pi(t_3) = a$. If at most one of s_4, t_4 is colored with a, then recolor u_4 with a and then go back to the previous above case. Hence, $\pi(s_4) = \pi(t_4) = a$. If $\pi(s_2) \neq a$, then switch the colors of u_1 and u_2, and assign color a to v_1 and b to v_2, v_3, v_4, respectively. So now assume $\pi(s_2) = a$. Notice that each of g_i and h_j is of degree at most 4 with $i, j = 1, 2, 3$. Moreover, for $i \in \{1, 2, 3, 4\}$, in H, there exists a path denoted by P_i connecting two vertices of $N_H(v_i)$ such that all vertices in P_i are colored with a. Similarly, for $j \in \{1, 2, 3, 4\}$, in H, there exists a path denoted by P'_j connecting two vertices of $N_H(u_j)$ such that all vertices in P'_j are colored with a. However, a monochromatic cycle C is formed in H by $\cup_{i=1}^{i=4} P_i$, $\cup_{j=1}^{j=4} P'_j$ and some edges x_1x_4, y_4x_3, y_3x_2, s_1s_4, t_4s_3 and t_3s_2 (if exist). This contradicts the choice of H. Thus, we complete the proof of Lemma 5.2.2. ∎

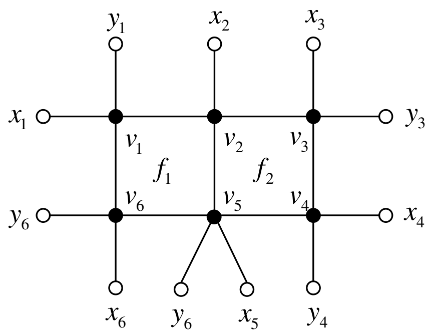


Figure 5.6: f_1 and f_2 are adjacent $(4, 4, 4, 5)$-faces.

Claim 5.2.7 *G does not contain two* $(4, 4, 4, 5)$*-faces* $f_1 = [v_2v_1v_6v_5]$ *and* $f_2 = [v_2v_3v_4v_5]$ *sharing a unique common edge* v_2v_5 *and* $d(v_5) = 5$.

Proof. Suppose on the contrary that G contains such adjacent $(4,4,4,5)$-faces f_1 and f_2, see Figure 5.6. Since there is no adjacent triangles, $v_1v_5 \notin E(G)$ and $v_2v_6 \notin E(G)$. It implies that $v_1v_2\cdots v_6v_1$ is a 6-cycle. Let $H = G - \{v_1, \cdots, v_6\}$. Then H admits a 2-forest-coloring π by the minimality of G. Let $S(a)$ denote the subset of $\{\{x_1,y_1\},\{x_3,y_3\},\{x_4,y_4\},\{x_6,y_6\}\}$ which satisfies that all vertices in $S(a)$ get the same color a in the coloring π. Thus $0 \leqslant |S(a)| \leqslant 4$. The following proof is divided into five cases as follows, depending on the value of $|S(a)|$.

Case 1 $|S(a)| = 4$.

It implies that $\pi(x_i) = \pi(y_i) = a$ for all $i = 1,3,4,6$. If at most one of x_5, y_5 is colored with b, color v_1, v_3, v_4, v_5, v_6 with b and v_2 with a. Otherwise, color v_1, v_3, v_4, v_6 with b and v_2, v_5 with a.

Case 2 $|S(a)| = 3$.

By symmetry, we have two possible cases below.

• Assume that $\pi(x_i) = \pi(y_i) = a$ for all $i = 1,3,4$. W,l.o.g., assume that $\pi(x_6) = b$. If $\pi(x_5) = \pi(y_5) = b$, then color v_1, v_2, v_3, v_4 with b and v_5, v_6 with a. Otherwise, color v_1, v_3, v_4, v_5 with b and v_2, v_6 with a.

• Assume that $\pi(x_i) = \pi(y_i) = a$ for all $i = 3,4,6$. W.l.o.g., assume that $\pi(x_1) = b$. We first color v_3, v_4, v_6 with b and v_1 with a. If the color a appears at most once on the set x_5, y_5, then further color v_2 with b and v_5 with a. Otherwise, we assign v_2 and v_5 with b to extend π to G successfully.

Case 3 $|S(a)| = 2$.

By symmetry, we have four possible cases below.

• Assume that $\pi(x_1) = \pi(y_1) = \pi(x_3) = \pi(y_3) = a$. W.l.o.g., suppose that $\pi(x_4) = \pi(x_6) = b$. We first color v_1, v_3 with b and v_4, v_6 with a. If at least one of x_5 and y_5 is colored with a, then further color v_2 with a and v_5 with b. Otherwise, suppose that $\pi(x_5) = \pi(y_5) = b$. In this case, we color v_2, v_5 with a. If the resulting coloring is not a 2-forest-coloring, we assert that at least one of y_4 and y_6 is colored with a, say y_6. And thus we can reassign color b to v_6 to derive a 2-forest-coloring of G, a contradiction.

• Assume that $\pi(x_1) = \pi(y_1) = \pi(x_4) = \pi(y_4) = a$. W.l.o.g., assume that $\pi(x_3) = \pi(x_6) = b$. We first color v_1, v_4 with b and v_3, v_6 with a. If $\pi(x_5) = \pi(y_5) = b$, then further color v_2 with b and v_5 with a. Otherwise, w.l.o.g., suppose that $\pi(x_5) = a$. We further color v_2, v_5 with b. Similarly, if the resulting coloring is not a 2-forest-coloring, we assert that $\pi(x_2) = \pi(y_5) = b$ and thus reassign v_2 with a to obtain a 2-forest-coloring of G. This contradicts the choice of G.

• Assume that $\pi(x_1) = \pi(y_1) = \pi(x_6) = \pi(y_6) = a$. W.l.o.g., assume that $\pi(x_3) = \pi(x_4) = b$. First assume that $\pi(y_3) = \pi(y_4) = b$. If at least one of x_5, y_5 is colored with a, then color v_1, v_5, v_6 with b and v_2, v_3, v_4 with a. Otherwise, assume that $\pi(x_5) = \pi(y_5) = b$ and thus color v_1, v_2, v_6 with b and v_3, v_4, v_5 with a. Next assume that $\pi(y_3) = b$ and $\pi(y_4) = a$. If at least one of x_5, y_5 is colored with b, then color v_1, v_2, v_4, v_6 with b and v_3, v_5 with a. Otherwise, assume that $\pi(x_5) = \pi(y_5) = a$ and hence we may color v_1, v_4, v_5, v_6 with b and v_2, v_3 with a. Finally assume that $\pi(y_3) = \pi(y_4) = a$. If at least one of x_5, y_5 is colored with b, then color v_1, v_2, v_4, v_6

with b and v_3, v_5 with a. Otherwise, assume that $\pi(x_5) = \pi(y_5) = a$ and hence we may color v_1, v_3, v_5, v_6 with b and v_2, v_4 with a.

• Assume that $\pi(x_4) = \pi(y_4) = \pi(x_6) = \pi(y_6) = a$. W.l.o.g., assume that $\pi(x_1) = \pi(x_3) = b$. If $\pi(x_5) = \pi(y_5) = a$, then color v_2, v_4, v_5, v_6 with b and v_1, v_3 with a. Otherwise, we may color v_2, v_4, v_6 with b and v_1, v_3, v_5 with a.

Case 4 $|S(a)| = 1$.

By symmetry, we have two possible cases below.

• Assume that $\pi(x_1) = \pi(y_1) = a$. Assume, w.l.o.g., that $\pi(x_3) = \pi(x_4) = \pi(x_6) = b$. Moreover, we may suppose that at most one of y_3, y_4, y_6 is colored with b. Otherwise, we reduce the proof to the previous Case 2 or Case 3. First assume that $\pi(y_3) = b$ and $\pi(y_4) = \pi(y_6) = a$. If $\pi(x_5) = \pi(y_5) = a$ then color v_1, v_2, v_5 with b and v_3, v_4, v_6 with a. If $\pi(x_5) = \pi(y_5) = b$ then color v_1, v_4, v_6 with b and v_2, v_3, v_5 with a. Otherwise, assume that $\{\pi(x_5), \pi(y_5)\} = \{a, b\}$. If $\pi(x_2) = a$, then color v_1, v_2, v_4, v_6 with b and v_3, v_5 with a. Otherwise, color v_1, v_4, v_6 with b and v_2, v_3, v_5 with a. Next assume that $\pi(y_4) = b$ and $\pi(y_3) = \pi(y_6) = a$. If $a \in \{\pi(x_5), \pi(y_5)\}$, then color v_1, v_3, v_5 with b and v_2, v_4, v_6 with a. Otherwise, assume that $\pi(x_5) = \pi(y_5) = b$ and thus color v_1, v_3, v_6 with b and v_2, v_4, v_5 with a. Finally assume that $\pi(y_3) = \pi(y_4) = a$ and $\pi(y_6) \in \{a, b\}$. If at least one of x_5, y_5 is colored with a, then color v_1, v_3, v_5 with b and v_2, v_4, v_6 with a. Otherwise, assume that $\pi(x_5) = \pi(y_5) = b$ and thus color v_1, v_2, v_4 with b and v_3, v_5, v_6 with a.

• Assume that $\pi(x_6) = \pi(y_6) = a$. The argument is similar to the above case.

Case 5 $|S(a)| = 0$.

Without loss of generality, we may assume that $\pi(x_i) = a$ and $\pi(y_i) = b$ for all $i = 1, 3, 4, 6$. If at least one of x_5, y_5 is colored with b, then color v_2, v_4, v_6 with b and v_1, v_3, v_5 with a. Otherwise, assume that $\pi(x_5) = \pi(y_5) = a$ and therefore we can color v_1, v_3, v_5 with b and v_2, v_4, v_6 with a.

Thus, we complete the proof of Claim 5.2.7. ∎

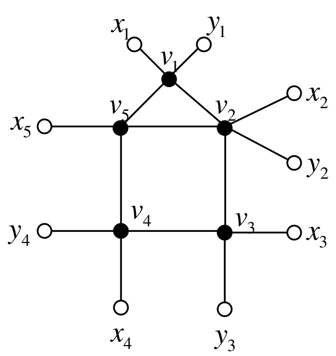


Figure 5.7: The configuration in Claim 5.2.8.

Claim 5.2.8 *G contains no a 5-cycle $C = v_1v_2 \cdots v_5v_1$ with a chord v_2v_5 such that $d(v_i) = 4$ for all $i = 1, 3, 4, 5$ and $d(v_2) = 5$.*

Proof. Suppose to the contrary that G contains a 5-cycle $C = v_1 v_2 \cdots v_5 v_1$ with a chord $v_2 v_5$ such that $d(v_i) = 4$ for all $i = 1, 3, 4, 5$ and $d(v_2) = 5$, see Figure 5.7. Let $H = G - \{v_1, \cdots, v_5\}$. Then H admits a 2-forest-coloring π by the minimality of G. For $a \in C$, let $S(a)$ denote the subset of $\{\{x_1, y_1\}, \{x_3, y_3\}, \{x_4, y_4\}\}$ which satisfies that all vertices in $S(a)$ get the same color a in the coloring π. Thus $0 \leqslant |S(a)| \leqslant 3$. The following proof is divided into four cases as follows, according to the value of $|S(a)|$.

Case 1 $|S(a)| = 3$.

It implies that $\pi(x_i) = \pi(y_i) = a$ for all $i = 1, 3, 4$. If at most one of x_2, y_2 is colored with b, then color v_1, v_2, v_3, v_4 with b and v_5 with a. Otherwise, color v_1, v_3, v_4, v_5 with b and v_2 with a.

Case 2 $|S(a)| = 2$.

By symmetry, we have three possible cases below.

• Assume that $\pi(x_1) = \pi(y_1) = \pi(x_3) = \pi(y_3) = a$. W.l.o.g., assume that $\pi(x_4) = b$. If $b \in \{\pi(x_2), \pi(y_2)\}$, color v_1, v_3, v_5 with b and v_2, v_4 with a. Otherwise, assume that $\pi(x_2) = \pi(y_2) = a$. We color v_1, v_2, v_3 with b and v_4, v_5 with a. If the resulting coloring is not a 2-forest-coloring, y_4 must be colored with a and thus reassign v_4 with b.

• Assume that $\pi(x_1) = \pi(y_1) = \pi(x_4) = \pi(y_4) = a$. W.l.o.g., assume that $\pi(x_3) = b$. If $\pi(x_2) = \pi(y_2) = b$, then color v_1, v_4, v_5 with b and v_2, v_3 with a. Otherwise, we color v_1, v_2, v_4 with b and v_3, v_5 with a.

• Assume that $\pi(x_3) = \pi(y_3) = \pi(x_4) = \pi(y_4) = a$. W.l.o.g., assume that $\pi(x_1) = b$. If $\pi(x_2) = \pi(y_2) = b$, then color v_3, v_4, v_5 with b and v_1, v_2 with a. So suppose that $\pi(x_2) = a$. If $\pi(y_1) = b$, then color v_2, v_3, v_4 with b and v_1, v_5 with a. Hence $\pi(y_1) = a$. If $\pi(x_2) = \pi(y_2) = a$, then color v_1, v_2, v_3, v_4 with b and v_5 with a. Otherwise, then color v_1, v_3, v_4, v_5 with b and v_2 with a.

Case 3 $|S(a)| = 1$.

By symmetry, we have three possible cases below.

• Assume that $\pi(x_1) = \pi(y_1) = a$. W.l.o.g., assume that $\pi(x_3) = \pi(x_4) = b$. Moreover, we may suppose that at most one of y_3, y_4 is colored with b. Otherwise, we reduce the proof to the previous Case 2. First assume that $\pi(y_3) = b$ and $\pi(y_4) = a$. If at least one of x_2, y_2 is colored with a, then color v_1, v_2, v_4 with b and v_3, v_5 with a. Otherwise, assume that $\pi(x_2) = \pi(y_2) = b$ and thus color v_1, v_5 with b and v_2, v_3, v_4 with a. Next assume that $\pi(y_3) = a$ and $\pi(y_4) = b$. If $\pi(x_2) = \pi(y_2) = a$, then color v_1, v_2, v_3 with b and v_4, v_5 with a. Otherwise, color v_1, v_3, v_5 with b and v_2, v_4 with a. Afterwards, assume that $\pi(y_3) = \pi(y_4) = a$. If $\pi(x_2) = \pi(y_2) = a$, then color v_1, v_2, v_4 with b and v_3, v_5 with a. Otherwise, color v_1, v_3, v_5 with b and v_2, v_4 with a.

