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Thèse de doctorat

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Introduction

The relationship between human capital accumulation and economic growth has sparked a huge literature in theoretical research. In spite of the fact that the literature on the different determinants of economic growth has shown powerful results, this dissertation focuses only on investment in education as the main source of human capital accumulation and consequently long-run economic growth.

According to Solow (1956), the contribution of the total factor productivity (TFP) is around 80 percent for the US economic growth. Solow explains long-run economic growth by considering capital accumulation, population growth and increases in total productivity, commonly referred to as technological progress. The Solow model uses the form of Cobb-Douglas production function: $Y_t = A_t K_t^{\alpha} N_t^{\beta}$, where A_t is exogenous technical progress; K_t, N_t are the number of machines and the number of workers, respectively. In this equation, the machines have the same quality, also the workers have the same skill. However, the restrictive assumptions of the contribution of Solow are the constant growth rate of savings and exogenous technological change. Furthermore, the barrier to long-run economic growth in the Solow model is the diminishing returns. Therefore, for long-run economic growth to take place, there has to be a way to overcome the diminishing returns so that the productive inputs accumulate over time.

Years after the study of Solow (1956), many works continue to introduce shocks on the technological progress and study their impact on the economy (see e.g., Brock & Mirman, 1973; Kydland & Prescott, 1982; King, Plosser, & Rebelo, 1988; King, 1999). For example, King et al. (1988) state the implications of capital accumulation and economic fluctuations initiated by impulses to technology. They show that substantial persistence in technological shocks is required if the economy is to spread through periods of economic activity that persistently diverge from a deterministic tendency.

Nevertheless, if the technological improvements over time still occur exogenously, then this way is quite a restriction and its reexamination has given rise to a new approach, i.e., the notion of effective inputs (physical and human capital). This also opens the door to different directions of endogenous growth theory.

In another circumstance, Mankiw, Romer, and Weil (1992) rediscover the seminal article of Solow (1956) and show that the exclusion of human capital accumulation from the model of Solow may potentially explain why the estimated impacts of savings and population growth on income per capita are two large. After introducing human capital accumulation as an additional explanatory variable into their cross-country regressions, they show that human capital is in fact correlated with the rate of savings and population growth. Particularly, they examine that the number of 80 percent for economic growth of the Solow model is correct if human capital formation is taken to account.

Inspired by Solow (1956) and Mankiw et al. (1992), among others, this thesis begins to introduce the notion of effective inputs (physical and labor). Let κ, h respectively denote by the average quality of the machines and the average skill of labors then the output should depend on the effective inputs. It turns out that the production output of Solow's model may turn out to be the new form: $Y_t = A(\kappa_t K_t)^\alpha (h_t N_t)^\beta \equiv [A\kappa_t^\alpha h_t^\beta] K_t^\alpha N_t^\beta$. This opens the door to two ways of endogenous growth. The first is to introduce technology as factor enhancing the capital productivity (Romer, 1990). The second is to introduce education and/or training to improve the productivity of labor and human capital (Romer, 1986; Lucas, 1988; Lucas, 2015; Krueger & Lindahl, 2001). These authors call for more attention to raising labor productivity via enhancing human capital accumulation. For instance, Lucas (1988) proves that human capital accumulation is determinant for economic growth with externalities. Interestingly, Lucas himself, Lucas (2015) calls for the need of placing human capital formation at the center of economic growth[1], instead of the external source. Even though investment in both new technology and human capital will provoke endogenous economic growth, the dissertation, however, studies only on education investment in generating human capital accumulation and ultimately economic growth.

Besides, this thesis also considers the two main following aspects: On the one hand, it places at the center of the role of the human capital accumulation in generating economic growth without an external source, such as new technology (Lucas, 2015). This is also in line with the work of Le Van, Luong, Nguyen, and Nguyen (2010), who suggest that a country needs to devote capitals to investment in education and training for enhancing human capital stock in the last stage of the economic development process. They explain the emergence of various development stages consistent with unbounded growth. On the other hand, it considers the impact of the interaction between physical capital and human capital on economic growth (McGrattan & Prescott, 2009; Galor & Moav, 2004; Schoellman, 2012; Manuelli & Seshadri, 2014; Manuelli, 2015).

Human capital, according to Smith (1776) and Becker (1964), accounts for labor productivity and depends on the worker's state of health and level of education. In equilibrium, when the labor market is perfect, human capital turns out to be equivalent to the discounted value of life-span labor incomes. Since the seminal Lucas' contribution (1988), the accumulation of human capital is also recognized as a crucial source of economic growth.

Investments in human capital (health and education) are either private or public choices. While the household can decide the share of income, altruistic investment and leisure devoted to schooling, nutrition, sports or medicines, a government can also implement on education policies and/or improve the health system via enhancing the accumulation of human capital. In this dissertation,

[1]This is the key motivation for the dissertation.

the first two chapters study the role of private investment in education in the accumulation of human capital and consequently economic growth. The third chapter considers this investment as a public choice, i.e., through government's expenditure on education, funded by taxation. The final chapter addresses an important technical issue, namely the non-concave technology, by considering a growth model with only human capital. The dissertation is constructed as follows.

The first chapter is co-authored with Cuong LE VAN, Thai HA-HUY, and Cao-Tung PHAM. This chapter revisits the theoretical framework of endogenous economic growth by considering models where human capital accumulation is placed at the center of the process of economic growth. Our work is in accordance with Lucas (2015) who calls for the need to avoid considering "too large a role to exogenous technological change" (p. 86), while highlighting that "the contribution of human capital accumulation to economic growth deserves a production function of its own" (p. 87). The novelty of this chapter is that we build a bridge between Ramsey (1928) and Lucas (1988, 2015).

In the second chapter, co-written with Cao-Tung PHAM, we study the dynamics of physical and human capital accumulation in a two-period overlapping generations (OLG) model with heterogeneous agents. In this model, parents are altruistic toward their children and invest in their offsprings' human capital. We suppose that there are two communities with distinct levels of altruism and patience. The former represents the parents' preference for investment in human capital, while the latter introduces their preference for investment in physical capital. In this study, we prove the existence and uniqueness of the balanced growth path (BGP) of the economy and characterize the equilibria around this BGP. We also study inequality in terms of labor income and consumption, by considering the GINI coefficient.

The third chapter studies the importance of the public sector to highlight the effect of public spending on the formation of human capital and economic growth in the framework of an OLG economy. In this sense, investment in education is financed by the government's expenditure through taxation on labor income and capital returns. The accumulation of human and physical capital follows a solution to a dynamic general equilibrium problem. The existence and uniqueness of the BGP are studied. The impact of taxation policy on long-run economic growth is studied both analytically and numerically. The intertemporal social welfare problem is also considered.

In the last chapter, co-authored with Cuong LE VAN and Thai HA-HUY, we consider an optimal growth model with non-concave technology to shed light on the issue of sustained growth. The study first establishes a general mathematical model of non-concave technology. It then follows with an application for an economy, where the instantaneous utility function is not concave in human capital (no physical capital). We demonstrate the existence of the poverty trap and the middle-income trap, and specify the conditions under which sustained growth is possible.

1 Review of literature on human capital and economic growth

In "The Wealth of Nations", Smith (1776) notes that "The acquisition of ... talents during ... education, study, or apprenticeship, costs a real expense, which is capital in [a] person. Those talents [are] part of his fortune [and] likewise that of society" (p. 32). Adam Smith also points out that one of the most important source of human capital is experience gained thanks to specialization via the division of labor. Moreover, another source is education, either in the form of formal schooling or apprenticeship.

The role of human capital was re-awakened in the late 50s and early 60s of the twentieth century primarily. Horvat (1958) introduces the concept of human capital and the difference between investment in physical capital and the "human factor" (p. 748). He also indicates that a human being depends on four basic elements: "personal consumption, health, knowledge, and economic and political organization" (p. 751-752). In addition, the incorporation of the human being into the concept of capital is explained by Fisher (1906), Kiker (1966), and Spengler (1977). They also assert that the investment in human education raises productivity and thus adds to national wealth.

To explore the different mechanisms for investment in education via enhancing human capital accumulation and consequently economic growth, I return to mention the workhorse model of growth theory that is the model of Solow (1956). Despite the valuable contributions, the main restrictive assumptions that underlie the model of Solow are the diminishing returns in factors of production and a constant growth rate of savings. This model predicts that because of the diminishing returns, there may be no long-run economic growth. However, this growth will take place if there exist technological improvements over time leading overcome the diminishing returns to the producible inputs, i.e., the notion of effective labor units coming from labor-augmenting technical change.

Although the limitation of diminishing returns is eliminated by the act of technical improvements, the drawback of the study of Solow (1956) that technological progress still occurs exogenously. This quite a restrictive assumption and its reconsideration has given rise to a new literature, known as endogenous growth theory with two main directions. On the one hand, capital productivity is generated through the advancement of new technology. In this sense, human capital is feature identified as a key to adoption of new technology. The technological diffusion is connected to the accumulation of human capital due to human capital is crucial for the effective use of new technology. For instance, Benhabib and Spiegel (1994) approach (that is built on Nelson and Phelps (1966)) according to which human capital does not enter into production process directly but it promotes the adoption and advancement of new technology and ultimately determinant of capital productivity. Especially, Romer (1990) asserts that human capital displays a dual role: as an input in the final production, and also as the key element of technical progress in the formulation of development of new product varieties, as well as a determinant of long-run growth. On the other hand, the labor productivity is improved through learning-by-doing (see Arrow, 1962; Romer, 1986), and education investment via generating human capital accumulation (as shown in Lucas, 1988; Lucas, 2015). Nevertheless, the dissertation only considers the latter form, with the aim to highlight the central role of human capital accumulation and consequently economic growth.

The benchmark of endogenous growth models is that the process of technological change or the accumulation of human capital is not constant and predetermined but is derived from model's elements. The endogenous growth approach, therefore, handle the drawback of the study of Solow (1956) and neo-classical models where technology is assumed to grow at an exogenously constant rate. In other words, the restrictive assumptions of the work of Solow (1956) and neo-classical models will be relaxed in models of endogenous growth where all variables of production progress are expressed in terms of effective inputs (physical and human capital).

In addition, another study also helps to open the door of different channels of the endogenous growth theory that is the skeletal model of Mankiw et al. (1992). In their work (that is built on the model of Solow (1956)), according to which human capital enters into the production process. Following this setting, Mankiw et al. (1992) show that if the main source of economic growth is again exogenous technological change, then the results are qualitatively similar to the model without human capital. Therefore, the only way that human capital formation may act directly as the main engine of sustainable economic growth is to help overcome the diminishing returns, which is achieved in the context of endogenous growth models.

As mentioned above, the endogenous growth approach treats human capital as an input in a manner symmetric to physical capital in production process. Furthermore, the accumulation of human capital has to do with the two ways of formation, as follows: the process of learning-by-doing and the investment in education. The former is known as a way to reach a positive constant growth rate because the externality from human capital in production overcomes the diminishing returns. In some sense, an individual's higher degree of human capital contributes to a rise in labor productivity for all workers in the economy (Lucas, 1988). Nonetheless, this dissertation studies only on the latter formation, where the investment in education is one of the most popular formulations of human capital accumulation within the endogenous growth models (Lucas, 2015).

Indeed, one of the most important channels for forming human capital is education (Schultz, 1960; Temple, 1999). For example, Schultz (1960) identifies human capital narrowly with investment in education and place forward the proposition that "important increases in national income are a consequence of additions to the stock of this form of [human] capital" (p. 571). Besides, according to Becker (1964), additional sources of the accumulation of human capital include both education[2] and other investments to improve "emotional and physical health". The numerous studies also support the view that human capital accumulation is generated by education investment (Barro, 1991; Barro, 2000; Mankiw et al., 1992; Lucas, 1988; Lucas, 2015; Krueger & Lindahl, 2001).

Consequently, the investment in education and the accumulation of human capital may help to eliminate the barrier of the diminishing returns and tend to a positive constant growth even in the absence of technical progress in the short run. However, in order to achieve a positive long-run growth without technological change, the production process has to do continuous improvement through the adoption of effective inputs.

Research on the role of human capital and its impacts on economic growth really sheds light at the skeletal article of Lucas (1988). According to Lucas's contribution, human capital is considered as the reproducible nature of capital and the attribute of externalities generated by human capital accumulation. In particular, knowledge would have an effect on the productivity not only of the individuals accumulating knowledge but also of their co-workers. Thenceforth,

[2]In my dissertation, I concentrate only on the education investment in generating the human capital accumulation, and ultimately economic growth.

education investment and human capital accumulation are placed at the center of the economic growth process (Barro, 1991; Barro, 2000; Mankiw et al., 1992; Krueger & Lindahl, 2001). More recently, Lucas (2015) again emphasizes the central role of human capital and its impact on economic growth, instead of external source, like new technology.

In fact, the main inspiration for the chapters of my dissertation is derived from the seminal articles of Lucas (1988, 2015). However, I sidestep the issue of externalities but highlight the impact of human capital formation on economic growth through the measurement of individuals' labor productivity. Furthermore, in this dissertation, I seek to understand the different channels of funding for investment in education in generating human capital accumulation that are restricted to two basic aspects as follows. First, it depends on private decisions, which are households' savings and leisure devoted to investment in education process (Lucas, 1988; Caballé & Santos, 1993; Galor & Moav, 2004). In addition, altruistic investment also determines the children's education in generating human capital accumulation (Barro, 1974; Abel & Warshawsky, 1988; Michel & Vidal, 2000; Galor & Moav, 2004). Second, investment in education as apublic choice, i.e., through government's spending on public education coming from the tax revenues (Glomm & Ravikumar, 1992; Bosi & Nourry, 2007).

As for the aspect of household investment decision, it returns to the explanation by the early neoclassical model of Solow (1956). Although the drawback of the Solow's model is the case of exogenous technological progress, the contribution of Solow's model with the importance of savings plays a non-trivial role. Moreover, many current studies continue to focus on the context of technological improvements over time lead to overcoming diminishing returns as labor input becomes more productive.

The relationship between the accumulation of human capital and the rate of savings is also considered by Mankiw et al. (1992). They revisit the seminal article of Solow (1956) by considering a large set of data of countries. They affirm that savings and population growth rates affect income in ways that Solow predicted. Nevertheless, the prediction of the Solow model in the context of testing of Mankiw et al. (1992) is roughly correct with more than half of the cross-country variation in income per capita. It means that the Solow model predicts the ways of effects of savings and population growth (the trend of growth), but it does not correctly predict the magnitudes. After that Mankiw et al. (1992) introduce a proxy for the accumulation of human capital as an additional explanatory variable into their cross-country regressions. They then find two main results. First, human capital accumulation is in fact correlated with savings and population growth. Second, their results (account for 80 percent of the cross-country variation in income per capita) are equivalent to the prediction of the study of Solow if human capital accumulation is taken into account.

Arguably, Mankiw et al. (1992), using the production function with three input factors (physical capital, human capital/the stock of knowledge and labor), develop the model of Solow (1956), and give a sound of departure from which endogenous growth models with the role of the share of savings and human capital accumulation are performed.

As a matter of fact that household's savings is a crucial source of funding for investment in education in generating human capital formation. However, the rate of savings is not necessary to ensure constant. It can be determined endogenously through the optimizing behavior of households, which is the main contribution of Ramsey (1928)[3], Cass (1965) and Koopmans (1965). In other words, an endogenous rate of savings comes from the context that consumers maximize lifetime utility from consumption taking into account preferences for consumption over time and trade-off between present and future consumption.

I now need to remind myself again of the siminal work of Lucas (1988), which shows that human capital accumulation is determinant for economic growth by

[3]In chapter one, we bridge the model of Ramsey into the models of Lucas. This study will divide individuals savings into two parts: one is devoted to physical capital and the rest is used for investing in human capital through education spending.

considering a constant return to scale production function with external effects of human capital in output production[4]. Lucas also demonstrates that the rate of economic growth is directly proportional to the investment in education and learning time devoted to the accumulation of human capital and the efficiency of human capital formation technology[5]. Therefore, the evolution of human capital may rely on both learning time and the share of savings devoted to investment in education process.

As mentioned at the begining of the introduction part, the interaction between physical and human capital is also studied[6] in this thesis. The interplay between two types of capital may lead to a large increase in productivity and consequently income (Schultz, 1960; Weisborod, 1961). In addition, Schultz (1962) argues that the investment in human beings and the formation of capital will change the usual measures of savings. That leads to a change in the structure of earnings. Moreover, the correlation among earnings, the returns on capital and aggregate investment is examined by Becker (1962) and Becker and Chiswick (1966). In particular, the contribution of Mankiw et al. (1992) also explain that the variation in cross-country income per capita can be explained by their differences in the levels of savings, education, and population growth.

This dissertation also considers the impact of the interaction between physical and human capital on economic growth in the process of the economic development. This is in accordance with the works of McGrattan and Prescott (2009), Schoellman (2012), Manuelli and Seshadri (2014), and Galor and Moav (2004). These authors use different ways to prove that the production factors have more important impacts on economic growth than external sources, like technology.

[4] The production output is transformed in the discrete-time version: $y_t = Ak_t^{\mu}(\tau_t h_t)^{1-\mu}(h_a)^{\gamma}$, where k_t and h_t are physical and human capital, respectively. The fraction of non-leisure time τ_t is devoted to current production (as opposed to formal learning time), and h_a is the average human capital in economy.

[5] The production function of human capital accumulation of Lucas's model is that $\dot{h} = \theta(1-\tau)h$, where θ denotes the efficiency parameter. If we take $dlog(h)/dt = \theta(1-\tau)$ then sustained growth arises because there are constant returns in production of human capital.

[6] Benhabib and Spiegel (1994), McGrattan and Prescott (2009), Schoellman (2012), Manuelli and Seshadri (2014), and Lucas (2015) show that the production inputs have more important impact on economic growth than the external effects.

McGrattan and Prescott (2009), using the "technology capital" in production, conclude that "there are gains to openness even for countries that do little or no investment in technology capital" (p. 2474). Schoellman (2012), in the analysis of schooling quality, suggests that "differences in education quality are roughly as important as differences in year of schooling in accounting for the difference in output per worker across countries" (p. 411). Manuelli and Seshadri (2014), using data on schooling and age-earnings profile, find that differences in quality of human capital rather than differences in levels of the total factor of productivity, better explain differences between countries in average income.

These above authors also suggest that the input factors (physical and human capital) may compete with each other in order to be the prime engine of economic growth. In particular, Galor and Moav (2004) show that at some stages of economic development process, human capital formation can surpass physical capital accumulation as the main source of economic growth[7].

Overall, the thesis literature begins with the workhorse model of Solow (1956) that has afforded a convenient point of starting from which to investigate the contribution of human capital in the context of endogenous growth. Next, this literature offers a more detailed picture of mechanics that allow for human capital enter explicitly as an input in production process. This has become known as *the human capital-extended Solow (1956) model* as rediscovered by Mankiw et al. (1992). This literature then introduces the benchmark of the seminal model of Lucas (1988) to highlight one of the most popular formations of human capital within the endogenous growth process. Finally, the dissertation literature arrives at a unified picture where human capital can be turned into an important instrument, and placed at the center of the role of sustained (endogenous) growth (as shown in Lucas, 2015).

[7] Galor and Moav (2004) consider a kind of asymmetry of investment opportunity in the physical and human capital. This asymmetry is characterized by the assumption that "human capital is inherently embodied in humans and the existence of physiological constraints subjects its accumulation at the individual level to diminishing returns" (p. 1002), preventing its accumulation from being "widely spread among individuals in society" (p. 1004).

2 Human capital accumulation: when Ramsey meets Lucas

Among research that investigate the engine and mechanisms of economic growth, as well as the causes of cross-country income differences, the model of Solow (1956) has become a reference framework. Solow (1957) uses the theoretical model to develop an empirical growth accounting study in which it is estimated that 87.5% of the growth of per capita output is due to residual sources, which are usually referred to as total factor productivity. Only 12.5% of the growth is accounted for by the production factors.

Reconsidering seriously the work of Solow (1956, 1957), Mankiw et al. (1992) conclude that the predictions of the Solow model are consistent with the evidence. However, though savings and population growth affect the income in the directions predicted by Solow, only a little more half of the cross-country variation in income per capita can be explained by these two variables alone. They play too large a role in accounting for growth.

In order to explain the magnitude of economic growth with more precision, Mankiw et al. (1992) consider a Solow model including human capital beside physical capital. Using a multi-country data set constructed for the period 1960-1985, they find that the augmented Solow model provides a better description of the cross-country data. Moreover, they argue that the "exclusion of human capital" (p. 408) is the reason for which the impact of the exogenous variables on economic growth in the original Solow model is too large.

Similarly to Solow, the work of Mankiw, Romer, and Weil (1992) consider exogenous rates of saving for physical capital and for human capital. One of the purposes of our paper is re-consider the works of these authors, by assuming endogenous rates of saving for two types of capital, in a context of the Ramsey (1928) and Lucas (1988) models with discrete time version.

In chapter one, we first introduce simple two-period models to highlight the role of human capital with or without physical capital. We then extend and enrich these models, in a gradual manner, to account for a more complicated setting where time and the interaction between human and physical capital play important roles in generating economic growth. In other words, the goal is to walk the audience from Ramsey (1928) to Lucas (1988, 2015).

In the first two-period model without physical capital, the formation of human capital depends on the initial human capital stock and the total saving of the first period. In this simplest setting, we find that given sufficiently high total factor of production (TFP) and households' level of the initial human capital, as well as efficiency in human capital formation technology, economic growth can be sustained. This result is also in line with Lucas (2015) who argues that "any model of sustained growth must assume that we will never run out of ideas" (p. 86).

Another purpose of our paper is discussing interactions between physical capital and human capital in different stages of the process of economic growth. For instance, McGrattan and Prescott (2009), Schoellman (2012), and Manuelli and Seshadri (2014) show that production factors like physical capital and human capital play an more important role in enhancing economic growth than external sources like technology. In this context, these factors will compete with each other in order to be the prime engine of economic growth.

For this reason, in the second model, we introduce physical capital and allow for an interaction between the two types of capital. The introduction of physical capital highlights the importance of human capital in economic growth. In particular, we find that if the formation of human capital is sufficiently efficient, human capital can surpass physical capital as the main source of economic growth. This conclusion is in accordance with the study done by Galor and Moav (2004) in a more complicated setting.[8]

[8]Galor and Moav (2004) considers a model with market imperfections caused by asymmetric educational investment opportunities between individuals to show that at some later stages of

From the third model, we relax the two-period assumption and allow time to run to infinity. The third model is a direct extension of the first one in infinite time, which is essentially Ramsey (1928) with physical capital replaced by human capital. The fourth model is a simplified version of Lucas (1988) in the sense that we restrict our attention to only human capital with depreciation but no externalities. The conclusions from these two models are not too different from the previous ones. In the fifth model, we again consider the interaction between the two types of capital (human and physical capital) without the accumulation of human capital. In this extended version of Ramsey (1928), we are able to prove the existence of the optimal sequences of physical and human capital, and their convergence to the steady states, which are also computed. Moreover, the steady state values of both types of capital increase with respect to the efficiency parameter of human capital formation technology.

The phrase *"Ramsey meets Lucas"* is justified in our last two models. In particular, the sixth model extends the fourth one in two aspects. First, it brings back physical capital but maintains that all savings go to investment in physical capital. Second, it introduces time into production and learning process. This model can be viewed as the discrete-time analog of Caballé and Santos (1993), but with depreciation in human capital. We recover their main results regarding the existence and uniqueness of a Balanced Growth Path (BGP) when the efficiency of human capital formation is sufficiently high. Finally, the grande finale enriches the sixth model by not only letting households determine the allocation of savings between investment in human and physical capital, but also allowing for non-linearity in the accumulation of human capital production function with respect to both time and the share of savings devoted to investment in human capital. In this much more complicated setting, we manange to establish the existence and uniqueness of a BGP. Another interesting point of the last model is that if the time factor is set to be a constant then this model becomes the special case of

development when the rate of return on investment in human capital is high enough, human capital accumulation can challenge physical capital accumulation as prime factor that fuels economic growth.

social welfare problem, which is discovered, in the third chapter.

To sum up, the novelty of this chapter is that we bridge the gap between Ramsey (1928) and Lucas (1988, 2015). By gradually adding different layers of complexity to a model of only physical capital, we arrive at a more unified picture of different sources of economic growth, allowing for the interaction between physical and human capital where time plays a non-trivial role.

Although this chapter has reached its aims, there is an unavoidable limitation. We have placed the central role of human capital accumulation on economic growth without the technological progress increasing the total factor productivity (TFP is a constant). Therefore, to generalize a fulfilled picture of economic growth, the chapter should have involved in the role of technological change.

3 From physical to human capital accumulation: Heterogenous intergenerational altruism and inequality

In order to construct the second chapter, we continue to follow the study of Lucas (1988), who explains that the formation of human capital accumulation comes from both "a private and a social activity"[9] (Michel & Vidal, 2000, p. 276). However, his view of the social activity is restricted by a population or a community or a country. Our study complements Lucas (1988) in allowing for different cohabiting communities in the same economy. In particular, in our model, one community differs from the other by the patience coefficient for their investment in physical capital and the degree of altruism for spending on their offspring's education. In our OLG economy, the future generation's stock of human capital is formed by the current generation's level of human capital and altruistic investment. In this sense, through the process of education, young people take into account their ability and

[9] The formation of human capital is a social activity, involving groups of people in a way that has no counterpart in physical capital accumulation.

altruistic investment to learn and develop skill and knowledge.

As mentioned by De la Croix (2001) that "each generation has more resources at its commands on reaching adulthood" (p. 1415). These additional sources come from both productivity connected to the accumulation of physical capital and the formation of human capital as offsprings inherit an altruistic investment, is devoted to children's education. The second chapter considers a kind of forward altruism and its impact on human capital accumulation. We hypothesize the utility function of individuals depend on their own consumption and spending on children's education. Indeed, individuals may extract pleasure from altruism via "joy of giving or warm glove" (Barro, 1974; Abel & Warshawsky, 1988; Michel & Vidal, 2000; Galor & Moav, 2004).

As is well-known, the Ramsey model in a certain way is equivalent to an OLG model with intergenerational altruism in the sense of Barro (1974). It is interesting to study other forms of intergenerational altruism in presence of human capital accumulation. The second chapter therefore considers the impact of paternalistic altruism in the sense of Abel and Warshawsky (1988) in a heterogeneous economy where the agents differ in their altruism degree, which is manifest in their manner of investment in the education of their descendant.

Like similar framework included in the previous chapter, the second chapter focuses also on the effect of the interaction between physical capital and human capital on economic growth. In an altruistic economy, altruism also affects individuals' choices, and thus entails the different impacts on the formation and properties of physical and human capital accumulation, and consequently heterogeneity among individuals coming from the interplay between the two kinds of capital (Galor & Moav, 2004; Turnovsky & Mitra, 2013).

In the first chapter, we have examined the role of human capital accumulation, and the impact of the interaction between physical and human capital on economic growth in an economy of homogeneous agents. In this chapter, we introduce, in the simplest manner some degree of heterogeneity among indidivuals in an overlapping generations (OLG) economy by hypothesizing that there exists two communities differing only in the degree of patience, and the degree of altruisim towards the next generation. This setup helps us understand the role of altruistic investment on the children's accumulation of human capital and consequently on economic growth and inequality.

In fact, the way we introduce heterogeneity can be best compared to that of the Galor and Moav (2004)[10]. In particular, while these authors hypothesize that children themeselves also make a decision on how much to investment in their own education, we assume that parents are the only decision-makers in children's education. The difference arises from our intention to focus on heterogeneity in the parents' generation, while Galor and Moav (2004) focus on the offsprings' generation. Overall, our study differs from Galor and Moav (2004) in two perpectives: First, an additional source of heterogeneity hypothesis is the degree of altruism. Second, an altruistic investment must be positive and totally devoted to finance their offspring's spending on education.

The setup of our model is directly inspired by Michel and Vidal (2000). Although our modeling on heterogeneity is much simpler with no cross-community externalities, we allow for non-linearity on the accumulation of human capital with respect to the share of savings devoted to education. Essentially, this implies decreasing marginal return on investment in education on human capital, which we believe is a more reasonable assumption to make. Furthermore, we assume that human capital can also depreciate to reflect the reality that knowledge could become obsolete.

In this setting, we prove the existence and uniqueness of a BGP. We find that in one community, the less altruistic the parents, the more they invest in physical capital. For each community, a coefficient (named the coefficient of relative pref-

[10] The theoretical setup of the second chapter is motivated by Galor and Moav (2004). Galor's contribution is based on the heterogeneity hypothesis of the asymmetric investment opportunity in the physical and human capital. They also assume that altruistic investment is allocated between an immediate finance of their children's spending on education and savings for the future wealth of their children.

erence in the paper) that is decreasing in the degree of altruism but increasing in the patience parameter captures the relative preference between human and physical capital. More specifically, the lower this coefficient, the higher the relative preference towards investment in human capital. At the steady state, the lower the coefficient of relative preference a community (the rich one) has, the higher its ratio of human capital relative to the other community (the poor one).

On the issue of inequality, we find that it is important for this economy not only to consider income inequality but also inequality in terms of consumption levels. We then describe the impact of this inequality of consumption on the interaction between human and physical capital accumulation. The understanding of the dynamics of inequality in the context of our model is limited since we have not been able to account for the heterogeneity in initial human and physical capital.

Even though our study has the notable results, it still has the following limitations. On the one hand, we can say little about the impact of the fundamental parameters of the model on the growth rate of the economy. On the other hand, our model does not take into account the mobility of households between communities.

4 Public investment in education, tax policy and endogenous economic growth

The first two chapters of my dissertation have explained the role of private investment (household investment decisions) in education in promoting the accumulation of human capital and economic growth. The third chapter considers this investment as a public choice, i.e., through government spending on education, financed by tax revenues.

In fact, the role of government is essential to public education and social benefits. According to Eckwert and Zilcha (2012), social benefits from public spending on education encompass the direct benefits of higher salary enjoyed by individuals, as well as the indirect gains that economy derives from the human capital accumulation produced via the education system. This social benefit, however, comes with social costs. The social costs of acquiring skills include expenses incurred by the society that performs the education and/or training (Viaene & Zilcha, 2013). Furthermore, Garrat and Marshall (1994), Fernandez and Rogerson (1995), and Gradstein and Justman (1995) assert that one of the contributors to the public education budget is workers whose incomes are taxed to finance all parts of public spending on education.

The third chapter is in the spirit of Bosi and Nourry (2007), who examine the effect of public spendings on economic growth. In addition, our study is directly related to the works by Garrat and Marshall (1994), Fernandez and Rogerson (1995), and Gradstein and Justman (1995). The main conclusion of these authors is that individuals who have more education would earn more income in the future and hence pay more income taxes. They also emphasize the role of the government, through investment in education, on reducing inequality and enhancing the benefits of social welfare.

The main motivation of the third chapter is the work of Glomm and Ravikumar (1992), who study human capital investment through formal schooling in an OLG economy with heterogeneous agents and twofold human capital accumulation formation, i.e., public and private investment in education. This chapter considers only the former of human capital accumulation through the government's investment in education, with the aim to highlight the importance of the role of taxation in the formation of human capital accumulation and ultimately on economic growth.

In order to understand the impact of taxation policies on economic growth in separation from the decisions of the households, I assume that the households only invest in physical capital. In addition, the government taxes both wage (labor income) and capital return (physical capital) on the whole population (both young and old). The government then uses tax revenues to finance all parts of public education.

This chapter is interesting in two aspects. Firstly, it allows having a general equilibrium perspective. Secondly, it permits to study the dynamics associated with the interaction between the accumulation of physical and human capital. In particular, two effects of taxation are noteworthy. On the one hand, it reduces the households' total income, hence lowering their investment in physical capital, which has a negative impact on production and output. On the other hand, it raises public funding, raising the government's expenditure on public education, raising human capital accumulation and ultimately enhancing the labor productivity of the future generations. Consequently, the net impact on economic growth is ambiguous.

This study also provides notable results regarding the BGP of the economy. The existence and uniqueness of a BGP are established. The growth rate on the BGP is also determined as a function of taxes. Based on these results, the impact of taxes on the long-run growth rate is analysed under several taxation policies. In addition, the issue of intertemporal social welfare is also explored. In particular, given a discount rate for all future generations' utilities, the existence and uniqueness of a BGP, as well as the long-run growth rate of the economy are again characterized.

The limitation of this chapter is that we consider only the government's investment in public education in an OLG economy with homogeneous agents. In the near future, we will investigate this investment in both private and public choice in generating human capital accumulation and economic growth. We will also study the trade-off between the two options for the investment in education.

5 Optimal growth with non-concave technology: Application to human capital model

The standard convexity assumption leads to extensive use of the convex structure in different domains of economic sciences. In fact, the large body of the literature on dynamic programming, especially on optimal growth rely on key assumptions of convexity, as follows.

Ramsey (1928), Cass (1965), and Koopmans (1965) show that the optimal intertemporal growth tends to a unique steady state that is based on the convexity technology. Benveniste and Scheinkman (1979) introduce the concave function to the initial capital stock. Becker (1965), Ben-Porath (1967), and Mincer (1974), using the concavity characteristic, study the investment in skills, including schooling (pre-labor investment) and training (on-the-job investment). Lucas (1988) and Azariadis and Drazen (1990) emphasize the externalities of human capital in the economic process. Benhabib and Spiegel (1994) and Foster and Rosenzweig (1995) illustrate not only the role of human being in the productivity of existing tasks but also the capacity of workers to cope with change, disruptions and especially new technologies and technological adoption. Furthermore, the technique of convex analysis is also discovered in a well-developed branch of applied mathematics wherein an excellent treatment can be found by Stokey (1989).

Nevertheless, imposing certain convexity assumptions could be restrictive. This calls for a need to develop a theoretical framework that tackles the class of problems where certain assumptions of convexity are violated. Ours is not the first attempt in this direction. Previous works include.

Non-concave technology is used for both one-sector and multi-sector models. Dechert and Nishimura (1983), focusing on one-sector of optimal programs, examine that the optimal paths are monotonic. Amir (1996), using multi-sector of non-classical optimal growth and the characteristic of super-modularity, refers that the optimal paths are monotonicity.

In particular, Dechert and Nishimura (1983) consider a convex-concave model with a $S-$shape production function. They prove that the optimal capital stock is monotonic, and give a proof of the existence of a poverty trap. In addition, the study of Dechert and Nishimura (1983) is revisited by Le Kama, Ha-Huy, Le Van, and Schubert (2014), who consider a convex-concave production function and show the explicitly determining a poverty trap.

Non-concave technology for a continuous time version is examined by Romer (1983, 2011). This technology in the discrete-time framework is addressed by Dimaria, Le Van, and Morhaim (2002). Both of them suppose that the technological changes are endogenous, and depend on capital. They assume that the production function of knowledge is increasing, in order to ensure that economic growth tends to infinity.

Hung, Le Van, and Michel (2009), using an aggregation of two separate concave production technologies, show the existence of two steady states if the discount rate is not too high or too low. They also prove that the convergence depends on either the initial state or the discount rate.

Kamihigashi and Roy (2007), developing the analysis of Majumdar and Nermuth (1982) and Dechert and Nishimura (1983), impose neither convexity properties nor continuity properties on the production function, and characterize the conditions for the neighborhood property and the existence of the poverty trap or sustained growth to infinity. The work of Kamihigashi and Roy (2007) deals with the classical situation of optimal growth with the separation of consumption and capital investment.

Although the literature on non-concave technology has been applied by many scholars, there has been a lack of application in the human capital research. Moreover, the convex structure may not be guaranteed when the instantaneous utility function is no longer concave in a variable of interest. Consequently, interesting results such as the policy functions (of physical and human capital), the existence and uniqueness of the BGP, and equilibria around this BGP are not addressed when this assumption is violated. The final chapter hence builds a general mathematical model with non-concave programming and an application to a human capital accumulation model.

In particular, we introduce non-concave technology to a growth model with only human capital (no physical capital). In this case, we assume that the concavity property of the utility function in human capital fails. We then attempt to tackle this issue by making use of the characteristic of super-modularity (satisfied for almost models). This means that the value function is twice differentiable and strictly supermodular (see e.g., Amir, 1996). Furthermore, we also impose several technical assumptions to ensure the convergence to infinity of the optimal path of human capital and the convergence to the maximal value of the growth rate of human capital.

In this setting, we find the poverty trap (the optimal path of human capital convergence to zero), and the middle-income trap (the optimal path of human capital bounded away from zero and infinity). In addition, we show that if the initial level of human capital is greater than a threshold and the discount rate is big enough, then the optimal path of human capital converges to infinity. Under the same assumption, the optimal growth rate of human capital also tends to the maximal value.

Besides the notable results, in this chapter, the non-concave technology applies to a growth model with only human capital accumulation. Therefore, this technique should mention about the co-evolution of human and physical capital accumulation. More importantly, the non-concave programming may apply in the previous chapters, also in a wider class of problems such as public debts, international aid, multi-sectors model.

Chapter I

Human Capital Accumulation: The Source of Economic Growth [1]

[1]This paper is co-written with Cuong LE-VAN, Thai HA-HUY and Cao-Tung PHAM

1 Introduction

The model of Solow (1956) is a seminal reference among the theories that seek to understand the cause of economic growth. One year after, Solow (1957) uses the theoretical model to develop an empirical growth accounting study in which it is estimated that there is around 80 percent of the growth of per capita output is due to residual sources, which are usually referred to as total factor productivity (TFP). The remainder of the growth is thus accounted for by the production factors (labor and physical capital).

While technological progress in the Solow model is exogenous, Mankiw et al. (1992) observe that the rate of saving and the population growth rate, both are also exogenous in the Solow model, play a too large role in accounting for growth. They augment the original Solow model introducing human capital accumulation. Using a multi-country data set constructed for the period 1960-1985, they find that the augmented Solow model provides a better description of the cross-country data. Moreover, they argue that the "exclusion of human capital" (see Mankiw et al., 1992, p. 408) is the reason for which the impact of the exogenous variables on economic growth in the original Solow model is too large.

Le Van et al. (2010), based on empirical data from countries like China, Korea, and Taiwan, show that "when income is under a critical level, there is no demand for investing in human capital" (p. 230). Only "when the country reaches this critical level of income, it must invest not only in physical capital but also in new technology and in higher education. Under some mild conditions on the quality of the production of the new technology and on the supply of skilled workers, the share of the investment in human capital and in new technology capital increases when the country becomes rich" (p. 224). Especially, they also find that the share of the investment in human capital can exceed the share in new technology capital in the long run.

Under these perspective, we understand why, in his recent paper Lucas (2015), calls for the need of placing human capital accumulation at the center of economic growth, instead of external channels like technology. In this work, Lucas, on the one hand, argues that there may be “a misinterpretation of the evidence, especially of census data on schooling and age-earnings profiles” (p. 85), that leads to recent empirical results strongly emphasizing the role of differences in technological productivity in explaining cross-country differences in growth rates. On the other hand, he cites several alternative research directions, in which economists like McGrattan and Prescott (2009), Schoellman (2012), and Manuelli and Seshadri (2014) prove that production factors accumulation plays more central role in promoting economic growth, compared to external sources like technology. Especially, the later work of Manuelli and Seshadri provides new methods for measuring human capital based on data on schooling and age-earnings profile. They find that differences in quality of human capital explain better differences between countries in average income, than differences in levels of total factor of productivity do.

The analysis and discussions in Lucas (2015) are ended with conclusion that “the contribution of human capital accumulation to economic growth deserves a production function of its own” (p. 87). Responding to this appeal, we revisit the endogenous growth problem in several models where human capital plays a central role (without external effects), and is accumulated in different ways: new stock of human capital may be made up of former stock of knowledge, devoted effort in term of time, and investment in term of output. We start our analysis by considering rather simple two-period models which actually provide very clear insights on how important human capital accumulation is in enhancing economic growth. We then generalize these findings into infinite-horizon economies, with or without physical capital accumulation.

Beside the generalization of the original linear function of human capital formation considered by Lucas (1988), one of our main contribution is proofs for existence, uniqueness, and convergence of optimal balanced growth paths in several economies. These facts are often ignored before characterizing optimal balanced paths. In these aspects, we are in line with Caballé and Santos (1993) who provide proofs for existence and convergence of balanced growth paths in a continuous time infinite horizon model. However, we depart from their result in not only requiring the technology for human capital formation to be linear with respect to h_t[2]. In fact, our production function of human capital will be generalized throughout the paper in order to encompass time as well as economic effort devoted to investment in human capital, and also the role of knowledge of the past in the formation of current human capital stock.

Another purpose of our paper is discussing interactions between physical capital and human capital in different stages of the process of economic growth. As a matter of fact, since the works by McGrattan and Prescott (2009), Schoellman (2012), and Manuelli and Seshadri (2014) suggest that production factors have more important impacts on economic growth than external sources like technology, it would be very valuable in this case to study how different production factors compete each other in order to be the prime engine of economic growth. In this aspect, our research is in line with Galor and Moav (2004) who show that at some stages of the process of economic development, human capital formation can outperform physical capital accumulation as the main source of economic growth. Nonetheless, our approach differs from theirs in not considering any specific credit constraint on investment in human capital. They actually consider a kind of asymmetry between physical capital accumulation and human capital formation, characterized by the assumption that "human capital is inherently embodied in humans and the existence of physiological constraints subjects its accumulation at the individual level to diminishing returns" (p. 1002), preventing its accumulation from being "widely spread among individuals in society" (p. 1004). Whereas

[2]Caballé and Santos (1993) consider an non linear function of human capital accumulation. However, associated to this function, they provide proofs of existence and local stability of solutions. Moreover, in their human capital formation function, they consider an external effect of current physical capital stock, omitting the role of the depreciation rate of human capital.

we reach interesting results concerning the relationship between physical capital and human capital and their relative roles in sustaining economic growth, without considering any asymmetric hypothesis.

