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Limite de champ moyen et propagation du chaos pour des systèmes de particules avec interaction discontinue

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pour obtenir le grade de docteur

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Résumé

Résumé

Dans cette thèse, on étudie des problèmes de propagation du chaos et de limite de champ moyen pour des modèles relatant le comportement collectif d'individus ou de particules. Particulièrement, on se place dans des cas où l'interaction entre ces individus/particules est discontinue. Le premier travail établit la propagation du chaos pour l'équation de Vlasov-Poisson-Fokker-Planck 1d. Plus précisément, on montre que la distribution des particules évoluant sur la droite des réels interagissant via la fonction signe, converge vers la solution de l'équation de VPFP 1d, en probabilité par des techniques de type grandes déviations, et en espérance par des techniques de loi des grands nombre. Dans le second travail, on étudie une variante du modèle de Cucker-Smale, où le noyau de communication est l'indicatrice d'un cône dont l'orientation dépend de la vitesse de l'individu. Une estimation de stabilité fort-faible en distance de M.K.W. est obtenue, qui implique la limite de champ moyen. Le troisième travail a consisté à introduire de la diffusion en vitesse dans le modèle précédemment cité. Cependant, il faut ajouter une diffusion tronquée afin de préserver un système dans lequel les vitesses restent uniformément bornées. Finalement, on étudie une variante de l'équation d'agrégation où l'interaction entre individus est donnée par un cône dont l'orientation dépend de la position de l'individu. Dans ce cas on peut seulement donner une estimation de stabilité fort-faible en distance W_∞, et le modèle doit être posé dans un domaine borné dans le cas avec diffusion.

Mots-clés : Propagation du chaos, limite de champ moyen, distance de Wasserstein, contrainte de vision géométrique, estimation de déviation.

Abstract

In this thesis, we study some propagation of chaos and mean field limit problems arising in modelisation of collective behavior of individuals or particles. Particularly, we set ourselves in the case where the interaction between the individuals/particles is discontinuous. The first work establihes the propagation of chaos for the 1d Vlasov-Poisson-Fokker-Planck equation. More precisely, we show that the distribution of particles evolving on the real line and interacting through the sign function converges to the solution of the 1d VPFP equation, in probability by large deviations-like techniques, and in expectation by law of large numbers-like techniques. In the second work, we study a variant of the Cucker-Smale, where the communication weight is the indicatrix function of a cone which orientation depends on the velocity of the individual. Some weak-strong stability estimate in M.K.W. distance is obtained for the limit equation, which implies the mean field limit. The third work consists in adding some diffusion in velocity to the model previously quoted. However one must add some truncated diffusion in order to preserve a system in which velocities remain unifomrly bounded. Finally we study a variant of the aggregation equation where the interaction between individuals is also given by a cone which orientation depends on the position of the individual. In this case we are only able to provide some weak-strong stability estimate in W_∞ distance, and the problem must be set in a bounded domain for the case with diffusion.

Key words : Propagation of chaos, mean field limit, Wasserstein distance, vision geometrical constrains, deviation estimate.

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Introduction

1. Stosszahlanatz

La théorie cinétique, dans laquelle s'inscrivent mes travaux de thèse, prend ses sources au cours de la seconde moitié du XIXème siècle. A cette époque, les mathématiciens s'attachent à essayer de démontrer le second principe de la thermodynamique qu'ont établi auparavant des scientifiques comme Clausius ou Kelvin. Sans vouloir donner de définition précise des termes employés, rappelons que ce principe stipule que tout système isolé qui évolue dans le temps voit son entropie - ou son désordre - augmenter, et que l'évolution n'est réversible dans le temps uniquement si l'entropie est constante.
A la suite notamment de travaux de Maxwell, Boltzmann publie en 1872 un article intitulé "Nouvelles études sur l'équilibre de la chaleur parmi des molécules gazeuses". Dans celui-ci, il se propose de déterminer, à partir de considérations sur les atomes qui le composent, le comportement d'un nuage de gaz parfait hors de l'équilibre. Un atome se meut en ligne droite jusqu'à ce qu'il en rencontre un second et qu'il se produise une collision dont il décrit les effets sur leurs dynamiques respectives. Grâce à ces considérations microscopiques, il établit sur la fonction de distribution cinétique $f(v,t)$, qui mesure la densité de particules à la vitesse v et au temps t, l'équation d'évolution qui porte maintenant son nom. Mais surtout, il introduit une quantité $E(t)$, qui est l'entropie du système et dont il démontre qu'elle ne peut que croître. C'est le fameux "théorème H" qui donne une explication à l'irréversibilité des phénomènes macroscopiques.
Parmi les controverses que suscita cet article, il est utile de s'arrêter sur une en particulier. C'est celle qui opposa Boltzmann à Loschmidt, autre scientifique autrichien, qui lui fait valoir que la dynamique des particules étant fondée sur les lois de Newton qui sont réversibles en temps, tout processus macroscopique construit à partir de cette dynamique particulaire devrait être lui aussi réversible. Cela oblige Boltzmann à préciser une hypothèse essentielle de son article de 1872, qui est le **Stosszahlanatz**, littéralement "hypothèse de comptage des chocs" ou encore "hypothèse du chaos moléculaire". Cette hypothèse stipule que le nombre de chocs entre une particule à la vitesse v et une particule à la vitesse v' au temps t est proportionnel $f(v,t)f(v',t)$, ce qui revient à dire en terme statistique que deux particules qui entrent en collision sont décorrélées en vitesse. Cependant, même si on peut supposer une telle dé-corrélation à l'instant initial, tout indique que cela sera perdu une fois que sera advenu le premier choc entre deux particules. L'irréversibilité du phénomène vient donc du fait que cette hypothèse de chaos moléculaire crée un "avant" et un "après" choc, et donc un sens sur la ligne du temps. Cette hypothèse de nature probabiliste au milieu des lois de Newton hautement déterministes n'est pas sans émouvoir les contemporains de Boltzmann.

2. Propagation du chaos et limite de champ moyen

Bien plus tard, en 1956 Mark Kac publie un article intitulé "Foundations of Kinetic Theory" [**15**] dans lequel il propose de dériver l'équation de Boltzmann en assumant pleinement le caractère stochastique de l'hypothèse de chaos moléculaire. Il y introduit la

notion de "propriété de Boltzmann", qui est maintenant appelée chaos et qui examinée dans la section suivante.

2.1. Chaos et propagation du chaos. On commence cette section en donnant la

DÉFINITION 2.1. *([17]) Soit E un espace polonais et soit $(X^N = (x_1^N, \cdots, x_N^N))_{N\geq 0}$ une suite de variables aléatoires sur E^N. On suppose que la loi u_N de X_N est symétrique, c'est à dire que pour toute permutation $\sigma \in S_N$, $(x_1^N, \cdots, x_N^N)$ a même loi que $(x_{\sigma(1)^1}, \cdots, x_{\sigma(N)^N})$. On dit que la suite de loi $(u_N)_{N\geq 0}$ est u-chaotique avec $u \in \mathcal{P}(E)$, si pour tout entier $k \geq 2$ et toutes fonctions tests $\phi_1, \cdots, \phi_k \in C_b(E)$ on a*

$$\int_{E^N} \prod_{i=1}^{k} \phi_i(x_i) u_N(dx_1, \cdots, dx_N) \underset{N\to\infty}{\longrightarrow} \prod_{i=1}^{k} \int_E \phi_i(x) u(dx).$$

Cette définition est donnée par Kac dans son article de 1956, qu'il appelle donc "propriété de Boltzmann". Une suite u -chaotique peut se voir au vu de cette définition comme un produit "asymptotiquement" tensorisé de u avec elle-même. Cette nouvelle notion asymptotique est une précision concernant l'hypothèse de chaos moléculaire de Boltzmann. En effet, si on a du mal à se convaincre de l'indépendance entre deux particules prêtes à entrer en collision quand trois particules sont impliquées, on en a moins quand il s'agit de milliers, et encore moins quand il s'agit de nombre d'Avogadro qui est l'échelle à considérer dans ce genre de problèmes.
Dès lors, lorsqu'une dynamique particulaire est proposée, il est essentiel afin de dériver rigoureusement la loi d'évolution macroscopique du système, de montrer qu'ayant initialement imposé au système une condition initiale chaotique, la dynamique préserve ce caractère chaotique. On l'a vu avec l'exemple de Boltzmann. On va l'illustrer à travers un autre exemple bien plus simple.
Soient $b : \mathbb{R}^d \to \mathbb{R}^d$ un champ de vecteurs Lipschitz, $(B^k)_{k\in\mathbb{N}}$ une suite de mouvements browniens indépendants, et $u_0 \in \mathcal{P}(\mathbb{R}^d)$. Considérons maintenant l'EDS

$$X_t^i = X_0^i + \int_0^t \frac{1}{N} \sum_{j=1}^{N} b(X_s^i - X_s^j)\, ds + B_t^i \quad , \quad i = 1, \cdots, N, \tag{1}$$

où les $(X_0^i)_{i=1,\cdots,N}$ sont supposés indépendants de la famille $(B^k)_{k\in\mathbb{N}}$ et indépendants identiquement distribués de loi u_0 (ce qui est une hypothèse plus forte que la u_0-chaoticité). Pour les nombres N de particules considérés dans la pratique, ce genre de systèmes n'a aucune chance d'être résolu numériquement, même si la théorie standard des EDS garantit l'existence d'une solution. L'étude du système particule par particule n'est d'ailleurs pas nécessaire pour déterminer le comportement macroscopique du système. Pour cela, il est plus utile d'introduire une quantité qui rend mieux compte de la globalité du système, la mesure empirique associée, qu'on notera généralement μ_t^N définie par

$$\mu_t^N := \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^i},$$

qui est une variable aléatoire à valeur dans $\mathcal{P}(\mathbb{R}^d)$. L'étude du comportement asymptotique (en N) de cette mesure de probabilité donne des informations sur le comportement macroscopique du système. Pour cela, citons la

PROPOSITION 2.1. *([17]) Soit $(X^N = (x_1^N, \cdots, x_N^N))_{N\geq 0}$ une suite de variables aléatoires sur E^N, telle que u_N la loi de X_N est symétrique. La suite $(u_N)_{N\geq 0}$ est u-chaotique si et seulement si la suite des μ^N définie par*

$$\mu^N := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i},$$

converge (au sens des mesures) en loi vers $u \in \mathcal{P}(E)$ quand N tend vers l'infini.

Il s'agit maintenant de trouver un candidat à être la limite de μ_t^N associée au système (1). Notons que celui-ci peut se réécrire

$$X_t^i = X_0^i + \int_0^t \int_{\mathbb{R}^d} b(X_s^i - y)\mu_s^N(dy)\, ds + B_t^i \quad , \quad i = 1, \cdots, N.$$

Si μ_t^N était amenée à converger (uniformément en temps) en loi vers un certain u_t, alors par la Proposition 2.1 la loi de chaque particule au temps t serait alors "asymptotiquement" u_t. Cette considération heuristique amène à s'intéresser à l'EDS non linéaire suivante

$$Y_t = Y_0 + \int_0^t \int_{\mathbb{R}^d} b(Y_s - y)u_s(dy)\, ds + B_t \quad , \quad u_s = \mathcal{L}(Y_s), \tag{2}$$

dans laquelle on a remplacé l'interaction avec les autres particules par l'interaction avec sa propre loi, et à considérer la loi de sa solution (si elle existe) comme un candidat à la limite recherchée. L'existence de solution à (1) n'est pas évidente à priori, même si pour un champ b tel qu'on l'a choisi, cela ne pose pas de problèmes particuliers. De plus, on note que la famille des marginales en temps (i.e. la famille des lois à chaque instant du processus) est solution au sens des distributions de l'EDP non linéaire

$$\begin{cases} \partial_t u_t + \nabla \cdot (u_t(b * u_t)) = \dfrac{1}{2}\Delta u_t \\ u_{t=0} = u_0, \end{cases} \tag{3}$$

Pour montrer que la solution de (1) est u-chaotique où u est donnée par (3), on utilise une méthode dite de couplage, qui consiste à construire la solution de (1) et N copies indépendantes de la solution de (2) sur le même espace de probabilité, c'est à dire avec le même aléa (i.e. mêmes conditions initiales et mêmes diffusions) :

$$Y_t^i = X_0^i + \int_0^t \int_{\mathbb{R}^d} b(Y_s^i - y)u_s(dy)\, ds + B_t^i \quad , \quad u_s = \mathcal{L}(Y_s^i). \tag{4}$$

Il est facile de voir que les processus $(Y_\cdot^i)_{i=1,\cdots,N}$ sont indépendants identiquement distribués dont les marginales en temps sont données par la solution de (3), car ils sont construits à partir de conditions initiales et de diffusions indépendantes.
Maintenant, quand on compare les trajectoires de (1) et de (4), il vient

$$\begin{aligned} |X_t^i - Y_t^i| &= \Big| \int_0^t \frac{1}{N}\Big(\sum_{j=1}^N b(X_s^i - X_s^j) - \int_{\mathbb{R}^d} b(Y_s^i - y)u_s(dy)\Big)\, ds\Big| \\ &\leq \int_0^t \frac{1}{N}\sum_{j=1}^N \big|b(X_s^i - X_s^j) - b(Y_s^i - Y_s^j)\big|\, ds \\ &+ \int_0^t \Big|\frac{1}{N}\sum_{j=1}^N b(Y_s^i - Y_s^j) - \int_{\mathbb{R}^d} b(Y_s^i - y)u_s(dy)\Big|\, ds \\ &\leq \int_0^t \|b\|_{Lip}\big(|X_s^i - Y_s^i| + \frac{1}{N}\sum_{j=1}^N \big|X_s^j - Y_s^j\big|\big)\, ds \\ &+ \int_0^t \beta_s^i\, ds, \end{aligned} \tag{5}$$

avec

$$\beta_s^i := \Big|\frac{1}{N}\sum_{j=1}^N b(Y_s^i - Y_s^j) - \int_{\mathbb{R}^d} b(Y_s^i - y)u_s(dy)\Big|.$$

Notons que puisque pour $j \neq i$, Y_s^i est indépendant de Y_s^j et que la loi de Y_s^j est u_s, on a

$$\mathbb{E}\big[b(Y_s^i - Y_s^j) \mid Y_s^i\big] = \int_{\mathbb{R}^d} b(Y_s^i - y)u_s(dy),$$

et donc, par les propriétés de l'espérance conditionnelle on a

$$\begin{aligned}\mathbb{E}\big[\beta_s^i\big] &= \mathbb{E}\big[\mathbb{E}\big[\beta_s^i \mid Y_s^i\big]\big] \\ &\leq \mathbb{E}\big[\mathbb{E}\big[(\beta_s^i)^2 \mid Y_s^i\big]^{1/2}\big] \\ &\leq \mathbb{E}\big[\mathbb{E}\big[\frac{1}{N^2}\sum_{k,j=1}^{N}\big(b(Y_s^i - Y_s^j) - \int_{\mathbb{R}^d} b(Y_s^i - y)u_s(dy)\big)\big(b(Y_s^i - Y_s^k) - \int_{\mathbb{R}^d} b(Y_s^i - y)u_s(dy)\big) \mid Y_s^i\big]^{1/2}\big],\end{aligned}$$

en dénotant

$$\hat{b}_s^j := b(Y_s^i - Y_s^j) - \int_{\mathbb{R}^d} b(Y_s^i - y)u_s(dy),$$

d'espérance nulle conditionné à Y_s^i, et comme conditionné à Y_s^i, $\hat{b}_s^j$ et $\hat{b}_s^k$ sont indépendants on a

$$\mathbb{E}\big[\hat{b}_s^j\hat{b}_s^k \mid Y_s^i\big] = 0,$$

dès que $k \neq j$. D'un autre côté, il est facile de voir que pour tout $j = 1, \cdots, N$, presque sûrement, on a

$$|\hat{b}_s^j| \leq 2\|b\|_\infty,$$

et donc il vient

$$\mathbb{E}\big[\beta_s^i\big] \leq \frac{2\|b\|_\infty}{\sqrt{N}}. \tag{6}$$

Maintenant, quand on somme les inégalités (5) pour $i = 1, \cdots, N$ et en divisant par N et en prenant l'espérance on obtient, au vu de (6)

$$\mathbb{E}\big[\frac{1}{N}\sum_{i=1}^{N}|X_t^i - Y_t^i|\big] \leq 2\|b\|_{Lip}\int_0^t \mathbb{E}\big[\frac{1}{N}\sum_{i=1}^{N}|X_s^i - Y_s^i|\big]\,ds + \frac{2\|b\|_\infty}{\sqrt{N}},$$

duquel on déduit par le Lemme de Gronwall que

$$\mathbb{E}\big[\frac{1}{N}\sum_{i=1}^{N}|X_t^i - Y_t^i|\big] \leq \frac{2\|b\|_\infty}{\sqrt{N}}e^{2t\|b\|_{Lip}}.$$

Maintenant en introduisant ν_t^N la mesure empirique associée aux N copies indépendantes (4) $\nu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{Y_t^i}$, on a

$$\begin{aligned}\mathbb{E}\big[W_1(\mu_t^N, u_t)\big] &\leq \mathbb{E}\big[W_1(\mu_t^N, \nu_t^N)\big] + \mathbb{E}\big[W_1(\nu_t^N, u_t)\big] \\ &\leq \mathbb{E}\big[\frac{1}{N}\sum_{i=1}^{N}|X_t^i - Y_t^i|\big] + \mathbb{E}\big[W_1(\nu_t^N, u_t)\big] \\ &\leq \frac{2\|b\|_\infty}{\sqrt{N}}e^{2t\|b\|_{Lip}} + \mathbb{E}\big[W_1(\nu_t^N, u_t)\big],\end{aligned} \tag{7}$$

où W_1 est la distance de Monge-Kantorovitch-Wasserstein (cf [**19**]). Le second terme du membre de droite tend également vers 0 quand N tend vers l'infini. Un Théorème de Fournier et Guillin [**9**] donne le taux optimal pour cette convergence, mais pour ne pas avoir à rappeler ici ses hypothèses techniques, on se contentera de dire que ν_t^N est une moyenne empirique de mesures aléatoires indépendantes dont l'intensité est u_t. En effet, pour toute fonction test $\phi \in C_b(\mathbb{R}^d)$, on a

$$\mathbb{E}\big[\int \phi(x)\delta_{Y_t^i}(dx)\big] = \mathbb{E}\big[\phi(Y_t^i)\big] = \int \phi(x)u_t(dx).$$

Donc la grandeur $W_1(\nu_t^N, u_t)$ mesure l'écart entre une moyenne empirique de variables aléatoires indépendantes identiquement distribuées et leur moyenne, et on se convainc sans mal que cette grandeur converge vers 0 en espérance. En fin de compte, on déduit du fait que $\mathbb{E}\big[W_1(\mu_t^N, u_t)\big]$ converge vers 0 et de la Proposition 2.1 que la suite de solutions au temps t de (1) est u_t-chaotique. Au passage, on a justifié rigoureusement que (3) découlait

de la dynamique particulaire, et qu'il rend bien compte de l'état macroscopique du système de particules et on dispose même d'un taux d'erreur.

2.2. Limite de champ moyen. Si l'on retire tout aléa de (1), alors l'approche abordée dans la section précédente n'est plus valable, cependant la question du comportement du système de particules quand le nombre de particules devient grand conserve son intérêt. Dans ce carde purement déterministe, on doit faire appel à des outils de la théorie du transport plutôt qu'au calcul stochastique. Considérons le système d'EDO suivant

$$\begin{cases} \dfrac{dx_t^i}{dt} = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} b(x_t^i - x_t^j) \quad , \quad i = 1, \cdots, N, \\ x_{t=0}^i = x_0^i, \end{cases} \tag{8}$$

où b est cette fois Lipschitz seulement. Encore une fois, la quantité à considérer pour déterminer le comportement macroscopique du système est la mesure empirique μ_t^N associée :

$$\mu_t^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{x_t^i}.$$

qui cette fois est une quantité déterministe. Soit maintenant une fonction test $\phi \in C_c^\infty(\mathbb{R}^d)$, on a alors

(9)

$$\begin{aligned} \frac{d}{dt}\int \phi(x)\mu_t^N(dx) = \frac{d}{dt}\frac{1}{N}\sum_{i=1}^{N}\phi(x_t^i) &= \frac{1}{N}\sum_{i=1}^{N}\nabla\phi(x_t^i)\cdot\frac{dx_t^i}{dt} \\ &= \frac{1}{N}\sum_{i=1}^{N}\nabla\phi(x_t^i)\cdot\big(b*\mu_t^N(x_t^i)\big) = \int_{\mathbb{R}^d}\nabla\phi(x)\cdot\big(b*\mu_t^N(x)\big)\mu_t^N(dx) \\ &= -\int \phi(x)\nabla\cdot\big(\big(b*\mu_t^N(x)\big)\mu_t^N(dx)\big). \end{aligned}$$

Donc la famille $(\mu_t^N)_{t\geq 0}$ est solution au sens des distributions de l'EDP non linéaire

$$\begin{cases} \partial_t\mu_t^N + \nabla\cdot\big(\big(b*\mu_t^N\big)\mu_t^N\big) = 0 \\ \mu_{t=0}^N = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\delta_{x_0^i}. \end{cases} \tag{10}$$

En reprenant l'idée de la section précédente, il est naturel de se demander ce qu'il advient quand la suite de conditions initiales $(\frac{1}{N}\sum_{i=1}^{N}\delta_{x_0^i})_{N\geq 0}$ converge au sens des mesures vers un profil continu $u_0 \in \mathcal{P}_1(\mathbb{R}^d)$. Est-ce que cette convergence est propagée pour les instants suivants ? Et si oui, vers quelle limite ? Le candidat naturel est la solution de (10) avec la condition initiale u_0, et la question devient alors une question de stabilité pour l'équation (10).

On présente ici l'approche développée dans [**4**]. Soient $(f_t)_{t\geq 0}$ et $(g_t)_{t\geq 0}$ deux solutions au sens des distributions de (10) pour les données initiales respectives $f_0, g_0 \in \mathcal{P}_1(\mathbb{R}^d)$. Il s'agit d'estimer l'écart entre ces deux solutions de la même équation en fonction de l'écart initial. Naturellement l'outil qu'on utilisera pour quantifier cet écart est la métrique de Wasserstein. Comme on souhaite faire jouer par une des deux solutions, disons g, le rôle de la mesure empirique associée au système (8), qui est une mesure atomique et n'a aucune régularité, on ne pourra utiliser aucune régularité sur g dans l'argument. Mais on peut supposer sans perte de généralité que f_0 est absolument continue par rapport à la mesure de Lebesgue (il faudrait montrer que cette propriété est propagée en temps), et donc d'après [**19**, Théorème 2.12] il existe une application de transport $\mathcal{T}_0$ transportant f_0 sur

g_0 telle que

$$W_1(f_0, g_0) = \int |x - \mathcal{T}_0(x)| f_0(dx).$$

Considérons maintenant les caractéristiques associées à chacune des deux solutions f et g, dénotées respectivement $X^t(x)$ et $Y^t(y)$ définies par

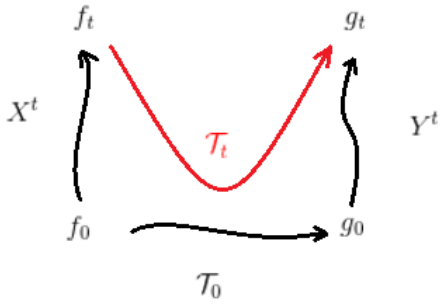
$$\begin{cases} \dfrac{dX^t(x)}{dt} = b * f_t(X^t(x)) \\ X^{t=0}(x) = x, \end{cases} \tag{11}$$

et

$$\begin{cases} \dfrac{dY^t(y)}{dt} = b * g_t(Y^t(y)) \\ Y^{t=0}(y) = y. \end{cases} \tag{12}$$

Notons que l'application X^t transporte f_0 sur f_t (de même Y^t, g_0 sur g_t). Il est donc possible grâce à ces trois applications de transport, d'en construire une quatrième transportant f_t sur g_t, qui est donnée par

$$\mathcal{T}_t := Y^t \circ \mathcal{T}_0 \circ (X^t)^{-1}.$$



Par la définition des distances de Wasserstein, on a

$$W_1(f_t, g_t) \leq \int |\mathcal{T}_t(x) - x| f_t(x) = \int \left|X^t(x) - Y^t(\mathcal{T}_0(x))\right| f_0(dx) := Q(t),$$

et il s'agit comme à la section précédente d'établir une inégalité de type de Gronwall sur la quantité $Q(t)$ afin de conclure à la stabilité. Ainsi,

$$\begin{aligned} \frac{d}{dt}Q(t) &\leq \int \frac{d}{dt}\left|X^t(x) - Y^t(\mathcal{T}_0(x))\right| f_0(dx) \\ &\leq \int \left|b * f_t(X^t(x)) - b * g_t(Y^t(\mathcal{T}_0(x)))\right| f_0(dx) \\ &\leq \int \Big| \int b(X^t(x) - y) f_t(dy) - \int b(Y^t(\mathcal{T}_0(x)) - y) g_t(dy) \Big| f_0(dx) \\ &\leq \int \int \left|b(X^t(x) - y) - b(Y^t(\mathcal{T}_0(x)) - \mathcal{T}_t(y))\right| f_t(dy) f_0(dx) \\ &\leq \int \|b\|_{Lip} |X^t(x) - Y^t(\mathcal{T}_0(x))| f_0(dx) + \int \|b\|_{Lip} |y - \mathcal{T}_t(y)| f_t(dy) = 2\|b\|_{Lip} Q(t). \end{aligned} \tag{13}$$

Ainsi l'utilisation du Lemme de Gronwall donne

$$W_1(f_t, g_t) \leq Q(t) \leq e^{2t\|b\|_{Lip}} Q(0) = e^{2t\|b\|_{Lip}} W_1(f_0, g_0).$$

On peut désormais faire jouer à f le rôle de la solution de (10) pour la condition initiale $u_0 \in \mathcal{P}_1(\mathbb{R}^d)$ et à g celui de la mesure empirique associée au système (8), qui est aussi

solution de (10) pour la condition initiale $\frac{1}{N}\sum_{i=1}^N \delta_{x_0^i}$ et obtenir

$$W_1\big(\frac{1}{N}\sum_{i=1}^N \delta_{x_t^i}, u_t\big) \leq e^{2t\|b\|_{Lip}} W_1\big(\frac{1}{N}\sum_{i=1}^N \delta_{x_0^i}, u_0\big).$$

Et donc, si le profil atomique initial converge vers un profil continu quand le nombre de particules tend vers l'infini, cette convergence du particulaire vers le continu se propage à tous les instants ultérieurs. On a donc rigoureusement dérivé l'équation (10) à partir de la dynamique des particules.

3. Interaction non Lipschitz

La question essentielle de la justification rigoureuse de modèle macroscopique à partir d'une dynamique particulaire se pose dans la majorité des cas revêtant un intérêt physique, hors du contexte très simple d'une interaction Lipschitz entre les particules qu'on a vu plus haut. On citera dans cette section quelques problèmes de propagation du chaos et de limite en champ moyen résolus plus ou moins récemment, ou encore ouverts, avec ce type d'interaction.

3.1. Cas stochastique.

- **Interaction Holder**
  Dans un article récent [**13**], Thomas Holding considère l'équation (1) ainsi que sa version cinétique, où le champ d'interaction est choisi Hölder. On voit que la stratégie classique ne fonctionne plus dès que la régularité est abaissée de Lipschitz à Hölder. L'astucieuse stratégie utilisée consiste à prendre à rebours cette stratégie classique pour obtenir de la décroissance en N avec le terme qui servait à fermer l'inégalité de Gronwall et inversement.
- **Navier-Stokes 2D**
  En dimension 2, il est possible de reformuler l'équation de Navier Stokes sous forme de vortex interagissant via le noyau Biot et Savart [**7**]. Celui-ci est défini par

  $$K_{BS} : (x,y) \in \mathbb{R}^2 \mapsto \big(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\big) \in \mathbb{R}^2.$$

  Un vortex peut être vu comme un point qui évolue dans le plan et qui induit une rotation dans l'espace qui influe sur les mouvements des autres vortex. On le définit avec un couple $(X_t^i, \mathcal{M}_i) \in \mathbb{R}^2 \times \mathbb{R}$ qui désigne sa position et son intensité. Leur interaction est décrite par l'EDS

  $$X_t^i = X_0^i + \int_0^t \frac{1}{N}\sum_{j\neq i} \mathcal{M}_j K_{BS}(X_s^i - X_s^j)\, ds + \sqrt{2\sigma} B_t^i \quad , \quad \forall i = 1, \cdots, N. \tag{14}$$

  Quand le nombre de vortex devient grand, on s'attend à ce que la mesure empirique associée à ce système converge vers la solution de l'EDP

  $$\partial_t w_t + (K_{BS} * w_t) \cdot \nabla w_t = \sigma \Delta w_t, \tag{15}$$

  qui est l'équation de Navier-Stokes en formulation rotationnelle en dimension 2.

  La stratégie utilisée est totalement différente de la méthode de couplage classique présentée plus haut. Les auteurs concluent à la convergence en loi de la mesure empirique associée au système (14) vers la solution de l'équation (15) , grâce à un argument de tension obtenu grâce à une estimation sur la dissipation d'entropie de l'équation de Liouville associée au système de particules. Le résultat est qualitatif et ne quantifie pas cette convergence. C'est à ma connaissance le premier résultat de propagation du chaos pour une interaction singulière et non bornée sans cut-off.

- **Equation de Keller Segel**
  Ce modèle issu de la biologie, rend compte du mouvement de cellules dans un milieu présentant une substance chimique, qu'elles sont capables d'émettre et d'absorber. La densité de cellules évolue selon l'équation

$$\partial_t \rho_t + \chi \nabla \cdot \big(\rho_t (K * \rho_t)\big) = \sigma \Delta \rho_t, \tag{16}$$

  où K est l'interaction Coulombienne définie par

$$K(x) = -\frac{x}{|x|^d},$$

  et $\chi \in \mathbb{R}$ est un paramètre décrivant la sensibilité des cellules à l'attraction de la substance chimique (pour $\chi > 0$ elles sont attirées par la substance, et repoussées pour $\chi < 0$). Le système de cellules en interaction s'écrit alors

$$X_t^i = X_0^i + \frac{\chi}{N} \int_0^t \sum_{j \neq i} K(X_s^i - X_s^j)\, ds + \sqrt{2\sigma} B_t^i \quad , \quad \forall i = 1, \cdots, N. \tag{17}$$

  En dimension 2, ce système de particules semble très similaire à celui de la formulation vortex de l'équation de Navier-Stokes. Cependant, contrairement au noyau de Biot et Savart, le noyau d'interaction Coulombienne n'est pas à divergence nulle. Ce qui rend la stratégie utilisée dans [**7**] inapplicable à ce problème. Cependant, dans [**10**], les auteurs s'en inspirent pour obtenir des résultats en dimension 2 dans le cas sous-Newtonien, c'est à dire avec une interaction du type

$$K(x) = \frac{x}{|x|^{1+\varepsilon}}\ , \ \varepsilon \in (0,1).$$

  Dans le cas Newtonien plus compliqué, on peut néanmoins montrer [**6**] que la loi du système de particules (17) est chaotique, et que le paramètre de chaoticité est donné par une solution de (16). Cependant ce n'est pas un résultat de limite macroscopique puisqu'il ne dit pas que toute solution de (16) est une limite de système de particules.

3.2. Cas déterministe.

- **Interaction bornée**
  Dans un récent article [**14**], Jabin et Wang considèrent le système de particules suivant

$$\begin{cases} \dfrac{dX_t^i}{dt} = V_t^i, \\ \dfrac{dV_t^i}{dt} = \dfrac{1}{N} \displaystyle\sum_{i=1}^{N} H(X_t^i - X_t^j), \end{cases} \tag{18}$$

  où H est seulement supposée dans $L^\infty(\mathbb{R}^d)$. Le résultat est obtenu à l'aide d'estimation de l'entropie relative de la solution de l'équation de Liouville associée (18) par rapport à la solution de l'équation limite.
- **Équation d'agrégation**
  Celle ci est donnée par

$$\partial_t \rho_t + \nabla \cdot \big(\rho_t (\nabla g_s * \rho_t)\big) = 0, \tag{19}$$

  où g_s est le potentiel de Riesz défini par

$$g_s(x) = \begin{cases} c_{d,s}^{-1} |x|^{-s}, & \text{si } s > 0 \\ -c_{d,0}^{-1} \ln(|x|), & \text{si } s = 0, \end{cases}$$

et $c_{d,s}$ est une constante de normalisation. Dans le cas $s = d-2 > 0$, on a

$$\nabla g_{d-2}(x) = -c_{d,s}^{-1}(d-2)\frac{x}{|x|^d},$$

et dans le cas $d = 2$

$$\nabla g_0(x) = -2c_{d,0}^{-1}\frac{x}{|x|^2},$$

et on reconnait le pendant déterministe du modèle de Keller-Segel dans le cas $\chi > 0$. Dans le cas $s < d-2, d \geq 2$, c'est à dire dans le cas sous-Newtonien on peut obtenir des estimations quantitatives de la convergence de la mesure empirique du système de particules vers l'équation limite (19), grâce à l'utilisation de la distance de Wasserstein d'ordre infini W_∞ et à un contrôle minutieux de la distance minimale inter-particule [**2**]. Cette distance minimale est un paramètre très important quand une interaction singulière à l'origine est considérée. Également dans [**5**], Duerinckx conclut dans le cas $s < 1$ et $d = 1, 2$ grâce à un argument d'énergie modulée qui consiste à estimer l'écart entre deux solutions de (19) avec la métrique

$$\mathcal{E}(\mu,\nu) = \int_{\mathbb{R}^{2d}} g_s(x-y)d(\mu-\nu)(x)d(\mu-\nu)(y),$$

bien adaptée à la structure du problème.

- **Équation de Vlasov-Poisson**
  C'est le principal modèle de la théorie cinétique, en tout cas celui pour lequel la question de la limite en champ moyen est la plus côté. Considérons un système de N particules chargées dans l'espace, toutes de même charge q et de même masse m. D'après les lois de la mécanique et de l'électrostatique, leur mouvement est régi par l'équation différentielle

$$\begin{cases} \dot{X_i^t} = V_i^t, \\ m\dot{V_i^t} = \displaystyle\sum_{j\neq i} \frac{q^2}{4\pi\varepsilon_0}\frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^3}, \end{cases}$$

où ε_0 est la permittivité diélectrique du vide. Si maintenant on introduit une longueur L et un temps T caractéristiques du système, on peut adimensionner celui-ci en définissant les nouvelles variables

$$x_i^\tau = \frac{X_i^{T\tau}}{L}, \quad v_i^\tau = T\frac{V_i^{T\tau}}{L}.$$

Ainsi, on obtient sur ces nouvelles variables le système différentiel suivant

$$\begin{cases} \dot{x_i^\tau} = v_i^\tau, \\ \dot{v_i^\tau} = \dfrac{T^2q^2}{m4\pi\varepsilon_0 L^3}\displaystyle\sum_{j\neq i}\frac{x_i^\tau - x_j^\tau}{|x_i^\tau - x_j^\tau|^3} = \frac{1}{N}\sum_{j\neq i} aN\frac{x_i^\tau - x_j^\tau}{|x_i^\tau - x_j^\tau|^3}, \end{cases}$$

où la constante

$$a = \frac{T^2q^2}{m4\pi\varepsilon_0 L^3},$$

est adimensionnelle. Maintenant si les grandeurs caractéristiques du système, qui dépendent du nombre de particules, sont telles que

$$\frac{q^2}{m4\pi\varepsilon_0}\frac{NT^2}{L^3} = \mathcal{O}_{N\sim\infty}(1),$$

on peut transformer le système adimensionné en l'EDO

$$\begin{cases} \dfrac{dx_i^t}{dt} = v_i^t, \\ \dfrac{dv_i^t}{dt} = \dfrac{1}{N}\displaystyle\sum_{j\neq i} -\nabla W(x_i^t - x_j^t), \end{cases} \tag{20}$$

avec $W(x) = |x|^{-1}$, le potentiel Coulombien. Ainsi la mesure empirique associée au système de particules (20) est solution de l'équation de Vlasov-Poisson (en dimension 3) donnée par

$$\partial_t f_t + v \cdot \nabla_x f_t - (\nabla W * \rho_t) \cdot \nabla_v f_t = 0, \tag{21}$$

où ρ_t est la marginale en position de f_t. Si W est choisi librement, on parle simplement d'équation de Vlasov ou de type Vlasov. Mais avec le choix qu'on a fait, on note que le champ électrique

$$E(t,x) = -\nabla W * \rho_t(x),$$

dérive du potentiel électrique

$$V(t,x) = W * \rho_t(x),$$

et comme le potentiel Coulombien est la solution fondamentale de l'équation de Poisson, on a

$$-\Delta V(t,x) = (\Delta W) * \rho_t(x) = \rho_t(x),$$

et on parle alors d'équation de Vlasov-Poisson. Ce modèle (ainsi que tout un tas de variantes) est largement utilisé en physique des plasmas pour déterminer par exemple le comportement d'un nuage d'ions et de particules chargées confinées magnétiquement dans un Tokamak, ou en cosmologie pour déterminer le comportement d'une galaxie (il faut alors changer le signe devant le potentiel pour changer la force répulsive en une force attractive). Cependant, tant que la question de la limite en champ moyen n'est pas résolue, ce modèle reste, comme le dit Villani, "un acte de foi" [**18**].
Tout cela en fait un problème particulièrement prisé. Le résultat majeur dans ce domaine a été obtenu par Hauray et Jabin dans [**11**]. Cependant, ils considèrent une interaction sous-Newtonienne entre les particules du type

$$K(x) = \frac{x}{|x|^{1+\alpha}}, \quad \alpha < 1.$$

La stratégie est similaire à celle décrite pour le cas de l'équation d'agrégation (d'ailleurs je ne sais pas exactement à quel cas elle a été appliquée en premier), c'est à dire contrôler la distance minimale entre particules et faire usage de la distance W_∞ du fait de la forte singularité. Cependant, comme ici on est dans le cas d'un système du second ordre, les estimations sont beaucoup plus compliquées.

4. Argument de la corde

Reprenons l'équation (8), dans laquelle on remplace le champ d'interaction Lipschitz $b(\cdot)$ par $\mathbf{1}_{B_r}(\cdot)e$, avec $\mathbf{1}_{B_r}$ l'indicatrice de la boule de rayon $r > 0$ dans $\mathbb{R}^2$ et $e \in \mathbb{S}^1$. L'équation devient alors

$$\begin{cases} \dfrac{dx_t^i}{dt} = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \mathbf{1}_{B_r}(x_t^i - x_t^j)e \quad , \quad i = 1, \cdots, N, \\ x_{t=0}^i = x_0^i. \end{cases} \tag{22}$$

L'existence de solution à cette EDO à coefficient discontinu n'est pas évidente à prouver. Admettons la pour l'instant, car on souhaite attirer l'attention sur un autre point. L'EDP

non linéaire de transport dont est solution la mesure empirique associée au système (22) est donnée par

$$\begin{cases} \partial_t f_t + \nabla \cdot \big(f_t F[f_t]\big) = 0, \\ F[f_t] = e\mathbf{1}_{B_r} * f_t. \end{cases} \tag{23}$$

Sous réserve encore une fois d'existence des caractéristiques associées, qu'on admet pour le moment, on peut reprendre la méthode et les notations employées à la section 2.2. On pourrait reprendre les estimations menées à (13) jusqu'à l'avant dernière ligne, on obtiendrait alors

$$\begin{aligned} \frac{d}{dt}Q(t) &\leq \int \big|F[f_t](X^t(x)) - F[g_t](Y^t(\mathcal{T}_0(x)))\big| f_0(dx) \\ &\leq \int \Big| \int \big(\mathbf{1}_{B_r}(X^t(x) - y) - \mathbf{1}_{B_r}(Y^t(\mathcal{T}_0(x)) - \mathcal{T}_t(y))\big) f_t(dy) \Big| f_0(dx). \end{aligned} \tag{24}$$

On voit que la discontinuité empêche de conclure à l'aide de l'argument classique. Cependant, notons que $X^t(x)$ est transporté par $\mathcal{T}^t$ en $Y^t(\mathcal{T}_0(x))$ et de même y est transporté en $\mathcal{T}_t(y)$. Qualitativement, les distances $|X^t(x) - Y^t(\mathcal{T}_0(x))|$ et $|y - \mathcal{T}_t(y)|$ sont donc relativement petites alors que les normes de $X^t(x) - y$ et $Y^t(\mathcal{T}_0(x)) - \mathcal{T}_t(y)$ qui peuvent être vues comme des variables d'intégration parcourant les supports de f_t et g_t respectivement, n'ont pas de raison de l'être.
Mettons nous dans le cas où $|X^t(x) - y|$ est très grand devant le rayon r. Les distances $|X^t(x) - Y^t(\mathcal{T}_0(x))|$ et $|y - \mathcal{T}_t(y)|$, étant données par un transport, on peut les considérer comme petites devant r, de sorte que par l'inégalité triangulaire renversée, on a

$$|Y^t(\mathcal{T}_0(x)) - \mathcal{T}_t(y)| \geq |X^t(x) - y| - |X^t(x) - Y^t(\mathcal{T}_0(x))| - |y - \mathcal{T}_t(y)| >> r.$$

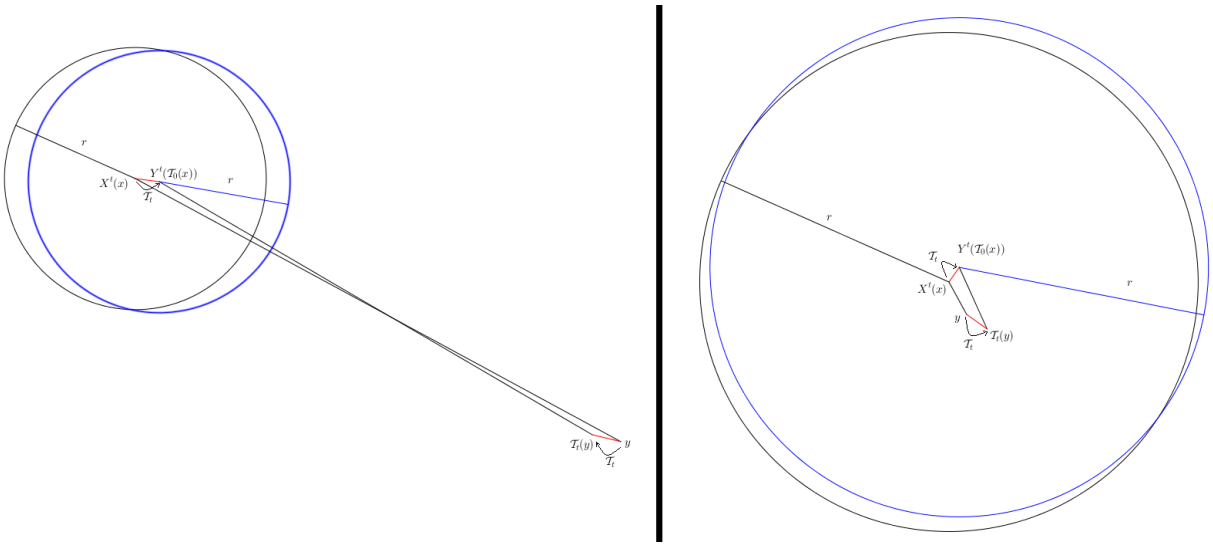
De même, si $|X^t(x) - y|$ est très petit devant r, on a également

$$|Y^t(\mathcal{T}_0(x)) - \mathcal{T}_t(y)| \leq |X^t(x) - y| + |X^t(x) - Y^t(\mathcal{T}_0(x))| + |y - \mathcal{T}_t(y)| << r.$$

Ainsi, dans ces deux cas marginaux on a

$$\mathbf{1}_{B_r}(X^t(x) - y) - \mathbf{1}_{B_r}(Y^t(\mathcal{T}_0(x)) - \mathcal{T}_t(y)) = 0,$$

et l'argument employé n'a utilisé que des considérations sur la région dans laquelle se trouve le vecteur $X^t(x) - y$.



Si maintenant on contrôlait le volume de la région dans laquelle $X^t(x) - y$ doit se trouver pour que cette différence puisse ne pas être nulle, comme elle est dans tous les cas

bornée par 1, on serait en mesure de borner le membre de droite de (24), à condition que la densité de f_t soit bornée. Aux vues de ce qu'on a dit plus haut, cette région semble répartie autour de la discontinuité sur une longueur d'ordre $|X^t(x) - Y^t(\mathcal{T}_0(x))| + |y - \mathcal{T}_t(y)|$ (la fameuse corde). Plus précisément, on a le

LEMME 4.1. *(Argument de la corde) Soient $x_1, x_2, y_1, y_2 \in \mathbb{R}^2$. Alors*

$$\big|\mathbf{1}_{B_r}(x_1 - y_1) - \mathbf{1}_{B_r}(x_2 - y_2)\big| \leq \mathbf{1}_{C_r^{2|x_1-x_2|}}(x_1 - y_1) + \mathbf{1}_{C_r^{2|y_1-y_2|}}(x_1 - y_1),$$

où $C_r^\varepsilon := B_{r+\varepsilon} \setminus B_{r-\varepsilon}$.

DÉMONSTRATION. Montrons que

$$\big|\mathbf{1}_{B_r}(x_1 - y_1) - \mathbf{1}_{B_r}(x_2 - y_2)\big| \leq \mathbf{1}_{B_{r+(|x_1-x_2|+|y_1-y_2|)}\setminus B_{r-(|x_1-x_2|+|y_1-y_2|)}}(x_1 - y_1).$$

Le résultat en découlera puisque

$$|x_1 - x_2| + |y_1 - y_2| \leq 2\max(|x_1 - x_2|, |y_1 - y_2|)$$

Supposons d'abord que $x_1 - y_1 \in B_{r-(|x_1-x_2|+|y_1-y_2|)}$. Alors

$$\begin{aligned} |x_2 - y_2| &\leq |x_1 - y_1| + |x_1 - x_2| + |y_1 - y_2| \\ &\leq r, \end{aligned}$$

et ainsi, dans ce cas on a

$$\big|\mathbf{1}_{B_r}(x_1 - y_1) - \mathbf{1}_{B_r}(x_2 - y_2)\big| = 0.$$

En raisonnant de la même manière, on aurait obtenu le même résultat si l'on était parti de l'hypothèse $x_1 - y_1 \notin B_{r+(|x_1-x_2|+|y_1-y_2|)}$. Ce qui achève la preuve. □

Revenons à l'estimation (24), en utilisant le Lemme 4.1, on obtient

$$\begin{aligned} \frac{d}{dt}Q(t) &\leq \int\int \big|\mathbf{1}_{B_r}(X^t(x) - y) - \mathbf{1}_{B_r}(Y^t(\mathcal{T}_0(x)) - \mathcal{T}_t(y))\big| f_t(dy) f_0(dx) \\ &\leq \int\int \mathbf{1}_{C_r^{2|X^t(x)-Y^t(\mathcal{T}_0(x))|}}(X^t(x) - y) f_t(dy) f_0(dx) + \int\int \mathbf{1}_{C_r^{2|y-\mathcal{T}_t(y)|}}(X^t(x) - y) f_0(dx) f_t(dy) \\ &:= J_t^1 + J_t^2. \end{aligned}$$

Maintenant, il est clair que si $\varepsilon \leq r$, on a

$$|C_r^\varepsilon| = \pi\big((r+\varepsilon)^2 - (r-\varepsilon)^2\big) = 4\pi r\varepsilon,$$

et donc en notant $A := \{x \in \mathbb{R}^2\ ,\ |X^t(x) - Y^t(\mathcal{T}_0(x))| \leq r/2\}$, on estime J_1^t ainsi

$$\begin{aligned} J_t^1 &\leq \int_A \int \mathbf{1}_{C_r^{2|X^t(x)-Y^t(\mathcal{T}_0(x))|}}(X^t(x) - y) f_t(dy) f_0(dx) + \int_{\mathbb{R}^2\setminus A} \int \mathbf{1}_{C_r^{2|X^t(x)-Y^t(\mathcal{T}_0(x))|}}(X^t(x) - y) f_t(dy) f_0(dx) \\ &\leq \int_A 8\pi r \|f_t\|_\infty |X^t(x) - Y^t(\mathcal{T}_0(x))| f_0(dx) + \int_{\mathbb{R}^2\setminus A} \frac{2}{r}|X^t(x) - Y^t(\mathcal{T}_0(x))| f_0(dx) \\ &\leq C\big(\|f_t\|_\infty + 1\big) \int |X^t(x) - Y^t(\mathcal{T}_0(x))| f_0(dx) =: C\big(\|f_t\|_\infty + 1\big) Q(t), \end{aligned}$$

où C est une constante qui ne dépend que de r. On estime J_2^t de la même manière, en réalisant préalablement un changement de variable le long de $(X^t)^{-1}$. Au final, après l'utilisation du Lemme de Gronwall, on obtient

$$W_1(f_t, g_t) \leq Q(t) \leq e^{C\int_0^t \big(\|f_s\|_\infty + 1\big) ds} Q(0) = W_1(f_0, g_0), \tag{25}$$

qui est l'estimation de stabilité désirée. Contrairement à l'estimation obtenue à la section 2.2, les rôles joués par f et g sont asymétriques ; c'est pourquoi on parlera parfois d'estimation de type fort-faible au sens où la régularité d'une des deux solutions peut être très faible (typiquement une solution de type mesure atomique), mais l'autre doit admettre une densité par rapport à la mesure de Lebesgue qui est dans $L^\infty(\mathbb{R}^2)$. L'argument de la

corde a été introduit par Maxime Hauray dans [**12**], pour traiter du cas de la limite de champ moyen pour l'équation de Vlasov-Poisson 1d.

