École doctorale IAEM Lorraine

# Analyse sur les espaces singuliers et théorie de l'indice

## THÈSE

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par

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# Organisation générale de la thèse

Cette thèse est organisée en quatre grandes parties, décrites en détail plus bas. Les résultats présentés sont adaptés de quatre articles écrits pendant la thèse [25, 26, 49, 55] : trois d'entre eux sont publiés ou acceptés, et le quatrième est soumis.

La partie I constitue l'introduction de la thèse. Celle-ci donne un bref aperçu des motivations, puis décrit de manière précise les résultats majeurs obtenus, en expliquant les outils et arguments essentiels des preuves qui seront présentées.

La partie II est dédiée aux préliminaires. Le chapitre 3 donne une introduction générale au contexte dans lequel s'inscrit cette thèse et aux outils utilisés, qui pourra intéresser une personne peu familière avec le domaine. Le chapitre 4 expose ensuite les points théoriques essentiels qui seront mobilisés dans la thèse : théorie des $C^*$-algèbres, opérateurs pseudodifférentiels et groupoïdes.

L'exposé des résultats commence partie III. Celle-ci est dédiée à l'étude de variétés non compactes et de domaines singuliers pouvant être modélisés par une certaine classe de groupoïdes favorables à l'analyse des opérateurs elliptiques. J'y présente les résultats des articles [49] (travail commun avec C. Carvalho et Y. Qiao) et [55], tous deux publiés. Je m'intéresserai notamment à la structure locale de ces groupoïdes et de leurs représentations.

La partie IV s'intéresse à l'indice équivariant d'opérateurs pseudodifférentiels sur une variété compacte, sur laquelle agit un groupe fini. J'y présente les résultats des articles [26] et [25], réalisés en commun avec A. Baldare, M. Lesch et V. Nistor. Le premier de ces deux articles est accepté pour publication, l'autre soumis.

Dans ces deux dernières parties, chaque chapitre présente un article différent. Le contenu mathématique des articles n'a pas été modifié, mais les notations, introductions et préliminaires ont été adaptées afin que l'ensemble du manuscrit forme un tout cohérent. Enfin, et bien que ce ne soit pas mon premier critère quant à leur agencement, ceux-ci sont présentés dans l'ordre chronologique de leur écriture.

Bonne lecture !

# Table des matières

# Partie I.

# Introduction

**Résumé**

Après un court aperçu des motivations et de l'organisation générale du manuscrit, cette partie donne une introduction détaillée aux principaux résultats de la thèse. Une personne peu familière avec le domaine pourra commencer sa lecture par l'*introduction générale* donnée chapitre 3, avant de revenir à cette partie.

# 1. Aperçu général

## 1.1. Motivations

Le fil directeur de cette thèse est l'étude des opérateurs *elliptiques* et l'obtention de conditions de Fredholm sur des espaces singuliers et des variétés non compactes. La motivation vient de résultats très bien compris sur les variétés lisses compactes (sans bords). Si $M$ est une telle variété et $P$ un opérateur différentiel d'ordre $m \geq 0$, on sait en effet que $P$ s'étend en un opérateur borné entre espaces de Sobolev :

$$P : H^s(M) \to H^{s-m}(M),$$

pour tout $s \in \mathbb{R}$. En outre, cet opérateur est de Fredholm[1] si, et seulement si, $P$ est elliptique [98]. Une quantité d'importance fondamentale associée à $P$ est alors son *indice*, défini par

$$\operatorname{index} P = \dim(\ker P) - \dim(\operatorname{coker} P).$$

L'étude de l'indice des opérateurs elliptiques a culminé dans les années 60 avec le théorème d'Atiyah et Singer, qui calcule l'indice de Fredholm de $P$ en fonction de certaines classes de cohomologie construites à partir du symbole principal de $P$, un objet local [16, 17]. L'ellipticité de $P$ implique également la régularité *a priori* des solutions de l'équation différentielle associée : si $f \in H^s(M)$ et $u$ est une distribution telle que $Pu = f$, alors $u \in H^{s+m}(M)$. Ces résultats de régularité sont d'un grand intérêt pratique, puisqu'ils assurent une certaine vitesse de convergence de méthodes numériques comme celle des Éléments Finis.

Ces différentes propriétés des opérateurs elliptiques ne sont plus vraies en général lorsqu'on considère des variétés non compactes ou des espaces dits *singuliers*. À titre d'exemple, ces derniers peuvent inclure des polyèdres de $\mathbb{R}^n$, le domaine extérieur à une sphère tangente à un plan, ainsi que des variétés algébriques complexes ou réelles [41, 45, 53, 88]. Dans ce cas, un opérateur elliptique $P$ n'est pas nécessairement Fredholm : il faut ajouter à l'ellipticité certaines condition *d'inversibilité à l'infini*, souvent d'une famille d'opérateurs limites associés à $P$ [85, 113, 136, 172]. En conséquence, les formules d'indices connues font apparaître des termes non locaux, construits à partir des opérateurs limites [13, 48, 117, 136]. Enfin, sur des domaines singuliers comme des polygones du plan, les solutions des équations elliptiques ne possèdent pas *a priori* la régularité qu'on a dans le cas lisse [108, 146].

[1] Un opérateur entre espaces de Banach est dit *de Fredholm* si son noyau et son conoyau sont tous deux de dimensions finies. Toutes les notions évoquées ici sont présentées avec plus de détails dans la partie préliminaire, page 19.

Outre l'étude des espaces singuliers et variétés non compactes, le théorème de l'indice d'Atiyah et Singer a reçu des raffinement et extensions dans d'autres directions : indices d'une familles d'opérateurs [18], d'opérateurs agissant le long d'un feuilletage [56, 62], indices « grossiers » [180], etc. Le cadre qui m'intéressera particulièrement dans cette thèse est celui de *l'indice équivariant.* Ainsi, si $G$ est un groupe de Lie compact agissant de manière lisse sur une variété compacte $M$ et $P$ un opérateur elliptique $G$-invariant, alors $\ker P$ et $\operatorname{coker} P$ sont des représentations de dimensions finies de $G$. L'indice équivariant de $P$ est défini comme la différence des caractères associées, soit

$$\operatorname{index}_G P = \chi_{\ker P} - \chi_{\operatorname{coker} P}.$$

On obtient ainsi une fonction lisse sur $G$, dont l'évaluation en l'élément neutre est l'indice de Fredholm de $P$. Atiyah et Singer ont calculé cette fonction en chaque point de $G$ par une formule cohomologique intégrée sur des variétés de points fixes [12, 17], formule qui a été étudiée par la suite en $K$-théorie et en cohomologie [30, 101, 103, 155].

Un enjeu essentiel de la thèse est donc de caractériser les opérateurs différentiels qui sont Fredholm dans une configuration géométrique donnée. La théorie des algèbres d'opérateurs constitue pour ça un outil fondamental, en particulier celle des *$C^*$-algèbres.* La justification vient d'un résultat classique d'Atkinson : pour un opérateur quelconque sur un espace de Hilbert $\mathcal{H}$, être Fredholm est équivalent à être inversible modulo les opérateurs compacts. On est donc amené à étudier le quotient $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, qui est une $C^*$-algèbre. De nombreux outils reliés au programme de géométrie non commutative de Connes sont ainsi particulièrement adaptés à l'étude de l'indice : un exemple est donné par la $K$-théorie des $C^*$-algèbres et ses extensions [59, 60, 102, 103, 190].

Bien souvent, la $C^*$-algèbre $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ est trop grande pour obtenir une caractérisation pertinente des opérateurs de Fredholm. Ainsi, sur une variété $M_0$ non compacte, il n'est *a priori* pas souhaitable d'étudier tous les opérateurs bornés sur $L^2(M_0)$, mais seulement une certaine sous-algèbre $A \subset \mathcal{B}(L^2(M_0))$ qui contient, par exemple, les résolvantes des opérateurs différentiels « intéressants » : en particulier les opérateurs géométriques tel le Laplacien ou les opérateurs de type Dirac. Un moyen d'obtenir une telle $C^*$-algèbre $A$ est de construire un *calcul pseudodifférentiel* qui prend en compte la géométrie de $M_0$ : l'algèbre $A$ est alors définie comme la fermeture des opérateurs d'ordre 0 dans $\mathcal{B}(L^2(M_0))$. C'est ensuite l'étude du quotient $A/\mathcal{K}$ et de ses représentations qui permet de caractériser les opérateurs de Fredholm, souvent en fonction d'une famille *d'opérateurs limites* à l'infini [9, 85, 127, 136, 172].

Dans cette thèse, ces calculs pseudodifférentiels sont construits à partir de *groupoïdes* qui encodent et résolvent naturellement les singularités d'un espace donné. Cette approche a été exploitée de manière fructueuse par Connes et Skandalis pour déterminer l'indice des opérateurs elliptiques « le long des feuilles » d'un feuilletage régulier [56, 57, 62, 96] et étendue plus récemment à l'étude des feuilletages singuliers [10, 11, 69]. Elle fournit un objet géométrique, le groupoïde, auquel est associé naturellement une $C^*$-algèbre [173]. L'étude des représentations de cette $C^*$-algèbre permet alors de caractériser les opérateurs de Fredholm et d'isoler les éléments nécessaires au calcul de leurs indices [50, 51, 73, 74, 202].

L'ensemble des notions exposées dans cette courte introduction seront reprises avec plus de détail partie II. Je donnerai notamment une introduction plus détaillée du contexte dans lequel s'inscrit cette thèse chapitre 3, et j'exposerai les résultats théoriques dont j'aurai besoin chapitre 4.

## 1.2. Aperçu des résultats de la thèse

Passons maintenant à l'exposé plus précis des résultats de la thèse. Ceux-ci sont adaptés de quatre articles rédigés pendant la thèse [25, 26, 49, 55]. Trois de ces articles sont publiés, le quatrième soumis. Chaque article est repris dans un chapitre à part. Le contenu mathématique des articles est reproduit inchangé, mais les notations, introductions et préliminaires ont été adaptés afin de former un tout cohérent. Ces chapitres forment deux grandes parties, présentées dans l'ordre chronologique d'écriture.

Dans la partie III, j'étudie des variétés non compactes et des espaces singuliers sur lesquels l'analyse est modélisée par une classe de groupoïdes ayant de « bonnes » propriétés. En particulier, le caractère de Fredholm des opérateurs différentiels y est déterminé par une famille d'opérateurs limites agissant sur les fibres du groupoïde. Le chapitre 5 présente ainsi un travail réalisé avec Catarina Carvalho et Yu Qiao. Nous y étudions de manière générale des groupoïdes obtenus par recollement de d'actions de groupes, que nous utilisons pour construire et étudier un groupoïde adapté à l'étude des opérateurs potentiels de couche sur les domaines à singularités coniques. Je complète cette étude dans le chapitre 6, dans lequel j'étudie les propriétés des représentations de ces groupoïdes d'un point de vue local, montrant notamment que les recollements de groupoïdes préservent les « bonnes » propriétés évoquées ci-dessus.

Dans la partie IV, je présente un travail réalisé avec Alexandre Baldare, Matthias Lesch et Victor Nistor qui porte sur l'indice équivariant des opérateurs elliptiques. On se place alors sur une variété compacte $M$, sur laquelle agit un groupe fini $\Gamma$. Étant donnée une représentation irréductible $\alpha \in \widehat{\Gamma}$, un opérateur pseudodifférentiel $\Gamma$-équivariant se restreint entre les composantes isotypiques des espaces de Sobolev associées à $\alpha$. La question posée est la suivante : à quelle condition cette restriction est-elle de Fredholm ? Nous construisons un *$\alpha$-symbole principal* en isolant une partie de l'information contenue dans le symbole principal usuel, et donnons une notion correspondante d'*$\alpha$-ellipticité* qui répond à la question ci-dessus. Dans ce contexte, le chapitre 7 présente les premiers éléments de la preuve et traite le cas où $\Gamma$ est abélien. Nous abordons le cas général dans le chapitre 8, détaillant en particulier le lien avec la théorie de l'indice.

Les prochaines pages visent à donner un aperçu plus détaillé de chacun de ces articles. Je donnerai dans chaque cas les résultats majeurs, ainsi que les principaux outils et les grandes lignes des preuves qui seront présentées plus loin.

# 2. Introduction aux résultats de la thèse

Comme expliqué dans les pages précédentes, les résultats de la thèse seront présentés dans les parties III et IV, dont je vais maintenant donner un aperçu.

## 2.1. Partie III : groupoïdes de Fredholm et applications à l'analyse

La partie III est adaptée des articles [49] et [55]. Le premier de ces deux travaux a été réalisé en commun avec Catarina Carvalho[1] et Yu Qiao[2].

### 2.1.1. Motivations : groupoïdes de Fredholm

On s'intéresse ici à l'analyse sur une variété Riemannienne complète $M_0$, généralement non compacte . Soit $E$, $F$ deux fibrés vectoriels sur $M_0$, et $P$ un opérateur différentiel d'ordre $m \geq 0$ agissant des sections de $E$ vers celles de $F$. L'introduction a expliqué le fait que l'ellipticité ne suffit plus à caractériser le caractère Fredholm de $P$ entre espaces de Sobolev. Ainsi, dans tous les cas particuliers étudiés, une condition nécessaire supplémentaire est l'inversibilité d'une famille d'opérateurs limites $P_\alpha$ sur des variétés $X_\alpha$, indexées par un $\alpha \in A$. Ceci apparaît dans les travaux de Kondrat′ev [108], Maz′ya et Plamenevskiĭ [129], Melrose [136, 137], Schulze [172, 184, 183] parmi beaucoup d'autres auteurs [20, 68, 71, 85, 115, 121, 127, 169]. Je l'expliquerai plus en détail Partie II.

La classe des groupoïdes de Lie dits *Fredholm* a été introduite par Carvalho, Nistor et Qiao [51] dans ce cadre. Dit brièvement, un groupoïde Fredholm $\mathcal{G} \rightrightarrows M$ a pour base une variété compacte à coins et s'écrit comme une union disjointe

$$\mathcal{G} = M_0 \times M_0 \bigsqcup \mathcal{G}_{\partial M},$$

avec $\mathcal{G}_{\partial M}$ la restriction de $\mathcal{G}$ à $\partial M$. Le caractère Fredholm d'un opérateur pseudodifférentiel $P \in \Psi^\infty(\mathcal{G})$ est alors déterminé par l'inversibilité de sa restriction à l'infini, qui vit dans $\Psi^\infty(\mathcal{G}_{\partial M})$. On requiert de plus d'un groupoïde Fredholm que l'inversibilité d'un élément de $\Psi^\infty(\mathcal{G}_{\partial M})$ soit déterminée par celle d'une famille d'opérateurs « concrets » sur des variétés, retrouvant la formulation en termes d'opérateurs limites évoquée ci-dessus. Plus précisément, les groupoïdes Fredholm vérifient les hypothèses minimales pour que le théorème suivant soit valide :

[1]University of Lisbon, Portugal
[2]Shaanxi Normal University, Chine

**Théorème 2.1.1** (Carvalho, Nistor, Qiao [51]). *Soit $\mathcal{G} \rightrightarrows M$ un groupoïde de Fredholm et $P \in \Psi^m(\mathcal{G}; E, F)$. Pour tout $s \in \mathbb{R}$, l'opérateur*

$$P : H^s(M_0; E) \to H^{s-m}(M_0; F)$$

*est Fredholm, si et seulement si :*

*(i) $P$ est uniformément elliptique, c'est-à-dire $\sigma_m(P)$ est inversible sur $S^*\mathcal{G}$, et*
*(ii) les opérateurs $P_x : H^s(\mathcal{G}_x; E) \to H^{s-m}(\mathcal{G}_x; F)$ sont tous inversibles, pour $x \in \partial M$.*

Dit autrement, le spectre essentiel d'un opérateur uniformément elliptique $P$ de $\Psi^m(\mathcal{G}; E, F)$ est déterminé par la famille d'opérateurs limites associée :

$$\mathrm{Spec}_{\mathrm{ess}}(P) = \bigcup_{x \in \partial M} \mathrm{Spec}(P_x).$$

Les groupoïdes Hausdorff et moyennables sont Fredholm, mais la réciproque n'est pas vraie (voir l'exemple 6.4.8) : un des buts de la partie 5 est de donner des caractérisations de ces groupoïdes Fredholm.

La classe des variétés complètes pour lesquelles l'analyse peut-être modélisée par un groupoïde de Lie est relativement large : dans la plupart des exemples mentionnés ci-dessus, il existe un groupoïde naturel sous-jacent. Pour un aperçu des travaux utilisant cette approche, citons parmi d'autres [1, 4, 10, 48, 70, 71, 73, 113, 143, 202, 203, 209, 161]. Parallèlement, Georgescu et Iftimovici ont contribué à l'étude du spectre essentiel du problème à $N$ corps sur $\mathbb{R}^n$ en utilisant des $C^*$-algèbres de produits croisés, qui sont des cas particulier de groupoïdes de Fredholm [84, 85] ; voir [8, 7, 142, 124] pour des travaux reliés. Măntoiu, Purice et Richard ont étendu cette approche à des $C^*$-algèbres de groupoïdes tordues par un cocycle, considérant ainsi des potentiels magnétiques [99, 125, 126, 127].

### 2.1.2. Chapitre 5 : recollements de groupoïdes et potentiels de couche

Le chapitre 5 est adapté de l'article publié [49], réalisé en commun avec Catarina Carvalho et Yu Qiao.

#### Recollements de groupoïdes d'action

Dans [49], nous avons étudié de manière générale des groupoïdes obtenus comme recollements. Ainsi, soit $(U_i)_{i \in I}$ un recouvrement ouvert d'un espace localement compact séparé $X$ et $(\mathcal{G}_i \rightrightarrows U_i)_{i \in I}$ une famille de groupoïdes localement compacts, isomorphes sur les intersections $U_i \cap U_j$, pour $i, j \in I$. On suppose que les isomorphismes satisfont une condition naturelle de cocycle. Le recollement

$$\mathcal{G} := \bigcup_{i \in I} \mathcal{G}_i,$$

est l'espace localement compact obtenu en prenant l'union disjointe des $\mathcal{G}_i$, quotienté par l'identification sur les intersections $U_i \cap U_j$, pour $i, j \in I$. Bien que muni d'applications

| Variétés... | Champs de vecteurs associés | Modèle local |
|---|---|---|
| ... à bouts cylindriques | $\mathcal{V}_b = \{X \mid X \cdot r = 0 \text{ sur } \partial M\}$ | $[0,\infty) \rtimes (0,\infty)$ |
| ... asymptotiquement euclidiennes | $\mathcal{V}_{\mathrm{sc}} = r\mathcal{V}_b$ | $\mathbb{S}^n_+ \rtimes \mathbb{R}^n$ |
| ... asymptotiquement hyperboliques | $\mathcal{V}_0 = \{X \mid X = 0 \text{ sur } \partial M\}$ | $X_n \rtimes G_n$ |

TABLE 2.1. : Modèle locaux de groupoïdes $\mathcal{G} \rightrightarrows M$ associés à des géométries particulières. Ici $M$ est une variété à bord et $r$ est une fonction lisse définissant $\partial M$. L'espace $\mathbb{S}^n_+$ est la compactification radiale de $\mathbb{R}^n$ introduite partie 3.3.3, et le groupe $G_n = (0,\infty) \ltimes \mathbb{R}^{n-1}$, avec l'action de $(0,\infty)$ par dilatation, et $X_n$ est sa compactification partielle $[0,\infty) \rtimes (0,\infty)$.

source et but $\mathcal{G} \rightrightarrows X$, le produit de deux éléments composables n'est *a priori* pas défini. Nous donnons ainsi une condition de recollement « faible », et étudions les propriétés générales des groupoïdes obtenus ainsi. Ceci étend une construction de Gualtieri et Li [94] utilisée pour la classification des intégrations de certains algébroïdes de Lie.

Nous étions motivés par l'observation suivante, que nous montrons dans l'article : beaucoup de groupoïdes apparaissant dans l'analyse de variétés non compactes peuvent être construit en recollant des briques élémentaires très simples, qui sont des groupoïdes d'action (voir le tableau 2.1). On peut considérer ce point de vue comme un lien entre les deux approches mentionnées à la fin de la partie 2.1.1 :

- l'approche « locale » de Georgescu, Iftimovici et autres auteurs [84, 85, 99, 142, 124, 127], où des résultats spectraux sont obtenus *via* l'étude d'une algèbre de produit croisée, donnée par un groupe de Lie agissant sur une compactification de lui-même ;
- l'approche « globale » où l'analyse est donnée par un groupoïde [6, 48, 71, 113, 209], qui ressemble alors localement aux groupoïdes d'actions mentionnés ci-dessus.

L'intérêt est que de nombreux résultats sont connus sur les représentations des algèbres de produits croisés [76, 162, 207]. Si $\mathcal{G}$ est un recollement de groupoïdes d'actions, on espère donc en déduire beaucoup d'information sur les représentations de la $C^*$-algèbre réduite de $\mathcal{G}$. On obtient un résultat partiel dans ce sens : sous l'hypothèse forte que $\mathcal{G}$ agit trivialement sur $\partial M$ et que les groupes qui agissent sont moyennables, le groupoïde résultant du recollement est Fredholm. Cette conclusion sera grandement améliorée par les résultats de la partie 6.

### Opérateurs potentiels de couche

Nous avons ensuite appliqué les résultats de la partie 2.1.2 à l'étude des problèmes à bords sur des domaines à points coniques $\Omega \subset \mathbb{R}^n$, avec $n \geq 2$. Pour des domaines bidimensionnels, le *potentiel de double couche* est l'opérateur $\mathcal{D} : C^\infty(\partial\Omega) \to C^\infty(\Omega)$ donné par

$$(\mathcal{D}f)(x) = \frac{1}{\pi} \int_{\partial\Omega} f(y) \frac{(x-y)\cdot\nu(y)}{|x-y|^2}\, \mathrm{d}y\,,$$

pour tout $x \in \Omega$, avec $\nu(y)$ le vecteur normal à $\partial\Omega$ en $y$, pointant vers l'intérieur du domaine. Si $u = \mathcal{D}f$ pour un $f \in C^\infty(\partial\Omega)$, et $x \in \partial\Omega$ est un point régulier, notons $u_-(z)$ et $u_+(z)$ les limites de $u$ quand $z \to x$ de manière non tangentielle, respectivement par l'intérieur et par l'extérieur de $\Omega$. Les résultats classiques donnent alors :

$$\begin{cases} \Delta u = 0 & \text{sur } \mathbb{R}^2 \setminus \partial\Omega, \\ u_\pm = (\mp 1 + K) f & \text{sur } \partial\Omega. \end{cases}$$

Ici, $K$ est l'opérateur de $C^\infty(\partial\Omega)$ dans lui-même défini par

$$(Kf)(x) = \frac{1}{\pi} \int_{\partial\Omega} f(y) \frac{(x-y)\cdot\nu(y)}{|x-y|^2} \, \mathrm{d}y \,,$$

pour tout $x \in \partial\Omega$ [193]. L'inversibilité des problèmes de Dirichlet sur $\Omega$ et $\mathbb{R}^2 \setminus \overline{\Omega}$, avec donnée $f \in L^2(\partial\Omega)$, est donc équivalente à celle des opérateurs $(\pm 1 + K)$ sur $L^2(\partial\Omega)$. Ces résultats restent vrais en dimension supérieure, en modifiant la formule du potentiel sous l'intégrale définissant $\mathcal{D}$ et $K$.

Lorsque $\partial\Omega$ est lisse, l'opérateur $K$ est dans $\Psi^{-1}(\partial\Omega)$, donc compact sur $L^2(\partial\Omega)$. L'opérateur $\pm I + K$ étant alors Fredholm d'indice 0, il suffit de montrer son injectivité pour établir la solvabilité du problème de Dirichlet. L'opérateur $K$ demeure compact si $\partial\Omega$ est de régularité $C^{1+\epsilon}$, pour un $\epsilon > 0$. Cependant, dès lors que $\partial\Omega$ est moins régulière, l'opérateur $K$ n'est en général plus compact sur $L^2(\partial\Omega)$ [80, 81, 108].

### Groupoïde des potentiels de couche

Dans le cas où $\Omega \subset \mathbb{R}^n$ est un domaine à points coniques, on peut espérer retrouver le caractère Fredholm de $K$ dans des espaces à poids, comme expliqué partie 3.2.1. Les opérateurs potentiels de couche sur les polyèdres et les problèmes proches de transmission ont ainsi été étudiés par de nombreux auteurs, parmi lesquels Fabes, Jodeit et Lewis [81], Elschner [80], Kellogg [104], ainsi que Costabel et Stephan [64].

Par « domaine à points coniques », on entend que $\partial\Omega$ est lisse partout en dehors d'un ensemble fini de points $p_1, \ldots, p_k$, près desquels un difféomorphisme local envoie $\Omega$ sur un secteur

$$\{r\theta \mid (r,\theta) \in (0,\epsilon) \times \omega_j\},$$

pour un $\epsilon > 0$ et un ouvert $\omega_j \subset \mathbb{S}^{n-1}$, avec $j = 1, \ldots, k$. Le domaine peut ainsi avoir des « cassures », comme illustré par la Figure 2.1. On se réfèrera à la partie 5.4 pour une définition précise.

Carvalho et Qiao ont construit dans [52] une variété à bords $M$ qui désingularise $\partial\Omega$, en incluant plusieurs copies des points singuliers (voir Figure 2.1). Ils ont également introduit un groupoïde $\mathcal{G} \rightrightarrows M$ adapté à l'étude des potentiels de couche. Le groupoïde $\mathcal{G}$ intègre l'algèbre de Lie $\mathcal{V}_b(M)$, mais est en général plus grand que le $b$-groupoïde $\mathcal{G}_b \rightrightarrows M$ : ceci est nécessaire de par la *non-localité* des opérateurs potentiels de couche.

En application des résultats de la partie 2.1.2, nous montrons que le groupoïde $\mathcal{G}$ est Fredholm. On en déduit un résultat sur le caractère de Fredholm des opérateurs

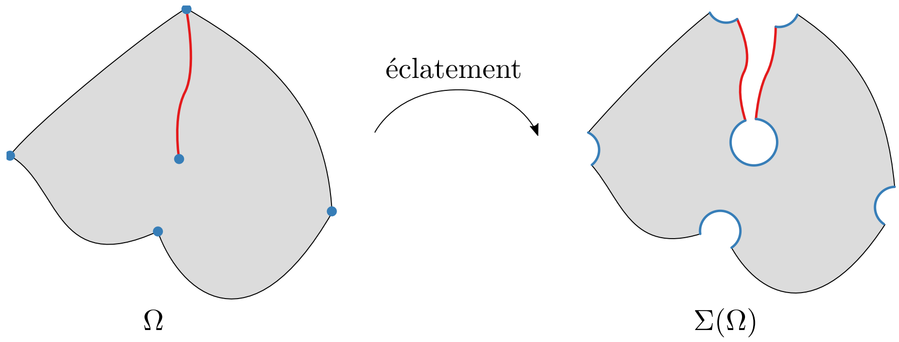


FIGURE 2.1. : Un domaine à points coniques $\Omega \subset \mathbb{R}^2$ exhibant une cassure (en rouge) et sa résolution en variété à coins $\Sigma(\Omega)$ : on double les cassures et on éclate chaque point conique. On définit $\Sigma(\partial\Omega)$ comme la fermeture de la partie lisse de $\partial\Omega$ dans $\partial\Sigma(\Omega)$.

engendrés par $\mathcal{G}$ dans les *espaces à poids*

$$\mathcal{K}_a^m(\partial\Omega) := \{u \in L^2_{\text{loc}}(\partial\Omega) \mid r^{|\alpha|-a}\partial^\alpha u \in L^2(\partial\Omega) \text{ pour tout } |\alpha| \leq m\},$$

avec $r$ la distance aux points singuliers. Pour plus de détails sur ces espaces, voir la partie 3.2.1 ainsi que [108, 110].

**Théorème 2.1.2** (Carvalho, Qiao [52] – Carvalho, C., Qiao [49])**.** *Supposons que $\Omega \subset \mathbb{R}^n$ soit un domaine conique sans cassure, et soit $p_1, \ldots, p_l$ les points singuliers de $\partial\Omega$. Soit $P \in \Psi^m(\mathcal{G})$. Pour tout $l \in \mathbb{Z}$, l'opérateur*

$$P : \mathcal{K}^l_{\frac{n-1}{2}}(\partial\Omega) \to \mathcal{K}^{l-m}_{\frac{n-1}{2}}(\partial\Omega)$$

*est Fredholm si, et seulement si :*

*(i) $P$ est elliptique, et*

*(ii) les opérateurs limites (de Mellin)*

$$P_j : H^l((0,\infty) \times \partial\omega_j) \to H^{l-m}((0,\infty) \times \partial\omega_j)$$

*sont tous inversibles, pour $j = 1, \ldots, k$.*

Le Théorème 2.1.2 avait déjà été prouvé par Carvalho et Qiao dans [52], en utilisant un résultat antérieur de Lauter et Nistor [113]. Établir que $\mathcal{G}$ est Fredholm permet de replacer ce résultat dans un contexte unifié et d'anticiper l'extension de ces méthodes à d'autres géométries. En particulier, comprendre la théorie des potentiels de couches sur des domaines à points cuspidaux (pour lesquels il y a toujours un groupoïde sous-jacent) fait partie des questions ouvertes posées par Maz′ya [128] (voir également [131]). Dans le cas des polyèdres, un problème général est de savoir si on a toujours $K \in C^*_r(\mathcal{G})$ [166].

Lorsque $\Omega$ est un polygone de $\mathbb{R}^2$, Qiao et Li [167] ont montré par un calcul explicite que $\pm 1 + K \in \overline{\Psi^0}(\mathcal{G})$ et déterminé la famille d'opérateurs limites : ils en déduisent le caractère bien posé du problème de Dirichlet dans les espaces à poids. Enfin, le résultat est toujours valable en présence de cassures : il faut alors ajouter plus d'opérateurs limites du fait de la résolution des cassures lors de la construction de $\Sigma(\Omega)$ (voir Figure 2.1).

### 2.1.3. Chapitre 6 : étude locale des groupoïdes de Fredholm

Le chapitre 6 est adapté de l'article [55]. Afin de généraliser les résultats sur les recollements présentés chapitre 5, j'étudie ici la structure locale des représentations irréductibles de la $C^*$-algèbre d'un groupoïde localement compact. Ces représentations, ou plutôt leurs noyaux, forment ensemble un espace topologique appelé *spectre primitif* (la définition précise est donnée en 4.1.2). L'un des résultats principaux de [55] est que le spectre primitif de l'algèbre d'un groupoïde a une structure locale, dans le sens donné par le Théorème 2.1.3 plus bas.

Si $\mathcal{G} \rightrightarrows X$ est un groupoïde avec source et but $d, r : \mathcal{G} \to X$, et si $U \subset X$, rappelons qu'on note $\mathcal{G}|_U := r^{-1}(U) \cap d^{-1}(U)$ la *restriction* de $\mathcal{G}$ à $U$, et $\mathcal{G} \cdot U = r(d^{-1}(U))$ la *saturation* de $U$ par $\mathcal{G}$.

**Théorème 2.1.3** (C. [55]). *Soit $\mathcal{G} \rightrightarrows X$ un groupoïde localement compact et dénombrable à l'infini. On considère une famille d'ouverts $(U_i)_{i\in I}$ de $X$ tels que la famille des saturations $(\mathcal{G} \cdot U_i)_{i\in I}$ forme un recouvrement ouvert de $X$. Alors*

$$\operatorname{Prim} C_r^*(\mathcal{G}) = \bigcup_{i\in I} \operatorname{Prim} C_r^*(\mathcal{G}|_{U_i}).$$

Le même résultat est vrai si on remplace les algèbres réduites de groupoïdes par les algèbres maximales, ou si on remplace le spectre primitif par le spectre total de $C_r^*(\mathcal{G})$. La preuve repose sur la construction d'un foncteur d'induction depuis la catégorie des classes d'équivalences unitaires de représentations de chaque $C_r^*(\mathcal{G}|_{U_i})$ vers celle de $C_r^*(\mathcal{G})$, en utilisant la machinerie classique de Mackey et Rieffel [171, 177]. Le résultat suit en exploitant les propriétés naturelles de ce foncteur ainsi qu'un argument de partition de l'unité.

Puisque la propriété Fredholm porte essentiellement sur les représentations de $C_r(\mathcal{G})$, on en déduit que cette dernière propriété est elle aussi locale. Notons que pour un groupoïde, être Fredholm est une propriété topologique : on travaille donc dans le cadre général des groupoïdes localement compacts. Cela a un intérêt pour l'analyse : certains des groupoïdes apparaissant dans les applications pratiques ne sont en effet pas de Lie, mais seulement longitudinalement lisses (voir l'exemple 6.4.14).

**Théorème 2.1.4** (C. [55]). *Sous les hypothèses du Theorème 2.1.3, le groupoïde $\mathcal{G}$ est Fredholm si, et seulement si, chaque réduction $\mathcal{G}|_{U_i}$ est Fredholm, pour tout $i \in I$.*

Une conséquence est que les groupoïdes obtenus comme recollements de groupoïdes Fredholm sont eux-mêmes Fredholm. Dans la continuité des résultats de [49] introduits ci-dessus, on considère des groupoïdes $\mathcal{G} \rightrightarrows M$ qui sont *localement des groupoïdes d'action* : cela signifie que tout point $x \in M$ admet un voisinage $U$ tel que $\mathcal{G}|_U$ soit isomorphe à une réduction $(X \rtimes G)|_V$, avec $V$ un ouvert de $X$. Cette classe est plus large que celle des groupoïdes obtenus par recollements de groupoïdes d'action : par exemple, le $b$-groupoïde est localement un groupoïde d'action (voir la Table 2.1), mais ne peut pas être obtenu comme recollement de tels groupoïdes en dimension $n \geq 2$.

Comme dans la partie précédente, l'objectif est de transférer au cas des groupoïdes les résultats « locaux » connus sur les algèbres de produits croisés, notamment en relation avec l'étude du problème à $N$ corps [8, 76, 85, 99, 124, 127, 142, 207]. On donne par exemple comme résultat que si le modèle local est donné par l'action d'un groupe moyennable, alors le groupoïde est Fredholm.

## 2.2. Partie IV : opérateurs équivariants et composantes isotypiques

La partie IV porte sur un travail réalisé conjointement avec Alexandre Baldare[1], Matthias Lesch[2] et Victor Nistor[3], et est adaptée des deux articles [25, 26].

### 2.2.1. Motivations : actions de groupes et indice équivariant

Dans toute la partie IV, on considère l'action lisse d'un groupe de Lie $G$ *compact* sur une variété lisse $M$ également compacte, sans bord. On suppose qu'on a deux fibrés vectoriels $E, F$ sur $M$ sur lesquels $G$ agit également de manière lisse, et on se donne un opérateur pseudodifférentiel elliptique $P$ d'ordre $m$, agissant des sections de $E$ vers celles de $F$. Lorsque $P$ est $G$-équivariant, les espaces $\ker P$ et $\operatorname{coker} P$ sont des représentations de dimensions finie de $G$. Atiyah et Singer [16] ont alors introduit *l'indice équivariant* de $P$ comme la différence des caractères correspondants :

$$\operatorname{index}_G P = \chi_{\ker P} - \chi_{\operatorname{coker} P}.$$

L'indice équivariant définit ainsi une fonction lisse sur $G$, et son évaluation en l'élément neutre est l'indice de Fredholm de $P$.

Puisque $G$ est compact, les représentations $\ker P$ et $\operatorname{coker} P$ se décomposent en sommes directes de représentations irréductibles, si bien qu'on peut écrire

$$\operatorname{index}_G P = \sum_{\alpha \in \widehat{G}} m_\alpha \chi_\alpha,$$

avec $\chi_\alpha$ le caractère d'une représentation irréductible $\alpha \in \widehat{G}$ et $m_\alpha \in \mathbb{Z}$ la différence des multiplicités de $\alpha$ dans $\ker P$ et $\operatorname{coker} P$.

L'indice équivariant a également un sens si $P$ est seulement elliptique *transversalement aux orbites* de $G$. Cela signifie que le symbole principal $\sigma_m(P)(\xi)$ est inversible seulement pour les covecteurs $\xi \in T^*M \setminus \{0\}$ qui annulent les champs de vecteurs fondamentaux de l'action de $G$. Dans ce cas, bien que $\ker P$ et $\operatorname{coker} P$ soient généralement de dimension

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[3] Intitut Élie Cartan de Lorraine, Metz, France

infinies, Atiyah et Singer ont montré que les multiplicités $(m_\alpha)_{\alpha\in\widehat{G}}$ sont chacune finies et que la somme $\sum_{\alpha\in\widehat{G}} m_\alpha\chi_\alpha$ converge au sens des distributions sur $G$ [12]. Les mêmes auteurs ont calculé cette distribution par une formule cohomologique intégrée sur des sous-variétés de points fixes. L'indice des opérateurs $G$-transversalement elliptiques a par la suite été étudié du point de vue de la $K$-théorie [22, 24, 101, 103], ainsi qu'en cohomologie équivariante [23, 29, 30, 155].

Le problème de calculer directement individuellement les multiplicités $m_\alpha$, pour $\alpha \in \widehat{G}$, a reçu moins d'attention. On peut reformuler le problème comme suit. Puisque le groupe $G$ agit sur l'espace de Sobolev $H^s(M;E)$, pour $s \in \mathbb{R}$, on peut le décomposer en composantes isotypiques :

$$H^s(M;E) = \bigoplus_{\alpha\in\widehat{G}} H^s(M;E)_\alpha,$$

où $H^s(M;E)_\alpha$ est une somme directe de représentations de $G$ isomorphes à $\alpha$, pour chaque $\alpha \in \widehat{G}$. L'opérateur $P$ étant équivariant, sa restriction à la composante isotypique associée à un $\alpha \in \widehat{G}$ définit un opérateur

$$\pi_\alpha(P) : H^s(M;E)_\alpha \to H^{s-m}(M;F)_\alpha.$$

Si $P$ est transversalement elliptique, alors $\pi_\alpha(P)$ est Fredholm et $\operatorname{index} \pi_\alpha(P) = m_\alpha \dim \alpha$.

La question du calcul des multiplicités $m_\alpha$ a été considérée par Paradan et Vergne, qui ont obtenu plusieurs résultats important [156, 157]. Elle apparaît également implicitement dans les travaux de Brüning, qui a initié l'étude de la « trace isotypique » du noyau de la chaleur, c'est-à-dire de $\operatorname{tr}\Big(\pi_\alpha(e^{-t\Delta})\Big)$, et de son développement asymptotique quand $t \to 0$ [38, 42, 43]. Le programme de Brüning mènerait en principe à un théorème d'indice pour $\pi_\alpha(D)$ *via* l'équation de la chaleur, lorsque $D$ est un opérateur de type Dirac. Ce dernier objectif est l'une des motivations de ce travail. En effet, comme on le verra, nous déterminons un symbole $\alpha$-principal dont la classe d'homotopie détermine exactement $\operatorname{index} \pi_\alpha(P)$. Nous espérons que nos résultats auront des conséquence quant à l'étude de la théorie de Hodge des quotients singuliers et des variétés algébriques [3, 36, 54, 89]. Calculer cet indice explicitement demandera sans doute d'utiliser des outils et idées issues de la géométrie non commutative, en particulier des calculs de cohomologie cyclique [59, 60, 102, 147] : nous laissons ce projet pour le futur.

### 2.2.2. Le résultat principal : ellipticité relativement à une représentation irréductible

La question que nous nous posons est la suivante : étant donné un opérateur $G$-équivariant $P \in \Psi^m(M;E,F)^G$, pas nécessairement elliptique, et une représentation irréductible $\alpha \in \widehat{G}$, à quelle condition la restriction

$$\pi_\alpha(P) : H^s(M;E)_\alpha \to H^{s-m}(M;F)_\alpha$$

est-elle de Fredholm ?

Nous construisons un symbole $\alpha$-principal $\sigma_m^\alpha(P)$, obtenu en sélectionnant une partie de l'information contenue dans le symbole principal $\sigma_m(P)$, et qui caractérise le caractère

Fredholm de $\pi_\alpha(P)$. Pour ça, nous avons besoin de supposer *G fini* (des commentaires sur les raisons de cette hypothèse sont inclus à la fin de la partie 2.2.4).

Remplaçons donc $G$ par un groupe $\Gamma$ fini. L'action de $\Gamma$ sur $M$ induit une action sur $T^*M$ ; dénotons par $\Gamma_\xi$ le sous-groupe d'isotropie d'un élément $\xi \in T^*M$. Si $x \in M$ est la projection de $M$ sur la base, alors $\Gamma_\xi \subset \Gamma_x$, donc $\Gamma_\xi$ agit sur les fibres $E_x$ et $F_x$. Puisque $P$ est $\Gamma$-équivariant, l'évaluation $\sigma_m(P)(\xi)$ définit un élément de $\mathrm{Hom}_{\Gamma_\xi}(E_x, F_x)$. Étant donnée une représentation irréductible $\rho$ de $\Gamma_\xi$, on définit alors

$$\sigma_m^\Gamma(P)(\xi,\rho) := \pi_\rho(\sigma_m(P)(\xi)) : E_{x\rho} \to F_{x\rho}$$

comme la restriction de $\sigma_m(P)(\xi)$ aux composantes isotypiques correspondant à $\rho$. Posant ainsi

$$X_{M,\Gamma} := \{(\xi,\rho) \mid \xi \in T^*M \setminus \{0\} \text{ et } \rho \in \widehat{\Gamma}_\xi\}, \tag{2.1}$$

on obtient une fonction $\sigma_m^\Gamma(P)$ définie sur $X_{M,\Gamma}$.

Pour prendre en compte la représentation $\alpha$, nous avons besoin de la notion suivante :

**Définition 2.2.1.** Soient $\Gamma_1, \Gamma_2$ deux groupes finis, et $H$ un sous-groupe de $\Gamma_1$ et $\Gamma_2$. Soient $\rho_1 \in \widehat{\Gamma}_1$ et $\rho_2 \in \widehat{\Gamma}_2$. On dit que $\rho_1$ et $\rho_2$ sont *$H$-associées* si $\mathrm{Hom}_H(\rho_1, \rho_2) \neq 0$.

Dit autrement, les représentations $\rho_1$ et $\rho_2$ sont $H$-associées si leur décompositions respectives sur les représentations irréductibles de $H$ comportent au moins un terme en commun ; c'est-à-dire qu'il existe $\beta \in \widehat{H}$ tel que $\mathrm{Hom}_H(\beta, \rho_1) \neq 0$ et $\mathrm{Hom}_H(\beta, \rho_2) \neq 0$. Si $\Gamma_1$ et $\Gamma_2$ sont abéliens, alors les caractères $\rho_1$ et $\rho_2$ sont $H$-associées si, et seulement si, $\rho_1|_H = \rho_2|_H$.

Quitte à restreindre l'étude à chaque composante, on peut supposer l'espace des orbites $M/\Gamma$ connexe. Sous cette hypothèse, un résultat classique donne l'existence d'un sous groupe d'isotropie *minimal* $\Gamma_0 \subset \Gamma$ : cela signifie que le stabilisateur $\Gamma_x$ contient un sous-groupe conjugué à $\Gamma_0$ pour tout $x \in M$, et que l'ensemble $M_{(\Gamma_0)}$ des points $x$ de $M$ tels que $\Gamma_x$ est conjugué à $\Gamma_0$ est un ouvert dense de $M$ [34].

On a $\Gamma_{g\xi} = g\Gamma_\xi g^{-1}$ pour tout $\xi \in T^*M$. Si $\rho \in \widehat{\Gamma}_\xi$, on définit ainsi une représentation $g \cdot \rho \in \widehat{\Gamma}_{g\xi}$ par $(g \cdot \rho)(h) = \rho(g^{-1}hg)$, pour tout $h \in \Gamma_{g\xi}$. Posons alors

$$X_{M,\Gamma}^\alpha := \{(\xi,\rho) \in X_{M,\Gamma} \mid \exists g \in \Gamma,\ g \cdot \rho \text{ et } \alpha \text{ sont } \Gamma_0\text{-associées}\}. \tag{2.2}$$

Le *symbole $\alpha$-principal* $\sigma_m^\alpha(P)$ est défini comme la restriction de $\sigma_m^\Gamma(P)$ à $X_{M,\Gamma}^\alpha$. L'opérateur $P$ est dit *$\alpha$-elliptique* si $\sigma_m^\alpha(P)$ est inversible, donc si

$$\pi_\rho(\sigma_m(P)(\xi)) : E_{x\rho} \to F_{x\rho}$$

est inversible pour tout $(\xi,\rho) \in X_{M,\Gamma}^\alpha$. Autrement dit, on ne sélectionne qu'une partie de l'information contenue dans le symbole principal de $P$, et on demande que cette composante du symbole soit inversible. Considérant l'opérateur trivial entre deux espaces vectoriels nuls comme étant inversible, la condition ne porte que sur les paires $(\xi,\rho)$ telles que $E_{x\rho} \neq 0$ ou $F_{x\rho} \neq 0$.

**Théorème 2.2.2** (Baldare, C., Lesch, Nistor [25, 26]). *Soit $\Gamma$ un groupe fini agissant par difféomorphismes sur une variété $M$ compacte, sans bord. Soit $P \in \Psi^m(M; E, F)^\Gamma$, $m \in \mathbb{R}$, un opérateur pseudodifférentiel classique $\Gamma$-équivariant agissant entre les sections de deux $\Gamma$-fibrés vectoriels $E$ et $F$. Étant donnée une représentation $\alpha \in \widehat{\Gamma}$, l'opérateur*

$$\pi_\alpha(P) : H^s(M; E)_\alpha \to H^{s-m}(M; F)_\alpha$$

*est Fredholm si, et seulement si, $P$ est $\alpha$-elliptique.*

Une reformulation de ce théorème en terme de variétés de points fixes est donné plus bas, par la Proposition 2.2.3. Si $P$ est elliptique, alors $P$ est $\alpha$-elliptique pour tout $\alpha \in \widehat{\Gamma}$ : on retrouve naturellement que chaque $\pi_\alpha(P)$ est de Fredholm. En outre, si $\Gamma$ agit librement sur un ouvert dense, alors le sous-groupe minimal d'isotropie $\Gamma_0$ est trivial, si bien que $X^\alpha_{M,\Gamma} = X_{M,\Gamma}$ pour tout $\alpha \in \widehat{\Gamma}$. Dans ce cas, être elliptique équivaut à être $\alpha$-elliptique : cela signifie que l'ellipticité de l'opérateur peut être vérifiée en se restreignant à une unique composante isotypique.

La signification pour l'indice est la suivante : comme on l'expliquera dans la partie suivante, le symbole $\alpha$-principal d'un opérateur d'ordre 0 vit naturellement dans une $C^*$-algèbre $\Sigma^\alpha$, obtenue comme quotient de la $C^*$-algèbre des symboles. Le théorème 2.2.2 donne une suite exacte de $C^*$-algèbres

$$0 \longrightarrow \mathcal{K}(L^2(M; E)_\alpha)^\Gamma \longrightarrow \pi_\alpha(\overline{\Psi^0}(M; E)^\Gamma) \xrightarrow{\sigma_0^\alpha} \Sigma^\alpha \longrightarrow 0, \tag{2.3}$$

L'indice de Fredholm de $\pi_\alpha(P)$, pour un $P \in \Psi^0(M; E)$, est alors donné par l'image de la classe de $\sigma_0^\alpha(P)$ par l'application de bord dans la suite de $K$-théorie associée à (2.3). Comme mentionné plus haut, calculer explicitement cet indice demande un calcul non trivial en cohomologie cyclique : nous laissons ce travail pour le futur.

### 2.2.3. Chapitre 7 : première partie de la preuve et cas abélien

On présente dans cet article la première partie de la preuve, adaptée de l'article [26] écrit conjointement avec Alexandre Baldare, Matthias Lesch et Victor Nistor. Bien que l'énoncé n'utilise pas d'algèbres d'opérateurs, la preuve du Théorème 2.2.2 repose sur l'étude de la $C^*$-algèbre $A_M^\Gamma := C(S^*M; \operatorname{End} E)^\Gamma$ qui contient les symboles des opérateurs de $\overline{\Psi^0}(M; E)^\Gamma$ (la fermeture étant prise dans $\mathcal{B}(L^2(M)))$. La suite exacte

$$0 \longrightarrow \overline{\Psi^{-1}}(M; E)^\Gamma \longrightarrow \overline{\Psi^0}(M; E)^\Gamma \xrightarrow{\sigma_0} A_M^\Gamma \longrightarrow 0$$

permet de définir un morphisme

$$\mathcal{R}_M : A_M^\Gamma \longrightarrow \pi_\alpha(\overline{\Psi^0}(M; E)^\Gamma)/\pi_\alpha(\overline{\Psi^{-1}}(M; E)^\Gamma).$$

Puisque $\pi_\alpha(\overline{\Psi^{-1}}(M; E))$ s'identifie aux opérateurs compacts sur $L^2(M; E)_\alpha$, le caractère Fredholm d'un opérateur $P$ est exactement déterminé par la projection de son symbole dans l'algèbre quotient $A_M^\Gamma / \ker \mathcal{R}_M$ : cette projection s'identifie précisément au symbole $\alpha$-principal $\sigma_0^\alpha(P)$.

Notre objectif devient alors de déterminer les représentations irréductibles de l'algèbre $A_M^\Gamma / \ker \mathcal{R}_M$. On commence par déterminer celles de $A_M^\Gamma$ ; plus précisément, on calcule le *spectre primitif* $\mathrm{Prim}\, A_M^\Gamma$, qui est un espace topologique construit à partir des représentations irréductibles de $A_M^\Gamma$ (une définition précise est donnée en 4.1.2). On montre ainsi que

$$\mathrm{Prim}\, A_M^\Gamma \simeq X_{M,\Gamma}/\Gamma,$$

avec $X_{M,\Gamma}$ l'ensemble défini en (2.1), sur lequel $\Gamma$ agit naturellement[1]. Qui plus est, l'espace $\mathrm{Prim}\, A_M^\Gamma$ vient avec une projection $\phi^* : \mathrm{Prim}\, A_M^\Gamma \to S^*M/\Gamma$. Cette dernière donne à $\mathrm{Prim}\, A_M^\Gamma$ une structure de fibré localement trivial au-dessus de l'ouvert dense $S^*M_0/\Gamma$ des orbites principales, avec une fibre naturellement isomorphe à un sous-espace de $\widehat{\Gamma}_0$. La fibre dégénère au-dessus d'une orbite $\Gamma\xi \notin S^*M_0/\Gamma$ : l'espace $\mathrm{Prim}\, A_M^\Gamma$ est donc en général non séparé.

Les représentations irréductibles de $A_M^\Gamma / \ker \mathcal{R}_M$ déterminent un espace fermé

$$\Xi := \mathrm{Prim}(A_M^\Gamma / \ker \mathcal{R}_M) \subset \mathrm{Prim}\, A_M^\Gamma. \tag{2.4}$$

Déterminer $\Xi$ est facilité lorsque $\Gamma$ est *abélien.* Dans ce cas le groupe $\Gamma_0$ est unique (sa classe de conjugaison est réduite à un seul élément), si bien que l'action se factorise pour donner une action de $\Gamma/\Gamma_0$ qui est libre sur l'ouvert dense $M_0$. En utilisant ce fait, on parvient à montrer que

$$\Xi \simeq X_{M,\Gamma}^\alpha/\Gamma,$$

où $X_{M,\Gamma}^\alpha$ est défini par (2.2). Cette détermination des représentations irréductibles de $A_M^\Gamma / \ker \mathcal{R}_M$ donne directement la condition d'$\alpha$-ellipticité et le théorème 2.2.2.

Notons que la définition de $X_{M,\Gamma}^\alpha$ se simplifie dans le cas abélien : puisque toutes les représentations irréductibles sont des caractères, on a

$$X_{M,\Gamma}^\alpha = \{(\xi, \rho) \in X_{M,\Gamma} \mid \rho_{|\Gamma_0} = \alpha_{|\Gamma_0}\}.$$

### 2.2.4. Chapitre 8 : spectre primitif de la $C^*$-algèbre des symboles

Nous complétons la preuve pour un groupe fini $\Gamma$ quelconque dans l'article [25], également commun avec Alexandre Baldare, Matthias Lesch et Victor Nistor. Les résultats établis partie 7 sur la structure du spectre $\mathrm{Prim}\, A_M^\Gamma$ restent vrais dans le cas général : il s'agit d'un fibré localement trivial au-dessus de l'ouvert dense $S^*M_0/\Gamma$, dont la fibre dégénère au-dessus des orbites extérieures à $S^*M_0/\Gamma$.

Afin de déterminer l'espace fermé $\Xi \subset \mathrm{Prim}\, A_M^\Gamma$ défini en (2.4), on procède en deux grandes étapes :

1. en exploitant la structure localement triviale de $\mathrm{Prim}\, A_M^\Gamma$ induite par l'application $\phi^* : \mathrm{Prim}\, A_M^\Gamma \to S^*M_0/\Gamma$ et un argument d'induction, on se ramène à un calcul en chaque point qui détermine la partie ouverte $\Xi_0 := \Xi \cap (\phi^*)^{-1}(S^*M_0)$ ;

[1] Plus précisément, on montre que $\mathrm{Prim}\, A_M^\Gamma \simeq \tilde{X}_{M,\Gamma}/\Gamma$, avec $\tilde{X}_{M,\Gamma}$ l'ensemble des $(\xi, \rho) \in X_{M,\Gamma}$ tels que $E_{x\rho} \neq 0$. L'identification avec $X_{M,\Gamma}/\Gamma$ se fait donc modulo des copies de la représentation triviale (nulle). Puisque l'opérateur nul sur l'espace vectoriel trivial est inversible, cet abus de notation ne pose pas de problème pour caractériser l'inversibilité d'un élément de $A_M^\Gamma$ (voir la Remarque 8.2.21).

2. on étudie précisément la topologie de $\operatorname{Prim} A_M^\Gamma$, ce qui permet de conclure à la fois que $\Xi$ est donné par la fermeture de $\Xi_0$ et qu'on a à nouveau

$$\Xi \simeq X_{M,\Gamma}^\alpha/\Gamma,$$

où $X_{M,\Gamma}^\alpha$ est défini par (2.2).

Comme précédemment, ceci donne directement la condition d'$\alpha$-ellipticité et le Théorème 2.2.2. Nous concluons l'article en donnant l'application à l'indice évoquée à la fin de la partie 2.2.1, en explicitant quelques cas particuliers et en établissant un lien avec une propriété d'inversion locale établie par Simonenko [189].

En outre, grâce à une suggestion de Paul-Émile Paradan, nous donnons une reformulation de la condition d'$\alpha$-ellipticité en terme de variétés de points fixes. Ce sont ces mêmes variétés qui apparaissent dans la formule de l'indice équivariant d'Atiyah et Singer [12, 14].

**Proposition 2.2.3** (Baldare, C., Lesch, Nistor [25])**.** *Soit $P \in \Psi^m(M; E, F)^\Gamma$ et $\alpha \in \widehat{\Gamma}$. Les deux propositions suivantes sont équivalentes :*

*(i) $P$ est $\alpha$-elliptique,*

*(ii) Le symbole principal $\sigma_m(P)$ définit par restriction un élément inversible*

$$(\sigma_m(P) \otimes 1)|_{(E\otimes\alpha)^{\Gamma_0}} \in C^\infty(T^*M^{\Gamma_0}; (E \otimes \alpha)^{\Gamma_0}).$$

Plusieurs éléments de la preuve restent vrai si on remplace l'action de $\Gamma$ par celle d'un groupe de Lie compact $G$. D'après les travaux d'Atiyah et Singer mentionnés partie 2.2.1, on sait dans ce cas qu'on doit remplacer $S^*M$ par sa composante $G$-transverse $S_G^*M$. Il y a néanmoins des obstacles techniques à une généralisation directe de la preuve : en particulier, les sous-espaces de $TM$ transverses à l'action ne sont plus des ouverts, ce qui pose des difficultés pour appliquer des arguments d'induction à l'algèbre des opérateurs pseudodifférentiels. Nous aborderons ces problèmes dans un travail futur.

# Partie II.

# Préliminaires

**Résumé**

Cette partie donne une introduction générale au contexte dans lequel s'inscrit cette thèse et aux outils utilisés, qui pourra intéresser une personne peu familière avec le domaine (chapitre 3). La langue d'écriture passe à *l'anglais* à partir du chapitre 4, qui présente les points théoriques essentiels pour la suite de cette thèse : $C^*$-algèbres, opérateurs pseudodifférentiels et groupoïdes.

# 3. Introduction générale

Dans ce chapitre, mon objectif est d'introduire de manière assez générale le contexte dans lequel s'inscrit cette thèse. Ainsi, la partie 3.1 présente les résultats classiques de la théorie elliptique sur des variétés lisses, en insistant sur les résultats de régularité et la théorie de l'indice. On explique en partie 3.2 ce qu'on entend par « espaces singuliers », et quels sont alors les phénomènes nouveaux en lien avec les opérateurs elliptiques.

Les parties suivantes présentent de manière générale les méthodes qui seront utilisées dans la thèse : je commencerai par motiver l'utilisation des algèbres d'opérateurs en 3.3, avant de montrer en 3.4 comment réaliser ces algèbres à travers la construction de calculs pseudodifférentiels adaptés aux singularités. Enfin, j'introduirai partie 3.5 l'usage des groupoïdes comme objets appropriés pour comprendre géométriquement ces différents calculs.

## 3.1. Théorie elliptique dans le cas lisse

Le but de cette partie est d'introduire la notion d'opérateurs *elliptiques* et les propriétés qui leur sont associées dans le cas lisse. Les points importants pour cette thèse sont la notion de *régularité elliptique* et la *théorie de l'indice* sur les variétés compactes sans bord.

### 3.1.1. Problèmes bien posés et régularité elliptique

En guise de motivation pour cette partie, considérons l'équation de Poisson sur un ouvert $\Omega$ de $\mathbb{R}^n$, qu'on suppose borné et de frontière lisse. On cherche alors une solution $u$ au problème

$$\begin{cases} \Delta u = f & \text{sur } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{cases} \tag{P}$$

avec $\Delta = -\sum_{i=1}^{n} \partial_i^2$ le Laplacien sur $\mathbb{R}^n$, et $f$ une donnée de classe $C^\infty$ sur $\mathbb{R}^n$. En termes physiques, trouver $u$ revient à calculer le potentiel électrique généré par une distribution de charge $f$ sur $\Omega$, le bord $\partial\Omega$ étant relié à la masse.

Le problème (P) est *bien posé* : cela signifie que pour toute donnée $f \in C^\infty(\overline{\Omega})$, il existe une unique solution $u \in C^\infty(\overline{\Omega})$, qui dépend continûment de $f$ (dans un sens qui reste à préciser). La preuve moderne de ce résultat repose sur l'utilisation d'espaces fonctionnels appropriés, dans ce cas les espaces de Sobolev[1] $H^s(\Omega)$, pour $s \in \mathbb{R}$. On

[1]Pour $k \in \mathbb{N}$, l'espace $H^k(\Omega)$ contient les fonctions (plus précisément les distributions) $u : \Omega \to \mathbb{C}$ dont toutes les dérivées jusqu'à l'ordre $k$ sont de carré intégrable. Plus $k$ est grand, et plus les éléments de $H^k(\Omega)$ ont des dérivées régulières.

procède classiquement en deux grandes étapes :

1. on montre qu'il existe une unique solution faible $u \in H_0^1(\Omega)$ au problème (P), *a priori* peu régulière,
2. on prouve que cette solution $u$ a toujours la régularité maximale possible : si $f \in H^s(\Omega)$, alors « $u$ possède deux dérivées de plus », c'est-à-dire $u \in H^{s+2}(\Omega)$.

La dépendance continue de $u$ en $f$ est ainsi mesurée par une estimation

$$\|u\|_{H^{s+2}(\Omega)} \leq C_s \|f\|_{H^s(\Omega)},$$

pour tout $s \geq 1$, et avec une constante $C_s > 0$ qui ne dépend pas de $f$. Cette propriété de régularité est typique des opérateurs dit *elliptiques*, dont le Laplacien est l'archétype.

### 3.1.2. Opérateurs elliptiques sur des variétés compactes

La théorie des opérateurs elliptiques est particulièrement bien comprise dans le cas des variétés lisses compactes sans bord. Soit $M$ une telle variété et $P$ un opérateur différentiel scalaire[1] d'ordre $m$ sur $M$. Cela signifie qu'on peut écrire en coordonnées locales

$$P = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha, \tag{3.1}$$

avec $a_\alpha \in C^\infty(M)$, et où on utilise la notation usuelle $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, pour tout $\alpha \in \mathbb{N}^n$.

Un objet important associé à $P$ est son *symbole principal*. Si $P$ est d'ordre $m$ et s'écrit comme en (3.1) dans une carte locale, alors son symbole principal est défini dans cette même carte par

$$\sigma_m(P)(x, \xi) := i^m \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha,$$

pour tout $(x, \xi) \in T^*M$, et où $\xi^\alpha$ est calculé à partir des coordonnées locales de $\xi$ comme $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}$. Il s'avère que cette quantité ne dépend pas de la carte dans laquelle on la regarde, si bien que le symbole principal définit une fonction $\sigma_m(P) \in C^\infty(T^*M)$.

**Définition 3.1.1.** Un opérateur différentiel est *elliptique* si son symbole principal est inversible en dehors de la section nulle de $T^*M$.

Une notion importante rattachée à celle des opérateurs elliptiques est la notion *d'opérateur de Fredholm*, que l'on rappelle ici.

**Définition 3.1.2.** Soient $\mathcal{E}, \mathcal{F}$ deux espaces de Banach. Un opérateur borné $T : \mathcal{E} \to \mathcal{F}$ est *de Fredholm* si $\ker T$ et $\operatorname{coker} T := \mathcal{F} / \operatorname{im} T$ sont tous deux de dimensions finies.

Les opérateurs de Fredholm sont « presque » inversible : plus précisément, Atkinson [19] a montré qu'un opérateur $T : \mathcal{E} \to \mathcal{F}$ est de Fredholm si, et seulement si, il est

[1] Pour simplifier les notations, on ne considère dans cette introduction que des opérateurs scalaires, agissant sur les fonctions lisses à valeurs complexes. Les notions s'étendent bien sûr aux opérateurs (pseudo)différentiels agissant entre les sections de fibrés vectoriels complexes sur $M$.

inversible modulo des opérateurs compacts. En d'autres termes, on peut trouver une « paramétrice », c'est-à-dire un opérateur $S : \mathcal{F} \to \mathcal{E}$ tel que

$$TS = 1 + R \quad \text{et} \quad ST = 1 + R',$$

avec $R$ et $R'$ des opérateurs compacts.

Un résultat fondamental des années 1950 dû au travail de nombreux auteurs sur les opérateurs d'intégrales singulières est que, sur les variétés compactes, l'ellipticité caractérise exactement les opérateurs différentiels qui sont Fredholm entre espaces de Sobolev [86, 98, 138, 139, 185].

**Théorème 3.1.3.** *Soit $M$ une variété compacte sans bord et $P$ un opérateur différentiel d'ordre $m$ sur $M$. Pour tout $s \in \mathbb{R}$, l'opérateur $P : H^s(M) \to H^{s-m}(M)$ est Fredholm si, et seulement si, $P$ est elliptique.*

La preuve du Théorème 3.1.3 et la construction des paramétrices correspondantes se comprend bien en mobilisant la notion de *calcul pseudodifférentiel* sur $M$, ce que nous allons faire dans la partie suivante.

### 3.1.3. Calcul pseudodifférentiel sur les variétés compactes

La définition précise des opérateurs pseudodifférentiels peut être attribuée à Kohn et Nirenberg en 1965 [107], ainsi qu'à Seeley, Hörmander, Bokobza et Unterberger [32, 97, 186]. Le but était alors de donner une base solide à l'étude des opérateurs d'intégrales singulières qui étaient étudiés depuis les années 1930 et d'obtenir une meilleure compréhension des propriétés liées à l'ellipticité[1].

Sur $\mathbb{R}^n$, un opérateur pseudodifférentiel $P$ est donné pour tout $f \in C_c^\infty(\mathbb{R}^n)$ et $x \in \mathbb{R}^n$ par une intégrale oscillante

$$(Pf)(x) = \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} a(x,\xi) f(y) \, \mathrm{d}y \, \mathrm{d}\xi \,. \tag{3.2}$$

La fonction $a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ est le symbole *total* de $P$ et doit vérifier certaines conditions de croissance sous-polynomiale en $\xi$ ; on se réfère au livre de Hörmander pour plus de détails [98]. Si $a$ est un polynôme en $\xi$, alors $P$ est un opérateur différentiel du même ordre et la formule (3.2) est simplement l'action de $P$ dans l'espace de Fourier.

Sur une variété compacte sans bord $M$, un opérateur pseudodifférentiel est un opérateur $P : C^\infty(M) \to C^\infty(M)$ qui s'écrit localement comme en (3.2). L'ensemble des opérateurs pseudodifférentiels[2] d'ordre $m$ est noté $\Psi^m(M)$, et $m$ peut alors être n'importe quel réel. Les opérateurs pseudodifférentiels forment ainsi une algèbre graduée qui contient celle des opérateurs différentiels.

[1] Une bonne introduction historique à l'étude des Équations aux Dérivées Partielles est donnée par Brezis et Browder dans une synthèse de 1998 [35].

[2] Plus précisément, on dénote par $\Psi^m(M)$ l'ensemble des opérateurs pseudodifférentiels *classiques* sur $M$, c'est-à-dire ceux dont le symbole admet une certaine expansion asymptotique (voir partie 4.2).

Les noyaux de Schwartz de ces opérateurs ont été caractérisés par Hörmander [98] : il s'agit des distributions sur $M \times M$ qui sont *conormales* par rapport à la diagonale

$$\Delta_M = \{(x,x) \mid x \in M\} \subset M \times M,$$

ce qui signifie qu'elles sont lisses en dehors de $\Delta_M$ et ont des singularités prescrites sur $\Delta_M$. Un rôle important est joué par la sous-algèbre des opérateurs *régularisants* donnés par les noyaux lisses sur $M \times M$. Un tel opérateur $R$ envoie toute distribution $f \in \mathcal{D}'(M)$ sur une fonction lisse $Rf \in C^\infty(M)$.

Comme pour les opérateurs différentiels, il existe une notion de *symbole principal* $\sigma_m(P)$ associé à un opérateur $P \in \Psi^m(M)$. Ce symbole définit une fonction sur le fibré sphérique cotangent $S^*M$, défini comme quotient de $T^*M \setminus \{0\}$ par l'action de $\mathbb{R}_+^*$, et induit une suite exacte

$$0 \longrightarrow \Psi^{m-1}(M) \longrightarrow \Psi^m(M) \xrightarrow{\sigma_m} C^\infty(S^*M) \longrightarrow 0$$

Si $\sigma_m(P)$ est inversible, donc si $P$ est elliptique, la théorie permet ainsi de construire une paramétrice $Q \in \Psi^{-m}(M)$ telle que

$$PQ = 1 + R \quad \text{et} \quad QP = 1 + R',$$

avec $R$ et $R'$ deux opérateurs régularisants. Puisque les opérateurs régularisants envoient $\mathcal{D}'(M)$ sur $C^\infty(M)$ et l'inclusion $C^\infty(M) \hookrightarrow H^s(M)$ est compacte pour tout $s \in \mathbb{R}$, on en déduit le caractère Fredholm de $P$ annoncé dans la partie précédente. On retrouve également la notion de régularité elliptique évoquée dans la partie 3.1.1 : si $Pf \in H^s(M)$ pour un $s \in \mathbb{R}$, alors $f = (QP - R')f$ est dans $H^{s+m}(M)$.

### 3.1.4. Le théorème de l'indice d'Atiyah et Singer

Une quantité essentielle associée à un opérateur de Fredholm $T$ est son *indice*, qui est l'entier relatif défini par

$$\operatorname{index} T = \dim(\ker T) - \dim(\operatorname{coker} T).$$

Une propriété remarquable de l'indice est son invariance par déformations continues de $T$ ; ceci n'est pas le cas pour $\dim(\ker T)$ ou $\dim(\operatorname{coker} T)$ pris indépendamment l'un de l'autre. Lorsque $P$ est un opérateur elliptique d'ordre $m$, son noyau et son conoyau ne contiennent que des fonctions lisses ; donc l'indice de $P$ comme opérateur $H^s(M) \to H^{s-m}(M)$ ne dépend pas de $s$. Cet indice ne dépend en fait que de la classe d'homotopie du symbole principal $\sigma_m(P)$.

Du point de vue des équations aux dérivées partielles, l'indice est une obstruction à ce que $P$ soit inversible, et donc à ce que l'équation

$$Pu = f$$

soit bien posée entre espaces de Sobolev. L'indice est également source d'invariants géométriques ou topologiques. Ainsi, la caractéristique d'Euler d'une variété orientée est l'indice de l'opérateur $d + d^*$ allant des formes différentielles de degré pair sur $M$ vers celles de degré impair. Sur des variétés complexes, la caractéristique d'Euler holomorphe est l'indice de l'opérateur de Dolbeault. La signature d'une variété, point de départ de la conjecture de Novikov importante en topologie, est un invariant d'homotopie orientée qui coïncide avec l'indice de l'opérateur de signature. Enfin, l'indice de l'opérateur de Dirac sur les variétés spin est une obstruction à l'existence de métriques de courbure scalaire positive [91, 161, 180, 182].