The rest of the chapter is organized as follows. Section 2 discusses the accumulation of human capital and its impact on economic growth in models with and without physical capital. Section 3 extends these discussions into infinite-horizon models in several manners. In fact, we use the models of Section 3 as a bridge that links our clear and intuitive findings in Section 2 to Section 4, where our main results are located. Section 5 is conclusion.

2 Human capital in two-period models

In this section, we study the formation of human capital in two two-period models with and without physical capital. Our very first examination of how human capital is accumulated across successive stages, without genuine surprise, basically relies on the way that households devote their saving to invest in education. However, these intentionally simple setups help us to easily emphasize the human capital accumulation process and its role as the key engine of economic growth.

Indeed, in the first model - without physical capital, we rediscover the basic ideas of Lucas (2015) that human capital on its own can guarantee a sustained economic growth under some conditions that he calls "enough new ideas out there to keep the economy growing" (p. 86).

Whereas, in the second model we prove that, even with the presence of physical capital, human capital can surpass physical capital to take the role of a prime engine of growth. We thus rediscover the result by Galor and Moav (2004) (when they consider a hypothesis of a fundamental asymmetry between human capital and physical capital based on credit constraints on human capital investment that do not exist with regard to physical capital investment) that physical capital can be challenged by human capital when it comes to sustain the economic growth.

2.1 Human capital accumulation in a two-period model without physical capital

We consider a two periods economy with homogeneous agents each of whom is endowed with a human capital stock denoted by h. This human capital h can be interpreted as the effective labor. Moreover, we assume for simplicity that the output is obtained by using only the effective labor through a production process captured by a production function F, which is assumed to be concave and strictly increasing.

For simplicity, we assume a constant and normalized population. At the period 0, the unique agent is endowed with an initial level of human capital h_0 to produce an amount $F(h_0)$ of output. She determines her current level of consumption c_0 and savings S_0 in the first period, as well as her consumption c_1 of the next period, where her effective labor supply is denoted by h_1, in order to maximize her life time utility $u(c_0) + \beta u(c_1)$.

The total savings S_0 will be devoted to the human capital formation according to the following dynamics:

$$\frac{h_1}{h_0} = \phi(\theta, S_0)$$

where the function ϕ is supposed to be increasing in both variables. The parameter θ represents the efficiency of human capital formation technology. It captures several features such as quality and appropriateness of syllabus, quality of teaching facilities, and so forth.

Moreover, we assume that the function ϕ also satisfies the following assumption:

Assumption 2.1 *There exists a critical value θ_0 such that when the efficiency of* *human capital formation technology θ is larger than this critical value, we have:*

$$\phi(\theta, 0) = 1 - \delta,$$
$$\phi(\theta, F(h_0)) > 1,$$

with $0 < \delta < 1$.

Since ϕ represents the growth rate of human capital across eras, this assumption means that, without investing in human capital, this one will be depreciated at rate δ. In addition, since $F(h_0)$ is the maximal level that S_0 can reach, the second condition $\phi(\theta, F(h_0)) > 1$ highlights the important role of the efficiency parameter θ and the initial human capital endowment h_0: θ is assumed to be high enough so that when S_0 reaches its maximal level $F(h_0)$, the growth rate of human capital is sustained (which means that the rate of growth is greater than 1). We thus only focus on the case where $\theta > \theta_0$.

The maximization program of the unique household can be re-written as follow:

$$\max[u(c_0) + \beta u(c_1)] \tag{I.1}$$

with $0 < \beta < 1$ represents the time preference parameter, subject to the constraints:

$$c_0 + S_0 = F(h_0), \tag{I.2}$$
$$c_1 = F(h_1). \tag{I.3}$$

In order to explicitly resolve this program, we assume that $u(c) = ln(c)$, $F(h) = Ah^{\mu}$, with $0 < \mu < 1$, and $\phi(\theta, S_0) = \theta S_0^{\alpha} + (1 - \delta)$, where $0 < \alpha < 1$. In this setting, we can easily calculate the critical value of efficiency parameter[3].

[3] The efficient threshold: $\theta_0 = \frac{\delta}{[F(h_0)]^{\alpha}}$

Furthermore, let us denote by

$$s = \frac{S_0}{F(h_0)}$$

the rate of saving. We then obtain the following results:

Proposition 2.1

(a) The optimal rate of investment in human capital, $s(A, h_0, \theta)$is increasing in both variables which are: the initial human capital endowment h_0, the total factor productivity A, the efficiency parameter of the human capital formation θ, and unsurprisingly the time preference parameter β.
(b) The rate of growth between the two periods is higher when we invest in human capital.
(c) The rate of growth is sustained only if the efficiency parameter θ is high enough and/or the initial human capital endowment is sufficiently large.

Proof: Appendix ■

Comment: This result, although being derived from a rather simple model, deserves to be commented on: by proving that the economic growth is sustained if the efficiency parameter θ is high enough and/or households are endowed with a sufficiently large initial human capital h_0, (by sustained growth, we simply mean that the rate of economic growth $\frac{F(h_1)}{F(h_0)}$ is greater than 1), we are in line with Lucas (2015) who argues that "any model of sustained growth must assume that we will never run out of ideas" (p. 86), considering in his model a parameter capturing "the quality of the individual's intellectual environment" (p.86).

2.2 A two-period model with human and physical capital

We introduce physical capital into the previous two-period model. Individual's saving S_0 is therefore no longer totally devoted to improving human capital stock but also to investment in physical capital.

The production technology is summarized by a function denoted by f, depending in both factors, so that at period date $t \in \{0, 1\}$, the output produced is $y_t = f(k_t, h_t)$, where k_t and h_t respectively denote physical capital and the human capital.

The unique agent of the economy maximizes the following utility

$$\max[u(c_0) + \beta u(c_1)] \tag{I.4}$$

subject to constraints:

$$c_0 + k_1 + e_1 \leq f(k_0, h_0), \tag{I.5}$$

$$c_1 \leq f(k_1, h_1), \tag{I.6}$$

where the level of initial physical and human capital k_0, h_0 are given, c_0 and c_1 represent the consumption, while k_1 and e_1 respectively denote amounts of saving used for investment in physical and human capital.

We assume that the production function of the economy f is concave and increasing, with $f(0) = 0$. In addition, the utility function u is supposed to be strictly concave, continuous and increasing, satisfying the Inada condition $u^{'}(0) = +\infty$.

The formation of human capital across stages is captured by a function ϕ, which now depends on economic effort (economic savings) spending on education and/or training:

$$\frac{h_1}{h_0} = \phi(\theta, e_1). \tag{I.7}$$

Assumption 2.2 *The economy's technology is represented by an augmented Cobb-Douglas production function, $f(k_t, h_t) = Ak_t^{1-\mu}h_t^{\mu}$, $0 < \mu < 1$. Whereas the production function of human capital from education expenditure is explicitly given by $\phi(\theta, e_1) = \theta e_1^{\alpha} + (1 - \delta), \alpha \in (0, 1)$, where θ and δ are interpreted in the same* *way as in the previous model. We also assume $u(c) = \ln(c)$.*

Proposition 2.2

1. *There is a unique optimal solution (k_1, e_1) given by*

$$f(k_0, h_0) = \left(1 + \frac{1-\mu}{\alpha\mu} + \frac{1}{\alpha\beta\mu}\right) e_1 + \frac{1-\delta}{\theta}\left(\frac{1-\mu}{\alpha\mu} + \frac{1}{\alpha\beta\mu}\right) e_1^{1-\alpha}, \quad \text{(I.8)}$$

$$k_1 = \frac{1-\mu}{\mu}\left(\frac{\theta e_1^{\alpha} + (1-\delta)}{\theta\alpha e_1^{\alpha-1}}\right). \quad \text{(I.9)}$$

2. *When θ tends to infinity, savings allocated to education e_1 and accumulated physical capital k_1 respectively converge to finite values $\bar{e}_1 = \frac{f(k_0,h_0)}{1+\chi}$ and $\frac{1-\mu}{\mu}\frac{\bar{e}_1}{\alpha}$, (where $\chi = \frac{1-\mu}{\alpha\mu} + \frac{1}{\alpha\beta\mu}$); while human capital stock h_1, as well as output y_1, both tend to infinity. In other words, if the formation of human capital is sufficiently efficient, human capital can surpass physical capital to play the role as the main source of economic growth.*

Proof: Appendix ■

Comment: In our two-period model with physical and human capital, we rediscover the result by Galor and Moav (2004) who prove that, at some later stages of development when the rate of return to human capital investment is high enough, human capital accumulation can challenge physical capital accumulation as the prime factor that fuels economic growth. However, we reach this result without considering market imperfection caused by asymmetric educational investment opportunities between heterogeneous individuals, and in a rather simple economy setup.

3 Human capital in infinite horizon models

The objective of this section is to introduce human capital into infinite horizon models in as simple manners as possible.

One of these, which is in fact subject of the two first following subsections, is considering infinite horizon economies without physical capital, but with human capital accumulation functions depending not only on individuals' saving decision but also on the stock of ideas or knowledge that is already "out there" - according to the terms used by (Lucas, 2015, p. 86). This saving decision can take the form of economic savings (as in the subsection 3.1), or time effort devoted to enhancing the effective labor (as in the subsection 3.2).

Whereas, in the last subsection 3.3, we consider a human capital formation function which only depends on households' saving decision, in a infinite horizon model with physical capital accumulation *à la* Ramsey.

3.1 Human capital accumulation as the main source of growth

As in the subsection 2.1, we assume that the rate of human capital formation across stages $\frac{h_{t+1}}{h_t}$ is dictated by a function $\phi(S_t, \theta)$ which depends on the agent's savings at t and satisfies Assumption 2.1, that is: $\phi(S_t, \theta) = 1-\delta+\theta S_t^{\alpha}$, where $1-\delta$ and θ respectively represent human capital depreciation rate and the efficiency of human capital formation technology.

Rather than a two periods life-time utility, the unique agent of the economy now maximizes an infinite intertemporal utility:

$$\max \sum_{t=0}^{+\infty} \beta^t u(c_t)$$

subject to the constrain:

$$c_t + S_t = Ah_t^{\mu}, \tag{I.10}$$

$$h_{t+1} = h_t\left[\theta S_t^{\alpha} + (1-\delta)\right], \tag{I.11}$$

where $h_0 > 0$ is given, and $f(h_t) = Ah_t^{\mu}$ represents the fact that the output is assumed to be obtained by using only the effective labor through the production function f, which is concave, increasing, continuous. In addition, as usual, the utility function is supposed to be strictly increasing, concave and continuous.

From (I.11) we get:

$$S_t = \frac{1}{\theta^{\frac{1}{\alpha}}}\left[\frac{h_{t+1}}{h_t} - (1-\delta)\right]^{\frac{1}{\alpha}}. \tag{I.12}$$

Substituting (I.12) into (I.10), we obtain

$$c_t = Ah_t^{\mu} - \frac{1}{\theta^{\frac{1}{\alpha}}}\left[\frac{h_{t+1}}{h_t} - (1-\delta)\right]^{\frac{1}{\alpha}}. \tag{I.13}$$

Since $c_t \geq 0$, the equation (I.13) leads to the following condition:

$$\frac{h_{t+1}}{h_t} \leq \theta A^{\alpha} h_t^{\alpha\mu} + 1 - \delta. \tag{I.14}$$

Taking into account (I.13) and (I.14), the above problem becomes the following program, denoted by $(P1)$:

$$\max \sum_{t=0}^{+\infty} \beta^t u\left(\left[Ah_t^{\mu} - \frac{1}{\theta^{\frac{1}{\alpha}}}\left[\frac{h_{t+1}}{h_t} - (1-\delta)\right]^{\frac{1}{\alpha}}\right]\right)$$

subject to the constraint:

$$(1-\delta)h_t \leq h_{t+1} \leq [\theta A^{\alpha} h_t^{\alpha\mu} + 1 - \delta]h_t. \tag{I.15}$$

Lemma 3.1

Let us define $V \equiv u\left(\left[Ah_t^{\mu} - \frac{1}{\theta^{\frac{1}{\alpha}}}\left[\frac{h_{t+1}}{h_t} - (1-\delta)\right]^{\frac{1}{\alpha}}\right]\right)$. *Then the utility function* V *is twice differentiable and strictly super-modular, for any* h_t, h_{t+1}, *we have:*

$$V_{12}(h_t, h_{t+1}) = \frac{\partial^2 V(h_t, h_{t+1})}{\partial h_t \partial h_{t+1}} > 0.$$

The set $\left\{(x,y) \in \mathbb{R}_+^2 : (1-\delta)x \leq y \leq [\theta A^{\alpha} x^{\alpha\mu} + 1 - \delta]x\right\}$ *is a sub-lattice. Hence, we have the monotonicity of the solutions.*

Resolving the problem $(P1)$, we obtain that associated Euler's equation is given by:

$$-\frac{u^{'}(c_t)}{\alpha h_t \theta^{\frac{1}{\alpha}}}\left[\frac{h_{t+1}}{h_t} - (1-\delta)\right]^{\frac{1}{\alpha}-1} + \beta u^{'}(c_{t+1})\left[\mu A h_{t+1}^{\mu-1} + \frac{\left(\frac{h_{t+2}}{h_{t+1}} - (1-\delta)\right)^{\frac{1}{\alpha}-1}}{\alpha\theta^{\frac{1}{\alpha}}}\frac{h_{t+2}}{h_{t+1}^2}\right] = 0. \tag{I.16}$$

Proposition 3.1

1. *If* $\delta = 0$, *which means human capital is not depreciated at all, then human capital converges to infinity. Moreover the rate of growth of human capital is unbounded from above if* $u(c) = \ln(c)$.

2. *If* $\delta > 0$, *let us introduce* $\bar{h} \equiv \left[\frac{1-\beta}{\beta}\left(\frac{1}{\alpha\theta^{\frac{1}{\alpha}}\mu A}\right)\delta^{\frac{1}{\alpha}-1}\right]^{\frac{1}{\mu}}$, *and* $\tilde{h}$ *satisfies* $\theta A^{\alpha}\tilde{h}^{\alpha\mu} = \delta$, *then, since the optimal sequence* $\{h_t^*\}$ *converges, there are only three cases with regards to the convergence of human capital accumulation:*

   (a) *If* $h_0 < \tilde{h}$ *then* $h_t^* \to 0$.

   (b) *Assume that* $h_0 > \tilde{h} > \bar{h}$. *If* $h_1^* > h_0$ *then the sequence* $\{h_t^*\}$ *converges to infinity. If* $h_1^* < h_0$ *then the sequence* $\{h_t^*\}$ *converges to* 0.

   (c) *In the case where* $\tilde{h} < \bar{h} < h_0$. *If* $h_1^* > h_0$ *then* $h_t^* \to +\infty$. *If* $h_1^* < h_0$ *then the sequence* $\{h_t^*\}$ *converges either to* 0 *or* $\bar{h}$.

Proof: Appendix. ■

Comment: Focusing on the point 2 (c) of Proposition 3.1, where $h_0 > \tilde{h} > \bar{h}$, which means that the patience coefficient β, the efficiency coefficient of human capital technology θ, and especially the initial human capital stock h_0 are sufficiently important with respect to human capital depreciation rate δ, we understand why Lucas (2015) requires an ideas stock to be always "out there" (p. 86) so that human capital accumulation can carry on its own shoulders the growth of economies.

3.2 A Lucas model without physical capital

In this subsection, we consider a discrete time version à la Lucas (1988) where there is no physical capital and no externality. We normalize to 1 the number of workers. We suppose therefore there is a single representative worker, who disposes a total time normalized by 1. This time can be divided into two parts: a working part, denoted by τ_t and another part devoted to human capital formation (as time for schooling, training, relaxing, leisure, etc).

Denote by h_t the human capital stock at date t. Assume that the accumulation of human capital over time is characterized by the equation

$$\frac{h_{t+1}}{h_t} = 1 - \delta + \theta(1 - \tau_t), \tag{I.17}$$

where $0 < \delta < 1$, which represents the depreciation rate of human capital and θ the productivity of training (or of recovery). With respect to the original Lucas function of human capital formation, we consider the depreciation rate of human capital.

The production of the consumption good is given by $f(L)$ where L is the efficient labor given by $L_t = \tau_t h_t$ and f is a production function. Assume that f is concave, differentiable, $f(0) = 0$ and satisfies Inada condition $f'(0) = +\infty$. There is no physical capital accumulation and the consumption at date t is equal to the production $c_t = f(L_t) = f(\tau_t h_t)$.

The representative agent solves the following problem:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t = f(\tau_t h_t),$$
$$h_{t+1} = [1 - \delta + \theta(1 - \tau_t)]h_t,$$
$$h_0 > 0 \text{ is given.}$$

Assume that $0 < \beta < 1$ and the uttility function u is strictly concave satisfying Inada condition $u'(0) = +\infty$.

We can re-write the human capital formation equation as follow:

$$\tau_t h_t = \frac{(1 - \delta + \theta)h_t - h_{t+1}}{\theta}.$$

For any $(1 - \delta)h \leq h' \leq (1 - \delta + \theta)h$, let us define

$$V(h, h') = u\left(f\left(\frac{(1 - \delta + \theta)h - h'}{\theta}\right)\right).$$

The maximization problem becomes the following one, denoted by (P):

$$\max \sum_{t=0}^{\infty} \beta^t V(h_t, h_{t+1})$$
$$\text{s.t } (1 - \delta)h_t \leq h_{t+1} \leq (1 - \delta + \theta)h_t,$$
$$h_0 \text{ is given.}$$

We can verify that the function V is supermodular.

Proposition 3.2

1. *There exists unique solution $\{h_t^*\}_{t=0}^{\infty}$ of Problem (P).*

2. *If* $\beta(1-\delta+\theta) > 1$ *then* $\{h_t^*\}_{t=0}^{\infty}$ *is strictly increasing and converges to infinity.*

3. *If* $\beta(1-\delta+\theta) < 1$ *then* $\{h_t^*\}_{t=0}^{\infty}$ *is strictly decreasing and converges to zero.*

Proof: Appendix. ■

Proposition 3.3 *Assume* $u(c) = \ln(c),\ f(L) = L^{\alpha}, \alpha \in (0,1), \beta(1+\lambda) > 1-\delta$.

1. *Then there exists a unique solution*

$$h_t^* = u^{*t} h_0, \text{ with } u^* = \beta(1-\delta+\theta).$$

2. $u^* > 1 \Leftrightarrow \beta > \frac{1}{1-\delta+\theta}$.

Proof: Appendix. ■

Comment: Since the condition $\beta > \frac{1}{1-\delta+\theta}$ can be equivalently expressed as $1-\delta+\theta > \frac{1}{\beta}$, one can argue that there is sustained economic growth in this Lucas model if human capital formation technology is relatively efficient (large θ and/or low δ) with respect to household's level of impatience ($1/\beta$).

3.3 Adding human capital in a Ramsey model: a first attempt

We now introduce human capital into an infinite horizon economy *à la* Ramsey (1928). While the population of infinite life-time households is normalized, each household of the economy is assumed to be able to invest in both physical and human capital by dividing her savings S_t into physical capital investment and education expenditure via enhancing human capital accumulation:

$$S_t = k_{t+1} + e_{t+1},$$

where S_t is the savings at date t, and k_{t+1}, e_{t+1} are respectively the investments in physical capital and human capital.

We respectively consider the following output and human capital production functions :

$$f(k_t, h_t) = Ak_t^{\mu} h_t^{1-\mu},$$
$$h_t = \phi(\theta, e_t),$$

where the effective labor is denoted by h_t, and its formation is given by $\phi(\theta, .)$, which is an increasing concave function only depending on education investment e_t, with $\phi(\theta, 0) = \underline{\mathrm{e}} > 0$.

Considering a strictly concave, increasing utility function u, the maximization program of the agent is written as follow:

$$\max \sum_{t=0}^{+\infty} \beta^t u(c_t) \tag{I.18}$$

subject to the following constraints:

$$c_t + k_{t+1} + e_{t+1} \leq Ak_t^{\mu}[\phi(\theta, e_t)]^{1-\mu}, \tag{I.19}$$
$$e_t \geq 0. \tag{I.20}$$

In order to solve this program, we assume that:

$$\phi(\theta, e_t) = \underline{\mathrm{e}} + \theta e_t^{\alpha},\ \ \alpha \in (0, 1),\ \ \theta > 0, \underline{\mathrm{e}} > 0.$$

Lemma 3.2 *Considering the following problem:*

$$\max\{Ak^{\mu}\left[\underline{e} + \theta e^{\alpha}\right]^{1-\mu} : k \geq 0, e \geq 0, k + e \leq s\}.$$

Let $F(\theta, s) = \max\{Ak^{\mu}\left[\underline{e} + \theta e^{\alpha}\right]^{1-\mu} : k \geq 0, e \geq 0, k + e \leq s\}$*, then we have:*

1. *The function* $F(\theta, .)$ *is concave, increasing.*

2. *The solution* $(\tilde{k}, \tilde{e})$ *is unique and satisfies:*

$$\mu \underline{e}\, \tilde{e}^{1-\alpha} + [\mu\theta + (1-\mu)\theta\alpha]\, \tilde{e} = (1-\mu)\theta\alpha s, \tag{I.21}$$

$$\theta\alpha(1-\mu)\tilde{k} = \mu(\underline{e} + \theta\tilde{e}^{\alpha})\tilde{e}^{1-\alpha}. \tag{I.22}$$

The function $F(\theta, .)$ *is differentiable.*

3. *If* $s = 0$ *then* $\tilde{e} = 0$, $\tilde{k} = 0$. *Hence* $F(\theta, 0) = 0$. *If* $s \to +\infty$ *then* $\tilde{e} \to +\infty$, $\tilde{k} \to +\infty$.

4. *If* s *increases then both* $\tilde{e}, \tilde{k}$ *increase.*

Proof: Appendix. ■

Using the previous Lemma 3.2, we get the following results with regards to the existence and uniqueness, as well as the convergence of the solution.

Proposition 3.4

1. *The sequence* $\{k^*_{t+1}, e^*_{t+1}\}_{t\geq 0}$ *solves the following problem denoted by* $\mathcal{P}_1$*:*

$$\max \sum_{t=0}^{+\infty} \beta^t u(c_t)$$

subject to the constraints:

$$c_t + k_{t+1} + e_{t+1} \leq A k_t^{\mu} [\phi(\theta, e_t)]^{1-\mu},$$

$$c_t \geq 0, e_t \geq 0, k_t \geq 0,\ k_0 > 0,\ e_0 > 0,$$

with

$$\phi(\theta, e_t) = \underline{e} + \theta e_t^{\alpha},\ \alpha \in (0,1),\ \theta > 0, \underline{e} > 0,$$

if, and only if, it solves the following problem denoted by $\mathcal{P}_2$:

$$\max \sum_{t=0}^{+\infty} \beta^t u(c_t)$$

subject to the constraints:

$$c_0 + s_1 \leq Ak_0^{\mu}[\phi(\theta, e_0)]^{1-\mu},$$
$$c_t + s_{t+1} \leq F(\theta, s_t), \ \forall t \geq 1,$$
$$c_t \geq 0, e_t \geq 0, k_t \geq 0, k_0 > 0, e_0 > 0,$$

with

$$\phi(\theta, e_t) = \underline{e} + \theta e_t^{\alpha}, \ \alpha \in (0,1), \ \theta > 0, \underline{e} > 0.$$

2. *The optimal human capital is $h_t^* = \phi(\theta, e_t^*)$. The optimal capitals k_t^*, h_t^* increase with the efficient parameter θ.*

Proof: Appendix. ■

Proposition 3.5

1. *There exists a unique optimal sequence $(k_t^*, h_t^*)_{t\geq 1}$.*

2. *This sequence is increasing and converges to a steady state.*

Proof: Appendix ■

Noticing that we can reach the previous Propositions 3.4 and 3.5 without any specification of the utility function u, however, in order to compute explicitly the steady state, we need more precise information on household's utility. Therefore, let us assume that $u(c) = \ln(c)$. We then obtain:

Proposition 3.6 *Let $\bar{e}, \bar{k}$ denote the values at the steady state of the investment for human capital and for physical capital. They satisfy the following equations:*

$$\bar{e} = \left[(\mu A\beta)^{\frac{1}{1-\mu}} \left(\frac{1-\mu}{\mu}\right) \alpha\right]^{\frac{1}{1-\alpha}} \theta^{\frac{1}{1-\alpha}},$$
$$\bar{k} = (\mu A\beta)^{\frac{1}{1-\mu}} \left[\underline{e} + \theta \bar{e}^{\alpha}\right].$$

The steady state human capital is

$$\bar{h} = \left[\underline{e} + \theta \bar{e}^{\alpha}\right].$$

The values of the variables of the steady state increase with respect to θ, the efficiency parameter of the human capital formation.

Proof: Appendix ■

Comment: In this economy where the function of human capital formation only builds upon economic saving, we can easily see that the ratio physical capital to human capital in steady state only depends on parameters that affect agent's portfolio choice between investment in physical capital and human capital: $\bar{k}/\bar{h} = (\mu A\beta)^{\frac{1}{1-\mu}}$. The intuition behind this is that, in presence of physical capital, the patience coefficient β represents the willingness of households to save, whereas μ represents physical capital's share in output production; and in addition, in the process of human capital formation, there is a part played by saving from final/capital good.

4 Human capital accumulation: when Ramsey meets Lucas

This section, generalizing the models discussed in the previous sections in several ways, contains the main results of our study, responding to calls of Lucas (2015) to place human capital at the center of economic growth, without the need for an external channel, like technology. Interestingly, it was Lucas, who earlier in his very influential paper Lucas (1988), considers an economy with production technology relies not only on the individual inputs of physical and human capital, but also on an average level of skill or human capital, "calling external effects of human capital" (p. 36). However, Lucas himself later on, citing the works by Schoellman (2012) and Manuelli and Seshadri (2014), among others, argues that we should avoid understanding the role of human capital accumulation as well as giving "too large a role to exogenous technological change" (Lucas, 2015, p. 86).

In order to study human capital accumulation in the model of Ramsey (1928) who considers identical infinitely-lived consumers that maximize an isoelastic intertemporal utility function in an economy where production depends on the individual inputs of physical capital (physical capital accumulation), we generalize the human capital production function *à la* Lucas (1988) which is linear with respect to individuals' time effort devoted to education process. As a matter of fact, in the first following subsection 4.1, we consider an affine generalization function of human capital formation by adding to the previous one the depreciation rate of human capital: $\frac{h_{t+1}}{h_t} = 1 - \delta + \theta(1 - \tau_t)$, with $1 - \tau_t$ denotes the level of time effort devoted to human capital accumulation. Whereas, in the second following subsection 4.2, we consider another generalization function of human capital accumulation, which is no longer linear with respect to neither the time effort spent on education, nor the current level of human capital stock h_t, and in addition, also depends on households' economic effort devoted to investment in education: $h_{t+1} = (1 - \delta)h_t + \theta e_t^{\gamma} h_t^{1-\gamma}(1 - \tau_t)^{\sigma}$. This later function generalizes not only the initial human capital production function considered by Lucas (1988)[4], but also all the human capital formation functions that we consider until then[5].

To sum up, the novel idea of this section is that we bridge the model of Ramsey

[4] It is worthy to mention that, our latest human capital production function also generalizes the human production function studied by Caballé and Santos (1993), which is "linearly homogeneous on $\tilde{h}(t)$: $\tilde{G} = \tilde{h}(t)G(1 - u(t))$ where $1 - u(t)$ denotes non-leisure time devoted to human capital accumulation" (p. 1046).

[5] In fact, the function of human capital production in the subsection 4.1 is a special case of the one in the subsection 4.2 with $\gamma = 0$ and $\sigma = 1$.

(1928) to the models of Lucas (1988, 2015) by considering (i) identical infinitely-lived agents endowed with both physical and human capital; (ii) non-linear human capital accumulation function taking into account both the depreciation rate of human capital, the role of current human capital stock, and the level of time effort as well as economic saving devoted to education; (iii) but without external effect so that human capital accumulation plays a central role in contributing to economic growth.

4.1 The first model (the linear human capital accumulation function)

In this subsection, we study human capital accumulation in a Ramsey-like model. We suppose the free disposal property for human capital. Namely, the representative agent has the choice to work with her maximum competence (measured in terms of time effort) or not. Hence, she solves the following maximization problem:

$$\begin{aligned}
&\max \sum_{t=0}^{\infty} \beta^t u(c_t)\\
&\text{s.t } \forall t \geq 0,\ c_t + k_{t+1} \leq f(k_t, \tau_t h_t),\\
&\frac{h_{t+1}}{h_t} = (1-\delta) + \theta(1-\tau_t),\\
&c_t \geq 0, k_{t+1} \geq 0, h_{t+1} \geq 0, 0 \leq \tau_t \leq 1,\\
&k_0 > 0,\ h_0 > 0, \quad \text{are given.}
\end{aligned}$$

In this program, c_t, k_t and h_t respectively denote consumption, capital stock and human capital stock; τ_t and $1-\tau_t$ are working time and learning time, respectively; $\delta \in (0,1)$ is the depreciation rate of the human capital; $\theta > 0$ represents the learning ability parameter. We also assume that the output production function and the utility function f and u are strictly increasing, strictly concave functions, satisfying Inada conditions. In fact, we assume for simplicity $u(c) = \ln(c)$.

As it will be shown later on in the proof of Proposition 4.3, if we denote by $1+\rho$

the rate of growth along a balanced growth path (BGP), then Euler conditions necessarily imply that $1+\rho=\beta\,(\theta+1-\delta)$.

This fact motivates us to distinguish different cases depending on whether or not $\beta(1-\delta+\theta)>1$, and firstly, analyze the case where $\beta(1-\delta+\theta)>1$, or equivalently $\theta>\frac{1}{\beta}-1+\delta$, which means the learning ability parameter is high enough, or in terms used by Lucas (2015), requires "the quality of the individual's intellectual environment" (p. 86).

When learning ability is sufficiently high: $\theta>\frac{1}{\beta}-1+\delta$**.**

At the optimum, we have

$$h_{t+1}=(1-\delta+\theta)h_t-\theta\tau_t h_t,$$

which is equivalent to

$$\tau_t h_t=\frac{(1-\delta+\theta)h_t-h_{t+1}}{\theta}.$$

For each $(k,h)\in\mathbb{R}_+^2$, define $\Gamma(k,h)$ the set of $x'=(k',h')\in\mathbb{R}_+^2$ such that:

$$\begin{aligned}&0\le h'\le(1-\delta+\theta)h,\\&0\le k'\le f\left(k,\frac{(1-\delta+\theta)h-h'}{\theta}\right).\end{aligned}$$

For $(k',h')\in\Gamma(x)=\Gamma(k,h)$, define:

$$V(k,h,k',h')=u\left(f\left(k,\frac{(1-\delta+\theta)h-h'}{\theta}\right)-k'\right).$$

For each value $1+\rho$ let us consider the following maximization problem, denoted by $\mathcal{P}(1+\rho)$:

$$\max \sum_{t=0}^{\infty} \beta^t V(k_t, h_t, k_{t+1}, h_{t+1})$$
$$\text{s.t } (k_{t+1}, h_{t+1}) \in \Gamma(k_t, h_t),$$
$$(k_0, h_0) \in \mathbb{R}_+^2 \text{ given.}$$

Lemma 4.1

1. *The correspondence Γ is compact convex.*
2. *Define by $\Pi(k_0, h_0)$ the set of feasible sequences from (k_0, h_0). This set $\Pi(k_0, h_0)$ is convex compact in product topology.*
3. *Define $W(k_0, h_0)$ as value function. W is strictly concave.*
4. *There exists unique solution for the optimization problems. The solution is in interior of $\Pi(k_0, h_0)$.*

Proof: Appendix ■

Based on Lemma (4.1), it is well known that the value function is solution of functional equation:

$$W(k_0, h_0) = \max_{(k_1,h_1)\in\Gamma(k_0,h_0)} \left[V(k_0, h_0, k_1, h_1) + \beta W(k_1, h_1)\right].$$

From the strictly concavity of function W, for any $(k_0, h_0) \in \mathbb{R}_{++}^2$, there exists unique (k_1, h_1) such that

$$(k_1, h_1) = \underset{\Gamma(k_0,h_0)}{\arg\max} \left[V(k_0, h_0, k_1, h_1) + \beta W(k_1, h_1)\right].$$

Let us define

$$(k_1, h_1) = \psi(k_0, h_0) = (\psi_k(k_0, h_0), \psi_h(k_0, h_0)).$$

We then obtain the following result with respect to the convergence of solutions:

Proposition 4.1 *For the general optimization problem $\mathcal{P}(1+\rho)$ and the optimal sequence $\{(k_t^*, h_t^*)\}_{t=0}^{\infty}$, there exist k^* and h^* such that*

$$\lim_{t\to\infty} \frac{k_t^*}{(1+\rho)^t} = k^*,$$
$$\lim_{t\to\infty} \frac{h_t^*}{(1+\rho)^t} = h^*.$$

Proof: Appendix. ■

Definition 4.1 *A Balanced Growth Path (BGP) of the model is a sequence $(c_t, k_{t+1}, h_{t+1}, e_t, \tau_t)_{t\geq 0}$ solution to the model which satisfies:*

$$k_0 > 0, h_0 > 0 \textit{ are given,}$$
$$\textit{for all } t \geq 0, \tfrac{c_{t+1}}{c_t} = \tfrac{k_{t+1}}{k_t} = \tfrac{h_{t+1}}{h_t},$$
$$\textit{for all } t \geq 0, \tau_t \in [0,1] \textit{ and is constant (denoted by } \tau).$$

Proposition 4.2 *We assume that $u(c) = \ln(c)$, $f(k_t, \tau_t h_t) = Ak_t^{\mu}(\tau_t h_t)^{1-\mu}$, $\mu \in (0,1)$ and $\theta > \frac{1}{\beta} + \delta - 1$ (or equivalently $\rho > 0$). If the parameter A is large enough then there exists a BGP : for all $t \geq 0$*

- $k_{t+1}^* = k_0(1+\rho^*)^{t+1}$,
- $h_{t+1}^* = h_0(1+\rho^*)^{t+1}$,
- $c_t^* = c_0(1+\rho^*)^{t+1}$,
- $\rho^* = \beta[\theta + 1 - \delta] - 1 > 0$,
- $\tau^* = \frac{(1-\delta+\theta)(1-\beta)}{\theta} < 1$,
- $c_0 = Ak_0^{\mu}\tau^{*1-\mu}h_0^{1-\mu} - k_0(1+\rho^*) > 0.$

Proof: Appendix. ■

Comments:

1. When the learning ability parameter θ increases, the rate of growth ρ^* increases. Also, the working time τ^* decreases, because the labor is more efficient and compensates the time for working.

2. When $\theta + 1 - \delta > \frac{1}{\beta}$, or in other words when the learning ability parameter is high enough and the depreciation rate of the stock of ideas that are already "out there" - according to the term used by (Lucas, 2015, p. 86) is sufficiently low with respect to the impatience parameter $(1/\beta)$, economic growth is fueled by both physical capital accumulation and human capital accumulation. We achieve this result with homogeneous agents, rather than heterogeneous agents like in Galor and Moav (2004).

When $\theta = \frac{1}{\beta} - 1 + \delta$

Secondly, we analysis the existence of steady states in the case $\rho = 0$, or equivalent to $\beta(1 - \delta + \theta) = 1$. This case is determinant for the proof in which in general cases, the optimal solution converges to balance growth path. Now, we will prove that if $\rho = 0$, there exist an infinite number of steady states, and the optimal solution converges to a steady state, depending on the initial state (k_0, h_0). For any (k_0, h_0), denote by $\{(k_t^*, h_t^*)\}_{t=0}^{\infty}$ the optimal sequence.

Proposition 4.3 *Assume that $\rho = 0$ and $\beta = \frac{1}{1-\delta+\theta}$. With the same functions u and f as in the previous proposition:*

1. *There exists a continuum number of steady states. The set of steady states is defined as the set of $(k, h) \in \mathbb{R}^2_{++}$ satisfying:*

$$f'_k\left(k, \frac{h(1-\beta)}{\beta\theta}\right) = \frac{1}{\beta},$$

which is equivalent to

$$\frac{k}{h} = \frac{A^{\frac{1}{1-\mu}} \mu^{\frac{1}{1-\mu}} (1-\beta) \beta^{\frac{\mu}{1-\mu}}}{\theta}.$$

2. *The optimal sequence converges.*

Proof: Appendix ■

Comment:

1. A very simple comparative statics computation in steady state easily shows that the optimal ratio of physical capital to human capital is increasing in Total Factor Productivity (A) of output production and decreasing in the efficiency parameter of education (θ). Moreover, this ratio is increasing in patience parameter (β) only if this parameter β is smaller than the total share of physical capital in ouput manufacturing (μ).

2. As we have previously shown in a model without physical capital but with the same technology of human capital formation (subsection 3.2), economic growth is not sustained (or in other words, the gross rate of growth is less than 1), if and only if $\theta + 1 - \delta < \frac{1}{\beta}$. Thus, we do not consider this case.

4.2 The second model (the non-linear human capital accumulation function)

We now generalize the previous model by considering a function of human capital accumulation which is no longer linear and also takes into account the role of economic saving: $h_{t+1} = (1-\delta)h_t + \theta e_t^{\gamma} h_t^{1-\gamma}(1-\tau_t)^{\sigma}$. Clearly, if the setting is that $\gamma = 0$ and $\sigma = 1$, we then return to the previous proposition's human capital formation function.

In addition, our definition of human capital accumulation rule is closely related to the one considered by Caballé and Santos (1993) in their section VI (p. 1061).

However, in a continuous time model of infinite horizon, they require that economic effort devoted to the formation of the next period's human capital stock to be proportional to the current capital stock, omitting potential effect of human capital depreciation.

In this economy, an infinite-lifetime representative agent chooses $(c_t, \tau_t, h_t, k_{t+1}, e_t)_{t\geq 0}$ in order to maximize her life-time utility:

$$\max \sum_{t=0}^{+\infty} \beta^t u(c_t)$$

subject to:

$$c_t + k_{t+1} + e_t \leq Ak_t^{\mu}(\tau_t h_t)^{1-\mu}, \tag{I.23}$$

$$0 \leq \tau_t \leq 1, \tag{I.24}$$

$$(1-\delta)h_t \leq h_{t+1} \leq (1-\delta)h_t + \theta e_t^{\gamma} h_t^{1-\gamma}(1-\tau_t)^{\sigma}, \tag{I.25}$$

$$h_0 > 0, k_0 > 0, \tag{I.26}$$

with $\mu < 1, \gamma < 1, \sigma \leq 1$.

Assumption 4.1 *We assume that* $u(c) = \ln(c)$.

Lemma 4.2 *In equilibrium,*

$$h_{t+1} = (1-\delta)h_t + \theta e_t^{\gamma} h_t^{1-\gamma}(1-\tau_t)^{\sigma}, \tag{I.27}$$

and $h_{t+1} \geq (1-\delta)h_t$. *In addition, we also obtain that* τ_t *is strictly less than* 1. *In other words, households always devote their time to improve their human capital stock.*

Proof: Appendix. ■

Definition 4.2 *A Balanced Growth Path (BGP) of the model is a sequence* $(c_t, k_{t+1}, h_{t+1}, e_t, \tau_t)_{t\geq 0}$ *solution to the model which satisfies:*

$$k_0 > 0, h_0 > 0 \text{ are given,}$$
$$\text{for all } t \geq 0, \tfrac{c_{t+1}}{c_t} = \tfrac{k_{t+1}}{k_t} = \tfrac{h_{t+1}}{h_t} = \tfrac{e_{t+1}}{e_t},$$
$$\text{for all } t \geq 0, \tau_t \in [0,1] \text{ and is constant.}$$

Moreover, let us respectively define the rate of economic growth and the working time along a BGP: $1 + \rho^* = \frac{k_{t+1}}{k_t}$, $\tau = \tau_t$.

We then obtain the following result:

Proposition 4.4 *For total factor productivity, A is large enough, there exists a unique BGP determined by*

$$\tfrac{1+\rho^*}{\beta} = A\mu k_0^{\mu-1} h_0^{1-\mu} \tau^{1-\mu}, \tag{I.28}$$
$$(\rho^* + \delta) h_0^{\gamma} = \theta e_0^{\gamma} (1-\tau)^{\sigma}, \tag{I.29}$$
$$\gamma(1-\tau)\tau^{-\mu} A k_0^{\mu} h_0^{1-\mu} = e_0 \sigma, \tag{I.30}$$
$$c_0 = A k_0^{\mu} (\tau h_0)^{1-\mu} - e_0 - k_0(1+\rho^*), \tag{I.31}$$

and $c_0 > 0$.

Proof: Appendix ■

Comment: A direct consequence of this result is that when the learning ability parameter θ increases, the optimal rate of growth ρ^* also increases.

More interestingly, when the agent's economic effort is also taken into account in the process of human capital formation, a high enough TFP of output production is required to ensure that economic growth can rely on both physical capital accumulation as well as human capital accumulation, regardless the level of learning ability parameter, in contrast with what we have found through Proposition

4.2 of the previous section. In other words, only requiring a sufficiently efficient technology of output production, we can prove that human capital plays a role as central as physical capital in powering economic growth - an idea supported by Lucas (2015), Manuelli and Seshadri (2014), and Galor and Moav (2004).

5 Conclusion

This chapter revisits the theoretical framework of endogenous economic growth in order to respond to the contribution of Lucas (2015), calling for the necessity of placing human capital accumulation at the center of economic growth, rather than external sources like technology. Our starting point is the consideration of intuitive two-period models which provide us very clear insights of what importance is the role of human capital accumulation for economic growth, and how it competes with physical capital accumulation in supporting economic growth.

We extend these discussions into infinite-horizon model in several ways, so that our production function of human capital accumulation is generalized throughout the paper, in order to encompass not only time and economic effort devoted to investment in education, but also the role of knowledge of the past in the formation of current human capital stock. We provide proofs for existence, uniqueness, and convergence of optimal balanced growth paths in several cases. Moreover, we show that human capital accumulation can challenge physical capital accumulation as a prime source of economic growth, without the hypothesis of market imperfection caused by asymmetric educational investment opportunities between heterogeneous households considered by Galor and Moav (2004).