5. Modèles étudiés dans cette thèse

5.1. Vlasov-Poisson-Fokker-Planck 1D. Le point de départ de mes travaux de thèse a donc été d'adapter cet argument de la corde à un modèle cinétique avec bruit, pour lesquels il n'y avait à l'époque pas de résultat de propagation du chaos pour des interactions non Lipschitz et sans cut-off (les récents résultats en la matière ont été cités plus haut). En dimension 1, la solution fondamentale de l'équation de Poisson

$$\partial_x^2 \Phi(x) = \delta_0,$$

est donnée par

$$\Phi(x) = \frac{|x|}{2}.$$

Ainsi, la force d'attraction coulombienne qui dérive de Φ est donnée par

$$F(x) = -\partial_x \Phi(x) = -\frac{x}{2|x|} = -2^{-1}\text{sign}(x) \quad \text{si } x \neq 0,$$

avec la condition naturelle de non auto-interaction $F(0) = 0$. La singularité de cette interaction est semblable à celle d'une fonction indicatrice et l'argument de la corde est adapté pour ce problème. Plus précisément le système étudié s'écrit

$$\begin{cases} X_t^i = X_0^i + \displaystyle\int_0^t V_s^i \, ds, \\ V_t^i = V_0^i + \chi \displaystyle\int_0^t \frac{1}{N} \sum_{j=1}^N F(X_s^i - X_s^j) \, ds + \sqrt{2\sigma} B_t^i, \end{cases} \tag{26}$$

où les (X_0^i, Y_0^i) sont i.i.d. de loi $f_0 \in \mathcal{P}_1(\mathbb{R}^{2d})$, les $(B_\cdot^i)_{i=1,\cdots,N}$ sont des mouvements browniens indépendants et indépendants des conditions initiales, $\chi \in \mathbb{R}$ (l'interaction peut être attractive ou répulsive) et $\sigma \in \mathbb{R}^+$. La limite attendue de la loi de la solution de (26) quand le nombre de particules tend vers l'infini est donnée par l'équation de Vlasov-Poisson-Fokker-Planck en dimension 1

$$\begin{cases} \partial_t f_t + v.\partial_x f_t + \chi(F * \rho_t)(x).\partial_v f_t = \sigma \partial_v^2 f_t, \\ \rho_t(x) = \displaystyle\int f_t(x,v) dv. \end{cases} \tag{27}$$

Tout d'abord, on montre l'existence de solutions à (26), qui n'est pas assurée par la théorie standard. Comme la diffusion est ici additive, on se contente de montrer l'existence faible au sens des EDS, c'est à dire qu'on construit ad hoc à l'aide du Théorème de Girsanov un espace de probabilité et un processus sur cet espace qui s'avère être solution de (26). On montre également l'existence et l'unicité de solution à (27) vérifiant

$$\int_0^t \|\rho_s\|_{L^\infty} \, ds < \infty,$$

en régularisant le noyau discontinu puis avec une estimation de complétude obtenue grâce au même argument utilisé pour la limite macroscopique. Pour établir cette limite, on procède classiquement comme présenté à la Section 2.1 en introduisant N copies indépendantes

$$\begin{cases} Y_t^i = X_0^i + \displaystyle\int_0^t W_s^i \, ds, \\ W_t^i = V_0^i + \chi \displaystyle\int_0^t \int_{\mathbb{R}} F(Y_s^i - y) \rho_s(dy) \, ds + \sqrt{2\sigma} B_t^i \quad , \quad \rho_s = \mathcal{L}(Y_s^i), \end{cases} \tag{28}$$

qui sont par construction i.i.d. En comparant les trajectoires de (26) et de (28), on s'aperçoit (en choisissant $\chi = 1$ pour la simplicité des notations) que la difficulté vient du terme

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N}\Big(|X_t^i - Y_t^i| + |V_t^i - W_t^i|\Big) \leq \int_0^t \frac{1}{N}\sum_{i=1}^{N}|V_s^i - W_s^i|\,ds \\ + \int_0^t \frac{1}{N}\sum_{i=1}^{N}\frac{1}{N}\sum_{j=1}^{N}\Big|F(X_s^i - X_s^j) - F(Y_s^i - Y_s^j)\Big|\,ds \\ + \int_0^t \underbrace{\frac{1}{N}\sum_{i=1}^{N}\frac{1}{N}\sum_{j=1}^{N}\Big|F(Y_s^i - Y_s^j) - \int_0^t\int_{\mathbb{R}} F(Y_s^i - y)\rho_s(dy)\Big|}_{=H_s^N}\,ds,\end{aligned}$$

et plus particulièrement du second terme de droite. Avec des considérations similaires à celles développées dans la Section 4, on obtient que pour tout $x_1, x_2, y_1, y_2 \in \mathbb{R}$

$$|\mathrm{sign}(x_1 - x_2) - \mathrm{sign}(y_1 - y_2)| \leq \mathbf{1}_{|y_1-y_2|\leq 2|x_1-y_1|} + \mathbf{1}_{|y_1-y_2|\leq 2|x_2-y_2|},$$

de sorte qu'on peut borner ce second terme problématique

(29)

$$\begin{aligned}\int_0^t \frac{1}{N}\sum_{i=1}^{N}\frac{1}{N}\sum_{j=1}^{N}\Big|F(X_s^i - X_s^j) - F(Y_s^i - Y_s^j)\Big|\,ds &\leq \int_0^t \frac{1}{N}\sum_{i=1}^{N}\frac{1}{N}\sum_{j=1}^{N}\mathbf{1}_{|Y_s^i-Y_s^j|\leq 2|X_s^i-Y_s^i|}\,ds \\ &\quad + \int_0^t \frac{1}{N}\sum_{j=1}^{N}\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}_{|Y_s^i-Y_s^j|\leq 2|X_s^j-Y_s^j|}\,ds \\ &= 2\int_0^t \frac{1}{N}\sum_{i=1}^{N}\rho_s^N\Big([Y_s^i - 2|X_s^i - Y_s^i|, Y_s^i + 2|X_s^i - Y_s^i|]\Big)\,ds,\end{aligned}$$

où $\rho_s^N := \frac{1}{N}\sum_{i=1}^{N}\delta_{Y_s^i}$ est la mesure empirique associée aux N copies i.i.d. de (28). Deux stratégies sont alors adoptées, donnant lieu à deux types de résultats différents. La première consiste à introduire une norme infinie discrète $\|\cdot\|_{\infty,\varepsilon}$ définie sur $\mathcal{P}(\mathbb{R})$ par

$$\|\nu\|_{\infty,\varepsilon} = \sup_{x\in\mathbb{R}}\frac{\nu\left([x-\varepsilon, x+\varepsilon]\right)}{2\varepsilon} \leq (2\varepsilon)^{-1}.$$

Ainsi, on obtient

$$\int_0^t \frac{1}{N}\sum_{i=1}^{N}\frac{1}{N}\sum_{j=1}^{N}\Big|F(X_s^i - X_s^j) - F(Y_s^i - Y_s^j)\Big|\,ds \leq \int_0^t \|\rho_s^N\|_{\infty,\varepsilon}\big(\frac{1}{N}\sum_{i=1}^{N}|X_s^i - Y_s^i| + 2\varepsilon\big)\,ds.$$

Au final, l'application de Lemme de Gronwall donne

$$\frac{1}{N}\sum_{i=1}^{N}\Big(|X_t^i - Y_t^i| + |V_t^i - W_t^i|\Big) \leq e^{\int_0^t\left(1+\|\rho_s^N\|_{\infty,\varepsilon}\right)ds}\int_0^t\left(2\varepsilon\|\rho_s^N\|_{\infty,\varepsilon} + H_s^N\right)ds.$$

Maintenant, comme ρ_s est la limite faible des $(\rho_s^N)_N$ (cf [**9**]), on s'attend à ce que

$$\int_0^t \|\rho_s^N\|_{\infty,\varepsilon}\,ds < \infty,$$

avec grande probabilité pour N et ε bien choisis. On quantifie cette probabilité à l'aide de techniques de grandes déviations. Pour $x \in \mathbb{R}$ fixé, la variable aléatoire

$$\frac{\rho_s^N\left([x-\varepsilon, x+\varepsilon]\right)}{2\varepsilon} = \frac{1}{2N\varepsilon}\sum_{i=1}^{N}\mathbf{1}_{Y_s^i\in[x-\varepsilon,x+\varepsilon]},$$

est proportionnelle à une variable de Binomiale de paramètres $N, \rho_s([x-\varepsilon, x+\varepsilon])$. Des estimations classiques donnent ainsi

$$\sup_{s\leq t}\sup_{x\in\mathbb{R}}\mathbb{P}\Big(\frac{\rho_s^N([x-\varepsilon,x+\varepsilon])}{2\varepsilon}\geq \|\rho_s\|_{L^\infty}+\alpha\Big)\leq 2e^{-2N\alpha^2}.$$

Il s'agit maintenant d'utiliser l'équation (28) ainsi que les moments en position de sa solution pour intervertir $\sup_{s\leq t}\sup_{x\in\mathbb{R}}$ et $\mathbb{P}$ dans l'estimation ci-dessus, afin de contrôler la probabilité que la variable aléatoire $\int_0^t \|\rho_s^N\|_{\infty,\varepsilon}\,ds$ dépasse un seuil raisonnable, par exemple 10 fois son espérance.
La seconde stratégie consiste à majorer le second terme problématique dans (29), ainsi

$$\begin{aligned}\rho_s^N\left([Y_s^i-2|X_s^i-Y_s^i|,Y_s^i+2|X_s^i-Y_s^i|]\right)&\leq \rho_s\left([Y_s^i-2|X_s^i-Y_s^i|,Y_s^i+2|X_s^i-Y_s^i|]\right)\\&\quad+\left|(\rho_s^N-\rho_s)\left([Y_s^i-2|X_s^i-Y_s^i|,Y_s^i+2|X_s^i-Y_s^i|]\right)\right|\\&\leq \|\rho_s\|_{L^\infty}|X_s^i-Y_s^i|+\sup_{u\geq 0}\left|(\rho_s-\rho_s^N)\left([Y_s^i-u,Y_s^i+u]\right)\right|.\end{aligned}$$

Il s'agit alors de voir le second terme comme un terme de fluctuation dont on peut obtenir sur l'espérance des taux de décroissance en N. En effet, on remarque qu'il implique la différence entre une mesure empirique et sa moyenne. On se tourne alors naturellement vers [**9**] où une technique de dépoissonisation est employée pour mesurer un tel écart. Celle-ci consiste à introduire une mesure aléatoire de Poisson M_s

$$M_s=N^{-1}\sum_{i=1}^{K_N}\delta_{Y_s^i},$$

avec K_N une variable aléatoire de Poisson de paramètre N. L'intensité M_s est alors ρ_s et on obtient

$$\begin{aligned}\sup_{u\geq 0}\left|(\rho_s-\rho_s^N)\left([Y_s^i-u,Y_s^i+u]\right)\right|&\leq \sup_{u\geq 0}\left|(\rho_s^N-M_s)\left([Y_s^i-u,Y_s^i+u]\right)\right|\\&\quad+\sup_{u\geq 0}|(\overbrace{M_s-\rho_s}^{=\bar{M}_s})\left([Y_s^i-u,Y_s^i+u]\right)|\\&\leq \frac{|K_N-N|}{N}+\sup_{u\geq 0}\mathcal{M}_u^{i,s},\end{aligned}$$

où $\bar{M}$ désigne la mesure de Poisson compensée (i.e. qu'on lui a retranché son intensité). Maintenant, les propriétés élémentaires des mesures aléatoires de Poisson font de $(\mathcal{M}_u^{i,s})_{u\geq 0}$ une martingale pour une filtration bien choisie, de sorte que les moments de son supremum sont estimables à l'aide de l'inégalité maximale de Doob. Ces deux stratégies mènent au

THÉORÈME 5.1. *(Hauray, S. 2015) Soit $f_0\in\mathcal{P}(\mathbb{R}^2)$ et ρ_0 sa marginale en position. On suppose $\rho_0\in L^\infty(\mathbb{R})$. Pour tout $T>0$, il existe alors une unique solution $(f_t)_{t\in[0,T]}$ à (27) avec pour condition initiale f_0, telle que la famille des marginales en position $(\rho_t)_{t\in[0,T]}$ vérifie*

$$\int_0^T \|\rho_s\|_{L^\infty}\,ds<\infty,$$

De plus, en notant $\mu_t^N:=\frac{1}{N}\sum_{i=1}^N\delta_{X_t^i,V_t^i}$ (respectivement $\nu_t^N:=\frac{1}{N}\sum_{i=1}^N\delta_{Y_t^i,W_t^i}$) la mesure empirique associée à la solution de (26) (respectivement la solution de (27)), on a

- *dans le cas où f_0 admet des moments d'ordre 1,*

$$\sup_{t\leq T}\mathbb{E}\left[W_1(\mu_t^N,\nu_t^N)\right]\leq\frac{9}{\sqrt{N}}e^{\int_0^t(1+\|\rho_s\|_{L^\infty})\,ds},$$

- *dans le cas où f_0 admet des moments exponentiels d'ordre $\lambda > 0$, pour tout $\varepsilon \geq \lambda N^{-1/2}$ il existe une constante $C > 0$ dépendant uniquement de λ, T et de f_0 telle que*

$$\mathbb{P}\left(\sup_{t\leq T} W_1(\mu_t^N, \nu_t^N) \geq C\varepsilon\right) \leq C(1+\varepsilon\sqrt{N})N^{3/2}e^{-2N\varepsilon^2}.$$

Bien sur on peut conjuguer à ce résultat les résultats de [**9**] pour donner des estimations sur les quantités

$$\mathbb{E}\left[W_1(\mu_t^N, f_t)\right], \quad \text{et} \quad \mathbb{P}\left(W_1(\mu_t^N, f_t) \geq C\varepsilon\right).$$

Ce travail en collaboration avec mon directeur de thèse Maxime Hauray, a donné lieu à la rédaction d'un article. Celui ci est actuellement soumis à la revue **Kinetic and Related Models**.

5.2. Équation de Cucker Smale avec cône de vision. Ce modèle issue de l'éthologie, qui rend compte du mouvement d'un nuage d'oiseaux ou d'un banc de poissons, a été proposé dans [**3**]. Celui-ci se lit

$$\begin{cases} \dfrac{dx_t^i}{dt} = v_t^i, \\ \dfrac{dv_t^i}{dt} = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \psi(x_t^i - x_t^j)(v_t^j - v_t^i), \end{cases}$$

où $\psi : \mathbb{R}^d \to \mathbb{R}$ est un poids de communication qui pondère l'intensité de l'action de l'agent j sur l'agent i. L'agent i aligne donc sa vitesse sur les vitesses des autres agents, avec une pondération qui dépend uniquement de la position relative. Des phénomènes dits de "flocking" se produisent en temps long sur les solutions de ce système, c'est à dire que les positions relatives sont asymptotiquement bornées, et les vitesses relatives tendent vers 0. Une propriété intéressante, et que l'on utilisera, est que les vitesses sont à tout temps bornées par le maximum des vitesses à l'état initial (puisque la dynamique aligne les vitesses). Précisément le système de particules étudié s'écrit

$$\begin{cases} \dfrac{dx_t^i}{dt} = v_t^i, \\ \dfrac{dv_t^i}{dt} = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \mathbf{1}_{K(v_t^i)}(x_t^j - x_t^i)(v_t^j - v_t^i), \end{cases} \tag{30}$$

où $K(v) \subset \mathbb{R}^d$ est le domaine de vision d'un agent à vitesse v (physiquement il est raisonnable de penser qu'un tel domaine dépende de la vitesse à laquelle se déplace l'agent). Un exemple typique de domaine de vision choisi est le cône de vision défini par

$$C(v) := \{x \in \mathbb{R}^d \,|\, |x| \leq r\,, |\frac{x\cdot v}{|x||v|}| \leq \cos^{-1}(\theta(|v|))\},$$

où $r > 0$ est fixé et $\theta : \mathbb{R}^+ \to [0, \pi]$ est une fonction décroissante telle qu'il existe $u_0 > 0$ tel que $\theta(u) = \pi$ pour $u \leq u_0$. De plus on suppose que θ est uniformément bornée par le bas par une valeur θ^*. Ces deux hypothèses techniques sur la fonction d'angle sont là pour contrôler ce qu'il se produit dans les deux régimes asymptotiques des vitesses très grandes et très petites, qui sont à exclure dans le cadre du modèle de Cucker-Smale. Pour généraliser le cône de vision on introduit une application $K(\cdot)$ à valeur dans l'espace des sous-ensembles de $\mathbb{R}^d$, pour laquelle il existe une application Θ à valeur dans l'espace des sous-ensembles de $\mathbb{R}^d$ de dimension de Hausdorff $d-1$ qui vérifie les hypothèses suivantes

- $\mathbf{H} - (i)$ $\partial K(v) \subset \Theta(v)$, pour tout $v \in \mathbb{R}^d$,
- $\mathbf{H} - (ii)$ $|\Theta(v)^{\varepsilon,+}| \leq C\varepsilon$, pour tout $\varepsilon \in (0,1)$,
- $\mathbf{H} - (iii)$ $K(v)\Delta K(w) \subset \Theta(v)^{C|v-w|,+}$, pour $v, w \in \mathbb{R}^d$,

- $\mathbf{H}-(iv)$ $\Theta(w)\subset\Theta(v)^{C|v-w|,+}$, pour $v,w\in\mathbb{R}^d$,

où pour un ensemble A on a noté

$$A^{\varepsilon,+}:=\{x\in\mathbb{R}^d\,|\,d(x,A)\leq\varepsilon\},$$

son ε-élargissement. L'application Θ peut être vue comme une régularisation de l'application $v\mapsto\partial K(v)$.

Comme à la Section 2.2, la question de la limite macroscopique pour le système (30) se résume à une question de stabilité dans la classe des solutions au sens des mesures de l'équation

$$\left\{\begin{array}{l}\partial_t f_t+v\cdot\nabla_x f_t+\nabla_v\cdot(f_tF[f_t])=0,\\ F[f](x,v)=\displaystyle\int_{\mathbb{R}^{2d}}\mathbf{1}_{K(v)}(y-x)(w-v)f(dy,dw).\end{array}\right. \tag{31}$$

Mais comme à la Section 4 l'estimation de stabilité qu'on peut espérer établir est du type fort-faible. C'est à dire qu'il faut qu'on choisisse la solution limite vers laquelle on veut faire tendre le système de particules, avec une certaine régularité. Comme on veut utiliser un argument similaire au Lemme 4.1, il faut que la densité spatiale de la solution limite soit bornée. Comme la dynamique impose à la solution un support compact en vitesse si la condition initiale est elle même à support compact en vitesse, il suffit de choisir une solution limite bornée pour que sa densité spatiale le soit. Cependant comme ici le champ de transport n'est pas à divergence nulle, on est seulement en mesure de prouver que pour $f_0\in\mathcal{P}_1\cap L^\infty(\mathbb{R}^{2d})$ il existe un temps $t_0>0$ et une unique solution à (31) $(f_t)_{t\leq t_0}$ telle que

$$\int_0^{t_0}\|f_s\|_{L^\infty}\,ds<\infty.$$

Soit alors $X^t=(X_1^t,X_2^t)$ la caractéristique associée à cette solution définie par

$$\left\{\begin{array}{l}\dfrac{dX_1^t(x,v)}{dt}=X_2^t(x,v),\\ \dfrac{dX_2^t(x,v)}{dt}=\displaystyle\int_{\mathbb{R}^{2d}}\mathbf{1}_{K(X_2^t(x,v))}(y-X_1^t(x,v))(w-X_2^t(x,v))f_t(y,w)dydw.\end{array}\right.$$

Soit maintenant $(g_t)_{t\leq t_0}$ une autre solution de (31) sur laquelle on ne suppose aucune régularité, $Y^t=(Y_1^t,Y_2^t)$ la caractéristique associée à $(g_t)_{t\leq t_0}$ et $\mathcal{T}_0$ l'application qui transporte f_0 sur g_0. De sorte que l'application $\mathcal{T}^t=(\mathcal{T}_1^t,\mathcal{T}_2^t):=Y^t\circ\mathcal{T}_0\circ(X^t)^{-1}$ transporte f_t sur g_t et il s'agit, pour obtenir la stabilité fort-faible, d'estimer la quantité

$$Q(t):=\int_{\mathbb{R}^{2d}}\left(|X_1^t(x,v)-Y_1^t\left(\mathcal{T}^0(x,v)\right)|+|X_2^t(x,v)-Y_2^t\left(\mathcal{T}^0(x,v)\right)|\right)\,f_0(x,v)dxdv.$$

La difficulté apparait lorsqu'on différencie le second terme de droite en temps. En introduisant des termes croisés et en se rappelant que les vitesses sont uniformément bornées par la vitesse maximale à l'état initial, on a

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^{2d}}|X_2^t(x,v)-Y_2^t\left(\mathcal{T}^0(x,v)\right)|\,f_0(x,v)dxdv\\ &\quad\leq\int_{\mathbb{R}^{4d}}\left|(w-X_2^t(x,v))-(\mathcal{T}_2^t(y,w)-Y_2^t\left(\mathcal{T}^0(x,v)\right)\right|\,f_t(y,w)dydw\,f_0(x,v)dxdv\\ &\quad+2V_{max}\Big(\int_{\mathbb{R}^{4d}}\left|\mathbf{1}_{K(X_2^t(x,v))}(y-X_1^t(x,v))-\mathbf{1}_{K(Y_2^t(\mathcal{T}^0(x,v)))}(y-X_1^t(x,v))\right|\,f_t(y,w)dydw\,f_0(x,v)dxdv\\ &\quad+\int_{\mathbb{R}^{4d}}\left|\mathbf{1}_{K(Y_2^t(\mathcal{T}^0(x,v)))}(y-X_1^t(x,v))-\mathbf{1}_{K(Y_2^t(\mathcal{T}^0(x,v)))}(\mathcal{T}_1^t(y,w)-Y_2^t\left(\mathcal{T}^0(x,v)\right))\right|\,f_t(y,w)dydw\,f_0(x,v)dxdv\Big).\end{aligned}$$

Le premier terme du membre de droite ne pose pas de difficulté. On réécrit le second

$$\int_{\mathbb{R}^{4d}} \left|\mathbf{1}_{K(X_2^t(x,v))}(y - X_1^t(x,v)) - \mathbf{1}_{K(Y_2^t(\mathcal{T}^0(x,v)))}(y - X_1^t(x,v))\right| f_t(y,w)dydw\, f_0(x,v)dxdv$$
$$= \int_{\mathbb{R}^{4d}} \mathbf{1}_{K(X_2^t(x,v))\Delta K(Y_2^t(\mathcal{T}^0(x,v)))}(y - X_1^t(x,v)) f_t(y,w)dydw\, f_0(x,v)dxdv,$$

où $A\Delta B$ désigne la différence symétrique entre deux ensembles. On peut traiter ce terme facilement l'aide des hypothèses $\mathbf{H}-(i)$ et $\mathbf{H}-(ii)$, puisque f_t est bornée pour tout $t \leq t_0$. Le dernier terme se traite de manière analogue à ce qui a été développé dans la Section 4. Il faut cependant modifier le Lemme 4.1 en le

LEMME 5.1. *Soit $K \subset \mathbb{R}^d$ et $x_1, x_2, y_1, y_2 \in \mathbb{R}^d$. Alors*

$$\left|\mathbf{1}_K(x_1 - y_1) - \mathbf{1}_K(x_2 - y_2)\right| \leq \mathbf{1}_{\partial K^{2|x_1-x_2|,+}}(x_1 - y_1) + \mathbf{1}_{\partial K^{2|y_1-y_2|,+}}(x_1 - y_1),$$

où $\partial K^{\varepsilon,+} := \{x \in \mathbb{R}^d \,|\, d(x, \partial K) \leq \varepsilon\}$.

Ici on a remplacé le pendant de la couronne C_r^ε pour la boule B_r par l'ε-élargissement de la frontière ∂K. Au final, on a le

THÉORÈME 5.2. *(Carrillo, Choi, Hauray, S., 2015) Soit $f_0 \in \mathcal{P}_1 \cap L^\infty(\mathbb{R}^{2d})$, qu'on suppose à support compact en vitesse. Il existe alors un temps $T > 0$, et une unique solution $(f_t)_{t\in[0,T]}$ à (31) où le domaine de vision $K(v)$ satisfait les hypothèses $\boldsymbol{H} - (i - iv)$, elle aussi à support compact en vitesse et qui vérifie*

$$\int_0^T \|f_s\|_{L^\infty}\, ds < \infty.$$

De plus si $(g_t)_{t\in[0,T]}$ est une autre solution à (31) pour la condition initiale $g_0 \in \mathcal{P}_1(\mathbb{R}^{2d})$ on a

$$W_1(f_t, g_t) \leq e^{C\int_0^t \|f_s\|_{L^1\cap L^\infty}\, ds} W_1(f_0, g_0),$$

où $\|\cdot\|_{L^1\cap L^\infty} = \|\cdot\|_{L^1} + \|\cdot\|_{L^\infty}$, et $C > 0$ est une constante qui dépend du support en vitesse de $f_.$.

Ce théorème prouve que la mesure empirique associée à toute solution de (30) converge vers la solution de (31) dont la condition initiale est donnée par la limite faible des mesures empiriques initiales. Cependant la discontinuité des coefficients rend difficile de trouver des solutions de (30). On remplace donc ce système d'EDO avec coefficients discontinus par un système d'inclusion différentielle pour lequel on est en mesure de trouver des solutions grâce à un Théorème de Fillipov. Comme la singularité est bornée, on parvient à montrer que les solutions de ce système d'inclusion différentielle de substitution convergent tout de même vers le profil macroscopique attendu.
Ce travail a été réalisé en collaboration avec José Carrillo et Young-Pil Choi de l'Imperial College of London, ainsi qu'avec mon directeur de thèse Maxime Hauray. Il a donné lieu à la rédaction d'un article accepté pour publication à la revue **Journal of the Europeean Mathematical Society**.

5.3. Équation de Cucker Smale avec cône de vision et diffusion tronquée. Une fois le projet précédent conclu, avec Young-Pil Choi nous avions eu l'intention de nous intéresser à la version stochastique du modèle de Cucker-Smale que nous venions d'étudier. Typiquement

$$\begin{cases} X_t^i = X_0^i + \displaystyle\int_0^t V_s^i\, ds, \\ V_t^i = V_0^i + \displaystyle\int_0^t \frac{1}{N}\sum_{j=1}^N \mathbf{1}_{K(V_s^i)}(X_s^j - X_s^i)(V_s^j - V_s^i)\, ds + \sqrt{2\sigma}B_t^i, \end{cases} \tag{32}$$

où les (B^i) sont des mouvements browniens indépendants. Cependant appliquer une telle diffusion en vitesse détruit la propriété de la version déterministe (30) qui conserve les

vitesses bornées par le maximum des vitesses à l'état initial. Cette propriété était fortement utilisée dans le travail précédent. Une stratégie envisageable [**1**] pouvait être de montrer que le système propage des moments exponentiels en vitesse, et d'utiliser ces moments pour montrer que les grandes vitesses ne pèsent pas sur les estimations. Cependant nous avons préféré ne pas introduire de diffusion additive, précisément nous nous sommes intéressés au système suivant

$$\begin{cases} X_t^i = X_0^i + \displaystyle\int_0^t V_s^i \, ds, \\ V_t^i = V_0^i + \displaystyle\int_0^t \frac{1}{N} \sum_{j=1}^N \mathbf{1}_{K(V_s^i)}(X_s^j - X_s^i)(V_s^j - V_s^i) \, ds + \sqrt{2\sigma} \int_0^t \mathcal{R}(V_s^i) dB_s^i, \end{cases} \tag{33}$$

où $\mathcal{R}$ est une fonction régulière à support compact. Celle ci joue le rôle de coefficient de troncature et annule la diffusion perçue par un agent si sa vitesse devient trop grande. Ainsi en choisissant les vitesses initiales dans le support de $\mathcal{R}$ on obtient qu'à tout instant ultérieur les vitesses restent dans ce support. L'équation limite devient dans ce cas

$$\begin{cases} \partial_t f_t + v \cdot \nabla_x f_t + \nabla_v \cdot (f_t F[f_t]) = \sigma \nabla_v \left(\mathcal{R}^2(v) f_t \right), \\ F[f](x,v) = \displaystyle\int_{\mathbb{R}^{2d}} \mathbf{1}_{K(v)}(y-x)(w-v) f(dy, dw). \end{cases} \tag{34}$$

Cependant avec ce bruit non additif, il n'est pas possible d'obtenir l'existence au moins au sens faible de solutions à l'équation (33) à l'aide du théorème de Girsanov. On arrive cependant à montrer l'existence de solutions au problème d'inclusion intégrale, c'est à dire l'équation (33) où l'égalité est remplacée par une inclusion et le drift discontinu par une fonction multi-valeurs, précisément

(35)
$$\begin{cases} X_t^i = X_0^i + \displaystyle\int_0^t V_s^i \, ds, \\ V_t^i = V_0^i + \displaystyle\int_0^t \frac{1}{N} \sum_{j=1}^N \alpha_s^{i,j}(V_s^j - V_s^i) \, ds + \sqrt{2\sigma} \int_0^t \mathcal{R}(V_s^i) dB_s^i, \quad \alpha_s^{i,j} \in F(X_s^i, X_s^j, V_s^i), \end{cases}$$

avec

$$F(x,y,v) = \begin{cases} \{0\} & \text{si } y - x \notin K(v) \\ \{1\} & \text{si } y - x \in \overset{\circ}{(K(v))} \\ [0,1] & \text{si } y - x \in \partial K(v) \end{cases}.$$

A l'aide des hypothèses introduites lors du travail précédent on établit une estimation de stabilité fort-faible similaire à celle obtenue dans le cas déterministe

THÉORÈME 5.3. *(Choi, S., 2017) Soit $\mathcal{R} \in \mathcal{C}^\infty(\mathbb{R}^d)$ une fonction à support compact, et $f_0 \in \mathcal{P}_1 \cap L^\infty(\mathbb{R}^{2d})$ à support compact en vitesse dans le support de $\mathcal{R}$. Il existe alors un temps $T > 0$, et une unique solution $(f_t)_{t\in[0,T]}$ à (31) où le domaine de vision $K(v)$ satisfait les hypothèses $\boldsymbol{H} - (i - iv)$, elle aussi à support compact en vitesse et qui vérifie*

$$\int_0^T \|f_s\|_{L^\infty} \, ds < \infty.$$

De plus si $(g_t)_{t\in[0,T]}$ est une autre solution à (31) pour la condition initiale $g_0 \in \mathcal{P}_1(\mathbb{R}^{2d})$ on a

$$W_1(f_t, g_t) \leq e^{C \int_0^t \|f_s\|_{L^1 \cap L^\infty} ds} W_1(f_0, g_0),$$

où $\|\cdot\|_{L^1\cap L^\infty} = \|\cdot\|_{L^1} + \|\cdot\|_{L^\infty}$, et $C > 0$ est une constante qui dépend du support en vitesse de $f_.$.

On montre également à l'aide de ce résultat et des techniques de martingale introduites lors des travaux précédents (et originellement trouvées pour ce travail ci) que les solutions de l'équation (35) avec des conditions initiales i.i.d. de loi f_0 supposée comme au Théorème ci-dessus sont $f_\cdot$-chaotique où $f_\cdot$ est la solution de (34) pour la condition initiale f_0. Ce travail en collaboration avec Young-Pil Choi entre temps passé par le Technische Universitat Muenchen et maintenant à l'Université Inha de Incheon (Corée du Sud) a donné lieu à la rédaction d'un article qui est actuellement soumis à **Journal of Differential Equations**.

5.4. Équation d'agrégation confinée avec cône de vision. Puis, toujours avec Young-Pil Choi, nous nous sommes intéressés au problème d'introduire l'interaction de contrainte de vision géométrique dans un modèle du premier ordre, c'est à dire n'impliquant que les positions. En effet du point de vue physique ce type d'interaction fait également sens dans la modélisation des mouvements de piétons, et du point de vue des estimations tout ne se passe pas exactement comme dans le cas cinétique.

Jusque là, l'argument de la corde n'avait été utilisé uniquement pour établir des estimations de stabilité fort-faible en distance W_1. Cette métrique n'est pas bien adaptée à l'interaction de type cône de vision pour un système du premier ordre. En effet, considérons le système de particules en interaction suivant

$$X_t^i = X_0^i + \int_0^t \frac{1}{N}\sum_{j=1}^N \mathbf{1}_{K(w(X_s^i))}(X_s^j - X_s^i)\nabla\varphi(X_s^i - X_s^j)\,ds,$$

et le système de N-copies indépendantes couplées au système de particules

$$Y_t^i = X_0^i + \int_0^t \int_{\mathbb{R}^d} \mathbf{1}_{K(w(Y_s^i))}(y - Y_s^i)\nabla\varphi(Y_s^i - y)\rho_s(dy)\,ds,$$

où $K(\cdot)$ est une application de $\mathbb{R}^d$ à valeurs dans les sous ensembles de $\mathbb{R}^d$, $w : \mathcal{O} \to \mathbb{R}^d$ est un champ d'orientation et $\nabla\varphi$ est un champ d'interaction. Un exemple typique d'application $K(\cdot)$ qu'on souhaite choisir est le cône de vision défini pour $y \in \mathbb{R}^d$

$$C(y) := \{z \in \mathbb{R}^d \,|\, |z| \le r,\, \big|\frac{z\cdot y}{|z||y|}\big| \le \cos^{-1}(\theta)\},$$

où $r > 0$ et $\theta \in (0,\pi)$ sont donnés. Si l'on souhaite reproduire la stratégie adaptée pour la distance W_1 développée dans ces sections il nous faudra estimer

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^N \frac{1}{N}\sum_{j=1}^N &\Big|\mathbf{1}_{K(w(X_s^i))}(X_s^j - X_s^i)\nabla\varphi(X_s^i - X_s^j) - \mathbf{1}_{K(w(Y_s^i))}(Y_s^j - Y_s^i)\nabla\varphi(Y_s^i - Y_s^j)\Big| \\
&\le \frac{1}{N}\sum_{i=1}^N \frac{1}{N}\sum_{j=1}^N \Big|\nabla\varphi(X_s^i - X_s^j) - \nabla\varphi(Y_s^i - Y_s^j)\Big| \\
&+ \frac{1}{N}\sum_{i=1}^N \frac{1}{N}\sum_{j=1}^N \|\nabla\varphi\|_{L^\infty}\Big|\mathbf{1}_{K(w(X_s^i))}(Y_s^j - Y_s^i) - \mathbf{1}_{K(w(Y_s^i))}(Y_s^j - Y_s^i)\Big| \\
&+ \frac{1}{N}\sum_{i=1}^N \frac{1}{N}\sum_{j=1}^N \|\nabla\varphi\|_{L^\infty}\Big|\mathbf{1}_{K(w(X_s^i))}(X_s^j - X_s^i) - \mathbf{1}_{K(w(X_s^i))}(Y_s^j - Y_s^i)\Big|.
\end{aligned}$$

Les deux premiers termes ne posent pas de difficulté, mais il n'est pas possible d'estimer le troisième à l'aide de la distance W_1 entre les deux mesures empiriques associées aux X^i

et aux (Y^i). En effet en utilisant l'argument de la corde (Lemme 5.1) on aurait

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^N \frac{1}{N}\sum_{j=1}^N \left|\mathbf{1}_{K(w(X_s^i))}(X_s^j - X_s^i) - \mathbf{1}_{K(w(X_s^i))}(Y_s^j - Y_s^i)\right| \\ \leq \frac{1}{N}\sum_{i=1}^N \frac{1}{N}\sum_{j=1}^N \mathbf{1}_{\partial K(w(X_s^i))^{2|X_s^i - Y_s^i|,+}}(Y_s^j - Y_s^i) \\ + \frac{1}{N}\sum_{j=1}^N \frac{1}{N}\sum_{i=1}^N \mathbf{1}_{\partial K(w(X_s^i))^{2|X_s^j - Y_s^j|,+}}(Y_s^j - Y_s^i),\end{aligned}$$

et on ne saurait pas comment traiter la seconde somme. En effet la somme en i ne peut pas être vue comme une intégration sur un ensemble déterminé par l'indicatrice, puisque cette indicatrice dépend elle même de i. Il apparait alors que pour éviter cette asymétrie entre les deux termes du membre de droite, il vaut mieux établir l'estimation de stabilité fort-faible avec la métrique W_∞. Auquel cas on obtient

$$\begin{aligned}\frac{d}{dt}\sup_{i=1,\cdots,N}|X_t^i - Y_t^i| &\leq \sup_{i=1,\cdots,N}\frac{1}{N}\sum_{j=1}^N \left|\mathbf{1}_{K(w(X_s^i))}(X_s^j - X_s^i) - \mathbf{1}_{K(w(X_s^i))}(Y_s^j - Y_s^i)\right| \\ &\leq \sup_{i=1,\cdots,N}\frac{1}{N}\sum_{j=1}^N \mathbf{1}_{\partial K(w(X_s^i))^{2|X_s^i - Y_s^i|,+}}(Y_s^j - Y_s^i) \\ &+ \sup_{i=1,\cdots,N}\frac{1}{N}\sum_{j=1}^N \mathbf{1}_{\partial K(w(X_s^i))^{2|X_s^j - Y_s^j|,+}}(Y_s^j - Y_s^i) \\ &\leq 2\sup_{i=1,\cdots,N}\frac{1}{N}\sum_{j=1}^N \mathbf{1}_{\partial K(w(X_s^i))^{2\sup_{k=1,\cdots,N}|X_s^k - Y_s^k|,+}}(Y_s^j - Y_s^i),\end{aligned}$$

et on peut mener les estimations. Mais pour utiliser la métrique W_∞, il faut s'assurer que les mesures considérées sont à support compact, ce qui n'est bien évidement pas le cas si l'on souhaite introduire une diffusion standard. Il aurait été possible d'introduire une diffusion tronquée comme à l'article précédent. Nous avons préféré nous orienter vers les EDS réfléchies à la Lions et Sznitman [**16**] pour lesquels il n'y avait à notre connaissance qu'un seul résultat de propagation du chaos. Pour un domaine borné $\mathcal{O}$ convexe, ces équations s'écrivent

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\cdot dB_s - K_t \in \mathcal{O}, \\ K_t = \displaystyle\int_0^t n(X_s)\,d|K|_s \quad , \quad |K|_t = \int_0^t \mathbf{1}_{\partial\mathcal{O}}(X_s)d|K|_s, \end{cases}$$

où n désigne la normale sortante de $\mathcal{O}$ et $|\cdot|_t$ désigne la variation totale du processus. Le processus $(K_t)_{t\geq 0}$ peut être vu comme un processus de confinement qui fait que la solution $(X_t)_{t\geq 0}$ reste à tout instant dans le domaine $\mathcal{O}$. On a ainsi un moyen pratique de confiner les particules dans un domaine borné, et donc on peut introduire la distance W_∞ entre deux solutions. Précisément le système de particules étudié s'écrit

$$\begin{cases} X_t^i = X_0^i + \displaystyle\int_0^t \frac{1}{N}\sum_{j=1}^N \mathbf{1}_{K(w(X_s^i))}(X_s^j - X_s^i)\nabla\varphi(X_s^i - X_s^j)\,ds + \sqrt{2\sigma}B_t^i - K_t^i, \\ K_t^i = \displaystyle\int_0^t n(X_s^i)\,d|K^i|_s \quad , \quad |K^i|_t = \int_0^t \mathbf{1}_{\partial\mathcal{O}}(X_s^i)d|K^i|_s, \end{cases} \tag{36}$$

et la limite pour ce système de particules est naturellement le système de N-copies indépendantes suivant

(37)
$$\begin{cases} Y_t^i = X_0^i + \displaystyle\int_0^t \int_{\mathbb{R}^d} \mathbf{1}_{K(w(Y_s^i))}(y - Y_s^i)\nabla\varphi(Y_s^i - y)\rho_s(dy)\,ds + \sqrt{2\sigma}B_t^i - \tilde{K}_t^i, \quad \rho_s = \mathcal{L}(Y_s^i) \\ \tilde{K}_t^i = \displaystyle\int_0^t n(Y_s^i)\,d|\tilde{K}^i|_s \quad , \quad |\tilde{K}^i|_t = \int_0^t \mathbf{1}_{\partial\mathcal{O}}(Y_s^i)d|\tilde{K}^i|_s, \end{cases}$$

dont la loi du processus solution $(\rho_t)_{t\geq 0}$ est donnée par de l'EDP

(38)
$$\begin{cases} \partial_t\rho_t + \nabla\cdot(V[\rho_t]\rho_t) = \sigma\Delta\rho_t, \quad x\in\mathcal{O}, \quad t>0, \\ V[\rho_t](x) = \displaystyle\int_{\mathcal{O}} \mathbf{1}_{K(w(x))}(y-x)\nabla\varphi(x-y)\rho_t(dy), \\ \langle\sigma\nabla\rho_t - \rho_t V[\rho_t], n\rangle = 0 \quad \text{on} \quad \partial\mathcal{O}. \end{cases}$$

Une fois prouvées l'existence et l'unicité de solutions aux problèmes ci dessus, on établit le

THÉORÈME 5.4. *(Choi, S., 2016) Soit $\mathcal{O}$ un sous ensemble borné convexe de $\mathbb{R}^d$. On suppose que le champ d'orientation $w : \mathcal{O} \to \mathbb{R}^d$ est Lipschitz et borné par le bas, que $K : \mathbb{R}^d \mapsto 2^{\mathbb{R}^d}$ est telle que $K\circ w : \mathcal{O} \to 2^{\mathbb{R}^d}$ satisfait les hypothèses $\boldsymbol{H} - (i-iv)$ (voir Section 5.2) et que $\varphi \in W^{2,\infty}(\mathbb{R}^d)$. Alors pour tout $\rho_0 \in \mathcal{P}(\mathcal{O})\cap L^\infty(\mathcal{O})$ il existe un temps $T>0$ et une unique solution $(\rho_t)_{t\in[0,T]}$ à l'équation* (38) *qui vérifie*

$$\int_0^T \|\rho_s\|_{L^\infty}\,ds < \infty.$$

De plus si $(\mu_t^N)_{t\in[0,T]}$ (respectivement $(\nu_t^N)_{t\in[0,T]}$) est la mesure empirique associée au système de particules (36) *(respectivement* (37)*) avec pour condition initiale i.i.d. de loi ρ_0 alors il existe une constante $C>0$ qui dépend des données du problème telle que pour tout $m, N \in \mathbb{N}^*$ avec $N \geq (2m)^2$ on a*

$$\sup_{t\in[0,T]} \mathbb{E}\left[W_\infty(\mu_t^N, \nu_t^N)\right] \leq Ce^{\int_0^t \|\rho_s\|_{L^1\cap L^\infty}\,ds}\left(((2m)!)^{\frac{1}{2m}}\sqrt{8m} + 8e^{3m}\right)N^{-\frac{1}{2}+\frac{1}{2m}}.$$

Naturellement on peut coupler ce résultat avec le résultat de [**9**] pour obtenir un taux de décroissance sur

$$\sup_{t\in[0,T]} \mathbb{E}\left[W_p(\mu_t^N, \rho_t)\right],$$

pour tout $p \in [1,\infty)$. Dans le cas sans diffusion, on peut montrer que la solution de l'équation (38) pour une donnée initiale à support compact reste à support compact en temps fini. Ainsi on peut dans ce cas établir une estimation de stabilité en distance W_∞ pour l'équation (38) restreinte au cas de conditions initiales à support compact. Cet article a été accepté à la revue **Mathematical Models and Methods in Applied Sciences**.

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CHAPTER 1

Propagation of chaos for the Vlasov-Poisson-Fokker-Planck system in 1D

1. Introduction

We consider here a one dimensional system of N particles, with position $X_i^N \in \mathbb{R}$ and velocity $V_i^N \in \mathbb{R}$, interacting via the Poisson interaction, and submitted to independent Brownian noises and friction. The associated system of Stochastic Differential Equation (SDE) is the following:

$$dX_{i,t}^N = V_{i,t}^N\,dt, \qquad dV_{i,t}^N = \Big(\frac{1}{N}\sum_{j=1}^N K\big(X_{i,t}^N - X_{j,t}^N\big) - V_{i,t}^N\Big)\,dt + \sqrt{2}\,dB_{i,t}, \tag{39}$$

where the $(B_{i,t})_{t\geq 0}$ are independent Brownian motions. The interaction kernel is defined (everywhere) by

$$K(x) := \pm\frac{1}{2}\,\mathrm{sign}(x) = \pm\frac{1}{2}\begin{cases} 1 & \text{if } x>0,\\ 0 & \text{if } x=0,\\ -1 & \text{if } x<0.\end{cases} \tag{40}$$

The case $K = \frac{1}{2}\,\mathrm{sign}$ corresponds to the repulsive case, while the interaction is attractive when $K = -\frac{1}{2}\,\mathrm{sign}$. This two cases lead of course to very different dynamics, but concerning the propagation of chaos in finite time the sign of K is not very relevant, so we will handle both cases in the same way.

Well-posedness of the particle system. The first non-obvious problem is raised by the particle system (39). Since the force field is only of bounded variation (BV in short) and not Lipschitz, the standard theory does not apply. Neither does the theory of existence and uniqueness developed for uniformly elliptic diffusions (see for instance[**15, 5, 2**] among many others) since the diffusion act here only on the velocities. However, due to the particular geometry of the problem, we can still get weak existence and uniqueness in law. The precise result is the following

THEOREM 1. *For any $N \geq 2$, and any (deterministic) initial condition $(X_{i,0}^N, V_{i,0}^N)_{i\leq N} \in \mathbb{R}^{2N}$, weak existence and uniqueness in law hold for SDE* (39).

Since (39) is linear, it also implies weak existence and uniqueness for any random initial condition. Theorem 1 is proved in Section 2 using the following strategy: we reformulate (39) in an SDE with memory involving the $(V_{i,t}^N)_{i\leq N, t\geq 0}$ only (simply because the $X_{i,t}^N$ are time integrals of the $V_{i,t}^N$). Then we apply to that new (non-markovian) SDE a standard technique relying on the Girsanov's theorem. To the best of our knowledge, the strong existence and uniqueness of solution to that system are yet unknown, and we were not able to prove it.

A non-linear SDE related to the Vlasov-Poisson-Fokker-Planck equation. When the number of particles is large, and under the assumption that two particles picked up among all the others are roughly independent at any time (in particular this should be true at $t = 0$), we expect the N particles to behave almost like N i.i.d copies of the (expected

unique) solution to the following non linear SDE, or McKean-Vlasov process:

$$dY_t = W_t\,dt, \qquad dW_t = \mathbb{E}_{\bar{Y}_t}\big[K(Y_t - \bar{Y}_t)\big]\,dt - W_t\,dt + \sqrt{2}\,dB_t, \tag{41}$$

where $(B_t)_{t\geq 0}$ is a Brownian motion (independent of the rest) and $\bar{Y}_t$ is an independent copy of Y_t.

We will prove the well-posedness of that nonlinear SDE for initial data with a uniform control on the velocity tails, and even a weak-strong stability estimate (here "strong solution" means that the law $\mathcal{L}(Y_t)$ of Y_t remains uniformly bounded in time). Before stating the precise result, we introduce two useful norms:

Definition 1. *For any $\lambda > 0$, and any $\gamma > 0$, we define for any $f \in L^\infty(\mathbb{R}^2)$ the two following norms:*

$$\|f\|_{e,\lambda} := \text{ess-sup}_{(x,v)\in\mathbb{R}^2}\, f(x,v)e^{\lambda|v|} \in [0,+\infty] \tag{42}$$

$$\|f\|_{p,\gamma} := \text{ess-sup}_{(x,v)\in\mathbb{R}^2}\, f(x,v)\langle v\rangle^{-\gamma} \in [0,+\infty] \tag{43}$$

where $\langle v\rangle = \sqrt{1+v^2}$, so that equivalently $\|f\|_{p,\gamma} = \text{ess-sup}_{(x,v)\in\mathbb{R}^2}\, f(x,v)\big(1+v^2\big)^{-\gamma/2}$, and the essential supremum are taken with respect to the Lebesgue measure.

Theorem 2. *(i) Existence of strong solutions:*

Assume that the law f_0 of the initial condition (Y_0, W_0) satisfies $f_0 \in \mathcal{P}_1(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)$ and also $\|f_0\|_{e,\lambda} < +\infty$, for some $\lambda > 0$ or $\|f_0\|_{p,\gamma} < +\infty$, for some $\gamma > 1$. Then, given any Brownian motion $(B_t)_{t\geq 0}$ (independent of the initial conditions) there exists a solution (Y_t, W_t) to (41) with initial condition (Y_0, W_0), and its law at time $t \geq 0$: $f_t = \mathcal{L}(Y_t, W_t)$ satisfies either

$$\|f_t\|_{e,\lambda} \leq 2\,e^{t+\lambda+\frac{\lambda^2}{2}}\|f_0\|_{e,\lambda e^{-t}}, \quad \text{or} \quad \|f_t\|_{p,\gamma} \leq C_\gamma e^{\gamma t}\|f_0\|_{p,\gamma}, \tag{44}$$

where C_γ is a constant depending only on γ.

(ii) Weak/strong stability for solution with bounded density (in y).

If (Y_t^1, W_t^1) and (Y_t^2, W_t^2) are two solutions to (41) built on the same probability space with the same Brownian motion $(B_t)_{t\in\mathbb{R}}$, if the density $\rho_t^1 = \mathcal{L}(Y_t^1)$ is uniformly bounded at any time: $\|\rho_t\|_\infty < +\infty$ for any $t \geq 0$, then the following stability estimate holds

$$\mathbb{E}\big[|Y_t^1 - Y_t^2| + |W_t^1 - W_t^2|\big] \leq e^{8\left(t+\int_0^t \|\rho_s^1\|_\infty\, ds\right)}\mathbb{E}\big[|Y_0^1 - Y_0^2| + |W_0^1 - W_0^2|\big]. \tag{45}$$

The proof of the weak-strong stability estimate relies on the crucial Lemma 4 that allows to control the singularity of the force, when comparing the evolution of two solutions, assuming only that one of them has a bounded density in position. That Lemma was already used in [**10**], to get similar results for the associated deterministic particle system: *i.e.* particles interacting via the same kernel but without noise and friction. The proof of the existence of solutions relies on an usual approximation procedure, see Section 3.3. The propagation of the bound on $\|f_t\|_{e,\lambda}$ or $\|f_t\|_{p,\gamma}$ is done using a standard argument that we may call the method of characteristics or Feynman-Kac's formula.

The stability result on the Vlasov-Poisson-Fokker-Planck equation. The stability results on the process (41) simply translate on the associated Fokker-Planck or Kolmogorov forward equation, which is here the Vlasov-Poisson-Fokker-Planck equation (VPFP in short):

$$\partial_t f_t + v\,\partial_x f_t + (\rho_t \star K)\partial_v f_t = \partial_v(\partial_v f_t + v\,f_t), \tag{46}$$

where f_t is the law at time t of the process (Y_t, W_t) and $\rho_t = \int f_t\,dv$ is the law at time t of Y_t. As the kernel K is bounded and defined everywhere by (40), remark that very few hypothesis are required to define solutions to (46) in the sense of distribution: if $f \in L^1_{loc}(\mathbb{R}^+, \mathcal{P}(\mathbb{R}^2))$, where $\mathcal{P}(\mathbb{R}^2)$ stand for the space of probability, then all the terms appearing in (46) define a distribution.

The Theorem 2 as the following consequence on VPFP:

COROLLARY 1 (of Theorem 2). *(i) Existence of strong solution. Let $f_0 \in \mathcal{P} \cap L^1(\mathbb{R}^2)$ with a finite order one moment: $\int \big(|x| + |v|\big) f_0(dx, dv)$, and satisfying either $\|f_0\|_{e,\lambda} < +\infty$, for some $\lambda > 0$ or $\|f_0\|_{p,\gamma} < +\infty$, for some $\gamma > 1$. Then, there exists a solution f_t to (46) with initial condition f_0, and it satisfies (44).*

(ii) Weak/strong uniqueness for solution with bounded density (in y).

If f_t^1 and f_t^2 are two solutions to (46) and if $\rho_t^1 = \int f_t^1\, dv$ is uniformly bounded for any time $t \geq 0$, then the following stability estimate holds:

$$W_1\big(f_t^1, f_t^2\big) \leq e^{8\left(t + \int_0^t \|\rho_s^1\|_\infty\, ds\right)} W_1\big(f_0^1, f_0^2\big) \tag{47}$$

This corollary is proved in Section 3.4. It is a direct consequence of Theorem 2, and of the fact that a weak solution f_t to the VPFP1D equation (46), can always be represented as the time marginals of a process $(Y_t, W_t)_{t\geq 0}$ solution to (41).

The quantitative propagation of chaos in the mean. When comparing a solution $(X_{i,t}^N, V_{i,t}^N)_{i\leq N}$ of the particle system (39) with the limit process (41) and its associated Fokker-Planck equation (46), a very natural strategy (which goes back to McKean [**12**]) is to introduce N independent copies $(Y_{i,t}^N, W_{i,t}^N)_{i\leq N}$ of the limit nonlinear SDE (41), constructed with the same Brownian motion as the $(X_{i,t}^N, V_{i,t}^N)$, and with initial conditions coupled in a optimal way. In our case, we are able to prove a sharp estimate on the average distance between these two systems.

To state our result properly, we recall that by definition exchangeable random variables have a law that is invariant under permutation, and that chaotic sequences of r.v. are defined as follows:

DEFINITION 2. *Let f be a probability on $\mathbb{R}^2$. A sequence $\big((X_i^N, V_i^N)_{i\leq N}\big)_{N\in\mathbb{N}}$ of exchangeable random variables is said to be f-chaotic, if one of the equivalent conditions below is satisfied:*

i) $\forall k \in \mathbb{N}, \quad \mathcal{L}\big((X_i^N, V_i^N)_{i\leq k}\big) \xrightarrow[N\to\infty]{w} f^{\otimes k}$,

ii) $\mathcal{L}\big((X_1^N, V_1^N), (X_2^N, V_2^N)\big) \xrightarrow[N\to\infty]{w} f \otimes f$,

iii) $\mu^N := \dfrac{1}{N}\displaystyle\sum_{i=1}^N \delta_{(X_i^N, V_i^N)} \xrightarrow[N\to\infty]{\mathcal{L}} f$.