Un résultat majeur de la seconde moitié du XX$^{\mathrm{e}}$ siècle, annoncé en 1963, est l'obtention par Atiyah et Singer d'une formule générale pour calculer l'indice. Celle-ci généralise des résultats obtenus dans les années 1950, notamment la formule de Chern-Gauss-Bonnet et le théorème de Hirzebruch-Riemann-Roch.

**Théorème 3.1.4** (Atiyah, Singer [15, 16])**.** *Soit $M$ une variété compacte sans bord de dimension $n$, et $P$ un opérateur elliptique d'ordre $m$ sur $M$. Alors*

$$\operatorname{index} P = (-1)^n \int_{T^*M} \operatorname{ch}(\sigma_m(P)) \wedge \operatorname{Td}(T^*M \otimes \mathbb{C}).$$

Les expressions sous l'intégrale dénotent certaines classes de cohomologie (les classes caractéristiques de Chern et de Todd) dont nous ne détaillerons pas la construction ici. L'intérêt conceptuel est le suivant : la quantité analytique *globale* qu'est l'indice de $P$ est exprimée comme l'intégrale d'une quantité géométrique *locale*, construite à partir du symbole principal.

La théorie de l'indice a donné lieu à de nombreux raffinements et extensions. Dans leur série d'articles originaux [14, 16, 17], Atiyah, Singer et Segal ont défini et étudié un indice *équivariant* pour les opérateurs invariants sous l'action d'un groupe de Lie compact : nous reviendrons dessus partie 2.2. Les même auteurs ont considéré dans [18] l'indice d'une famille continue d'opérateurs indexée par un espace topologique $Y$ : l'indice est alors une différence formelle de fibrés vectoriels sur $Y$. Une généralisation directe consiste à étudier d'un opérateur qui agit le long d'un feuilletage sur une variété compacte : son indice vit alors dans la $K$-théorie du feuilletage, et a été calculé par Connes et Skandalis [56, 62]. Roe a introduit un indice « grossier » sur les variétés non compactes, qui donne une obstruction à l'existence de métriques de courbure scalaire positive [180]. Boutet de Monvel [33] a étudié l'indice de problèmes aux limites sur des domaines lisses. Enfin, l'étude de l'indice est liée à la conjecture de Baum-Connes, fondamentale au sein de la recherche sur les $C^*$-algèbres et dont la résolution, même partielle, a des conséquences géométriques profondes [27, 95, 153, 154, 191, 197].

Par ailleurs, de nombreux travaux [13, 46, 48, 50, 109, 115, 117, 136] ont porté sur l'extension du théorème d'Atiyah et Singer à des variétés non compactes. Un phénomène nouveau dans ce cas est l'apparition de termes non locaux dans les formules d'indice : c'est ce que nous allons tâcher d'expliquer maintenant.

## 3.2. Espaces singuliers et variété non compactes

La partie 3.1 illustre les propriétés essentielles des opérateurs elliptiques sur des domaines bornés lisses ou des variétés lisse compactes sans bord : les opérateurs elliptiques sont

Fredholm entre espaces de Sobolev, et les solutions de l'équation aux dérivées partielles associées ont toujours la régularité maximale possible. Nous allons voir maintenant que ces deux propriétés fondamentales ne sont plus vraies sur des espaces « singuliers ».

### 3.2.1. Perte de régularité sur les domaines à coins

En guise d'illustration, considérons un domaine polygonal $\Omega \subset \mathbb{R}^2$ (voir figure 3.1). Au vu de l'importance de ce type de domaine pour des problèmes d'ingénierie, ceux-ci ont été étudiés dès la fin des années 1960 notamment par Kondrat′ev [108], Maz′ya et Plamenevskiĭ [129], Grisvard [90], Dauge [68], ainsi que de nombreux autres auteurs.

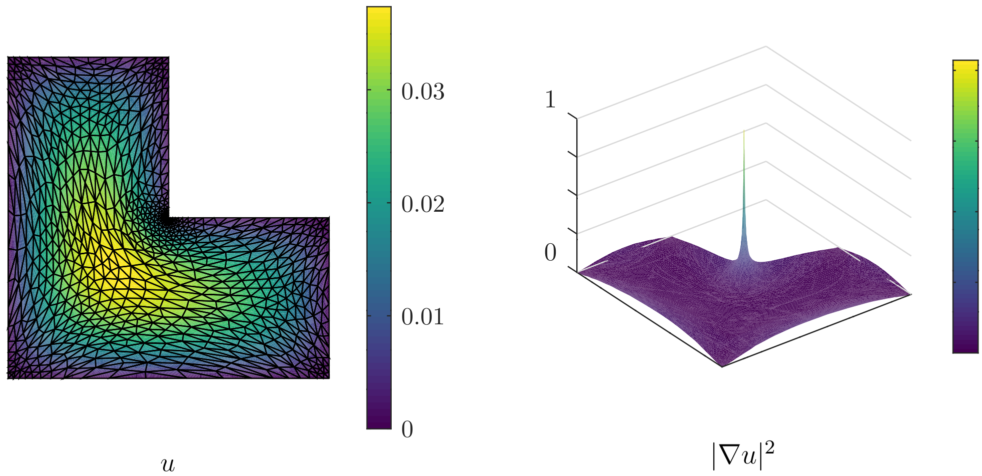


Figure 3.1. : Un domaine polygonal $\Omega$ avec un coin rentrant d'angle $3\pi/2$. La solution $u$ du problème de Poisson pour $f = 1$ a été approchée par la méthode des Éléments Finis. À gauche, la solution $u$ et la triangulation associée, qui a été raffinée près du coin rentrant. À droite, le carré de la norme du gradient de $u$ diverge près du coin rentrant, ce qui est lié au fait qu'on a seulement $u \in H^{5/3^-}(\Omega)$.

Considérons à nouveau le problème de Poisson (P) sur $\Omega$, c'est-à-dire l'équation

$$\begin{cases} \Delta u = f & \text{sur } \Omega, \\ u = 0 & \text{sur } \partial\Omega. \end{cases} \tag{P}$$

Si $f \in C^\infty(\overline{\Omega})$, les méthodes variationnelles classiques s'appliquent toujours pour montrer que (P) admet une unique solution faible $u \in H_0^1(\Omega)$. Cependant cette solution n'est a priori pas lisse sur $\overline{\Omega}$ : en fait, les travaux de Kondrat′ev montre qu'en général on a seulement $u \in H^{1+\pi/\alpha-\epsilon}(\Omega)$ pour tout $\epsilon > 0$, où $\alpha$ est le plus grand angle entre deux côtés adjacents du polygone.

Ce défaut de régularité, par rapport à ce que l'on pourrait attendre de la théorie elliptique classique, pose problème pour les méthodes numériques d'approximation de la solution par Éléments Finis. En effet, une connaissance a priori de la régularité de la solution $u$ est essentielle pour estimer la vitesse de convergence du schéma numérique : si $u \in H^m(\Omega)$ pour un $m \geq 0$, alors la méthode converge à vitesse $h^m$, avec $h$ la taille typique des éléments de la triangulation.

Les résultats de régularité se retrouvent cependant si on remplace les espaces de Sobolev usuels par des espaces *à poids*. Pour $m \in \mathbb{N}$ et $a \in \mathbb{R}$, on définit ainsi les espaces de Hilbert

$$\mathcal{K}_a^m(\Omega) = \{v : \Omega \to \mathbb{C} \mid r^{|\alpha|-a}\partial^\alpha v \in L^2(\Omega) \text{ pour tout } |\alpha| \leq m\}.$$

Le Laplacien s'étend en un opérateur borné $\Delta : \mathcal{K}_{a+1}^{m+1}(\Omega) \to \mathcal{K}_{a-1}^{m-1}(\Omega)$ pour tout $m, a$. Un phénomène important est que le caractère bien posé du problème (P) dans les espaces $\mathcal{K}_a^m(\Omega)$ est déterminé par une famille *d'opérateurs limites* près des coins du polygone. Si on dénote par $\alpha_1, \ldots, \alpha_k$ les angles que forment ces coins, cette famille est donnée par

$$I_j(\lambda) := \Delta_{[0,\alpha_j]} + \lambda^2,$$

pour $j = 1, \ldots, k$ et $\lambda \in \mathbb{C}$, et avec $\Delta_{[0,\alpha_j]}$ le Laplacien sur l'intervalle $[0, \alpha_j]$. À $j$ fixé, la famille $(I_j(\lambda))_{\lambda\in\mathbb{C}}$ est appelée crayon d'opérateurs[1] dans les travaux de Maz′ya et Plamenevskiĭ et *famille indicielle* par Melrose : je reviendrai dessus partie 3.4.1.

**Théorème 3.2.1** (Kondrat′ev [108] – Maz′ya, Plamenevskiĭ [129])**.** *Pour tout $m \geq 1$, l'opérateur*

$$\Delta : \mathcal{K}_{a+1}^{m+1}(\Omega) \cap H_0^1(\Omega) \to \mathcal{K}_{a-1}^{m-1}(\Omega) \tag{3.3}$$

*est Fredholm si, et seulement si, les opérateurs*

$$I_j(\lambda) : H^{m+1}[0, \alpha_j] \cap H_0^1[0, \alpha_j] \to H^{m-1}[0, \alpha_j]$$

*sont inversibles pour tout $j = 1, \ldots, k$ et pour tout $\lambda \in \mathbb{C}$ sur la droite* $\operatorname{Im}\lambda = b$.

Le caractère bien posé de (P) dans les espaces à poids dépend ainsi des spectres des problèmes de Poisson sur les intervalles $[0, \alpha_j]$, pour $j = 1, \ldots, k$, qui sont connus explicitement. Du point de vue des méthodes numériques, l'utilisation de ces espaces $\mathcal{K}_a^m(\Omega)$ permet de retrouver des taux de convergence quasi-optimaux dans les méthodes aux éléments finis, en utilisant des raffinements appropriés des triangulations du domaine près des coins [21, 120].

Dans le cas du problème de Poisson pour le Laplacien, il y a des manières plus directes d'obtenir le caractère bien posé dans les espaces $\mathcal{K}_a^m(\Omega)$ [5, 130, 133]. Néanmoins le Théorème 3.2.1 est très général et illustre un phénomène qu'on retrouvera sous de nombreuses formes plus loin : sur un domaine singulier, l'ellipticité seule ne suffit plus pour caractériser les opérateurs de Fredholm et doit s'accompagner d'une condition d'inversibilité de certains opérateurs limites, localisés près des singularités.

[1] « *operator pencil* »

### 3.2.2. Correspondance entre espaces singuliers et variétés complètes

L'approche usuelle pour obtenir des résultats tels le théorème 3.2.1 sur des domaines en deux dimensions, qu'ils soient à coins ou avec des singularités plus générales, repose sur un changement conforme de métrique près des coins qui envoie la singularité à l'infini et transforme le domaine en une variété à bords non compacte, mais complète.

On peut illustrer ceci sur l'exemple général des variétés *à singularité coniques isolées*. L'étude générale de la théorie des opérateurs elliptiques sur ces espaces, en particulier de la théorie de Hodge, a été initiée par Cheeger dès la fin des années 1970 [53, 54]. Lorsque la singularité est unique, il s'agit d'un espace topologique compact $X$ qui se représente comme sur la Figure 3.2 : on a une décomposition $X = M_0 \cup \{x_0\}$, où $M_0$ est une variété lisse et $x_0$ admet un voisinage $V$ homéomorphe à

$$V \simeq [0,1) \times L/\{0\} \times L$$

Ici $L$ est une variété lisse sans bord, et on suppose que l'identification ci-dessus se restreint en un difféomorphisme $V \cap M_0 \simeq (0,1) \times L$. On munit $M_0$ d'une métrique riemannienne (incomplète) $g_0$ qui s'écrit $dr^2 + r^2 g_L$ sur $(0,1) \times L$, avec $r$ la coordonnée dans $(0,1)$ et $g_L$ une métrique sur $L$.

Le changement conforme $g := r^{-2} g_0$ définit alors une métrique complète sur $M_0$. En fait, un changement de variable $t := \ln r$ envoie le voisinage $(0,1) \times L$ sur un cylindre infini $C = (-\infty, 0) \times L$, où l'image de $g$ est la métrique produit $dt^2 + g_L$. On peut ainsi dire que l'étude des variétés coniques est équivalente à celle des variétés *à bouts cylindriques*, comme l'illustre la Figure 3.2.

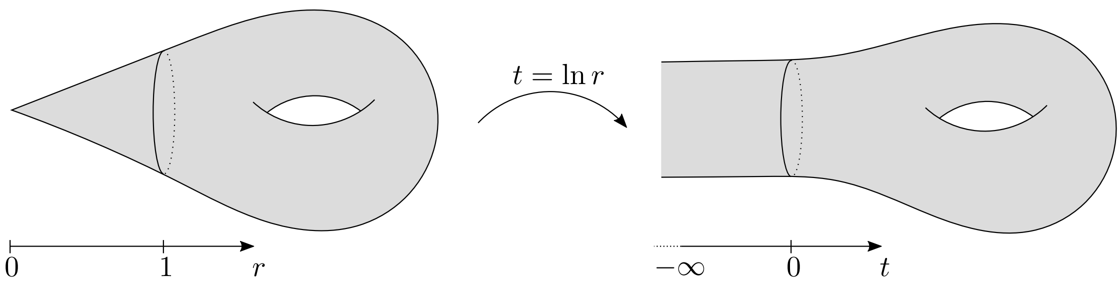


FIGURE 3.2. : Variétés coniques et à bouts cylindriques

On peut imaginer beaucoup d'autres classes de singularités que les coins évoqués jusque là : on parle par exemple de variétés à *cusps*, *edges*, ou de *pseudovariétés stratifiées* qui mélangent plusieurs types de singularités de dimensions différentes ; voir la Figure 3.3. Outre les applications à des problèmes physiques, l'étude de ces différents types de singularités est motivée par leur occurrence dans l'étude des variétés algébriques singulières [41, 88, 179]. Via un changement conforme de métrique, l'étude d'un tel domaine singulier $(M_0, g_0)$ se ramène à l'étude de l'analyse sur une variété riemannienne $(M_0, g)$ complète et non compacte, avec ou sans bord. Les espaces à poids sur $(M_0, g_0)$ évoqués dans la partie 3.2.1 correspondent aux espaces de Sobolev usuels sur $(M_0, g)$, le poids étant donné par le facteur conforme qui permet de passer de $g_0$ à $g$. La question générale est donc la suivante :

**Question.** *Dans quelle mesure peut-on étendre les résultats de la théorie elliptique classique vus en partie 3.1 (caractérisation des opérateurs de Fredholm, théorie d'indice, etc.) à des variétés non compactes ?*

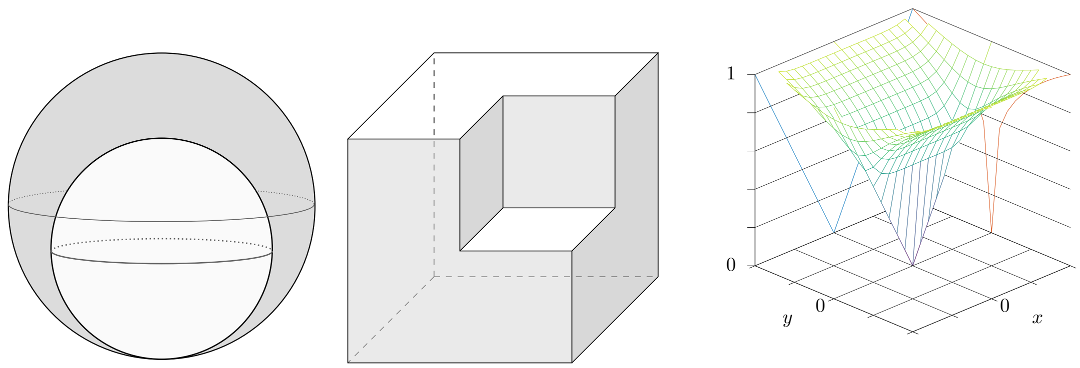


FIGURE 3.3. : De gauche à droite : deux boules imbriquées l'une dans l'autre forment un cusp « fibré » ; un polyèdre est un exemple de pseudovariété stratifiée avec des singularités de types « coins » et « bords » ; la variété algébrique réelle donnée par $x^8 + y^2 = z^8$ possède un cusp anisotropique : son intersection avec le plan $y = 0$ est un cône, celle avec le plan $x = 0$ est un cusp.

Pour avoir une chance raisonnable d'obtenir des résultats, on se restreint à certaines classes particulières de variétés riemannienne. On suppose usuellement que $(M_0, g)$ est de *géométrie bornée*[1], ce qui permet d'obtenir des estimées uniformes [92, 93]. La plupart des résultats fins supposent de plus que $M_0$ est l'intérieur d'une variété $M$ à bord, ou plus généralement à coins, et que la métrique $g$ prend une forme bien particulière sur un voisinage de $\partial M$ : c'est le cas par exemple des variétés à bouts cylindriques, sur lesquelles je reviendrai partie 3.4

Comme évoqué plus haut, plusieurs phénomènes nouveaux distinguent particulièrement et compliquent la théorie des opérateurs elliptiques sur les variétés non compactes du cas compact :

1. les opérateurs régularisants n'étant pas nécessairement compacts, l'ellipticité ne suffit plus à caractériser les opérateurs de Fredholm : il faut ajouter une condition « d'inversibilité à l'infini » qui est souvent non locale ;
2. en conséquence, le spectre des opérateurs elliptique comporte toujours une partie continue, c'est-à-dire que leur spectre essentiel est non-vide ;

[1] c'est-à-dire que toutes les dérivées covariantes du tenseur de courbure sont bornées et que le rayon d'injectivité de $g$ est strictement positif.

3. l'indice des opérateurs de Fredholm étant en partie déterminé par leur action « à l'infini », les formules d'indice connues font apparaître des termes non locaux (voir partie 3.4.2).

## 3.3. L'utilisation des algèbres d'opérateurs

La théorie des algèbres d'opérateurs constitue un outil majeur de cette thèse, en particulier l'utilisation des $C^*$-algèbres. Cette partie vise à justifier pourquoi ces dernières apparaissent naturellement en lien avec l'étude des opérateurs elliptiques et des problèmes d'indice.

### 3.3.1. $C^*$-algèbre des symboles sur une variété compacte

La théorie des $C^*$-algèbres s'est développée dans la première moitié du XX[e] siècle à partir notamment des travaux de Gelfand et Naimark [83]. On peut les définir de manière simple en disant que, si $\mathcal{H}$ est un espace de Hilbert, alors toute sous-algèbre involutive de $\mathcal{B}(\mathcal{H})$ qui est fermée pour la topologie de la norme d'opérateur constitue une $C^*$-algèbre. Une propriété remarquable est que ces algèbres peuvent être définies de manière abstraite par un nombre réduit d'axiomes, et possèdent ainsi une structure extrêmement rigide[1]. Les $C^*$-algèbres occupent une place d'importance au sein du programme de géométrie non commutative d'Alain Connes : elles y constituent les « espaces topologiques non commutatifs » [59, 60].

La théorie des opérateurs de Fredholm et de leur indice est fortement liée à celles des $C^*$-algèbres. La raison est la suivante : l'idéal des opérateurs compact $\mathcal{K}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ forme une $C^*$-algèbre, ainsi que l'algèbre quotient $\mathcal{Q}(\mathcal{H}) := \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Comme évoqué en partie 3.1.2, un opérateur $T \in \mathcal{B}(\mathcal{H})$ est Fredholm si, et seulement si, il est inversible modulo $\mathcal{K}(\mathcal{H})$ ; donc si, et seulement si, son image dans l'algèbre $\mathcal{Q}(\mathcal{H})$ est inversible. Qui plus est, les $C^*$-algèbres possèdent une théorie cohomologique dont on peut dire qu'elle est « faite » pour l'étude de l'indice : c'est la $K$-théorie[2]. L'indice de $T$ est ainsi donné par un morphisme naturel d'un groupe de $K$-théorie dans un autre :

$$\partial : K_1(\mathcal{Q}(\mathcal{H})) \to K_0(\mathcal{K}(\mathcal{H})) \simeq \mathbb{Z}.$$

Considérons à nouveau une variété compacte sans bord $M$. Pour étudier les opérateurs différentiels sur $M$, la $C^*$-algèbre intéressante est la clôture $\overline{\Psi^0}(M)$ des opérateurs pseudodifférentiels d'ordre 0 dans $\mathcal{B}(L^2(M))$. En effet, si $P \in \Psi^m(M)$, alors $P(1+P^*P)^{1/2} \in \overline{\Psi^0}(M)$ : ceci justifie de toujours se ramener à l'ordre 0. Puisque la fermeture de $\Psi^{-1}(M)$ coïncide avec les compacts sur $L^2(M)$, le symbole principal induit une suite exacte de $C^*$-algèbres

$$0 \longrightarrow \mathcal{K} \longrightarrow \overline{\Psi^0}(M) \xrightarrow{\sigma_0} C(S^*M) \longrightarrow 0, \tag{3.4}$$

[1]Les points essentiels de la théorie des $C^*$-algèbre sont donnés partie 4.1.

[2]Introduite par Grothendieck en géométrie algébrique, celle-ci a été étendu au cadre topologique par Atiyah et Hirzebruch précisément pour attaquer la preuve du théorème de l'indice [16].

où $\mathcal{K}$ dénote simplement $\mathcal{K}(L^2(M))$.

Une manière d'interpréter cette suite exacte est de dire que le symbole principal mesure la distance aux opérateurs compacts dans $\Psi^0(M;E)$. Autrement dit, si $P \in \Psi^0(M;E)$, alors

$$\inf_{K\in\mathcal{K}(L^2(M))} \|P-K\| = \|\sigma_0(P)\|_\infty.$$

La suite exacte (3.4) est une nouvelle façon de dire que l'ellipticité caractérise les opérateurs de Fredholm sur une variété compacte sans bord : le symbole principal $\sigma_0(P)$ est inversible si, et seulement si, l'opérateur $P$ est inversible dans l'algèbre $\overline{\Psi^0}(M)/\mathcal{K}$.

### 3.3.2. Algèbres de comparaison et variétés non compactes

Il est difficile d'étendre directement cette approche à une variété riemannienne $M_0$ non compacte, pour la simple raison que les opérateurs différentiels d'ordre 0 ne sont pas *a priori* bornés sur $L^2(M_0)$, et les opérateurs régularisants ne sont pas nécessairement compacts. On doit ainsi se restreindre à une certaine sous-algèbre de $\overline{\Psi^0}(M_0)$ en imposant un contrôle sur le comportement des opérateurs à l'infini.

Puisqu'il semble déraisonnable d'étudier tous les opérateurs différentiels sur $M_0$ : il suffit de se restreindre à une sous-algèbre d'opérateurs « intéressants ». Supposons par exemple que $M_0$ soit une variété à bout cylindrique, c'est-à-dire comme en partie 3.2.2 que $M_0$ est l'intérieur d'une variété à bord $M$ munie d'une métrique $g$, qui dans un voisinage tubulaire du bord $\partial M \times [0,1)$ s'écrit

$$g = \frac{dr^2}{r^2} + g_{\partial M},$$

avec $r$ la coordonnée dans $[0,1)$.

Dans ce cas, les opérateurs géométriques tels le Laplacien, les opérateurs de type Dirac, etc. sont engendrés par l'algèbre de Lie des champs de vecteurs uniformément bornés sur $M_0$ et lisses jusqu'au bord, soit

$$\mathcal{V}_b := \{X \in C^\infty(M;TM) \mid dr(X) = 0 \text{ sur } \partial M\}.$$

Autrement dit, il s'agit des champs de vecteurs qui s'annulent dans la direction normale sur $\partial M$. Dans des coordonnées locales $(r, y_1, \ldots, y_n)$ sur $[0,1) \times \partial M$, les opérateurs différentiels (scalaires) correspondants s'écrivent comme

$$P = \sum_{|\alpha|\leq m} a_\alpha (r\partial_r)^{\alpha_0} \partial_{y_1}^{\alpha_1} \cdots \partial_{y_n}^{\alpha_n}, \tag{3.5}$$

avec des coefficients $a_\alpha \in C^\infty(M)$.

Une idée naturelle pour étudier les opérateurs ci-dessus, est de considérer la $C^*$-algèbre $A$ engendrée dans $\mathcal{B}(L^2(M_0))$ par les opérateurs de type $P(1+\Delta)^{-m/2}$, avec $P$ d'ordre $m$ qui s'écrit comme en (3.5) et $\Delta$ le Laplacien de la métrique $g$. L'algèbre obtenue joue alors le rôle des opérateurs d'ordre 0 et a été appelée *algèbre de comparaison* dans les travaux de Cordes et McOwen [63]. Une des premières occurrence de ces algèbres peut se trouver dans la thèse de Taylor [192]. Une approche distincte, mais reliée, consiste à ne construire que les opérateurs d'ordre $-1$ : on obtient ainsi une $C^*$-algèbre $A \subset \mathcal{B}(L^2(M_0))$ qui contient les résolvantes des opérateurs uniformément elliptiques[1] s'écrivant comme en (3.5). On retrouve cette idée dans les travaux de Bellissard [28], Georgescu et Iftimovici [84, 85], ainsi que Măntoiu, Purice et leurs collaborateurs [9, 99, 124, 125, 126, 127], parmi d'autres [7, 8, 142].

Quelle que soit l'approche choisie pour construire cette $C^*$-algèbre $A$, caractériser les opérateurs de Fredholm revient dans les deux cas à comprendre le quotient $A/\mathcal{K}$, en particulier ses représentations. Une difficulté importante est que ce quotient n'est en général pas commutatif, contrairement au cas des variétés compactes sans bord. On peut bien sûr considérer d'autre types de métriques que celle des variétés à bouts cylindriques : l'algèbre des opérateurs différentiels « intéressants » est alors modifiée en conséquence, et la théorie des représentations de $A/\mathcal{K}$ reflète la géométrie de $M_0$ à l'infini.

### 3.3.3. Produits croisés et opérateurs de Schrödinger

En guise d'exemple, considérons une version simplifiée du problème à $N$ corps sur $\mathbb{R}^n$. Comme on va le voir, la détermination du quotient $A/\mathcal{K}$ introduit dans la section précédente redonne la notion *d'opérateurs limites* discutée partie 3.2.1.

Soit $H$ un opérateur de Schrödinger donné par

$$H = \Delta + V,$$

avec $\Delta$ le Laplacien sur $\mathbb{R}^n$ et $V \in C^\infty(\mathbb{R}^n)$ qui joue le rôle d'un potentiel. Faisons l'hypothèse que $\lim_{\lambda \to +\infty} V(\lambda\omega)$ existe pour tout $\omega \in \mathbb{S}^{n-1}$, et que ces limites dépendent continûment de $\omega$.

Une $C^*$-algèbre $A$ qui va contenir les résolvantes des opérateurs $H$ ci-dessus peut être construite à partir de la structure de groupe sur $\mathbb{R}^n$. Considérons ainsi la compactification sphérique $\mathbb{S}^n_+ \simeq \mathbb{R}^n \sqcup \mathbb{S}^{n-1}$ de $\mathbb{R}^n$. On obtient alors $A$ comme un *produit croisé*

$$A = C(\mathbb{S}^n_+) \rtimes \mathbb{R}^n,$$

induit par l'action de $\mathbb{R}^n$ sur lui-même par translations, qui s'étend en une action triviale sur la frontière $\partial\mathbb{S}^n_+ \simeq \mathbb{S}^{n-1}$. Pour une action de groupe aussi simple, la structure de $A$ est très bien comprise [76, 207]. En particulier, on a un isomorphisme

$$A/\mathcal{K} \simeq C(\partial\mathbb{S}^n_+) \rtimes \mathbb{R}^n,$$

et comme $\mathbb{R}^n$ agit trivialement sur $\partial\mathbb{S}^n_+$, cette dernière $C^*$-algèbre est commutative et isomorphe à $C_0(\partial\mathbb{S}^n_+ \times \mathbb{R}^n)$.

[1] c'est-à-dire tels que $\sigma_m(P)(\xi)$ est inversible pour tout $\xi \in S^*M_0$ et $|\sigma(P)(\xi)|^{-1}$ est uniformément bornée sur $S^*M_0$.

Cette structure algébrique de l'algèbre des résolvantes a des conséquences directes sur l'opérateur $H$. En particulier, son spectre essentiel[1] est déterminé par

$$\mathrm{Spec}_{\mathrm{ess}}(H) = \bigcup_{\omega \in \mathbb{S}^{n-1}} \mathrm{Spec}(H_\omega),$$

avec $H_\omega$ *l'opérateur limite* sur $\mathbb{R}^n$ donné par $H_\omega = \Delta + \tilde{V}(\omega)$ et $\tilde{V}(\omega) = \lim_{\lambda \to +\infty} V(\lambda\omega)$. Ce résultat, proche du théorème HVZ bien connu en physique mathématique, s'applique à une classe très large d'opérateurs qui inclut certains opérateurs hypoelliptiques [85]. En remplaçant $\mathbb{S}^n_+$ par d'autres compactifications de $\mathbb{R}^n$, on peut étendre ce résultat à des potentiels bien plus généraux [84, 85, 124, 125, 126, 127, 142].

## 3.4. Calculs pseudodifférentiels adaptés aux singularités

La partie précédente illustre l'utilisation de la théorie des $C^*$-algèbres dans l'étude de l'analyse sur une variété non compacte. Une difficulté est de construire, dans une situation géométrique donnée, une $C^*$-algèbre appropriée à l'étude des opérateurs elliptiques. On va voir dans cette partie comment la construction d'un calcul pseudodifférentiel adapté à la géométrie répond à ce problème.

### 3.4.1. Un exemple : le $b$-calcul sur une variété à bouts cylindriques

Comme précédemment, l'exemple qui va nous guider sera celui des variétés à bouts cylindriques, notamment car cette géométrie est très bien comprise depuis les années 1990. Comme en partie 3.2.2 et 3.3.2, on considère une variété $M$ à bord, et on munit son intérieur $M_0$ d'une métrique $g$ qui s'écrit dans un voisinage tubulaire $\partial M \times [0,1)$ du bord comme

$$g = \frac{dr^2}{r^2} + g_{\partial M},$$

avec $r$ la coordonnée dans $[0,1)$.

J'ai souligné partie 3.3.2 que les opérateurs différentiels « intéressants » sur $M_0$, en particulier les opérateurs géométriques, sont engendrés par l'algèbre de Lie des champs de vecteurs

$$\mathcal{V}_b(M_0) := \{X \in \Gamma(TM) \mid dr(X) = 0 \text{ sur } \partial M\}. \tag{3.6}$$

La question de la construction d'un calcul pseudodifférentiel contenant $\mathcal{V}_b(M_0)$ et ayant de bonnes propriétés a été posée dès les années 1980 dans les travaux de Melrose [134, 136], Plamenevskiĭ [163], Rempel et Schulze [172, 184], ainsi qu'Unterberger [199]. Les relations précises entre les calculs de Melrose et Schulze ont été étudiées en détails par Lauter et Seiler [114].

Je choisis de me concentrer sur l'approche de Melrose, car c'est celle qui présente le plus de similarité avec l'utilisation des groupoïdes dans cette thèse. Celle-ci repose sur

[1] Le spectre essentiel de $H$ est l'ensemble $\mathrm{Spec}_{\mathrm{ess}}(H) \subset \mathrm{Spec}(H)$ des $\lambda \in \mathbb{C}$ tels que $H - \lambda$ n'est pas Fredholm. Le complémentaire de $\mathrm{Spec}_{\mathrm{ess}}(H)$ dans le spectre de $H$ est l'ensemble des valeurs propres isolées de multiplicités finies.

l'utilisation d'un espace « éclaté[1] » $M_b^2$ qui agrandit $M^2$ pour inclure plus d'informations près du coin $(\partial M)^2$. Dans le cas où $\partial M$ est connexe, on définit cet espace comme

$$M_b^2 := M^2 \setminus (\partial M)^2 \bigsqcup (\partial M)^2 \times \mathbb{R}_+^*,$$

c'est-à-dire qu'on élargit $(\partial M)^2$ en le remplaçant par $(\partial M)^2 \times \mathbb{R}_+^*$. L'espace $M_b^2$ est une variété à coins qui possède trois faces, comme illustré Figure 3.4 : les deux « anciennes » faces rb et lb, et la nouvelle face bf qui est la fermeture de $(\partial M)^2 \times \mathbb{R}_+^*$.

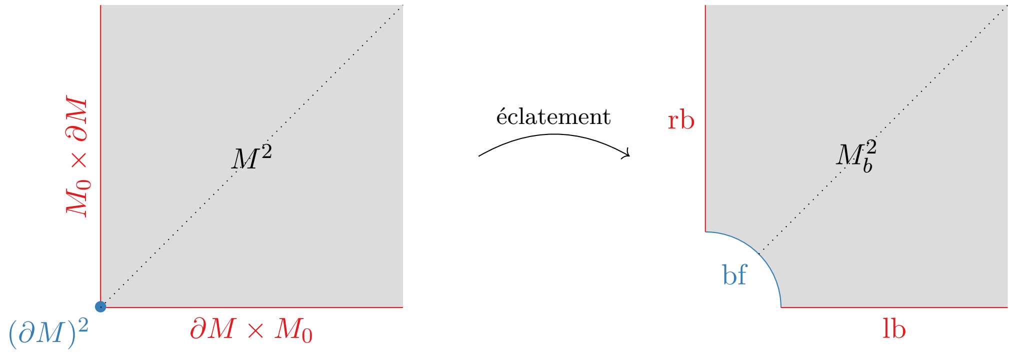


FIGURE 3.4. : L'espace éclaté $M_b^2$, obtenu en remplaçant $(\partial M)^2$ par $(\partial M)^2 \times \mathbb{R}_+^*$. L'inclusion de la diagonale $\Delta_M$ dans $M_b^2$ est soulignée en pointillés.

Par analogie avec le cas compact, le (petit) $b$-calcul de Melrose est défini comme l'ensemble $\Psi_b^\infty(M)$ des opérateurs $P : C_c^\infty(M_0) \to C_c^\infty(M_0)$, dont le noyau de Schwartz s'étend en une distribution sur $M_b^2$ conormale à la diagonale $\Delta_M$ (en pointillés sur la Figure 3.4), et s'annulant à tout ordre sur les faces lb et rb.

### 3.4.2. Famille indicielle et formule de l'indice

Le $b$-calcul est construit de telle sorte qu'un opérateur uniformément elliptique sur $M_0$ (par exemple, le Laplacien pour la métrique $g$) admette un inverse modulo les opérateurs régularisants $\Psi_b^{-\infty}(M)$, c'est-à-dire les opérateurs dont le noyau de Schwartz est une fonction lisse sur $M_b^2$. Un tel opérateur n'est compact sur $L^2(M_0)$ que « s'il n'agit pas à l'infini » au sens où la restriction de son noyau à la nouvelle face bf est nulle.

Étant donné un opérateur $P \in \Psi_b^\infty(M)$, Melrose exploite cette observation pour définir une famille holomorphe d'opérateurs $I(P,\lambda)_{\lambda\in\mathbb{C}} \subset \Psi^m(\partial M)$, induite par la restriction du noyau de Schwartz de $P$ à bf $\simeq (\partial M)^2 \times \mathbb{R}_+^*$, et qui détermine le caractère Fredholm de $P$. Si $P$ est un opérateur différentiel s'écrivant dans des coordonnées locales

$$P = \sum_{|\alpha|\leq m} a_\alpha(r,y)(r\partial_r)^{\alpha_0}\partial_{y_1}^{\alpha_1}\cdots\partial_{y_n}^{\alpha_n},$$

[1] « *blown-up space* »

comme en (3.5), alors sa *famille indicielle* est donnée par

$$I(P, \lambda) = \sum_{|\alpha| \leq m} a_\alpha(0, y) \lambda^{\alpha_0} \partial_{y_1}^{\alpha_1} \cdots \partial_{y_n}^{\alpha_n}, \tag{3.7}$$

pour tout $\lambda \in \mathbb{C}$.

**Théorème 3.4.1** (Melrose [136]). *Pour tout $P \in \Psi_b^m(M)$, l'opérateur*

$$P : r^a H^s(M_0, g) \to r^a H^{s-m}(M_0, g)$$

*est Fredholm si, et seulement si :*

*(i) $P$ est uniformément elliptique sur $M_0$,*
*(ii) l'opérateur $I(P, \lambda) : C^\infty(\partial M) \to C^\infty(\partial M)$ est inversible pour tout $\lambda \in \mathbb{C}$ sur la droite* $\operatorname{Re} \lambda = a$.

Ce théorème est similaire au résultat classique de Kondrat′ev sur les domaines polygonaux donné partie 3.2.1. Si par exemple $P = \Delta_g$, alors

$$I(P, \lambda) = \Delta_{\partial M} + \lambda^2,$$

avec $\Delta_{\partial M}$ le Laplacien de la métrique $g_{\partial M}$. Puisque le spectre de ce dernier opérateur est discret, le Laplacien $\Delta_g$ est Fredholm sur presque tous les espaces à poids. Qui plus est, la théorie donne des formules « d'indices relatifs » lors du franchissement d'un point critique en fonction des résidus de la famille méromorphe $\lambda \mapsto (I(P, \lambda))^{-1}$.

Quant à l'indice global d'un opérateur $P \in \Psi_b^m(M)$, il est déterminé à la fois par le symbole principal de $P$ et sa famille indicielle, puisque ce sont exactement ces objets qui déterminent l'inversibilité de $P$ modulo $\mathcal{K}$. Pour les opérateurs d'ordre 1 de type Direca, ceci est illustré par le théorème de l'indice d'Atiyah, Patodi et Singer établi dans les années 1970 [13]. Ainsi, supposons que $(M_0, g)$ possède une structure spin « exacte » et soit $\not{D}^+$ l'opérateur de Dirac correspondant. Dans ce cas $\not{D}^+$ est Fredholm si, et seulement si, l'opérateur de Dirac $\not{D}_\partial^+$ induit par la structure spin sur $\partial M$ est inversible.

**Théorème 3.4.2** (Atiyah, Patodi, Singer [13] – Melrose [136]). *Supposons que $\not{D}_\partial^+$ soit inversible. Alors*

$$\operatorname{index}(\not{D}^+) = \int_M \hat{A}(M) - \frac{1}{2} \eta(\not{D}_\partial^+). \tag{3.8}$$

La formule ci-dessus est un modèle de ce qu'on peut espérer pour un théorème d'indice sur des variétés non compactes exhibant d'autres géométries : l'intégrande de gauche est le terme local qu'on obtient par la formule d'indice classique sur une variété compacte sans bord (Théorème 3.1.4) ; le terme correctif $\eta$ est un invariant non local qui dépend uniquement du spectre de $\not{D}_\partial^+$, donc du comportement de $\not{D}^+$ à l'infini. L'obtention de formules analogues à (3.8) est une motivation essentielle de la théorie elliptique sur les variétés non compactes [46, 50, 82, 109, 115, 117, 200, 201].

La preuve du théorème 3.4.2 par Atiyah, Patodi et Singer passe par l'étude de la trace du noyau de la chaleur [13]. Il est à noter qu'une version purement cohomologique de la formule (3.8) a été obtenue récemment par Carrillo-Rouse, Lescure et Monthubert [48], en utilisant une réinterprétation du $b$-calcul comme calcul sur un groupoïde : ceci devrait suffire à motiver la partie 3.5.

## 3.5. Les groupoïdes comme désingularisation

Le $b$-calcul de Melrose introduit partie 3.4 n'est qu'un exemple des nombreux types de calculs pseudodifférentiels qui ont été construits sur des variétés non compactes, chacun pour une géométrie particulières. Le recours aux groupoïdes permet de comprendre ces constructions dans un cadre géométrique unifié, qui s'articule particulièrement bien avec l'utilisation des algèbres d'opérateurs. Un bon aperçu de cette approche peut se trouver dans une série de conférences données par Nistor [149].

### 3.5.1. Intégration des algébroïdes de Lie

Melrose a initié en 1990 [135] le projet qui consiste à étendre la construction du $b$-calcul dans un cadre géométrique général : étant donnée une variété à coins $M$ et une algèbre de Lie de champs de vecteurs $\mathcal{V} \subset \Gamma(TM)$ vérifiant certaines hypothèses naturelles, peut-on construire un calcul pseudodifférentiel qui contient $\mathcal{V}$ et vérifie de bonnes propriétés ?

Ammann, Lauter et Nistor [6] ont donné à cette question une réponse affirmative. Leur construction repose sur un résultat algébrique de Serre et Swan qui affirme l'existence d'un fibré $A_\mathcal{V} \to M$ dont les sections sont isomorphes à $\mathcal{V}$. Le crochet de Lie sur $\mathcal{V}$ donne à $A_\mathcal{V}$ une structure d'*algébroïde de Lie*, un objet introduit par Pradines dans les années 1960 [164]. Sa motivation était de construire la théorie infinitésimale d'objets définis antérieurement par Ehresmann : les *groupoïdes de Lie*[1] [78]. Dit brièvement, un groupoïde de Lie $\mathcal{G}$ sur $M$ est une variété dont la structure possède de fortes similarité avec celle d'un groupe et munie de deux submersions vers $M$ (« source » et « but »), qui encodent certaines relations entre les éléments de $M$ : on note ainsi $\mathcal{G} \rightrightarrows M$. Les définitions précises seront données en partie 4.3.

**Exemple 3.5.1.** Le graphe de l'action d'un groupe de Lie $G$ sur $M$ est un groupoïde de Lie, difféomorphe comme variété à $M \times G$.

**Exemple 3.5.2.** L'ensemble des classes d'homotopies à extrémités fixées de chemins sur $M$ forme un groupoïde de Lie sur $M$, qu'on appelle *groupoïde fondamental.*

À chaque groupoïde de Lie $\mathcal{G} \rightrightarrows M$ est associé une algébroïde de Lie $A\mathcal{G} \to M$ qui encode le comportement infinitésimal du groupoïde. Une question naturelle est de savoir s'il existe un troisième théorème de Lie dans ce contexte : tous les algébroïdes de Lie sont-ils intégrables par un groupoïde ? Crainic et Fernandes ont répondu par la négative au début des années 2000 en exhibant une obstruction à l'intégrabilité [66]. Mais les algébroïdes considérés dans cette thèse sont particuliers : Debord [69] a montré qu'ils sont toujours intégrables et qu'il existe un plus petit groupoïde $\mathcal{G}_\mathcal{V} \rightrightarrows M$ tel que $A\mathcal{G}_\mathcal{V} \simeq A_\mathcal{V}$.

L'utilisation des groupoïdes dans cette thèse est proche du rôle qu'ils occupent dans l'étude des variétés feuilletées. Le groupoïde important dans ce contexte est le *groupoïde d'holonomie*[2] qui encode le feuilletage dans un objet lisse : voir la Figure 3.5. Connes et

[1] appelés « groupoïdes différentiables » par Ehresmann.

[2] introduit dans les années 1960 par Ehresmann [79], sa structure lisse a été construite ultérieurement par Pradines et Winkelnkemper [165, 208].

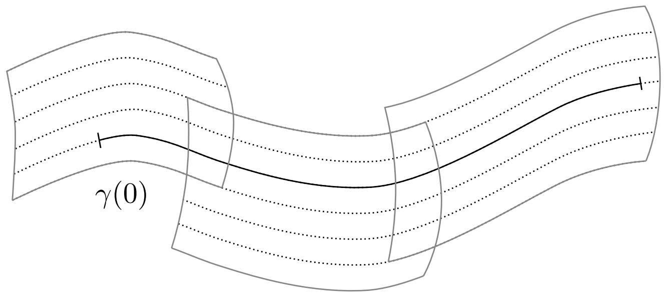


FIGURE 3.5. : Un feuilletage régulier est une partition localement triviale en sous-variétés de même dimension. Le groupoïde d'holonomie est constitué de classes d'équivalences de chemins continus tracés le long des feuilles.

Skandalis ont montré comment utiliser ce groupoïde pour étudier l'analyse « le long des feuilles » et en ont déduit un théorème de l'indice pour les feuilletages réguliers dans les années 1980 [56, 62]. Le groupoïde $\mathcal{G}_{\mathcal{V}}$ qui nous intéresse est le groupoïde d'holonomie du feuilletage *singulier* défini par $\mathcal{V}$ sur $M$ : il s'agit d'un cas particulier d'une construction générale donnée plus récemment par Androulidakis et Skandalis [10, 11].

Les groupoïdes entretiennent également des liens forts avec les formules d'indices. Ainsi, Connes a donné dans les années 1990 une preuve purement géométrique du théorème de l'indice d'Atiyah et Singer, en construisant un groupoïde *tangent* qui encode en un seul objet la déformation d'un opérateur pseudodifférentiel vers son symbole principal [60]. Un fait remarquable observé récemment par Debord et Skandalis [72] et étendu par van Erp et Yuncken [202, 203] est qu'il est possible de *définir* les opérateurs pseudodifférentiels classiques sur une variété à partir de son groupoïde tangent, sans utiliser de coordonnées.

### 3.5.2. Produits de convolution et opérateurs pseudodifférentiels

Étant donnée une algèbre de Lie $\mathcal{V}$ de champs de vecteurs correspondant à une géométrie donnée, la structure de groupoïde de Lie de l'objet $\mathcal{G}_{\mathcal{V}}$ qui intègre $\mathcal{V}$ offre plusieurs avantages. Premièrement, il existe un calcul pseudodifférentiel défini de manière général sur les groupoïdes de Lie. Celui-ci a été d'abord introduit par Connes [56] pour le cas particulier du groupoïde d'holonomie, puis généralisé indépendamment par Monthubert [140] et Nistor, Weinstein et Xu [152]. Le calcul pseudodifférentiel sur $\mathcal{G}_{\mathcal{V}}$ contient en particulier les opérateurs différentiels engendrés par $\mathcal{V}$. Le fibré dual $(A\mathcal{G}_{\mathcal{V}})^*$ prend le rôle joué usuellement par le fibré cotangent $T^*M$ : le symbole principal donne ainsi une suite exacte

$$0 \longrightarrow \Psi^{-1}(\mathcal{G}_{\mathcal{V}}) \longrightarrow \Psi^{0}(\mathcal{G}_{\mathcal{V}}) \xrightarrow{\sigma_0} C^{\infty}(S^*\mathcal{G}_{\mathcal{V}}) \longrightarrow 0,$$

avec $S^*\mathcal{G}_{\mathcal{V}}$ le fibré sphérique de $(A\mathcal{G}_{\mathcal{V}})^*$.

Deuxièmement, la structure de groupoïde indique la façon dont les noyaux des opérateurs pseudodifférentiels doivent être composés entre eux. Elle spécifie en particulier un produit de convolution[1] sur $C_c^\infty(\mathcal{G}_\mathcal{V})$, qu'on peut définir de manière formelle par analogie avec la convolution sur un groupe :

$$(f * g)(\gamma) := \int_{\gamma_1\gamma_2=\gamma} f(\gamma_1)g(\gamma_2),$$

pour $f, g \in C_c^\infty(\mathcal{G}_\mathcal{V})$ et $\gamma \in \mathcal{G}_\mathcal{V}$. Ce produit s'étend aux noyaux des opérateurs pseudo-différentiels et donne ainsi une représentation de $\Psi^0(M_0)$ comme opérateurs bornés sur $L^2(M_0)$.

**Exemple 3.5.3.** Si $M$ est une variété compacte sans bord et $\mathcal{V} = \Gamma(TM)$ est l'algèbre de Lie de tous les champs de vecteurs, alors $A_\mathcal{V} = TM$. Le groupoïde qui intègre $TM$ est le groupoïde des pairs $\mathcal{G}_\mathcal{V} := M \times M$. Le calcul pseudodifférentiel du groupoïde $M \times M$ est alors exactement le calcul pseudodifférentiel usuel sur $M$. Le produit de convolution de deux fonction $f, g \in C^\infty(M \times M)$ induit par la structure de groupoïde est simplement

$$(f * g)(x, y) = \int_M f(x, z)g(z, y)\,\mathrm{d}z\,,$$

pour tout $(x, y) \in M \times M$.

**Exemple 3.5.4.** Lorsque $M$ est une variété à bords et $\mathcal{V}_b$ l'algèbre des champs de vecteurs tangents au bord comme en partie 3.4.1. Le groupoïde $\mathcal{G}_b \rightrightarrows M$ correspondant ressemble fortement à l'espace éclaté $M_b^2$ de Melrose. Plus précisément :

$$\mathcal{G}_b \simeq M_b^2 \setminus (\mathrm{lb} \cup \mathrm{rb}).$$

Comme pour le $b$-calcul, les opérateurs pseudodifférentiels sur $\mathcal{G}_b$ sont des distributions conormales à l'inclusion de la diagonale $\Delta_M \subset \mathcal{G}_b$ (voir Figure 3.6). Si l'on impose une condition de décroissance rapide à l'infini de ces noyaux, Monthubert [140] a montré que l'on retrouve exactement le petit $b$-calcul de Melrose. Cet exemple est discuté plus en détail partie 4.3.4.

Troisièmement, la structure géométrique de $\mathcal{G}_\mathcal{V}$ est utile pour caractériser les opérateurs de Fredholm. Le caractère quasi-injectif de $\mathcal{V}$ induit une décomposition

$$\mathcal{G}_\mathcal{V} = M_0 \times M_0 \bigsqcup \mathcal{G}_{\partial M},$$

avec $\mathcal{G}_{\partial M} \rightrightarrows \partial M$ la restriction du groupoïde à $\partial M$. Or la fermeture de $\Psi^{-1}(M_0 \times M_0)$ dans $\mathcal{B}(L^2(M_0))$ est exactement l'algèbre $\mathcal{K}$ des opérateurs compacts. On obtient, *sous certaines conditions sur* $\mathcal{G}_\mathcal{V}$, une suite exacte de $C^*$-algèbres

$$0 \longrightarrow \mathcal{K} \longrightarrow \overline{\Psi^0}(\mathcal{G}_\mathcal{V}) \xrightarrow{\sigma_0, \mathcal{R}_\partial} C(S^*\mathcal{G}_\mathcal{V}) \times_{\partial M} \overline{\Psi^0}(\mathcal{G}_{\partial M}) \longrightarrow 0. \qquad (3.9)$$

[1]Cette affirmation est volontairement un peu inexacte : le produit de convolution sur $C_c^\infty(\mathcal{G}_\mathcal{V})$ dépend d'un choix, par exemple celui d'une métrique invariante à droite sur les fibres de $\mathcal{G}_\mathcal{V}$. L'obtention d'un produit de convolution canonique se fait usuellement en considérant les demi-densités sur $\mathcal{G}_\mathcal{V}$, une approche que nous ne suivons pas dans cette thèse. En effet, l'un des objectif est d'obtenir des applications à l'analyse sur les variétés riemannienne, auquel cas une métrique est déjà présente.

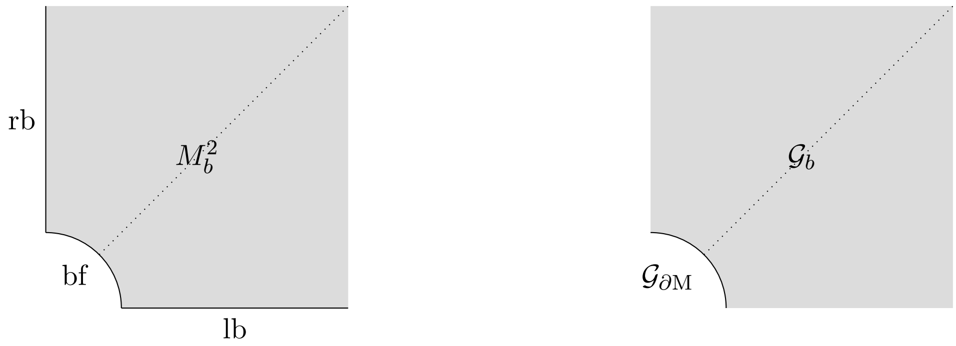


FIGURE 3.6. : À gauche, l'espace éclaté $M_b^2$ de Melrose (voir Figure 3.4) ; à droite, le groupoïde minimal $\mathcal{G}_b$ intégrant $\mathcal{V}_b$. Ce dernier s'identifie à $M_b^2 \setminus (\mathrm{rb} \cup \mathrm{lb})$.

L'algèbre de droite est donnée par les paires $(f, P)$ telle que $\sigma_0(P) = f|_{\partial M}$, le morphisme $\mathcal{R}_\partial$ étant donné par la restriction d'un opérateur pseudodifférentiel au bord de $M$.

La suite (3.9) redonne de manière géométrique le point central de cette introduction. Comme précédemment, le caractère Fredholm d'un opérateur $P \in \Psi^0(\mathcal{G}_\mathcal{V})$ est contrôlé par l'inversibilité de deux objets : son symbole principal $\sigma_0(P) \in C^\infty(S^*\mathcal{G}_\mathcal{V})$, et une famille d'opérateurs limites $\mathcal{R}_\partial(P) \in \Psi^0(\mathcal{G}_{\partial M})$. L'indice de Fredholm de $P$ est donc déterminé par la classe de $K$-théorie que définit le couple $(\sigma_0(P), \mathcal{R}_\partial(P))$. Les conditions qu'il faut imposer à $\mathcal{G}_\mathcal{V}$ pour obtenir la suite exacte (3.9) ainsi qu'une bonne description des opérateurs limites ont été étudiées par Carvalho, Qiao et Nistor [51] et constituent l'objet central de la partie III de ma thèse.

# 4. Preliminaries

We introduce in this chapter the main tools and general results that will be used throughout the manuscript. Section 4.1 thus explains the basic facts of $C^*$-algebra theory, Section 4.2 deals with pseudodifferential operators and Section 4.3 introduces groupoids.

## 4.1. $C^*$-algebras and their representations

The theory of $C^*$-algebras will be used extensively throughout this manuscript, hence the decision to recall the main facts and definition in this section. For the general theory, we rely mostly on Dixmier's monograph [75].

### 4.1.1. $C^*$-algebras

If $\mathcal{H}$ is a Hilbert space and $\mathcal{B}(\mathcal{H})$ the set of bounded operators on $\mathcal{H}$, then any involutive subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed for the norm topology is a $C^*$-algebra. The starting point of the theory may be attributed to Gelfand and Naimark, who showed in 1943 that such algebras can actually be characterized abstractly as follows [83].

**Definition 4.1.1.** A *$C^*$-algebra* is a Banach algebra $A$ endowed with an antilinear involution $a \mapsto a^*$ such that $\|a^*a\| = \|a\|^2$, for any $a \in A$.

The identity $\|a^*a\| = \|a\|^2$ forces a very rigid structure on $A$. An illustration of this fact is that a Banach algebra has at most one $C^*$-algebra norm. A *morphism* between two $C^*$-algebras is a map that is compatible with the structure of involutive algebras; such maps turn out to be automatically continuous.

**Example 4.1.2.** As mentioned earlier, any closed involutive subalgebra of $\mathcal{B}(\mathcal{H})$ is a $C^*$-algebra for the operator norm. An important example is the ideal of compact operators $\mathcal{K}(\mathcal{H})$.

**Example 4.1.3.** Let $X$ be a locally compact space, and let $C_0(X)$ be the space of complex-valued continuous functions on $X$ that vanish at infinity. We endow $C_0(X)$ with the supremum norm $\|\cdot\|_\infty$ and the involution induced by complex conjugation. Then $C_0(X)$ is a commutative $C^*$-algebra.

Examples 4.1.2 and 4.1.3 turn out to be generic. Indeed Gelfand, Naimark and Segal have proven that any $C^*$-algebra embeds as a closed involutive subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ [83]. Moreover, any commutative $C^*$-algebra $A$ is isomorphic to an algebra $C_0(X)$ for some locally compact space $X$; the space $X$ in this case is the *spectrum* of $A$, whose definition will be given in Section 4.1.2. The latter isomorphism gives rise to an equivalence between the categories of commutative algebras and locally compact spaces (with morphisms satisfying a properness condition). This fundamental fact may be seen as the starting point of Connes' program of *noncommutative topology* [60].

**Example 4.1.4.** Given a $C^*$-algebra $A$ and a closed two-sided ideal $I \subset A$, the quotient $A/I$ is also a $C^*$-algebra. If $\pi : A \to A/I$ denotes the projection, then the norm on $A/I$ is naturally given by $\|\pi(a)\| = \sup_{b\in I} \|a + b\|$. An important special case is the *Calkin algebra* of a Hilbert space $\mathcal{H}$, given as $\mathcal{B}(H)/\mathcal{K}(H)$. A theorem of Atkinson states that the Fredholm operators on $\mathcal{H}$ are exactly the invertible elements of $\mathcal{B}(H)/\mathcal{K}(H)$ [19].

A closed, two-sided ideal $I \subset A$ is called *essential* if for any $a \in A$, the equality $aI = 0$ implies that $a = 0$.

**Example 4.1.5.** If $A$ is a non-unital $C^*$-algebra, then the smaller unital $C^*$-algebra containing $A$ as an essential ideal is its *unitisation*, denoted $A^+$. As a vector space, we have $A^+ \simeq A \oplus \mathbb{C}$, and the product is defined as

$$(a \oplus z)(b \oplus w) := ab + zb + wa \oplus zw,$$

which makes $0_A \oplus 1$ the unit of $A^+$. If $A$ is commutative, then the discussion following Example 4.1.3 gives a locally compact space $X$ such that $A = C_0(X)$. The unitisation $A^+$ then identifies with $C(X^+)$, with $X^+$ the one-point compactification of $X$.

**Example 4.1.6.** For any $C^*$-algebra $A$, there is a maximal $C^*$-algebra containing $A$ as an essential ideal: this is the *multiplier algebra* of $A$, denoted $M(A)$ [2, 44]. It may be constructed as the set of pairs $(L, R)$ of bounded linear maps on $A$ such that $aL(b) = R(a)b$ for any $a, b \in A$. If $A$ is commutative and given by $A = C_0(X)$ for some locally compact space $X$, then $M(A)$ identifies with the algebra $C_b(X)$ of bounded continuous functions on $X$. Its spectrum is the *Stone-Čech compactification* $\beta X$ of $X$, which means that we have $M(A) \simeq C(\beta X)$.

We conclude this sections by mentioning two important results of the theory. The first one is the existence of *approximate identities*: if $A$ is a $C^*$-algebra, then there exists a generalized sequence $(u_\lambda)_{\lambda\in\Lambda}$ of elements $u_\lambda \in A$ such that $u_\lambda a \to a$ and $au_\lambda \to a$ for any $a \in A$. The second one may be stated as follows.

**Theorem 4.1.7.** *Let $A \subset B$ be two $C^*$-algebras and $a \in A$. The element $a$ is invertible in $B$ if, and only if, it is invertible in $A$.*

### 4.1.2. The primitive spectrum

The study of a $C^*$-algebra $A$ is tightly related with the study of its representations. The primitive spectrum of $A$ is a topological space that aggregates a lot of information regarding the irreducible representations of $A$.

**Definition 4.1.8.** A *representation* of $A$ is a morphism of $C^*$-algebras $\pi : A \to \mathcal{B}(\mathcal{H}_\pi)$, with $\mathcal{H}_\pi$ a Hilbert space.

As mentioned in Section 4.1.1, representations of $C^*$-algebras are automatically continuous. A representation $\pi : A \to \mathcal{B}(\mathcal{H})$ is said to be *non-degenerate* if $\pi(A)\mathcal{H}$ is dense in $\mathcal{H}$. The representation $\pi$ is *irreducible* if the only closed subspaces of $\mathcal{H}$ that are stable by $\pi(A)$ are 0 and $\mathcal{H}$ itself.

**Definition 4.1.9.** An ideal $J \subset A$ is called *primitive* if it is the kernel of a non-zero irreducible representation of $A$. The *primitive spectrum* of $A$, denoted $\operatorname{Prim} A$, is the set of all primitive ideals in $A$.

Let $I \subset A$ be a closed, two-sided ideal. A well-known fact of the theory is that any non-degenerate representation $\pi$ of $I$ on a Hilbert space $\mathcal{H}$ extends uniquely as a representation $\operatorname{Ind}_I^A \pi$ of $A$ on the same space $\mathcal{H}$. The representation $\operatorname{Ind}_I^A \pi$ is irreducible if, and only if, $\pi$ is. This induction functor thus induces an injective map $\operatorname{Prim} I \to \operatorname{Prim} A$ mapping $\ker \pi$ to $\ker(\operatorname{Ind}_I^A \pi)$. Its image is the set

$$\operatorname{Prim}^I A := \{J \in \operatorname{Prim} A \mid I \not\subset J\} \simeq \operatorname{Prim} I.$$

The complement of $\operatorname{Prim} I$ in $\operatorname{Prim} A$ is

$$\operatorname{Prim}_I A := \{J \in \operatorname{Prim} A \mid I \subset J\} \simeq \operatorname{Prim} A/I.$$

The sets $\operatorname{Prim} I \subset \operatorname{Prim} A$, where $I$ ranges through the closed two-sided ideals of $A$, are precisely the open sets in the *Jacobson topology* on $\operatorname{Prim} A$. An ideal $I \subset A$ is *essential* (see Section 4.1.1) if, and only if, the space $\operatorname{Prim} I$ is dense in $\operatorname{Prim} A$ for this topology.

**Example 4.1.10.** Let $X$ be a locally compact space, and $A = C_0(X)$. The irreducible representations of $A$ are the characters $\mathrm{ev}_x : f \mapsto f(x)$, given by pointwise evaluation at a point $x \in X$. The primitive spectrum of $A$ is then

$$\operatorname{Prim} C_0(X) \simeq X,$$

where the isomorphism is given by the map $\ker(\mathrm{ev}_x) \mapsto x$.

### 4.1.3. Exhaustive families of representations

This section is adapted from [151, 178]. The main use of the primitive spectrum for our purposes will be to characterize families of morphisms of $A$ having certain "nice" properties. Given a $C^*$-algebra $B$ and a morphism $\phi : A \to B$, its *support* is the closed subset

$$\operatorname{supp} \phi := \operatorname{Prim}(A/\ker \phi) = \{J \in \operatorname{Prim} A \mid \ker \phi \subset J\}.$$

**Definition 4.1.11.** A family $\mathcal{F}$ of $C^*$-algebra morphisms $\phi : A \to B_\phi$ is *faithful* if for any $a \in A$, we have that $a = 0$ if, and only if, $\phi(a) = 0$ for any $\phi \in \mathcal{F}$.

**Proposition 4.1.12.** *A family $\mathcal{F}$ of morphisms $\phi : A \to B_\phi$ is faithful if, and only if,*

$$\operatorname{Prim} A = \overline{\bigcup_{\phi \in \mathcal{F}} \operatorname{supp} \phi}.$$

If $\mathcal{F}$ is a faithful family of morphisms $\phi : A \to B_\phi$, and $a \in A$, then $\|a\| = \sup_{\phi\in\mathcal{F}} \|\phi(a)\|$. Moreover, the element $1 + a \in A^+$ is invertible if, and only if, the following conditions both hold:

(i) $1 + \phi(a)$ is invertible in $B_\phi^+$ for any $\phi \in \mathcal{F}$, and

(ii) $\sup_{\phi\in\mathcal{F}} \|1 + \phi(a)\|^{-1} < \infty$.

Te second condition is often uneasy to check in practice. This motivates the following stronger notion:

**Definition 4.1.13.** A family $\mathcal{F}$ of $C^*$-algebra morphisms $\phi : A \to B_\phi$ is *exhaustive* if

$$\operatorname{Prim} A = \bigcup_{\phi\in\mathcal{F}} \operatorname{supp}\phi.$$

A $C^*$-algebra $A$ is *separable* if it is separable as a topological space, i.e. $A$ possesses a dense countable set. All the $C^*$-algebras considered in this thesis are separable. For this class of $C^*$-algebras, exhaustive families are exactly the families that characterize invertibility, in the following sense.

**Theorem 4.1.14** (Nistor, Prudhon [151])**.** *Let $A$ be a $C^*$-algebra and $\mathcal{F}$ an exhaustive family of morphisms $\phi : A \to B_\phi$. For any $a \in A$, we have that*

*(i) $1 + a$ is invertible in $A^+$ if, and only if, $1 + \phi(a)$ is invertible in $B_\phi^+$ for any $\phi \in \mathcal{F}$,*

*(ii) there is a $\phi \in \mathcal{F}$ such that $\|a\| = \|\phi(a)\|$.*

*Moreover, if $A$ is separable, then any family of morphisms satisfying (i) or (ii) is also exhaustive.*

Exhaustive families also enjoy a useful localization property. Recall from Section 4.1.2 that to any ideal $I \subset A$ corresponds a decomposition

$$\operatorname{Prim} A = \operatorname{Prim} I \bigsqcup \operatorname{Prim}(A/I),$$

with $\operatorname{Prim} I$ open and $\operatorname{Prim}(A/I)$ closed. Thus, let $\mathcal{F}_I$ and $\mathcal{F}_{A/I}$ be two exhaustive families of representations for $I$ and $A/I$, respectively. If we identify representations in $\mathcal{F}_I$ and $\mathcal{F}_{A/I}$ with the representations they induce on $A$, then the family $\mathcal{F} := \mathcal{F}_I \cup \mathcal{F}_{A/I}$ is automatically exhaustive for $A$. This gives the following useful result.

**Proposition 4.1.15.** *Let $A$ be a unital $C^*$-algebra and $I \subset A$ be a closed, two-sided ideal. Consider an exhaustive family of representations $\mathcal{F}_I$ of $I$. An element $a \in A$ is invertible if, and only if, $a$ is invertible in $A/I$ and $\pi(a)$ is invertible for any $\pi \in \mathcal{F}_I$.*

### 4.1.4. Induction of representations

Let again $A$ be a $C^*$-algebra. We have seen in Section 4.1.2 how a closed two-sided ideal $I \subset A$ determines an induction functor $\operatorname{Ind}_I^A$ from the category of unitary equivalence classes of representations of $I$ to that of $A$. This construction may be extended to more general pairs of $C^*$-algebras through the notion of *$C^*$-correspondences*, which originated in the work of Rieffel [177]. We give the essential facts here and refer to the monograph of Raeburn and Williams [171] for more details.

**Definition 4.1.16.** A (right) *Hilbert $A$-module* $X$ is an $A$-module endowed with an inner product
$$\langle\cdot,\cdot\rangle : X \times X \to A,$$
that is $A$-linear on the right, positive definite and antisymmetric. We moreover require the space $X$ to be complete for the norm $\|x\|_A := \|\langle x,x\rangle\|^{1/2}$.

The module $X$ is *full* if the ideal spanned by the products $\langle x,y\rangle$, where $x$ and $y$ range through $X$, is dense in $A$.

**Example 4.1.17.** A Hilbert space is naturally a Hilbert $\mathbb{C}$-module.

**Example 4.1.18.** If $Y$ is a locally compact space and $E \to Y$ a hermitian vector bundle, then the space $C_0(Y;E)$ of continuous sections that vanish at infinity is a Hilbert $C_0(Y)$-module for the inner product
$$\langle f,g\rangle(y) := (f(y),g(y))_{E_y}$$
for any $f,g \in C_0(Y;E)$ and $y \in Y$.

General Hilbert modules enjoy less structure than Hilbert spaces. In particular, not every closed subspace has an orthogonal complement, and not every operator is adjointable.

**Definition 4.1.19.** Let $X$ and $Y$ be Hilbert $A$-modules. A map $T : X \to Y$ is *adjointable* if there exists $T^* : Y \to X$ such that
$$\langle Tx,y\rangle = \langle x,T^*y\rangle,$$
for any $x \in X$ and $y \in Y$.

Any adjointable map $T : X \to Y$ is bounded and linear. We denote by $\mathcal{L}(X,Y)$ the space of adjointable maps from $X$ to $Y$, with the topology of the operator norm. An operator $T \in \mathcal{L}(X,Y)$ is of *rank one* if there exist $x \in X$ and $y \in Y$ such that $T = y\langle x,\cdot\rangle$. The space of *compact operators* $\mathcal{K}(X,Y)$ is the closure in $\mathcal{L}(X,Y)$ of the subspace generated by rank-one operators. Note that $\mathcal{L}(X)$ is a $C^*$-algebra, and $\mathcal{K}(X)$ a closed two-sided ideal.

A very important and useful construction is the tensor product of Hilbert modules. Assume given a Hilbert $A$-module $X_A$ and a Hilbert $B$-module $Y_B$, together with a $C^*$-algebra morphism $\phi : A \to \mathcal{L}(Y)$. If $a \in A$ and $y \in X$, we denote by $a \cdot y := \phi(a)(y)$. The *(internal) tensor product* $X \otimes_\phi Y$ is the completion of the algebraic tensor product $X \odot Y$ with the topology of the $B$-valued inner product
$$\langle x_1 \odot y_1, x_2 \odot y_2\rangle_B := \langle\langle x_1,x_2\rangle_A \cdot y_1, y_2\rangle_B.$$
This makes $X \otimes Y$ a Hilbert $B$-module.