6 Appendix

Proof of Proposition 2.1

(a) Since

$$\frac{h_1}{h_0} = \phi(\theta, S_0)$$

or equivalently

$$\frac{h_1}{h_0} = \theta S_0^\alpha + (1-\delta).$$

We solve the following problem

$$\max_{S_0}[u(F(h_0) - S_0) + \beta u(F(h_0\phi(\theta, S_0)))]. \tag{I.32}$$

The first order condition is

$$\begin{aligned}
\frac{1}{F(h_0) - S_0} &= \beta \frac{1}{F(h_1)} F^{'}(h_1) h_0 \theta \alpha S_0^{\alpha-1} \\
\frac{1}{F(h_0) - S_0} &= \beta \frac{\mu h_1^{\mu-1}}{h_1^\mu} h_0 \theta \alpha S_0^{\alpha-1} \\
\frac{1}{F(h_0) - S_0} &= \beta\mu \frac{h_0}{h_1} \theta \alpha S_0^{\alpha-1} \\
\frac{1}{F(h_0) - S_0} &= \beta\mu \frac{\theta \alpha S_0^{\alpha-1}}{\theta S_0^\alpha + 1 - \delta} \\
\frac{(1-\delta)S_0^{1-\alpha} + \theta S_0}{F(h_0) - S_0} &= \alpha\beta\mu\theta.
\end{aligned}$$

Let

$$s = \frac{S_0}{F(h_0)}$$

then the latter becomes

$$\frac{(1-\delta)s^{1-\alpha}(F(h_0))^{1-\alpha}+\theta sF(h_0)}{F(h_0)-sF(h_0)} = \alpha\beta\mu\theta$$
$$\frac{(1-\delta)s^{1-\alpha}}{\theta[F(h_0)]^{\alpha}} = \alpha\beta\mu(1-s)-s.$$

It would be the equivalent of

$$\frac{(1-\delta)s^{1-\alpha}}{\theta[Ah_0^{\mu}]^{\alpha}} + s = \alpha\beta\mu(1-s). \tag{I.33}$$

The RHS of (I.33) is decreasing, while the LHS is concave increasing. The solution is thus unique. In addition, we observe that LHS (1) $>$ RHS (1) $=0$. Hence, the optimal value $s(A, h_0, \theta)$ is strictly smaller than 1.

Moreover, when θ increases, the LHS decreases. Therefore, $s(A, h_0, \theta)$ increases. Similarly when h_0 or A increase the LHS decreases and thus the optimal value $s(A, h_0, \theta)$ increases. Finally, when β increases, RHS moves upward (but we still have RHS (1) $=0$.), while LHS does not change. Hence, $s(A, h_0, \theta)$ increases.

(b) When we do not invest in human capital ($S_0 = 0$), we have $h_1 = h_0(1-\delta)$ and $F(h_1) = F(h_0(1-\delta))$. Whereas we invest in human capital, the optimal value is $h_1^* = h_0(\phi(\theta, S_0^*) + 1 - \delta) > h_1$ and hence $F(h_1^*) > F(h_1)$. In other words,

$$\frac{F(h_1^*)}{F(h_0)} > \frac{F(h_1)}{F(h_0)}.$$

(c) The rate of GDP growth will be greater than 1, iff

$$[\theta s(A, h_0, \theta)^{\alpha}(Ah_0^{\mu})^{\alpha}] > \delta. \tag{I.34}$$

From (I.33), we see that the optimal value $s(A, h_0, \theta)$ tends to $\frac{\alpha\beta\mu}{1+\alpha\beta\mu}$ when $[\theta Ah_0^{\mu}]$ tend to infinity. Hence, if θ or/and A or/and h_0 is/are large enough or/and β is close to 1, then condition (I.34) is fulfilled.■

Proof of Proposition 2.2

(1) Taking the Lagrange equation from (I.4), (I.5), (I.6) and (I.7) with the following remark: $\phi(\theta, e_1) = \theta e_1^\alpha + (1-\delta)$. We then obtain

$$\mathcal{L} = \ln(c_0) + \beta \ln(c_1) + \lambda[Ak_0^{1-\mu}h_0^\mu - c_0 - k_1 - e_1] + \eta A k_1^{1-\mu}[h_0(\theta e_1^\alpha + (1-\delta))]^\mu - c_1. \tag{I.35}$$

First order conditions are:

$$\frac{1}{c_0} = \lambda, \tag{I.36}$$

$$\frac{\beta}{c_1} = \eta, \tag{I.37}$$

$$\frac{\lambda}{\eta} = (1-\mu)Ak_1^{-\mu}h_0^\mu[\theta e_1^\alpha + (1-\delta)]^\mu, \tag{I.38}$$

$$\frac{\lambda}{\eta} = \mu A k_1^{1-\mu}h_0^\mu[\theta e_1^\alpha + (1-\delta)]^{\mu-1}\theta\alpha e_1^{\alpha-1}. \tag{I.39}$$

Moreover, the budget constraints (I.5)and (I.6) are now binding ($\lambda > 0$) and ($\eta > 0$).

From equation (I.6), we have

$$\frac{c_1}{k_1} = Ak_1^{-\mu}h_0^\mu[\theta e_1^\alpha + (1-\delta)]^\mu. \tag{I.40}$$

Combining (I.38) and (I.40), we get

$$\frac{\lambda}{\eta} = (1-\mu)\frac{c_1}{k_1}. \tag{I.41}$$

Incorporating (I.36), (I.37) and (I.41), we receive

$$c_0 = \frac{1}{\beta(1-\mu)}k_1. \tag{I.42}$$

The equation (I.38) is divided by (I.39). We then have

$$k_1 = \frac{1-\mu}{\mu} \times \frac{\theta e_1^{\alpha} + (1-\delta)}{\theta \alpha e_1^{\alpha-1}}. \tag{I.43}$$

Substituting (I.42) and (I.43) into equation (I.5), we will have (I.8). From (I.8), we see that when $\theta = +\infty$ the equation becomes

$$(1+\chi)e_1 = f(k_0, h_0),$$

remark $\chi = \frac{1-\mu}{\alpha\mu} + \frac{1}{\alpha\beta\mu}$.

(2) Therefore $e_1 \to \bar{e}_1 = \frac{f(k_0,h_0)}{(1+\chi)}$ when $\theta \to +\infty$. From (I.43), $k_1 \to \frac{1-\mu}{\mu} \times \frac{\bar{e}_1}{\alpha}$. Now, since $h_1 = h_0(\theta e_1^{\alpha} + 1 - \delta)$, we have $h_1 \to +\infty$ when $\theta \to +\infty$. The output y_1 converges to infinity too since $y_1 = Ak_1^{1-\mu}h_1^{\mu}$. ■

Proof of Proposition 3.1

1. If $\delta = 0$ then from (I.11), $h_{t+1} \geq h_t$ for every t which implies h_t must converge. Suppose h_t tends to $\bar{h}$, with $+\infty > \bar{h} \geq h_0 > 0$ then $\frac{h_{t+1}}{h_t}$ tends to 1, and then from (I.13) $c_t \to \bar{c} = A\bar{h}^{\mu}$.

   From (I.16), we obtain $\beta u'(\bar{c})\mu A\bar{h}^{\mu-1} = 0$: a contradiction. Therefore $\bar{h} = +\infty$.

   The Euler equation becomes

$$\left(\frac{1}{Ah_t^{\mu} - S_t}\right)\left(\frac{1}{\alpha}\right)\left[\frac{h_{t+1}}{h_t} - (1-\delta)\right]^{\frac{1}{\alpha}-1} \frac{1}{h_t}\frac{1}{\theta^{\frac{1}{\alpha}}} =$$

$$\beta\left(\frac{1}{Ah_{t+1}^{\mu} - S_{t+1}}\right)\left[\mu A h_{t+1}^{\mu-1} + \left(\frac{h_{t+2}}{h_{t+1}} - (1-\delta)\right)^{\frac{1}{\alpha}-1} \frac{h_{t+2}}{h_{t+1}^2}\frac{1}{\alpha}\frac{1}{\theta^{\frac{1}{\alpha}}}\right].$$

   Suppose the ratio $\frac{h_{t+1}}{h_t}$ is uniformly bounded from above. Then one can check that the term $h_{t+1}^{\mu+1}\times$ LHS of the relation above is bounded while the term $h_{t+1}^{\mu+1}\times$ RHS converges to infinity. Thus the ratio $\frac{h_{t+1}}{h_t}$ cannot be uniformly bounded from above.

2. Consider the case $\delta > 0$. Observe that $\tilde{h}$ verifies $\tilde{h} = [\theta A^\alpha \tilde{h}^{\alpha\mu} + 1 - \delta]\tilde{h}$. If the sequence $\{h_t^*\}$ converges, if its limit $\bar{h}$ is finite then this one equals $\left[\frac{1-\beta}{\beta}\left(\frac{1}{\alpha\theta^{\frac{1}{\alpha}}\mu A}\right)\delta^{\frac{1}{\alpha}-1}\right]^{\frac{1}{\mu}}$ by Euler equation and since $S_t^* \to 0$.
   (a) The sequence $\{h_t^*\}$ satisfies the constraint $h_{t+1}^* \leq [\theta A^\alpha h_t^{*\alpha\mu} + 1 - \delta]h_t^*$. In this case, we have: for any t, $h_t^* \leq x_t$ where the sequence $\{x_t\}$ satisfies $\forall t, x_{t+1} = [\theta A^\alpha x_t{}^{\alpha\mu} + 1 - \delta]x_t, x_0 = h_0$. Since the function in x, $[\theta A^\alpha x^{\alpha\mu} + 1 - \delta]x$ is strictly convex, we can easily see that $x_0 < \tilde{h} \Rightarrow x_t \to 0$. Hence $h_t^* \to 0$ if $h_0 < \tilde{h}$.
   (b) Assume $h_0 > \tilde{h} > \bar{h}$. If $h_1^* > h_0$ then the sequence $\{h_t^*\}$ is increasing. It cannot converge to a finite value because this one will be $\bar{h}$ and $\bar{h} < h_0$: a contradiction. Hence the sequence will converge to infinity. Now, if $h_1^* < h_0$ then the sequence $\{h_t^*\}$ is decreasing. If it converges to a finite limit then this limit will be $\bar{h}$. Since $\bar{h} < \tilde{h}$, there will be T such that $h_T^* < \tilde{h}$. From (1), it will converge to 0.
   (c) Assume $h_0 > \bar{h} > \tilde{h}$. If $h_1^* > h_0$ then the sequence $\{h_t^*\}$ is increasing. The same argument as in (2) gives the result $h_t^* \to +\infty$. If $h_1^* < h_0$ then the sequence $\{h_t^*\}$ is decreasing. It converges either to $\bar{h}$ or 0. ■

Proof of Proposition 3.2

1. Uniqueness is a direct corollary of strictly concavity of u and f.

2. For the case $\beta(1 - \delta + \theta) > 1$. We can consider a variation of problem (P), say (P'):

$$\max \sum_{t=0}^{\infty} \beta^t V(h_t, h_{t+1})$$
$$\text{s.t } 0 \leq h_{t+1} \leq (1 - \delta + \theta)h_t,$$
$$h_0 \text{ is given.}$$

Now, we will prove that the optimal sequence of problem (P') is feasible for problem (P), hence it is also optimal for problem (P).

Consider the optimal sequence of problem (P'), $\{h_t^*\}_{t=0}^{\infty}$. From Inada conditions, one has for any t:

$$0 < h_{t+1} < (1-\delta+\theta)h_t.$$

Consider the Euler equation which is the same for (P) or (P'):

$$\begin{aligned}
&\frac{1}{\theta}u'\left(f\left(\frac{(1-\delta+\theta)h_t^*-h_{t+1}^*}{\theta}\right)\right)f'\left(\frac{(1-\delta+\theta)h_t^*-h_{t+1}^*}{\theta}\right)\\
&=\frac{\beta(1-\delta+\theta)}{\theta}u'\left(f\left(\frac{(1-\delta+\theta)h_{t+1}^*-h_{t+2}^*}{\theta}\right)\right)f'\left(\frac{(1-\delta+\theta)h_{t+1}^*-h_{t+2}^*}{\theta}\right),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&u'\left(f\left(\frac{(1-\delta+\theta)h_t^*-h_{t+1}^*}{\theta}\right)\right)f'\left(\frac{(1-\delta+\theta)h_t^*-h_{t+1}^*}{\theta}\right)\\
&=\beta(1-\delta+\theta)u'\left(f\left(\frac{(1-\delta+\theta)h_{t+1}^*-h_{t+2}^*}{\theta}\right)\right)f'\left(\frac{(1-\delta+\theta)h_{t+1}^*-h_{t+2}^*}{\theta}\right).
\end{aligned}$$

Since $\beta(1-\delta+\theta) > 1$, the sequence $\{(1-\delta+\theta)h_t^*-h_{t+1}^*\}_{t=0}^{\infty}$ is increasing and converges to infinity. From the supermodularity property, the optimal sequence $\{h_t^*\}_{t=0}^{\infty}$ is monotonic. Since $(1-\delta+\theta)h_t^*-h_{t+1}^*$ converges to infinity, this implies that $\{h_t^*\}_{t=0}^{\infty}$ is strictly increasing and converges to infinity.

Since $\{h_t^*\}_{t=0}^{\infty}$ is increasing, for any t we have

$$h_t^* < h_{t+1}^* < (1-\delta+\theta)h_t^*.$$

The first inequality is corollary of increasing property, the second one is corollary of Inada conditions.

Hence $\{h_t^*\}_{t=0}^{\infty}$ is feasible sequence of problem (P'). Obviously, this implies that $\{h_t^*\}_{t=0}^{\infty}$ is solution of problem (P).

3. For the case $\beta(1-\delta+\theta) < 1$, consider the optimal sequence $\{h_t^*\}_{t=0}^{\infty}$ of problem (P'). By super-modularity property, this sequence is monotonic.

   By Inada condition, for any t, $h_{t+1}^* < (1-\delta+\theta)h_t^*$. If $h_0 \leq h_1^*$, then $\{h_t^*\}_{t=0}^{\infty}$ is strictly increasing and is an interior solution, which is in contradiction with Euler equation. Hence the optimal sequence $\{h_t^*\}_{t=0}^{\infty}$ is strictly decreasing. If there exists an infinite number of t such that $h_{t+1}^* = (1-\delta)h_t^*$, then $\lim_{t\to\infty} h_t^* = 0$. Suppose that for t sufficiently big, $h_{t+1}^* > (1-\delta)h_t^*$, then by Euler equation, $(1-\delta+\theta)h_t^* - h_{t+1}^*$ is decreasing. The limit of $(1-\delta+\theta)h_t^* - h_{t+1}^*$ cannot be strictly positive, since this case contradicts the Euler equation. Hence $\lim_{t\to\infty}(1-\delta+\theta)h_t^* - h_{t+1}^* = 0$. This implies $\lim_{t\to\infty} h_t^* = 0$. ■

Proof of Proposition 3.3

1. The proof of (1) is shown, as follows:

   First, observe that $v(f(L)) = \alpha \ln(L)$. Let the sequence $\{h_t\}$ be defined by $h_t = {u^*}^t h_0$ where $u^* = \beta(1+\lambda)$. It is easy to see that

   - $1-\delta < u^* < 1+\lambda$,
   - The sequence $\{h_t\}$ satisfies the Euler equation and the transversality condition.

   That proves this sequence is optimal.

2. The proof of (2) is obvious. ■

Proof of Lemma 3.2

1. It is easy to check that F is concave and increasing.

2. The FOCs are given by equations (I.21) and (I.22). The LHS of (I.21) is increasing, while the RHS is constant. Hence the solution is unique for $\tilde{e}$.

Hence $\tilde{k}$ is uniquely determined by (I.22).

We see from these equations that $\tilde{e}, \tilde{k}$ are differentiable in s. Since $F(\theta, s) = A\tilde{k}^{\alpha}\left[\underline{e} + \theta\tilde{e}^{\gamma}\right]^{1-\alpha}$, it is differentiable in s.

3. When $s = 0$, from (I.21), we see that $\tilde{e} = 0$. From (I.22), one gets $\tilde{k} = 0$. Hence $F(\theta, 0) = 0$.

   From (I.21) we have $\tilde{e} \to +\infty$ if $s \to +\infty$. Equation (I.22) implies that $\tilde{k}$ goes to infinity too.

4. Obviously, from (I.21), one can deduce easily that $\tilde{e}$ increases with s and hence $\tilde{k}$ is increasing by (I.22). ■

Proof of Proposition 3.4

1. Let $(k_t^*, e_t^*)_{t\geq 1}$ solve $\mathcal{P}_1$ and let (c_t^*) be the associated sequence of consumptions. And let $(\tilde{k}, \tilde{e})$ solve $\mathcal{P}_2$ with the associated consumptions $\{\tilde{c}_t\}$. Define $s_t^* = k_t^* + e_t^*$, $\forall t \geq 1$. Then obviously the sequence $\{s_t^*\}$ satisfies the constraints of $\mathcal{P}_2$. Hence,

$$\sum_{t=0}^{\infty} \beta^t u(c_t^*) \leq \sum_{t=0}^{\infty} \beta^t u(\tilde{c}_t).$$

   We have $\tilde{s}_t = \tilde{k}_t + \tilde{e}_t$. Since $F(\theta, \tilde{s}_t) = A\tilde{k}_t^{\alpha}\phi(\theta, \tilde{e}_t)^{1-\alpha}$, the sequence $\{\tilde{c}_t, \tilde{k}_t, \tilde{e}_t\}$ satisfies the constraints of $\mathcal{P}_1$. Therefore

$$\sum_{t=0}^{\infty} \beta^t u(\tilde{c}_t) \leq \sum_{t=0}^{\infty} \beta^t u(c_t^*).$$

   Consequently,

$$\sum_{t=0}^{\infty} \beta^t u(\tilde{c}_t) = \sum_{t=0}^{\infty} \beta^t u(c_t^*).$$

   Since the utility is concave, the constraints are convex, our claim is true.

2. The result is a consequence of the fact that $\theta' > \theta$. It turns out that $F(\theta', .) > F(\theta, .)$. ■

Proof of Proposition 3.5

1. The problem actually is

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t u(F(\theta, s_t) - s_{t+1}) \\ \text{st} \quad & 0 \leq s_{t+1} \leq F(\theta, s_t), \\ & s_0 > 0. \end{aligned}$$

   The utility function is strictly concave, the function $F(\theta, .)$ is also concave. Hence there exists a unique solution $\{s_y^*\}_{t\geq 1}$. Furthermore, equations (I.21) and (I.22) give the values of $\{k_t^*, e_t^*\}_{t\geq 1})$.

2. We will show that $F_s'(\theta, 0) = +\infty$. We have

$$\frac{F(\theta, s)}{s} = \frac{A\mu^\mu[\underline{e} + \theta e^{*\alpha}]e^{*\mu(1-\alpha)}}{[(1-\mu)\theta\alpha + \mu\theta]e^* + \mu\ \underline{e}\ e^{*1-\alpha}}.$$

   Since $e^* \to 0$ when $s \to 0$, then $\frac{F(\theta,s)}{s}$ is equivalent to

$$\frac{A\mu^\mu[\underline{e} + \theta e^{*\alpha}]e^{*\mu(1-\alpha)}}{\mu\ \underline{e}\ e^{*1-\alpha}} = A\mu^\mu[\underline{e} + \theta e^{*\alpha}]e^{*(\mu-1)(1-\alpha)} \to +\infty,$$

   since $\mu - 1 < 0$ and $e^* \to 0$. Hence, the sequence $\{s_t^*\}$ is increasing. Moreover, k_t^*, e_t^* are increasing with respect to s_t^*, they increase too.

   Observe that $F(\theta, s_{t+1}^*) = A\mu^\mu[\underline{e} + \theta e^{*\alpha}_{t+1}]e^{*\mu(1-\alpha)}_{t+1}$ and

$$s_{t+2}^* \leq F(\theta, s_{t+1}^*) = A\mu^\mu[\underline{e} + \theta e^{*\alpha}_{t+1}]e^{*\mu(1-\alpha)}_{t+1}.$$

   The RHS $F(\theta, s_{t+1}^*)$ is concave in s_{t+1}^* and $\frac{F(\theta,s)}{s}$ converges to zero if $s \to +\infty$. The sequence $\{s_t^*\}$ is therefore bounded from above. Since it is increasing, it will converge to a finite value. Similarly, the sequences $\{k_t^*\}, \{e_t^*\}$ converge to finite values. ■

Proof of Proposition 3.6

At the steady state is, we have

$$F(\theta, \bar{s}) = \max_{k,e}\{Ak^{\mu}[\underline{e} + \theta e^{\alpha}]^{1-\mu} - \lambda(k + e - \bar{s})\}, \lambda \geq 0.$$

From the Envelop Theorem, we will have $F'(\theta, \bar{s}) = \lambda$.

Moreover, the first order conditions give

$$\mu A\bar{k}^{\mu-1}[\underline{e} + \theta\bar{e}^{\alpha}]^{1-\mu} = \lambda,$$
$$A\bar{k}^{\mu}(1-\mu)[\underline{e} + \theta\bar{e}^{\alpha}]^{-\mu}\theta\alpha\bar{e}^{\alpha-1} = \lambda.$$

We also have $F'(\theta, \bar{s}) = \frac{1}{\beta}$. Therefore,

$$\mu A\bar{k}^{\mu-1}[\underline{e} + \theta\bar{e}^{\alpha}]^{1-\mu} = \tfrac{1}{\beta}, \tag{I.44}$$
$$\tfrac{\mu}{(1-\mu)\theta\alpha}\bar{e}^{1-\alpha}[\underline{e} + \theta\bar{e}^{\alpha}] = \bar{k}. \tag{I.45}$$

These equations yield

$$\bar{e} = \left[(\mu A\beta)^{\frac{1}{1-\mu}}\left(\frac{1-\mu}{\mu}\right)\alpha\right]^{\frac{1}{1-\alpha}}\theta^{\frac{1}{1-\alpha}}, \tag{I.46}$$
$$\bar{k} = (\mu A\beta)^{\frac{1}{1-\mu}}\left[\underline{e} + \theta\bar{e}^{\alpha}\right]. \tag{I.47}$$

There are the steady state solutions. ■

Proof of Lemma 4.1

1. Use the concavity of functions u and f.
2. Use the compact convex properties of correspondence Γ.
3. The strictly concavity of W comes from the strictly concavity of u and f.

4. We can prove that the sum $\sum_{t=0}^{\infty} \beta^t V(k_t, h_t, k_{t+1}, h_{t+1})$ is upper semicontinuous on $\Pi(k_0, h_0)$, which is compact in product topology. The uniqueness comes from the strict concavity property. The interior property is consequence of Inada conditions.■

Proof Proposition 4.1

Consider the optimization problem with $\rho \neq 0$. We will prove that the optimal sequence converges to balance growth path (BGP). For any feasible sequence $\{k_t, h_t\}_{t=0}^{\infty} \in \Pi(k_0, h_0)$, define the sequence $\{\tilde{k}_t, \tilde{h}_t\}_{t=0}^{\infty}$ as

$$\tilde{k}_t = \frac{k_t}{1+\rho^t},$$
$$\tilde{h}_t = \frac{h_t}{1+\rho^t}.$$

Observe that for any t,

$$0 \leq \tilde{h}_{t+1} \leq \frac{1}{\beta}\tilde{h}_t,$$

and

$$\begin{aligned}
\tilde{k}_{t+1} &= \frac{k_{t+1}}{1+\rho^{t+1}} \\
&\leq \frac{f\left(k_t, \frac{(1-\delta+\theta)h_t - h_{t+1}}{\theta}\right)}{1+\rho^{t+1}} \\
&\leq f\left(\frac{k_t}{1+\rho^{t+1}}, \frac{(1-\delta+\theta)h_t - h_{t+1}}{\theta(1+\rho^{t+1})}\right) \\
&= f\left(\frac{1}{1+\rho}\tilde{k}_t, \frac{\tilde{h}_t - \beta\tilde{h}_{t+1}}{\beta\theta}\right).
\end{aligned}$$

Define $\tilde{f}(k,h) = f\left(\frac{1}{1+\rho}k, h\right)$. For the optimal sequence $\{k_t^*, h_t^*\}_{t=0}^{\infty}$ of problem

$P(1+\rho)$, and also define that

$$\tilde{k}_t^* = \frac{k_t^*}{1+\rho^t},$$
$$\tilde{h}_t^* = \frac{h_t^*}{1+\rho^t},$$

for all t.

Observe that since $u(c) = \ln(c)$, for any t, any feasible sequence $\{(k_t, h_t)\}_{t=0}^{\infty}$,

$$u\left(f\left(k_t, \frac{(1-\delta+\theta)h_t - h_{t+1}}{\theta}\right) - k_{t+1}\right) =$$
$$= u\left((1+\rho^{t+1})f\left(\frac{1}{1+\rho}\tilde{k}_t, \frac{\tilde{h}_t - \beta\tilde{h}_{t+1}}{\beta\theta}\right) - (1+\rho^{t+1})\tilde{k}_{t+1}\right)$$
$$= u(1+\rho^{t+1}) + u\left(f\left(\frac{1}{1+\rho}\tilde{k}_t, \frac{\tilde{h}_t - \beta\tilde{h}_{t+1}}{\beta\theta}\right) - \tilde{k}_{t+1}\right).$$

It is easy to verify that $\{\tilde{k}_t^*, \tilde{h}_t^*\}_{t=0}^{\infty}$ is solution of problem:

$$\max \sum_{t=0}^{\infty} \beta^t u\left(f\left(\frac{1}{1+\rho}\tilde{k}_t, \frac{\tilde{h}_t - \beta\tilde{h}_{t+1}}{\beta\theta}\right) - \tilde{k}_{t+1}\right)$$
$$0 \leq \tilde{h}_{t+1} \leq \frac{1}{\beta}\tilde{h}_t,$$
$$0 \leq \tilde{k}_{t+1} \leq \tilde{f}\left(\tilde{k}_t, \frac{\tilde{h}_t - \beta\tilde{h}_{t+1}}{\beta\theta}\right).$$

Using Proposition 4.3, the sequences $\{(\tilde{k}_t^*, h_t^*)\}_{t=0}^{\infty}$ converges to (k^*, h^*). Hence

$$\lim_{t\to\infty} \frac{k_t^*}{1+\rho^t} = k^*,$$
$$\lim_{t\to\infty} \frac{h_t^*}{1+\rho^t} = h^*.$$

■

Proof Proposition 4.2

The Lagrange is given by

$$\sum_{t=0}^{\infty}\left\{\beta^t u(c_t) + \lambda_t[Ak_t^{\mu}(\tau_t h_t)^{1-\mu} - c_t - k_{t+1}] + \eta_t[(1-\delta)h_t + \theta(1-\tau_t)h_t - h_{t+1} - \zeta_t(\tau_t - 1)\right\}.$$

The first oder conditions are

$$\beta^t u' c_t = \lambda_t, \tag{I.48}$$

$$\lambda_{t+1}R_{t+1} = \lambda_t, \tag{I.49}$$

$$\lambda_{t+1}w_{t+1}\tau_{t+1} + \eta_{t+1}[1-\delta+\theta(1-\tau_t)] = \eta_t, \tag{I.50}$$

$$\eta_t\theta h_t = \lambda_t w_t h_t - \zeta_t, \tag{I.51}$$

$$h_{t+1} = h_t\left[1-\delta+\theta(1-\tau_t)\right], \tag{I.52}$$

where $R_{t+1} = \mu A(\frac{k_{t+1}}{\tau_{t+1}h_{t+1}})^{\mu-1}$, $\zeta_t(\tau_t - 1) = 0$, and $w_{t+1} = (1-\mu)A(\frac{k_{t+1}}{\tau_{t+1}h_{t+1}})^{\mu}$.

Assume that $\tau_t < 1$ then $\zeta_t = 0$. Furthermore, if there are solutions of the form: $\frac{c_{t+1}}{c_t} = \frac{k_{t+1}}{k_t} = 1+\rho$, $\forall t$, then combining equations (I.48)and (I.49) we obtain:

$$R_{t+1} = \frac{\lambda_t}{\lambda_{t+1}} = \frac{\beta^t u'(c_t)}{\beta^{t+1}u'(c_{t+1})}.$$

And then we get

$$\begin{aligned} R_{t+1} &= \frac{1+\rho}{\beta} \equiv R \\ \frac{k_{t+1}}{\tau_{t+1}h_{t+1}} &= (\frac{A\mu\beta}{\rho})^{1/(1-\mu)}, \\ w_{t+1} &= (1-\mu)A(\frac{A\mu\beta}{\rho})^{\mu/(1-\mu)} \equiv w. \end{aligned}$$

From equation (I.51), we have

$$\eta_t = \frac{w}{\theta}\lambda_t. \tag{I.53}$$

Substituting (I.53) into (I.50), we receive the steady state of growth, as follows:

$$1 + \rho = \beta[\theta + 1 - \delta]. \tag{I.54}$$

From equation (I.52), we get

$$1 + \rho = 1 - \delta + \theta(1 - \tau). \tag{I.55}$$

Combining (I.55) and (I.49), we receive

$$\tau = \frac{(1 - \delta + \theta)(1 - \beta)}{\theta}. \tag{I.56}$$

It is easy to check that if we have $\theta > \frac{1}{\beta} + \delta - 1$ then $\rho > 0$ and $\tau < 1$. Observe these values are independent of A, k_0, h_0. Now since

$$c_0 = Ak_0^{\mu}\tau^{1-\mu}h_0^{1-\mu} - k_0(1 + \rho).$$

If

$$A > \frac{k_0^{1-\mu}(1 + \rho)}{(h_0\tau)^{1-\mu}},$$

then $c_0 > 0$. ■

Proof Proposition 4.3

1. Since $\rho = 0$ then $\beta = \frac{1}{1-\delta+\theta}$, the optimization problem becomes

$$\begin{aligned}
&\max \sum_{t=0}^{\infty} \beta^t u\left(f\left(k_t, \frac{(1 - \delta + \theta)h_t - h_{t+1}}{\theta}\right) - k_{t+1}\right) \\
&0 \leq h_{t+1} \leq \frac{1}{\beta}h_t, \\
&0 \leq k_{t+1} \leq f\left(k_t, \frac{h_t - \beta h_{t+1}}{\beta\theta}\right).
\end{aligned}$$

One can verify that any sequence $\{k_t, h_t\}_{t=0}^{\infty}$ satisfying $k_t = k$ and $h_t = h$ with

$$f'_k\left(k, \frac{h(1-\beta)}{\beta\theta}\right) = \frac{1}{\beta}$$

satisfies Euler equations and tranversality condition. For $f(k,h) = Ak^{\mu}h^{1-\mu}$, one has

$$f'_k\left(k, \frac{h(1-\beta)}{\beta\theta}\right) = \frac{1}{\beta}$$

if and only if

$$\frac{k}{h} = \frac{A^{\frac{1}{1-\mu}}\mu^{\frac{1}{1-\mu}}(1-\beta)\beta^{\frac{\mu}{1-\mu}}}{\theta}.$$

2. Using the Euler equations, we will give an explicit form of the ratio $\frac{k_t^*}{\beta h_t^* - h_{t+1}^*}$.

Consider Euler equations, the first one for k_{t+1}^* and the second one for h_{t+1}^*. Recall that $u(c) = \ln(c)$ and $f(k,h) = Ak^{\mu}h^{1-\mu}$ and $\beta(1-\delta+\theta) = 1$.

$$\frac{1}{A(k_t^*)^{\mu}\left(\frac{(1-\delta+\theta)h_t^* - h_{t+1}^*}{\theta}\right)^{1-\mu} - k_{t+1}^*} = \beta \times \frac{A\mu(k_{t+1}^*)^{\mu-1}\left(\frac{(1-\delta+\theta)h_{t+1}^* - h_{t+2}^*}{\theta}\right)^{1-\mu}}{A(k_{t+1}^*)^{\mu}\left(\frac{(1-\delta+\theta)h_{t+1}^* - h_{t+2}^*}{\theta}\right)^{1-\mu} - k_{t+2}^*}$$

$$\frac{A(1-\mu)(k_t^*)^{\mu}\left(\frac{(1-\delta+\theta)h_t^* - h_{t+1}^*}{\theta}\right)^{-\mu}}{A(k_t^*)^{\mu}\left(\frac{(1-\delta+\theta)h_t^* - h_{t+1}^*}{\theta}\right)^{1-\mu} - k_{t+1}^*} = \beta(1-\delta+\theta) \times \frac{A(1-\mu)(k_{t+1}^*)^{\mu}\left(\frac{(1-\delta+\theta)h_{t+1}^* - h_{t+2}^*}{\theta}\right)^{-\mu}}{A(k_{t+1}^*)^{\mu}\left(\frac{(1-\delta+\theta)h_{t+1}^* - h_{t+2}^*}{\theta}\right)^{1-\mu} - k_{t+2}^*}.$$

Divide the second equation by the first one, we get:

$$A(1-\mu)(k_t^*)^{\mu}\left(\frac{(1-\delta+\theta)h_t^* - h_{t+1}^*}{\theta}\right)^{-\mu} = \frac{(1-\mu)k_{t+1}^*}{\mu\left(\frac{(1-\delta+\theta)h_{t+1}^* - h_{t+2}^*}{\theta}\right)}(1-\delta+\theta).$$

This implies that

$$\begin{aligned}\frac{k_{t+1}^*}{(1-\delta+\theta)h_{t+1}^* - h_{t+2}^*} &= \frac{\mu A}{\theta^{1-\mu}} \frac{1}{1-\delta+\theta} \times \left(\frac{k_t^*}{(1-\delta+\theta)h_t^* - h_{t+1}^*}\right)^{\mu} \\ &= \frac{\mu A}{\theta^{1-\mu}} \frac{1}{1-\delta+\theta} \times \left(\frac{\mu A}{\theta^{1-\mu}} \frac{1}{1-\delta+\theta}\right)^{\mu} \left(\frac{k_{t-1}^*}{(1-\delta+\theta)h_{t-1}^* - h_t^*}\right)^{\mu^2} \\ &= \cdots \\ &= \left(\frac{\mu A}{\theta^{1-\mu}} \frac{1}{1-\delta+\theta}\right)^{\frac{1-\mu^{t+1}}{1-\mu}} \left(\frac{k_0^*}{(1-\delta+\theta)h_0^* - h_1^*}\right)^{\mu^{t+1}}.\end{aligned}$$

Hence, we obtain

$$\lim_{t\to\infty} \frac{k_{t+1}^*}{(1-\delta+\theta)h_{t+1}^* - h_{t+2}^*} = \left(\frac{\mu A}{\theta^{1-\mu}} \frac{1}{1-\delta+\theta}\right)^{\frac{1}{1-\mu}}.$$

In addition, from the first Euler equation, we have

$$\begin{aligned}\frac{c_t^*}{c_{t+1}^*} &= \frac{1}{\mu\beta A} \times \left(\frac{k_{t+1}^*}{(1-\delta+\theta)h_{t+1}^* - h_{t+2}^*}\right)^{1-\mu} \times \theta^{1-\mu} \\ &= \frac{1}{\beta\mu A} \times \left(\frac{\mu\beta A}{\theta^{1-\mu}}\right)^{1-\mu^{t+1}} \times \theta^{1-\mu} \times \left(\frac{k_0^*}{(1-\delta+\theta)h_0^* - h_1^*}\right)^{(1-\mu)\mu^{t+1}} \\ &= \frac{\theta^{\mu^{t+1}}}{(\mu\beta A)^{\mu^{t+1}}} \times \left(\frac{k_0^*}{(1-\delta+\theta)h_0^* - h_1^*}\right)^{(1-\mu)\mu^{t+1}}.\end{aligned}$$

This refers

$$\begin{aligned}c_{t+1}^* &= \frac{(\mu\beta A)^{\mu^{t+1}}}{\theta^{\mu^{t+1}} \left(\frac{k_0^*}{(1-\delta+\theta)h_0^* - h_1^*}\right)^{(1-\mu)\mu^{t+1}}} \times c_t^* \\ &= \cdots \\ &= \left(\frac{\mu\beta A}{\theta}\right)^{\frac{1-\mu^{t+2}}{1-\mu}} \times \frac{1}{\left(\frac{k_0^*}{(1-\delta+\theta)h_0^* - h_1^*}\right)^{1-\mu^{t+2}}} \times c_0^*.\end{aligned}$$

Let t tends to infinity, we obtain

$$\lim_{t\to\infty} c_t^* = \left(\frac{\mu\beta A}{\theta}\right)^{\frac{1}{1-\mu}} \times \frac{(1-\delta+\theta)h_0^* - h_1^*}{k_0^*} \times c_0^*.$$

■

Proof of Lemma 4.2

At the optimum, the income at any t must be at its maximal level. Hence, $h_{t+1} = (1-\delta)h_t + \theta e_t^\gamma h_t^{1-\gamma}(1-\tau_t)^\sigma$. We can remove the constraint $h_{t+1} \geq (1-\delta)h_t$. The Lagrange associated with the optimization program is that

$$\sum_{t=0}^{\infty}\{\beta^t u(c_t) + \lambda_t[Ak_t^\mu(\tau_t h_t)^{1-\mu} - c_t - k_{t+1} - e_t] + \eta_t(1-\tau_t) +$$
$$+ \kappa_t(-h_{t+1} + (1-\delta)h_t + \theta e_t^\gamma h_t^{1-\gamma}(1-\tau_t)^\sigma)\}.$$

The necessary conditions are

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{\lambda_t}{\lambda_{t+1}}, \tag{I.57}$$

$$A\mu k_{t+1}^{\mu-1}(\tau_{t+1}h_{t+1})^{1-\mu} = \frac{\lambda_t}{\lambda_{t+1}}, \tag{I.58}$$

$$\gamma\theta e_t^{\gamma-1}h_t^{1-\gamma}(1-\tau_t)^\sigma = \frac{\lambda_t}{\kappa_t}, \tag{I.59}$$

$$\kappa_t\theta e_t^\gamma h_t^{1-\gamma}\sigma(1-\tau_t)^{\sigma-1} + \eta_t = \lambda_t A k_t^\mu(1-\mu)(\tau_t h_t)^{-\mu}h_t, \tag{I.60}$$

$$\lambda_t(1-\mu)Ak_t^\mu(\tau_t h_t)^{-\mu}\tau_t = \kappa_{t-1} + \kappa_t\left((1-\gamma)B\left(\frac{e_t}{h_t}\right)^\gamma(1-\tau_t)^\sigma + (1-\delta)\right), \tag{I.61}$$

$$\lambda_t[Ak_t^\mu(\tau_t h_t)^{1-\mu} - c_t - k_{t+1} - e_t] = 0, \tag{I.62}$$

$$\eta_t[1-\tau_t] = 0, \tag{I.63}$$

$$\kappa_t[-h_t + (1-\delta)h_{t-1} + \theta e_{t-1}^\gamma h_{t-1}^{1-\gamma}(1-\tau_{t-1})^\sigma] = 0. \tag{I.64}$$

From (I.57), we have $\lambda_t > 0$, $\forall t$. From (I.61) we have $\kappa_t > 0$, $\forall t$. Consequently, (I.59) implies that $\tau_t < 1$, $\forall t$. ■

Proof Proposition 4.4

Write $k_t = k_0(1+\rho)^t, h_t = h_0(1+\rho)^t, e_t = e_0(1+\rho)^t, c = c_0(1+\rho)^t$. We will determine ρ, τ, e_0, c_0.

Since $u(c) = \ln(c)$, from (I.57) gives $\frac{\lambda_t}{\lambda_{t+1}} = \frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{\rho}{\beta}$. Combining with (I.58), we get:

$$\frac{1+\rho}{\beta} = A\mu k_0^{\mu-1} h_0^{1-\mu} \tau^{1-\mu},$$

which is equation (I.28).

From (I.27), we have:

$$(\rho+\delta) h_0^{\gamma} = \theta e_0^{\gamma} (1-\tau)^{\sigma},$$

which is (I.29).

Dividing (I.59) by (I.60), and since $\eta_t = 0$, we obtain:

$$\gamma(1-\tau)\tau^{-\mu}(1-\mu) A k_0^{\mu} h_0^{1-\mu} = e_0 \sigma,$$

which is (I.30).

Balance between demand and supply of good gives (I.31). Moreover, pludging in (I.29) the value of ρ given by (I.28), we receive:

$$e_0^{\gamma} = \frac{1}{\theta} \frac{\beta A\mu k_0^{\mu-1} h_0^{1-\mu} \tau^{1-\mu} + \delta - 1}{(1-\tau)^{\sigma}} h_0^{\gamma}. \tag{I.65}$$

Let $\tilde{A}$ satisfy $\beta \tilde{A} \mu k_0^{\mu-1} h_0^{1-\mu} \tau^{1-\mu} + \delta - 1 = 0$.

Assume that

$$A > \tilde{A}.$$

Let $\tilde{\tau}$ satisfy $\beta A\mu k_0^{\mu-1}h_0^{1-\mu}\tilde{\tau}^{1-\mu}+\delta-1=0$.

If $A>\tilde{A}$ then $\tilde{\tau}<1$.

Equations (I.30) and (I.65) will give the equilibrium values of (e_0^*,τ^*).

Consider (I.65). This gives a function $e_0^1(\tau)$ which is increasing. When $\tau=\tilde{\tau}$, $e_0^1(\tau)=0$. And when $\tau=1$, $e_0^1(\tau)=+\infty$.

Consider (I.30). It gives a function $e_0^2(\tau)$ which is decreasing. It equals $+\infty$ when $\tau=0$ and equals 0 when $\tau=1$.

The equilibrium value τ^* solves $e_0^1(\tau^*)=e_0^2(\tau^*)$. It is unique.