• Assume that $\pi(x_3) = \pi(y_3) = a$. W.l.o.g., assume that $\pi(x_1) = \pi(x_4) = b$. At first, assume that $\pi(y_1) = b$. If $\pi(y_4) = b$, then reduce to the previous Case 2. Otherwise, assume $\pi(y_4) = a$. If at least one of x_2, y_2 is colored with b, then color v_3, v_5 with b and v_1, v_2, v_4 with a. Otherwise, assume that $\pi(x_2) = \pi(y_2) = a$ and thus color v_2, v_3, v_5 with b and v_1, v_4 with a. Now assume that $\pi(y_1) = a$.

If $\pi(x_2) = \pi(y_2) = b$, then color v_3, v_5 with b and v_1, v_2, v_4 with a. If $\pi(x_2) = \pi(y_2) = a$, then color v_2, v_3, v_5 with b and v_1, v_4 with a. Otherwise, assume that $\{\pi(x_2), \pi(y_2)\} = \{a, b\}$. If $\pi(x_5) = a$, then color v_2, v_3, v_5 with b and v_1, v_4 with a. Otherwise, assume that $\pi(x_5) = b$. Then color v_1, v_3 with b and v_2, v_4, v_5 with a. If such coloring is not a 2-forest-coloring, y_4 must be assigned with a and thus reassign v_4 with b to extend π to G successfully.

• Assume that $\pi(x_4) = \pi(y_4) = a$. W.l.o.g., assume that $\pi(x_1) = \pi(x_3) = b$. Similarly, we deduce that at most one of y_1 and y_3 can be colored with b. If at least one of x_2, y_2 is colored with a, then color v_2, v_4 with b, v_1, v_3 with a and finally color v_5 with a color different from $\pi(x_5)$. Otherwise, assume that $\pi(x_2) = \pi(y_2) = b$. If $\pi(y_1) = \pi(y_3) = a$, then color v_1, v_3, v_4 with b and v_2, v_5 with a. Otherwise, we can extend π to G by coloring v_1, v_2, v_3 with a and v_4, v_5 with b.

Case 4 $|S(a)| = 0$.

Without loss of generality, assume that $\pi(x_i) = a$ and $\pi(y_i) = b$ for all $i = 1, 3, 4$. If x_2, y_2 are both colored with a, then color v_1, v_2, v_4 with b and v_3, v_5 with a. If x_2, y_2 are both colored with b, then color v_1, v_2, v_4 with a and v_3, v_5 with b. Otherwise, assume that $\pi(x_2) = a$ and $\pi(y_2) = b$. If $\pi(x_5) = a$, then color v_1, v_3, v_5 with b and v_2, v_4 with a. If $\pi(x_5) = b$, then color v_1, v_3, v_5 with a and v_2, v_4 with b.

Thus, we complete the proof of Claim 5.2.8. ∎

Claim 5.2.9 *G does not contain the configuration **(F1)**, as shown in Figure 5.8, where f_1, f_2, f_3 are all faces.*

Proof. Assume G contains (F1). By Claim 5.2.8, $d(v_8) \geqslant 5$. By Claim 5.2.5, we deduce that f_1 and f_2 are normally adjacent. In other words, $V(f_1) \cap V(f_2) = \{v_1, v_4\}$. First we claim that $V(f_1) \cap V(f_3) = \{v_1\}$. It suffices to show that $v_9 \notin \{v_2, v_3, v_4\}$. It is easy to see that $v_9 \neq v_4$. If $v_9 = v_2$, a 3-cycle $v_1v_{10}v_2v_1$ is adjacent to a 3-cycle $v_7v_1v_2v_1$, a contradiction. If $v_9 = v_3$, then $v_8 = v_4$, a contradiction since $d(v_4) = 4$. Next we claim that $V(f_2) \cap V(f_3) = \{v_1, v_7\}$. To see that, we only need to show that $v_9 \neq v_5$ and $v_{10} \notin \{v_5, v_6\}$. If $v_9 = v_5$, then $v_8 = v_4$, a contradiction. If $v_{10} = v_5$, then a 5-cycle $v_1v_2v_3v_4v_5v_1$ with a chord v_1v_4 such that $d(v_i) = 4$ for all $i = 1, \cdots, 5$ exists in G, contradicting to Claim 5.2.3. If $v_{10} = v_6$, then a 3-cycle $v_1v_7v_6v_1$ is adjacent to a 3-cycle $v_9v_7v_6v_9$, a contradiction. Thus, in what follows, we assume that all vertices in the set $\{v_1, v_2, \cdots, v_{10}\}$ are mutually distinct.

Let $H = G - V(f_1)$. By the minimality of G, H admits a 2-forest-coloring π. If π cannot be extended to G, by (1) of Lemma 5.2.1, we deduce that all vertices in $\bigcup_{i=1}^{i=4} N_H(v_i)$ get the same color in the coloring π. Without loss of generality, suppose that $v_7, v_{10}, x_2, y_2, x_3, y_3, x_4, v_5$ are all colored with a. We have to consider two cases below by the color of v_6.

Case 1 Assume $\pi(v_6) = a$.

If at most one of x_5 and y_5 is colored with b, we recolor v_5 with b, color v_1, v_2, v_3 with b and v_4 with a. So suppose that $\pi(x_5) = \pi(y_5) = b$. If at most one of x_6 and y_6 is colored with b, we recolor v_6 with b, color v_1, v_2, v_3 with b and v_4 with a. Now suppose that $\pi(x_6) = \pi(y_6) = b$. If at most one of x_7, v_8, v_9 is colored with b, then

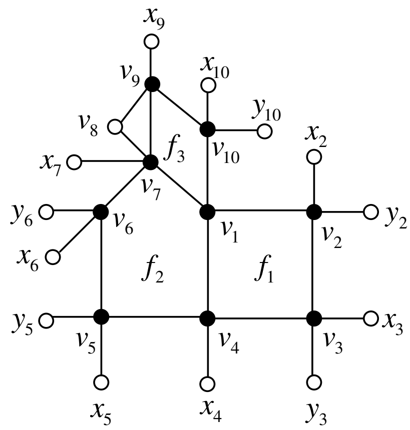


Figure 5.8: The configuration (F1) in Claim 5.2.9.

recolor v_7 with b and thus we can color v_1 with a and finally color v_2, v_3, v_4 with b. If the color a did not appear on the set $\{x_7, v_8, v_9\}$, we can extend π to G by coloring v_1 with a and v_2, v_3, v_4 with b. Thus, in what follows, assume that exactly two of x_7, v_8, v_9 are colored with b and one is colored with a. We need to discuss three possibilities below.

• $\pi(x_7) = a$ and $\pi(v_8) = \pi(v_9) = b$. It is easy to derive that one of x_{10} and y_{10} is colored with a. Otherwise, we may give the color a to v_1 and the color b to other three remaining uncolored vertices. Therefore, we can first recolor v_7, v_{10} with b, v_9 with a and then extend π to G by coloring v_1 with a and v_2, v_3, v_4 with b.

• $\pi(v_8) = a$ and $\pi(x_7) = \pi(v_9) = b$. Similarly, we deduce that one of x_{10} and y_{10} is colored with a. Otherwise, we can color v_1 with a and v_2, v_3, v_4 with b to derive a 2-forest-coloring of G, a contradiction. Thus, we recolor v_{10} with b, color v_1 with a and v_2, v_3, v_4 with b. If the resulting coloring is not a 2-forest-coloring, x_9 must be colored with b. Then we further switch the colors of v_7 and v_9.

• $\pi(v_9) = a$ and $\pi(x_7) = \pi(v_8) = b$. If at most one of x_{10} and y_{10} is colored with b, we recolor v_{10} with b, color v_1 with a and v_2, v_3, v_4 with b. Now suppose that $\pi(x_{10}) = \pi(y_{10}) = b$. If $\pi(x_9) = b$, we color v_1, v_2, v_3 with b and v_4 with a. Otherwise, recolor v_9 with b and then color v_1 with a and v_2, v_3, v_4 with b.

Case 2 Assume $\pi(v_6) = b$.

One can easily observe that one of x_5, y_5 is assigned with a. Otherwise, we may color v_4 with a and v_1, v_2, v_3 with b. If the color b did not appear on the set $\{x_5, y_5\}$, we first recolor v_5 with b and color v_4 with a and v_1, v_2, v_3 with b. So, w.l.o.g., assume that $\pi(x_5) = a$ and $\pi(y_5) = b$. By a similar argument, we can deduce that $\{\pi(x_6), \pi(y_6)\} = \{a, b\}$. If at most one of x_7, v_8, v_9 is colored with b, then recolor v_7, v_5 with b, v_6 with a, and thus color v_1 with a and finally color v_2, v_3, v_4 with b. If the color a did not appear on the set $\{x_7, v_8, v_9\}$, we can extend π to G by coloring v_1 with a and v_2, v_3, v_4 with b. Thus, in what follows, assume that exactly two of x_7, v_8, v_9 are colored with b and one is colored with a. The following proof is similar to the previous Case 1.

Therefore, we complete the proof of Claim 5.2.9. ∎

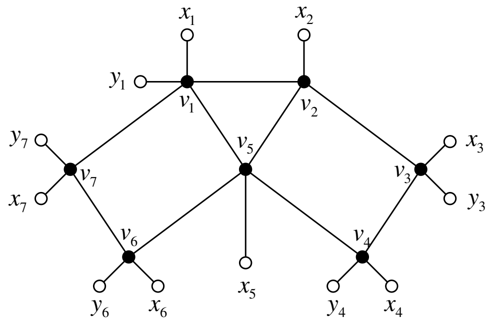


Figure 5.9: The configuration (F2) in Claim 5.2.10.

Claim 5.2.10 *G does not contain the configuration **(F2)**, as shown in Figure 5.9.*

Proof. Assume G contains (F2). Clearly, $\{v_3, v_4\} \cap \{v_6, v_7\} = \varnothing$, since G contains no adjacent triangles. It follows that $C = v_1v_2 \cdots v_7v_1$ is a 7-cycle. Moreover, it is easy to see that $x_2 \notin C$. By the minimality of G, $G - \{v_2\}$ admits a 2-forest-coloring π. It is easy to observe that if there exists a color c appearing at most once on the set $\{x_2, v_1, v_3, v_5\}$, we can color v_2 with c to obtain a 2-forest-coloring of G. So, in the following, we always assume that the colors a and b appear exactly twice on the set $\{x_2, v_1, v_3, v_5\}$, respectively. We need to handle the following cases.

Case 1 $\pi(x_2) = \pi(v_3) = a$ and $\pi(v_1) = \pi(v_5) = b$.

First consider the case that $\pi(v_4) = a$. If $a \in \{\pi(x_3), \pi(y_3)\}$, recolor v_3 with b and color v_2 with a. So assume $\pi(x_3) = \pi(y_3) = b$. If neither x_4 nor y_4 is colored with a, we color v_2 with a. If neither x_4 nor y_4 is colored with b, recolor v_4 with b and color v_2 with a. Thus, in what follows, w.l.o.g., assume that $\pi(x_4) = a$ and $\pi(y_4) = b$. If at most one of x_5, v_6 is colored with a, then recolor v_5 with a, v_4 with b, and color v_2 with b. Otherwise, suppose that $\pi(x_5) = \pi(v_6) = a$. If x_1, y_1, v_7 are all colored with a, then recolor v_4 with b and thus we can color v_2 with a. If at least two of x_1, y_1, v_7 are colored with b, then recolor v_1 with a and then color v_2 with b. Otherwise, assume that exactly two of x_1, y_1, v_7 are colored with a and one is colored with b. By symmetry, we need to consider two subcases as follows.

- $\pi(v_7) = b$ and $\pi(x_1) = \pi(y_1) = a$. If neither x_6 nor y_6 is colored with a, switch the colors of v_4 and v_5 and color v_2 with b and afterwards color v_2 with b. If neither x_7 nor y_7 is colored with b, recolor v_4 with b and color v_2 with a. So, w.l.o.g., assume that $\pi(x_6) = a$ and $\pi(x_7) = b$. In this case, we may first switch the colors of v_6 and v_7 and then go back to the previous case.
- $\pi(x_1) = b$ and $\pi(y_1) = \pi(v_7) = a$. If one of x_6, y_6 is colored with a, recolor v_4, v_6 with b, v_5 with a and color v_2 with b. So assume that $\pi(x_6) = \pi(y_6) = b$. Similarly, if one of x_7, y_7 is colored with a, recolor v_7 with b, v_1 with a and color v_2 with b. So assume that $\pi(x_7) = \pi(y_7) = b$. Now, we can recolor v_5 with a, v_4 with b, and color v_2 with b to extend π to G successfully.

Now consider the case that $\pi(v_4) = b$. If $\pi(x_3) = \pi(y_3) = a$, recolor v_3 with b and color v_2 with a. If $\pi(x_3) = \pi(y_3) = b$, color v_2 with a. So, w.l.o.g., assume that $\pi(x_3) = a$ and $\pi(y_3) = b$. If $\pi(x_4) = \pi(y_4) = b$, recolor v_4 with a and then go back to the previous case. If $\{\pi(x_4), \pi(y_4)\} = \{a, b\}$, switch the colors of v_3 and v_4 and color v_2 with a. Now, suppose that $\pi(x_4) = \pi(y_4) = a$. If at most one of x_5, v_6 is colored with a, then recolor v_5 with a, and color v_2 with b. Otherwise, suppose that $\pi(x_5) = \pi(v_6) = a$. The following proof is similar to the first case.

Case 2 $\pi(x_2) = \pi(v_5) = a$ and $\pi(v_1) = \pi(v_3) = b$.

We first consider the case that $\pi(v_4) = a$. If $\pi(x_3) = \pi(y_3) = a$, color v_2 with b. If $\pi(x_3) = \pi(y_3) = b$, recolor v_3 with a and color v_2 with b. So, assume that $\pi(x_3) = a$ and $\pi(y_3) = b$. If $a \in \{\pi(x_4), \pi(y_4)\}$, recolor v_4 with b, v_3 with a and color v_2 with b. Now, suppose that $\pi(x_4) = \pi(y_4) = b$. If neither v_6 nor x_5 is colored with a, then color v_2 with a. If neither v_6 nor x_5 is colored with b, then recolor v_5 with b and color v_2 with a. So, assume that $\{\pi(x_5), \pi(v_6)\} = \{a, b\}$. We have two cases below.