**Example 4.1.20.** Let $C_0(Y;E)$ be the $C_0(Y)$-module of Example 4.1.18 and denote by $\mathrm{ev}_y : C_0(Y) \to \mathbb{C}$ the character given by the evaluation at a point $y \in Y$. Then the tensor product $C_0(Y;E) \otimes_{\mathrm{ev}_y} \mathbb{C}$ is isomorphic to $E_y$ through the map induced by $f \otimes 1 \mapsto f(y)$.

**Definition 4.1.21.** Let $A, B$ be $C^*$-algebras. An $(A,B)$*-correspondence* $X_B$ is a full right Hilbert $B$-module with a morphism $\phi : A \to \mathcal{L}(X)$.

The notion of $C^*$-correspondences was introduced by Rieffel under the name of $A$-rigged $B$-module [177]. To any $(A,B)$-correspondence $X_B$ is associated an induction functor

$$X\text{–Ind} : \mathcal{R}(B) \to \mathcal{R}(A)$$

from the category of unitary equivalence classes of representations of $B$ (with intertwining morphisms) to the one of $A$. If $\pi$ is a representation of $B$ on a Hilbert space $\mathcal{H}$, the corresponding induced representation is given by the action of $A$ on the Hilbert space $X \otimes_\pi \mathcal{H}$.

Correspondences may be composed by taking their tensor product. An invertible $(A,B)$-correspondence yields a *Morita equivalence* between $A$ and $B$. The corresponding module ${}_AX_B$ is called an *imprimitivity bimodule*: it is a full right Hilbert $B$-module as well as a full left Hilbert $A$-module, such that the two inner products satisfy some compatibility conditions. In that case, the functor $X$–Ind induces an equivalence of category between $\mathcal{R}(B)$ and $\mathcal{R}(A)$, which descends to a homeomorphism

$$X\text{–Ind} : \operatorname{Prim} B \to \operatorname{Prim} A.$$

## 4.2. Pseudodifferential operators

We recall here the definition and main properties of (classical) pseudodifferential operators, on $\mathbb{R}^n$ and on manifolds. The introduction of the pseudodifferential calculus may be attributed to Kohn and Nirenberg [107] (and also to Hörmander, Seeley, Bokobza and Unterberger [32, 97, 186]), although singular integral operators had been extensively studied since the 1930's. For this introduction, we refer mostly to Hörmander's monograph [98].

### 4.2.1. Definitions and notations on $\mathbb{R}^n$

A (scalar) *pseudodifferential operator* on $\mathbb{R}^n$ is a linear map $P : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ defined for any $f \in C_c^\infty(\mathbb{R}^n)$ and any $x \in \mathbb{R}^n$ by an oscillatory integral

$$(Pf)(x) := \int_{\mathbb{R}^{2n}} e^{(x-y)\cdot\xi} a(x,\xi) f(y)\, \mathrm{d}y\, \mathrm{d}\xi\,. \tag{4.1}$$

Here $a$ is the *symbol* of $P$, which is a function of $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ that must satisfy some growth properties. Thus for any $m \in \mathbb{R}$, we denote by $S^m(\mathbb{R}^n \times \mathbb{R}^n)$ the space of *order-$m$ symbols*, consisting of smooth functions $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|} \tag{4.2}$$

for any $\alpha, \beta \in \mathbb{N}^n$ and $x, \xi \in \mathbb{R}^n$, with some constants $C_{\alpha,\beta} > 0$. The space $S^m(\mathbb{R}^n \times \mathbb{R}^n)$ is a Fréchet space for the family of semi-norms defined by the smallest admissible constant $C_{\alpha,\beta}$ in (4.2), when $\alpha$ and $\beta$ range over $\mathbb{N}^n$.

In this manuscript, we consider only *classical* (or polyhomogeneous) symbols: this means that $a$ admits an asymptotic expansion

$$a \sim \sum_{j=0}^{\infty} a_j, \tag{4.3}$$

where $a_j \in S^{m-j}(\mathbb{R}^n \times \mathbb{R}^n)$ is positively homogeneous of order $m-j$, meaning that $a_j(x, t\xi) = t^{m-j} a_j(x, \xi)$ for any $|\xi| \geq 1$ and $t \geq 1$. The meaning of the asymptotic expansion above is that $a - \sum_{j=0}^{k} a_j$ should be a symbol of order $m-k-1$, for any $k \geq 0$.

The space $\Psi^m(\mathbb{R}^n)$ of pseudodifferential operators of order $m$ consists of those operators that may be defined as in (4.1), with a *classical* symbol $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$. Note that the integral (4.1) does not a priori converge and must be understood as a limit in some sense (this is Hörmander's theory of oscillatory integrals). When the symbol $a$ is a polynomial in $\xi$, then $P$ is a differential operator of the same order and the integral (4.1) illustrates the action of $P$ in Fourier space.

### 4.2.2. Pseudodifferential operators on manifolds

Let $M$ be a smooth Riemannian manifold without boundary (not necessarily compact). A pseudodifferential operator of order $m$ on $M$ is a map $P : C_c^\infty(M) \to C^\infty(M)$ that may be written like (4.1) in local coordinates. This definition extends to pseudodifferential operators acting between sections of vector bundles $E, F$ over $M$, by replacing the symbol $a$ in (4.1) with an element of $\mathrm{Hom}(E, F)$.

As a distribution on $M \times M$, the Schwartz kernel of a pseudodifferential operator is said to be *conormal* relatively to the diagonal $\Delta_M := \{(x, x) \mid x \in M\}$. Such distributions are smooth everywhere outside $\Delta_M$, where their singularities are prescribed by (4.1). Moreover, Hörmander showed that conormal distributions admit a coordinate-free description.

We say that a pseudodifferential operator is *compactly supported* if its Schwartz kernel has a compact support. Thus, we will denote by $\Psi^m(M; E, F)$ the space of classical, compactly supported pseudodifferential operators of order $m$ on $M$, acting from the space of compactly supported sections $C_c^\infty(M; E)$ to $C_c^\infty(M; F)$. We also set

$$\Psi^{-\infty}(M; E, F) := \bigcap_{m \in \mathbb{R}} \Psi^m(M; E, F) \quad \text{and} \quad \Psi^{\infty}(M; E, F) := \bigcup_{m \in \mathbb{R}} \Psi^m(M; E, F).$$

An operator $R \in \Psi^{-\infty}(M; E, F)$ has a kernel that is everywhere smooth, and as such induces a map $R : \mathcal{D}'(M) \to C^\infty(M)$. Such operators are called *regularizing*.

The symbol of an operator $P \in \Psi^m(M; E, F)$ is not a well-defined quantity on $M$, since it depends on a choice of a local coordinate system. Nevertheless, as a function of $(x, \xi) \in T^*M$, its principal part $a_0$ in the asymptotic expansion (4.3) is invariant under coordinate change, modulo some lower-order terms. Because of this fact and the positive homogeneity of $a_0$, the restriction of $a_0$ to the unit sphere bundle $S^*M = T^*M/\mathbb{R}_+^*$ is a well-defined section with value in[1] $\mathrm{Hom}(E, F)$.

[1]More precisely, the principal symbol is a section of $\pi^* \mathrm{Hom}(E; F)$, with $\pi$ the projection $S^*M \to M$. By a slight abuse of notation, we denote this space of sections by $C^\infty(S^*M; \mathrm{Hom}(E, F))$.

**Definition 4.2.1.** For any $P \in \Psi^m(M; E, F)$, the principal symbol of $P$ is the section

$$\sigma_m(P) \in C_c^\infty(S^*M; \mathrm{Hom}(E, F))$$

defined by the restriction of the leading term $a_0$ in the asymptotic expansion (4.3) of the symbol of $P$, in any coordinate system.

By definition, the equality $\sigma_m(P) = 0$ means that $P \in \Psi^{m-1}(M; E, F)$. Thus there is an exact sequence of vector spaces

$$0 \longrightarrow \Psi^{m-1}(M; E, F) \longrightarrow \Psi^m(M; E, F) \xrightarrow{\sigma_m} C_c^\infty(S^*M; \mathrm{Hom}(E; F)) \longrightarrow 0.$$

Furthermore, compactly supported pseudodifferential operators may be composed : if $P \in \Psi^m(M; E, F)$ and $Q \in \Psi^l(M; F, G)$ for another vector bundle $G$, then $PQ$ is in $\Psi^{m+l}(M; E, G)$. If $E$ and $F$ are hermitian, then the $L^2$-adjoint $P^*$ given by

$$(Pu, v)_{L^2(F)} = (u, P^*v)_{L^2(E)},$$

for $u \in C_c^\infty(M; E)$ and $v \in C_c^\infty(M; F)$, is a well-defined operator in $\Psi^m(M; E)$. In that case, we have $\sigma_{m+l}(PQ) = \sigma_m(P)\sigma_l(Q)$ and $\sigma_m(P^*) = (\sigma_m(P))^*$.

**Definition 4.2.2.** A pseudodifferential operator in $\Psi^m(M; E, F)$ is *elliptic* if its principal symbol is invertible.

If $P \in \Psi^m(M; E, F)$ is elliptic, then we may find a $Q \in \Psi^{-m}(M; E, F)$ such that $\sigma_0(PQ) = 1$. With an appropriate refinement, we may actually choose $Q$ so that

$$PQ - 1 \in \Psi^{-\infty}(M; F) \quad \text{and} \quad QP - 1 \in \Psi^{-\infty}(M; E).$$

An operator $Q$ as above is called a *parametrix* for $P$.

### 4.2.3. Boundedness properties on closed manifolds

Assume now that $M$ is a closed Riemannian manifold, i.e. compact, smooth and without boundary, and let $E, F$ be two hermitian vector bundles on $M$, as earlier. For an $s \in [0, \infty)$, the Sobolev spaces on $M$ may be defined using local coordinate charts, or as

$$H^s(M; E) := \mathrm{dom}(1 + \Delta_E)^{s/2},$$

with $\Delta_E = \nabla_E^* \nabla_E$ the (positive) Laplace operator induced by a metric-preserving connection $\nabla_E$ on $E$. For $s < 0$, we define $H^s(M; E)$ as the dual of $H^{-s}(M; E)$ with respect to the pairing defined by the $L^2$-inner product.

**Theorem 4.2.3.** *A pseudodifferential operator $P \in \Psi^m(M; E, F)$ induces for any $s \in \mathbb{R}$ a bounded operator*

$$P : H^s(M; E) \to H^{s-m}(M; F).$$

It follows in particular that $\Psi^0(M;E)$ is a subalgebra of $\mathcal{B}(L^2(M;E))$, stable under taking adjoints. According to Section 4.1, its closure $\overline{\Psi^0}(M;E)$ for the operator norm is a $C^*$-algebra. The closure of $\Psi^{-1}(M;E)$ then coincides with the ideal $\mathcal{K}$ of compact operators on $L^2(M;E)$. Moreover, the principal symbol map is continuous for this topology, which induces an exact sequence of $C^*$-algebras:

$$0 \longrightarrow \mathcal{K} \longrightarrow \overline{\Psi^0}(M;E) \xrightarrow{\sigma_0} C(S^*M;\mathrm{End}(E)) \longrightarrow 0. \tag{4.4}$$

As explained in the introduction, Section 3.3.1, this exact sequence is related with the Fredholm property of elliptic operators.

**Theorem 4.2.4.** *Let $M$ be a closed manifold and $P \in \Psi^m(M;E,F)$. The operator*

$$P : H^s(M;E) \to H^{s-m}(M;F)$$

*is Fredholm if, and only if, $P$ is elliptic.*

When $P$ is elliptic, its Fredholm index is defined as in Section 3.1.4 by

$$\begin{aligned}\operatorname{index} P &:= \dim(\ker P) - \dim(\operatorname{coker} P)\\ &= \dim(\ker P) - \dim(\ker P^*).\end{aligned}$$

As emphasized in the introduction, the index is constant under compact perturbations and continuous deformations of the operator $P$. It also does not depend on $s$: indeed elliptic regularity implies that $\ker P \subset C^\infty(M;E)$ and $\ker P^* \subset C^\infty(M;F)$, hence both spaces are independent of $s$. The index of an elliptic operator on a closed manifold is computable in terms of its principal symbol: this is the content of Atiyah and Singer's index formula, see Section 3.1.4.

Regarding Fredholm or invertibility properties, the study of pseudodifferential operators of arbitrary order may often be reduced to that of order-zero operators, as explained by the following result.

**Lemma 4.2.5.** *If $P \in \Psi^m(M;E)$, then*

$$\widetilde{P}_s := (1+\Delta_E)^{(s-m)/2} P (1+\Delta_E)^{-s/2} \in \overline{\Psi^0}(M;E),$$

*for any $s \in \mathbb{R}$. Moreover, the operator $P : H^s(M;E) \to H^{s-m}(M;E)$ is Fredholm if, and only if, $\widetilde{P}_s$ is.*

*Proof.* The first part follows from the well-known fact that the powers of the Laplace operator are classical pseudodifferential operators [187] and the continuity of the multiplication in the operator norm. The second part follows from the fact $(1+\Delta_E)^{s/2}$ is an isomorphism from $H^m(M;E)$ to $H^{m-s}(M;E)$ for all $m, s \in \mathbb{R}$, since $M$ is compact. □

The case of operators in $\Psi^m(M;E,F)$ acting between two vector bundles can be reduced to the case of a single vector bundle by considering $\Psi^m(M;E\oplus F)$. In that case, the operator $P \in \overline{\Psi^0}(M;E,F)$ is Fredholm in Sobolev spaces if, and only if,

$$\begin{pmatrix} 0 & P \\ P^* & 0 \end{pmatrix} \in \overline{\Psi^0}(M;E\oplus F)$$

is Fredholm. Therefore it is sufficient to consider only the case of order-zero operators acting on a single vector bundle.

## 4.3. Groupoids

Groupoids form the main object of study in Part III of this manuscript. We recall here the main definitions and notations that will be used throughout the text.

### 4.3.1. Definitions and notations

We recall here the basic notions regarding groupoids and give the definitions of the different structures we will be dealing with in this thesis. The main references for this section are the monographs of Mackenzie [122, 123].

A groupoid may be defined in a very concise way using the language of category theory:

**Definition 4.3.1.** A groupoid is a small category in which every morphism is invertible.

Let us elaborate a bit. A groupoid consists of two sets: a set of *arrows* or *morphisms* $\mathcal{G}$ and a set of *units* $X$, together with five structural maps the domain and range $d, r : \mathcal{G} \to X$, the inverse $\iota : \mathcal{G} \to \mathcal{G}$, the inclusion of units $u : X \to \mathcal{G}$ and the product $\mu$ from the space

$$\mathcal{G}^{(2)} := \{(g, h) \in \mathcal{G} \times \mathcal{G}, d(g) = r(h)\}$$

of *compatible pairs* to $\mathcal{G}$. These five morphisms satisfy the obvious compatibility conditions coming from the categorical definition: e.g. $r(\iota(g)) = d(g)$ for any $g \in \mathcal{G}$, $s \circ u = id_X \ldots$

Throughout the paper, we denote by $\mathcal{G} \rightrightarrows X$ a groupoid with set of units $X$. If $A \subset X$, we denote $\mathcal{G}^A := r^{-1}(A)$ and $\mathcal{G}_A := d^{-1}(A)$. When $x \in X$, we will simply denote $\mathcal{G}_x := \mathcal{G}_{\{x\}}$ and $\mathcal{G}^x := \mathcal{G}^{\{x\}}$. The groupoid $\mathcal{G}|_A := \mathcal{G}^A \cap \mathcal{G}_A$, with units $A$, is the *reduction* of $\mathcal{G}$ to $A$.

A groupoid acts on its unit space: if $x \in X$ and $g \in \mathcal{G}_x$, then $g \cdot x := r(g)$. The orbit of the point $x$ is thus $\mathcal{G} \cdot x := r(\mathcal{G}_x)$. More generally, we denote by $\mathcal{G} \cdot A := r(\mathcal{G}_A)$ the *saturation* of the subset $A \subset X$ through the action of $\mathcal{G}$.

**Definition 4.3.2.** A *locally compact groupoid* is a groupoid $\mathcal{G} \rightrightarrows X$ such that

(i) the sets $\mathcal{G}$ and $X$ are locally compact spaces, with $X$ Hausdorff,
(ii) all five structural maps $d, r, \iota, u$ and $\mu$ are continuous,
(iii) the domain map $d$ is open.

It follows from Definition 4.3.2 that $\iota$ is a homeomorphism and $r$ is open as well. In addition to being locally compact, we shall usually require the topological space $\mathcal{G}$ to be second-countable and locally Hausdorff (in the sense that each element of $\mathcal{G}$ should have a Hausdorff neighborhood).

The groupoids occurring in analysis usually exhibit a smooth structure. Since the theory of manifolds with singularities often deals with blown-up spaces, we allow the unit space to be a manifold with corners, i.e. locally modelled on open subsets of $[0, 1[^n$ for an $n \geq 0$ (see [152]). Such manifolds may be non-Hausdorff. A *submersion* of manifolds with corners $f : M \to N$ is a smooth map such that $df_x : T_xM \to T_{f(x)}N$ is onto for any $x \in M$ and, if $df_x(v)$ is inward-pointing for a $v \in T_xM$, then so is $v$. This implies that $f^{-1}(y)$ is a smooth manifold *without* corners for any $y \in N$.

**Definition 4.3.3.** A *Lie groupoid* is a locally compact groupoid $\mathcal{G} \rightrightarrows M$ such that

(i) the sets $\mathcal{G}$ and $M$ are smooth manifolds with corners, with $M$ Hausdorff,
(ii) all five structural maps $d, r, \iota, u$ and $\mu$ are smooth,
(iii) the domain map $d$ is a submersion of manifolds with corners.

The last point implies that the space of composable pairs $\mathcal{G}^{(2)}$ is also a manifold with corners (which allows to talk about the smoothness of the product map $\mu$) and that $\mathcal{G}_x$ is a smooth manifold *without* corners, for any $x \in M$. The inverse map is then a diffeomorphism, hence $r$ is also a submersion and $\mathcal{G}^x$ a smooth manifold.

As for the theory of Lie groups, the tangent space of $\mathcal{G}$ to the units exhibits a rich structure.

**Definition 4.3.4.** Let $M$ be a manifold with corners. A *Lie algebroid* on $M$ is a vector bundle $A \to M$ such that

(i) the space of smooth sections $\Gamma(A)$ is a Lie algebra,
(ii) there is a vector bundle map $\rho : A \to M$ that induces a Lie algebra morphism $\Gamma(A) \to \Gamma(TM)$,
(iii) for any $X, Y \in \Gamma(A)$, we have $[X, fY] = f[X, Y] + (\rho(X) \cdot f)Y$.

The map $\rho$ of Definition 4.3.4 is the *anchor map* of the algebroid. Given a Lie groupoid $\mathcal{G} \rightrightarrows M$, its Lie algebroid is the vector bundle

$$A\mathcal{G} := (\ker d_*)|_M = \bigcup_{x \in M} T_x\mathcal{G}_x.$$

The anchor map is given by the differential of the range map $r_* : A\mathcal{G} \to TM$.

### 4.3.2. Groupoid $C^*$-algebras

We now recall the definition of the $C^*$-algebras associated with a locally compact groupoid, following Renault [173]. In what follows, we consider a locally compact, locally Hausdorff groupoid $\mathcal{G} \rightrightarrows X$.

When $\mathcal{G}$ is Hausdorff, we denote by $C_c(\mathcal{G})$ be the space of $\mathbb{C}$-valued continuous functions with compact support in $\mathcal{G}$. If $\mathcal{G}$ is only locally Hausdorff, then we use Connes' definition[1] : the algebra $C_c(\mathcal{G})$ is the linear space generated by continuous functions that are compactly supported in a Hausdorff subset of $\mathcal{G}$ [57]. Defining a convolution product on $C_c(\mathcal{G})$ requires the notion of a *Haar system*, which we recall from [173]. In the following definition, we denote by $R_g : \mathcal{G}_{r(g)} \to \mathcal{G}_{d(g)}$ the homeomorphism induced by right-multiplication by an element $g \in \mathcal{G}$.

[1]There is an alternate definition of $C_c(\mathcal{G})$ that is due to Crainic [65]. Connes' algebra is a quotient of Crainic's algebra. Note that both algebras separate points of $\mathcal{G}$.

**Definition 4.3.5.** Let $\mathcal{G} \rightrightarrows X$ be a locally compact groupoid. A continuous, right-invariant *Haar system* on $\mathcal{G}$ is a family of Borel measures $(\lambda_x)_{x \in X}$ such that

(i) for all $x \in X$, the support of $\lambda_x$ is $\mathcal{G}_x := d^{-1}(x)$,
(ii) (right-invariance) for any $g \in \mathcal{G}$, we have $(R_g)_* \lambda_{r(g)} = \lambda_{d(g)}$,
(iii) (continuity) for any $f \in C_c(\mathcal{G})$, the function

$$x \mapsto \int_{\mathcal{G}_x} f \,\mathrm{d}\lambda_x$$

is continuous.

Assume that $\mathcal{G}$ is endowed with a continuous, right-invariant Haar system. If $f, g$ are two functions in $C_c(\mathcal{G})$, we define their convolution product $f * g \in C_c(\mathcal{G})$ by

$$f * g(\gamma) = \int_{\mathcal{G}_{d(\gamma)}} f(\gamma\delta^{-1}) g(\delta) \,\mathrm{d}\lambda_{d(\gamma)}(\delta)\,.$$

This allows to define the *regular representation* $\pi_x$ of $C_c(\mathcal{G})$ at a point $x \in X$. Given $f \in C_c(\mathcal{G})$ and $\xi \in L^2(\mathcal{G}_x, \lambda_x)$, we set

$$(\pi_x(f)\xi)(\gamma) := f * \xi(\gamma) = \int_{\mathcal{G}_x} f(\gamma\delta^{-1}) g(\delta) \,\mathrm{d}\lambda_x(\delta)\,.$$

The *reduced* $C^*$-algebra of $\mathcal{G}$, denoted $C_r^*(\mathcal{G})$, is the completion of $C_c(\mathcal{G})$ for the norm

$$\|f\|_{C_r^*(\mathcal{G})} = \sup_{x \in X} \|\pi_x(f)\|.$$

The involution is defined for any $f \in C_c(\mathcal{G})$ by $f^*(\gamma) := \overline{f(\gamma^{-1})}$. The *full* $C^*$-algebra of $\mathcal{G}$, denoted $C^*(\mathcal{G})$, is the completion of $C_c(\mathcal{G})$ for the norm

$$\|f\|_{C^*(\mathcal{G})} = \sup_{\pi} \|\pi(f)\|,$$

where $\pi$ ranges over all representations of $C_c(\mathcal{G})$ that are bounded by the norm

$$\|f\|_I := \max\left\{ \sup_{x \in X} \int_{\mathcal{G}_x} |f| \,\mathrm{d}\lambda_x \,, \sup_{x \in X} \int_{\mathcal{G}_x} |f^*| \,\mathrm{d}\lambda_x \right\}.$$

**Definition 4.3.6.** The groupoid $\mathcal{G}$ is said to be *metrically amenable* if the canonical surjective $*$-homomorphism $C^*(\mathcal{G}) \to C_r^*(\mathcal{G})$, induced by the definitions above, is also injective.

Let $\mathcal{G} \rightrightarrows M$ be a second countable, locally compact groupoid with a Haar system. Let $U \subset M$ be an open $\mathcal{G}$-invariant subset, $F := M \smallsetminus U$. There is then a decomposition $\mathcal{G} = \mathcal{G}_U \sqcup \mathcal{G}_F$, where $\mathcal{G}_U$ and $\mathcal{G}_F$ are subgroupoids of $\mathcal{G}$. By the classic results of [144, 145, 175], the algebra $C^*(\mathcal{G}_U)$ embeds as a closed two-sided ideal of $C^*(\mathcal{G})$, and the restriction map $\mathcal{R}_F : C_c(\mathcal{G}) \to C_c(\mathcal{G}_F)$ yields the short exact sequence

$$0 \longrightarrow C^*(\mathcal{G}_U) \longrightarrow C^*(\mathcal{G}) \xrightarrow{\mathcal{R}_F} C^*(\mathcal{G}_F) \longrightarrow 0$$

If $\mathcal{G}_F$ is metrically amenable, then one also has the exact sequence

$$0 \longrightarrow C_r^*(\mathcal{G}_U) \longrightarrow C_r^*(\mathcal{G}) \xrightarrow{\mathcal{R}_F} C_r^*(\mathcal{G}_F) \longrightarrow 0$$

It follows from the Five Lemma that if the groupoids $\mathcal{G}_F$ and $\mathcal{G}_U$ (respectively, $\mathcal{G}$) are metrically amenable, then $\mathcal{G}$ (respectively, $\mathcal{G}_U$) is also metrically amenable.

The notion of correspondence introduced for $C^*$-algebras in Section 4.1.4 has a counterpart for groupoids [176, 96]. We will not need the general notion of correspondences, and thus choose to focus on Morita equivalences.

**Definition 4.3.7.** A *Morita equivalence* of two locally compact groupoids $\mathcal{G} \rightrightarrows X$ and $\mathcal{H} \rightrightarrows Y$ is a locally compact space $Z$ with a left $\mathcal{G}$-action and a right $\mathcal{H}$-action, such that both actions are free and proper, commute with each other and the anchors $Z \to X$ and $Z \to Y$ induces bijections $\mathcal{G}\backslash Z \simeq Y$ and $Z/\mathcal{H} \simeq X$.

Morita equivalent (locally compact, locally Hausdorff) groupoids have Morita equivalent $C^*$-algebras [144, 198]. It follows from Subsection 4.1.4 that there are homeomorphisms $\operatorname{Prim} C^*(\mathcal{G}) \simeq \operatorname{Prim} C^*(\mathcal{H})$ and $\operatorname{Prim} C_r^*(\mathcal{G}) \simeq \operatorname{Prim} C_r^*(\mathcal{H})$.

### 4.3.3. Pseudodifferential operators on groupoids

Pseudodifferential operators on Lie groupoids were introduced by Connes in the late 1970's for the specific case of the holonomy groupoid of a foliation [56], and subsequently generalized by Monthubert [140] and Nistor, Weinstein and Xu [152] to any Lie groupoid. This calculus gives a very broad geometric setting to understand a number of different constructions (Melrose's $b$-calculus, scattering operators on $\mathbb{R}^n$, Schulze's calculi on edges and cusps. . . ). Moreover, it may be generalized to groupoids that are only "longitudinally smooth" [113, 158, 159], which is needed for practical applications in analysis (see Example 6.4.14).

We outline here the main facts and definitions regarding this calculus. Throughout this section, we consider a Lie groupoid $\mathcal{G} \rightrightarrows M$, with $M$ a manifold with corners as in Section 4.3.1. We also assume given two complex vector bundles $E$ and $F$ over $M$.

**Definition 4.3.8.** Let $m \in \mathbb{R}$. A *pseudodifferential operator* of order $m$ on $\mathcal{G}$ with coefficients in $\operatorname{Hom}(r^*E, r^*F)$ is a linear map $P : C_c^\infty(\mathcal{G}; r^*E) \to C^\infty(\mathcal{G}; r^*F)$ such that:

(i) *($C^\infty(M)$-linearity)* there is a family $(P_x)_{x\in M}$ of order-$m$ pseudodifferential operators on $\mathcal{G}_x$, acting from $r^*E$ to $r^*F$, such that

$$(Pf)|_{\mathcal{G}_x} = P_x f|_{\mathcal{G}_x},$$

for any $f \in C_c^\infty(\mathcal{G})$;

(ii) *(smoothness)* for any open chart $U$, diffeomorphic to an open subset of $d(U) \times \mathbb{R}^n$, and any $\varphi \in C_c^\infty(U)$, the family $(\varphi P_x \varphi)_{x\in d(U)}$ is given by a smooth family of symbols $a_x \in S^m(d(U) \times \mathbb{R}^n; \operatorname{Hom}(r^*E, r^*F))$;

(iii) *(right invariance)* for any $g \in \mathcal{G}$, the diffeomorphism $R_g : \mathcal{G}_{r(g)} \to \mathcal{G}_{d(g)}$ given by right multiplication induces an equality

$$P_{r(g)} R_g^* = R_g^* P_{d(g)}.$$

It should be noted that pseudodifferential operators on groupoids were characterized in a coordinate-free way by Debord and Skandalis [72], and van Erp and Yuncken [203]. A pseudodifferential operator $P = (P_x)_{x \in M}$ on $\mathcal{G}$ has a *reduced kernel*, which is the distribution $k_P$ on $\mathcal{G}$ defined by the formula

$$k_P(gh^{-1}) := k_{d(g)}(g, h),$$

with $k_x$ the Schwartz kernel of $P_x$, for any $x \in M$. The operator $P$ is said to be *uniformly supported* if $k_P$ has compact support.

**Definition 4.3.9.** We denote by $\Psi^m(\mathcal{G}; E, F)$ the space of order-$m$, uniformly supported classical pseudodifferential operators.

As for the calculus on manifolds, the pseudodifferential calculus on $\mathcal{G}$ is stable under composition and adjoints. Moreover, the composition of two reduced kernels is given by a convolution product extending the one on $C_c(\mathcal{G})$ that we defined in Section 4.3.2 (see [118] for more details on that point).

The principal symbol of an operator $P \in \Psi^m(\mathcal{G}; E, F)$ is defined as the restriction of the principal symbols $\sigma_m(P_x)$ to the unit space $M \subset \mathcal{G}$; because of the right-invariance, this restriction contains as much information as the whole family $(\sigma_m(P_x))_{x \in M}$. We thus obtain a section

$$\sigma_m(P) \in C_c^\infty(S^*\mathcal{G}; \mathrm{Hom}(E, F)),$$

with $S^*\mathcal{G} := S(A\mathcal{G})^*$ the spherical bundle of the dual of the Lie algebroid of $\mathcal{G}$, see Section 4.3.1.

Given a right-invariant metric on $r^*E$, there is an associated Laplace operator $\Delta_E$ in $\Psi^2(\mathcal{G}; E)$. We define[1] the Sobolev space $H^s(\mathcal{G}_x; E)$ as the domain of $(1 + (\Delta_E)_x)^{s/2}$ for $s \geq 0$, and as the dual of $H^{-s}(\mathcal{G}_x; E)$ relatively to the $L^2$-inner product for $s < 0$.

**Theorem 4.3.10** (Lauter, Monthubert, Nistor [112] – Vassout [204])**.** *If $P \in \Psi^m(\mathcal{G}; E, F)$, then for any $x \in M$ and any $s \in \mathbb{R}$, the operator $P_x$ extends as a bounded operator*

$$P_x : H^s(\mathcal{G}_x; E) \to H^{s-m}(\mathcal{G}_x; F),$$

*with uniformly bounded operator norm when $x$ ranges over $M$.*

We denote by $\pi_x$ the corresponding representation of $\Psi^0(\mathcal{G}; E)$ on $L^2(\mathcal{G}_x; E)$. Similarly to the introduction of the reduced groupoid $C^*$-algebra in 4.3.2, the reduced norm of an operator $P \in \Psi^0(\mathcal{G}; E)$ is defined as

$$\|P\|_r := \sup_{x \in M} \|\pi_x(P)\|_{L^2},$$

[1] An alternative definition is given by Vassout [204], who defines $H^s(\mathcal{G}; E)$ as a Hilbert $C_r^*(\mathcal{G})$-module.

which is finite. This allows to define $\overline{\Psi^0}(\mathcal{G}; E)$ as the closure of $\Psi^0(\mathcal{G}; E)$ for this norm. When $E$ is trivial, the closure of $\Psi^{-1}(\mathcal{G})$ is exactly $C_r^*(\mathcal{G})$. Therefore, we obtain an exact sequence that is a direct generalization of (4.4):

$$0 \longrightarrow C_r^*(\mathcal{G}) \longrightarrow \overline{\Psi^0}(\mathcal{G}) \xrightarrow{\sigma_0} C(S^*\mathcal{G}) \longrightarrow 0.$$

**Remark 4.3.11** (Differential operators)**.** An important subalgebra of $\Psi^\infty(\mathcal{G}; E, F)$ is given by the subspace $\mathrm{Diff}(\mathcal{G}; E, F)$ of *differential operators* on $\mathcal{G}$, which contains in particular the geometric operators associated to right-invariant metrics (e.g. Laplace or Dirac-type operators). These are generated by right-invariant smooth vector fields on the fibers $\mathcal{G}_x$, for $x \in M$. Thus the algebra $\mathrm{Diff}(\mathcal{G})$ may be alternatively described as the universal enveloping algebra of the Lie algebroid $A\mathcal{G}$ [152].

### 4.3.4. Examples of Lie groupoids

Let us now give a few common examples of Lie groupoids. All of them will play a role in Part III.

**Example 4.3.12** (The pair groupoid)**.** Let $M$ be a smooth manifold, and consider the Lie groupoid $\mathcal{G} = M \times M$ as the groupoid having exactly one arrow between any two units, with structural morphisms as follow: the domain is $d(x, y) = y$, the range $r(x, y) = x$, and the product is given by $(x, y)(y, z) = (x, z)$. Thus $u(x) = (x, x)$ and $\iota(x, y) = (y, x)$. This example is called the *pair groupoid* of $M$. The Lie algebroid of $\mathcal{G}$ is isomorphic to $TM$.

In this case, we have $\Psi^m(\mathcal{G}) \simeq \Psi^m(M)$, the algebra of compactly supported, classical pseudodifferential operators on $M$, as defined in 4.2.2. For any $x \in M$, the regular representation $\pi_x$ defines an isomorphism between $C^*(M \times M)$ and the algebra of compact operators $\mathcal{K}(L^2(M))$. In particular, all pair groupoids are metrically amenable.

**Example 4.3.13** (Bundles of Lie groups)**.** Any Lie group $G$ can be regarded as a Lie groupoid $\mathcal{G} = G$ with exactly one unit $M = \{e\}$, the identity element of $G$, and obvious structure maps. Its Lie algebroid is the Lie algebra of the group. In that case $\Psi^m(\mathcal{G}) \simeq \Psi^m_{\mathrm{prop}}(G)^G$, the algebra of right translation invariant and properly supported pseudodifferential operators on $G$ (an operator $P : C_c^\infty(G) \to C^\infty(G)$ is *properly supported* if the two coordinate projections $G \times G \to G$ induce proper maps on the support of the Schwartz kernel of $P$).

More generally, we can let $\mathcal{G} \rightrightarrows B$ be a locally trivial bundle of groups, with fiber a Lie group $G$. In that case $d = r$, and $\mathcal{G}$ is metrically amenable if, and only if, the group $G$ is amenable.

**Example 4.3.14** (Actions groupoids)**.** Let $X$ be a smooth manifold and $G$ a Lie group acting on $X$ smoothly and from the right. The *action groupoid* generated by this action is the graph of the action, denoted by $X \rtimes G$. Its set of arrows is $X \times G$, together with the structural morphisms $r(x, g) := x$, $d(x, g) := x \cdot g^{-1}$ and $(x, h)(x \cdot h^{-1}, g) := (x, gh)$.

The Lie algebroid of $X \rtimes G$ is denoted by $X \rtimes \mathfrak{g}$. As a vector bundle, it is simply $X \times \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of $G$. Its Lie bracket is generated by the one of $\mathfrak{g}$:

namely, if $\tilde{\xi}, \tilde{\eta}$ are constant sections of $X \times \mathfrak{g}$ such that $\tilde{\xi}(x) = \xi$ and $\tilde{\eta}(x) = \eta$ for all $x \in X$, then $[\tilde{\xi}, \tilde{\eta}]_{X \rtimes \mathfrak{g}}$ is the constant section on $X$ everywhere equal to $[\xi, \eta]_{\mathfrak{g}}$. The anchor $\rho : X \rtimes \mathfrak{g} \to TX$ is given by the *fundamental vector fields* generated by the action:

$$\rho(x, \xi) = \frac{d}{dt}\bigg|_{t=0} (x \cdot \exp(t\xi))$$

for all $x \in X$ and $\xi \in \mathfrak{g}$. The differential operators on $X \rtimes G$ act on $C_c^\infty(G)$: the corresponding image of $\mathrm{Diff}(X \rtimes G)$ in $\mathrm{Diff}(G)$ is the subalgebra generated by the fundamental vector fields of the action and $C^\infty(X)$.

The study of action groupoids relates to that of crossed-product algebras, which have been much studied in the literature [76, 207]; see also [9, 84, 85, 99, 124, 126, 127, 142] for applications to spectral theory. One case of interest here is when $\mathcal{G} := [0, \infty) \rtimes (0, \infty)$ is the transformation groupoid with the action of $(0, \infty)$ on $[0, \infty)$ by dilation. Then the $C^*$-algebra associated to $\mathcal{G}$ is the algebra of Wiener-Hopf operators on $(0, \infty)$, and its unitisation is the algebra of Toeplitz operators [143].

**Example 4.3.15** (Fibered pullback groupoids)**.** Let $M, N$ be manifolds with corners, and $f : M \to N$ a surjective tame submersion. Assume that we have a Lie groupoid $\mathcal{H} \rightrightarrows N$. An important generalization of the pair groupoid is the *fibered pullback* of $\mathcal{H}$ along $f$, defined by

$$f^{\downarrow\downarrow}(\mathcal{H}) := \{(x, g, y) \in M \times \mathcal{H} \times M \mid r(g) = f(x) \text{ et } d(g) = f(y)\}$$

with units $M$. The domain and range are given by $d(x, g, y) = y$ and $r(x, g, y) = x$. The product is $(x, g, y)(y, g', y') =: (x, gg', y')$.

The groupoid $f^{\downarrow\downarrow}(\mathcal{H})$ is a Lie groupoid, which is a subgroupoid of the product of the pair groupoid $X \times X$ and $\mathcal{H}$, and whose Lie algebroid is given by the *thick pullback*

$$f^{\downarrow\downarrow}(A\mathcal{H}) := \{(\xi, X) \in A\mathcal{H} \times TM \mid \rho(\xi) = f_*(X)\}\,.$$

See [122, 123, 51] for more details. The groupoids $f^{\downarrow\downarrow}(\mathcal{H})$ and $\mathcal{H}$ are Morita equivalent [67], so $f^{\downarrow\downarrow}(\mathcal{H})$ is metrically amenable if, and only, the groupoid $\mathcal{H}$ also is.

**Example 4.3.16** ($b$-groupoid)**.** Let $M$ be a manifold with smooth boundary, and denote by $\mathcal{V}_b$ the vector fields on $M$ that are tangent to the boundary $\partial M$, as in Section 3.4.1. The $b$-groupoid $\mathcal{G}_b \rightrightarrows M$ is the holonomy groupoid associated to the singular foliation defined by $\mathcal{V}_b$, following the general construction of Debord [69]. It was explicitly constructed in [136, 141, 152], and is given as a disjoint union:

$$\mathcal{G}_b := M_0 \times M_0 \bigsqcup \Big( \bigcup_j \partial_j M \times \partial_j M \times (0, \infty) \Big)$$

where $M_0 \times M_0$ denotes the pair groupoid of the interior $M_0 := M \setminus \partial M$ and $\partial_j M$ denotes the connected components of $\partial M$.

To specify the Lie groupoid structure on $\mathcal{G}_b$, one can construct it by a gluing procedure, in two steps (this is related with the constructions of Section 5.3.1).

(i) Consider the groupoid

$$\mathcal{H}_j := (\partial_j M \times \partial_j M) \times ([0,\infty) \rtimes (0,\infty)) \rightrightarrows \partial_j M \times [0,\infty),$$

obtained as the direct product of the pair groupoid over $\partial_j M$ with the groupoid associated with the action of $(0,\infty)$ on $[0,\infty)$ by dilation. Restrict this groupoid to obtain $\mathcal{H}'_j := \mathcal{H}_j|_{\partial_j M \times [0,1)}$.

(ii) Choose tubular neighborhoods $U_j \simeq \partial_j M \times [0,1)$ of each boundary component $\partial_j M$ inside $M$, and use them to define $\mathcal{G}_b$ as a gluing:

$$\mathcal{G}_b := M_0 \times M_0 \cup \Big( \bigcup_j \mathcal{H}'_j \Big),$$

where $\mathcal{H}'_j$ is identified with the pair groupoid over $U_j \cap M_0$.

More explicitly, a sequence $(x_n, y_n) \in M_0 \times M_0$, with $n \in \mathbb{N}$, converges to a point $(x, y, \lambda)$ in $\partial_j M \times \partial_j M \times (0,\infty)$ if, and only if, $x_n \to x$, $y_n \to y$ and $r_j(x_n)/r_j(y_n) \to \lambda$, with $r_j$ the coordinate in $[0,1)$ given by the identification $U_j \simeq \partial_j M \times [0,1)$.

The Lie algebroid $A\mathcal{G}_b$ of $\mathcal{G}_b$ identifies with Melrose's *b-tangent bundle*, usually denoted ${}^bTM$; its anchor map yields an isomorphism $\Gamma(A\mathcal{G}_b) \simeq \mathcal{V}_b$. The $b$-groupoid itself is closely related with Melrose's blown-up space, as emphasized in the introduction and Figure 3.6. The pseudodifferential calculus obtained coincides with the small $b$-calculus with compact support: see [136, 141, 152] for details. The $b$-groupoid is a particular case of the class of *blup groupoids* recently introduced by Debord and Skandalis [73].

# Part III.

# Fredholm groupoids and analysis on singular spaces

**Abstract**

We present here the first part of the results of this thesis, based on the two papers [49] and [55]. The former is a joint work with Catarina Carvalho and Yu Qiao. Other than homogenization of notations and the addition of Example 6.4.8, the mathematical content of these articles has not been changed. The introduction and background material have been adapted, in order to avoid any repetition.

# 5. Gluing action groupoids: Fredholm conditions and layer potentials

The present chapter is adapted from the published paper [49] of the same name, which is a joint work with Catarina Carvalho[1] and Yu Qiao[2].

## 5.1. Introduction

We refer to Section 2.1 for a contextualized introduction, and only recall the main results here. The aim of this chapter is to study some constructions related with the notion of Fredholm Lie groupoids, as introduced by Carvalho, Nistor and Qiao in [51]. All the required notions and notations regarding groupoids are recalled in Section 4.3.

The first half of the chapter studies a special class of groupoids, which we call *boundary action groupoids*, that are obtained by gluing action groupoids. We will show that this setting recovers many interesting algebras of pseudodifferential operators. Moreover, we rely on the results in [51] to obtain the following Fredholm condition:

**Theorem 5.1.1.** *Let $\mathcal{G} \rightrightarrows M$ be a boundary action groupoid with unique open dense orbit $U \subset M$, and let $P \in \Psi^m(\mathcal{G})$. Assume that the action of $\mathcal{G}$ on $\partial M$ is trivial and that, for each $x \in \partial M$, the isotropy group $\mathcal{G}_x^x$ is amenable. Then $P : H^s(U) \to H^{s-m}(U)$ is Fredholm if, and only if*

*(i) $P$ is elliptic, and*
*(ii) each limit operator $P_x : H^s(\mathcal{G}_x) \to H^{s-m}(\mathcal{G}_x)$ is invertible for all $x \in \partial M$.*

For the second part of the chapter, we relate both the Fredholm groupoid and boundary groupoid approaches to the study of *layer potential operators* on domains with *conical singularities*. We consider here bounded domains with conical points $\Omega$ in $\mathbb{R}^n$, $n \geq 2$, that is, $\overline{\Omega}$ is locally diffeomorphic to a cone with smooth, *possibly disconnected*, base. If $n = 2$, we allow $\Omega$ to be a domain with cracks, see Section 5.4.

In [52], Carvalho and Qiao introduced a *layer potentials groupoid*, which is a groupoid over a desingularization of the boundary. The layer potentials groupoid is related to the $b$-groupoid introduced in Section 3.5.2 (see also Example 4.3.16) ; if the boundaries of the local cones bases are connected, then the two groupoids coincide (note that it is often the case that the boundaries are disconnected, for instance when $n = 2$). In general, the main difference at the groupoid level is that, in the usual $b$-calculus, there is

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no interaction between the different faces at each conical point, whereas this interaction is needed when one considers layer potential operators.

We show that the layer potential groupoid is both a boundary action groupoid *and* a Fredholm groupoid, which implies a Fredholm criterion in weighted Sobolev spaces :

**Theorem 5.1.2.** *Let $\Omega \subset \mathbb{R}^n$ be a conical domain without cracks with the set of conical points $\Omega^{(0)} = \{p_1, p_2, \cdots, p_l\}$, with possibly disconnected cone bases $\omega_i \subset S^{n-1}$.*

*Let $\mathcal{G} \rightrightarrows M = \partial'\Sigma(\Omega)$ be the layer potential groupoid as in Definition 5.4.2. Let $P \in \Psi^m(\mathcal{G})$ and $s \in \mathbb{R}$. Then $P : \mathcal{K}^s_{\frac{n-1}{2}}(\partial\Omega) \to \mathcal{K}^{s-m}_{\frac{n-1}{2}}(\partial\Omega)$ is Fredholm if, and only if,*

*(i) $P$ is elliptic, and*

*(ii) the Mellin convolution operators*

$$P_i : H^s((0,\infty) \times \partial\omega_i; g) \to H^{s-m}((0,\infty) \times \partial\omega_i; g),$$

*are invertible for all $i = 1, ..., l$, with $g$ the product of a Haar metric on the multiplicative group $(0,\infty)$ and the Euclidean metric on $\partial\omega_i$.*

The above theorem also holds, with suitable modifications, for polygonal domains with ramified cracks. We expect it to have applications for layer potential methods on singular domains. For instance in [167], Li and Qiao applied the techniques of pseudodifferential operators on Lie groupoids to the method of layer potentials on plane polygons (without cracks) to obtain the invertibility of operators $I \pm K$ on suitable weighted Sobolev spaces on the boundary, where $K$ is the double layer potential operators associated to the Laplacian and the polygon. This yields a well-posedness result for the Dirichlet problem on polygons. Moreover, Qiao [166] used a similar idea to make a connection between the double layer potential operators on three-dimensional wedges and action Lie groupoids.

### 5.1.1. Overview of the chapter

We start in Section 5.2 with reviewing the definition of Fredholm groupoids and their characterization, relying on exhaustive families of representations, resulting on Fredholm criteria for operators on Fredholm groupoids.

Section 5.3 introduces one of the main constructions of the chapter, which is the gluing of a family of locally compact groupoids $(\mathcal{G}_i)_{i\in I}$. We give two different conditions that are sufficient to define a groupoid structure on the gluing $\mathcal{G} = \bigcup_{i\in I} \mathcal{G}_i$, and show some properties of the gluing. When each $\mathcal{G}_i$ is a Lie groupoid, we describe the Lie algebroid of the glued groupoid $\mathcal{G}$.

We define the class of *boundary action groupoids* in Subsection 5.3.3. We give some examples of boundary action groupoids which occur naturally when dealing with analysis on open manifolds. We then explain the construction of the algebra of differential operators generated by a Lie groupoid $\mathcal{G}$, and prove the Fredholm condition given by Theorem 5.1.1.

In the remaining sections, we consider the case of layer potential groupoids on conical domains. In Section 5.4, we describe the construction of the relevant groupoids and give their main properties in the case with no cracks. We check that the layer potential groupoid is always a boundary action groupoid, and show moreover that such groupoids are Fredholm. As a consequence, we obtain Fredholm criteria for operators on layer potential groupoids.

## 5.2. Preliminaries : Fredholm Lie groupoids

We recall in this section the definition of Fredholm Lie groupoids as introduced by Carvalho, Nistor and Qiao in [51], as well as the main related results. Regarding groupoids, we follow the notations of Section 4.3. In particular, the unit space of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is assumed to be a manifold with corners, in the loose sense discussed in 4.3.1. For simplicity, all the groupoids considered in this chapter will be *Hausdorff* (some adjustments to the definition must be considered in the non-Hausdorff case, see [57] and Chapter 6).

### 5.2.1. Definitions

Fredholm groupoids were introduced in [51] as the groupoids whose pseudodifferential operators are Fredholm if, and only if, its principal symbol and a family of "limit operators" are invertible (in a sense to be made precise below). We review their definition and properties in this subsection.

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with $M$ compact. For any $x \in M$, we denote by $\pi_x$ the regular representation of $C_r^*(\mathcal{G})$ on $L^2(\mathcal{G}_x)$, as in 4.3.2. Assume that $U \subset M$ is an open, $\mathcal{G}$-invariant subset such that $\mathcal{G}_U \simeq U \times U$ (the pair groupoid, see Example 4.3.12). For any $x \in U$, the representation $\pi_x$ and the isomorphism $\mathcal{G}_x \simeq U$ determine a morphism

$$\pi_0 : C_r^*(\mathcal{G}) \longrightarrow \mathcal{B}(L^2(U)),$$

which does not depend on the choice of $x$. The representation $\pi_0$ will be called the *vector representation*. We recall the following definition from [51].

**Definition 5.2.1.** A Lie groupoid $\mathcal{G} \rightrightarrows M$ is called a *Fredholm Lie groupoid* if

(i) there exists an open, dense, $\mathcal{G}$-invariant subset $U \subset M$ such that $\mathcal{G}_U \simeq U \times U$;
(ii) for any $a \in C_r^*(\mathcal{G})$, we have that $1 + \pi_0(a)$ is Fredholm if, and only if, all $1 + \pi_x(a)$, for $x \in F := M \backslash U$, are invertible.

As an open dense $\mathcal{G}$-orbit, the set $U$ is uniquely determined by $\mathcal{G}$. Moreover, a simple observation is that $F := M \backslash U$ is closed and $\mathcal{G}$-invariant. We shall keep this notation throughout the chapter. Note also that two regular representations $\pi_x$ and $\pi_y$ are unitarily equivalent if, and only if, there is $g \in \mathcal{G}$ such that $d(g) = x$ and $r(g) = y$, that is, if $x, y$ are in the same orbit of $\mathcal{G}$ acting on $M$. In particular, one only needs to verify (ii) for a representative of each orbit of $\mathcal{G}_F$.

In [51], we gave easier-to-check conditions for a groupoid $\mathcal{G}$ to be Fredholm, depending on properties of representations of $C_r^*(\mathcal{G})$. We review briefly the main notions, see [151, 178] for details.

Let $A$ be a $C^*$-algebra. As detailed in Section 4.1.2, a two-sided ideal $I \subset A$ is said to be *primitive* if it is the kernel of an non-zero irreducible representation of $A$. We denote by $\mathrm{Prim}(A)$ the set of primitive ideals of $A$ and we equip it with the Jacobson topology: see Section 4.1.2 and [75, 207] for more details. Let $\phi$ be a representation of $A$. The *support* $\operatorname{supp}\phi \subset \mathrm{Prim}(A)$ is defined to be the set of primitive ideals of $A$ that contain $\ker\phi$.

As is also introduced in 4.1.2, a family of representation $\mathcal{F}$ of $A$ is called *exhaustive* if

$$\mathrm{Prim}\, A = \bigcup_{\phi\in\mathcal{F}} \operatorname{supp}\phi,$$

that is, if any irreducible representation of $A$ is weakly contained in some $\phi \in \mathcal{F}$. For separable $C^*$-algebras, exhaustive families characterize invertible elements in $A$: given an $a \in A$, we have that $1 + a$ is invertible in the unitisation $A^+$ if, and only if, the images $1 + \phi(a)$ are all invertible, for any $\phi \in \mathcal{F}$.

The next result was given in [51] and gives a characterization of Fredholm groupoids. For a groupoid $\mathcal{G}$, we usually denote by $\mathcal{R}(\mathcal{G})$ the set of its regular representations.

**Theorem 5.2.2.** *Let $\mathcal{G} \rightrightarrows M$ be a Hausdorff Lie groupoid and $U$ an open, dense, $\mathcal{G}$-invariant subset such that $\mathcal{G}_U \simeq U \times U$. Set $F = M\backslash U$. If $\mathcal{G}$ is a Fredholm groupoid, we have that:*

*(i) the restriction map $\mathcal{R}_F : C_r^*(\mathcal{G}) \to C_r^*(\mathcal{G}_F)$ induces an exact sequence*

$$0 \longrightarrow C_r^*(\mathcal{G}_U) \longrightarrow C_r^*(\mathcal{G}) \xrightarrow{\mathcal{R}_F} C_r^*(\mathcal{G}_F) \longrightarrow 0.$$

*(ii) $\mathcal{R}(\mathcal{G}_F) := \{\pi_x \mid x \in F\}$ is an exhaustive set of representations of $C_r^*(\mathcal{G}_F)$.*

*Conversely, if $\mathcal{G} \rightrightarrows M$ satisfies (i) and (ii), then, for any unital $C^*$-algebra $\mathbf{\Psi}$ containing $C_r^*(\mathcal{G})$ as an essential ideal, and for any $a \in \mathbf{\Psi}$, we have that $a$ is Fredholm on $L^2(U)$ if, and only if, $\pi_x(a)$ is invertible for each $x \notin U$* and *the image of $a$ in $\mathbf{\Psi}/C_r^*(\mathcal{G})$ is invertible.*

In [51], we dubbed condition (ii) as *Exel's property* (for $\mathcal{G}_F$). If $\mathcal{R}(\mathcal{G}_F) = \{\pi_x \mid x \in F\}$ is an exhaustive set of representations of $C^*(\mathcal{G}_F)$, then $\mathcal{G}_F$ is said to have the *strong Exel property*. In this case, it is metrically amenable. We will use the sufficient conditions in Theorem 5.2.2 in the following form, also from [51].

**Proposition 5.2.3.** *Let $\mathcal{G} \rightrightarrows M$ be a Hausdorff Lie groupoid and $U$ an open, dense, $\mathcal{G}$-invariant subset such that $\mathcal{G}_U \simeq U \times U$. Let $F = M\backslash U$. Assume $\mathcal{R}(\mathcal{G}_F) = \{\pi_x \mid x \in F\}$ is an exhaustive set of representations of $C^*(\mathcal{G}_F)$. Then $\mathcal{G}$ is Fredholm and metrically amenable.*

This characterization of Fredholm groupoids, together with the properties of exhaustive families, allows us to show that large classes of groupoids are Fredholm. See for instance Corollary 5.2.8 below and all the examples in Subsection 5.3.3, and more generally [51].

**Remark 5.2.4.** The notion of exhaustive family can be linked to that of EH-amenability and to the Effros-Hahn conjecture [51, 151]. Let $\mathcal{G} \rightrightarrows F$ be an EH-amenable locally compact groupoid. Then the family of regular representations $\{\pi_y \mid y \in F\}$ of $C^*(\mathcal{G})$ is exhaustive. Thus if $U$ is a dense invariant subset such that $\mathcal{G}_U$ is the pair groupoid and $\mathcal{G}_F$ is $EH$-amenable, then $\mathcal{G}$ is Fredholm. Combining with the proof of the generalized EH conjecture [100, 174, 175] for amenable, Hausdorff, second countable groupoids, we get a set of sufficient conditions for $\mathcal{G}$ to be Fredholm.

### 5.2.2. Examples and application to Fredholm conditions

We give some examples of Fredholm groupoids that occur in applications. Recall that all the groupoids considered in this chapter are assumed to be Hausdorff.

**Example 5.2.5.** Let $\overline{\mathcal{H}} = [0, \infty] \rtimes (0, \infty)$ be the transformation groupoid with the action of $(0, \infty)$ on $[0, \infty]$ by dilation, as in Example 4.3.14. Then $\overline{\mathcal{H}}$ is Fredholm. Indeed, it is clear that $(0, \infty) \subset [0, \infty]$ is an invariant open dense subset, and $\overline{\mathcal{H}}|_{(0,\infty)} \simeq (0, \infty)^2$, the pair groupoid of $(0, \infty)$. Then $F = \{0, \infty\}$ and $\overline{\mathcal{H}}_F \simeq (0, \infty) \sqcup (0, \infty)$, the disjoint union of two amenable Lie groups, so $C^*(\overline{\mathcal{H}}_F) \simeq C_0(0, \infty) \oplus C_0(0, \infty)$. Therefore $\overline{\mathcal{H}}_F$ has Exel's property (the regular representations at 0 and $\infty$ are induced from the regular representation of the group, which is just convolution). So $\overline{\mathcal{H}}$ is Fredholm.

In the next example, we study an important class of Lie groupoids for which the set of regular representations is an exhaustive set of representations of $C^*(\mathcal{G})$. The point is that locally, our groupoid is the product of a group $G$ and a space, so its $C^*$-algebra is of the form $C^*(G) \otimes \mathcal{K}$, where $\mathcal{K}$ are the compact operators. See [51, Proposition 3.10] for a complete proof.

**Example 5.2.6.** Let $\mathcal{H} \rightrightarrows B$ be a locally trivial bundle of groups, so $d = r$, with fiber a locally compact group $G$. Then $\mathcal{H}$ has Exel's property, that is, the set of regular representations $\mathcal{R}(\mathcal{H})$ is exhaustive for $C^*_r(\mathcal{H})$, since any irreducible representation of $C^*_r(\mathcal{H})$ factors through evaluation at $\mathcal{H}_x \simeq G$, and the regular representations of $\mathcal{H}$ are obtained from the regular representation of $G$. It is exhaustive for the full algebra $C^*(\mathcal{H})$ if, and only if, the group $G$ is amenable.

More generally, let $f : M \to B$ be a continuous surjective map. The *pullback groupoid*

$$\mathcal{G} = f^{\downarrow\downarrow}(\mathcal{H}) := \{(x, g, y) \in M \times \mathcal{H} \times M \mid r(g) = f(x) \text{ and } d(g) = f(y)\}$$

is a locally compact groupoid with a Haar system that also has Exel's property. The set $\mathcal{R}(\mathcal{G})$ is exhaustive for $C^*(\mathcal{G})$ if, and only if, the group $G$ is amenable (note that $G$ is isomorphic to the isotropy group $\mathcal{H}^x_x$, for $x \in M$). This stems from the fact that $\mathcal{H}$ and $f^{\downarrow\downarrow}(\mathcal{H})$ are *Morita equivalent* groupoids, hence have homeomorphic primitive spectra (see Subsection 4.1.4 and [51]).

**Remark 5.2.7.** In fact, $f^{\downarrow\downarrow}(\mathcal{H})$ satisfies the generalized EH conjecture, and hence it has the weak-inclusion property. It will be EH-amenable if, and only if, the group $G$ is amenable (see [51]).

Putting together the previous example and Proposition 5.2.3, we conclude the following:

**Corollary 5.2.8.** *Let $\mathcal{G} \rightrightarrows M$ is a Hausdorff Lie groupoid with $U \subset M$ an open, dense, invariant subset. Set $F = M \setminus U$ and assume that we have a decomposition $\mathcal{G}_U \simeq U \times U$ and $\mathcal{G}_F \simeq f^{\downarrow\downarrow}(\mathcal{H})$; in particular,*

$$\mathcal{G} = (U \times U) \sqcup f^{\downarrow\downarrow}(\mathcal{H}),$$

*where $f : F \to B$ is a tame submersion and $\mathcal{H} \rightrightarrows B$ is a bundle of amenable Lie groups. Then $\mathcal{G}$ is Fredholm.*

Corollary 5.2.8 is enough to obtain the Fredholm property for many groupoids used in applications. Several examples can be found in [51, Section 5]. They include the $b$-groupoid modelling manifolds with poly-cylindrical ends, groupoids modelling analysis on asymptotically Euclidean spaces, asymptotically hyperbolic spaces, and the edge groupoids. Some of these examples will be discussed in Subsection 5.3.3.

We consider Fredholm groupoids because of their applications to Fredholm conditions. Let $\Psi^m(\mathcal{G})$ be the space of order $m$, classical pseudodifferential operators $P = (P_x)_{x\in M}$ on $\mathcal{G}$: we refer to Section 4.3.3 for the precise definitions. Recall that each element $P_x$ is a pseudodifferential operator on $\mathcal{G}_x$, for any $x \in M$ and that $P_x = \pi_x(P)$, with $\pi_x$ the regular representation of $\mathcal{G}$ at $x \in M$. The operator $P$ acts on $U$ via the (injective) vector representation

$$\pi_0 : \Psi^m(\mathcal{G}) \to \mathcal{B}(H^s(U), H^{s-m}(U)).$$

We denote by $L^m_s(\mathcal{G})$ the norm closure of $\Psi^m(\mathcal{G})$ in the topology of continuous operators $H^s(U) \to H^{s-m}(U)$.

Recall that a differential operator $P : C^\infty(U) \to C^\infty(U)$ is called *elliptic* if its principal symbol $\sigma(P) \in \Gamma(T^*U)$ is invertible outside the zero section [98]. The following Fredholm condition is one of the main results of [51].

**Theorem 5.2.9** (Carvalho, Nistor, Qiao [51, Theorem 4.17])**.** *Let $\mathcal{G} \rightrightarrows M$ be a Fredholm Lie groupoid and let $U \subset M$ be the dense, $\mathcal{G}$-invariant subset such that $\mathcal{G}_U \simeq U \times U$. Let $s \in \mathbb{R}$ and $P \in L^m_s(\mathcal{G}) \supset \Psi^m(\mathcal{G})$. We have*

$$\begin{aligned} P : H^s(U) \to H^{s-m}(U) \ \text{ is Fredholm } \ &\Leftrightarrow \ P \text{ is elliptic and} \\ &P_x : H^s(\mathcal{G}_x) \to H^{s-m}(\mathcal{G}_x) \ \text{ is invertible for all } x \in F := M \smallsetminus U\,. \end{aligned}$$

*Proof.* This theorem is proved by considering $a := (1+\Delta)^{(s-m)/2} P (1+\Delta)^{-s/2}$, which belongs to the $C^*$-algebra $\overline{\Psi}(\mathcal{G}) =: L^0_0(\mathcal{G})$ by the results in [112, 113]. Since $\overline{\Psi}(\mathcal{G})$ contains $C^*_r(\mathcal{G})$ as an essential ideal, the conclusion follows from Theorem 5.2.2. See [51] for more details. □

Theorem 5.2.9 extends straightforwardly to operators acting between sections of vector bundles. The operators $P_x$, for $x \in M \setminus U$, are called *limit operators* of $P$. Note that $P_x$ is invariant under the action of the isotropy group $\mathcal{G}^x_x$ on the fiber $\mathcal{G}_x$. Similar characterizations of Fredholm operators were obtained in different contexts in [68, 71, 85, 136, 184], to cite a few examples.

## 5.3. Boundary action groupoids

We describe in this section a procedure for gluing locally compact groupoids. This extends a construction of Gualtieri and Li that was used to classify the Lie groupoids integrating certain Lie algebroids [94] (see also [150]).

### 5.3.1. The gluing construction

Let $X$ be a locally compact Hausdorff space, covered by a family of open sets $(U_i)_{i\in I}$. Recall that, if $\mathcal{G} \rightrightarrows X$ is a locally compact groupoid and $U \subset X$ an open set, then the *reduction* of $\mathcal{G}$ to $U$ is the open subgroupoid $\mathcal{G}|_U := \mathcal{G}_U^U = d^{-1}(U) \cap r^{-1}(U)$.

Now, for each $i \in I$, let $\mathcal{G}_i \rightrightarrows U_i$ be a locally compact groupoid with domain $d_i$ and range $r_i$. Assume that we are given a family of isomorphisms between all the reductions

$$\phi_{ji} : \mathcal{G}_i|_{U_i\cap U_j} \to \mathcal{G}_j|_{U_i\cap U_j},$$

such that $\phi_{ij} = \phi_{ji}^{-1}$ and $\phi_{ij}\phi_{jk} = \phi_{ik}$ on the common domains. Our aim is to glue the groupoids $\mathcal{G}_i$ to build a groupoid $\mathcal{G} \rightrightarrows X$ such that, for all $i \in I$,

$$\mathcal{G}|_{U_i} \simeq \mathcal{G}_i.$$

As a topological space, the groupoid $\mathcal{G}$ is defined as the quotient

$$\mathcal{G} = \bigsqcup_{i\in I} \mathcal{G}_i \Big/ \sim, \tag{5.1}$$

where $\sim$ is the equivalence relation generated by $g \sim \phi_{ji}(g)$, for all $i, j \in I$ and $g \in \mathcal{G}_i$. Since each $\mathcal{G}_i$ is a locally compact space, the space $\mathcal{G}$ is also locally compact for the quotient topology. If $g \in \mathcal{G}$ is the equivalence class of a $g_i \in \mathcal{G}_i$, we define

$$d(g) = d_i(g_i) \qquad \text{and} \qquad r(g) = r_i(g_i).$$

Because the groupoids $\mathcal{G}_i$ are isomorphic on common domains $U_i \cap U_j$, for $i, j \in I$, this definition is independent on the choice of the representative $g_i$. The unit $u : X \to \mathcal{G}$ and inverse maps are defined in the same way. Therefore, the subsets $\mathcal{G}|_{U_i} = r^{-1}(U_i) \cap d^{-1}(U_i)$ are well defined, for each $i \in I$.

**Lemma 5.3.1.** *For each $i \in I$, the quotient map $\pi_i : \mathcal{G}_i \to \mathcal{G}$ induces an homeomorphism (of topological spaces)*

$$\pi_i : \mathcal{G}_i \to \mathcal{G}|_{U_i}.$$

*Proof.* The topology on $\mathcal{G}$ is the coarsest one such that each quotient map $\pi_i$ is open and continuous, for every $i \in I$. Moreover, for any $i \in I$, the definition of the equivalence relation $\sim$ in Equation (5.1) implies that $\pi_i$ is injective. Therefore, the map $\pi_i$ is a homeomorphism onto its image, which is obviously contained in $\mathcal{G}|_{U_i}$.

To prove that $\pi_i(\mathcal{G}_i) = \mathcal{G}|_{U_i}$, let $g \in \mathcal{G}|_{U_i}$ be represented by an element $g_j \in \mathcal{G}_j$, for $j \in I$. Then $g_j \in \mathcal{G}_j|_{U_i\cap U_j}$, which is isomorphic to $\mathcal{G}_i|_{U_i\cap U_j}$ through $\phi_{ij}$ : thus $g$ also has a representative in $\mathcal{G}_i$. This shows that $\pi_i(\mathcal{G}_i) = \mathcal{G}|_{U_i}$. □

In particular, Lemma 5.3.1 implies that the structural maps $d, r, u$ and $\iota$ are continuous and that the domain and range maps $d, r : \mathcal{G} \to X$ are open. The only missing element to have a groupoid structure on $\mathcal{G}$ is a well-defined product. Therefore, define the set of composable arrows by

$$\mathcal{G}^{(2)} = \{(g, h) \in \mathcal{G} \mid d(g) = r(g)\}.$$

A problem is that there are a priori no relation between the two groupoids $\mathcal{G}_i$ and $\mathcal{G}_j$, for $i \neq j$. Thus, if $(g_i, g_j) \in \mathcal{G}^{(2)}$ with $g_i \in \mathcal{G}_i$ and $g_j \in \mathcal{G}_j$, then there is a priori no obvious way of defining the product $g_i g_j$ in $\mathcal{G}$. A way around this issue is to introduce a "gluing condition", so that any composable pair $(g, h) \in \mathcal{G}^{(2)}$ is actually contained in a single groupoid $\mathcal{G}_k$, for a $k \in I$.

**Definition 5.3.2.** We say that a family $(\mathcal{G}_i \rightrightarrows U_i)_{i \in I}$ of locally compact groupoids satisfy the *weak gluing condition* if for every composable pair $(g, h) \in \mathcal{G}^{(2)}$, there is an $i \in I$ such that both $g$ and $h$ have a representative in $\mathcal{G}_i$.

An equivalent statement of Definition 5.3.2 is to say that the family $(\mathcal{G}_i^{(2)})_{i \in I}$ is an open cover of the space of composable arrows $\mathcal{G}^{(2)}$.

**Lemma 5.3.3.** *Assume that the family $(\mathcal{G}_i)_{i \in I}$ satisfy the weak gluing condition. Then there is a unique groupoid structure on*

$$\mathcal{G} = \bigsqcup_{i \in I} \mathcal{G}_i \Big/ \sim$$

*such that the projection maps $\pi_i : \mathcal{G}_i \to \mathcal{G}|_{U_i}$ are isomorphisms of locally compact groupoids, for every $i \in I$.*

*Proof.* Let $(g, h) \in \mathcal{G}^{(2)}$ be a composable pair. The weak gluing condition implies that there is an $i \in I$ such that $g$ and $h$ have representatives $g_i$ and $h_i$ in $\mathcal{G}_i$. We thus define the product $gh$ as the class of $g_i h_i$ in $\mathcal{G}$, and we check at once that this does not depend of a choice of representative for $g$ and $h$. Lemma 5.3.1 and the definition of the structural maps on $\mathcal{G}$ imply that each $\pi_i : \mathcal{G}_i \to \mathcal{G}|_{U_i}$ is an isomorphism of locally compact groupoids, for each $i \in I$.