It remains to show with the values of τ^*, e_0^*, the consumption c_0^* in (I.31) is positive. We claim that when A is large, it will be true.

Observe that when A goes to infinity, then τ^* goes to 1. Tedious computations give

$$c_0^*=Ak_0^{\mu}h_0^{1-\mu}\tau^{*1-\mu}\left[\left(1-\frac{\gamma}{\sigma}\frac{1-\tau^*}{\tau^*}(1-\mu)\right)-\beta\mu\right].$$

When $A\rightarrow+\infty$, we have $1-\frac{\gamma}{\sigma}\frac{1-\tau^*}{\tau^*}(1-\mu)-\beta\mu\rightarrow 1-\beta\mu>0$. Hence $c_0^*>0$ when A is large enough.

When θ increases, the graph of the function e_0^1 moves downward while the graph of e_0^2 does not change. Hence τ^* increases. From (I.28), ρ^* increases.■

Chapter II

From Physical to Human Capital Accumulation: Heterogeneous Intergenerational Altruism and Inequality [1]

[1]This chaper is co-written with Cao-Tung PHAM

1 Introduction

There is an extensive literature in both the area of development economics and the endogenous growth theory which relies on overlapping generation (OLG) model. More recently, scholars agree that education is understood as a key somehow evoking human capital, allow individuals to accumulate human capital at the beginning of the life cycle and is crucially related to inter-generational matters. In spite of the large body of current studies on education and human capital in OLG model, this research area still leaves many dimensions, such as inter-generational culture transmission, diversity of communities, heterogeneous agents and integration between two communities/countries.

In his influential work, Lucas (1988) shows that the economic growth rate depends on the rate of growth of human capital stock. Nonetheless, the question whether or not human capital accumulation plays a central role in supporting economic growth is still opened up to debate. Recently, Lucas (2015) himself later raises on doubt that we consider "too large a role to exogenous technological change" (p. 86). In fact, as it is proved in Manuelli and Seshadri (2014), with alternative methods for measuring human capital based on data on schooling and age-earnings profile, differences between countries in average wealth are better explained by differences in levels of the accumulation of human capital, rather than levels of total factor productivity. This is the reason why Lucas (2015) argues that "the contribution of human capital accumulation to economic growth deserves a production function of its own" (p. 87). In line with these works, we consider a human capital accumulation function that captures the idea that, future generation's stock of human capital is formed by current generation's level of human capital and altruistic investment in education.

Together with Manuelli and Seshadri (2014), the works by McGrattan and Prescott (2009) and Schoellman (2012) show that production factors like physical capital and human capital play an more important role in enhancing economic growth than external sources like technology. In this context, the question that should be posed is whether or not human capital accumulation can compete with physical capital accumulation in the process of economic growth, if externalities are ignored[2]. By considering a credit constraint on investment in human capital, Galor and Moav (2004) not only show that human capital can compete with physical capital in this process, but also find out that at some stages, human capital can surpass physical capital in supporting economic growth. Our study differs from theirs in many points: rather than a heterogeneity hypothesis based on asymmetric investment opportunities, our heterogeneity hypothesis is based on differences in levels of altruism. Moreover, our form of paternalistic altruism is also different from theirs. We actually require that altruistic investment from parents must be positive and totally devoted to finance their offspring's expenditure on education.

This idea that parents are also concern about their children's education is far from new. In this aspect, our work are in line with Galor and Moav (2004) who suppose that altruistic investment from parents "are allocated between an immediate finance of their offspring's expenditure on education and saving for the future wealth of their offspring" (p. 1006), and Michel and Vidal (2000) who introduce in parents' utility function their children's human capital level. However, Michel and Vidal (2000) discuss "endogenous growth in a two-country overlapping-generations world" (p. 275) by exploring "the influence of cross-border external effects in human capital on growth" (p. 275), whereas we consider an economy with heterogeneous agents and study interactions between different production factors in the process of economic growth. Moreover, our form of altruism is closer (compared to theirs) to the original form of paternalistic altruism developed by

[2]The interaction between physical capital and human capital is also studied by Turnovsky and Mitra (2013) in a two-sector growth model, but unlike their work, ours does not consider any external source like technology.

Abel and Warshawsky (1988), who integrate the level of altruistic investment from parents into their utility function. In our overlapping generations model, the link between parents and their offspring is twofold. Firstly, children's education is financed by their parents' altruistic investment. Secondly, through an educational link, young people take control of the stock of knowledge of their ascendants.

In such a context, we study the dynamic of physical capital and human capital accumulation in an overlapping generations economy with heterogeneous agents. In this economy, there are two different communities, denoted by 1 and 2, cohabit. Each type of household $i \in \{1, 2\}$ differs from the other by the patience degree $0 < \beta_i < 1$, as well as the parameter of intergenerational altruism $0 < \gamma_i < 1$. These communities share the same technology of human capital formation. However, since they do not share the same degree of altruism (with respect to their willingness to invest in physical capital), it turns out that people of each community has their own level of human capital accumulation, and therefore, workers issued form different communities provide different levels of effective labor supply.

Our first result is that, we study the dynamic of physical and human capital accumulation, and prove the existence of a balanced growth path. We then find out that, along this steady growth path, human capital formation can play an as important role as physical capital accumulation in supporting growth. Another one of our findings is that, along this balanced growth path, if an agent's own intertemporal consumption is weighted more importantly than the amount of educational altruism that she intends to give to her children, then this agent invests more in physical capital. Furthermore, considering for example an agent of type 1, if her relative weigh assigned to her own intertemporal consumption (with respect to her educational altruism) is lower compared to an agent of type 2's one, then the relative ratio of human capital intensities $\frac{h_1}{h_2}$ increases.

As to the issue of inequality, we find out that it is important in this economy to not only consider inequality of labor income, but also inequality in terms of consumption levels. We then describe the impact of this inequality of consumption on the interaction between human and physical capital accumulation.

The remainder of this chapter is organized as follows, section 2 presents the fundamentals of our model. In section 3, we reveal the descriptions of global equilibrium dynamics. In section 4, we analyze the steady-growth path. In section 5, we study inequality of incomes as well as consumption levels. Conclusion is the section 6. The proofs are gathered in Appendix.

2 Fundamentals

We consider a two-period OLG model with physical and human capital accumulation, as follows:

Firms. The economy's technology is represented by an augmented Cobb-Douglas production function:

$$F(K_t, L_{1,t}, L_{2,t}) \equiv AK_t^{1-\alpha-\sigma} L_{1,t}^{\alpha} L_{2,t}^{\sigma}, \tag{II.1}$$

where K_t denotes the aggregate capital, and for $i \in \{1, 2\}$, $L_{i,t}$ is the labor force of type i.

A representative firm chooses K_t, $L_{1,t}$ and $L_{2,t}$ to maximize its profits:

$$F(K_t, L_{1,t}, L_{2,t}) - R_t K_t - w_{1,t} L_{1,t} - w_{2,t} L_{2,t}.$$

The firm's profit maximization implies:

$$R_t = (1 - \alpha - \sigma) AK_t^{-\alpha-\sigma} L_{1,t}^{\alpha} L_{2,t}^{\sigma}, \tag{II.2}$$

$$w_{1,t} = \alpha AK_t^{1-\alpha-\sigma} L_{1,t}^{\alpha-1} L_{2,t}^{\sigma}, \tag{II.3}$$

$$w_{2,t} = \sigma AK_t^{1-\alpha-\sigma} L_{1,t}^{\alpha} L_{2,t}^{\sigma-1}, \tag{II.4}$$

where R_t represents the return on capital, while, for $i \in \{1, 2\}$, $w_{i,t}$ denotes the wage rate on labor force of type i.

Generations. In this model, agents live for two periods, young and old. At time t, the young people's population is denoted by N_t, while the demographic growth factor is supposed to be a greater-than-one constant n: $n \equiv \frac{N_{t+1}}{N_t}$.

Workers. Each individual, as usual, is supposed to work only when young, and when old, he only consumes. For $i \in \{1, 2\}$, a worker of type i born at time t provides service flow $h_{i,t}$.

Young workers born at of each type i learn and develop their skill through an education process, and thanks to help from paternalistic parents:

$$h_{i,t+1} = Bh_{i,t}^{1-\theta} e_{i,t}^{\theta} + (1 - \delta)h_{i,t}, \tag{II.5}$$

where $0 \leq \delta \leq 1$ denotes by the depreciation rate of human capital, $e_{i,t}$ represents the paternalistic investment intensity from parents born at time t to their offspring, and $0 < \theta < 1$.

It turns out that, in this economy with paternalistic altruism and human capital accumulation, the link between generations is twofold. Firstly, parents finance their children's education by leaving an educational fund. Secondly, through the process of education, not only young people learn and develop their skill, but also take control of the stock of knowledge of their ascendants.

Consumers. Taking the price of consumption good as given, each household of type i determines her saving portfolio $(s_{i,t}, e_{i,t})$ of investment in physical capital and her children's education, as well as her intertemporal consumption choice $(c_{i,t}, d_{i,t+1})$, in order to maximize his intertemporal utility:

$$\max \quad u\left(c_{i,t}, d_{i,t+1}\right) + v\left(e_{i,t}\right) \equiv \ln c_{i,t} + \beta_i \ln d_{i,t+1} + \gamma_i \ln e_{i,t} \tag{II.6}$$

subject to the following budget constraints:

$$c_{i,t} + s_{i,t} + e_{i,t} \leq w_{i,t} h_{i,t}, \tag{II.7}$$

$$d_{i,t+1} \leq R_{t+1} s_{i,t}. \tag{II.8}$$

It turns out that, each type of household $i \in \{1, 2\}$ differs from the other by the patience parameter $0 < \beta_i < 1$, as well as the degree of paternalistic altruism $0 < \gamma_i < 1$. We suppose that, at each date t, the proportion in total population of households of type 1 is π, and $1 - \pi$ represents that of type 2.

Different communities share the same function of human capital accumulation. Nonetheless, since the degrees of altruism are different from each other, agents of different communities do not share the same level of human capital accumulation, and therefore, workers of type 1 and 2 provide different levels of effective labor supply.

Markets clearing conditions are given by:

$$L_{1,t} = \pi N_t h_{1,t}, \tag{II.9}$$

$$L_{2,t} = (1 - \pi) N_t h_{2,t}, \tag{II.10}$$

$$n k_{t+1} = \pi s_{1,t} + (1 - \pi) s_{2,t}, \tag{II.11}$$

$$A K^{1-\alpha-\sigma} L_{1,t}^{\alpha} L_{2,t}^{\sigma} = N_t (c_t + s_t + e_t) + N_{t-1} d_t, \tag{II.12}$$

where $k_t \equiv \frac{K_t}{N_t}$ denotes the capital intensity at each date t.

Definition 2.1 *A positive list* $(w_{1,t}, w_{2,t}, R_t, c_{1,t}, c_{2,t}, d_{1,t+1}, d_{2,t+1}, s_{1,t}, s_{2,t}, e_{1,t}, e_{2,t}, h_{1,t}, h_{2,t}, k_{t+1})$ *is said to be an equilibrium for the economy if: (1) the allocations* $(c_{1,t}, c_{2,t}, d_{1,t+1}, d_{2,t+1}, s_{1,t}, s_{2,t}, e_{1,t}, e_{2,t})$ *maximize (II.6), given* $(w_{1,t}, w_{2,t}, R_{t+1}, h_{1,t}, h_{2,t})$*, subject to constraints (II.7) and (II.8); (2) the markets clearing conditions (II.9), (II.10), (II.11) and the formation equation of human capital (II.5) are satisfied.*

3 Equilibrium dynamics

Let us introduce the income intensity of the economy at date t, $y_t \equiv \frac{F(K_t, L_t)}{N_t}$. In equilibrium, we obtain:

$$\begin{aligned} y_t &= A k_t^{1-\alpha-\sigma} \left(\pi h_{1,t}\right)^{\alpha} \left((1-\pi) h_{2,t}\right)^{\sigma} \\ &\equiv f(k_t, h_{1,t}, h_{2,t}). \end{aligned} \tag{II.13}$$

Furthermore, at equilibrium, individuals' level of consumption $c_{i,t}$, value of transfer to offspring $e_{i,t}$ and savings in physical capital $s_{i,t}$ are proportional to the income intensity y_t:

Lemma 3.1 *For each agent of type i, we have:*

$$c_{i,t} = \frac{1}{1+\beta_i+\gamma_i} w_{i,t} h_{i,t}, \tag{II.14}$$

$$e_{i,t} = \frac{\gamma_i}{1+\beta_i+\gamma_i} w_{i,t} h_{i,t}, \tag{II.15}$$

$$s_{i,t} = \frac{\beta_i}{1+\beta_i+\gamma_i} w_{i,t} h_{i,t}. \tag{II.16}$$

Moreover, individual labor income, $w_{i,t} h_{i,t}$ can be expressed in term of the output intensity, y_t as follows:

$$w_{1,t} h_{1,t} = \frac{\alpha}{\pi} y_t, \tag{II.17}$$

$$w_{2,t} h_{2,t} = \frac{\sigma}{1-\pi} y_t. \tag{II.18}$$

Proof: Appendix ■

Remark 3.1 *We can equivalently refer to the sequence $(h_{1,t}, h_{2,t}, k_{t+1})_{t\geq 0}$ as an equilibrium.*

Proposition 3.1 *The dynamic system of the economy is given by:*

$$k_{t+1} = \eta f(k_t, h_{1,t}, h_{2,t}), \tag{II.19}$$

$$h_{1,t+1} = B\left[\frac{\gamma_1}{1+\beta_1+\gamma_1}\frac{\alpha}{\pi}f(k_t, h_{1,t}, h_{2,t})\right]^{\theta} h_{1,t}^{1-\theta} + (1-\delta)h_{1,t}, \tag{II.20}$$

$$h_{2,t+1} = B\left[\frac{\gamma_2}{1+\beta_2+\gamma_2}\frac{\sigma}{1-\pi}f(k_t, h_{1,t}, h_{2,t})\right]^{\theta} h_{2,t}^{1-\theta} + (1-\delta)h_{2,t}, \tag{II.21}$$

where $\eta \equiv \frac{\alpha\frac{\beta_1}{1+\beta_1+\gamma_1}+\sigma\frac{\beta_2}{1+\beta_2+\gamma_2}}{n}$ *can be interpreted as the propensity to save of the economy.*

a **Proof**: Appendix ■

Remark 3.2 *This propensity to save of the economy increases with individuals' patience parameters β_i, and decreases with the degrees of altruism γ_i. At a constant level of β_i, the more altruistic parents are, the more altruistic investment in their children they are, and consequently, the less they are concerned about their future level of consumption.*

Furthermore, if we consider individuals' ratios of human capital to physical capital, $u_t \equiv \frac{h_{1,t}}{k_t}$ and $v_t \equiv \frac{h_{2,t}}{k_t}$ then the above three-dimensional global dynamic system can be reduced into the following two-dimensional dynamic system:

Corollary 3.1 *Let us define* $C_u \equiv \left(\frac{\alpha}{\eta\pi}\frac{\gamma_1}{1+\beta_1+\gamma_1}\right)^{\theta}$ *and* $C_v \equiv \left(\frac{\sigma}{\eta(1-\pi)}\frac{\gamma_2}{1+\beta_2+\gamma_2}\right)^{\theta}$. *Then the system (II.19)- (II.21) becomes:*

$$u_{t+1} = BC_u\left[\frac{1}{\eta A\pi^{\alpha}(1-\pi)^{\sigma}}\frac{u_t^{1-\alpha}}{v_t^{\sigma}}\right]^{1-\theta} + (1-\delta)\frac{1}{\eta A\pi^{\alpha}(1-\pi)^{\sigma}}\frac{u_t^{1-\alpha}}{v_t^{\sigma}}, \tag{II.22}$$

$$v_{t+1} = BC_v\left[\frac{1}{\eta A\pi^{\alpha}(1-\pi)^{\sigma}}\frac{v_t^{1-\sigma}}{u_t^{\alpha}}\right]^{1-\theta} + (1-\delta)\frac{1}{\eta A\pi^{\alpha}(1-\pi)^{\sigma}}\frac{v_t^{1-\sigma}}{u_t^{\alpha}}, \tag{II.23}$$

where $u_t \equiv \frac{h_{1,t}}{k_t}$ *and* $v_t \equiv \frac{h_{2,t}}{k_t}$.

Proof: Appendix ■

4 Steady-growth path analysis

In this section, we focus on the balanced-growth paths (BGP) along which long-run physical capital and human capital grow at the same positive constant rate: $\rho \equiv \frac{k_{t+1}}{k_t} = \frac{h_{1,t+1}}{h_{1,t}} = \frac{h_{2,t+1}}{h_{2,t}}$.

Proposition 4.1 *There exists an unique balanced-growth path, and the rate of growth of the economy is given by:*

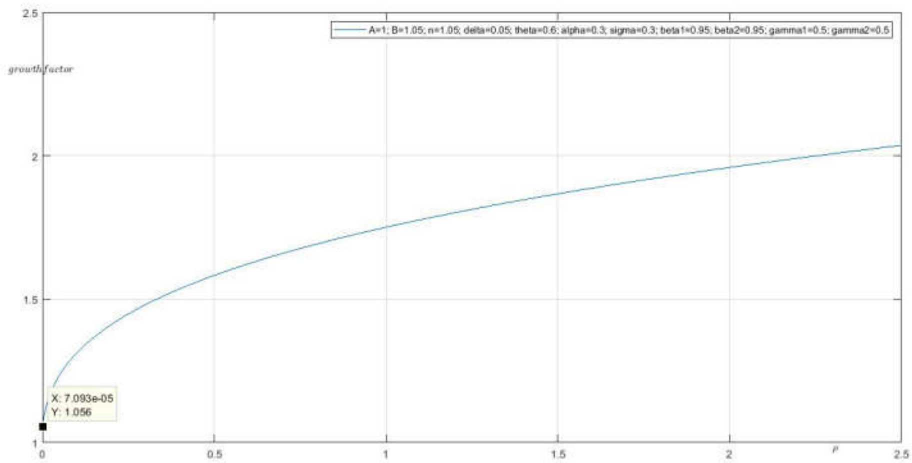
$$\rho = \varphi^{-1}\left[\eta\left(AB^{\frac{\alpha+\sigma}{\theta}}\left(\alpha\frac{\gamma_1}{1+\beta_1+\gamma_1}\right)^{\alpha}\left(\sigma\frac{\gamma_2}{1+\beta_2+\gamma_2}\right)^{\sigma}\right)^{\frac{1}{1-\alpha-\sigma}}\right], \tag{II.24}$$

where $\varphi(\rho) \equiv \rho(\rho-1+\delta)^{\frac{\alpha+\sigma}{\theta(1-\alpha-\sigma)}}$, *which is an increasing function in* ρ.

Proof: Appendix ■

For the intuition, we present the numerical solution of the BGP growth rate of the economy which the following parameters are fixed: $A = 1$, $B = 1.05$, $n = 1.05$, $\delta = 0.05$, $\theta = 0.6$ $\alpha = 0.3$, $\sigma = 0.3$, $\beta_1 = 0.95$, $\beta_2 = 0.95$ $\gamma_1 = 0.5$, $\gamma_2 = 0.5$.

Figure 2.1. The growth rate of economy on the balanced-growth path



The growth factor of economy on the BGP is approximately 1.056 (it turns out that the growth rate is equal to 5.6%), as shown in Figure 2.1 above.

Furthermore, noticing that the propensity to save of the economy is given by $\eta \equiv \frac{\alpha \frac{\beta_1}{1+\beta_1+\gamma_1} + \sigma \frac{\beta_2}{1+\beta_2+\gamma_2}}{n}$, we are interested in the impact of the patience parameters β_i as well as the altruisms degrees γ_i on economic growth. For this purpose, we assume homogeneous agents, that is $\beta_1 = \beta_2 = \beta$ and $\gamma_1 = \gamma_2 = \gamma$. We then have:

Proposition 4.2

1. *For a given parameter of patience, there exists a critical degree of altruism, under which the growth rate is increasing with respect to the altruism degree, otherwise it is decreasing.*

2. *For a given degree of altruism, there exists a critical level of patience, under which the growth rate is increasing with respect to the patience parameter, otherwise it is decreasing.*

Proof: Appendix ■

To illustrate Proposition 4.2, let us proceed with a numerical exercise with $A = 1$, $B = 1.05$, $n = 1.05$, $\delta = 0.05$, $\theta = 0.6$ $\alpha = 0.3$, $\sigma = 0.3$, $\beta = 0.95$. In the first case, we assume that these parameters remain unchanged, whereas we let the parameter of altruism degree, γ increase from 0.28 to 0.47.

With reference to the above setting, we see that in this case, the growth rate of economy, ρ increases from 0.33 to 0.36 when the parameter of altruism degree respectively is equal to 0.28 and 0.34 (as shown in Figure 2.2 (a) and 2.2 (b)). Nevertheless, when the degree of altruism approximately reaches at 0.42 then the rate of growth of economy, ρ is only reduced to 0.32 (see Figure 2.2 (c)). And then this growth rate of economy, ρ is only about 1.030 if the degree altruism parameter is arround 0.47 (see Figure 2.2 (d)).

Figure 2.2 (a). The BGP growth rate of economy with the role of altruism degree ($\gamma = 0.28$, the growth factor: 1.033).

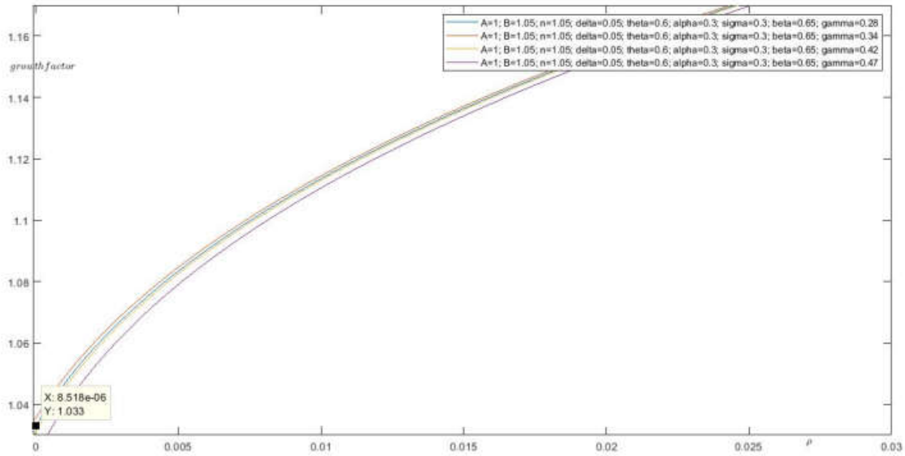


Figure 2.2 (b). The BGP growth rate of economy with the role of altruism degree ($\gamma = 0.34$, the growth factor: 1.036).

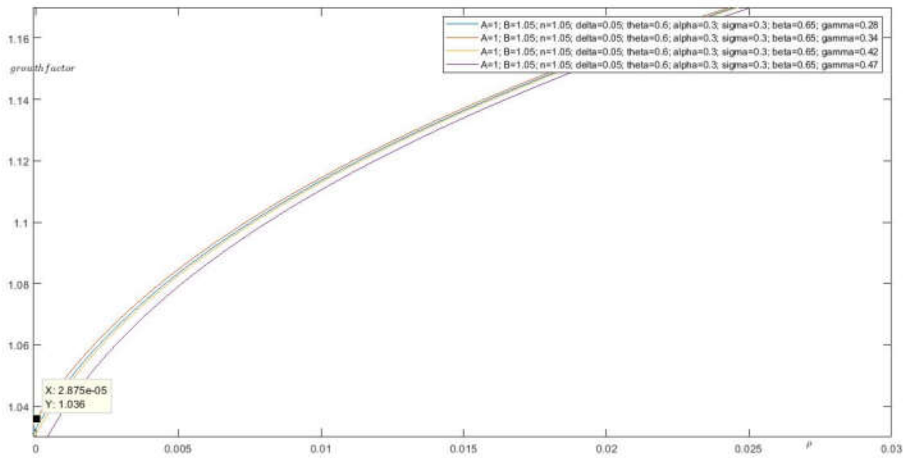


Figure 2.2 (c). The BGP growth rate of economy with the role of altruism degree ($\gamma = 0.42$, the growth factor: 1.032).

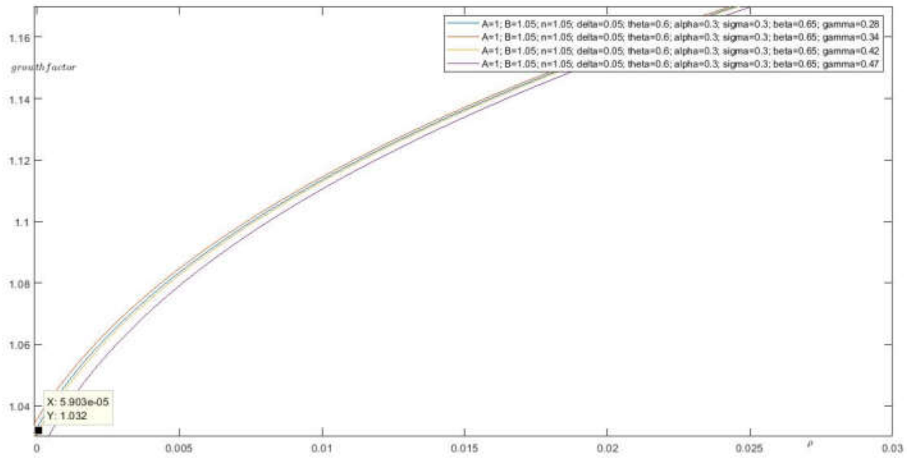


Figure 2.2 (d). The BGP growth rate of economy with the role of altruism degree ($\gamma = 0.47$, the growth factor: 1.030).

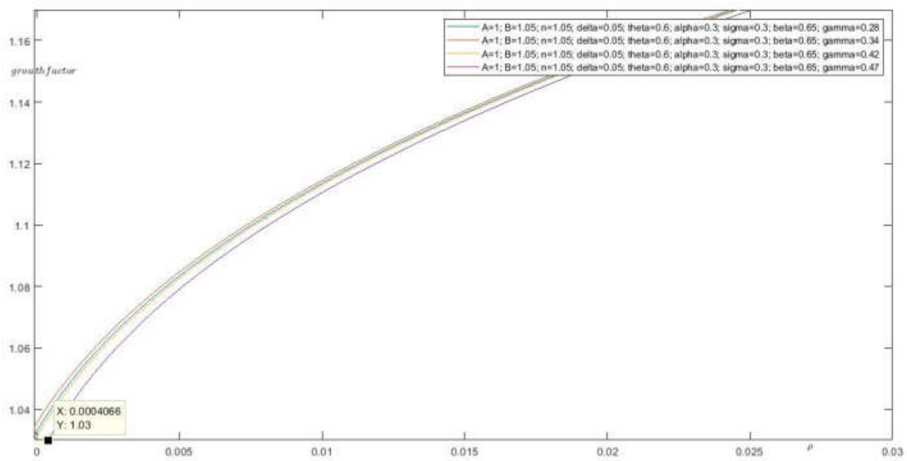


Figure 2.2 (a,b,c,d) show that the importance of parameter of altruism degree, γ on the growth of the economy.

By contrast, in the second case, we let the patience parameter β increase from 0.65 to 1.75 while the degree of altruism parameter is fixed at the value of 0.35 with the unchanged parameters as in the first case. As illustrated in Figure 2.3 (a) and 2.3 (b), if the patience parameter, β increases from 0.65 to 0.95 then the growth rate of economy, ρ attains at 0.35 and 0.37, respectively. While if β continue to increase to 1.25 then ρ decreases to 0.33 (see Figure 2.3 (c)). The number of economic growth rate is around 0.30 when β continues to increase to 1.75 (as shown in Figure 2.3 (d)).

Figure 2.3 (a). The BGP growth rate of economy with the role of patience degree ($\beta = 0.65$, the growth factor: 1.035).

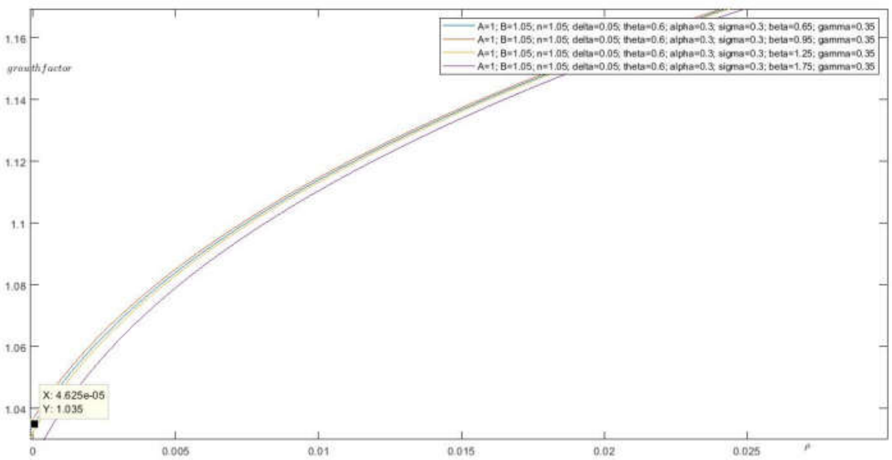


Figure 2.3 (b). The BGP growth rate of economy with the role of patience degree ($\beta = 0.95$, the growth factor: 1.037).

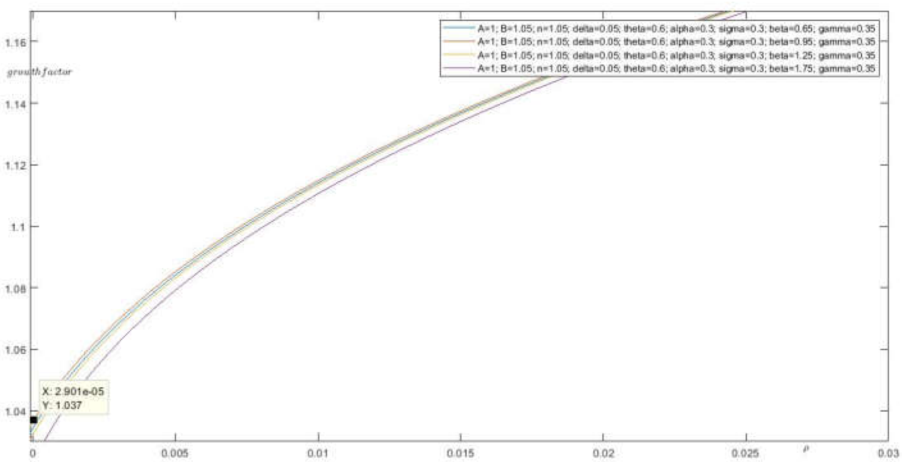


Figure 2.3 (c). The BGP growth rate of economy with the role of patience degree ($\beta = 1.25$, the growth factor: 1.033).

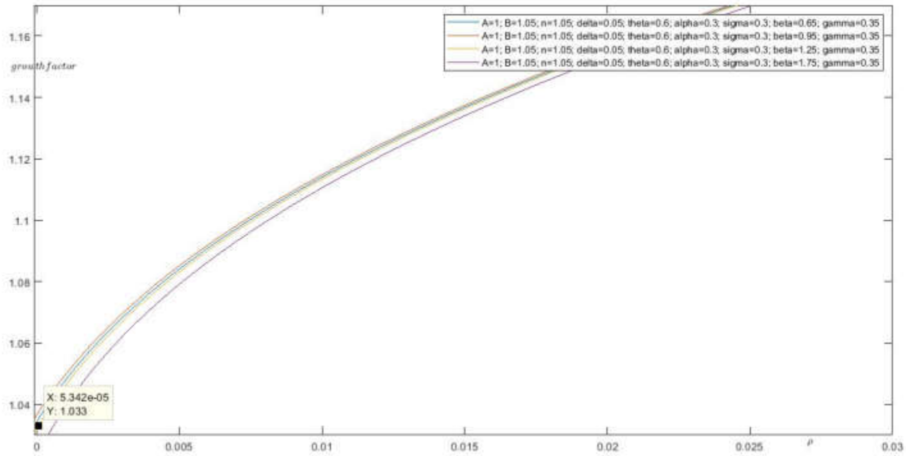


Figure 2.3 (d). The BGP growth rate of economy with the role of patience degree ($\beta = 1.75$, the growth factor: 1.030).

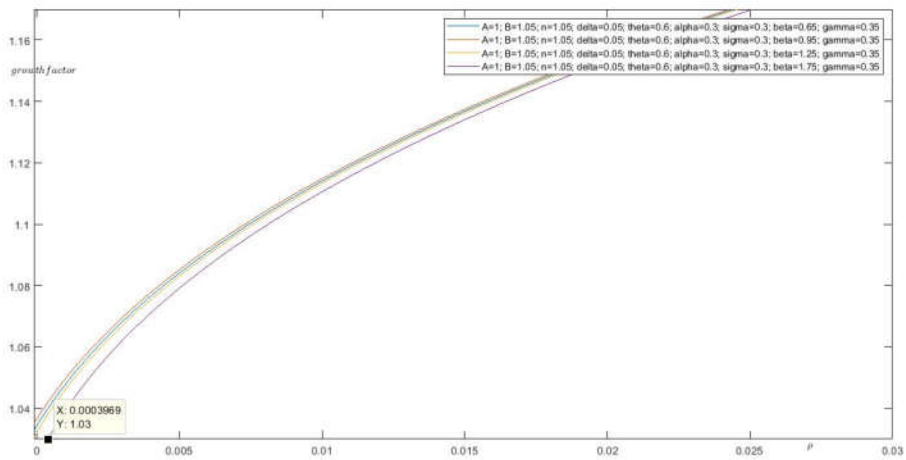


Figure 2.3 (a,b,c,d) exhibits the importance of parameter of patience, β on the growth of the economy.

4.1 Comparative statics

Corollary 4.1 *The steady state ratio of human capital stocks* $\frac{h_1}{h_2}$ *is given by:*

$$\frac{h_1}{h_2} = \frac{\alpha}{\sigma}\frac{1-\pi}{\pi}\frac{1+[(1+\beta_2)/\gamma_2]}{1+[(1+\beta_1)/\gamma_1]}. \tag{II.25}$$

Meanwhile, the ratios physical capital to human capital are given by:

$$\frac{k}{h_1} = \pi(\eta A)^{1/(\alpha+\sigma)}\left[\frac{\sigma}{\alpha}\frac{1+[(1+\beta_1)/\gamma_1]}{1+[(1+\beta_2)/\gamma_2]}\right]^{\frac{\sigma}{\alpha+\sigma}}, \tag{II.26}$$

$$\frac{k}{h_2} = (1-\pi)(\eta A)^{1/(\alpha+\sigma)}\left[\frac{\alpha}{\sigma}\frac{1+[(1+\beta_2)/\gamma_2]}{1+[(1+\beta_1)/\gamma_1]}\right]^{\frac{\alpha}{\alpha+\sigma}}. \tag{II.27}$$

Proof: Appendix ■

Moreover, for each type of agent $i \in \{1, 2\}$, let us introduce the following parameter:

$$\begin{aligned}\kappa_i &\equiv \frac{1+\beta_i+\gamma_i}{\gamma_i}\\ &= 1+\frac{1+\beta_i}{\gamma_i}.\end{aligned}$$

The parameter κ_i captures the relation between the patience parameter β_i and the altruism degree γ_i of an agent i. Noticing that 1 and β_i are the weights assigned to current period's consumption and future consumption, while altruistic investment is weighted by γ_i in parents' utility, therefore κ_i can be considered as a relative ratio between one's own consumption and one's altruistic investment to offspring. An increase in κ_i may mean that the agent i become more concerned about her future consumption, or less altruistic toward her children, and inversely. Therefore, it would be valuable to compare different behaviors of agents by comparing κ_1 with κ_2. That's why we will measure the impact of the ratio κ_1/κ_2 to the relative ratio h_1/h_2.

Definition 4.1 *Let us define:*

$$\epsilon_{kh_i} \equiv \frac{\Delta(\frac{k}{h_i})/\frac{k}{h_i}}{\Delta\kappa_i/\kappa_i}$$

the elasticity of physical capital to human capital of an agent i with respect to her willingness in physical capital saving and altruistic investment, and

$$\epsilon_{hh} \equiv \frac{\Delta(\frac{h_1}{h_2})/\frac{h_1}{h_2}}{\Delta(\frac{\kappa_1}{\kappa_2})/\frac{\kappa_1}{\kappa_2}}$$

the elasticity of human capital of type 1 to human capital of type 2 with respect to different levels of willingness in physical capital saving and altruistic investment.

Proposition 4.3 *We obtain that:*

$$\epsilon_{kh_1} = \frac{\sigma}{\alpha + \sigma}, \tag{II.28}$$

$$\epsilon_{kh_2} = \frac{\alpha}{\alpha + \sigma}, \tag{II.29}$$

$$\epsilon_{hh} = -1. \tag{II.30}$$

Proof: Appendix ■

Comment: An increase in κ_i leads to a raise in the relative ratio physical - human capital, $\frac{k}{h_i}$: If one weights her own consumption more than the altruistic investment to her offspring then one invests more in physical capital. In another hand, since ϵ_{hh} is negative, if an agent of type 1's relative ratio of consumption to altruistic investment is lower compared to the other type's one, then the relative ratio of human capital levels $\frac{h_1}{h_2}$ increases.

4.2 Local dynamics

Proposition 4.4 *The equilibrium dynamics are approximated around the steady state by the linear system*

$$\begin{bmatrix} \frac{du_{t+1}}{u} \\ \frac{dv_{t+1}}{v} \end{bmatrix} = \begin{bmatrix} (1-\alpha)a & -\sigma a \\ -\alpha b & (1-\sigma)b \end{bmatrix} \begin{bmatrix} \frac{du_t}{u} \\ \frac{dv_t}{v} \end{bmatrix} \tag{II.31}$$

where $a \equiv 1 - \frac{\theta B C_u \left(\frac{1}{\eta A \pi^\alpha (1-\pi)^\sigma} \frac{u^{1-\alpha}}{v^\sigma} \right)^{1-\theta}}{u}$ *and* $b \equiv 1 - \frac{\theta B C_v \left(\frac{1}{\eta A \pi^\alpha (1-\pi)^\sigma} \frac{v^{1-\sigma}}{u^\alpha} \right)^{1-\theta}}{v}$.

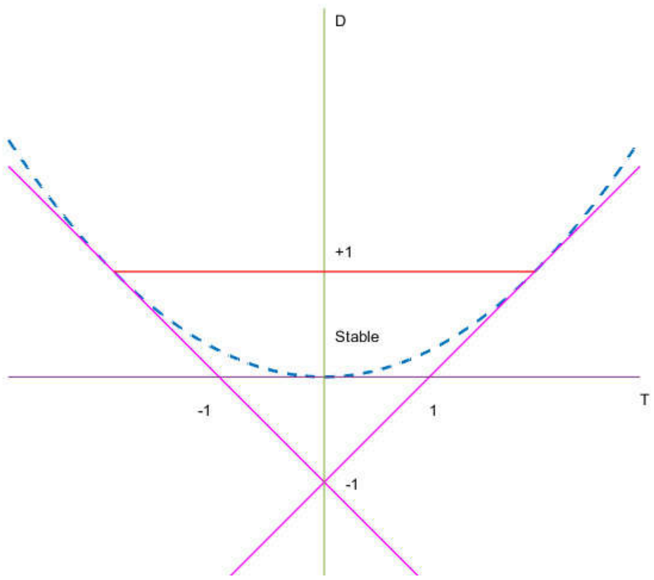
Proof: Appendix ■

As proven in Appendix, $0 < a, b < 1$, and moreover we have:

Proposition 4.5 *The steady state is locally determinate.*

Proof: Appendix ■

Figure 2.4. The local dynamic system of the economy.



Remark 4.1 *As proved by the proof of Proposition 4.5 (see Appendix), if we denote the characteristic polynomial of system II.31 by* $P(\lambda) \equiv \lambda^2 - T\lambda + D$*, then we always have:*

$$\begin{aligned} D &< 1, \\ P(-1) &= 1 + T + D > 0, \\ P(1) &= 1 - T + D > 0. \end{aligned}$$

5 Inequality and the interaction between physical capital and human capital

5.1 Inequality in terms of wage incomes

Wage incomes of the two types of agent are respectively $w_{1,t}h_{1,t} = \frac{\alpha}{\pi}y_t$ and $w_{2,t}h_{2,t} = \frac{\sigma}{1-\pi}y_t$. We suppose that an agent of type 1 has a higher wage income than that of an agent of type 2, or equivalently:

Assumption 5.1 *Assume that the share of labor income of type 1,* α*, is high enough, as well as the population density of community 1,* π*, is low enough such that* $\frac{\alpha}{\pi} > \frac{\sigma}{1-\pi}$.

This inequality condition can be explained by the fact that the population density of community 1 - the richer community, π is much more fewer than that of community 2, $1-\pi$; while the share of labor income of community 1, α is considerably high with respect to that of community 2, σ.

Under this assumption 5.1, we obtain the Gini coefficient of wage income distribution, denoted by G_w, can be expressed as follow:

Proposition 5.1

$$G_w = \frac{\alpha}{\alpha + \sigma} - \pi. \tag{II.32}$$

Income inequality between communities is increasing in $\frac{\alpha}{\sigma}$, and decreasing (respectively increasing) in π (respectively $1 - \pi$).

Proof: Appendix ■

Remark 5.1 *Income inequality index G_w can be expressed in terms of elasticities ϵ_{kh_i}. Indeed, equation (II.32), combined with equations (II.28) and (II.29) of proposition 4.3, turns out equivalent to be:*

$$G_w = (1 - \pi)\epsilon_{kh_2} - \pi\epsilon_{kh_1}. \tag{II.33}$$

Noticing that $\epsilon_{kh_1} + \epsilon_{kh_2} = 1$. We therefore find out a perfect positive (negative) correlation between income inequality G_w and the elasticity of physical capital saving to altruistic investment of the poorer (richer) community.