If Y is a random variable of law f, we will equivalenty say that a sequence is f-chaotic or Y-chaotic. We refer to [**28**] for the equivalence of the three conditions above, and to [**11**] for a quantitative version of that equivalence.

We also state the following proposition, that reformulate propagation of chaos in term of coupling. It is a consequence of [**11**, Theorem 1.2]

PROPOSITION 1. *Assume that $\big((X_i^N, V_i^N)_{i\leq N}\big)_{N\in\mathbb{N}}$ is a sequence of exchangeable random variables with uniformly bounded order two moment: $\sup_{N\in\mathbb{N}} \mathbb{E}\big[|X_1^N|^2 + |V_1^N|^2\big] < +\infty$. Let f be a probability on $\mathbb{R}^2$. Then, the following statements are equivalent:*

i) *The sequence $\big((X_i^N, V_i^N)_{i\leq N}\big)_{N\in\mathbb{N}}$ is f-chaotic;*

ii) $\mathbb{E}\Big[W_1\big(\mu^N, f\big)\Big] \xrightarrow[N\to\infty]{} 0$, *where W_1 stands for the order one Monge-Kantorovich-Wasserstein distance, and $\mu^N = \frac{1}{N}\sum_{i=1^N} \delta_{(X_i^N, V_i^N)}$ is the associated empirical measure;*

iii) *If the $(Y_i, W_i)_{i\in\mathbb{N}}$ are i.i.d.r.v. with commun law f, independent of the $(X_i^N, V_i^N)_{i\leq N}$ for all N, then*

$$\mathbb{E}\Big[\big|X_1^N - Y_1\big| + \big|V_1^N - W_1\big|\Big] = \mathbb{E}\Big[\frac{1}{N}\sum_i \big|X_i^N - Y_i\big| + \big|V_i^N - W_i\big|\Big] \xrightarrow[N\to\infty]{} 0;$$

iv) $\min\limits_{coupling} \mathbb{E}\Big[\dfrac{1}{N}\displaystyle\sum_i \big|X_i^N - Y_i\big| + \big|V_i^N - W_i\big|\Big] \xrightarrow[N\to\infty]{} 0$, *where the minimum is taken on all the exchangeable coupling with (Y_i, V_i) i.i.d. with common law f.*

The above Proposition allows to state a precise result of propagation of molecular chaos. We emphasize that this result is a true result of propagation: it does not apply only to i.i.d. initial conditions, but to any chaotic initial conditions (with finite second order moment). However, the general case is somewhat more technical, so we warn the reader that the following theorem is simpler to understand if we only consider the case of i.i.d. initial conditions.

THEOREM 3. *Let $f_0 \in \mathcal{P}(\mathbb{R}^2)$ with finite order two moment: $\int \big(|x|^2+|v|^2\big)\, f_0(dx,dv) < \infty$, and such that there exists a (necessary unique by Corollary 1) solution f_t to (46) with initial condition f_0 satisfying $\int_0^t \|\rho_s\|_\infty\, ds < +\infty$ for any time $t \geq 0$, where ρ_s stands for the density in position: $\rho_s(x) := \int f_s(x,dv)$. We also denote by $(Y_t, W_t)_{t\geq 0}$ the unique solution to (41) such that $\mathcal{L}(Y_0, W_0) = f_0$.*

Let $(X^N_{i,0}, V^N_{i,0})_{i\leq N}$ be a sequence of f_0-chaotic random variable with uniformly bounded order two moment: $\sup_{N\in\mathbb{N}} \mathbb{E}\big[|X^N_{1,0}|^2+|V^N_{1,0}|^2\big] < +\infty$. By Theorem 1, we may find a probability space together with a N-dimensional Brownian motion and a process $(X^N_{i,t}, V^N_{i,t})_{i\leq N}$ solution to (39). Then, the sequence $\big((X^N_{i,t}, V^N_{i,t})_{i\leq N}\big)_{n\in\mathbb{N}}$ is $(Y_t, W_t)_{t\geq 0}$ chaotic.

More precisely, by standard arguments, we may also construct on that probability space N i.i.d. copies $(Y^N_{i,t}, W^N_{i,t})_{i\leq N}$ of the solutions of (41) with the same Brownian motion, and with initial conditions of law f_0, coupled with $(X^N_{i,0}, V^N_{i,0})_{i\leq N}$ in an exchangeable way. Then the following estimate holds:

$$\mathbb{E}\left[\sup_{s\in[0,t]} \big|X^N_{1,s}-Y^N_{1,s}\big|+\big|V^N_{1,s}-W^N_{1,s}\big|\right] \leq \left(\mathbb{E}\Big[\big|X^N_{1,0}-Y^N_{1,0}\big|+\big|V^N_{1,0}-W^N_{1,0}\big|\Big]+\frac{9}{\sqrt{N}}\right) e^{\left(t+8\int_0^t \|\rho_s\|_\infty\, ds\right)}.$$

In particular, if $(X^N_{i,0}, V^N_{i,0})_{i\leq N}$ are i.i.d. with law f_0, then

$$\mathbb{E}\left[\sup_{s\in[0,t]} \big|X^N_{1,s}-Y^N_{1,s}\big|+\big|V^N_{1,s}-W^N_{1,s}\big|\right] \leq \frac{9}{\sqrt{N}} e^{\left(t+8\int_0^t \|\rho_s\|_\infty\, ds\right)}.$$

REMARK 1. *Thanks to Corollary 1, the last hypothesis on f_0 is satisfied if $\|f_0\|_{e,\lambda} < \infty$ for some $\lambda > 0$, or if $\|f_0\|_{p,\gamma} < \infty$ for some $\gamma > 1$.*

The proof of that result is performed in Section 4: it relies in a crucial way on a "rope argument" introduced in Section 3.1. A second key argument is the introduction of an ad-hoc Poisson Random Measure (PRM), which allows to conclude using standard properties of PRMs.

Using standard results on the convergence on empirical measures toward their mean [**14**, Theorem 1], we may deduce convergence results between the empirical measure of the particle system and the expected limit profile.

COROLLARY 2 (of Theorem 3). *Under the same assumptions than in Theorem 3, and assuming moreover that $\int \big(|x|+|v|\big)^q\, f_0(dx,dv) < \infty$ for some $q > 2$ we obtain that for any time $t \geq 0$,*

$$\mathbb{E}\big[W_1(\mu^N_t, f_t)\big] \leq C_t\left(\frac{\ln(1+N)}{\sqrt{N}} + \mathbb{E}\big[W_1(\mu^N_0, f_0)\big]\right),$$

for some constant $C_t > 0$ depending on t,q and f_0.

Exponential concentration inequalities for the particle system. In the case were the initial conditions are i.i.d., we also prove concentration inequalities for the solutions of the particle system (39), precisely:

THEOREM 4. *Let $f_0 \in \mathcal{P}(\mathbb{R}^2)$ with some finite exponential moment: $\int e^{\lambda(|x|+|v|)}\, f_0(dx,dv) < \infty$ for some $\lambda > 0$, and such that there exists a (necessary unique by Corollary 1) solution f_t to (46) with initial condition f_0 satisfying $\kappa_t := \sup_{s\leq t} \|\rho_s\|_\infty\, ds < +\infty$ for any time $t \geq 0$, where ρ_s stands for the density in position: $\rho_s(x) := \int f_s(x,dv)$. Let $(X^N_{i,0}, V^N_{i,0})_{i\leq N}$ be a sequence of N i.i.d random variables with common law f_0.*

By Theorem 1, we may find a probability space together with a N-dimensional Brownian motion and a process $(X^N_{i,t}, V^N_{i,t})_{i\le N}$ solution to (39). By standard arguments, we may also construct on that probability space N copies $(Y^N_{i,t}, W^N_{i,t})_{i\le N}$ of the solutions of (41) with the same initial conditions $(X^N_{i,0}, V^N_{i,0})_{i\le N}$ and Brownian motion.

Then the following concentration inequality holds for $\lambda N^{-1/2} \le \varepsilon \le (5\kappa_t \wedge 1)\min\big(\frac{1}{16}, \frac{\lambda}{2}, \lambda^{-2}\big)$:

$$\mathbb{P}\bigg(\frac{1}{N}\sum_{i=1}^N \sup_{s\in[0,t]} \big(|X^N_{i,s} - Y^N_{i,s}| + |V^N_{i,s} - W^N_{i,s}|\big) \ge \boldsymbol{B_t}\varepsilon \bigg) \le (t+\varepsilon)\big(\boldsymbol{A_t} + \boldsymbol{A'_t}\sqrt{N}\varepsilon\big) N^{\frac{3}{2}} e^{-2N\varepsilon^2},$$

where the three constants depend on t, λ and the initial conditions f_0. See the end of Section 6 for precise values.

Remark 2. *Thanks to Corollary 1, the hypothesis on f_0 are satisfied if $\|f_0\|_{e,\lambda'} < \infty$ for some $\lambda' > \lambda$.*

The proof of Theorem 4 relies on a different technique than the one used in the proof of Theorem 3. Here, we rather use exponential concentration inequalities on discrete infinite norms of empirical measures, and on some fluctuation terms appearing naturally when comparing solutions to (39) to copies of solutions to the nonlinear SDE (41).

Using deviation upper bounds for the approximation of probability measure by random empirical measures associated to i.i.d sample, as for instance in [**14**, Theorem 2], we can also obtain the following corollary.

Corollary 3. *Under the same assumption as in Theorem 4, and if moreover the initial positions and velocities $(X^{\cdot}_{i,0} V^N_{i,0})$ are i.i.d random variables with law f_0, then for any $T \ge 0$ and any $\lambda > 0$ there exists two constants C_1, C_2 such that for any $\varepsilon \ge 0$ satisfying $\lambda N^{-12} \le \varepsilon \le (5\kappa_t \wedge 1)\min\big(\frac{1}{16}, \lambda, \lambda^{-2}\big)$, we have:*

$$\sup_{t\in[0,T]} \mathbb{P}\Big[\, W_1(\mu^N_{X,t}, f_t) \ge \varepsilon \Big] \le C_1\Big(N^4 e^{-C_2 N\varepsilon^2} + e^{-C_2(N\varepsilon)^{1-\alpha}} + e^{-C_2 N\varepsilon^2/(1-\ln\varepsilon)^2}\Big),$$

where f_t is the unique solution of the VPFP equation (46) with initial condition f_0.

Exponential concentration for discrete infinite norms. One of the key ingredient of the proof of Theorem 4 is a deviation inequality on time integral (or supremum) of discrete infinite norms for the $\mu^N_{Y,t} := N^{-1}\sum_i \delta_{Y^{N,i}_t}$, where the $Y^{N,i}_t$ are N i.i.d. r.v. solutions to a given SDE. Such a result have a interest by itself so we state it below

Proposition 2. *Let $(Y_i, W_i)_{i_l eN}$ be N i.i.d copies of a solution of (41) driven by independent Brownian motions $(B_{i,t})_{i\le N, t\ge 0}$, and assume that for $t \ge 0$:*

- *the common initial condition has a law f_0 which admits a finite exponential moment (in position and velocity) for some $\lambda > 0$: $\mathbb{E}\Big[e^{\lambda(|Y_0|+|W_0|)}\Big] < +\infty$. In particular we denote $c_\lambda := \frac{5}{2} + \frac{1}{\lambda}\ln \mathbb{E}\big[e^{\lambda|W_0|}\big]$;*
- *$\kappa_t := \sup_{0\le s\le t}\|\rho_s\|_\infty < +\infty$ where ρ_s stands for the time marginal of Y_i at time s.*

Then provided that $\lambda N^{-1/2} \le \varepsilon\gamma \le 5\kappa_t \min\Big(\frac{1}{16}, \lambda^{-2}\Big)$, the following bound holds with $\boldsymbol{C_t} := 10 + \kappa_t^2\lambda^{-1} + e^{\lambda(1/2+\lambda)t}\,\mathbb{E}\Big[e^{\lambda(|Y_0|+|W_0|)}\Big]$:

$$\mathbb{P}\bigg(\sup_{s\in[0,t]} \big\|\rho^N_s\big\|_{\infty,\varepsilon} \ge \kappa_t + \gamma \bigg) \le \boldsymbol{C_t}\bigg(1 + t\frac{2\kappa_t + \gamma}{\lambda}\big(c_\lambda + \sqrt{N}\varepsilon\gamma\big)\bigg) N^{\frac{3}{2}}\, e^{-2N(\varepsilon\gamma)^2},$$

Essentially, the bound behave like $N^{3/2}e^{-2N(\varepsilon\gamma)^2}$, in the most interesting range, when $\varepsilon\gamma \sim N^{-1/2}$.

Some related works. The literature on the convergence of particle systems towards non linear mean-field models is quite huge, so we will restrict ourselves to one dimensional models. The usual strategy, valid for smooth interaction, is well explained in Lecture notes by Sznitman [**28**]. In [**3**] Cépa and Lépingle prove the propagation of chaos for the Dyson model: an order one model (*i.e.* without velocities) with a strongly singular interaction $K(x) \sim |x|^{-1}$ modeling the behavior of eigenvalues of large hermitian matrices. Their proof relies on the use of maximal monotone operators. Recently, that convergence result was extended to similar systems with even stronger interaction $K(x) \sim |x|^{-1-\alpha}$ with $\alpha \in [0,1)$, by Berman and Önnheim [**1**] using Wasserstein gradient flows. For second oder models (involving positions and velocities), our result is to the best of our knowledge the first result of propagation of chaos with the Poisson singularity in the stochastic case. In the deterministic case, *i.e.* when the system under study is (39) without the Brownian motions, then the mean field limit was proved originally by [**14**], and then by [**4**] as a special case of semi-geostrophic equations, and again by the first author [**10**].

Plan of the paper. The paper is organized in the following way. In section 2, we prove Theorem 1 (weak existence and uniqueness of the particle system). In Section 3, we focus on the nonlinear limit SDE and prove Theorem 2 and its corollary 1, and also a useful proposition about propagation of moments (Proposition 3). Section 4 is devoted to the proof of Theorem 3 on the propagation of chaos in the mean, Section 5 to the proof of Proposition 2 and Section 6 to the proof of Theorem 4 on the exponential concentration. A useful regularity lemma is proved in the Appendix.

2. Proof of Theorem 1

A related SDE with memory. First note that a weak solution to equation (39) is some stochastic basis, together with a N dimensional Brownian motion $(B_t^N)_{t\geq 0}$ on it and a $\mathbb{R}^{2N}$ valued processes $(X_t^N, V_t^N)_{t\geq 0}$ satisfying for all $t \geq 0$

$$X_t^N = X_0^N + \int_0^t V_s^N \, ds, \qquad V_t^N = V_0^N + \int_0^t K^N(V_s^N) ds - \int_0^t V_s^N \, ds + \sqrt{2}\, B_t^N,$$

where we denoted $K^N(x_1, \cdots, x_N)$ the vector valued field which i-th component is $\frac{1}{N}\sum_{j\neq i} K(x_i - x_j)$. But, from that system, we may write a SDE "with memory" involving V^N only:

$$\forall\, t \geq 0, \quad V_t^N = V_0^N + \int_0^t K^N\left(X_0^N + \int_0^s V_u^N \, du\right) ds - \int_0^t V_s^N \, ds + \sqrt{2}\, B_t^N. \tag{48}$$

Conversely, given a solution to the SDE (48), it is not difficult to construct a solution to the original system. So it will be enough to prove weak existence and uniqueness in law for the delayed SDE (48).

Weak existence. Let $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P})$ be a stochastic basis and $(B_t^N)_{t\geq 0}$ be a N-dimensional on it. We define

$$U_t^N := -\frac{1}{\sqrt{2}} \int_0^t K^N\Big(X_0^N + sV_0^N + \sqrt{2}\int_0^s B_u^N \, du\Big) ds + \int_0^t \left(B_s^N + \frac{1}{\sqrt{2}} V_0^N\right) ds + B_t^N, \quad V_t^N := V_0^N + \sqrt{2}\, B_t^N.$$

The above definition of the two r.v. $(U_t^N, V_t^N)_{t\geq 0}$ implies that for any $t \geq 0$,

$$V_t^N = V_0^N + \int_0^t K^N\left(X_0^N + \int_0^s V_u^N \, du\right) ds - \int_0^t V_s^N \, ds + \sqrt{2}\, U_t^N,$$

which is exactly (48) with B_t^N replaced by U_t^N. So, it remains to apply Cameron-Martin-Girsanov (CMG) theorem: with an appropriate change of the reference probability measure, $(U_t^N)_{t\geq 0}$ can be considered has a N-dimensional Brownian motion. For this, remark that $dU_t^N = -H_t^N \, dt + dB_t^N$, where

$$H_t^N := \frac{1}{\sqrt{2}} K^N\left(X_0^N + tV_0^N + \int_0^t \sqrt{2} B_u^N \, du\right) - B_t^N - \frac{1}{\sqrt{2}} V_0^N,$$

is $\mathcal{F}_t$-adapted and progressively measurable, and that for $0 < \gamma < \frac{1}{6t}$

$$\mathbb{E}\Big[e^{\gamma|H_t^N|^2}\Big] \leq e^{\frac{3}{2}\gamma\left(N\|K\|_\infty^2 + |V_0^N|^2\right)}\mathbb{E}\Big[e^{3\gamma(B_t^N)^2}\Big] < +\infty.$$

Therefore we deduce from classical results about exponential martingales that the process Z_t^N defined by

$$Z_t^N = \exp\left(\int_0^t H_s^N \cdot dB_s^N - \frac{1}{2}\int_0^t |H_s^N|^2 ds\right)$$

(where $\cdot$ stands for the scalar product) is a martingale, and due to CMG theorem $(U_t^N)_{t\geq 0}$ is a N-dimensional Brownian motion under the probability $\mathbb{Q}$ defined for any A in $\mathcal{F}_t$ by $\mathbb{Q}(A) = \int_A Z_t^N \, d\mathbb{P}$. Therefore $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q}, (U_t^N, V_t^N)_{t\geq 0}\big)$ is a weak solution to SDE (48).

Uniqueness in law. Suppose that $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P}, (V_t^N, B_t^N)_{t\geq 0}$ is a solution to equation (48) with initial condition V_0^N. We may use CMG theorem again: in fact $2^{-1/2}dV_t^N = -\tilde{H}_t^N\, dt + dB_t^N$, where

$$\tilde{H}_t^N := -\frac{1}{\sqrt{2}}K^N\left(X_0^N + \int_0^t V_u^N\, du\right) + \frac{1}{\sqrt{2}}V_t^N.$$

Moreover by (48), V_t^N also satisfies

$$V_t^N = e^{-t}V_0^N + \int_0^t e^{s-t}K^N\left(X_0^N + \int_0^s V_u^N\, du\right)ds + \sqrt{2}\int_0^t e^{s-t}dB_s^N,$$

which implies that

$$\big|\tilde{H}_t^N\big| \leq \big|V_t^N\big| + \frac{\sqrt{N}}{2} \leq \big|V_0^N\big| + \frac{3\sqrt{N}}{2} + \big|M_t^N\big|,$$

where M_t^N is a Gaussian r.v. with law $\mathcal{N}\big(0, (1-e^{2t})\mathrm{Id}\big)$. It follows that $\mathbb{E}\big[e^{\gamma|\tilde{H}_t^N|^2}\big] < \infty$, for $t \geq 0$ and $\gamma < 1/2$. This implies that the process $\tilde{Z}_t^N$ defined by

$$\tilde{Z}_t^N = \exp\left(\int_0^t \tilde{H}_s^N \cdot dB_s^N - \frac{1}{2}\int_0^t \big|\tilde{H}_s^N\big|^2\, ds\right) = \exp\left(\frac{1}{\sqrt{2}}\int_0^t \tilde{H}_s^N \cdot dV_s^N + \frac{1}{2}\int_0^t \big|\tilde{H}_s^N\big|^2\, ds\right),$$

is a martingale, and by Cameron-Martin-Girsanov theorem, $2^{-1/2}(V_t^N - V_0^N)_{t\geq 0}$ is a N-dimensional Brownian motion on the filtered space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \tilde{\mathbb{Q}})$ where $\tilde{\mathbb{Q}}$ is defined for any A in $\mathcal{F}_t$ by $\tilde{\mathbb{Q}}(A) = \int_A \tilde{Z}_t^N \, d\mathbb{P}$.

Now for any "cylindrical" function ϕ on $C(\mathbb{R}^+, \mathbb{R}^N)$ of the form $\phi\big((V_s^N)_{s\geq 0}\big) = \varphi_1(V_{t_1}^N) \times \cdots \times \varphi(V_{t_k}^N)$, we get for $t \geq t_k$:

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\Big[\phi\big((V_s^N)_{s\geq 0}\big)\Big] &= \mathbb{E}_{\tilde{\mathbb{Q}}}\Big[\phi\big((V_s^N)_{s\geq 0}\big)\left(\tilde{Z}_t^N\right)^{-1}\Big] \\ &= \mathbb{E}_{\tilde{\mathbb{Q}}}\left[\phi\big((V_s^N)_{s\geq 0}\big)\exp\left(-\frac{1}{\sqrt{2}}\int_0^t \tilde{H}_s^N \cdot dV_s^N - \frac{1}{2}\int_0^t \big|\tilde{H}_s^N\big|^2\, ds\right)\right].\end{aligned}$$

The expression on the last line does not involve $(B_t^N)_{t\geq 0}$ anymore. Since the law of $(V_t^N)_{t\geq 0}$ under $\tilde{\mathbb{Q}}$ is the law of the Brownian motion, and since $\tilde{Z}_t^N$ can be expressed in term of V_t^N only, that last expression does not depend on the specific solution we selected at the beginning of this paragraph: we will obtain exactly the same formula starting from a second solution. Since, solution to (48) have continuous trajectories, this implies the uniqueness in law of the solutions to the SDE (48) and also to (39).

3. Proof of Theorem 2

3.1. Weak-strong stability and uniqueness.

A key bound. Here we will use a "rope" argument which has already been used in [**10**] to treat the propagation of chaos for deterministic VP1D equation. It consists in noticing that:

$$K(x-\bar{x}) - K(y-\bar{y}) = 0$$

as soon as $|y-\bar{y}| > |x-y| + |\bar{x}-\bar{y}|$. It is also interesting to replace the later condition by the stronger one $|y-\bar{y}| > 2\max(|x-y|, |\bar{x}-\bar{y}|)$. Since K is bounded by $1/2$, it implies the following bound, that we will use many times in the sequel

$$|K(x-\bar{x}) - K(y-\bar{y})| \leq \mathbb{1}_{|y-\bar{y}|\leq 2\,|x-y|} + \mathbb{1}_{|y-\bar{y}|\leq 2\,|\bar{x}-\bar{y}|} \tag{49}$$

That one-sided condition (the indicator functions in the r.h.s. take only $y-\bar{y}$ as argument) allows to prove a simple but important Lemma, where discrete infinite norms are introduced:

DEFINITION 3. *For any $\varepsilon > 0$, and any $f \in \mathcal{P}(\mathbb{R})$, we define the infinite norm at scale ε, denoted $\|f\|_{\infty,\varepsilon}$ by:*

$$\|f\|_{\infty,\varepsilon} := \sup_{x\in\mathbb{R}} \frac{f\big([x-\varepsilon, x+\varepsilon]\big)}{2\varepsilon} = \Big\| f \star \frac{1}{2\varepsilon}\mathbb{1}_{[-\varepsilon,\varepsilon]} \Big\|_\infty .$$

LEMMA 4. *Assume that (X,Y) is a random couple of real numbers and that $(\bar{X},\bar{Y})$ is an independent copy of that couple.*

(i) Then,

$$\mathbb{E}\big[K(X-\bar{X}) - K(Y-\bar{Y})\big] \leq 8\,\min\big(\|\rho_X\|_\infty, \|\rho_Y\|_\infty\big)\mathbb{E}\big[|X-Y|\big],$$

where ρ_X, ρ_Y denote respectively the density (with respect to the Lebesgue measure) of the law of X and Y.

(ii) For $\varepsilon > 0$, we also have a similar estimate involving the discrete infinite norms $\|\cdot\|_{\infty,\varepsilon}$ defined in Definition 3:

$$\mathbb{E}\big[\big|K(X-\bar{X}) - K(Y-\bar{Y})\big|\big] \leq 8\,\min\big(\|\rho_X\|_{\infty,\varepsilon}, \|\rho_Y\|_{\infty,\varepsilon}\big)\Big(\mathbb{E}\big[|X-Y|\big] + \frac{\varepsilon}{2}\Big).$$

(iii) In the case where (X,Y) and $(\bar{X},\bar{Y})$ are still independent but with possibly different distributions, we get the more general estimate: for $\varepsilon, \bar{\varepsilon} \geq 0$ (in case set $\varepsilon = 0$ or $\bar{\varepsilon} = 0$, set $\|\cdot\|_{\infty,0} = \|\cdot\|_\infty$),

$$\mathbb{E}\big[\big|K(X-\bar{X}) - K(Y-\bar{Y})\big|\big] \leq 4\,\|\rho_{\bar{Y}}\|_{\infty,\bar{\varepsilon}}\Big(\mathbb{E}\big[|X-Y|\big] + \bar{\varepsilon}/2\Big) + 4\,\|\rho_Y\|_{\infty,\varepsilon}\Big(\mathbb{E}\big[|\bar{X}-\bar{Y}|\big] + \varepsilon/2\Big).$$

PROOF. We first prove *i*). Starting from (49), we may bound

$$\begin{aligned}
\mathbb{E}\big[\big|K(X-\bar{X}) - K(Y-\bar{Y})\big|\big] &\leq \mathbb{E}\big[\mathbb{1}_{|Y-\bar{Y}|\leq 2\,|X-Y|} + \mathbb{1}_{|Y-\bar{Y}|\leq 2\,|\bar{X}-\bar{Y}|}\big] \\
&\leq 2\,\mathbb{E}\big[\mathbb{1}_{|Y-\bar{Y}|\leq 2\,|X-Y|}\big] \\
&\leq 2\,\mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{|Y-\bar{Y}|\leq 2\,|X-Y|}\big|(X,Y)\big]\Big] \\
&\leq 2\,\mathbb{E}\Big[4\,\|\rho_Y\|_\infty\,|X-Y|\Big] \\
&\leq 8\,\|\rho_Y\|_\infty \mathbb{E}\big[|X-Y|\big].
\end{aligned}$$

In the second line, we used that the expectation remain unchanged if we permute (X,Y) with $(\bar{X},\bar{Y})$. In the fourth, we used that the law of $\bar{Y}$ has density ρ_Y. If we apply the previous calculation with the couple (Y,X) and $(\bar{Y},\bar{X})$, we obtain a similar result, with $\|\rho_X\|_\infty$ in place of $\|\rho_Y\|_\infty$.

To obtain *ii*), remark that for any interval $[a,b] \subset \mathbb{R}$:

$$\mathbb{E}\big[\mathbb{1}_A(X)\big] = \int_a^b \rho_X(dx) \leq \|\rho_X\|_{\infty,\varepsilon}\big(|b-a| + 2\varepsilon\big).$$

Indeed, to use discrete infinite norm, we need to cover $[a,b]$ by a union of small intervals of length 2ε. For this at most (the integer part of) $|b-a|/(2\varepsilon)+1$ such intervals are requested.

For the third point, we cannot use the permutation $(X,Y)\leftrightarrow(\bar X,\bar Y)$ in the previous calculation and have to estimate the two terms separately. The necessary adaptations are straightforward. □

A simple Grönwall lemma. That bound allows us to prove the weak-strong stability part of Theorem 2. We introduce $(X_t,V_t)_{t\in\mathbb{R}_+}$ and $(Y_t,W_t)_{t\in\mathbb{R}_+}$ two solutions of the non-linear SDE (41) constructed on the same Brownian motion $(B_t)_{t\in\mathbb{R}}$, and also $(\bar X_t,\bar V_t,\bar Y_t,\bar W_t)_{t\in\mathbb{R}}$ an independent copy of the previous coupled processes. We also assume that $\rho_t:=\mathcal{L}(X_t)$ has a bounded density for all $t\geq 0$. Then $(X_t-Y_t,V_t-W_t)_{t\in\mathbb{R}}$ solves the following ODE system:

$$\frac{d}{dt}(X_t-Y_t)=V_t-W_t,\qquad \frac{d}{dt}(V_t-W_t)=-(V_t-W_t)+\mathbb{E}_{(\bar X_t,\bar Y_t)}\big[K(X_t-\bar X_t)-K(Y_t-\bar Y_t)\big],$$

which naturally leads to

$$\sup_{s\in[0,t]}|X_s-Y_s|\leq|X_0-Y_0|+\int_0^t|V_s-W_s|\,ds,$$
$$\sup_{s\in[0,t]}|V_s-W_s|\leq|V_0-W_0|+\int_0^t\big|K(X_s-\bar X_s)-K(Y_s-\bar Y_s)\big|\,ds.$$

If we take the expectation in the previous system, and apply the point (i) of Lemma 4 to the couple (Y_t,X_t) and $(\bar Y_t,\bar X_t)$, we may write:

$$\mathbb{E}\bigg[\sup_{s\in[0,t]}|X_s-Y_s|\bigg]\leq\mathbb{E}[|X_0-Y_0|]+\int_0^t\mathbb{E}\big[|V_s-W_s|\big]\,ds$$
$$\mathbb{E}\bigg[\sup_{s\in[0,t]}|V_s-W_s|\bigg]\leq\mathbb{E}\big[|V_0-W_0|\big]+8\int_0^t\|\rho_s\|_\infty\mathbb{E}\big[|X_s-Y_s|\big]\,ds.$$

Summing up the two inequalities and applying the Grönwall lemma lead to the requested estimate

$$\mathbb{E}\bigg[\sup_{s\in[0,t]}\big(|X_s-Y_s|+|V_s-W_s|\big)\bigg]\leq\mathbb{E}\big[\,|X_0-Y_0|+|V_0-W_0|\bigg]\exp\bigg(t+8\int_0^t\|\rho_s\|_\infty\,ds\bigg).$$

Remark that it is not completely straightforward to take advantage of the restoring force in order to improve the above bound, especially because of the supremum in time.

3.2. Propagation of moments. Here, we show that order one moments, and exponential moments are propagated by the SDE (41). We emphasize that our results apply not only for K given by (40), but as soon as $\|K\|_\infty\leq 1/2$. In particular, we will apply it later to nonlinear SDE where K is replaced by a smooth mollification.

Proposition 3. *Let be $(Y_t,W_t)_{t\geq0}$ be a weak solution to 41, with given (random) initial condition (Y_0,W_0), and with an interaction kernel K which is not necessary given by* (40) *but satisfies $\|K\|_\infty\leq\frac12$. If (Y_0,W_0) has an exponential moment of order $\lambda>0$, it holds for $t\geq0$:*

$$(i)\qquad \mathbb{E}\Big[e^{\lambda|W_t|}\Big]\leq e^{\frac{\lambda}{2}(3+\lambda)}\mathbb{E}\Big[e^{\lambda e^{-t}|W_0|}\Big],$$
$$(ii)\qquad \mathbb{E}\Big[e^{\lambda|Y_t|}\Big]\leq 2\,e^{\lambda t\left(\frac12+\lambda\right)}\,\mathbb{E}\Big[e^{\lambda(|Y_0|+|W_0|)}\Big]:=2\,e^{\lambda t\left(\frac12+\lambda\right)}\,\mathcal{M}_\lambda^{x,v}(f_0).$$

It also holds for any $0\leq s<t\leq s+\min\Big(\frac{1}{16},\lambda^{-2}\Big)$:

$$(iii)\qquad \mathbb{E}\Big[e^{\lambda(t-s)^{-1}\sup_{s\leq u\leq t}|Y_u-Y_s|}\Big]\leq e^{\frac{\lambda}{2}(5+\lambda)}\mathbb{E}\Big[e^{\lambda|W_0|}\Big].$$

Lastly, simpler estimates on the order one moments also hold: for any $0 \le s < t \le s + \frac{1}{4}$,

$$(iv) \quad \mathbb{E}\big[|W_t|\big] \le e^{-t}\,\mathbb{E}\big[|W_0|\big] + 2, \qquad \mathbb{E}\big[|Y_t - Y_s|\big] \le |s-t|\Big(\mathbb{E}\big[|W_0|\big] + 3\Big).$$

PROOF. *Point* (i). First, introducing the notation f_t for the time marginal of (Y_t, W_t) and $F(t,x) := \int K(x-y) f_t(dy, dw)$ we have by (41)

$$e^t W_t = W_0 + \int_0^t e^s F(s, Y_s)\, ds + \sqrt{2}\int_0^t e^s dB_s. \tag{50}$$

Next, since $|K|$ and also $|F|$ are bounded by $1/2$, we get a simple inequality

$$|W_t| \le e^{-t}|W_0| + \frac{1}{2} + |M_t|, \quad \text{with} \quad M_t := \sqrt{2}\int_0^t e^{s-t} dB_s. \tag{51}$$

M_t is a centered Gaussian random variable with variance $1 - e^{-2t} \le 1$, for which exponential moments are simple to obtain. In fact for a Gaussian variable $Z \sim \mathcal{N}(0, \sigma^2)$ we have a simple bound

$$E\left(e^{\lambda|Z|}\right) = \frac{2}{\sqrt{2\pi}\sigma}\int_0^{+\infty} e^{\lambda x - \frac{x^2}{2\sigma^2}}\, dx = \frac{2}{\sqrt{\pi}} e^{\frac{\lambda^2\sigma^2}{2}} \int_{-\lambda\sigma\, 2^{-1/2}}^{+\infty} e^{-x^2}\, dx = e^{\frac{\lambda^2\sigma^2}{2}}\left(1 + \mathrm{erf}\left(\frac{\lambda\sigma}{\sqrt{2}}\right)\right),$$

where we used for the error function erf the definition $\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}\, dx$. Using that $\mathrm{erf}(x) \le \min\big(1, e^x\big)$, we finally get the following bound, that will be very useful in the sequel:

$$E\left(e^{\lambda|Z|}\right) \le \min\Big(2, e^{\frac{1}{\sqrt{2}}\lambda\sigma}\Big) e^{\frac{\lambda^2\sigma^2}{2}} \tag{52}$$

Together with the independence of the Brownian motion $(B_t)_{t\ge 0}$ and the initial condition W_0, it leads here to

$$\begin{aligned}\mathbb{E}\Big[e^{\lambda|W_t|}\Big] &\le e^{\frac{\lambda}{2}}\,\mathbb{E}\Big[e^{\lambda e^{-t}|W_0|}\Big]\mathbb{E}\Big[e^{\lambda|M_t|}\Big] \le e^{\frac{\lambda}{2}}\,\mathbb{E}\Big[e^{\lambda e^{-t}|W_0|}\Big] e^{\frac{1}{\sqrt{2}}\lambda + \frac{1}{2}\lambda^2} \\ &\le e^{\frac{\lambda}{2}(3+\lambda)}\mathbb{E}\Big[e^{\lambda e^{-t}|W_0|}\Big],\end{aligned}$$

which is exactly (i).

Point (ii). For the second point, we integrate the inequality (50) and get:

$$Y_t = Y_0 + \int_0^t W_s\, ds = Y_0 + (1 - e^{-t})W_0 + \int_0^t (1 - e^{s-t}) F(s, Y_s)\, ds + N_t, \tag{53}$$

$$|Y_t| \le |Y_0| + |W_0| + \frac{t}{2} + |N_t|, \tag{54}$$

$$\text{with} \quad N_t := \int_0^t e^{-s} M_s\, ds = \sqrt{2}\int_0^t (1 - e^{u-t})\, dB_u,$$

where we have used a stochastic version of Fubini's Theorem in the last line. N_t is a centered random Gaussian variable with variance $\sigma_t^2 := 4e^{-t} - e^{-2t} - 3 + 2t \le 2t$. The bound (52) and the independence of the initial condition and N_t then lead to

$$\begin{aligned}\mathbb{E}\Big[e^{\lambda|Y_t|}\Big] &\le e^{\lambda\frac{t}{2}}\,\mathbb{E}\Big[e^{\lambda(|Y_0| + |W_0|)}\Big]\mathbb{E}\Big[e^{\lambda|N_t|}\Big], \\ &\le e^{\lambda\frac{t}{2}}\,\mathbb{E}\Big[e^{\lambda(|Y_0| + |W_0|)}\Big]\, 2\, e^{\frac{1}{2}\lambda^2\sigma_t^2} \le 2\, e^{\lambda\frac{t}{2} + \lambda^2 t}\,\mathbb{E}\Big[e^{\lambda(|Y_0| + |W_0|)}\Big],\end{aligned}$$

which leads to the claimed result.

Point (*iii*). We first integrate equality (50) between s and u, and take a supremum in time:

$$Y_u = Y_s + (1-e^{s-u})W_s + \int_s^u (1-e^{v-u})F(v,Y_v)\,dv + N'_u,$$

(55)

$$\sup_{s\le u\le t}|Y_u - Y_s| \le (t-s)|W_s| + \frac{t-s}{2} + \sup_{s\le u\le t}|N'_u|, \quad \text{with} \quad N'_u := \sqrt{2}\int_s^u (1-e^{v-u})\,dB_v,$$

But, thanks to the properties of Brownian motion, by a change of time $\tau = \sigma^2_{t-s} = 4e^{s-t} - e^{2(s-t)} - 3 + 2(t-s) \le (t-s)^3$:

$$\sup_{s\le u\le t}|N'_u| \overset{\mathcal{L}}{=} \sup_{0\le u\le \tau}|B_u| \overset{\mathcal{L}}{=} \sqrt{\tau}\sup_{0\le u\le 1}|B_u|.$$

The law of supremum in time of the absolute value of a 1D Brownian motion is explicitly known, see for instance [**6**, p. 342]. Here, we will use only simple estimates on the exponential moments:

$$\begin{aligned}\mathbb{E}\Big[e^{\lambda\sup_{0\le u\le 1}|B_u|}\Big] &\le \mathbb{E}\Big[e^{\lambda\sup_{0\le u\le 1}B_u}\Big] + \mathbb{E}\Big[e^{\lambda\sup_{0\le u\le 1}(-B_u)}\Big]\\ &\le 2\,\mathbb{E}\Big[e^{\lambda|B_1|}\Big] \le 4e^{\frac{1}{2}\lambda^2}.\end{aligned}$$

In the second line, we use the well-known equality $\sup_{0\le u\le\tau}B_u \overset{\mathcal{L}}{=} \sup_{0\le u\le\tau}(-B_u) \overset{\mathcal{L}}{=} |B_\tau|$, and then the exponential moments given by (52). The constant 4 appearing above will raise some difficulties so we will perform a little optimization to get rid of it. For any $\theta \ge 1$, we may also bound

$$\mathbb{E}\Big[e^{\lambda\sup_{0\le u\le 1}|B_u|}\Big] \le \mathbb{E}\Big[e^{\lambda\theta\sup_{0\le u\le 1}|B_u|}\Big]^{\frac{1}{\theta}} \le \Big(4e^{\frac{1}{2}(\theta\lambda)^2}\Big)^{\frac{1}{\theta}} = e^{\frac{1}{\theta}\ln 4 + \frac{\theta}{2}\lambda^2}$$

The optimal θ seems to be $\theta = \frac{2\sqrt{\ln 2}}{\lambda}$. It is admissible when $\lambda \le 2\sqrt{\ln 2}$. It leads to

$$\mathbb{E}\Big[e^{\lambda\sup_{0\le u\le 1}|B_u|}\Big] \le e^{2\lambda\sqrt{\ln 2}} \le e^{2\lambda}.$$

Finally for $t-s \le \lambda^{-2}$, the upper bound on τ leads to $\lambda(t-s)^{-1}\sqrt{\tau} \le \lambda\sqrt{t-s} \le 1$ and

$$\mathbb{E}\Big[e^{\lambda(t-s)^{-1}\sup_{s\le u\le t}|N'_u|}\Big] = \mathbb{E}\Big[e^{\lambda(t-s)^{-1}\sqrt{\tau}\sup_{0\le u\le 1}|B_u|}\Big] \le e^{2\lambda\sqrt{t-s}}.$$

Using the point (ii) of the Proposition and (55), we write

$$\begin{aligned}\mathbb{E}\Big[e^{\lambda(t-s)^{-1}\sup_{s\le u\le t}|Y_u-Y_s|}\Big] &\le e^{\frac{\lambda}{2}}\,\mathbb{E}\Big[e^{\lambda|W_s|}\Big]\,\mathbb{E}\Big[e^{\lambda(t-s)^{-1}\sup_{s\le u\le t}|N'_u|}\Big]\\ &\le e^{\frac{\lambda}{2}}\,e^{\frac{\lambda}{2}(3+\lambda)}\mathbb{E}\Big[e^{\lambda|W_0|}\Big]\,e^{2\lambda\sqrt{t-s}},\end{aligned}$$

which concludes the proof, using that $\sqrt{t-s} \le \frac{1}{4}$ by assumption.

Point (*iv*). Taking the expectation in (51),

$$\mathbb{E}\big[|W_t|\big] \le e^{-t}\mathbb{E}\big[|W_0|\big] + \frac{1}{2} + \mathbb{E}\big[|M_t|\big].$$

and using that M_t is $\mathcal{N}(0, 1-e^{-2t})$ distributed, the expectation is simply bounded: $\mathbb{E}\big[|M_t|\big] \le \sqrt{2/\pi} \le 1$ and it implies the first bound of point (*iv*). The second bound uses (54) written with s instead of 0, which leads in average to

$$\mathbb{E}\big[|Y_t - Y_s|\big] \le (1-e^{s-t})\mathbb{E}\big[|W_s|\big] + \frac{t-s}{2} + \mathbb{E}\big[|N_{t-s}|\big].$$

Using, $1-e^{s-t} \le t-s$ and the previous bound on $\mathbb{E}[|W_s|]$, and the fact that N_t is $\mathcal{N}(0,\sigma^2_{t-s})$ distibuted, with $\sigma^2_{t-s} \le (t-s)^3$, it comes

$$\mathbb{E}\big[|Y_t - Y_0|\big] \le (t-s)\bigg(\mathbb{E}\big[|W_0|\big] + 2 + \frac{1}{2} + \sqrt{\frac{2(t-s)}{\pi}}\bigg),$$

and the conclusion follows when $t-s \le \frac{1}{4}$. □

3.3. Strong existence via regularization and Feymann-Kac type estimates. We introduce a smoothing kernel $\chi \in C^\infty(\mathbb{R}, \mathbb{R}^+)$ with support included in $[-1,1]$ and satisfying $\int_{\mathbb{R}} \chi(y)\, dy = 1$. And then standardly for $\eta > 0$, $\chi_\eta := \eta^{-1}\chi(\frac{\cdot}{\eta})$, and the approximated kernel

$$K_\eta := K * \chi_\eta, \quad \text{which satisfies} \quad |(K_\eta - K)(x)| \le \mathbb{1}_{[-\eta,\eta]}(x), \tag{56}$$

for all $x \in \mathbb{R}$. Given a stochastic basis, and a Brownian $(B_t)_{t\ge 0}$ motion on it, we consider the following non linear SDE:

$$Y_t^\eta = Y_0^\eta + \int_0^t W_s^\eta\, ds, \qquad W_t^\eta = W_0^\eta + \int_0^t \mathbb{E}_{\overline{Y}}\big[K_\eta(Y_s^\eta - \overline{Y})\big]\, ds - \int_0^t W_s^\eta\, ds + \sqrt{2}\, B_t, \tag{57}$$

where $\overline{Y}$ is an independent copy of Y^η and the initial condition (Y_0^η, W_0^η) is defined as

$$Y_0^\eta := Y_0 + \eta U, \qquad W_0^\eta := W_0 + \eta V, \tag{58}$$

where (Y_0, W_0) has law f_0 and is independent of (U,V) of law $\chi \otimes \chi$ (and both are independent of the Brownian motion $(B_t)_{t\ge 0}$). Then (Y_0^η, W_0^η) has for law $\mu_0^\eta := f_0 * \tilde{\chi}_\eta$ with $\tilde{\chi}_\eta(y,w) := \chi_\eta(y)\chi_\eta(w)$, and is independent of the Brownian motion $(B_t)_{t\ge 0}$. Introducing the notation

$$\widetilde{K}_\eta[\mu](x) = \int_{\mathbb{R}^2} K_\eta(x-y)\mu(dy, dw),$$

this system can be written in an equivalent manner:

$$Y_t^\eta = Y_0^\eta + \int_0^t W_s^\eta\, ds, \qquad W_t^\eta = W_0^\eta + \int_0^t \widetilde{K}_\eta[\mu_s^\eta](Y_s^\eta)\, ds - \int_0^t W_s^\eta\, ds + \sqrt{2}\, B_t,$$

where μ_t^η is the time marginal at time t, *i.e.* the law of (Y_t^η, W_t^η).

Since the kernel K_η is globally Lipschitz, [**28**, Thm 1.1] implies the strong existence and uniqueness of the process $(Y_t^\eta, W_t^\eta)_{t\ge 0}$ solving (57). And by an application of Ito's rule the family of the time marginals $(\mu_t^\eta)_{t\ge 0}$ of that process is a weak solution of the following regularized Vlasov-Poisson-Fokker-Planck equation:

$$\frac{\partial}{\partial t}\mu_t^\eta + v\, \partial_x \mu_t^\eta + \widetilde{K}_\eta[\mu_t^\eta]\, \partial_v \mu_t^\eta = \partial_v(\partial_v \mu_t^\eta + v\mu_t^\eta), \tag{59}$$

with the initial condition $\mu_0^\eta = \mathcal{L}(Y_0, W_0) = f_0 * \tilde{\chi}_\eta$. We begin by proving some η independent estimates on μ_t^η, for $t \ge 0$.

Feynmann-Kac type estimates.

LEMMA 5. *Assume that the law of the initial condition of equation* (41) $f_0 \in \mathcal{P}_1 \cap L^1(\mathbb{R}^2)$ *satisfies either* $\|f_0\|_{e,\lambda} < \infty$ *for some* $\lambda > 0$ *or* $\|f_0\|_{p,\gamma} < \infty$ *for some* $\gamma > 1$. *Then for all* $t > 0$, *the unique (measure) solution to the smoothed VPFP equation* (59) *with initial condition* μ_0^η *satisfies respectively*

$$\|\mu_t^\eta\|_{e,\lambda} \le 2\, e^{t+\lambda\eta+\frac{\lambda}{2}+\frac{\lambda^2}{2}}\, \|f_0\|_{e,\lambda e^{-t}}, \qquad \|\mu_t^\eta\|_{p,\gamma} \le \|f_0\|_{p,\gamma}\, \langle\eta\rangle^\gamma\, C_\gamma e^{(1+\gamma)t}, \tag{60}$$

where C_γ *is a constant depending explicitly on* γ.

In particular, the associated spatial density $\rho_t^\eta := \int_{\mathbb{R}} \mu_t^\eta(x,v)\, dv$ *satisfies respectively*

$$\|\rho_t^\eta\|_\infty \le \frac{4}{\lambda}\, e^{t+\frac{\lambda}{2}+\frac{\lambda^2}{2}} e^{\lambda\eta}\, \|f_0\|_{e,\lambda e^{-t}}, \qquad \|\rho_t^\eta\|_\infty \le \frac{2\gamma}{\gamma-1} C_\gamma e^{(1+\gamma)t} \langle\eta\rangle^\gamma \|f_0\|_{p,\gamma}. \tag{61}$$

PROOF. *Step 1. Regularization and Feynmann-Kac's formula.* Fix $t \ge 0$ and consider the following "backward" SDE:

$$Y_s^{x,v} = x - \int_0^s W_u^{x,v}\, du, \qquad W_s^{x,v} = v - \int_0^s \widetilde{K}_\eta[\mu_{t-u}^\eta](Y_u^{x,v})\, du + \int_0^s W_u^{x,v}\, du + \sqrt{2}\, B_s, \tag{62}$$

First note that $\tilde{K}_\eta[\mu^\eta]$ is uniformly Lipschitz in position on $\mathbb{R}^+ \times \mathbb{R}^2$. So strong existence and uniqueness of solution to the (linear) SDE (62) are guaranteed by standard results. We set:

$$\theta_s = e^s\, \mu^\eta_{t-s}\big(Y^{x,v}_s, W^{x,v}_s\big).$$

Moreover the initial condition $\mu^\eta_0 = f_0 * \tilde{\chi}_\eta$ fulfills the hypothesis of Proposition 4 of the Appendix: $\partial^k_x \partial^l_v \mu^\eta_0 \in L^2(\mathbb{R}^2)$ for any $k, l \geq 0$. This implies that $\mu^\eta_t(x,v)$ possesses one continuous derivative in time, and two (continuous) derivative in position and velocity. So, we may apply Ito's rule to θ: we get

$$\begin{aligned} e^{-s} d\theta_s &= \mu^\eta(t-s, Y^{x,v}_s, W^{x,v}_s)ds - \partial_t\mu^\eta(t-s, Y^{x,v}_s, W^{x,v}_s)ds + \partial_x\mu^\eta(t-s, Y^{x,v}_s, W^{x,v}_s)dY^{x,v}_s \\ &\qquad + \partial_v\mu^\eta(t-s, Y^{x,v}_s, W^{x,v}_s)dW^\eta_s + \Delta_v\mu^\eta(t-s, Y^{x,v}_s, W^{x,v}_s)\,\langle dW^{x,v}_s\rangle^2 \\ &= \Big[-\partial_t\mu^\eta - v\,\partial_x\mu^\eta - \widetilde{K}_\eta[\mu^\eta]\,\partial_v\mu^\eta + \partial_v(v\mu^\eta) + \Delta_v\mu^\eta\Big]\,(t-s, Y^{x,v}_s, W^{x,v}_s)ds \\ &\qquad + \partial_v\mu^\eta(t-s, Y^{x,v}_s, W^{x,v}_s)dB_s, \end{aligned}$$

and since μ^η is a strong solution of (59), we get precisely that for any $0 \leq s \leq s' \leq t$:

$$\theta_{s'} - \theta_s = \int_s^{s'} e^u \partial_v\mu^\eta(t-u, Y^{x,v}_u, W^{x,v}_u)\, dB_u.$$

In particular, $(\theta_s)_{0\leq s\leq t}$ is a martingale, so that

$$\mu^\eta(t,x,v) = \theta_0 = \mathbb{E}[\theta_t] = e^t \mathbb{E}\big[f_0 * \tilde{\chi}_\eta(Y^{x,v}_t, W^{x,v}_t)\big]. \tag{63}$$

Step 2. Proof in the case of uniform exponential tails.

But by the hypothesis on f_0, and since χ has support in $[-1,1]$,

$$\begin{aligned} f_0 * \tilde{\chi}_\eta(x,v) &= \int f_0(x-y, v-w)\tilde{\chi}_\eta(y,w)\, dydw \leq \|f_0\|_{e,\lambda} \int_{\mathbb{R}} e^{-\lambda|v-\eta w'|}\chi(w')\, dw' \\ &\leq \|f_0\|_{e,\lambda} e^{-\lambda|v|} \int_{\mathbb{R}} e^{\lambda\eta|w'|}\chi(w')\, dw' \leq \|f_0\|_{e,\lambda} e^{-\lambda|v|+\lambda\eta}. \end{aligned} \tag{64}$$

Moreover, the definition (62) of W^η also implies that for $0 \leq s \leq t$:

$$W^{x,v}_s = e^s v - \int_0^s e^{s-u}\widetilde{K_\eta}[\mu^\eta_u]\, du + M_s, \quad \text{with } M_s := \sqrt{2}\int_0^s e^{s-u} dB_u,$$

Remark that M_s is in fact a centered Gaussian variable with variance $e^{2s}-1$. Since $\big\|\widetilde{K^\eta}[\mu^\eta_u]\big\|_\infty \leq 1/2$, it leads to the following lower bound

$$|W^{x,v}_t| \geq e^t|v| - |M_t| - \frac{e^t}{2}. \tag{65}$$

Using all of this in the representation formula (63) leads to

$$\mu^\eta_t(x,v) \leq \|f_0\|_{e,\lambda} e^{t+\lambda\eta}\, \mathbb{E}\Big[e^{-\lambda|W^{x,v}_t|}\Big] \leq \|f_0\|_{e,\lambda} e^{t+\lambda(e^t/2+\eta)-\lambda e^t|v|}\, \mathbb{E}\left[e^{\lambda|M_t|}\right]$$

An application of (52) to M_t leads to the following bound that is uniform in η (for η small) :

$$\|\mu^\eta_t\|_{e,\lambda e^t} \leq 2\, e^{t+\lambda\eta+\frac{\lambda}{2}e^t+\frac{\lambda^2}{2}}\, \|f_0\|_{e,\lambda}. \tag{66}$$

The conclusion follow by replacing λ by λe^{-t} in the above bound. And the estimate on $\|\rho^\eta_t\|_\infty$ is simply obtained by integration on v.