To show the uniqueness of the groupoid structure on $\mathcal{G}$, let us assume conversely that each map $\pi_i : \mathcal{G}_i \to \mathcal{G}|_{U_i}$ is a groupoid isomorphism. Since the reductions $(\mathcal{G}|_{U_i})_{i \in I}$ cover $\mathcal{G}$, the domain, range, identity and inverse maps of $\mathcal{G}$ are prescribed by those of each $\mathcal{G}_i$. Moreover, the weak gluing condition implies that, for each composable pair $(g, h) \in \mathcal{G}^{(2)}$, both $g$ and $h$ lie in a same reduction $\mathcal{G}|_{U_i}$. Thus the product on $\mathcal{G}$ is also determined by those of each groupoid $\mathcal{G}_i$, for $i \in I$. □

**Definition 5.3.4.** The groupoid $\mathcal{G}$ of Lemma 5.3.3 defines the *gluing* (or *glued groupoid*) of a family of locally compact groupoids $(\mathcal{G}_i)_{i \in I}$ satisfying the gluing condition. We denote it

$$\mathcal{G} = \bigcup_{i \in I} \mathcal{G}_i,$$

when there is no ambiguity about the family of isomorphisms $(\phi_{ij})_{i,j}$ involved.

**Remark 5.3.5.** The glued groupoid can also be defined by a universal property. Assume we only have two groupoids $\mathcal{G}_1 \rightrightarrows U_1$ and $\mathcal{G}_2 \rightrightarrows U_2$, and let $\mathcal{G}_{12} := \mathcal{G}_1|_{U_1 \cap U_2} \simeq \mathcal{G}_2|_{U_1 \cap U_2}$. Then $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$ is the *pushout* of the inclusions morphisms $j_i : \mathcal{G}_{12} \hookrightarrow \mathcal{G}_i$, for $i = 1, 2$. It is the "smallest" groupoid such that there is a commutative diagram

$$\begin{tikzcd} \mathcal{G} & \mathcal{G}_2 \arrow[l, hook'] \\ \mathcal{G}_1 \arrow[u, hook] & \mathcal{G}_{12}. \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

When we have a general family $(\mathcal{G}_i)_{i \in I}$ satisfying the gluing condition, the glued groupoid can similarly be defined as the colimit relative to the inclusions $\mathcal{G}|_{U_i \cap U_j} \hookrightarrow \mathcal{G}_i$, for all $i, j \in I$.

**Remark 5.3.6.** It is possible for a family $(\mathcal{G}_i)_{i \in I}$ to satisfy the weak gluing condition, even though there is a pair $(\mathcal{G}_{i_0}, \mathcal{G}_{j_0})$ that do not satisfy the gluing condition, for some $i_0, j_0 \in I$. For instance, let $X$ be a locally compact, Hausdorff space and $U_1, U_2$ two distinct open subsets in $X$ with non-empty intersection $U_{12}$. Let

$$\mathcal{G}_0 = X \times X, \quad \mathcal{G}_1 = U_1 \times U_1 \quad \text{and} \quad \mathcal{G}_2 = U_2 \times U_2$$

be pair groupoids over $X$, $U_1$ and $U_2$ respectively. The family $(\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2)$ satisfies the weak gluing condition of Definition 5.3.2, and may be glued to obtain the groupoid $\mathcal{G} = X \times X = \mathcal{G}_0$. However, the pair $(\mathcal{G}_1, \mathcal{G}_2)$ does not satisfy the weak gluing condition.

**Lemma 5.3.7.** *Let $(\mathcal{G}_i)$ be a family of groupoids satisfying the weak gluing condition. If each $\mathcal{G}_i$, for $i \in I$, is a Hausdorff groupoid, then the gluing $\mathcal{G} = \bigcup_{i \in I} \mathcal{G}_i$ is also Hausdorff.*

*Proof.* Let $g, h \in \mathcal{G}$. There are two cases.

- Assume $d(g) = d(h)$ and $r(g) = r(h)$. Then, because of the gluing condition, there is an $i \in I$ such that $g$ and $h$ are both in the Hausdorff groupoid $\mathcal{G}|_{U_i}$.
- Otherwise, either $d(g) \neq d(h)$ or $r(g) \neq r(h)$. Let us assume the former. Then, since $X$ is Hausdorff, there are open sets $U, V \subset X$ such that $d(g) \in U$, $d(h) \in V$ and $U \cap V = \emptyset$. Thus $g \in \mathcal{G}_U$ and $h \in \mathcal{G}_V$, which are disjoint open subsets of $\mathcal{G}$. $\square$

We also introduce the *strong gluing condition*, which is often easier to check.

**Definition 5.3.8.** We say that the family $(\mathcal{G}_i \rightrightarrows U_i)_{i \in I}$ of locally compact groupoids satisfy the *strong gluing condition* if, for each $x \in X$, there is an $i_x \in I$ such that

$$\mathcal{G}_j \cdot x \subset U_{i_x}$$

for all $j \in I$.

In other words, the orbit of a point through the action of $\mathcal{G}$ should always be induced by a single element of the family $(\mathcal{G}_i)_{i \in I}$.

**Lemma 5.3.9.** *Let $(\mathcal{G}_i)_{i\in I}$ be a family of groupoids which satisfies the strong gluing condition. Then the family $(\mathcal{G}_i)_{i\in I}$ also satisfies the weak gluing condition.*

*Proof.* Let $(g,h) \in \mathcal{G}^{(2)}$, and assume that $g$ has a representative $g_i \in \mathcal{G}_i$ and $h$ a representative $h_j \in \mathcal{G}_j$. Let $x = d(g) = r(h)$. The gluing condition implies that there is an $i_x \in I$ such that $\mathcal{G}_i \cdot x \subset U_{i_x}$ and $\mathcal{G}_j \cdot x \subset U_{i_x}$. Thus $r_i(g_i) \in U_{i_x}$, so $g_i \in \mathcal{G}_i|_{U_i\cap U_{i_x}}$. But there is an isomorphism

$$\phi_{i_x i} : \mathcal{G}_i|_{U_i\cap U_{i_x}} \to \mathcal{G}_{i_x}|_{U_i\cap U_{i_x}}$$

so that $g$ actually has a representative $g_{i_x}$ in $\mathcal{G}_{i_x}$. The same arguments show that $h$ also has a representative $h_{i_x} \in \mathcal{G}_{i_x}$. ☐

We conclude this subsection with a condition for which a groupoid $\mathcal{G} \rightrightarrows X$ may be written as the gluing of its reductions. This definition was introduced by Gualtieri and Li for Lie algebroids [94].

### 5.3.2. Gluing Lie groupoids

Let $M$ be a manifold with corners, and $(U_i)_{i\in I}$ an open cover of $M$. Let $(\mathcal{G}_i)_{i\in I}$ be a family of *Lie groupoids* satisfying the weak gluing condition of Definition 5.3.2. Assume that the morphisms $\phi_{ji} : \mathcal{G}_i|_{U_i\cap U_j} \to \mathcal{G}_j|_{U_i\cap U_j}$ are Lie groupoid morphisms, and let $\mathcal{G} := \bigcup_{i\in I} \mathcal{G}_i$ be the glued groupoid over $M$.

**Lemma 5.3.10.** *If each $\mathcal{G}_i$, for $i \in I$, is a Lie groupoid, then there is a unique Lie groupoid structure on $\mathcal{G}$ such that $\pi_i : \mathcal{G}_i \to \mathcal{G}|_{U_i}$ is an isomorphism of Lie groupoids, for all $i \in I$.*

*Proof.* By Definition 5.3.4, the reductions $\mathcal{G}|_{U_i} \simeq \mathcal{G}_i$, for $i \in I$, provide an open cover of $\mathcal{G}$. Since each $\mathcal{G}_i$ is a Lie groupoid, and all $\phi_{ij}$ are smooth, this induces a manifold structure on $\mathcal{G}$. Each structural map of $\mathcal{G}$ coincides locally with a structural map of one of the groupoids $\mathcal{G}_i$, hence is smooth. This gives the Lie groupoid structure. ☐

**Remark 5.3.11.** A similar statement holds when each $\mathcal{G}_i$ is a continuous family groupoids, for all $i \in I$: then $\mathcal{G}$ is also a continuous family groupoid [112, 158].

To specify the Lie algebroid of $\mathcal{G}$, we need first study the gluing of Lie algebroids. For each $i \in I$, let $A_i \to U_i$ be a Lie algebroid. Assume that there are Lie algebroid isomorphisms $\psi_{ij} : A_i|_{U_i\cap U_j} \to A_j|_{U_i\cap U_j}$ covering the identity, such that $\psi_{ij}^{-1} = \psi_{ji}$ and $\psi_{ij}\psi_{jk} = \psi_{ik}$ on common domains. As vector bundles, the family $(A_i)_{i\in I}$ is in particular a family of groupoids that satisfies the strong gluing condition of Definition 5.3.8 (the orbit of any $x \in M$ is reduced to $\{x\}$). Thus, the gluing $A = \bigcup_{i\in I} A_i$ is a smooth vector bundle on $M$, with inclusion maps $\pi_i : A_i \hookrightarrow A$.

**Lemma 5.3.12.** *There is a unique Lie algebroid structure on $A = \bigcup_{i\in I} A_i$ such that each map $\pi_i : A_i \to A$ is a morphism of Lie algebroids.*

*Proof.* By definition, the Lie algebroids $A|_{U_i} \simeq A_i$, for all $i \in I$, provide an open cover of $A$. Let $X, Y \in \Gamma(A)$, and define $[X,Y] \in \Gamma(A)$ by

$$[X,Y]|_{U_i} := [X|_{U_i}, Y|_{U_i}]_i,$$

where $[.,.]_i$ is the Lie bracket on $A_i$. Since $A_i|_{U_i \cap U_j}$ and $A_j|_{U_i \cap U_j}$ are isomorphic as Lie algebroid, the section $[X,Y]$ is well-defined on $U_i \cap U_j$, for all $i, j \in I$. This defines the Lie bracket on $\Gamma(A)$. The anchor map is similarly defined as $\rho(X)|_{U_i} := \rho_i(X|_{U_i})$, with $\rho_i$ the anchor map of $A_i$. Because the family $(A_i)_{i\in I}$ covers $A$, this it is the unique Lie algebroid structure on $A$ such that each map $\pi : A_i \to A|_{U_i}$ is a Lie algebroid isomorphism. □

**Lemma 5.3.13.** *Let $(\mathcal{G}_i \rightrightarrows U_i)_{i\in I}$ be a family of Lie groupoids satisfying the gluing condition, with isomorphisms $\phi_{ij} : \mathcal{G}_j|_{U_i\cap U_j} \to \mathcal{G}_i|_{U_i\cap U_j}$. The Lie algebroid of the resulting glued groupoid $\mathcal{G} = \bigcup_{i\in I} \mathcal{G}_i$ is isomorphic to the gluing of the family $(A\mathcal{G}_i)_{i\in I}$, with Lie algebroid isomorphisms $(\phi_{ij})_* : A\mathcal{G}_i|_{U_i\cap U_j} \to A\mathcal{G}_j|_{U_i\cap U_j}$, for $i, j \in I$.*

*Proof.* By definition of the quotient maps $\pi_i : \mathcal{G}_i \to \mathcal{G}$, the map $\pi_j^{-1} \circ \pi_i$ coincides with the isomorphism $\phi_{ji} : \mathcal{G}_i|_{U_i\cap U_j} \to \mathcal{G}_j|_{U_i\cap U_j}$, for all $i, j \in I$. Let $\xi \in A\mathcal{G}_i|_{U_i\cap U_j}$. Then

$$(\pi_i)_*(\xi) = (\pi_j)_* \circ (\pi_j^{-1} \circ \pi_i)_*(\xi) = (\pi_j)_* \circ (\phi_{ji})_*(\xi) \in A\mathcal{G}|_{U_i\cap U_j} \tag{5.2}$$

Let $\Psi : \bigsqcup_{i\in I} A\mathcal{G}_i \to A\mathcal{G}$ bet the map given by $\Psi(\xi) := (\pi_i)_*(\xi)$, whenever $\xi \in A\mathcal{G}_i$. Equation (5.2) implies that $\Psi$ induces a map from the quotient $A = \bigcup_{i\in I} A\mathcal{G}_i$, which is the glued algebroid, to $A\mathcal{G}$. Each map $\pi_i : \mathcal{G}_i \to \mathcal{G}$ gives an isomorphism $(\pi_i)_* : A\mathcal{G}_i \to A\mathcal{G}|_{U_i}$, so $\Psi : A \to A\mathcal{G}$ is also a Lie algebroid isomorphism. □

### 5.3.3. Boundary action groupoids

Our aim is to study Fredholm conditions for algebras of differential operators generated by Lie groupoids $\mathcal{G} \rightrightarrows M$. To this end, we define the class of *boundary action groupoids*, which are obtained by gluing reductions of action groupoids. We will show that many examples of groupoids arising in analysis on open manifold belong to this class, and obtain Fredholm condition for the associated differential operators.

Recall that gluing conditions were discussed in Subsection 5.3.1.

**Definition 5.3.14.** A Lie groupoid $\mathcal{G} \rightrightarrows M$ is *a boundary action groupoid* if

(i) there is an open dense $\mathcal{G}$-invariant subset $U \subset M$ such that $\mathcal{G}_U \simeq U \times U$;

(ii) there is an open cover $(U_i)_{i\in I}$ of $M$ such that for all $i \in I$, we have a Hausdorff manifold $X_i$, a Lie group $G_i$ acting smoothly on $X_i$ on the right and an open subset $U_i' \subset X_i$ diffeomorphic to $U_i$ satisfying

$$\mathcal{G}|_{U_i} \simeq (X_i \rtimes G_i)|_{U_i'};$$

(iii) the family of groupoids $(\mathcal{G}|_{U_i})_{i\in I}$ satisfy the weak gluing condition, with the obvious identifications of $\mathcal{G}|_{U_i}$ and $\mathcal{G}|_{U_j}$ with $\mathcal{G}|_{U_i\cap U_j}$ over common domains.

In other words, boundary action groupoids are groupoids that are obtained by gluing reductions of action groupoids, and that are simply the pair groupoid over a dense orbit. Note that, as an open dense $\mathcal{G}$-orbit in $M$, the subset $U$ in Definition 5.3.14 is uniquely determined by $\mathcal{G}$.

**Example 5.3.15.** If $M_0$ is a smooth manifold (without corners), then the pair groupoid $\mathcal{G} = M_0 \times M_0$ is a boundary action groupoid. Indeed, for any triple $(x, y, z) \in M_0^3$, we can choose an open subset $U_{x,y,z} \subset M_0$ that contains $x$, $y$, and $z$ and is such that $U_{x,y,z}$ is diffeomorphic to an open subset $U'_{x,y,z} \subset \mathbb{R}^n$ (just choose three disjoint, relatively compact coordinate charts near each point $x$, $y$ and $z$ and take $U_{x,y,z}$ to be their disjoint union). Then

$$\mathcal{G}|_{U_{x,y,z}} \simeq (\mathbb{R}^n \times \mathbb{R}^n)|_{U'_{x,y,z}} \simeq (\mathbb{R}_n \rtimes \mathbb{R}^n)|_{U'_{x,y,z}},$$

where $\mathbb{R}^n$ acts on itself by translation. Moreover the family of groupoids $(\mathcal{G}|_{U_{x,y,z}})$, for $x, y, z \in M_0$, satisfy the weak gluing condition: for any composable pair $(x, y)$ and $(y, z)$ in $\mathcal{G}$, both $(x, y)$ and $(y, z)$ are contained in $\mathcal{G}|_{U_{x,y,z}}$. This shows (ii) and (iii) from Definition 5.3.14, whereas (i) is trivially satisfied.

Other practical examples will be introduced in Subsection 5.3.4 below. One of the main points of this definition is to have a good understanding of how $\mathcal{G}_U$ and $\mathcal{G}_{M \setminus U}$ are glued together near the boundary. In particular:

**Lemma 5.3.16.** *Boundary action groupoids are Hausdorff.*

*Proof.* We keep the notations of Definition 5.3.14 above. Note that all $(X_i \rtimes G_i)|_{U'_i}$ are Hausdorff groupoids (as subsets of the Hausdorff spaces $X_i \times G_i$). Since the groupoids $(\mathcal{G}|_{U_i})_{i \in I}$ satisfy the weak gluing condition, the groupoid $\mathcal{G}$ is obtained by gluing Hausdorff groupoids. The result then follows from Lemma 5.3.7. □

Lemmas 5.3.17 to 5.3.19 give some possible combinations of boundary action groupoids that preserve the local structure.

**Lemma 5.3.17.** *Let $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ be boundary action groupoids. Then $\mathcal{G} \times \mathcal{H} \rightrightarrows M \times N$ is a boundary action groupoid.*

*Proof.* First, let $U, V$ be the respective open dense orbits of $\mathcal{G}$ and $\mathcal{H}$. Then $U \times V$ is an open dense orbit for $\mathcal{G} \times \mathcal{H}$, and $(\mathcal{G} \times \mathcal{H})_{U \times V}$ is the pair groupoid $(U \times V)^2$. Secondly, let $(U_i)_{i \in I}$ and $(V_j)_{j \in J}$ be respective open covers of $M$ and $N$ such that we have isomorphisms

$$\mathcal{G}|_{U_i} \simeq (X_i \rtimes G_i)_{U'_i} \qquad \text{and} \qquad \mathcal{H}|_{V_j} \simeq (Y_j \rtimes H_j)|_{V'_j}$$

and both families $(\mathcal{G}|_{U_i})_{i \in I}$ and $(\mathcal{H}|_{V_j})_{j \in J}$ satisfy the weak gluing condition. Then the family $\{(\mathcal{G} \times \mathcal{H})|_{U_i \times V_j}\}$, for $i \in I$ and $j \in J$, satisfy the weak gluing condition over $M \times N$ and we have

$$(\mathcal{G} \times \mathcal{H})|_{U_i \times V_j} \simeq (X_i \times Y_j) \rtimes (G_i \times H_j),$$

for all $i \in I$ and $j \in J$, where the action of $G_i \times H_j$ is the product action. □

**Lemma 5.3.18.** *Let $\mathcal{G} \rightrightarrows M$ be a boundary action groupoid and $V$ an open subset of $M$. Then $\mathcal{G}|_V$ is a boundary action groupoid.*

*Proof.* Let $U$ be the unique open dense orbit of $\mathcal{G}$. Then $U \cap V$ is the unique open dense orbit of $\mathcal{G}|_V$, and $(\mathcal{G}|_{U\cap V}) \simeq (U \times U)|_{U\cap V}$ is the pair groupoid of $U \cap V$. Moreover, there is an open cover $(U_i)_{i\in I}$ of $M$ with isomorphisms

$$\mathcal{G}|_{U_i} \simeq (X_i \rtimes G_i)|_{U_i'}$$

for all $i \in I$, and such that the family $(\mathcal{G}|_{U_i})_{i\in I}$ satisfy the weak gluing condition. For all $i \in I$, let $V_i = U_i \cap V$ and $V_i'$ be the image of $V_i$ in $U_i'$. The weak gluing condition imply that, for any pair $(g, h)$ of composable arrows in $\mathcal{G}|_V$, there is an $i \in I$ such that $g, h$ are both in $\mathcal{G}|_{U_i}$. Then $g, h \in \mathcal{G}|_{V_i}$, which shows that the family $(\mathcal{G}|_{V_i})_{i\in I}$ satisfy the weak gluing condition. Finally, we have isomorphisms

$$\mathcal{G}|_{V_i} = \mathcal{G}|_{U_i\cap V} \simeq (X_i \rtimes G_i)|_{V_i'}$$

for all $i \in I$, which concludes the proof. ☐

**Lemma 5.3.19.** *Let $M$ be a manifold with corners, and assume that we have two open subsets $U, V \subset M$ such that*

*(i) the set $U$ is dense in $M$ and $M = U \cup V$,*
*(ii) there is a boundary action groupoid $\mathcal{H} \rightrightarrows V$ whose unique open dense orbit is $U \cap V$.*

*Then the glued groupoid $\mathcal{G} = \mathcal{H} \cup (U \times U)$ over $M$ is a boundary action groupoid.*

Lemma 5.3.19 should be thought as a way of "attaching ends" to a pair groupoid, which will model the geometry at infinity.

*Proof.* First, the definition of boundary action groupoids gives $\mathcal{H}_{U\cap V} \simeq (U \cap V)^2$, so that $U \times U$ and $\mathcal{H}$ are isomorphic over $U \cap V$. The pair $(\mathcal{H}, U \times U)$ satisfies the strong gluing condition (the $\mathcal{G}$-orbit of any point in $M$ is either $U$ or contained in $V \setminus U$), so the gluing has a well-defined groupoid structure.

It follows from the properties of the gluing (Lemma 5.3.3) that $U$ is the unique open dense $\mathcal{G}$-orbit in $M$, and that $\mathcal{G}_U \simeq U \times U$. We know from Example 5.3.15 that $U \times U$ is a boundary action groupoid. Therefore, there is an open cover $(U_i)_{i\in I}$ of $U$ and an open cover $(V_j)_{j\in J}$ of $V$ with isomorphisms

$$(U \times U)_{U_i} \simeq (\mathbb{R}^n \rtimes \mathbb{R}^n)|_{U_i'} \qquad \text{and} \qquad \mathcal{H}|_{V_j} \simeq (X_j \rtimes G_j)_{V_j'}, \tag{5.3}$$

and such that the respective families of reductions satisfy the weak gluing condition. Because $\mathcal{G}|_V \simeq \mathcal{H}$ and $\mathcal{G}_U \simeq U \times U$, the isomorphisms of Equation (5.3) also hold for the reductions $\mathcal{G}|_{U_i}$ and $\mathcal{G}|_{V_j}$. Besides, any two composable arrows for $\mathcal{G}$ are either in $\mathcal{G}_U$ or in $\mathcal{G}|_V$, so the family $(\mathcal{G}|_{U_i})_{i\in I} \cup (\mathcal{G}|_{V_j})_{j\in J}$ also satisfies the weak gluing condition. ☐

### 5.3.4. Examples

We will show here that many groupoids occurring in the study of analysis on singular manifolds are boundary action groupoids. We will explain in Subsection 5.3.5 how this class of groupoids allows to obtain Fredholm conditions for many interesting differential operators. Our examples are based on the following result:

**Theorem 5.3.20.** *Let $M$ be a manifold with corners and $M_0 := M \setminus \partial M$. Let $A \to M$ be a Lie algebroid, such that the anchor map $\rho$ induces an isomorphism $A|_{M_0} \simeq TM_0$ and $\rho(A|_F) \subset TF$ for any face $F$ of $M$. Then there is a unique Lie groupoid $\mathcal{G} \rightrightarrows M$ integrating $A$, such that $\mathcal{G}_{M_0} \simeq M_0 \times M_0$ and $\mathcal{G}_{\partial M}$ is d-simply-connected.*

*Proof.* The existence of such a groupoid has been proven by Debord [69] and Nistor [148]. The uniqueness follows from Lie's second theorem for Lie algebroid, i.e. there exists (up to isomorphism) at most one $d$-connected groupoid integrating $A|_{\partial M}$. If $\mathcal{H}$ is another groupoid satisfying the assumptions of Theorem 5.3.20, then $\mathcal{H}_{\partial M}$ and $\mathcal{G}_{\partial M}$ are isomorphic. The main result in [148] implies that $\mathcal{G}$ is then isomorphic to $\mathcal{H}$. $\square$

The groupoid $\mathcal{G}$ in Theorem 5.3.20 will be called the *maximal integration* of $A$. Based on this theorem, we give several examples of boundary action groupoids which occur naturally in the context of analysis on open manifolds : see [51] for more details.

**Example 5.3.21** (Zero-groupoid)**.** Consider $G_n := (0, \infty) \ltimes \mathbb{R}^{n-1}$, where $(0, \infty)$ acts by dilation on $\mathbb{R}^{n-1}$. The right action of $G_n$ upon itself extends uniquely to an action on $X_n := [0, \infty) \times \mathbb{R}^{n-1}$, by setting

$$(x_1, \ldots, x_n) \cdot (t, \xi_2, \ldots, \xi_n) = (tx_1, x_2 + x_1\xi_2, \ldots, x_n + x_1\xi_n).$$

The Lie algebra of fundamental vector fields for this action (recall Example 4.3.14) is the one spanned by $(x_1\partial_1, \ldots, x_1\partial_n)$ on $X_n$.

To generalize this setting, let $M$ be a manifold with boundary and let $\mathcal{V}_0$ be the Lie algebra of all vector fields on $M$ vanishing on $\partial M$. In a local coordinate system $[0, \infty) \times \mathbb{R}^{n-1}$ near $\partial M$, we have

$$\mathcal{V}_0 = \mathrm{Span}(x_1\partial_1, \ldots, x_1\partial_n),$$

as a $C^\infty(M)$-module.

It follows from Serre-Swan's Theorem that there is a unique Lie algebroid $A_0 \to M$ such that the anchor map induces an isomorphism $\Gamma(A_0) \simeq \mathcal{V}_0$. The *zero-groupoid* $\mathcal{G}_0 \rightrightarrows M$ is the maximal integration of $A_0$, as given by Theorem 5.3.20: it is the natural space for the Schwartz kernels of differential operators that are induced by *asymptotically hyperbolic* metrics on $M_0$ [137].

**Theorem 5.3.22.** *The 0-groupoid $\mathcal{G}_0 \rightrightarrows M$ is a boundary action groupoid. Moreover, for each $p \in \partial M$, there is a neighborhood $U$ of $p$ in $M$, and an open set $V \subset [0, \infty) \times \mathbb{R}^{n-1}$, such that*

$$\mathcal{G}_0|_U \simeq (X_n \rtimes G_n)|_V.$$

*Proof.* For each $p \in \partial M$, there is a neighborhood $U$ of $p$ in $M$ that is diffeomorphic to an open subset $V \subset [0, \infty) \times \mathbb{R}^{n-1}$, through $\phi : U \to V$. The diffeomorphism $\phi$ maps $\partial U$ to $\partial V$, so $\phi_*(\mathcal{V}_0(U)) = \mathcal{V}_0(V)$. This implies that there is an isomorphism $A_0(U) \simeq A_0(V)$ covering $\phi$. Both groupoids $\mathcal{G}_0|_U$ and $(X_n \rtimes G_n)|_V$ are maximal integrations of $A_0(U) \simeq A_0(V)$, so Theorem 5.3.20 implies that $\mathcal{G}_0|_U \simeq (X_n \rtimes G_n)|_V$.

To prove that $\mathcal{G}_0$ is a boundary action groupoid, let $(U_i)_{i=1}^n$ be an open cover of $\partial M$, such that each $\mathcal{G}_0|_{U_i}$ is isomorphic to a reduction of $X_n \rtimes G_n$, for all $i = 1, \dots, n$. Let $U_0 = M_0$. Then $(U_i)_{i=0}^n$ is an orbit cover of $M$ that satisfies the assumptions of Definition 5.3.14. □

**Example 5.3.23.** Example 5.3.21 can be slightly generalized by replacing $\mathcal{V}_0$ with a Lie algebra $\mathcal{V} \subset \Gamma(TM)$ such that, for any point $p \in \partial M$, there is a $n$-tuple $\alpha \in \mathbb{N}^n$ and a local coordinate system $[0, \infty) \times \mathbb{R}^{n-1}$ near $p$ with

$$\mathcal{V} = \mathrm{Span}(x_1^{\alpha_1}\partial_1, \dots, x_1^{\alpha_n}\partial_n).$$

If $\alpha_1 = 1$ and every $\alpha_i \geq 1$, for $i = 2, \dots, n$, then the maximal integration $\mathcal{G} \rightrightarrows M$ of $\mathcal{V}$ is again a boundary action groupoid. Indeed, consider the action of $(0, \infty)$ on $\mathbb{R}^{n-1}$ given by

$$t \cdot (x_2, \dots, x_n) = (t^{\alpha_2}x_2, \dots, t^{\alpha_n}x_n),$$

and form the semidirect product $G_\alpha = (0, \infty) \ltimes_\alpha \mathbb{R}^{n-1}$ given by this action. As in Example 5.3.21, the right action of $G_\alpha$ upon itself extends uniquely to an action on $X_n := [0, \infty) \times \mathbb{R}^{n-1}$, by setting

$$(x_1, \dots, x_n) \cdot (t, \xi_2, \dots, \xi_n) = (tx_1, x_2 + x_1^{\alpha_2}\xi_2, \dots, x_n + x_1^{\alpha_n}\xi_n).$$

An argument analogous to that of Theorem 5.3.22 shows that $\mathcal{G}$ is obtained by gluing reductions of actions groupoids $X_n \rtimes G_\alpha$, for some $n$-tuples $\alpha \in \mathbb{N}^n$.

**Example 5.3.24** (Scattering groupoid)**.** Let $\mathbb{S}^n_+$ be the stereographic compactification of $\mathbb{R}^n$. Consider the action of $\mathbb{R}^n$ upon itself by translation, and extend it to $\mathbb{S}^n_+$ in the only possible way, by a trivial action on $\partial\mathbb{S}^n_+$. The action groupoid $\mathcal{G}_{sc} = \mathbb{S}^n_+ \rtimes \mathbb{R}^n$ has been much studied in the literature, and is related to the study of the spectrum of the $N$-body problem in Euclidean space [85, 142].

As in Example 5.3.24, we can generalize this setting to any manifold with boundary $M$. Let $\mathcal{V}_b$ be the Lie algebra of vector fields on $M$ which are tangent to the boundary, and let $x \in C^\infty(M)$ be a defining function for $\partial M$. We define the Lie algebra of *scattering vector fields* on $M$ as $\mathcal{V}_{sc} := x\mathcal{V}_b$. In a local coordinate system $[0, \infty) \times \mathbb{R}^{n-1}$ near $\partial M$, we have

$$\mathcal{V}_{sc} = \mathrm{Span}(x_1^2\partial_1, x_1\partial_2, \dots, x_1\partial_n),$$

as a $C^\infty(M)$-module. One can check that, when $M = \mathbb{S}^n_+$ as above, then $\mathcal{V}_{sc}$ is the Lie algebra of fundamental vector fields induced by the action of $\mathbb{R}^n$ on $\mathbb{S}^n_+$.

As in Example 5.3.21, there is a unique Lie algebroid $A_{sc} \to M$ whose sections are isomorphic to $\mathcal{V}_{sc}$ through the anchor map. The *scattering groupoid* $\mathcal{G}_{sc} \rightrightarrows M$ is the maximal integration of $A_{sc}$, and it generates the algebra of differential operators on $M_0$ that are induced by *asymptotically Euclidean* metrics [137]. The proof of Theorem 5.3.22 can be adapted to this context to give:

**Theorem 5.3.25.** *The scattering groupoid $\mathcal{G}_{sc} \rightrightarrows M$ is a boundary action groupoid. Moreover, for each $p \in \partial M$, there is a neighborhood $U$ of $p$ in $M$, and an open set $V \subset \mathbb{S}^n_+$, such that*

$$\mathcal{G}_{sc}|_U \simeq (\mathbb{S}^n_+ \rtimes \mathbb{R}^n)|_V.$$

We introduce and study in Section 5.4 another example of boundary action groupoid, used to model layer potentials methods on conical domains.

### 5.3.5. Fredholm conditions

Let $\mathcal{G} \rightrightarrows M$ be a boundary action groupoid, with $U$ its unique dense $\mathcal{G}$-orbit. Our aim in this Subsection is to obtain some conditions under which $\mathcal{G}$ is a Fredholm groupoid, as introduced in Subsection 5.2.1.

We again denote by $\Psi^\infty(\mathcal{G})$ the algebra of pseudodifferential operators on the groupoid $\mathcal{G}$, introduced in details in Section 4.2. As before, let us also denote by $L^m_s(\mathcal{G})$ the closure of the space of order-$m$ pseudodifferential operators in $\mathcal{B}(H^s(U), H^{s-m}(U))$.

**Theorem 5.3.26.** *Let $\mathcal{G} \rightrightarrows M$ be a boundary action groupoid, and $U \subset M$ its unique dense orbit. Assume that the action of $\mathcal{G}$ on $F := M \setminus U$ is trivial, and that for all $x \in \partial M$, the group $\mathcal{G}^x_x$ is amenable. Let $P$ be an operator in $L^m_s(\mathcal{G})$. Then for all $s \in \mathbb{R}$, the operator $P : H^s(U) \to H^{s-m}(U)$ is Fredholm if, and only if:*

*(i) $P$ is elliptic, and*
*(ii) $P_x : H^s(\mathcal{G}^x_x) \to H^{s-m}(\mathcal{G}^x_x)$ is invertible for all $x \in F$.*

Under the assumptions of Theorem 5.3.26, the characterization of Fredholm operators in $L^m_s(\mathcal{G})$ reduces to the study of right-invariant operators $P_x$ on the amenable groups $\mathcal{G}^x_x$, for $x \in M \setminus U$. It should be emphasized that, if $P$ is a geometric operator (Dirac, Laplacian…) for a metric on $U$ which is "compatible" with $\mathcal{G}$ (in a sense made precise in [113]), then each $P_x$ is an operator of the same type induced by a right-invariant metric on the amenable group $\mathcal{G}^x_x$. Theorem 5.3.26 extends straightforwardly to pseudodifferential operators acting between sections of vector bundles.

*Proof of Theorem 5.3.26.* First, according to Lemma 5.3.16, the groupoid $\mathcal{G}$ is Hausdorff. Let $(U_i)_{i\in I}$ be an open cover satisfying the conditions of Definition 5.3.14, and let $F_i = U_i \cap F$. Because the family $(\mathcal{G}|_{U_i})_{i\in I}$ satisfies the weak gluing condition over $M$ and $F$ is $\mathcal{G}$-invariant, the family $(\mathcal{G}|_{F_i})_{i\in I}$ also satisfies the weak gluing condition over $F$. In other words, the groupoid $\mathcal{G}_F$ is the gluing of the family $(\mathcal{G}|_{F_i})_{i\in I}$. Moreover, the action of $\mathcal{G}$ on $F$ is trivial, so each $\mathcal{G}|_{F_i}$ is isomorphic to $F_i \times G_i$, for every $i \in I$.

These local trivializations show that $\mathcal{G}_F$ is a Lie group bundle over each connected component of $F$. Each $G_i$ is amenable, for $i \in I$, so we can conclude from Corollary 5.2.8 that $\mathcal{G}$ is a Fredholm groupoid. Theorem 5.3.26 is then a consequence of Theorem 5.2.9. ☐

**Example 5.3.27.** The scattering groupoid $\mathcal{G}_{sc}$ of Example 5.3.24 satisfies the assumptions of Theorem 5.3.26. When $P \in \Psi^m(\mathcal{G}_{sc})$, the limit operators $(P_x)_{x\in\partial M}$ are translation-invariant operators on $\mathbb{R}^n$. In that case, the operator $P_x$ is simply a Fourier multiplier on $C^\infty(\mathbb{R}^n)$, whose invertibility is easy to study: see [50].

**Example 5.3.28.** The 0-groupoid $\mathcal{G}_0$ of Example 5.3.21, which models asymptotically hyperbolic geometries, also satisfies the assumptions of Theorem 5.3.26. If $P \in \Psi^m(\mathcal{G}_0)$, the limit operators $P_x$ are order-$m$, right-invariant pseudodifferential operators on the non-commutative groups $G_n = (0,\infty) \ltimes \mathbb{R}^{n-1}$.

**Remark 5.3.29.** We will show in Chapter 6 that a result similar to Theorem 5.3.26 holds without the assumption of a trivial action of $\mathcal{G}$ on $\partial M$ (the proof requires a more involved study of the representations of $\mathcal{G}$). Thus all boundary action groupoids that are obtained by gluing actions by amenable groups are Fredholm groupoids.

## 5.4. Layer potentials groupoids

In this section, we review the construction of layer potentials groupoids for conical domains in [52]. In order to study layer potentials operators, which are operators on the boundary, we consider a groupoid over the desingularized boundary. Our aim is to relate this groupoid with the boundary action groupoids defined in the previous section in an explicit way, which we shall do in Section 5.4.4.

### 5.4.1. Conical domains and desingularization

We begin with the definition of domains with conical points [45, 52, 133].

**Definition 5.4.1.** Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be an open connected bounded domain. We say that $\Omega$ is a *domain with conical points* if there exists a finite number of points $\{p_1, p_2, \cdots, p_l\} \subset \partial\Omega$, such that

(1) $\partial\Omega\backslash\{p_1, p_2, \cdots, p_l\}$ is smooth;

(2) for each point $p_i$, there exist a neighborhood $V_{p_i}$ of $p_i$, a possibly disconnected domain $\omega_{p_i} \subset S^{n-1}$, $\omega_{p_i} \neq S^{n-1}$, with smooth boundary, and a diffeomorphism $\phi_{p_i} : V_{p_i} \to B^n$ such that

$$\phi_{p_i}(\Omega \cap V_{p_i}) = \{rx' : 0 < r < 1, x' \in \omega_{p_i}\}.$$

We assume always that $\overline{V_{p_i}} \cap \overline{V_{p_j}} = \emptyset$, for $i \neq j$, $i, j \in \{1, 2, \cdots, l\}$.

If $\partial\Omega = \partial\overline{\Omega}$, then we say that $\Omega$ is a *domain with no cracks*. The points $p_i$, $i = 1, \cdots, l$ are called *conical points* or *vertices*. If $n = 2$, $\Omega$ is said to be a *polygonal domain*.

We distinguish two cases: *conical domains without cracks*, $n \in \mathbb{N}$, and *polygonal domains with ramified cracks*. Note that if $n \geq 3$ then domains with cracks have edges, and are no longer conical. For simplicity, we assume $\Omega$ to be a subset of $\mathbb{R}^n$.

In applications to boundary value problems in $\Omega$, it is often useful to regard smooth boundary points as artificial vertices, representing for instance a change in boundary

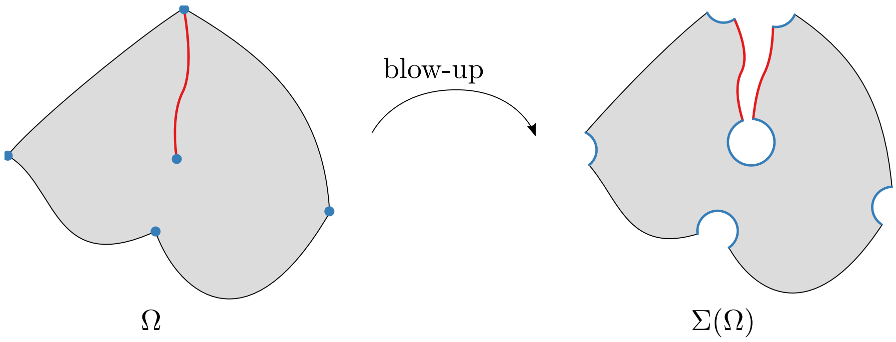


Figure 5.1.: A domain with conical points $\Omega \subset \mathbb{R}^2$, having one crack (in red). The desingularization as a manifold with corners $\Sigma(\Omega)$ is on the right: the crack has been "doubled" and each conical point is replaced by a small line.

conditions. Then a conical point $x$ is a smooth boundary point if, and only if, $\omega_x \simeq S_+^{n-1}$. The minimal set of conical points is unique and coincides with the singularities of $\partial\Omega$; these are *true conical points* of $\Omega$. Here we will give our results for true vertices, but the constructions can easily be extended to artificial ones.

For the remainder of the chapter, we keep the notation as in Definition 5.4.1. Moreover, for a conical domain $\Omega$, we always denote by

$$\Omega^{(0)} = \{p_1, p_2, \cdots, p_l\},$$

the set of true conical points of $\Omega$, and by $\partial_0\Omega$ be the smooth part of $\partial\Omega$, i.e., $\partial_0\Omega = \partial\Omega\backslash\{p_1, p_2, \cdots, p_l\}$. We remark that we allow the bases $\omega_{p_i}$ and $\partial\omega_{p_i}$ to be disconnected (in fact, if $n = 2$, $\partial\omega_{p_i}$ is always disconnected).

We now recall the definition of the *desingularization* $\Sigma(\Omega)$ of $\Omega$ of a conical domain without cracks, which is obtained from $\Omega$ by removing a, possibly non-connected, neighborhood of the singular points and replacing each connected component by a cylinder. We refer to [45] for details on this construction, see also [52, 108, 136]. Note first that the map $\phi_{p_i}$ introduced above induces, for any $i = 1, \ldots, l$, a diffeomorphism of manifolds with boundary

$$\phi_{p_i} : (\overline{\Omega} \setminus \{p_i\}) \cap V_{p_i} \simeq (0,1) \times \overline{\omega_{p_i}}.$$

Assuming as before that the neighborhoods $V_{p_i}$ are disjoint from one another, let $\phi := \sqcup_{i=1}^{l} \phi_{p_i}$. We then define the manifold with corners $\Sigma(\Omega)$ as the gluing

$$\Sigma(\Omega) := \left( \bigsqcup_{p_i \in \Omega^{(0)}} [0,1) \times \overline{\omega_{p_i}} \right) \bigcup_{\phi} \left( \overline{\Omega} \setminus \Omega^{(0)} \right),$$

see Figure 5.1.

In the terminology of [45], the hyperfaces which are *not at infinity* correspond to actual faces of $\Omega$, denoted by $\partial'\Sigma(\Omega)$. In the non-crack case, i.e. when $\partial\overline{\omega_{p_i}} = \partial\omega_{p_i}$ for any

$i = 1, \ldots, l$, we thus have

$$\partial'\Sigma(\Omega) \simeq \left( \bigsqcup_{p_i \in \Omega^{(0)}} [0,1) \times \partial\omega_{p_i} \right) \bigcup_{\phi} \partial_0\Omega. \tag{5.4}$$

A hyperface *at infinity* corresponds to a singularity of $\Omega$. Let $\partial''\Sigma(\Omega)$ denote the union of hyperfaces at infinity. Hence

$$\partial''\Sigma(\Omega) \simeq \bigsqcup_{p_i \in \Omega^{(0)}} \{0\} \times \overline{\omega_{p_i}}. \tag{5.5}$$

The boundary $\partial\Sigma(\Omega)$ can be identified with the union of $\partial'\Sigma(\Omega)$ and $\partial''\Sigma(\Omega)$, i.e. we can write

$$\begin{aligned} \partial\Sigma(\Omega) &= \partial'\Sigma(\Omega) \cup \partial''\Sigma(\Omega) \\ &\simeq \left( \bigsqcup_{p_i \in \Omega^{(0)}} [0,1) \times \partial\omega_{p_i} \cup \{0\} \times \overline{\omega_{p_i}} \right) \bigcup_{\phi} \partial_0\Omega. \end{aligned} \tag{5.6}$$

We denote by $M := \partial'\Sigma(\Omega)$. Note that $M$ coincides with the closure of $\partial_0\Omega$ in $\Sigma(\Omega)$. It is a compact manifold with (smooth) boundary

$$\partial M = \bigsqcup_{p_i \in \Omega^{(0)}} \{0\} \times \partial\omega_{p_i}.$$

In fact, we regard $M := \partial'\Sigma(\Omega)$ as a desingularization of the boundary $\partial\Omega$. Operators on $M$ will be related to (weighted) operators on $\partial\Omega$, as we shall see in Subsection 5.4.4. See [45, 52] for more details.

### 5.4.2. Groupoid construction for conical domains without cracks

Let $\Omega$ be a conical domain without cracks, $\Omega^{(0)} = \{p_1, p_2, \cdots, p_l\}$ be the set of (true) conical points of $\Omega$, and $\partial_0\Omega$ be the smooth part of $\partial\Omega$. We will review the definition of the *layer potentials groupoid* $\mathcal{G} \rightrightarrows M$, with $M := \partial'\Sigma(\Omega)$ a compact set, as in the previous subsection, following [52].

Let $\mathcal{H} := [0,\infty) \rtimes (0,\infty)$ be the transformation groupoid with the action of $(0,\infty)$ on $[0,\infty)$ by dilation (see Example 4.3.14 for more details). To each $p_i \in \Omega^{(0)}$, we first associate a groupoid $\mathcal{H} \times (\partial\omega_{p_i})^2 \rightrightarrows [0,\infty) \times \partial\omega_{p_i}$, where $(\partial\omega_{p_i})^2$ is the pair groupoid of $\partial\omega_{p_i}$ (see Example 4.3.12). We then take its reduction to $[0,1) \times \partial\omega_{p_i}$ to define

$$\mathcal{J}_i := \left( \mathcal{H} \times (\partial\omega_{p_i})^2 \right)\Big|_{[0,1)\times\partial\omega_{p_i}} \rightrightarrows [0,1) \times \partial\omega_{p_i}.$$

We now want to glue the pair groupoid $\partial_0\Omega \times \partial_0\Omega = (\partial_0\Omega)^2$ and the family $(\mathcal{J}_i)_{i=1,2,\ldots,l}$ in a suitable way. First, let $V_i \subset \mathbb{R}^n$ be a neighborhood of $p_i$ such that there is a diffeomorphism $\varphi_i : V_i \cap \partial_0\Omega \simeq (0,1) \times \partial\omega_i$. Assuming that the neighborhoods $V_i$ are disjoint from one another, we let $\varphi = \sqcup_{i=1}^{l}\varphi_i$ and set

$$M = \left( \bigsqcup_{p_i \in \Omega^{(0)}} [0,1) \times \partial\omega_{p_i} \right) \bigcup_{\varphi} \partial_0\Omega = \partial'\Sigma(\Omega),$$

as above.

Note that $\mathcal{J}_i|_{(0,1)\times\partial\omega_{p_i}}$ is the pair groupoid $((0,1) \times \partial\omega_{p_i})^2$, so that

$$\mathcal{J}_i|_{(0,1)\times\partial\omega_{p_i}} \simeq (\partial_0\Omega \times \partial_0\Omega)|_{V_i}.$$

Moreover, it is easy to see that the family $(\mathcal{J}_i)_{i=1}^{l} \cup \{\partial_0\Omega \times \partial_0\Omega\}$ satisfies the strong gluing condition of Subsection 5.3.1 (the orbit of any point in $M$ is either $\partial_0\Omega$ or one of the $\partial\omega_{p_i}$, for $i = 1, \dots, l$). Therefore the gluing in the following definition is a well defined Hausdorff Lie groupoid.

**Definition 5.4.2.** Let $\Omega$ be a conical domain without cracks. The *layer potentials groupoid* associated to $\Omega$ is the Lie groupoid $\mathcal{G} \rightrightarrows M := \partial'\Sigma(\Omega)$ defined by

$$\mathcal{G} := \left( \bigsqcup_{p_i \in \Omega^{(0)}} \mathcal{J}_{p_i} \right) \bigcup_{\varphi} \partial_0\Omega \times \partial_0\Omega \rightrightarrows M \tag{5.7}$$

where $\varphi = \sqcup_{i=1}^{l}\varphi_i$ as above, with space of units

$$M := \partial'\Sigma(\Omega) \simeq \left( \bigsqcup_{p_i \in \Omega^{(0)}} [0,1) \times \partial\omega_{p_i} \right) \bigcup_{\varphi} \partial_0\Omega,$$

where $\partial'\Sigma(\Omega)$, defined in Equation (5.4), denotes the union of hyperfaces which are not at infinity of the desingularization.

Clearly, the space $M$ of units is compact. We have that $\partial_0\Omega$ coincides with the interior of $M$, so $\partial_0\Omega$ is an open dense subset of $M$. The following proposition summarizes the properties of the layer potentials groupoid and its groupoid $C^*$-algebra.

**Proposition 5.4.3.** *Let $\mathcal{G}$ be the layer potentials groupoid* (5.7) *associated to a domain with conical points $\Omega \subset \mathbb{R}^n$. Let $\Omega^{(0)} = \{p_1, p_2, \cdots, p_l\}$ be the set of conical points and $\partial_0\Omega = \partial\Omega\backslash\Omega^{(0)}$ be the smooth part of $\partial\Omega$. Then, $\mathcal{G}$ is a Lie groupoid with units $M = \partial'\Sigma(\Omega)$ (defined in Equation (5.4)) such that*

*(i) $\partial_0\Omega$ is an open, dense invariant subset with $\mathcal{G}_{\partial_0\Omega} \simeq \partial_0\Omega \times \partial_0\Omega$ and $\Psi^m(\mathcal{G}_{\partial_0\Omega}) \simeq \Psi^m(\partial_0\Omega)$.*

*(ii) For each conical point $p \in \Omega^{(0)}$, the subset $\{p\} \times \partial\omega_p$ is $\mathcal{G}$-invariant and*

$$\mathcal{G}_{\partial M} = \bigsqcup_{i=1}^{l} (\partial\omega_i \times \partial\omega_i) \times (0,\infty) \times \{p_i\}$$

*(iii) If $P \in \Psi^m(\mathcal{G}_{\partial M})$ then for each $p_i \in \Omega^{(0)}$, $P$ defines a Mellin convolution operator on $(0,\infty) \times \partial\omega_i$.*

*(iv) $\mathcal{G}$ is (metrically) amenable, i.e. $C^*(\mathcal{G}) \simeq C_r^*(\mathcal{G})$.*

*(v) We have that*

$$C^*(\mathcal{G}_{\partial M}) \simeq \begin{cases} \bigoplus_{i=1}^{l} C_0(0,\infty) \otimes \mathcal{K} & \text{if } n \geq 3 \\ \bigoplus_{i=1}^{l} M_{k_i}(C_0(0,\infty)) & \text{if } n = 2, \end{cases}$$

*where $k_i$ is the number of elements of $\partial\omega_i$, the integer $l$ is the number of conical points, and $\mathcal{K}$ is the algebra of compact operators on an infinite dimensional separable Hilbert space.*

To explain Item (iii) above in greater detail, note that if $P \in \Psi^m(\mathcal{G})$ then, at the boundary, the regular representation yields an operator

$$P_i := \pi_{p_i}(P) \in \Psi^m((0,\infty) \times (\partial\omega_i)^2),$$

where $(0,\infty) \times (\partial\omega_i)^2$ is regarded as a groupoid (see (ii) above). The operator $P_i$ is defined by a distribution kernel $\kappa_i$ in $(0,\infty) \times (\partial\omega_i)^2$ that is invariant by the dilation action of $(0,\infty)$, hence is a Mellin convolution operator on $(0,\infty) \times \partial\omega_i$ with kernel $\tilde{\kappa_i}(r,s,x',y') := \kappa_i(r/s,x',y')$. If $P \in \Psi^{-\infty}(\mathcal{G})$, that is, if $\kappa_i$ is smooth, then it defines a smoothing Mellin convolution operator on $(0,\infty) \times \partial\omega_i$ (see [119, 168]). This is one of the motivations in our definition of $\mathcal{G}$.

**Remark 5.4.4.** Recall the definition of $b$-groupoid in Example 4.3.16, which in our case comes down to

$$\mathcal{G}_b = \left( \bigsqcup_{i,j} (0,\infty) \times (\partial_j\omega_i)^2 \right) \bigcup \partial_0\Omega \times \partial_0\Omega \rightrightarrows M$$

where $\partial_j\omega_i$ denote the connected components of $\partial\omega_i$. If $\partial\omega_i$ is connected, for all $i = 1, ..., l$, then $\mathcal{G} = \mathcal{G}_b$. In many cases of interest, $\partial\omega$ is not connected: for instance, if $n = 2$, that is if we have a polygonal domain, then $\partial\omega$ is always disconnected. In general, the groupoid $\mathcal{G}$ is larger and not $d$-connected, and $\mathcal{G}_b$ is an open, wide subgroupoid of $\mathcal{G}$. The main difference is that here we allow the different connected components of the boundary, corresponding to the same conical point, to interact, in that there are arrows between them. The Lie algebroids of these two groupoids coincide, as $A(\mathcal{G}) \simeq {}^bTM$, the $b$-tangent bundle of $M$. Moreover, $\Psi(\mathcal{G}) \supset \Psi(\mathcal{G}_b)$, and the latter is the algebra of (compactly supported) $b$-pseudodifferential operators on $M$.

**Remark 5.4.5.** The construction of the layer potential groupoid can be extended to polygonal domains with cracks, when $n = 2$, that is domains $\Omega \subset \mathbb{R}^2$ such that $\partial\Omega \neq \partial\overline{\Omega}$. This construction was done in [52].

The point is that in two dimensions, the actual cracks, given by $\partial\Omega \backslash \partial\overline{\Omega}$, are a collection of smooth crack lines and boundary points that behave like conical singularities (in higher dimensions we get "edges"). To each polygonal domain with cracks we can associate a generalized conical domain with *no cracks*, the so-called unfolded domain,

$$\Omega^u = \Omega \cup \partial^u \Omega,$$

where $\partial^u \Omega$ is the set of inward pointing unit normal vectors to the smooth part of $\partial\Omega$. The main idea is that a smooth crack point should be covered by two points, which correspond to the two sides of the crack, see Figure 5.1. At the boundary we get a double cover of each smooth crack line, and a $k$-cover of each singular crack point, $k$ being the ramification number (see [52] for details). The vertices of $\Omega$ are still vertices of the generalized domain, but now each singular crack point yields $k$ new vertices. The groupoid construction defined above still applies to this case, as do all the results in this subsection and the next.

### 5.4.3. Desingularization and weighted Sobolev spaces for conical domains

An important class of function spaces on singular manifolds are weighted Sobolev spaces. Let $\Omega$ be a conical domain, and $r_\Omega$ be the smoothened distant function to the set of conical points $\Omega^{(0)}$ as in [45, 52]. The space $L^2(\Sigma(\Omega))$ is defined using the volume element of a compatible metric on $\Sigma(\Omega)$. A natural choice of compatible metrics is $g = r_\Omega^{-2} g_0$, where $g_0$ is the Euclidean metric. Then the Sobolev spaces $H^m(\Sigma(\Omega))$ are defined in the usual way. These Sobolev spaces can be identified with weighted Sobolev spaces.

Let $m \in \mathbb{N}_0$ and $a \in \mathbb{R}$. The $m$-th Sobolev space on $\Omega$ with weight $r_\Omega$ and index $a$ is defined by

$$\mathcal{K}_a^m(\Omega) = \{u \in L^2_{\mathrm{loc}}(\Omega) \mid r_\Omega^{|\alpha|-a} \partial^\alpha u \in L^2(\Omega), \text{ for all } |\alpha| \le m\}. \tag{5.8}$$

We define similarly the spaces $\mathcal{K}_a^m(\partial\Omega)$. Note that in this case, as $\partial\Omega$ has no boundary, these spaces can be defined for any $m \in \mathbb{R}$ by complex interpolation [45].

The following result is taken from [45, Proposition 5.7 and Definition 5.8].

**Proposition 5.4.6.** *Let $\Omega \subset \mathbb{R}^n$ be a domain with conical points, $\Sigma(\Omega)$ be its desingularization, and $\partial'\Sigma(\Omega)$ be the union of the hyperfaces that are not at infinity. We have that*

*(i) $\mathcal{K}^m_{n/2}(\Omega) \simeq H^m(\Sigma(\Omega), g)$, for all $m \in \mathbb{N}_0$;*

*(ii) $\mathcal{K}^s_{\frac{n-1}{2}}(\partial\Omega) \simeq H^s(\partial'\Sigma(\Omega), g)$, for all $s \in \mathbb{R}$.*

*where the metric $g = r_\Omega^{-2}\, g_0$ with $g_0$ the Euclidean metric.*

### 5.4.4. Fredholm conditions for layer potentials

In this section, we relate the layer potential groupoids for conical domains constructed in Section 5.4 with boundary action groupoids. Moreover, we also show that they fit in the framework of Fredholm groupoids, so we can apply the Fredholm criteria obtained in the previous sections operators on layer potential groupoids. All the results hold also for polygonal domains with ramified cracks, as in Remark 5.4.5.

Recall the definition of boundary action groupoids from Subsection 5.3.3.

**Theorem 5.4.7.** *The layer potentials groupoid defined in Definition 5.4.2 is a boundary action groupoid.*

*Proof.* The layer potential groupoids is build in several steps. First, the groupoid $\mathcal{H} = [0,\infty) \rtimes (0,\infty)$ is obviously a boundary action groupoid. If $p_i$ is a conical point of $\Omega$, then $\partial\omega_{p_i} \times \partial\omega_{p_i}$ is also a boundary action groupoid (see Example 5.3.15). Hence $\mathcal{H} \times (\partial\omega_{p_i})^2$ is a boundary action groupoid by Theorem 5.3.17, and so is its reduction $\mathcal{J}_i = (\mathcal{H} \times (\partial\omega_{p_i})^2)\,|_{[0,1)\times\partial\omega_{p_i}}$, according to Lemma 5.3.18. The layer potential groupoids is then obtained by gluing boundary action groupoids $\mathcal{J}_i$, for $i = 1, \ldots, l$, with the pair groupoid $\partial_0\Omega \times \partial_0\Omega$, therefore it is again a boundary action groupoid by Lemma 5.3.19. □

Let us now show that the layer potentials groupoid is Fredholm. The results of Subsection 5.3.5 do not apply here, so we use a more direct method. Let us first see the case of straight cones. Let $\omega \subset S^{n-1}$ be an open subset with smooth boundary (note that we allow $\omega$ to be *disconnected*) and

$$\Omega := \{ty' \mid y' \in \omega, t \in (0,\infty)\}$$

be the (open, unbounded) cone with base $\omega$. The desingularization becomes in this case an half-infinite solid cylinder

$$\Sigma(\Omega) = [0,\infty) \times \overline{\omega}$$

with boundary $\partial\Sigma(\Omega) = [0,\infty) \times \partial\omega \cup \{0\} \times \omega$, so that $M = \partial'\Sigma(\Omega) = [0,\infty) \times \partial\omega$ the union of the hyperfaces not at infinity. Taking the one-point compactification $[0,\infty]$ of $[0,\infty)$, we can consider the groupoid $\overline{\mathcal{H}}$ as in Example 5.2.5. Then the layer potentials groupoid associated to a straight cone $\Omega \simeq \mathbb{R}^+\omega$ is the product Lie groupoid with units $M = [0,\infty] \times \partial\omega$, corresponding to a desingularization of $\partial\Omega$, defined as

$$\mathcal{J} := \overline{\mathcal{H}} \times (\partial\omega)^2.$$

Now, we have seen in Example 5.2.5 that $\overline{\mathcal{H}}$ is a Fredholm groupoid, hence $\mathcal{J}$ is also a Fredholm groupoid.

In the general case, we can proceed in several ways: we can use the same argument as in the straight cone case (that is, as in Example 5.2.5 ), or we can use the fact that the gluing (along the interior) of Fredholm groupoids is also a Fredholm groupoid. By analogy with the classes of Fredholm groupoids studied in [51], we chose to check that $\mathcal{G}$ is actually given by a fibered pair groupoid over the boundary.

**Theorem 5.4.8.** *The layer potentials groupoid defined in Definition 5.4.2 is a Fredholm groupoid.*

*Proof.* It is clear that $\partial_0\Omega$ is an open, dense, $\mathcal{G}$-invariant subset of $M = \partial'\Sigma(\Omega)$, with $\mathcal{G}_{\Omega_0}$ is the pair groupoid. Let

$$F := M \backslash \partial_0\Omega = \partial M = \bigcup_{p\in\Omega^{(0)}} \{p\} \times \partial\omega_p \simeq \bigsqcup_{i=1}^{l} \partial\omega_{p_i}.$$

We have

$$\mathcal{G}_F = \bigsqcup_{i=1}^{l} (\partial\omega_i \times \partial\omega_i) \times (0, \infty)$$

For any $x \in F$, we have $(\mathcal{G}_F)_x^x = \mathcal{G}_x^x \simeq \{x\} \times (0, \infty) \simeq (0, \infty)$. Since the group $(0, \infty)$ is commutative, it is amenable. Thus $\mathcal{G}_F$ is amenable as well, which shows that $\mathcal{R}(\mathcal{G}_F) = \{\pi_x,\, x \in F\}$ is an exhaustive set of representations of $C^*(\mathcal{G}_F)$. This can be proved directly, using the description in (iv) of Proposition 5.4.3.

We show alternatively that $\mathcal{G}_F$ can be given as a fibered pair groupoid. Indeed, let $\mathcal{P} := \{\partial\omega_i\}_{i=1,\dots,l}$ be a finite partition of the smooth manifold $F$ and let $f : F \to \mathcal{P}$, $x \in \partial\omega_i \mapsto \partial\omega_i$. Let $\mathcal{H} := \mathcal{P} \times \mathbb{R}^+$, as a product of a manifold and a Lie group. Since $f$ is a locally constant fibration, we obtain

$$f^{\downdownarrows}(\mathcal{H}) = \bigsqcup_{i=1}^{l} (\partial\omega_i \times \partial\omega_i) \times (0, \infty) = \mathcal{G}_F.$$

Hence, by Corollary 5.2.8, the result is proved. □

If we apply Theorem 5.2.9 [51, Theorem 4.17] to our case, we obtain the main theorems as follows. Recall that the regular representations $\pi_x$ and $\pi_y$ are unitarily equivalent for $x, y$ in the same orbit of $\mathcal{G}_F$, so that for $P \in \Psi^m(\mathcal{G})$ we obtain a family of Mellin convolution operators $P_i := \pi_x(P)$ on $(0, \infty) \times \partial\omega_i$, $i = 1, ..., p$, with $x = (p_i, x') \in \partial M$, $x' \in \partial\omega_{p_i}$.

Recall that the space $L^m_s(\mathcal{G})$ is the *norm closure* of $\Psi^m(\mathcal{G})$ in the topology of continuous operators $H^s(M) \to H^{s-m}(M)$. By the results in [167, 168], if $P \in L^m_s(\mathcal{G})$, then $\pi_{p_i}(P)$ is also a Mellin convolution operator.

**Theorem 5.4.9.** *Suppose that $\Omega \subset \mathbb{R}^n$ is a conical domain without cracks and $\Omega^{(0)} = \{p_1, p_2, \cdots, p_l\}$ is the set of conical points. Let $\mathcal{G} \rightrightarrows M = \partial'\Sigma(\Omega)$ be the layer potentials groupoid as in Definition 5.4.2. Let $P \in L^m_s(\mathcal{G}) \supset \Psi^m(\mathcal{G})$ and $s \in \mathbb{R}$. Then*

$$P : \mathcal{K}^s_{\frac{n-1}{2}}(\partial\Omega) \to \mathcal{K}^{s-m}_{\frac{n-1}{2}}(\partial\Omega)$$

*is Fredholm if, and only if,*

*(i) $P$ is elliptic and*

*(ii) the Mellin convolution operators*

$$P_i : H^s((0, \infty) \times \partial\omega_i; g) \to H^{s-m}((0, \infty) \times \partial\omega_i; g)$$

*are invertible, for $i = 1, \dots, p$, where the metric $g$ is a product of the Haar metric on $(0, \infty)$ with the Euclidean metric on $\partial\omega_i$.*

**Remark 5.4.10.** Fredholm conditions similar to those of Theorem 5.4.9 also hold for polygonal domains with cracks. In that case, some extra limit operators arise from the fact that the boundary $\partial\Omega$ should be desingularized near the crack points.

We expect that these results will have applications to layer potentials.

# 6. The Fredholm property for groupoids is a local property

The present chapter is adapted from the published paper of the same name [55].

## 6.1. Introduction and notations

We refer to Section 2.1.3 for a contextualized introduction, and only recall the main results and notations here.

### 6.1.1. Short introduction

We continue here the general study of Fredholm groupoids initiated Chapter 5. Compared to the last chapter that was only dealing with the Lie case, the definition of Fredholm groupoids will be recalled in Section 6.3 in a slightly generalized setting. The main result of the paper is the following:

**Theorem 6.1.1.** *Let $\mathcal{G} \rightrightarrows X$ be a locally compact, second-countable and locally Hausdorff groupoid, endowed with a right-invariant Haar system. Assume that*

*(i) there is an open dense $\mathcal{G}$-invariant subset $V \subset X$ with $\mathcal{G}_V \simeq V \times V$, and*

*(ii) we have a family $(U_i)_{i \in I}$ of open subsets of $X$ such that the saturations $(\mathcal{G} \cdot U_i)_{i \in I}$ provide an open cover of $X$.*

*Then $\mathcal{G}$ is a Fredholm groupoid if, and only if, each reduction $\mathcal{G}|_{U_i}$ is also a Fredholm groupoid, for every $i \in I$.*

Here $\mathcal{G}|_U := \mathcal{G}^U \cap \mathcal{G}_U$ is the *reduction* or *restriction* of $\mathcal{G}$ to the open subset $U$ of $X$, following the notations of Section 4.3.1. The main element of the proof is the construction of an induction functor $\mathrm{Ind}_{U_i}$ from the category of unitary equivalence classes of representations of $C_r^*(\mathcal{G}|_{U_i})$ to the one of $C_r^*(\mathcal{G})$. As an important intermediate step, we establish the following result.

**Theorem 6.1.2.** *Under assumption (ii) of Theorem 6.1.1, we have that*

$$\mathrm{Prim}\, C_r^*(\mathcal{G}) = \bigcup_{i \in I} \mathrm{Prim}\, C_r^*(\mathcal{G}|_{U_i}),$$

*where we identify $\mathrm{Prim}\, C^*(\mathcal{G}|_{U_i})$ with its image through $\mathrm{Ind}_{U_i}$.*

### 6.1.2. Outline of the chapter

We begin in Section 6.2 with introducing our main tool, which is the induction functor from the $C^*$-algebra of a reduction to an open subset. We define this functor and establish some important properties, and then use it to prove the decomposition of the primitive spectrum stated in Theorem 6.1.2.

It is in Section 6.3 that we start dealing with Fredholm groupoids. We introduce our definition of Fredholm groupoids in the locally compact case and prove Theorem 6.1.1. We then prove a few consequences. Among them, we define the notion of *local isomorphisms* of two groupoids and the class of *local action groupoids*.

Finally, Section 6.4 gives some concrete examples of Fredholm and local action groupoids. To motivate their construction, we step back into the setting of Lie groupoids and recall the link with the study of differential operators on manifolds. We then show how Theorem 6.1.1 may be used to prove that the groupoids under study are Fredholm.

## 6.2. Primitive spectrum and groupoid reductions

In this section, we show that the primitive spectrum of a groupoid $C^*$-algebra can be investigated locally: this is the content of Theorem 6.2.10, which will be our main tool for Section 6.3. Throughout the section, we shall consider a locally compact, second-countable and locally Hausdorff groupoid $\mathcal{G} \rightrightarrows X$ that is endowed with a right-invariant continuous Haar system. We refer again to Section 4.3 for all definitions and notations regarding groupoids.

### 6.2.1. Representations induced from a reduction

We will show that each reduction of the groupoid to an open subset $U \subset X$ defines an induction functor between the categories of representations of $\mathcal{G}|_U = d^{-1}(U) \cap r^{-1}(U)$ and $\mathcal{G}$. The starting point for our construction is the Mackey-Green-Rieffel induction mechanism recalled in Sections 4.1.4 and 4.3.2, as well as Remark 6.2.1 below.

**Remark 6.2.1.** If $U \subset X$ is an open subset and $W := \mathcal{G} \cdot U$ its saturation, then the reduction $\mathcal{G}|_U$ is Morita equivalent to the groupoid $\mathcal{G}_W$. The equivalence is implemented by the $(\mathcal{G}_W, \mathcal{G}|_U)$-space $\mathcal{G}_U = d^{-1}(U)$, acted upon by left and right multiplication. Both actions are free and proper, and the domain and range maps induce isomorphisms $\mathcal{G}_W \backslash \mathcal{G}_U \simeq U$ and $\mathcal{G}_U / \mathcal{G}|_U \simeq W$.

According to the results recalled in Section 4.3.2, it follows that $C^*(\mathcal{G}|_U)$ and $C^*(\mathcal{G}_W)$ are Morita equivalent [144]. The corresponding $(C^*(\mathcal{G}_W), C^*(\mathcal{G}|_U))$-imprimitivity bimodule $\mathcal{E}_U$ is the completion of $C_c(\mathcal{G}_U)$ for the norm

$$\|f\|_{\mathcal{E}_U} = \|f^* * f\|^{1/2}_{C^*(\mathcal{G}|_U)},$$

with $C^*(\mathcal{G}_W)$ and $C^*(\mathcal{G}|_U)$ acting by right and left multiplication respectively. For any $C^*$-algebra $A$, we denote by $\mathcal{R}(A)$ the category of unitary equivalence classes of representations, with intertwining morphisms. The imprimitivity bimodule $\mathcal{E}_U$ induces an induction functor

$$\mathcal{E}_U\text{–Ind} : \mathcal{R}(C^*(\mathcal{G}|_U)) \to \mathcal{R}(C^*(\mathcal{G}_W)),$$

given as the tensor product with $\mathcal{E}_U$. Similarly, there is a Morita equivalence between the reduced algebras $C^*_r(\mathcal{G}_U)$ and $C^*_r(\mathcal{G}_W)$. We choose to stick to the study of the full algebras for now, but all results of this section apply to their reduced counterparts, as pointed out by Remark 6.2.7.

It is well-known that for any $\mathcal{G}$-invariant open subset $W \subset X$, the $C^*$-algebra $C^*(\mathcal{G}_W)$ embeds as an ideal in $C^*(\mathcal{G})$ [145]. As explained in Section 4.1.4, this implies the existence of an induction functor

$$\mathrm{Ind}_W : \mathcal{R}(C^*(\mathcal{G}_W)) \to \mathcal{R}(C^*(\mathcal{G})).$$

**Definition 6.2.2.** Let $\mathcal{G} \rightrightarrows X$ be a locally compact, second-countable and locally Hausdorff groupoid. Let $U$ be an open subset of $X$, and $W := \mathcal{G} \cdot U$ be its saturation. The *induced representation* functor

$$\mathrm{Ind}_U : \mathcal{R}(C^*(\mathcal{G}|_U)) \to \mathcal{R}(C^*(\mathcal{G}))$$

is defined as the composition $\mathrm{Ind}_U = \mathrm{Ind}_W \circ \mathcal{E}_U\text{–Ind}$.