5.2 Inequality in terms of consumptions

Motivation: G_w does not depend on the parameters β_i and γ_i, which are important since they respectively capture the willingness of an agent in investment in physical capital and altruistic investment. Furthermore, by considering the Gini coefficient of consumption, we will emphasize that these behavior parameters, β_i and γ_i, importantly matter, when it comes to compare the consumption levels of different agents.

As above, first, we consider inequality conditions between communities:

Assumption 5.2 *Assume that the share of labor income of type 1, α, is high enough, as well as the population density of community 1, π, is low enough such* *that both of the following inequalities hold:*

$$\frac{\alpha}{\pi} > \frac{\sigma}{1-\pi}\frac{1+\beta_1+\gamma_1}{1+\beta_2+\gamma_2}, \tag{II.34}$$

$$\frac{\alpha}{\pi} > \frac{\sigma}{1-\pi}\frac{\beta_2}{\beta_1}\frac{1+\beta_1+\gamma_1}{1+\beta_2+\gamma_2}. \tag{II.35}$$

Under the above assumption, individuals of the community 1 can be qualified as richer compared to the others.

We can easily see that, if $\beta_1 = \beta_2$, then the two above consumption inequality conditions are the same. Moreover, if $\beta_2 < \beta_1$, the inequality condition (II.34) automatically implies (II.35): Intuitively, consumption inequality between communities among old individuals is worse compared to that of younger generation.

We respectively denote G_c^y the Gini index of consumption between young workers, and G_c^o the Gini index of consumption between old agents. While reminding that $d_{i,t+1} = R_{t+1}s_{i,t}$, and $c_{i,t}$ and $s_{i,t}$ are given by equations (II.14) and (II.16), we compute G_c^y and G_c^o as follow:

Proposition 5.2 *Under assumption 5.2, consumption inequality indexes G_c^y and G_c^o are given by:*

$$G_c^y = \frac{\alpha}{\alpha + \sigma\frac{1+\beta_1+\gamma_1}{1+\beta_2+\gamma_2}} - \pi, \tag{II.36}$$

$$G_c^o = \frac{\alpha}{\alpha + \sigma\frac{\beta_2}{\beta_1}\frac{1+\beta_1+\gamma_1}{1+\beta_2+\gamma_2}} - \pi. \tag{II.37}$$

Furthermore, if $\beta_2 < \beta_1$, that is the preference for physical capital saving of the poorer community is lower than that of the richer community, then consumption inequality between communities among young workers is amplified when these workers become retired.

Proof: Appendix ■

This result leads to interesting interpretations.

Remark 5.2

1. *As seen in Proposition 5.1, these inequality indexes are also increasing in* $\frac{\alpha}{\sigma}$*, and decreasing (respectively increasing) in* π *(respectively* $1-\pi$*).*

2. *We notice that*

$$\frac{\beta_2}{\beta_1}\frac{1+\beta_1+\gamma_1}{1+\beta_2+\gamma_2} = \left[1+\frac{1+\gamma_1}{\beta_1}\right] / \left[1+\frac{1+\gamma_2}{\beta_2}\right]$$

and $\forall i$*,* $\frac{1+\gamma_i}{\beta_i}$ *captures the degree of educational altruism relative to the parameter of physical capital saving. Therefore, inequality is not only a problem for individuals when young, but when they become older retired, this kind of inequality between rich and poor communities/classes/countries may be even worse. Indeed,* $\frac{1+\gamma_2}{\beta_2}$ *can be considerably greater than* $\frac{1+\gamma_1}{\beta_1}$*: the poorer an individual is, the less preference she shows for investment in physical capital, and meanwhile, the higher effort will be required to finance her children's education.*

Comment: The impact of consumption inequality on the interaction between physical capital and human capital can be expressed by the following note:

Since consumption inequality indexes are increasing in $\frac{\alpha}{\sigma}$, these indexes are also decreasing with respect to $\frac{\sigma}{\alpha+\sigma}$, which is proved by Proposition 4.3 to be equal to ϵ_{kh_1} - representing the elasticity of physical capital to human capital of an individual of the richer community with respect to her willingness in physical capital savings and altruistic investment: In other words, the worse inequality is, the more the richer community's agents raise their level of human capital accumulation (at a given level of physical capital), compared to agents of the other community, if we assume that the two communities share a same degree of willingness in altruistic investment with respect to investment in physical capital.

6 Conclusion

This chapter studies the dynamics of human capital accumulation in a two-period overlapping generations model with heterogeneous agents. In this model, parents are altruistic toward their children and invest in their offspring's human capital. We suppose there are two communities with distinct levels of altruism degree as well as different preferences for investments in physical capital. We prove the uniqueness of the balanced growth path (BGP) of the economy and characterize equilibria around this BGP. We also study the inequality in terms of labor income and consumption, by considering the GINI coefficient.

In spite of the fact that our study has results in interesting and worthwhile conclusions, there are two limitations: On the one hand, we can say little about the impact of the parameters on the rate of economic growth such as the patience coefficient, the degree of altruism and the share of two types of labor. On the other hand, the mobility of individuals between two communities is restricted. It means that only agent 1 gives birth to agent 1 and so on. Therefore, analyzing more detail about the effects of fundamental parameters on the rate of economic growth and introducing the probability distribution of the mobility of households between two communities which are the future research.

7 Appendix

Proof of Lemma 3.1.

We solve the program of consumer's utility, by considering the following Lagrange:

$$\mathcal{L} = \ln c_{i,t} + \beta_i \ln d_{i,t+1} + \gamma_i \ln e_{i,t} + \lambda[w_{i,t}h_{i,t} - c_{i,t} - s_{i,t} - e_{i,t}] + \mu[R_{t+1}s_{i,t} - d_{i,t+1}].$$

The necessary conditions are

$$\frac{1}{c_{i,t}} = \lambda, \tag{II.38}$$

$$\frac{\beta_i}{d_{i,t+1}} = \mu, \tag{II.39}$$

$$\mu R_{t+1} = \lambda, \tag{II.40}$$

$$\frac{\gamma_i}{e_{i,t}} = \lambda. \tag{II.41}$$

Moreover, the budget constraints are binded:

$$c_{i,t} + s_{i,t} + e_{i,t} = w_{i,t} h_{i,t}, \tag{II.42}$$

$$d_{i,t+1} = R_{t+1} s_{i,t}. \tag{II.43}$$

Combining (II.38) and (II.39), we have:

$$\frac{\lambda}{\mu} = \frac{d_{i,t+1}}{\beta_i c_{i,t}}. \tag{II.44}$$

From (II.40) and (II.41), we get:

$$\mu R_{t+1} = \frac{\gamma_i}{e_{i,t}}. \tag{II.45}$$

Meanwhile, combining (II.38) and (II.41) gives:

$$e_{i,t} = \gamma_i c_{i,t}. \tag{II.46}$$

With reference to the equations (II.43) and (II.44), we receive:

$$s_{i,t} = \beta_i c_{i,t}. \tag{II.47}$$

Finally, we substitute (II.46) and (II.47) into the budget constraint (II.42), we respectively obtain the individual's consumption and savings, as well as the level of expenditure on education. ■

Proof of Proposition 3.1.

Combining the capital market clearing condition (II.11) with equations (II.16)-(II.18) from the Lemma 1, we get (II.19).

In order to achieve the results of II.20 and II.21, we simultaneously combine the human capital accumulation equation (II.5) with equations (II.15), (II.17) and (II.18).■

Proof of Corrolary 3.1.

Dividing (II.20) and (II.21) by k_{t+1}, we get the following system:

$$\frac{h_{1,t+1}}{k_{t+1}} = BC_u \left(\frac{h_{1,t}}{k_{t+1}}\right)^{1-\theta} + (1-\delta)\frac{h_{1,t}}{k_{t+1}}, \tag{II.48}$$
$$\frac{h_{2,t+1}}{k_{t+1}} = BC_v \left(\frac{h_{2,t}}{k_{t+1}}\right)^{1-\theta} + (1-\delta)\frac{h_{2,t}}{k_{t+1}}, \tag{II.49}$$

where

$$C_u \equiv \left(\frac{\gamma_1}{1+\beta_1+\gamma_1}\frac{\alpha}{\pi}\right)^{\theta}\frac{1}{\eta^{\theta}},$$
$$C_v \equiv \left(\frac{\gamma_2}{1+\beta_2+\gamma_2}\frac{\sigma}{1-\pi}\right)^{\theta}\frac{1}{\eta^{\theta}}.$$

Noticing that $k_{t+1} = \eta A k_t^{1-\alpha-\sigma}(\pi h_{1,t})^{\alpha}[(1-\pi)h_{2,t}]^{\sigma}$, we rewrite $\frac{h_{1,t}}{k_{t+1}}$ and $\frac{h_{2,t}}{k_{t+1}}$ as follow:

$$\frac{h_{1,t}}{k_{t+1}} = \frac{1}{\eta A}\frac{1}{\pi^{\alpha}}\frac{1}{(1-\pi)^{\sigma}}\left(\frac{h_{1,t}}{k_t}\right)^{1-\alpha-\sigma}\left(\frac{h_{1,t}}{h_{2,t}}\right)^{\sigma},$$
$$\frac{h_{2,t}}{k_{t+1}} = \frac{1}{\eta A}\frac{1}{\pi^{\alpha}}\frac{1}{(1-\pi)^{\sigma}}\left(\frac{h_{2,t}}{k_t}\right)^{1-\alpha-\sigma}\left(\frac{h_{2,t}}{h_{1,t}}\right)^{\alpha}.$$

Let us define $u_t \equiv \frac{h_{1,t}}{k_t}$ and $v_t \equiv \frac{h_{2,t}}{k_t}$. We obtain:

$$\frac{h_{1,t}}{k_{t+1}} = \frac{1}{\eta A} \frac{1}{\pi^\alpha} \frac{1}{(1-\pi)^\sigma} \frac{u_t^{1-\alpha}}{v_t^\sigma}, \tag{II.50}$$

$$\frac{h_{2,t}}{k_{t+1}} = \frac{1}{\eta A} \frac{1}{\pi^\alpha} \frac{1}{(1-\pi)^\sigma} \frac{v_t^{1-\sigma}}{u_t^\alpha}. \tag{II.51}$$

Replacing (II.50) and (II.51) respectively into (II.48) and (II.49), we have:

$$u_{t+1} = BC_u \left[\frac{1}{\eta A \pi^\alpha (1-\pi)^\sigma} \frac{u_t^{1-\alpha}}{v_t^\sigma} \right]^{1-\theta} + (1-\delta) \frac{1}{\eta A \pi^\alpha (1-\pi)^\sigma} \frac{u_t^{1-\alpha}}{v_t^\sigma},$$

$$v_{t+1} = BC_v \left[\frac{1}{\eta A \pi^\alpha (1-\pi)^\sigma} \frac{v_t^{1-\sigma}}{u_t^\alpha} \right]^{1-\theta} + (1-\delta) \frac{1}{\eta A \pi^\alpha (1-\pi)^\sigma} \frac{v_t^{1-\sigma}}{u_t^\alpha}.$$

■

Proof of Proposition 4.1.

Along the balanced-growth path, physical and human capital grow at the same positive constant rate: $\rho \equiv \frac{k_{t+1}}{k_t} = \frac{h_{1,t+1}}{h_{1,t}} = \frac{h_{2,t+1}}{h_{2,t}}$.

Combining equations (II.16) and (II.11) gives:

$$nk_{t+1} = \left(\alpha \frac{\beta_1}{1+\beta_1+\gamma_1} + \sigma \frac{\beta_2}{1+\beta_2+\gamma_2} \right) y_t. \tag{II.52}$$

In another hand, by transforming equation (II.5), we have:

$$\frac{h_{i,t+1}}{h_{i,t}} = B \left(\frac{e_{i,t}}{h_{i,t}} \right)^\theta + (1-\delta)$$

$$\frac{e_{i,t}}{h_{i,t}} = \left(\frac{\rho - 1 + \delta}{B} \right)^{\frac{1}{\theta}}. \tag{II.53}$$

Incorpotating equations (II.15), (II.17) and (II.18), we get:

$$\begin{aligned}
e_{1,t} &= \frac{\gamma_1}{1+\beta_1+\gamma_1} w_{1,t} h_{1,t} = \frac{\alpha}{\pi} \frac{\gamma_1}{1+\beta_1+\gamma_1} y_t, \\
e_{2,t} &= \frac{\gamma_2}{1+\beta_2+\gamma_2} w_{2,t} h_{2,t} = \frac{\sigma}{1-\pi} \frac{\gamma_2}{1+\beta_2+\gamma_2} y_t.
\end{aligned}$$

Moreover, combining the two latter equations with (II.53), we obtain:

$$\begin{aligned}
\left(\frac{\rho-1+\delta}{B}\right)^{\frac{1}{\theta}} h_{1,t} &= \frac{\alpha}{\pi} \frac{\gamma_1}{1+\beta_1+\gamma_1} y_t, \\
\left(\frac{\rho-1+\delta}{B}\right)^{\frac{1}{\theta}} h_{2,t} &= \frac{\sigma}{1-\pi} \frac{\gamma_2}{1+\beta_2+\gamma_2} y_t.
\end{aligned}$$

The above equations allow us to express human capital stock, $h_{i,t}$ in term of the output intensity, y_t, as follows:

$$h_{1,t} = \frac{\alpha}{\pi} \left(\frac{B}{\rho-1+\delta}\right)^{\frac{1}{\theta}} \frac{\gamma_1}{1+\beta_1+\gamma_1} y_t, \tag{II.54}$$

$$h_{2,t} = \frac{\sigma}{1-\pi} \left(\frac{B}{\rho-1+\delta}\right)^{\frac{1}{\theta}} \frac{\gamma_2}{1+\beta_2+\gamma_2} y_t. \tag{II.55}$$

Substituting (II.54) and (II.55) into (II.13), we receive:

$$\begin{aligned}
y_t &= A k_t^{1-\alpha-\sigma} \left[\alpha \frac{\gamma_1}{1+\beta_1+\gamma_1} \left(\frac{B}{\rho-1+\delta}\right)^{\frac{1}{\theta}}\right]^{\alpha} \left[\sigma \frac{\gamma_2}{1+\beta_2+\gamma_2} \left(\frac{B}{\rho-1+\delta}\right)^{\frac{1}{\theta}}\right]^{\sigma} y_t^{\alpha+\sigma} \\
y_t^{1-\alpha-\sigma} &= A k_t^{1-\alpha-\sigma} \left[\alpha \frac{\gamma_1}{1+\beta_1+\gamma_1} \left(\frac{B}{\rho-1+\delta}\right)^{\frac{1}{\theta}}\right]^{\alpha} \left[\sigma \frac{\gamma_2}{1+\beta_2+\gamma_2} \left(\frac{B}{\rho-1+\delta}\right)^{\frac{1}{\theta}}\right]^{\sigma}
\end{aligned}$$

or, equivalently

$$y_t = \left[A \left(\alpha \frac{\gamma_1}{1+\beta_1+\gamma_1} \left(\frac{B}{\rho-1+\delta}\right)^{\frac{1}{\theta}}\right)^{\alpha} \left(\sigma \frac{\gamma_2}{1+\beta_2+\gamma_2} \left(\frac{B}{\rho-1+\delta}\right)^{\frac{1}{\theta}}\right)^{\sigma}\right]^{\frac{1}{1-\alpha-\sigma}} \times k_t. \tag{II.56}$$

Combining (II.19) and (II.56), we get:

$$\frac{k_{t+1}}{k_t} = \eta \left[A \left(\alpha \frac{\gamma_1}{1+\beta_1+\gamma_1} \left(\frac{B}{\rho - 1 + \delta} \right)^{\frac{1}{\theta}} \right)^{\alpha} \left(\sigma \frac{\gamma_2}{1+\beta_2+\gamma_2} \left(\frac{B}{\rho - 1 + \delta} \right)^{\frac{1}{\theta}} \right)^{\sigma} \right]^{\frac{1}{1-\alpha-\sigma}}.$$

Notice that $\eta = \frac{\alpha \frac{\beta_1}{1+\beta_1+\gamma_1} + \sigma \frac{\beta_2}{1+\beta_2+\gamma_2}}{n}$, we finally have:

$$\rho(\rho - 1 + \delta)^{\frac{\alpha+\sigma}{\theta(1-\alpha-\sigma)}} = \frac{1}{n} A^{\frac{1}{1-\alpha-\sigma}} \left(\alpha \frac{\beta_1}{1+\beta_1+\gamma_1} + \sigma \frac{\beta_2}{1+\beta_2+\gamma_2} \right) \left(\alpha \frac{\gamma_1}{1+\beta_1+\gamma_1} \right)^{\frac{\alpha}{1-\alpha-\sigma}}$$
$$\left(\sigma \frac{\gamma_2}{1+\beta_2+\gamma_2} \right)^{\frac{\sigma}{1-\alpha-\sigma}} B^{\frac{\alpha+\sigma}{\theta(1-\alpha-\sigma)}}. \tag{II.57}$$

■

Proof of Proposition 4.2.

When $\beta_1 = \beta_2 = \beta$ and $\gamma_1 = \gamma_2 = \gamma$, $\varphi(\rho)$ can be rewritten as:

$$\varphi(\rho) = \text{constant} \times \frac{\beta}{1+\beta+\gamma} \left(\frac{\gamma}{1+\beta+\gamma} \right)^{\frac{\alpha+\sigma}{1-\alpha-\sigma}}.$$

Furthermore, $\varphi(\rho)$ can be linearized as follow:

$$\begin{aligned} \ln[\varphi(\rho)] &= \ln(\beta) - \ln(1+\beta+\gamma) + \frac{\alpha+\sigma}{1-\alpha-\sigma} [\ln(\gamma) - \ln(1+\beta+\gamma)] + \text{constant} \\ &= \ln(\beta) + \frac{\alpha+\sigma}{1-\alpha-\sigma} \ln(\gamma) - \frac{1}{1-\alpha-\sigma} \ln(1+\beta+\gamma) + \text{constant}. \end{aligned}$$

1. Taking the derivative of $\ln(\varphi(\rho))$ with respect to γ:

$$\frac{\partial \ln(\varphi(\rho))}{\partial \gamma} = \frac{(\alpha+\sigma)(1+\beta) - (1-\alpha-\sigma)\gamma}{\gamma(1-\alpha-\sigma)(1+\beta+\gamma)}.$$

For a given value of β, the critical level of γ is $\bar{\gamma} = \frac{(1+\beta)(\alpha+\sigma)}{1-\alpha-\sigma}$.

2. Taking the derivative of $\ln(\varphi(\rho))$ with respect to β:

$$\frac{\partial \ln(\varphi(\rho))}{\partial \beta} = \frac{(1-\alpha-\sigma)(1+\gamma)-(\alpha+\sigma)\beta}{\beta(1-\alpha-\sigma)(1+\beta+\gamma)}.$$

For a given value of γ, the critical level of β is $\bar{\beta} = \frac{(1-\alpha-\sigma)(1+\gamma)}{\alpha+\sigma}$.

■

Proof of Corollary 4.1.

Along the steady growth path, the equations of (II.48) and (II.49) become:

$$\begin{aligned}
\frac{h_1}{k} &= BC_u \left(\frac{h_1}{k}\right)^{1-\theta} + (1-\delta)\frac{h_1}{k}, \\
\frac{h_2}{k} &= BC_v \left(\frac{h_2}{k}\right)^{1-\theta} + (1-\delta)\frac{h_2}{k},
\end{aligned}$$

which imply that:

$$\begin{aligned}
\delta\frac{h_1}{k} &= BC_u \left(\frac{h_1}{k}\right)^{1-\theta}, \\
\delta\frac{h_2}{k} &= BC_v \left(\frac{h_2}{k}\right)^{1-\theta}.
\end{aligned}$$

Consequently, it turns out that

$$\begin{aligned}
\frac{h_1}{h_2} &= \left[\frac{C_u}{C_v}\right]^{\frac{1}{\theta}} \\
&= \frac{\alpha}{\sigma}\frac{1-\pi}{\pi}\frac{1+[(1+\beta_2)/\gamma_2]}{1+[(1+\beta_1)/\gamma_1]}.
\end{aligned}$$

Furthermore, we can obtain (II.26) and (II.27) by combining (II.52), (II.54) and (II.55). ■

Proof of Proposition 4.3.

In order to calculate ϵ_{kh_1}, we linearize equation (II.26) as follow:

$$\ln\left(\frac{k}{h_1}\right) = \frac{\sigma}{\alpha+\sigma}\ln(\kappa_1) + \text{constant}.$$

Consequently, we easily obtain (II.28).

As for ϵ_{kh_2}, ϵ_{hh}, they can be computed in the same manner. ■

Proof of Proposition 4.4.

Around the steady state, we linearize (II.22) as follow:

$$\begin{aligned}
&du_{t+1} : 1,\\
&\quad du_t : BC_u(1-\theta)\left[\frac{1}{\eta A\pi^\alpha(1-\pi)^\sigma}\frac{u^{1-\alpha}}{v^\sigma}\right]^{-\theta}\frac{1}{\eta A\pi^\alpha(1-\pi)^\sigma}(1-\alpha)\frac{u^{-\alpha}}{v^\sigma}\\
&\qquad\quad + (1-\delta)\frac{1}{\eta A\pi^\alpha(1-\pi)^\sigma}(1-\alpha)\frac{u^{-\alpha}}{v^\sigma},\\
&\quad dv_t : BC_u(1-\theta)\left[\frac{1}{\eta A\pi^\alpha(1-\pi)^\sigma}\frac{u^{1-\alpha}}{v^\sigma}\right]^{-\theta}(-\sigma)\frac{1}{\eta A\pi^\alpha(1-\pi)^\sigma}\frac{u^{1-\alpha}}{v^{\sigma+1}}\\
&\qquad\quad - (1-\delta)\frac{1}{\eta A\pi^\alpha(1-\pi)^\sigma}\sigma\frac{u^{1-\alpha}}{v^{\sigma+1}}.
\end{aligned}$$

Consequently, we have:

$$\begin{aligned}
\frac{udu_{t+1}}{u} &= \left[BC_u(1-\theta)\left(\frac{1}{\eta A\pi^\alpha(1-\pi)^\sigma}\frac{u^{1-\alpha}}{v^\sigma}\right)^{1-\theta} + (1-\delta)\frac{1}{\eta A\pi^\alpha(1-\pi)^\sigma}\frac{u^{1-\alpha}}{v^\sigma}\right](1-\alpha)\frac{du_t}{u}\\
&\quad + \left[BC_u(1-\theta)\left(\frac{1}{\eta A\pi^\alpha(1-\pi)^\sigma}\frac{u^{1-\alpha}}{v^\sigma}\right)^{1-\theta} + (1-\delta)\frac{1}{\eta A\pi^\alpha(1-\pi)^\sigma}\frac{u^{1-\alpha}}{v^\sigma}\right](-\sigma)\frac{dv_t}{v}.
\end{aligned} \tag{II.58}$$

Noticing that at the steady state, we also have:

$$
\begin{aligned}
BC_u(1-\theta)\left(\frac{1}{\eta A\pi^{\alpha}(1-\pi)^{\sigma}}\frac{u^{1-\alpha}}{v^{\sigma}}\right)^{1-\theta} &+ (1-\delta)\frac{1}{\eta A\pi^{\alpha}(1-\pi)^{\sigma}}\frac{u^{1-\alpha}}{v^{\sigma}} \\
&= u - \theta BC_u\left(\frac{1}{\eta A\pi^{\alpha}(1-\pi)^{\sigma}}\frac{u^{1-\alpha}}{v^{\sigma}}\right)^{1-\theta}. \qquad \text{(II.59)}
\end{aligned}
$$

Combining (II.58) and (II.59) we obtain:

$$
\begin{aligned}
\frac{udu_{t+1}}{u} &= \left[u - \theta BC_u\left(\frac{1}{\eta A\pi^{\alpha}(1-\pi)^{\sigma}}\frac{u^{1-\alpha}}{v^{\sigma}}\right)^{1-\theta}\right]\left[(1-\alpha)\frac{du_t}{u} - \sigma\frac{dv_t}{v}\right] \\
\frac{du_{t+1}}{u} &= \left[1 - \frac{\theta BC_u\left(\frac{1}{\eta A\pi^{\alpha}(1-\pi)^{\sigma}}\frac{u^{1-\alpha}}{v^{\sigma}}\right)^{1-\theta}}{u}\right]\left[(1-\alpha)\frac{du_t}{u} - \sigma\frac{dv_t}{v}\right]. \qquad \text{(II.60)}
\end{aligned}
$$

Let us define:

$$
a \equiv 1 - \frac{\theta BC_u\left(\frac{1}{\eta A\pi^{\alpha}(1-\pi)^{\sigma}}\frac{u^{1-\alpha}}{v^{\sigma}}\right)^{1-\theta}}{u}.
$$

Because of (II.59), we can easily see that $0 < a < 1$. Moreover, (II.60) becomes:

$$
\frac{du_{t+1}}{u} = a\left[(1-\alpha)\frac{du_t}{u} - \sigma\frac{dv_t}{v}\right]. \qquad \text{(II.61)}
$$

Around the steady state, we linearize (II.23) as follow:

$$
\begin{aligned}
& dv_{t+1} : 1, \\
& du_t : BC_v(1-\theta)\left[\frac{1}{\eta A\pi^{\alpha}(1-\pi)^{\sigma}}\frac{v^{1-\sigma}}{u^{\alpha}}\right]^{-\theta}\frac{1}{\eta A\pi^{\alpha}(1-\pi)^{\sigma}}(-\alpha)\frac{v^{1-\sigma}}{u^{\alpha+1}} \\
& \qquad - (1-\delta)\frac{1}{\eta A\pi^{\alpha}(1-\pi)^{\sigma}}\alpha\frac{v^{1-\sigma}}{u^{\alpha+1}}, \\
& dv_t : BC_v(1-\theta)\left[\frac{1}{\eta A\pi^{\alpha}(1-\pi)^{\sigma}}\frac{v^{1-\sigma}}{u^{\alpha}}\right]^{-\theta}(1-\sigma)\frac{1}{\eta A\pi^{\alpha}(1-\pi)^{\sigma}}\frac{v^{-\sigma}}{u^{\alpha}} \\
& \qquad + (1-\delta)\frac{1}{\eta A\pi^{\alpha}(1-\pi)^{\sigma}}(1-\sigma)\frac{v^{-\sigma}}{u^{\alpha}}.
\end{aligned}
$$

Consequently, we have:

$$\frac{vdv_{t+1}}{v} = \left[BC_v(1-\theta)\left(\frac{1}{\eta A\pi^\alpha(1-\pi)^\sigma}\frac{v^{1-\sigma}}{u^\alpha}\right)^{1-\theta} + (1-\delta)\frac{1}{\eta A\pi^\alpha(1-\pi)^\sigma}\frac{v^{1-\sigma}}{u^\alpha}\right](-\alpha)\frac{du_t}{u}$$
$$+ \left[BC_v(1-\theta)\left(\frac{1}{\eta A\pi^\alpha(1-\pi)^\sigma}\frac{v^{1-\sigma}}{u^\alpha}\right)^{1-\theta} + (1-\delta)\frac{1}{\eta A\pi^\alpha(1-\pi)^\sigma}\frac{v^{1-\sigma}}{u^\alpha}\right](1-\sigma)\frac{dv_t}{v}. \tag{II.62}$$

Noticing that at the steady state:

$$BC_v(1-\theta)\left(\frac{1}{\eta A\pi^\alpha(1-\pi)^\sigma}\frac{v^{1-\sigma}}{u^\alpha}\right)^{1-\theta} + (1-\delta)\frac{1}{\eta A\pi^\alpha(1-\pi)^\sigma}\frac{v^{1-\sigma}}{u^\alpha} =$$
$$= v - \theta BC_v\left(\frac{1}{\eta A\pi^\alpha(1-\pi)^\sigma}\frac{v^{1-\sigma}}{u^\alpha}\right)^{1-\theta}. \tag{II.63}$$

Combining (II.62) and (II.63) we obtain:

$$\frac{vdv_{t+1}}{v} = \left[v - \theta BC_v\left(\frac{1}{\eta A\pi^\alpha(1-\pi)^\sigma}\frac{v^{1-\sigma}}{u^\alpha}\right)^{1-\theta}\right]\left[(-\alpha)\frac{du_t}{u} + (1-\sigma)\frac{dv_t}{v}\right]$$
$$\frac{dv_{t+1}}{v} = \left[1 - \frac{\theta BC_v\left(\frac{1}{\eta A\pi^\alpha(1-\pi)^\sigma}\frac{v^{1-\sigma}}{u^\alpha}\right)^{1-\theta}}{v}\right]\left[(-\alpha)\frac{du_t}{u} + (1-\sigma)\frac{dv_t}{v}\right]. \tag{II.64}$$

Let us define:

$$b \equiv 1 - \frac{\theta BC_v\left(\frac{1}{\eta A\pi^\alpha(1-\pi)^\sigma}\frac{v^{1-\sigma}}{u^\alpha}\right)^{1-\theta}}{v}.$$

Because of (II.63), we can easily see that $0 < b < 1$. Moreover, (II.64) becomes:

$$\frac{dv_{t+1}}{v} = b\left[(-\alpha)\frac{du_t}{u} + (1-\sigma)\frac{dv_t}{v}\right]. \tag{II.65}$$

Combining the results of (II.61) and (II.65), we finally obtain the system (II.31).

■

Proof of Proposition 4.5.

The characteristic polynomial of the system (II.31) is given by:

$$P(\lambda) = \lambda^2 - T\lambda + D,$$

where T and D respectively denote by the trace and the determinant of the Jacobian matrix. They are addressed by

$$\begin{aligned} T &\equiv (1-\alpha)a + (1-\sigma)b, \\ D &= (1-\alpha-\sigma)ab. \end{aligned}$$

Furthermore, we have:

$$\begin{aligned} \Delta &\equiv T^2 - 4D \\ &= [(1-\alpha)a + (1-\sigma)b]^2 - 4(1-\alpha-\sigma)ab \\ &= [(1-\alpha)a]^2 + [(1-\sigma)b]^2 + 2(1-\alpha)(1-\sigma)ab - 4(1-\alpha-\sigma)ab \\ &= [(1-\alpha)a - (1-\sigma)b]^2 + 4\alpha\sigma ab. \end{aligned}$$

Consequently, we can easily see that $\Delta > 0$ and $T = \lambda_1 + \lambda_2 > 0$, $D = \lambda_1\lambda_2 > 0$. This means that the eigenvalues λ_1 and λ_2 are distinct and positive. Moreover, because the sum of these eigenvalues $T = (1-\alpha)a + (1-\sigma)b$ is less than 2, one of the eigenvalues is smaller than 1: If $\lambda_1 < \lambda_2$ then necessarily $\lambda_1 < 1$.

Now, in order to prove that λ_2 is also less than 1, it is sufficient to show that $P(1) > 0$. Indeed, since

$$\begin{aligned} P(1) &= 1 - T + D \\ &= 1 - [(1-\alpha)a + (1-\sigma)b] + (1-\alpha-\sigma)ab \\ &= [1-(1-\alpha)a][1-(1-\sigma)b] - \alpha\sigma ab, \end{aligned}$$

proving that $P(1) > 0$ is equivalent to point out that $[1-(1-\alpha)a][1-(1-\sigma)b] > \alpha\sigma ab$, or equivalently $\frac{[1-(1-\alpha)a][1-(1-\sigma)b]}{\alpha\sigma ab} > 1$. The later inequality is always true, because it is obvious that:

$$1-(1-\alpha)a = 1-a+\alpha a > \alpha a,$$

and

$$1-(1-\sigma)b = 1-b+\sigma b > \sigma b.$$

■

Proof of Proposition 5.1

Since wage incomes of the two types of agent are respectively $w_{1,t}h_{1,t} = \frac{\alpha}{\pi}y_t$ and $w_{2,t}h_{2,t} = \frac{\sigma}{1-\pi}y_t$, we have:

$$\begin{aligned} G_w &= \frac{\pi(1-\pi)(\frac{\alpha}{\pi}y_t - \frac{\sigma}{1-\pi}y_t)}{(\alpha+\sigma)y_t} \\ &= (1-\pi)\frac{\alpha}{\alpha+\sigma} - \pi\frac{\sigma}{\alpha+\sigma} \\ &= \frac{\alpha}{\alpha+\sigma} - \pi. \end{aligned}$$

■

Proof of Proposition 5.2

By the reason of the consumption of the two types of young workers of two communities respectively are

$$c_{1,t} = \frac{1}{1+\beta_1+\gamma_1}\frac{\alpha}{\pi}y_t,$$

and

$$c_{2,t} = \frac{1}{1+\beta_2+\gamma_2}\frac{\sigma}{1-\pi}y_t.$$

we obtain:

$$
\begin{aligned}
G_c^y &= \frac{(1-\pi)\frac{\alpha}{1+\beta_1+\gamma_1} - \pi\frac{\sigma}{1+\beta_2+\gamma_2}}{\frac{\alpha}{1+\beta_1+\gamma_1} + \frac{\sigma}{1+\beta_2+\gamma_2}} \\
&= \frac{\alpha}{\alpha + \sigma\frac{1+\beta_1+\gamma_1}{1+\beta_2+\gamma_2}} - \pi.
\end{aligned}
$$

Similarly, we also had the consumption of an agent of type 1 is

$$
d_{1,t+1} = R_{t+1}\frac{\beta_1}{1+\beta_1+\gamma_1}\frac{\alpha}{\pi}y_t,
$$

and the consumption of an agent of type 2 is:

$$
d_{2,t+1} = R_{t+1}\frac{\beta_2}{1+\beta_2+\gamma_2}\frac{\sigma}{1-\pi}y_t.
$$

We now achieve the consumption GINI coefficient index when individuals are old:

$$
\begin{aligned}
G_c^o &= \frac{(1-\pi)\alpha\frac{\beta_1}{1+\beta_1+\gamma_1} - \pi\sigma\frac{\beta_2}{1+\beta_2+\gamma_2}}{\alpha\frac{\beta_1}{1+\beta_1+\gamma_1} + \sigma\frac{\beta_2}{1+\beta_2+\gamma_2}} \\
&= \frac{\alpha}{\alpha + \sigma\frac{\beta_2}{\beta_1}\frac{1+\beta_1+\gamma_1}{1+\beta_2+\gamma_2}} - \pi.
\end{aligned}
$$

■

Chapter III

Human Capital Accumulation, Tax Policy and Endogenous Economic Growth

1 Introduction

According to standard definitions Smith (1776) and Becker (1964), human capital accounts for labor productivity and depends on the worker's state of health and the level of education. In equilibrium, when the labor market is perfect, human capital turns out to be equivalent to the discounted value of life-span labor incomes. Since the seminal Lucas' contribution (1988), the accumulaiton of human capital is also recognized to contribute to economic growth to a large extent.

Investments in human capital (health and education) are either private or public choices. While households can decide the share of income and leisure devoted to schooling, nutrition, medecines or sports, a government also can implement educational policies and/or improve the health system. In this chapter, I will focus only on public spending on human capital accumulation through investment in education and consider the impact of this policy on (endogenous) growth and social welfare.

To this purpose I build a simple OLG model with two overlapping generations. This approach is interesting in two respects. On the one hand, it allows to have a general equilibrium perspective. On the other hand, it permits to study the (complex) dynamics associated to the interplay between the accumulation of physical and human capital.

In this study, I provide a study of the dynamic properties at the steady state and during the transition. Besides, I address the impact of a two-sided fiscal policy on human capital accumulation, economic growth, and social welfare. More precisely, I assume that the government finances the young's education by levying taxes on labor and capital income following a simple balanced-budget rule: two distinct but constant tax rates on labor and capital income are considered.

There is a large existing body of literature on the accumulation of human capital. The classical work of Lucas (1988) focuses on human capital and considers the trade-off between learning time and working time. Interestingly, a recent work of this author, Lucas (2015) argues that “the contribution of human capital accumulation to economic growth deserves a production function of its own” (p. 87), and emphasizes the importance of human capital as source of economic growth. Manuelli and Seshadri (2014), by measuring human capital according to schooling and age-earnings data, show that the variation in average wealths among countries is well-explained by the differences in the level of human capital accumulation.

Directly related to this work are the studies by Garrat and Marshall (1994), Fernandez and Rogerson (1995), and Gradstein and Justman (1995). The main conclusion of these authors is that individuals who attend institutions of education would earn more income in the future and hence pay more income taxes. They also emphasize the role of the government, through investment in education, on reducing inequality and enhancing the benefits of social welfare. In addition, the main motivation for this article is the work of Glomm and Ravikumar (1992), who consider an OLG economy with heterogeneous agents and two-fold human capital formation, i.e., public and private investment in education. I consider only the former, with the aim to highlight the importance of the role of taxation on human capital accumulation and ultimately on economic growth.

This research is also in the spirit of Barro (1990) and Bosi and Nourry (2007), who examine the effect of public spendings on economic growth. Differently from the first two chapters of my dissertation, where the investment in education is decentralized (the role of private investment in education), this chapter considers this variable as a government choice. In order to understand the impact of tax policies on economic growth in separation from the decisions of the households, I assume that the households invest only in physical capital. One of the purposes of this research is to study the co-evolution of human and physical capital in an OLG model. This work not only completes the work in Chapter 1, where the problem is analysed in a Ramsey model, but also sheds light on the impact of taxation on long-run econmic growth.

The main idea runs as follows. I consider an overlapping generation (OLG) model with homogeneous economic agents. The newborn agents live in date t, when they are young, and in date $t+1$, when they are old. They work when young, endowed with inelastic working time and a skill level. The latter is understood as the *human capital level* characterizing their generation. They make decisions on consumption and investment in physical capital (savings) when young and consume what they get from savings when old.

The government taxes both wage (labor income) and capital returns on the whole population (both young and old). Two effects of taxation are noteworthy. First, it reduces the households' total income, hence lowering their investment in physical capital, which has a negative impact on output. Second, it raises tax revenues and the government's investment in education, thus increasing the productivity of the future generations. Consequently, the net impact on economic growth is ambiguous.

Given taxes on labor and capital incomes, I characterize the balanced-growth path (BGP) of the economy. I prove the existence and uniqueness of the BGP and their convergence to this BGP in the long run. The growth rate on the BGP is also determined as a function of taxes. Based on these results, I analyse the impact of taxes on the long-run growth rate. In particular, I show that if the labor income tax rate belongs to a certain interval, then the growth rate on the BGP is positive, and the economy enjoys stable growth to infinity. On the contrary, if this tax is either too low or too high, the BGP growth rate is negative and the economy collapses to zero. The question on intertemporal welfare is also considered. Furthermore, if I suppose that the government fixes a discount rate for all future generations' utilities, then I again compute the BGP and the long-run growth rate.

The remainder of this chapter is organized as follows, section 2 outlines the fundamentals of the economy. Section 3 gives the descriptions of global equilibrium dynamics. In section 4, the study analyzes steady-growth path. This research also introduces an analysis of social welfare issue in section 5. Section 6 is conclusion. The proofs are gathered in Appendix.

2 Fundamentals

This study considers a two-period overlapping generations model with physical and human capital accumulation. Both goods and factors markets are competitive.

Firms. In each period, a representative firm uses physical capital and labor. The economy's technology is represented by the Cobb-Douglas production function as follows:

$$F(K_t, L_t) \equiv AK_t^{\alpha} L_t^{1-\alpha}, \tag{III.1}$$

where K_t and L_t denote the aggregate capital and the effective labor force, respectively.

Given the return on physical capital R_t and the wage rate for effective labor w_t, the problem of the representative firm is:

$$\max_{(K_t, L_t)} [F(K_t, L_t) - R_t K_t - w_t L_t].$$

The first order conditions of the optimization yields:

$$R_t = \alpha A K_t^{\alpha-1} L_t^{1-\alpha}, \tag{III.2}$$

$$w_t = (1-\alpha) A K_t^{\alpha} L_t^{-\alpha}. \tag{III.3}$$

Consumers. Each individual lives for two periods: young at time t and old at time $t+1$. She works and saves when young and consumes from the return on savings when old.

At time t, the young population is N_t. We assume that the population grows at a constant rate $n = \frac{N_{t+1}}{N_t}$, for all t.

Let the level of human capital stock of generation N_t be h_t. The effective labor force is therefore $L_t = h_t N_t$. In this economy, a generation-t young worker supplies labor and earns income $w_t h_t$, which is divided into consumption and savings at time t. Consumptions when young and old are denoted by c_t and d_{t+1}, respectively.

The household's optimization problem is the following:

$$\max_{(c_t, d_{t+1}, s_t)} \{u(c_t) + \beta u(d_{t+1})\}, \tag{III.4}$$

subject to the following budget constraints:

$$c_t + s_t \leq (1-\tau) w_t h_t, \tag{III.5}$$

$$d_{t+1} \leq (1-\tau_k) R_{t+1} s_t, \tag{III.6}$$

taking as given the income tax rate τ, the capital income tax rate τ_k, the return on captial R_{t+1} and the labor wage rate w_t.

Government. The government in this economy generates revenues from taxes and invests all these revenues in public education. In addition, the three kinds of intensity form are considered as follows. The intensity of capital at time t: $k_t \equiv K_t/N_t$, the income intensity of the economy at date t: $y_t \equiv Y_t/N_t$, and the intensity of government spending for public education at period t: $g_t \equiv G_t/N_t$.

Furthermore, the individual labor income $w_t h_t$, the return of physical capital $R_t k_t$ and the intensity of government spending for public education g_t can be expressed in term of the income intensity of economy y_t:

$$w_t h_t = (1-\alpha) y_t, \tag{III.7}$$

$$R_t k_t = \alpha y_t, \tag{III.8}$$

$$g_t = [\alpha \tau_k + (1-\alpha)\tau] y_t. \tag{III.9}$$

Besides the accumulation of physical and human capital in the economy evolve according to the following difference equations:

$$nk_{t+1} = s_t, \tag{III.10}$$

$$h_{t+1} = Bg_t^{\gamma}h_t^{1-\gamma} + (1-\delta)h_t, \tag{III.11}$$

where $0 \leq \delta \leq 1$ is the depreciation rate of human capital, the parameter B represents the efficiency of human capital formation technology, and γ is the share of the contribution of the government spending (while $1-\gamma$ expresses the current human capital stock of individual) to the generation of new human capital.[1]

Market clearing conditions.