- $\pi(v_6) = a$ and $\pi(x_5) = b$. If x_1, y_1, v_7 are all colored with a, then recolor v_5 with b and color v_2 with a. If at least two of x_1, y_1, v_7 are colored with b, then recolor v_1, v_3 with a, v_5 with b, and v_2 with b. Otherwise, assume that exactly two of x_1, y_1, v_7 are colored with a and one is colored with b. By symmetry, we need to handle the following two possibilities.
  - $\pi(x_1) = b$ and $\pi(y_1) = \pi(v_7) = a$. If $a \in \{\pi(x_6), \pi(y_6)\}$, recolor v_6 with b and then reduce the proof to the former case. Otherwise, set $\pi(x_6) = \pi(y_6) = b$. If $a \in \{\pi(x_7), \pi(y_7)\}$, recolor v_5, v_7 with b, v_1, v_3 with a, and color v_2 with b. Now we assert that $\pi(x_7) = \pi(y_7) = b$. In this case, we can color v_2 with a. It is easy to verify that the resulting coloring of G is a 2-forest-coloring, a contradiction.
  - $\pi(v_7) = b$ and $\pi(x_1) = \pi(y_1) = a$. If $a \notin \{\pi(x_6), \pi(y_6)\}$, recolor v_3 with a and color v_2 with b. If $b \notin \{\pi(x_7), \pi(y_7)\}$, color v_2 with b. Otherwise, w.l.o.g., assume that $\pi(x_6) = a$ and $\pi(x_7) = b$. We may first switch the colors of v_6 and v_7, and then color v_2 with a.
- $\pi(v_6) = b$ and $\pi(x_5) = a$. Similarly, we deduce that exactly two of x_1, y_1, v_7 are colored with a and one is colored with b. By symmetry, we need to handle the following two possibilities.
  - $\pi(x_1) = b$ and $\pi(y_1) = \pi(v_7) = a$. If $a \notin \{\pi(x_7), \pi(y_7)\}$, recolor v_1, v_3 with a, v_5 with b and color v_2 with a. If $b \notin \{\pi(x_6), \pi(y_6)\}$, recolor v_5 with b and color v_2 with a. Otherwise, recolor v_1, v_6 with a, v_5, v_7 with b and color v_2 with b.
  - $\pi(v_7) = b$ and $\pi(x_1) = \pi(y_1) = a$. If $b \in \{\pi(x_7), \pi(y_7)\}$, recolor v_7 with a and then go back to the previous case. Now, assume $\pi(x_7) = \pi(y_7) = a$. Similarly, if $b \in \{\pi(x_6), \pi(y_6)\}$, then color v_6 with a, v_5 with b and color v_2 with a. So, assume $\pi(x_6) = \pi(y_6) = a$. Therefore, we may color v_2 with b successfully.

Now we consider the case that $\pi(v_4) = b$. If $b \in \{\pi(x_3), \pi(y_3)\}$, recolor v_3 with a and color v_2 with b. So assume $\pi(x_3) = \pi(y_3) = a$. If $\pi(x_4) = \pi(y_4) = a$, color v_2 with b. If $\pi(x_4) = \pi(y_4) = b$, recolor v_4 with a and color v_2 with b. Now, suppose that $\{\pi(x_4), \pi(y_4)\} = \{a, b\}$. If neither v_6 nor x_5 is colored with a, then color v_2 with a. If neither v_6 nor x_5 is colored with b, then recolor v_5 with b, v_4 with a, and color v_2 with a. So, assume that both colors a and b appear exactly once on the set $\{x_5, v_6\}$. The following discussion is similar to the previous case.

Case 3 $\pi(x_2) = \pi(v_1) = a$ and $\pi(v_3) = \pi(v_5) = b$.

First consider the case that $\pi(v_4) = a$. If $\pi(x_3) = \pi(y_3) = a$, color v_2 with b and color v_2 with b. If $\pi(x_3) = \pi(y_3) = b$, recolor v_3 with a and color v_2 with b. So, assume that $\{\pi(x_3), \pi(y_3)\} = \{a, b\}$. If $\pi(x_4) = \pi(y_4) = a$, recolor v_4 with b, v_3 with a and color v_2 with b. If $\pi(x_4) = \pi(y_4) = b$, recolor v_3 with a, and color v_2 with b. So, now assume that $\{\pi(x_4), \pi(y_4)\} = \{a, b\}$. If neither v_6 nor x_5 is colored with a, then recolor v_3, v_5 with a, v_4 with b, and color v_2 with b. If neither v_6 nor x_5 is colored with b, then color v_2 with b. So, assume that both colors a and b appear on the set $\{x_5, v_6\}$. We have two cases below.

- $\pi(v_6) = a$ and $\pi(x_5) = b$. If x_1, y_1, v_7 are all colored with b, then color v_2 with a. If at least two of x_1, y_1, v_7 are colored with a, then recolor v_1, v_4 with b, v_3, v_5 with a, and v_2 with b. Otherwise, assume that exactly two of x_1, y_1, v_7 are colored with b and one is colored with a. By symmetry, we need to deal with the following two possibilities.
  - $\pi(v_7) = a$ and $\pi(x_1) = \pi(y_1) = b$. If at least one of x_7, y_7 is colored with a, then recolor v_7 with b and color v_2 with a. Otherwise, assume $\pi(x_6) = \pi(y_6) = b$. If $a \notin \{\pi(x_6), \pi(y_6)\}$, color v_2 with a. Otherwise, recolor v_4, v_6 with b and v_3, v_5 with a and color v_2 with b.
  - $\pi(x_1) = a$ and $\pi(v_7) = \pi(y_1) = b$. If $b \notin \{\pi(x_7), \pi(y_7)\}$, recolor v_1, v_4 with b, v_3, v_5 with a, and color v_2 with b. If $a \notin \{\pi(x_6), \pi(y_6)\}$, recolor v_3, v_5 with a, v_4 with b, and color v_2 with b. Otherwise, we can first recolor v_3, v_5, v_7 with a and v_1, v_4, v_6 with b.
- $\pi(v_6) = b$ and $\pi(x_5) = a$. By a similar argument as above, we may suppose that exactly two of x_1, y_1, v_7 are colored with b and one is colored with a. By symmetry, we need to deal with the following two possibilities.
  - $\pi(v_7) = a$ and $\pi(x_1) = \pi(y_1) = b$. If either $a \notin \{\pi(x_7), \pi(y_7)\}$ or $b \notin \{\pi(x_6), \pi(y_6)\}$, then color v_2 with a or b. Otherwise, set $\pi(x_7) = a$ and $\pi(x_6) = b$. Then, switch the colors of v_6 and v_7 and then color v_2 with a successfully.
  - $\pi(x_1) = a$ and $\pi(v_7) = \pi(y_1) = b$. If $b \in \{\pi(x_6), \pi(y_6)\}$, recolor v_6 with a and color v_2 with b. Hence $\pi(x_6) = \pi(y_6) = a$. If $b \in \{\pi(x_7), \pi(y_7)\}$, recolor v_7 with a, v_1 with b, and color v_2 with a. Otherwise, color v_2 with b easily.

Now consider the case that $\pi(v_4) = b$. If $b \in \{\pi(x_3), \pi(y_3)\}$, recolor v_3 with a and color v_2 with b. Otherwise, assume that $\pi(x_3) = \pi(y_3) = a$. If $b \in \{\pi(x_4), \pi(y_4)\}$, recolor v_4 with a and then go back to the previous case. So we may assume that $\pi(x_4) = \pi(y_4) = a$. If v_5 can be given a new color a without arising any monochromatic cycle, we can further color v_2 with b successfully. Otherwise, we have the following two cases.

First assume that $\pi(v_6) = \pi(x_5) = a$. If x_1, y_1, v_7 are all colored with b, then color v_2 with a. If at least two of x_1, y_1, v_7 are colored with a, then recolor v_1 with b, and color v_2 with a. Otherwise, assume that exactly two of x_1, y_1, v_7 are colored with b and one is colored with a. By symmetry, we need to deal with two possibilities below.

- $\pi(v_7) = a$ and $\pi(x_1) = \pi(y_1) = b$. If $a \in \{\pi(x_7), \pi(y_7)\}$, recolor v_7 with b and color v_2 with a. So assume that $\pi(x_7) = \pi(y_7) = b$. If $a \in \{\pi(x_6), \pi(y_6)\}$, recolor v_6 with b, v_5 with a and color v_2 with b. Thus, $\pi(x_6) = \pi(y_6) = b$. In this case, we can color v_2 with a to derive a 2-forest-coloring of G, a contradiction.
- $\pi(x_1) = a$ and $\pi(v_7) = \pi(y_1) = b$. If $b \notin \{\pi(x_7), \pi(y_7)\}$, recolor v_1 with b and color v_2 with a. So, w.l.o.g., assume $\pi(x_7) = b$. If $a \notin \{\pi(x_6), \pi(y_6)\}$, recolor v_7 with a, v_1 with b and finally color v_2 with a. Otherwise, recolor v_1, v_6 with b v_7 with a, and color v_2 with a.

Now assume that $\{\pi(v_6), \pi(x_5)\} = \{a, b\}$. The proof is similar to the previous case.

Therefore, we complete the proof of Claim 5.2.10. ∎

5.2.2 Discharging argument

We complete the proof with a discharging procedure. Similarly, we define a weight function ω on the vertices and faces of G by letting $\omega(v) = 2d(v) - 6$ if $v \in V(G)$ and $\omega(f) = d(f) - 6$ if $f \in F(G)$. Before showing discharging rules, we need to give some notation used the following argument.

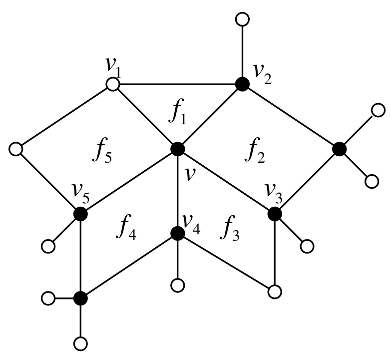


Figure 5.10: v is a special 5-vertex and f_5 is a special 4-face.

Suppose v is a 5-vertex. Let $v_1, v_2, \cdots, v_5$ be the neighbors of v in a cyclic order. Let f_i be the face with vv_i and vv_{i+1} as two boundary edges for $i = 1, 2, \cdots, 5$, where indices are taken modulo 5. We call v a *special 5-vertex of* f_5 if the following conditions hold:

(1) $d(f_1) = 3$;
(2) $d(f_i) = 4$ for all $i = 2, 3, 4, 5$;
(3) f_2 and f_4 are both $(5, 4, 4, 4)$-faces.

Moreover, we call f_5 a *special* 4-*face* with respect to v. Figure 5.10 shows a special 5-vertex v. By Claim 5.2.8 and Claim 5.2.10, we have to notice that such special 4-face is either a $(5, 4, 4, 6^+)$-face or a $(5, 4, 5^+, 5^+)$-face. These two observations will be used directly in the following proof. Recall that $l(x)$ denotes the number of light 4-faces incident or adjacent to x. Our discharging rules are as follows:

(R1) Every 6^+-vertex sends 1 to each incident 3^+-face.
(R2) Let v be a 5-vertex incident to a face f. Then
(R2.1) $\tau(v \to f) = 1$, if f is either a 3-face or $(5, 4, 4, 4)$-face;
(R2.2) $\tau(v \to f) = \frac{2}{3}$, if f is either a non-special 4-face or a bad 5-face.
(R2.3) $\tau(v \to f) = \frac{1}{3}$, if f is either a special 4-face or a good 5-face.
(R3) Let v be a 4-vertex and f_1, f_2, f_3, f_4 denote the faces of G incident to v in a cyclic order.
(R3.1) Assume $m_3(v) = 0$. Then
(R3.1.1) If $l(v) = 0$, then $\tau(v \to f_i) = \frac{1}{2}$ for each $i = 1, 2, 3, 4$.
(R3.1.2) If $l(v) = 1$, say f_1, then $\tau(v \to f_1) = \frac{2}{3}$, $\tau(v \to f_3) = \frac{1}{3}$ and $\tau(v \to f_i) = \frac{1}{2}$ for each $i = 2, 4$.
(R3.1.3) If $l(v) = 2$, then v sends $\frac{2}{3}$ to each incident light 4-face and $\frac{1}{3}$ to each other incident face.
(R3.2) Assume $m_3(v) = 1$ and f_1 is a 3-face. Then v sends 1 to f_1. Moreover,
(R3.2.1) If $l(v) = 0$, then $\tau(v \to f_i) = \frac{1}{3}$ for each $i = 2, 3, 4$.
(R3.2.2) Assume f_2 is a light 4-face. Then
(a1) If f_3 is a 4-face, then $\tau(v \to f_i) = \frac{1}{3}$ for each $i = 2, 3, 4$.
(a2) If f_3 is a 6^+-face, then $\tau(v \to f_2) = \frac{2}{3}$ and $\tau(v \to f_4) = \frac{1}{3}$.
(a3) Assume f_3 is a 5-face. Then
(a3.1) If either f is a good 5-face or $m_{5^+}(f_2) = 1$, then $\tau(v \to f_2) = \frac{2}{3}$ and $\tau(v \to f_4) = \frac{1}{3}$.
(a3.2) Assume f_3 is a bad 5-face and f_2 is adjacent to an another 5^+-face f^* different from f_3.
(a3.2.1) If f^* is a bad 5-face, then $\tau(v \to f_2) = \frac{1}{2}$, $\tau(v \to f_3) = \frac{1}{6}$ and $\tau(v \to f_4) = \frac{1}{3}$.
(a3.2.2) Otherwise, $\tau(v \to f_i) = \frac{1}{3}$ for each $i \in \{2, 3, 4\}$.
(R3.2.3) Assume f_3 is a light 4-face. Then
(b1) If one of f_2 and f_4 is of degree at least 6, say f_2, then $\tau(v \to f_3) = \frac{2}{3}$ and $\tau(v \to f_4) = \frac{1}{3}$.
(b2) If $m_5(v) = 0$, then $\tau(v \to f_i) = \frac{1}{3}$ for each $i = 2, 3, 4$.
(b3) Assume $m_5(v) = 2$ such that f_2 and f_4 are both 5-faces.
(b3.1) If one of f_2, f_4 is a good 5-face, say f_2, then $\tau(v \to f_3) = \frac{2}{3}$ and $\tau(v \to f_4) = \frac{1}{3}$.
(b3.2) Otherwise, $\tau(v \to f_2) = \tau(v \to f_4) = \frac{1}{6}$ and $\tau(v \to f_3) = \frac{2}{3}$.
(b4) Assume $m_5(v) = 1$ such that f_2 is a 4-face and f_4 is a 5-face.