**Remark 6.2.3.** A possibly more direct way to define the functor $\mathrm{Ind}_U$ is to observe that $\mathcal{E}_U$ is a $(C^*(\mathcal{G}), C^*(\mathcal{G}|_U))$-correspondence, as detailed in Subsection 4.1.4. Correspondingly, the space $\mathcal{G}_U$ is a $(\mathcal{G}, \mathcal{G}|_U)$-correspondence (or Hilsum-Skandalis morphism) in the sense of [176, 96]. We do not emphasize this approach too much however, because the factorization of $\mathrm{Ind}_U$ through $\mathcal{R}(C^*(\mathcal{G}_W))$ given by Definition 6.2.2 will be of use below.

**Remark 6.2.4.** The algebra $C^*(\mathcal{G}|_U)$ is actually a hereditary subalgebra of $C^*(\mathcal{G})$. A hereditary subalgebra $B \subset A$ is always Morita equivalent to the closed, two-sided ideal $I_B$ it generates: in our case this ideal is $I_{C^*(\mathcal{G}|_U)} = C^*(\mathcal{G}_W)$, with $W = \mathcal{G} \cdot U$. There is therefore an induction functor

$$\mathrm{Ind}_B^A : \mathcal{R}(B) \to \mathcal{R}(A),$$

which factorizes through $\mathcal{R}(I_B)$, just as in Definition 6.2.2. This recasts our construction in a more general setting.

For any open subset $U \subset X$, set

$$\mathrm{Prim}_U C^*(\mathcal{G}) := \{J \in \mathrm{Prim}\, C^*(\mathcal{G}) \mid C^*(\mathcal{G}|_U) \subset J\},$$

and let $\mathrm{Prim}^U C^*(\mathcal{G})$ be its complementary subset in $\mathrm{Prim}\, C^*(\mathcal{G})$.

**Lemma 6.2.5.** *Let $U \subset X$ be open, and $W = \mathcal{G} \cdot U$ be its saturation. Then*

$$\mathrm{Prim}_U C^*(\mathcal{G}) = \mathrm{Prim}_W C^*(\mathcal{G}) \quad \textit{and} \quad \mathrm{Prim}^U C^*(\mathcal{G}) = \mathrm{Prim}^W C^*(\mathcal{G}).$$

*Proof.* The algebra $C^*(\mathcal{G}_W)$ is the closed, two-sided ideal generated by $C^*(\mathcal{G}|_U)$ in $C^*(\mathcal{G})$. Thus, if $J$ is a primitive ideal of $C^*(\mathcal{G})$ that contains $C^*(\mathcal{G}|_U)$, then $J$ also contains all of $C^*(\mathcal{G}_W)$. On the other hand, it is obvious that if $J$ contains $C^*(\mathcal{G}_W)$, then it also contains the subalgebra $C^*(\mathcal{G}|_U)$. This proves that $\mathrm{Prim}_U C^*(\mathcal{G}) = \mathrm{Prim}_W C^*(\mathcal{G})$, and therefore that $\mathrm{Prim}^U C^*(\mathcal{G}) = \mathrm{Prim}^W C^*(\mathcal{G})$. ☐

We can now record the main properties of $\mathrm{Ind}_U$.

**Proposition 6.2.6.** *Let $\mathcal{G} \rightrightarrows X$ be a locally compact, second-countable and locally Hausdorff groupoid. Let $U$ be an open subset of $X$.*

*(i) The functor $\mathrm{Ind}_U$ descends to a continuous map*

$$\mathrm{Ind}_U : \mathrm{Prim}\, C^*(\mathcal{G}|_U) \to \mathrm{Prim}\, C^*(\mathcal{G}),$$

*which is an homeomorphism onto $\mathrm{Prim}^U C^*(\mathcal{G})$.*

*(ii) Let $\pi, \rho$ be two non-degenerate representations of $C^*(\mathcal{G}|_U)$ such that $\pi$ is weakly contained in $\rho$. Then $\mathrm{Ind}_U \pi$ is weakly contained in $\mathrm{Ind}_U \rho$.*

*(iii) If $\pi$ is a non-degenerate representation of $C^*(\mathcal{G}|_U)$, then*

$$\mathrm{Ind}_U(\mathrm{supp}\,\pi) \subset \mathrm{supp}(\mathrm{Ind}_U \pi).$$

*(iv) For $x \in U$, let $\pi_x^U$ be the corresponding regular representation of $C^*(\mathcal{G}|_U)$ and $\pi_x$ the one of $C^*(\mathcal{G})$. Then $\mathrm{Ind}_U \pi_x^U = \pi_x$.*

*Proof.* According to Definition 6.2.2, the functor $\mathrm{Ind}_U$ is defined as the composition $\mathrm{Ind}_W \circ \mathcal{E}_U$–Ind. As highlighted in Section 4.1.4, both $\mathrm{Ind}_W$ and $\mathcal{E}_U$–Ind induce continuous maps between primitive spectra, that are homeomorphisms onto their respective images. Therefore, the map

$$\begin{aligned} \mathrm{Ind}_U : \mathrm{Prim}\, C^*(\mathcal{G}|_U) &\to \mathrm{Prim}\, C^*(\mathcal{G}) \\ \ker \pi &\mapsto \ker(\mathrm{Ind}_U \pi) \end{aligned} \tag{6.1}$$

is well-defined, continuous, and an homeomorphism onto $\mathrm{Prim}^W C^*(\mathcal{G})$. The latter coincides with $\mathrm{Prim}^U C^*(\mathcal{G})$ by Lemma 6.2.5, which proves Assertion (i).

Assertion (ii) is a well-known property of the Rieffel induction procedure, whose proof can be found in [171]. Assertion (iii) is a direct consequence. Indeed, let $J = \ker \rho$ be a primitive ideal contained in $\mathrm{supp}\,\pi$. By definition, this is equivalent to $\rho \prec \pi$. Then $\mathrm{Ind}_U \rho \prec \mathrm{Ind}_U \pi$, which means that $\ker(\mathrm{Ind}_U \pi) \subset \ker(\mathrm{Ind}_U \rho)$. Since $\mathrm{Ind}_U J = \ker(\mathrm{Ind}_U \rho)$ by Equation (6.1), we conclude that $\mathrm{Ind}_U J \in \mathrm{supp}(\mathrm{Ind}_U(\pi))$. This proves the inclusion $\mathrm{Ind}_U(\mathrm{supp}\,\pi) \subset \mathrm{supp}(\mathrm{Ind}_U \pi)$.

To prove Assertion (iv), let $\mathcal{H} = \mathcal{E}_U \otimes_{\pi_x^U} L^2(\mathcal{G}_x^U)$. We need to show that the map $\Phi : \mathcal{H} \to L^2(\mathcal{G}_x)$ defined by

$$\Phi : f \otimes \xi \mapsto f * \xi$$

extends to a Hilbert space isomorphism. Let $f, g \in C_c(\mathcal{G}_U)$ and $\xi, \eta \in C_c(\mathcal{G}_x^U)$. By definition

$$\begin{aligned} \langle f \otimes \xi, g \otimes \eta \rangle_{\mathcal{H}} &= \langle (g^* * f) * \xi, \eta \rangle_{L^2(\mathcal{G}_x^U)} = \langle (g^* * f) * \xi, \eta \rangle_{L^2(\mathcal{G}_x)} \\ &= \langle f * \xi, g * \eta \rangle_{L^2(\mathcal{G}_x)}, \end{aligned}$$

so $\Phi$ is an isometry. To show that $\Phi$ is onto, let $f \in C_c(\mathcal{G}_U)$, and let $(\xi_n)_{n\in\mathbb{N}}$ be an approximate unit in $C_c(\mathcal{G}|_U)$. Then $(f * \xi_n)|_{\mathcal{G}_x}$ converges to $f|_{\mathcal{G}_x}$ in $L^2(\mathcal{G}_x)$. This proves that the image of $\Phi$ contains the dense subset $C_c(\mathcal{G}_x)$, hence $\Phi$ is onto. The map $\Phi$ is therefore an isomorphism. Now, let $\rho = \operatorname{Ind}_U \pi_x^U$. If $g \in C_c(\mathcal{G})$, then

$$\begin{aligned}\Phi(\rho(g)(f \otimes \xi)) &:= \Phi((g * f) \otimes \xi) = g * (f * \xi) \\ &= \pi_x(g)(f * \xi) = \pi_x(g)\Phi(f \otimes \xi).\end{aligned}$$

Since $C_c(\mathcal{G})$ is dense in $C^*(\mathcal{G})$ and $\rho$ and $\pi_x$ are continuous, this proves that $\operatorname{Ind}_U \pi_x^U$ and $\pi_x$ define the same class in $\mathcal{R}(C^*(\mathcal{G}))$. □

**Remark 6.2.7.** All the constructions of this section can be made in the same way by replacing every full groupoid algebras by their *reduced* counterparts. More explicitly, if $\mathcal{G} \rightrightarrows X$ is a groupoid satisfying the assumptions of Definition 6.2.2 and $U \subset X$ an open subset, then there is an induction functor

$$\operatorname{Ind}_U : \mathcal{R}(C_r^*(\mathcal{G}|_U)) \to \mathcal{R}(C_r^*(\mathcal{G})).$$

All the properties from Proposition 6.2.6 follow if we replace each full algebra by its reduced counterpart.

### 6.2.2. Decomposition of the spectrum

As in the previous sections, let $\mathcal{G} \rightrightarrows X$ be a locally compact, second-countable and locally Hausdorff groupoid, endowed with a right-invariant continuous Haar system. If $f \in C_c(\mathcal{G})$ and $\varphi \in C_0(X)$, we follow the notation of [173] and denote by $\varphi f$ the function $(\varphi \circ r) \cdot f$ (the central dots denotes scalar multiplication, and not convolution).

**Lemma 6.2.8.** *Let $A$ be a $C^*$-algebra and $(I_\lambda)_{\lambda\in\Lambda}$ a family of ideals in $A$ such that $\sum_{\lambda\in\Lambda} I_\lambda = A$. Then*

$$\operatorname{Prim} A = \bigcup_{\lambda\in\Lambda} \operatorname{Prim} I_\lambda,$$

*where we identify $\operatorname{Prim} I_\lambda$ with its image $\operatorname{Prim}^{I_\lambda} A$ through $\operatorname{Ind}_{I_\lambda}^A$.*

The reader should refer to Section 4.1.2 for the definition of $\operatorname{Prim}^{I_\lambda} A$ and the induction map $\operatorname{Ind}_{I_\lambda}^A$.

*Proof.* For all $J \in \operatorname{Prim}(A)$, there is a $\lambda \in \Lambda$ such that $I_\lambda \not\subset J$. Indeed, if that were not the case, then we would have $A = \sum_{\lambda\in\Lambda} I_\lambda \subset J$ so $J = A$, which is not a primitive ideal. Therefore there is a $\lambda \in \Lambda$ such that $J \in \operatorname{Prim}^{I_\lambda}(A)$, which proves the proposition. □

**Lemma 6.2.9.** *Let $\varphi \in C_0(X)$, and define $M_\varphi : C_c(\mathcal{G}) \to C_c(\mathcal{G})$ by $M_\varphi(f) = \varphi f$. Then $M_\varphi$ extends as a continuous linear map from $C^*(\mathcal{G})$ to itself. Moreover, if $U \subset X$ is a $\mathcal{G}$-invariant open subset of $X$ such that $\operatorname{supp} \varphi \subset U$, then $f \mapsto \varphi f$ extends as a continuous map from $C^*(\mathcal{G})$ to $C^*(\mathcal{G}_U)$.*

Using the first part of the lemma, one can further check that $C_0(X)$ embeds into the multiplier algebra of $C^*(\mathcal{G})$.

*Proof.* The first statement was proven in [173, Proposition 1.14], in which it was shown that

$$\|\varphi f\|_{C^*(\mathcal{G})} \leq \|\varphi\|_\infty \|f\|_{C^*(\mathcal{G})}.$$

If $U \subset X$ is an open subset such that $\varphi \in C_c(U)$, then $\varphi \circ r \in C_c(\mathcal{G}^U)$. If $U$ is moreover $\mathcal{G}$-invariant, then $\mathcal{G}^U = \mathcal{G}_U$, so $\varphi f \in C_c(\mathcal{G}_U)$. We know from [173] that $C^*(\mathcal{G}_U)$ is an ideal in $C^*(\mathcal{G})$, so

$$\|\varphi f\|_{C^*(\mathcal{G}_U)} = \|\varphi f\|_{C^*(\mathcal{G})} \leq \|\varphi\|_\infty \|f\|_{C^*(\mathcal{G})}.$$

This proves the continuity as a map to $C^*(\mathcal{G}_U)$. □

We are ready to prove one of the main theorems of this chapter. Again, recall that the definition of $\operatorname{Prim}^U C^*(\mathcal{G})$ and $\operatorname{Ind}_U$ were introduced in the previous subsection.

**Theorem 6.2.10.** *Let $\mathcal{G} \rightrightarrows X$ be a locally compact, second-countable and locally Hausdorff groupoid. Assume that we have a family of open subsets $(U_i)_{i\in I}$ such that their saturations $(\mathcal{G} \cdot U_i)_{i\in I}$ form an open cover of $X$. Then*

$$\operatorname{Prim} C^*(\mathcal{G}) = \bigcup_{i\in I} \operatorname{Prim} C^*(\mathcal{G}|_{U_i}),$$

*where we identify $\operatorname{Prim} C^*(\mathcal{G}|_{U_i})$ with its image $\operatorname{Prim}^{U_i} C^*(\mathcal{G})$ through $\operatorname{Ind}_{U_i}$.*

Theorem 6.2.10 is a localization result: the primitive spectrum of $C^*(\mathcal{G})$ can be fully described by restricting our attention to sufficiently many reductions of $\mathcal{G}$ to open subsets of the unit space.

*Proof.* Put $W_i := \mathcal{G} \cdot U_i$, for each $i \in I$. The assumption is that $(W_i)_{i\in I}$ is an open cover of $X$, so let $(\varphi_i)_{i\in I}$ be a partition of unity subordinate to that cover. If $a \in C^*(\mathcal{G})$, then it follows from Lemma 6.2.9 that $\varphi_i a$ is well defined for all $i$ and belongs to $C^*(\mathcal{G}_{W_i})$. Since $\sum_{i\in I} \varphi_i = 1$, we have $a = \sum_{i\in I} \varphi_i a$. Thus

$$C^*(\mathcal{G}) = \sum_{i\in I} C^*(\mathcal{G}_{W_i}),$$

which is all we need to apply Lemma 6.2.8. Therefore

$$\operatorname{Prim} C^*(\mathcal{G}) = \bigcup_{i\in I} \operatorname{Prim}^{W_i} C^*(\mathcal{G}),$$

and we conclude with the identification $\operatorname{Prim}^{W_i} C^*(\mathcal{G}) = \operatorname{Prim}^{U_i} C^*(\mathcal{G})$ established in Proposition 6.2.6. □

**Remark 6.2.11.** As was already highlighted in Remark 6.2.7, all results from this chapter remain valid if we replace the full groupoid algebras with their *reduced* counterparts. More explicitly, under the assumptions of Theorem 6.2.10, there is a decomposition

$$\operatorname{Prim} C_r^*(\mathcal{G}) = \bigcup_{i\in I} \operatorname{Prim} C_r^*(\mathcal{G}|_{U_i}),$$

where we identify $\operatorname{Prim} C_r^*(\mathcal{G}|_{U_i})$ with its image $\operatorname{Prim}^{U_i} C_r^*(\mathcal{G})$ through $\operatorname{Ind}_{U_i}$. Note that the technical Lemma 6.2.9 is much easier to prove in the reduced case. Indeed there is no need to use Renault's disintegration theorem here, since we only have to deal with the regular representations of $\mathcal{G}$.

**Remark 6.2.12.** It should also be noted that a decomposition similar to that of Theorem 6.2.10 holds for the full spectrum of $C^*(\mathcal{G})$ (i.e. equivalence classes of irreducible representations as defined in [75]). Under the assumptions of Theorem 6.2.10, we may write

$$\widehat{C^*(\mathcal{G})} = \bigcup_{i\in I} \widehat{C^*(\mathcal{G}|_{U_i})}.$$

where $\widehat{C^*(\mathcal{G}|_{U_i})}$ is identified with its image through $\operatorname{Ind}_{U_i}$. The same statement holds for the spectra of the reduced $C^*$-algebras.

### 6.2.3. Families of representations

The main motivation for Theorem 6.2.10 is to study the representations of $C^*(\mathcal{G})$ from the representations of its reductions. In particular, recall from Section 4.1.2 that a family of representations $\mathcal{F}$ of a $C^*$-algebra $A$ is faithful if, and only if

$$\operatorname{Prim} A = \overline{\bigcup_{\pi\in\mathcal{F}} \operatorname{supp} \pi}.$$

**Corollary 6.2.13.** *We follow the assumptions of Theorem 6.2.10. For each $i \in I$, let $\mathcal{F}_i$ be a faithful family of non-degenerate representations of $C^*(\mathcal{G}|_{U_i})$. Then the family*

$$\mathcal{F} := \{\operatorname{Ind}_{U_i} \pi \mid i \in I, \pi \in \mathcal{F}_i\}$$

*is faithful for $C^*(\mathcal{G})$.*

*Proof.* By assumption, for all $i \in I$ we have

$$\operatorname{Prim} C^*(\mathcal{G}|_{U_i}) = \overline{\bigcup_{\pi\in\mathcal{F}_i} \operatorname{supp} \pi_i}.$$

Using Theorem 6.2.10, we get

$$\operatorname{Prim} C^*(\mathcal{G}) = \bigcup_{i\in I} \operatorname{Prim}^{U_i} C^*(\mathcal{G}) = \bigcup_{i\in I} \overline{\bigcup_{\pi\in\mathcal{F}_i} \operatorname{Ind}_{U_i}(\operatorname{supp} \pi)}. \tag{6.2}$$

It was proven in Proposition 6.2.6 that $\operatorname{Ind}_{U_i}(\operatorname{supp}\pi) \subset \operatorname{supp}(\operatorname{Ind}_{U_i}\pi)$ for any non-degenerate representation $\pi$ of $C^*(\mathcal{G}|_{U_i})$. Thus

$$\bigcup_{\pi\in\mathcal{F}_i} \operatorname{Ind}_{U_i}(\operatorname{supp}\pi) \subset \bigcup_{\pi\in\mathcal{F}_i} \operatorname{supp}(\operatorname{Ind}_{U_i}\pi)$$

Together with Equation (6.2), we obtain

$$\operatorname{Prim} C^*(\mathcal{G}) \subset \bigcup_{i\in I} \overline{\bigcup_{\pi\in\mathcal{F}_i} \operatorname{supp}(\operatorname{Ind}_{U_i}\pi)} \subset \overline{\bigcup_{i\in I}\bigcup_{\pi\in\mathcal{F}_i} \operatorname{supp}(\operatorname{Ind}_{U_i}\pi)} = \overline{\bigcup_{\pi\in\mathcal{F}} \operatorname{supp}\pi}.$$

The converse inclusion is trivial. This shows that $\mathcal{F}$ is a faithful family. □

As a direct application, recall that a groupoid $\mathcal{G}$ is called *metrically amenable* if the canonical morphism $C^*(\mathcal{G}) \to C^*_r(\mathcal{G})$ is an isomorphism [173].

**Corollary 6.2.14.** *Under the assumptions of Theorem 6.2.10, assume that each groupoid $\mathcal{G}|_{U_i}$ is metrically amenable, for all $i \in I$. Then $\mathcal{G}$ is metrically amenable.*

*Proof.* For each $i \in I$, let $\mathcal{F}_i = (\pi_x^{U_i})_{x\in U_i}$ be the family of all regular representations of $\mathcal{G}|_{U_i}$. The groupoid $\mathcal{G}|_{U_i}$ is metrically amenable if, and only if, the family $\mathcal{F}_i$ is faithful for $C^*(\mathcal{G}|_{U_i})$. Now recall from Proposition 6.2.6 that $\operatorname{Ind}_{U_i}\pi_x^{U_i} = \pi_x$, which is the regular representation of $\mathcal{G}$ at $x$. Corollary 6.2.13 implies that the family $\mathcal{F} = (\pi_x)_{x\in X}$ is faithful for $C^*(\mathcal{G})$. This in turn is equivalent to $\mathcal{G}$ being metrically amenable. □

**Definition 6.2.15** (Nistor, Prudhon [151])**.** Let $A$ be a $C^*$-algebra. A family $\mathcal{F}$ of representations of $A$ is called *exhaustive* if

$$\operatorname{Prim} A = \bigcup_{\pi\in\mathcal{F}} \operatorname{supp}\pi.$$

As is also explained in 4.1.2, exhaustive families provide a refinement of faithful families and will be used in Section 6.3.

**Corollary 6.2.16.** *We follow the assumptions of Theorem 6.2.10. For each $i \in I$, let $\mathcal{F}_i$ be an exhaustive family of representations of $C^*(\mathcal{G}|_{U_i})$. Then the family*

$$\{\operatorname{Ind}_{U_i}\pi \mid i \in I, \pi \in \mathcal{F}_i\}$$

*is exhaustive for $C^*(\mathcal{G})$.*

The proof is the same as that of Corollary 6.2.13.

## 6.3. Fredholm groupoids

As explained in Chapter 5, the class of Fredholm Lie groupoid was introduced by Carvalho, Nistor and Qiao in [51] as an important tool to study differential equations on manifolds with singularities. Our main result (Theorem 6.3.5) is that a groupoid $\mathcal{G}$ is Fredholm if, and only if, for any family of open sets $(U_i)_{i\in I}$ such that the saturations $(\mathcal{G}\cdot U_i)_{i\in I}$ form an open cover of the unit space, each reductions $\mathcal{G}|_{U_i}$ is Fredholm. This justifies our point that the Fredholm property is a *local* property. Furthermore, it motivates the definition of *local action groupoids* in Subsection 6.3.3, which occur naturally in many practical cases.

### 6.3.1. Definitions

The original definition of Fredholm groupoid is done in the setting of Lie groupoid, both in [51] and Chapter 5, but this definition is topological in nature. Note in particular that groupoids that are only longitudinally smooth are useful in applications, see e.g. Example 6.4.14. For this reason, we shall work in the topological setting.

Let thus $\mathcal{G} \rightrightarrows X$ be a locally compact, second-countable, locally Hausdorff groupoid with a continuous right-invariant Haar system. Throughout this subsection, we will assume that there is a $\mathcal{G}$-invariant, open and dense orbit $V \subset X$ such that $\mathcal{G}_V \simeq V \times V$. Such a set $V$ is necessarily unique. Define the *vector representation*

$$\pi_0 : C_r^*(\mathcal{G}) \to \mathcal{B}(L^2(V)),$$

as the equivalence class of any regular representation $\pi_x$, for any $x \in V$ (all those representations are conjugated through the action of $\mathcal{G}$ on its fibers $\mathcal{G}_x = d^{-1}(x)$).

Fredholm groupoid are tailored to study differential operators on $V$, so one usually asks $V$ to have a smooth structure: this is the case, for example, when $\mathcal{G}$ is a Lie groupoid, or more generally a continuous family groupoid [112, 158]. However, the differential setting is not needed for the results we seek; thus our definition of a Fredholm groupoids is a strict extension of the original one from [51].

**Definition 6.3.1.** A *Fredholm groupoid* is a locally compact, second-countable, locally Hausdorff groupoid $\mathcal{G} \rightrightarrows X$, endowed with a continuous right-invariant Haar system, such that

(i) there is an open, dense $\mathcal{G}$-orbit $V$ such that $\mathcal{G}_V \simeq V \times V$,
(ii) the vector representation $\pi_0 : C_r^*(\mathcal{G}) \to \mathcal{B}(L^2(V))$ is injective, and
(iii) for any $a \in C_r^*(\mathcal{G})$, the operator $1 + \pi_0(a)$ is Fredholm in $\mathcal{B}(L^2(V))$ if, and only if, each operator $1 + \pi_x(a)$ is invertible, for every $x \in X \setminus V$.

As in Section 5.2.1, we also have the following equivalent definition, originally stated in [51]. Recall the concept of an exhaustive family of representations from Definition 6.2.15 and Section 4.1.2.

**Proposition 6.3.2.** *Let $\mathcal{G} \rightrightarrows X$ be a locally compact, second-countable and locally Hausdorff groupoid, endowed with a continuous right-invariant Haar system. Then $\mathcal{G}$ is a Fredholm groupoid if, and only if, all the following conditions are met:*

*(i) there is an open, dense $\mathcal{G}$-orbit $V$ such that $\mathcal{G}_V \simeq V \times V$,*
*(ii) the vector representation $\pi_0 : C_r^*(\mathcal{G}) \to \mathcal{B}(L^2(V))$ is injective,*
*(iii) the restriction map $C_r^*(\mathcal{G}) \to C_r^*(\mathcal{G}_F)$ induces an isomorphism*

$$C_r^*(\mathcal{G})/C_r^*(\mathcal{G}_V) \simeq C_r^*(\mathcal{G}_F),$$

*where $F = X \setminus V$, and*
*(iv) the family of representations $(\pi_x)_{x\in F}$ is exhaustive for $C_r^*(\mathcal{G}_F)$.*

Proposition 6.3.2 was proven in [51] for Lie groupoids, but without making any use of the smooth structure: thus it extends without any modification to our setting. Note that Conditions (iii) and (iv) may be checked at once by stating that the family $(\pi_x)_{x\in F}$ is exhaustive for the quotient algebra $C^*_r(\mathcal{G})/C^*_r(\mathcal{G}_V)$.

Recall the definition of a *metrically amenable* groupoid from Subsection 6.2.3. The following Theorem was stated for Lie groupoids in Chapter 5, Theorem 5.2.3.

**Theorem 6.3.3.** *Let $\mathcal{G} \rightrightarrows X$ be a locally compact and second-countable groupoid endowed with a continuous right-invariant Haar system. Assume that there is an open, dense and $\mathcal{G}$-invariant subset $V \subset X$ such that $\mathcal{G}_V \simeq V \times V$, and put $F = X \setminus V$. Assume moreover that $\mathcal{G}$ is Hausdorff and $\mathcal{G}_F$ metrically amenable. Then $\mathcal{G}$ is Fredholm.*

Theorem 6.3.3 gives a sufficient condition for Fredholmness which is satisfied by many groupoids encountered in practical cases (see Subsection 6.4.1 for examples). When $\mathcal{G}$ is moreover a Lie groupoid, its dense orbit $V$ is called a manifold with *amenable ends* [51].

*Proof.* This result was proven in [51] for Lie groupoids, so we will only give a sketch of the proof here. First, it follows from a lemma of Khoskham and Skandalis [105] (and the density of $V$ in $X$) that the vector representation is always injective when $\mathcal{G}$ is Hausdorff. This proves Condition (ii) of Proposition 6.3.2.

The amenability of $\mathcal{G}_F$ and $\mathcal{G}_V \simeq V \times V$ imply that $\mathcal{G}$ is also metrically amenable. It is then a standard fact that the restriction map induces an isomorphism $C^*_r(\mathcal{G})/C^*_r(\mathcal{G}_U) \simeq C^*_r(\mathcal{G}_F)$ [173], which proves Condition (iii). Condition (iv) is a result of Nistor and Prudhon: if $\mathcal{G}_F$ is metrically amenable, then its set of regular representations $(\pi_x)_{x\in F}$ is exhaustive for $C^*_r(\mathcal{G}_F)$ [151, Theorem 3.18]. This follows from the Effros-Hahn conjecture, which was proven for amenable groupoids [100]. □

Many examples of Fredholm groupoids (as well as their relation with the study of differential equations on open manifolds) will be given in Section 6.4.

### 6.3.2. The Fredholm property is local

Our aim in this section is to use the results of Section 6.2 to prove our main result, Theorem 6.1.1. In a nutshell, we show that a groupoid $\mathcal{G}$ is Fredholm if, and only if, all its reductions to any family of open subsets generating the units are Fredholm.

**Lemma 6.3.4.** *Let $\mathcal{G} \rightrightarrows X$ be a Fredholm groupoid. Then, for any open set $U \subset X$, the reduction $\mathcal{G}|_U$ is also a Fredholm groupoid.*

*Proof.* Let $V \subset X$ be the unique open dense $\mathcal{G}$-orbit such that $\mathcal{G}_V \simeq V \times V$, and put $F = X \setminus V$. Then $V' := U \cap V$ is the unique open dense $\mathcal{G}|_U$-orbit such that $\mathcal{G}|_{V'} \simeq V' \times V'$.

Let $a \in C^*_r(\mathcal{G}|_U)$. Because $\pi_0$ is injective on $C^*_r(\mathcal{G})$ and $\pi_0(C^*_r(V' \times V')) \simeq \mathcal{K}(L^2(V'))$, there is an induced isomorphism

$$\pi_0 : C^*_r(\mathcal{G}|_U)/C^*_r(\mathcal{G}|_{V'}) \simeq \pi_0(C^*_r(\mathcal{G}|_U))/\mathcal{K}(L^2(V')).$$

Therefore, for any $a \in C_r^*(\mathcal{G}|_U)$, the operator $1 + \pi_0(a)$ is Fredholm in $\mathcal{B}(L^2(V'))$ if, and only if, the class of $1 + a$ is invertible in the unitisation of $C_r^*(\mathcal{G}|_U)/C_r^*(\mathcal{G}|_{V'})$. But $C_r^*(\mathcal{G}|_U)$ is a subalgebra of $C_r^*(\mathcal{G})$, and $C_r^*(\mathcal{G}|_{V'}) = C_r^*(\mathcal{G}_V) \cap C_r^*(\mathcal{G}|_U)$. Thus

$$C_r^*(\mathcal{G}|_U)/C_r^*(\mathcal{G}|_{V'}) \subset C_r^*(\mathcal{G})/C^*(\mathcal{G}_V).$$

Hence, $1 + a$ is invertible in the unitisation of $C_r^*(\mathcal{G}|_U)/C_r^*(\mathcal{G}|_{V'})$ if, and only if, it is invertible as an element of the unitisation of $C_r^*(\mathcal{G})/C^*(\mathcal{G}_V)$.

Now, since $\mathcal{G}$ is a Fredholm groupoid, we deduce that $1+\pi_0(a)$ is Fredholm in $\mathcal{B}(L^2(V'))$ if, and only if, the operator $1 + \pi_x(a)$ is invertible for each $x \in F$. But $\pi_x(a) = 0$ for all $x \notin U$. Therefore, the operator $1 + \pi_0(a)$ is Fredholm if, and only if, the operator $1 + \pi_x(a)$ is invertible for each $x \in F \cap U = U \setminus V'$. This proves that $\mathcal{G}|_U$ is a Fredholm groupoid. □

We now establish the converse of Lemma 6.3.4.

**Theorem 6.3.5.** *Let $\mathcal{G} \rightrightarrows X$ be a locally compact, second-countable and locally Hausdorff groupoid, endowed with a right-invariant Haar system. Assume that*

*(i) there is an open dense $\mathcal{G}$-invariant subset $V \subset X$ with $\mathcal{G}_V \simeq V \times V$, and*

*(ii) we have a family $(U_i)_{i\in I}$ of open subsets of $X$ such that the saturations $(\mathcal{G} \cdot U_i)_{i\in I}$ provide an open cover of $X$.*

*Then $\mathcal{G}$ is a Fredholm groupoid if, and only if, each reduction $\mathcal{G}|_{U_i}$ is also a Fredholm groupoid, for every $i \in I$.*

Theorem 6.3.5 is the main result of this chapter. It emphasizes the fact that the Fredholmness of a groupoid $\mathcal{G}$ is determined by its local structure. In particular, what really matters is the local structure in a neighborhood of the closed set $F = X \setminus V$, or in other words how the groupoid $\mathcal{G}_F$ is glued to the pair groupoid $\mathcal{G}_V = V \times V$.

*Proof of Theorem 6.3.5.* Assume that each reduction $\mathcal{G}|_{U_i}$ is a Fredholm groupoid, and let $V_i \subset U_i$ be the unique open dense $\mathcal{G}|_{U_i}$-orbit such that $\mathcal{G}|_{V_i} \simeq V_i \times V_i$. We only have to prove that $\mathcal{G}$ satisfies the assumptions (ii), (iii) and (iv) of Proposition 6.3.2.

First, for any $i \in I$, let $\pi_0^i : C_r^*(\mathcal{G}|_{V_i}) \to \mathcal{B}(L^2(V_i))$ be the vector representation of $\mathcal{G}|_{V_i}$. We know from Proposition 6.2.6 that $\mathrm{Ind}_{U_i} \pi_0^i$ is the vector representation $\pi_0$ of $C_r^*(\mathcal{G})$ on $\mathcal{B}(L^2(V))$. Moreover, because $\mathcal{G}|_{U_i}$ is Fredholm, the representation $\pi_0^i$ is faithful. Corollary 6.2.13 implies that $\pi_0$ is a faithful representation of $C^*(\mathcal{G})$, which proves Assumption (ii).

Now, because $(\mathcal{G} \cdot U_i)_{i\in I}$ is an open cover of $X$, Theorem 6.2.10 implies that

$$\mathrm{Prim}\, C_r^*(\mathcal{G}) = \bigcup_{i\in I} \mathrm{Prim}\, C_r^*(\mathcal{G}|_{U_i}).$$

Since $V_i$ is a $\mathcal{G}|_{U_i}$-invariant open subset of $U_i$, we may expand this decomposition:

$$\begin{aligned}\mathrm{Prim}\, C_r^*(\mathcal{G}) &= \bigcup_{i\in I} \big(\mathrm{Prim}\, C_r^*(\mathcal{G}|_{V_i}) \bigsqcup \mathrm{Prim}\,(C_r^*(\mathcal{G}|_{F_i}))\big) \\ &= \left(\bigcup_{i\in I} \mathrm{Prim}\, C_r^*(\mathcal{G}|_{V_i})\right) \bigcup \left(\bigcup_{i\in I} \mathrm{Prim}\,(C_r^*(\mathcal{G}|_{F_i}))\right),\end{aligned} \tag{6.3}$$

where we have put $F_i := U_i \setminus V_i$ and used the isomorphism $C_r^*(\mathcal{G}|_{U_i})/C_r^*(\mathcal{G}|_{V_i}) \simeq C_r^*(\mathcal{G}_{F_i})$ given by the fact that $\mathcal{G}|_{U_i}$ is a Fredholm groupoid. But the family $(\mathcal{G} \cdot V_i)_{i\in I}$ is an open cover of $V$, so another application of Theorem 6.2.10 yields

$$\operatorname{Prim} C_r^*(\mathcal{G}_V) = \bigcup_{i\in I} \operatorname{Prim} C_r^*(\mathcal{G}|_{V_i}).$$

By substituting this last expression in Equation (6.3), we obtain

$$\operatorname{Prim} C_r^*(\mathcal{G}) = \operatorname{Prim} C_r^*(\mathcal{G}_V) \bigcup \left( \bigcup_{i\in I} \operatorname{Prim} C_r^*(\mathcal{G}|_{F_i}) \right), \tag{6.4}$$

On the other hand, because $V$ is a $\mathcal{G}$-invariant open subset, there is also a decomposition

$$\operatorname{Prim} C_r^*(\mathcal{G}) = \operatorname{Prim} C_r^*(\mathcal{G}_V) \bigsqcup \operatorname{Prim}(C_r^*(\mathcal{G})/C_r^*(\mathcal{G}_V)) \tag{6.5}$$

Combining Equations (6.4) and (6.5) proves the inclusion

$$\operatorname{Prim}(C_r^*(\mathcal{G})/C_r^*(\mathcal{G}_V)) \subset \bigcup_{i\in I} \operatorname{Prim} C_r^*(\mathcal{G}|_{F_i}).$$

For $i \in I$ and $x \in U_i$, let us denote by $\pi_x^i$ the regular representation of $\mathcal{G}|_{U_i}$ at $x$. Recall from Proposition 6.2.6 that $\operatorname{Ind}_{U_i}(\pi_x^i) = \pi_x$ (with $\pi_x$ the regular representation of $\mathcal{G}$ at $x$), so $\operatorname{Ind}_{U_i}(\operatorname{supp} \pi_x^i) \subset \operatorname{supp} \pi_x$. Since $\mathcal{G}|_{U_i}$ is a Fredholm groupoid, the family $(\pi_x^i)_{x\in F_i}$ is exhaustive for $C_r^*(\mathcal{G}|_{F_i})$; in other words $\operatorname{Prim} C_r^*(\mathcal{G}|_{F_i})$ is the union of the support of every $\operatorname{supp}(\pi_x^i)$, for $x \in F_i$. Therefore

$$\operatorname{Prim}(C_r^*(\mathcal{G})/C_r^*(\mathcal{G}_V)) \subset \bigcup_{i\in I} \bigcup_{x\in F_i} \operatorname{Ind}_{U_i}(\operatorname{supp} \pi_x^i) \subset \bigcup_{x\in F} \operatorname{supp} \pi_x,$$

with $F := X \setminus V = \bigcup_{i\in I} F_i$. On the other hand, the representation $\pi_x$ vanishes on $C_r^*(\mathcal{G}_V)$ for any $x \in F$, so that $\operatorname{supp} \pi_x$ is contained in $\operatorname{Prim}(C_r^*(\mathcal{G})/C_r^*(\mathcal{G}_V))$. This proves the equality

$$\operatorname{Prim}(C_r^*(\mathcal{G})/C_r^*(\mathcal{G}_V)) = \bigcup_{x\in F} \operatorname{supp} \pi_x,$$

which by definition indicates that the family $(\pi_x)_{x\in F}$ is exhaustive for the quotient algebra $C_r^*(\mathcal{G})/C_r^*(\mathcal{G}_V)$. This proves Assumptions (iii) and (iv) of Proposition 6.3.2 and concludes the proof that $\mathcal{G}$ is a Fredholm groupoid. Finally, the “only if” part of Theorem 6.3.5 is a consequence of Lemma 6.3.4 above. □

### 6.3.3. Consequences

We give here several corollaries of Theorem 6.3.5, which may be used as tools to check the Fredholm property for a given groupoid. Some concrete examples of groupoids and applications of these results will be shown in Section 6.4.

**Gluing groupoids**

Let $(U_i)_{i\in I}$ be an open cover of a locally compact, Hausdorff space $X$. Assume that we are being given a family of locally compact groupoids $(\mathcal{G}_i \rightrightarrows U_i)_{i\in I}$ with isomorphisms

$$\phi_{ji} : \mathcal{G}_i|_{U_i\cap U_j} \to \mathcal{G}_j|_{U_i\cap U_j},$$

for all $i, j \in I$, satisfying a cocyle condition on common domains. We assume that the family $(\mathcal{G}_i)_{i\in I}$ satisfies the weak gluing condition defined in Section 5.3, which allows to define the *glued groupoid*

$$\mathcal{G} := \bigcup_{i\in I} \mathcal{G}_i \rightrightarrows X.$$

**Corollary 6.3.6.** *Let $(U_i)_{i\in I}$ be an open cover of a locally compact Hausdorff space $X$, and let $(\mathcal{G}_i \rightrightarrows U_i)_{i\in I}$ be a family of groupoids satisfying the weak gluing condition. Let $\mathcal{G} = \bigcup_{i\in I} \mathcal{G}_i$ be the glued groupoid, and assume that*

*(i) there is an open dense $\mathcal{G}$-invariant subset $V \subset X$ with $\mathcal{G}_V \simeq V \times V$, and*
*(ii) each groupoid $\mathcal{G}_i$ is Fredholm, for $i \in I$.*

*Then the groupoid $\mathcal{G}$ is Fredholm.*

*Proof.* By construction, each reduction $\mathcal{G}|_{U_i}$ is isomorphic to $\mathcal{G}_i$, hence Fredholm. Since the family $(U_i)_{i\in I}$ is an open cover of $X$, the result is a direct consequence of Theorem 6.3.5. $\square$

**Local isomorphisms**

Theorems 6.2.10 and 6.3.5 state that the primitive spectrum of a groupoid's $C^*$-algebra can be studied locally. This suggests the following notion of *local isomorphisms* between groupoids.

**Definition 6.3.7.** Let $\mathcal{G} \rightrightarrows X$ and $\mathcal{H} \rightrightarrows Y$ be two locally compact groupoids, and let $p \in X$.

(i) A *local isomorphism* between $\mathcal{G}$ and $\mathcal{H}$ around $p$ is a triplet $(U, \phi, V)$, where $U \subset X$ and $V \subset Y$ are open subsets, with $p \in U$ and

$$\phi : \mathcal{G}|_U \to \mathcal{H}|_V$$

is an isomorphism of locally compact groupoids.

(ii) We say that $\mathcal{G}$ is *locally isomorphic* to $\mathcal{H}$ around $p$, and we write $\mathcal{G} \sim_p \mathcal{H}$, if there exists an isomorphism between $\mathcal{G}$ and $\mathcal{H}$ around $p$.

Recall that the *direct product* of two groupoids $\mathcal{G} \rightrightarrows X$ and $\mathcal{H} \rightrightarrows Y$ is the groupoid $\mathcal{G} \times \mathcal{H}$, with units $X \times Y$, whose structural morphisms are the direct products of those of $\mathcal{G}$ and $\mathcal{H}$.

**Lemma 6.3.8.** *Let $\mathcal{G}_1 \rightrightarrows X_1$ and $\mathcal{G}_2 \rightrightarrows X_2$ be two locally compact groupoids. Let $p_1 \in X_1$ and $p_2 \in X_2$. Assume that there are groupoids $\mathcal{H}_1, \mathcal{H}_2$ such that $\mathcal{G}_1 \sim_{p_1} \mathcal{H}_1$ and $\mathcal{G}_2 \sim_{p_2} \mathcal{H}_2$. Then $\mathcal{G}_1 \times \mathcal{G}_2$ is locally isomorphic to $\mathcal{H}_1 \times \mathcal{H}_2$ near $(p_1, p_2)$.*

*Proof.* By assumptions, there are isomorphisms $\phi_1 : \mathcal{G}_1|_{U_1} \to \mathcal{H}_1|_{V_1}$ and $\phi_2 : \mathcal{G}_2|_{U_2} \to \mathcal{H}_2|_{V_2}$, with $p_1 \in U_1$ and $p_2 \in U_2$. Then $(p_1, p_2) \in U_1 \times U_2$ and

$$\phi_1 \times \phi_2 : (\mathcal{G}_1 \times \mathcal{G}_2)|_{U_1 \times U_2} \to (\mathcal{H}_1 \times \mathcal{H}_2)|_{V_1 \times V_2}$$

is an isomorphism, which proves the lemma. □

Section 5.3 introduced the gluing construction of a family of groupoids. We show that gluing groupoids preserves their local structure.

**Lemma 6.3.9.** *Let $(U_i)_{i\in I}$ be an open cover of a topological space $X$, and let $(\mathcal{G}_i \rightrightarrows U_i)_{i\in I}$ be a family of locally compacts groupoids satisfying the weak gluing condition. Let $i \in I$ and $p \in U_i$, and assume that there is a groupoid $\mathcal{H}$ such that $\mathcal{G}_i \sim_p \mathcal{H}$. Then*

$$\bigcup_{i\in I} \mathcal{G}_i \sim_p \mathcal{H}$$

*Proof.* Let $\mathcal{G} = \bigcup_{i\in I} \mathcal{G}_i$ be the glued groupoid. By definition, we have $\mathcal{G}|_{U_i} \simeq \mathcal{G}_i$; hence any local isomorphism $\phi : \mathcal{G}_i|_U \to \mathcal{H}|_V$ around $p$ induces a local isomorphism $\mathcal{G}|_{U_i \cap U} \simeq \mathcal{H}|_V$ around $p$. □

**Corollary 6.3.10.** *Let $\mathcal{G} \rightrightarrows X$ be a locally compact, second-countable and locally Hausdorff groupoid. Assume that*

*(i) there is an open dense $\mathcal{G}$-invariant subset $V \subset X$ with $\mathcal{G}_V \simeq V \times V$, and*
*(ii) for each $p \in X$, there is a Fredholm groupoid $\mathcal{H}_p$ such that $\mathcal{G} \sim_p \mathcal{H}_p$.*

*Then $\mathcal{G}$ is a Fredholm groupoid.*

The point of Corollary 6.3.10 is to emphasize again that only the local structure is important to characterize Fredholm groupoids.

*Proof.* Following the assumptions, there is for each $p \in X$ an open set $U_p$ containing $p$ and such that $\mathcal{G}|_{U_p}$ is isomorphic to a reduction $\mathcal{H}_p|_{V_p}$, with $\mathcal{H}_p$ a Fredholm groupoid. Lemma 6.3.4 implies that $\mathcal{H}_p|_{V_p}$ is Fredholm, so $\mathcal{G}|_{U_p}$ is also Fredholm. The conclusion follows from Theorem 6.3.5 applied to the open cover $(U_p)_{p\in X}$. □

### Local action groupoids

Many Fredholm groupoids occurring in the study of differential equation on singular spaces are very simple on a local scale: they are locally isomorphic to action groupoids. To formalize this, we introduce here the class of *local action groupoids.* Many examples of such groupoids will be found in Subsection 6.4.1 below.

**Remark 6.3.11.** Recall that, if $G$ is a group acting on a space $X$ on the right, then the corresponding *action groupoid* (or transformation groupoid) is written $X \rtimes G$ and defined as follows. As a set, we put $X \rtimes G := X \times G$. The domain and range maps are given by

$$d(x, g) = x \cdot g^{-1} \quad \text{and} \quad r(x, g) = x,$$

whereas the product is $(x, g)(x \cdot g^{-1}, h) = (x, hg)$ (see Example 4.3.14 for more details).

If $G$ and $X$ are both locally compact, second-countable and Hausdorff, and if moreover the action is continuous, then $X \rtimes G$ is a locally compact, second-countable, Hausdorff groupoid. The groupoid $X \rtimes G$ is endowed with a natural Haar system (induced by the Haar measure on $G$), and the reduced groupoid $C^*$-algebra $C_r^*(X \rtimes G)$ is then isomorphic to the crossed-product algebra $C_0(X) \rtimes_r G$.

**Theorem 6.3.12.** *Let $G$ be a topological group acting continuously on a space $X$. Assume that $G$ and $X$ are locally compact, Hausdorff and second-countable. Assume moreover that:*

*(i) there is an open, dense $G$-orbit $V \subset X$ such that the action of $G$ on $V$ is free, transitive and proper,*

*(ii) the group $G$ is amenable.*

*Then the groupoid $X \rtimes G$ is Fredholm.*

Action groupoids of the sort occur naturally in the study of the essential spectrum of differential operators on groups, a notable example being that of the quantum $N$-body problem on Euclidean space [85, 142, 127].

*Proof.* Let $\mathcal{G} = X \rtimes G$. Firstly, the assumptions on the action of $G$ on $V$ imply that the map

$$\begin{aligned} \alpha : V \rtimes G &\to V \times V \\ (x, g) &\mapsto (x, x \cdot g^{-1}) \end{aligned}$$

is continuous and bijective. Moreover $\alpha$ is proper with value in a Hausdorff space, hence closed. Therefore $\alpha$ is an homeomorphism, which shows that $\mathcal{G}_V \simeq V \times V$ Secondly, the amenability of $G$ implies that the groupoid $\mathcal{G}_F = F \rtimes G$ is metrically amenable [207], where $F = X \setminus G$. The result follows from Theorem 6.3.3. □

**Definition 6.3.13.** A locally compact and second-countable groupoid $\mathcal{G} \rightrightarrows X$ is said to be a *local action groupoid* if, for each $p \in X$, there is an action groupoid $X_p \rtimes G_p$ such that $\mathcal{G}$ is locally isomorphic to $X_p \rtimes G_p$ near $p$, where $G_p$ and $X_p$ are both locally compact, Hausdorff and second-countable.

The main point of Definition 6.3.13 is that the local structure of a such groupoids is very well understood: indeed, the $C^*$-algebras of action groupoids and their representations have been much studied in the literature [76, 162, 207]. Several examples of local action groupoids shall be given in Section 6.4.

**Proposition 6.3.14.** *Let $\mathcal{G}$ and $\mathcal{H}$ be local action groupoids. Then $\mathcal{G} \times \mathcal{H}$ is also a local action groupoid.*

*Proof.* This follows from Lemma 6.3.8 and the fact that, if $\mathcal{G}_1 = X_1 \rtimes G_1$ and $\mathcal{G}_2 = X_2 \rtimes G_2$ are action groupoids, then

$$\mathcal{G}_1 \times \mathcal{G}_2 \simeq (X_1 \times X_2) \rtimes (G_1 \times G_2),$$

where $G_1 \times G_2$ acts on $X_1 \times X_2$ by the product action. □

**Proposition 6.3.15.** *Let $(\mathcal{G}_i)_{i\in I}$ be a family of local action groupoids satisfying the weak gluing condition of Subsection 6.3.3. Then the glued groupoid $\mathcal{G} = \bigcup_{i\in I} \mathcal{G}_i$ is also a local action groupoid.*

*Proof.* This is a direct consequence of Lemma 6.3.9. □

Since "the Fredholm property is local", it is only natural that Theorem 6.3.12 generalizes to local action groupoids.

**Corollary 6.3.16.** *Let $\mathcal{G} \rightrightarrows X$ be a local action groupoid. Assume that*

*(i) there is an open, dense and $\mathcal{G}$-invariant subset $V \subset X$ such that $\mathcal{G}_V \simeq V \times V$;*
*(ii) for each $p \in X$, there is a local isomorphism $\mathcal{G} \sim_p X_p \rtimes G_p$, with $X_p$, $G_p$ locally compact, second-countable and Hausdorff, and $G_p$* amenable.

*Then $\mathcal{G}$ is a Fredholm groupoid.*

In other words, if $\mathcal{G}$ is locally given by the action of an *amenable* group, then $\mathcal{G}$ is Fredholm. Section 6.4 will provide many examples of such groupoids.

*Proof.* Set $p \in X$, and consider the local isomorphism $\phi_p : \mathcal{G}|_{U_p} \simeq (X_p \rtimes G_p)|_{U'_p}$, where $p \in U_p$. Let $V_p = V \cap U_p$ and $V'_p = \phi_p(V_p)$. Because $\mathcal{G}_V \simeq V \times V$, we have $\mathcal{G}|_{V_p} \simeq V_p \times V_p$, hence

$$(X_p \rtimes G_p)|_{V'_p} \simeq V'_p \times V'_p.$$

Now, because $G_i$ is amenable, the groupoid $X_p \rtimes G_p$ is Hausdorff and amenable. It follows that its reduction $(X_p \rtimes G_p)|_{U'_p}$ is also Hausdorff and amenable. Theorem 6.3.3 therefore implies that each groupoid $(X_p \rtimes G_p)|_{U_p}$ is Fredholm. We conclude using Theorem 6.3.10 that $\mathcal{G}$ is a Fredholm groupoid. □

## 6.4. Examples and applications

We conclude this chapter with many examples of Fredholm and local action groupoids. All our examples are motivated by the study of differential equations on singular spaces, so we begin in Subsection 6.4.1 by recalling the interest of Fredholm groupoids when dealing with analysis.

### 6.4.1. A source of examples: Lie manifolds

An important source of Fredholm groupoids is the following class of manifolds, introduced by Ammann, Lauter and Nistor in [6].

**Definition 6.4.1.** A *Lie manifold* is a pair $(M, \mathcal{V})$, where $M$ is a compact manifold with corners and $\mathcal{V}$ a Lie subalgebra of $\Gamma(TM)$ such that

(i) $\mathcal{V}$ is a Lie subalgebra of $\mathcal{V}_b$, with $\mathcal{V}_b$ the algebra of vector fields tangent to all faces of $M$,
(ii) $\mathcal{V}$ contains the compactly supported vector fields on $M_0$,
(iii) $\mathcal{V}$ is a finitely generated and projective $C^\infty(M)$-module.

Let $(M, \mathcal{V})$ be a Lie manifold as above and denote by $M_0$ the interior of $M$. We are interested in the algebra $\mathrm{Diff}(\mathcal{V})$ of differential operators generated by $\mathcal{V}$, seen as a subalgebra of $\mathrm{Diff}(M_0)$.

We know from Serre-Swan's theorem that there is a unique Lie algebroid $A_{\mathcal{V}} \to M$ whose anchor map induces an isomorphism $\Gamma(A_{\mathcal{V}}) \simeq \mathcal{V}$ (definitions of Lie algebroids and Lie groupoids are recalled in Section 4.3.1). Such algebroids are known to always be integrable, i.e. there is a Lie groupoid $\mathcal{G} \rightrightarrows M$ such that $A\mathcal{G} \simeq A_{\mathcal{V}}$ [69]. It follows that the algebra of differential operators generated by $\mathcal{G}$ identifies with $\mathrm{Diff}(\mathcal{V})$.

As emphasized in Chapter 5, the interest in characterizing Fredholm groupoids stems from the following result.

**Theorem 6.4.2** (Carvalho, Nistor, Qiao [51])**.** *Let $\mathcal{G} \rightrightarrows X$ be a Fredholm Lie groupoid with compact unit space $X$, and set $V \subset X$ its unique dense, open $\mathcal{G}$-orbit. Let $P$ be an order-$m$ differential operator on $\mathcal{G}$. Then for any $s \in \mathbb{R}$, the operator*

$$P : H^s(V) \to H^{s-m}(V)$$

*is Fredholm if, and only if,*

*(i) $P$ is elliptic and*

*(ii) for any $x \in X \setminus V$, the operator $P_x : H^s(\mathcal{G}_x) \to H^{s-m}(\mathcal{G}_x)$ is invertible.*

The operators $P_x$, for $x \in X \setminus V$, are called *limit operators* for $P$: they are invariant under the action of $\mathcal{G}$ on $\mathcal{G}_x$, and are of the same type as $P$ (e.g. if $P$ is the Laplacian on $V$, then $P_x$ is the Laplacian on $\mathcal{G}_x$). Note that Theorem 6.4.2 remains true if we consider pseudodifferential operators on $\mathcal{G}$ or operators acting between sections of vector bundles. Many similar results were known in particular cases, see [71, 184, 68, 108, 110, 85, 127, 124] and the reference therein.

**Remark 6.4.3.** There are several possible choices for the groupoid $\mathcal{G}_{\mathcal{V}}$, and not all of them are equally suited to obtain results in analysis. Two extremal cases should be distinguished:

- The "maximal" integration $\mathcal{G}_{max} \rightrightarrows M$ of Crainic and Fernandes [66] is the unique $d$-simply-connected groupoid integrating $A_{\mathcal{V}}$. The groupoid $\mathcal{G}_{max}$ has the property that, for any other integration $\mathcal{H} \rightrightarrows M$ of $A_{\mathcal{V}}$, there is a unique groupoid morphism $\mathcal{G}_{max} \to \mathcal{H}$.
- The "minimal" integration $\mathcal{G}_{min} \rightrightarrows M$ of Debord [69], or holonomy groupoid, is the unique $d$-connected *quasi-graphoid* integrating $A_{\mathcal{V}}$ (see Definition 6.4.4 below). For any other $d$-connected integration $\mathcal{H} \rightrightarrows M$ of $A_{\mathcal{V}}$, there is a unique groupoid morphism $\mathcal{H} \to \mathcal{G}_{min}$, and this morphism is onto.

**Definition 6.4.4** (Bigonnet, Pradines [31])**.** We say that a groupoid $\mathcal{G} \rightrightarrows M$ is a *quasi-graphoid* if, for any open subset $U \subset M$, the only continuous map $\nu : U \to \mathcal{G}$ that is a section to both $d$ and $r$ is the restriction of the unit map $u : M \to \mathcal{G}$ to $U$.

We will see in Examples 6.4.9 to 6.4.13 below that in our setting, the minimal groupoid integrating $\mathcal{V}$ can often be constructed in an elementary way by gluing reductions of action groupoids. The identification of our constructions with the groupoid of Debord is facilitated by the following lemma.

**Lemma 6.4.5.** *Let $\mathcal{G} \rightrightarrows M$ be a Hausdorff Lie groupoid. Assume that there is a dense open subset $V \subset M$ such that $\mathcal{G}_V \simeq V \times V$. Then $\mathcal{G}$ is a quasi-graphoid.*

*Proof.* Let $U \subset M$ be open and $\nu : U \to \mathcal{G}$ be a continuous section for both $d$ and $r$. Because $\mathcal{G}|_{U\cap V} \simeq (U \cap V)^2$, we have $\nu|_{U\cap V} = u|_{U\cap V}$. If $x \in U \setminus V$, then by continuity

$$\nu(x) = \lim_{\substack{y\to x \\ y\in U\cap V}} u(y) = u(x).$$

The fact that $\mathcal{G}$ is Hausdorff ensures that the above limit is unique. □

**Remark 6.4.6.** The maximal integration $\mathcal{G}_{max}$ is often too big to be a Fredholm groupoid, as is illustrated by Example 6.4.11. The minimal groupoid $\mathcal{G}_{min}$ is Fredholm in most practical cases, but there may be other groupoids integrating $\mathcal{V}$ that are Fredholm (typically they are not $d$-connected, see Remark 6.4.10).

### 6.4.2. Examples of Fredholm groupoids

**Example 6.4.7** (The pair groupoid)**.** Let $M$ be a closed manifold, i.e. a compact smooth manifold without boundary. Assume that $M$ is connected. Then the minimal groupoid integrating $TM$ is the pair groupoid $\mathcal{G} = M \times M$. It is clear that $\mathcal{G}$ is Fredholm: indeed, the vector representation

$$\pi_0 : C^*_r(M \times M) \to \mathcal{B}(L^2(M))$$

is an isomorphism onto the ideal $\mathcal{K}$ of compact operators on $L^2(M)$. Thus the operators $1 + \pi_0(a)$, for $a \in C^*_r(\mathcal{G})$, are all Fredholm. Assumption (iii) of Definition 6.3.1 is trivially satisfied in that case, because the boundary set $M \setminus M$ is empty. If $M$ is connected, then the pair groupoid $M \times M$ is the minimal groupoid (in the sense of Remark 6.4.3) integrating $TM$.

The algebra of differential operators on $\mathcal{G}$ identifies with $\mathrm{Diff}(M)$. Theorem 6.4.2 then recovers the classical Fredholmness result: a differential operator on $M$ is Fredholm if, and only if, it is elliptic. The groupoid $\mathcal{G}$ is a local action groupoid. Indeed, any point $p \in M$ has a neighborhood $U \subset M$ diffeomorphic to an open subset $U' \subset \mathbb{R}^n$. Then

$$\mathcal{G}|_U \simeq U' \times U' \simeq (\mathbb{R}^n \rtimes_\alpha \mathbb{R}^n)|_{U'},$$

where $\alpha$ is the action of $\mathbb{R}^n$ on itself by translation.

**Example 6.4.8** (A non-amenable example)**.** To highlight the fact that the Fredholm property for groupoids is distinct from any amenability property, let us give a simple example of a Fredholm groupoid that is *not* metrically amenable (see Section 4.3.2 for the definition of metrical amenability). Let $G$ be a locally compact, exact[1] and non-amenable group, and denote by $X := G \sqcup \{\infty\}$ its one-point compactification. The action of $G$ on itself by right translation extends to a continuous action on $X$ with two orbits: $G$ and $\{\infty\}$. We claim that the action groupoid $\mathcal{G} := X \rtimes G$ is Fredholm. Indeed, we may check that all the conditions of Theorem 5.2.2 are satisfied: first, the groupoid $\mathcal{G}$ is Hausdorff, and the $C^*$-algebra $C^*_r(\mathcal{G}_{\{\infty\}}) \simeq C^*_r(G)$ obviously satisfies Exel's property. Moreover, the exactness of $G$ guarantees that the sequence

$$0 \longrightarrow C(G) \rtimes_r G \longrightarrow C(X) \rtimes_r G \longrightarrow C^*_r(G) \longrightarrow 0,$$

induced by the open dense orbit $G \subset X$, is exact. Therefore $\mathcal{G}$ is a Fredholm groupoid. If we choose a group $G$ that is exact but not amenable, for instance $G = SL_3(\mathbb{R})$, then $\mathcal{G}$ is not metrically amenable.

**Example 6.4.9** (Cylindrical ends)**.** Let $M$ be a compact manifold with boundary, and let $\mathcal{V}_b$ be the Lie algebra of vector fields on $M$ that are tangent to $\partial M$. Assume that both $M$ and $\partial M$ are connected. Then the minimal groupoid integrating $\mathcal{V}_b$ in the sense of Remark 6.4.3 is

$$\mathcal{G}_b = M_0 \times M_0 \bigsqcup \partial M \times \partial M \times \mathbb{R},$$

where $M_0$ is the interior of $M$. Given a tubular neighborhood $U \simeq [0,1) \times \partial M$ of $\partial M$, the smooth structure of $\mathcal{G}_b$ is given by the isomorphism

$$\mathcal{G}_b|_U \simeq (\partial M \times \partial M) \times ([0,\infty) \rtimes (0,\infty))|_{[0,1)},$$

where $(0,\infty)$ acts on $[0,\infty)$ by multiplication (more details are given in Section 4.3.4).

We call $\mathcal{G}_b$ the *b-groupoid* of $M$. Any metric choice of metric on $A\mathcal{G}_b$ induces a metric $g_0$ on $M_0$ that is bi-Lipschitz equivalent to the product metric

$$\frac{dx^2}{x^2} + h_{\partial M}$$

on $U \simeq [0,1[\times \partial M$, with $h_{\partial M}$ a metric on $\partial M$. Thus $\mathcal{G}_b$ models *manifolds with cylindrical ends*. The algebra $\mathrm{Diff}_{\mathcal{G}_b}(M_0)$ is that of every differential operator $P$ on $M_0$ which can be written as

$$P = \sum_{|\alpha| \leq m} a_\alpha (x\partial_x)^{\alpha_1} \partial_{y_2}^{\alpha_2} \ldots \partial_{y_n}^{\alpha_n},$$

locally near any point of $\partial M$, with $a_\alpha \in C^\infty(M)$ and $(\partial_{y_i})_{i=2}^n$ a local basis of $\Gamma(T\partial M)$. It contains in particular any geometric operator associated to the metric $g_0$. The algebra $\mathrm{Diff}_b(M_0)$ has been extensively studied, and is closely related to the $b$-calculus of Melrose and the Atiyah-Patodi-Singer index theorem of manifolds with boundaries [136, 13].

[1] A group $G$ is *exact* if, for any dynamical system $(A, G, \alpha)$ and any ideal $I \subset A$ stable by $\alpha$, the sequence

$$0 \longrightarrow I \rtimes_r G \longrightarrow A \rtimes_r G \longrightarrow (A/I) \rtimes_r G \longrightarrow 0$$

is exact. "Many" groups are exact, for instance all locally compact almost-connected groups [106, 207].

The groupoid $\mathcal{G}_b$ is obtained gluing together several local action groupoids: it follows from Propositions 6.3.14 and 6.3.15 that $\mathcal{G}_b$ is also a local action groupoid. The local structure is very simple: for any $p \in M$, we have a local isomorphism

$$\mathcal{G} \sim_p ([0,\infty) \times \mathbb{R}^{n-1}) \rtimes (\mathbb{R} \times \mathbb{R}^{n-1})$$

The action is given by the product action of $\mathbb{R}^{n-1}$ on itself (by translation) and $\mathbb{R}$ on $[0,\infty)$ (given by $(x,t) \mapsto xe^t$). Since the acting groups is amenable, we conclude from Theorem 6.3.16 that $\mathcal{G}_b$ is a Fredholm groupoid. This is by no mean a new result [51], but should serve as a case in point to emphasizes the relevance of local action groupoids in practical cases.

**Remark 6.4.10.** The holonomy groupoid may not always be the best choice for applications to analysis: for instance, the layer potentials groupoid introduced by Carvalho and Qiao [52] and considered in Chapter 5 also integrates $\mathcal{V}_b$, but is not $d$-connected in general.

**Example 6.4.11** (A non-example on the disk)**.** Let $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ be the unit disk and denote by $D_0$ its interior. Consider again the Lie algebra $\mathcal{V}_b$ of vector fields on $D$ that are tangent to the boundary $\partial D \simeq S^1$. As in Example 6.4.9, the minimal groupoid integrating $\mathcal{V}_b$ is the groupoid

$$\mathcal{G}_b = D_0 \times D_0 \bigsqcup S^1 \times S^1 \times \mathbb{R}.$$

The maximal integration of $\mathcal{V}_b$, on the other hand, is given by

$$\mathcal{G}_{max} = D_0 \times D_0 \bigsqcup \mathcal{P}(S^1) \times \mathbb{R}.$$

Here $\mathcal{P}(S^1) \simeq S^1 \times \mathbb{R}$ is the path groupoid of $S^1$, whose elements are the homotopy classes of paths in $S^1$. The topology on $\mathcal{G}_{max}$ is the initial topology with respect to the quotient map $q : \mathcal{G}_{max} \to \mathcal{G}_b$. In particular $\mathcal{G}_{max}$ is not Hausdorff; for instance the points $(0,0,0)$ and $(0,2\pi,0)$ in $S^1 \times \mathbb{R} \times \mathbb{R}$ cannot be separated by open sets in $\mathcal{G}_{max}$.

The groupoid $\mathcal{G}_{max}$ is not a Fredholm groupoid, because the representation $\pi_0$ is not injective. To see this, let $g, h \in \mathcal{G}_{max}$ be such that $g \neq h$ and $q(g) = q(h)$. Then, because $q$ is a covering map, we can choose two Hausdorff open sets $U$ and $V$ in $\mathcal{G}_{max}$ such that $g \in U$, $h \in V$ and $q(U) = q(V) = W$. Let now $f \in C_c(W)$ be such that $f(q(g)) \neq 0$. Define $f_U \in C_c(U)$ and $f_V \in C_c(V)$ by $f_U = f \circ q|_U$ and $f_V = f \circ q|_V$. Though $f_U$ and $f_V$ do not extend continuously on $\mathcal{G}_{max}$, they both define elements of $C^*(\mathcal{G}_{max})$ (see [57] on that point).

Now $q$ is a homeomorphism over $D_0 \times D_0$, so $f_U$ and $f_V$ coincide over $D_0 \times D_0$. Therefore $\pi_0(f_U) = \pi_0(f_V)$. Since $h \notin U$ (because $U$ is Hausdorff), we have $f_U(h) = 0$ whereas $f_V(h) \neq 0$. Hence $f_U \neq f_V$, which shows that $\pi_0$ is not injective.

**Example 6.4.12** (Scattering manifolds)**.** Let $M$ be a connected, compact manifold with boundary and interior $M_0$. Let $x$ be a defining function for $\partial M$, and consider the Lie algebra of vector fields $\mathcal{V}_{sc} = x\mathcal{V}_b$.

It was shown in Chapter 5 that the minimal groupoid $\mathcal{G}_{sc} \rightrightarrows M$ integrating $\mathcal{V}_{sc}$ can be constructed by gluing reductions of the action groupoid $\mathbb{S}^n_+ \rtimes \mathbb{R}^n$. Here $\mathbb{S}^n_+$ is the stereographic compactification of $\mathbb{R}^n$ into a half-sphere, and the action of $\mathbb{R}^n$ on $\mathbb{S}^n_+$ is the only smooth one that extends the action of $\mathbb{R}^n$ on itself by translation. Thus $\mathcal{G}_{sc}$ is a local action groupoid that is locally isomorphic to $\mathbb{S}^n_+ \rtimes \mathbb{R}^n$ around any point. It follows that $\mathcal{G}_{sc}$ is Fredholm by Theorem 6.3.16.

The groupoid $\mathcal{G}_{sc}$ and closely related ones were studied in [205, 142] for instance, in relation with the study of the spectrum of the $N$-body problem on Euclidean space. The compatible metrics are called *scattering metrics.* In a tubular neighborhood $U \simeq [0,1) \times \partial M$ of $\partial M$ in $M$, such a metric can be written

$$g_0(x,p) = \frac{dx^2}{x^4} + \frac{h_{\partial M}}{x^2},$$

for any $(x,p) \in (0,1) \times \partial M$, and with $h_{\partial M}$ a metric on $\partial M$. A typical example is given by the euclidean metric on $\mathbb{R}^n$, seen as the interior of $\mathbb{S}^n_+$ [137, 205]. For this reason, manifolds with scattering metrics are also called *asymptotically euclidean.*

The algebra of scattering differential operators, written $\mathrm{Diff}_{sc}(M_0)$, is the one containing all differential operators $P$ on $M_0$ that can be written

$$P = \sum_{|\alpha| \leq m} a_\alpha (x^2 \partial_x)^{\alpha_1} (x\partial_{y_2})^{\alpha_2} \dots (x\partial_{y_n})^{\alpha_n},$$

locally near any point of $\partial M$, with $a_\alpha \in C^\infty(M)$ and $(\partial_{y_i})_{i=2}^n$ a local basis of $\Gamma(T\partial M)$. It contains in particular the Laplacian associated to $g_0$.