1. In the physical capital market:

$$K_{t+1} = N_t s_t. \tag{III.12}$$

2. In the labor market:

$$L_t = N_t h_t. \tag{III.13}$$

3. In the consumption goods market:

$$AK_t^{\alpha}L_t^{1-\alpha} = N_t c_t + N_t s_t + N_{t-1}d_t + G_t, \tag{III.14}$$

where $G_t \equiv \tau w_t h_t N_t + \tau_k R_t s_{t-1} N_{t-1}$ is the total government tax revenues, which are all invested in public schooling in our model.

[1] According to the evolution of human capital (equation III.11), the parameter B captures the efficiency of human capital formation technology, including several features such as quality and appropriateness of syllabus, quality of teaching facilities, and so forth. Additionally, the shares of γ and $1-\gamma$ reveal the effectiveness of parental human capital in their efforts towards educating their children, and the efficiency of public schooling in generating human capital, respectively. Moreover, γ is affected by home education and family background; whereas $1-\gamma$ is affected by the schooling and classes system, teachers, syllabus, facilities.

A competitive equilibrium in the economy is given by the following definition:

Definition 2.1 *An intertemporal competitive equilibrium of the economy is a sequence* $(w_t, R_t, c_t, d_{t+1}, s_t, y_t, h_t, k_{t+1}, g_t)_{t=0}^{\infty}$ *where* (w_t, R_{t+1}) *satisfy (III.2) and (III.3), households maximize intertemporal utility (III.4) subject to the constraints (III.5) and (III.6), markets clear (III.12), (III.13) and (III.14), evolution of physical and human capital satisfy equations (III.10), (III.11), and government spending satisfies (III.9).*

3 Equilibrium dynamics

Note that we can write:

$$\begin{aligned} y_t &= A k_t^{\alpha} h_t^{1-\alpha} \\ &\equiv f(k_t, h_t). \end{aligned}$$

Furthermore, as we show in the following lemma, equilibrium consumption and savings are proportional to output.

Lemma 3.1 *For each agent we have:*

$$c_t = \frac{1}{1+\beta}(1-\tau) w_t h_t, \tag{III.15}$$

$$s_t = \frac{\beta}{1+\beta}(1-\tau) w_t h_t, \tag{III.16}$$

$$d_{t+1} = \frac{\beta}{1+\beta}(1-\tau_k) R_{t+1} (1-\tau) w_t h_t.$$

Proof: Appendix ■

According to equations (III.12) and (III.16), the equilibrium in capital markets is given by

$$nk_{t+1} = \frac{\beta}{1+\beta}(1-\tau)(1-\alpha)y_t. \tag{III.17}$$

It is also equivalent to

$$nk_{t+1} = \frac{\beta}{1+\beta}(1-\tau)(1-\alpha)Ak_t^{\alpha}h_t^{1-\alpha}. \tag{III.18}$$

Proposition 3.1 *The dynamic system of the economy is given by*

$$k_{t+1} = \eta f(k_t, h_t), \tag{III.19}$$

$$h_{t+1} = B[\chi f(k_t, h_t)]^{\gamma}h_t^{1-\gamma} + (1-\delta)h_t, \tag{III.20}$$

where $\eta = \frac{\beta(1-\tau)(1-\alpha)}{n(1+\beta)}$, $\chi = [(1-\alpha)\tau + \alpha\tau_k]$ *and* $f(k_t, h_t) \equiv Ak_t^{\alpha}h_t^{1-\alpha}$.

Proof: Appendix ■

In order to analyze this two-dimensional equation system, we make a transformation of this system to a one-dimensional equation.

Let the ratio of physical capital to human capital be $x_t = \frac{k_t}{h_t}$, then the above two-dimensional global dynamical system is reduced to a one-dimensional global dynamical system as follows.

Corollary 3.1 *Let us define*

$$a = A\beta(1-\tau)(1-\alpha),$$

$$b = n(1+\beta)A^{\gamma}B\left[(1-\alpha)\tau + \alpha\tau_k\right]^{\gamma},$$

$$c = n(1+\beta)(1-\delta).$$

Then the system (III.19) - (III.20) becomes

$$\begin{aligned} x_{t+1} &= \frac{a}{bx_t^{-\alpha(1-\gamma)} + cx_t^{-\alpha}} \\ &= \frac{ax_t^{\alpha}}{bx_t^{\alpha\gamma} + c}. \end{aligned}$$

For any $x_0 > 0$, there exists the limit of sequence $\{x_t\}_{t=0}^{\infty}$

$$x^* = \lim_{t\to\infty} x_t,$$

where x^* is solution to the equation

$$x = \frac{ax^{\alpha}}{bx^{\alpha\gamma} + c}.$$

Proof: Appendix ■

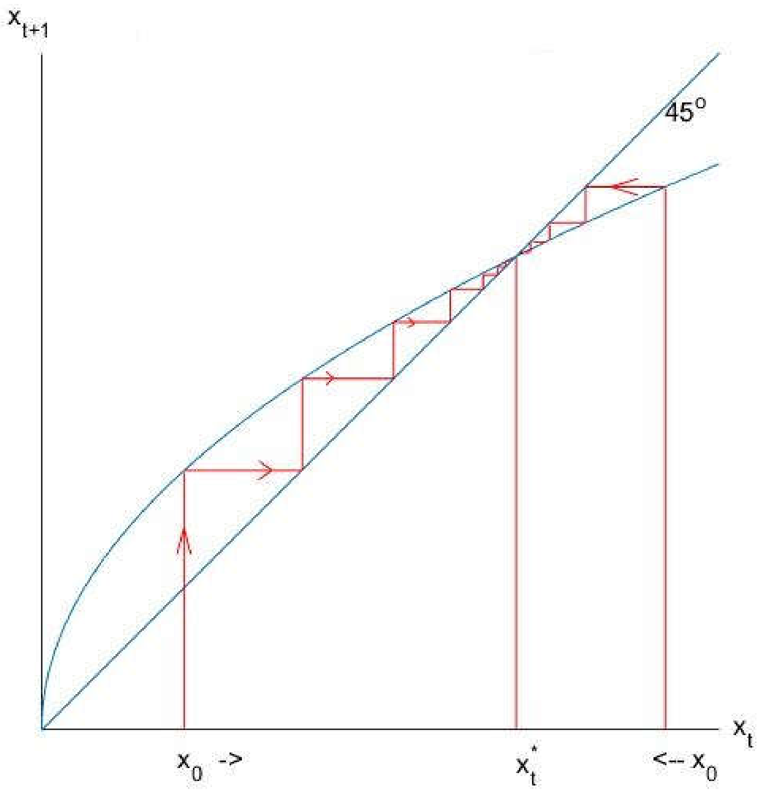
Proposition 3.2 *Given the initial pair of human and physical capital of (k_0, h_0), the solution of the dynamical system converges to a unique balanced growth path (BGP). Furthermore, on the BGP, human and physical capital grow at the same rate:*

$$\lim_{t\to\infty} \frac{k_{t+1}}{k_t} = \lim_{t\to\infty} \frac{h_{t+1}}{h_t} \equiv \rho.$$

Proof: Appendix ■

The one-dimensional global dynamic system can be expressed by the following graph:

Figure 3.1. The global dynamic system of the economy.



4 Steady-growth path analysis

If we know exactly the value of x^*, we can calculate the growth rate ρ. However, it is a difficult task. In order to calculate the value of the balanced-growth path growth rate of the economy, we must follow another approach.

In this section, I focus on the balanced-growth path along which long-run physical capital and human capital grow at the same positive constant growth rate of the economy, ρ.

Proposition 4.1 *There exists a unique BGP, on which the rate of growth of the* *economy is given by*

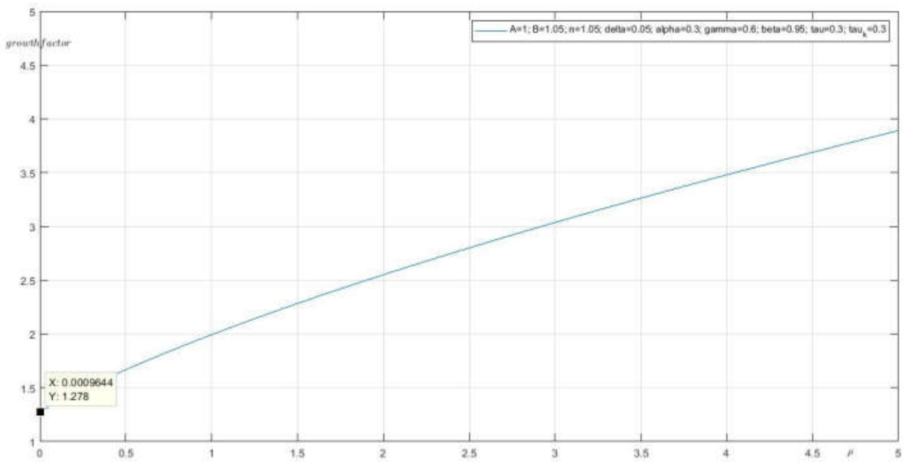
$$
\begin{aligned}
\rho &= \phi^{-1}\left(\psi(\tau,\tau_k)\right) \qquad \text{(III.21)}\\
&= \phi^{-1}\left[\frac{1}{n^{\alpha}} A B^{\frac{1-\alpha}{\gamma}} \left(\frac{\beta}{1+\beta}(1-\tau)(1-\alpha)\right)^{\alpha} \left((1-\alpha)\tau + \alpha\tau_k\right)^{1-\alpha}\right],
\end{aligned}
$$

where $\phi(\rho) \equiv \rho^{\alpha}(\rho - 1 + \delta)^{\frac{1-\alpha}{\gamma}}$*, which is an increasing function in* ρ*.*

Proof: Appendix ■

Next, I conduct a numerical exercise to illustrate Proposition 4.1 above. Assume that $A = 1$, $B = 1.05$, $n = 1.05$, $\delta = 0.05$, $\alpha = 0.3$, $\beta = 0.95$, $\gamma = 0.6$, $\tau = 0.3$, $\tau_k = 0.3$. Then the growth factor of economy is approximately $1,278$ (the growth rate of the economy is around 0.278), as shown in Figure 3.2 below.

Figure 3.2. The growth rate of economy on the balanced-growth path (the growth factor: 1.278).



4.1 Tax policy and growth

I now study the impact of tax rates on long-run growth. More precisely, the following lemma shows the conditions under which the growth factor of the economy, ρ is smaller or greater than zero.

Lemma 4.1 *According to Proposition 4.1, we have*

$$\psi(\tau,\tau_k) = \frac{1}{n^\alpha} AB^{\frac{1-\alpha}{\gamma}} \left(\frac{\beta}{1+\beta}(1-\tau)(1-\alpha)\right)^\alpha ((1-\alpha)\tau + \alpha\tau_k)^{1-\alpha}.$$

Define $\hat{\tau}$ to be a solution to

$$\underset{0\leq\tau\leq 1}{\arg\max} \left\{(1-\tau)^\alpha \left[(1-\alpha)\tau + \alpha\right]^{1-\alpha}\right\}.$$

We have

1. *If $0 < \alpha < \frac{1}{2}$ then the optimal labor income tax is $0 < \hat{\tau} < 1$,*
2. *If $\frac{1}{2} \leq \alpha < 1$ then the optimal labor income tax is $\hat{\tau} = 0$.*

Proof: Appendix ■

Proposition 4.2 *Suppose that $0 < \alpha < \frac{1}{2}$. We have two subcases:*

1. *If*

$$\delta^{\frac{1-\alpha}{\gamma}} > \frac{1}{n^\alpha} \times AB^{\frac{1-\alpha}{\gamma}} \times \left(\frac{\beta}{1+\beta}(1-\alpha)(1-\hat{\tau})\right)^\alpha \times ((1-\alpha)\hat{\tau}) + \alpha)^{1-\alpha},$$

   then the growth rate $\rho < 0$ and the economy converges to zero, for any τ, τ_k.

2. *If*

$$\delta^{\frac{1-\alpha}{\gamma}} < \frac{1}{n^\alpha} \times AB^{\frac{1-\alpha}{\gamma}} \times \left(\frac{\beta}{1+\beta}(1-\alpha)(1-\hat{\tau})\right)^\alpha \times ((1-\alpha)\hat{\tau}) + \alpha)^{1-\alpha},$$

   then there exists $0 \leq \underline{\tau} < \overline{\tau} < 1$ such that

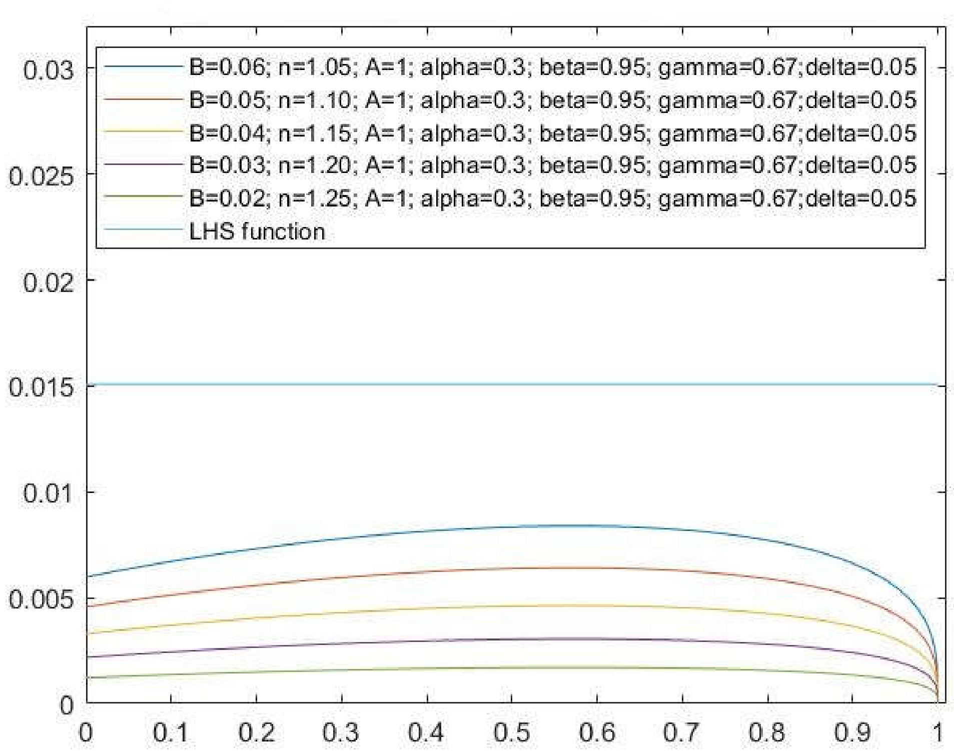
   (a) *If $\underline{\tau} < \tau < \overline{\tau}$, then there exists $\hat{\tau}_k$ such that for $\rho > 0$ if and only if $\hat{\tau}_k < \tau_k < 1$.*

   (b) *If $\tau \notin [\underline{\tau}, \overline{\tau}]$, then for any τ_k, we have $\rho < 0$.*

Proof: Appendix ■

To illustrate Proposition 4.2, let us proceed with a numerical exercise with $\alpha = 0.3, \beta = 0.95, \delta = 0.05$ and $\gamma = 0.67$. In the first case, we let the parameter of efficiency B decrease from 0.06 to 0.02 and the population growth rate increase from 1.05 to 1.25. We see that in this case, for any τ and τ_k, the growth rate of economy $\rho < 0$ and the economy collapses to zero, as shown in Figure 4.2.

Figure 3.3. The BGP growth rate factor of economy is strictly less than one.

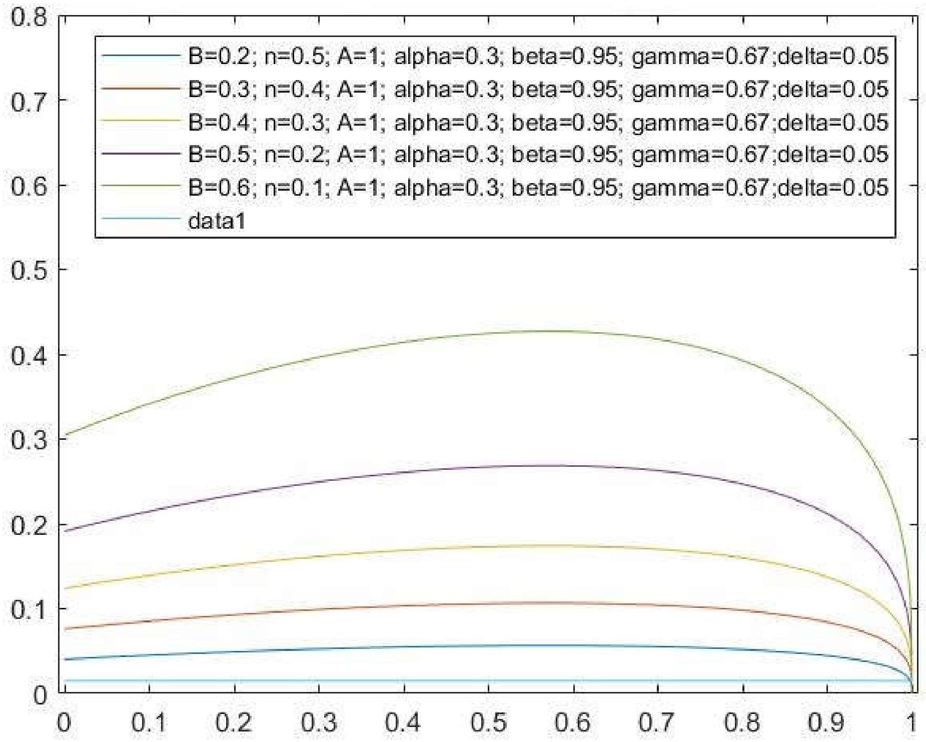


The graph above shows that the efficiency parameter of human capital formation B is too low, while the population growth rate n is too high.

By contrast, in the second case, we let B increase from 0.2 to 0.6 while the population growth decrease from 0.5 to 0.1. As illustrated in Figure 4.3, for any τ

and τ_k, when the efficiency parameter of human capital formation is high enough and the population growth rate is too low, the growth rate of economy is positive.

Figure 3.4. The BGP growth rate factor of economy is strictly greater than one.



The graph above exhibits that the parameter of efficiency of education process B is increasing, while the population growth rate n is decreasing.

Comment: Proposition 4.2 reveals that there are two impacts of taxation on long-run economic growth with respect to the depreciation rate of human capital (that is sufficiently low). On the one hand, if the labor income tax rate belongs to a certain interval with respect to the capital returns tax rate, then the BGP growth rate is positive, and the economy enjoys stable growth to infinity. On the other hand, if this tax rate is either too high or too low, then the rate of growth on the BGP is negative, and the economy collapses to zero.

Proposition 4.3 *Suppose that $\frac{1}{2} \leq \alpha < 1$. We again have two subcases:*

1. *If*

$$\delta^{\frac{1-\alpha}{\gamma}} > \frac{1}{n^\alpha} \times AB^{\frac{1-\alpha}{\gamma}} \times \left(\frac{\beta}{1+\beta}(1-\alpha)\right)^\alpha \times \alpha^{1-\alpha},$$

then the growth rate $\rho < 0$ and the economy collapses to zero, for any τ, τ_k.

2. *If*

$$\delta^{\frac{1-\alpha}{\gamma}} < \frac{1}{n^\alpha} \times AB^{\frac{1-\alpha}{\gamma}} \times \left(\frac{\beta}{1+\beta}(1-\alpha)\right)^\alpha \times \alpha^{1-\alpha},$$

then there exists $0 < \overline{\tau} < 1$ such that

   (a) *For any $\overline{\tau} < \tau < 1$, the growth rate $\rho < 0$ and the economy collapses to zero.*

   (b) *For any $0 < \tau < \overline{\tau}$, there exists $0 < \hat{\tau}_k < 1$ such that the growth rate $\rho > 0$ if and only if $\tau_k > \hat{\tau}_k$.*

Proof: Appendix ■

Comment: With reference to the case of $\frac{1}{2} \leq \alpha < 1$, and also the depreciation of human capital is low enough, for the BGP growth rate to takes place ($\rho > 0$), the labor income tax rate may only need to be less than the threshold of its value, with respect to the capital returns tax rate. It turns out that this tax rate should not be too high.

4.2 Comparative statics

Lemma 4.2 *The elasticities of the growth rate with respect to taxes are given by*

$$\frac{\partial\rho}{\rho} / \frac{\partial\tau}{\tau} = \left[\alpha + \frac{1-\alpha}{\gamma}\frac{\rho}{\rho-1+\delta}\right]^{-1}\left[-\frac{\alpha\tau}{1-\tau} + \frac{(1-\alpha)^2\tau}{(1-\alpha)\tau+\alpha\tau_k}\right] \quad \text{(III.22)}$$

$$\frac{\partial\rho}{\rho} / \frac{\partial\tau_k}{\tau_k} = \left[\alpha + \frac{1-\alpha}{\gamma}\frac{\rho}{\rho-1+\delta}\right]^{-1}\frac{\alpha(1-\alpha)\tau_k}{(1-\alpha)\tau+\alpha\tau_k} \quad \text{(III.23)}$$

Proof: Appendix ■

Proposition 4.4 *Given τ_k, the optimal tax on labor income of economy is given by $\tau^*(\tau_k) = 1 - \alpha - \frac{\alpha^2}{1-\alpha}\tau_k$. Furthermore, the rate of growth of economy attains its maximum value at τ^*.*

Proof: Appendix ■

Let us conduct a numerical test to interpret the Proposition 4.4. We assume that the capital returns tax rate, $\tau_k = 1/3$, and the share of capital in production, $\alpha = 1/3$. Following this setting, we obtain the optimal tax rate on labor income, $\tau^*(\tau_k) = 0.61$. Base on this result, we also compute the BGP growth rate of the economy which is at the maximum value, $\rho = 0.336$ (it turns out to be that the BGP growth factor is around 1.336).

Proposition 4.5 *If the ratio of taxes to GDP increases (decreases)* 1 *percent then the rate of growth of the economy increases (decreases) less than a corresponding value.*

Proof: Appendix ■

To depict Proposition 4.5, let us proceed with a numerical exercise with the following values: $\rho = 0.336$, $\tau_k = 1/3$ and $\tau^*(\tau_k) = 0.61$ (these results come from the simulation of Proposition 4.4). Under this setting and the equations (III.36) and (III.38), we calculate the value of the ratio taxes to GDP as follow. First, we obtain the value of $\xi = 0.52$ (remark: $\xi = (1-\alpha)[(1-\alpha) + \alpha\tau_k])$. Second, We compute the elasticity of the ratio taxes to GDP with respect to tax on capital: $\frac{\Delta\rho/\rho}{\Delta\xi/\xi} = -4.6$ (remark: $\frac{\Delta\rho/\rho}{\Delta\xi/\xi} = \left[\alpha + \frac{1-\alpha}{\gamma}\frac{\rho}{\rho-1+\delta}\right]^{-1}$). It turns out that if the ratio taxes to GDP increases (decreases) 1 percent, then the rate of growth of the economy increases (decreases) less than 4.6 percent.

5 Social welfare and the problem of planner

This section considers the social welfare problem under a meritocratic regime where the distribution of consumption and capital is taken by the government.

The government maximizes the following social welfare function

$$W = \sum_{t=-1}^{\infty} \zeta^t U(c_t, d_{t+1}), \tag{III.24}$$

where $0 < \zeta < 1$ is a fixed discount rate factor. This factor is determined by social planner.

Under the assumption that the utility function is separable and logarithmic, we can write:

$$U(c_t, d_{t+1}) = u(c_t) + \beta u(d_{t+1}) = \ln c_t + \beta \ln d_{t+1}.$$

Obviously, since the consumption of the young generation -1 has been done before the government's decision, the social welfare function (III.24) becomes

$$W = \sum_{t=0}^{\infty} \zeta^t \left(\ln c_t + \frac{\beta}{\zeta} \frac{\ln d_t}{n} \right). \tag{III.25}$$

At each time t, the government divides the intensive output $y_t = f(k_t, h_t)$ into three parts, as follows:

1. The consumption: $c_t + \frac{d_t}{n}$,
2. Investment in physical capital: $s_t = nk_{t+1}$,
3. Investment in human capital: e_t.

The social welfare problem faced by the government is thus:

$$\max_{(c_t,d_{t+1},e_t)} W = \sum_{t=0}^{\infty} \zeta^t \left(\ln c_t + \frac{\beta}{\zeta} \frac{\ln d_t}{n} \right) \tag{III.26}$$
$$\text{s.t. } c_t + \frac{d_t}{n} + nk_{t+1} + e_t \leq Ak_t^{\alpha} h_t^{1-\alpha},$$
$$h_{t+1} \leq Be_t^{\gamma} h_t^{1-\gamma} + (1-\delta)h_t.$$

Proposition 5.1

1. *The balanced-growth path is unique.*

2. *The growth rate of the economy is positive if and only if:*

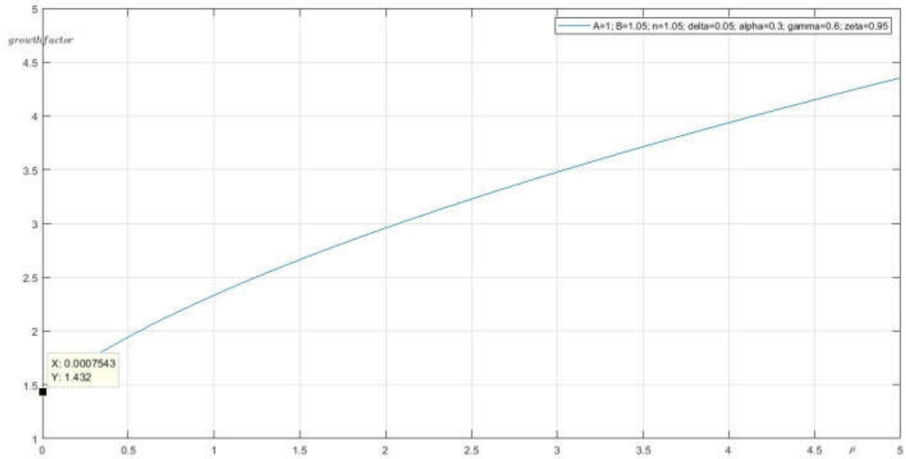
$$(1-\alpha)A^{\frac{1}{1-\alpha}} B^{\frac{1}{\gamma}} \left(\frac{\alpha}{n}\right)^{\frac{\alpha}{1-\alpha}} \zeta^{\frac{\alpha}{1-\alpha}} > \delta^{\frac{1-\gamma}{\gamma}} \left(\frac{1}{\zeta} - (1-\gamma) - \gamma(1-\delta) \right).$$

3. *The growth rate of the economy is increasing with respect to the efficiency parameters of production and human capital formation A, B and decreasing with respect to the population growth rate n.*

Proof: Appendix ■

Comment: In this economy, when the government fixes the discount rate factor for all future generation's utilities, then the existence and uniqueness of the BGP, as well as the long-run growth rate of the economy, are characterized. Furthermore, we can easily see that the BGP growth rate depends on the model's parameters and population growth, as well as the depreciation rate of stock of human capital that is sufficiently low. Arguably, if the total factor productivity, A and the efficiency of human capital formation technology, B are high enough, also the depreciation rate of human capital stock is low enough then the growth rate on the BGP will take place.

Figure 3.5. The BGP growth rate of economy under the regime of social planner problem (the growth factor: 1.432)



Again, a simulation is done with $A = 1$, $B = 1.05$, $n = 1.05$, $\delta = 0.05$, $\alpha = 0.3$, $\zeta = 0.95$, and $\gamma = 0.6$. In this case, the value of growth rate is approximately 0.432 (the growth factor of the economy is around 1.432), as represented in figure 5.1 above.

In the analysis of social welfare problem, I touched upon the comparision of steady state to section 4. This case is studied both analytically and numerically. This is also a promising topic for future research. In particular, an analysis of optimal tax rate which maximizes intergenerational welfare is called for a task. I will attempt to tackle this issue in the nearest future.

6 Conclusion

This study considers the important role of the public sector for investment in education in promoting human capital accumulation, i.e., through government's expenditure on education, funded by taxation. The results are interesting in a general equilibrium perspective. It also illustrates the dynamics associated with the interaction between the accumulation of human and physical capital, and ultimately economic growth. In particular, two impacts of taxation policy are notable. Firstly, it reduces the household's total income and the level of investment in physical capital, hence having a negative impact on output. Secondly, it raises tax revenues and the government's spending on public education and consequently enhancing the labor productivity of future generations. In addition, this research provides notable findings regarding the BGP of the economy and characterizes equilibria around this BGP. The issue of intertemporal social welfare is also studied.

7 Appendix

Proof of Lemma 3.1.

The Lagrange equation is given by

$$\ln c_t + \beta \ln d_{t+1} - \lambda_t(c_t + s_t - (1-\tau)w_t h_t) - \mu_t(d_{t+1} - (1-\tau_k)R_{t+1}s_t).$$

The first order conditions are

$$\begin{aligned} \frac{1}{c_t} &= \lambda_t, \\ \frac{\beta}{d_{t+1}} &= \mu_t, \\ \lambda_t &= (1-\tau_k)R_{t+1}\mu_t, \end{aligned}$$

which lead to

$$\begin{aligned} \frac{\lambda_t}{\mu_t} &= \frac{d_{t+1}}{\beta c_t} \\ &= (1-\tau_k)R_{t+1}. \end{aligned}$$

Moreover, the budget constraints are now binding, We therefore obtain

$$\begin{aligned} d_{t+1} &= (1-\tau_k)R_{t+1}s_t, \\ \beta c_t &= s_t. \end{aligned}$$

Since $c_t + s_t = (1-\tau)w_t h_t$, one has

$$\frac{1}{\beta}s_t + s_t = (1-\tau)w_t h_t,$$

which implies

$$s_t = \frac{\beta}{1+\beta}(1-\tau)w_t h_t,$$

and

$$c_t = \frac{1}{1+\beta}(1-\tau)w_t h_t.$$

■

Proof of proposition 3.1.

According to the equations of (III.10), (III.11), (III.17) and (III.18), the dynamics will be the form of equilibrium system, which is as following:

$$\begin{aligned} k_{t+1} &= \frac{\beta(1-\tau)(1-\alpha)}{n(1+\beta)} \times Ak_t^{\alpha}h_t^{1-\alpha}, \\ h_{t+1} &= B\left[(1-\alpha)\tau + \alpha\tau_k\right]^{\gamma}\left(Ak_t^{\alpha}h_t^{1-\alpha}\right)^{\gamma}h_t^{1-\gamma} + (1-\delta)h_t \\ &= B\left[(1-\alpha)\tau + \alpha\tau_k\right]^{\gamma}A^{\gamma}k_t^{\alpha\gamma}h_t^{1-\alpha\gamma} + (1-\delta)h_t. \end{aligned}$$

■

Proof of Corollary 3.1.

Let us remark that

$$x_t = \frac{k_t}{h_t}.$$

From Proposition 3.1, one can re-write the dynamical system as

$$\begin{aligned}
\frac{k_{t+1}}{k_t} &= \frac{A\beta(1-\tau)(1-\alpha)}{n(1+\beta)} \left(\frac{h_t}{k_t}\right)^{1-\alpha} \\
&= \frac{A\beta(1-\tau)(1-\alpha)}{n(1+\beta)x_t^{1-\alpha}}, \\
\frac{h_{t+1}}{h_t} &= A^\gamma B \left[(1-\alpha)\tau + \alpha\tau_k\right]^\gamma \left(\frac{k_t}{h_t}\right)^{\alpha\gamma} + (1-\delta) \\
&= A^\gamma B \left[(1-\alpha)\tau + \alpha\tau_k\right]^\gamma x_t^{\alpha\gamma} + (1-\delta).
\end{aligned}$$

This implies

$$\frac{x_{t+1}}{x_t} = \frac{A\beta(1-\tau)(1-\alpha)}{n(1+\beta)x_t^{1-\alpha}\left(A^\gamma B\left[(1-\alpha)\tau+\alpha\tau_k\right]^\gamma x_t^{\alpha\gamma} + (1-\delta)\right)}.$$

The relation between x_{t+1} and x_t is

$$x_{t+1} = \frac{A\beta(1-\tau)(1-\alpha)}{n(1+\beta)\left(A^\gamma B\left[(1-\alpha)\tau+\alpha\tau_k\right]^\gamma x_t^{-\alpha(1-\gamma)} + (1-\delta)x_t^{-\alpha}\right)}.$$

Proof of proposition 3.2.

From the Corollary 3.1, let us define

$$f(x) = \frac{ax^\alpha}{bx^{\alpha\gamma}+c},$$

We then have

$$\frac{x}{f(x)} = \frac{b}{a}x^{1-\alpha(1-\gamma)} + \frac{c}{a}x^{1-\alpha}.$$

This also implies that

$$\lim_{x \to 0} \frac{x}{f(x)} = 0,$$
$$\lim_{x \to \infty} \frac{x}{f(x)} = +\infty.$$

Hence for x small we have $f(x) > x$ and for x big we have $f(x) < x$. It is easy to verify that f is increasing. There is thus unique x^* solution to $x = f(x)$. It is equivalent to be

$$x^* = \frac{a(x^*)^\alpha}{b(x^*)^{\alpha\gamma} + c}.$$

We now prove that for any $x_0 > 0$,

$$\lim_{t \to \infty} x_t = x^*.$$

Indeed, if $x_0 < x^*$, then $f(x_0) > x_0$. By the monotonicity of f, we have $x_0 < x_1 = f(x_0) < f(x^*) = x^*$. By induction, we have the sequence $\{x_t\}_{t=0}^\infty$ is strictly increasing and $x_t < x^*$ for any t.

This implies that this sequence converges to a solution $f(x) = x$. Since x^* is the unique solution, we have

$$\lim_{t \to \infty} x_t = x^*.$$

For the case $x_0 > x^*$, we use the same arguments and get

$$\lim_{t \to \infty} x_t = x^*.$$

Given the pair of (k_0, h_0), we obtain

$$\lim_{t \to \infty} \frac{k_t}{h_t} = x^*.$$

In addition, since

$$\frac{k_{t+1}}{k_t} = \frac{A\beta(1-\tau)(1-\alpha)}{n(1+\beta)} \left(\frac{h_t}{k_t}\right)^{1-\alpha},$$

and then

$$\begin{aligned}\lim_{t\to\infty} \frac{k_{t+1}}{k_t} &= \frac{A\beta(1-\tau)(1-\alpha)}{n(1+\beta)x^*} \\ &= \rho.\end{aligned}$$

Moreover, we also have

$$\lim_{t\to\infty} \frac{h_{t+1}}{h_t} = \rho.$$

Therefore, given the initial pair of physical and human capital of (k_0, h_0), we have the solution of dynamical system converges to the balance growth path. ■

Proof of proposition 4.1.

Noticing that $\frac{k_{t+1}}{k_t} = \frac{h_{t+1}}{h_t} = \rho$, then the equation (III.11) gives

$$h_t = \left(\frac{B}{\rho - 1 + \delta}\right)^{\frac{1}{\gamma}} g_t. \tag{III.27}$$

Combining (III.10) and (III.17), we obtain

$$g_t = n\frac{1+\beta}{\beta}\frac{1}{(1-\tau)(1-\alpha)}[(1-\alpha)\tau + \alpha\tau_k]k_{t+1}. \tag{III.28}$$

Also, combining (III.27) and (III.28), we have

$$h_t = n\left(\frac{B}{\rho - 1 + \delta}\right)^{\frac{1}{\gamma}} \frac{1+\beta}{\beta}\frac{1}{(1-\tau)(1-\alpha)}[(1-\alpha)\tau + \alpha\tau_k]k_{t+1}. \tag{III.29}$$

Incorporating (III.18) and (III.29), we get

$$
\begin{aligned}
nk_{t+1} &= \frac{\beta}{1+\beta}(1-\tau)(1-\alpha)\times \\
& \quad An^{1-\alpha}k_t^{\alpha}\left(\frac{B}{\rho-1+\delta}\right)^{\frac{1-\alpha}{\gamma}}\left\{\frac{1+\beta}{\beta}\frac{1}{(1-\tau)(1-\alpha)}[(1-\alpha)\tau+\alpha\tau_k]\right\}^{1-\alpha}k_{t+1}^{1-\alpha}. \\
n &= An^{1-\alpha}\left(\frac{B}{\rho-1+\delta}\right)^{\frac{1-\alpha}{\gamma}}\left[\frac{\beta}{1+\beta}(1-\tau)(1-\alpha)\right]^{\alpha}[(1-\alpha)\tau+\alpha\tau_k]^{1-\alpha}(1+\rho)^{-\alpha}. \\
\rho^{\alpha}(\rho-1+\delta)^{\frac{1-\alpha}{\gamma}} &= An^{-\alpha}B^{\frac{1-\alpha}{\gamma}}\left[\frac{\beta}{1+\beta}(1-\tau)(1-\alpha)\right]^{\alpha}[(1-\alpha)\tau+\alpha\tau_k]^{1-\alpha}.
\end{aligned}
\tag{III.30}
$$

Let us define:

- $\phi(\rho) \equiv \rho^{\alpha}(\rho-1+\delta)^{\frac{1-\alpha}{\gamma}}$,
- $\psi(\tau,\tau_k) = \frac{1}{n^{\alpha}}AB^{\frac{1-\alpha}{\gamma}}\left(\frac{\beta}{1+\beta}(1-\tau)(1-\alpha)\right)^{\alpha}((1-\alpha)\tau+\alpha\tau_k)^{1-\alpha}$.

By logarithmizing the equation (III.30). We take the derivation of LHS calculus with respect to ρ, it is easy to see that $\phi'(\rho) = \frac{\alpha}{\rho} + \frac{1-\alpha}{\gamma}\frac{1}{\rho-1+\delta} > 0$. Moreover, the function ϕ is strictly increasing. The rate of growth of economy hence is $\rho = \phi^{-1}[\psi(\tau,\tau_k)]$. ■

Proof of lemma 4.1.

1. Suppose that $0 < \alpha < \frac{1}{2}$. We consider the following case

$$f(\tau) = (1-\tau)^{\alpha}[(1-\alpha)\tau+\alpha]^{1-\alpha}.$$

Since

$$g(\tau) = \frac{d\ln f(\tau)}{d\tau} = -\frac{\alpha}{1-\tau} + \frac{(1-\alpha)^2}{(1-\alpha)\tau+\alpha},$$

this is an decreasing function with respect to τ. Since $0 < \alpha < \frac{1}{2}$, we have $g(0) > 0$ and $g(1) = -\infty$. This implies that the existence of $0 < \hat{\tau} < 1$ such that $g(\tau) > 0$, for $0 < \tau < \hat{\tau}$ and $g(\tau) < 0$, for $\hat{\tau} < \tau < 1$.

The function f hence is strictly increasing in $[0, \hat{\tau}]$ and strictly decreasing in $[\hat{\tau}, 1]$.

2. Since $\frac{1}{2} \leq \alpha < 1$, for any $0 < \tau < 1$, we have $g(\tau) < g(0) < 0$. The function f is thus strictly decreasing in $[0, 1]$, and $\hat{\tau} = 0$. ■

Proof of proposition 4.2.

1. We consider the first case. We have $\phi(0) = 0$, and $\phi(1) = \delta^{\frac{1-\alpha}{\gamma}} > \max_{\tau,\tau_k} \psi(\tau, \tau_k)$.

   We observe that ϕ is strictly increasing. Since $\phi(1) > \psi(\tau, \tau_k)$, the value of optimal growth ρ lies between 0 and 1. The economy, hence converges to zero.

2. We consider the second case. By Lemma 4.1, the function $\psi(\tau, 1)$ is strictly increasing in $[0, \hat{\tau}]$ and strictly decreasing in $[\hat{\tau}, 1]$.

   Since $\phi(1) < \psi(\hat{\tau}, 1)$, there exists $0 \leq \underline{\tau} < \overline{\tau} < 1$ such that for $\underline{\tau} < \tau < \overline{\tau}$, $\psi(\tau, 1) > \phi(1)$ and for $\tau \notin [\underline{\tau}, \overline{\tau}]$, $\psi(\tau, 1) < \phi(1)$.

   For fixed value of τ lying between $\underline{\tau}$ and $\overline{\tau}$, denote by $\hat{\tau}_k$ the solution to $\phi(1) = \psi(\tau, \tau_k)$. Recall that $\hat{\tau}_k$ depends on τ.

   Since $\psi(\tau, \cdot)$ is strictly increasing with respect to τ_k, for any $0 < \tau_k < \hat{\tau}_k$, we have $\phi(1) > \psi(\tau, \tau_k)$, and for any $\hat{\tau}_k < \tau_k < 1$, we have $\phi(1) < \psi(\tau, \tau_k)$.

   Hence there exists growth if and only if $\hat{\tau}_k < \tau_k$.

   If $\tau \notin [\underline{\tau}, \overline{\tau}]$, then

   $$\phi(1) > \psi(\tau, 1) > \psi(\tau, \tau_k).$$

   This implies the growth rate $\rho < 0$. Hence the economy converges to zero.

   ■

Proof of proposition 4.3.

From Lemma 4.1 we have $\hat{\tau} = 0$, for any τ, τ_k, and get

$$\psi(\tau, \tau_k) < \psi(\tau, 1) \leq \psi(0, 1).$$

1. In the first case, we use the same arguments as in the proof of Proposition 4.2, the economy therefore collapses.

2. In the second case, by using Lemma 4.1, the function $\psi(\cdot, 1)$ is strictly decreasing with respect to τ. Since $\psi(0, 1) > \phi(1)$, we define $\overline{\tau}$ the unique solution to $\phi(1) = \psi(\tau, 1)$.