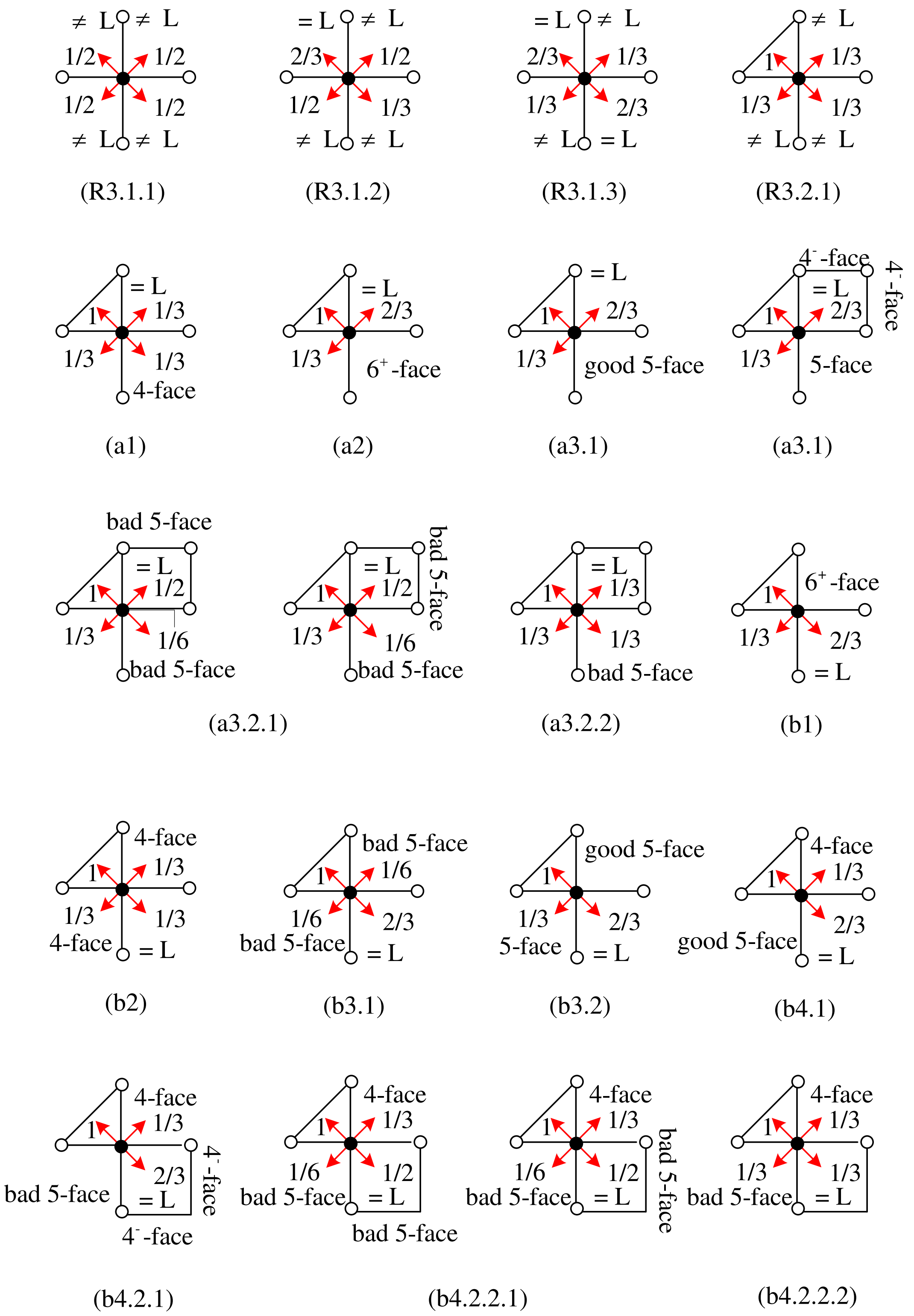


Figure 5.11: Discharging rule (R3).

(b4.1) If f_4 is a good 5-face, then $\tau(v \to f_2) = \frac{1}{3}$ and $\tau(v \to f_3) = \frac{2}{3}$.
(b4.2) Assume f_4 is a bad 5-face.
(b4.2.1) If $m_{5^+}(f_3) = 1$, then $\tau(v \to f_2) = \frac{1}{3}$ and $\tau(v \to f_3) = \frac{2}{3}$.
(b4.2.2) Assume f_3 is adjacent to an another 5^+-face f^* different from f_4.
(b4.2.2.1) If f^* is a bad 5-face, then $\tau(v \to f_2) = \frac{1}{3}$, $\tau(v \to f_3) = \frac{1}{2}$ and $\tau(v \to f_4) = \frac{1}{6}$.
(b4.2.2.2) Otherwise, $\tau(v \to f_i) = \frac{1}{3}$ for each $i \in \{2, 3, 4\}$.

For simplicity, in Figure 5.11, we use the notation "$= L$" to denote a light 4-face. By a careful observation, (R3) includes all possible incident cases for any vertex of degree 4. Thus, combining (R1) and (R2), the following statement holds.

Observation 5.2.3 *Every 4^+-vertex sends at least $\frac{1}{3}$ to each incident 4-face.*

We only need to show $\omega^*(x) \geqslant 0$ for all $x \in V(G) \cup F(G)$.

Let $v \in V(G)$. Since $\delta(G) \geqslant 4$, $d(v) \geqslant 4$. In what follows, let $v_1, v_2, \cdots, v_{d(v)}$ denote the neighbors of v in a cyclic order, and let f_i denote the incident face of v with vv_i and vv_{i+1} as two boundary edges for $i = 1, 2, \cdots, d(v)$, where indices are taken modulo $d(v)$. We have to handle the following cases, depending on the size of $d(v)$.

Case 1 If $d(v) \geqslant 6$, then it is trivial that $\omega^*(v) = 2d(v) - 6 - d(v) = d(v) - 6 \geqslant 0$ by (R1).

Case 2 If $d(v) = 5$, then $\omega(v) = 4$. Let $m_4^*(v)$ be the number of incident $(5, 4, 4, 4)$-faces. By Claim 5.2.7, $m_4^*(v) \leqslant 2$. Moreover, $m_3(v) \leqslant 1$ by the absence of intersecting triangles. If $m_3(v) = 0$, then $\omega^*(v) \geqslant 4 - m_4^*(v) - \frac{2}{3}(5 - m_4^*(v)) = \frac{2}{3} - \frac{1}{3}m_4^*(v) \geqslant 0$ by (R2).

Now, without loss of generality, assume that $f_1 = [vv_1v_2]$ is a 3-face. By (R2.1), $\tau(v \to f_1) = 1$. If $m_4^*(v) \leqslant 1$, then $\omega^*(v) \geqslant 4 - 1 - m_4^*(v) - \frac{2}{3}(4 - m_4^*(v)) = \frac{1}{3} - \frac{1}{3}m_4^*(v) \geqslant 0$ by (R2). So, in the following, we assume that $m_4^*(v) = 2$. By Claim 5.2.7 and Claim 5.2.5, there is only one possible case that f_2 and f_4 are both $(5, 4, 4, 4)$-faces. Note that $d(v_1) \geqslant 5$ by Claim 5.2.8. This fact implies that f_5 cannot be a $(5, 4, 4, 4, 4)$-face.

- If $d(f_3) \geqslant 6$, then v sends nothing to f_3 by (R2) and hence $\omega^*(v) \geqslant 4 - 1 - 1 \times 2 - \frac{2}{3} = \frac{1}{3}$.
- If $d(f_3) = 5$, then f_3 cannot be adjacent to any light 4-face by Claim 5.2.9. It follows immediately from the definition that f_3 is not a bad 5-face. So, by (R2.3), $\tau(v \to f_3) = \frac{1}{3}$. Therefore, we derive that $\omega^*(v) \geqslant 4 - 1 - 1 \times 2 - \frac{2}{3} - \frac{1}{3} = 0$.
- Now, suppose that $f_3 = [vv_3wv_4]$ is a 4-face. Moreover, f_2 is a $(5, 4, 5^+, 4)$-face and thus it gets at most $\frac{2}{3}$ from v by (R2.2). If we can show that f_5 gets at most $\frac{1}{3}$ from v and thus we obtain that $\omega^*(v) \geqslant 4 - 1 - 1 \times 2 - \frac{2}{3} - \frac{1}{3} = 0$. To see that, we have two cases. If f_5 is not a 4-face, then v sends at most $\frac{1}{3}$ to f_5 since f_5 cannot be a $(5, 4, 4, 4, 4)$-face. Now we assume that f_5 is a 4-face. It implies that f_5 is a special face with respect to v and therefore v sends $\frac{1}{3}$ to f_5 by (R2.3).

Case 3 If $d(v) = 4$, then $\omega(v) = 2$. Clearly, $m_3(v) \leqslant 1$. First assume that $m_3(v) = 0$. By Claim 5.2.2, v is incident to at most two light 4-faces. It is easy to derive that $\omega^*(v) \geqslant 2 - \frac{1}{2} \times 4 = 0$ by (R3.1.1), or $\omega^*(v) \geqslant 2 - \frac{2}{3} - \frac{1}{3} - \frac{1}{2} \times 2 = 0$ by (R3.1.2), or $\omega^*(v) \geqslant 2 - \frac{2}{3} \times 2 - \frac{1}{3} \times 2 = 0$ by (R3.1.3).

Now assume that $m_3(v) = 1$ and f_1 is a 3-face. By (R3.2), $\tau(v \to f_1) = 1$. By (R2), we notice that v only sends charge to incident face. So, in the following each case, it remains to show that $\sum_{i=2}^{i=4} \tau(v \to f_i) \leqslant 1$ and therefore we have that $\omega^*(v) \geqslant 2 - 1 - 1 = 0$. For simplicity, we write τ for $\sum_{i=2}^{i=4} \tau(v \to f_i)$. By Claim 5.2.2 and Claim 5.2.3, we obtain that $l(v) \leqslant 1$. In other words, v is incident to at most one light 4-face. If $l(v) = 0$, then $\tau(v \to f_i) = \frac{1}{3}$ for each $i = 2, 3, 4$ by (R3.2.1) and thus $\tau = \frac{1}{3} \times 3 = 1$. Now assume that $l(v) = 1$. By symmetry, the following proof is divided into two cases, depending on the situation of the incident light 4-face.

- Assume that f_2 is a light 4-face. If f_3 is a 4-face, by (a1), we have $\tau = \frac{1}{3} \times 3 = 1$. If f_3 is a 6^+-face, by (a2), we have $\tau = \frac{2}{3} + \frac{1}{3} = 1$. Now assume $d(f_3) = 5$. If either $m_{5^+}(f_2) = 1$ or f_3 is a good 5-face, then $\tau = \frac{2}{3} + \frac{1}{3} = 1$ by (a3.1). Otherwise, assume that f_3 is a bad 5-face and f_2 is adjacent to an another 5^+-face f^* different from f_3. We also obtain that $\tau = \frac{1}{3} \times 3 = 1$ by (a3.2.2) or $\tau = \frac{1}{2} + \frac{1}{6} + \frac{1}{3} = 1$ by (a3.2.1).

- Assume that f_3 is a light 4-face. If at least one of f_2 and f_4 is a 6^+-face, say f_2, then by (b1), we have that $\tau = \frac{2}{3} + \frac{1}{3} = 1$. So, in the following, suppose that f_i is either a 4-face or a 5-face for each $i \in \{2, 4\}$. If $d(f_2) = d(f_4) = 4$, then $\tau = \frac{1}{3} \times 3 = 1$ by (b2). Assume that $d(f_2) = d(f_4) = 5$. If at least one of f_2, f_4 is a good 5-face, then $\tau = \frac{2}{3} + \frac{1}{3} = 1$ by (b3.2). Otherwise, $\tau = \frac{1}{6} \times 2 + \frac{2}{3} = 1$ by (b3.1). Now, by symmetry, assume that $d(f_2) = 4$ and $d(f_4) = 5$. If f_4 is a good 5-face, then $\tau = \frac{1}{3} + \frac{2}{3} = 1$ by (b4.1). Now assume f_2 is a bad 5-face. If $m_{5^+}(f_3) = 1$, then $\tau = \frac{1}{3} + \frac{2}{3} = 1$ by (b4.2.1). Otherwise, assume that f_3 is adjacent to an another 5^+-face f^* different from f_4. If f^* is bad, then $\tau = \frac{1}{3} + \frac{1}{2} + \frac{1}{6} = 1$ by (b4.2.2). Otherwise, we deduce that $\tau = \frac{1}{3} \times 3 = 1$ by (b4.2.2).

It remains to show that $\omega^*(f) \geqslant 0$ for $f \in F(G)$. The proof is divided into four cases below according to the value of $d(f)$.

Case 4 If $d(f) \geqslant 6$, then $\omega^*(f) = d(f) - 6 \geqslant 0$ by (R1) to (R3).

Case 5 If $d(f) = 3$, then $\omega(f) = -3$. By Claim 5.2.1, f is incident to three 4^+-vertices and thus $\omega^*(f) = -3 + 1 \times 3 = 0$ by (R1)-(R3).

Case 6 If $d(f) = 4$, then $\omega(f) = -2$. By Claim 5.2.1, we see that $d(v_i) \geqslant 4$ for all $i = 1, 2, 3, 4$. Moreover, for $i \in \{1, 2, 3, 4\}$, v_i sends at least $\frac{1}{3}$ to f by Observation 5.2.3. This observation will be used frequently without further notice. If f is incident to at least one 6^+-vertex, say v_1, then $\tau(v_1 \to f) = 1$ by (R1) and thus $\omega^*(f) \geqslant -2 + 1 + \frac{1}{3} \times 3 = 0$. Now, in the following, we assume that $4 \leqslant d(v_i) \leqslant 5$ for all $i = 1, 2, 3, 4$. By symmetry, we only need to consider six subcases below.