**Example 6.4.13** (Asymptotically hyperbolic manifolds)**.** As before, let $M$ be a connected, compact manifold with boundary and $M_0$ its interior. Consider the Lie algebra $\mathcal{V}_0 \subset \mathcal{V}_b$ of vector fields vanishing on $\partial M$. As in Example 6.4.12, the minimal groupoid $\mathcal{G}_0 \rightrightarrows M$ integrating $\mathcal{V}_0$ can be constructed by gluing reductions of an action groupoid

$$X_n \rtimes G_n := ([0,\infty) \times \mathbb{R}^{n-1}) \rtimes ((0,\infty) \ltimes \mathbb{R}^{n-1}).$$

Here $(0,\infty)$ acts on $\mathbb{R}^{n-1}$ by dilation, and the action of $G_n := (0,\infty) \ltimes \mathbb{R}^{n-1}$ on itself by right multiplication extends smoothly to the boundary by the formula

$$(x_1, \dots, x_n) \cdot (t, \xi_2, \dots, \xi_n) = (tx_1, x_2 + x_1\xi_2, \dots, x_n + x_1\xi_n),$$

for all $(x_1, \dots, x_n) \in \mathbb{R}^n$ and $(t, \xi_2, \dots, \xi_n) \in G_n$. Therefore $\mathcal{G}_0$ is a local action groupoid that is locally isomorphic to $X_n \rtimes G_n$ around each point of $M$. Because $G_n$ is amenable, Theorem 6.3.16 again implies that $\mathcal{G}_0$ is a Fredholm groupoid.

The metrics on $M_0$ induced from metrics on $A\mathcal{G}_0$ are bi-Lipschitz equivalent to the *asymptotically hyperbolic metric*

$$g_0(x,p) = \frac{dx^2 + h_{\partial M}}{x^2}$$

where $(x, p)$ is in a tubular neighborhood $[0, 1[\times \partial M$, for $x > 0$ (here $h_{\partial M}$ is a metric on $\partial M$, as before). A typical example is the hyperbolic space $\mathbb{H}^n$ with its usual metric, compactified into the Poincaré ball. The operators generated by $\mathcal{V}_0$ are those that can be written

$$P = \sum_{|\alpha| \leq m} a_\alpha (x\partial_x)^{\alpha_1} (x\partial_{y_2})^{\alpha_2} \dots (x\partial_{y_n})^{\alpha_n},$$

locally near any point of $\partial M$, with $a_\alpha \in C^\infty(M)$ and $(\partial_{y_i})_{i=2}^n$ a local basis of $\Gamma(T\partial M)$. These operators and related pseudodifferential calculi were studied in [132, 184, 111, 71] for instance.

**Example 6.4.14** (Cusp metrics)**.** Let us give an example of Fredholm groupoid that does not come from the integration of a Lie algebroid. Example 6.4.9 can be generalized by replacing the action of $(0, \infty)$ on $[0, \infty)$ by a more general one. For example, let $\varphi$ be a non-negative function in $C[0, \infty)$, vanishing only at 0 and such that

(i) $\varphi \in C^\infty(0, \infty)$, and
(ii) $\varphi'$ is bounded on $(0, \infty)$.

Let $\alpha : \mathbb{R} \times [0, \infty) \to [0, \infty)$ be the flow associated to the continuous vector field $x \mapsto \varphi(x)\partial_x$ on $[0, \infty)$. The function $\varphi$ is globally Lipschitz, so this flow is well-defined for any time $t \in \mathbb{R}$. A typical example is any function $\varphi \in C[0, \infty) \cap C^\infty(0, \infty)$ satisfying

$$\begin{cases} \varphi(x) = x^r & \text{if } x \in [0, 1], \text{ and} \\ \varphi(x) = 1 & \text{if } x \geq 2, \end{cases}$$

for any $r \in [1; +\infty)$. If $r = 1$, then for small $x$, $t$ we have $\alpha(t, x) = e^t x$; this recovers the action by dilation of Example 6.4.9, considering the group isomorphism $\mathbb{R} \to (0, \infty)$ given by the exponential map.

We thus consider the action of $\mathbb{R}$ on $[0, \infty)$ given by $\alpha$. This action has an orbit $(0, \infty)$ on which the action is free and transitive, so we may construct a groupoid $\mathcal{G}_\varphi \rightrightarrows M$ by following the same gluing procedure as in Example 6.4.9. The groupoid $\mathcal{G}_\varphi$ is a local action groupoid that is locally isomorphic to $([0, \infty) \times \mathbb{R}^{n-1}) \rtimes_\alpha (\mathbb{R} \times \mathbb{R}^{n-1})$, hence it is Fredholm by Theorem 6.3.16.

The compatible metrics are bi-Lipschitz equivalent to the complete metric

$$g_0(x, p) = \frac{dx^2}{(\varphi(x))^2} + h_{\partial M}$$

when $(x, p)$ is in a tubular neighborhood $[0, 1[\times \partial M$ of $\partial M$, with $x > 0$. This models *manifolds with cusps*, see e.g. [170, 110, 68]. The differential operators $P \in \mathrm{Diff}_{\mathcal{G}_\varphi}(M_0)$ can be written

$$P = \sum_{|\alpha| \leq m} a_\alpha (\varphi(x)\partial_x)^{\alpha_1} \partial_{y_2}^{\alpha_2} \dots \partial_{y_n}^{\alpha_n},$$

locally near any point of $\partial M$, with $a_\alpha \in C^\infty(M)$ and $(\partial_{y_i})_{i=2}^n$ a local basis of $\Gamma(T\partial M)$. The function $\varphi$ may not be smooth at $x = 0$, in which case $\mathcal{G}_\varphi$ would only be a continuous family groupoid [158, 112].

# Part IV.

# Fredholm conditions for invariant operators restricted to isotypical components

**Abstract**

We present here the second part of the results of this thesis, based on the two papers [25] and [26]. Both are joint works with Alexandre Baldare, Matthias Lesch and Victor Nistor. Other than homogenization of notations, the mathematical content of these articles has not been changed. The introduction and background material have been adapted, in order to avoid any repetition.

# 7. First part of the proof and abelian case

This chapter is adapted from the paper "Fredholm conditions for invariant operators: finite abelian groups and boundary value problems" [26], to appear in *Journal of Operator Theory*, and which is a joint work with Alexandre Baldare[1], Matthias Lesch[2] and Victor Nistor[3].

## 7.1. Introduction

As for the previous chapters, we refer to Section 2.2.3 for a detailed introduction and only recall the main objective here.

### 7.1.1. Short introduction and notations

Throughout Chapters 7 and 8, we consider a smooth, closed manifold $M$, together with a *finite* group $\Gamma$ that acts by diffeomorphisms on $M$. We assume given two $\Gamma$-vector bundles $E, F \to M$. Let $P \in \Psi^m(M; E)^\Gamma$, i.e. $P$ is a $\Gamma$-invariant classical pseudodifferential operator acting from the space of sections $C^\infty(M; E)$ to $C^\infty(M; F)$. Given an irreducible representation $\alpha \in \widehat{\Gamma}$, the operator $P$ restricts between the corresponding isotypical components of the Sobolev spaces. We denote this restriction by

$$\pi_\alpha(P) : H^s(M; E)_\alpha \to H^{s-m}(M; F)_\alpha, \tag{7.1}$$

for any $s \in \mathbb{R}$. We answer in these two chapters the following question: under which condition is $\pi_\alpha(P)$ Fredholm?

The present chapter deals with the case when $\Gamma$ is *abelian*, while giving important intermediate results that will be useful for the general case discussed in Chapter 8. To state our main theorem, we introduce the space

$$X_{M,\Gamma} := \{(\xi, \rho) \mid \xi \in T^*M \setminus \{0\} \text{ and } \rho \in \widehat{\Gamma}_\xi\}. \tag{7.2}$$

If $\xi \in T^*M$ has for base point $x \in M$, then the stabilizer $\Gamma_\xi$ acts on $E_x$. The principal symbol $\sigma_m(P)$ is $\Gamma$-equivariant, so we may consider its restriction

$$\sigma_m^\Gamma(P)(\xi, \rho) := \pi_\rho(\sigma_m(P)) \in \mathrm{Hom}(E_{x\rho}, F_{x\rho})^{\Gamma_\xi}, \tag{7.3}$$

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for any $\rho \in \widehat{\Gamma}_\xi$, and where $\pi_\rho$ is the restriction as a map between the isotypical components of $E_x$ and $F_x$ associated with $\rho$, as in Equation (7.1).

The characterization of Fredholm pseudodifferential operators can be reduced to each connected component of the orbit space, so we assume for the rest of this introduction that $M/\Gamma$ is connected. There exists then a *minimal isotropy group* $\Gamma_0$ (see Subsection 7.2.3), which is unique if $\Gamma_0$ is abelian. Given a character $\alpha \in \widehat{\Gamma}$, we define the subset

$$X^{\alpha}_{M,\Gamma} := \{(\xi, \rho) \in X_{M,\Gamma} \mid \alpha_{|\Gamma_0} = \rho_{|\Gamma_0}\}. \tag{7.4}$$

We can now define the *$\alpha$-principal symbol* $\sigma^{\alpha}_m(P)$ of $P$.

**Definition 7.1.1.** The *$\alpha$-principal symbol* $\sigma^{\alpha}_m(P)$ of $P$ is the restriction of $\sigma^{\Gamma}_m(P)$ to the subset $X^{\alpha}_{M,\Gamma}$, that is:

$$\sigma^{\alpha}_m(P) := \sigma^{\Gamma}_m(P)|_{X^{\alpha}_{M,\Gamma}}.$$

We shall say that $P \in \Psi^m(M; E)^{\Gamma}$ is *$\alpha$-elliptic* if its $\alpha$-principal symbol $\sigma^{\alpha}_m(P)$ is invertible everywhere on its domain of definition.

To summarize, the $\alpha$-principal symbol is given as a restriction of the principal symbol, to isotypical components that are related with $\alpha$. Note that if $(\xi, \rho) \in X^{\alpha}_{M,\Gamma}$ is such that $E_{x\rho} = F_{x\rho} = 0$, then $\sigma^{\Gamma}_m(P)(\xi, \rho)$ is the null operator between two trivial spaces, hence is invertible. Our main result is the following:

**Theorem 7.1.2.** *Let $\Gamma$ be a finite* abelian *group acting on a smooth, closed manifold $M$ and let $P \in \Psi^m(M; E, F)^{\Gamma}$ be a $\Gamma$-invariant classical pseudodifferential operator acting between sections of two $\Gamma$-equivariant bundles $E, F \to M$, $m \in \mathbb{R}$, and $\alpha \in \widehat{\Gamma}$. We have that*

$$\pi_\alpha(P) : H^s(M; E)_\alpha \to H^{s-m}(M; F)_\alpha$$

*is Fredholm if, and only if, $P$ is $\alpha$-elliptic.*

If $\Gamma$ acts without fixed points on a *dense* open subset of $M$, then $\Gamma_0 = 1$, and hence $X_{M,\Gamma} = X^{\alpha}_{M,\Gamma}$ for all $\alpha \in \widehat{\Gamma}$. Hence, in this case, $P$ is $\alpha$-elliptic if, and only if, it is elliptic. The ellipticity of $P$ can thus be checked in this case simply by looking at a single isotypical component. We stress, however, that if $\Gamma$ is not discrete, then this statement and our main result (Theorem 7.1.2 above) are not true anymore. Theorem 7.1.2 holds for a general finite group with an appropriate reformulation, although the proof is much more involved: see Chapter 8.

### 7.1.2. Contents of the chapter

Let us quickly describe here the contents of the chapter. We start in Section 7.2 with some preliminaries. We thus recall some facts about groups, most notably the Frobenius reciprocity (for finite groups) and the definitions of induced representations, of minimal isotropy groups and of the principal orbit bundle. We also review some notions concerning the primitive ideal spectrum of $C^*$-algebras, as well as basic facts concerning (equivariant) pseudodifferential operators.

In Section 7.3, we compute the image of the algebra $\overline{\Psi^{-1}}(M; E)$ of regularizing operators via $\pi_\alpha$. We do this by proving some general results on the structure of $C^*$-algebras with an inner action of our group $\Gamma$. When the action of the group $\Gamma$ is inner, the results and their proofs become simpler.

The main difficulties arise in Section 7.4. There, we set $A_M := C_0(M; \mathrm{End}(E))$ and identify the primitive spectrum of the $C^*$-algebra $A_M^\Gamma$ of $\Gamma$-invariant symbols with the set $X_{M,\Gamma}/\Gamma$, with $X_{M,\Gamma}$ described in (7.2) above. We then consider the projection from $A_M^\Gamma$ to the Calkin algebra of $L^2(M; E)_\alpha$ and show that the closed subset of $\mathrm{Prim}\, A_M^\Gamma$ associated to its kernel is $X_{M,\Gamma}^\alpha/\Gamma$. These results are used in Section 7.5 to prove the main result of the chapter, Theorem 7.1.2. We also discuss an application to mixed boundary value problems and explain why our result is not true when the group $\Gamma$ is not discrete.

## 7.2. Preliminaries

We begin by setting up the terminology and the notation used in this chapter and the next. Throughout the chapter, $\Gamma$ will be a compact group acting on a locally compact space $M$. For the most part, $M$ will be a smooth Riemannian manifold and $\Gamma$ will be a *compact Lie group* acting smoothly and isometrically on $M$. The final result holds only for discrete (thus finite) groups and $M$ compact, but many intermediate results hold in greater generality, so we have tried to state the results in the greatest generality possible when this did not involve too much extra work.

### 7.2.1. Group representations

We follow the standard terminology and conventions. See, for instance, [196, 188], where one can find further details.

If $x \in M$, then $\Gamma x$ is the $\Gamma$ orbit of $x$ and

$$\Gamma_x := \{\gamma \in \Gamma \mid \gamma x = x\} \subset \Gamma \tag{7.5}$$

the isotropy group of the action at $x$. We shall write $H \sim H'$ if the subgroups $H$ and $H'$ are conjugated in $\Gamma$. If $H \subset \Gamma$ is a subgroup, then $M_{(H)}$ will denote the set of elements of $M$ whose isotropy $\Gamma_x$ is conjugated to $H$ in $\Gamma$, that is, the set of elements $x \in M$ such that $\Gamma_x \sim H$.

Assuming that $\Gamma$ acts on a space $X$, we denote by $\Gamma \times_H X$ the space

$$\Gamma \times_H X := (\Gamma \times X)/\sim, \tag{7.6}$$

where $(\gamma h, x) \sim (\gamma, hx)$, $\forall \gamma \in \Gamma, h \in H$ and $x \in X$.

Let $V$ be a normed complex vector space and $\mathcal{L}(V)$ be the set of bounded operators on $V$. A representation of $\Gamma$ on $V$ is a group morphism $\Gamma \to \mathcal{L}(V)$; in that case we also call $V$ a $\Gamma$-module. For any two $\Gamma$-modules $\mathcal{H}$ and $\mathcal{H}_1$, we shall denote by

$$\mathrm{Hom}_\Gamma(\mathcal{H}, \mathcal{H}_1) = \mathrm{Hom}(\mathcal{H}, \mathcal{H}_1)^\Gamma = \mathcal{L}(\mathcal{H}, \mathcal{H}_1)^\Gamma$$

the set of continuous linear maps $T : \mathcal{H} \to \mathcal{H}_1$ that commute with the action of $\Gamma$, that is, $T(\gamma\xi) = \gamma T(\xi)$ for all $\xi \in \mathcal{H}$ and $\gamma \in \Gamma$.

Let $\mathcal{H}$ be a $\Gamma$-module and $\alpha$ an irreducible $\Gamma$-module. Then $p_\alpha$ will denote the $\Gamma$-invariant projection onto the $\alpha$-isotypical component $\mathcal{H}_\alpha$ of $\mathcal{H}$, defined as the largest (closed) $\Gamma$ submodule of $\mathcal{H}$ that is isomorphic to a multiple of $\alpha$. In other words, $\mathcal{H}_\alpha$ is the sum of all $\Gamma$-submodules of $\mathcal{H}$ that are isomorphic to $\alpha$. Notice that $\mathcal{H}_\alpha \simeq \alpha \otimes \mathrm{Hom}_\Gamma(\alpha, \mathcal{H})$.

Since $\Gamma$ is compact, we have

$$\mathcal{H}_\alpha \neq 0 \ \Leftrightarrow \ \mathrm{Hom}_\Gamma(\alpha, \mathcal{H}) \neq 0 \ \Leftrightarrow \ \mathrm{Hom}_\Gamma(\mathcal{H}, \alpha) \neq 0. \tag{7.7}$$

If $T \in \mathcal{L}(\mathcal{H})^\Gamma$ (i.e. $T$ is $\Gamma$-equivariant), then $T(\mathcal{H}_\alpha) \subset \mathcal{H}_\alpha$ and we let

$$\pi_\alpha : \mathcal{L}(\mathcal{H})^\Gamma \to \mathcal{L}(\mathcal{H}_\alpha)\,, \quad \pi_\alpha(T) := T|_{\mathcal{H}_\alpha}\,, \tag{7.8}$$

be the associated morphism, as in Equation (7.1) of the Introduction. The morphism $\pi_\alpha$ will play an essential role in what follows.

### 7.2.2. Induction and Frobenius reciprocity

We now review some basic definitions and results for induced representations. We will use induction *for finite groups only,* so we assume in this discussion of the Frobenius reciprocity (i.e. in this subsection) that $\Gamma$ is finite.

#### Definition of the induced module

Since we are assuming in this subsection that $\Gamma$ is finite, we have that $C^*(\Gamma) = C(\Gamma) = \mathbb{C}[\Gamma]$, the group algebra of $\Gamma$. We will use the standard notation $V^{(I)} := \{f : I \to V\}$, valid for $I$ finite. If $H \subset \Gamma$ is a subgroup (hence also finite) and $V$ is a $H$-module, we let

$$\mathrm{Ind}_H^\Gamma(V) := \mathbb{C}[\Gamma] \otimes_{\mathbb{C}[H]} V \ \simeq \ \{\, \xi : \Gamma \to V \mid f(gh^{-1}) = hf(g) \,\} \ \simeq \ V^{(\Gamma/H)} \tag{7.9}$$

be the *induced representation* from $V$. The last isomorphism is obtained by choosing a set of representatives of the right cosets $\Gamma/H$. The action of $\Gamma$ on $\mathrm{Ind}_H^\Gamma(V)$ is by left multiplication on $\mathbb{C}[\Gamma]$, and the indicated isomorphism is an isomorphism of $\Gamma$-modules. The $\Gamma$-module $\mathrm{Ind}_H^\Gamma(V)$ depends functorially on $V$.

**Remark 7.2.1.** If $V$ is an algebra and the group $H$ acts on $V$ by algebra homomorphisms, then the isomorphism $\mathrm{Ind}_H^\Gamma(V) \simeq \{\xi : \Gamma \to V \mid f(gh^{-1}) = hf(g)\}$ shows that $\mathrm{Ind}_H^\Gamma(V)$ is an algebra for the pointwise product. If $V_1$ is a left $V$-module (with a structure compatible with the action of $\Gamma$), then $\mathrm{Ind}_H^\Gamma(V_1)$ is a $\mathrm{Ind}_H^\Gamma(V)$ module, again with the pointwise multiplication. The induction is moreover compatible with morphisms of modules and algebras (change of scalars), again by the function picture of the induced representation. In particular, if $\phi : V \to W$ is a $H$-morphism of algebras, if $V_1$ and $W_1$ are modules over these algebras, and $\psi : V_1 \to W_1$ is a $H$-module morphism such that $\psi(ab) = \phi(a)\psi(b)$, then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Ind}_H^\Gamma(V) \otimes \mathrm{Ind}_H^\Gamma(V_1) & \xrightarrow{\phi\otimes\psi} & \mathrm{Ind}_H^\Gamma(W) \otimes \mathrm{Ind}_H^\Gamma(W_1) \\ \downarrow & & \downarrow \\ \mathrm{Ind}_H^\Gamma(V_1) & \xrightarrow{\quad\psi\quad} & \mathrm{Ind}_H^\Gamma(W_1), \end{array} \tag{7.10}$$

with vertical arrows being given by the multiplication. All maps, including the multiplications, are assumed to be compatible with the action of $\Gamma$.

### Explicit isomorphisms

We will use the following form of the *Frobenius reciprocity*: the map

$$\begin{aligned} \Phi = \Phi_{H,V}^{\Gamma,\mathcal{H}} : \mathrm{Hom}_H(\mathcal{H}, V) &\to \mathrm{Hom}_\Gamma(\mathcal{H}, \mathrm{Ind}_H^\Gamma(V)), \\ \Phi(f)(\xi) := \frac{1}{|H|} &\sum_{g\in\Gamma} g \otimes_{\mathbb{C}[H]} f(g^{-1}\xi), \end{aligned} \tag{7.11}$$

is an isomorphism ($\xi \in \mathcal{H}$, $f \in \mathrm{Hom}_H(\mathcal{H}, V)$). This version of the Frobenius reciprocity is not valid in general, but is valid for finite groups [188, 196]. Often one writes $\mathrm{Hom}_H(\mathrm{Res}_{\mathrm{H}}^\Gamma(\mathcal{H}), V)$ instead of $\mathrm{Hom}_H(\mathcal{H}, V)$. Let $\alpha$ be an irreducible representation of $\Gamma$, let $H \subset \Gamma$ be a subgroup and $\beta$ an irreducible representation of $H$. Frobenius reciprocity gives, in particular, that the multiplicity of $\alpha$ in $\mathrm{Ind}_H^\Gamma(\beta)$ is the same as the multiplicity of $\beta$ in the restriction of $\alpha$ to $H$. In particular, $\alpha$ is contained in $\mathrm{Ind}_H^\Gamma(\beta)$ if, and only if, $\beta$ is contained in the restriction of $\alpha$ to $H$. Furthermore, by taking $\mathcal{H}$ to be the trivial $\Gamma$-module $\mathbb{C}$, we obtain an isomorphism

$$\begin{aligned} \Phi : V^H = \mathrm{Hom}_H(\mathbb{C}, V) &\simeq \mathrm{Hom}_\Gamma(\mathbb{C}, \mathrm{Ind}_H^\Gamma(V)) = \mathrm{Ind}_H^\Gamma(V)^\Gamma, \\ \Phi(\xi) := \frac{1}{|H|} &\sum_{g\in\Gamma} g \otimes_{\mathbb{C}[H]} \xi = \sum_{x\in\Gamma/H} x \otimes \xi. \end{aligned} \tag{7.12}$$

The chosen normalization in the definition of $\Phi$ is such that it is an isomorphism of algebras if $V$ is an algebra.

### Inducing endomorphism modules

Let $\beta_1, \dots, \beta_N$, be non-isomorphic irreducible $H$-modules (with $H$ a subgroup of $\Gamma$, as above), and

$$\beta := \oplus_{j=1}^N \beta_j^{k_j}. \tag{7.13}$$

Let $\alpha$ be an irreducible $\Gamma$-module. We want to study the algebra $\mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta))^\Gamma$ acting on $\mathrm{Ind}_H^\Gamma(\beta)$ and on its isotypical component $p_\alpha(\mathrm{Ind}_H^\Gamma(\beta)) = \mathrm{Ind}_H^\Gamma(\beta)_\alpha$ (see 7.2.1 for the definition of the projection $p_\alpha$). We have, by the Frobenius isomorphism and by the form of the $H$-module $\beta$, that

$$\mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta))^\Gamma \simeq \mathrm{End}(\beta)^H \simeq \oplus_{j=1}^N \mathrm{End}(\beta_j^{k_j})^H \simeq \oplus_{j=1}^N M_{k_j}(\mathbb{C}), \tag{7.14}$$

which is a semi-simple algebra. Moreover, we have that $\mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta))^\Gamma = \Phi(\mathrm{End}(\beta)^H)$, where $\Phi$ is the map of Equation (7.12). From the properties of the induction functor $\mathrm{Ind}_H^\Gamma$, we also have that $\mathrm{Ind}_H^\Gamma(\beta) = \oplus_{j=1}^N \mathrm{Ind}_H^\Gamma(\beta_j^{k_j})$.

**Lemma 7.2.2.** *Let* $\beta := \oplus_{j=1}^N \beta_j^{k_j}$ *be as in Equation* (7.13)*, let*

$$T = (T_j) \in \mathrm{End}(\beta)^H \simeq \oplus_{j=1}^N \mathrm{End}(\beta_j^{k_j})^H ,$$

*with* $T_j \in \mathrm{End}(\beta_j^{k_j})^H$*, and let* $\xi_j \in \mathrm{Ind}_H^\Gamma(\beta_j^{k_j})$*. We let*

$$\xi := (\xi_j) \in \oplus_{j=1}^N \mathrm{Ind}_H^\Gamma(\beta_j^{k_j}) \simeq \mathrm{Ind}_H^\Gamma(\beta) .$$

*Then* $\Phi(T)(\xi) = (\Phi(T_j)\xi_j)_{j=1,\dots,N}$.

In other words, the Frobenius isomorphism $\Phi$ of Equation (7.12) is compatible with direct sums and with the action of morphisms on modules.

*Proof.* This follows from the naturality of the product, the isomorphism

$$\mathrm{Ind}_H^\Gamma(\oplus_{j=1}^N \mathrm{End}(\beta_j^{k_j}))^\Gamma \simeq \mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta))^\Gamma, \tag{7.15}$$

and Remark 7.2.1 (especially Equation (7.10)). □

Put differently, the simple factor of the algebra $\mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta))^\Gamma$ corresponding to $\mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta_j^{k_j}))^\Gamma$ acts only on the $j$th component of $\oplus_{i=1}^N \mathrm{Ind}_H^\Gamma(\beta_i^{k_i}) = \mathrm{Ind}_H^\Gamma(\beta)$.

### The abelian case

Assume now that $\Gamma$ is *abelian.* If $V$ is an irreducible $H$-module, then the action of $H$ on $V$ is via scalars: $h \cdot v = \chi_V(h)v$, for some group morphism (i.e. character) $\chi_V : H \to \mathbb{C}^*$. If $\chi$ is a character of $\Gamma$, we shall denote by $V_\chi$ the $H$-module equal to $\mathbb{C}$ as a vector space, with the action of $h \in H$ given by $h \cdot v = \chi(h)v$.

**Lemma 7.2.3.** *Assume that* $\Gamma$ *is a finite abelian group and that* $H$ *is a subgroup of* $\Gamma$*. Let* $V$ *be an irreducible* $H$*-module corresponding to the character* $\chi_V : H \to \mathbb{C}^*$*. Then*

$$\mathrm{Ind}_H^\Gamma(V) \simeq \bigoplus_{\substack{\chi \in \hat{\Gamma}, \\ \chi_{|H} = \chi_V}} \mathrm{Ind}_H^\Gamma(V)_\chi .$$

*Moreover, by writing* $\mathrm{Ind}_H^\Gamma(V) \simeq \mathbb{C}[\Gamma/H] \otimes V$ *as vector spaces, we obtain an action of* $\widehat{\Gamma/H}$ *on* $\mathrm{Ind}_H^\Gamma(V)$ *by the formula*

$$\rho \cdot (\gamma H \otimes v) := \rho(\gamma H)\gamma H \otimes v, \quad \gamma \in \Gamma,\ v \in V, \quad \textit{and} \quad \rho \in \widehat{\Gamma/H} .$$

*This action maps* $\mathrm{Ind}_H^\Gamma(V)_\chi$ *to* $\mathrm{Ind}_H^\Gamma(V)_{\rho\chi}$.

*Proof.* Given $\chi \in \widehat{\Gamma}$, we have by the Frobenius isomorphism that $\chi$ is contained in $\mathrm{Ind}_H^\Gamma(V)$ if, and only if, $\chi|_H = \chi_V$. If $A$ is a finite abelian group, we have the (non-canonical) isomorphism $\widehat{A} \simeq A$ of groups. There are $|\widehat{\Gamma/H}| = |\Gamma/H|$-many characters $\chi$ of $\Gamma$ with the property $\chi|_H = \chi_V$. It follows, by counting dimensions, that they all appear with multiplicity one in $\mathrm{Ind}_H^\Gamma(V)$. The statement about the action of $\widehat{\Gamma/H}$ is proved by a direct computation. This proof is complete. $\square$

Let again $\beta_1, \ldots, \beta_N$, be non-isomorphic irreducible $H$-modules (with $H$ a subgroup of $\Gamma$, as above), and

$$\beta := \oplus_{j=1}^N \beta_j^{k_j}. \tag{7.16}$$

If $H$ is abelian, then each $\beta_j$ is one dimensional, and hence $H$ acts by scalars on each $\beta_j^{k_j}$. The following proposition will play a crucial role in what follows.

**Proposition 7.2.4.** *Assume that $\Gamma$ is abelian, and let $\beta := \oplus_{j=1}^N \beta_j^{k_j}$ be as in Equation (7.16). Let $J$ be the set of indices $j \in \{1, 2, \ldots, N\}$ such that $\alpha_{|H} = \beta_j$. Then the morphism*

$$\pi_\alpha : \mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta))^\Gamma \simeq \oplus_{j=1}^N \mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta_j^{k_j}))^\Gamma \to \mathrm{End}(p_\alpha \mathrm{Ind}_H^\Gamma(\beta))^\Gamma$$

*is such that we have natural isomorphisms*

$$\ker(\pi_\alpha) \simeq \oplus_{j \notin J} \mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta_j^{k_j}))^\Gamma \quad \textit{and} \quad \mathrm{Im}(\pi_\alpha) \simeq \oplus_{j \in J} \mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta_j^{k_j}))^\Gamma .$$

*Proof.* By Lemma 7.2.2, we can assume that $N = 1$. By Lemma 7.2.3, $\mathrm{Ind}_H^\Gamma(\beta_1^{k_1}) \simeq \oplus_{\chi \in \widehat{\Gamma}} V_\chi$, with $\chi|_H = \beta_1$. We obtain

$$p_\alpha \mathrm{Ind}_H^\Gamma(\beta_1^{k_1}) \simeq \begin{cases} \alpha^{k_1} & \text{if } \alpha|_H = \beta_1 \\ 0 & \text{otherwise.} \end{cases} \tag{7.17}$$

Thus, if $\alpha|_H \neq \beta_1$, $J = \emptyset$, $\pi_\alpha = 0$, and hence

$$\ker(\pi_\alpha) = \mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta_1^{k_1}))^\Gamma = \oplus_{j \notin J} \mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta_j^{k_j}))^\Gamma \quad \text{and}$$
$$\mathrm{Im}(\pi_\alpha) = 0 = \oplus_{j \in J} \mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta_j^{k_j}))^\Gamma ,$$

as claimed.

Let us assume now that $\alpha|_H = \beta_1$. The morphism $\pi_\alpha$ can then be written as the composition

$$\begin{aligned}\pi_\alpha : \mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta_1^{k_1}))^\Gamma \simeq \mathrm{End}(\beta_1^{k_1})^H \simeq M_{k_1}(\mathbb{C}) &\xrightarrow{\Psi_\alpha} M_{k_1}(\mathbb{C}) \\ &\simeq \mathrm{End}(\alpha^{k_1})^\Gamma \simeq \mathrm{End}(p_\alpha \mathrm{Ind}_H^\Gamma(\beta_1^{k_1}))^\Gamma .\end{aligned}$$

To prove our proposition in this case ($\chi|_H = \beta_1$, and hence in general, by Lemma 7.2.2), it thus suffices to show that $\Psi_\alpha = id$, which equivalent to the fact that $\Psi_\alpha \neq 0$ for $\alpha|_H = \beta_1$.

To prove that $\Psi_\alpha \neq 0$ for $\alpha|_H = \beta_1$, let us begin by noticing that the morphism $\mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta)) \to \mathrm{End}(\mathrm{Ind}_H^\Gamma(\beta))$ is injective. Hence $\mathrm{Ind}_H^\Gamma(\mathrm{End}(\beta))^\Gamma \to \mathrm{End}(\mathrm{Ind}_H^\Gamma(\beta))^\Gamma$ is injective as well. This means that not all of the maps $\Psi_\rho$, with $\rho|_H = \beta_1$, can be zero, by Lemma 7.2.3. On the other hand, it can be checked by a direct calculation that the action of $\widehat{\Gamma/H}$ on $\mathrm{Ind}_H^\Gamma(\beta)$ of the same lemma permutes the morphisms $\Psi_\rho$. It follows either that they are all zero or that they are all non-zero. As their direct sum is non-zero, it follows that all $\Psi_\rho \neq 0$, $\rho|_H = \beta_1$. $\square$

### 7.2.3. Group actions on manifolds

As before, we consider a general finite group $\Gamma$ acting by isometries on a compact Riemannian manifold $M$.

#### Slices and tubes

Given $x \in M$, the isotropy group $\Gamma_x$ acts linearly and isometrically on $T_xM$. For $r > 0$, let $U_x := (T_xM)_r$ denote the set of vectors of length $< r$ in $T_xM$. It is known then that, for $r > 0$ small enough, the exponential map gives a $\Gamma$-equivariant isometric diffeomorphism

$$W_x = \exp(\Gamma \times_{\Gamma_x} U_x) \simeq \Gamma \times_{\Gamma_x} U_x \tag{7.18}$$

where $W_x$ is a $\Gamma$-invariant neighborhood of $x$ in $M$ and $\Gamma \times_{\Gamma_x} U_x$ is defined in equation (7.6). More precisely, $W_x$ is the set of $y \in M$ at distance $< r$ to the orbit $\Gamma x$, if $r > 0$ is small enough. The set $W_x$ is called a *tube* around $x$ (or $\Gamma x$) and the set $U_x$ is called the *slice* at $x$. When $M$ is compact, the injectivity radius is bounded from below, so we may assume that the constant $r$ does not depend on $x$.

#### Equivariant vector bundles

Let us consider now a $\Gamma$-equivariant smooth vector bundle $E \to M$. Let us fix $x \in M$ and consider as above the tube $W_x \simeq \Gamma \times_{\Gamma_x} U_x$ around $x$, see Equation (7.18). We use this diffeomorphism to identify $U_x$ to a subset of $M$, in which case, we can also assume the restriction of $E$ to the slice $U_x$ to be trivial. Therefore, there exists a $\Gamma_x$-module $\beta$ such that

$$\begin{aligned} E|_{U_x} &\simeq U_x \times \beta \quad \text{and} \\ E|_{W_x} &\simeq \Gamma \times_{\Gamma_x} (U_x \times \beta)\,, \end{aligned} \tag{7.19}$$

The second isomorphism is $\Gamma$-equivariant.

Assume $E$ is endowed with a $\Gamma$-invariant hermitian metric. We then have isomorphisms of $\Gamma$-modules:

$$\begin{aligned} L^2(W_x; E|_{W_x}) &\simeq \mathrm{Ind}_{\Gamma_x}^\Gamma(L^2(U_x;\beta)) \quad \text{and} \\ C_0(W_x; E|_{W_x}) &\simeq \mathrm{Ind}_{\Gamma_x}^\Gamma(C_0(U_x;\beta))\,. \end{aligned} \tag{7.20}$$

In view of the previous isomorphism, we will often identify $W_x$ and $\Gamma \times_{\Gamma_x} U_x$, making no distinction between them to simplify notations.

**The principal orbit bundle**

Recall that $M_{(H)}$ denotes the set of points of $M$ whose stabilizer is conjugated in $\Gamma$ to $H$. Recall also that we have assumed that $M/\Gamma$ is *connected*. It is known then [196] that there exists a *minimal isotropy* subgroup $\Gamma_0 \subset \Gamma$, in the sense that $M_{(\Gamma_0)}$ is a dense open subset of $M$, with measure zero complement in $M$.

In particular, the fact that $M/\Gamma$ is connected gives that there exist minimal elements for the set of isotropy groups of points in $M$ (with respect to inclusion) and all minimal isotropy groups are conjugated to a fixed subgroup $\Gamma_0 \subset \Gamma$. By the definition, the set $M_{(\Gamma_0)}$ consists of the points whose stabilizer is conjugated to that minimal subgroup. The set $M_{(\Gamma_0)}$ is called the *principal orbit bundle* of $M$, and we will denote it by $M_0 := M_{(\Gamma_0)}$ in the sequel.

The principal orbit bundle $M_0$ has the following useful property. If $x \in M_0$, then $\Gamma_x$ acts *trivially* on the slice $U_x$ at $x$, by the minimality of $\Gamma_0$. Hence $\Gamma_0$ acts trivially on $T_x^*M$ as well, which implies that $\Gamma_0 \subset \Gamma_\xi$ for any $\xi \in T_x^*M$. If, on the other hand, $x \in M$ is arbitrary (not necessarily in the principal orbit bundle), then the isotropy of $\Gamma_x$ will contain a subgroup conjugated to $\Gamma_0$.

When $\Gamma$ is *abelian*, there is only one minimal isotropy group $\Gamma_0$ (recall that we are assuming $M$ to be connected). Moreover, we can then factor the action of $\Gamma$ to an action of $\Gamma/\Gamma_0$ on $M$, which has trivial minimal isotropy, that is, it is free on a dense, open subset of $M$.

### 7.2.4. The primitive ideal spectrum

We shall need a few basic concepts and facts about $C^*$-algebras, for which we refer to Section 4.1. Recall that a two-sided ideal $I \subset A$ of a $C^*$-algebra $A$ is called *primitive* if it is the kernel of a non-zero, irreducible $*$-representation of $A$. Hence $A$ is *not* a primitive ideal of itself. By $\mathrm{Prim}(A)$ we shall denote the set of primitive ideals of $A$, called the *primitive ideal spectrum* of $A$. The space $\mathrm{Prim}(A)$ is a topological space for the Jacobson topology: we refer to Section 4.1.2 for more details, and will recall some facts about this topology when needed.

If $A$ is a type I $C^*$-algebra, then $\mathrm{Prim}(A)$ identifies with the set $\widehat{A}$ of unitary equivalence classes of irreducible representations of $A$. Any $C^*$-algebra with only finite dimensional irreducible representations is a type I algebra [75]. Most of the algebras considered in this chapter have this property, a notable exception being the algebras of compact operators.

The following example will be used several times.

**Remark 7.2.5.** Let $H$ be a finite group and $\beta = \oplus_{j=1}^N \beta_j^{k_j}$ be a finite dimensional $H$-module with $\beta_j$ non-isomorphic simple $H$-modules. We obtain that

$$\begin{aligned}\mathrm{End}_H(\beta) \simeq \mathrm{Hom}_H(\oplus_i \beta_i^{k_i}, \oplus_j \beta_j^{k_j}) \simeq \bigoplus_{i,j} \mathrm{Hom}_H(\beta_i^{k_i}, \beta_j^{k_j})\\ \simeq \bigoplus_j \mathrm{End}_H(\beta_j^{k_j}) \simeq \bigoplus_j M_{k_j}(\mathbb{C})\,.\end{aligned}$$

The algebra $\mathrm{End}_H(\beta)$ is thus a $C^*$-algebra with only finite dimensional representations and we have natural homeomorphisms

$$\mathrm{Prim}(\mathrm{End}_H(\beta)) \simeq \{\beta_1, \beta_2, \dots, \beta_N\} \simeq \{1, 2, \dots, N\}.$$

Another important tool will be the following "central character map".

**Remark 7.2.6.** Let $Z$ be a commutative $C^*$-algebra and $\phi : Z \to M(A)$ be a $*$-morphism to the multiplier algebra $M(A)$ of $A$ (see Example 4.1.6). Assume that $\phi(Z)$ commutes with $A$ and $\phi(Z)A = A$. Then Schur's lemma gives that every irreducible representation of $A$ restricts to (a multiple of) a character of $Z$ and hence there exists a natural continuous map

$$\phi^* : \mathrm{Prim}(A) \to \mathrm{Prim}(Z)\,, \tag{7.21}$$

which we shall call also the *central character map* (associated to $\phi$).

We conclude our discussion with the following simple result.

**Lemma 7.2.7.** *We freely use the notation of Example 7.2.5. The inclusion of the unit* $\mathbb{C} \to \mathrm{End}_H(\beta)$ *induces a morphism* $j : C_0(X) \to C_0(X; \mathrm{End}_H(\beta)) \simeq C_0(X) \otimes \mathrm{End}_H(\beta)$. *The resulting central character map is the first projection*

$$j^* : \mathrm{Prim}(C_0(X; \mathrm{End}_H(\beta))) \simeq X \times \{1, 2, \dots, N\} \to X \simeq \mathrm{Prim}(C_0(X))\,. \tag{7.22}$$

### 7.2.5. Pseudodifferential operators

We continue to assume that $\Gamma$ is a compact Lie group that acts smoothly and isometrically on a smooth Riemannian manifold $M$. As in the rest of this manuscript, we let $\Psi^m(M; E)$ denote the space of order $m$, *classical* pseudodifferential operators on $M$ with *compactly supported* distribution kernel, acting on the sections of the $\Gamma$-equivariant vector bundle $E$: see Section 4.2 for the precise definitions. We let $\overline{\Psi^0}(M; E)$ and $\overline{\Psi^{-1}}(M; E)$ denote the norm closures of $\Psi^0(M; E)$ and $\Psi^{-1}(M; E)$ in $L^2(M; E)$, respectively. We will moreover denote by $\mathcal{K}(\mathcal{H})$ the algebra of compact operators acting on a Hilbert space $\mathcal{H}$, and write $\mathcal{K}$ instead of $\mathcal{K}(\mathcal{H})$ when the Hilbert space $\mathcal{H}$ is clear from the context. We have

$$\overline{\Psi^{-1}}(M; E) = \mathcal{K}(L^2(M; E)), \tag{7.23}$$

since we have considered only pseudodifferential operators with compactly supported distribution kernels.

Let $S^*M$ denote the *unit cosphere bundle* of $M$, that is, the set of unit vectors in $T^*M$, as usual. As we did before, we will denote by $C_0(S^*M; \mathrm{End}(E))$ the set of continuous sections of the *lift* of the vector bundle $\mathrm{End}(E) \to M$ to $S^*M$. The action of $\Gamma$ then extends to an action on $E$ and on $\Psi^m(M; E)$, $\overline{\Psi^0}(M; E)$, and $\overline{\Psi^{-1}}(M; E)$.

**Lemma 7.2.8.** *We have an exact sequence*

$$0 \longrightarrow \mathcal{K}^\Gamma \longrightarrow \overline{\Psi^0}(M; E)^\Gamma \xrightarrow{\sigma_0} C_0(S^*M; \mathrm{End}(E))^\Gamma \longrightarrow 0.$$

*Proof.* This comes from the well-known exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \overline{\Psi^0}(M;E) \xrightarrow{\sigma_0} C(S^*M;\mathrm{End}(E)) \longrightarrow 0.$$

discussed in 4.2.3, together with the fact that the functor $\mathcal{H} \to \mathcal{H}^\Gamma$ is exact on the category of $\Gamma$-modules with continuous $\Gamma$-action, since $\Gamma$ is compact. □

As argued in 4.2.3, studying the Fredholm properties of operators in $\Psi^m(M;E,F)$ acting between *two* vector bundles $E, F \to M$ can be reduced to the case of order-zero operators acting on a single vector bundle $E \oplus F$. This reduces our study to that of the $C^*$-algebra $\overline{\Psi^0}(M;E)^\Gamma$.

## 7.3. The structure of regularizing operators

We continue to assume that $M$ is a complete Riemannian manifold and that $\Gamma$ is a compact Lie group acting by isometries on $M$. From now on, all our vector bundles will be $\Gamma$-equivariant vector bundles.

As explained in the Introduction, we want to identify the structure of the restrictions of $\Gamma$-invariant pseudodifferential operators on $M$ to the isotypical components of $L^2(M;E)$. Let $\pi_\alpha$ be this restriction morphism to the $\alpha$-isotypical component. More precisely, we want to understand the structure of the algebra $\pi_\alpha(\overline{\Psi^0}(M;E)^\Gamma)$, for any fixed $\alpha \in \Gamma$. See Equations (7.1) and (7.8) for the definition of the restriction morphism $\pi_\alpha$ and of the projectors $p_\alpha \in C^*(\Gamma)$.

In this section, we study two basic cases: that of inner actions and that of free actions of $\Gamma$.

### 7.3.1. Inner actions of Γ: the abstract case

In this subsection we deal with the case when the action of $\Gamma$ is implemented by unitaries in the multiplier algebra (the case of inner actions). This allows us, in particular, to settle the case of regularizing operators. We shall need the following notion of direct sum. Recall that $M(A)$ denotes the multiplier algebra of a $C^*$-algebra $A$. Recall the following standard definition.

**Definition 7.3.1.** Let $\phi : \Gamma \to \mathrm{Aut}(A)$ be the action of a group $\Gamma$ by automorphisms on a $C^*$-algebra $A$. We shall say that this action is *inner* if the morphism $\phi$ lifts to a morphism $\psi : \Gamma \to U(M(A))$ to the group of unitary elements of $M(A)$ such that

$$\phi_g(a) = \psi(g)a\psi(g)^{-1}\,, \quad a \in A,\ g \in \Gamma\,.$$

**Remark 7.3.2.** If $\alpha : \Gamma \to \mathrm{Aut}(A)$ is an inner action as in Definition 7.3.1, then we obtain, in particular, a morphism $C^*(\Gamma) \to M(A)$. If, moreover, $\pi : A \to \mathcal{L}(\mathcal{H})$ is a $*$-representation, then it extends to a representation of $M(A)$. This induces naturally a unitary representation of $\Gamma$ on $\mathcal{H}$. This representation is uniquely determined if $\pi$ is non-degenerate (i.e. if $\pi(A)\mathcal{H}$ is dense in $\mathcal{H}$).

If $A_n$, $n \geq 1$ is a sequence of $C^*$-algebras, we shall denote by $c_0\text{–}\oplus_{n=1}^{\infty} A_n$ the inductive limit $\lim_{N\to\infty} \oplus_{n=1}^{N} A_n$. This definition extends immediately to countable families of $C^*$-algebras. Recall that $\widehat{\Gamma}$ denotes the set of isomorphism classes of irreducible unitary representations of $\Gamma$. We shall need then the following general result.

**Proposition 7.3.3.** *Let $A$ be a $C^*$-algebra with a *-morphism $C^*(\Gamma) \to M(A)$ and let $p_\alpha \in C^*(\Gamma)$ denote the central projector corresponding to $\alpha \in \widehat{\Gamma}$. Let $A^\Gamma := \{a \in A \mid af = fa,\ f \in C^*(\Gamma)\}$. Then*

$$A^\Gamma \simeq c_0\text{–}\bigoplus_{\alpha\in\hat{\Gamma}} p_\alpha A^\Gamma. \tag{7.24}$$

*If $I \subset A$ is a closed two-sided ideal, then $C^*(\Gamma)I \subset I$ and hence we obtain *-morphisms $C^*(\Gamma) \to M(I)$ and $C^*(\Gamma) \to M(A/I)$ such that the induced sequence*

$$0 \longrightarrow p_\alpha I^\Gamma \longrightarrow p_\alpha A^\Gamma \longrightarrow p_\alpha (A/I)^\Gamma \longrightarrow 0$$

*is exact for any $\alpha \in \widehat{\Gamma}$.*

*Proof.* Let us arrange the elements of $\widehat{\Gamma}$ in a sequence $\rho_n$, $n \geq 1$. Then, for any $a \in A$, $\lim_{N\to\infty} \sum_{n=1}^{N} p_{\rho_n} a = a$. Since, for any $a \in A^\Gamma$, we have $p_\alpha a = a p_\alpha$, the isomorphism $A^\Gamma \simeq c_0\text{–}\oplus_{\alpha\in\hat{\Gamma}} A^\Gamma p_\alpha$ follows. Finally, it is known that if $I$ is a two-sided ideal of $A$, then $M(A)I \subset I$. □

If $A \subset \mathcal{L}(\mathcal{H})$ is a sub-$C^*$-algebra of the algebra of bounded operators on a Hilbert space $\mathcal{H}$ together with a compatible $\Gamma$-module structure on $\mathcal{H}$, we let $\pi_\alpha : A^\Gamma \to \mathcal{L}(\mathcal{H}_\alpha)$ be the restriction morphism to the $\alpha$-isotypical component, as before, see Equations (7.1) and (7.8).

**Proposition 7.3.4.** *In addition to the assumptions of Proposition 7.3.3, let us suppose that $A \subset \mathcal{L}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$. Let $\alpha \in \widehat{\Gamma}$, as before. Then the morphism $\pi_\alpha : A^\Gamma \to \mathcal{L}(\mathcal{H}_\alpha)$ restricts to an isomorphism $p_\alpha A^\Gamma \to \pi_\alpha(A^\Gamma)$.*

*Proof.* Let us recall that, as explained in Remark 7.3.2, the group $\Gamma$ will be represented on $\mathcal{H}$. The rest follows from Proposition 7.3.3, whose notation we shall use freely. Indeed, if $\alpha \neq \beta \in \widehat{\Gamma}$, then $\pi_\alpha(p_\beta) = 0$. On the other hand $\pi_\alpha(p_\alpha) = p_\alpha$. The result follows by combining this property with $\mathcal{H}_\alpha = p_\alpha \mathcal{H}$ and with Proposition 7.3.3. □

### 7.3.2. Inner actions of Γ: pseudodifferential operators

We now apply the results of the previous subsection to the algebra of pseudodifferential operators. In particular, this gives a rather complete picture for the case of negative order operators (recall that the closure of regularizing operators coincides with that of negative order operators). Since we are eventually interested only in the case $\Gamma$ finite, we discuss only briefly the issues related to the non-discrete case (such as the continuity of the action of $\Gamma$).

A crucial first observation is that if $\gamma \in \Gamma$ and $P \in \Psi^{-\infty}(M; E)$, then $\gamma P$ and $P\gamma \in \Psi^{-\infty}(M; E)$. This leads to several interesting consequences. We record this as a lemma.

**Lemma 7.3.5.** *We have $\gamma\Psi^{-\infty}(M;E) = \Psi^{-\infty}(M;E)\gamma = \Psi^{-\infty}(M;E)$, for all $\gamma \in \Gamma$, and the induced actions (to the right and to the left) of $\Gamma$ on $\Psi^{-\infty}(M;E)$ are continuous and unitary in the operator norm. Consequently, we have*

$$C^*(\Gamma)\overline{\Psi^{-1}}(M;E) + \overline{\Psi^{-1}}(M;E)C^*(\Gamma) \subset \overline{\Psi^{-1}}(M;E)\,.$$

*Proof.* Since smoothing operators have smooth kernels $k(x,y) \in \mathrm{Hom}(E_y, E_x)$, the action of $\Gamma$ induces an action on $\Psi^{-\infty}(M;E)$. Thanks to the $\Gamma$-invariance of the metric on $M$, we have $\|\gamma P\| = \|P\gamma\| = \|P\|$ in the $L^2$-operator norm, for any $P \in \Psi^{-\infty}(M;E)$. The second part follows from the first part. ☐

We then have the following:

**Proposition 7.3.6.** *The multiplication by $\Gamma$ on $\overline{\Psi^{-1}}(M;E)$ defines a *-morphism $C^*(\Gamma) \to U(M(\overline{\Psi^{-1}}(M;E)))$ to the multiplier algebra of $\overline{\Psi^{-1}}(M;E)$.*

*Proof.* This follows from Lemma 7.3.5. ☐

We obtain the following corollary.

**Corollary 7.3.7.** *Let $A = \overline{\Psi^{-1}}(M;E)$. If $\Gamma$ acts trivially on $M$ (so that the action of $C^*(\Gamma)$ extends to the algebra $\overline{\Psi^0}(M;E)$), we also allow $A = \overline{\Psi^0}(M;E)$ or $A = C_0(S^*M;\mathrm{End}(E))$. We then have isomorphisms*

$$A^\Gamma \simeq c_0\text{-}\bigoplus_{\alpha\in\widehat{\Gamma}} p_\alpha A^\Gamma\,.$$

*Moreover, if $\Gamma$ acts trivially on $M$ and $\alpha \in \hat{\Gamma}$, we have an exact sequence*

$$0 \longrightarrow p_\alpha\overline{\Psi^{-1}}(M;E)^\Gamma \longrightarrow p_\alpha\overline{\Psi^0}(M;E)^\Gamma \longrightarrow C_0(S^*M;\mathrm{End}(p_\alpha E)^\Gamma) \longrightarrow 0.$$

*Proof.* The first part follows from Proposition 7.3.3 applied to the algebra $\overline{\Psi^0}(M;E)$ and to its ideal $\mathcal{K}$. The second part follows from the exactness of the functors $V \to V^\Gamma$ and $V \to p_\alpha V$ on the category of $\Gamma$-modules. ☐

Let $\alpha \in \hat{\Gamma}$ and let $\pi_\alpha$ be the representation of $\overline{\Psi^0}(M;E)^\Gamma$ on $L^2(M;E)_\alpha$ defined by restriction as before, Equations (7.1) and (7.8). The assumptions of Proposition 7.3.4 are satisfied for $A = \overline{\Psi^{-1}}(M;E)$, so we obtain the following.

**Corollary 7.3.8.** *The morphism $\pi_\alpha$ restricts to an isomorphism from the algebra $p_\alpha\overline{\Psi^{-1}}(M;E)^\Gamma$ to $\pi_\alpha(\overline{\Psi^{-1}}(M;E)^\Gamma)$.*

We also have the following simple result, which makes the last corollary more precise. Recall that $\mathcal{K} \simeq \overline{\Psi^{-1}}(M;E)$. This allows us to better describe the structure of $\overline{\Psi^{-1}}(M;E)^\Gamma$.

**Proposition 7.3.9.** *The algebra $\pi_\alpha(\mathcal{K}^\Gamma)$ is the algebra of $\Gamma$-equivariant compact operators on $L^2(M;E)_\alpha$.*

*Proof.* Let $T \in \mathcal{K}$ commute with $\Gamma$. Then its restriction to a $\Gamma$-invariant subspace is still compact and still commutes with $\Gamma$. This shows that $\pi_\alpha(\mathcal{K}^\Gamma)$ is contained in the set $K_\alpha$ of $\Gamma$-invariant compact operators on $L^2(M;E)_\alpha$. Conversely, $K_\alpha \subset \mathcal{K}^\Gamma$ and $\pi_\alpha$ acts as the identity on $K_\alpha$. ☐

### 7.3.3. The case of free actions

Let us now tackle the opposite case, that is when $\Gamma$ acts freely on $M$. We shall assume that $\Gamma$ is finite, for simplicity (we only need this case), and hence, in particular, the action of $\Gamma$ is proper. We have then the following well-known result (see [60, 147] and the references therein).

**Proposition 7.3.10.** *Let us assume that $\Gamma$ is a finite group acting freely on $M$ and let $\alpha(\gamma) = 1$ be the trivial representation, so $\pi_\alpha = \pi_1$. Let us denote by $F := E/\Gamma \to M/\Gamma$ the resulting vector bundle and $\phi : C_0(S^*M; \mathrm{End}(E))^\Gamma \to C_0(S^*M/\Gamma; \mathrm{End}(E/\Gamma))$ the resulting isomorphism. Then we have the following morphism of exact sequences, with the vertical arrows surjective:*

$$\begin{tikzcd}
0 \arrow[r] & \overline{\Psi^{-1}}(M;E)^\Gamma \arrow[r] \arrow[d, "\pi_1"] & \overline{\Psi^{0}}(M;E)^\Gamma \arrow[r] \arrow[d, "\pi_1"] & C_0(S^*M;\mathrm{End}(E))^\Gamma \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & \overline{\Psi^{-1}}(M/\Gamma;F) \arrow[r] & \overline{\Psi^{0}}(M/\Gamma;F) \arrow[r] & C_0(S^*M/\Gamma;\mathrm{End}(F)) \arrow[r] & 0.
\end{tikzcd}$$

For an open set $\mathcal{O} \subset M$, let $A_\mathcal{O} := C_0(S^*\mathcal{O}; \mathrm{End}(E))$ and consider the *surjective* map

$$\begin{aligned}
\mathcal{R}_\mathcal{O} \,:\, A_\mathcal{O}^\Gamma := C_0(S^*\mathcal{O}; \mathrm{End}(E))^\Gamma \;\simeq\; & \overline{\Psi^0}(\mathcal{O};E)^\Gamma / \overline{\Psi^{-1}}(\mathcal{O};E)^\Gamma \\
& \to \pi_\alpha(\overline{\Psi^0}(\mathcal{O};E)^\Gamma)/\pi_\alpha(\overline{\Psi^{-1}}(\mathcal{O};E)^\Gamma)\,. \quad (7.25)
\end{aligned}$$

**Proposition 7.3.11.** *Let $\Gamma$ be a finite group acting on a smooth compact manifold $M$ (without boundary). Assume that the action of $\Gamma$ is free on a dense, open subset of $M$. Let $E \to M$ be an equivariant vector bundle. Then the map*

$$\mathcal{R}_M : C_0(S^*M; \mathrm{End}(E))^\Gamma \to \pi_1(\overline{\Psi^0}(M;E)^\Gamma)/\pi_1(\overline{\Psi^{-1}}(M;E)^\Gamma)$$

*of Equation* (7.25) *is injective, and hence an isomorphism of algebras.*

*Proof.* Let $M_0 \subset M$ be an open, dense subset on which $\Gamma$ acts freely. Proposition 7.3.10 for $M$ replaced with $M_0$ shows that $\mathcal{R}_{M_0}$ is injective. Hence $\mathcal{R}_M$ is injective on $C_0(S^*M_0; \mathrm{End}(E))^\Gamma$, because the restriction of $\mathcal{R}_M$ to $C_0(S^*M_0; \mathrm{End}(E))^\Gamma$ is $\mathcal{R}_{M_0}$. Since the later is an essential ideal in $C_0(S^*M; \mathrm{End}(E))^\Gamma$, it follows that $\mathcal{R}_M$ is also injective (everywhere on $C_0(S^*M; \mathrm{End}(E))^\Gamma$). ☐

## 7.4. The principal symbol

Let us fix an irreducible representation $\alpha$ of $\Gamma$ and consider the fundamental restriction morphism $\pi_\alpha$ of Equation (7.1). See also Subsection 7.2.1, especially Equation (7.8), for more details on the morphism $\pi_\alpha$. We are mostly concerned with the morphism $\pi_\alpha : \overline{\Psi^0}(M;E)^\Gamma \to \mathcal{L}(L^2(M;E)_\alpha)$ and, in this section, we identify the quotient

$$\pi_\alpha(\overline{\Psi^0}(M;E)^\Gamma)/\pi_\alpha(\overline{\Psi^{-1}}(M;E)^\Gamma)\,.$$

Since $\pi_\alpha(\overline{\Psi^{-1}}(M;E)^\Gamma)$ was identified in the previous section, information on the above quotient algebra will give further insight into the structure of the algebra $\pi_\alpha(\overline{\Psi^0}(M;E)^\Gamma)$ and will provide us, eventually, with Fredholm conditions.

In the beginning of this section, we continue to assume that $M$ is a complete Riemannian manifold and that $\Gamma$ is a compact Lie group acting by isometries on $M$. Since the results are different in the discrete and non-discrete case, we will assume beginning with Subsection 7.4.2 that $\Gamma$ is finite. Moreover, for the main result, we shall assume that $\Gamma$ is abelian, since the abelian case is simpler and presents some additional features. Some intermediate results are true only in the abelian case. It is also the abelian case that will be used for our application to boundary value problems.

### 7.4.1. The primitive ideal spectrum of the symbol algebra

We now turn to the description of the primitive ideal spectrum of the algebra of symbols $A_M^\Gamma := C_0(S^*M;\mathrm{End}(E))^\Gamma$. For simplicity, given an open subset $\mathcal{O} \subset M$, we denote $A_\mathcal{O} := C_0(S^*\mathcal{O};\mathrm{End}(E))$, as in the definition of the morphism $\mathcal{R}_\mathcal{O}$ of Equation (7.25). We shall be mostly concerned with the cases $\mathcal{O} = M$ and $\mathcal{O} = M_0 := M_{(\Gamma_0)}$. We have the following standard result.

**Proposition 7.4.1.** *The algebra $Z_M := C_0(S^*M)^\Gamma = C_0(S^*M/\Gamma)$ identifies with a central subalgebra of $A_M^\Gamma := C_0(S^*M;\mathrm{End}(E))^\Gamma$. Let $z_\xi$ be the maximal ideal of $Z_M$ associated to the orbit $\Gamma\xi$ for some $\xi \in S_x^*M$, let $E_x$ be a fiber of $E$ corresponding to $\xi$, and let $E_x \simeq \oplus_{j=1}^N \beta_j^{k_j}$ be its decomposition into $\Gamma_\xi$-isotypical components, with $\beta_j$ simple, non-isomorphic $\Gamma_\xi$ modules. Then $A_M^\Gamma/z_\xi A_M^\Gamma \simeq \mathrm{End}(E_x)^{\Gamma_\xi}$ is a semi-simple, finite-dimensional (complex) algebra with $N$ simple factors $\mathrm{End}_{\Gamma_\xi}(\beta_j^{k_j}) \simeq M_{k_j}(\mathbb{C})$, $j \in \{1, 2, \dots, N\}$.*

*Proof.* We have $C_0(M) \subset C_0(M;\mathrm{End}(E)) \subset \mathrm{End}(C_0(M;E))$, with $f \in C_0(M)$ acting as scalar on each fiber $E_x$. In fact, this identifies $C_0(M)$ with the center $Z(C_0(M;\mathrm{End}(E)))$ of $C_0(M;\mathrm{End}(E))$. By considering $\Gamma$ invariant functions, we obtain that $Z_M := C_0(S^*M)^\Gamma = C_0(S^*M/\Gamma)$ is contained in the center $Z(A_M^\Gamma)$ of $A_M^\Gamma := C_0(S^*M;\mathrm{End}(E))^\Gamma$. Let $\Gamma\xi$ denote the orbit in $S^*M$ that we consider and let $J$ be the (non-maximal, in general) ideal of $C_0(S^*M)$ corresponding to functions vanishing on this orbit. Then $J$ is $\Gamma$ invariant and $J^\Gamma = z_\xi$. By taking the $\Gamma$ invariants in the exact sequence $0 \to JA_M \to A_M \to A_M/JA_M \to 0$ and using Frobenius reciprocity for $A_M/JA_M \simeq \mathrm{Ind}_{\Gamma_\xi}^\Gamma(\mathrm{End}(E_x))$, we obtain that

$$A_M^\Gamma/z_\xi A_M^\Gamma \simeq (A_M/JA_M)^\Gamma \simeq \mathrm{End}(E_x)^{\Gamma_\xi}\,. \tag{7.26}$$

The proof is completed using Remark 7.2.5 for $H = \Gamma_\xi$. □

See [37, 77, 87, 181, 207] for similar results. Recall from Remark 7.2.6 that if $\phi : Z \to M(A)$ is a central *-morphism (i.e. $\phi(z)a = a\phi(z)$, for $a \in A$ and $z \in Z$) such that $\phi(Z)A = A$, then it defines a natural "central character" map $\phi^* : \mathrm{Prim}(A) \to \mathrm{Prim}(Z)$ by Schur's Lemma. The same proof yields the following.

**Corollary 7.4.2.** *There is a one-to-one correspondence between the primitive ideals of $A_M^\Gamma := C_0(S^*M; \mathrm{End}(E))^\Gamma$ and the $\Gamma$-orbits of the pairs $(\xi, \rho)$, where $\xi \in S_x^*M$ and $\rho \in \widehat{\Gamma}_\xi$ appears in $E_x$ (i.e. $\mathrm{Hom}_{\Gamma_\xi}(\rho, E_x) \neq 0$). The group $\Gamma$ acts by joint conjugation on both $\xi$ and $\rho$. The inclusion $Z_M := C_0(S^*M)^\Gamma \to A_M^\Gamma$ is such that the associated canonical central character map of spectra*

$$\mathrm{Prim}(A_M^\Gamma) \ \to \ \mathrm{Prim}(Z_M) = S^*M/\Gamma$$

*is continuous, finite-to-one, and maps the orbit $\Gamma(\xi, \rho)$ to the orbit $\Gamma\xi$.*

*Proof.* Let $\pi$ be an irreducible representation of $A_M^\Gamma := C_0(S^*M; \mathrm{End}(E))^\Gamma$. Then $\pi$ is a multiple of a character on $Z_M := C_0(S^*M)^\Gamma \subset Z(A_M^\Gamma)$, by Schur's lemma. Let this character correspond to the orbit $\Gamma\xi \in S^*M/\Gamma$, with $\xi \in S_x^*M$ and denote by $z_\xi$ the corresponding maximal ideal of $Z_M$, as in the proof of the last lemma. In other words, $z_\xi$ is the value of the central character map corresponding to the inclusion $Z_M \subset A_M^\Gamma$ applied to $\pi$. Then $\pi$ factors out through an irreducible representation of $A_M^\Gamma/z_\xi A_M^\Gamma \simeq \mathrm{End}(E_x)^{\Gamma_\xi}$. Let us write $E_x \simeq \oplus_{j=1}^N \beta_j^{k_j}$ with $\beta_j$ non-isomorphic simple $\Gamma_\xi$ modules, as in the statement of Proposition 7.4.1. Then $\mathrm{End}(E_x)^{\Gamma_\xi} \simeq \oplus_{j=1}^N \mathrm{End}_{\Gamma_\xi}(\beta_j^{k_j})$, a direct sum of simple algebras. Thus $\pi$ factors through one of the simple algebras $\mathrm{End}_{\Gamma_\xi}(\beta_j^{k_j})$. This associates to $\pi$ the pair $(\xi, \rho) = (\xi, \beta_j)$, as desired. This pair is not unique, but depends on the choice of $\xi$. It becomes unique modulo the action of $\Gamma$, however. Conversely, given such a pair $(\xi, \rho)$, we obtain an irreducible representation of $A_M^\Gamma$ following exactly the same procedure in reverse order. The first part of the result follows.

To prove that $\phi^*$ is finite to one, we notice that, by construction, $\phi^*(\xi, \rho) = \xi$. Since only a finite number of (isomorphism classes of) simple $\Gamma_\xi$ modules appears in $E_x$, the finiteness follows. □

**Remark 7.4.3.** Let us denote by $\tilde{X}_{M,\Gamma}$ the set of pairs $(\xi, \rho)$, where $\xi \in T_x^*M \setminus \{0\}$ and $\rho \in \widehat{\Gamma}_\xi$ appears in $E_x$ (i.e. $\mathrm{Hom}_{\Gamma_\xi}(\rho, E_x) \neq 0$), as in the statement of Corollary 7.4.2. The main result of that corollary is a natural bijection

$$\tilde{X}_{M,\Gamma}/\Gamma \ \simeq \ \mathrm{Prim}(A_M^\Gamma)\,. \tag{7.27}$$

This bijection can be explicitly described as follows: to an orbit $\Gamma(\xi, \rho)$ in $\tilde{X}_{M,\Gamma}$ is associated $\ker(\pi_{\xi,\rho})$ in $\mathrm{Prim}\, A_M^\Gamma$, where for any $f \in A_M^\Gamma$ we define $\pi_{\xi,\rho}(f)$ as the restriction of $f(\xi)$ to the $\rho$-isotypical component of $E_x$.

Recall the definition of $X_{M,\Gamma}$ in (7.2) as the set of *all* pairs $(\xi, \rho)$ such that $\xi \in T^*M \setminus \{0\}$ and $\rho \in \widehat{\Gamma}_\xi$. If $(\xi, \rho) \in X_{M,\Gamma} \setminus \tilde{X}_{M,\Gamma}$, then $E_{x\rho} = 0$, hence $\pi_{\xi,\rho}$ is the null representation. Thus $\mathrm{Prim}(A_M^\Gamma)$ may be identified with the (possibly larger) set $X_{M,\Gamma}/\Gamma$, modulo some null representations on trivial vector spaces. The latter make no difference from the point of view of characterizing invertible elements of $A_M^\Gamma$, since the null operator on a trivial vector space is always invertible.

### 7.4.2. Factoring out the minimal isotropy

Recall that we are assuming $M/\Gamma$ to be connected; in that case there is a minimal isotropy type for the action of $\Gamma$. We shall also assume from now on that $\Gamma$ is abelian, for the reasons discussed in the Introduction. In particular, it is the case needed for our applications to boundary value problems and, moreover, some results are not true in the non-abelian case.

Let $\alpha \in \widehat{\Gamma}$ as before, and recall that we want to determine the structure of the quotient $\pi_\alpha(\overline{\Psi^0}(M;E)^\Gamma)/\pi_\alpha(\overline{\Psi^{-1}}(M;E)^\Gamma)$ of the restricted algebras to the $\alpha$-isotypical component. To this end, recall the morphism

$$\mathcal{R}_M : A_M^\Gamma := C_0(S^*M;\mathrm{End}(E))^\Gamma \to \pi_\alpha(\overline{\Psi^0}(M;E)^\Gamma)/\pi_\alpha(\overline{\Psi^{-1}}(M;E)^\Gamma)$$

of Equation (7.25). The main result of this subsection is to determine the kernel of this morphism.

The main reason why the abelian case is simpler than the general case is that in the abelian case all minimal isotropy subgroups of $\Gamma$ acting on $M$ coincide. The (unique) minimal isotropy subgroup of $\Gamma$ acting on $M$ will be denoted by $\Gamma_0$, as before. Recall that the set $M_0 := M_{(\Gamma_0)}$ of points $x \in M$ with isotropy $\Gamma_x = \Gamma_0$ is called the principal orbit bundle of $M$; it is a dense, open subset of $M$. For every (other) $x \in M$, we have $\Gamma_0 \subset \Gamma_x$.