   For $\overline{\tau} < \tau < 1$, we have $\psi(\tau, \tau_k) \leq \psi(\tau, 1) < \phi(1)$, hence $\rho < 0$ and the economy collapses.

   Suppose that if $0 < \tau < \overline{\tau}$, then $\psi(\tau, 1) > \phi(1)$. We also define $\hat{\tau}_k$ is solution to $\phi(1) = \psi(\tau, \tau_k)$. For $0 < \tau_k < \hat{\tau}_k$, we have $\psi(\tau, \tau_k) < \phi(1)$, and $\rho < 0$.

   In contrast, if $\hat{\tau}_k < \tau_k < 1$, $\psi(\tau, \tau_k) > \phi(1)$, then $\rho > 0$. the growth is possible. ■

Proof of lemma 4.2.

First, by logarithmizing the equation (III.30) that gives

$$\alpha \ln \rho + \frac{1-\alpha}{\gamma} \ln(\rho - 1 + \delta) = \ln A - \alpha \ln n + \frac{1-\alpha}{\gamma} \ln B + \alpha \ln \left[\frac{\beta}{1+\beta}\alpha\right] + \alpha \ln(1-\tau) + (1-\alpha) \ln[(1-\alpha)\tau + \alpha\tau_k], \tag{III.31}$$

where

- $\phi(\rho) = \alpha \ln \rho + \frac{1-\alpha}{\gamma} \ln(\rho - 1 + \delta)$,
- $\mathcal{C}(\tau, \tau_k) \equiv \alpha \ln(1-\tau) + (1-\alpha) \ln[(1-\alpha)\tau + \alpha\tau_k]$.

Second, let us calculate

$$\frac{\partial \mathcal{C}}{\partial \tau} = -\frac{\alpha}{1-\tau} + \frac{(1-\alpha)^2}{(1-\alpha)\tau + \alpha\tau_k},$$
$$\frac{\partial \mathcal{C}}{\partial \tau_k} = \frac{\alpha(1-\alpha)}{(1-\alpha)\tau + \alpha\tau_k}.$$

Differentiating the equation (III.31) with respect to labor income tax rate, τ and capital returns tax rate, τ_k. We therefore get the following derivatives:

$$\begin{aligned}
\frac{\partial \rho}{\partial \tau} &= (\phi^{-1})' \left[\mathcal{C}(\tau, \tau_k)\right] \frac{\partial \mathcal{C}}{\partial \tau} \\
&= \frac{1}{\phi'(\rho)} \frac{\partial \mathcal{C}}{\partial \tau} \\
&= \left[\frac{\alpha}{\rho} + \frac{1-\alpha}{\gamma} \frac{1}{\rho - 1 + \delta}\right]^{-1} \left[-\frac{\alpha}{1-\tau} + \frac{(1-\alpha)^2}{(1-\alpha)\tau + \alpha\tau_k}\right], \qquad \text{(III.32)} \\
\frac{\partial \rho}{\partial \tau_k} &= (\phi^{-1})'[\mathcal{C}(\tau, \tau_k)] \frac{\partial \mathcal{C}}{\partial \tau_k} \\
&= \frac{1}{\phi'(\rho)} \frac{\partial \mathcal{C}}{\partial \tau_k} \\
&= \left[\frac{\alpha}{\rho} + \frac{1-\alpha}{\gamma} \frac{1}{\rho - 1 + \delta}\right]^{-1} \frac{\alpha(1-\alpha)}{(1-\alpha)\tau + \alpha\tau_k}. \qquad \text{(III.33)}
\end{aligned}$$

These imply the above elasticites of the growth rate to taxes. ■

Proof of proposition 4.4.

Obviously, $\frac{\partial \rho}{\partial \tau} = 0$ iff

$$\begin{aligned}
\frac{\alpha}{1-\tau} &= \frac{(1-\alpha)^2}{(1-\alpha)\tau + \alpha\tau_k} \\
\alpha(1-\alpha)\tau + \alpha^2\tau_k &= (1-\alpha)^2(1-\tau) \\
\alpha(1-\alpha) + \alpha^2\tau_k &= (1-\alpha)^2(1-\tau) + \alpha(1-\alpha) - \alpha(1-\alpha)\tau \\
&= (1-\alpha)(1-\tau) \\
1 - \tau &= \alpha + \frac{\alpha^2}{1-\alpha}\tau_k.
\end{aligned}$$

When τ_k is given, the optimal tax on labor income is

$$\tau^*(\tau_k) = 1 - \alpha - \frac{\alpha^2}{1-\alpha}\tau_k. \tag{III.34}$$

Since $(\tau^*)'(\tau_k) = -\frac{\alpha^2}{1-\alpha} < 0$. Therefore, $\frac{\partial \rho}{\partial \tau} > 0$ iff $\tau < \tau^*(\tau_k)$. The rate of growth, ρ hence attains its maximum value at τ^*.

Moreover, the positivity of τ^* requires $1 - \alpha - \frac{\alpha^2}{1-\alpha}\tau_k > 0$, which is always ensured by the following assumption: $\frac{1-\alpha}{\alpha} > 1$. ■

Proof of proposition 4.5.

According to the equation (III.23), the elasticity of growth rate with respect to tax on capital, evaluated at $(\tau_k, \tau^*(\tau_k))$ is given by:

$$\frac{\partial \rho}{\rho} / \frac{\partial \tau_k}{\tau_k} = \left[\alpha + \frac{1-\alpha}{\gamma}\frac{\rho}{\rho - 1 + \delta}\right]^{-1} \frac{\alpha}{\frac{1-\alpha}{\tau_k} + \alpha}. \tag{III.35}$$

Additionally, we get the elasticity of $\tau^*(\tau_k)$ with respect to τ_k:

$$\frac{\partial \tau^*}{\tau^*} / \frac{\partial \tau_k}{\tau_k} = -\frac{\alpha^2 \tau_k}{(1-\alpha)^2 - \alpha^2 \tau_k}.$$

With reference to the equation (III.9), the ratio taxes to GDP is define as follows:

$$\xi \equiv g_t / y_t = \alpha \tau_k + (1-\alpha)\tau. \tag{III.36}$$

Furthermore, evaluated at $\tau^*(\tau_k), \tau_k$, the taxes to GDP ratio is equal to

$$\xi = (1-\alpha)[(1-\alpha) + \alpha \tau_k].$$

Then the elasticity of the ratio taxes to GDP with respect to tax on capital that is computed as follows:

$$\begin{aligned}\frac{\Delta\xi/\xi}{\Delta\tau_k/\tau_k} &= \frac{\alpha\tau_k}{1-\alpha+\alpha\tau_k}\\ &= \left[\alpha+\frac{1-\alpha}{\gamma}\frac{\rho}{\rho-1+\delta}\right]\frac{\partial\rho/\rho}{\partial\tau_k/\tau_k}. \end{aligned} \tag{III.37}$$

Combining (III.23) and (III.37). Hence, the elasticity of growth rate with respect to the ratio of taxes to GDP is given by

$$\frac{\Delta\rho/\rho}{\Delta\xi/\xi} = \left[\alpha+\frac{1-\alpha}{\gamma}\frac{\rho}{\rho-1+\delta}\right]^{-1}. \tag{III.38}$$

Obviously, $\frac{\Delta\rho/\rho}{\Delta\xi/\xi} < 1$. ■

Proof of proposition 5.1.

For each $x > 0$. From the equation (III.25), we define $w(x)$ as the following function:

$$w(x) = \max_{c,d}\left(\ln c + \frac{\beta}{\zeta n}\ln d\right), \tag{III.39}$$

$$\text{s.t } c + \frac{d}{n} \leq x. \tag{III.40}$$

Taking FOCs by (III.39) under the constraint (III.40), we then receive:

$$\begin{aligned} c &= \frac{x}{1+\frac{\beta}{n\zeta}},\\ d &= \frac{\beta}{\zeta}\frac{x}{1+\frac{\beta}{n\zeta}}. \end{aligned}$$

From the later caculus, we can re-write the problem (III.26), as follows:

$$\begin{aligned}\max W &= \sum_{t=0}^{\infty} \zeta^t w(x_t) \\ &= \sum_{t=0}^{\infty} \zeta^t \left(\ln\left(\frac{x_t}{1+\frac{\beta}{n\zeta}} \right) + \frac{\beta}{\zeta} \ln\left(\frac{\beta}{\zeta} \frac{x_t}{1+\frac{\beta}{n\zeta}} \right) \right) \\ &= \left(1+\frac{\beta}{\zeta}\right) \sum_{t=0}^{\infty} \zeta^t \ln(x_t) - \sum_{t=0}^{\infty} \zeta^t \left[\left(1+\frac{\beta}{n\zeta}\right) \ln\left(1+\frac{\beta}{n\zeta}\right) - \frac{\beta}{\zeta} \ln \frac{\beta}{\zeta} \right].\end{aligned}$$

The problem above becomes

$$\max \sum_{t=0}^{\infty} \zeta^t \ln(x_t) \tag{III.41}$$

$$\text{s.t } x_t + nk_{t+1} + e_t \leq Ak_t^{\alpha} h_t^{1-\alpha}, \tag{III.42}$$

$$h_{t+1} \leq Be_t^{\gamma} h_t^{1-\gamma} + (1-\delta)h_t. \tag{III.43}$$

Next, considering the the following Lagrange

$$L = \sum_{t=0}^{\infty} \zeta^t \ln x_t - \sum_{t=0}^{\infty} \lambda_t \left(x_t + nk_{t+1} + e_t - Ak_t^{\alpha} h_t^{1-\alpha} \right) - \sum_{t=0}^{\infty} \mu_t \left(h_{t+1} - Be_t^{\gamma} h_t^{1-\gamma} - (1-\delta)h_t \right).$$

Then, solving this above system, we therefore obtain

$$\frac{\partial L}{\partial x_t} = \frac{\zeta^t}{x_t} - \lambda_t, \tag{III.44}$$

$$\frac{\partial L}{\partial k_{t+1}} = -n\lambda_t + \lambda_{t+1} \alpha A \left(\frac{h_{t+1}}{k_{t+1}} \right)^{1-\alpha}, \tag{III.45}$$

$$\frac{\partial L}{\partial h_{t+1}} = \lambda_{t+1}(1-\alpha)A \left(\frac{k_{t+1}}{h_{t+1}} \right)^{\alpha} + \mu_{t+1} \left((1-\gamma)B \left(\frac{e_{t+1}}{h_{t+1}} \right)^{\gamma} + (1-\delta) \right) - \mu_t, \tag{III.46}$$

$$\frac{\partial L}{\partial e_t} = -\lambda_t + \mu_t \gamma B \left(\frac{h_t}{e_t} \right)^{1-\gamma}. \tag{III.47}$$

We continue to analyze the problem at BGP with the BGP growth rate of the economy, ρ.

The equation (III.44) gives

$$\lambda_t = \frac{\zeta^t}{x_t},$$

which implies

$$\lambda_t = \frac{\zeta^t}{x_0 \rho^t}. \tag{III.48}$$

From the equations (III.44) and (III.45), we get

$$\alpha A \left(\frac{h_{t+1}}{k_{t+1}} \right)^{1-\alpha} = n \frac{\rho}{\zeta},$$

which is equivalent to

$$\frac{k_0}{h_0} = \left(\frac{\alpha A \zeta}{n \rho} \right)^{\frac{1}{1-\alpha}}. \tag{III.49}$$

According to the equation (III.47), we have

$$\lambda_t = \mu_t \gamma B \left(\frac{h_t}{e_t} \right)^{1-\gamma},$$

which refers

$$\mu_t = \lambda_t \frac{1}{\gamma B} \left(\frac{e_0}{h_0} \right)^{1-\gamma}. \tag{III.50}$$

Combining (III.46), (III.48) and (III.50), the program receives

$$\begin{aligned} \frac{\zeta^t}{x_0 \rho^t} \times \frac{1}{\gamma B} \left(\frac{e_0}{h_0} \right)^{1-\gamma} &= \frac{\zeta^{t+1}}{x_0 \rho^{t+1}} \times (1-\alpha) A \left(\frac{k_0}{h_0} \right)^{\alpha} + \\ &+ \frac{\zeta^{t+1}}{x_0 \rho^{t+1}} \times \frac{1}{\gamma B} \left(\frac{e_0}{h_0} \right)^{1-\gamma} \times \left[(1-\gamma) B \left(\frac{e_0}{h_0} \right)^{\gamma} + (1-\delta) \right]. \end{aligned}$$

This implies

$$\frac{\rho}{\zeta} = (1-\alpha)\gamma AB\left(\frac{k_0}{h_0}\right)^{\alpha} \times \left(\frac{h_0}{e_0}\right)^{1-\gamma} + (1-\gamma)B \times \left(\frac{e_0}{h_0}\right)^{\gamma} + (1-\delta). \quad \text{(III.51)}$$

Subtituting (III.49) into (III.51), we get

$$\frac{\rho}{\zeta} = (1-\alpha)\gamma AB\left(\frac{\alpha A}{n}\right)^{\frac{\alpha}{1-\alpha}}\left(\frac{\zeta}{\rho}\right)^{\frac{\alpha}{1-\alpha}} \times \left(\frac{h_0}{e_0}\right)^{1-\gamma} + (1-\gamma)B \times \left(\frac{e_0}{h_0}\right)^{\gamma} + (1-\delta). \quad \text{(III.52)}$$

In addition, from equation of $h_{t+1} = Be_t^{\gamma}h_t^{1-\gamma} + (1-\delta)h_t$, we have

$$\rho = B\left(\frac{e_0}{h_0}\right)^{\gamma} + (1-\delta),$$

it turns out to be equivalent to

$$\frac{e_0}{h_0} = \left(\frac{\rho - 1 + \delta}{B}\right)^{\frac{1}{\gamma}}. \quad \text{(III.53)}$$

Replacing (III.53) into (III.52) with the definition of $\hat{\phi}(\rho) = \frac{\rho}{\zeta}$, yields

$$\begin{aligned}\hat{\phi}(\rho) &= \frac{(1-\alpha)\gamma AB\left(\frac{\alpha A}{n}\right)^{\frac{\alpha}{1-\alpha}}\left(\frac{\zeta}{\rho}\right)^{\frac{\alpha}{1-\alpha}}}{\left(\frac{\rho-1+\delta}{B}\right)^{\frac{1-\gamma}{\gamma}}} + (1-\gamma)B\left(\frac{\rho-1+\delta}{B}\right) + (1-\delta)\\ &= \frac{(1-\alpha)A^{\frac{1}{1-\alpha}}B^{\frac{1}{\gamma}}\left(\frac{\alpha}{n}\right)^{\frac{\alpha}{1-\alpha}}\left(\frac{\zeta}{\rho}\right)^{\frac{\alpha}{1-\alpha}}}{(\rho-1+\delta)^{\frac{1-\gamma}{\gamma}}} + (1-\gamma)\rho + \gamma(1-\delta).\end{aligned}$$

Let us define

$$\begin{aligned}\phi(\rho) &= \hat{\phi}(\rho) - (1-\gamma)\rho - \gamma(1-\delta)\\ &= \frac{(1-\alpha)A^{\frac{1}{1-\alpha}}B^{\frac{1}{\gamma}}\left(\frac{\alpha}{n}\right)^{\frac{\alpha}{1-\alpha}}\left(\frac{\zeta}{\rho}\right)^{\frac{\alpha}{1-\alpha}}}{(\rho-1+\delta)^{\frac{1-\gamma}{\gamma}}}.\end{aligned}$$

The BGP growth rate is thus solution to

$$\begin{aligned}\phi(\rho) &= \frac{\rho}{\zeta} - (1-\gamma)\rho - \gamma(1-\delta)\\ &= \left(\frac{1}{\zeta} - (1-\gamma)\right)\rho - \gamma(1-\delta)\\ &= \psi(\rho),\end{aligned}$$

where $\psi(\rho) = \frac{(1-\alpha)A^{\frac{1}{1-\alpha}}B^{\frac{1}{\gamma}}\left(\frac{\alpha}{n}\right)^{\frac{\alpha}{1-\alpha}}\left(\frac{\zeta}{\rho}\right)^{\frac{\alpha}{1-\alpha}}}{(\rho-1+\delta)^{\frac{1-\gamma}{\gamma}}}$.

1. Since ϕ is strictly increasing and ψ is strictly decreasing with respect to ρ. We therefore have

$$\lim_{\rho\to(1-\delta)^+}\phi(\rho) = +\infty,$$
$$\lim_{\rho\to+\infty}\phi(\rho) = 0.$$

   Hence there exists $\rho^* > 1-\delta$ such that $\phi(\rho^*) = \psi(\rho^*)$.

2. The BGP growth rate, ρ^* is bigger than 1 if and only if

$$\psi(1) > \phi(1),$$

   which is equivalent to

$$\frac{(1-\alpha)A^{\frac{1}{1-\alpha}}B^{\frac{1}{\gamma}}\left(\frac{\alpha}{n}\right)^{\frac{\alpha}{1-\alpha}}\zeta^{\frac{\alpha}{1-\alpha}}}{\delta^{\frac{1-\gamma}{\gamma}}} > \frac{1}{\zeta} - (1-\gamma) - \gamma(1-\delta).$$

   This also implies that

$$(1-\alpha)A^{\frac{1}{1-\alpha}}B^{\frac{1}{\gamma}}\left(\frac{\alpha}{n}\right)^{\frac{\alpha}{1-\alpha}}\zeta^{\frac{\alpha}{1-\alpha}} > \delta^{\frac{1-\gamma}{\gamma}}\left(\frac{1}{\zeta} - (1-\gamma) - \gamma(1-\delta)\right).$$

3. This property is obvious, since ϕ is increasing with respect of parameters A and B and decreasing with respect to n. ■

Chapter IV

Optimal Growth with Non-Concave Technology: Application to Human Capital Model [1]

[1]This paper is co-written with Cuong LE-VAN and Thai HA-HUY

1 Introduction

A large number of economic theories are based on the property of decreasing marginal utility and marginal productivity. This property leads to an extensive convex structure in different domains of economic sciences. Particularly, in dynamic programming, a wide class of issues need have to be satisfied with a strong convex structure. This approach is also used in a well-developed branch of applied mathematics, such as an excellent treatment can be found in Stokey (1989).

Additionally, the convexity properties are commonly applied in the macroeconomic area. In the first line, Ramsey (1928) bases on the convexity technology and examines that the optimal intertemporal growth tends to a unique steady state (see Cass, 1965; Koopmans, 1965). While Benveniste and Scheinkman (1979) exhibit the initial level of capital stock is a concave function. In the second line, Becker (1965), Ben-Porath (1967), and Mincer (1974) focus on researching the incentive investment in skills, including schooling (pre-labor investment) and training (on-the-job investment). In addition to the externalities of human capital in the process of economic growth that are emphasized by Lucas (1988, 2015) and Azariadis and Drazen (1990). In the third stream, the role of human capital is to introduce into productivity contribution, and the capacity of workers to cope with changes, disruptions and new technologies. In this sense, Benhabib and Spiegel (1994) and Foster and Rosenzweig (1995) suggest that the important role of human capital is to facilitate technology adoption.

Nonetheless, imposing certain convexity assumptions could be restrictive. This calls for a need to advance a theoretical framework that tackles the class of issues where the restrictive assumptions of convexity properties are violated. Our study is not the first attempt in this direction.

As a matter of fact that non-concave technology displays a key role in the optimal growth problems. Some studies indicate that the function of planner is concave utility, while the function of producers and constraints are non-concave function. In addition, non-concave technology is also used to study in one-sector and multi-sector models where the optimal programs are monotonic (see e.g., Dechert & Nishimura, 1983; Amir, 1996).

Non-concave technology, according to Dechert and Nishimura (1983), consider a model with convex-concave, $S-$shape production function. They prove the monotonicity of an optimal capital stock and give a proof about the existence of a poverty trap. They also examine that the economy collapses to zero if the initial level of capital stock is below a certain threshold, the economy otherwise converges to a positive steady state. This problem is also revisited by Le Kama et al. (2014), who consider a convex-concave production function and show explicitly determinant of a poverty trap. An additional example can be found by Hung et al. (2009), who study the production function of an optimal growth model is an aggregation of two separate concave production technologies. They show the existence of two steady states when the discount rate is not too high or too low. Their study also examines the optimal path can converge to the lower or the higher one; Otherwise, there exists unique a steady state which is low or high value if the discount rate is high or low value.

Non-concave programming for a continuous time version is addressed by Romer (1983, 2011). Whereas this technique in the discrete-time version is examined by Dimaria et al. (2002). Both of them work with the law of diminishing returns. However, the production function of knowledge exhibits increasing return for ensuring sustainable growth. Their model supposes that technological change is endogenous and depend on capital. In another aspect, the concavity of utility function is accepted in the almost models about economic dynamics, the concavity of the production function does not have this agreement (as Clark, 1971; Majumdar & Mitra, 1983).

Although the literature on the non-concave technology has been shown by some economists and researchers, there has been a lack of research on macroeconomics area, especially on the field of human capital. Therefore, this chapter aims to propose a general mathematical model with non-concave technology and an application to human capital model.

Our study is related to the work of Kamihigashi and Roy (2007). Their study is neither convexity nor continuity properties of the production function. To overcome the lack of smoothness properties, Kamihigashi and Roy (2007) use a notion of *discounted net return on investment*, which appeared in the analysis of Majumdar and Nermuth (1982) and Dechert and Nishimura (1983). They prove that the optimal paths increase in the future with the net return on investment. Their work also provides in-depth insights into economic dynamics and characterizes conditions for poverty trap or sustainable growth.

Apart from the study of Kamihigashi and Roy (2007) deals only with the classical situation of optimal growth. It means that their outcome is split between consumption and capital investment. This may lead to restrictions in human capital research. This chapter, therefore, works with an indirect utility function and fill this gap by considering an optimal growth model with human capital.

This study has two contributions. On the one hand, it builds a general mathematical model about non-concave technology. On the other hand, it shows an application to an optimal growth model in which education is the prime source of human capital formation. In the application section, the study pays attention to the two roles of human capital: as a prime engine of endogenous growth and as a force behind new technology.

The rest of this chapter is organized as follows. Section 2 shows a general mathematical model of non-concave technology. Section 3 studies an application in infinite-horizon optimal-growth model with human capital accumulation. Conclusion is Section 4 .The proofs and computations are relegated to Appendix.

2 Fundamentals-mathematical preparations

2.1 Value function and Bellmann equation

The technology of the economy is characterized by $G : \mathbb{R}_+ \to \mathbb{R}_+$, a strictly increasing function. Given capital stock $x \geq 0$, the interval $\Gamma(x) = [0, G(x)]$ denotes the set of possible capital stock x' invested for the next period. When x, x' are determined, the instantaneous utility (or felicity) value is $V(x, x')$, in which V is an increasing function with respect to the first argument and decreasing function with respect to the second one.

For each initial value $x_0 \geq 0$, denote by $\pi(x_0)$, the set of feasible sequences $\chi = \{x_t\}_{t=0}^{\infty}$: for any $t \geq 0$, $x_{t+1} \in \Gamma(x_t)$.

Define the value function:

$$v(x_0) = \sup_{\chi \in \Pi(x_0)} \sum_{t=0}^{\infty} \beta^t V(x_t, x_{t+1}).$$

The following assumption supposes that the productivity is not very high when the level of capital stock is low.

Assumption G1

1. *The function G is strictly increasing, differentiable with $G'(0) < 1$,*
2. *For x big enough, $x \in int\Gamma(x)$, i.e., $0 < x < G(x)$.*

This assumption is not satisfied in standard model with concave technology function, where the productivity is very high for low level of capital and $x \notin \Gamma(x)$ for big values of x. According to Dechert and Nishimura (1983), for x big, the possible production is bigger than x. This opens room for economic growth.

In this section, we suppose that the value function V is bounded from below. Obviously, the unbounded from below case must be considered with the importance of the logarithmic functions.

Assumption V1 *The function V is bounded from below: $V(0,0) = 0$.*

The properties of technical level can allows the possibility for sustainable growth. We assume that the rate of growth cannot overcome effect of discount factor, otherwise the value function could take infinity values.

Assumption V2 *For fixed $x_0 > 0$, for any $\epsilon > 0$, there exists T big enough such that for any $\chi \in \Pi(x_0)$ we have*

$$\sum_{t=T}^{\infty} \beta^t V(x_t, x_{t+1}) < \epsilon.$$

Under assumption **V2**, for any $x_0 \geq 0$, we have $v(x_0) < +\infty$ and the function v is continuous on $[0, +\infty)$.

Proposition 2.1

1. *For any $x_0 \geq 0$, $v(x_0) < +\infty$.*
2. *For any $x_0 \geq 0$, there exists $\chi^* \in \Pi(x_0)$ such that*

$$v(x_0) = \sum_{t=0}^{\infty} \beta^t V(x_t^*, x_{t+1}^*).$$

3. *The value function v is continuous.*
4. *The value function v satisfies the functional equation: for any $x_0 > 0$,*

$$v(x_0) = \sup_{0 \leq x_1 \leq G(x_0)} \left[V(x_0, x_1) + \beta v(x_1) \right].$$

5. *For any $x_0 > 0$, there exists $x_1^* \in \Gamma(x_0)$ such that*

$$v(x_0) = V(x_0, x_1) + \beta v(x_1).$$

Proof: Appendix ■

2.2 The monotonicity and continuity properties of optimal policy functions

The compactness of Γ and the continuity of v ensure the maximum problem in part (5) of Proposition 2.1. It means that optimal solution exists. By the lack of concavity of V and G, the solution can not be unique. We will consider the optimal policy correspondence and focus on the maximal and the minimal values of this correspondence. In order to analysis the properties, we must add some supplementary structure. The assumption V3 is standard property in optimal growth theory. If we increase the capital stock of future, we diminish the instantaneous utility. The assumption V4 supposes that the utility function satisfies *strict increasing differences* property (see Amir, 1996).

Assumption V3

1. *The function V is strictly increasing in the first variable and strictly decreasing in the second one.*

2. Inada condition:
$$\lim_{y \to G(x)} \frac{\partial V(x,y)}{\partial y} = -\infty.$$

Assumption V4 *The utility function V is twice differentiable and strictly supermodular: for any x, y we have*

$$V_{12}(x,y) = \frac{\partial^2 V(x,y)}{\partial x \partial y} > 0.$$

Since the functions V and v are not concave, the arg max of problem

$$\sup_{0 \leq x_1 \leq G(x_0)} \big[V(x_0,x_1) + \beta v(x_1)\big]$$

could have multitude number of solutions.

Let define

$$\phi(x_0) = \arg\max_{x_1 \in \Gamma(x_0)} \big[V(x_0, x_1) + \beta v(x_1)\big].$$

Under the Super-modularity property, the optimal policy correspondence satisfies some "increasing property-like", as in Proposition 2.2 below.

Proposition 2.2 *Assume G1, V1, V2, V3, V4, for all $x_0 < x_0'$, and $x_1 \in \phi(x_0)$, $x_1' \in \phi(x_0')$, we have $x_1 < x_1'$.*

Proof: Appendix ■

Base on the continuity properties of V and v, the set $\phi(x_0)$ is closed. The minimum and maximum of $\phi(x_0)$ are $\underline{\varphi}(x_0)$ and $\overline{\varphi}(x_0)$, respectively.

We now describe some properties of two functions $\underline{\varphi}$ and $\overline{\varphi}$ by using Lemma 2.1, as follows.

Lemma 2.1 *Assume G1, V1, V2, V3, V4.*

1. *The functions $\underline{\varphi}$ and $\overline{\varphi}$ are strictly increasing .*

2. *The function $\overline{\varphi}$ is upper semi-continuous and continuous from the right,*

$$\lim_{x \to x_0^+} \overline{\varphi}(x) = \overline{\varphi}(x_0).$$

3. *The function $\underline{\varphi}$ is lower semi-continuous and continuous from the left,*

$$\lim_{x \to x_0^-} \underline{\varphi}(x) = \underline{\varphi}(x_0).$$

Proof: Appendix ■

2.3 Poverty trap, middle income trap and sustainable growth

The existence of poverty trap is obvious. Because the productivity is low when the capital stock is low.

Proposition 2.3 *Assume G1. The poverty trap exists.*

Proof: Appendix ■

Next, we will study the possibility of existence of middle income trap and necessary conditions for the existence of sustainable growth. We present here a preparation lemma, which is similar with the result in Hung et al. (2009).

Lemma 2.2 *Assume G1, V1, V2, V3, V4. The following statements are equivalents:*

1. *There exists $\tilde{x}_0$ such that there exists an optimal path in $\Pi(\tilde{x}_0)$ satisfying $\lim_{t\to\infty} x_t^* = +\infty$.*
2. *There exists $\tilde{x}_0$ such that for any $x \geq \tilde{x}_0$ we have $x_0 < \underline{\varphi}(x_0)$.*

Proof: Appendix ■

This lemma states that the existence of an optimal path converging to inifnity is equivalent to the existence of threshold value beyond every optimal path converges to infinity.

Even if the case of the productivity function G is high when capital stock converges to infinity, a low value of β could trap the optimal path into a bounded set. In the next sections, we will discuss about conditions under which, there exists an optimal sequence which converges to infinity. For instance, we call this is *sustainable growth condition.*

Let us define:

$$\underset{\sim}{x} = \sup\{\tilde{x}_0 \geq 0 \text{ such that } \overline{\varphi}(x_0) < x_0 \text{ for every } x_0 \leq \tilde{x}_0\},$$
$$\tilde{x} = \inf\{\tilde{x}_0 \geq 0 \text{ such that } \underline{\varphi}(x_0) > x_0 \text{ for every } x_0 \geq \tilde{x}_0\}.$$

Before presenting Proposition 2.4, we provide a preparation lemma. This is similar to the result of Hung et al. (2009).

Lemma 2.3 *Suppose that $x_1 \in \phi(x_0)$ for some $x_0 \geq 0$. Then $\underline{\varphi}(x_1) = \overline{\varphi}(x_1)$.*

Proof: Appendix ■

Proposition 2.4

Assume G1, V1, V2, V3, V4 and sustainable growth condition, *we obtain*

1. *$\underset{\sim}{x}$ and $\tilde{x}$ exist.*

2. *If $x_0 < \underset{\sim}{x}$, then any optimal path from x_0 converges to zero.*

3. *If $x_0 > \tilde{x}$, then any optimal path from x_0 converges to infinity.*

4. *There exist $\underline{x}_0$ and $\overline{x}_0$ such that*

   - $\underset{\sim}{x} \leq \underline{x}_0 \leq \overline{x}_0 \leq \tilde{x}$,

   - *For any $\underline{x}_0 \leq x_0 \leq \overline{x}_0$, the optimal path beginning from x_0 is bounded away from zero and infinity.*

5. *There exists x^* such that $x^* = \underline{\varphi}(x^*) = \overline{\varphi}(x^*)$. Furthermore, $\underline{x} \leq x^* \leq \overline{x}$.*

Proof: Appendix ■

Now, we will discuss a sufficient condition such that the *sustainable growth condition* is satisfied. Therefore, we need to make the following lemma:

Assumption V5 *1. For any x, y, the functions $V(\cdot, y)$ and $V(x, \cdot)$ are concave.*

2. For x big enough we have $V_2(x,x) + \beta V_1(x,0) > 0$.

In the first condition, one argument is fixed, the utility level in function of the other satisfies the decreasing marginal utility property. The second condition is to ensure a sustainable growth.

Comment. In standard growth model where technological function is concave, the function $V(x,x)$ has a single-peak form. There usually exists golden rule x^G which maximizes $V(x,x)$. In this chapter, we find conditions under which the optimal paths can converge to infinity. Naturally, a good candidate is a function V with no golden rule, by example V satisfying $V(x,x)$ is increasing, or $V_1(x,x) + V_2(x,x) > 0$. This condition ensures that there are at most a finite number of steady states. We suppose that the utility function satisfies a condition which is stronger: $V_2(x,x) + \beta V_1(x,0) > 0$ for large values of x.

For instance, the assumption V5 is only a technical condition. We hope that we can find a good economic meaning for them. We will see that this condition is satisfied in Section 3, which considers a problem with human capital.

Proposition 2.5 *Assume G1, V1, V2, V3, V4, V5. There exists $\tilde{x}_0 \geq 0$ such that for any $x_0 \geq \tilde{x}_0$, every optimal path from x_0 converges to infinity.*

Proof: Appendix ■

2.4 Sensitivity analysis

Proposition 2.6 *Assume G1, V1, V2, V3, V4 and* sustainable growth condition. *For each $\beta \in (0,1)$, define $\underset{\sim}{x}_0(\beta)$, $\tilde{x}_0(\beta)$ like in proposition 2.4. Then $\underset{\sim}{x}_0(\beta)$ and $\tilde{x}_0(\beta)$ are decreasing functions with respect to β.*

Proof: Appendix ■

3 Applications in infinite horizon optimal growth with human capital

3.1 Fundamentals

In order to understand the non-concave technology, we begin with a case in the following Lemma:

Lemma 3.1 *The poverty trap and middle-income trap from convex-concave production function which could be as a result of a fixed cost or corruption.*

Proof: Appendix ■

Following Lemma 3.1 above, we introduce the non-concave technology into a growth model, in which human capital investment may bring on a sustainable economic growth. Our model supposes that there is no physical capital. In addition to the agent, or the social planer divides the production into consumption and investment in human capital, in order to maximize the intertemporal sum of utility:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s.c } c_t + s_{t+1} \leq f(h_t N_t),$$

$$h_0 > 0 \text{ given.}$$

c_t, s_t are respectively the consumption and the savings at period t. N_t denotes the exogenous number of workers at period t, h_t is their human capital at the same period. For simplicity, we suppose that $N_t = 1$ for every t. We suppose the output is obtained by using only the effective labor through a production function f which is concave, increasing, continuous, as usual. The utility function is strictly increasing, concave, continuous.

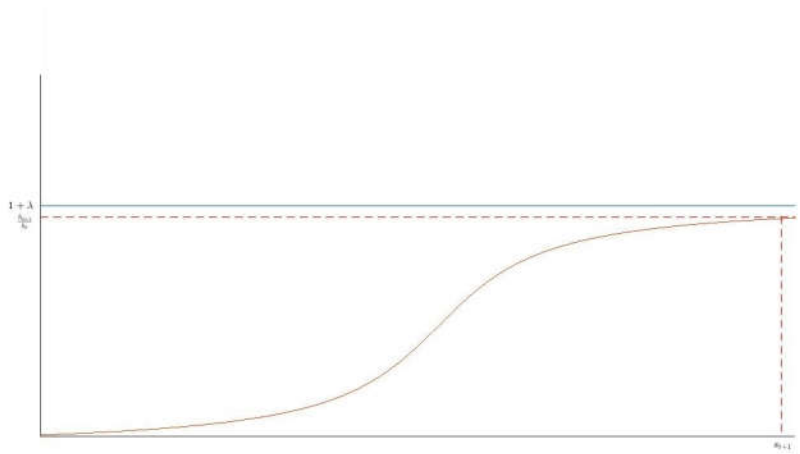
The rate of growth of the human capital depends on the investment s_{t+1},

$$h_{t+1} = h_t \phi(s_{t+1}),$$

where ϕ is convex-concave and diffrentiable function. The parameter δ represents the depreciation rate of human capital. The maximal rate of growth, λ which human capital can attaints.

Furthermore, the evolution of human capital accumulation can be expressed by the following:

Figure 4.1. The convex-concave function of human capital formation.



Let us denote $\psi = \phi^{-1}$. Since the function ψ is increasing then $\psi(1+\lambda) = +\infty$. At the optimum, $c_t = f(h_t) - \psi\left(\frac{h_{t+1}}{h_t}\right)$. We can re-write the problem as follow

$$v(h_0) = \max \sum_{t=0}^{\infty} \beta^t u\left(f(h_t) - \psi\left(\frac{h_{t+1}}{h_t}\right)\right)$$

$$0 \le h_{t+1} \le h_t \phi(f(h_t)),$$

$$h_0 > 0 \text{ given.}$$

Let $\Gamma(x) = \{(x,y) \in \mathbb{R}^2_+ : 0 \leq y \leq G(x)\}$, with $G(x) = x\phi(f(x))$. Define

$$V(x,y) = u\left(f(x) - \psi\left(\frac{y}{x}\right)\right).$$

For each $h_0 \geq 0$, define $\Pi(h_0)$ the set of sequence: $\{h_t\}_{t=0}^{\infty}$ such that

$$0 \leq h_{t+1} \leq G(h_t).$$

3.2 Dynamical analysis

In this section, we begin with the following assumption that resumes the most important properties of the utility function, production function and discount factor.

Assumption H1

1. *The utiity function u is stirctly increasing, strictly concave and bounded from below: $u(0) = 0$. Assume that $u'(0) = +\infty$.*

2. *The human capital production function satisfies:*

   - $\phi(0) = 1 - \delta$,
   - $\phi(+\infty) = 1 + \lambda$,
   - $\lim_{x \to 0} x\phi'(f(x)) = 0$.

3. *For any $h_0 \geq 0$, for any $\epsilon > 0$, there exists $T_0 \geq 0$ such that for any $T \geq T_0$, and $\{h_t\}_{t=0}^{\infty} \in \Pi(h_0)$, we have*

$$\sum_{t=T}^{\infty} \beta^t V(h_t, h_{t+1}) < \epsilon.$$

Next, in order to establish the properties of G and V, we need to have two additional assumptions. Under these assumptions, there exists not only optimal converging to ininifty, but aso the growth rate converging to λ (the maximal rate of growth).

Assumption H2 *Suppose that for every $\rho > 0$ we have*

$$\lim_{x\to\infty} xf'(\rho x) = +\infty.$$

Assumption H3 *Suppose that for any $\rho \geq 1$, $M, N > 0$ we have*

$$\liminf_{x\to\infty} \frac{u'(f(\rho x) - M)}{u'(f(x) - N)} > 0.$$

The assumption H2 is standard form in the literature framework. The following one, the assumption H3 says that when $\rho > 1$, $f(\rho x)$ increase to infinity faster than $f(x)$, but the speed is not "too fast". We can verify easily that these assumptions can be satisfied for the forms: $f(x) = Ax^{\alpha}$, and $u(x) = \ln x$ or $u(x) = x^{\gamma}$ with $0 < \gamma < 1$.

H1, H2 and H3 allow to establish G and V. Moreover, these assumptions permit to prove the convergence of growth rate to the maximal value, λ.

Lemma 3.2 *Assume H1, H2 and H3, we have*

1. *G satisfies assumption G1.*
2. *V satisfies assumptions V1, V2, V3, V4, V5.*

Proof: Appendix ■

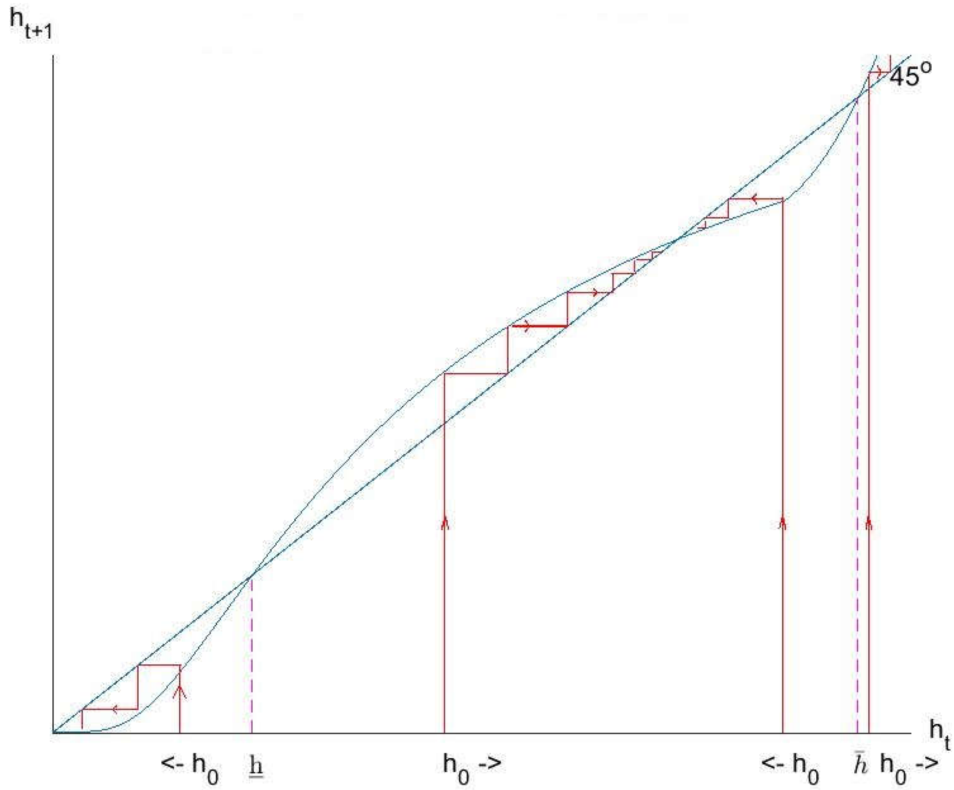
Proposition 3.1

Assume assumptions H1, H2 and H3. There exists poverty trap, middle income trap, and threshold for convergence to infinity of optimal path, as in Proposition 2.4.

Proof: Appendix ■

We present the following figure to illustrate the above Proposition 3.1.

Figure 4.2. The poverty trap and middle-income trap.



Proposition 3.2

Assume assumptions H1, H2 and H3. Suppose that optimal path $\{h_t^*\}_{t=0}^{\infty}$ from x_0 converges to infinity, it also satisfies

$$\lim_{t\to\infty} \frac{h_{t+1}^*}{h_t^*} = 1 + \lambda.$$

Proof: Appendix ■

4 Conclusion

This chapter considers an optimal growth model with non-concave technology. We first build a general mathematical model in order to create fundamentals via applying in a growth model with the accumulation of human capital. In this application section, we prove the existence of poverty trap (the optimal path of human capital converges to zero) and middle-income trap (the optimal sequence of human capital bounded away from zero and infinity). In addition, we point out the conditions under which sustainable growth is possible. Under the same assumption, the optimal growth rate also tends to the maximal value.