First assume that $d(v_i) = 4$ for all $i = 1, 2, 3, 4$. Namely, f is a light 4-face. By (2) of Lemma 5.2.1, f is adjacent to at least one 5^+-face. Without loss of generality, assume that f_1 is a 5^+-face. If $d(f_1) \geqslant 6$, then $\tau(v_1 \to f) = \tau(v_2 \to f) = \frac{2}{3}$ by

(R3.1), (a2) and (b1). Therefore, $\omega^*(f) \geqslant -2 + \frac{2}{3} \times 2 + \frac{1}{3} \times 2 = 0$. So assume that $f_1 = [v_1u_1u_2u_3v_2]$ is a 5-face. If f_1 is a good 5-face, by (R3.1), (a3.1), (b3.2) and (b4.1), we see that each of v_1 and v_2 sends $\frac{2}{3}$ to f, respectively. Thus $\omega^*(f) \geqslant -2 + \frac{2}{3} \times 2 + \frac{1}{3} \times 2 = 0$. Now assume f_1 is a bad 5-face. If f_2, f_3, f_4 are all 4^--faces, then similarly we obtain that $\omega^*(f) \geqslant -2 + \frac{2}{3} \times 2 + \frac{1}{3} \times 2 = 0$ by (R3.1), (a3.1), (b3.1) and (b4.2.1). So, in the following, we may suppose that f_i is a 5^+-face for some fixed $i \in \{2, 3, 4\}$. Moreover, we may suppose that f_i is a bad 5-face. If not, we can reduce the argument to the previous cases. By symmetry, we have two cases below.

- Assume f_3 is a bad 5-face. It follows from (R3.1), (a3.2.1), (b1), (b3.1), (b3.2) and (b4.2.2) that $\tau(v_i \to f) \geqslant \frac{1}{2}$ for each $i = 1, 2, 3, 4$. Thus, $\omega^*(f) \geqslant -2 + \frac{1}{2} \times 4 = 0$.
- Assume f_4 is a bad 5-face. It implies that v_1 is a 4-vertex which is incident to two opposite bad 5-faces. By (b3.1), $\tau(v_1 \to f) = \frac{2}{3}$. Similarly, by (R3.1), (a3.2.1), (b1), (b3.1), (b3.2) and (b4.2.2) again, $\tau(v_2 \to f) = \tau(v_4) = \frac{1}{2}$. Therefore $\omega^*(f) \geqslant -2 + \frac{2}{3} + \frac{1}{2} \times 2 + \frac{1}{3} = 0$.

Next assume that $d(v_1) \geqslant 5$ and $d(v_i) = 4$ for all $i = 2, 3, 4$. By (R1) and (R2), v_1 sends 1 to f. Hence, $\omega^*(f) \geqslant -2 + 1 + \frac{1}{3} \times 3 = 0$.

Next assume that $d(v_1) = d(v_2) = 5$ and $d(v_3) = d(v_4) = 4$. Since each special 4-face is either a $(5, 4, 4, 6^+)$-face or a $(5, 4, 5^+, 5^+)$-face, neither v_1 nor v_2 can be a special 5-vertex of f. Thus $\omega^*(f) \geqslant -2 + \frac{2}{3} \times 2 + \frac{1}{3} \times 2 = 0$ by (R2.2).

Next assume that $d(v_1) = d(v_3) = 5$ and $d(v_2) = d(v_4) = 4$. The discussion is similar to the above case.

Now assume that $d(v_1) = d(v_2) = d(v_3) = 5$ and $d(v_4) = 4$. We first notice that v_2 cannot be a special vertex since neither f_1 nor f_2 is a $(5, 4, 4, 4)$-face. If at most one of v_1, v_3 is a special vertex, then it is easy to derive that $\omega^*(f) \geqslant -2 + \frac{2}{3} \times 2 + \frac{1}{3} + \frac{1}{3} = 0$ by (R2.2) and (R2.3). Otherwise, suppose that v_1 and v_3 are both special 5-vertices. By the definition, we obtain immediately that f_1 and f_2 are both 3-faces while f_3, f_4 are both $(5, 4, 4, 4)$-faces. This contradicts the assumption on G.

Finally assume that $d(v_i) = 5$ for all $i = 1, 2, 3, 4$. Notice again that none of v_1, v_2, v_3, v_4 is a special 5-vertex. Consequently, $\omega^*(f) \geqslant -2 + \frac{2}{3} \times 4 = \frac{2}{3}$ by (R2.2).

Claim 5.2.11 *Suppose that v is a 4-vertex. Let f_1, f_2, f_3, f_4 denote the faces of G incident to v in a cyclic order such that f_1 is a 5-face. If neither f_2 nor f_4 is a light 4-face, then $\tau(v \to f_1) \geqslant \frac{1}{3}$.*

Proof. First assume that $l(v) = 0$. It follows immediately from (R3.1.1) and (R3.2.1) that $\tau(v \to f_1) \geqslant \frac{1}{3}$ and thus we are done. Otherwise, assume that f_3 is a light 4-face. By (a1), (a2) and (a3), it is easy to deduce that $\tau(v \to f_1) \geqslant \frac{1}{3}$. Thus, we complete the proof of Claim 5.2.11. ∎

Case 7 If $d(f) = 5$, then $\omega(f) = -1$. Notice that $d(v_i) \geqslant 4$ by Claim 5.2.1. If f is incident to at least one 6^+-vertex, then $\omega^*(f) \geqslant -1 + 1 = 0$ by (R1). So, in the following, assume that $4 \leqslant d(v_i) \leqslant 5$ for all $i = 1, \cdots, 5$. In what follows, let $n_5(f)$

denote the number of 5-vertices incident to f. First assume that $n_5(f) \geqslant 3$. It is trivial that $\omega^*(f) \geqslant -1 + \frac{1}{3} \times 3 = 0$ by (R2).

Next assume that $n_5(f) = 2$. By (R2), each 5-vertex sends at least $\frac{1}{3}$ to f. It suffices to show that f gets at least $\frac{1}{3}$ from the remaining 4-vertices in total. By symmetry, we have two possibilities:

- Assume $d(v_1) = d(v_2) = 5$. It implies that $d(v_3) = d(v_4) = d(v_5) = 4$. So there are at most two light 4-faces adjacent to f. If $l(f) = 2$, i.e., f_3, f_4, then $\tau(v_4 \to f) = \frac{1}{3}$ by (R3.1.3). Suppose $l(f) = 1$. By symmetry, suppose that f_3 is a light 4-face and f_4 is not. By Claim 5.2.11, it is easy to deduce that $\tau(v_5 \to f) \geqslant \frac{1}{3}$ since f_5 is not a light 4-face. Finally suppose that $l(f) = 0$. We obtain immediately that v_4 sends at least $\frac{1}{3}$ to f by Claim 5.2.11.

- Assume $d(v_1) = d(v_3) = 5$. Then $d(v_2) = d(v_4) = d(v_5) = 4$. Obviously, neither f_1 nor f_2 is a light 4-face. Thus, v_2 sends at least $\frac{1}{3}$ to f by Claim 5.2.11.

Now assume $n_5(f) = 1$, say v_1. Then $d(v_i) = 4$ for all $i = 2, 3, 4, 5$ and $l(f) \leqslant 3$. If $l(f) = 3$, then $\tau(v_3 \to f) = \tau(v_4 \to f) = \frac{1}{3}$ by (R3.1.3) and $\tau(v_1 \to f) \geqslant \frac{1}{3}$ by (R2.2) and (R2.3). Thus, $\omega^*(f) \geqslant -1 + \frac{1}{3} + \frac{1}{3} \times 2 = 0$. If $l(f) \leqslant 1$, then there exist v_i and v_j whose incident light 4-face must be opposite to f. By Claim 5.2.11, each of them sends $\frac{1}{3}$ to f and hence $\omega^*(f) \geqslant -1 + \frac{1}{3} \times 3 = 0$. Now, assume that $l(f) = 2$. If f_2, f_3 are light 4-faces and f_4 is not, then $\tau(v_3 \to f) = \frac{1}{3}$ by (R3.1.3) and $\tau(v_5 \to f) \geqslant \frac{1}{3}$ by Claim 5.2.11. So we have that $\omega^*(f) \geqslant -1 + \frac{1}{3} \times 3 = 0$. If f_2, f_4 are light 4-faces and f_3 is not, then f is a light 5-face. By Lemma 5.2.2, at least one of f_2 and f_4 is adjacent to a 5^+-face different from f, say f_2. By (R3.1), (a.3.2.1), (a3.2.2), (b1), (b3.1), (b3.2), (b4.2.2), we assert that each of v_2, v_3 sends at least $\frac{1}{6}$ to f. Therefore, $\omega^*(f) \geqslant -1 + \frac{2}{3} + \frac{1}{6} \times 2 = 0$ by (R2.2).

Finally assume that $n_5(f) = 0$. Namely, $d(v_i) = 4$ for all $i = 1, \cdots, 5$. In other words, f is a light 5-face. By Claim 5.2.4, none of $f_1, \cdots, f_5$ is a light 4-face. It follows directly from Claim 5.2.11 that each v_i sends at least $\frac{1}{3}$ to f. Therefore, we conclude that $\omega^*(f) \geqslant -1 + \frac{1}{3} \times 5 = \frac{2}{3}$. ∎

5.3 Further research

By Appel and Haken's Four Color Theorem [AH76], every planar graph G has a partition (V_1, V_2, V_3, V_4) such that each V_i induces an independent set. However, Wegner [Weg73] showed that there exists a planar graph which cannot be partitioned into (V_1, V_2, V_3) such that V_1, V_2 are independent sets and V_3 is a forest; and even earlier, Chartrand and Kronk [CK69] showed that there exists a planar graph which cannot be partitioned into two forests. On the other hand, Voigt [Voi93] and independently, by Mirzakhani [Mir96] proved that not all planar graphs are 4-choosable. All of these facts imply that it is impossible to strengthen the Four Color Theorem.

A natural problem arises of finding sufficient conditions for planar graphs to be 4-choosable, as well as for planar graphs to have vertex-arboricity 2, list vertex-arboricity 2, and so on.

A graph G is *k-degenerate* if every subgraph H of G has a vertex of degree at most k in H. It is well known that every k-degenerate graph is $(k+1)$-choosable. It is easy to prove that every planar triangle-free graph is 3-degenerate by using Euler's formula. Wang and Lih [WL02] proved that planar graphs without 5-cycles are 3-degenerate, while Fijavž et al. [FJMv02] showed that planar graphs without 6-cycles are also 3-degenerate. The lack of 4-cycles does not imply the 3-degeneracy of a planar graph, i.e., the line graph of a dodecahedron. However, Lam, Xu and Liu [LXL99] proved that planar graphs without 4-cycles are 4-choosable. Recently, Farzad [Far09] proved the conjecture proposed by Fijavz et al. [FJMv02] and independently, Wang and Lih [WL01] that planar graphs without 7-cycles are 4-choosable.

Combining these facts, we have the following:

Theorem 5.3.1 *If G is a planar graph without i-cycles for some fixed $i \in \{3,4,5,6,7\}$, then G is 4-choosable.*

Borodin and Ivanova [BI08b] improved the above-mentioned result in [LXL99] by showing that every planar graph without 4-cycles adjacent to 3-cycles is 4-choosable. Moreover, in [BI09b], they extend this result in terms of covering the vertices of a graph by induced subgraphs of variable degeneracy. In particular, they proved that every planar graph without 4-cycles adjacent to 3-cycles can be partitioned into two induced forests.

It is natural to ask the following question: is a planar graph G 4-choosable if G can be partitioned into two induced forests? We give a negative answer basing on the example constructed by Mirzakhani in [Mir96]. In other words, there exists a non-4-choosable planar graph which has vertex-arboricity 2.

To conclude this chapter, we would like to propose the following problem.

Problem 5.3.2 *Does every planar graph without chordal k-cycles have vertex-arboricity at most 2, where $4 \leqslant k \leqslant 7$?*

The case $k = 6$ was handled by Huang and Wang in [HW10]. So, we leave the case of $k \in \{4, 5, 7\}$ as an open problem.

Chapter 6

Fractional coloring

Contents

In this chapter, we study the fractional coloring of graphs by considering (n, k)-colorings which are generalizations of the conventional vertex coloring problem. As a special case, a graph is $(5, 2)$-colorable if and only if it has a homomorphism to the Petersen graph. In Section 6.2, we will consider the relationship between the Petersen graph and the sparse graphs, i.e., graphs with maximum average degree less than c. More precisely, we prove that every triangle-free graph with $\mathrm{Mad}(G) < 5/2$ is homomorphic to the Petersen graph. In other words, such a graph is $(5, 2)$-colorable. Moreover, we show that the bound on the maximum average degree in our result is sharp.

6.1 Introduction

For positive integers k and $n \geqslant 2k$, an (n, k)-*coloring* of a graph G is a mapping $c\colon\ V(G) \to \binom{\{1, 2, \cdots, n\}}{k}$ such that for any two adjacent vertices x and y, $c(x)$ and $c(y)$ are disjoint. The concept of (n, k)-coloring is a generalization of the conventional vertex coloring problem. In fact, an $(n, 1)$-coloring is exactly an ordinary proper n-coloring.

The *fractional chromatic number*, denoted $\chi_f(G)$, of a graph G is the infimum of the fractions n/k for which there exists an (n, k)-coloring of G. The *Kneser graph*, denoted by $K_{n:k}$, is defined to be the graph in which vertices represent subsets of cardinality k taken from $\{1, 2, \cdots, n\}$ and two vertices are adjacent if and only if the corresponding subsets are disjoint. Note that $K_{5:2}$ is the famous Petersen graph. We recall that a *homomorphism* from G to H is a mapping $h : V(G) \rightarrow V(H)$ such that if $xy \in E(G)$ then $h(x)h(y) \in E(H)$. It is easy to observe that a graph G has an (n, k)-coloring if and only if there exists a homomorphism from G to $K_{n:k}$. As a special case, a graph is $(5, 2)$-colorable if and only if it has a homomorphism to the Petersen graph. Some background and more details about fractional coloring can be found in the monograph of Scheinerman and Ullman [SU97].

Fractional coloring has been investigated by Klostermeyer and Zhang [KZ02]. Some results related to planar graphs are collected as follows:

Theorem 6.1.1 [KZ02] *Let G be a planar graph.*

(1) *If the odd girth of G is at least $10k - 7$ with $k \geqslant 2$, then $\chi_f(G) \leqslant 2 + \frac{1}{k}$.*

(2) *If $g(G) \geqslant 10k - 9$ with $k \geqslant 2$ and $\Delta(G) \leqslant 3$, then $\chi_f(G) \leqslant 2 + \frac{1}{k}$.*

(3) *There exists a planar graph G with odd girth $2k + 1$ such that $\chi_f(G) > 2 + \frac{1}{k}$.*

We have to notice that the conclusion (1) in Theorem 6.1.1 was improved by Pirnazar and Ullman [PU02] as follows:

Theorem 6.1.2 [PU02] *Every planar graph with $g \geqslant 8k - 4$ ($k \geqslant 1$) has fractional chromatic number at most $2 + \frac{1}{k}$.*

Theorem 6.1.2 for $k = 2$ implies that planar graphs with girth at least 12 have the fractional chromatic number at most $\frac{5}{2}$. Dvořák et al.[DvV08] improved this result by showing that every planar graph with odd girth at least 9 is $(5, 2)$-colorable. Moreover, they left the case of odd girth 7 as an open problem.