We obtain that the group $\Gamma_0$ acts trivially on $M$. Moreover, there exists a unitary group morphism (representation) $\Gamma_0 \to \mathrm{End}(E)$ that implements the action of $\Gamma_0$ on $\overline{\Psi^0}(M;E)$. Let $p_\beta^{(0)} \in C^*(\Gamma_0)$, $\beta \in \widehat{\Gamma}_0$, be the central projectors associated to the irreducible representations of $\Gamma_0$ (the additional exponent is to differentiate them from the projectors $p_\alpha$, $\alpha \in \widehat{\Gamma}$). Corollary 7.3.7 then gives the exact sequence

$$0 \to p_\beta^{(0)}\overline{\Psi^{-1}}(M;E)^{\Gamma_0} \to p_\beta^{(0)}\overline{\Psi^0}(M;E)^{\Gamma_0} \to C_0(S^*M;\mathrm{End}(p_\beta^{(0)}E))^{\Gamma_0} \to 0\,.$$

Moreover,

$$\overline{\Psi^0}(M;E)^{\Gamma_0} \;\simeq\; \oplus_{\beta\in\widehat{\Gamma}_0}\, p_\beta^{(0)}\overline{\Psi^0}(M;E)^{\Gamma_0}\,.$$

Here the direct sum is finite, so there is no need to include the "$c_0$"-specification like in Corollary 7.3.7. Since the actions of $\Gamma$ and $\Gamma_0$ commute, we can further take the $\Gamma$-invariants to obtain:

$$0 \to p_\beta^{(0)}\overline{\Psi^{-1}}(M;E)^{\Gamma} \to p_\beta^{(0)}\overline{\Psi^0}(M;E)^{\Gamma} \to C_0(S^*M;\mathrm{End}(p_\beta^{(0)}E))^{\Gamma} \to 0\,. \tag{7.28}$$

Moreover,

$$\overline{\Psi^0}(M;E)^{\Gamma} \simeq \bigoplus_{\beta\in\widehat{\Gamma}} p_\beta^{(0)}\overline{\Psi^0}(M;E)^{\Gamma}\,. \tag{7.29}$$

In particular, we have that

**Lemma 7.4.4.** *Let $\Gamma$ be a finite abelian group and $E \to M$ a $\Gamma$-equivariant vector bundle over a smooth, compact, connected manifold $M$ (thus without boundary). Let $\Gamma_0 \subset \Gamma$ be the minimal isotropy group. We have*

$$C_0(S^*M;\mathrm{End}(E))^\Gamma \simeq \bigoplus_{\beta\in\widehat{\Gamma}_0} C_0(S^*M;\mathrm{End}(p_\beta^{(0)}E))^\Gamma$$

*Proof.* We successively have

$$\begin{aligned} C_0(S^*M;\mathrm{End}(E))^\Gamma &\simeq \left(C_0(S^*M;\mathrm{End}(E))^{\Gamma_0}\right)^{\Gamma/\Gamma_0} \simeq \left(C_0(S^*M;\mathrm{End}(E)^{\Gamma_0})\right)^{\Gamma/\Gamma_0} \\ &\simeq \bigoplus_{\beta\in\widehat{\Gamma}_0} \left(C_0(S^*M;\mathrm{End}(p_\beta^{(0)}E)^{\Gamma_0})\right)^{\Gamma/\Gamma_0} \\ &\simeq \bigoplus_{\beta\in\widehat{\Gamma}_0} C_0(S^*M;\mathrm{End}(p_\beta^{(0)}E))^\Gamma, \end{aligned}$$

where we have used that $\mathrm{Hom}(p_\beta^{(0)}E, p_{\beta'}^{(0)}E)^{\Gamma_0}$ for $\beta \neq \beta' \in \widehat{\Gamma}_0$. ☐

Let us record now the following corollary of Proposition 7.3.11.

**Corollary 7.4.5.** *Let $\Gamma$ be a finite abelian group acting on a smooth, connected compact manifold $M$ (without boundary). Let $E \to M$ be an equivariant vector bundle. Assume that minimal isotropy is trivial: $\Gamma_0 = 1$. Then the map*

$$\mathcal{R}_M : C_0(S^*M;\mathrm{End}(E))^\Gamma \to \pi_\alpha(\overline{\Psi^0}(M;E)^\Gamma)/\pi_\alpha(\overline{\Psi^{-1}}(M;E)^\Gamma)$$

*of Equation* (7.25) *is injective, and hence an isomorphism of algebras.*

*Proof.* By replacing the action $\pi$ of $\Gamma$ on $E$ with $\pi_0 := \pi\alpha^{-1}$, that is, with, $\pi_0(g)\xi := \alpha^{-1}(g)\pi(g)\xi$, we can assume that $\alpha = 1$. The action of $\Gamma$ is moreover free on the dense open subset $M_{(1)} = M_{\Gamma_0}$ of $M$. Proposition 7.3.11 then allows us to conclude. ☐

We now turn to the main result of this section.

**Theorem 7.4.6.** *Let $\Gamma$ be a finite abelian group acting on a smooth compact, connected manifold $M$ (without boundary) and let $E \to M$ be a $\Gamma$-equivariant vector bundle. Then the kernel of the morphism*

$$\begin{aligned} \mathcal{R}_M : \oplus_{\beta\in\widehat{\Gamma}_0} C_0(S^*M;\mathrm{End}(p_\beta^{(0)}E))^\Gamma \ \simeq\ C_0(S^*M;\mathrm{End}(E))^\Gamma \ =: A_M^\Gamma \\ \to\ \pi_\alpha(\overline{\Psi^0}(O;E)^\Gamma)/\pi_\alpha(\overline{\Psi^{-1}}(M;E)^\Gamma) \end{aligned}$$

*of Equation* (7.25) *is* $\bigoplus_{\beta\in\widehat{\Gamma}_0,\beta\neq\alpha'} C_0(S^*M;\mathrm{End}(p_\beta^{(0)}E))^\Gamma$*, where $\alpha' := \alpha|_{\Gamma_0}$. In particular,* $\mathcal{R}_M(C_0(S^*M;\mathrm{End}(E))^\Gamma) \simeq C_0(S^*M;\mathrm{End}(p_{\alpha'}^{(0)}E))^\Gamma$.

*Proof.* It is enough to identify the action of $\mathcal{R}_M$ on each one of the direct summands $C_0(S^*M;\mathrm{End}(p_\beta^{(0)}E))^\Gamma$ of $C_0(S^*M;\mathrm{End}(E))^\Gamma$. We can thus study the action of the morphism $\mathcal{R}_M$ one isotypical component $\beta \in \widehat{\Gamma}_0$ at a time.

The relation between the central projectors of $C^*(\Gamma_0)$ and $C^*(\Gamma)$ is that $p_\beta^{(0)} = \sum_{\alpha|_{\Gamma_0}=\beta} p_\alpha$, $\beta \in \widehat{\Gamma}_0$. Of course, $p_\alpha p_{\alpha'} = 0$ if $\alpha \neq \alpha' \in \widehat{\Gamma}$. It follows that

$$\pi_\alpha(p_\beta^{(0)}P) = p_\alpha p_\beta^{(0)} P|_{L^2(M;E)_\alpha} = \begin{cases} p_\alpha P|_{L^2(M;E)_\alpha} & \text{if } \alpha|_{\Gamma_0} = \beta \\ 0 & \text{otherwise.} \end{cases}$$

This shows that $\mathcal{R}_M = 0$ on $C_0(S^*M; \mathrm{End}(p_\beta^{(0)}E))^\Gamma$ if $\alpha|_{\Gamma_0} \neq \beta$.

On the other hand, for $\alpha|_{\Gamma_0} = \beta$, we shall show that $\mathcal{R}_M$ is an isomorphism on $C_0(S^*M; \mathrm{End}(p_\beta^{(0)}E))^\Gamma$. By replacing the action $\pi$ of $\Gamma$ on $E$ with $\pi_0 := \alpha^{-1}\pi$, that is, with, $\pi_0(g)\xi := \alpha^{-1}(g)\pi(g)\xi$, we can assume that $\Gamma_0$ acts trivially on $E_\beta = p_\beta^{(0)}E$ (we already know that $\Gamma_0$ acts trivially on $M$). We can then factor the action of $\Gamma$ to an action of $\Gamma/\Gamma_0$, and thus assume that the minimal isotropy is trivial: $\Gamma_0 = 1$.

After these reductions, the orbit bundle of $M$ is $M_0 := M_{(1)}$, and the action of $\Gamma$ on $M_0$ is free (and proper since $\Gamma$ is compact). Corollary 7.4.5 then shows that $\mathcal{R}_M$ is injective on $C_0(S^*M; \mathrm{End}(p_\beta^{(0)}E))^\Gamma$. □

We note that the component of $\sigma_m(P)$ in $C_0(S^*M; \mathrm{End}(p_\beta^{(0)}E))^\Gamma$ is $\sigma_m^\alpha(P)$, for any $\alpha \in \widehat{\Gamma}$ such that $\alpha_{|\Gamma_0} = \beta$; in other words, the restriction of $\sigma_0^\Gamma(P)$ to $X_{M,\Gamma}^\alpha$. In this regard, we notice that

$$\begin{cases} X_{M,\Gamma}^\alpha = X_{M,\Gamma}^{\alpha'} & \text{if } \alpha|_{\Gamma_0} = \alpha'|_{\Gamma_0} \\ X_{M,\Gamma}^\alpha \cap X_{M,\Gamma}^{\alpha'} = \emptyset & \text{otherwise.} \end{cases}$$

Let us denote $X_{M,\Gamma}^\beta := X_{M,\Gamma}^\alpha$ if $\alpha|_{\Gamma_0} = \beta$. This gives the *disjoint union* decomposition

$$X_{M,\Gamma} = \bigsqcup_{\beta \in \widehat{\Gamma}_0} X_{M,\Gamma}^\beta.$$

It would be interesting to establish an analogous relation in the non-abelian case.

## 7.5. Applications and extensions

We now prove the main result of the chapter on the characterization of Fredholm operators and discuss some extensions of our results.

### 7.5.1. Fredholm conditions

We now turn to the proof of our main result. We assume that $M$ is a compact smooth manifold. We have the following $\Gamma$-equivariant version of Atkinson's theorem.

**Proposition 7.5.1.** *Let $V$ be a unitary $\Gamma$-module and $P$ be a $\Gamma$-equivariant bounded operator on $V$. We have that $P$ is Fredholm if, and only if, it is invertible modulo $\mathcal{K}(V)^\Gamma$, in which case, we can choose the parametrix (i.e. the inverse modulo the compacts) to also be $\Gamma$-invariant.*

*Proof.* This follows from the inclusion of $C^*$-algebras

$$\mathcal{L}(V)^\Gamma/\mathcal{K}(V)^\Gamma \subset \mathcal{L}(V)/\mathcal{K}(V).$$

It is a standard fact that, if $B \subset A$ is an inclusion of unital $C^*$-algebras, then an element $a \in B$ is invertible in $A$ if, and only if, it is invertible in $B$ [75, Proposition 1.3.10].

Therefore if $P \in \mathcal{L}(V)^\Gamma$, then its projection in $\mathcal{L}(V)^\Gamma/\mathcal{K}(V)^\Gamma$ is invertible if, and only if, it is invertible in the greater algebra $\mathcal{L}(V)/\mathcal{K}(V)$. By Atkinson's theorem, the latter is equivalent to $P$ being Fredholm. $\square$

Since $\pi_\alpha(\mathcal{K}^\Gamma) = \pi_\alpha(\mathcal{K}(L^2(M;E)_\alpha))^\Gamma$ and $\overline{\Psi^{-1}}(M;E) = \mathcal{K} := \mathcal{K}(L^2(M;E)_\alpha)$, we obtain the following corollary.

**Corollary 7.5.2.** *Let $P \in \overline{\Psi^0}(M;E)^\Gamma$ and $\alpha \in \hat{\Gamma}$. We have that $\pi_\alpha(P)$ is Fredholm on $L^2(M;E)_\alpha$ if, and only if, $\pi_\alpha(P)$ is invertible modulo $\pi_\alpha(\mathcal{K}^\Gamma)$ in $\pi_\alpha(\overline{\Psi^0}(M;E)^\Gamma)$.*

We are now in a position to prove the main result of this chapter, Theorem 7.1.2.

*Proof ot Theorem 7.1.2.* The standard argument discussed in Section 4.2.3 implies that we may assume $P \in \overline{\Psi^0}(M;E)^\Gamma$. Corollary 7.5.2 then states that $\pi_\alpha(P)$ is Fredholm if, and only if, the image of its principal symbol $\sigma_0(P)$ is invertible in the quotient algebra

$$\mathcal{R}_M(A_M^\Gamma) = \pi_\alpha(\overline{\Psi^0}(M;E)^\Gamma)/\pi_\alpha(\mathcal{K}^\Gamma).$$

We have shown the isomorphism $\mathrm{Prim}(A_M^\Gamma) \simeq \tilde{X}_{M,\Gamma}/\Gamma$ in Equation (7.27). Moreover, Remark 7.4.3 states that the latter set identifies with $X_{M,\Gamma}/\Gamma$, modulo some null representations. Now Theorem 7.4.6 and the discussion following it identify the primitive spectrum of $\mathcal{R}_M(A_M^\Gamma)$, which is a closed subset of $\mathrm{Prim}(A_M^\Gamma)$, with the set $X_{M,\Gamma}^\alpha/\Gamma$. Recall that

$$X_{M,\Gamma}^\alpha = \{(\xi,\rho) \in T^*M \setminus \{0\} \times \widehat{\Gamma}_\xi \mid \rho_{|\Gamma_0} = \alpha_{|\Gamma_0}\},$$

as defined in the introduction. We know that an element of a unital $C^*$-algebra is invertible if, and only if, $\pi(a)$ is invertible for any irreducible representation $\pi$ of $A$. Therefore $\mathcal{R}_M(\sigma(P))$ is invertible if, and only if, the endomorphism $\pi_{\xi,\rho}(\sigma(P))$ is invertible for all $(\xi,\rho) \in X_{M,\Gamma}^\alpha$, i.e. if and only if $P$ is $\alpha$-elliptic. $\square$

### 7.5.2. Boundary value problems

In this subsection, we very briefly indicate an application to mixed boundary value problems. Let $M$ be a smooth compact manifold with boundary and choose a tubular neighborhood $U \simeq [0,1) \times \partial M$ of the boundary. Let $M^d$ be the *double* of $M$ along $\partial M$: as a topological space, the space $M^d$ is the quotient of $M \times \{-1,1\}$ by the subspace $\partial M \times \{-1,1\}$. We shall denote $M_\pm := M \times \{\pm 1\}$. On $M^d$ we consider the smooth structure such that

(i) the projections $p_\pm : M_\pm \to M$ are smooth maps, and
(ii) the map $U^d \simeq (0,1) \times \partial M$ is smooth.

Thus the smooth structure on $M^d$ thus depends on our choice of tubular neighborhood. For any $x = (x', i) \in M^d$, we denote by $-x$ its symmetrical counterpart, i.e. $-x = (x', -i)$. Then the map $x \mapsto -x$ gives a natural smooth action of $\mathbb{Z}_2$ on $M^d$.

If $E \to M$ is a smooth vector bundle, then we define $E^d \to M^d$ as the smooth vector bundle obtained by gluing two copies of $E$ on $M_+$ and $M_-$ along $\partial M$. Then the $\mathbb{Z}_2$-action on $M^d$ extends to an action on $E^d$, which maps an element $v \in E^d_x$ to its copy in $E^d_{-x}$, for any $x \in M^d$.

We generalize this construction to the case when we have a disjoint union decomposition of the boundary $\partial M = \partial_D M \cup \partial_N M$ into two disjoint, closed and open subsets. Then we double first with respect to the "Dirichlet" part of the boundary $\partial_D M$ and then with respect to the "Neumann" part of the boundary $\partial_N M$. We obtain accordingly an action of $\mathbb{Z}_2^2$ on the resulting manifold $M^{dd}$. We let this group act on the resulting vector bundle $E^{dd}$ such that the action of the first component of $\mathbb{Z}_2$ is twisted (i.e. tensored) with its only non-trivial character, namely $-1$. We have the following standard lemma.

**Lemma 7.5.3.** *The restriction map $r_+ : C^\infty(M^{dd}; E^{dd}) \to C^\infty(M_+; E)$ induces a isomorphisms*

*(i)* $L^2(M^{dd}; E^{dd})^{\mathbb{Z}_2^2} \simeq L^2(M; E)$,

*(ii)* $H^2(M^{dd}; E^{dd})^{\mathbb{Z}_2^2} \simeq H^2(M; E) \cap \{\, u|_{\partial_D M} = 0,\ \partial_\nu u|_{\partial_N M} = 0 \,\}$.

An order-2, $\mathbb{Z}_2^2$-invariant pseudodifferential operator $P$ on $M^{dd}$ will map invariant sections to invariant sections; this means that we consider the case $\alpha = 1$ in Theorem 7.1.2. Because the action of $\mathbb{Z}_2^2$ is free on a dense subset of $M^{dd}$, Theorem 7.1.2 implies that $P$ is Fredholm from $H^2(M^{dd}; E^{dd})^{\mathbb{Z}_2^2}$ to $L^2(M^{dd}; E^{dd})^{\mathbb{Z}_2^2}$ if, and only if, it is elliptic. This then yields Fredholm conditions for the restriction of $P$ to $M$, with mixed Dirichlet/Neumann boundary conditions on $\partial_D M$ and $\partial_N M$.

### 7.5.3. The case of non-discrete groups

If $\Gamma$ is not discrete, then it is enough for our operators to be *transversally* elliptic. Indeed, let us assume that $M$ is a compact smooth manifold and that $\Gamma$ is a compact Lie group acting on $M$. Denote by $\mathfrak{g}$ the Lie algebra of $\Gamma$. Recall that any $X \in \mathfrak{g}$ defines as usual the vector field $X^*_M$ given by $X^*_M(m) = \frac{d}{dt}_{|t=0} e^{tX} \cdot m$. Let first introduce the $\Gamma$-transversal space

$$T^*_\Gamma M := \{\alpha \in T^*M \mid \alpha(X^*_M(\pi(\alpha))) = 0, \forall X \in \mathfrak{g}\}.$$

A $\Gamma$-invariant classical pseudodifferential operator $P$ is said to be $\Gamma$-*transversally elliptic* if its principal symbol is invertible on $T^*_\Gamma M \setminus \{0\}$. Let $P \in \Psi^m(M; E_0, E_1)$ be $\Gamma$-transversally elliptic. Recall the now classical result of Atiyah and Singer [12, Corollary 2.5]

**Theorem 7.5.4.** *Assume $P$ is $\Gamma$-transversally elliptic. Then for every irreducible representation $\alpha \in \widehat{\Gamma}$,*

$$\pi_\alpha(P) : H^s(M; E_0)_\alpha \to H^{s-m}(M; E_1)_\alpha,$$

*is Fredholm.*

Note that this implies that Theorem 7.1.2 is not true anymore for $\Gamma$-transversally elliptic operators if $\Gamma$ is non-discrete.

# 8. The general case

The current chapter is adapted from the paper "Fredholm conditions and index for restrictions of invariant pseudodifferential operators to isotypical components" [25], currently a preprint, and which is a joint work with Alexandre Baldare, Matthias Lesch and Victor Nistor.

## 8.1. Introduction

As for the previous chapters, we refer to Section 2.2.4 for a detailed introduction and only recall the main goals and results here.

### 8.1.1. Short introduction and notations

We pursue in this chapter the work initiated in Chapter 7. Thus, we consider again a smooth, closed manifold $M$ and a *finite* group $\Gamma$ acting on $M$ by diffeomorphisms. We assume given two $\Gamma$-vector bundles $E, F \to M$, and let $P$ be a $\Gamma$-invariant pseudodifferential operator acting from $E$ to $F$. Given an irreducible representation $\alpha \in \widehat{\Gamma}$, we consider as before the restriction

$$\pi_\alpha(P) : H^s(M; E)_\alpha \longrightarrow H^{s-m}(M; F)_\alpha \tag{8.1}$$

between the corresponding $\alpha$-isotypical components, for any $s \in \mathbb{R}$, with $m$ the order of $P$. Our aim is to answer the same question as in the previous chapter: when is $\pi_\alpha(P)$ Fredholm? This time, we focus on the general case (i.e. non necessarily abelian), for which the proof differs significatively from the abelian case.

Recall from Chapter 7 that we introduced the space

$$X_{M,\Gamma} := \{(\xi, \rho) \mid \xi \in T^*M \setminus \{0\} \text{ and } \rho \in \widehat{\Gamma}_\xi\}, \tag{8.2}$$

whose quotient $X_{M,\Gamma}/\Gamma$ identifies with the primitive spectrum of the $C^*$-algebra of equivariant symbols, modulo some null representations (see Remarks 7.4.3 and 8.2.21). The space $X_{M,\Gamma}$ was used to define the $\Gamma$-*principal symbol* of $P$ as

$$\sigma_m^\Gamma(P)(\xi, \rho) := \pi_\rho(\sigma_m(P)) \in \operatorname{Hom}(E_{x\rho}, F_{x\rho})^{\Gamma_\xi}, \tag{8.3}$$

for any $(\xi, \rho) \in X_{M,\Gamma}$, and where $\pi_\rho$ is the restriction as a map between the isotypical components of $E_x$ and $F_x$ associated with $\rho$, as in Equation (8.1).

We assume for simplicity that the orbit space $M/\Gamma$ is *connected* and denote by $\Gamma_0$ a minimal isotropy subgroup, see Section 7.2.3. As in Chapter 7, we need to select those pairs $(\xi, \rho) \in X_{M,\Gamma}$ for which $\rho$ is $\Gamma_0$-associated to $\alpha$, in the following sense.

**Definition 8.1.1.** Let $\Gamma_1$ and $\Gamma_2$ be finite groups and let $H$ a subgroup of both $\Gamma_1$ and $\Gamma_2$. Let $\rho_1 \in \widehat{\Gamma}_1$ and $\rho_2 \in \widehat{\Gamma}_2$. We say that $\rho_1$ and $\rho_2$ are *$H$-disjoint* if $\mathrm{Hom}_H(\alpha, \beta) = 0$. Otherwise we say that they are *$H$-associated* (to each other).

Put otherwise, the representations $\rho_1$ and $\rho_2$ are $H$-associated if, and only if, there is a term $\beta \in \widehat{H}$ that appears both of the isotypical decompositions of $\rho_{1|H}$ and $\rho_{2|H}$, as representations of $H$.

Recall now that $\Gamma_{g\xi} = g\Gamma_\xi g^{-1}$ for any $\xi \in T^*M$ and $g \in \Gamma$, and that this defines an action of $\Gamma$ on the set of stabilizer subgroups $\mathrm{Stab}_\Gamma(T^*M) := \{\Gamma_\xi \mid \xi \in T^*M\}$ given by $g \cdot \Gamma_\xi = \Gamma_{g\xi}$. For $\rho \in \widehat{\Gamma}_\xi$, define $g \cdot \rho \in \widehat{\Gamma}_{g\xi}$ by $(g \cdot \rho)(h) = \rho(g^{-1}hg)$, for all $h \in \Gamma_{g\xi}$. We then introduce

$$X^\alpha_{M,\Gamma} := \{(\xi, \rho) \in X_{M,\Gamma} \mid \exists g \in \Gamma,\ \mathrm{Hom}_{\Gamma_0}(g \cdot \rho, \alpha) \neq 0\ \}\,. \tag{8.4}$$

Note that it is implicit in the definition of $X^\alpha_{M,\Gamma}$ that $\Gamma_0 \subset \Gamma_{g\xi} = g \cdot \Gamma_\xi$. When the groups $\Gamma_1$ and $\Gamma_2$ of Definition 8.1.1 are both abelian, then the characters $\rho_1$ and $\rho_2$ are $H$-associated if, and only if, we have $\rho_{1|H} = \rho_{2|H}$. Thus the definition of $X^\alpha_{M,\Gamma}$ above is really a generalization of the one given by Equation (7.4), and the definition of the *$\alpha$-principal symbol* follows analogously.

**Definition 8.1.2.** The *$\alpha$-principal symbol* $\sigma^\alpha_m(P)$ of $P$ is the restriction of the $\Gamma$-*principal symbol* $\sigma^\Gamma_m(P)$ to $X^\alpha_{M,\Gamma}$:

$$\sigma^\alpha_m(P) := \sigma^\Gamma_m(P)|_{X^\alpha_{M,\Gamma}}\,.$$

We say that $P \in \Psi^m(M; E, F)^\Gamma$ is *$\alpha$-elliptic* if its $\alpha$-principal symbol $\sigma^\alpha_m(P)$ is invertible everywhere on its domain of definition.

Note that, as in Chapter 7, when $(\xi, \rho) \in X^\alpha_{M,\Gamma}$ is such that $E_{x\rho} = F_{x\rho} = 0$, then the null operator $\sigma^\Gamma_m(P)(\xi, \rho) : 0 \to 0$ is always invertible. We explain in Section 8.3.1 how the previous definition needs to be modified when $M/\Gamma$ is not connected. The main result of the present chapter is the following theorem.

**Theorem 8.1.3.** *Let $\Gamma$ be a finite group acting on a smooth, compact manifold $M$ and let $P \in \Psi^m(M; E, F)^\Gamma$ be a $\Gamma$-invariant classical pseudodifferential operator acting between sections of two $\Gamma$-equivariant bundles. Let moreover $\alpha \in \widehat{\Gamma}$. For any $s \in \mathbb{R}$, we have that*

$$\pi_\alpha(P) : H^s(M; E)_\alpha \longrightarrow H^{s-m}(M; F)_\alpha$$

*is Fredholm if, and only if, $P$ is $\alpha$-elliptic.*

We discuss in the present Chapter the connections with the index theory of $\Gamma$-equivariant operators, see Section 8.3.2. Moreover, we give the following reformulation of the $\alpha$-ellipticity condition in terms of the fixed-points manifold $T^*M^{\Gamma_0}$, following a suggestion of Paul-Émile Paradan.

**Proposition 8.1.4.** *If $P \in \Psi^m(M; E, F)^\Gamma$ and $\alpha \in \widehat{\Gamma}$, then the following are equivalent:*

*(i) $P$ is $\alpha$-elliptic,*

*(ii) the principal symbol $\sigma_m(P)$ defines by restriction an invertible element*

$$(\sigma_m(P) \otimes 1)|_{(E\otimes\alpha)^{\Gamma_0}} \in C^\infty(T^*M^{\Gamma_0}; (E\otimes\alpha)^{\Gamma_0}).$$

Such fixed-points manifolds appear notably in the formulation of Atiyah and Singer's equivariant index theorem [12, 30, 155].

### 8.1.2. Contents of the chapter

Although some results from Chapter 7 remain true in the general case, the main part of the proof relies on some completely different arguments. As before, we will denote by $A_M^\Gamma := C(S^*M; \mathrm{End}(E))^\Gamma$ the $C^*$-algebra of symbols of $\Gamma$-invariant pseudodifferential operators. In Chapter 7, we identified the primitive spectrum of $A_M^\Gamma$ with the set $X_{M,\Gamma}/\Gamma$ described above.

The most substantial technical results are in Section 8.2. There, we carefully describe the corresponding topology on $X_{M,\Gamma}/\Gamma$. We then consider the canonical map from $A_M^\Gamma$ to the Calkin algebra of $L^2(M; E)_\alpha$ and show that the closed subset of $\mathrm{Prim}(A_M^\Gamma)$ associated to its kernel is $X_{M,\Gamma}^\alpha/\Gamma$.

These descriptions are used in Section 8.3 to prove the main result of the chapter, Theorem 8.1.3. We explain the consequences for the index theory of equivariant operators, and prove the equivalent formulation of the $\alpha$-ellipticity condition stated in Proposition 8.1.4 This section also addresses some particular cases of the Theorem, gives a few examples, and explains the relation with previously known results, namely:

- the particular formulation in the abelian case, which was established in Chapter 7,
- Fredholm conditions for transversally elliptic operators when the group $\Gamma$ is not discrete,
- Simonenko's local principle for Fredholm operators.

## 8.2. The principal symbol

As discussed above, we assume that $M$ is a smooth, compact manifold acted upon by a finite group $\Gamma$ and that $M/\Gamma$ is connected. We refer to Section 7.2 for most of the preliminaries needed for the current chapter.

Let us fix an irreducible representation $\alpha$ of $\Gamma$ and consider the restriction morphism $\pi_\alpha$ to the $\alpha$-isotypical component of $L^2(M; E)$. Recall that this morphism was first introduced in Equation (7.1) and discussed in detail in Section 7.2.1. As in Chapter 7, we now turn to the identification of the quotient

$$\pi_\alpha(\overline{\Psi^0}(M;E)^\Gamma)/\pi_\alpha(\overline{\Psi^{-1}}(M;E)^\Gamma) = \pi_\alpha(\overline{\Psi^0}(M;E)^\Gamma)/\mathcal{K}(L^2(M;E)_\alpha)^\Gamma$$

The methods used in this chapter diverge, however, drastically from those of the previous one.

Since $\pi_\alpha(\overline{\Psi^{-1}}(M;E)^\Gamma)$ was identified in the previous section with the algebra of $\Gamma$-invariant compact operators on $L^2(M;E)_\alpha$ , the promised identification of the quotient

$\pi_\alpha(\overline{\Psi^0}(M;E)^\Gamma)/\pi_\alpha(\overline{\Psi^{-1}}(M;E)^\Gamma)$ will give further insight into the structure of the algebra $\pi_\alpha(\overline{\Psi^0}(M;E)^\Gamma)$ and will provide us, eventually, with Fredholm conditions. Recall that, in this chapter, we are assuming $\Gamma$ to be finite. Nevertheless, a several intermediate results hold also in the case $\Gamma$ compact.

### 8.2.1. The primitive ideal spectrum of $A_M^\Gamma$

As before, $S^*M$ denotes the unit cosphere bundle of $M$. For the simplicity of the notation, we shall write

$$A_M := C(S^*M; \mathrm{End}(E)),$$

as in the Introduction. Recall from Section 7.2.5 that we have an algebra isomorphism

$$\overline{\Psi^0}(M;E)^\Gamma/\overline{\Psi^{-1}}(M;E)^\Gamma \simeq A_M^\Gamma. \tag{8.5}$$

In our case, the inclusion $j : C(S^*M/\Gamma) \subset A_M^\Gamma$ as a central subalgebra induces, as in Equation (7.21), a central character map

$$\phi^* : \mathrm{Prim}(A_M^\Gamma) \to S^*M/\Gamma,$$

that underscores the local nature of the structure of the primitive ideal spectrum of $A_M^\Gamma$. For any $\xi \in S_x^*M$ and $\rho \in \widehat{\Gamma}_\xi$, we introduce again the representation $\pi_{\xi,\rho}$ defined for any $f \in A_M^\Gamma$ by

$$\pi_{\xi,\rho}(f) = \pi_\rho(f(\xi)),$$

that is $\pi_{\xi,\rho}(f)$ is the restriction of $f(\xi) \in \mathrm{End}(E_x)$ to the $\rho$-isotypical component of $E_x$. The central character map $\phi^*$ was used in Corollary 7.4.2 to obtain the following identification of $\mathrm{Prim}(A_M^\Gamma)$.

**Proposition 8.2.1.** *Let $\tilde{X}_{M,\Gamma}$ be the set of pairs $(\xi, \rho)$, where $\xi \in S_x^*M$, $x \in M$, and $\rho \in \widehat{\Gamma}_\xi$ appears in $E_x$ (i.e. $\mathrm{Hom}_{\Gamma_\xi}(\rho, E_x) \neq 0$).*

*(i) The map $\tilde{X}_{M,\Gamma}/\Gamma \ni \Gamma(\xi,\rho) \mapsto \ker(\pi_{\xi,\rho}) \in \mathrm{Prim}(A_M^\Gamma)$ is bijective.*

*(ii) The central character map $\tilde{X}_{M,\Gamma}/\Gamma \simeq \mathrm{Prim}(A_M^\Gamma) \to S^*M/\Gamma$ maps $\Gamma(\xi,\rho) \in \tilde{X}_{M,\Gamma}/\Gamma$ to $\Gamma\xi$ and is continuous and finite-to-one.*

The space $\mathrm{Prim}(A_M^\Gamma)$ is endowed with the Jacobson topology, whose definition is recalled in Section 4.1.2; thus Proposition 8.2.1 allows us to obtain a topology on $\tilde{X}_{M,\Gamma}/\Gamma$, which together with the central character map $\phi^*$ will play a crucial role in what follows. We thus now turn to the study of this topology on $\tilde{X}_{M,\Gamma}/\Gamma$. We begin with the following standard lemma.

**Lemma 8.2.2.** *Let $A$ be a $C^*$-algebra. The family $(V_a)_{a\in A}$ defined by*

$$V_a = \{J \in \mathrm{Prim}\, A \mid a \notin J\},$$

*for any $a \in A$, is a basis of open sets for* $\mathrm{Prim}(A)$.

*Proof.* Following [75], we know that the open, non-empty subsets of $\mathrm{Prim}(A)$ are exactly the sets

$$\{J \in \mathrm{Prim}(A) \mid I \not\subset J\} \simeq \mathrm{Prim}(I)$$

where $I$ ranges through the closed, non-zero, two-sided ideals of $A$. If $a \in A$, let us denote by $I_a := \overline{AaA}$ the closed, two-sided ideal generated by $a$. Then $a \notin J \Leftrightarrow I_a \not\subset J$, and hence $V_a = \mathrm{Prim}(I_a)$. This shows that $V_a$ is open.

Next, let $V \subset \mathrm{Prim}(A)$ be a non-empty open subset and $J_0 \in V$. We know then that there exists a closed, two-sided ideal $I$, $0 \neq I \subset A$, such that $V = \mathrm{Prim}(I)$. We have $I \not\subset J_0$, and hence we can choose $a \in I \setminus J_0$. If $J \subset A$ is a primitive ideal such that $a \notin J$, then *a fortiori* $I \not\subset J$. Therefore $V_a \subset \mathrm{Prim}(I)$. This shows that $J_0 \in V_a \subset V$. Therefore the family $(V_a)_{a\in A}$ is a basis for the topology on $\mathrm{Prim}(A)$. □

We shall use the bijection of Proposition 8.2.1 to conclude the following.

**Corollary 8.2.3.** *A basis for the induced topology on $\tilde{X}_{M,\Gamma}/\Gamma \simeq \mathrm{Prim}(A_M^\Gamma)$ is given by the sets*

$$V_f := \{\, \Gamma(\xi, \rho) \in \tilde{X}_{M,\Gamma}/\Gamma \ \mid \ \pi_{\xi,\rho}(f) \neq 0 \,\},$$

*where $f$ ranges through the non-zero elements of $A_M^\Gamma$.*

### 8.2.2. The restriction morphisms

Let $\mathcal{O} \subset M$ be an open subset. Then $S^*\mathcal{O}$ is the restriction of $S^*M$ to $\mathcal{O}$. We shall need the algebras

$$A_{\mathcal{O}} := C_0(S^*\mathcal{O}; \mathrm{End}(E)) \quad \text{and} \quad B_{\mathcal{O}} := \overline{\Psi^0}(\mathcal{O}; E)\,. \tag{8.6}$$

Assume that $\mathcal{O} \subset M$ is $\Gamma$-invariant. The group $\Gamma$ does not act, in general, as multipliers on the $C^*$-algebra $B_{\mathcal{O}} := \overline{\Psi^0}(\mathcal{O}; E)$ (it does however act by conjugation), so the method used in Chapter 7 to compute $\overline{\Psi^{-1}}(\mathcal{O}; E)^\Gamma \simeq \mathcal{K}(L^2(\mathcal{O}; E))^\Gamma$ does not extend to compute $B_{\mathcal{O}}^\Gamma$. We shall thus consider the natural, surjective map

$$\begin{aligned}\mathcal{R}_{\mathcal{O}} \,:\, A_{\mathcal{O}}^\Gamma := C_0(S^*\mathcal{O}; \mathrm{End}(E))^\Gamma \,\simeq\, & B_{\mathcal{O}}^\Gamma / \overline{\Psi^{-1}}(\mathcal{O}; E)^\Gamma \\ & \to \pi_\alpha(B_{\mathcal{O}}^\Gamma)/\pi_\alpha(\overline{\Psi^{-1}}(\mathcal{O}; E)^\Gamma).\end{aligned} \tag{8.7}$$

Recall from Proposition 7.3.9 that $\pi_\alpha(\overline{\Psi^{-1}}(M; E)^\Gamma) = \mathcal{K}(L^2(M; E)_\alpha)^\Gamma$. Therefore, for a given $P \in \overline{\Psi^0}(M; E)$, we have that $\pi_\alpha(P)$ is Fredholm if, and only if, the principal symbol of $P$ is invertible in $A_M^\Gamma / \ker(\mathcal{R}_M)$. This will be discussed in more detail in the next section.

We shall approach the computation of $\ker(\mathcal{R}_M) \subset A_M^\Gamma$ by determining the closed subset

$$\Xi := \mathrm{Prim}(A_M^\Gamma / \ker(\mathcal{R}_M)) \,\subset\, \mathrm{Prim}(A_M^\Gamma) \tag{8.8}$$

of the primitive ideal spectrum of $A_M^\Gamma$ corresponding to $\ker(\mathcal{R}_M)$. Once we will have determined $\Xi$, we will also have determined $\ker(\mathcal{R}_M)$, in view of the definitions recalled in Section 4.1.2 that put in bijection the closed, two-sided ideals of a $C^*$-algebra with the closed subsets of its primitive ideal spectrum.

Since $C(M/\Gamma) \subset B_M$, it follows from the definition of $\mathcal{R}_M$ that it is a $C(M/\Gamma)$-module morphism, and hence that $\ker(\mathcal{R}_M)$ is a $C(M/\Gamma)$-module. Let us also recall that

$$C(M/\Gamma) = C(M)^\Gamma \subset Z_M := C(S^*M)^\Gamma \subset Z(A_M^\Gamma) \subset A_M^\Gamma \subset A_M\,.$$

The local nature of $\ker(\mathcal{R}_M)$ and of the space $\Xi$ is explained in the following remark.

**Remark 8.2.4.** Let $M/\Gamma = \cup V_k$ be an open cover and

$$\ker(\mathcal{R}_M)_{V_k} := C_0(V_k)\ker(\mathcal{R}_M) = \ker(\mathcal{R}_{V_k}).$$

If we determine each $\ker(\mathcal{R}_M)_{V_k}$, then we determine $\ker(\mathcal{R}_M)$ using a partition of unity through:

$$\ker(\mathcal{R}_M) = \sum_k{}' \phi_k \ker(\mathcal{R}_{V_k})\,, \tag{8.9}$$

where $\sum'$ refers to sums with only finitely many non-zero terms and $(\phi_k)$ is a partition of unity of $M/\Gamma$ with continuous functions subordinated to the covering $(V_k)$ (thus, in particular, $\operatorname{supp}(\phi_k) \subset V_k$). Since $M$ is compact, we can assume the covering to be finite (otherwise, we would need to take the closure of the right hand side in Equation (8.9)). To determine $\mathcal{R}_M$, we can therefore replace $M$ by any of the open sets $V_k$ in the covering and study $\ker(\mathcal{R}_{V_k})$. We shall do that for the covering of $M/\Gamma$ with the tubes $W_x \simeq \Gamma \times_{\Gamma_x} U_x$ considered in 7.2.3, see Equation (7.18).

### 8.2.3. Local calculations

In view of Remark 8.2.4, we shall concentrate now on the local structure of $\ker(\mathcal{R}_M)$, that is, on the structure of $\ker(\mathcal{R}_\mathcal{O})$ for suitable ("small") open, $\Gamma$-invariant subsets $\mathcal{O} \subset M$. Let us fix then $x \in M$ and let $W_x \simeq \Gamma \times_{\Gamma_x} U_x$ be the tube around $x$, as given by Equation (7.18). For simplicity, we shall write

$$A_x := A_{U_x} := C_0(S^*U_x; \operatorname{End}(E)) \quad\text{and}\quad Z_x := Z(A_x^{\Gamma_x})\,. \tag{8.10}$$

For these algebras, the role of $\Gamma$ will be played by $\Gamma_x$. For the statement of the following lemma, recall the definitions in Subsection (7.2.3), especially Equation (7.18).

**Lemma 8.2.5.** *Let $W_x \simeq \Gamma \times_{\Gamma_x} U_x$. Then $S^*W_x \simeq \Gamma \times_{\Gamma_x} S^*U_x$ and we have $\Gamma$-equivariant algebra isomorphisms*

$$A_{W_x} := C_0(S^*W_x; \operatorname{End}(E)) \simeq \operatorname{Ind}_{\Gamma_x}^\Gamma \left(C_0(S^*U_x; \operatorname{End}(E))\right) =: \operatorname{Ind}_{\Gamma_x}^\Gamma (A_x)\,.$$

*Consequently, the Frobenius isomorphism $\Phi$ of Equation* (7.12) *induces an isomorphism*

$$\Phi^{-1} : A_{W_x}^\Gamma \to A_x^{\Gamma_x}\,.$$

*Proof.* We have that $E|_{W_x} \simeq \Gamma \times_{\Gamma_x} (E|_{U_x})$, hence $\operatorname{End}(E)|_{W_x} \simeq \Gamma \times_{\Gamma_x} (\operatorname{End}(E)|_{U_x})$. Equation (7.20) then gives that $C_0(W_x, \operatorname{End}(E)) \simeq \operatorname{Ind}_{\Gamma_x}^\Gamma (C_0(U_x, \operatorname{End}(E)))$. The rest follows right away from the Frobenius reciprocity (more precisely, from Equation (7.12)) and from Equation (7.20), with $\beta$ replaced with $\operatorname{End}(E)$. $\square$

**Remark 8.2.6.** In view of Equation (7.12), the isomorphism $\Phi$ of Lemma 8.2.5 can be written explicitly as follows. Let $f \in A_x^{\Gamma_x}$. Then, for any equivalence class $[\gamma, \xi] := \Gamma_x(\gamma, \xi) \in \Gamma \times_{\Gamma_x} S^*U_x \simeq S^*W_x$ we have

$$\Phi(f)([\gamma, \xi]) = [\gamma, f(\xi)],$$

where $[\gamma, f(\xi)] \in \Gamma \times_{\Gamma_x} (U_x \times \mathrm{End}(E_x))^{\Gamma_x} \simeq \Gamma \times_{\Gamma_x} \mathrm{End}(E|_{U_x})^{\Gamma_x} \simeq \mathrm{End}(E|_{W_x})^{\Gamma}$. This defines $\Phi(f) \in C_0(S^*W_x; \mathrm{End}(E|_{W_x}))^{\Gamma} = A_{W_x}^{\Gamma}$.

Lemma 8.2.5 together with the following remark will allow us to reduce the study of the algebra $A_M^{\Gamma}$ to that of its analogues defined for slices.

**Remark 8.2.7.** Let $U$ be an open set of some euclidean space and $W = U \times \{1, 2, \ldots, N\}$, where the space on the second factor is endowed with the discrete topology. For simplicity, we identify $L^2(W)$ with $L^2(U)^N$ using the map $f \mapsto (f(i))_{i=1\cdots N}$. Then

$$\begin{aligned} \Psi^{-1}(W) = M_N(\Psi^{-1}(U)) \simeq \Psi^{-1}(U) \otimes M_N(\mathbb{C}) \text{ and hence} \\ \overline{\Psi^{-1}}(W) = M_N(\overline{\Psi^{-1}}(U)) \simeq \overline{\Psi^{-1}}(U) \otimes M_N(\mathbb{C})\,. \end{aligned} \tag{8.11}$$

On the other hand, if $A^N$ denotes the direct sum of $N$-copies of the algebra $A$, then we have the following inclusions of algebras

$$\begin{aligned} \Psi^0(U)^N \subset \Psi^0(W) \subset M_N(\Psi^0(U)) \simeq \Psi^0(U) \otimes M_N(\mathbb{C}), \text{ and hence} \\ \overline{\Psi^0}(U)^N \subset \overline{\Psi^0}(W) \subset M_N(\Psi^0(U)) \simeq \overline{\Psi^0}(U) \otimes M_N(\mathbb{C})\,. \end{aligned} \tag{8.12}$$

The following lemma makes explicit the group actions in the isomorphisms of the last remark. Thus, in analogy with the definitions of the algebras $A_{W_x} = C_0(S^*W_x; \mathrm{End}(E))$ and $A_x = C_0(S^*U_x; \mathrm{End}(E))$, we consider the algebras

$$B_{W_x} := \overline{\Psi^0}(W_x; E) \quad \text{and} \quad B_x := \overline{\Psi^0}(U_x; E)\,. \tag{8.13}$$

We shall also use the standard notation $V^{(I)} := \{f : I \to V\}$ for $I$ finite, as before.

**Lemma 8.2.8.** *We keep the notation of Lemma 8.2.5 and of Equation* (8.13) *above. Then we have* $\Gamma$*-equivariant algebra isomorphisms*

$$B_{W_x} \simeq \mathrm{Ind}_{\Gamma_x}^{\Gamma}(B_x) + \overline{\Psi^{-1}}(W_x; E)\,.$$

*Consequently,* $B_{W_x}^{\Gamma} \simeq \Phi(B_x^{\Gamma_x}) + \overline{\Psi^{-1}}(W_x; E)^{\Gamma}$.

*Proof.* Since $B_y = B_{U_y} \subset B_{W_x}$ for all $y \in \Gamma x$ and since $U_x$ and $U_y$ are diffeomorphic through any $\gamma \in \Gamma$ such that $\gamma x = y$ we obtain the inclusion $B_x^{(\Gamma/\Gamma_x)} \subset B_{W_x}$, as in Remark 8.2.7. Similarly, since $B_x \to A_x$ is surjective, we obtain the equality $B_{W_x} = B_x^{(\Gamma/\Gamma_x)} + \overline{\Psi^{-1}}(W_x; E)$ as in the same remark. From Equation (8.5) and Lemma 8.2.5 we know that $B_{W_x}/\overline{\Psi^{-1}}(W_x; E) \simeq A_{W_x} \simeq A_{U_x}^{(\Gamma/\Gamma_x)} = \mathrm{Ind}_{\Gamma_x}^{\Gamma}(A_x)$, and hence we obtain $B_{W_x} \simeq \mathrm{Ind}_{\Gamma_x}^{\Gamma}(B_x) + \overline{\Psi^{-1}}(W_x; E)$. The last isomorphism follows from the Frobenius reciprocity (more precisely, from Equation (7.20), with $\beta$ replaced with $B_x$) and from the exactness of the functor $V \to V^{\Gamma}$. □

To be able to make further progress, it will be convenient to look first at the case when $x \in M$ has minimal isotropy $\Gamma_x \sim \Gamma_0$, that is, when $x$ belongs to the principal orbit bundle $M_0 := M_{(\Gamma_0)}$. The notation $\Gamma_0$ will remain fixed from now on.

### 8.2.4. Calculations for the principal orbit bundle

We assume as before that $M/\Gamma$ is connected. Let $\Gamma_0$ be a minimal isotropy group (which, we recall, is unique up to conjugation). Let $x \in M$ be our fixed point and $\Gamma_x$ its isotropy, as before. The case when $\Gamma_x$ is conjugated to $\Gamma_0$ is simpler since, as noticed already, then $\Gamma_x$ acts trivially on $U_x$.

Let us fix $x \in M$ with isotropy group $\Gamma_x = \Gamma_0$. As before, we let

$$W_x \simeq \Gamma \times_{\Gamma_0} U_x \text{ and } E_{|W_x} \simeq \Gamma \times_{\Gamma_0} (U_x \times \beta)\,,$$

where $\beta$ is some $\Gamma_0$-module, as in Equations (7.18) and (7.19). We decompose $\beta$ into a direct sum of representations of the form $\beta_j^{k_j}$ for some non-isomorphic irreducible module (or representation) $\beta_j$ of $\Gamma_0$, again as before:

$$E_x = \beta \simeq \oplus \beta_j^{k_j}\,.$$

**Remark 8.2.9.** We have noticed earlier that $\Gamma_0$ acts trivially on $U_x$, hence on $T_x^*M$. In particular $S^*M$ also has $\Gamma_0$ as minimal isotropy subgroup, and $S^*M_0$ is a dense subset of the principal bundle of $S^*M$.

**Corollary 8.2.10.** *Let $x \in M$ be such that $\Gamma_x = \Gamma_0$ and $\beta = \oplus_{j=1}^N \beta_j^{k_j}$, for some non-isomorphic, irreducible $\Gamma_0$-modules $\beta_j$. Then*

$$A_{W_x}^{\Gamma} \simeq A_x^{\Gamma_x} \simeq C_0(S^*U_x) \otimes \mathrm{End}_{\Gamma_0}(\beta) \simeq \oplus_{j=1}^N M_{k_j}(C_0(S^*U_x)).$$

*In particular, the canonical central character map*

$$\mathrm{Prim}(A_x^{\Gamma_0}) \to S^*U_x \simeq \mathrm{Prim}(C_0(S^*U_x)^{\Gamma_0})$$

*of Proposition 8.2.1 corresponds to the trivial finite covering $S^*U_x \times \mathrm{Prim}(\mathrm{End}_{\Gamma_0}(\beta)) \to S^*U_x$.*

*Proof.* The first isomorphism is repeated from Lemma 8.2.5. The second one is obtained from the following:

(i) from the definition of $A_x = A_{U_x}$,
(ii) from the assumption that $\Gamma_x = \Gamma_0$,
(iii) from the fact that $\Gamma_0$ acts trivially on $U_x$, and
(iv) from the identifications

$$A_x^{\Gamma_0} := C_0(S^*U_x; \mathrm{End}(E))^{\Gamma_0} \simeq C_0(S^*U_x) \otimes \mathrm{End}(\beta)^{\Gamma_0}.$$

The last isomorphism follows from Example 7.2.5 and the isomorphism $M_n(\mathbb{C}) \otimes A \simeq M_n(A)$, valid for any algebra $A$. The rest follows from Lemma 7.2.7.

Indeed, since both $C_0(S^*U_x)$ and $\mathrm{End}(\beta)^{\Gamma_0}$ have only finite dimensional irreducible representations, we obtain $\mathrm{Prim}(A_x^{\Gamma_0}) = S^*U_x \times \mathrm{Prim}(\mathrm{End}_{\Gamma_0}(\beta)) \simeq S^*U_x \times \{1, 2, \ldots, N\}$, where we use the identification $\mathrm{Prim}(C_0(S^*U_x)) \simeq S^*U_x$ and where the set $\{1, 2, \ldots, N\}$ is in natural bijection with the primitive ideal spectrum of the algebra $\mathrm{End}_{\Gamma_0}(\beta) \simeq$

$\oplus_{j=1}^N M_{k_j}(\mathbb{C})$. The inclusion $C_0(S^*U_x) = C_0(S^*U_x)^{\Gamma_0} \to A_x^{\Gamma_0}$ is given by the unital inclusion $\mathbb{C} \to \oplus_{j=1}^N M_{k_j}(\mathbb{C})$. Hence the map $\mathrm{Prim}(A_x^{\Gamma_0}) \to S^*U_x$ identifies with the first projection in $S^*U_x \times \{1, 2, \dots, N\} \to S^*U_x$. That is, it is a trivial covering, as claimed. □

The fibers of $\mathrm{Prim}(A_{M_0}^{\Gamma}) \to M_0/\Gamma$ are thus the simple factors of $\mathrm{End}(E_x)^{\Gamma_0}$, whose structure was determined in Example 7.2.5. We focus first on a single fiber and consider the $\Gamma_0$-representation $\beta := E_x$. As before, we write

$$\beta = \bigoplus_{j=1}^{N} \beta_j^{k_j}, \tag{8.14}$$

where $\beta_1, \dots, \beta_N$ are irreducible representations of $\Gamma_0$. For the abelian case, the following elementary result was proved in Proposition 7.2.4. That proof *does not* generalize to our case, for which we need the notion of $\Gamma_0$-associated representations given in Definition 8.1.1.

**Proposition 8.2.11.** *Let $\beta := \oplus_{j=1}^N \beta_j^{k_j}$ be as in Equation* (8.14)*. Let $J \subset \{1, 2, \dots, N\}$ be the set of indices $j$ such that $\alpha$ and $\beta_j$ are $\Gamma_0$-disjoint (i.e. $\beta_j$ is not contained in the restriction of $\alpha$ to $\Gamma_0$). Then the morphism*

$$\pi_\alpha : \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\mathrm{End}(\beta))^{\Gamma} \to \mathrm{End}(p_\alpha \, \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\beta))$$

*is such that*

$$\ker(\pi_\alpha) = \bigoplus_{j \in J} \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\mathrm{End}(\beta_j^{k_j}))^{\Gamma} \text{ and } \mathrm{Im}(\pi_\alpha) \simeq \bigoplus_{j \notin J} \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\mathrm{End}(\beta_j^{k_j}))^{\Gamma}.$$

*Proof.* By Lemma 7.2.2, we can assume that $N = 1$. Therefore the algebra $\mathrm{End}(\beta)^{\Gamma_0}$ is simple (more precisely, isomorphic to the matrix algebra $M_q(\mathbb{C})$, with $q := k_1$). We shall use the isomorphism $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\mathrm{End}(\beta))^{\Gamma} \simeq \mathrm{End}(\beta)^{\Gamma_0} \simeq M_q(\mathbb{C})$ of Equation (7.14). This isomorphism shows in particular that the action of $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\mathrm{End}(\beta))^{\Gamma}$ on $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\beta)$ is unital (i.e. non-degenerate), so the morphism

$$M_q(\mathbb{C}) \simeq \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\mathrm{End}(\beta))^{\Gamma} \longrightarrow \mathrm{End}(p_\alpha \, \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\beta)) \tag{8.15}$$

is injective if, and only if, $p_\alpha \, \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\beta) \neq 0$. By the Frobenius isomorphism of Equation (7.11), we have the following equivalences

$$p_\alpha \, \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\beta) \neq 0 \Leftrightarrow \mathrm{Hom}(\alpha, \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\beta))^{\Gamma} \neq 0 \Leftrightarrow \mathrm{Hom}(\alpha, \beta)^{\Gamma_0} \neq 0,$$

which is again equivalent with stating that the $\beta$-isotypical component of $\alpha_{|\Gamma_0}$ is non-zero, see Equation (7.7). This proves the proposition. □

We shall need the following remark similar to Remark 8.2.7, but simpler.

**Remark 8.2.12.** Let $U$ be an open subset of a euclidean space, let $V$ be a finite dimensional vector space and let $V$ denote, by abuse of notation, also the trivial, vector bundle with fiber $V$. Then we have *natural* isomorphisms

$$\begin{aligned}\Psi^{-1}(U;V) \;\simeq\; \Psi^{-1}(U)\otimes \mathrm{End}(V) \quad \text{and}\\ \Psi^{0}(U;V) \;\simeq\; \Psi^{0}(U)\otimes \mathrm{End}(V)\,.\end{aligned}$$

Consequently, we also have the analogous isomorphisms for the completions

$$\begin{aligned}\overline{\Psi^{-1}}(U;V) \;\simeq\; \overline{\Psi^{-1}}(U)\otimes \mathrm{End}(V) \quad \text{and}\\ \overline{\Psi^{0}}(U;V) \;\simeq\; \overline{\Psi^{0}}(U)\otimes \mathrm{End}(V)\,.\end{aligned}$$

We are in position now to determine the kernel of $\mathcal{R}_{W_x}$, when $x$ is in the principal orbit bundle. We will use the notation of Subsection 7.2.3 that was recalled at the beginning of this subsection as well as the notation of Subsection 7.2.2. In particular, recall that $\beta_j \in \widehat{\Gamma}_0$ and $\alpha \in \widehat{\Gamma}$ are said to be $\Gamma_0$-disjoint if $\beta_j$ is *not* contained in the restriction of $\alpha$ to $\Gamma_0$. Also, $\Phi$ is the Frobenius isomorphism, Equations (7.11) and (7.12) and Corollary 8.2.10.

**Proposition 8.2.13.** *Let $\Gamma_x = \Gamma_0$, let $E_x = \beta = \oplus_{j=1}^{N}\beta_j^{k_j}$, and $\Phi : C_0(S^*U_x)\otimes \mathrm{End}_{\Gamma_0}(\beta) \simeq A_x^{\Gamma_0} \to A_{W_x}^{\Gamma}$ be the Frobenius isomorphism of Corollary 8.2.10. Then*

*(i) $C_0(S^*U_x)\otimes \mathrm{End}_{\Gamma_0}(\beta_j^{k_j}) \subset \Phi^{-1}(\ker(\mathcal{R}_{W_x}))$ if $\beta_j$ and $\alpha$ are $\Gamma_0$-disjoint, and*

*(ii) $C_0(S^*U_x)\otimes \mathrm{End}_{\Gamma_0}(\beta_j^{k_j}) \cap \Phi^{-1}(\ker(\mathcal{R}_{W_x})) = 0$ if $\beta_j$ and $\alpha$ are $\Gamma_0$-associated.*

*In particular, Also, let $J \subset \{1,2,\ldots,N\}$ be the set of indices $j$ such that $\beta_j$ and $\alpha$ are $\Gamma_0$-disjoint, then*

$$\begin{aligned}\ker(\mathcal{R}_{W_x}) = \Phi\big(\oplus_{j\in J}\, C_0(S^*U_x)\otimes \mathrm{End}_{\Gamma_0}(\beta_j^{k_j})\big) \quad \textit{and}\\ \pi_\alpha(B_M^{\Gamma})/\pi_\alpha(\overline{\Psi^{-1}}(M;E)^{\Gamma}) \;\simeq\; \Phi\big(\oplus_{j\notin J}\, C_0(S^*U_x)\otimes \mathrm{End}_{\Gamma_0}(\beta_j^{k_j})\big)\,.\end{aligned}$$

*Proof.* The proof is essentially a consequence of Proposition 8.2.11 by including $U_x$ as a parameter, using also Lemma 8.2.8. To see how this is done, we will use the notation of that lemma, in particular, $W_x \simeq \Gamma\times_{\Gamma_0} U_x \simeq (\Gamma/\Gamma_0)\times U_x$ and $E \simeq \Gamma\times_{\Gamma_0}(U_x\times\beta)$. We identify $W_x$ with $\Gamma\times_{\Gamma_x} U_x$, i.e. we work with $W_x = \Gamma\times_{\Gamma_x} U_x$.

Let $\pi_\alpha$ the fundamental morphism of restriction to the $\alpha$-isotypical component, see Equations (7.1) and (7.8). Recall that $B_x := \overline{\Psi^0}(U_x;E)$. Since $\Gamma_x$ acts trivially on $U_x$, Remark 8.2.12 yields the $\Gamma$-equivariant isomorphisms

$$\mathrm{Ind}_{\Gamma_0}^{\Gamma}(B_x) \;\simeq\; \overline{\Psi^0}(U_x)\otimes \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\mathrm{End}(\beta)) \subset B_{W_x}\,, \tag{8.16}$$

where the last inclusion is modulo the trivial identification given by $P\otimes f(s)(\gamma,x) = P(f(\gamma)s(\gamma))(x)$, $P\in\overline{\Psi^0}(U_x)$, $f\in \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\mathrm{End}(\beta))$ and $s\in C_c(W_x,\mathrm{End}(E))$. Combining further Remark 8.2.12 with Remark 8.2.7, we further obtain the isomorphism

$$\overline{\Psi^{-1}}(W_x;E) \;\simeq\; \overline{\Psi^{-1}}(U_x)\otimes \mathrm{End}(\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\beta))\,.$$

Lemma 8.2.8 and the exactness of the functor $V \to V^\Gamma$ give $\pi_\alpha(B_{W_x}^\Gamma) = \pi_\alpha \circ \Phi(B_x^{\Gamma_x}) + \pi_\alpha(\overline{\Psi^{-1}}(W_x)^\Gamma)$. Hence we obtain

$$\pi_\alpha(B_{W_x}^\Gamma)/\pi_\alpha(\overline{\Psi^{-1}}(W_x)^\Gamma) = \pi_\alpha \circ \Phi(B_x^{\Gamma_x})/\pi_\alpha \circ \Phi(B_x^{\Gamma_x}) \cap \pi_\alpha(\overline{\Psi^{-1}}(W_x)^\Gamma).$$

Let $\mathfrak{A}$ and $\mathfrak{J}$ be the image and, respectively, the kernel of $\pi_\alpha : \operatorname{Ind}_{\Gamma_0}^\Gamma(\operatorname{End}(\beta))^\Gamma \to \operatorname{End}(p_\alpha \operatorname{Ind}_{\Gamma_0}^\Gamma(\beta))$, which have been identified in Proposition 8.2.11 in terms of the set $J$. Recall next from Equation (7.20) that $L^2(W_x; E) = L^2(U_x) \otimes \operatorname{Ind}_{\Gamma_0}^\Gamma(\beta)$, again $\Gamma$-equivariantly. Each time, the action is on the second component, since $\Gamma_0 = \Gamma_x$ acts trivially on $\overline{\Psi^0}(U_x)$. The action of $\operatorname{Ind}_{\Gamma_0}^\Gamma(B_x) \subset B_{W_x}$ on $L^2(W_x; E) = L^2(U_x) \otimes \operatorname{Ind}_{\Gamma_0}^\Gamma(\beta)$ is compatible with the tensor product decomposition of Equation (8.16), in the sense that $\overline{\Psi^0}(U_x)$ acts on $L^2(U_x)$ and $\operatorname{Ind}_{\Gamma_0}^\Gamma(\operatorname{End}(\beta))$ acts on $\operatorname{Ind}_{\Gamma_0}^\Gamma(\beta)$. Also, $\operatorname{Ind}_{\Gamma_0}^\Gamma(B_x)^\Gamma \simeq \overline{\Psi^0}(U_x) \otimes \operatorname{Ind}_{\Gamma_0}^\Gamma(\operatorname{End}(\beta))^\Gamma$, (we use this isomorphism to identify them). We obtain that

$$\pi_\alpha \circ \Phi(B_x^{\Gamma_x}) = \pi_\alpha(\operatorname{Ind}_{\Gamma_0}^\Gamma(B_x)^\Gamma) = \overline{\Psi^0}(U_x) \otimes \mathfrak{A}\,. \tag{8.17}$$

On the other hand, Proposition 7.3.9 then gives that $\pi_\alpha(\overline{\Psi^{-1}}(W_x; \operatorname{End}(E))^\Gamma)$ is the algebra of $\Gamma$-invariant compact operators acting on the space $p_\alpha(L^2(W_x, \operatorname{End}(E)))$. Therefore, $\overline{\Psi^{-1}}(U_x) \otimes \mathfrak{A} \subset \pi_\alpha(\overline{\Psi^{-1}}(W_x; \operatorname{End}(E))^\Gamma)$, since $\overline{\Psi^{-1}}(U_x) \otimes \mathfrak{A}$ consists of compact, $\Gamma$-invariant operators acting on $p_\alpha(L^2(W_x, E))$. Consequently,

$$\begin{aligned}\overline{\Psi^{-1}}(U_x) \otimes \mathfrak{A} \subset \pi_\alpha(\operatorname{Ind}_{\Gamma_0}^\Gamma(B_x)^\Gamma) \cap \pi_\alpha(\overline{\Psi^{-1}}(W_x)^\Gamma)&\\ \subset \overline{\Psi^0}(U_x) \otimes \mathfrak{A} \cap \mathcal{K}(p_\alpha L^2(W_x; E))^\Gamma \subset \overline{\Psi^{-1}}(U_x) \otimes \mathfrak{A}\,,&\end{aligned} \tag{8.18}$$

and hence we have equalities everywhere.

Recall from Corollary 8.2.10 that $A_{W_x}^\Gamma \simeq A_x^{\Gamma_x}$. We obtain that the map

$$\mathcal{R}_{W_x} : A_{W_x}^\Gamma \simeq B_{W_x}^\Gamma/\overline{\Psi^{-1}}(W_x; E)^\Gamma \to \pi_\alpha(B_{W_x}^\Gamma)/\pi_\alpha(\overline{\Psi^{-1}}(W_x; E)^\Gamma) \tag{8.19}$$

becomes, up to the canonical isomorphisms above, the map

$$\begin{aligned}A_x^{\Gamma_x} \simeq C_0(S^*U_x) \otimes \operatorname{End}_{\Gamma_0}(\beta) &\to \pi_\alpha(B_{W_x}^\Gamma)/\pi_\alpha(\overline{\Psi^{-1}}(W_x)^\Gamma)\\ &= \pi_\alpha \circ \Phi(B_x^{\Gamma_x})/\pi_\alpha \circ \Phi(B_x^{\Gamma_x}) \cap \pi_\alpha(\overline{\Psi^{-1}}(W_x)^\Gamma)\\ &\simeq \overline{\Psi^0}(U_x) \otimes \mathfrak{A}/\overline{\Psi^{-1}}(U_x) \otimes \mathfrak{A} \simeq C_0(S^*U_x) \otimes \mathfrak{A}\,,\end{aligned} \tag{8.20}$$

with all maps being surjective and preserving the tensor product decompositions. This identifies the kernel of $\mathcal{R}_{W_x}$ with $C_0(S^*U_x) \otimes \mathfrak{J}$ and the image of $\mathcal{R}_{W_x}$ with $C_0(S^*U_x) \otimes \mathfrak{A}$. The rest of the statement follows from the identification of $\mathfrak{J}$ and $\mathfrak{A}$ in Proposition 8.2.11. □

Proposition 8.2.13 above and its proof give the following corollary.

**Corollary 8.2.14.** *We use the notation of Proposition 8.2.13 and we identify the space* $\operatorname{Prim}(\operatorname{End}(\beta))$ *with* $\{1, 2, \ldots, N\}$ *as in Remark 7.2.5. Then the homeomorphism* $\operatorname{Prim}(A_{W_x}^\Gamma) \simeq S^*U_x \times \{1, 2, \ldots, N\}$ *maps the set* $\Xi \cap \operatorname{Prim}(A_{W_x}^\Gamma)$ *to* $S^*U_x \times J$*. In particular, the restriction* $\Xi \cap \operatorname{Prim}(A_{W_x}^\Gamma) \to S^*U_x$ *of the central character is a covering as well.*

*Proof.* Using the notations of the proof of Proposition 8.2.13, we have that $\ker(\mathcal{R}_{W_x})$ has primitive ideal spectrum $S^*U_x \times \mathrm{Prim}(\mathfrak{J})$. We have $\Xi \cap \mathrm{Prim}(A^\Gamma_{W_x}) = S^*U_x \times \mathrm{Prim}(\mathfrak{A})$. $\square$

The same methods yield the following result (recall that $M_0 = M_{(\Gamma_0)}$ is the principal orbit bundle).

**Corollary 8.2.15.** *Let $M_0 := M_{(\Gamma_0)}$, the principal orbit bundle. The central character map $\mathrm{Prim}(A^\Gamma_{M_0}) \to S^*M_0/\Gamma$ defined by the inclusion $C_0(S^*M_0/\Gamma) \subset Z(A^\Gamma_{M_0})$ is a covering with typical fiber $\mathrm{Prim}(\mathrm{End}(E_x)^{\Gamma_0})$ such that $\Xi \cap \mathrm{Prim}(A^\Gamma_{M_0}) \to S^*M_0/\Gamma$ is a subcovering, see (8.8) for the definition of $\Xi$. In particular, $\Xi \cap \mathrm{Prim}(A^\Gamma_{M_0})$ is open and closed in $\mathrm{Prim}(A^\Gamma_{M_0})$.*

*Proof.* The first statement is true locally, by Corollary 8.2.10, and hence it is true globally. Indeed, let $x \in M_0$, let $\xi \in S^*_x M_0$, and let $\rho \in \widehat{\Gamma}_x$ that appears in $E_x$ (so $(\xi, \rho) \in \tilde{X}_{M,\Gamma}$). We let $W_x \subset M_0 \subset M$ be the typical tube with minimal isotropy $\Gamma_x = \Gamma_0$, as before. Let $Z_x := C_0(S^*W_x)^\Gamma \subset Z_M = C(S^*M)^\Gamma$. Then $\mathrm{Prim}(Z_x A^\Gamma_M)$ is an open neighborhood in $\mathrm{Prim}(A^\Gamma_{M_0})$ of the primitive ideal $\ker(\pi_{\xi,\rho})$, see Proposition 8.2.1 for notation and details. We have that $Z_x A^\Gamma_M = A^\Gamma_{W_x}$ and hence, on $\mathrm{Prim}(Z_x A^\Gamma_M)$, the central character is a covering, by Corollary 8.2.10. Similarly, its restriction to $\Xi \cap \mathrm{Prim}(Z_x A^\Gamma_M)$ is a covering by Corollary 8.2.14. $\square$

Putting Corollary 8.2.15 and Proposition 8.2.13 together we obtain the following results.