Step 3. Proof in the case of uniform polynomial tails.

The simple inequality $\langle v+w\rangle \le \sqrt{2}\langle v\rangle\langle w\rangle$ (recall that $\langle v\rangle^2 := 1+v^2$) implies that $\langle v-w\rangle^{-1} \le \sqrt{2}\langle v\rangle^{-1}\langle w\rangle$, which allows to bound

$$\begin{aligned} f_0 * \tilde\chi_\eta(x,v) &= \int f_0(x-y,v-w)\tilde\chi_\eta(y,w)\,dydw \le \|f_0\|_{p,\gamma}\int_{\mathbb{R}} \langle v-\eta w'\rangle^{-\gamma}\chi(w')\,dw', \\ &\le 2^{\gamma/2}\|f_0\|_{p,\gamma}\langle v\rangle^{-\gamma}\int_{\mathbb{R}}\langle \eta w'\rangle^{\gamma}\chi(w')\,dw' \le 2^{\gamma/2}\|f_0\|_{p,\gamma}\langle \eta\rangle^{\gamma}\langle v\rangle^{-\gamma}. \end{aligned}$$

Plugging it into the representation formula (63), using the lower bound (65) and the simple inequality, $\langle v+\frac12 e^t\rangle^\gamma \le 2^{\gamma/2}\big(e^{\gamma t}+\langle 2v\rangle^\gamma\big)$ (simply separate the cases $v\le \frac12 e^t$ and $v\ge \frac12 e^t$),

$$\begin{aligned} \mu^\eta(t,x,v) &= e^t\,\mathbb{E}\big[f_0 * \tilde\chi_\eta(Y_t^{x,v},W_t^{x,v})\big] \le 2^{\gamma/2}\|f_0\|_{p,\gamma}\langle\eta\rangle^\gamma\mathbb{E}\big[\langle W_t^{x,v}\rangle^{-\gamma}\big] \\ &\le 2^{\gamma/2}\|f_0\|_{p,\gamma}\langle\eta\rangle^\gamma\mathbb{E}\Big[\Big\langle e^t|v|-|M_t|-\frac{e^t}{2}\Big\rangle^{-\gamma}\Big] \\ &\le 2^{\gamma}\|f_0\|_{p,\gamma}\langle\eta\rangle^\gamma\big\langle e^t v\big\rangle^{-\gamma}\,\mathbb{E}\Big[\Big\langle |M_t|+\frac{e^t}{2}\Big\rangle^{\gamma}\Big] \\ &\le 2^{3\gamma/2}\|f_0\|_{p,\gamma}\langle\eta\rangle^\gamma\big\langle v\big\rangle^{-\gamma}\,\Big(e^{\gamma t}+2^\gamma\mathbb{E}\big[|M_t|^\gamma\big]\Big). \end{aligned}$$

Since $M_t\sim\mathcal{N}(0,e^{2t}-1)$ we have $\mathbb{E}\big[|M_t|^\gamma\big] = (e^{2t}-1)^{\frac\gamma2}\,m_\gamma \le e^{\gamma t}m_\gamma$, where m_γ stands for the moment of order γ of the law $\mathcal{N}(0,1)$. This implies the claimed bound on $\|\mu_t^\eta\|_{p,\gamma}$ with $C_\gamma := 2^{3\gamma/2}(1+2^\gamma m_\gamma)$. The bound on $\|\rho_t^\eta\|_\infty$ follows from the simple bound $\langle v\rangle^{-\gamma} \le \min\big(1,|v|^{-\gamma}\big)$ and an integration in v. □

Completeness estimates. Thanks to the propagation of the uniform estimates on the tails in velocity and the crucial Lemma 4, we are in position to show that the family $(Y_t^\eta,W^\eta)_{\eta>0}$ of solutions to (57) has the Cauchy property.

LEMMA 6. *For η,$\eta'>0$, let $(Y_t^\eta,W_t^\eta)_{t\ge0}$ and $(Y_t^{\eta'},W_t^{\eta'})_{t\ge0}$ be two (unique) solutions of the nonlinear SDE* (57)*, constructed on a given probability basis, with a common Brownian motion $(B_t)_{t\ge0}$, and initial condition chosen as* (58)*. If the law of the initial condition satisfies either $\|f_0\|_{e,\lambda}<\infty$ for some $\lambda>0$ or $\|f_0\|_{e,\gamma}<\infty$ for some $\gamma>1$, then the following stability estimate holds for any $t\ge0$:*

$$\mathbb{E}\left[\sup_{s\in[0,t]}\Big(\Big|Y_s^\eta-Y_s^{\eta'}\Big|+\Big|W_s^\eta-W_s^{\eta'}\Big|\Big)\right] \le \frac32(\eta+\eta')\exp\Big((1+8K_{t,\eta+\eta'})t\Big), \tag{67}$$

where $K_{t,\eta+\eta'}$ is the constant appearing in the r.h.s of (61) *with η replaced by $\eta+\eta'$,* i.e. *respectively*

$$K_{t,\eta+\eta'} = \frac{4}{\lambda}\,e^{t+\frac\lambda2+\frac{\lambda^2}{2}}\,e^{\lambda(\eta+\eta')}\,\|f_0\|_{e,\lambda e^{-t}}, \quad \text{or} \quad K_{t,\eta+\eta'} = \frac{2\gamma}{\gamma-1}C_\gamma e^{\gamma t}\langle\eta+\eta'\rangle^\gamma\|f_0\|_{p,\gamma}.$$

PROOF. Our strategy is the same as in the proof of the weak strong stability estimate in subsection 3.1. For $t\in[0,T]$ we have:

$$\begin{aligned} \sup_{s\in[0,t]}\Big|Y_t^\eta-Y_t^{\eta'}\Big| &\le \int_0^t\Big|W_s^\eta-W_s^{\eta'}\Big|\,ds\,+|\eta-\eta'||U| \\ \sup_{s\in[0,t]}\Big|W_s^\eta-W_s^{\eta'}\Big| &\le \int_0^t\Big|\widetilde K_\eta[\mu_s^\eta](Y_s^\eta)-\widetilde K_{\eta'}[\mu_s^{\eta'}](Y_s^{\eta'})\Big|\,ds\,+|\eta-\eta'||V| \\ &\le \int_0^t\Big(\Big|\widetilde K_\eta[\mu_s^\eta](Y_s^\eta)-\widetilde K[\mu_s^\eta](Y_s^\eta)\Big|+\Big|\widetilde K[\mu_s^\eta](Y_s^\eta)-\widetilde K[\mu_s^{\eta'}](Y_s^{\eta'})\Big| \\ &\qquad +\Big|\widetilde K[\mu_s^{\eta'}](Y_s^{\eta'})-\widetilde K_{\eta'}[\mu_s^{\eta'}](Y_s^{\eta'})\Big|\Big)\,ds+|\eta-\eta'|, \end{aligned}$$

since $|V|$ and $|U|$ are always bounded by 1 (we recall that χ has its support included in $[-1,1]$). But thanks to (56), for any $y \in \mathbb{R}$:

$$\left|\widetilde{K}_\eta[\mu_s^\eta](y) - \widetilde{K}[\mu_s^\eta](y)\right| = \left|\int (K_\eta - K)(y-y')\,\rho_s^\eta(dy')\right| \leq \int \mathbb{1}_{[-\eta,\eta]}(y-y')\,\rho_s^\eta(dy') \leq 2\eta\,\|\rho_s^\eta\|_\infty\,.$$

This allows to bound the the first and third term, in the r.h.s. of the second inequality by $2\int_0^t \big(\eta\|\rho_s^\eta\|_\infty + \eta'\|\rho_s^{\eta'}\|_\infty\big)\,ds$. And the second term is estimated in expectation with the help of Lemma 4:

$$\begin{aligned}\mathbb{E}\Big[\Big|\widetilde{K}[\mu_s^\eta](Y_s^\eta) - \widetilde{K}[\mu_s^{\eta'}](Y_s^{\eta'})\Big|\Big] &\leq 8\min\big(\|\rho_t^\eta\|_\infty, \|\rho_t^{\eta'}\|_\infty\big)\,\mathbb{E}\big[|Y_t^\eta - Y_t^{\eta'}|\big]\\ &\leq 4\big(\|\rho_t^\eta\|_\infty + \|\rho_t^{\eta'}\|_\infty\big)\,\mathbb{E}\big[|Y_t^\eta - Y_t^{\eta'}|\big].\end{aligned}$$

Gathering all of this leads to

$$\mathbb{E}\left[\sup_{s\in[0,t]}\Big(\big|Y_s^\eta - Y_s^{\eta'}\big| + \big|W_s^\eta - W_s^{\eta'}\big|\Big)\right] \leq \int_0^t \alpha(s)\left(\mathbb{E}\Big[\big|Y_s^\eta - Y_s^{\eta'}\big| + \big|W_s^\eta - W_s^{\eta'}\big|\Big] + \frac{3}{2}(\eta+\eta')\right) ds,$$

with $\alpha(s) := 1 + 4\|\rho_s^\eta\|_\infty + 4\|\rho_s^{\eta'}\|_\infty$. A simple application of the Gronwall's lemma to $\mathbb{E}\big[\sup_{s\in[0,t]}\big(|Y_s^\eta - Y_s^{\eta'}| + |W_s^\eta - W_s^{\eta'}|\big)\big] + 3(\eta+\eta')/2$ leads to

$$\mathbb{E}\left[\sup_{s\in[0,t]}\Big(\big|Y_s^\eta - Y_s^{\eta'}\big| + \big|W_s^\eta - W_s^{\eta'}\big|\Big)\right] \leq \frac{3}{2}(\eta+\eta')\exp\left(\int_0^t \alpha(s)\,ds\right).$$

The conclusion follows from (61), which implies that $\|\rho_s^\eta\|_\infty \leq K_{s,\eta} \leq K_{t,\eta+\eta'}$ for any $s \in [0,t]$ (and a similar inequality with η' replacing η). □

The Cauchy property in the space of path. We now consider the space $\mathcal{A}$ of measurable applications (or random variables) from $\Omega = C(\mathbb{R}^+;\mathbb{R}^2)$ (with the Wiener measure) into itself. We endow it with the topology of uniform convergence on compact (in time) subsets. An associated distance to that topology is for instance

$$\forall\, Y, Z \in \mathcal{A}, \quad d(Y,Z) = \sum_{n\in\mathbb{N}^*} \frac{1}{2^n}\mathbb{E}\left(1 \wedge \sup_{t\in[0,2^n]}\big(|Y_t^1 - Z_t^1| + |Y_t^2 - Z_t^2|\big)\right), \tag{68}$$

for which $\mathcal{A}$ is complete (the symbol $\wedge$ stand for the minimum). Let us consider the "sequence" $(Y^\eta, W^\eta)_{\eta>0}$ (the correct denomination is "net"). By Lemma 6 it is a Cauchy "sequence" (or net) in $(\mathcal{A}, d)$, and then it converges towards a certain $(Y_t, W_t)_{t\geq 0}$ in $(\mathcal{A}, d)$.

At a fixed time t, this implies the convergence in probability and then in law of (Y_t^η, W_t^η) toward (Y_t, W_t): *i.e.* the time marginals μ_t^η weakly converge (as measures) towards f_t. Using a standard argument, we can pass in the limit in the uniform bound (60) obtained in Lemma 5. So that for any time t, the density of the law f_t of (Y_t, W_t) satisfies one of the bound of (44).

Identification of the limit. In order to prove that $(Y_t, W_t)_{t\geq 0}$ is a solution to (41) we have to show that for any $t \geq 0$:

$$\mathbb{E}\left[\sup_{s\in[0,t]}\left|Y_s - Y_0 - \int_0^s W_u\,du\right|\right] = 0, \quad \mathbb{E}\left[\sup_{s\in[0,t]}\left|W_s - W_0 + \int_0^s \big(W_u - \widetilde{K}[\mu_u](Y_u)\big)\,du - B_s\right|\right] = 0. \tag{69}$$

But this is something we know for the approximated process $(Y_t^\eta, W_t^\eta)_{t\geq 0}$. Precisely, by Definition of strong solution to (57):

$$\begin{aligned}&\mathbb{E}\left[\sup_{s\in[0,t]}\left|Y_s^\eta - Y_0 - \eta U - \int_0^s W_u^\eta\,du\right|\right] = 0,\\ &\qquad\qquad \mathbb{E}\left[\sup_{s\in[0,t]}\left|W_s^\eta - W_0 - \eta V + \int_0^s \big(W_u^\eta - \widetilde{K}_\eta[\mu_u^\eta](Y_u^\eta)\big)\,du - B_s\right|\right] = 0.\end{aligned}$$

But the convergence in $(\mathcal{A}, d)$ allows to pass to the limit in the first equality above , and we obtain the first equality in (69). In order to pass to the limit in the second inequality, and get the second part of (69), the main difficulty is to handle the non-linear term. Precisely, we will show in the rest of the proof that

$$\sup_{s\in[0,t]} \mathbb{E}\left(\left|\widetilde{K}_\eta[\mu_s^\eta](Y_s^\eta) - \widetilde{K}[\mu_s](Y_s)\right|\right) \xrightarrow[\eta\to 0]{} 0.$$

This will conclude the proof.

For this, we introduce a independent copy $(\bar{Y}_t^\eta, \bar{Y}_t)$ of the couple (Y_t^η, Y_t). We then rewrite the force with the help of that independent copy and estimate

$$\begin{aligned}\mathbb{E}\Big[\Big|\widetilde{K}_\eta[\mu_s^\eta](Y_s^\eta) - \widetilde{K}[\mu_s](Y_s)\Big|\Big] &= \mathbb{E}\Big[\big|K_\eta(Y_s^\eta - \bar{Y}_s^\eta) - K(Y_s - \bar{Y}_s)\big|\Big] \\ &\le \mathbb{E}\Big[|K_\eta - K|(Y_s^\eta - \bar{Y}_s^\eta)\Big] + \mathbb{E}\Big[\big|K(Y_s^\eta - \bar{Y}_s^\eta) - K(Y_s - \bar{Y}_s)\big|\Big]\end{aligned}$$

The first term in the r.h.s is bounded thanks to (56):

$$\begin{aligned}\mathbb{E}\big[|K_\eta - K|(Y_s^\eta - \bar{Y}_s^\eta)\big] &\le \mathbb{E}\big[\mathbb{1}_{[-\eta,\eta]}(Y_s^\eta - \bar{Y}_s^\eta)\big] \le \mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{[-\eta,\eta]}(Y_s^\eta - \bar{Y}_s^\eta)\big|Y_s^\eta\big]\Big] \\ &\le \mathbb{E}\big[2\eta\|\rho_s^\eta\|_\infty\big] = 2\eta\|\rho_s^\eta\|_\infty\end{aligned}$$

The second term in the r.h.s. is bounded thanks to Lemma 4 and we get

$$\mathbb{E}\Big[\Big|\widetilde{K}_\eta[\mu_s^\eta](Y_s^\eta) - \widetilde{K}[\mu_s](Y_s)\Big|\Big] \le 2\eta\|\rho_s^\eta\|_\infty + 8\,\|\rho_s\|_\infty \mathbb{E}\big[|Y_s^\eta - Y_s|\big].$$

Since $\|\rho_s^\eta\|_\infty$ and $\|\rho_s\|_\infty$ are bounded uniformly in time on $[0,t]$ and for $\eta \in (0,1)$, and thanks to the convergence of (Y^η, W^η) towards (Y, W) in $(\mathcal{A}, d)$, it is simple to conclude that the requested term goes to zero, as η goes to zero.

3.4. Proof of Corollary 1. The existence part is a simple consequence of the existence of the process solution to the non linear SDE (41), for a given initial condition with a polynomial or exponential decays of the velocity tails (use Ito's rule). So we only prove the weak-strong stability estimate.

Let $(g_t)_{t\ge 0}$ be weak solution to (46) starting from $g_0 \in \mathcal{P}_1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ and $(f_t)_{t\ge 0}$ be another weak solution to (46) starting from $f_0 \in \mathcal{P}_1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ satisfying for any $t > 0$

$$\sup_{s\le t} \|\rho_s\|_\infty < \infty, \qquad \text{where} \quad \rho_s(x) = \rho_s^f := \int_{\mathbb{R}} f_s(x, dv).$$

By [**13**, Theorem 2.6] or [**15**, Proposition B.1], there exists a stochastic basis $\left(\Omega, \mathbb{P}, (\mathcal{F}_t)_{t\ge 0}, \mathcal{F}\right)$ and a Brownian motion $(B_t)_{t\ge 0}$ on this basis and a process $(X_t, V_t)_{t\ge 0}$ solution to (41), which has exactly the time marginal g_t at any time $t \ge 0$.

Next, remark that the force field $\widetilde{K}$ created by f, precisely $\widetilde{K}_s(x) = \int_{\mathbb{R}} K(x-y)\rho_s^f(dy)$, is Lipschitz in position. In fact,

$$\big|\widetilde{K}_s(x) - \widetilde{K}_s(y)\big| \le \int_{\mathbb{R}} \big|K(x-z) - K(y-z)\big|\rho_s(dz) \le \int_{\mathbb{R}} \mathbb{1}_{[x,y]}(z)\rho_s(dz) \le \|\rho_s\|_\infty |x-y|.$$

Extending the probability space, we may choose a r.v. $(Y_0, W_0) \sim f_0$ such that

$$W_1(f_0, g_0) = \mathbb{E}\Big[\big|Y_0 - X_0\big| + \big|W_0 - V_0\big|\Big],$$

and since $\tilde{K}$ is regular, standard results allow us to build on that probability space a stochastic process $(Y_t, W_t)_{t\ge 0}$ solution to

$$Y_t = Y_0 + \int_0^t W_s\, ds, \qquad W_t = W_0 + \int_0^t \widetilde{K}_s(Y_s)\, ds - \int_0^t W_s\, ds + \sqrt{2}\, B_t.$$

Note that the family of time marginals $(h_t)_{t\geq 0}$ of (Y_t, W_t) is a solution to the following linear Vlasov-Fokker-Planck equation

$$\partial_t h_t + v\,\partial_x h_t + \widetilde{K}_t(x)\,\partial_v h_t = \partial_v(\partial_v h_t + v h_t), \quad \text{where} \quad \widetilde{K}_s(x) = K * \rho_s^f(x), \tag{70}$$

with initial condition $h_{t=0} = f_0$. Of course, that equation is also satisfied by f by assumption. But since $\tilde{K}$ is globally Lipschitz in the space variable, uniformly in the time variable, uniqueness holds for equation (70) in the class $L^\infty_t(L^1)$ by standard results. So, we have $h_t = f_t$ for all time $t \geq 0$ and $(Y_t, W_t)_{t\geq 0}$ is actually a solution to (41) defined on the same probability space as $(X_t, V_t)_{t\geq 0}$. Then applying the point (ii) of Theorem 2 to those processes and recalling that

$$W_1(f_t, g_t) \leq \mathbb{E}\Big[\big|Y_t - X_t\big| + \big|W_t - V_t\big|\Big],$$

we get the expected estimate.

4. Proof of Theorem 3

Before going to the proof, we will prove a useful lemma.

LEMMA 7. *Let $(X_1, \dots, X_N)$ be N i.i.d random variables of law $\rho \in \mathcal{P}(\mathbb{R}^d)$, and $\rho_N = \frac{1}{N}\sum_i \delta_{X_i}$ be the associated empirical measure. Then, for all $a \in \mathbb{R}^d$ we have:*

$$\mathbb{E}\Big[\sup_{u\in\mathbb{R}_+}\Big|\int_{\mathbb{R}^d} \mathbb{1}_{|a-y|\leq u}\,(\rho_N - \rho)(dy)\Big|\Big] \;\leq\; \frac{3}{\sqrt{N}}$$

PROOF. In fact, we choose a sequence $(X_n)_{n\in\mathbb{N}}$ of i.i.d random variables of law $\rho \in \mathcal{P}(\mathbb{R}^d)$ and L some Poisson random variable of parameter N independent of the $(X_n)_{n\in\mathbb{N}}$. We define the two point process by

$$M_N = \sum_{i=1}^{L} \delta_{Y_i},$$

M_N is in fact a Poisson Random Measure (PRM) with intensity measure $N\rho$. Remark that $\|M_N - N\rho_N\|_{TV} = |L - N|$. For all $a \in \mathbb{R}^d$, we have

$$\begin{aligned}\Big|\int \mathbb{1}_{|a-y|\leq u}(\rho_N - \rho)(dy)\Big| &\leq \Big|\int \mathbb{1}_{|a-y|\leq u}\Big(\frac{1}{N}M_N - \rho\Big)(dy)\Big| + \Big|\int \mathbb{1}_{|a-y|\leq u}\Big(\frac{1}{N}M_N - \rho_N\Big)(dy)\Big| \\ &\leq \frac{1}{N}\Big|\int \mathbb{1}_{|a-y|\leq u}\big(M_N - N\rho\big)(dy)\Big| + \frac{1}{N}\big\|M_N - N\rho_N\big\|_{TV}\end{aligned}$$

$$\sup_{u\in\mathbb{R}_+}\Big|\int \mathbb{1}_{|a-y|\leq u}(\rho_N - \rho)(dy)\Big| \leq \frac{1}{N}\sup_{u\in\mathbb{R}_+}\Big|\mathcal{M}_u^{N,a}\Big| + \frac{|L-N|}{N}, \tag{71}$$

$$\text{where} \quad \mathcal{M}_u^{N,a} = \int \mathbb{1}_{|a-y|\leq u}\big(M_N - N\rho\big)(dy).$$

Since M_N is a PRM, $(\mathcal{M}_u^{N,a})_{u\geq 0}$ is a martingale with respect to the filtration $(\mathcal{F}_u^a)_{u\geq 0} = \big(\sigma\big(M_N \mathbb{1}_{B(a,u)}\big)\big)_{u\geq 0}$, where $B(a,u)$ the ball of center a and radius u in $\mathbb{R}^d$. So using Doob's inequality and the fact that $\mathcal{M}_\infty^{N,a} = L - N$ we find

$$\mathbb{E}\Big[\sup_{u\geq 0}\big|\mathcal{M}_u^{N,a}\big|\Big] \leq \Big(\mathbb{E}\Big[\sup_{u\geq 0}\big|\mathcal{M}_u^{N,a}\big|^2\Big]\Big)^{1/2} \leq 2\left(\mathbb{E}\Big[\big|\mathcal{M}_\infty^{N,a}\big|^2\Big]\right)^{1/2} \leq 2\,\mathcal{V}[L] = 2\sqrt{N}.$$

Taking now the expectation in (71), using the above bound and $\mathbb{E}|L-N| \leq \sqrt{\mathcal{V}(L)} = \sqrt{N}$, we conclude the proof. □

We are now in position to prove Theorem 3.

Step 1. Coupling estimates.

We begin by a calculation valid for any fixed realization of these processes (*i.e.* given any initial conditions and Brownian paths): for all $i = 1, \cdots, N$ we have:

$$\sup_{s\in[0,t]} \left|X^N_{i,s} - Y^N_{i,s}\right| \le \left|X^N_{i,0} - Y^N_{i,0}\right| + \int_0^t \left|V^N_{i,s} - W^N_{i,s}\right| ds$$

$$\sup_{s\in[0,t]} \left|V^N_{i,s} - W^N_{i,s}\right| \le \left|V^N_{i,0} - W^N_{i,0}\right| + \int_0^t \left| \frac{1}{N}\sum_{j=1}^N K(X^N_{i,s} - X^N_{j,s}) - \left(\int_{\mathbb{R}\times\mathbb{R}} K(Y^N_{i,s} - x)\mu_t(dx,dv)\right)\right|$$

$$\le \left|V^N_{i,0} - W^N_{i,0}\right| + \int_0^t \left| \frac{1}{N}\sum_{j=1}^N K(X^N_{i,s} - X^N_{j,s}) - K(Y^N_{i,s} - Y^N_{j,s})\right| ds + \frac{t}{N-1} + \int_0^t \Lambda^N_{i,s}\, ds$$

$$\text{with} \quad \Lambda^N_{i,s} := \left| \frac{1}{N-1}\sum_{j\neq i}^N K(Y^N_{i,s} - Y^N_{j,s}) - \int_{\mathbb{R}\times\mathbb{R}} K(Y^N_{i,s} - x)\mu_s(dx,dv)\right|$$

We sum these inequalities over $i = 1, \cdots, N$, divide by $N \ge 2$ and then get:

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^N \sup_{s\in[0,t]} \left|X^N_{i,s} - Y^N_{i,s}\right| &\le \frac{1}{N}\sum_{i=1}^N \left|X^N_{i,0} - Y^N_{i,0}\right| + \int_0^t \frac{1}{N}\sum_{i=1}^N |V^N_{i,s} - W^N_{i,s}| ds \\
\frac{1}{N}\sum_{i=1}^N \sup_{s\in[0,t]} \left|V^N_{i,s} - W^N_{i,s}\right| &\le \frac{1}{N}\sum_{i=1}^N \left|V^N_{i,0} - W^N_{i,0}\right| + \frac{2t}{N} + \int_0^t \Lambda^N_s\, ds \\
&\quad + \frac{1}{N^2}\sum_{i\neq j}^N \int_0^t \left|K(X^N_{i,s} - X^N_{j,s}) - K(Y^N_{i,s} - Y^N_{j,s})\right| ds \\
&\quad \text{where} \quad \Lambda^N_s := \frac{1}{N}\sum_{i=1}^N \Lambda^N_{i,s}\, ds.
\end{aligned} \tag{72}$$

Using equality (49) we find with the notation $\widehat{\rho}^{i,N}_s = \frac{1}{N-1}\sum_{j\neq i} \delta_{Y^N_{j,s}}$,

$$\begin{aligned}
&\frac{1}{N^2}\sum_{i\neq j}^N \left|K(X^N_{i,s} - X^N_{j,s}) - K(Y^N_{i,s} - Y^N_{j,s})\right| \\
&\quad \le \frac{2}{N^2}\sum_{i\neq j}^N \mathbb{1}_{|Y^N_{i,s} - Y^N_{j,s}| \le 2|X^N_{i,s} - Y^N_{i,s}|} = \frac{2(N-1)}{N^2}\sum_{i=1}^N \int_{\mathbb{R}} \mathbb{1}_{|Y^N_{i,s} - y| \le 2|X^N_{i,s} - Y^N_{i,s}|} \widehat{\rho}^{i,N}_s(dy) \\
&\quad \le \frac{2}{N}\sum_{i=1}^N \int_{\mathbb{R}} \mathbb{1}_{|Y^N_{i,s} - y| \le 2|X^N_{i,s} - Y^N_{i,s}|} \rho_s(dy) + \frac{2}{N}\sum_{i=1}^N \left| \int_{\mathbb{R}} \mathbb{1}_{|Y^N_{i,s} - y| \le 2|X^N_{i,s} - Y^N_{i,s}|} (\widehat{\rho}^{i,N}_s - \rho_s)(dy)\right| \\
&\quad \le \frac{8}{N}\,\|\rho_s\|_\infty \sum_{i=1}^N |X^N_{i,s} - Y^N_{i,s}| + \Gamma^N_s
\end{aligned}$$

where

$$\Gamma^N_s := \frac{2}{N}\sum_{i=1}^N \Gamma^N_{i,s}, \qquad \Gamma^N_{i,s} := \sup_{u\in\mathbb{R}_+} \left| \int_{\mathbb{R}} \mathbb{1}_{|Y^N_{i,s} - y| \le u} (\widehat{\rho}^{i,N}_s - \rho_s)(dy)\right|. \tag{73}$$

Then, $\beta(t) := \frac{1}{N}\sum_{i=1}^N \big(\sup_{s\in[0,t]} |X^N_{i,s} - Y^N_{i,s}| + \sup_{s\in[0,t]} |V^N_{i,s} - W^N_{i,s}|\big)$ satisfies the integral inequality

$$\beta(t) \le \beta(0) + \int_0^t \left(\big(1 + 8\|\rho_s\|_\infty\big)\beta(s) + \Lambda^N_s + 2\,\Gamma^N_s + \frac{2}{N}\right) ds.$$

An application of Grönwall's Lemma leads to

$$\beta(t) \leq e^{\left(t+8\int_0^t \|\rho_s\|_\infty\, ds\right)}\left(\beta(0)+\int_0^t e^{-s}\left(\Lambda_s^N+2\,\Gamma_s^N+\frac{2}{N}\right)ds\right).$$

Taking the expectation and using the symmetry of the laws of the of $(X_{i,t}^N, V_{i,t}^N)_{i=1,\dots,N}$ and $(Y_{i,t}^N, W_{i,t}^N)_{i=1,\dots,N}$ we find

$$\begin{aligned}(74)\quad \mathbb{E}\Big[\sup_{s\in[0,t]}\Big|X_{1,s}^N-Y_{1,s}^N\Big|+\Big|V_{1,s}^N-W_{1,s}^N\Big|\Big] &\leq e^{\left(t+8\int_0^t \|\rho_s\|_\infty\, ds\right)}\\ &\left(E\Big[\Big|X_{1,0}^N-Y_{1,0}^N\Big|+\Big|V_{1,0}^N-W_{1,0}^N\Big|\Big]+\int_0^t e^{-s}\Big(\mathbb{E}\big[\Lambda_{1,s}^N+2\,\Gamma_{1,s}^N\big]+\frac{2}{N}\Big)\,ds\right).\end{aligned}$$

We will bound the expectation of stochastic terms appearing in the r.h.s. with the help of Lemma 7.

Step 2. Conclusion of the proof.

We recall that $\widehat{\rho}_t^{i,N}=\frac{1}{N-1}\sum_{j\neq i}\delta_{Y_{j,t}^N}$ is the empirical measure associated to the $(Y_{i,t}^N)_{2\leq i\leq N}$, and is then independent of $Y_{1,t}^N$. By the definition (73) of Γ and Lemma 7, we have

$$\begin{aligned}\mathbb{E}\left[\Gamma_{1,t}^N\right] &= \mathbb{E}\left[\sup_{u\geq 0}\left|\int \mathbb{1}_{|Y_{1,t}^N-y|\leq u}\big(\widehat{\rho}_t^{1,N}-\rho_t\big)(dy)\right|\right],\\ &= \mathbb{E}\left[\mathbb{E}\left[\sup_{u\geq 0}\left|\int \mathbb{1}_{|Y_{1,t}^N-y|\leq u}\big(\widehat{\rho}_t^{1,N}-\rho_t\big)(dy)\right|\,\middle|\, Y_{1,t}^N\right]\right]\leq\frac{3}{\sqrt{N}},\end{aligned}$$

Moreover, using again the fact that the $(Y_{i,t}^N)_{1\leq i\leq N}$ are i.i.d and that $\|K\|_\infty\leq 1/2$, we find for $N\geq 2$

$$\begin{aligned}\mathbb{E}\big[\Lambda_{1,t}^N\big] &= \mathbb{E}\left[\mathbb{E}\left[\left|\frac{1}{N-1}\sum_{j=2}^N K(Y_{1,t}^N-Y_{j,t}^N)-\int_{\mathbb{R}^2}K(Y_{1,t}^N-x)\mu_s(dx,dv)\right|\,\middle|\, Y_{1,t}^N\right]\right],\\ &\leq \mathbb{E}\left[\left(\mathcal{V}\left[\frac{1}{N-1}\sum_{j=2}^N K(Y_{1,t}^N-Y_{j,t}^N)\,\middle|\, Y_{1,t}^N\right]\right)^{1/2}\right]\leq\frac{1}{2\sqrt{N-1}}\leq\frac{1}{\sqrt{2N}}.\end{aligned}$$

Then applying these results to equation (74) leads to

$$\mathbb{E}\Big[\sup_{s\in[0,t]}\Big|X_{1,s}^N-Y_{1,s}^N\Big|+\Big|V_{1,s}^N-W_{1,s}^N\Big|\Big]\leq e^{\left(t+8\int_0^t \|\rho_s\|_\infty\, ds\right)}\left(\frac{1}{\sqrt{N}}(1+2\times 3)+\frac{2}{N}\right),$$

which leads to the claimed bound.

5. Proof of Proposition 2

We distinguish here the deviation upper bounds for $\sup_{t\in[0,T]}\left\|\mu_{Y,t}^N\right\|_{\infty,\varepsilon}$ because such a result have an interest by itself. Moreover, we will see it is used in the proof of the concentration inequalities. In the present proof, we apply the following strategy: we first find prove concentration inequalities for the quantity $\left\|\mu_{Y,t}^N\right\|_{\infty,\varepsilon}$ at a fixed time t, then for the variation of this quantity on small time intervals, and finally conclude by mixing both estimates in an optimal way. That proof could be extended to dimension larger than 1, but we prefer to restrict here to the case of interest.

5.1. A uniform deviation upper bound for Binomial variables. We begin with a general result about deviation upper bounds for Binomial variable.

Lemma 8. *Let $p\in[0,1]$. If X is binomial variable of parameter (N,p), then for any $\alpha>0$:*

$$\mathbb{P}\left(|X-Np|\geq N\alpha\right)\leq 2e^{-2\alpha^2 N}.$$

PROOF. We may write $X = \sum_{i=1}^N X_i$, with a family $(X_i)_{i\leq N}$ of N i.i.d. Bernoulli variables of parameter p. For all $\lambda > 0$ we have:

$$\mathbb{E}\big[e^{\lambda X}\big] = \prod_{i=1}^N \mathbb{E}\big[e^{\lambda X_i}\big] = \Big(\mathbb{E}\big[e^{\lambda X_1}\big]\Big)^N = \Big(1-p+pe^{\lambda}\Big)^N.$$

Using Markov's inequality we get:

$$\mathbb{P}\left(X \geq N(p+\alpha)\right) \leq \left(\frac{1-p+pe^{\lambda}}{e^{\lambda(p+\alpha)}}\right)^N = \Big((1-p)e^{-\lambda(p+\alpha)} + pe^{\lambda(1-(p+\alpha))}\Big)^N,$$

The optimal λ turns out to be $ln\left(\frac{(p+\alpha)(1-p)}{(1-(p+\alpha))p}\right)$, and we get after some calculations:

$$\mathbb{P}\left(X \geq N(p+\alpha)\right) \leq e^{-Ng_p(\alpha)},$$

where

$$g_p(\alpha) := H\left(\mathcal{B}(p+\alpha)\middle|\mathcal{B}(p)\right) = (1-(p+\alpha))ln\left(\frac{1-(p+\alpha)}{1-p}\right) + (p+\alpha)ln\left(\frac{p+\alpha}{p}\right),$$

is the relative entropy with respect to $\mathcal{B}(p)$. By the properties of relative entropy, we have $g_p(0) = 0$ and $g_p(\alpha) > 0$ if $\alpha \neq 0$. So $g_p'(0) = 0$. Moreover a straight calculation gives for all α such that $p + \alpha < 1$:

$$g_p''(\alpha) = \frac{1}{p+\alpha} + \frac{1}{1-(p+\alpha)} \geq 4,$$

since for all $x \in (0,1)$, $x(1-x) \leq \frac{1}{4}$. Using Taylor's formula with integral rest at order 2 we have:

$$g_p(\alpha) = g_p(0) + g_p'(0)\alpha + \int_0^\alpha g_p''(u)(\alpha - u)du \geq 2\alpha^2.$$

So combining all these estimates we get for $\alpha + p < 1$:

$$\mathbb{P}\left(X \geq N(p+\alpha)\right) \leq e^{-Ng_p(\alpha)} \leq e^{-2N\alpha^2}.$$

It is still valid for $\alpha + p = 1$: just pass to the limit, and there is nothing to prove for $p + \alpha > 1$, since X cannot be larger than N. Moreover, since $\mathbb{P}\left(X \leq N(p-\alpha)\right) = \mathbb{P}\left(N - X \geq N\left((1-p)+\alpha\right)\right)$, an application of the above bound to the $\mathcal{B}(N, 1-p)$-Binomial variable $N - X$ leads to

$$\mathbb{P}\left(X \leq N(p-\alpha)\right) \leq e^{-2N\alpha^2},$$

and this concludes the proof. □

5.2. Concentration inequalities at fixed time. Thanks to lemma 8 we are able to give some concentration inequalities for the empirical measure $\rho^N = \frac{1}{N}\sum_{i=1}^N \delta_{Y_i}$, for i.i.d.r.v. $(Y_i)_{i_l eN}$.

LEMMA 9. *Let $\alpha, \varepsilon > 0$. Assume that $(Y_1, \ldots, Y_N)$ are N independent random variables, all with law $\rho \in L^\infty$ (we identify the law and its density). Assume also that ρ has an exponential moment of order $\lambda > 0$: $\mathcal{M}_\lambda(\rho) := \int e^{\lambda|y|}\rho(dy) < +\infty$. We denote by $\rho^N := \frac{1}{N}\sum_i \delta_{Y_i}$ the associated empirical measure. Then for any $\varepsilon, \alpha > 0$,*

$$\mathbb{P}\big(\|\rho^N\|_{\infty,\varepsilon} \geq \|\rho\|_\infty + \alpha\big) \leq \left(\frac{4\|\rho\|_\infty N(\varepsilon\alpha)}{\lambda} + 2 + N\mathcal{M}_\lambda(\rho)\right)e^{-2N(\varepsilon\alpha)^2}$$

PROOF. *Step 1. A first bound valid on compact subset.* For any $0 < \delta < \varepsilon$ we set $k = \lfloor \frac{R}{2\delta} \rfloor + 1$. It is clear that for all $x \in [-R, R]$, there exists $\ell \in \{-k, \cdots, k\}$ such that $B(x, \varepsilon) \subset B(2\ell\delta, \varepsilon + \delta)$. It implies

$$\mathbb{P}\Big(\sup_{x\in[-R,R]} \frac{\rho^N[B(x,\varepsilon)]}{2\varepsilon} \geq \|\rho\|_\infty + \alpha \Big) \leq \mathbb{P}\left(\sup_{\ell=-k,\cdots,k} \rho^N\big[B(2\ell\delta, \varepsilon+\delta)\big] \geq 2\varepsilon\big(\|\rho\|_\infty + \alpha\big)\right)$$
$$\leq \sum_{\ell=-k}^{k} \mathbb{P}\Big(\rho^N\left[B(2\ell\delta, \varepsilon+\delta)\right] \geq 2(\varepsilon+\delta)\|\rho\|_\infty + 2\big(\alpha\varepsilon - \delta\|\rho\|_\infty\big)\Big).$$

Since for any ℓ, $N\rho^N\big(B(2\ell\delta, \varepsilon+\delta)\big)$ is a Binomial variable of parameter N and $p_\ell = \int_{B(2\ell\delta,\varepsilon+\delta)} \rho_t(dx) \leq 2(\varepsilon+\delta)\|\rho_t\|_\infty$, we may apply Lemma 8 and bound each term in the r.h.s. by $\exp\big(-8N(\alpha\varepsilon - \delta\|\rho\|_\infty)^2\big)$. By the definition of k, and provided that $\varepsilon\alpha \geq \delta\|\rho\|_\infty$ it leads to

$$\mathbb{P}\left(\sup_{x\in[-R,R]} \frac{\rho^N\left[B(x,\varepsilon)\right]}{2\varepsilon} \geq \|\rho\|_\infty + \alpha \right) \leq \Big(\frac{R}{\delta} + 2\Big) e^{-8N(\alpha\varepsilon - \delta\|\rho\|_\infty)^2}.$$

We now have to choose δ in order to minimize the right hand side. The particular choice $\delta\|\rho\|_\infty = (\varepsilon\alpha)/2$ satisfies the previous restriction and already provides an interesting bound:

$$\mathbb{P}\left(\sup_{x\in[-R,R]} \frac{\rho^N\left[B(x,\varepsilon)\right]}{2\varepsilon} \geq \|\rho\|_\infty + \alpha \right) \leq \left(\frac{2R\|\rho\|_\infty}{\varepsilon\alpha} + 2 \right) e^{-2N(\varepsilon\alpha)^2}. \tag{75}$$

Step 2. Extension to the whole space. It is clear that

$$\mathbb{P}\big(\|\rho^N\|_{\infty,\varepsilon} \geq \|\rho\|_\infty + \alpha\big) \leq \mathbb{P}\left(\sup_{x\in[-R,R]} \frac{\rho^N\left[B(x,\varepsilon)\right]}{2\varepsilon} \geq \|\rho_t\|_\infty + \alpha \right) + \mathbb{P}\big(\exists\, i \leq N, |Y_i| > R\big)$$
$$\leq \left(\frac{2R\|\rho\|_\infty}{\varepsilon\alpha} + 2 \right) e^{-2N(\varepsilon\alpha)^2} + N\mathcal{M}_\lambda(\rho) e^{-\lambda R},$$

where we used (75) to bound the first term in the r.h.s, and a simple application of Chebyshev's inequality to bound the second term: precisely that $\mathbb{P}\left(\exists\, i \leq N, |Y_i| > R\right) \leq N\mathbb{P}\left(|Y_1| > R\right) \leq N\mathcal{M}_\lambda(\rho)e^{-\lambda R}$. We choose $R = \frac{2}{\lambda}N(\alpha\varepsilon)^2$, and finally get:

$$\mathbb{P}\big(\|\rho^N\|_{\infty,\varepsilon} \geq \|\rho\|_\infty + \alpha\big) \leq \left(\frac{4\|\rho\|_\infty N(\varepsilon\alpha)}{\lambda} + 2 \right) e^{-2N(\varepsilon\alpha)^2} + N\mathcal{M}_\lambda(\rho) e^{-2N(\varepsilon\alpha)^2},$$

and the result is proved. □

5.3. Conclusion of the proof of Proposition 2. We begin with the following corollary of Proposition 3, which will be useful in the sequel

COROLLARY 10. *Let be $(Y_t, W_t)_{t\in[0,T]}$ be a solution of 41 for some initial condition (Y_0, W_0) of law f_0, having an exponential moment of order $\lambda > 0$. We define $c_\lambda := \frac{5}{2} + \frac{1}{\lambda}\ln\Big(\mathbb{E}\big[e^{\lambda|W_0|}\big]\Big)$ and denote by f_t the law of (Y_t, W_t). For $0 \leq s < t \leq s + \min\Big(\frac{1}{16}, \lambda^{-2}\Big)$, and $\beta > 0$, it holds:*

$$\mathbb{P}\Big(\sup_{s\leq u\leq t} |Y_u - Y_s| \geq (t-s)(c_\lambda + \beta) \Big) \leq e^{-\frac{\beta}{2}\min\left(\beta,\lambda\right)}.$$

And if the (Y_i^N, W_i^N) are N independent copies of the previous process, with the same notation

$$\mathbb{P}\left(\frac{1}{N}\sum_{i=1}^{N} \sup_{s\leq u\leq t} |Y_{i,u}^N - Y_{i,s}^N| \geq (t-s)(c_\lambda + \beta) \right) \leq e^{-\frac{\beta}{2}\min\left(\beta,\lambda\right)N}.$$

PROOF. Using point (iii) of the Proposition 3, and Chebyshev's inequality, and the definition of c_λ we get for any $0 < \lambda' \leq \lambda$:

$$\begin{aligned}\mathbb{P}\Big(\sup_{s\leq u\leq t}|Y_u - Y_s| \geq (t-s)(c_\lambda+\beta)\Big) &\leq e^{-\lambda'(c_\lambda+\beta)}\mathbb{E}\Big[e^{\lambda'(t-s)^{-1}\sup_{s\leq u\leq t}|Y_u-Y_s|}\Big]\\ &\leq e^{-\lambda'(c_\lambda+\beta)}e^{\frac{1}{2}\lambda'(5+\lambda')}\,\mathbb{E}\Big[e^{\lambda'|W_0|}\Big]\\ &\leq e^{-\lambda'\beta+\frac{1}{2}(\lambda')^2}\mathbb{E}\Big[e^{\lambda|W_0|}\Big]^{-\frac{\lambda'}{\lambda}}\mathbb{E}\Big[e^{\lambda'|W_0|}\Big] \leq e^{-\lambda'\beta+\frac{1}{2}(\lambda')^2}.\end{aligned}$$

Optimization in λ' leads to the particular choice $\lambda' = \beta$, when $\beta \leq \lambda$, and to the choice $\lambda' = \lambda$ otherwise. If we use $\beta - \frac{\lambda}{2} \geq \frac{\beta}{2}$ in the later case, we obtain the expected bound.

The proof of the second bound involving N independent copies follows the same lines: by independence

$$\mathbb{P}\Big(\frac{1}{N}\sum_{i=1}^N \sup_{s\leq u\leq t}|Y^N_{i,u}-Y^N_{i,s}| \geq (t-s)(c_\lambda+\beta)\Big) \leq \Big(e^{-\lambda'(c_\lambda+\beta)}\mathbb{E}\Big[e^{\lambda'(t-s)^{-1}\sup_{s\leq u\leq t}|Y_u-Y_s|}\Big]\Big)^N,$$

and the conclusion follows with the same optimization on λ'. □

We are now in position to prove Proposition 2. We fix $\gamma > 0$, define $\alpha := \frac{\gamma}{2}$, and recall from Corollary 10 the notation $c_\lambda = \frac{5}{2} + \frac{1}{\lambda}\ln\Big(\mathbb{E}\big[e^{\lambda|W_0|}\big]\Big)$ together with $\kappa_t := \sup_{0\leq s\leq t}\|\rho_s\|_\infty$.

$$\beta := \sqrt{N}\varepsilon\gamma\,\max\Big(1, \frac{\sqrt{N}\varepsilon\gamma}{\lambda}\Big), \qquad \Delta t := \frac{\varepsilon\alpha}{(\kappa_t+\alpha)(c_\lambda+\beta)}.$$

Remark that β satisfies $\frac{\beta}{2}\min\big(\beta,\lambda\big) = 2N(\varepsilon\alpha)^2$ and that Δt is always smaller than $\min\big(\frac{1}{16}, \lambda^{-2}\big)$ by the assumptions in Proposition 2, so that we may apply Corollary 10. We then choose $K = \big\lfloor \frac{t}{\Delta t}\big\rfloor + 1$, define $t_k = k\Delta t$ for all $0 \leq k \leq K$ (remark that $t_K \geq t$), and the two following events Ω^1 and Ω^2 as:

$$\begin{aligned}\Omega^1 &= \Big\{\exists\, 0\leq k\leq K-1,\ \ \Big\|\rho^N_{t_k}\Big\|_{\infty,\varepsilon+\Delta t(c_\lambda+\beta)} > \kappa_t+\alpha\Big\},\\ \Omega^2 &= \Big\{\exists\, 0\ \leq k\leq K-1,\ \exists\, i\leq N,\ \sup_{s\in[t_k,t_{k+1}]}\big|Y^N_{i,s}-Y^N_{i,t_k}\big| > \Delta t(c_\lambda+\beta)\Big\}.\end{aligned}$$

If the events Ω_1^c and Ω_2^c are realized, then for any $0 \leq s \leq t$, we choose k such that $s \in [t_k, t_{k+1})$ and get for any $x \in \mathbb{R}$

$$\begin{aligned}\rho^N_s\big(B(x,\varepsilon)\big) &\leq \rho^N_{t_k}\big(B(x,\varepsilon+\Delta t(c_\lambda+\beta)\big) \leq 2(\kappa_t+\alpha)\big(\varepsilon+\Delta t(c_\lambda+\beta)\big)\\ &= 2\varepsilon(\kappa_t+\alpha)\Big(1+\frac{\alpha}{(\kappa_t+\alpha)}\Big) = 2\varepsilon(\kappa_t+\gamma).\end{aligned}$$

The last equalities follow from the definition of α and Δ_t. It means that if Ω_1^c and Ω_2^c are realized, then

$$\sup_{s\in[0,t]}\big\|\rho^N_s\big\|_{\infty,\varepsilon} \leq \kappa_t+\gamma.$$

Next, we can bound $\mathbb{P}(\Omega_1)$ and $\mathbb{P}(\Omega_2)$ with the help of Lemma 9, Corollary 10, and point (ii) of Proposition 3 (on the control of the exponential moments of ρ_t): it leads precisely

to

$$\begin{aligned}\mathbb{P}\Big(\sup_{s\in[0,t]}\big\|\rho_s^N\big\|_{\infty,\varepsilon}\geq\kappa_t+\gamma\Big)&\leq\mathbb{P}\big(\Omega_1\big)+\mathbb{P}\big(\Omega_2\big)\\&\leq\sum_{k=0}^{K}\bigg(\frac{4\|\rho_{t_k}\|_\infty N(\varepsilon\alpha)}{\lambda}+2+N\mathcal{M}_\lambda(\rho_{t_k})\bigg)e^{-2N(\varepsilon\alpha)^2}+8\,NK\,e^{-\frac{\beta}{2}\min\left(\beta,\lambda\right)}\\&\leq\bigg(\frac{t}{\Delta t}+1\bigg)\bigg(\frac{4\kappa_t N(\varepsilon\alpha)}{\lambda}+2+2Ne^{\lambda(\frac{1}{2}+\lambda)t}\,\mathcal{M}_\lambda^{x,v}(f_0)+8N\bigg)e^{-2N(\varepsilon\alpha)^2}\\&=\bigg(\frac{(2\kappa_t+\gamma)(c_\lambda+\beta)t}{\varepsilon\gamma}+1\bigg)\bigg(\frac{2\kappa_t(\varepsilon\gamma)}{\lambda}+2e^{\lambda(\frac{1}{2}+\lambda)t}\,\mathcal{M}_\lambda^{x,v}(f_0)+10\bigg)N\,e^{-\frac{1}{2}N(\varepsilon\gamma)^2}.\end{aligned}$$

Using that by assumption $\varepsilon\gamma\leq\frac{1}{2}\kappa_t$, and the expression of β, leads to with

$$\mathbb{P}\Big(\sup_{s\in[0,t]}\big\|\rho_s^N\big\|_{\infty,\varepsilon}\geq\kappa_t+\gamma\Big)\leq\boldsymbol{C_1}\bigg(1+t\frac{2\kappa_t+\gamma}{\varepsilon\gamma}\Big(c_\lambda+\sqrt{N}\varepsilon\gamma\max(1,\sqrt{N}\varepsilon\gamma\lambda^{-1}\Big)\bigg)N\,e^{-\frac{1}{2}N(\varepsilon\gamma)^2},$$

$\boldsymbol{C_1}:=10+\kappa_t^2\lambda^{-1}+2e^{\lambda(\frac{1}{2}+\lambda)t}\,\mathcal{M}_\lambda^{x,v}(f_0)$. The conclusion follows using that $\lambda N^{-1/2}\leq\varepsilon\gamma$ by assumption.