5 Appendix

Proof of Proposition 2.1

The proof of this proposition requires only standard arguments in dynamic programming literature. ■

Proof of Proposition 2.2

We use the characteristics of Super-modularity of V, which is in line with the study of Dechert and Nishimura (1983).

Suppose the contrary, there exist $x_0 < x'_0$, $x_1 \in \phi(x_0)$, $x'_1 \in \phi(x'_0)$ and $x_1 \geq x'_1$. Hence we have $x'_1 \in \Gamma(x_0)$. This is in acordance with the definition of Amir (1996):

$$(x'_0, x_1) = (x_0, x_1) \vee (x'_0, x'_1),$$
$$(x_0, x'_1) = (x_0, x_1) \wedge (x'_0, x'_1).$$

We also have

$$V(x_0, x_1) + \beta v(x_1) \geq V(x_0, x_1') + \beta v(x_1'),$$
$$V(x_0', x_1') + \beta v(x_1') \geq V(x_0', x_1) + \beta v(x_1).$$

From these two equations above we obtain

$$V(x_0, x_1) + V(x_0', x_1') \geq V(x_0, x_1') + V(x_0', x_1),$$

which is contradictory to the strict Super-modularity assumption. ■

Proof of Lemma 2.1

1. The increasing property of functions $\underline{\varphi}$ and $\overline{\varphi}$ are direct consequence of the Super-modularity of V (Amir (1996)).

2. First, we prove that $\overline{\varphi}$ is upper semi-continuous. Take any sequence $\{x_0^n\}_{n=0}^{\infty}$ converging to x_0 satisfies $x_0^n \geq x_0$, for any n. For the sake of simplicity, define $x_1^n = \overline{\varphi}(x_0^n)$. By the increasing property of $\underline{\varphi}$, the sequence $\{x_1^n\}_n$ is bounded from above and below. Without loss of generality, suppose that $\lim_{n\to\infty} x_1^n = x_1^*$. By the continuity of G, we have $x_1^* \in \Gamma(x)$. Take any $0 \leq x_1 < G(x_0)$, for n sufficiently big, we have

$$\begin{aligned} V(x_0, x_1) + \beta v(x_1) &\leq V(x_0^n, x_1) + \beta v(x_1) \\ &\leq V(x_0^n, x_1^n) + \beta v(x_1^n). \end{aligned}$$

By the continuity of v, let n converges to infinity, we get for any $0 \leq x_1 < G(x_0)$,

$$V(x_0, x_1) + \beta v(x_1) \leq V(x_0, x_1^*) + \beta v(x_1^*).$$

By the continuity of V and v, we also get $v(x_0) \leq V(x_0, x_1^*) + \beta v(x_1^*)$, which implies $x_1^* \in \phi(x_0)$. Hence $x_1^* \leq \overline{\varphi}(x_0)$, or $\limsup_{n\to\infty} \overline{\varphi}(x_1^n) \leq \varphi(x_0)$. From the monotonicity of $\overline{\varphi}$, and recall that $x_0^n \geq x_0$ for any n, we obtain

$$\lim_{n\to\infty} \varphi(x_0^n) = \overline{\varphi}(x_0).$$

This implies the function $\overline{\varphi}$ is continuous from the right.

Consider the sequence $\{x_0^n\}_{n=0}^{\infty}$ converging to x_0 satisfying $x_0^n \leq x_0$ for any n. Obviously, $\limsup_{n\to\infty} \overline{\varphi}(x_0^n) \leq \overline{\varphi}(x_0)$. The function $\overline{\varphi}$ is upper semi-continuous.

3. From the continuity of value function v, we can use the same arguments as in the proof of the part 2. Remark that in the case v is only upper semi-continuous, the same arguments for the value of $\overline{\varphi}$ cannot be applied for $\underline{\varphi}$. We cannot be sure about the continuity properties of $\underline{\varphi}$. ■

Proof of Proposition 2.3

Since $G'(0) < 1$, for x small enough, we have $G(x) < x$. There exists $x_0 > 0$ such that for any $0 < x \leq x_0$ we get $G(x) < x$. Consider any feasible sequence $\{x_t\}_{t=0}^{\infty}$ belonging to $\Pi(x_0)$, we obtain for any t, $0 \leq x_{t+1} < G(x_t)$. The sequence hence converges to a value $x^* \leq x_0$. By the definition of x_0, one gets $x_0 = 0$, which is equivalent to $\lim_{t\to\infty} x_t = 0$. ■

Proof of Lemma 2.2

Assume that (1) is true. Take any $x_0 \geq \tilde{x}_0$. Suppose that $\underline{\varphi}(x_0) \leq x_0$. By induction, we get $\underline{\varphi}^t(x_0) \leq x_0$ for any $t \geq 0$. By Proposition 2.2, and by induction, we get also for any $t \geq 0$, $\overline{\varphi}^t(\tilde{x}_0) \leq \underline{\varphi}^t(x_0)$, which is in contradiction with the hypothesis that $\lim_{t\to\infty} x_t^* = +\infty$, since this implies $\lim_{t\to\infty} \overline{\varphi}^t(\tilde{x}_0) = \infty$.

Assume that (2) is true. We will prove that for any $x_0 \geq \tilde{x}_0$, any optimal sequence with initial value x_0 converges to infinity. Suppose the contrary. Since $x_0 < \underline{\varphi}(x_0)$, by induction we have the sequence $\{\underline{\varphi}^t(x_0)\}_{t=0}^{\infty}$ is stricly increasing. If this sequence is bounded, then $\lim_{t\to\infty} \varphi^t(x_0) = x^*$. Since $\underline{\varphi}$ is continuous at the left, this implies $\underline{\varphi}(x^*) = \lim_{t\to\infty} \underline{\varphi}(\underline{\varphi}^t(x_0)) = x^*$: a contradiction.■

Proof of Lemma 2.3

Obviously, the lemma is true for $x_0 = 0$ or $x_1 = 0$. We consider the case $x_0, x_1 > 0$. Define $x_2 = \underline{\varphi}(x_1)$ and $x_2' = \overline{\varphi}(x_1)$. By *Inada condition*, we have $x_1 < G(x_0)$ and $0 < x_2, x_2' < G(x_1)$.

Suppose the contrary case $x_2 < x_2'$, from Euler equation we have

$$V_2(x_0, x_1) + \beta V_1(x_1, x_2) = 0.$$

By the supermodularity, this implies $V_2(x_0, x_1) + \beta V_1(x_1, x_2') > 0$, hence for ϵ sufficiently small:

$$V(x_0, x_1 + \epsilon) + \beta V(x_1 + \epsilon, x_2') > V(x_0, x_1) + \beta V(x_1, x_2'),$$

a contradiction. ■

Proof of Proposition 2.4

1. The existence of $\underset{\sim}{x}$ is a consequence of properties of function G. The existence of $\overline{x}$ is direct consequence of Proposition 2.5.

2. This is a consequence of Proposition 2.3.

3. This is a consequence of Proposition 2.5.

4. First, observe that $\underset{\sim}{x} \le \overline{\varphi}(\underset{\sim}{x})$. Indeed, suppose that the contrary: $\overline{\varphi}(\underset{\sim}{x}) < \underset{\sim}{x}$. Since $\overline{\varphi}$ is upper semi-continuous, there exists $\epsilon > 0$ such that for any $\underset{\sim}{x} < x < \underset{\sim}{x} + \epsilon$, we have $\overline{\varphi}(x) < \underset{\sim}{x} < x$: a contradiction with the definition of $\underset{\sim}{x}$.

   By the same arguments, we have $\tilde{x} \ge \underline{\varphi}(\tilde{x})$. Obviously, $\underset{\sim}{x} \le \tilde{x}$.

Define $\underline{x}_0 = \overline{\varphi}(\underset{\sim}{x})$, and $\overline{x}_0 = \underline{\varphi}(\tilde{x})$. By Proposition 2.2 and Lemma 2.4,

$$\underline{\varphi}(\underline{x}_0) = \overline{\varphi}(\underline{x}_0) \leq \underline{\varphi}(\overline{x}_0) = \overline{\varphi}(\overline{x}_0).$$

Moreover, we have

$$\underset{\sim}{x} \leq \underline{x}_0 \leq \overline{x}_0 \leq \tilde{x}.$$

Since $\underset{\sim}{x} \leq \overline{\varphi}(\underset{\sim}{x})$, by the monotonicity of $\overline{\varphi}$, we have $\underline{x}_0 \leq \overline{\varphi}(\underline{x}_0) = \underline{\varphi}(\underline{x}_0)$.

By the same arguments, we have $\overline{x}_0 \geq \overline{\varphi}(\overline{x}_0)$. And we have

$$\underline{x}_0 \leq \underline{\varphi}(\underline{x}_0) \leq \overline{\varphi}(\overline{x}_0) \leq \overline{x}_0.$$

For any $\underline{x}_0 \leq x_0 \leq \overline{x}_0$, for any optimal path $\{x_t^*\}_{t=0}^{\infty}$ from initial state x_0, so by induction, we obtain

$$\underline{x}_0 \leq \underline{\varphi}^t(\underline{x}_0) \leq x_t^* \leq \overline{\varphi}^t(\overline{x}_0) \leq \overline{x}_0.$$

The optimal path $\{x_t^*\}_{t=0}^{\infty}$ is hence bounded away from zero and infinity.

5. Consider the sequence $\{x_t^*\}_{t=0}^{\infty}$ such that $x_0^* = \overline{x}_0$ and for any t, $x_{t+1}^* = \underline{\varphi}(x_t^*)$. Since $\overline{x}_0 \leq \overline{\varphi}(\overline{x}_0)$, the sequence $\{x_t^*\}_{t=0}^{\infty}$ is decreasing and converges to x^*. By the continuity from the right of $\overline{\varphi}$, we have $x^* = \overline{\varphi}(x^*)$. By Proposition 2.4, $x^* = \underline{\varphi}(x^*) = \overline{\varphi}(x^*)$. ■

Proof of Proposition 2.5

Resulting from the Lemma 2.2, we have only to prove that there exists $x_0 \geq 0$ such that there is one optimal path beginning from x_0 converges to infinity.

Suppose the contrary, there exists a sequence x_0^n which converges to infinity satisfying $\underline{\varphi}(x_0^n) \leq x_0^n$ for any n. Define $x_1^n = \underline{\varphi}(x_0^n)$ and $x_2^n = \underline{\varphi}(x_1^n)$. We have

$x_2^n \le x_1^n \le x_0^n$. By Euler equation we obtain

$$V_2(x_0^n, x_1^n) + \beta V_1(x_1^n, x_2^n) = 0,$$

which implies

$$\beta = \frac{-V_2(x_0^n, x_1^n)}{V_1(x_1^n, x_2^n)}.$$

By the concavity of $V(x_0^n, \cdot)$, we have $V_2(x_0^n, x_0^n) \le V_2(x_0^n, x_1^n)$, which implies $-V_2(x_0^n, x_1^n) \le -V_2(x_0^n, x_0^n)$.

Similarly, by the concavity of $V(\cdot, x_2^n)$, and the super-modularity of V ($V_{12}(x, x') \ge 0$ for any x, x'), we also have

$$V_1(x_1^n, x_2^n) \ge V_1(x_0^n, x_2^n) \ge V_1(x_0^n, 0).$$

Hence for any n we get

$$\beta \le \frac{-V_2(x_0^n, x_0^n)}{V_1(x_0^n, 0)},$$

which implies $V_2(x_0^n, x_0^n) + \beta V_1(x_0^n, 0) \le 0$, a contradiction with $\lim_{n\to\infty} x_0^n = +\infty$ and assumption V5. The proof is completed. ■

Proof of Proposition 2.6

According to Amir (1996), the optimal policy functions is written by function of x_0 and β: $\underline{\varphi}(x, \beta)$ and $\overline{\varphi}(x, \beta)$ are strictly increasing with respect to β.

Considering $\beta_1 \le \beta_2$. We prove that $\underset{\sim}{x}(\beta_1) \ge \underset{\sim}{x}(\beta_2)$ and $\tilde{x}(\beta_1) \ge \tilde{x}(\beta_2)$.

By the strict monotonocity of $\overline{\varphi}(x, \cdot)$, if $\overline{\varphi}(x, \beta_2) < x$, then $\overline{\varphi}(x, \beta_1) < x$. This implies $[0, \underset{\sim}{x}(\beta_2)] \subset [0, \underset{\sim}{x}(\beta_1)]$, which is equivalent to $\underset{\sim}{x}(\beta_2) \le \underset{\sim}{x}(\beta_1)$.

Using the same arguments, we have $\tilde{x}(\beta_2) \le \tilde{x}(\beta_1)$. The values of $\underset{\sim}{x}(\cdot)$, $\tilde{x}(\cdot)$ are decreasing functions with respect to β. ■

Proof of Lemma 3.1

The Harrod-like model is considered, as follows.

$$
\begin{aligned}
c_t + I_t &= F(k_t),\\
c_t &= \gamma F(k_t),\\
I_t &= k_{t+1} - (1-\delta)k_t.\\
k_0 &> 0 \text{ given.}
\end{aligned}
$$

Let c_t, k_t denote by the consumption and the physical capital at period t, respectively. I_t be the total investment of the economy at period t. The depreciation rate of physical capital is δ.

The model above can be re-written as

$$
\begin{aligned}
k_{t+1} &= (1-\gamma)F(k_t) + (1-\delta)k_t,\\
k_0 &> 0 \text{ given.}
\end{aligned}
$$

If $F(k) = Ak$ (Ak model), we then have:

$$k_{t+1} = [A(1-\gamma) + 1 - \delta]k_t.$$

Assume that $A(1-\gamma)+1-\delta > 1$. In this case, the sequence of capital, k_t will converges to infinity, $\forall t$.

Now, we also suppose that the economy exists a fix cost $\tilde{k}$, i.e., $F(k) = 0$ if $k \leq \tilde{k}$ and $F(k) = A(k - \tilde{k})$ if $k > \tilde{k}$. The model is thus

$$
\begin{aligned}
k_{t+1} &= (1-\delta)k_t \text{ when } k_t \leq \tilde{k},\\
k_{t+1} &= (1-\gamma)A(k-\tilde{k}) + (1-\delta)k_t \text{ when } k_t > \tilde{k}.
\end{aligned}
$$

Let $\bar{k}$ satisfy:

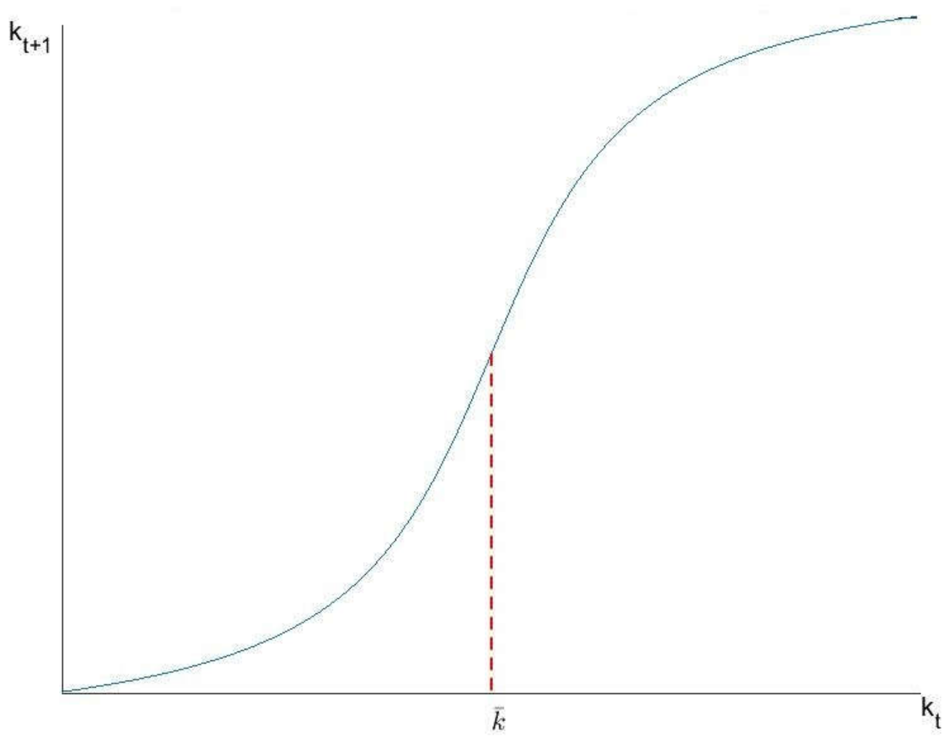
$$\bar{k} = (1-\gamma)A(\bar{k} - \tilde{k}) + (1-\delta)\bar{k}.$$

Following this setting, it is easy to see that

1. $\bar{k} > \tilde{k}$,
2. If $k_0 < \bar{k}$, then k_t converges to zero,
3. If $k_0 > \bar{k}$, then k_t converges to infinity,
4. If $k_0 = \bar{k}$, then $k_t = \bar{k}$, $\forall t$.

Therefore, the economy exists an poverty trap which is $\bar{k}$. For the intuition, we present this problem as shown in Figure 3.1 below.

Figure 4.3. The poverty trap and non-concave technology function.



Proof of Lemma 3.2

1. This is obvious, because of using the properties of function ϕ.

2. The assumption V1, V2 are satisfied by the condition (3) in H1. The assumption V3 is satisfied by the monotonicity of u, f and ϕ.

   For the partial concavity of V, considering its partial derivatives:

$$V_1(x,y) = u'\left(f(x) - \psi\left(\frac{y}{x}\right)\right)\left(f'(x) + \psi'\left(\frac{y}{x}\right)\frac{y}{x^2}\right),$$
$$V_2(x,y) = -u'\left(f(x) - \psi\left(\frac{y}{x}\right)\right)\psi'\left(\frac{y}{x}\right) \times \frac{1}{x}.$$

   From the increasing property of ψ, fix y and $f(x)$, $f(x) - \psi\left(\frac{y}{x}\right)$ is increasing with respect to x, and $\left(f'(x) + \psi'\left(\frac{y}{x}\right)\frac{y}{x^2}\right)$ is decreasing in respect to x. This implies $V_{11}(x,y) \leq 0$, or $V(\cdot, y)$ is concave with respect to the first argument.

   For the concavity with respect to the second argument, we have

$$\frac{\beta V_1(x,0)}{-V_2(x,x)} = \beta \times \frac{u'(f(x))}{u'\left(f(x) - \psi(1)\right)} \times \frac{xf'(x)}{\psi'(1)}.$$

   By using the assumption H2 and H3, we also obtain

$$\lim_{t\to\infty} \frac{\beta V_1(x,0)}{-V_2(x,x)} = +\infty.$$

   Hence, for x big enough we have $\beta V_1(x,0) > -V_2(x,x)$, which is $V_2(x,x) + \beta V_1(x,0) > 0$. ■

Proof of Proposition 3.1

The result is direct consequence of Proposition 2.4 and Lemma 3.2. ■

Proof of Proposition 3.2

First, we prove that for any $1 < \rho < 1 + \lambda$, there exists $\tilde{h}_0$ big enough such that for any $h_0 > \tilde{h}_0$ we have

$$\underline{\varphi}(h_0) > \rho h_0.$$

Suppose the contrary, there exists a sequence h_0^n which converges to infinity and satisfies $\underline{\varphi}(h_0^n) \leq \rho h_0^n$ for any n. For the sake of simplicity, denote $h_1^n = \underline{\varphi}(h_0), h_2^n = \underline{\varphi}(h_1^n)$. From the Euler equation, we get

$$\frac{V_1(h_1^n, h_2^n)}{-V_2(h_0^n, h_1^n)} = \frac{1}{\beta}.$$

Since h_0 is big enough, we have $h_0 < \underline{\varphi}(h_0)$, Assume that for any n, $h_n^1 > h_0^n$ and hence $h_2^n > h_1^n$.

Since $h_2^n > h_1^n$ we obtain

$$\begin{aligned} u'\left(f(h_1^n) - \psi\left(\frac{h_2^n}{h_1^n}\right)\right) &\geq u'\big(f(h_1^n) - \psi(1)\big) \\ &\geq u'\big(f(\rho h_0^n) - \psi(1)\big). \end{aligned}$$

We also have

$$u'\left(f(h_0^n) - \psi\left(\frac{h_1^n}{h_0^n}\right)\right) \leq u'\big(f(h_0^n) - \psi(\rho)\big).$$

Hence, we get

$$\begin{aligned} \liminf_{n\to\infty} \frac{u'\left(f(h_1^n) - \psi(\frac{h_2^n}{h_1^n})\right)}{u'\left(f(h_0^n) - \psi\left(\frac{h_1^n}{h_0^n}\right)\right)} &\geq \liminf_{n\to\infty} \frac{u'(f(h_1^n) - \psi(1))}{u'\left(f(h_0^n) - \psi(\rho)\right)} \\ &\geq \liminf_{n\to\infty} \frac{u'(f(\rho h_0^n) - \psi(1))}{u'\left(f(h_0^n) - \psi(\rho)\right)} \\ &> 0. \end{aligned}$$

Now, let us consider

$$
\begin{aligned}
\frac{V_1(h_1^n, h_2^n)}{-V_2(h_0^n, h_1^n)} &= \frac{u'\left(f(h_1^n) - \psi(\frac{h_2^n}{h_1^n})\right)\left[f'(h_1^n) + \psi'\left(\frac{h_2^n}{h_1^n}\right)\frac{h_2^n}{(h_1^n)^2}\right]}{u'\left(f(h_0^n) - \psi\left(\frac{h_1^n}{h_0^n}\right)\right)\psi'\left(\frac{h_1^n}{h_0^n}\right) \times \frac{1}{h_0^n}} \\
&\geq \frac{u'\left(f(h_1^n) - \psi\left(\frac{h_2^n}{h_1^n}\right)\right)}{u'\left(f(h_0^n) - \psi\left(\frac{h_1^n}{h_0^n}\right)\right)} \times \frac{f'(h_1^n)}{\psi'\left(\frac{h_1^n}{h_0^n}\right) \times \frac{1}{h_0^n}} \\
&\geq \frac{u'\left(f(h_1^n) - \psi\left(\frac{h_2^n}{h_1^n}\right)\right)}{u'\left(f(h_0^n) - \psi\left(\frac{h_1^n}{h_0^n}\right)\right)} \times \frac{h_0^n f'(h_1^n)}{\psi'\left(\frac{h_1^n}{h_0^n}\right)} \\
&\geq \frac{1}{\psi'(\rho)} \frac{u'\left(f(h_1^n) - \psi\left(\frac{h_2^n}{h_1^n}\right)\right)}{u'\left(f(h_0^n) - \psi\left(\frac{h_1^n}{h_0^n}\right)\right)} \times h_0^n f'(h_1^n) \\
&\geq \frac{1}{\psi'(\rho)} \frac{u'\left(f(h_1^n) - \psi(\frac{h_2^n}{h_1^n})\right)}{u'\left(f(h_0^n) - \psi\left(\frac{h_1^n}{h_0^n}\right)\right)} \times h_0^n f'(\rho h_0^n).
\end{aligned}
$$

Furthermore, since

$$
\liminf_{n\to\infty} \frac{u'\left(f(h_1^n) - \psi\left(\frac{h_2^n}{h_1^n}\right)\right)}{u'\left(f(h_0^n) - \psi\left(\frac{h_1^n}{h_0^n}\right)\right)} > 0,
$$

and $\lim_{n\to\infty} h_0^n f'(\rho h_0^n) = +\infty$, we have for n big enough,

$$
\frac{V_1(h_1^n, h_2^n)}{-V_2(h_0^n, h_1^n)} > \frac{1}{\beta},
$$

a contradiction. For any $1 < \rho < 1 + \lambda$ and the optimal path $\{h_t^*\}_{t=0}^{\infty}$ from h_0 sufficiently big, we have $h_{t+1}^* \geq \rho h_t^*$, $\forall_t$. For each value ρ, define h_0^ρ the infimum of such initial value. Consider an optimal sequence which converges to infinity, for any $1 < \rho < 1 + \lambda$, there exists T_ρ such that for any $t \geq T_\rho$, $h_t^* \geq h_0^\rho$, hence $h_{t+1}^* \geq \rho h_t^*$. This implies

$$
\lim_{t\to\infty} \frac{h_{t+1}^*}{h_t^*} = 1 + \lambda.
$$

■

Bibliography

Abel, A. B., & Warshawsky, M. (1988). Specification of the joy of giving: insights from altruism, *The Review of Economics and Statistics, 70*, 145-149.

Andreoni, J. (1990). Impure altruism and donations to public goods: A theory of warm-glow giving. *Economic Journal, 100*(401), 464-477.

Amir, R. (1996). Sensitivity analysis of multisector optimal of economic dynamics. *Journal of Mathematical Economics, 25*(1), 123-141.

Arrow, K. J. (1962). The economic implications of learning by doing. *The review of economic studies, 29*(3), 155-173.

Azariadis, C., & Drazen, A. (1990). Threshold Externalities in economic development. *The Quarterly Journal of Economics, 105*(2), 501-526.

Barro, R. J. (1974). Are government bonds net wealth? *Journal of Political Economy, 82*(6), 1095-1117.

Barro, R. J. (1990). Government spending in a simple of endogenous growth. *Journal of Political Economy, 98*(5, part 2), S103-S125.

Barro, R. J. (1991). Economic growth in a cross section of countries. *Quarterly Journal of Economics, 106*(2), 407-433.

Barro, R. J. (2000). Inequality and growth in a panel of countries. *Journal of Economic Growth, 51*(1), 5-32.

Barro, R. J. (2001). Human capital and growth. *American Economic Review, 91*(2), 12-17.

Becker, G. S. (1962). Investment in Human Capital: A theoretical analysis. *Journal of Political Economy, 70*(5, Part 2), 9-49.

Becker, G. S. (1964). Human capital: A theoretical and empirical analysis with special reference to education. *New York: Columbia University Press.*

Becker, G. S. (1965). Human Capital: A theoretical and empirical analysis, with special reference to education. *University of Chicago Press.*

Becker, G. S., & Chiswick, B. R. (1966). Education and the Distribution of Earning. *American Economic Review, 56*(1/2), 358-369.

Becker, G. S., & Tomes, N. (1979). An equilibrium theory of the distribution of income and intergenerational mobility. *Journal of Political Economy, 87*(6), 1153-1189.

Becker, G. S. (1991). A treatise on the family. *Enlarged edition Cambridge, Mass. and London: Harvard University Press.*

Benhabib, J., & Spiegel, M. M. (1994). The role of human capital in economic development evidence from aggregate cross-country data. *Journal of Monetary Economics, 34*(2), 143-173.

Benveniste, L. M., & Scheinkman, J. A. (1979). On the differentiability of the value function in dynamic models of economics. *Econometrica, 47*(3), 727-732.

Ben-Porath, Y. (1967). The production of human capital and the life cycle of earnings. *Journal of Political Economy, 75*(4, Part 1), 352-365.

Bils, M., & Klenow, P. J. (2000). Does schooling cause growth?. *American Economic Review, 90*(5), 1160-1183.

Borissov, K., Bosi, S., Ha-Huy, T., & Modesto, L. (2016). Inequality and Growth: The Role of Human Capital with Heterogeneous Skills. *Institute for the Study of Labor (IZA).*

Bevia, C., & Iturbe-Ormaetxe, I. (2002). Redistribution and subsidies for higher education. *Scandinavian Journal of Economics, 104*(2), 321-340.

Bosi, S., & Nourry, C. (2007). Growth and fluctuations: The role of public dividends and public spending. *Journal of Mathematical Economics, 43*(3-4), 420-445.

Brock, W. A., & Mirman, L. J. (1973). Optimal economic growth and uncertainty: the no discounting case. *International Economic Review*, 560-573.

Cass, D. (1965). Optimum growth in an aggregative model of capital accumulation. *The Review of Economic Studies, 32*(3), 233-240.

Caballé, J., & Santos, M. S. (1993). On endogenous growth with physical and human capital. *Journal of Political Economy, 101*(6), 1042-1067.

Clark, C. W. (1971). Economically optimal policies for the utilization of biologically renewable resources. *Mathematical Biosciences, 12*(3-4), 245-260.

De la Croix, D. (2001). Growth dynamics and education spending: the role of inherited tastes and abilities. *European Economic Review, 45*(8), 1415-1438.

De La Croix, D., & Michel, P. (2001). Altruism and self-restraint. *Annales d'Economie et de Statistique, 63*, 233-259.

De La Croix, D., & Michel, P. (2002). A theory of economic growth: dynamics and policy in overlapping generations. *Cambridge University Press.*

De la Croix, D., & Delavallade, C. (2009). Growth, public investment and corruption with failing institutions. *Economics of Governance, 10*(3), 187.

Dechert, W. D., and Nishimura. K. (1983). A complete characterization of optimal growth paths in an aggregated model with a non-concave production function. *Journal of Economic Theory, 31*, 332-354.

Diamond, P. A. (1965). National debt in a neoclassical growth model. *American Economic Review, 55*(5), 1126-1150.

Eckwert, B., & Zilcha, I. (2012). Private investment in higher education: Comparing alternative funding schemes. *Economica, 79*(313), 76-96.

Elmendorf, D. W., & Mankiw, N. G. (1999). Government debt. *Handbook of Macroeconomics, 1*, 1615–1669.

Fernandez, R., & Rogerson, R. (1995). On the political economy of education subsidies. *The Review of Economic Studies, 62*(2), 249-262.

Foster, A. D., & Rosenzweig, M. R. (1995). Learning by doing and learning from others: Human capital and technical change in agriculture. *Journal of Political Economy, 103*(6), 1176-1209.

Galor, O., & Moav, O. (2004). From physical to human capital accumulation: Inequality and the process of development. *The Review of Economic Studies, 71*(4), 1001-1026.

Garrat, R., & Marshall, J. M. (1994). Public finance of private goods: The case of college education. *Journal of Political Economy, 102*(3), 566-582.

Glomm, G., & Ravikumar, B. (1992). Public versus private investment in human capital: endogenous growth and income inequality. *Journal of Political Economy, 100*(4), 818-834.

Gradstein, M., & Justman, M. (1995). Competitive investment in higher education: The need for policy coordination. *Economics Letters, 47*(3-4), 393-400.

Hall, R. E., & Jones, C. I. (1999). Why do some countries produce so much more output per worker than others?. *The Quarterly Journal of Economics, 114*(1), 83-116.

Horvat, B. (1958). The optimum rate of investment. *Economic Journal, 68*(272), 747-767.

Klenow, P. J., & Rodriguez-Clare, A. (1997). The neoclassical revival in growth economics: Has it gone too far?. *NBER Macroeconomics Annual, Cambridge: MIT Press, 12*, 73-103.

Hung, N. M., Le Van, C., & Michel, P. (2009). Non-convex aggregate technology and optimal economic growth. *Economic Theory, 40*(3), 457-471.

Kamihigashi, T., & Roy, S. (2007). A nonsmooth, nonconvex model of optimal growth. *Journal of Econonomic Theory, 132*(1), 435-460.

Kiker, B. F. (1966). The historical roots of the concept of human capital. *Journal of Political Economy, 74*(5), 481-499.

King, R. G., Plosser, C. I., & Rebelo, S. T. (1988). Production, growth and business cycles: I. The basic neoclassical model. *Journal of monetary Economics, 21*(2-3), 195-232.

King, R. G., & Rebelo, S. T. (1999). Resuscitating real business cycles. *Handbook of macroeconomics, 1*, 927-1007.

Koopmans, T. C. (1965). On the concept of optimal economic growth. *In The Econometric Approach to Development Planning, Amsterdam: North Holland.*

Krugman, P. (2011). Self-defeating austerity. *New York Times, 7.*

Krueger, A. B., & Lindahl, M. (2001). Education for growth: Why and for whom?. *Journal of economic literature, 39*(4), 1101-1136.

Kydland, F. E., & Prescott, E. C. (1982). Time to build and aggregate fluctuations. *Econometrica: Journal of the Econometric Society*, 1345-1370.

Le Van, C., Morhaim, L., & Dimaria, C. H. (2002). The discrete time version of the Romer model. *Economic Theory, 20*(1), 133-158.

Le Van, C., & Maurel, M. (2007). Education, Corruption and Growth in developing countries. *Maison des Sciences Economiques*, 1624-0340.

Le Van, C., Nguyen, T. A., Nguyen, M. H., & Luong, T. B. (2010). New technology, human capital, and growth in a developing country. *Mathematical Population Studies, 17*(4), 215-241.

Le Kama, A. A., Ha-Huy, T., Le Van, C., & Schubert, K. (2014). A never-decisive and anonymous criterion for optimal growth models. *Economic Theory, 55*(2), 281-306.

Lin, S., & Zhang, W. (2009). The effect of corruption on capital accumulation. *Journal of Economics, 97*(1), 67-93.

Lucas Jr, R. E. (1988). On the mechanics of economic development. *Journal of Monetary Economics, 22*(1), 3-42.

Lucas Jr, R. E. (2015). Human capital and growth. *American Economic Review, 105*(5), 85-88.

Majumdar, M., & Nermuth, M. (1982). Dynamic optimization in non-convex models with irreversible investment: monotonicity and turnpike results. *Journal of Economic Theory, 42*(4), 339-362.

Majumdar, M., & Mitra, T. (1983). Dynamic optimization with a non-convex technology: the case of a linear objective function. *Review of Economic Studies, 50*(1), 143-151.

Mankiw, N. G., Romer, D., & Weil, D. N. (1992). A contribution to the empirics of economic growth. *Quarterly Journal of Economics, 107*(2), 407-437.

Manuelli, R. E., & Seshadri, A. (2014). Human capital and the wealth of nations. *American Economic Review, 104*(9), 2736-2762.

Manuelli, R. E. (2015). Human Capital and Development. *Federal Reserve Bank of St. Louis Review, Third Quarter, 97,*(2), 197-216.

Mincer, J. (1974). Introduction to schooling, experience, and earnings. *National Bureau of Economic Research, 4*, 1-4.

Michel, P., & Vidal, J. P. (2000). Economic integration and growth under intergenerational financing of human-capital formation. *Journal of Economics, 72*(3), 275-294.

McGrattan, E. R., & Prescott, E. C. (2009). Openness, technology capital, and development. *Journal of Economic Theory, 144*(6), 2454-2476.

Nelson, R. R., & Phelps, E. S. (1966). Investment in humans, technological diffusion, and economic growth. *American Economic Review, 56*(1/2), 69-75.

Parente, S. L., & Prescott, E. C. (2000). *Cambridge, MA: MIT Press.*

Pritchett, L., & Filmer, D. (1997). What education production function really show: A positive theory of education expenditures. *Economics of Education Review, 18*(2), 223-39.

Ramsey, F. P. (1928). A mathematical theory of saving. *The Economic Journal, 38*(152), 543-559.

Rebelo, S. (1991). Long-run policy analysis and long-run growth. *Journal of Political Economy, 99*(3), 500-521.

Romer, P. M. (1983). Dynamic competitive equilibria with externalities, increasing returns and unbounded growth. *University of Chicago.*

Romer, P. M. (1986). Increasing returns and long-run growth. *Journal of Political Economy, 94*(5), 1002-1037.

Romer, P. M. (1990). Endogenous technological change. *Journal of Political Economy, 98*(5, Part 2), S71-S102.

Romer, P. M. (2011). Advanded macroeconomic. *McGraw-Hill Higher Education edition 4.*

Schoellman, T. (2012). Education quality and development accounting. *Review of Economic Studies, 79*,(1) 388-417.

Schultz, T. W. (1960). Capital formation by education. *Journal of Political Economy, 68*(6), 571-583.

Schultz, T. W. (1961). Investment in human capital. *American Economic Review, 51*(1), 1-17.

Schultz, T. W. (1962). Reflections on investment in man. *Journal of Political Economy, 70*(5, Part 2), 1-8.

Solow, R. M. (1956). A contribution to the theory of economic growth. *The quarterly journal of economics, 70*(1) 65-94.

Solow, R. M. (1957). Technical change and the aggregate production function. *The review of Economics and Statistics, 39*(3), 312-320.

Spengler, J. J. (1977). Adam Smith on human capital. *American Economic Review, 67*(1), 32-36.

Smith, A. (1776). The wealth of nations, *book 1. London, Methuen & Co.*

Stokey, N. L. (1989). Recursive methods in economic dynamics. *Harvard University Press.*

Temple, J. (1999). A positive effect of human capital on growth. *Economics Letters, 65*(1), 131-134.

Turnovsky, S. J., & Mitra, A. (2013). The interaction between human and physical capital accumulation and the growth-inequality trade-off. *Journal of Human Capital, 7*(1), 26-75.

Viaene, J. M., & Zilcha, I. (2003). Human capital formation, income inequality, and growth. *Cambridge MA: MIT Press*, 89-117.

Viaene, J. M., & Zilcha, I. (2013). Public funding of higher education. *Journal of Public Economics, 108*, 78-89.

Weisbrod, B. A. (1961). The valuation of human capital. *Journal of Political Economy, 69*(5), 425-436.

ÉCOLE DOCTORALE
Sciences de l'homme et de la société (SHS)

Titre : Un essai sur l'accumulation du capital humain et la croissance économique

Mots clés : éducation, capital humain, croissance économique

Résumé Le modèle de *Solow (1956)* est une référence parmi les théories qui cherchent la cause de la croissance économique. Dans ce modèle, ce ne sont pas les deux facteurs de production (travail et capital physique) mais le progrès technique qui parait être le moteur principal de la croissance. *Mankiw, Romer et Weil (1992)* améliorent ce modèle de Solow en y introduisant l'accumulation du capital humain, et montrent empiriquement pourquoi les variables qui sont initialement supposées comme exogènes dans le modèle de Solow varient de façon aussi remarquable entre pays. Leur résultat s'appuie sur les rôles plus importants des facteurs de production, en particulier du capital humain. Cette thèse s'inspire des arguments développés par *Lucas (2015)* qui réclame la nécessité de placer le capital humain au centre de la croissance économique, sans aucune source d'externalité.

La nouveauté de la dissertation est l'introduction du capital humain à la *Lucas (1988)* dans un modèle de type *Ramsey (1928)*, en ajoutant progressivement différents niveaux de complexité, afin d'aboutir à un modèle unifiant les différents sources de croissance économique, permettant non seulement d'étudier l'interaction entre le capital physique et le capital humain, mais aussi de mettre en évidence le rôle central du procès d'accumulation du capital humain.

Comme nous le savons bien, un modèle à la Ramsey est, dans un certain sens, équivalent à un modèle à générations imbriquées augmenté par l'altruisme intergénérationnel à la *Barro (1974)*. Il est intéressant de considérer d'autres formes d'altruisme intergénérationnel en présence d'accumulation du capital humain. Cette thèse explore l'impact de l'altruisme paternaliste au sens *d'Abel et Warshawsky (1988)* dans une société hétérogène où les ménages diffèrent par leur degré d'altruisme, ce qui se traduit par le fait que chaque agent économique a sa propre manière d'investir dans l'éducation des descendants.

L'éducation ne concerne pas seulement les individus, mais également les institutions publiques. L'investissement en l'éducation pour générer du capital humain, est par la suite, supposé n'être qu'un choix public, c'est-à-dire par le biais des dépenses publiques en faveur de l'éducation, financées par les recettes fiscales. L'étude est intéressante dans aux dynamiques associées à l'interaction entre l'accumulation du capital physique et humain et, par conséquent, à la croissance économique. Dans ce contexte, l'impact des politiques fiscales sur la croissance est également étudié.

Plusieurs enjeux économiques, par exemple le piège de pauvreté, ou celui du revenu intermédiaire, peuvent être modélisés à l'aide des modèles dans lesquels on n'a plus la propriété de concavité de la fonction d'utilité. Pour cette raison, il serait utile d'explorer des modèles avec technologies non concaves, en présence d'accumulation du capital humain.

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ÉCOLE DOCTORALE
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Title : An Essay on Human Capital Accumulation and Economic Growth

Keywords : education, human capital, economic growth

Abstract The model of *Solow (1956)* is a seminal reference among the theories that seek to understand the cause of economic growth. In this model, it is not the factors of production (labor and capital) but the technical progress that gives rise to economic growth. *Mankiw, Romer and Weil (1992)* augment this model by introducing human capital accumulation and show empirically why the variables considered exogenous in Solow's model vary in such a remarkable manner among countries. Their results emphasize the importance of factors of production, particularly of human capital. This thesis is inspired by the arguments of *Lucas (2015)* who calls for the necessity of putting human capital at the center of economic growth without any source of externalities.

The novelty of the dissertation is the formation of human capital *à la Lucas (1988) in Ramsey (1928) model*. By gradually adding different layers of complexity, the dissertation arrives at a unified picture of different source of economic growth, allowing for the interaction between physical and human capital where savings and time play a non-trivial role.

As is well-known, the Ramsey model in a certain way is equivalent to an OLG model with intergenerational altruism in the sense of *Barro (1974)*. It is interesting to consider other forms of intergenerational altruism in presence of human capital accumulation. This thesis explores the impact of paternalistic altruism in the sense of *Abel and Warshawsky (1988)* in a heterogeneous economy where the agents differ in their degree of altruism, which is manifest in their manner of investment in the education of their offspring.

Education concerns not only the individuals but also public institutions. Investment in education to generate human capital can therefore be considered a public choice, that is to say, by the bias in public spending on education financed by tax revenues. The study is interesting in the dynamics associated with the interaction between the accumulation of physical and human capital, and consequently in economic growth. In this context, the impact of taxation policy on growth is also studied.

Many economic phenomena, for example, poverty trap and middle-income trap, can be analyzed in models where the concave property of the utility function no longer holds. For this reason, it would be useful to explore these models of non-concave technology in presence of human capital accumulation.

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