**Corollary 8.2.16.** *Let $M_0$ be the principal orbit type of $M$. The ideal $\ker(\mathcal{R}_{M_0}) = A^\Gamma_{M_0} \cap \ker(\mathcal{R}_M)$ is defined by the closed subset $\Xi_0 := \Xi \cap \mathrm{Prim}(A^\Gamma_{M_0})$ of $\mathrm{Prim}(A^\Gamma_{M_0})$ consisting of the sheets of $\mathrm{Prim}(A^\Gamma_{M_0}) \to S^*M_0/\Gamma$ that correspond to the simple factors $\mathrm{End}(E_{x\rho})^{\Gamma_0}$ of $\mathrm{End}(E_x)^{\Gamma_0}$ with $\rho$ and $\alpha$ $\Gamma_0$-associated.*

If $\Gamma$ is abelian, then $\rho$ and $\alpha$ are characters and saying that they are $\Gamma_0$-associated means, simply, that their restrictions to $\Gamma_0$ coincide: $\rho|_{\Gamma_0} = \alpha_{\Gamma_0}$. This is consistent with the definition given in Chapter 7.

### 8.2.5. The non-principal orbit case

As in the rest of the chapter, we assume $M/\Gamma$ to be connected. We will show in Theorem 8.2.18 that $\Xi$ is the closure of $\Xi_0$ in $\mathrm{Prim}(A^\Gamma_M)$. To that end, we first construct a suitable basis of neighborhoods of $\mathrm{Prim}(A^\Gamma_M)$ using Lemma 8.2.2.

**Remark 8.2.17.** Let $\Gamma(\xi, \rho) \in \mathrm{Prim}(A^\Gamma_M)$, where we have used the description of $\mathrm{Prim}(A^\Gamma_M)$ provided in Proposition 8.2.1 as orbits of pairs $\xi \in S^*M$ and suitable $\rho \in \widehat{\Gamma}_\xi$. We construct a basis of neighborhoods $(V_{\xi,\rho,n})_{n\in\mathbb{N}}$ of $\Gamma(\xi, \rho)$ in $\mathrm{Prim}(A^\Gamma_M)$ as follows. Let $\xi \in S^*_x M$ (that is, $\xi$ sits above $x \in M$) and we use the notation $U_x$ and $W_x$ of Equation (7.18), as always.

First, by choosing a different point $\xi$ in its orbit, if necessary, we may assume that $\Gamma_0 \subset \Gamma_\xi$. Now let $(\mathcal{O}_n)_{n\in\mathbb{N}}$ be a family of $\Gamma_\xi$-invariant neighborhoods of $\xi$ in $S^*U_x$ such that:

(i) for all $n$ and $\gamma \in \Gamma \setminus \Gamma_\xi$, we have $\gamma \mathcal{O}_n \cap \mathcal{O}_n = \emptyset$,
(ii) $\mathcal{O}_{n+1} \subset \mathcal{O}_n$ and $\bigcap_{n\in\mathbb{N}} \mathcal{O}_n = \{\xi\}$.

For any $n \in \mathbb{N}$, we choose a function $\varphi_n \in C_c(\mathcal{O}_n)^{\Gamma_\xi}$ such that $\varphi_n \equiv 1$ on $\mathcal{O}_{n+1}$. Let $p_\rho \in \mathrm{End}(E_x)^{\Gamma_\xi}$ be the projection onto $E_{x\rho}$. We can assume the bundle $E$ to be trivial on $U_x$ and, using that, we first extend $p_\rho$ constantly on $\mathcal{O}_n$ and then as an element $q_n \in C_c(S^*U_x; \mathrm{End}(E_x))^{\Gamma_x}$ defined as

$$q_n := \begin{cases} \Phi_{\Gamma_\xi,\Gamma_x}(\varphi_n p_\rho) & \text{on } \Gamma_x \mathcal{O}_n \\ 0 & \text{on } S^*U_x \setminus \Gamma_x \mathcal{O}_n, \end{cases}$$

with $\Phi_{\Gamma_\xi,\Gamma_x}$ the Frobenius isomorphism of Equation (7.12). Let us set $\tilde{q}_n := \Phi_{\Gamma_x,\Gamma}(q_n) \in A_M^\Gamma$, where $\Phi_{\Gamma_x,\Gamma}$ is the Frobenius isomorphism of Equation (7.12). Finally, we associate to $\tilde{q}_n$ the open set

$$V_{\xi,\rho,n} := \{J \in \mathrm{Prim}(A_M^\Gamma) \mid \tilde{q}_n \notin J\}.$$

Recall from 8.2.2 that $V_{\xi,\rho,n}$ is an open subset of $\mathrm{Prim}(A_M^\Gamma)$. Moreover, it follows from our definition that $V_{\xi,\rho,n+1} \subset V_{\xi,\rho,n}$ and that $\bigcap_{n\in\mathbb{N}} V_{\xi,\rho,n} = \{\Gamma(\xi,\rho)\}$.

Recall that we are assuming that $M/\Gamma$ is connected.

**Theorem 8.2.18.** *Let* $\Xi := \mathrm{Prim}(A_M^\Gamma/\ker(\mathcal{R}_M)) \subset \mathrm{Prim}(A_M^\Gamma)$ *be the closed subset defined by the ideal* $\ker(\mathcal{R}_M)$*. Then* $\Xi$ *is the closure in* $\mathrm{Prim}(A_M^\Gamma)$ *of the set* $\Xi_0 := \Xi \cap \mathrm{Prim}(A_{M_0}^\Gamma)$*, where* $M_0$ *is the principal orbit bundle of* $M$*.*

*Proof.* We have that $\overline{\Xi}_0 \subset \Xi$ since $\Xi_0 \subset \Xi$ and the latter is a closed set. Conversely, let $\mathfrak{P} \in \mathrm{Prim}(A_M^\Gamma) \setminus \overline{\Xi}_0$. We will show that $\mathfrak{P} \notin \Xi$. Let $\mathfrak{P}$ correspond to $(\xi,\rho) \in \tilde{X}_{M,\Gamma}$, as in Proposition 8.2.1. We may assume that $\Gamma_0 \subset \Gamma_\xi$. Let $x$ be projection of $\xi$ onto $M$. Since the problem is local, we may also assume that $U_x \subset T_xM$, that $M = W_x := \Gamma \times_{\Gamma_x} U_x$, and that $E := \Gamma \times_{\Gamma_x} (U_x \times \beta)$ for some $\Gamma_x$-module $\beta$.

Using the notations of Remark 8.2.17, there exists $n > 0$ such that $V_{\xi,\rho,n} \cap \Xi_0 = \emptyset$. Let $\tilde{q}_n = \Phi_{\Gamma_x,\Gamma}(q_n)$ be the symbol defined in Remark 8.2.17. The description of $\Xi_0$ provided in Corollary 8.2.16, the definition of $V_{\xi,\rho,n}$, and the definition of $\tilde{q}_n$ imply that $\pi_{\zeta,\rho'}(\tilde{q}_n) = 0$ for any $\zeta \in S^*M_0$ and $\rho' \in \widehat{\Gamma}_0$ such that $\Gamma(\zeta,\rho') \in \Xi_0$, that is, such that $\rho'$ and $\alpha$ are $\Gamma_0$-associated.

We next “quantize $\tilde{q}_n$” in an appropriate way, that is, we construct an operator $\widetilde{Q}_n \in B_{W_x}^\Gamma$ with symbol $\tilde{q}_n$ and with other convenient properties as follows. First, let $\chi \in C_c^\infty(U_x)^{\Gamma_x}$ be such that $\chi\varphi_n = \varphi_n$, which is possible since $\varphi_n$ has compact support. Then let $\psi \in C^\infty(T_x^*M)^{\Gamma_x}$ be such that $\psi(0) = 0$ if $|\eta| < 1/2$ and $\psi(\eta) = 1$ whenever $|\eta| \geq 1$. Recall that in this proof $U_x \subset T_xM$ is identified with its image in $M = \Gamma \times_{\Gamma_x} U_x$ through the exponential map. Let for any symbol $a$

$$Op(a)f(y) := \int_{T_x^*M} \int_{U_x} e^{i(y-z)\cdot\eta}\, a(y,z,\eta) f(z)\, dz d\eta.$$

We shall use this for $a_n(y,z,\eta) := \chi(y)\psi(\eta)\tilde{q}_n\left(\frac{\eta}{|\eta|}\right)\chi(z)$, then set

$$Q_n := Op(a_n)\,, \quad \text{that is}$$
$$Q_n f(y) := \int_{T_x^*M}\int_{U_x} e^{i(y-z)\cdot\eta}\chi(y)\psi(\eta)\tilde{q}_n\left(\frac{\eta}{|\eta|}\right)\chi(z)f(z)dzd\eta$$

to be the standard pseudodifferential operator on $U_x$, associated to the symbol

$$a_n(y,z,\eta) := \chi(y)\psi(\eta)\tilde{q}_n\left(\frac{\eta}{|\eta|}\right)\chi(z).$$

The operator $Q_n$ is $\Gamma_x$-invariant by construction. Using the Frobenius isomorphism of Equation (7.12), we extend $Q_n$ to the operator $\widetilde{Q}_n := \Phi(Q_n)$, which acts on $M = W_x = \Gamma \times_{\Gamma_x} U_x$ (see also Equation (7.20) with regards to this isomorphism). Then $\widetilde{Q}_n \in \Psi^0(M;E)^\Gamma$, that is, it is $\Gamma$-invariant, by construction, and has principal symbol $\sigma_0(\widetilde{Q}_n) = \tilde{q}_n$.

Now let $x_0 \in M_0 \cap U_x$, where, we recall, $M_0 := M_{(\Gamma_0)}$ denotes the principal orbit bundle. We have

$$L^2(W_{x_0};E) = \mathrm{Ind}_{\Gamma_0}^{\Gamma}\left(L^2(U_{x_0};\beta)\right) = L^2(U_{x_0};\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\beta)),$$

where $\beta = E_{x_0} = E_x$ by the assumption that $E := \Gamma \times_{\Gamma_x}(U_x \times \beta)$.

Let $\beta_j \in \widehat{\Gamma_0}$ be the isomorphism classes of the $\Gamma_\xi$-submodules of $\beta$ and $k_j \geq 0$ is the dimension of the corresponding $\beta_j$-isotypical component in $\beta$, so that $\beta \simeq \oplus_{j=1}^N \beta_j^{k_j}$, as $\Gamma_0$-modules, as before. Thus

$$L^2(W_{x_0};E) \simeq \bigoplus_{j=1}^N L^2(U_{x_0};\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\beta_j^{k_j}))\,.$$

Recall that the $\alpha$-isotypical component of $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\beta_j^{k_j})$ is $\alpha\otimes\mathrm{Hom}_\Gamma(\alpha,\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\beta_j^{k_j}))$, which is non-zero if, and only if, $\alpha$ and $\beta_j$ are $\Gamma_0$-associated, by the Frobenius isomorphism. Hence, passing to the $\alpha$-isotypical components, we have

$$L^2(W_{x_0};E)_\alpha := p_\alpha L^2(W_{x_0};E) = \bigoplus_{j\in J^c} L^2(U_{x_0};\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\beta_j^{k_j}))_\alpha\,, \tag{8.21}$$

where $J \subset \{1,\dots,N\}$ is the set of indices such that $\beta_j \in \widehat{\Gamma}_0$ and $\alpha$ are $\Gamma_0$-disjoint; $J^c$ is its complement (i.e. $\beta_j \in \widehat{\Gamma}_0$ and $\alpha$ are $\Gamma_0$-associated).

Let $p_J \in \mathrm{End}(\beta)^{\Gamma_0}$ be the projector onto $\bigoplus_{j\in J^c}\beta_j^{k_j}$. Recall that $\pi_{\zeta,\beta_j}(\tilde{q}_n) = 0$ for any $(\zeta,\beta_j) \in S^*M_0 \times \widehat{\Gamma}_0$ with $j \notin J$. Therefore $\tilde{q}_n(\zeta)p_J = 0$, for all $\zeta \in S^*M_0$. Since $S^*M_0$ is dense in $S^*M$, this implies that $\tilde{q}_n p_J = 0$. Thus

$$\widetilde{Q}_n p_J = Op(\chi\psi\tilde{q}_n\chi)p_J = Op(\chi\psi\tilde{q}_n\chi p_J) = 0\,.$$

Hence for any $f \in L^2(W_{x_0};E)_\alpha$, we have that $\widetilde{Q}_n f = 0$. This is true for any $x_0 \in M_0$, so we conclude that $\widetilde{Q}_n$ is zero on $L^2(M_0;E)_\alpha$. Since $M_0$ has measure zero complement in $M$, we have $L^2(M_0;E)_\alpha = L^2(M;E)_\alpha$; therefore $\pi_\alpha(\widetilde{Q}_n) = 0$. This implies that $\mathcal{R}_M(\tilde{q}_n) = 0$, while $\pi_{\xi,\rho}(\tilde{q}_n) = 1$. Thus $\Gamma(\xi,\rho) \notin \Xi$, which concludes the proof. $\square$

Our question now is to decide whether some given $\Gamma(\xi, \rho)$ is in $\Xi$ or not. Recall that $\rho$ and $\alpha$ are said to be $\Gamma_0$-associated if $\mathrm{Hom}_{\Gamma_0}(\rho, \alpha) \neq 0$. The set $X^{\alpha}_{M,\Gamma}$ was defined in the introduction as the set of pairs $(\xi, \rho) \in T^*M \setminus \{0\} \times \widehat{\Gamma}_\xi$ for which there is an element $g \in \Gamma$ such that $g \cdot \rho$ and $\alpha$ are $\Gamma_0$-associated.

**Remark 8.2.19.** Let us highlight the following interesting fact, implied by the proof of Theorem 8.2.18. We have that $E_{x\rho} = 0$ for any $(\xi, \rho) \in X^{\alpha}_{M_0,\Gamma}$ (with $x$ the projection of $\xi$ on $M_0$) if, and only if, $L^2(M; E)_\alpha = 0$.

Indeed, for any $x \in M_0$ with $\Gamma_x = \Gamma_0$, we have noted in Equation (8.21) that

$$L^2(W_x; E)_\alpha = \bigoplus_{\rho} L^2(U_x; \mathrm{Ind}_{\Gamma_0}^{\Gamma}(E_{x\rho}))_\alpha,$$

where the direct sum is indexed by the representations $\rho \in \widehat{\Gamma}_0$ that are $\Gamma_0$-associated to $\alpha$. If $E_{x\rho} = 0$ for any such representation, then $L^2(W_x; E)_\alpha = 0$. Such open sets $W_x$ cover $M_0$, so $L^2(M_0; E)_\alpha = 0$. Since $M_0$ has measure zero complement, we conclude that $L^2(M; E)_\alpha = 0$.

**Proposition 8.2.20.** *We use the notation in the last two paragraphs. We have* $\Gamma(\xi, \rho) \in \Xi$ *if, and only if, there is a* $g \in \Gamma$ *such that* $g \cdot \rho$ *and* $\alpha$ *are* $\Gamma_0$*-associated.*

*Proof.* Let $\Gamma(\xi, \rho) \in \mathrm{Prim}(A^{\Gamma}_M)$, with $x \in M$ the base point of $\xi$. We can assume (by choosing a different element in the orbit if needed) that $\Gamma_0 \subset \Gamma_\xi$. Let $\tilde{q}_n \in A^{\Gamma}_M$ be the element defined in Remark 8.2.17 and $V_{\xi,\rho,n}$ the corresponding neighbourhood of $\Gamma(\xi, \rho)$ in $\mathrm{Prim}(A^{\Gamma}_M)$.

There is a $\Gamma_x$-equivariant isomorphism $E|_{U_x} \simeq U_x \times \beta$, where $\beta = E_x$ is a $\Gamma_x$-module. Since $\Gamma_0 \subset \Gamma_x$, we may decompose $\beta$ into $\Gamma_0$-isotypical components, i.e. $\beta = \bigoplus_{j=1}^{N} \beta_j^{k_j}$, with the usual notation. If $\eta \in \mathcal{O}_n$, then $\pi_{\eta,\beta_j}(\tilde{q}_n) = \varphi_n(\eta)\pi_{\beta_j}(p_\rho)$. Therefore, for any $\eta \in S^*M$, we have

$$\pi_{\eta,\beta_j}(\tilde{q}_n) = 0 \Leftrightarrow \mathrm{Hom}_{\Gamma_0}(\beta_j, \rho) = 0 \text{ or } \tilde{q}_n(\eta) = 0.$$

This implies that

$$V_{\xi,\rho,n} \cap \Xi_0 = \{\Gamma(\eta, \beta) \in \Xi_0 \mid \tilde{q}_n(\eta) \neq 0 \text{ and } \mathrm{Hom}_{\Gamma_0}(\beta, \rho) \neq 0\}$$

It follows from the determination of $\Xi_0$ in Corollary 8.2.16 that $V_{\xi,\rho,n} \cap \Xi_0 \neq \emptyset$ if, and only if, we have $\mathrm{Hom}_{\Gamma_0}(\rho, \alpha) \neq 0$. Now $\Xi = \overline{\Xi}_0$ by Theorem 8.2.18. Since the open sets $(V_{\xi,\rho,n})_{n\in\mathbb{N}}$ form a basis of neighborhoods of $\Gamma(\xi, \rho)$, we conclude that $\Gamma(\xi, \rho) \in \Xi$ if, and only if, we have $\mathrm{Hom}_{\Gamma_0}(\rho, \alpha) \neq 0$. □

**Remark 8.2.21.** Our definition of $\alpha$-ellipticity for an operator $P \in \overline{\Psi^0}(M; E)^{\Gamma}$ was stated in terms of the set $X^{\alpha}_{M,\Gamma}$, defined in Equation (8.4). Proposition 8.2.20 establishes that $\Xi \simeq \tilde{X}^{\alpha}_{M,\Gamma}/\Gamma$, where $\tilde{X}^{\alpha}_{M,\Gamma}$ is the (possibly smaller) subset of pairs $(\xi, \rho) \in X^{\alpha}_{M,\Gamma}$ such that $E_{x\rho} \neq 0$ (with $x$ the projection of $\xi$ on $M$). Keeping in mind the fact that the null operator on a trivial vector space is invertible, we have that $\sigma_0^{\Gamma}(P)(\xi, \rho)$ is invertible for any $(\xi, \rho) \in X^{\alpha}_{M,\Gamma}$ if, and only if, it is invertible for any $(\xi, \rho) \in \tilde{X}^{\alpha}_{M,\Gamma}$. The pathological case $\Xi = \emptyset$, for which $E_{x\rho} = 0$ for any $(\xi, \rho) \in X^{\alpha}_{M,\Gamma}$, causes no problem: indeed, as noticed in Remark 8.2.19, we then have $L^2(M; E)_\alpha = 0$. In that case $\pi_\alpha(P)$ is Fredholm for any $P \in \overline{\Psi^0}(M; E)^\Gamma$, which is consistent with the invertibility of $\sigma_0^\alpha(P)(\xi, \rho) : 0 \to 0$ for any $(\xi, \rho) \in X^{\alpha}_{M,\Gamma}$.

We summarize part of the above discussions in the following proposition.

**Proposition 8.2.22.** *Let $\tilde{X}^{\alpha}_{M,\Gamma}$ be as in Remark 8.2.21. The primitive ideal spectrum $\Xi = \mathrm{Prim}(A^\Gamma_M / \ker(\mathcal{R}_M))$ is canonically homeomorphic to $\tilde{X}^{\alpha}_{M,\Gamma}/\Gamma$ via the restriction map from $A^\Gamma_M := C(S^*M; \mathrm{End}(E))^\Gamma$ to sections over $X^{\alpha}_{M,\Gamma}$.*

## 8.3. Applications and extensions

We now prove the main result of the chapter, Theorem 8.1.3, on the characterization of Fredholm operators, and discuss some extensions of our results. We first explain how to reduce the proof to the case $M/\Gamma$ connected and we discuss in more detail the $\Gamma$-principal symbol and $\alpha$-ellipticity (this discussion can be skipped at a first lecture).

### 8.3.1. Reduction to the connected case and $\alpha$-ellipticity

In this subsection, unlike most of the rest of the chapter, we do *not* assume that $M/\Gamma$ is connected in order to explain how to reduce the general case to the connected one. We do assume however, as always, that $M$ is compact. We also provide some other reductions of our proof.

Let $\pi_{M,\Gamma} : M \to M/\Gamma$ be the quotient map and let us write then $M/\Gamma = \cup_{i=1}^N C_i$ as the *disjoint* union of its connected components. We let $M_i := \pi^{-1}_{M,\Gamma}(C_i)$ be the preimages of these connected components. Note that, in general, the submanifolds $M_i$ are not connected, but, for each $i$, $M_i/\Gamma = C_i$ *is connected*. In particular, a similar discussion applies to yield the definition of the space

$$X^{\alpha}_{M,\Gamma} := \bigsqcup_{i=1}^{N} X^{\alpha}_{M_i,\Gamma} \tag{8.22}$$

as a disjoint union of the spaces $X^{\alpha}_{M_i,\Gamma}$, which makes sense since each of the spaces $M_i$ is invariant for $\Gamma$ and $M_i/\Gamma$ is connected (see Equation (8.4) of the Introduction for the definition of the spaces $X^{\alpha}_{M_i,\Gamma}$).

We shall decorate with the index $i$ the restrictions of objects on $M$ to $M_i$. Thus, $E_i := E|_{M_i}$, and so on and so forth. This almost works for an operator $P \in \overline{\Psi^0}(M; E)^\Gamma$. Indeed, we first notice that

$$L^2(M; E) \simeq \bigoplus_{i=1}^{N} L^2(M_i; E_i) \quad \text{and} \quad \bigoplus_{i=1}^{N} \overline{\Psi^0}(M_i; E_i) \subset \overline{\Psi^0}(M; E). \tag{8.23}$$

Recall that $\mathcal{K}(V)$ denotes the algebra of compact operators on a Hilbert space $V$. The following proposition provides the desired reduction to the connected case.

**Proposition 8.3.1.** *Let $p_i : L^2(M;E) \to L^2(M_i;E_i)$ be the canonical orthoghonal projection. For $P \in \overline{\Psi^0}(M;E)$, we let $P_i := p_i P p_i \in \overline{\Psi^0}(M_i;E_i)$. Then $P - \sum_{i=1}^N P_i \in \mathcal{K}(L^2(M;E))$. If we regard $\sum_{i=1}^N P_i = \bigoplus_{i=1}^N P_i$ as an element of $\bigoplus_{i=1}^N \overline{\Psi^0}(M_i;E_i)$, then we see that*

$$\overline{\Psi^0}(M;E) = \bigoplus_{i=1}^{N} \overline{\Psi^0}(M_i;E_i) + \mathcal{K}(L^2(M;E)).$$

*Moreover, $\pi_\alpha(P) - \oplus_{i=1}^N \pi_\alpha(P_i)$ is compact and hence $\pi_\alpha(P)$ is Fredholm if, and only if, each $\pi_\alpha(P_i)$ is Fredholm, for $i = 1, \ldots, N$.*

*Proof.* If $i \neq j$, $p_i P p_j$ has zero principal symbol, and hence it is compact. Therefore $P - \sum_{i=1}^N P_i = \sum_{i\neq j} p_i P p_j$ is compact. The rest follows from Equation (8.23), its corollary $L^2(M;E)_\alpha \simeq \bigoplus_{i=1}^N L^2(M_i;E_i)_\alpha$, and the fact that $\pi_\alpha$ respects these direct sum decompositions. $\square$

**Remark 8.3.2.** The $\Gamma$-principal symbol $\sigma_m^\Gamma(P)$ was defined in (8.3), and we stress that the definition of the space $X_{M,\Gamma}$ did not require that $M/\Gamma$ be connected. The disjoint union definition of the space $X_{M,\Gamma} = \bigsqcup_{i=1}^N X_{M_i,\Gamma}$ means that

$$\sigma_m^\Gamma(P)|_{X_{M_i,\Gamma}} = \sigma_m^\Gamma(P_i)$$

for each $i = 1, \ldots, N$. The analogous disjoint union decomposition of $X_{M,\Gamma}^\alpha := \bigsqcup_{i=1}^N X_{M_i,\Gamma}^\alpha$ gives that $P$ is $\alpha$-elliptic if, and only if, for each $i$, $P_i$ is $\alpha$-elliptic.

This allows us to reduce the proof of our main theorem, Theorem 8.1.3 to the connected case since, assuming that the connected case has been proved, we have

$$\begin{aligned} \pi_\alpha(P) \text{ is Fredholm} &\Leftrightarrow \forall i, \pi_\alpha(P_i) \text{ is Fredholm} \\ &\Leftrightarrow \forall i, P_i \text{ is } \alpha\text{-elliptic} \\ &\Leftrightarrow P \text{ is } \alpha\text{-elliptic}, \end{aligned}$$

where the first equivalence is by Proposition 8.3.1, the second equivalence is by the assumption that our main theorem has been proved in the connected case, and the last equivalence is by the first part of this remark.

We now *resume our assumption that $M/\Gamma$ is connected,* for convenience. In particular, $\Gamma_0$ will be a minimal isotropy group, which is unique up to conjugacy (since we are again assuming that $M/\Gamma$ is connected). We shall take a closer look next at the $\Gamma$- and $\alpha$-principal symbols, so the following simple discussion will be useful. Recall that if $K \subset \Gamma$, $\rho \in \widehat{\Gamma}$, and $g \in \Gamma$, then $g \cdot K := gKg^{-1}$ and $(g \cdot \rho)(\gamma) := \rho(g^{-1}\gamma g)$, so that $g \cdot \rho$ is an irreducible representation of $g \cdot K$ (i.e. $g \cdot \rho \in \widehat{g \cdot K}$).

**Remark 8.3.3.** Let $\xi \in T^*M \setminus \{0\}$ and $\rho \in \widehat{\Gamma}_\xi$ (that is, $(\xi, \rho) \in X_{M,\Gamma}$). Then the following three statements are equivalent:

(i) the pair $(\xi, \rho) \in X_{M,\Gamma}^\alpha$;
(ii) there is $g \in \Gamma$ such that $\Gamma_0 \subset g \cdot \Gamma_\xi = \Gamma_{g\xi}$ and such that $g \cdot \rho$ and $\alpha$ are $\Gamma_0$-associated;

(iii) There is $\gamma \in \Gamma$ such that $\gamma \cdot \Gamma_0 \subset \Gamma_\xi$ and $\mathrm{Hom}_{\gamma\cdot\Gamma_0}(\rho, \alpha) \neq 0$.

Indeed, if (i) is satisfied, then the definition of $X^\alpha_{M,\Gamma}$ (see Equation (8.4) and Definition 8.1.1) is equivalent to the existence of $g$, i.e. (i) $\Leftrightarrow$ (ii). Recalling that $g \cdot \rho \in \widehat{\Gamma_{g\xi}}$, we stress then that we need $\Gamma_0 \subset g \cdot \Gamma_\xi = \Gamma_{g\xi}$ for $\alpha$ and $g \cdot \rho$ to be associated.

To prove (ii) $\Leftrightarrow$ (iii), let $\gamma = g^{-1}$. We have $\Gamma_0 \subset g \cdot \Gamma_\xi$ and $\mathrm{Hom}_{\Gamma_0}(g \cdot \rho, \alpha) \neq 0$ if, and only if, $\gamma \cdot \Gamma_0 \subset \Gamma_\xi$ and $\mathrm{Hom}_{\gamma\cdot\Gamma_0}(\rho, \gamma \cdot \alpha) \neq 0$. The result follows from the fact that $\alpha$ and $\gamma \cdot \alpha$ are equivalent (since $\gamma \in \Gamma$ and $\alpha$ is a representation of $\Gamma$).

We include next below, in Proposition 8.3.5, a reformulation of our $\alpha$-ellipticity condition in terms of the fixed point manifold $S^*M^{\Gamma_0}$, with $\Gamma_0$ a minimal isotropy subgroup as before. This result was suggested by some discussions with P.-É. Paradan, whom we thank for his useful input.

In the following, $\mathrm{Stab}_\Gamma(M)$ will denote the set of stabilizer subgroups $K$ of $\Gamma$, that is, the set of subgroups $K \subset \Gamma$ such that there is $m \in M$ with $K = \Gamma_m$. It is a finite set, since $\Gamma$ is finite. Similarly, we let

$$\mathrm{Stab}_\Gamma^{\Gamma_0}(M) := \{K \in \mathrm{Stab}_\Gamma(M) \mid \Gamma_0 \subset K\}.$$

Note that $\mathrm{Stab}_\Gamma(T^*M) = \mathrm{Stab}_\Gamma(M)$. Recall also that $(T^*M)^K = T^*(M^K)$, where $M^K$ is the submanifold of fixed points of $M$ by $K$, as usual. For a $\Gamma$-space $X$ and $K \subset \Gamma$ a subgroup, we shall let $X_K := \{x \in X \mid \Gamma_x = K\} \subset X^K$ denote the set of points of $X$ with isotropy $K$. Note that, in general, $T^*(M_K) \neq (T^*M)_{(K)}$.

**Lemma 8.3.4.** *The set $M_K := \{m \in M \mid \Gamma_m = K\}$ is a submanifold.*

*Proof.* Let $x \in M_K$, that is, $\Gamma_x = K$. The problem is local, so, using, [196, Proposition 5.13], we see that it suffices to consider the case $M = \Gamma \times_K V$, where $V$ is a $K$-representation. Then, if $z = (\gamma, y) \in \Gamma \times_K V$, we have $\Gamma_z = \gamma K_y \gamma^{-1}$ and hence, if $\Gamma_z = K$, we obtain $K = \gamma K_y \gamma^{-1}$, which, in turn, gives $K_y = K$ and $\gamma \in N(K) := \{g \in \Gamma \mid gKg^{-1} = K\}$. We thus obtain that

$$M_K = \{(\gamma, y) \in \Gamma \times_K M \mid K_y = \gamma^{-1} K \gamma\} = N(K) \times_K V^K,$$

which is a submanifold of $M$. □

Let $K \subset \Gamma$ be a subgroup and $\rho \in \widehat{K}$. Then $E_\rho := \bigsqcup_{x\in M^K} E_{x\rho}$ is a smooth vector bundle over $M^K$, the set of fixed points of $M$ with respect to $K$. Similarly, $(E \otimes \rho)^K \to M^K$ is a smooth vector bundle (over $M^K$). Moreover, we have an isomorphism

$$\mathrm{End}(E_\rho)^K \simeq \mathrm{End}((E \otimes \rho)^K \otimes \rho)^K \simeq \mathrm{End}((E \otimes \rho)^K), \tag{8.24}$$

of vector bundles over $M^K$, where the last isomorphism comes from the fact that $\mathrm{End}(\rho)^K = \mathbb{C}$. In view of this discussion, we choose to state the following result in terms of the vector bundle $(E \otimes \rho)^K$ over $M^K$ rather than in terms of $E_\rho$. This discussion shows also that it is enough in our proofs to assume that $\alpha$ is the trivial (one-dimensional) representation.

**Proposition 8.3.5.** *Let $\alpha \in \widehat{\Gamma}$ and $P \in \Psi^m(M; E)$, for some $m \in \mathbb{R}$. Recall the vector bundle $(M \otimes \rho)^K \to M^K \supset M_K$. The following are equivalent:*

*(i) $P$ is $\alpha$-elliptic (Definition 8.1.2).*

*(ii) For all $K \in \mathrm{Stab}_\Gamma^{\Gamma_0}(M)$ and all $\rho \in \widehat{K}$ that are $\Gamma_0$-associated with $\alpha$, we have that $(\sigma_m(P) \otimes id_\rho)|_{(E\otimes\rho)^K}$ defines by restriction an invertible element of*

$$C^\infty((T^*M \setminus \{0\})_K, \mathrm{End}((E \otimes \rho)^K)) .$$

*(iii) The principal symbol $(\sigma_m(P) \otimes id_\alpha)|_{(E\otimes\alpha)^{\Gamma_0}}$ defines by restriction an invertible element in*

$$C^\infty(T^*M^{\Gamma_0} \setminus \{0\}; \mathrm{End}((E \otimes \alpha)^{\Gamma_0}) .$$

Recall that for representations $\alpha$ and $\beta$ to be $H$-associated, they have to be defined, after restriction, on $H$. See Definition 8.1.1.

*Proof.* Recall that $P$ is $\alpha$-elliptic if the restriction of $\sigma_m^\Gamma(P)$ to $X_{M,\Gamma}^\alpha$ is invertible (see Remark 8.3.3 for a detailed definition and discussion of the space $X_{M,\Gamma}^\alpha$ appearing in the definition of $\alpha$-ellipticity).

Let $K \in \mathrm{Stab}_\Gamma^{\Gamma_0}(M)$ (so $\Gamma_0 \subset K$), $\rho \in \widehat{K}$, and $\xi \in T_x^*M \setminus \{0\}$ with $\Gamma_\xi = K$. We have that $(\sigma_m(P) \otimes id_\rho)|_{(E\otimes\rho)^K}$ is invertible at $\xi \in (T^*M)_K$ if, and only if, the restriction of $\sigma_m(P)(\xi)$ to $E_{x\rho}$ is invertible, since they correspond to each other under the isomorphism of Equation (8.24). The relation (ii) thus means that the restriction of the principal symbol $\sigma_m(P)$ is invertible on a *subset* of $X_{M,\Gamma}^\alpha$, so (i) implies (ii) right away.

Let us show next that (ii) implies (i), let $\xi \in T^*M \setminus \{0\}$ and let $K' := \Gamma_\xi$. By definition, $\xi$ belongs to $(T^*M)_{K'}$. Assume that $(\xi, \rho) \in X_{M,\Gamma}^\alpha$. This means that there exists $g \in \Gamma$ such that $\rho' := g \cdot \rho$ and $\alpha$ are $\Gamma_0$ associated (see Equation (8.4) and Definition 8.1.1; alternatively, this is also recalled in Remark 8.3.3). For this to make sense, it is implicit that

$$\Gamma_0 \subset \Gamma_{g\xi} = g \cdot \Gamma_\xi = g \cdot K' =: K$$

(again, see Remark 8.3.3). Then $g : (T^*M)_{K'} \to (T^*M)_K$ is a diffeomorphism. Condition (ii) for the group $K$ gives that $\pi_{g\xi,\rho'}(\sigma_m(P))$ is invertible, since the irreducible representation $\rho'$ of $\Gamma_{g\xi}$ is $\Gamma_0$-associated to $\alpha$ (we have used here again the isomorphism (8.24)). Furthermore, $g : E_{\xi,\rho} \to E_{g\xi,\rho'}$ is an isomorphism. Now, by the $\Gamma$-invariance of $\sigma := \sigma_m(P)$, we have $(g^{-1}\sigma)(\xi) = g^{-1}(\sigma(g\xi))g = \sigma(\xi)$ therefore $\pi_{\xi,\rho'}(\sigma)$ is invertible if, and only if, $\pi_{g\xi,\rho}(\sigma)$ is.

For the equivalence of (i) and (iii), we can assume that $m = 0$. Recall first that the density of $\Xi_0$ in $\Xi$ established in Theorem 8.2.18 gives that the family of representations

$$\mathcal{F}_0 := \{\pi_{\xi,\rho} \mid (\xi, \rho) \in X_{M,\Gamma}^\alpha, \, \Gamma_\xi = \Gamma_0\}$$

is faithful for the $C^*$-algebra $A_M^\Gamma / \ker(\mathcal{R}_M)$ (see e.g. [178, Theorem 5.1]). In other words, the restriction morphism

$$A_M^\Gamma / \ker(\mathcal{R}_M) \to \bigoplus_{\substack{\rho \in \widehat{\Gamma}_0, \\ \rho \subset \alpha|_{\Gamma_0}}} C((T^*M \setminus \{0\})_{\Gamma_0}, \mathrm{End}(E_\rho)^{\Gamma_0})$$

is injective. Since $(T^*M)_{\Gamma_0}$ is dense in $T^*M^{\Gamma_0}$, it follows that the restriction morphism

$$R_M : A_M^{\Gamma}/\ker(\mathcal{R}_M) \to \bigoplus_{\substack{\rho\in\widehat{\Gamma}_0,\\ \rho\subset\alpha|_{\Gamma_0}}} C(T^*M^{\Gamma_0}\setminus\{0\}, \mathrm{End}(E_\rho)^{\Gamma_0})$$

is also injective.

Let us write $\alpha_{|\Gamma_0} = \bigoplus_{\rho\in\widehat{\Gamma}_0} m_\rho\rho$, with multiplicities $m_\rho \geq 0$. By considering the representations $\rho$ with $m_\rho > 0$, we see that there is an injective vector bundle morphism over the manifold $M^{\Gamma_0}$ defined by

$$\Psi : \bigoplus_{\substack{\rho\in\widehat{\Gamma}_0,\\ \rho\subset\alpha|_{\Gamma_0}}} \mathrm{End}(E_\rho)^{\Gamma_0} \simeq \bigoplus_{\substack{\rho\in\widehat{\Gamma}_0,\\ \rho\subset\alpha|_{\Gamma_0}}} \mathrm{End}((E\otimes\rho)^{\Gamma_0}) \hookrightarrow \mathrm{End}((E\otimes\alpha)^{\Gamma_0})\,, \tag{8.25}$$

where the last morphism maps any element $T \in \mathrm{End}((E\otimes\rho)^{\Gamma_0})$ to a direct sum of copies of $T$ acting on the direct summand $\left[(E\otimes\rho)^{\Gamma_0}\right]^{m_\rho} \subset (E\otimes\alpha)^{\Gamma_0}$.

Condition (iii) amounts to the fact that

$$\Psi(R_M(\sigma_0^\Gamma(P))) \in C^\infty(T^*M^{\Gamma_0}\setminus\{0\}; \mathrm{End}((E\otimes\alpha)^{\Gamma_0})$$

is invertible. To establish that (i) $\Leftrightarrow$ (iii), we thus need to prove that $P$ is $\alpha$-elliptic if, and only if, $\Psi(R_M(\sigma_0^\Gamma(P)))$ is invertible.

Recall the definition of the symbol algebras $A_M$ from Equation (8.6). We have that $P \in \overline{\Psi^0}(M;E)$ is $\alpha$-elliptic if, and only if, the image of $\sigma_0^\Gamma(P)$ in the quotient algebra $A_M^\Gamma/\ker(\mathcal{R}_M)$ is invertible (by the determination of $\ker(\mathcal{R}_M)$ in Remark 8.2.21 or Proposition 8.2.22). But since both $\Psi$ and $R_M$ are injective, $\Psi\circ R_M$ is injective on $A_M^\Gamma/\ker(\mathcal{R}_M)$. Thus $\sigma_0^\Gamma(P)$ is invertible in the quotient algebra $A_M^\Gamma/\ker(\mathcal{R}_M)$ if, and only if, $\Psi(R_M(\sigma_0^\Gamma(P)))$ is invertible. As we have seen above, this amounts to (i) $\Leftrightarrow$ (iii). ☐

### 8.3.2. Fredholm conditions and Hodge and index theory

We continue to assume that $M$ is a compact smooth manifold. Let us recall the following corollary of Atkinson's Theorem, see Corollary 7.5.2

**Corollary 8.3.6.** *Let $P \in \overline{\Psi^0}(M;E)^\Gamma$ and $\alpha \in \hat{\Gamma}$. We have that $\pi_\alpha(P)$ is Fredholm on $L^2(M;E)_\alpha$ if, and only if, $\pi_\alpha(P)$ is invertible modulo $\pi_\alpha(\mathcal{K}^\Gamma)$ in $\pi_\alpha(\overline{\Psi^0}(M;E)^\Gamma)$.*

We are now in a position to prove the main result of this chapter, Theorem 8.1.3.

*Proof ot Theorem 8.1.3.* Following the standard reduction to order-zero operator (see Section 4.2.3), we may assume that $P \in \overline{\Psi^0}(M;E)^\Gamma$. Corollary 8.3.6 then states that $\pi_\alpha(P)$ is Fredholm if, and only if, the image of its symbol $\sigma(P)$ is invertible in the quotient algebra

$$\mathcal{R}_M(A_M^\Gamma) = \pi_\alpha(\overline{\Psi^0}(M;E)^\Gamma)/\pi_\alpha(\mathcal{K}(L^2(M;E))^\Gamma).$$

According to Proposition 8.2.20 and Remark 8.2.21 following it, the primitive spectrum $\Xi$ of $\mathcal{R}_M(A_M^\Gamma)$ identifies with $X_{M,\Gamma}^\alpha$. Therefore $\mathcal{R}_M(\sigma(P))$ is invertible if, and only if, the endomorphism $\pi_{\xi,\rho}(\sigma(P))$ is invertible for all $(\xi, \rho) \in X_{M,\Gamma}^\alpha$, i.e. if, and only if, $P$ is $\alpha$-elliptic (see Definition 8.1.2). $\square$

**Remark 8.3.7.** Let $P : H^s(M; E) \to H^{s-m}(M; E)$ be an order $m$, classical pseudodifferential operator. Since the index of Fredholm operators is invariant under small perturbations and under compact perturbations, we obtain that the index of $\pi_\alpha(P)$ depends only on the homotopy class of its $\alpha$-principal symbol $\sigma_m^\alpha(P)$.

An alternative approach to the Fredholm property (Theorem 8.1.3) can be obtained from the following theorem. Recall that $X_{M,\Gamma}^\alpha$ was defined in (8.4). Below, by $\partial$ we shall denote the connecting morphism in the six-term $K$-theory exact sequence associated to a short exact sequence of $C^*$-algebras. Recall that $\sigma_0^\alpha$ is the $\alpha$-principal symbol map, see Definition 8.1.2.

**Theorem 8.3.8.** *Let us denote by $C(X_{M,\Gamma}^\alpha)$ the algebra of restrictions of $A_M := C(S^*M; \mathrm{End}(E))^\Gamma$ to $X_{M,\Gamma}^\alpha$. Using the notation of Corollary 8.3.6, we have an exact sequence*

$$0 \longrightarrow \mathcal{K} \longrightarrow \pi_\alpha(\overline{\Psi^0}(M; E)^\Gamma) \xrightarrow{\sigma_0^\alpha} C(X_{M,\Gamma}^\alpha) \longrightarrow 0.$$

*Let $\partial : K_1(C(X_{M,\Gamma}^\alpha)) \to \mathbb{Z} \simeq K_0(\mathcal{K})$ be the associated connecting morphism and let $P \in \overline{\Psi^0}(M; E)^\Gamma$ such that $\pi_\alpha(P)$ is Fredholm. Then* $\mathrm{index}(\pi_\alpha(P)) = \dim(\alpha)\partial[\sigma_0^m(P)]$.

*Proof.* The exactness of the sequence follows from the proof of Corollary 8.3.6 and the fact that $\mathcal{K}(L^2(M; E)_\alpha)^\Gamma \simeq \mathcal{K}$, the algebra of compact operators on a model separable Hilbert space $\mathcal{H}$. Under this isomorphism, the resulting representation of $\mathcal{K}$ on $L^2(M; E)_\alpha$ is isomorphic to $\dim(\alpha)$ times the standard representation of $\mathcal{K}$ on $\mathcal{H}$. This justifies the factor $\dim(\alpha)$. The rest follows from the fact that the index is the connecting morphism in $K$-theory for the Calkin exact sequence. See [147] for more details. $\square$

**Remark 8.3.9.** As in [147], it follows that the index of $\pi_\alpha(P)$ with $P \in \Psi^0(M; E)^\Gamma$ is the pairing between a cyclic cocycle $\phi$ on $C^\infty(X_{M,\Gamma})$ (the algebra of principal symbols of operators in $\Psi^0(M; E)^\Gamma$) and the $K$-theory class of the $\alpha$-principal symbol of $P$ [58]. See also [47, 59, 60, 102]. Lemma 8.2.8 gives that the restriction of this cyclic cocycle to the principal orbit bundle is the usual Atiyah-Singer cocycle (i.e. the cocycle that yields the Atiyah-Singer index theorem in cyclic homology [61, 116, 147, 160], which thus corresponds, after suitable rescaling, to the Todd class). The full determination of the class of the index cyclic cocycle $\phi$ require, however, a non-trivial use of cyclic homology, since the quotient algebra $C^\infty(X_{M,\Gamma})$ is non-commutative, in general.

**Remark 8.3.10.** As for the case of compact complex varieties [89, 206], we can consider complexes of operators [39] and the corresponding notion of $\alpha$-ellipticity. In particular, we obtain the finiteness of the corresponding cohomology groups if the complex is $\alpha$-elliptic.

This is related to the Hodge theory of singular spaces [3, 36, 40, 54, 194, 195]. Moreover, since a general $P$ may act between different bundles $E_0$ and $E_1$, it would be convenient to extend our framework to Connes' tangent groupoid [60, 11]. However, this goes beyond the scope of this chapter.

### 8.3.3. Special cases

We now specialize our main result to some particular cases.

#### The abelian case of Chapter 7

Many statements and definitions become easier in the case of abelian groups. In particular, if $\Gamma_i$, $i = 1, 2$, are both abelian, then the irreducible representations $\alpha_i \in \widehat{\Gamma}_i$ are characters, that is, morphisms $\alpha_i : \Gamma_i \to \mathbb{C}^*$, and we have that they are $H$-associated for some subgroup $H$ if, and only if, $\alpha_1|_H = \alpha_2|_H$.

Let $\alpha$ be an irreducible representation of $\Gamma$. When $\Gamma$ is abelian, the conjugacy class of isotropy subgroups corresponding to the principal orbit type of the action has only one element, namely $\Gamma_0$. In that case, the set $X^{\alpha}_{M,\Gamma}$ defined in Equation (8.4) of the introduction has the simpler expression:

$$X^{\alpha}_{M,\Gamma} = \{(\xi, \rho) \mid \xi \in T^*M \setminus \{0\},\, \rho \in \widehat{\Gamma}_\xi, \rho|_{\Gamma_0} = \alpha|_{\Gamma_0}\}.$$

As a consequence, it is easier to check the $\alpha$-ellipticity for an operator $P$ in the abelian case. Let $E, F$ be $\Gamma$-equivariant vector bundles over $M$ and set $\alpha_0 := \alpha|_{\Gamma_0}$. Recall that, for any $x \in M$, we denote by $E_{x\alpha_0}$ the $\alpha_0$-isotypical component of $E_x$, seen as a $\Gamma_0$-representation. We then recover the main result of Chapter 7, Theorem 7.1.2.

#### Scalar operators

Our main theorem becomes quite explicit when we are dealing with scalar operators, i.e. when the vector bundles $E_i = M \times \mathbb{C}$, where $\mathbb{C}$ denotes the trivial representation of $\Gamma$.

**Proposition 8.3.11.** *Let $P : H^s(M) \to H^{s-m}(M)$ be a $\Gamma$-invariant pseudodifferential operator. Let $\alpha \in \widehat{\Gamma}$. Then $P$ is $\alpha$-elliptic if, and only if, $\sigma(P)(\xi)$ is invertible for all $\xi \in T^*M \setminus \{0\}$ such that $\alpha$ is $\Gamma_0$-associated to the trivial (constant 1) representation of $\Gamma_\xi$.*

*Proof.* Let $\widehat{1}_{\Gamma_\xi}$ denote the trivial representation of $\Gamma_\xi$ and let $(\xi, \rho) \in X^{\alpha}_{M,\Gamma}$. If $\rho \neq \widehat{1}_{\Gamma_\xi}$ then $\mathbb{C}_\rho = 0$ and then $\pi_\rho(\sigma(P)(\xi)) : 0 \to 0$ is invertible. Now if $\rho = \widehat{1}_{\Gamma_\xi}$ then $(\xi, \rho) \in X^{\alpha}_{M,\Gamma}$ if, and only if, $\alpha$ is $\Gamma_0$-associated to $\widehat{1}_{\Gamma_\xi}$. □

#### Trivial actions

Assume that $\Gamma$ acts trivially on $M$ (in particular, $M$ is then also connected). Our assumption implies that $\Gamma_0 = \Gamma_\xi = \Gamma$, for all $\xi \in T^*M \setminus \{0\}$. It follows that $\rho \in \widehat{\Gamma}_\xi$ is $\Gamma_0$-associated to $\alpha \in \widehat{\Gamma}$ if, and only if, $\alpha = \rho$.

Let $E \to M$ be a $\Gamma$-equivariant vector bundle. For any $x \in M$, recall that we denote $E_{x\alpha}$ the $\alpha$-isotypical component of $E_x$. Assuming $M$ to be connected, we have that $E_\alpha = \bigcup_{x \in M} E_{x\alpha}$ is a $\Gamma$-equivariant sub-vector bundle of $E$. Our main result then becomes the following statement.

**Proposition 8.3.12.** *Assume that $\Gamma$ acts trivially on $M$ and let $\alpha \in \widehat{\Gamma}$. Let $E, F$ be two $\Gamma$-equivariant vector bundles over $M$ and let $P \in \Psi^m(M; E, F)^\Gamma$. Then for any $s \in \mathbb{R}$, the following are equivalent*

*(i)* $\pi_\alpha(P) : H^s(M; E_\alpha) \to H^{s-m}(M; F_\alpha)$ *is Fredholm,*

*(ii) for all $(x, \xi) \in T^*M \setminus \{0\}$, the morphism*

$$\pi_\alpha(\sigma(P)(x, \xi)) : E_{x\alpha} \to F_{x\alpha}$$

*is invertible,*

*(iii) for all $(x, \xi) \in T^*M \setminus \{0\}$, the morphism*

$$\sigma_m(P) \otimes \mathrm{id}_{\alpha^*}(x, \xi) : \mathrm{Hom}_\Gamma(\alpha, E_x) \to \mathrm{Hom}_\Gamma(\alpha, F_x)$$

*is invertible.*

Of course, the above result is nothing but the classical condition that the elliptic operator $p_{F_\alpha} P p_{E_\alpha} \in \Psi^m(M; E_\alpha, F_\alpha)$ be Fredholm.

*Proof.* The equivalence between (i) and (ii) is a direct consequence of Theorem 8.1.3. Let us check the equivalence of (i) and (iii). First note that

$$(H^s(M, E) \otimes \alpha)^\Gamma = H^s(M, (E \otimes \alpha)^\Gamma),$$

since the action of $\Gamma$ on $M$ is trivial. The operator $\pi_\alpha(P)$ is Fredholm if, and only if, the pseudodifferential operator $P_\alpha : H^s(M, \mathrm{Hom}(\alpha, E)^\Gamma) \to H^{s-m}(M, \mathrm{Hom}(\alpha, F)^\Gamma)$ defined for any $v^* \in \alpha^*$ and $s \in C^\infty(M, E)$ by $P_\alpha(v^* s) = v^* P s$ is Fredholm. Furthermore, the operator $P_\alpha$ is Fredholm if, and only if, it is elliptic, that is if, and only if, $\sigma_m(P) \otimes \mathrm{id}_{\alpha^*}(x, \xi) : \mathrm{Hom}_\Gamma(\alpha, E_x) \to \mathrm{Hom}_\Gamma(\alpha, F_x)$ is invertible for any $(x, \xi) \in T^*M \setminus \{0\}$. Note that the invertibility of $\sigma_m(P) \otimes \mathrm{id}_{\alpha^*}(x, \xi)$ is equivalent to the invertibility of $\pi_\alpha(\sigma_m(P)(x, \xi))$ by definition, which is consistent with (ii). □

### Free action on a dense subset

As in the previous sections, the group $\Gamma$ is finite and acts continuously on the manifold $M$. We consider vector bundles $E, F \to M$.

We have following corollary of the last few results in Section 8.2.

**Corollary 8.3.13.** *Let us assume that $\Gamma$ acts freely on a dense open subset of $M$. Then* $\Xi = \mathrm{Prim}(A_M^\Gamma)$.

*Proof.* The assumption on the action implies that $\Gamma_0 = \{1\}$. If $\xi \in T^*M \setminus \{0\}$ and $\rho \in \widehat{\Gamma}_\xi$, then $\rho$ and $\alpha$ are always $\{1\}$-associated. The Corollary then follows from Proposition 8.2.20. □

Similarly, we have the following result.

**Proposition 8.3.14.** *Assume that* $\Gamma$ *acts freely on a dense subset in* $M$*, and let* $P \in \Psi^m(M; E, F)^\Gamma$*. For any* $\alpha \in \widehat{\Gamma}$*, we have that* $P$ *is* $\alpha$*-elliptic if, and only if,* $P$ *is elliptic.*

*Proof.* It follows from Corollary 8.3.13 that $X^\alpha_{M,\Gamma} = X_{M,\Gamma}$. Thus the operator $P_\alpha$ is $\alpha$-elliptic if, and only if, the sum $\bigoplus_{\rho\in\widehat{\Gamma}_\xi} \pi_\rho(\sigma_m(P)(\xi)) = \sigma_m(P)(\xi)$ is invertible for all $\xi \in T^*M \setminus \{0\}$, that is, if, and only if, $P$ is elliptic. □

### 8.3.4. Simonenko's localization principle

In this section, we obtain an equivariant version of Simonenko's principle [189]. In this subsection and the rest of the chapter, we consider a compact Lie group $G$ instead of $\Gamma$.

#### Simonenko's general principle

Let $A$ be a unital $C^*$-algebra and $Z \simeq C(\Omega_Z)$ a unital sub-$C^*$-algebra in $A$, i.e. $1_Z = 1_A$. An element $a \in A$ is said to *have the strong Simonenko local property* with respect to $Z$ if, for every $\phi, \psi \in Z$ with compact disjoint supports, $\phi a \psi = 0$.

**Lemma 8.3.15.** *The set* $B \subset A$ *of elements* $a$ *satisfying the strong Simonenko local property is the set of elements of* $A$ *commuting with* $Z$*.*

*Proof.* We are going to show that the set of elements $a \in A$ with the strong Simonenko local type property is a $C^*$-algebra $B$ containing $Z$ and that every irreducible representation of $B$ restricts to a scalar valued representation on $Z$, and hence that $Z$ commutes with $B$.

Let us show first that $B$ is a sub-$C^*$-algebra of $A$. Note that $B$ is not empty since $Z \subset B$. To show that $B$ is a sub-$C^*$-algebra, the only fact that is non-trivial to prove is that $ab \in B$, for all $a, b \in B$. Let $\phi$ and $\psi \in Z$ with disjoint compact supports and let $\theta$ be a function equal to 1 on $\mathrm{supp}(\psi)$ and 0 on $\mathrm{supp}(\phi)$, which exists by Urysohn's lemma. Then we have

$$\phi ab\psi = \phi a(\theta + 1 - \theta)b\psi = \phi a\theta b\psi + \phi a(1-\theta)b\psi = 0, \tag{8.26}$$

since $\phi a\theta = 0$ and $(1-\theta)b\psi = 0$.

Let $\pi : B \to \mathcal{L}(H)$ be an irreducible representation of $B$. First, let us show that for any $\phi, \psi \in Z$ with disjoint support, we either have $\pi(\phi) = 0$ or $\pi(\psi) = 0$. Indeed we have $\pi(\phi)\pi(a)\pi(\psi) = 0$ since $\phi a\psi = 0$, for any $a \in B$. Assume that $\pi(\psi) \neq 0$ then there is $\eta \in H$ such that $\pi(\psi)\eta \neq 0$. Now, $\pi$ is irreducible so we get that the set $\{\pi(a)\pi(\psi)\eta,\ a \in B\}$ is dense in $H$. Thus $\pi(\phi) = 0$ on a dense subspace of $H$ and so on $H$.

Assume now that $\pi(Z) \neq \mathbb{C}1_H$. Then there exist two distinct characters $\chi_0, \chi_1 \in \mathrm{Spec}(\pi(Z))$. Denote by $h_\pi : \mathrm{Spec}(\pi(Z)) \to \mathrm{Spec}(Z)$ the injective map adjoint to $\pi$, and choose $\phi, \psi \in C(\mathrm{Spec}(Z))$ with disjoint supports such that $\phi(h_\pi(\chi_0)) = 1$ and $\psi(h_\pi(\chi_1)) = 1$. Then $\pi(\phi)(\chi_0) = 1$ and $\pi(\psi)(\chi_1) = 1$, which contradicts the fact that either $\pi(\phi) = 0$ or $\pi(\psi) = 0$. □

Recall that a family $(\varphi_i)_{i\in I}$ of morphisms of a $C^*$-algebra $A$ is said to be *exhaustive* if any primitive ideal contains some $\ker(\varphi_i)$ for a suitable $i \in I$, see Section 4.1.3. Then Remark 7.2.6 gives that the family of morphisms

$$\chi_\omega : A \to A/\omega A, \tag{8.27}$$

for $\omega \in \Omega_Z$, is exhaustive for $A$.

**Definition 8.3.16.** Denote by $\mathcal{H} = L^2(M)$. An operator $P \in \mathcal{L}(\mathcal{H})$ is said to be *locally invertible* at $x \in M$ if there are a neighbourhood $V_x$ of $x$ and operators $Q_1^x$ and $Q_2^x \in \mathcal{L}(\mathcal{H})$ such that

$$Q_1^x P\phi = \phi \quad \text{and} \quad \phi P Q_2^x = \phi, \qquad \text{for any } \phi \in C_c(V_x). \tag{8.28}$$

The operator $P$ is said to be *locally invertible* if it is locally invertible at any $x \in M$.

Let $\Psi_M \subset \mathcal{L}(\mathcal{H})$ be the $C^*$-algebra of all $P \in \mathcal{L}(\mathcal{H})$ such that $\phi P \psi \in \mathcal{K}(\mathcal{H})$, for all $\phi, \psi \in C(M)$ with disjoint support. We denote by $\mathcal{B}_M$ the image of $\Psi_M$ in the Calkin algebra $\mathcal{Q}(\mathcal{H}) := \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. We know by Lemma 8.3.15 that

$$\mathcal{B}_M = \{P \in \mathcal{Q}(H) \mid \phi P = P\phi \text{ for all } \phi \in C(M)\}.$$

Simonenko's principle is then [189]:

**Proposition 8.3.17** (Simonenko's principle)**.** *If $P \in \Psi_M$, then $P$ is locally invertible if, and only if, it is Fredholm.*

We shall prove, in fact, a stronger version of this result, see Proposition 8.3.19.

### 8.3.5. Compact (non-finite) groups

We now allow for compact groups and try to see to what extent our results remain valid. To that end, we turn to an analog of Simonenko's principle. Let then $G$ be a compact Lie group acting smoothly on $M$. We continue to study Fredholm conditions for $\pi_\alpha(P)$, $\alpha \in \widehat{G}$. Denote by $\mathcal{H} := L^2(M, E)$ and by $\mathcal{H}_\alpha$ the $\alpha$-isotypical component associated to $\alpha \in \widehat{G}$.

**Definition 8.3.18.** We shall say that $P \in \mathcal{L}(\mathcal{H})$ is *locally $\alpha$-invertible at $x \in M$* if there are a $G$-invariant neighbourhood $V_x$ of $\Gamma x$ and operators $Q_1^x$ and $Q_2^x \in \mathcal{L}(\mathcal{H}_\alpha)$ such that

$$Q_1^x \pi_\alpha(P)\phi = \phi \qquad \text{and} \qquad \phi\pi_\alpha(P)Q_2^x = \phi, \tag{8.29}$$

as operators on $\mathcal{H}_\alpha$, for any $\phi \in C(M)^G$ supported in $V_x$.

We denote by $\Psi_M^G$ the $G$-invariant elements in the $C^*$-algebra $\Psi_M$, which was defined in the previous subsection.

**Proposition 8.3.19** (Simonenko's equivariant principle)**.** *Let $P \in \Psi_M^G$. Then $P$ is locally $\alpha$-invertible if, and only if, $\pi_\alpha(P)$ is Fredholm.*

*Proof.* We now use the results of the last section for $Z = C(M)^G = C(M/G)$. Let $\mathcal{B}_M^\alpha$ be the image of $\Psi_M^G$ in the Calkin algebra $\mathcal{Q}(\mathcal{H}_\alpha)$. We know from Lemma 8.3.15 that

$$\mathcal{B}_M^\alpha = \{P \in \mathcal{Q}(\mathcal{H}_\alpha) \mid \phi P = P\phi,\ \forall \phi \in C(M)^G\}.$$

Assume that $P$ is locally $\alpha$-invertible, i.e. $\forall x \in M$, there are a neighborhood $V_x$ of $Gx$ and operators $Q_1^x, Q_2^x \in \mathcal{L}(\mathcal{H}_\alpha)$ such that $Q_1^x \pi_\alpha(P)\phi = \phi$ and $\phi\pi_\alpha(P)Q_2^x = \phi$, for any $\phi \in C(M)^G$ supported in $V_x$. Let $\chi_x$, be the family of representations of $\mathcal{B}_G^\alpha$ introduced in Equation (8.27). We use the same notation for $\pi_\alpha(P)$ and its image in $\mathcal{Q}(\mathcal{H}_\alpha)$. We have that

$$\chi_x(Q_1^x\pi_\alpha(P)\phi) = \chi_x(Q_1^x)\chi_x(\pi_\alpha(P))\chi_x(\phi) = \chi_x(\phi).$$

Since $\chi_x(\phi) = 1$, we get:

$$\chi_x(Q_1^x)\chi_x(\pi_\alpha(P)) = 1.$$

And similarly,

$$\chi_x(\pi_\alpha(P))\chi_x(Q_2^x) = 1.$$

Therefore, $\chi_x(\pi_\alpha(P))$ is invertible for all $x$. Since the family $\chi_x$ is exhaustive, it follows that that $\pi_\alpha(P)$ is invertible in $\mathcal{B}_M^\alpha$ and so it is Fredholm.

Now assume that $\pi_\alpha(P)$ is Fredholm and let $Q$ be an inverse modulo $\mathcal{K}(\mathcal{H}_\alpha)$ for $\pi_\alpha(P)$, i.e. $\pi_\alpha(P)Q = id + K$ and $Q\pi_\alpha(P) = id + K'$, with $K, K' \in \mathcal{K}(\mathcal{H}_\alpha)$. Using Corollary 8.3.6, we can assume that $K = \pi_\alpha(k)$ and $K' = \pi_\alpha(k')$, with $k, k' \in \mathcal{K}(\mathcal{H})^G$. Let $\chi \in C(M)^G$ be equal to 1 on a $G$-invariant neighbourhood $V_x$ of $Gx$ and let $\phi \in C(M)^G$ be supported in $V_x$ then

$$\phi\chi\pi_\alpha(P)Q\chi = \phi\chi^2 + \phi\chi K\chi \qquad \text{and} \qquad \chi\pi_\alpha(P)Q\chi\phi = \chi^2\phi + \chi K'\chi\phi.$$

Since $\phi$ is supported in $V_x$, we have $\phi\chi = \phi$ and so

$$\phi\pi_\alpha(P)Q\chi = \phi(1 + \chi K\chi) \qquad \text{and} \qquad \pi_\alpha(P)Q\chi\phi = (1 + \chi K'\chi)\phi.$$

As $V_x$ becomes smaller and smaller, we have that $\chi$ converges strongly to 0. Since $K$ is compact, we obtain that $\|\chi K\chi\| \to 0$. Thus, by choosing $V_x$ small enough, we may assume that $\|\chi K\chi\| < 1$ and $\|\chi K'\chi\| < 1$.

It follows that $(1 + \chi K\chi)$ and $(1 + \chi K'\chi)$ are invertible and this implies

$$\phi\pi_\alpha(P)(Q\chi(1 + \chi K\chi)^{-1}) = \phi \quad \text{and} \quad ((1 + \chi K'\chi)^{-1}\chi Q)\pi_\alpha(P)\phi = \phi,$$

i.e. $P$ is locally $\alpha$-invertible. □

**Corollary 8.3.20.** *Assume that $M$ is compact, $\Gamma$ is a finite group and $M/\Gamma$ connected. Let $P \in \Psi(M; E, F)^\Gamma$ and $\alpha \in \widehat{\Gamma}$. Then the following are equivalent:*

*(i)* *$\pi_\alpha(P) : H^s(M; E)_\alpha \to H^{s-m}(M; F)_\alpha$ is Fredholm for any $s \in \mathbb{R}$,*

*(ii)* *$P$ is $\alpha$-elliptic,*

*(iii)* *$P$ is locally $\alpha$-invertible.*

*Proof.* The first equivalence is given by Theorem 8.1.3. Now since a finite group is compact Proposition 8.3.19 implies that (i) is equivalent to (iii). □

### Transversally elliptic operators

As in Chapter 7, we conclude with a discussion of transversally elliptic operators. Assume that $M$ is a compact smooth manifold and that $G$ is a compact Lie group acting on $M$. Denote by $\mathfrak{g}$ the Lie algebra of $G$. Then any $X \in \mathfrak{g}$ defines as usual the vector field $X_M^*$ given by $X_M^*(m) = \frac{d}{dt}_{|t=0} e^{tX} \cdot m$. Denote by $\pi : T^*M \to M$ the canonical projection and let us introduce as in [12] the $G$-transversal space

$$T_G^*M := \{\alpha \in T^*M \mid \alpha(X_M^*(\pi(\alpha))) = 0, \forall X \in \mathfrak{g}\}.$$

Recall that a $G$-invariant classical pseudodifferential operator $P$ of order $m$ is said *$G$-transversally elliptic* if its principal symbol is invertible on $T_G^*M \setminus \{0\}$ [12, 155].

We may now state the classical result of Atiyah and Singer [12, Corollary 2.5].

**Theorem 8.3.21** (Atiyah-Singer [12]). *Assume $P$ is $G$-transversally elliptic. Then, for every irreducible representation $\alpha \in \widehat{G}$, $\pi_\alpha(P) : H^s(M; E_0)_\alpha \to H^{s-m}(M; E_1)_\alpha$, is Fredholm.*

Note that this implies that Theorem 8.1.3 is not true anymore for if $G$ is non-discrete. In particular, we obtain the following consequence of the localization principle.

**Corollary 8.3.22.** *Assume that $M$ is compact and that $G$ is a compact Lie group and let $P \in \Psi^m(M; E)^G$ be a $G$-transversally elliptic operator. Then $P$ is locally $\alpha$-invertible for any $\alpha \in \widehat{G}$, as in Definition 8.3.18.*

*Proof.* Using Theorem 8.3.21 we obtain that $\pi_\alpha(P)$ is Fredholm. Therefore by Proposition 8.3.19 $P$ is $\alpha$-invertible. $\square$

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# Index

# Index of notations

## Analyse sur les espaces singuliers et théorie de l'indice

**Mots-clefs :** analyse globale, opérateurs elliptiques, algèbres d'opérateurs, théorie de l'indice.

Le contexte général de cette thèse est celui de l'extension de la théorie des opérateurs elliptiques, bien connue dans le cadre lisse, à des domaines dits *singuliers*. Les méthodes utilisées reposent d'une part sur l'emploi d'algèbres d'opérateurs et d'outils issus de la géométrie non commutative, d'autre part sur l'introduction de calculs pseudodifférentiels adaptés à la géométrie du domaine, souvent *via* un groupoïde qui résout les singularités.

La première partie de la thèse s'intéresse à l'étude d'une classe particulière de ces groupoïdes, dits Fredholm, qui donnent un cadre très favorable à l'analyse des opérateurs elliptiques. Un des résultats majeurs obtenu est que cette propriété de Fredholm est locale, au sens où elle ne dépend que des restrictions du groupoïde à un nombre suffisant d'ouverts. Dans le même esprit, nous considérons avec C. Carvalho et Y. Qiao des groupoïdes obtenus comme recollements d'actions de groupes, et étudions en particulier un groupoïde adapté à l'étude des opérateurs potentiels de couche. Je conclus cette partie avec la résolution d'un problème aux limites pour un domaine à singularité de type cusp rotationnel.

La seconde partie s'intéresse aux opérateurs équivariants sur des variétés compactes, sous l'action d'un groupe fini. On répond à la question suivante : étant donnée une représentation irréductible du groupe, à quelle condition un opérateur différentiel est-il Fredholm entre les composantes isotypiques correspondantes des espaces de Sobolev ? Dans un travail commun avec A. Baldare, M. Lesch et V. Nistor, nous définissons une notion correspondante d'ellipticité associée à une représentation irréductible fixée et montrons qu'elle caractérise les opérateurs de Fredholm.

## Analysis on singular spaces and index theory

**Keywords:** global analysis, elliptic operators, operator algebras, index theory.

This thesis is set in the general context of extending the theory of elliptic operators, well-understood in the smooth setting, to so-called *singular domains*. The methods used rely on operator algebras and tools coming from non commutative geometry, together with suitable pseudodifferential calculi that are often built from a groupoid adapted to the particular geometry of the problem.

The first part of the thesis deals with the general investigation of a particular class of such groupoids, called Fredholm, that provide a very good setting for the study of elliptic operators. One of the major results proved here is that this Fredholm property is local, in the sense that it only depends on the restrictions of the groupoid to sufficiently many open subsets. In the same spirit, we study with C. Carvalho and Y. Qiao groupoids whose local structure is given by gluing group actions, and consider in particular a groupoid suited to the study of layer potential operators. This part concludes with a well-posedness result for a boundary value problem on a domain with a rotational cusp.

The second part deals with equivariant operators on a compact manifold, acted upon by a finite group. We answer the following question: given an irreducible representation of the group, under which condition is a differential operator Fredholm between the corresponding isotypical components of the Sobolev spaces? In a joint work with A. Baldare, M. Lesch and V. Nistor, we introduce a corresponding notion of ellipticity associated with some fixed irreducible representation, and show that it characterizes Fredholm operators.