6. Proof of Theorem 4

6.1. MKW estimates on devitation between particle and coupled systems. The first step of the proof is similar to the first step of the proof of Theorem 3. Precisely we start with equation (72), which reads now

$$\begin{aligned}&\frac{1}{N}\sum_{i=1}^{N}\sup_{s\in[0,t]}\Big|X_{i,s}^N-Y_{i,s}^N\Big|\leq\int_0^t\frac{1}{N}\sum_{i=1}^{N}|V_{i,s}^N-W_{i,s}^N|ds\\&\frac{1}{N}\sum_{i=1}^{N}\sup_{s\in[0,t]}\Big|V_{i,s}^N-W_{i,s}^N\Big|\leq\int_0^t\frac{1}{N^2}\sum_{i,j=1}^{N}\Big|K(X_{i,s}^N-X_{j,s}^N)-K(Y_{i,s}^N-Y_{j,s}^N)\Big|\,ds+\frac{t}{N-1}+\int_0^t\Lambda_s^N\,ds,\\&\quad\text{where}\quad\Lambda_s^N:=\frac{1}{N}\sum_{i=1}^{N}\Lambda_{i,s}^N\,ds=\frac{1}{N}\sum_{i=1}^{N}\bigg|\frac{1}{N-1}\sum_{j=1}^{N}K(Y_{i,s}^N-Y_{j,s}^N)-\int_{\mathbb{R}\times\mathbb{R}}K(Y_{i,s}^N-x)\mu_s(dx,dv)\bigg|,\end{aligned}$$

since the initial conditions are now equal. Remark that if we introduce σ,τ two independent random variables with uniform law on $\{1,2,\dots,N\}$, then the sum involving K becomes

$$\frac{1}{N^2}\sum_{i,j=1}^{N}\Big|K(X_{i,s}^N-X_{j,s}^N)-K(Y_{i,s}^N-Y_{j,s}^N)\Big|=\mathbb{E}_{\sigma,\tau}\Big[\Big|K(X_{\sigma,s}^N-X_{\tau,s}^N)-K(Y_{\sigma,s}^N-Y_{\tau,s}^N)\Big|\Big],$$

where we emphasize that the expectation is taken only with respect to (σ,τ). So, we are in position to apply point (iii) Lemma 4 ; *i.e.* the part involving discrete uniform norms, and get:

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N}\sup_{s\in[0,t]}\Big|V_{i,s}^N-W_{i,s}^N\Big|&\leq8\int_0^t\|\rho_s^N\|_{\infty,\varepsilon}\bigg(\frac{1}{N}\sum_{i=1}^{N}\Big|X_{i,s}^N-Y_{i,s}^N\Big|+\frac{\varepsilon}{2}\bigg)ds+\frac{t}{N-1}+\int_0^t\Lambda_s^N\,ds\\&\leq\int_0^t\big(1+8\|\rho_s^N\|_{\infty,\varepsilon}\big)\bigg(\frac{1}{N}\sum_{i=1}^{N}\Big|X_{i,s}^N-Y_{i,s}^N\Big|+\frac{\varepsilon}{2}+\frac{1}{N-1}+\sup_{u\leq t}\Lambda_u^N\bigg)ds\end{aligned}$$

where ρ_s^N is the empirical measure of the $(Y_{i,s}^N)_{1\leq i\leq N}$: $\rho_s^N:=\frac{1}{N}\sum_{i=1}^{N}\delta_{Y_{i,s}^N}$. Applying finally Gronwall's Lemma on the interval $[0,t]$ where the quantity $\sup_{u\leq t}\Lambda_u^N$ may be

considered as fixed, we get:

$$\frac{1}{N}\sum_{i=1}^{N}\sup_{s\in[0,t]}\big(|X_{i,s}^N-Y_{i,s}^N|+|V_{i,s}^N-W_{i,t}^N\big)$$

$$\leq\left(\frac{\varepsilon}{2}+\frac{1}{N-1}+\sup_{s\leq t}\Lambda_s^N\right)\exp\left(t+8\int_0^t\left\|\rho_s^N\right\|_{\infty,\varepsilon}\,ds\right), \tag{76}$$

$$\text{where}\quad \Lambda_s^N:=\frac{1}{N}\sum_{i=1}^{N}\left|\frac{1}{N-1}\sum_{j=1}^{N}K(Y_{i,s}^N-Y_{j,s}^N)-\int_{\mathbb{R}\times\mathbb{R}}K(Y_{i,s}^N-x)\mu_s(dx,dv)\right|. \tag{77}$$

We now focus on finding some concentration inequalities for the random variable $\sup_{t\in[0,T]}\Lambda_t^N$. In order to prove concentration inequalities for these supremum in time, we follow the same steps as in the proof of Proposition 2. Once it is be done, we will combine them with the concentration inequalities on $\sup_{t\in[0,T]}\left\|\rho_t^N\right\|_{\infty,\varepsilon}$ given by Proposition 2, and we will obtain some deviation upper bounds for

$$\frac{1}{N}\sum_{i=1}^{N}\sup_{s\in[0,t]}\left(|X_{i,s}^N-Y_{i,s}^N|+|V_{i,s}^N-W_{i,t}^N|\right).$$

6.2. Estimation of the fluctuations term Λ_t^N at fixed time t. We first establish the

LEMMA 11. *Let $(Y_i^N)_{1\leq i\leq N}$ be i.i.d. random variables, with a diffuse common law (*i.e. *a law that does not charge any atom). For $i=1,\dots,N$ define the random variable Λ^N as follows:*

$$\Lambda^N=\frac{1}{N}\sum_{i=1}^{N}\left|\frac{1}{N-1}\sum_{j\neq i}K(Y_i^N-Y_j^N)-\mathbb{E}\big[K(Y_i^N-Y_j^N)\big|Y_i^N\big]\right|$$

Then for all $\alpha>0$,

$$\mathbb{P}\left(|\Lambda^N|\geq\alpha\right)\leq 2N\,e^{-2\alpha^2(N-1)}.$$

PROOF. *Step 1. A calculation with Y_1^N frozen.* To begin, we "freeze" $Y_1^N=a$ and define

$$\Lambda_1^N(a):=\frac{1}{N-1}\sum_{j\geq 2}K(a-Y_j^N)-\mathbb{E}\big[K(a-Y_j^N)\big].$$

By the definition (40) of K, and since $\mathbb{P}(Y_j^N=a)=0$ by assumption the random variable ζ_j^a defined by:

$$\zeta_j^a=K(a-Y_j^N)+\frac{1}{2}=\begin{cases}0 & \text{si } a<Y_j^N\\ 1/2 & \text{si } a=Y_j^N\\ 1 & \text{si } a>Y_j^N\end{cases}$$

is a Bernoulli variable with parameter $p_a:=\mathbb{P}(Y_j^N>a)$. But since $p_a-\frac{1}{2}=\mathbb{E}[K(a-Y_j^N)]$, we have $\sum_{j\geq 2}\zeta_j^a-(N-1)p_a=\Lambda_1^N(a)$. So, by an application of Lemma 8 to the binomial variable $\sum_{j\leq 2}\zeta_j^a$:

$$\mathbb{P}\left(\left|\Lambda_1^N(a)\right|\geq\alpha\right)=\mathbb{P}\left(\left|\sum_{j\neq i}\zeta_j^a-(N-1)p_a\right|\geq(N-1)\alpha\right)\leq 2\,e^{-\alpha(N-1)\alpha^2},$$

Step 2. Summing up on N. Using the notation introduced in the previous step, we can rewrite $\Lambda^N = \frac{1}{N}\sum_{i=1}^N \left|\Lambda_i^N(Y_i^N)\right|$, and then

$$\begin{aligned}\mathbb{P}\left(\Lambda^N \geq \alpha\right) &\leq \mathbb{P}\left(\sup_{i=1,\dots,N} |\Lambda_i^N(Y_i^N)| \geq \alpha\right) \leq \sum_{i=1}^N \mathbb{P}\left(|\Lambda_i^N(Y_i^N)| \geq \alpha\right)\\ &= N\,\mathbb{E}\Big[\mathbb{P}\big(|\Lambda_1^N(Y_1^N)| \geq \alpha \big| Y_1^N\big)\Big],\end{aligned}$$

where we have used the fact that the variables $(Y_i^N)_{1\leq i\leq N}$ are exchangeable. But by independence of Y_1^N and $(Y_i^N)_{i\geq 2}$, we obtain using the previous step that $\mathbb{P}\big(|\Lambda_1^N(Y_1^N)| \geq \alpha\big|Y_1^N\big) \leq 2\,e^{-2(N-1)\alpha^2}$ and the conclusion follows. □

6.3. Estimation on the supremum in time of the fluctuations term.

LEMMA 12. *Assume that (Y_i^N, W_i^N) are N independent copies of a solution (Y, W) to (41) of initial law f_0 satisfying $\mathcal{M}_\lambda^{x,v}(f_0) < +\infty$ for some $\lambda > 0$, and Λ_t^N is defined by (77). Then, with $\rho_s := \mathcal{L}(Y_s)$ $\kappa_t := \sup_{0\leq s\leq t}\|\rho_s\|_\infty$ we have provided that $\varepsilon \leq \min\Big(\frac{1}{16}, \frac{\lambda}{2}, \lambda^{-2}\Big)$.*

$$\mathbb{P}\left(\sup_{s\in[0,t]} |\Lambda_s^N| \geq \boldsymbol{C_t}\varepsilon\right) \leq (\varepsilon + t)\left(\frac{5}{\varepsilon} + 4\kappa_t N\lambda^{-1} + \frac{N}{\varepsilon}e^{\lambda t\left(\frac{1}{2}+\lambda\right)}\mathbb{E}\Big[e^{\lambda(|Y_0|+|W_0|)}\Big]\right)e^{-2N\varepsilon^2},$$

with $\boldsymbol{C_t} := 36 + 80\kappa_t + (1+3\kappa_t)\frac{8}{\lambda}\ln\mathbb{E}\Big[e^{\lambda|W_0|}\Big]$.

PROOF. *Step 1. Bounding the time fluctuations.* Let $(Z_t)_{t\geq 0} = (Y_t, W_t)_{t\geq 0}$ be a solution to the non linear SDE (41). We show in that step that for $N \geq 2$:

$$\begin{aligned}\sup_{s\in[t,t+\Delta t]} |\Lambda_s^N - \Lambda_t^N| \leq \left(16\left\|\rho_t^N\right\|_{\infty,\varepsilon} + 4\|\rho_t\|_\infty\right) &\sup_{s\in[t,t+\Delta t]} \frac{1}{N}\sum_{i=1}^N \big|Y_{i,s}^N - Y_{i,t}^N\big|\\ &+ 4\left\|\rho_t^N\right\|_{\infty,\varepsilon}|s-t|\Big(\mathbb{E}\big[|W_0|\big] + 2\Big) + 10\left\|\rho_t^N\right\|_{\infty,\varepsilon}\varepsilon.\end{aligned} \tag{78}$$

Indeed, by the definition (77) of Λ_s^N:

$$\begin{aligned}|\Lambda_N^s - \Lambda_N^t| \leq \frac{1}{N(N-1)}\sum_{i=1}^N\sum_{j=1}^N &\Big|K(Y_{i,s}^N - Y_{j,s}^N) - K(Y_{i,t}^N - Y_{j,t}^N)\Big|\\ &+ \frac{1}{N}\sum_{i=1}^N\left|\int_{\mathbb{R}} K(Y_{i,s}^N - x)\rho_s(dx) - \int_{\mathbb{R}} K(Y_{i,t}^N - x)\,\rho_t(dx)\right|\end{aligned}$$

Using the second point of Lemma 4 with two copies of a vector of joint law $\frac{1}{N}\sum_i \delta_{(Y_{i,t}^N, Y_{i,s}^N)}$, we may bound the first term in the r.h.s. by

$$\frac{1}{N(N-1)}\sum_{i=1}^N\sum_{j=1}^N\Big|K(Y_{i,s}^N - Y_{j,s}^N) - K(Y_{i,t}^N - Y_{j,t}^N)\Big| \leq 8\frac{N}{N-1}\left\|\rho_t^N\right\|_{\infty,\varepsilon}\left(\frac{1}{N}\sum_{i=1}^N |Y_{i,s}^N - Y_{i,t}^N| + \frac{\varepsilon}{2}\right).$$

To estimate the second term in the r.h.s, we use the third point of Lemma 4, applied to independent couples: The first one with law $\frac{1}{N}\sum_i \delta_{(Y_{i,t}^N, Y_{i,s}^N)}$ and (Y_t, Y_s). It leads to the estimate

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^N\left|\int_{\mathbb{R}} K(Y_{i,s}^N - x)\rho_s(dx) - \int_{\mathbb{R}} K(Y_{i,t}^N - x)\,\rho_t(dx)\right|&\\ \leq 4\Big(\|\rho_t\|_\infty\frac{1}{N}\sum_{i=1}^N |Y_{i,s}^N - Y_{i,t}^N| + \left\|\rho_t^N\right\|_{\infty,\varepsilon}\Big(\mathbb{E}\big[|Y_s - Y_t|\big] + \varepsilon/2\Big)\Big).&\end{aligned}$$

Putting these two estimates together, using point (iv) of Proposition 3 in order to bound $\mathbb{E}\big[|Y_t - Y_s|\big]$ and taking the supremum in time leads to (78).

Step 2. Controlling the deviation. We define $t_k = k\varepsilon$ for $0 \le k \le k_M := \lfloor t\varepsilon^{-1} \rfloor$ and recall that by assumption $\varepsilon \le \min\Big(\frac{1}{16}, \frac{\lambda}{2}, \lambda^{-2}\Big)$. Consider the events

$$\Omega_1 := \big\{\|\rho^N_{t_k}\|_{\infty,\varepsilon} \le \kappa_t + 1 \text{ for all } 0 \le k \le k_M\big\},$$
$$\Omega_2 := \Big\{\sup_{s\in[t_k,t_k+\varepsilon]} \frac{1}{N}\sum_{i=1}^N \big|Y^N_{i,s} - Y^N_{i,t_k}\big| \le \varepsilon(c_\lambda + 2\varepsilon) \text{ for all } 0 \le k \le k_M\Big\},$$
$$\Omega_3 := \big\{|\Lambda^N_{t_k}| \le 4\varepsilon \text{ for all } 0 \le k \le k_M\big\}.$$

By Lemma 9 and the point (ii) of Lemma 3,

$$\mathbb{P}(\Omega_1^c) \le (k_M+1)\Big(4\kappa_t N\varepsilon\lambda^{-1} + 2 + Ne^{\lambda t\left(\frac12+\lambda\right)}\mathbb{E}\Big[e^{\lambda(|Y_0|+|W_0|)}\Big]\Big)e^{-2N\varepsilon^2}.$$

By Corollary 10, $\mathbb{P}(\Omega_2^c) \le (k_M+1)e^{-2N\varepsilon^2}$. By Lemma 11, $\mathbb{P}(\Omega_3^c) \le 2(k_M+1)\,e^{-8(N-1)\varepsilon^2} \le 2(k_M+1)\,e^{-2N\varepsilon^2}$ if $N \ge 2$. When the three above events are realized, we get with the help of the bound (78)

$$\begin{aligned}\sup_{s\in[0,t]}|\Lambda^N_s| &\le \sup_{k=0,\dots,k_M}|\Lambda^N_{t_k}| + \sup_{k=0,\dots,k_M,\ s\in[t_k,t_k+\varepsilon]}|\Lambda^N_s - \Lambda^N_{t_k}| \\ &\le 4\varepsilon + (20\kappa_t+4)\varepsilon(c_\lambda+\varepsilon) + 4\,(\kappa_t+1)\varepsilon\Big(\mathbb{E}\big[|W_0|\big]+2\Big) + 10\,(\kappa_t+1)\varepsilon \\ &\le \boldsymbol{C'_t}\varepsilon,\end{aligned}$$

with $\boldsymbol{C'_t} := 26 + 4c_\lambda + 4\mathbb{E}[|W_0|] + \kappa_t(20c_\lambda + 4\mathbb{E}[|W_0|] + 30)$. Using the expression of c_λ and the fact that $\lambda\mathbb{E}[|W_0|] \le \ln E\big[e^{\lambda|W_0|}\big]$, it is not difficult to show that $\boldsymbol{C'_t} \le \boldsymbol{C_t}$, where $\boldsymbol{C_t}$ is the constant introduced in Lemma 12. Moreover, gathering the three previous estimates on $\mathbb{P}(\Omega_i^c)$ for $i = 1, 2, 3$, we get

$$\mathbb{P}\big(\Omega_1^c \cup \Omega_2^c \cup \Omega_3^c\big) \le \frac{\varepsilon+t}{\varepsilon}\left(5 + 4\kappa_t N\varepsilon\lambda^{-1} + Ne^{\lambda t\left(\frac12+\lambda\right)}\mathbb{E}\Big[e^{\lambda(|Y_0|+|W_0|)}\Big]\right)e^{-2N\varepsilon^2},$$

which concludes the proof. □

6.4. Conclusion of the proof of Theorem 4. We now consider the two events

$$\Omega_1 := \Big\{\sup_{s\in[0,t]}|\Lambda^N_s| \ge \boldsymbol{C_t}\varepsilon\Big\}, \qquad \Omega_2 := \Big\{\sup_{s\in[0,t]}\big\|\rho^N_s\big\|_{\infty,\varepsilon} \ge \kappa_t + 1\Big\},$$

where $\boldsymbol{C_t}$ is the constant defined in 12. By Lemma 12, and since we also assume $\varepsilon \ge \lambda N^{-1/2}$

$$\begin{aligned}\mathbb{P}\left(\Omega_1^c\right) &\le (\varepsilon+t)\left(\frac{5}{\varepsilon} + 4\kappa_t N\lambda^{-1} + \frac{N}{\varepsilon}e^{\lambda t\left(\frac12+\lambda\right)}\mathbb{E}\Big[e^{\lambda(|Y_0|+|W_0|)}\Big]\right)e^{-2N\varepsilon^2}, \\ &\le (\varepsilon+t)\lambda^{-1}N^{\frac32}\left(5 + 4\kappa_t + e^{\lambda t\left(\frac12+\lambda\right)}\mathbb{E}\Big[e^{\lambda(|Y_0|+|W_0|)}\Big]\right)e^{-2N\varepsilon^2}\end{aligned}$$

and by Proposition 2, provided that $\varepsilon \le 5\kappa_t \min\Big(\frac{1}{16}, \frac{\lambda}{2}, \lambda^{-2}\Big)$, and since we also assume $\varepsilon \ge \lambda N^{-1/2}$, the following bound holds with $\boldsymbol{D_t} := 10 + \kappa_t^2\lambda^{-1} + e^{\lambda(\frac12+\lambda)t}\,\mathbb{E}\Big[e^{\lambda(|Y_0|+|W_0|)}\Big]$

$$\begin{aligned}\mathbb{P}\big(\Omega_2^c\big) &\le \boldsymbol{D_t}\left(1 + t\frac{2\kappa_t+1}{\varepsilon}\Big(c_\lambda + \sqrt{N}\varepsilon\max(1, \sqrt{N}\varepsilon\lambda^{-1})\Big)\right)N\,e^{-\frac12 N\varepsilon^2}, \\ &\le \boldsymbol{D_t}\left(1 + t\frac{2\kappa_t+1}{\varepsilon}\Big(c_\lambda + N\varepsilon^2\lambda^{-1}\Big)\right)N\,e^{-\frac12 N\varepsilon^2}, \\ &\le \boldsymbol{D_t}\lambda^{-1}\left(\varepsilon + t(2\kappa_t+1)\Big(c_\lambda + \sqrt{N}\varepsilon\Big)\right)N^{\frac32}\,e^{-2N\varepsilon^2}.\end{aligned}$$

If the two events are satisfied, then thanks to (76) we get

$$\frac{1}{N}\sum_{i=1}^{N}\sup_{s\in[0,t]}\big(|X^N_{i,s}-Y^N_{i,s}|+|V^N_{i,s}-W^N_{i,t}|\big)\le\bigg(\Big(\boldsymbol{C_t}+\frac{1}{2}\Big)\varepsilon+\frac{1}{N-1}\bigg)\exp\bigg(t+8\int_0^t\|\rho_s\|_\infty\,ds\bigg),$$
$$\le\bigg(\boldsymbol{C_t}+\frac{1}{2}+\frac{2}{\lambda}\bigg)\exp\bigg(t+8\int_0^t\|\rho_s\|_\infty\,ds\bigg)\varepsilon,$$

where we have used that $\varepsilon\ge\lambda N^{-1/2}\ge\lambda\big(2(N-1)\big)^{-1}$ when $N\ge 2$. which concludes the proof since

$$\boldsymbol{B_t}:=\boldsymbol{C_t}+\frac{1}{2}+\frac{2}{\lambda}.$$

Defining also

$$\boldsymbol{A''_t}:=\lambda^{-1}\bigg(\boldsymbol{D_t}\big(1+(2\kappa_t+1)c_\lambda\big)+5+4\kappa_t+e^{\lambda t\left(\frac{1}{2}+\lambda\right)}\mathbb{E}\Big[e^{\lambda(|Y_0|+|W_0|)}\Big]\bigg),\quad \boldsymbol{A'_t}:=\lambda^{-1}\boldsymbol{D_t}(2\kappa_t+1),$$

we obtain from the previous bound $\mathbb{P}\big(\Omega_1^c\cup\Omega_2^c\big)\le(t+\varepsilon)\big(\boldsymbol{A_t}+\boldsymbol{A'_t}\sqrt{N}\varepsilon\big)N^{\frac{3}{2}}e^{-2N\varepsilon^2}$, and that concludes the proof. In particular it can be checked that $\boldsymbol{A''_t}\le\boldsymbol{A_t}$ where $\boldsymbol{A_t}$ is defined by

$$\boldsymbol{A_t}:=\lambda^{-1}\bigg[12+\kappa_t^2\lambda^{-1}+2\,e^{\lambda(\frac{1}{2}+\lambda)t}\,\mathbb{E}\Big[e^{\lambda(|Y_0|+|W_0|)}\Big]\bigg]\big(1+(2\kappa_t+1)c_\lambda\big),$$
$$\boldsymbol{A'_t}:=\lambda^{-1}\bigg[10+\kappa_t^2\lambda^{-1}+e^{\lambda(\frac{1}{2}+\lambda)t}\,\mathbb{E}\Big[e^{\lambda(|Y_0|+|W_0|)}\Big]\bigg](2\kappa_t+1),$$
$$\boldsymbol{B_t}:=37+\frac{2}{\lambda}+80\kappa_t+(1+3\kappa_t)\frac{8}{\lambda}\ln\mathbb{E}\Big[e^{\lambda|W_0|}\Big].$$

7. Appendix

PROPOSITION 4. *Let $K:(t,x)\in\mathbb{R}^+\times\mathbb{R}\mapsto K_t(x)\in\mathbb{R}$ be a function belonging to $L^\infty_{loc}\big(\mathbb{R}^+,C^k(\mathbb{R})\big)$ for all $k\in\mathbb{N}$ and to $C^1(\mathbb{R}^+\times\mathbb{R}^2)$. Consider the unique (in the class of measures) solution to the following linear PDE*

$$\partial_t f_t+v\,\partial_x f_t+K_t(x)\,\partial_v f_t=\partial_v(\partial_v f_t+v f_t),\tag{79}$$

for an initial condition $f_0\in\mathcal{P}(\mathbb{R}^2)$ satisfying $\partial_x^k\partial_v^l f_0\in L^2(\mathbb{R}^2)$ for all $k,l\in\mathbb{N}$. Then $f\in C^1(\mathbb{R}^+\times\mathbb{R}^2)$ and is even two times continuously differentiable in (x,v).

PROOF. Differentiating equation (79) k times in x variable, and l times in v variable leads to

$$\partial_t(\partial_x^k\partial_v^l f_t)+v\,\partial_x(\partial_x^k\partial_v^l f_t)+K_t(x)\,\partial_v(\partial_x^k\partial_v^l f_t)-\partial_v(v\,\partial_x^k\partial_v^l f_t)-\partial_v^2(\partial_x^k\partial_v^l f_t)$$
$$=-\sum_{k'=0}^{k-1}\binom{k}{k'}\partial_x^{k-k'}K_t(x)(\partial_x^{k'}\partial_v^{l+1}f_t)-\partial_x^{k+1}\partial_v^{l-1}f_t+l\,\partial_x^k\partial_v^l f_t,\tag{80}$$

with the convention that a term containing a derivative with a negative power vanishes. Multiplying the above equation by $\partial_x^k\partial_v^l f_t$ and performing an integration by part, we obtain

$$\frac{d}{dt}\big\|\partial_x^k\partial_v^l f_t\big\|_2^2\le\big\|\partial_x^k\partial_v^l f_t\big\|_2\big\|\partial_x^{k+1}\partial_v^{l-1}f_t\big\|_2+C_{t,k}\sum_{k'=0}^{k-1}\binom{k}{k'}\big\|\partial_x^{k'}\partial_v^{l+1}f_t\big\|_2\big\|\partial_x^k\partial_v^l f_t\big\|_2+\Big(l+\frac{1}{2}\Big)\big\|\partial_x^k\partial_v^l f_t\big\|_2^2,$$
$$\text{where}\quad C_{t,k}=\sup_{k'=1,\cdots,k}\big\|\partial_x^{k'}K_t\big\|_\infty.$$

Then summing these inequalities over the k,l such that $k+l\le m$, we find

$$\frac{d}{dt}H_m(t)\le\tilde{C}_{m,t}H_m(t),\quad\text{where}\quad H_m(t)=\sum_{k+l\le m}\Big\|\partial_x^k\partial_v^l f_t\Big\|_2^2,$$

and the constant $\tilde{C}_{m,t}$ is locally bounded in time. Since by assumption $H_m(0) < \infty$ for any $m \in \mathbb{N}$, we get that for any time $t \geq 0$, $\sup_{s\in[0,t]} H_m(s) < \infty$. Then by Morrey's inequality for m large enough ($m = 4$), $f \in L^\infty([0,t], C^2(\mathbb{R}^2))$, and so does $\partial_x^k \partial_v^l f$ for all $k, l \geq 0$. Using (80), we deduce that $\partial_t \partial_x^k \partial_v^l f \in L^\infty_{loc}(\mathbb{R}^+ \times \mathbb{R}^2)$, and in particular $\partial_x^k \partial_v^l f$ is continuous. So f is two times continuously differentiable in (x, v). And using finally (79), we see that $\partial_t f$ is itself continuous in all the variables, as a sum of continuous functions. This concludes the proof. □

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CHAPTER 2

Mean-field limit for collective behavior models with sharp sensitivity regions

1. Introduction

In this work, we deal with the rigorous derivation of kinetic models of collective motion when sharp sensitivity regions are considered. Most of the minimal particle-like models for swarming include the effects of attraction, repulsion, and alignment by means of local averages of these kind of forces around the location or velocity of the individuals. These Individual Based Models (IBMs) with local sensitivity regions have been proposed in the theoretical biology literature [**25, 26**] as well as in the applied mathematical community [**28, 1**]. For instance, a basic model introducing the three effects combines the repulsive-attractive forces modeled by an effective pairwise interaction potential $\varphi(x)$ and the gregariousness behavior of individuals by locally averaging their relative velocities with weights depending on the interindividual distances, see [**12, 1**] for instance. These effects lead to pattern formations such as localized flocks, which are stable for the particle dynamics as proven in [**1, 13**]. This model for alignment is usually referred as the Cucker-Smale model for velocity consensus [**16**] that has been throughly studied in the last years, see for instance [**12**]. The Cucker-Smale model averages the relative velocities through a coupling function $h(v)$, linear in the original model, and a communication weight function $\psi(x)$ weighting the relative importance in the orientation and speed averaging according to the interparticle distances. In summary, given N particles the evolution of their positions $X_i(t)$ and velocities $V_i(t)$ is determined by the 2nd order ODE system:

$$\begin{cases} \dot{X}_i(t) = V_i(t), \quad i = 1, \cdots, N, \quad t > 0, \\ \dot{V}_i(t) = \dfrac{1}{N} \displaystyle\sum_{j \neq i} \left[\psi(X_i - X_j) h(V_j - V_i) + \nabla_x \varphi(X_i - X_j) \right] \mathbf{1}_{K(V_i)}(X_i - X_j) \\ (X_i(0), V_i(0)) =: \left(X_i^0, V_i^0 \right), \quad i = 1, \cdots, N. \end{cases} \tag{81}$$

Here, $\mathbf{1}_{K(v)}$ is the indicator function on the set $K(v) \subset \mathbb{R}^d$. The local averaging for attraction, repulsion and reorientation is done on velocity dependent sets $K(v)$ that we refer as the sensitivity regions, a typical case being vision cones for animals such as birds or mammals. However, sensitivity regions can highly vary across animals due to their different sensory organs, for instance pressure waves through lateral lines for fish or visual field for birds could depend a lot on the particular specie, see for instance [**27, 29**].

In this work, we are interested in the continuum mean-field models obtained from these IBMs in the large particle limit, i.e. when $N \to \infty$. The mean-field limit in statistical mechanics is obtained when the total force felt by each individual is of unit order when $N \to \infty$, this means that the forces exerted by each individual on others have to be scaled inversely proportional to the number of individuals, see [**34**] for further explanations about the mean-field scaling. The mean-field limit leads as usual to kinetic equations of Vlasov-type. Their formal derivation has been done for IBMs of collective behavior introducing these kind of sensitivity regions by different authors [**20, 12, 1, 4, 31, 31, 21**] using BBGKY hierarchies arguments. These models include locally averaged interactions in the metric sense [**20, 12, 1, 31, 31**] and in the topological sense [**4, 21**].

On the other hand, trying to prove rigorously that this mean-field limit holds or equivalently trying to show rigorously the validity of the kinetic equations obtained from the formal BBGKY hierarchy procedure is a very challenging question. The lack of enough regularity of the velocity field, due to the sharpness of the sensitivity regions, prevents classical arguments for deriving rigorously the mean-field limit such as [**12, 6, 32, 34, 35, 18**] to be applicable. The different variants of these strategies introduced in [**22, 3, 7, 4**] deal with locally Lipschitz velocity fields to allow swarming systems such as the Cucker-Smale alignment models [**16, 33, 1**]. However, these arguments do not apply either for discontinuous sharp interaction regions between individuals. There have been several important works dealing with singular interaction forces either at the first order or the second order models [**16, 23, 24, 5, 7**] but none dealing with discontinuous averaging regions.

Our work is the first in showing the rigorous proof of the mean-field limit of a system of interacting particles where each particle only interacts with those inside a local region whose shape depends on the position and velocity of the particle. Even the meaning of the particle model is not clear due to the discontinuous right-hand side of the dynamical system. We can show that Filippov's theory [**17**] allows us to have a well-defined notion of solutions via differential inclusions. We will exemplify our main results in the Cucker-Smale model [**16**] with local sensitivity regions, given by

$$\partial_t f + v\cdot\nabla_x f + \nabla_v\cdot(F(f)f) = 0, \quad (x,v)\in\mathbb{R}^d\times\mathbb{R}^d,\ t>0, \tag{82}$$

with the initial data:

$$f(x,v,t)|_{t=0} = f_0(x,v). \tag{83}$$

Here, $F(f)$ is an alignment force trying to average the relative velocities as

$$F(f)(x,v,t) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbf{1}_{K(v)}(x-y)(w-v)f(y,w)\,dydw\,.$$

Under suitable assumptions of the local sensitivity regions $K(v)$, we will show that the velocity field associated to an L^1-L^∞ density f is locally Lipschitz continuous. This is the basic property that allows for a weak-strong quantitative stability estimate between weak solutions of the kinetic continuum model (82) and the particle solutions of the discrete differential inclusion system. This estimate implies readily the mean-field limit derivation of the kinetic equation (82). These results are easily generalizable for the general kinetic model associated to (81):

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + \nabla_v\cdot(F(f)f) = 0, & (x,v)\in\mathbb{R}^d\times\mathbb{R}^d, \quad t>0,\\ f(x,v,t)|_{t=0} = f_0(x,v) & (x,v)\in\mathbb{R}^d\times\mathbb{R}^d, \end{cases}$$

with

$$F(f)(x,v,t) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \left[\psi(x-y)h(w-v) + \nabla_x\varphi(x-y)\right]\mathbf{1}_{K(v)}(x-y)f(y,w)\,dydw.$$

This paper is organized as follows: we first show the existence of the differential inclusion system and the crucial Lipschitz property of $F(f)$ for L^1-L^∞ densities f under very reasonable but technical assumptions on the sensitivity regions. We explain carefully the necessary assumptions on the sensitivity sets in Section 2 before we state the main results of this work. Section 3 is devoted to show the weak-strong stability estimate and the mean-field limit. Section 4 shows the well-posedness of weak solutions for system (82)-(83) under the same assumptions on the sensitivity regions. Section 5 gives some typical examples of local sensitivity regions verifying our assumptions commonly used in applications in mathematical biology. Finally, we extend these results to general locally averaged potential and alignment models on sharp sensitivity regions in Section 6.

2. Preliminaries & Main Results

In order to deal with the differential inclusion system associated to the IBMs discussed in the introduction and their associated kinetic equations, we first need some notation to treat rigorously the dependence on the sensitivity sets.

2.1. Sensitivity Regions: Notation.

DEFINITION 2.1. *Let $K \subset \mathbb{R}^d$ be a non-empty compact set and $\varepsilon > 0$. We define the ε-boundary of K by:*

$$\partial^\varepsilon K := \partial K + \overline{B(0,\varepsilon)} = \{x + y \mid x \in \partial K, |y| \leq \varepsilon\},$$

and also the ε-enlargement(resp. ε-reduction) $K^{\varepsilon,+}$ (resp. $K^{\varepsilon,-}$) by

$$K^{\varepsilon,+} := \partial^\varepsilon K \cup K = K + \overline{B(0,\varepsilon)} \quad \text{and} \quad K^{\varepsilon,-} := K \setminus \partial^\varepsilon K.$$

Note that $K^{\varepsilon,-}$ can be the empty set and that $\partial^\varepsilon K = K^{\varepsilon,+} \setminus K^{\varepsilon,-}$. We sketch these definitions in Figure 1. We denote the Lebesgue measure of set A in $\mathbb{R}^d$ by $|A|$.

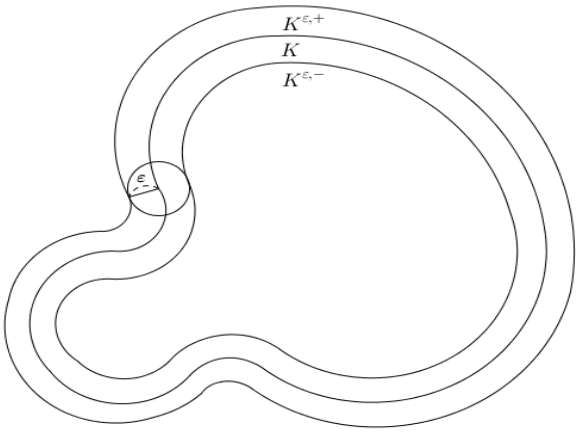


FIGURE 1. A sketch of the sets $K^{\varepsilon,-}, K$, and $K^{\varepsilon,+}$.

We are going to show several elementary estimates, some of them obvious by geometrical arguments.

LEMMA 2.1. *For $\varepsilon > 0$ and $\delta > 0$, it holds*

$$(\partial^\varepsilon K)^{\delta,+} \subset \partial^{\varepsilon+\delta} K.$$

PROOF. For $x \in (\partial^\varepsilon K)^{\delta,+}$, there exists a $y \in \partial^\varepsilon K$ such that $d(x,y) \leq \delta$ by definition. This yields

$$d(x,K) \leq d(x,y) + d(y,K) \leq \delta + \varepsilon,$$

i.e., $x \in K^{(\varepsilon+\delta),+}$. We now suppose $x \in K^{(\varepsilon+\delta),-}$, i.e., $d(x,K^c) > \varepsilon + \delta$. Then for all $y \in \partial^\varepsilon K$ we obtain

$$\varepsilon + \delta < d(x,K^c) \leq d(x,y) + d(y,K^c) \leq d(x,y) + \varepsilon.$$

This is a contradiction to $d(x,y) \leq \delta$. Hence we have $x \notin K^{(\varepsilon+\delta),-}$, and this completes the proof. □

LEMMA 2.2. *For $K \subset \mathbb{R}$ and $x_1, y_1, x_2, y_2 \in \mathbb{R}^d$, define $\varepsilon_1 = |x_1 - x_2|$, $\varepsilon_2 = |y_1 - y_2|$, we get*

$$|\mathbf{1}_K(y_1 - x_1) - \mathbf{1}_K(y_2 - x_2)| \leq \mathbf{1}_{\partial^{2\max(\varepsilon_1,\varepsilon_2)}K}(y_1 - x_1) \leq \mathbf{1}_{\partial^{2\varepsilon_1}K}(y_1 - x_1) + \mathbf{1}_{\partial^{2\varepsilon_2}K}(y_1 - x_1).$$

PROOF. Since $\partial^{2\max(\varepsilon_1,\varepsilon_2)}K = \partial^{2\varepsilon_1}K \cup \partial^{2\varepsilon_2}K$, we deduce

$$\mathbf{1}_{\partial^{2\max(\varepsilon_1,\varepsilon_2)}K}(y_1 - x_1) \leq \mathbf{1}_{\partial^{2\varepsilon_1}K}(y_1 - x_1) + \mathbf{1}_{\partial^{2\varepsilon_2}K}(y_1 - x_1).$$

Thus, it is enough to show

$$|\mathbf{1}_K(y_1 - x_1) - \mathbf{1}_K(y_2 - x_2)| \leq \mathbf{1}_{\partial^{2\max(\varepsilon_1,\varepsilon_2)}K}(y_1 - x_1).$$

We separate cases:

- If $y_1 - x_1 \notin K^{2\max(\varepsilon_1,\varepsilon_2),+}$, then $d(y_1 - x_1, K) > 2\max(\varepsilon_1, \varepsilon_2)$ and
$$|y_1 - x_1 - (y_2 - x_2)| \leq \varepsilon_1 + \varepsilon_2 \leq 2\max(\varepsilon_1, \varepsilon_2)\,,$$
from which we deduce that
$$y_2 - x_2 \notin K \quad \text{and} \quad \mathbf{1}_K(y_1 - x_1) = \mathbf{1}_K(y_2 - x_2) = 0.$$
- If $y_1 - x_1 \in K^{2\max(\varepsilon_1,\varepsilon_2),-} \subset K$, we get
$$|y_1 - x_1 - (y_2 - x_2)| \leq 2\max(\varepsilon_1, \varepsilon_2) < d(y - x, K^c),$$
and this implies $y_2 - x_2 \in K$. Thus we obtain
$$\mathbf{1}_K(y_1 - x_1) - \mathbf{1}_K(y_2 - x_2) = 0.$$
- If $y_1 - x_1 \in K^{2\max(\varepsilon_1,\varepsilon_2),+} \setminus K^{2\max(\varepsilon_1,\varepsilon_2),-}$, it is trivial to get
$$|\mathbf{1}_K(y_1 - x_1) - \mathbf{1}_K(y_2 - x_2)| \leq \mathbf{1}_{\partial^{2\max(\varepsilon_1,\varepsilon_2)}K}(y_1 - x_1).$$

This concludes the desired result. □

2.2. Sensitivity Regions: Assumptions. Let us denote by $A\Delta B$ the symmetric difference between two sets $A \subset \mathbb{R}^d$ and $B \subset \mathbb{R}^d$, i.e.,

$$A\Delta B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

It is obvious that $|\mathbf{1}_A(x) - \mathbf{1}_B(x)| = \mathbf{1}_{A\Delta B}(x)$.

In this work, we will assume that the sensitivity region set-valued function $K(v)$ depending on the velocity variable satisfies the following conditions:

- (**H1**): $K(\cdot)$ is globally compact, i.e., $K(v)$ is compact and there exists a compact set $\mathcal{K}$ such that $K(v) \subseteq \mathcal{K}\,, \forall v \in \mathbb{R}^d$.
- (**H2**): There exists a family of closed sets $v \mapsto \Theta(v)$ and a constant C such that:
  - (i) $\partial K(v) \subset \Theta(v)$, for all $v \in \mathbb{R}^d$,
  - (ii) $|\Theta(v)^{\varepsilon,+}| \leq C\varepsilon$, for all $\varepsilon \in (0,1)$,
  - (iii) $K(v)\Delta K(w) \subset \Theta(v)^{C|v-w|,+}$, for $v, w \in \mathbb{R}^d$,
  - (iv) $\Theta(w) \subset \Theta(v)^{C|v-w|,+}$, for $v, w \in \mathbb{R}^d$.

REMARK 2.1. *Note that* (**H2**)*-*(ii) *is satisfied if* $\Theta(v)$ *is made for piecewise* $\mathcal{C}^1$*-compact hypersurfaces. That will be the case in all our applications. In general it roughly means that* $\Theta(v)$ *has finite* $(d-1)$*-area, i.e. it has a* $(d-1)$*-dimensional Hausdorff measure bounded by some constant.*

We will discuss several examples of sets satisfying these hypotheses in Section 5: a fixed ball, a ball with a radius depending on the speed, a vision cone in which the angle depends on the speed. Let us just mention here that the hypotheses are fulfilled in the constant case $K(v) = K_0$ when ∂K_0 is made of compact $\mathcal{C}^1$ hypersurfaces. In fact, in this case it is enough to choose $\Theta(v) = \partial K_0$, and all the hypothesis are satisfied (many of them are empty in this particular case), see Section 5.1.

REMARK 2.2. *One of most common choices for sensitivity region in applications is a vision cone. Let us define*

$$C(r, v, \theta) = \left\{ x : |x| \leq r, \quad \frac{x \cdot v}{|x||v|} \geq \cos\theta \right\},$$

with $\theta \in (0,\pi)$, $r>0$. Note that a cone with fixed radius and angle is a 0-homogeneous set-valued function in v, that is,

$$C(r,v,\theta) = C\left(r, \frac{v}{|v|}, \theta\right) \quad \forall\, v \in \mathbb{R}^d \setminus \{0\}.$$

Then, it is not difficult to see that $v \mapsto C(r,v,\theta)$ is discontinuous at the origin, and (**H2**)-$(ii), (iii)$ *which implies that $|K(v)\Delta K(w)| \leq C|v-w|$ cannot be satisfied except in the case where the cone is a ball. This discontinuity due to the vanishing velocity is not important from the modelling view point since most of swarming models have additional terms preventing individuals to stop* [**12, 1, 4**]*. We will discuss in Section* 5 *very simple and mild variations of vision cone regions that do satisfy* (**H2**)-(ii) *and that are innocuous from the modelling viewpoint in swarming.*

Remark 2.3. *The additional set valued function $\Theta(v)$ is an enlargement of the boundary $\partial K(v)$ for any $v \in \mathbb{R}^d$, and it is requested because we cannot work only with the boundary set $\partial K(v)$ for two reasons. In fact, there is first a natural set, larger than $\partial K(v)$, that appears in many calculations. Let us define it as*

$$\widetilde{\partial} K(v) := \left\{ x \in \mathbb{R}^d, (v,x) \in \partial \widetilde{K} \right\}, \quad \text{where} \quad \widetilde{K} := \{(v,x),\ x \in K(v)\}\,. \tag{84}$$

Thus it is a slice of the boundary of the set $\widetilde{K}$. Note that $\partial K(v) \subset \widetilde{\partial} K(v)$.

The set $\widetilde{\partial} K(v)$ is in fact important in order to define the particle system associated to discontinuous kernels. More precisely, we need to give a sense to the derivative in time of the particle paths when they cross the boundary of the set $K(v)$. In order to do that, we will work with differential inclusions, and thus we need to take into account the set of admissible slopes at the boundary of the set $K(v)$. We define the set-valued function of admissible slopes as

$$\mathcal{A}(x,v) = Conv\left\{\alpha \in [0,1] : \exists\, (x^n, v^n) \to (x,v) \text{ such that } \mathbf{1}_{K(v^n)}(x^n) \to \alpha\right\},$$

for $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$, where $Conv$ denotes the convex hull of a set. We remark that there are only three different values for $\mathcal{A}(x,v)$: two singleton sets $\{0\}$, $\{1\}$, and the unit interval $[0,1]$. Indeed, we have

$$\widetilde{\partial} K(v) = \left\{ x \in \mathbb{R}^d : \mathcal{A}(x,v) = [0,1] \right\}.$$

Note that $\widetilde{\partial} K(v)$ can be strictly larger than $\partial K(v)$, see for instance the example of the vision cone in Section 5*. For example, if we take into account the cone $K(v) = C(r,v,\alpha(|v|))$ with fixed r but with varying angles $\alpha(|v|)$ such that $\alpha(|v|) \to \pi$ as $|v| \searrow 1$, then the generalized boundary set $\widetilde{\partial} K(v)$ is strictly larger than $\partial K(v)$ (see Figure* 2*).*

On the other hand, we may also need to enlarge a little bit $\widetilde{\partial} K(v)$, since it may not satisfy the hypothesis (**H2**)-(iv)*, that is more or less a kind of Lipschitz property for the set valued function Θ. This is why we need to introduce the additional set $\Theta(v)$ which does contain $\widetilde{\partial} K(v)$, as we prove next, even if it is not explicitly written in the assumption* (**H2**)*.*

Lemma 2.3. *Under the assumptions* (**H2**)-$(i), (ii)$*, and (iii), the set valued function $\widetilde{\partial} K$ defined in* (84) *satisfies*

$$\widetilde{\partial} K(v) \subseteq \Theta(v)\ , \forall v \in \mathbb{R}^d.$$

Proof. Let $v \in \mathbb{R}^d$ and $x \in \widetilde{\partial} K(v)$. By definition of $\widetilde{\partial} K(v)$, there exists two sequences $(x_n^1, v_n^1)_{n\in\mathbb{N}}$ and $(x_n^2, v_n^2)_{n\in\mathbb{N}}$ both converging to (x,v) such that $x_n^1 \in K(v_n^1)$ and $x_n^2 \notin K(v_n^2)$.

If $x \in \partial K(v)$ then it is also in $\Theta(v)$ by the assumption (**H2**)-(i). If $x \in \overset{\circ}{K}(v)$, then for n large enough $x_n^2 \in \overset{\circ}{K}(v)$. Therefore $x_n^2 \in K(v_n^2)\Delta K(v) \subset \Theta(v)^{C|v_n^2 - v|,+}$ due to

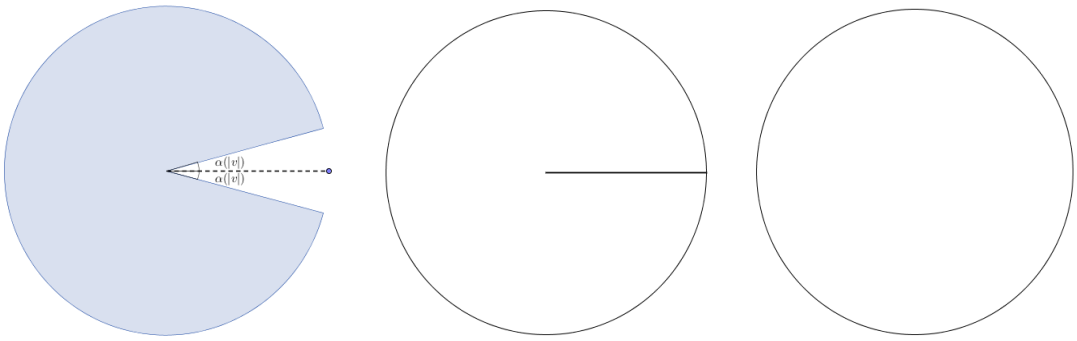


FIGURE 2. Evolution of $\widetilde{\partial}K(v)$ when $K(v) = C(r, v, \theta(|v|)$ for (from left to right) $|v| > 1$, $|v| = 1$ and $|v| < 1$. $\widetilde{\partial}K(v)$ is strictly larger than $\partial K(v)$ when $|v| = 1$.

(**H2**)-(*iii*). Thus $x \in \Theta(v)^{C|v_n^2 - v| + |x_n^2 - x|, +}$, and letting n go to infinity we deduce that $x \in \Theta(v)$ since $\Theta(v)$ is closed. If $x \in K(v)^c$, we consider the $(x_n^1, v_n^1)_{n\in\mathbb{N}}$ and repeat the previous step. □

2.3. Particle system: Differential inclusions. A particle approximation of the kinetic equation (82) should read as the following system:

$$\begin{cases} \dot{X}_i(t) = V_i(t), \\ \dot{V}_i(t) = -\displaystyle\sum_{j\neq i} m_j \mathbf{1}_{K(V_i)}(X_i - X_j)(V_i - V_j), \quad i = 1, \cdots, N, \quad t > 0, \\ (X_i(0), V_i(0)) =: \left(X_i^0, V_i^0\right) \qquad\qquad\qquad\qquad i = 1, \cdots, N. \end{cases} \tag{85}$$

Here $\{(X_i, V_i)\}_{i=1}^N$ and $\{m_i\}_{i=1}^N$ are the position, velocity, and weight of i-th particles, respectively. We will always assume the normalization condition $\sum_{i=1}^N m_i = 1$, with $m_i > 0$, $i = 1, \cdots, N$. However, this particle system is not well-defined due to the discontinuous character of the right-hand side. For the mean-field limit, we will use the global solutions to the following differential inclusion system generalizing the ODE system (85):

$$\dot{\mathcal{Z}}^N(t) \in \mathcal{B}^N(\mathcal{Z}^N(t)), \tag{86}$$

where $\mathcal{B}^N : \mathbb{R}^{2dN} \to P(\mathbb{R}^{2dN})$ is a set-valued given by

$$\mathcal{B}^N(x_1, \cdots, x_N, v_1, \cdots, v_N) := \{v_1\} \times \cdots \times \{v_N\} \times \mathcal{F}_1(x, v) \times \cdots \times \mathcal{F}_N(x, v),$$

with

$$\mathcal{F}_i(x, v) := \left\{ \sum_{j\neq i} \alpha_{ij} m_j (v_j - v_i) : \alpha_{ij} \in \mathcal{A}(x_i - x_j, v_i) \right\},$$

for $i \in \{1, \cdots, N\}$. A function $\mathcal{Z}^N(t)$ is called a solution of the differential inclusion system (86) with initial condition $\mathcal{Z}^N(0) = \mathcal{Z}_0^N$ if

$$\mathcal{Z}^N(t) = \mathcal{Z}_0^N + \int_0^t \mathcal{F}^N(s)\, ds, \quad t \geq 0, \quad \text{and} \quad \mathcal{F}^N(t) \in \mathcal{B}^N(\mathcal{Z}^N(t)) \quad \text{a.e.} \quad t \geq 0,$$

which is equivalent to

$$\dot{\mathcal{Z}}^N(t) = \mathcal{F}^N(t) \quad \text{and} \quad \mathcal{F}^N(t) \in \mathcal{B}^N(\mathcal{Z}^N(t)) \quad \text{a.e.} \quad t \geq 0. \tag{87}$$

Using the following Filippov theory of discontinuous dynamical systems [**17**], we will show that the differential inclusion system (86) has global-in-time solutions.

THEOREM 2.1. [**17**] *Let $F : (x,t) \in \mathbb{R}^n \times \mathbb{R}_+ \mapsto F(x,t) \in P(\mathbb{R}^n)$ be a set-valued function such that*

- *$F(x,t)$ is nonempty convex and closed for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$.*
- *There exists a bounded set $K \subset \mathbb{R}^n$ such that $F(x,t) \subseteq K$ for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$.*
- *F is upper continuous with respect to the inclusion.*

Then, for all $(x_0,t_0) \in \mathbb{R}^n \times \mathbb{R}_+$, there exists at least one solution to the problem:

$$\dot{x}(t) \in F(x(t),t) \quad \text{with} \quad x(t_0) = x_0.$$

Thanks to this result, we provide global existence of solutions in time to the differential inclusion system (86) in the proposition below.

PROPOSITION 2.1. *For any initial data $\mathcal{Z}^N(0)$, there exists at least one global solution to the differential inclusion system* (86).