Recall that the *maximum average degree* of G, denoted $\mathrm{Mad}(G)$, is defined as:

$$\mathrm{Mad}(G) = \max\{\tfrac{2|E(H)|}{|V(H)|} : H \subseteq G\}.$$

This is a conventional measure of sparseness of an arbitrarary graph (not necessarily planar). For more details on this invariant see [JT95a] where properties of the maximum average degree are exhibited, and where it is proved that maximum average degree may be computed by a polynomial algorithm. In this chapter, we are interested in homomorphisms of sparse graphs with given maximum average degree to the Petersen graph. More precisely, we will prove the following result:

Theorem 6.1.3 *Every triangle-free graph G with* $\mathrm{Mad}(G) < \frac{5}{2}$ is homomorphic to the Petersen graph.

Consider the following graph H_2, depicted in Figure 6.1, which was constructed by Klostermeyer and Zhang in [KZ02]. Obviously, H_2 has eight vertices and ten edges, yielding average degree $\frac{5}{2}$. Moreover, all its proper subgraphs have smaller average degree. Hence, we have that $\mathrm{Mad}(H_2) = \frac{5}{2}$. Suppose H_2 admits a homomorphism h to the Petersen graph, depicted in Figure 6.2. Obviously, a 5-cycle uu_1u_2wvu in H_2 is mapped to a 5-cycle in the Petersen graph. By symmetry of the

Petersen graph, we may assume that $h(u) = x_1$, $h(u_1) = x_2$, $h(u_2) = x_3$, $h(w) = y_3$ and $h(v) = y_1$. Clearly, $h(z)$ must be y_1. So, $h(v) = h(z)$. This implies that vv_1v_2z cannot be mapped to the Petersen graph properly, which is a contradiction.

The above argument implies that the bound on maximum average degree in Theorem 6.1.3 is sharp. On the other hand, since the girth of the Petersen graph is 5, any triangle cannot be mapped to the Petersen graph. It means that the assumption in Theorem 6.1.3 that G is triangle-free is necessary.

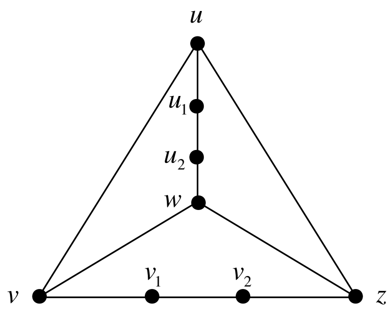


Figure 6.1: An example H_2.

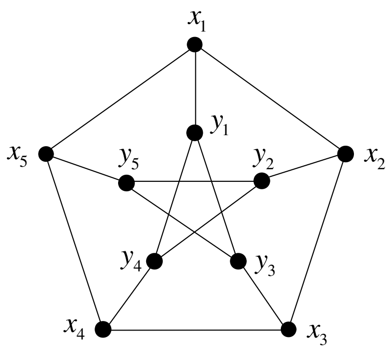


Figure 6.2: The Petersen graph.

In light of the fact that the Petersen graph is the Kneser graph $K_{5:2}$, our main result is equivalent to the following:

Theorem 6.1.4 *If G is a triangle-free graph with* $\mathrm{Mad}(G) < \frac{5}{2}$, *then G is* $(5,2)$-*colorable.*

A distinctive feature of the proof of Theorem 6.1.3 is that a charge of vertices can be transferred along "feeding paths" to an unlimited distance. This kind of "global" discharging was introduced by Borodin, Ivanova, and Kostochka in [BIK06]. We remark that our result has been published in Discrete Mathematics [CR10a].

6.2 Proof of Theorem 6.1.3

In what follows, if there is no confusion about the context, we write $K_{5:2}$ for the Petersen graph. The proof of Theorem 6.1.3 is proceeded by a contradiction. Suppose to the contrary that G is a counterexample with the least number of vertices, i.e., a triangle-free graph with $\text{Mad}(G) < \frac{5}{2}$, without any homomorphism of G to $K_{5:2}$, but there exists a homomorphism of its any subgraph H with $|H| < |G|$ to $K_{5:2}$. It is easy to see that G is connected. Moreover, G is 2-connected, since $K_{5:2}$ is vertex transitive. We first show some reducible configurations of G in Section 6.2.1, then use Euler's formula and the technique to derive a contradiction in Section 6.2.2.

6.2.1 Reducible configurations

For $x \in V(K_{5:2})$, we define $L_i(x) = \{y |$ there is a walk of length i in $K_{5:2}$ joining x and $y\}$ and $F_i(x) = V(K_{5:2}) \backslash L_i(x)$. It is easy to obtain the following Claim 6.2.1.

Claim 6.2.1 *For $x \in V(K_{5:2})$, we have that $|F_1(x)| = 7$, $|F_2(x)| = 3$, $|F_3(x)| = 1$, and $|F_4(x)| = 0$.*

Claim 6.2.2 *There is no 3-thread in G.*

Proof. Suppose the claim is false. Let $v_1v_2v_3v_4v_5$ be a 3-thread in G such that $d(v_2) = d(v_3) = d(v_4) = 2$. Since G contains no triangles, $v_1 \neq v_4$ and $v_2 \neq v_5$. By the minimality of G, there is a homomorphism h from $G - \{v_2, v_3, v_4\}$ to $K_{5:2}$. By Claim 6.2.1, we see that $|F_4(x)| = 0$ for any $x \in V(K_{5:2})$. It means that there always exists a walk of length 4 connecting $h(v_1)$ and $h(v_5)$ in $K_{5:2}$. Therefore, we can map v_2, v_3, v_4 successfully to $K_{5:2}$ and thus extend the homomorphism h to the whole graph G. This contradicts the choice of G. ∎

Suppose that h is a homomorphism of G to $K_{5:2}$ and x, y are any two unmapped vertices in G. We will say that *y allows k vertices for x* if for any given mapping choice of y we have at least k vertices in $K_{5:2}$ for mapping x. Similarly, we will say that y *forbids k vertices* for x if for any given mapping choice of y we have $10 - k$ vertices in $K_{5:2}$ for mapping x.

Remark 1: For any two distinct vertices u, v in $K_{5:2}$, one can easily observe that there always exists a walk of length 3 connecting u and v. Basing on this fact, we can map any 3-path $v_1v_2v_3v_4$ with $d(v_2) = d(v_3) = 2$ to $K_{5:2}$ if v_1 and v_4 have been mapped to $K_{5:2}$ and the images of v_1, v_4 are different. Then we have the following claim, which plays an important role in Claim 6.2.7.

Claim 6.2.3 *Let $P = v_1v_2v_3v_4$ be a path of G with $d(v_2) = d(v_3) = 2$. If h is a homomorphism of G to $K_{5:2}$ with v_2, v_3 both unmapped and $h(v_1) \neq h(v_4)$, then there exist two internally disjoint walks in $K_{5:2}$ connecting $h(v_1)$ and $h(v_4)$.*

Proof. First suppose that $h(v_1)$ is adjacent to $h(v_4)$. W.l.o.g., suppose that $h(v_1) = x_1$ and $h(v_2) = x_2$, see Figure 6.2. Then $x_1x_5x_1x_2$ and $x_1x_2x_3x_2$ are two internally disjoint walks. Otherwise, by symmetry, suppose that $h(v_1) = x_1$ and $h(v_2) = x_3$. Then $x_1x_5x_4x_3$ and $x_1y_1y_3x_3$ are the desired walks. ∎

In the proofs of Claims 6.2.4-6.2.12, we use $\mathcal{B}$ to denote the set of all solid vertices, depicted in Figure 6.3 to Figure 6.10. Moreover, we call $\mathcal{B}$ *unmapped* if none of the vertices in $\mathcal{B}$ is mapped to $K_{5:2}$.

Claim 6.2.4 *There is no* $(1^+, 1^+, 1^+)$*-vertex in* G.

Proof. Suppose to the contrary that G contains a $(1^+, 1^+, 1^+)$-vertex v, depicted in Figure 6.3. Since G is triangle-free, we have that $\{x, y, z\} \cap \{v, x_1, y_1, z_1\} = \varnothing$. By the choice of G, there exists a homomorphism h from $G - \mathcal{B}$ to $K_{5:2}$. It follows that x, y, z have been all mapped to $K_{5:2}$. By Claim 6.2.1, each of x, y, z forbids, respectively, at most three vertices for v. Thus, there is one possible vertex for v to be mapped in $K_{5:2}$. Hence, h can be extended to the whole graph G. This contradiction completes the proof of Claim 6.2.4. ∎

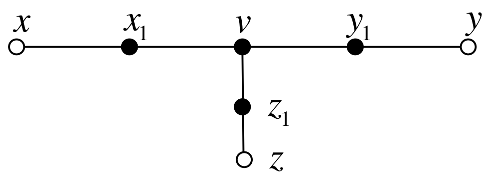


Figure 6.3: v is a $(1^+, 1^+, 1^+)$-vertex.

Claim 6.2.5 *There is no* $(2, 2, 0^+)$*-vertex in* G.

Proof. Assume to the contrary that G contains a $(2, 2, 0^+)$-vertex v shown by Figure 6.4. By Claim 6.2.2, we see that $d(x) \geqslant 3$ and $d(y) \geqslant 3$. It follows immediately that $\{x, y, z\} \cap \{v, x_1, x_2, y_1, y_2\} = \varnothing$ by the absence of 3-cycles in G. Obviously, there is a homomorphism h from $G - \mathcal{B}$ to $K_{5:2}$ by the minimality of G. By Claim 6.2.1, each of x, y forbids, respectively, one vertex for v and z forbids seven vertices for v. It implies that there is one possible vertex for v to be mapped in $K_{5:2}$. Therefore, we extend h to the whole graph G, which is a contradiction. ∎

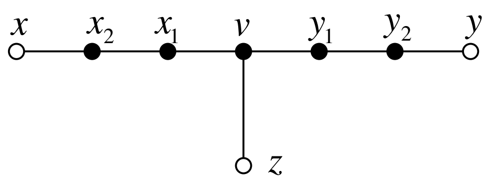


Figure 6.4: v is a $(2, 2, 0^+)$-vertex.

Claim 6.2.6 *There is no* $(1^+, 1^+, 2, 2)$*-vertex in* G.

Proof. Suppose to the contrary that there exists a $(1^+, 1^+, 2, 2)$-vertex v in G, depicted in Figure 6.5. It is easy to inspect that $\{x, y, z, w\} \cap \{v, x_1, x_2, z_1, z_2, y_1, w_1\} = \varnothing$, since there is no 3-cycles in G and $d(x), d(z) \geqslant 3$ by Claim 6.2.2. By the minimality of G, $G - \mathcal{B}$ admits a homomorphism h to $K_{5:2}$. By Claim 6.2.1, each of y and w forbids at most three vertices for v while each of x and z forbids at most one vertex for v. It follows that there are at least two mapping choices for v. Therefore, h can be extended to the whole graph G, which contradicts the choice of G. ∎

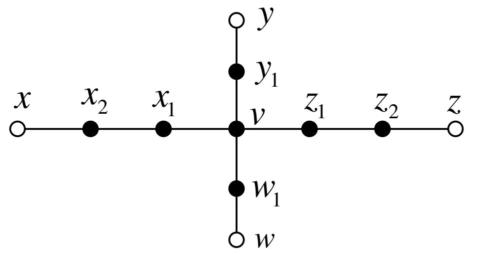


Figure 6.5: v is a $(1^+, 1^+, 2, 2)$-vertex.

Let v be a $(k_1, k_2, \cdots, k_m)$-vertex and P_{k_i} be the maximal k_i-thread incident to v with $i \in \{1, 2, \cdots, m\}$. Denote $P_{k_1} = vx_1 \cdots x_{k_1}x$ and $P_{k_m} = vy_1 \cdots y_{k_m}y$. If $x \neq y$, then we say that P_{k_1} and P_{k_m} have a *united thread structure* with a *knot* v, denoted by $\mathcal{P} = P_{k_1}(k_1, k_2, \cdots, k_m)P_{k_m}$, see Figure 6.6. Otherwise, we say that P_{k_1} and P_{k_m} have a *united thread-cycle structure* with a *head-knot* x, denoted by $\mathcal{Q}_x = P_{k_1}(k_1, k_2, \cdots, k_m)P_{k_m}$. Furthermore, if x is an (i, j, k)-vertex then we simply denote by $\mathcal{Q}_{(i,j,k)} = P_{k_1}(k_1, k_2, \cdots, k_m)P_{k_m}$. Noting that such united thread (thread-cycle) structure can also be obtained by concatenating several threads.

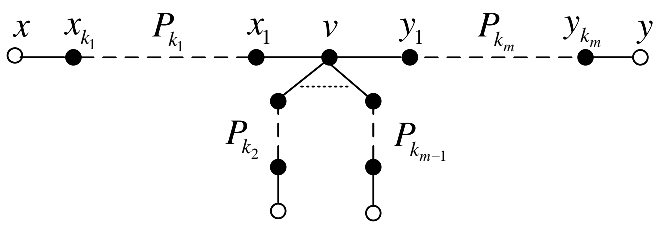


Figure 6.6: $P_{k_1}(k_1, \cdots, k_m)P_{k_m}$ with a knot v which is a $(k_1, \cdots, k_m)$-vertex.

For simplicity, we write P_1^i instead of writing $P_1(1,0,1)P_1(1,0,1)\cdots(1,0,1)P_1$ which contains exactly $i-1$ knots that are all $(1,0,1)$-vertices (if exist), where i is a positive integer.

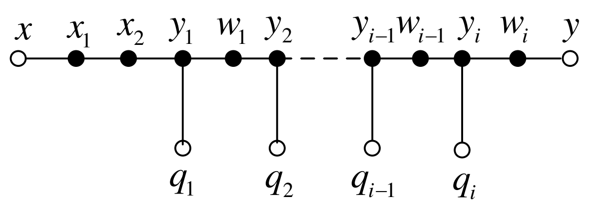


Figure 6.7: $P_2(2,0,1)P_1^i$.