PROOF. It follows from the definition of $\mathcal{B}^N$ that $\mathcal{B}^N$ is bounded, closed, and convex for each $\mathcal{Z}^N \in \mathbb{R}^{2dN}$. We next easily get the upper continuity of $\mathcal{B}^N$ with respect to the inclusion from our construction of the differential inclusion system. For $R > 0$, we consider the truncated field $\mathcal{B}_R^N$ defined as

$$\mathcal{B}_R^N := \{v_1^R\} \times \cdots \times \{v_N^R\} \times \mathcal{F}_1^R(x,v) \times \cdots \times \mathcal{F}_N^R(x,v),$$

where

$$v_i^R := \begin{cases} \dfrac{R \wedge |v_i|}{|v_i|} v_i & \text{if } \ v_i \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \mathcal{F}_i^R(x,v) := \left\{ \sum_{j \neq i} \alpha_{ij} m_j (v_j^R - v_i^R) : \alpha_{ij} \in \mathcal{A}(x_i - x_j, v_i) \right\}.$$

Here $a \wedge b := \min\{a,b\}$ for $a,b \in \mathbb{R}_+$. Then, by Theorem 2.1, we have global solutions to the following differential inclusion system:

$$\dot{\mathcal{Z}}^N(t) \in \mathcal{B}_R^N(\mathcal{Z}^N(t)) \quad \text{with the initial data} \quad \mathcal{Z}_0^N. \tag{88}$$

Then we use the solution to the system (88) to show that the fields $\mathcal{B}^N$ and $\mathcal{B}_R^N$ generate the same solutions for $R > 0$ large enough, i.e., $\mathcal{Z}^N(t)$ is also the solution to (86). Note that if we choose an index i such that $|V_i^N(t)| = \max_{1 \le j \le N} |V_j^N(t)|$ for each t, then it follows from (86) that for some $a_{ij}(t) \in \mathcal{A}(X_i^N(t) - X_j^N(t), V_i^N(t))$

$$\frac{d}{dt}|V_i^N(t)|^2 = \sum_{j \neq i} a_{ij}(t) m_j (V_j^N(t) - V_i^N(t)) \cdot V_i^N(t) \le 0,$$

due to the positivity of a_{ij} and the choice of the index i. This yields

$$\max_{1 \le i \le N} |V_i^N(t)| \le \max_{1 \le i \le N} |V_i^N(0)| \quad \text{for a.e. } t \ge 0,$$

Finally, by choosing $R = \max_{1 \le i \le N} |V_i(0)|$, we conclude the existence of solutions to the system (86). □

2.4. MKR-distance. In order to establish quantitative estimates between solutions of the kinetic equation (82) and the differential inclusion (86), we need some basic tools of optimal transportation.

DEFINITION 2.2. *(Monge-Kantorovich-Rubinstein distance) Let ρ_1, ρ_2 be two Borel probability measures on $\mathbb{R}^d$. Then the Monge-Kantorovich-Rubinstein distance distance between ρ_1 and ρ_2 is defined as*

$$d_1(\rho_1,\rho_2) := \inf_{\gamma} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y| \, d\gamma(x,y),$$

where the infimum runs over all transference plans, i.e., all probability measures γ on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals ρ_1 and ρ_2 respectively,

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x) d\gamma(x,y) = \int_{\mathbb{R}^d} \phi(x) \, d\rho_1(x),$$

and

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}\phi(y)d\gamma(x,y)=\int_{\mathbb{R}^d}\phi(y)\,d\rho_2(y),$$

for all $\phi\in\mathcal{C}_b(\mathbb{R}^d)$.

Note that $\mathcal{P}_1(\mathbb{R}^d)$, the set of probability measures in $\mathbb{R}^d$ with first bounded moment, is a complete metric space endowed with the Monge-Kantorovich-Rubinstein distance. The Monge-Kantorovich-Rubinstein distance, also called 1-Wasserstein distance, is also equivalent to the Bounded Lipschitz distance

$$d_1(\rho_1,\rho_2)=\sup\left\{\left|\int_{\mathbb{R}^d}\varphi(\xi)d\rho_1(\xi)-\int_{\mathbb{R}^d}\varphi(\xi)d\rho_2(\xi)\right|\;\Big|\;\varphi\in\mathrm{Lip}(\mathbb{R}^d),\mathrm{Lip}(\varphi)\le 1\right\},\tag{89}$$

where $\mathrm{Lip}(\mathbb{R}^d)$ and $\mathrm{Lip}(\varphi)$ denote the set of Lipschitz functions on $\mathbb{R}^d$ and the Lipschitz constant of a function φ, respectively. We also remind the definition of the push-forward of a measure by a mapping in order to give the relation between Wasserstein distances and optimal transportation.

Definition 2.3. *Let ρ_1 be a Borel measure on $\mathbb{R}^d$ and $\mathcal{T}:\mathbb{R}^d\to\mathbb{R}^d$ be a measurable mapping. Then the push-forward of ρ_1 by $\mathcal{T}$ is the measure ρ_2 defined by*

$$\rho_2(B)=\rho_1(\mathcal{T}^{-1}(B))\quad\text{for}\quad B\subset\mathbb{R}^d,$$

and denoted as $\rho_2=\mathcal{T}\#\rho_1$.

We recall in the next proposition some classical properties, which proofs may be found in [**36**].

Proposition 2.2. *(i) The definition of $\rho_2=\mathcal{T}\#\rho_1$ is equivalent to*

$$\int_{\mathbb{R}^d}\phi(x)\,d\rho_2(x)=\int_{\mathbb{R}^d}\phi(\mathcal{T}(x))\,d\rho_1(x)$$

for all $\phi\in\mathcal{C}_b(\mathbb{R}^d)$. Given a probability measure with first bounded moment ρ_0, consider two measurable mappings $X_1,X_2:\mathbb{R}^d\to\mathbb{R}^d$, then the following inequality holds:

$$d_1(X_1\#\rho_0,X_2\#\rho_0)\le\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|d\gamma(x,y)=\int_{\mathbb{R}^d}|X_1(x)-X_2(x)|d\rho_0(x).$$

Here, we used as transference plan $\gamma=(X_1\times X_2)\#\rho_0$ in Definition 2.2.

(ii) Given $\{\rho_k\}_{k=1}^\infty$ and ρ in $\mathcal{P}_1(\mathbb{R}^d)$, the following statements are equivalent:

- *$d_1(\rho_k,\rho)\to 0$ as $k\to+\infty$.*
- *ρ_k converges to ρ weakly-* as measures and*

$$\int_{\mathbb{R}^d}|\xi|d\rho_k(\xi)\to\int_{\mathbb{R}^d}|\xi|d\rho(\xi),\quad\text{as}\quad k\to+\infty.$$

2.5. Limiting Kinetic Equation: Well-posedness. Let us define the notion of weak solution to the expected limiting kinetic equation (82).

Definition 2.4. *For a given $T\in(0,\infty)$, f is a weak solution of the equation* (82) *on the time-interval $[0,T)$ with the sensitivity region set-valued function $K(v)$ satisfying* (**H1**)-(**H2**) *if and only if the following conditions are satisfied:*

(1) $f\in L^\infty(0,T;(L^1_+\cap L^\infty)(\mathbb{R}^d\times\mathbb{R}^d))\cap L^\infty(0,T;\mathcal{P}_1(\mathbb{R}^d\times\mathbb{R}^d))$.

(2) *For all $\Psi\in\mathcal{C}^\infty_c(\mathbb{R}^d\times\mathbb{R}^d\times[0,T])$,*

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}f(x,v,T)\Psi(x,v,T)\,dxdv-\int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d}f(\partial_t\Psi+\nabla_x\Psi\cdot v+\nabla_v\Psi\cdot F(f))\,dxdvdt$$
$$=\int_{\mathbb{R}^d\times\mathbb{R}^d}f_0(x,v)\Psi_0(x,v)\,dxdv,$$

where $\Psi_0(x,v):=\Psi(x,v,0)$.

The main theorem regarding the well-posedness of the limiting kinetic equation is based on a weak-strong stability estimate in d_1. Similar estimates were recently obtained in [**16**] in the case of the Vlasov-Poisson system in one dimension.

THEOREM 2.2. *Given an initial data satisfying*

$$f_0 \in (L^1_+ \cap L^\infty)(\mathbb{R}^d \times \mathbb{R}^d) \cap \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)\,, \tag{90}$$

and assume further that f_0 is compactly supported in velocity. Then there exists a positive time $T > 0$ such that the system (82)-(83) *with the sensitivity region set-valued function $K(v)$ satisfying* (**H1**)-(**H2**) *admits a unique weak solution f in the sense of Definition* 2.4 *on the time interval $[0, T]$, which is also compactly supported in velocity. Moreover, f is determined as the push-forward of the initial density through the flow map generated by the local Lipschitz velocity field $(v, F(f))$ in phase space. Furthermore if $f_i, i = 1, 2$ are two such solutions to the system* (82) *with initial data $f_i(0)$ satisfying* (90)*, we have*

$$d_1(f_1(t), f_2(t)) \leq d_1(f_1(0), f_2(0))e^{Ct}, \quad \text{for} \quad t \in [0, T], \tag{91}$$

where C is a positive constant that depends only on the $L^\infty(\mathbb{R}^d \times \mathbb{R}^d \times (0, T))$ norm of f_1.

We will show the existence of weak solutions satisfying the quantitative stability estimate (91) in Section 4.

REMARK 2.4. *Theorem* 2.2 *can be also proved by taking into account slightly weaker assumptions on K. More precisely, we may replace* (**H2**)-(iv) *by*

- (**H2**)-$(iv)'$:

$$\widetilde{\partial} K(v) \subset \Theta(w)^{C|v-w|,+} \quad \text{for } v, w \in \mathbb{R}^d$$

where $C > 0$ is independent of v, and w.

Moreover, let us mention here that (**H2**)-(i) *and* (iii) *are direct consequences of the condition*

$$(x + K(v))\,\Delta K(w) \subset \Theta(w)^{C|v-w|,+} \quad \text{for } x, v, w \in \mathbb{R}^d\,,$$

leading to an easier verifiable condition for the sensitivity regions.

To give an idea of the strategy of the proof, we provide the following proposition that includes the key argument of the weak-strong stability in d_1 and for which the assumptions on the sensitivity regions $K(v)$ are tailor-made.

Let us point out that the classical arguments for the mean-field limit [**12, 6, 32, 34, 35, 18**] and their variants [**22, 3, 7, 4**] using well-posedness of the associated Vlasov-like equations for measures initial data are doomed to failure for sharp sensitivity regions due to the lack of uniqueness of the associated differential inclusion system as seen in Section 2.3 above. The weak-strong stability in d_1 replaces the argument of continuity with respect to initial data both for the well-posednes of the kinetic equation and the mean-field limit as already remarked for the Vlasov-Poisson equation in one dimension in [**16**].

PROPOSITION 2.3. *Let f be a solution to* (82) *given by Theorem* 2.2*. Then the force field $F(f)$ generated by f is linearly growing at infinity and locally Lipschitz continuous in phase space uniformly on $[0, T]$. More precisely, there exists a constant $C > 0$ which depends on $\|f\|_{L^1 \cap L^\infty}$ and support of f_0 in velocity such that*

$$F(f)(x, v, t) \leq C(1+|v|)\|f\|_{L^1} \quad \text{and} \quad |F(f)(x, v) - F(f)(\tilde{x}, \tilde{v})| \leq C(1+|v|)|(x, v) - (\tilde{x}, \tilde{v})|,$$

for all $x, v, \tilde{x}, \tilde{v} \in \mathbb{R}^d$ and $t \in [0, T]$.

PROOF. First, it is clear to find

$$\begin{aligned} F(f)(x, v, t) &\leq \int_{\mathbb{R}^{2d}} |w - v| f(y, w, t) dy dw \leq \int_{\mathbb{R}^{2d}} |w| f(y, w, t) dy dw + |v| \int_{\mathbb{R}^{2d}} f(y, w, t) dy dw \\ &\leq C(1 + |v|)\|f\|_{L^1}, \end{aligned}$$

where we used the compact support of f in velocity. For the local Lipschitz continuity of the force with respect to x and v, we estimate

$$\begin{aligned}
&F(f)(x,v) - F(f)(\tilde{x},\tilde{v}) \\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbf{1}_{K(v)}(x-y)(w-v)f(y,w)\,dydw - \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbf{1}_{K(\tilde{v})}(\tilde{x}-y)(w-\tilde{v})f(y,w)\,dydw \\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d} \left(\mathbf{1}_{K(v)}(x-y) - \mathbf{1}_{K(\tilde{v})}(x-y)\right)(w-v)f(y,w)\,dydw \\
&\quad + \int_{\mathbb{R}^d\times\mathbb{R}^d} \left(\mathbf{1}_{K(\tilde{v})}(x-y) - \mathbf{1}_{K(\tilde{v})}(\tilde{x}-y)\right)(w-v)f(y,w)\,dydw \\
&\quad + \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbf{1}_{K(\tilde{v})}(\tilde{x}-y)(\tilde{v}-v)f(y,w)\,dydw \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

By using that f is compactly supported in velocity together with (**H2**)-(i), (ii), and (iii), we obtain that for $|x-\tilde{x}| \le 1/2$ and $C|v-\tilde{v}| \le 1$

$$\begin{aligned}
I_1 &\le C(1+|v|)\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbf{1}_{K(v)\Delta K(\tilde{v})}(x-y)f(y,w)\,dydw \\
&\le C(1+|v|)\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbf{1}_{\Theta(v)^{C|v-\tilde{v}|,+}}(x-y)f(y,w)\,dydw \\
&\le C\|\rho\|_{L^\infty}(1+|v|)|v-\tilde{v}| \le C(1+|v|)|v-\tilde{v}|, \\
I_2 &\le C(1+|v|)\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbf{1}_{\partial^{2|x-\tilde{x}|}K(\tilde{v})}(x-y)f(y,w)\,dydw \\
&\le C\|\rho\|_{L^\infty}(1+|v|)|\Theta(\tilde{v})^{2|x-\tilde{x}|,+}| \le C(1+|v|)|x-\tilde{x}|,
\end{aligned} \tag{92}$$

where $\rho := \int_{\mathbb{R}^d} f\,dv$. On the other hand, for $|x-\tilde{x}| \ge 1/2$ and $C|v-\tilde{v}| \ge$, we estimate I_1 and I_2 as

$$\begin{aligned}
I_1 &\le C(1+|v|)\|f\|_{L^1} \le C(1+|v|)|v-\tilde{v}|, \\
I_2 &\le C(1+|v|)\|f\|_{L^1} \le C(1+|v|)|x-\tilde{x}|.
\end{aligned}$$

Note that I_3 can be easily estimated by

$$I_3 \le C\|f\|_{L^1}|v-\tilde{v}|.$$

Combining the above estimates, we have

$$|F(f)(x,v) - F(f)(\tilde{x},\tilde{v})| \le C(1+|v|)|(x,v)-(\tilde{x},\tilde{v})|. \tag{93}$$

This completes the proof. □

By Proposition 2.3, we obtain that the flow $\mathcal{Z}$ given in (94) is well-defined for weak solutions in the sense of Definition 2.4. Indeed, we can define flows $\mathcal{Z}(t;s,x,v) := (X(t;s,x,v), V(t;s,x,v)) : \mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d\times\mathbb{R}^d$ generated from (82) satisfying

$$\begin{cases}
\dfrac{d}{dt}X(t;s,x,v) = V(t;s,x,v), \\
\dfrac{d}{dt}V(t;s,x,v) = F(f)(t,\mathcal{Z}(t;s,x,v)), \\
(X(s;s,x,v), V(s;s,x,v)) = (x,v),
\end{cases} \tag{94}$$

for all $s,t \in [0,T]$.

2.6. Particles to Continuum: Mean-Field Limit. The existence of a quantitative bound in d_1-distance where only the L^∞-bound of one of the solutions is needed is a strong indication that the mean-field limit from particles to continuum descriptions can be carried over. We define the empirical measure $\mu^N(t)$ associated to a solution to the differential inclusion system (86) as

$$\mu^N(t) = \sum_{i=1}^N m_i \delta_{X_i^N(t), V_i^N(t)}. \tag{95}$$

The additional work to take care concerns the control of the error term in d_1 between weak solutions and empirical measures associated to differential inclusions. The main result of this paper can be summarized as:

THEOREM 2.3. *Suppose that the sensitivity region set-valued function K satisfies* (**H1**) *and* (**H2**)*, and let f be a weak solution to the Cauchy problem* (82)-(83) *in the sense of Definition 2.4 up to time $T > 0$ with initial data $f_0 \in (\mathcal{P}_1 \cap L^\infty)(\mathbb{R}^d \times \mathbb{R}^d)$ compactly supported in velocity. Then we have:*

$$d_1(f(t), \mu^N(t)) \le e^{Ct} d_1(f(0), \mu^N(0)) \quad \textit{for} \quad t \in [0, T],$$

where C is a positive constant that depends only on the $L^\infty(\mathbb{R}^d \times \mathbb{R}^d \times (0,T))$ norm of f.

It is worth emphazing that our result can not be obtained in other Wasserstein distances d_p, with $p > 1$, see [**36**] for its definition. We will explain this in details in Remark 3.1 based on technical considerations about its proof.

3. Mean-field limit: Proof

In this section, we give the proof of the mean-field limit from the N-particle system (86) to the kinetic equation (82) as stated in Theorem 2.3. For a fixed $t_0 \in [0,T]$, we choose an optimal transport map for d_1 denoted by $\mathcal{T}^0 = (\mathcal{T}_1^0, \mathcal{T}_2^0)$ between $f(t_0)$ and $\mu^N(t_0)$, i.e., $\mu^N(t_0) = \mathcal{T}^0 \# f(t_0)$. Then we can find that $f(t) = \mathcal{Z}(t; t_0, \cdot, \cdot) \# f(t_0)$, and it follows from the definition of the solutions to the differential inclusion system $\mu^N(t) = \mathcal{Z}^N(t; t_0, \cdot, \cdot) \# \mu^N(t_0)$ for $t \ge t_0$, where $\mathcal{Z}^N$ is a solution to the system (86) with $\mathcal{Z}^N(t_0, t_0, x, v) = (x, v)$. Moreover, we obtain

$$\mathcal{T}^t \# f(t) = \mu^N(t) \quad \text{where} \quad \mathcal{T}^t = \mathcal{Z}^N(t; t_0, \cdot, \cdot) \circ \mathcal{T}^0 \circ \mathcal{Z}(t_0; t, \cdot, \cdot),$$

for $t \in [t_0, T]$.

We then use the Wasserstein 1-distance to get

$$d_1(f, \mu^N) \le \int_{\mathbb{R}^d \times \mathbb{R}^d} |\mathcal{Z}(t; t_0, x, v) - \mathcal{Z}^N(t; t_0, \mathcal{T}^0(x, v))| f(x, v, t_0)\, dxdv.$$

Set

$$Q(t) := \int_{\mathbb{R}^d \times \mathbb{R}^d} |\mathcal{Z}(t; t_0, x, v) - \mathcal{Z}^N(t; t_0, \mathcal{T}^0(x, v))| f(x, v, t_0)\, dxdv.$$

By the definition of solutions to the differential inclusion system, we find that the solution $\mathcal{Z}^N$ is differentiable at almost every t with $\mathcal{Z}^N \in \mathcal{B}^N(\mathcal{Z}^N(t))$. Thus we can compute the derivative of Q with respect to almost every time as follows.

$$\begin{aligned}
\frac{d}{dt} Q(t)\Big|_{t=t_0+} &\le \int_{\mathbb{R}^d \times \mathbb{R}^d} |V(t; t_0, x, v) - V^N(t; t_0, \mathcal{T}^0(x, v))| f(x, v, t_0) dxdv \Big|_{t=t_0+} \\
&\quad + \int_{\mathbb{R}^d \times \mathbb{R}^d} \Big| F(f)(\mathcal{Z}(t; t_0, x, v), t) - F^N(\mu^N)(\mathcal{Z}^N(t; t_0, \mathcal{T}^0(x, v)), t) \Big| f(x, v, t_0) dxdv \Big|_{t=t_0+} \\
&=: I + J.
\end{aligned}$$

$\diamond$ Estimate of I: We clearly obtain

$$I = \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - \mathcal{T}_2^0(x, v)| f(x, v, t_0)\, dxdv \le d_1(f(t_0), \mu^N(t_0)).$$

$\diamond$ Estimate of J: We notice that

$$\begin{aligned}J &= \int_{\mathbb{R}^d\times\mathbb{R}^d}\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}_{K(v)}(x-y)(w-v)f(y,w,t_0)dydw\right.\\&\quad\left.-\int_{\mathbb{R}^d\times\mathbb{R}^d}\alpha(\mathcal{T}^0(x,v),y)(w-\mathcal{T}_2^0(x,v))d\mu^N(y,w,t_0)\right|f(x,v,t_0)dxdv\\&=\int_{\mathbb{R}^d\times\mathbb{R}^d}\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}_{K(v)}(x-y)(w-v)f(y,w,t_0)dydw\right.\\&\quad\left.-\int_{\mathbb{R}^d\times\mathbb{R}^d}\alpha(\mathcal{T}^0(x,v),\mathcal{T}_1^0(y,w))(\mathcal{T}_2^0(y,w)-\mathcal{T}_2^0(x,v))f(y,w,t_0)dydw\right|f(x,v,t_0)dxdv,\end{aligned}$$

where $\alpha(\mathcal{T}^0(x,v),\mathcal{T}_1^0(y,w))\in\mathcal{A}(\mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w),\mathcal{T}_2^0(x,v))$ and given by

$$\alpha(\mathcal{T}^0(x,v),\mathcal{T}_1^0(y,w))=\begin{cases}\mathbf{1}_{K(\mathcal{T}_2^0(x,v))}(\mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w)) & \text{for } \mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w)\notin\widetilde{\partial}K(\mathcal{T}_2^0(x,v)),\\ \alpha\in[0,1] & \text{for } \mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w)\in\widetilde{\partial}K(\mathcal{T}_2^0(x,v)).\end{cases}$$

For notational simplicity, we omit the time dependency t_0 in the following computations for J. We divide it into three terms as follows:

$$J=\int_{\mathbb{R}^d\times\mathbb{R}^d}\left|\sum_{i=1}^3 J_i\right|f(x,v)\,dxdv,$$

where

$$\begin{aligned}J_1&=\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\mathbf{1}_{K(v)}(x-y)-\mathbf{1}_{K(\mathcal{T}_2^0(x,v))}(\mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w))\right)(w-v)f(y,w)\,dydw,\\J_2&=\int_{\mathbb{R}^d\times\mathbb{R}^d}(\mathbf{1}_{K(\mathcal{T}_2^0(x,v))}(\mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w))-\alpha(\mathcal{T}^0(x,v),\mathcal{T}_1^0(y,w)))\,(w-v)\,f(y,w)\,dydw,\end{aligned}$$

and

$$J_3=\int_{\mathbb{R}^d\times\mathbb{R}^d}\alpha(\mathcal{T}^0(x,v),\mathcal{T}_1^0(y,w))\left(w-v-(\mathcal{T}_2^0(y,w)-\mathcal{T}_2^0(x,v))\right)\,f(y,w)\,dydw.$$

$\diamond$ (Estimate of J_1): We decompose J_1 into two terms:

$$\begin{aligned}J_1&=\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\mathbf{1}_{K(v)}(x-y)-\mathbf{1}_{K(v)}(\mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w))\right)(w-v)f(y,w)\,dydw\\&\quad+\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\mathbf{1}_{K(v)}(\mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w))-\mathbf{1}_{K(\mathcal{T}_2^0(x,v))}(\mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w))\right)(w-v)f(y,w)\,dydw\\&=:J_1^1+J_1^2.\end{aligned}$$

Then using compact support in velocity and Lemma 1.1 we find

$$|J_1^1|\le\int\left(\mathbf{1}_{\partial^{2|x-\mathcal{T}_1^0(x,v)|}K(v)}(x-y)+\mathbf{1}_{\partial^{2|y-\mathcal{T}_1^0(y,w)|}K(v)}(x-y)\right)f(y,w)\,dydw.$$

Therefore, taking into account (**H2**)-(ii), we deduce that

$$\begin{aligned}\int_{\mathbb{R}^{2d}}|J_1^1|f(x,v)dxdv&\le\int\mathbf{1}_{\partial^{2|x-\mathcal{T}_1^0(x,v)|}K(v)}(x-y)f(y,w)\,f(x,v)\,dydw\,dxdv\\&\quad+\int\mathbf{1}_{\partial^{2|y-\mathcal{T}_1^0(y,w)|}K(v)}(x-y)f(x,v)\,f(y,w)\,dxdv\,dydw\\&\le C(1+\|f\|_\infty)\int|x-\mathcal{T}_1^0(x,v)|f(x,v)\,dxdv.\end{aligned}$$

Now, we proceed with the term J_1^2 by using (**H2**)-(iii) and the fact that for any $K\subset\mathbb{R}^d$ and $x,y\in\mathbb{R}^d$

$$\mathbf{1}_K(x)\le\mathbf{1}_{K^{|x-y|,+}}(y)$$

since $x \in K$ implies that $y \in K^{|x-y|,+}$, to find

$$\begin{aligned}\left|\mathbf{1}_{K(v)}(\mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w))-\ \mathbf{1}_{K(\mathcal{T}_2^0(x,v))}(\mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w))\right| \\ \leq \mathbf{1}_{K(v)\Delta K(\mathcal{T}_2^0(x,v))}(\mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w)) \\ \leq \mathbf{1}_{\Theta(v)^{|v-\mathcal{T}_2^0(x,v)|,+}}(\mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w)) \\ \leq \mathbf{1}_{\Theta(v)^{C|v-\mathcal{T}_2^0(x,v)|+|x-\mathcal{T}_1^0(x,v)|+|y-\mathcal{T}_1^0(y,w)|,+}}(x-y) \\ \leq \mathbf{1}_{\Theta(v)^{2|x-\mathcal{T}_1^0(x,v)|+2C|v-\mathcal{T}_2^0(x,v)|,+}}(x-y)+\mathbf{1}_{\Theta(v)^{2|y-\mathcal{T}_1^0(y,w)|,+}}(x-y).\end{aligned}$$

And thus using assumption (**H2**)-(ii) we conclude

$$\int_{\mathbb{R}^{2d}}|J_1^2|f(x,v)\,dxdv \leq C(1+\|f\|_\infty)\int\left(|x-\mathcal{T}_1^0(x,v)|+|v-\mathcal{T}_2^0(x,v)|\right)f(x,v)\,dxdv.$$

$\diamond$ Estimate of J_2: Note that by the definition of $\alpha(\mathcal{T}^0(x,v),\mathcal{T}_1^0(y,w))$, we infer that

$$\mathbf{1}_{K(\mathcal{T}_2^0(x,v))}(\mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w))-\alpha(\mathcal{T}^0(x,v),\mathcal{T}_1^0(y,w))\neq 0,$$

only if

$$\mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w)\in\widetilde{\partial}K(\mathcal{T}_2^0(x,v))\subset\Theta(\mathcal{T}_2^0(x,v)),$$

due to Lemma 2.3. Therefore we deduce from (**H2**)-(iv), or equivalently by using (**H2**)-$(iv)'$ in Remark 2.4, that

$$x-y\in\Theta(v)^{C|v-\mathcal{T}_2^0(x,v)|+|x-\mathcal{T}_1^0(x,v)|+|y-\mathcal{T}_2^0(x,v)|,+}.$$

Hence

$$\begin{aligned}|\mathbf{1}_{K(\mathcal{T}_2^0(x,v))}(\mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w))-\alpha(\mathcal{T}^0(x,v),\mathcal{T}_1^0(y,w))| \\ \leq \mathbf{1}_{\Theta(v)^{C|v-\mathcal{T}_2^0(x,v)|+|x-\mathcal{T}_1^0(x,v)|+|y-\mathcal{T}_1^0(y,w)|,+}}(x-y) \\ \leq \mathbf{1}_{\Theta(v)^{2C|v-\mathcal{T}_2^0(x,v)|+2|x-\mathcal{T}_1^0(x,v)|,+}}(x-y)+\mathbf{1}_{\Theta(v)^{2|y-\mathcal{T}_1^0(y,w)|,+}}(x-y).\end{aligned}$$

Combining the above estimates and being t_0 arbitrary in $[0,T]$, we conclude

$$\frac{d^+}{dt}d_1(f(t),\mu^N(t))\leq Cd_1(f(t),\mu^N(t)),$$

for all $t\in[0,T]$, where C is positive constant independent of N. This completes the proof.

REMARK 3.1. *We are now able to explain the reason we are not able to consider other Wasserstein distances d_p, with $p>1$. To estimate the d_p distance, we need to control*

$$Q_p(t):=\int_{\mathbb{R}^d\times\mathbb{R}^d}|\Xi(t;t_0,x,v)|^p f(x,v,t_0)\,dxdv\quad \textit{with } \Xi(t;t_0,x,v)=\mathcal{Z}(t;t_0,x,v)-\mathcal{Z}^N(t;t_0,\mathcal{T}^0(x,v)).$$

To control this quantity by the classical use of Gronwall's inequality, we need to estimate

$$p\int_{\mathbb{R}^d\times\mathbb{R}^d}|V(t;t_0,x,v)-V^N(t;t_0,\mathcal{T}^0(x,v))|\,|\Xi(t;t_0,x,v)|^{p-1}\,f(x,v)dxdv$$

and

$$p\int_{\mathbb{R}^d\times\mathbb{R}^d}\left|F(f)(\mathcal{Z}(t;t_0,x,v),t)-F^N(f)(\mathcal{Z}^N(t;t_0,\mathcal{T}^0(x,v)),t)\right|\,|\Xi(t;t_0,x,v)|^{p-1}\,f(x,v)dxdv.$$

Following the idea of the proof above, we would have to estimate (we drop all reference to t_0)

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}|\Xi(x,v)|^{p-1}\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\mathbf{1}_{K(v)}(x-y)-\mathbf{1}_{K(v)}(\mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w))\right)(w-v)\,f(y,w)\,dydw\right|f(x,v)dxdv.$$

It is not possible to use Fubini's theorem and perform an integration with respect to x in the second characteristic of the set as before. Moreover, the use of any Hölder inequalities will not lead to the requested linear estimate at the end. Thus our arguments fail to estimate d_p distances, $p>1$.

4. Local-in-time existence and uniqueness of weak solutions

In this section, we study the local existence and uniqueness of weak solutions to the system (82).

4.1. Regularization. In this part, we introduce a regularized system, and show the uniform boundedness of solutions to the regularized system in regularization parameters.

We consider a mollified indicator function with respect to phase space $\mathbf{1}_K^{\eta,\varepsilon}$ as $\mathbf{1}_K^{\eta,\varepsilon} := \mathbf{1}_K *_{(x,v)} (\psi_\varepsilon, \phi_\eta)$, i.e.,

$$\mathbf{1}_{K(v)}^{\eta,\varepsilon}(x) = \int_{\mathbb{R}^d} \mathbf{1}_{K(v-w)}(x-y)\phi_\eta(w)\psi_\varepsilon(y)\,dydw,$$

where $\phi_\eta(w) := \frac{1}{\eta^d}\phi\left(\frac{w}{\eta}\right)$ with

$$\phi(v) = \phi(-v) \geq 0, \quad \phi \in \mathcal{C}_0^\infty(\mathbb{R}^d), \quad \text{supp } \phi \subset B(0,1), \quad \text{and} \quad \int_{\mathbb{R}^d} \phi(v)\,dv = 1.$$

Here $B(0,r) := \{x \in \mathbb{R}^d : |x| < r\}$ for $r > 0$. Similarly, the mollifier function $\psi_\varepsilon(y)$ is also defined. Classical theory implies that there exists a unique global solution $f^{\eta,\varepsilon}$ compactly supported in velocity to the regularized system:

(96)
$$\begin{cases} \partial_t f^{\eta,\varepsilon} + v\cdot\nabla_x f^{\eta,\varepsilon} + \nabla_v \cdot \left[F^{\eta,\varepsilon}(f^{\eta,\varepsilon}) f^{\eta,\varepsilon}\right] = 0, & (x,v) \in \mathbb{R}^d \times \mathbb{R}^d, \quad t > 0, \\ F^{\eta,\varepsilon}(f^{\eta,\varepsilon})(x,v,t) := \displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbf{1}_{K(v)}^{\eta,\varepsilon}(x-y)(w-v) f^{\eta,\varepsilon}(y,w,t)dydw, & (x,v) \in \mathbb{R}^d \times \mathbb{R}^d, \quad t > 0, \\ f^{\eta,\varepsilon}(x,v,0) =: f_0(x,v), & (x,v) \in \mathbb{R}^d \times \mathbb{R}^d, \end{cases}$$

given by the initial data pushed forward through the characteristic system, see for instance [**4**].

Before we proceed, we first remark the uniform boundedness, with respect to the regularization parameter, of the velocity support of $f^{\eta,\varepsilon}$ using the method of characteristics. More precisely, we consider the forward characteristics $Z^{\eta,\varepsilon}(s) := (X^{\eta,\varepsilon}(s;0,x,v), V^{\eta,\varepsilon}(s;0,x,v))$ satisfying the following ODE system:

(97)
$$\begin{aligned} \frac{dX^{\eta,\varepsilon}(s)}{ds} &= V^{\eta,\varepsilon}(s), \\ \frac{dV^{\eta,\varepsilon}(s)}{ds} &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbf{1}_{K(V^{\eta,\varepsilon}(s))}^{\eta,\varepsilon}\left(X^{\eta,\varepsilon}(s) - y\right)\left(w - V^{\eta,\varepsilon}(s)\right) f^{\eta,\varepsilon}(y,w,s)dydw. \end{aligned}$$

Set $\Omega^{\eta,\varepsilon}(t)$ and $R_v^{\eta,\varepsilon}(t)$ the v-projection of compact $\text{supp} f^{\eta,\varepsilon}(\cdot,t)$ and maximum value of v in $\Omega^{\eta,\varepsilon}(t)$, respectively:

$$\Omega^{\eta,\varepsilon}(t) := \overline{\{v \in \mathbb{R}^d : \exists (x,v) \in \mathbb{R}^d \times \mathbb{R}^d \text{ such that } f^{\eta,\varepsilon}(x,v,t) \neq 0\}}, \quad R_v^{\eta,\varepsilon}(t) := \max_{v\in\Omega^{\eta,\varepsilon}(t)} |v|.$$

Then we have the uniform boundedness of support of $f^{\eta,\varepsilon}$ in velocity as follows.

LEMMA 4.1. *Let $Z^{\eta,\varepsilon}(t)$ be the solution to the particle trajectory* (97) *issued from the compact $supp_{(x,v)} f_0$ at time* 0. *Then we have*

$$R_v^{\eta,\varepsilon}(t) \leq R_v^{\eta,\varepsilon}(0) = R_v^0 := \max_{v\in\Omega(0)} |v|,$$

i.e., given $\tilde{\Omega}_0 := B(0, R_v^0) \subset \mathbb{R}^d$, then $\Omega^{\eta,\varepsilon}(t) \subset \tilde{\Omega}_0$ for $t \geq 0$, $\epsilon, \eta > 0$.

The proof was given in [**11**] and [**7**, Lemma 3.1].

REMARK 4.1. *It follows from Lemma 4.1 that all velocity moments of $f^{\eta,\varepsilon}$ and all space moments of $f^{\eta,\varepsilon}$ are uniformly bounded in $[0,T]$ with respect to the regularization parameters ε, η.*

PROPOSITION 4.1. *Let $f^{\eta,\varepsilon}$ be the solution to the system* (96). *Then there exists a positive time $T > 0$ such that the $L^1 \cap L^\infty$-estimate of $f^{\eta,\varepsilon}$:*

$$\sup_{0\le t\le T} \|f^{\eta,\varepsilon}(\cdot,\cdot,t)\|_{L^1\cap L^\infty} \le C,$$

holds, where C is a positive constant independent of ε and η.

PROOF. This is essentially a consequence of the local Lipschitz character of the velocity field (92) observed in Section 2. Due to conservation of mass, we only need to deduce the L^∞-estimate of $f^{\eta,\varepsilon}$, we rewrite the system (96) in the non-conservative form:

$$\partial_t f^{\eta,\varepsilon} + v\cdot\nabla_x f^{\eta,\varepsilon} + F^{\eta,\varepsilon}(f^{\eta,\varepsilon})\cdot\nabla_v f^{\eta,\varepsilon} = -f^{\eta,\varepsilon}\nabla_v\cdot(F^{\eta,\varepsilon}(f^{\eta,\varepsilon}))$$

Note that

$$\begin{aligned}\nabla_v\cdot(F^{\eta,\varepsilon}(f^{\eta,\varepsilon})) = &-d\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbf{1}^{\eta,\varepsilon}_{K(v)}(x-y) f^{\eta,\varepsilon}(y,w)\,dydw\\ &+\int_{\mathbb{R}^d\times\mathbb{R}^d}\Big(\nabla_v \mathbf{1}^{\eta,\varepsilon}_{K(v)}(x-y)\Big)\cdot(w-v) f^{\eta,\varepsilon}(y,w)\,dydw.\end{aligned}$$

The first term is obviously bounded by the L^1 norm of f_0. In order to estimate the second term in the right hand side of the above equation, we find that for $i = 1,\cdots,d$ and $h > 0$

$$\begin{aligned}&\int_{\mathbb{R}^d\times\mathbb{R}^d}\Big(\mathbf{1}^{\eta,\varepsilon}_{K(v+e_ih)}(x-y) - \mathbf{1}^{\eta,\varepsilon}_{K(v)}(x-y)\Big)(w_i-v_i) f^{\eta,\varepsilon}(y,w)\,dydw\\ &= \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}}\left(\mathbf{1}_{K(v+e_ih-v')}(x-y-x') - \mathbf{1}_{K(v-v')}(x-y-x')\right)(w_i-v_i) f^{\eta,\varepsilon}(y,w)\phi_\eta(v')\psi_\varepsilon(x')\,dydwdx'dv'\\ &\le C\|f^{\eta,\varepsilon}\|_{L^\infty}\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\left|\mathbf{1}_{K(v+e_ih-v')}(x-y-x') - \mathbf{1}_{K(v-v')}(x-y-x')\right|dy\right)\phi_\eta(v')\psi_\varepsilon(x')\,dx'dv'\\ &\le C\|f^{\eta,\varepsilon}\|_{L^\infty}h,\end{aligned}$$

where we used Lemma 4.1 and a similar argument to (92). This yields that

$$\|\nabla_v\cdot(F^{\eta,\varepsilon}(f^{\eta,\varepsilon}))\|_{L^\infty} \le C\|f^{\eta,\varepsilon}\|_{L^1\cap L^\infty}$$

and

$$\frac{d}{ds} f^{\eta,\varepsilon}(X^{\eta,\varepsilon}(s),V^{\eta,\varepsilon}(s),s) \le C(1+\|f^{\eta,\varepsilon}\|_{L^\infty}) f^{\eta,\varepsilon}(X^{\eta,\varepsilon}(s),V^{\eta,\varepsilon}(s),s).$$

We conclude that

$$\frac{d}{dt}\|f^{\eta,\varepsilon}(\cdot,\cdot,t)\|_{L^1\cap L^\infty} \le \|f^{\eta,\varepsilon}(\cdot,\cdot,t)\|^2_{L^1\cap L^\infty}$$

Hence we conclude that there exists a finite time $T > 0$ such that

$$\sup_{0\le t\le T}\|f^{\eta,\varepsilon}(\cdot,\cdot,t)\|_{L^\infty} \le C,$$

where C is a positive constant independent of ε and η. □

In the following lemma, we provide elementary properties of the mollified indicator function and several useful estimates of the differences between mollified indicator functions. $\mathbf{1}^\varepsilon_K$ denotes the mollified indicator function with respect to x, i.e., $\mathbf{1}^\varepsilon_K = \mathbf{1}_K *_x \phi_\varepsilon$. Similarly, $\mathbf{1}^\eta_K$ denotes the mollified indicator function with respect to v, i.e., $\mathbf{1}^\eta_K = \mathbf{1}_K *_v \phi_\eta$.

LEMMA 4.2.

(i) For any $\varepsilon > 0$ and compact set $K \subset \mathbb{R}^d$, we have

$$\mathbf{1}^\varepsilon_K(x) \le \mathbf{1}_{K^{\varepsilon,+}}(x). \tag{98}$$

(ii) For all $v \in \mathbb{R}^d$ and all $\varepsilon > 0$, it holds

$$\int_{\mathbb{R}^d}\left|\mathbf{1}^{\eta,\varepsilon}_{K(v)}(x) - \mathbf{1}^\eta_{K(v)}(x)\right|dx \le \sup_{v\in\mathbb{R}^d}|\partial^{2\varepsilon}K(v)|. \tag{99}$$

(iii) For all $v \in \mathbb{R}^d$ and $0 < \eta \leq 1$, it holds

$$\int_{\mathbb{R}^d} \left| \mathbf{1}^{\eta,\varepsilon}_{K(v)}(x) - \mathbf{1}^{\varepsilon}_{K(v)}(x) \right| dx \leq C\eta, \tag{100}$$

where C is a positive constant independent of ε and η.

(iv) For all $\eta, \varepsilon > 0$ and $x_1, y_1, x_2, y_2, v \in \mathbb{R}^d$ we have

$$|\mathbf{1}^{\eta,\varepsilon}_{K(v)}(y_1 - x_1) - \mathbf{1}^{\eta,\varepsilon}_{K(v)}(y_2 - x_2)| \leq \mathbf{1}^{\eta,\varepsilon}_{\partial^{2|x_1 - x_2|}K(v)}(y_1 - x_1) + \mathbf{1}^{\eta,\varepsilon}_{\partial^{2|y_1 - y_2|}K(v)}(y_1 - x_1). \tag{101}$$

Proof.

(i) By the definition of the mollification, it is clear to get the desired inequality.

(ii) A straightforward computation yields

$$\begin{aligned}
|\mathbf{1}^{\eta,\varepsilon}_{K(v)}(x) - \mathbf{1}^{\eta}_{K(v)}(x)| &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |\mathbf{1}_{K(w)}(x - y) - \mathbf{1}_{K(w)}(x)| \phi_\eta(v - w)\psi_\varepsilon(y)\, dydw \\
&= \int_{\{y \in \mathbb{R}^d : |y| \leq \varepsilon\} \times \mathbb{R}^d} |\mathbf{1}_{K(w)}(x - y) - \mathbf{1}_{K(w)}(x)| \phi_\eta(v - w)\psi_\varepsilon(y)\, dydw \\
&\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_{\partial^{2\varepsilon}K(w)}(x) \phi_\eta(v - w)\psi_\varepsilon(y)\, dydw \\
&= \mathbf{1}^{\eta}_{\partial^{2\varepsilon}K(v)}(x).
\end{aligned}$$

This concludes

$$\begin{aligned}
\int_{\mathbb{R}^d} |\mathbf{1}^{\eta,\varepsilon}_{K(v)}(x) - \mathbf{1}^{\eta}_{K(v)}(x)| dx &\leq \int_{\mathbb{R}^d} \mathbf{1}^{\eta}_{\partial^{2\varepsilon}K(v)}(x)\, dx = \int_{\mathbb{R}^d \times \mathbb{R}^d} |\partial^{2\varepsilon}K(w)| \phi_\eta(v - w)\, dwdx \\
&\leq \sup_{v \in \mathbb{R}^d} |\partial^{2\varepsilon}K(v)|.
\end{aligned}$$

(iii) In a similar fashion as above, we find

$$\begin{aligned}
|\mathbf{1}^{\eta,\varepsilon}_{K(v)}(x) - \mathbf{1}^{\varepsilon}_{K(v)}(x)| &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |\mathbf{1}_{K(v-w)}(x - y) - \mathbf{1}_{K(v)}(x - y)| \phi_\eta(w)\psi_\varepsilon(y)\, dydw \\
&= \int_{\mathbb{R}^d \times \{w \in \mathbb{R}^d : |w| \leq \eta\}} |\mathbf{1}_{K(v-w)}(x - y) - \mathbf{1}_{K(v)}(x - y)| \phi_\eta(v - w)\psi_\varepsilon(y)\, dydw.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left| \mathbf{1}^{\eta,\varepsilon}_{K(v)}(x) - \mathbf{1}^{0,\varepsilon}_{K(v)}(x) \right| dx \\
&\quad \leq \int_{\mathbb{R}^d \times \{w \in \mathbb{R}^d : |w| \leq \eta\}} \left(\int_{\mathbb{R}^d} |\mathbf{1}_{K(v-w)}(x - y) - \mathbf{1}_{K(v)}(x - y)| dx \right) \phi_\eta(v - w)\psi_\varepsilon(y)\, dydw \\
&\quad \leq \int_{\mathbb{R}^d \times \{w \in \mathbb{R}^d : |w| \leq \eta\}} |K(v - w)\Delta K(v)| \phi_\eta(v - w)\psi_\varepsilon(y)\, dydw \\
&\quad \leq C\eta,
\end{aligned}$$

where C is a positive constant independent of ε and η, due to (**H2**)-(ii) and (iii).

(iv) As a direct consequence of Lemma 1.1, we obtain

$$\begin{aligned}
&|\mathbf{1}^{\eta,\varepsilon}_{K(v)}(y_1 - x_1) - \mathbf{1}^{\eta,\varepsilon}_{K(v)}(y_2 - x_2)| \\
&\quad \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \left| \mathbf{1}_{K(v-w)}(y_1 - x_1 - z) - \mathbf{1}_{K(v-w)}(y_2 - x_2 - z) \right| \psi_\varepsilon(z)\phi_\eta(w)\, dzdw \\
&\quad \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_{\partial^{2|x_1 - x_2|}K(v-w)}(y_1 - x_1 - z)\psi_\varepsilon(z)\phi_\eta(w)\, dzdw \\
&\qquad + \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_{\partial^{2|y_1 - y_2|}K(v-w)}(y_1 - x_1 - z)\psi_\varepsilon(z)\phi_\eta(w)\, dzdw \\
&\quad = \mathbf{1}^{\eta,\varepsilon}_{\partial^{2|x_1 - x_2|}K(v)}(y_1 - x_1) + \mathbf{1}^{\eta,\varepsilon}_{\partial^{2|y_1 - y_2|}K(v)}(y_1 - x_1),
\end{aligned}$$

leading to the desired estimate. □

We now show the growth estimate of $d_1(f^{\eta,\varepsilon}(t), f^{\eta',\varepsilon'}(t))$ leading to a weak-strong stability which is the core of this section.

PROPOSITION 4.2. *Let $f^{\eta,\varepsilon}$ and $f^{\eta',\varepsilon'}$ be solutions to the system* (96). *Then we have*

$$d_1(f^{\eta,\varepsilon}(t), f^{\eta',\varepsilon'}(t)) \le C\left(\eta+\varepsilon+\eta'+\varepsilon'\right),$$

for $\frac{1}{2} \ge \varepsilon, \eta, \varepsilon', \eta' > 0$ with C independent of η, ε, η', and ε', only depending on the uniform L^∞-norm of the solutions ensured by Propostion 4.1 and T. Moreover, given two solutions $f_1^{\eta,\varepsilon}$ and $f_2^{\eta,\varepsilon}$ to the system (96) *with two different initial data, then*

$$d_1(f_1^{\eta,\varepsilon}(t), f_2^{\eta,\varepsilon}(t)) \le C\left(d_1(f_1^{\eta,\varepsilon}(0), f_2^{\eta,\varepsilon}(0)) + \eta + \varepsilon\right),$$

for $\frac{1}{2} \ge \varepsilon, \eta > 0$ with C independent of η, ε, only depending on the uniform L^∞-norm of **one** *of the solutions, $f_1^{\eta,\varepsilon}$ for instance, and T.*

PROOF. Let us consider the flows $Z^{\eta,\varepsilon} := (X^{\eta,\varepsilon}, V^{\eta,\varepsilon})$ and $Z^{\eta',\varepsilon'} := (X^{\eta',\varepsilon'}, V^{\eta',\varepsilon'})$ generated from the solutions to (96) satisfying the corresponding characteristic systems (97) well-defined for $s, t \in [0,T]$ due to their smoothness. We now choose an optimal transport map $\mathcal{T}^0 = (\mathcal{T}_1^0(x,v), \mathcal{T}_2^0(x,v))$ between $f^{\eta,\varepsilon}(t_0)$ and $f^{\eta',\varepsilon'}(t_0)$ for fixed $t_0 \in [0,T)$, i.e., $f^{\eta',\varepsilon'}(t_0) = \mathcal{T}^0 \# f^{\eta,\varepsilon}(t_0)$. It is classically known from [**36**] that such an optimal transport map exists when $f^{\eta,\varepsilon}(t_0)$ is absolutely continuous with respect to the Lebesgue measure. The solutions of these approximated problems can be expressed by the characteristic method with initial data at $t = t_0$ as $f^{\eta,\varepsilon}(t) = Z^{\eta,\varepsilon}(t; t_0, \cdot, \cdot) \# f^{\eta,\varepsilon}(t_0)$ and $f^{\eta',\varepsilon'}(t) = Z^{\eta',\varepsilon'}(t; t_0, \cdot, \cdot) \# f^{\eta',\varepsilon'}(t_0)$. We also notice that

$$\mathcal{T}^t \# f^{\eta,\varepsilon}(t) = f^{\eta',\varepsilon'}(t), \quad \text{where} \quad \mathcal{T}^t = Z^{\eta',\varepsilon'}(t; t_0, \cdot, \cdot) \circ \mathcal{T}^0 \circ Z^{\eta,\varepsilon}(t_0; t, \cdot, \cdot).$$

By Definition 2.2 of d_1-distance, we obtain

$$d_1(f^{\eta,\varepsilon}(t), f^{\eta',\varepsilon'}(t)) \le \int_{\mathbb{R}^d\times\mathbb{R}^d} |Z^{\eta,\varepsilon}(t; t_0, x, v) - Z^{\eta',\varepsilon'}(t; t_0, \mathcal{T}^0(x,v))| f^{\eta,\varepsilon}(x, v, t_0) dx dv.$$

Set

$$Q_{\eta,\varepsilon,\eta',\varepsilon'}(t) := \int_{\mathbb{R}^d\times\mathbb{R}^d} |Z^{\eta,\varepsilon}(t; t_0, x, v) - Z^{\eta',\varepsilon'}(t; t_0, \mathcal{T}^0(x,v))| f^{\eta,\varepsilon}(x, v, t_0) dx dv.$$

Then straightforward computations yield

$$\begin{aligned}
&\frac{d}{dt} Q_{\eta,\varepsilon,\eta',\varepsilon'}(t)\Big|_{t=t_0+} \\
&\le \int_{\mathbb{R}^d\times\mathbb{R}^d} |V^{\eta,\varepsilon}(t; t_0, x, v) - V^{\eta',\varepsilon'}(t; t_0, \mathcal{T}^0(x,v))| f^{\eta,\varepsilon}(x, v, t_0) dx dv\Big|_{t=t_0+} \\
&\quad + \int_{\mathbb{R}^d\times\mathbb{R}^d} \left| F^{\eta,\varepsilon}(f^{\eta,\varepsilon})(Z^{\eta,\varepsilon}(t; t_0, x, v), t) - F^{\eta',\varepsilon'}(f^{\eta',\varepsilon'})(Z^{\eta',\varepsilon'}(t; t_0, \mathcal{T}^0(x,v)), t) \right| f^{\eta,\varepsilon}(x, v, t_0) dx dv\Big|_{t=t_0+} \\
&=: I + J.
\end{aligned}$$

It is easy to estimate I as

$$I = \int_{\mathbb{R}^d\times\mathbb{R}^d} |v - \mathcal{T}_2^0(x,v)| f^{\eta,\varepsilon}(x, v, t_0) dx dv \le C d_1(f^{\eta,\varepsilon}(t_0), f^{\eta',\varepsilon'}(t_0)).$$

For the estimate of J, we notice that

$$\begin{aligned}
J &= \int_{\mathbb{R}^d\times\mathbb{R}^d}\Big|\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}^{\eta,\varepsilon}_{K(v)}(x-y)(w-v)f^{\eta,\varepsilon}(y,w,t_0)dydw\\
&\quad-\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}^{\eta,\varepsilon'}_{K(\mathcal{T}_2^0(x,v))}(\mathcal{T}_1^0(x,v)-y)(w-\mathcal{T}_2^0(x,v))f^{\eta',\varepsilon'}(y,w,t_0)dydw\Big|f^{\eta,\varepsilon}(x,v,t_0)dxdv\\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d}\Big|\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}^{\eta,\varepsilon}_{K(v)}(x-y)(w-v)f^{\eta,\varepsilon}(y,w,t_0)dydw\\
&\quad-\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}^{\eta,\varepsilon'}_{K(\mathcal{T}_2^0(x,v))}(\mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w))(\mathcal{T}_2^0(y,w)-\mathcal{T}_2^0(x,v))f^{\eta,\varepsilon}(y,w,t_0)dydw\Big|f^{\eta,\varepsilon}(x,v,t_0)dxdv.
\end{aligned}$$

We omit the time dependency on t_0 in the rest of computations for notational simplicity.
We now decompose J into four parts:

$$J=\int_{\mathbb{R}^d\times\mathbb{R}^d}\left|\sum_{i=1}^4 J_i\right|f^{\eta,\varepsilon}(x,v)\,dxdv,$$

where

$$\begin{aligned}
J_1&:=\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\mathbf{1}^{\eta,\varepsilon}_{K(v)}(x-y)-\mathbf{1}^{\eta',\varepsilon'}_{K(v)}(x-y)\right)(w-v)f^{\eta,\varepsilon}(y,w)\,dydw,\\
J_2&:=\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\mathbf{1}^{\eta',\varepsilon'}_{K(v)}(x-y)-\mathbf{1}^{\eta',\varepsilon'}_{K(v)}(\mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w))\right)(w-v)f^{\eta,\varepsilon}(y,w)\,dydw,\\
J_3&:=\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\mathbf{1}^{\eta',\varepsilon'}_{K(v)}(\mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w))-\mathbf{1}^{\eta',\varepsilon'}_{K(\mathcal{T}_2^0(x,v))}(\mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w))\right)(w-v)f^{\eta,\varepsilon}(y,w)\,dydw,\\
J_4&:=\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}^{\eta',\varepsilon'}_{K(\mathcal{T}_2^0(x,v))}(\mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w))(w-v-\left(\mathcal{T}_2^0(y,w)-\mathcal{T}_2^0(x,v)\right))f^{\eta,\varepsilon}(y,w)\,dydw.
\end{aligned}$$

$\diamond$ (Estimate of J_1): We decompose J_1 into two terms:

$$\begin{aligned}
J_1&=\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\mathbf{1}^{\eta,\varepsilon}_{K(v)}(x-y)-\mathbf{1}^{\eta,\varepsilon'}_{K(v)}(x-y)\right)(w-v)f^{\eta,\varepsilon}(y,w)\,dydw\\
&\quad+\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\mathbf{1}^{\eta,\varepsilon'}_{K(v)}(x-y)-\mathbf{1}^{\eta',\varepsilon'}_{K(v)}(x-y)\right)(w-v)f^{\eta,\varepsilon}(y,w)\,dydw\\
&=:J_1^1+J_1^2.
\end{aligned}$$

For the estimate of J_1^1, we obtain that for $0<\varepsilon\le 1/2$

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}}\left|\mathbf{1}^{\eta,\varepsilon}_{K(v)}(x-y)-\mathbf{1}^{\eta}_{K(v)}(x-y)\right||w-v|f^{\eta,\varepsilon}(x,v)f^{\eta,\varepsilon}(y,w)\,dxdydvdw\\
&\le C\int_{\mathbb{R}^{2d}\times\tilde\Omega_0^2}\left|\mathbf{1}^{\eta,\varepsilon}_{K(v)}(x-y)-\mathbf{1}^{\eta}_{K(v)}(x-y)\right|f^{\eta,\varepsilon}(x,v)f^{\eta,\varepsilon}(y,w)\,dxdydvdw\\
&\le C\|f^{\eta,\varepsilon}\|_{L^\infty}\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\int_{\mathbb{R}^d\times\tilde\Omega_0}\left|\mathbf{1}^{\eta,\varepsilon}_{K(v)}(x-y)-\mathbf{1}^{\eta}_{K(v)}(x-y)\right|dydw\right)f^{\eta,\varepsilon}(x,v)\,dxdv\\
&\le C\|f^{\eta,\varepsilon}\|_{L^\infty}\sup_{v\in\mathbb{R}^d}|\partial^{2\varepsilon}K(v)|\int_{\mathbb{R}^d\times\mathbb{R}^d}f^{\eta,\varepsilon}(x,v)\,dxdv\\
&\le C\|f^{\eta,\varepsilon}\|_{L^\infty}\varepsilon,
\end{aligned}$$

due to (**H2**) and (3.138). Similarly we also get

$$\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}}\left|\mathbf{1}^{\eta,\varepsilon'}_{K(v)}(x-y)-\mathbf{1}^{\eta}_{K(v)}(x-y)\right||w-v|f^{\eta,\varepsilon}(x,v)f^{\eta,\varepsilon}(y,w)\,dxdydvdw\le C\|f^{\eta,\varepsilon}\|_{L^\infty}\varepsilon'$$

for $0<\varepsilon'\le 1/2$. This yields

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}|J_1|f^{\eta,\varepsilon}(x,v)\,dxdv\le C\|f^{\eta,\varepsilon}\|_{L^\infty}(\varepsilon+\varepsilon').$$

We next use (3.140) together with the compact support in velocity similarly to (92) to find

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \left|\mathbf{1}^{\eta,\varepsilon'}_{K(v)}(x-y) - \mathbf{1}^{\varepsilon'}_{K(v)}(x-y)\right| |w-v| f^{\eta,\varepsilon}(x,v) f^{\eta,\varepsilon}(y,w)\,dxdydvdw \\
&\quad \le C\|f^{\eta,\varepsilon}\|_{L^\infty} \int_{\mathbb{R}^d\times\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \left|\mathbf{1}^{\eta,\varepsilon'}_{K(v)}(x-y) - \mathbf{1}^{\varepsilon'}_{K(v)}(x-y)\right| dy \right) f^{\eta,\varepsilon}(x,v)\,dxdv \\
&\quad \le C\|f^{\eta,\varepsilon}\|_{L^\infty}\eta.
\end{aligned}$$

Hence, we have

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |J_1^2| f^{\eta,\varepsilon}(x,v)\,dxdv \le C\|f^{\eta,\varepsilon}\|_{L^\infty}(\eta+\eta'),$$

and this concludes

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |J_1| f^{\eta,\varepsilon}(x,v)\,dxdv \le C\|f^{\eta,\varepsilon}\|_{L^\infty}(\eta+\eta'+\varepsilon+\varepsilon'),$$

where C is a positive constant independent of η, η', ε, and ε'.

$\diamond$ (Estimate of J_2): We first use (101) to find

$$|J_2| \le \int_{\mathbb{R}^d\times\mathbb{R}^d} \left(\mathbf{1}^{\eta',\varepsilon'}_{\partial^{2|x-\mathcal{T}_1^0(x,v)|}K(v)}(x-y) + \mathbf{1}^{\eta',\varepsilon'}_{\partial^{2|y-\mathcal{T}_1^0(y,w)|}K(v)}(x-y) \right) |w-v| f^{\eta,\varepsilon}(y,w)\,dydw$$

Moreover, by using (98) we obtain

$$\begin{aligned}
|J_2| &\le \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} \mathbf{1}_{\partial^{2|x-\mathcal{T}_1^0(x,v)|+\varepsilon'}K(v-v')}(x-y)|w-v| f^{\eta,\varepsilon}(y,w)\phi_{\eta'}(v')\,dydwdv' \\
&\quad + \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} \mathbf{1}_{\partial^{2|y-\mathcal{T}_1^0(y,w)|+\varepsilon'}K(v-v')}(x-y)|w-v| f^{\eta,\varepsilon}(y,w)\phi_{\eta'}(v')\,dydwdv' \\
&=: J_2^1 + J_2^2.
\end{aligned}$$

We now choose $0<\varepsilon'\le 1/2$ and consider a set $D := \{(x,v)\in\mathbb{R}^{2d} : |x-\mathcal{T}_1^0(x,v)| \le 1/4\}$. Then we obtain

$$\begin{aligned}
&\int_D |J_2^1| f^{\eta,\varepsilon}(x,v)\,dxdv \\
&\quad \le C\int_{D\times\mathbb{R}^d\times\tilde{\Omega}_0\times\mathbb{R}^d} \mathbf{1}_{\partial^{2|x-\mathcal{T}_1^0(x,v)|+\varepsilon'}K(v-v')}(x-y) f^{\eta,\varepsilon}(x,v) f^{\eta,\varepsilon}(y,w)\phi_{\eta'}(v')\,dxdydvdwdv' \\
&\quad \le C\|f^{\eta,\varepsilon}\|_{L^\infty}\int_{\mathbb{R}^d\times\mathbb{R}^d} \sup_{u\in\mathbb{R}^d}\left(|\partial^{2|x-\mathcal{T}_1^0(x,v)|+\varepsilon'}K(u)|\right) f^{\eta,\varepsilon}(x,v)\,dxdv \\
&\quad \le C\|f^{\eta,\varepsilon}\|_{L^\infty}\int_{\mathbb{R}^d\times\mathbb{R}^d} \left(|x-\mathcal{T}_1^0(x,v)|+\varepsilon'\right) f^{\eta,\varepsilon}(x,v)\,dxdv \\
&\quad \le C\|f^{\eta,\varepsilon}\|_{L^\infty}\left(\varepsilon' + d_1(f^{\eta,\varepsilon}, f^{\eta',\varepsilon'})\right),
\end{aligned}$$

due to (**H2**)-(ii). On the other hand, for $(x,v)\in D^c$, we have

$$\begin{aligned}
\int_{D^c} |J_2^1| f^{\eta,\varepsilon}(x,v)\,dxdv &\le C\int_{D^c\times\mathbb{R}^{2d}} f^{\eta,\varepsilon}(x,v) f^{\eta,\varepsilon}(y,w)\,dxdydvdw \\
&\le C\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} |x-\mathcal{T}_1^0(x,v)| f^{\eta,\varepsilon}(x,v) f^{\eta,\varepsilon}(y,w)\,dxdydvdw \\
&\le C d_1(f^{\eta,\varepsilon}, f^{\eta',\varepsilon'}).
\end{aligned}$$

This yields that for $0<\varepsilon'\le 1/2$

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |J_2^1| f^{\eta,\varepsilon}(x,v)\,dxdv \le C\|f^{\eta,\varepsilon}\|_{L^\infty}\varepsilon' + C(1+\|f^{\eta,\varepsilon}\|_{L^\infty}) d_1(f^{\eta,\varepsilon}, f^{\eta',\varepsilon'}).$$

Similarly, for J_2^2, we find

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\int_{D\times\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}_{\partial^{2|y-\mathcal{T}_1^0(y,w)|+\varepsilon'}K(v-v')}(x-y)|w-v|f^{\eta,\varepsilon}(y,w)\phi_{\eta'}(v')\,dydwdv'\right)f^{\eta,\varepsilon}(x,v)\,dxdv\\
&\quad\le C\int_{D\times\mathbb{R}^d\times\tilde{\Omega}_0\times\mathbb{R}^d}\mathbf{1}_{\partial^{2|y-\mathcal{T}_1^0(y,w)|+\varepsilon'}K(v-v')}(x-y)f^{\eta,\varepsilon}(x,v)f^{\eta,\varepsilon}(y,w)\phi_{\eta'}(v')\,dxdydvdwdv'\\
&\quad\le C\|f^{\eta,\varepsilon}\|_{L^\infty}\int_{\mathbb{R}^d\times\mathbb{R}^d}\sup_{u\in\mathbb{R}^d}\left(|\partial^{2|y-\mathcal{T}_1^0(y,w)|+\varepsilon'}K(u)|\right)f^{\eta,\varepsilon}(y,w)\,dydw\\
&\quad\le C\|f^{\eta,\varepsilon}\|_{L^\infty}\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(|y-\mathcal{T}_1^0(y,w)|+\varepsilon'\right)f^{\eta,\varepsilon}(y,w)\,dydw\\
&\quad\le C\|f^{\eta,\varepsilon}\|_{L^\infty}\left(\varepsilon'+d_1(f^{\eta,\varepsilon},f^{\eta',\varepsilon'})\right),
\end{aligned}$$

and this and together with the estimate of J_2^1 implies

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}|J_2|f^{\eta,\varepsilon}(x,v)\,dxdv\le C\|f^{\eta,\varepsilon}\|_{L^\infty}\varepsilon'+C(1+\|f^{\eta,\varepsilon}\|_{L^\infty})d_1(f^{\eta,\varepsilon},f^{\eta,\varepsilon'})$$

$\diamond$ (Estimate of J_3): Note that J_3 can be rewritten as

$$\begin{aligned}
J_3=&\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^{2d}}\mathbf{1}_{K(v-v')}(\mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w)-z)(w-v)f^{\eta,\varepsilon}(y,w)\,\phi_{\varepsilon'}(z)\phi_{\eta'}(v')\,dydw\,dzdv'\\
&-\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^{2d}}\mathbf{1}_{K(\mathcal{T}_2^0(x,v)-v')}(\mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w)-z)(w-v)f^{\eta,\varepsilon}(y,w)\,\phi_{\varepsilon'}(z)\phi_{\eta'}(v')\,dydw\,dzdv'.
\end{aligned}$$

Then using assumption (**H2**)-(*iii*), (98) and (2.1) we find

$$\begin{aligned}
|J_3|&\le\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^{2d}}\mathbf{1}_{K(v-v')\Delta K(\mathcal{T}_2^0(x,v)-v')}(\mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w)-z)f^{\eta,\varepsilon}(y,w)\,\phi_{\varepsilon'}(z)\phi_{\eta'}(v')\,dydw\,dzdv'\\
&\le C\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^{2d}}\mathbf{1}_{\Theta(v-v')^{C|v-\mathcal{T}_2^0(x,v)|,+}}(\mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w)-z)f^{\eta,\varepsilon}(y,w)\,\phi_{\varepsilon'}(z)\phi_{\eta'}(v')\,dydw\,dzdv'\\
&\le C\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}_{\Theta(v-v')^{C|v-\mathcal{T}_2^0(x,v)|+\varepsilon',+}}(\mathcal{T}_1^0(x,v)-\mathcal{T}_1^0(y,w))f^{\eta,\varepsilon}(y,w)\,\phi_{\eta'}(v')\,dydw\,dv'.
\end{aligned}$$

It follows from Lemmas 2.1 and 1.1 that for any compact set $K\subset\mathbb{R}^d$, $x,y\in\mathbb{R}^d$, we have

$$\mathbf{1}_K(x)\le\mathbf{1}_{K^{|x-y|,+}}(y).$$

Thus we deduce

$$\begin{aligned}
|J_3|&\le C\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}_{\Theta(v-v')^{C|v-\mathcal{T}_2^0(x,v)|+|x-\mathcal{T}_1^0(x,v)|+|y-\mathcal{T}_1^0(y,w)|+\varepsilon',+}}(x-y)f^{\eta,\varepsilon}(y,w)\,\phi_{\eta'}(v')\,dydwdv'\\
&\le C\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}_{\Theta(v-v')^{2C|v-\mathcal{T}_2^0(x,v)|+2|x-\mathcal{T}_1^0(x,v)|}}(x-y)f^{\eta,\varepsilon}(y,w)\,\phi_{\eta'}(v')\,dydwdv\\
&\quad+C\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}_{\Theta(v-v')^{2|y-\mathcal{T}_1^0(y,w)|+2\varepsilon',+}}(x-y)f^{\eta,\varepsilon}(y,w)\,\phi_{\eta'}(v')\,dydwdv\\
&:=J_3^1+J_3^2.
\end{aligned}$$

Introducing a set $F:=\{(x,v)\in\mathbb{R}^{2d}\ ,\ 2|x-\mathcal{T}_1^0(x,v)|+2C|v-\mathcal{T}_2^0(x,v)|\le 1\}$, we deduce from (**H2**) that

$$\begin{aligned}
\int_F|J_3^1|f^{\eta,\varepsilon}(x,v)\,dxdv&\le C\|f^{\eta,\varepsilon}\|_\infty\int_F\sup_{u\in\mathbb{R}^d}|\Theta(u)^{2|x-\mathcal{T}_1^0(x,v)|+2C|v-\mathcal{T}_2^0(x,v)|}|f^{\eta,\varepsilon}(x,v)\,dxdv\\
&\le C\|f^{\eta,\varepsilon}\|_\infty\int_F\left(|x-\mathcal{T}_1^0(x,v)|+|v-\mathcal{T}_2^0(x,v)|\right)f^{\eta,\varepsilon}(x,v)\,dxdv.
\end{aligned}$$

On the other hand, we obtain that for $(x,v)\in F^c$

$$\int_{F^c}|J_3^1|f^{\eta,\varepsilon}(x,v)\,dxdv\le C\int_{F^c}\left(|x-\mathcal{T}_1^0(x,v)|+|v-\mathcal{T}_2^0(x,v)|\right)f^{\eta,\varepsilon}(x,v)\,dxdv.$$

Using the similar fashion as the above, we estimate J_3^2 for $\varepsilon' \le 1/4$ as

$$\int_{\mathbb{R}^{2d}} |J_3^2| f^{\eta,\varepsilon}(x,v)\,dxdv \le C\|f^{\eta,\varepsilon}\|_\infty + C\big(\|f^{\eta,\varepsilon}\|_\infty + 1\big)\int_{\mathbb{R}^{2d}} |x - \mathcal{T}_1^0(x,v)| f^{\eta,\varepsilon}(x,v)\,dxdv.$$

This concludes that

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |J_3| f^{\eta,\varepsilon}(x,v)\,dxdv \le C\|f^{\eta,\varepsilon}\|_{L^\infty}\varepsilon' + C(1+\|f^{\eta,\varepsilon}\|_{L^\infty}) d_1(f^{\eta,\varepsilon}, f^{\eta',\varepsilon'}).$$

$\diamond$ (Estimate of J_4): Since $\mathbf{1}_K^{\eta,\varepsilon} \le 1$, we easily have

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d} |J_4| f^{\eta,\varepsilon}(x,v)\,dxdv &\le \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \big(|w - \mathcal{T}_2^0(y,w)| + |v - \mathcal{T}_2^0(x,v)|\big)\, f^{\eta,\varepsilon}(x,v) f^{\eta,\varepsilon}(y,w)\,dxdydvdw \\ &\le 2\int_{\mathbb{R}^d\times\mathbb{R}^d} |v - \mathcal{T}_2^0(x,v)| f^{\eta,\varepsilon}(x,v)\,dxdv \\ &\le C d_1(f^{\eta,\varepsilon}, f^{\eta',\varepsilon'}).\end{aligned}$$

We now combine the all estimates above to have

$$J \le C\|f^{\eta,\varepsilon}\|_{L^\infty}(\varepsilon + \varepsilon' + \eta + \eta') + C(1+\|f^{\eta,\varepsilon}\|_{L^\infty}) d_1(f^{\eta,\varepsilon}, f^{\eta',\varepsilon'}) \quad \text{for} \quad \eta,\eta',\varepsilon,\varepsilon' \le \frac{1}{2},$$

using Proposition 4.1. We conclude that

$$\frac{d^+}{dt} d_1(f^{\eta,\varepsilon}(t), f^{\eta',\varepsilon'}(t)) \le C\left(d_1(f^{\eta,\varepsilon}(t), f^{\eta',\varepsilon'}(t)) + \eta + \varepsilon + \eta' + \varepsilon'\right),$$

for $\frac{1}{2} \ge \varepsilon, \eta, \varepsilon', \eta' > 0$ with C independent of η, ε, η', and ε', only depending on the uniform L^∞-norm of the solutions ensured by Propostion 4.1. Here $\frac{d^+}{dt}$ denotes the time-derivative from the right. This implies that

$$d_1(f^{\eta,\varepsilon}(t), f^{\eta',\varepsilon'}(t)) \le C\left(\eta + \varepsilon + \eta' + \varepsilon'\right),$$

for $\frac{1}{2} \ge \varepsilon, \eta, \varepsilon', \eta' > 0$ with C independent of η, ε, η', and ε'. It is straightforward to check that in the previous argument the fact that both solutions have the same initial data does not play any role. Moreover, it is evident going through the proof that we only used the L^∞-norm on one of the solutions, actually $f^{\eta,\varepsilon}$. □

4.2. Passing to the limit as $\eta, \varepsilon \to 0$ and proof of Theorem 2.2. It follows from Proposition 4.2 that $\{f^{\eta,\varepsilon}\}_{\eta,\varepsilon>0}$ is a Cauchy sequence in $\mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}^d\times\mathbb{R}^d))$, and this implies that there exists a limit curve of measures $f \in \mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}^d\times\mathbb{R}^d))$, actually densities $f \in L^\infty(0,T;(L^1_+\cap L^\infty)(\mathbb{R}^d\times\mathbb{R}^d))$ due to Proposition 4.1. Thus it only remains to show that f is a solution of the equations (82). Choose a test function $\Psi(x,v,t) \in \mathcal{C}_c^\infty(\mathbb{R}^d\times\mathbb{R}^d\times[0,T])$, then $f^{\eta,\varepsilon}$ satisfies

(102)
$$\begin{aligned}&\int_{\mathbb{R}^d\times\mathbb{R}^d} \Psi_0(x,v) f_0(x,v) dxdv \\ &\quad= \int_{\mathbb{R}^d\times\mathbb{R}^d} \Psi(x,v,T) f^{\eta,\varepsilon}(x,v,T) dxdv + \int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d} f^{\eta,\varepsilon}(x,v,t)\partial_t\Psi(x,v,t) dxdvdt \\ &\qquad- \int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d} (\nabla_x\Psi)\cdot v f^{\eta,\varepsilon} dxdvdt - \int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d} (\nabla_v\Psi)\cdot F^{\eta,\varepsilon}(f^{\eta,\varepsilon}) f^{\eta,\varepsilon} dxdvdt.\end{aligned}$$

We can easily show that the first, second, and third terms in the rhs of (102) converge to their corresponding limits with f instead of $f^{\eta,\varepsilon}$ due to their linearity and $f^{\eta,\varepsilon} \to f$ in $\mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}^d\times\mathbb{R}^d))$. In order to show

$$\int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d} (\nabla_v\Psi)\cdot F^{\eta,\varepsilon}(f^{\eta,\varepsilon}) f^{\eta,\varepsilon} dxdvdt \to \int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d} (\nabla_v\Psi)\cdot F(f) f dxdvdt, \tag{103}$$

we provide two results.

LEMMA 4.3. *$F(f)(x,v,t)$ is locally Lipschitz continuous in $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$ for all $t \geq 0$.*

Its proof was given in subsection 2.5. The next property follows by similar arguments.

LEMMA 4.4. *The vector field*

$$G(f^{\eta,\varepsilon})(y,w) := \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_{K(v)}(x-y)\left(\nabla_v \Psi(x,v)\cdot(w-v)\right) f^{\eta,\varepsilon}(x,v)\,dxdv$$

is Lipschitz continuous in $(x,v) \in \mathbb{R}^d \times \tilde{\Omega}_0$ for all $t \geq 0$. Here the Lipschitz constant is independent of the regularization parameters η and ε.

PROOF. For $(y,w),(y',w') \in \mathbb{R}^d \times \tilde{\Omega}_0$, we obtain

$$\begin{aligned}
&|G(f^{\eta,\varepsilon})(y,w) - G(f^{\eta,\varepsilon})(y',w')| \\
&\quad = \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v \Psi(x,v)\cdot\left(\mathbf{1}_{K(v)}(x-y)(w-v) - \mathbf{1}_{K(v)}(x-y')(w'-v)\right) f^{\eta,\varepsilon}(x,v)\,dxdv \right| \\
&\quad \leq \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_{K(v)}(x-y)\left(\nabla_v \Psi(x,v)\cdot\left(w-w'\right)\right) f^{\eta,\varepsilon}(x,v)\,dxdv \right| \\
&\qquad + \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \left(\mathbf{1}_{K(v)}(x-y) - \mathbf{1}_{K(v)}(x-y')\right)\left(\nabla_v \Psi(x,v)\cdot(w'-v)\right) f^{\eta,\varepsilon}(x,v)\,dxdv \right| \\
&\quad =: I + J,
\end{aligned}$$

where I is easily estimated by.

$$I \leq \|\nabla_v \Psi\|_{L^\infty} |w-w'|.$$

For the estimate of J, we first consider the case $|y-y'| \leq 1/2$. In this case, we obtain that

$$J \leq C\|\nabla_v \Psi\|_{L^\infty} \int_{\mathbb{R}^d \times \tilde{\Omega}_0} \mathbf{1}_{\partial^{2|y-y'|}K(v)}(x-y) f^{\eta,\varepsilon}(x,v)\,dxdv \leq C\|\nabla_v \Psi\|_{L^\infty} \|f^{\eta,\varepsilon}\|_{L^\infty} |y-y'|,$$

due to the compact support in velocity, (**H2**) and Lemma 1.1 . On the other hand, if $|y-y'| \geq 1/2$ we get

$$J \leq C\|\nabla_v \Psi\|_{L^\infty} \leq C\|\nabla_v \Psi\|_{L^\infty} |y-y'|.$$

This yields that for any $y, y' \in \mathbb{R}^d$ we have

$$J \leq C\|\nabla_v \Psi\|_{L^\infty}(1 + \|f^{\eta,\varepsilon}\|_{L^\infty})|y-y'|,$$

concluding the proof. □

We are now in a position to estimate the convergence (103). Note that

$$\begin{aligned}
&\left| \int_0^T \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v \Psi \cdot (F^{\eta,\varepsilon}(f^{\eta,\varepsilon}) f^{\eta,\varepsilon} - F(f) f)\,dxdvdt \right| \\
&\quad = \left| \int_0^T \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v \Psi \cdot (F^{\eta,\varepsilon}(f^{\eta,\varepsilon}) f^{\eta,\varepsilon} - F(f^{\eta,\varepsilon}) f^{\eta,\varepsilon})\,dxdvdt \right| \\
&\qquad + \left| \int_0^T \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v \Psi \cdot (F(f^{\eta,\varepsilon}) f^{\eta,\varepsilon} - F(f) f^{\eta,\varepsilon})\,dxdvdt \right| \\
&\qquad + \left| \int_0^T \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v \Psi \cdot (F(f) f^{\eta,\varepsilon} - F(f) f)\,dxdvdt \right| \\
&\quad =: \sum_{i=1}^3 K_i.
\end{aligned}$$

$\diamond$ Estimate of K_1: A straightforward computation yields that

$$\begin{aligned}
K_1 &\le \left| \int_0^T \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \nabla_v \Psi \cdot \left(\mathbf{1}^{\eta,\varepsilon}_{K(v)}(x-y) - \mathbf{1}^{\eta}_{K(v)}(x-y) \right) (w-v) f^{\eta,\varepsilon}(y,w) f^{\eta,\varepsilon}(x,v)\, dydwdxdvdt \right| \\
&\quad + \left| \int_0^T \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \nabla_v \Psi \cdot \left(\mathbf{1}^{\eta}_{K(v)}(x-y) - \mathbf{1}_{K(v)}(x-y) \right) (w-v) f^{\eta,\varepsilon}(y,w) f^{\eta,\varepsilon}(x,v)\, dydwdxdvdt \right| \\
&\le C\|\nabla_v \Psi\|_{L^\infty} \int_0^T \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \mathbf{1}^{\eta}_{\partial^{2\varepsilon} K(v)}(x-y) f^{\eta,\varepsilon}(y,w) f^{\eta,\varepsilon}(x,v)\, dydwdxdvdt \\
&\quad + CT\|\nabla_v \Psi\|_{L^\infty} \|f^{\eta,\varepsilon}\|_{L^\infty} \eta \\
&\le CT\|\nabla_v \Psi\|_{L^\infty} \|f^{\eta,\varepsilon}\|_{L^\infty} (\varepsilon + \eta).
\end{aligned}$$

Here, we have used the same arguments as in previous Lemma and in (92) using the compact support in velocity, (3.138), (3.140) and (**H2**). Thus we have $K_1 \to 0$ as $\varepsilon \to 0$ and $\eta \to 0$.

$\diamond$ Estimate of K_2: We first notice that

$$\begin{aligned}
K_2 &= \left| \int_0^T \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \nabla_v \Psi \cdot \mathbf{1}_{K(v)}(x-y)(w-v)(f^{\eta,\varepsilon}(y,w) - f(y,w)) f^{\eta,\varepsilon}(x,v)\, dxdydvdwdt \right| \\
&= \left| \int_0^T \int_{\mathbb{R}^{d}\times\mathbb{R}^{d}} G(f^{\eta,\varepsilon})(y,w)(f^{\eta,\varepsilon}(y,w) - f(y,w)) dydwdt \right|,
\end{aligned}$$

and $\|G(f^{\eta,\varepsilon})\|_{W^{1,\infty}(\mathbb{R}^d\times\tilde{\Omega}_0)} \le C\|\nabla_v \Psi\|_{L^\infty}$. Then it follows from duality in (89) that

$$K_2 \le C d_1(f^{\eta,\varepsilon}, f) \to 0 \quad \text{as} \quad \eta, \varepsilon \to 0.$$

$\diamond$ Estimate of K_3: Similarly, we deduce from Lemma 4.3 that

$$K_3 = \left| \int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_v \Psi \cdot F(f)(x,v) \left(f^{\eta,\varepsilon}(x,v) - f(x,v) \right) dxdvdt \right| \le C d_1(f^{\eta,\varepsilon}, f) \to 0,$$

as $\eta, \varepsilon \to 0$. Hence we conclude that f is a solution of the system (82).

Uniqueness of weak solutions $f^{\eta,\varepsilon}$ just follows from the second statement in Proposition 4.2. To be more precise, let $f_1, f_2 \in L^\infty(0,T;(L^1_+ \cap L^\infty)(\mathbb{R}^d \times \mathbb{R}^d)) \cap \mathcal{C}([0,T], \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d))$ be the weak solutions to the system (82) with same initial data $f^0 \in (L^1_+ \cap L^\infty)(\mathbb{R}^d \times \mathbb{R}^d) \cap \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$. Then Proposition 4.2 yields that

$$\frac{d}{dt} d_1(f_1(t), f_2(t)) \le C(1 + \|f_1\|_{L^\infty_{x,v}}) d_1(f_1(t), f_2(t)), \quad \text{for} \quad t \in [0,T].$$

This completes the proof of Theorem 2.2. Let us obtain further properties of the solutions.

COROLLARY 4.1. *Let f be a weak solutions to* (82) *on the time-interval $[0,T)$. Then f is determined as the push-forward of the initial density through the flow map generated by $(v, F(f))$.*

PROOF. Consider the following flow map:

$$\begin{cases} \dfrac{d}{dt} X(t;s,x,v) = V(t;s,x,v), \\ \dfrac{d}{dt} V(t;s,x,v) = F(f)(X(t;s,x,v), V(t;s,x,v), t), \\ (X(s;s,x,v), V(s;s,x,v)) = (x,v), \end{cases} \tag{104}$$

for all $s, t \in [0,T]$. Then since f has compact support in v, the flow map (104) is well-defined using the local Lispchitzianity of the velocity fields as in the proof of Proposition 4.2. Standard duality arguments using suitable test functions in the weak formulation of the solution leads to

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} h(x,v) f(x,v,t) dxdv = \int_{\mathbb{R}^d\times\mathbb{R}^d} h(X(0;t,x,v), V(0;t,x,v)) f_0(x,v) dxdv,$$

for $t \in [0,T]$ for all test functions h. This yields that f is determined as the push-forward of the initial density through the flow map (104). □

COROLLARY 4.2. *Suppose that f is a solution to the system* (82) *satisfying $\|f\|_{L^\infty(0,T;L^1\cap L^\infty)} < \infty$ for some $T>0$. Then for any global measure solution μ to the same system* (82) *with finite first order moment in (x,v) we have*

$$d_1(f(t),\mu(t)) \le d_1(f_0,\mu_0)\exp\left(C\left(1+\|f\|_{L^\infty(0,T;L^1\cap L^\infty)}\right)t\right),$$

for all $t\in[0,T]$.

PROOF. The proof can be easily obtained from the observation made in the second part of the statement in Proposition 4.2 since we only require to use $L^1\cap L^\infty$-norm of one of solutions to the system (82). □

REMARK 4.2. *Our strategies can also be applied to various other communicate weights. For example, we can consider the linear combination of different communication weights:*

$$\tilde{\mathbf{1}}^m_{K(v)}(x-y) := \sum_{j=1}^m w_j \mathbf{1}_{K^j(v)}(x-y) \qquad \text{for some} \quad m\ge 1,$$

where $w_j\in\mathbb{R}_+$ and $K^j(v)$ satisfies the assumptions **(H1)**-**(H2)**.

5. Examples of Sensitivity Regions

5.1. A ball in $\mathbb{R}^d$. We consider the case $K(v)=B(0,r):=\{x\in\mathbb{R}^d : |x|\le r\}$ with $r>0$.

In that case, we choose $\Theta(v)=\partial B(0,r)$. We notice that it is enough to check the condition (**H2**)-(ii) for the existence of solutions and the mean-field limit since the influence set K is independent of the velocity variable. We can easily check that for $0<\varepsilon\le 1$

$$|\partial^\varepsilon B(0,r)| = \begin{cases} \alpha(d)(r+\varepsilon)^d \le \alpha(d)2^d\varepsilon^d \le C(d)\varepsilon & \text{if } r\le\varepsilon,\\ \alpha(d)\left((r+\varepsilon)^d-(r-\varepsilon)^d\right) = \alpha(d)2\varepsilon\sum_{k=0}^{d-1}(r+\varepsilon)^k(r-\varepsilon)^{d-1-k} \le C(d,r)\varepsilon & \text{if } \varepsilon<r, \end{cases}$$

where $\alpha(d)$ is the volume of unit ball in $\mathbb{R}^d$.

5.2. A ball with radius evolving regularly with respect to velocity in $\mathbb{R}^d$. Let $\tilde r:\mathbb{R}_+\to\mathbb{R}_+$ be bounded and Lipschitz function, and consider the case $K(v)=B(0,\tilde r(|v|))$.

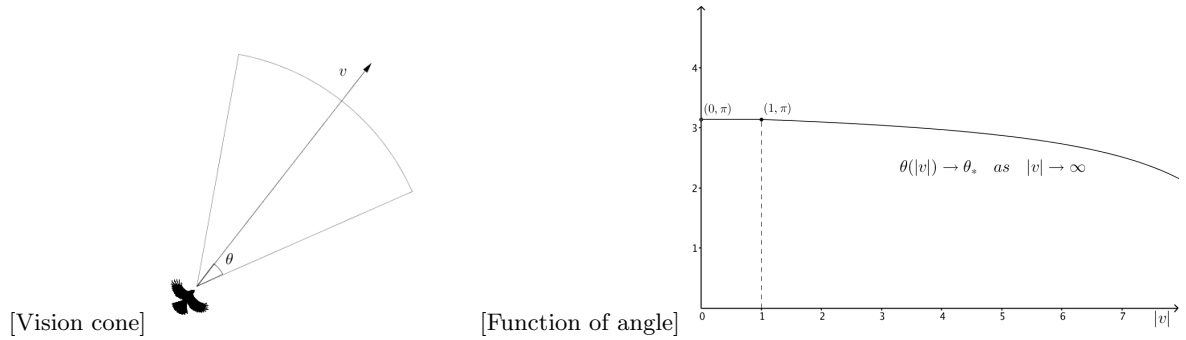
In this case, it is clear to show that $B(0,\tilde r(|v|))$ satisfies (**H1**) due to the boundedness of function $\tilde r$. Moreover, by choosing $\Theta(v)=\partial B(0,\tilde r(|v|))$, we can easily verify that $B(0,\tilde r(|v|))$ satisfies (**H2**)-(i) and (ii)(see also previous section). Concerning the conditions (**H2**)-(iii) and (iv), we notice that the symmetric difference $B(0,\tilde r(|v|))\Delta B(0,\tilde r(|w|))$ has a form of torus which can also be expressed by the enlargement of $\partial B(0,\tilde r(|v|))$. Thus it is very clear that $B(0,\tilde r(|v|))$ satisfies (**H2**)-(iii) and (iv) with the constant $C=\|\tilde r\|_{Lip}$.

5.3. A vision cone in $\mathbb{R}^d$ with $d=2,3$. We consider $K(v)=C(r,v,\theta(|v|))$ which is given by

$$C(r,v,\theta(|v|)) := \left\{x : |x|\le r \quad\text{and}\quad -\theta(|v|)\le \cos^{-1}\left(\frac{x\cdot v}{|x||v|}\right)\le\theta(|v|)\right\},$$

with $0<\theta(z)\in\mathcal{C}^\infty(\mathbb{R}_+)$ satisfying $\theta(z)=\pi$ for $0\le z\le 1$, $\theta(z)$ is decreasing for $z\ge 1$, and $\theta(z)\to\theta_*>0$ as $|z|\to+\infty$.

In the remaining part of this subsection, we focus on checking the conditions (**H1**) – (**H2**) for the case of vision cone.



[Vision cone] [Function of angle]

FIGURE 3. The vision cone is getting larger in covered angle as the speed gets smaller being a full ball at some chosen speed, for instance $|v| = 1$.

• (**H1**): It is obvious to get that $C(r, v, \theta(|v|)) \subseteq B(0, r)$ for all $v \in \mathbb{R}^d$ and $B(0, r)$ is compact set.

We now define the family set Θ as

$$\Theta(v) := \begin{cases} \partial C(r, v, \theta(|v|)) \cup R(v) & \text{if } |v| \in (1/2, 1), \\ \partial C(r, v, \theta(|v|)) & \text{else }, \end{cases}$$

where $R(v) = [a(v), b(v)]$ with

$$a(v) = -r\frac{v}{|v|} \quad , \quad b(v) = 2r(|v| - 1)\frac{v}{|v|}.$$

In other words, $R(v)$ is a segment with varies linearly from $\left\{-r\frac{v}{|v|}\right\}$ when $|v| = 1/2$, to the whole segment $\left[0, -r\frac{v}{|v|}\right]$ when $|v| = 1$. The family of sets $\Theta(v)$ interpolates continuously in some sense between the boundary of the ball for $|v| = 1/2$ to the limit of the boundary of the cone as $\theta(|v|) \searrow \pi$ for $|v| \searrow 1$ that includes the interval $R(v) = [-rv, 0]$, see middle picture in Figure 2.

• (**H2**)-(i): Due to definition of Θ, $\partial C(r, v, \theta(v)) \subset \Theta(v)$ for all $v \in \mathbb{R}^d$.

• (**H2**)-(ii): This is also satisfied since $\Theta(v)$ is made of surfaces and lines, whose total area is bounded uniformly in v. We give more detailed estimates below.

Remark first that for all $|v| \leq 1$ and $\varepsilon \in (0, 1)$

$$|R(v)^{\varepsilon,+}| \leq C\left(\varepsilon|R(v)| + \varepsilon^d\right) \leq C\varepsilon.$$

Since for any sets $A, B \subset \mathbb{R}^d$, $(A \cup B)^{\varepsilon,+} \subset A^{\varepsilon,+} \cup B^{\varepsilon,+}$, in order to check (**H2**)-(ii), it is enough to prove that

$$|(\partial C(r, v, \theta(|v|)))^{\varepsilon,+}| \leq C\varepsilon.$$

We consider the following two cases:

⋄ Case (i) $0 \leq \theta(|v|) < \pi/2$: By definition of θ, we have that $0 < \theta_* \leq \theta(|v|) < \pi/2$. In two dimensions, we consider the two cones $C^- := C(r - \varepsilon - \varepsilon/\sin(\theta(|v|)), v, \theta(|v|))$ and $C^+ := C(r + \varepsilon + \varepsilon/\sin(\theta(|v|)), v, \theta(|v|))$ to measure the ε-boundary of $C(r, v, \theta(|v|))$ as

$$\begin{aligned} |\partial^\varepsilon C(r, v, \theta(|v|))| \leq |C^+| - |C^-| &= \theta(|v|)\left(r + \varepsilon + \frac{\varepsilon}{\sin\theta(|v|)}\right)^2 - \theta(|v|)\left(r - \varepsilon - \frac{\varepsilon}{\sin\theta(|v|)}\right)^2 \\ &= 4r\varepsilon\theta(|v|)\left(1 + \frac{1}{\sin\theta(|v|)}\right) \leq 2r\pi\varepsilon\left(1 + \frac{1}{\sin\theta_*}\right), \end{aligned}$$

due to $C^{\varepsilon,+}(r, v, \theta(|v|)) \subseteq C^+$ and $C^- \subseteq C^{\varepsilon,-}(r, v, \theta(|v|))$, see Figure 4.

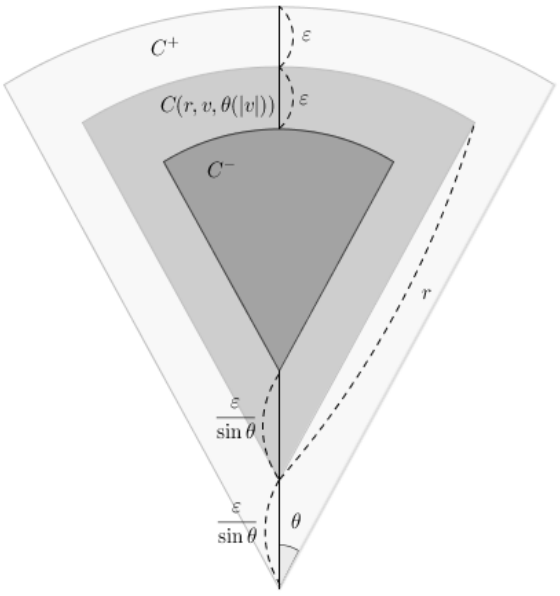


FIGURE 4. Case (i) $\theta(|v|) \leq \frac{\pi}{2}$.

In a similar way, for the three-dimensional case, we obtain

$$\begin{aligned}|\partial^\varepsilon C(r,v,\theta(|v|))| &\leq \frac{2\pi}{3}\left(1-\cos\theta(|v|)\right)\left[\left(r+\varepsilon+\frac{\varepsilon}{\sin\theta(|v|)}\right)^3-\left(r-\varepsilon-\frac{\varepsilon}{\sin\theta(|v|)}\right)^3\right]\\ &= \frac{4\pi}{3}\varepsilon\left(1-\cos\theta(|v|)\right)\left(1+\frac{1}{\sin\theta(|v|)}\right)\left(3r^2+\left(1+\frac{1}{\sin\theta(|v|)}\right)^2\right)\\ &\leq \frac{4\pi}{3}\varepsilon\left(1+\frac{1}{\sin\theta_*}\right)\left(3r^2+\left(1+\frac{1}{\sin\theta_*}\right)^2\right).\end{aligned}$$

Therefore we deduce that

$$\sup_{\left\{v\in\mathbb{R}^d\,:\,\theta(|v|)\leq\pi/2\right\}}|\partial^\varepsilon C(r,v,\theta(|v|))| \leq C\varepsilon$$

$\diamond$ Case (ii) $\pi \geq \theta(|v|) \geq \pi/2$: Using the same argument as above, in two dimensions, we find

$$C(r,v,\theta(|v|))^{\varepsilon,+} \subseteq C(r+\varepsilon,v,\theta(|v|)) \cup R_1 \cup \tilde{R}_1,$$

and

$$C(r-\varepsilon,v,\theta(|v|)) \setminus (R_2 \cup \tilde{R}_2) \subseteq C(r,v,\theta(|v|))^{\varepsilon,-},$$

where R_i and $\tilde{R}_i$, $i=1,2$ are given in Figure 5. Then this yields

$$\begin{aligned}|\partial^\varepsilon C(r,v,\theta(|v|))| &\leq |C(r+\varepsilon,v,\theta(|v|)) \cup R_1 \cup \tilde{R}_1| - |C(r-\varepsilon,v,\theta(|v|)) \setminus (R_2 \cup \tilde{R}_2)|\\ &\leq \theta(|v|)(r+\varepsilon)^2 + 2\varepsilon(r+\varepsilon) - \theta(|v|)(r-\varepsilon)^2 + 4\varepsilon r = 4r\varepsilon\theta(|v|) + 2\varepsilon(\varepsilon+3r) \leq C\varepsilon.\end{aligned}$$

Let us now consider $d=3$ and denote by $S(R_i)$ for $i=1,2$ the volume of the set obtained by rotating the rectangle R_i about the v-vector. We set R_0 a rectangle with length $r+\varepsilon$ and width ε. Then we notice that R_1 can be obtained by rotating R_0 with respect to one of edges of R_0. If we consider that R_0 is placed in the right of yz plane,

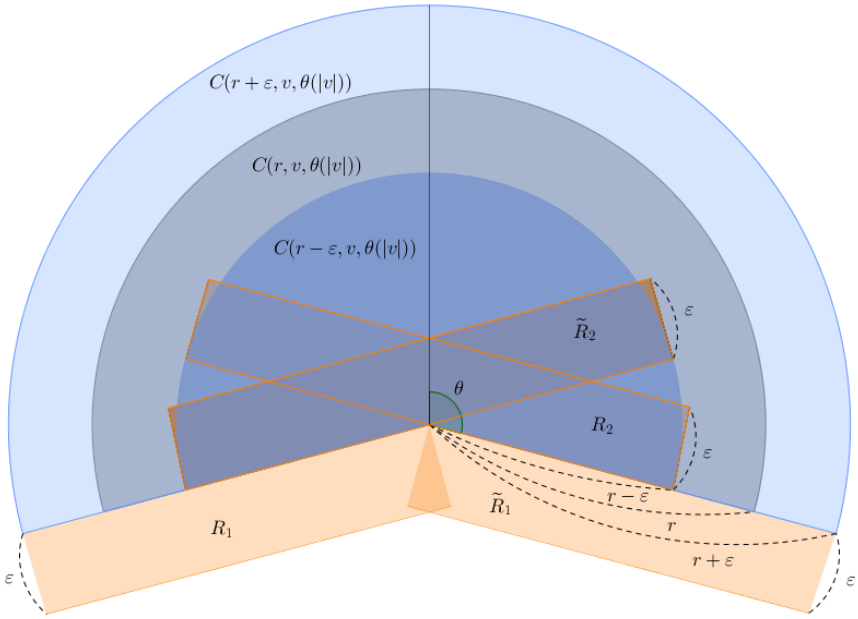


FIGURE 5. Case (ii) $\theta(|v|) \geq \frac{\pi}{2}$.

then it is clear that the volume of the solid obtained by rotating the R_0 about the y-axis is given by $2\pi \int_{R_0} y\, dy\, dz$. Then it follows from the definition of R_0 that there exists a $\tilde{\theta} \geq 0$ such that

$$S(R_1) \leq C \int_{R_0} |y \cos\tilde{\theta} + z \sin\tilde{\theta}|\, dy\, dz \leq \varepsilon \int_{R_0} |y|\, dy + \varepsilon |R_0| \leq C\varepsilon,$$

for some positive constant $C > 0$. Since this argument can be applied to other rectangles $\tilde{R}_1, R_2, \tilde{R}_2$, we get

$$S(R_1) + S(R_2) + S(\tilde{R}_1) + S(\tilde{R}_2) \leq C\varepsilon.$$

Then, in a similar fashion as the above, we find

$$|\partial^\varepsilon C(r, v, \theta(|v|))| \leq C\left((r+\varepsilon)^3 - (r-\varepsilon)^3\right) + C\varepsilon \leq 2C\varepsilon(3r^2 + \varepsilon^2) + C\varepsilon \leq C\varepsilon.$$

This concludes

$$\sup_{\{v \in \mathbb{R}^d\,:\, \theta(|v|) \geq \pi/2\}} |\partial^\varepsilon C(r, v, \theta(|v|))| \leq C\varepsilon.$$

Combining the estimates in Case (i) and (ii), we conclude that

$$\sup_{v \in \mathbb{R}^d} |\partial^\varepsilon C(r, v, \theta(|v|))| \leq C\varepsilon.$$

• (**H2**)-(*iii*) and (*iv*) : The vision cone is an interior domain, delimited by the union of some smooth surfaces. In that case, the symmetric difference between two different cone $K(v)\Delta K(w)$ is the set in-between the two boundaries $\partial K(v)$ and $\partial K(w)$. Since $\partial K(w) \subset \Theta(w)$ by (**H2**)-(*i*) if $\partial K(v) \subset \Theta(w)^{\varepsilon,+}$, we also have $K(v)\Delta K(w) \subset \Theta(w)^{\varepsilon,+}$, or equivalently in this particular case (**H2**)-(*iii*) is a consequence of (**H2**)-(*iv*). Thus we focus on the hypothesis (**H2**)-(*iv*) in the rest of the proof. For this we consider three steps as follows.

Step 1. The case $|v| = |w|$. We first check the properties in the simpler case where $|v| = |w|$. It allows to define the rotation $R_{v,w}$ with the only rotation (with an axis orthogonal to v and w if $d = 3$) that maps v to w. In view of the definition of Θ, it is clear that $\Theta(w) = R_{v,w}\Theta(v)$.

Now, inside the ball $B(0, r)$, the maximal displacement d_{max} made when we use $R_{v,w}$ is

$$d_{max} := \sup_{x\in B_r} |R_{v,w}x - x| = \left|\frac{r}{|v|}v - \frac{r}{|v|}w\right| = \frac{r}{|v|}|v - w|.$$

This implies that $\Theta(v) \subset \Theta(w)^{d_{max},+}$. When $|v| = |w| \geq 1/2$, $d_{max} \leq 2r|v - w|$ and we get

$$\Theta(v) \subset \Theta(w)^{2r|v-w|,+}.$$

This is also true when $|v| = |w| \leq 1/2$ because $\Theta(v) = \Theta(w) = \partial B(0, r)$ in that case. In fact, it is especially for that reason that we need a vision cone equals to the full ball for small velocities.

Step 2. The case $v = \lambda w$ with $\lambda > 0$. When both $|v|, |w| \leq 1$, it is clear to get $R(v) \subset R(w)^{2r|v-w|,+}$ since the function $b(v)$ is 2r-Lipschitz, and subsequently this implies $\Theta(v) \subset \Theta(w)^{2r|v-w|,+}$.

When $|v|, |w| \geq 1$, the set $\Theta(v)$ and $\Theta(w)$ are boundaries of cones with different angles, and we get

$$\Theta(v) \subset \Theta(w)^{r|\theta(v)-\theta(w)|,+} \subset \Theta(w)^{r\|\theta\|_{Lip}|v-w|,+}.$$

In the remaining cases, we can combine the two above case introducing $w' = \frac{v}{|v|} = \frac{w}{|w|}$. Since the three points v, w', w are aligned in that order, $|v - w| = |v - w'| + |w' - w|$, the following inclusions hold with $L = (2 \vee \|\theta\|_{Lip})r$:

$$\Theta(v) \subset \Theta(w')^{2r|v-w'|,+} \subset \Theta(w)^{2r|v-w'|+\|\theta\|_{Lip}r|w'-w|,+} \subset \Theta(w)^{L|v-w|,+}.$$

In fact, the above inclusions are valid in the case $|v| \leq 1 \leq |w|$, but the final inclusion is valid in both case.

Step 3. The general case. For any $v, w \in \mathbb{R}^d$, we introduce the middle point $\tilde{v}$ such that

$$|\tilde{v}| = |w| \quad \text{and} \quad \tilde{v} = \lambda v \quad \text{with} \quad \lambda \in (0, 1),$$

when $|v| \geq |w|$ and the point defined in the same way but exchanging the role of v and w when $|w| \geq |v|$. This middle point satisfies

$$|v - \tilde{v}| \leq |v - w|, \quad \text{and} \quad |\tilde{v} - w| \leq |v - w|.$$

Using the two steps above, we conclude that

$$\Theta(v) \subset \Theta(\tilde{v})^{L|v-\tilde{v}|,+} \subset \Theta(w)^{L|v-\tilde{v}|+2r|\tilde{v}-w|,+} \subset \Theta(w)^{2L|v-w|,+},$$

which concludes the verification of (**H2**)-(iv).

6. Further extensions

In this section, we present generalizations of main results with suitable modifications for much more general models:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (F(f)f) = 0, \qquad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d, \quad t > 0, \\ f(x, v, t)|_{t=0} = f_0(x, v) \qquad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d, \end{cases} \tag{105}$$

where the force term $F(f)$ can be chosen from two different types:

(106)
$$F(f)(x,v,t)=\begin{cases}\displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^d}\psi(x-y)\mathbf{1}_{K(v)}(x-y)h(w-v)f(y,w)\,dydw & \text{(Cucker-Smale type)}\\ \displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^d}\nabla_x\varphi(x-y)\mathbf{1}_{K(v)}(x-y)f(y,w)\,dydw & \text{(Attractive-Repulsive type).}\end{cases}$$

Here ψ, h, and φ denote the communication weight, velocity coupling, and interaction potential, respectively. We refer to [**20, 12, 1, 31, 31**].