Claim 6.2.7 *Suppose G contains a united thread structure $P_2(2,0,1)P_1^i$. If h is a homomorphism of $G - \mathcal{B}$ to $K_{5:2}$ such that $h(y) \neq h(q_i)$, then h can be extended to G, see Figure 6.7.*

Proof. We may suppose that $\mathcal{B}$ is unmapped to $K_{5:2}$. Notice that $q_iy_iw_iy$ is a path of length 3 with $h(y) \neq h(q_i)$. It follows directly from Claim 6.2.3

that there are two possible choices for the path $q_iy_iw_iy$ to be mapped to $K_{5:2}$. So, we may first map $q_iy_iw_iy$ to $K_{5:2}$ such that the image of y_i is different from $h(q_{i-1})$ according to Claim 6.2.3. Then, similarly, we can map, successively, that $q_{i-1}y_{i-1}w_{i-1}y_i, \cdots, q_2y_2w_2y_3, q_1y_1w_1y_2$ to $K_{5:2}$. It is obvious that we can choose $h(y_1)$ such that $h(y_1) \neq h(x)$. So by Remark 1, we may further map x_1, x_2 to $K_{5:2}$ successfully. This completes the proof of Claim 6.2.7. ∎

In the following, we will show some other reducible configurations of G.

Claim 6.2.8 *$P_2(2,0,1)P_1^i(1,0,2)P_2$ is reducible, where i is a positive integer.*

Proof. Suppose to the contrary that G contains a united thread structure $P_2(2,0,1)P_1^i(1,0,2)P_2$ depicted in Figure 6.8. Let u_0, u_i be two $(2,0,1)$-vertices and $u_1, \cdots, u_{i-1}$ be $i-1$ $(1,0,1)$-vertices. Let $\mathcal{B}$ denote the set of solid vertices in Figure 6.8. By the minimality of G, there is a homomorphism h from $G - \mathcal{B}$ to $K_{5:2}$. So, we may first map w_i to a vertex belonging to $N(h(w)) \setminus \{h(t_i)\}$, since $w_i \neq t_i$ by the absence of 3-cycles in G. Then extend the resulting homomorphism to G by Claim 6.2.7, which is a contradiction. ∎

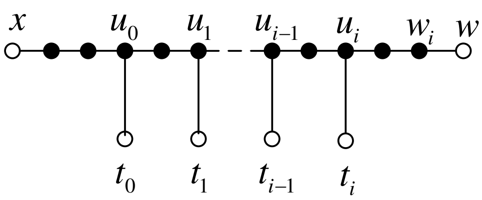


Figure 6.8: $P_2(2,0,1)P_1^i(1,0,2)P_2$.

Claim 6.2.9 *$P_2(2,0,1)P_1^i(1,1^+,1,1)P_1^j(1,0,2)P_2$ is reducible, where i and j are both positive integers.*

Proof. Assume to the contrary that G contains a united thread structure $P_2(2,0,1)P_1^i(1,1^+,1,1)$ $P_1^j(1,0,2)P_2$, depicted in Figure 6.9. Let $H = G - \mathcal{B}$. By the minimality of G, H has a homomorphism h to $K_{5:2}$. Clearly, $z \notin \{s,t\}$ by the absence of 3-cycles in G. So by Claim 6.2.1, each of s,t forbids one vertex for z and each of p,q forbids three vertices for z. It means that there are at most eight forbidden vertices for z in total. So, we first map pp_1z and qq_1z to $K_{5:2}$ by choosing one mapping choice for z. Obviously, $h(z) \neq h(s)$ and $h(z) \neq h(t)$. Then, by Claim 6.2.7, we can extend the resulting homomorphism to the whole graph G. This contradicts the choice of G. ∎

Obviously, in the proofs of Claim 6.2.8 and Claim 6.2.9, we do not require that the vertices x and w are different. In other words, they may coincide. So the proofs of Claim 6.2.8 and Claim 6.2.9 are also valid when x coincides w in G. We obtain the following Claim 6.2.10 and Claim 6.2.11.

Claim 6.2.10 *$\mathcal{Q}_{v_{x,w}} = P_2(2,0,1)P_1^i(1,0,2)P_2$ is reducible, where i is a positive integer and $v_{x,w} = x = w$ in Figure 6.8.*

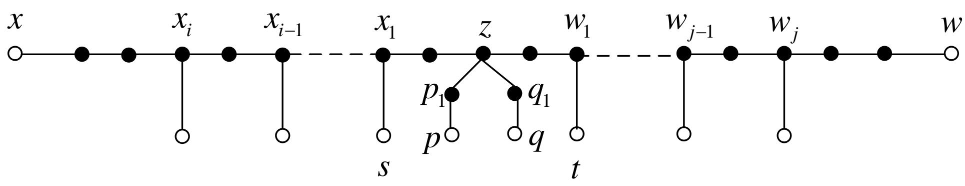


Figure 6.9: $P_2(2,0,1)P_1^i(1,1^+,1,1)P_1^j(1,0,2)P_2$.

Claim 6.2.11 *$\mathcal{Q}_{v_{x,w}} = P_2(2,0,1)P_1^i(1,1^+,1,1)P_1^j(1,0,2)P_2$ is reducible, where i and j are both positive integers and $v_{x,w} = x = w$ in Figure 6.9.*

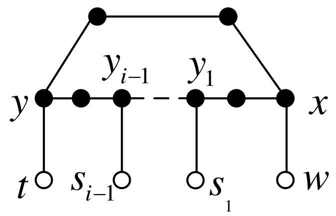


Figure 6.10: $Q_{(2,0,1)} = P_2(2,0,1)P_1^i$.

Claim 6.2.12 *$\mathcal{Q}_{(2,0,1)} = P_2(2,0,1)P_1^i$ is reducible, where i is a positive integer.*

Proof. Assume to the contrary that G contains a united thread-cycle structure $\mathcal{Q}_{(2,0,1)} = P_2(2,0,1)P_1^i$ as shown in Figure 6.10. Let $\mathcal{B}$ denote the set of solid vertices in Figure 6.10 and let y be its head-knot, i.e., y is a $(2,0,1)$-vertex. Let $H = G - \mathcal{B}$. By the minimality of G, H admits a homomorphism h to $K_{5:2}$. So, all t, w and $s_1, \cdots, s_{i-1}$ are already mapped to $K_{5:2}$. Now, we can first map y to a vertex in $N(h(t)) \setminus \{h(s_{i-1})\}$, since $y \neq s_{i-1}$ by the absence of 3-cycles in G, then extend the resulting homomorphism to the remaining vertices by Claim 6.2.7 and thus obtain a homomorphism of G to $K_{5:2}$, which contradicts the choice of G. ∎

6.2.2 Discharging argument

We begin with Definition 6.2.1 which was introduced by Borodin et al. [BHI+08].

Definition 6.2.1 A *compensatory path* for a $(2,0,1)$-vertex v is chosen as any shortest path F formed by concatenating threads in the following way. First, F starts along the unique 1-thread at v. Then F traversed some number of 1-threads by $(1,0,1)$-vertices. Let v^* be the first vertex reached which is not a $(1,0,1)$-vertex. We further say that v^* is a *slave* of v and v is a *master* of v^*.

Let $v \in V(G)$. Since G is 2-connected, $d(v) \geqslant 2$ for any $v \in V(G)$. Since there is no $(1,1,1)$-vertex in G, we deduce that the slave of a $(2,0,1)$-vertex always exists. We start from the following Lemma 6.2.2, which is crucial in the following discharging argument.

Lemma 6.2.2 *Suppose v is a $(2,0,1)$-vertex. Let v^* be the slave of v. Then the following hold:*

(1) *v^* is neither a 2-vertex nor a $(1,0,1)$-vertex;*

(2) *If $d(v^*) = 3$ then v^* is a $(1,0,0)$-vertex.*

Proof. It suffices to show (2) since (1) holds by Definition 6.2.1. Suppose v is such a $(2,0,1)$-vertex that it is incident to one 2-thread vx_1x_2x, one 1-thread vy_1y, and one 0-thread vz. This means that $d(x_1) = d(x_2) = d(y_1) = 2$ and $d(x), d(y), d(z) \geqslant 3$. Let v^* be the slave of v. By the definition of the compensatory path, we see that there exists one compensatory path F starting along the unique 1-thread vy_1y at v.

Suppose $d(v^*) = 3$. By Definition 6.2.1, v^* is incident to at least one 1-thread. So, in the following, we further suppose that v^* is a $(1,i,j)$-vertex. Clearly, $i,j \in \{0,1,2\}$ because of the absence of 3-threads in G by Claim 6.2.2. By symmetry, we assume that $\{i,j\} \in \{\{0,0\},\{0,1\},\{0,2\},\{1,1\},\{1,2\},\{2,2\}\}$. Note that $\{1,1\},\{1,2\}$ and $\{2,2\}$ are impossible by Claim 6.2.4. Moreover, v^* cannot be a $(1,0,1)$-vertex by (1). So $\{i,j\} \in \{\{0,0\},\{0,2\}\}$. Next, we will show that $\{i,j\} \neq \{0,2\}$.

Suppose to the contrary that $\{i,j\} = \{0,2\}$, then v^* is a $(2,0,1)$-vertex. We have to handle the following two cases:

(i) $v^* = x$.

For simplicity, denote $w = v^* = x$. Then a united thread-cycle structure $\mathcal{Q}_{(2,0,1)} = P_2(2,0,1)P_1^i$ with a head-knot w is formed by $P_2 = wx_2x_1v$ and F which is a compensatory path connecting v and w, which is a contradiction to Claim 6.2.12.

(ii) $v^* \neq x$.

Let $P_2 = v^*t_1t_2t$ denote the unique 2-thread incident to v^*. If $t \neq x$, then a united thread structure $P_2(2,0,1)P_1^i(1,0,2)P_2$ is constructed by $P_2 = xx_2x_1v$, $F = P_1^i$ and $P_2 = v^*t_1t_2t$, which is impossible by Claim 6.2.8. Otherwise, a united thread-cycle structure $\mathcal{Q}_{v_{t,x}} = P_2(2,0,1)P_1^i(1,0,2)P_2$ is formed by a similar discussion as previous case for $t \neq x$, where $v_{t,x} = t = x$, which contradicts Claim 6.2.10.

Therefore, we complete the proof of Lemma 6.2.2. ∎

Now we use a discharging argument with initial charge $\omega(v) = d(v)$ at each vertex v and with the following three discharging rules (R1)-(R3). We write ω^* to denote the charge at each vertex v after we apply the discharging rules. To complete the proof, we show that $\omega^*(v) \geqslant \frac{5}{2}$ for all $v \in V(G)$. This leads to the following obvious contradiction:

$$\frac{5}{2} \leqslant \frac{\sum_{v\in V(G)} \omega^*(v)}{|V(G)|} = \frac{\sum_{v\in V(G)} \omega(v)}{|V(G)|} = \frac{2|E(G)|}{|V(G)|} \leqslant \mathrm{Mad}(G) < \frac{5}{2}.$$

Hence no counterexample can exist.

Our discharging rules are defined as follows:

(R1) Each 2-vertex in a 2-thread gets a charge equal to $\frac{1}{2}$ from its 3^+-vertex neighbor.

(R2) Each 2-vertex in a 1-thread gets a charge equal to $\frac{1}{4}$ from each of its neighbors.

(R3) Each $(2,0,1)$-vertex gets a charge equal to $\frac{1}{4}$ from its slave.

Let us check that $\omega^*(v) \geqslant \frac{5}{2}$ for each $v \in V(G)$. In the sequel, we use P_i to denote a maximal i-thread. The proof is divided into four cases below:

Case 1 $d(v) = 2$.

Then $\omega(v) = 2$. Clearly, v does not have any master by Lemma 6.2.2 (1). Let v_1, v_2 be the neighbors of v. By Claim 6.2.2, we see that there is no 3-thread in G. It implies that v is in an i-thread with $i \in \{1, 2\}$. By (R1) and (R2), we have that $\omega^*(v) \geqslant 2 + \frac{1}{2} = \frac{5}{2}$.

For $v \in V(G)$, in what follows, let $|P_0(v)|$, $|P_1(v)|$, $|P_2(v)|$ denote the number of incident 0-threads, 1-threads, 2-threads of v, respectively. Clearly, $|P_0(v)| + |P_1(v)| + |P_2(v)| = d(v)$. We use $m(v)$ to denote the number of masters of v. By Definition 6.2.1, we see that each slave must be incident to a 1-thread. Furthermore, compensatory paths do not intersect internally, since there is no $(1,1,1)$-vertex in G by Claim 6.2.4. Basing on these two facts, we have:

Observation 6.2.3 *For $v \in V(G)$, $m(v) \leqslant |P_1(v)|$.*

Case 2 $d(v) = 3$.

Then $\omega(v) = 3$. Let v_1, v_2 and v_3 be the neighbors of v. Suppose v is an (i,j,k)-vertex with $i, j, k \in \{0, 1, 2\}$ in light of Claim 6.2.2. We need to deal with the following four subcases, depending on the situation of v.

(2.1) v is a $(0,0,0)$-vertex.

It is obvious that $\omega^*(v) \geqslant 3 - 0 \times 3 = 3 > \frac{5}{2}$ by (R1) to (R3).

(2.2) v is a $(1^+,0,0)$-vertex.

Assume, without loss of generality that $d(v_1) = 2$ and $d(v_i) \geqslant 3$ for each $i \in \{2, 3\}$. By Observation 6.2.3, $m(v) \leqslant |P_1(v)| \leqslant 1$. If $m(v) = 1$, that is to say v is a $(1,0,0)$-vertex, then $\tau(v \to v_1) = \frac{1}{4}$ and thus $\omega^*(v) \geqslant 3 - \frac{1}{4} - \frac{1}{4} = \frac{5}{2}$ by (R2) and (R3). Otherwise, v is a $(2,0,0)$-vertex. According to (R1), we have $\omega^*(v) \geqslant 3 - \frac{1}{2} = \frac{5}{2}$.

(2.3) v is a $(1^+,1^+,0)$-vertex.

By Claim 6.2.4, v cannot be any $(2,2,0)$-vertex. Thus, we have to consider the following two possibilities:

(2.3.1) Suppose v is a $(1,1,0)$-vertex. Let v_1 and v_2 be 2-vertices and v_3 be a 3^+-vertex. Let w_i be the other neighbor of v_i different from v for $i = 1, 2$. Noting that w_1 and w_2 are both 3^+-vertices. It follows from Lemma 6.2.2 (1) that $m(v) = 0$. Hence, $\omega^*(v) \geqslant 3 - \frac{1}{4} - \frac{1}{4} = \frac{5}{2}$ by (R2).

(2.3.2) Suppose v is a $(2,1,0)$-vertex. Let $P_2 = vv_1w_1u_1$, $P_1 = vv_2w_2$, and $P_0 = vv_3$ be the 2-thread, 1-thread, and 0-thread incident to v, respectively. Obviously, u_1, w_2, v_3 are all 3^+-vertices. By Lemma 6.2.2 (2), we see that v cannot be a slave of other $(2,1,0)$-vertex. In other words, $m(v) = 0$. Thus, $\tau(v \to v_1) = \frac{1}{2}$, $\tau(v \to v_2) = \frac{1}{4}$, and $\tau(v^* \to v) = \frac{1}{4}$ by (R1)-(R3), where v^* is the slave of v. Therefore, $\omega^*(v) \geqslant 3 - \frac{1}{2} - \frac{1}{4} + \frac{1}{4} = \frac{5}{2}$.

(2.4) v is a $(1^+,1^+,1^+)$-vertex. This contradicts Claim 6.2.4.

Case 3 $d(v) = 4$.

Clearly, $\omega(v) = 4$. If $|P_1(v)| + |P_2(v)| \leqslant 3$, then according to (R1)-(R3), we obtain that

$$\begin{aligned}\omega^*(v) &\geqslant 4 - \frac{1}{2}|P_2(v)| - \frac{1}{4}|P_1(v)| - \frac{1}{4}m(v) \\ &\geqslant 4 - \frac{1}{2}|P_2(v)| - \frac{1}{4}|P_1(v)| - \frac{1}{4}|P_1(v)| \\ &= 4 - \frac{1}{2}(|P_1(v)| + |P_2(v)|) \\ &\geqslant 4 - \frac{3}{2} \\ &= \frac{5}{2}.\end{aligned}$$

Now we may suppose that v is a $(1^+, 1^+, 1^+, 1^+)$-vertex. By Claim 6.2.2 and Claim 6.2.6, it is easy to infer that v is either a $(1, 1, 1, 1)$-vertex or a $(2, 1, 1, 1)$-vertex. For each $i \in \{1, 2, 3, 4\}$, let v_i be the neighbor of v and w_i be the other neighbor of v_i distinct from v. We further suppose that all of w_2, w_3, w_4 are 3^+-vertices. We have to consider two cases as follows:

(3.1) If $m(v) \leqslant 1$, then $\omega^*(v) \geqslant 4 - \frac{1}{2} - \frac{1}{4} \times 3 - \frac{1}{4} = \frac{5}{2}$ by (R1)-(R3).

(3.2) Now we may suppose that v has at least two masters. Let v^* and v^{**} be two such masters. One can observe that $v^* \neq v^{**}$ since each master must be incident to only one 1-thread. Thus, there exist two different compensatory paths F_1, F_2, each of which starts along the unique 1-thread at v^*, v^{**}, respectively. Obviously, $V(F_1) \cap V(F_2) = v$. Denote $v^*t_1t_2t$, $v^{**}s_1s_2s$ be the unique 2-thread incident to v^*, v^{**}, respectively. If $s \neq t$, then a united thread structure $P_2(2, 0, 1)P_1^i(1, 1^+, 1, 1)P_1^j(1, 0, 2)P_2$ is established by $tt_2t_1v^*$, F_1, F_2 and $v^{**}s_1s_2s$, which is a contradiction to Claim 6.2.9. Now we set $s = t$. Using a similar argument as the case for $s \neq t$, it is easy to see that a united thread-cycle structure $\mathcal{Q}_{v_{s,t}} = P_2(2, 0, 1)P_1^i(1, 1^+, 1, 1)P_1^j(1, 0, 2)P_2$ established, where $v_{s,t} = s = t$. This contradicts Claim 6.2.11.

Case 4 $d(v) \geqslant 5$.

Applying (R1) to (R3), we have

$$\begin{aligned}\omega^*(v) &\geqslant d(v) - \frac{1}{2}|P_2(v)| - \frac{1}{4}|P_1(v)| - \frac{1}{4}m(v) \\ &\geqslant d(v) - \frac{1}{2}|P_2(v)| - \frac{1}{4}|P_1(v)| - \frac{1}{4}|P_1(v)| \\ &= d(v) - \frac{1}{2}(|P_1(v)| + |P_2(v)|) \\ &\geqslant d(v) - \frac{1}{2}d(v) \\ &= \frac{1}{2}d(v) \\ &\geqslant \frac{5}{2}.\end{aligned}$$

∎

6.3 Concluding remarks

We would like to propose the following conjecture:

Conjecture 6.3.1 *Every graph G with odd girth $2k+1$ and* $\text{Mad}(G) < 2 + \frac{1}{k}$ has a fractional $(2k+1, k)$-coloring, where k is a positive integer.

In fact, if this conjecture is proved then the bound on maximum average degree is tight. We recall the example depicted in Figure 6.11 which was constructed by Klostermeyer and Zhang in [KZ02]. It is easy to see that the odd girth of H_k is $2k+1$. Moreover, H_k has $4k+2$ edges and $4k$ vertices, yielding average degree $2+\frac{1}{k}$, where k is a positive integer. Furthermore, all its proper subgraphs have smaller average degree. So we have that $\text{Mad}(H_k) = 2 + \frac{1}{k}$. However, it is proved in [KZ02] that H_k cannot be $(2k+1, k)$-colored by the fact that $K_{2k+1:k}$ has odd girth $2k+1$.

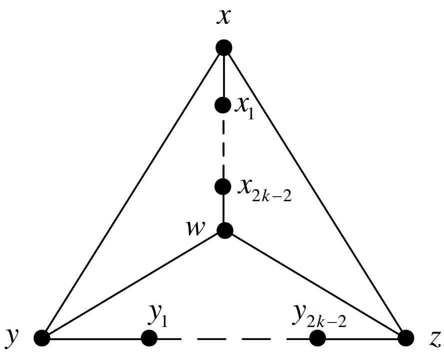


Figure 6.11: An example H_k.

A graph G is called *k-degenerate* if every subgraph H of G has $\delta(G) \leqslant k$. It is well known that a k-degenerate graph has chromatic number at most $k+1$. Moreover, for any graph G, the fractional chromatic number of G is always bounded by chromatic number of G. So, for Conjecture 6.3.1, the case $k = 1$ is obviously obtained since every graph G with $\text{Mad}(G) < 3$ is 2-degenerate. And we have handled the case $k = 2$ in this chapter. Therefore, we leave the case of $k \geqslant 3$ as an open problem.

On the other hand, there is a close relationship between fractional coloring and circular coloring. A *circular (k, d)-coloring* of a graph G, introduced by Vince [Vin88], is a map $c : V(G) \to \{0, \cdots, k-1\}$ such that $d \leqslant |c(u) - c(v)| \leqslant k - d$ for every edge $uv \in E(G)$. The *circular chromatic number* of G, denoted by $\chi_c(G)$, is defined as $\chi_c(G)$ =min$\{\frac{k}{d}$: G has a circular (k, d)-coloring$\}$. More details about circular coloring can be found in [Zhu01a].

For planar graphs, the flow problem (see [Tut54a], [Tut54b]) can be dualized to the circular coloring problem. More precisely, a circular (k, d)-coloring of a planar graph G corresponds to a (k, d)-flow of the dual graph of G. Therefore, the restriction of a Jaeger's conjecture for flow [Jae84] to planar graphs is equivalent to the following.

Conjecture 6.3.2 *Every planar graph G with girth at least $4k$ has a circular $(2k+1, k)$-coloring, where k is a positive integer.*

As far as we know, this conjecture is still open for any integer $k \geqslant 2$. Many results approaching the bound are mainly obtained in [BHI+08, BKKW04, Zhu01b]. Since for any graph G we have that $\chi_f(G) \leqslant \chi_c(G)$, we would like to propose a weaker version:

Conjecture 6.3.3 *Every planar graph G with girth at least $4k$ has a fractional $(2k+1,k)$-coloring, where k is a positive integer.*

The case $k = 1$ reduces to Grötzsch's Theorem. The case $k = 2$ was handled by Dvořák et al. [DvV08]. So, we also leave the case of $k \geqslant 3$ as an open problem.

Conclusion

In this thesis, we mainly investigated various vertex coloring of planar graphs and sparse graphs. More specifically, we studied proper list coloring, acyclic list coloring, star coloring, star list coloring, forest coloring and fractional coloring.

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In **Chapter 2**, we obtained some sufficient conditions for planar graphs to be 3-choosable. As mentioned in **Chapter 2**, the best known upper bound k^* for planar graphs without j-cycles for $4 \leqslant j \leqslant k^*$ to be 3-choosable is 9, given by Borodin [Bor96] in 1996. As far as we know, it is still unknown whether such upper bound 9 can be improved or not. On the other hand, Borodin, Glebov, Raspaud, and Salavatipour [BGRS05] proved that every planar graph without 4 to 7-cycles is 3-colorable. The following question naturally arises:

Question 1: *Is it true that every planar graph without* 4 *to* 8*-cycles is* 3*-choosable?*

******

In **Chapter 3**, our focus was on acyclic list coloring. The notion of acyclic list coloring of planar graphs was introduced by Borodin, Fon-Der Flaass, Kostochka, Raspaud, and Sopena [BFDFK+02]. Moreover, they proposed the following challenging conjecture:

Conjecture 2: *Every planar graph is acyclically* 5*-choosable.*

This conjecture attracted much attention recently. Obviously, if this conjecture were true, then it would strengthen the Borodin's acyclic 5-color theorem [Bor79] and the Thomassen's 5-choosable theorem [Tho94] about planar graphs. However, this challenging conjecture seems to be very difficult. In **Chapter 3**, we established some new sufficient conditions for planar graphs to be acyclically k-choosable for each $k \in \{3, 4, 5\}$.

As Borodin et al. proposed Conjecture 2 in [BFDFK+02], they also proved that every planar graph is acyclically 7-choosable. Recently, Wang and Chen [WC09] proved that every planar graph without 4-cycles is acyclically 6-choosable. Together with other known sufficient conditions for planar graphs to be acyclically 3-choosable or acyclically 4-choosable, one can easily observe that the cycles of length 4 are always forbidden. It means that the existence of 4-cycles is the main obstacle in acyclic list coloring problem. So first we would like to propose the following weaker conjecture of Conjecture 2.

Conjecture 3: *Every planar graph without* 4*-cycles is acyclically* 5*-choosable.*

We remark that Conjecture 3 was already mentioned in [CW08a]. Moreover, one of our results in **Chapter 3**, which states that every planar graph without 4-cycles and intersecting triangles is acyclically 5-choosable, partially confirms Conjecture 3. We also propose the following conjecture.

Conjecture 4: *Every planar graph is acyclically* 6*-choosable.*

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The notion of star coloring of graphs was introduced by Grünbaum [Grü73] in 1973. In Chapter 4, we proved that every subcubic graph is 6-star-colorable and this result is best possible by the fact that the Wagner graph is not 5-star-colorable.

In addition, we obtained several upper bounds for planar subcubic graphs with given girth. More precisely, we proved that if G is a planar subcubic graph, then (1) $\chi_s^l(G) \leqslant 6$; (2) $\chi_s^l(G) \leqslant 5$ if $g(G) \geqslant 8$; and (3) $\chi_s^l(G) \leqslant 4$ if $g(G) \geqslant 12$. In proving these results, we introduced a useful concept *L-in-coloring* and formalized the connection of *L-in-coloring* and star list coloring. The idea of using L-in-coloring to control the number of colors is due to [ACK+04]. The following question is our main concern.

Question 5: *Does there exist planar subcubic graphs that cannot be* 5*-star-choosable?*

If the answer to Question 5 is positive, then our result, which states that planar subcubic graphs are 6-star-choosable, is best possible. Actually, in proving this result, we feel that it is indeed difficult to decrease the upper bound 6. Moreover, constructing an example that satisfies Question 5 seems to be not easy.

On the other hand, Albertson, Chappell, Kierstead, Kündgen, and Ramamurthi [ACK+04] proved that the star chromatic number of planar graphs is between 10 and 20; but this gap remains open. So it is natural to ask:

Question 6: *What is the smaller integer k such that every planar graph is k-star-colorable?*

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A k-forest-coloring of a graph G is a mapping π from $V(G)$ to the set $\{1, \cdots, k\}$ such that each color class induces a forest. The vertex-arboricity of G is the smallest integer k such that G has a k-forest-coloring. It is well-known that the vertex-arboricity of planar graphs is at most 3 and this upper bound is optimal. In **Chapter 5**, we studied the vertex-arboricity of planar graphs and our main purpose was to give a positive answer to the conjecture of Raspaud and Wang in [RW08]. More precisely, we proved that every planar graph without intersecting triangles has vertex-arboricity at most 2. We are more interested in the following question:

Question 7: *Does every planar graph without adjacent triangles have vertex-arboricity at most 2?*

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Finally, in **Chapter 6**, we investigated homomorphism problems of sparse graphs to the Petersen graph. We proved that every triangle-free graph with maximum average degree less than $5/2$ admits a homomorphism to the Petersen graph. The bound on maximum average degree is sharp, based on the example constructed by Klostermeyer and Zhang in [KZ02]. On the other hand, since the girth of the Petersen graph is 5, any triangle cannot be mapped to the Petersen graph. So the assumption in our result that G is triangle-free cannot be dropped.

A distinctive feature of the proof of this result is that a charge of vertices can be transferred along “feeding paths” to an unlimited distance. This kind of “global” discharging was introduced by Borodin, Ivanova, and Kostochka in [BIK06].

To conclude the thesis, we would like to propose the following conjecture:

Conjecture 8: *Every graph G with odd girth $2k+1$ and $\mathrm{Mad}(G) < 2 + \frac{1}{k}$ has a fractional $(2k+1, k)$-coloring, where k is a positive integer.*

In fact, if this conjecture is proved then the bound on maximum average degree is best possible, based on the example constructed by Klostermeyer and Zhang [KZ02], see Figure 6.11. On the other hand, the case $k = 1$ obviously holds, since every graph G with $\mathrm{Mad}(G) < 3$ is 2-degenerate. The case $k = 2$ was handled in **Chapter 6**. Therefore, we leave the case of $k \geqslant 3$ as an open problem.

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