Similarly as before, we introduce the particle approximation of the kinetic equation (105):

(107)
$$\begin{cases}\dot{X}_i(t)=V_i(t), \quad i=1,\cdots,N, \quad t>0,\\ \dot{V}_i(t)=\begin{cases}\displaystyle\sum_{j\neq i}m_j\psi(X_i-X_j)\mathbf{1}_{K(V_i)}(X_i-X_j)h(V_j-V_i) & \text{(Cucker-Smale type),}\\ \displaystyle\sum_{j\neq i}m_j\nabla_x\varphi(X_i-X_j)\mathbf{1}_{K(V_i)}(X_i-X_j) & \text{(Attractive-Repulsive type),}\end{cases}\\ (X_i(0),V_i(0))=:\left(X_i^0,V_i^0\right), \quad i=1,\cdots,N.\end{cases}$$

Then we define the differential inclusion system with respect to the above ODE system (107) and the empirical measure μ^N associated to a solution to this differential inclusion system in a similar fashion with (86) and (95).

By a similar strategy as in Theorems 2.2 and 2.3, we have the existence of weak solutions to the kinetic equations and mean-field limit.

THEOREM 6.1. *Given an initial data compactly supported in velocity satisfying* (90) *and assume that the sensitivity region set-valued function $K(v)$ satisfies* (**H1**)-(**H2**). *Then there exists a positive time $T>0$ such that the system* (105)-(106) *with $\psi,h,\nabla_x\varphi\in W^{1,\infty}(\mathbb{R}^d)$ admits a unique weak solution f in the sense of Definition* 2.4 *on the time interval $[0,T]$, which is also compactly supported in velocity. Moreover, f is determined as the push-forward of the initial density through the flow map generated by the Lipschitz velocity field $(v,F(f))$ in phase space and the solutions satisfies the stability estimate* (91).

Furthermore, we have the estimate of mean-field limit such that

$$d_1(f(t),\mu^N(t))\le e^{Ct}d_1(f(0),\mu^N(0)) \quad \textit{for all} \quad t\in[0,T],$$

there exists a constant C depending only on T, f_0, and d.

PROOF. Since the proof is very similar to that of Theorems 2.2 and 2.3 as mentioned before, we only give essential parts of the proof. Let us also consider the Cucker-Smale type force in (106) for $F(f)$ being the attractive-repulsive force treated analogously.

• *Support estimate of the density in velocity:* For the existence of weak solutions to the kinetic equation (105)-(106), we regularize the solution similarly as in (96) and define the forward $Z^{\eta,\varepsilon}(s):=(X^{\eta,\varepsilon}(s;0,x,v),V^{\eta,\varepsilon}(s;0,x,v))$ with regularization parameters η and ε satisfying the following ODE system:

$$\frac{dX^{\eta,\varepsilon}(s)}{ds}=V^{\eta,\varepsilon}(s),$$
$$\frac{dV^{\eta,\varepsilon}(s)}{ds}=F^{\eta,\varepsilon}(X^{\eta,\varepsilon},V^{\eta,\varepsilon},s)=\int_{\mathbb{R}^d\times\mathbb{R}^d}\psi(x-y)\mathbf{1}^{\eta,\varepsilon}_{K(V^{\eta,\varepsilon})}(X^{\eta,\varepsilon}-y)h(w-V^{\eta,\varepsilon})f^{\eta,\varepsilon}(y,w)\,dydw.$$

Note that the force term $F^{\eta,\varepsilon}$ is bounded from above by $\|(\psi,h)\|_{L^\infty}\|f_0\|_{L^1}$ and this yields

$$R_v^{\eta,\varepsilon}(t)\le R_v^0+\|(\psi,h)\|_{L^\infty}\|f_0\|_{L^1}t,$$

i.e., the support of $f^{\eta,\varepsilon}$ in velocity linearly increases in time t, however it does not depend on the regularization parameters η and ε.

• *Lipschitz continuity of the force field $F(f)$*: Let us assume that there exists a solution f to the equation (105)-(106) in the sense of Definition (2.4) and f has compact support in velocity. Then we obtain

$$
\begin{aligned}
&F(f)(x,v) - F(f)(\tilde{x},\tilde{v}) \\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d} \Big(\psi(x-y)\mathbf{1}_{K(v)}(x-y)h(w-v) - \psi(\tilde{x}-y)\mathbf{1}_{K(\tilde{v})}(\tilde{x}-y)h(w-\tilde{v})\Big) f(y,w)\,dydw \\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d} \left(\psi(x-y)-\psi(\tilde{x}-y)\right)\mathbf{1}_{K(v)}(x-y)h(w-v)f(y,w)\,dydw \\
&\quad + \int_{\mathbb{R}^d\times\mathbb{R}^d} \psi(\tilde{x}-y)\left(\mathbf{1}_{K(v)}(x-y) - \mathbf{1}_{K(\tilde{v})}(x-y)\right)h(w-v)f(y,w)\,dydw \\
&\quad + \int_{\mathbb{R}^d\times\mathbb{R}^d} \psi(\tilde{x}-y)\left(\mathbf{1}_{K(\tilde{v})}(x-y) - \mathbf{1}_{K(\tilde{v})}(\tilde{x}-y)\right)h(w-v)f(y,w)\,dydw \\
&\quad + \int_{\mathbb{R}^d\times\mathbb{R}^d} \psi(\tilde{x}-y)\mathbf{1}_{K(\tilde{v})}(\tilde{x}-y)\left(h(w-v)-h(w-\tilde{v})\right)f(y,w)\,dydw \\
&=: \sum_{i=1}^4 I_i.
\end{aligned}
$$

In comparsion to (93), we just need to estimate an additional term. By using the similar argument as in (92) together with the regularity of communication weight ψ and velocity coupling h, we estimate each term $I_i, i = 1,\cdots,4$ as

$$
\begin{aligned}
&I_1 \le \|(\nabla_x\psi, h)\|_{L^\infty}\|f_0\|_{L^1}|x-\tilde{x}|, \\
&I_2 \le C(\|\rho\|_{L^\infty} + \|f_0\|_{L^1})(\psi,h)\|_{L^\infty}|v-\tilde{v}|, \\
&I_3 \le C(\|\rho\|_{L^\infty} + \|f_0\|_{L^1})(\psi,h)\|_{L^\infty}|x-\tilde{x}|, \\
&I_4 \le \|(\psi, \nabla_x h)\|_{L^\infty}\|f_0\|_{L^1}|v-\tilde{v}|,
\end{aligned}
$$

where (**H1**)-(**H2**) have been used. Hence we have

$$
|F(f)(x,v) - F(f)(\tilde{x},\tilde{v})| \le C|(x,v)-(\tilde{x},\tilde{v})|,
$$

and this concludes the proof. □

REMARK 6.1. *Our strategy is also applicable to first-order swarming models with sensing zones* [**11**]. *More precisely, let $\rho = \rho(x,t)$ be the probability density of individuals at position $x \in \mathbb{R}^d$ and time $t > 0$ satisfying*

$$
\begin{cases}
\partial_t\rho + \nabla_x\cdot(\rho u) = 0, & x\in\mathbb{R}^d, \quad t>0, \\
u(x,t) := \displaystyle\int_{\mathbb{R}^d} \mathbf{1}_{K(w(x))}(x-y)\nabla_x\varphi(x-y)\rho(y)\,dy, & x\in\mathbb{R}^d, \quad t>0, \\
\rho(x,t)|_{t=0} = \rho_0(x), & x\in\mathbb{R}^d,
\end{cases} \tag{108}
$$

where $\varphi \in W^{1,\infty}(\mathbb{R}^d)$ and w is a given orientational field satisfying $w \in W^{1,\infty}(\mathbb{R}^d)$ and $|w| \ge w_0 > 0$. Here the velocity field u is non-locally computed in terms of the density ρ. Suppose the sensing zone $K(w(x))$ satisfies our assumptions (**H1**)-(**H2**)*, then the continuity equation* (108) *is well approximated by*

$$
\begin{cases}
\dot{X}_i(t) = \displaystyle\sum_{j\neq i} m_j\nabla_x\varphi(X_i-X_j)\mathbf{1}_{K(w(X_i))}(X_i-X_j), \\
X_i(0) = X_i^0, \qquad i = 1,\cdots,N.
\end{cases}
$$

via the associated differential inclusion system in the sense of Theorem 6.1. Note that taking K to be $C(r,w(x),\theta_0)$ with $\theta_0 \in (0,\pi]$ satisfies these conditions.

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CHAPTER 3

Kinetic model with vision geometrical constraints: truncated noises

1. Introduction

In this paper, we are interested in the propagation of chaos for stochastic integral equations (in short, SIEs) describing collective behavior of individuals with vision geometrical constraints. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a probability space endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$. Here Ω is the random set, $\mathbb{P}$ and $\mathcal{F}$ are measure and σ-algebra on that set, respectively. On that probability space, let $\{B_t^i\}_{i=1}^N$ be N independent d-dimensional Brownian motions. In this setting, our main SIEs are given by

(1.109)
$$\begin{cases} X_t^i = X_0^i + \displaystyle\int_0^t V_s^i \, ds, \quad i = 1, \cdots, N, \quad t \geq 0, \\ V_s^i = V_0^i + \displaystyle\int_0^t F[\mu_s^N](X_s^i, V_s^i) \, ds + \sqrt{2\sigma} \int_0^t \mathcal{R}(V_s^i) \, dB_s^i, \quad \mu_s^N := \dfrac{1}{N} \displaystyle\sum_{i=1}^N \delta_{(X_s^i, V_s^i)}, \end{cases}$$

where $\mathcal{R} \in \mathcal{C}^2(\mathbb{R}^d)$ is a truncation function compactly supported in $\mathbb{B}_{V_{max}}^d := \{x \in \mathbb{R}^d : |x| < V_m\}$. Here X_t^i and V_t^i are position and velocity of i-th particle at time $t \geq 0$, respectively, and $F[\mu]$ denotes the velocity alignment force given by

$$F[\mu](x, v) := \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_{K(v)}(y - x)(w - v)\mu(dy, dw) \quad \text{for} \quad \mu \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d),$$

where $\mathbf{1}_{K(v)}$ is the indicator function on the vision set $K(v)$, which is called a communication weight.

The system (1.109) without the noise and vision geometrical constrains is refereed as the Cucker-Smale model [**16**], which is proposed to describe velocity alignment behaviors of individuals, such as flocks of birds or schools of fish, etc. Due to the presence of vision geometrical constraints in (1.109), which comes from $\mathbf{1}_K$ in the interaction force F, the individuals are only interacting with others in the velocity dependent region K and trying to align their velocities with others. The system (1.109) has a diffusion however it smoothly degenerates when the speed of individuals increases to V_m. This enables us to consider the situation where the particles moving with higher velocities are less affected by the noises. From this consideration, we can show that the uniform boundedness of speed in time for the system (1.109) almost surely under suitable assumption on the initial data, see Lemma 3.2. Thus, the system have a generic property of the velocity alignment models that the speed of individuals cannot be too high.

In the original Cucker-Smale model [**16**], a bounded Lipschitz communication weight is considered and obtained the flocking estimate under some condition on the initial configurations. In [**31**], a normalized communication weight is taken into account to deal with interactions between individuals through not only the distance between them but also their relative distance. Recently, in [**1, 7, 9, 32**], the collision avoidance between individuals is observed by considering a singular communication weight. The influence of noises in Cucker-Smale type models showing flocking or non-flocking behavior is studied in [**2, 12, 18, 21, 33**]. We refer the reader to [**7, 9**] and the references therein for general survey of flocking models.

Mean field limit and propagation of chaos are some challenging topics in the analysis of partial differential equations(in short, PDEs), which arise in the context of interacting multi-agent systems. Indeed, it provides a rigorous justification of the continuum models. For the original Cucker-Smale model which has regular force fields, the rigorous mean-field limit, existence of weak

solutions are studied in [**4, 11, 22**], see also [**14, 12, 21**] for more general types of equations. More recently, Vlasov systems with bounded kernels are taken into account in [**28**], however, their result cannot be directly applied for the system (1.109) since the interaction between particles not only depend on their relative distance but also on some topological considerations. For the particle system with singular or non-Lipschitz kernels, the rigorous derivation of continuum descriptions is studied in [**3, 5, 17, 27, 29**].

For the deterministic case, i.e., the system (1.109) without noises, the rigorous derivation of mean-field limit model is studied in [**6**] in the large particle limit $N \to \infty$. Since the force fields in (1.109) are not continuous, the differential inclusion system together with the extended boundary set is introduced, and the quantitative error estimate between solutions to that system and weak solutions to the limiting kinetic equation is obtained. Our main purpose of this paper is to extend the result in [**6**] to the stochastic case under the same assumption on the sensitivity regions K. To be more precise, we will show that the N interacting processes (X_t^i, V_t^i) of the system (1.109) respectively well approximates as $N \to \infty$ the processes (Y_t^i, W_t^i) to the following kinetic McKean-Vlasov type system:

$$\begin{cases} Y_t^i = Y_0^i + \displaystyle\int_0^t W_s^i \, ds, \quad i = 1, \cdots, N, \quad t > 0, \\ W_t^i = W_0^i + \displaystyle\int_0^t F[f_s](Y_s^i, W_s^i) \, ds + \sqrt{2\sigma} \int_0^t \mathcal{R}(W_s^i) \, dB_s^i, \quad \mathcal{L}(Y_t^i, W_t^i) = f_t, \\ \left(Y_0^i, W_0^i\right) = \left(X_0^i, V_0^i\right), \quad i = 1, \cdots, N. \end{cases} \tag{1.110}$$

Then, by applying Itô formula, we find that the probability density function f_t is governed by

$$\partial_t f_t + v \cdot \nabla_x f_t + \nabla_v \cdot (F[f_t] f_t) = \sigma \Delta_v \left(\mathcal{R}^2(v) f_t \right), \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d, \quad t > 0, \tag{1.111}$$

with the initial data

$$f_t(x, v)|_{t=0} =: f_0(x, v), \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{1.112}$$

We emphasize that types of SIEs (1.109), SIEs with discontinuous force fields and truncated diffusion, have not been treated so far to the best of authors' knowledge. It is also worth noticing that since the force fields are discontinuous, the existence of solutions to the SIEs (1.109) cannot be obtained by the classical theory of SIEs. Furthermore, it is not possible to use Girsanov's theorem to get a weak notion of solutions as in [**18**] due to the consideration of additive noises. Thus our strategy is to replace the SIEs (1.109) by a system of stochastic integral inclusion system. For this, we need to use the generalized boundary set $\widetilde{\partial} K$ defined in Definition 2.2 below as in [**6**] where the differential inclusion system is introduced for the existence of particle system without noises. To the best of authors' knowledge, there exist no available general integral inclusion theories in the stochastic framework as convenient as Fillipov's theory [**19**] in the deterministic setting. Thus, in the current work, we introduce the corresponding stochastic inclusion system to (1.109) and give details of constructing the global-in-time existence of solutions to that system.

Notations.- We introduce several notations used throughout the paper. $|\cdot|$ and $\langle \cdot, \cdot \rangle$ denote the Euclidean distance and the standard inner product on $\mathbb{R}^d$, respectively. We also use the notation $|\cdot|$ for the Lebesgue measure of some set or the cardinal of finite index sets when there is no confusion. $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ and $\mathcal{P}_p(\mathbb{R}^d \times \mathbb{R}^d)$ stand for the sets of all probability measures and probability measures with finite moments of order $p \in [1, \infty)$ on $\mathbb{R}^d \times \mathbb{R}^d$, respectively. For a function $f(x, v)$, $\|f\|_{L^p}$ represents the usual $L^p(\mathbb{R}^d \times \mathbb{R}^d)$-norm. For $p \in [1, \infty]$ and $T > 0$, $L^p(0, T; E)$ is the set of the L^p functions from an interval $(0, T)$ to a Banach space E. We denote by C a generic positive constant. For a set $A \subset \mathbb{R}^d$, $\mathrm{int}(A)$ and $\mathrm{cl}(A)$ represent the interior and closure of A, respectively, and $Tr(M)$ denotes the trace of a matrix $M \in \mathbb{R}^d \times \mathbb{R}^d$.

Organization of the paper.- In Section 2, we discuss our main mathematical tool, Wasserstein distance. We also present our main assumptions on the sensitivity regions and main results on the existence of solutions to the SIEs and PDEs, and the propagation of chaos. As mentioned before, the deterministic case is already studied in [**6**], thus we provide several examples of sensitivity sets K satisfying our main assumptions **(H1)**-**(H2)** below without giving the details of proof. In Section 2, we present a global-in-time existence of solutions to corresponding stochastic integral inclusion system (2.114) to the SIEs (1.109). In Section 3, we show the existence and uniqueness of the PDE and its associated nonlinear SIEs. Finally, in Section 4, we provide the details of proof for the propagation of chaos for the systems (1.109).

2. Preliminaries and main results

2.1. Wasserstein distance. In this part, we introduce the Wasserstein distance, which is our main mathematical tool to estimate the convergence of the empirical measure for the particle system to the probability measure in law. We also discuss an issue of making use of 1-Wasserstein distance for our case, and we finally recall from [**14**] the estimate of convergence rate of an empirical measure in Wasserstein distance.

For $p \geq 1$ and $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^n)$, the Wasserstein distance is defined by

$$\mathcal{W}_p^p(\mu,\nu) := \inf_{\xi \in \Gamma(\mu,\nu)} \int_{\mathbb{R}^n \times \mathbb{R}^n} |x-y|^p \xi(dx,dy) = \inf_{(X,Y)\sim(\mu,\nu)} \mathbb{E}\left[|X-Y|^p\right],$$

where $\Gamma(\mu,\nu)$ is the set of all probability measures on $\mathbb{R}^n \times \mathbb{R}^n$ with first and second marginals μ and ν, respectively, i.e.,

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \phi(x) d\xi(dx,dy) = \int_{\mathbb{R}^n} \phi(x)\mu(dx) \quad \text{and} \quad \int_{\mathbb{R}^n\times\mathbb{R}^n} \phi(y) d\xi(dx,dy) = \int_{\mathbb{R}^n} \phi(y)\nu(dy),$$

for all $\phi \in \mathcal{C}_b(\mathbb{R}^n)$, and (X,Y) are all possible couples of random variables with μ and ν as respective laws. Note that when $p=1$, the 1-Wasserstein distance is equivalent to the bounded Lipschitz distance:

$$\mathcal{W}_1(\mu,\nu) = \sup\left\{ \left| \int_{\mathbb{R}^n} \varphi(x)\mu(dx) - \int_{\mathbb{R}^n} \varphi(x)\nu(dx) \right| : \varphi \in \mathrm{Lip}(\mathbb{R}^n),\, \mathrm{Lip}(\varphi) \leq 1 \right\},$$

where $\mathrm{Lip}(\mathbb{R}^n)$ and $\mathrm{Lip}(\varphi)$ represent the set of Lipschitz functions on $\mathbb{R}^n$ and the Lipschitz constant of a function φ, respectively.

In the current work, it seems convenient to consider p-Wasserstein distance with $p \in 2\mathbb{N}$ due to the multiplicative noises. However it has already been pointed out in [**6**, Remark 3.1] by the authors and their collaborators that our strategy does not work in Wasserstein distance of order p with $p \in (1,\infty)$, thus the make use of $\mathcal{W}_1$ or $\mathcal{W}_\infty$ is essential in the framework because of the form of force fields, see [**15**]. For those reasons, we introduce a modified Wasserstein 1 distance $\mathcal{W}_1^\gamma$ as

$$\mathcal{W}_1^\gamma(\mu,\nu) := \inf_{(X,Y)\sim(\mu,\nu)} \mathbb{E}\left[\sqrt{\gamma^2 + |X-Y|^2}\right].$$

Note that $\mathcal{W}_1^\gamma$ is not a metric. Employing that quantity enables us to establish stability like estimates for any $\gamma > 0$ for the both diffusion and the singularity of the interaction kernel. Finally, by letting $\gamma \to 0$, we provide the results in the desired 1-Wasserstein metric.

Before closing this subsection, we recall from [**14**, Theorem 1] the result on the rate of convergence of the empirical measure in Wasserstein distance, which will be crucially used to obtain our result on propagation of chaos.

PROPOSITION 2.1. *Let $\mu \in \mathcal{P}(\mathbb{R}^n)$ and $p \geq 1$. Suppose that $M_q(\mu) := \int_{\mathbb{R}^n} |x|^q \mu(dx) < \infty$ for some $q > p$. Then we have*

$$\mathbb{E}\left[\mathcal{W}_p^p(\mu^N,\mu)\right] \leq C M_q^{p/q}(\mu) \begin{cases} N^{-1/2} + N^{-(q-p)/q} & \text{if } 2p > n \text{ and } q \neq 2p, \\ N^{-1/2}\log(1+N) + N^{-(q-p)/q} & \text{if } 2p = n \text{ and } q \neq 2p, \\ N^{-p/n} + N^{-(q-p)/q} & \text{if } 2p < n \text{ and } q \neq n/(n-p), \end{cases}$$

where $\mu^N = \frac{1}{N}\sum_{k=1}^N \delta_{X^k}$ and $C > 0$ depends only on p, d and q.

2.2. Sharp sensitivity regions. In this subsection, we introduce several notations for the set K and its properties. We also discuss our main assumptions for K.

DEFINITION 2.1. *Let $K \subset \mathbb{R}^d$ be a non-empty compact set and $\varepsilon > 0$. We define the ε-boundary of K by:*

$$\partial^\varepsilon K := \{x + y \mid x \in \partial K, |y| \leq \varepsilon\},$$

and also the ε-enlargement(resp. ε-reduction) $K^{\varepsilon,+}$ (resp. $K^{\varepsilon,-}$) by

$$K^{\varepsilon,+} := K \cup \partial^\varepsilon K \quad \text{and} \quad K^{\varepsilon,-} := K \setminus \partial^\varepsilon K$$

Note that $\partial^\varepsilon K = K^{\varepsilon,+} \setminus K^{\varepsilon,-}$ and $(\partial^\varepsilon K)^{\delta,+} \subset \partial^{\varepsilon+\delta} K$ for $\varepsilon > 0$ and $\delta > 0$.

We next provide the so called *rope argument* used in [**16, 18**] for the propagation of chaos of Vlasov-Poission or Vlasov-Poisson-Fokker-Planck systems in one dimension whose proof can be found in [**6**, Lemma 2.2].

LEMMA 2.1. *For $K \subset \mathbb{R}$ For $x_1, y_1, x_2, y_2 \in \mathbb{R}^d$, we have*

$$|\mathbf{1}_K(y_1 - x_1) - \mathbf{1}_K(y_2 - x_2)| \leq \mathbf{1}_{\partial^{2|x_1-x_2|}K}(y_1 - x_1) + \mathbf{1}_{\partial^{2|y_1-y_2|}K}(y_1 - x_1).$$

We now give our main assumption on the compact set $K(v)$:

(**H1**) $K(\cdot)$ is globally compact, i.e., $K(v)$ is compact and there exists a compact set $\mathcal{K}$ such that $K(v) \subseteq \mathcal{K}, \forall\, v \in \mathbb{R}^d$.

(**H2**) There exist a family of closed sets $v \mapsto \Theta(v)$ and a constant C such that:
- (i) $\partial K(v) \subset \Theta(v)$, for all $v \in \mathbb{R}^d$,
- (ii) $|\Theta(v)^{\varepsilon,+}| \leq C\varepsilon$, for all $\varepsilon \in (0,1)$,
- (iii) $K(v)\Delta K(w) \subset \Theta(v)^{C|v-w|,+}$ for $v, w \in \mathbb{R}^d$,
- (iv) $\Theta(w) \subset \Theta(v)^{C|v-w|,+}$ for $v, w \in \mathbb{R}^d$,

where Δ denotes the symmetric difference, i.e., $A\Delta B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$ for $A, B \subset \mathbb{R}^d$.

Before giving some comments on the set-valued function $\Theta(v)$ given in (**H2**), we introduce a generalized boundary of the set $\tilde{\partial} K(v)$ in the definition below.

DEFINITION 2.2. *For $v \in \mathbb{R}^d$, we define the generalized boundary set of $K(v)$, $\widetilde{\partial} K(v)$ by*

$$\widetilde{\partial} K(v) := \left\{ x \in \mathbb{R}^d : \mathcal{A}(x, v) = [0, 1] \right\},$$

where $\mathcal{A} : \mathbb{R}^d \times \mathbb{R}^d \to [0, 1]$ is given by

$$\mathcal{A}(x, v) := Conv\left\{ \alpha \in [0, 1] : \exists\, (x^n, v^n) \to (x, v) \text{ such that } \mathbf{1}_{K(v^n)}(x^n) \to \alpha \right\} \quad \text{for} \quad (x, v) \in \mathbb{R}^{2d}.$$

Introducing the above boundary set $\widetilde{\partial} K(v)$ is required to give a sense to the time-derivative of the particle trajectories when they cross the boundary of $K(v)$. The set-valued function $\Theta(v)$ is a kind of a regularization of the set valued function $\widetilde{\partial} K(v)$. Note that there are inclusion relations for the sets ∂K, $\widetilde{\partial} K$, and Θ:

$$\partial K(v) \subset \widetilde{\partial} K(v) \subset \Theta(v) \quad \text{for} \quad v \in \mathbb{R}^d.$$

We refer to [**6**] for details of its proof. We next provide several examples of sets satisfying the above conditions (**H1**)-(**H2**) that are studied in [**6**, Section 5].

(Example 1) A fixed closed ball in $\mathbb{R}^d$:

$$K(v) = \mathrm{cl}(\mathbb{B}^d_r) \quad \text{with} \quad r > 0.$$

(Example 2) A closed ball with radius evolving regularly with respect to velocity in $\mathbb{R}^d$:

$$K(v) = \mathrm{cl}(\mathbb{B}^d_{r(|v|)}) \quad \text{with a bounded Lipschitz function } r : \mathbb{R}_+ \to \mathbb{R}_+.$$

(Example 3) A vision cone in $\mathbb{R}^d$ with $d = 2, 3$:

$$K(v) = \left\{ x \,:\, |x| \leq r \quad \text{and} \quad -\theta(|v|) \leq \cos^{-1}\left(\frac{x \cdot v}{|x||v|} \right) \leq \theta(|v|) \right\}$$

with $0 < \theta(z) \in \mathcal{C}^\infty(\mathbb{R}_+)$ satisfying $\theta(z) = \pi$ for $0 \leq z \leq 1$, $\theta(z)$ is decreasing for $z \geq 1$ and $\theta(z) \to \theta_* > 0$ as $|z| \to +\infty$.

For the first two examples, we can choose the generalized boundary set $\Theta(v)$ as $\Theta(v) = \partial K(v)$. For the third one, if we define the family set $\Theta(v)$ as

$$\Theta(v) := \begin{cases} \partial C(r, v, \theta(|v|)) \cup R(v) & \text{if } |v| \in (1/2, 1), \\ \partial C(r, v, \theta(|v|)) & \text{else,} \end{cases} \tag{2.113}$$

where $R(v) = [a(v), b(v)]$ with

$$a(v) = -r\frac{v}{|v|} \quad , \quad b(v) = 2r(|v| - 1)\frac{v}{|v|},$$

then the set $\Theta(v)$ defined in (2.113) satisfies the condition (**H2**).

2.3. Main results. In this part, we present our main results of this paper. First, we establish the global-in-time existence of solutions to the stochastic particles system.

THEOREM 2.1. *There exists some stochastic basis $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}_t})_{t\geq 0}, \overline{\mathbb{P}})$, and on this basis a dN-dimensional Brownian motion $(B_t^1, \cdots, B_t^N)_{t\geq 0}$ of $2dN$ dimensional $(X_0^1, V_0^1, \cdots, X_0^N, V_0^N)$ random variables with law $f_0^N \in \mathcal{P}_2(\mathbb{R}^{2dN})$ and some $\mathcal{F}_t$-adapted $2dN$ dimensional $(X_t^1, V_t^1, \cdots, X_t^N, V_t^N)_{t\geq 0}$ process solution to the following inclusion integral equation*

$$\begin{cases} X_t^i = X_0^i + \displaystyle\int_0^t V_s^i \, ds, \quad i = 1, \cdots, N, \quad t > 0, \\ V_s^i = V_0^i + \dfrac{1}{N}\displaystyle\sum_{j=1}^N \int_0^t \alpha_s^{i,j}(V_s^j - V_s^i)\, ds + \sqrt{2\sigma}\int_0^t \mathcal{R}(V_s^i)\, dB_s^i, \\ \alpha_s^{i,j} \in \mathcal{I}(X_s^j - X_s^i, V_s^i), \quad \forall\, s \geq 0, \end{cases} \tag{2.114}$$

where $\mathcal{I}$ is the set valued function defined as

$$\mathcal{I}(x,v) = \begin{cases} \{1\} & \text{if } x \in int(K(v)) \setminus \widetilde{\partial} K(v), \\ \{0\} & \text{if } x \in K(v)^c \setminus \widetilde{\partial} K(v), \\ [0,1] & \text{if } x \in \widetilde{\partial} K(v). \end{cases}$$

For notational simplicity, we define the set valued function $\tilde{F}[\mu]$ as

$$\tilde{F}[\mu](x,v) := \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{I}(y-x, v)(w-v)\mu(dydw),$$

which makes sense at least when μ is an atomic measure.

We next state the theorem on the existence of solutions to the nonlinear SIEs (1.110) and its associated PDEs (1.111).

THEOREM 2.2. *Let f_0 be a probability measure on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying $f_0 \in (L^1 \cap L^\infty)(\mathbb{R}^d \times \mathbb{R}^d) \cap \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$ and let $(X_0^i, V_0^i)_{i=1,\cdots,N}$ be N independent variables with law f_0. Suppose the initial data f_0 is compactly supported in velocity in $\mathbb{B}^d_{V_{max}}$. Then, for some $T > 0$, there exists a unique strong solution $(Y_t^i, W_t^i)_{i=1,\cdots,N}$ to the nonlinear SIEs (1.110) and a unique $f \in L^\infty(0,T; (L^1 \cap L^\infty)(\mathbb{R}^d \times \mathbb{R}^d)) \cap \mathcal{C}([0,T]; \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d))$ weak solution to (1.111) which is the law of the process solution to (1.110) and compactly supported in velocity in $\mathbb{B}_{V_m}$ up to time $T > 0$. Moreover, if $\tilde{f} \in L^\infty(0,T; \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d))$ is another solution starting from $\tilde{f}_0 \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$ then*

$$\mathcal{W}_1(f_t, \tilde{f}_t) \leq \mathcal{W}_1(f_0, \tilde{f}_0) e^{\int_0^t \|f_s\|_{L^1 \cap L^\infty} \, ds}.$$

REMARK 2.1. *A straightforward computation yields that for $q \geq 1$*

$$\frac{d}{dt}\int_{\mathbb{R}^d \times \mathbb{R}^d} |x|^q f \, dxdv \leq q \int_{\mathbb{R}^d \times \mathbb{R}^d} |x|^{q-1}|v| f \, dxdv \leq C \int_{\mathbb{R}^d \times \mathbb{R}^d} |x|^{q-1} f \, dxdv \leq C \int_{\mathbb{R}^d \times \mathbb{R}^d} |x|^q f \, dxdv + C.$$

Thus the q-th moment of f is estimated as

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |x|^q f_t \, dxdv \leq C \int_{\mathbb{R}^d \times \mathbb{R}^d} |x|^q f_0 \, dxdv + C.$$

Our final result is on the propagation of chaos. For this, we recall the definition of a chaotic sequence and remark the reformulation of the notation of the propagation of chaos in terms of coupling. We refer to [**25**] for more details on that.

DEFINITION 2.3. *Let f be a probability on $\mathbb{R}^{2d}$. A sequence $\left((X_i^N, V_i^N)_{i\leq N}\right)_{N\in\mathbb{N}}$ of exchangeable random variables is f-chaotic if*

$$\mu^N := \frac{1}{N}\sum_{i=1}^N \delta_{(X_i^N, V_i^N)} \xrightarrow{\mathcal{L}} f \quad as \quad N \to \infty.$$

REMARK 2.2. *Assume that $f \in \mathcal{P}_p(\mathbb{R}^{2d})$ endowed with the $\mathcal{W}_p$ metric. Then a sufficient condition for the sequence $\left((X_i^N, V_i^N)_{i\leq N}\right)_{N\in\mathbb{N}}$ to be f-chaotic is*

$$\mathbb{E}\left[\mathcal{W}_p(\mu^N, f)\right] \to 0 \quad as \quad N \to \infty.$$

THEOREM 2.3. *Suppose that the set-valued function K satisfies* (**H1**) *and* (**H2**)*, and let f be a solution to the system* (1.111)-(1.112) *up to time $T > 0$, such that $f \in L^\infty(0,T;(L^1 \cap L^\infty)(\mathbb{R}^d \times \mathbb{R}^d)) \cap \mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d))$ with initial data $f_0 \in (L^1 \cap L^\infty)(\mathbb{R}^d \times \mathbb{R}^d) \cap \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$ compactly supported in velocity in $\mathbb{B}_{V_m}$. Let $(X_0^i, V_0^i)_{i=1,\cdots,N}$ be N independent variables with law f_0. Furthermore, we assume that $|x|^q f_0 \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$ for $q > 1$. Then there exists a constant $C > 0$ depending only on V_m, f_0, q and T such that*

$$\mathbb{E}\left[\mathcal{W}_1(\mu_t^N, f_t)\right] \le C \begin{cases} N^{-1/2} + N^{-(q-1)/q} & \text{if } 2 > d \text{ and } q \neq 2, \\ N^{-1/2}\log(1+N) + N^{-(q-1)/q} & \text{if } 2 = d \text{ and } q \neq 2, \\ N^{-1/d} + N^{-(q-1)/q} & \text{if } 2 < d \text{ and } q \neq d/(d-1), \end{cases}$$

for all $t \in [0,T]$, where $\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{(X_t^i, V_t^i)}$ is the empirical measure associated to the particle system (2.114).

3. Interacting stochastic particle system: Proof of Theorem 1.1

In this section, we construct a global-in-time solution to the stochastic integral inclusion system (2.114) which corresponds to the system (1.109) . For this, we regularize the indicator function with respect to phase space (x, v):

$$\mathbf{1}_{K(v)}^{\eta,\varepsilon}(x) = \mathbf{1}_K *_{(x,v)} (\phi_\varepsilon, \psi_\eta) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_{K(v-w)}(x-y)\phi_\eta(w)\psi_\varepsilon(y)\, dydw,$$

where $\phi_\eta(w) := (1/\eta^d)\phi(w/\eta)$ with

$$\phi(v) = \phi(-v) \ge 0, \quad \phi \in \mathcal{C}_0^\infty(\mathbb{R}^d), \quad \text{supp } \phi \subset B(0,1), \quad \text{and} \quad \int_{\mathbb{R}^d} \phi(v)\, dv = 1.$$

Using this newly defined function $\mathbf{1}_K^{\eta,\varepsilon}$, we define $F^{\eta,\varepsilon}[\mu]$ as

$$F^{\eta,\varepsilon}[\mu](x,v) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_{K(v)}^{\eta,\varepsilon}(y-x)(w-v)\mu(dy,dw). \tag{3.115}$$

We next recall from [**6**, Lemma 4.2] some some basic properties of the regularized indicator function in the lemma below.

LEMMA 3.1. *(i) For all $\varepsilon > 0$, it holds*

$$\int |\mathbf{1}_K^\varepsilon(x) - \mathbf{1}_K(x)|\, dx \le |\partial^{2\varepsilon} K|. \tag{3.116}$$

(ii) For all $x \in \mathcal{O}$ and $0 < \eta \le 1$, it holds

$$\int_{\mathbb{R}^d} \left|\mathbf{1}_{K(w(x))}^{\eta,\varepsilon}(y-x) - \mathbf{1}_{K(w(x))}^{\varepsilon}(y-x)\right| dy \le C\eta, \tag{3.117}$$

where C is a positive constant independent of ε and η.

We now consider the following SIEs with smooth force fields and diffusion:

(3.118)

$$X_t^{i,\eta,\varepsilon} = X_0^i + \int_0^t V_s^{i,\eta,\varepsilon}\, ds, \quad i = 1,\cdots,N, \quad t > 0,$$

$$V_t^{i,\eta,\varepsilon} = V_0^i + \frac{1}{N}\sum_{j=1}^N \int_0^t \mathbf{1}_{K(V_s^{i,\eta,\varepsilon})}^{\eta,\varepsilon}(X_s^{j,\eta,\varepsilon} - X_s^{i,\eta,\varepsilon})(V_s^{j,\eta,\varepsilon} - V_s^{i,\eta,\varepsilon})\, ds + \sqrt{2\sigma}\int_0^t \mathcal{R}(V_s^{i,\eta,\varepsilon})\, dB_s^i.$$

Then it is clear that strong existence and uniqueness hold for equation (3.118). Let us denote by

$$(\mathcal{X}_t^{N,\eta,\varepsilon}, \mathcal{V}_t^{N,\eta,\varepsilon}) := (X_t^{i,\eta,\varepsilon}, V_t^{i,\eta,\varepsilon})_{i=1,\cdots,N} \quad \text{and} \quad \mu_t^{N,\eta,\varepsilon} := \frac{1}{N}\sum_{j=1}^N \delta_{(X_t^{j,\eta,\varepsilon}, V_t^{j,\eta,\varepsilon})}.$$

In the lemma below, we estimate the upper bound of the velocity in (3.118) which is useful to control the linear velocity coupling term in the force fields.

LEMMA 3.2. *Let* $(\mathcal{X}_t^{N,\eta,\varepsilon}, \mathcal{V}_t^{N,\eta,\varepsilon})$ *be the solution to the system* (3.118) *on the time interval* $[0,T]$. *Suppose that*

$$\max_{1\le i\le N} |V_0^i| \le V_m, \quad \mathbb{P}\text{- a.s.}$$

Then it holds

$$\max_{1\le i\le N} |V_t^{i,\eta,\varepsilon}| \le V_m, \quad \mathbb{P}\text{- a.s.},$$

for $\eta,\varepsilon>0$ *and* $t\in[0,T]$.

PROOF. We divide the proof into two steps:

- In **Step A**, we show that the maximal value of $|V_t^{i,\eta,\varepsilon}|$ over $i=1,\cdots,N$ has a finite speed of growth in time t.
- In **Step B**, we show that this maximal value can not exceed the ball of radius V_m almost surely, and complete the desired result.

$\diamond$ **Step A.-** We set

$$A_t^{\eta,\varepsilon} := \max_{1\le i\le N} |V_t^{i,\eta,\varepsilon}|.$$

We also notice that if we set

$$\Omega_0 = \left\{\omega\in\Omega \,|\, s\in[0,T]\mapsto\ V_s^{i,\eta,\varepsilon}(\omega) \text{ is continuous}\right\}\in\mathcal{F},$$

then $\mathbb{P}(\Omega_0)=1$ since the paths of the Bwonian motion are almost surely continuous. For $t\in[0,T]$, we define the random (but not stopping) time $\tau_t^{i,\eta,\varepsilon}$ as

$$\tau_t^{i,\eta,\varepsilon} = \sup_{s\le t}\left\{|V_s^{i,\eta,\varepsilon}| = V_m\right\}.$$

Then we obtain

$$\begin{aligned}|V_t^{i,\eta,\varepsilon}| &= \left|\int_{\tau_t^{i,\eta,\varepsilon}}^t F^{\eta,\varepsilon}[\mu_s^{N,\eta,\varepsilon}](X_s^{i,\eta,\varepsilon},V_s^{i,\eta,\varepsilon})\,ds + \sqrt{2\sigma}\int_{\tau_t^{i,\eta,\varepsilon}}^t \mathcal{R}(V_s^{i,\eta,\varepsilon})\,dB_s + V_{\tau_t^{i,\eta,\varepsilon}}^{i,\eta,\varepsilon}\right| \\ &\le \int_0^t \left|F^{\eta,\varepsilon}[\mu_s^{N,\eta,\varepsilon}](X_s^{i,\eta,\varepsilon},V_s^{i,\eta,\varepsilon})\right| ds + V_m + \sqrt{2\sigma}\left|\int_{\tau_t^{i,\eta,\varepsilon}}^t \mathcal{R}(V_s^{i,\eta,\varepsilon})\,dB_s\right|\end{aligned}$$

Let us define two event sets Ω_A and Ω_B by

$$\Omega_A := \left\{[\tau_t^{i,\eta,\varepsilon},t]\subset\left\{s\in[0,T]\mid |V_s^{i,\eta,\varepsilon}|\le V_m\right\}\right\},$$

and

$$\Omega_B := \left\{]\tau_t^{i,\eta,\varepsilon},t]\subset\left\{s\in[0,T]\mid |V_s^{i,\eta,\varepsilon}|> V_m\right\}\right\}.$$

Then by definition of $\tau_t^{i,\eta,\varepsilon}$ and the fact that $s\mapsto V_s^{i,\eta,\varepsilon}$ is almost surely continuous, we get $\mathbb{P}(\Omega_A)+\mathbb{P}(\Omega_B)=1$. For the event Ω_A, it is clear to get

$$|V_s^{i,\eta,\varepsilon}|\le V_m.$$

For the event Ω_B,

$$\int_{\tau_t^{i,\eta,\varepsilon}}^t \mathcal{R}_\varepsilon(V_s^{i,\eta,\varepsilon})dB_s = 0,$$

and this yields

$$|V_i^{i,\eta,\varepsilon}| \le \int_0^t \left|F^{\eta,\varepsilon}[\mu_s^{N,\eta,\varepsilon}](X_s^{i,\eta,\varepsilon},V_s^{i,\eta,\varepsilon})\right| ds + V_m.$$

On the other hand, the alignment force term in the above inequality is estimated as

$$\left|F^{\eta,\varepsilon}[\mu_s^{N,\eta,\varepsilon}](X_s^{i,\eta,\varepsilon},V_s^{i,\eta,\varepsilon})\right| \le \int_{\mathbb{R}^d\times\mathbb{B}^d_{A_s^{\eta,\varepsilon}}} \left(|v|+|V_s^{i,\eta,\varepsilon}|\right)\mu_s^{N,\eta,\varepsilon}(dx,dv) \le A_s^{\eta,\varepsilon}+|V_s^{i,\eta,\varepsilon}|.$$

This implies

$$|V_t^{i,\eta,\varepsilon}| \le V_m e^t + \int_0^t A_s^{\eta,\varepsilon} e^{t-s}\,ds, \quad \mathbb{P}\text{-a.s.},$$

and subsequently, by definition of $A_t^{\eta,\varepsilon}$, we obtain

$$A_t^{\eta,\varepsilon} \le V_m e^t + \int_0^t A_s^{\eta,\varepsilon} e^{t-s} ds,$$

and by applying Gronwall's inequality we finally have

$$A_t^{\eta,\varepsilon} \le V_m e^t(1+t) \le V_m e^T(1+T), \quad \mathbb{P}\text{-a.s.}$$

$\diamond$ **Step B.-** It follows from the assumption that $A_0 \le V_m$. Suppose that for some $\omega \in \Omega$ it holds $\widetilde{A}(\omega) := \sup_{t\in[0,T]} A_t(\omega) > V_m$. Then, for some $i = 1, \cdots, N$, there exists $t_0 \in [0,T]$ such that

$$\widetilde{A}(\omega) = |V_{t_0}^{i,\eta,\varepsilon}(\omega)| > V_m.$$

We now choose a neighborhood V_ω of t_0 such that $|V_s^{i,\eta,\varepsilon}(\omega)| > V_m$ for all $s \in V_\omega$. Then $V_t^{i,\eta,\varepsilon}(\omega)$ is differentiable in that neighborhood and

$$\begin{aligned}
\frac{1}{2}\frac{d|V_s^{i,\eta,\varepsilon}(\omega)|^2}{ds} &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbf{1}_{K(V_s^{i,\eta,\varepsilon}(\omega))}(x - X_s^{i,\eta,\varepsilon}(\omega))(v - V_s^{i,\eta,\varepsilon}(\omega)) \cdot V_s^{i,\eta,\varepsilon}(\omega)\mu_s^{N,\eta,\varepsilon}(\omega)(dx,dv) \\
&\le \left| \int_{\mathbb{R}^d\times\{|v|\le\widetilde{A}(\omega)\}} \underbrace{(v - V_s^{i,\eta,\varepsilon}(\omega)) \cdot V_s^{i,\eta,\varepsilon}(\omega)}_{\le(\widetilde{A}(\omega)-|V_s^{i,\eta,\varepsilon}(\omega)|)|V_s^{i,\eta,\varepsilon}(\omega)|} \mu_s^{N,\eta,\varepsilon}(\omega)(dx,dv) \right| \\
&\le (\widetilde{A}(\omega) - |V_s^{i,\eta,\varepsilon}(\omega)|)|V_s^{i,\eta,\varepsilon}(\omega)|.
\end{aligned}$$

Set

$$Z_s^{i,\eta,\varepsilon}(\omega) := \frac{(\widetilde{A}(\omega) - |V_s^{i,\eta,\varepsilon}(\omega)|)^2}{2}. \tag{3.119}$$

Then it is straightforward to get

$$Z_{t_0}^{i,\eta,\varepsilon}(\omega) = 0 \quad \text{and} \quad \left|\frac{dZ_s^{i,\eta,\varepsilon}(\omega)}{ds}\right| = \left| -\frac{d|V_s^{i,\eta,\varepsilon}(\omega)|}{ds}\left(\widetilde{A}(\omega) - |V_s^{i,\eta,\varepsilon}(\omega)|\right)\right| \le 2Z_s^{i,\eta,\varepsilon}(\omega),$$

This yields $Z_t^{i,\eta,\varepsilon}(\omega) = 0$ for $t \in V_\omega$, and in particular V_ω does not depend on ω and thus $V_\omega = [0,T]$. Thus $V_t^{i,\eta,\varepsilon}(\omega) > V_m$ for all $t \in [0,T]$ due to (3.119). This subsequently implies the event

$$\left\{ \sup_{t\in[0,T]} \sup_{1\le i\le N} |V_t^{i,\eta,\varepsilon}| > V_m \right\}, \tag{3.120}$$

is of probability one. On the other hand, by the assumption, the event

$$\left\{ \sup_{t\in[0,T]} \sup_{1\le i\le N} |V_0^i| \le V_m \right\},$$

is of probability one, which contradicts to (3.120). This completes the proof. □

We are now in position to give the proof of Theorem 1.1. As mentioned before, we are going to take into account the generalized boundary set $\widetilde{\partial}K$ defined in Definition 2.2 to construct a global-in-time weak solution to the stochastic integral inclusion system (2.114). We remind the reader that the similar strategy is used for the system (1.109) without noises, i.e., $\sigma = 0$ in [**6**].

Proof of Theorem 1.1. We divide the proof into two steps.

$\diamond$ **Step A: Tightness.-** It follows from Lemma 3.2 that if the $(V_i^0)_{i=1,\cdots,N}$ are distributed with a law compactly supported in $\mathbb{B}^d_{V_{max}}$, then $(V_t^{i,\eta,\varepsilon})_{i=1,\cdots,N}$ lie inside $\mathbb{B}^d_{V_{max}}$ $\mathbb{P}$-a.s. Using this fact, we find that for $0 \le s,t \le T$

$$\begin{aligned}
&|\mathcal{X}_t^{N,\eta,\varepsilon} - \mathcal{X}_s^{N,\eta,\varepsilon}| \le CV_m|t-s| \\
&|\mathcal{V}_t^{N,\eta,\varepsilon} - \mathcal{V}_s^{N,\eta,\varepsilon}| \le CV_m|t-s| + \sqrt{2\sigma} \sup_{1\le i\le N} \sup_{0\le s<r<t\le T} \left| \int_s^r \mathcal{R}(V_u^{i,\eta,\varepsilon})\, dB_u^i \right|,
\end{aligned}$$

for some positive constant C. Thus we obtain

$$|\mathcal{X}_t^{N,\eta,\varepsilon} - \mathcal{X}_s^{N,\eta,\varepsilon}| + |\mathcal{V}_t^{N,\eta,\varepsilon} - \mathcal{V}_s^{N,\eta,\varepsilon}| \le (C + U_T^{\eta,\varepsilon})|t-s|^{1/3},$$

where

$$U_T^{\eta,\varepsilon} := \sqrt{2\sigma} \sup_{1\le i\le N} \sup_{0\le s<r<t\le T} \left| \int_s^r \mathcal{R}(V_u^{i,\eta,\varepsilon})\, dB_u^i \right| |t-s|^{-1/3}.$$

On the other hand, by using Burkholder-Davis-Gundy inequality, we get that for any $p > 1$

$$\begin{aligned}\mathbb{E}\left[\left(\sup_{1\le i\le N}\sup_{s<r<t}\left|\int_s^r \mathcal{R}(V_u^{i,\eta,\varepsilon})\,dB_u^i\right|\right)^{2p}\right] &\le C\sum_{i=1}^N \mathbb{E}\left[\sup_{s<r<t}\left|\int_s^r \mathcal{R}(V_u^{i,\eta,\varepsilon})\,dB_u^i\right|^{2p}\right]\\ &\le C\sum_{i=1}^N C_p\mathbb{E}\left[\left|\int_s^t \mathcal{R}^2(V_s^{i,\eta,\varepsilon})\,ds\right|^p\right]\\ &\le C_{N,p}\|\mathcal{R}\|_{L^\infty}^2|t-s|^p.\end{aligned}$$

This together with Kolmogorov's continuity theorem gives that for $p > 1$ the process

$$t \mapsto \sup_{1\le i\le N}\int_s^t \mathcal{R}(V_u^{i,\eta,\varepsilon})\,dB_u^i \quad \text{is } \gamma\text{-Hölder } \mathbb{P}\text{-a.s. for any } \gamma \in \left(0, \frac{p-1}{2p}\right).$$

This subsequently implies $U_T^{\eta,\varepsilon} < \infty$ $\mathbb{P}$-almost surely. Let us now denote by

$$K(R,a) := \left\{ f \in \mathcal{C}^{1/3}([0,T],\mathbb{R}^{2dN}) : \sup_{0\le s<t\le T}\frac{|f(t)-f(s)|}{|t-s|^{1/3}} \le R \quad \text{and} \quad f(0) \in \mathbb{B}_a^{2dN}\right\},$$

which is a compact subset of $C([0,T],\mathbb{R}^{2dN})$ due to Arzelá-Ascoli theorem. Then, for all $\eta, \varepsilon > 0$, we obtain that if $(\mathcal{X}_t^{N,\eta,\varepsilon}, \mathcal{V}_t^{N,\eta,\varepsilon})_{t\in[0,T]} \notin K(R,a)$, then either $C + U_T^{\eta,\varepsilon} \ge R$ or $|(\mathcal{X}_0^N, \mathcal{V}_0^N)| \ge a$. This yields

$$\mathbb{P}\left((\mathcal{X}_t^{N,\eta,\varepsilon}, \mathcal{V}_t^{N,\eta,\varepsilon})_{t\in[0,T]} \notin K(R,a)\right) \le \mathbb{P}\left(C + U_T^{\eta,\varepsilon} \ge R\right) + \mathbb{P}\left(|(\mathcal{X}_0^N, \mathcal{V}_0^N)| \ge a\right).$$

On the other hand, since $U_T^{\eta,\varepsilon}$ is almost surely finite, we can find some $R > 0$ such that $\mathbb{P}\left(C + U_T^{\eta,\varepsilon} \ge R\right) \le a^{-1}$. Moreover, it follows from Chebyshev's inequality that

$$\mathbb{P}\big(|(\mathcal{X}_0^N, \mathcal{V}_0^N)| \ge a\big) \le \frac{m_2(f_0^N)}{a^2},$$

where $m_2(f_0^N)$ denotes the second-order moment of f_0^N. Hence we have

$$\sup_{\eta,\varepsilon>0}\mathbb{P}\left((\mathcal{X}_t^{N,\eta,\varepsilon}, \mathcal{V}_t^{N,\eta,\varepsilon}) \notin K(R,a)\right) \le Ca^{-1}(1 \vee a^{-1}),$$

and this concludes that the family of law $(P^{\eta,\varepsilon})_{\eta,\varepsilon>0}$ under $\mathbb{P}$ of $(\mathcal{X}_t^{N,\eta,\varepsilon}, \mathcal{V}_t^{N,\eta,\varepsilon}, B_t)_{t\in[0,T]}$ is tight.

◇ **Step B: Identification of the limit.-** By **Step A** and Prokhorov's theorem, we can choose a subsequence $(P^{\eta(n),\varepsilon(n)})_n$ converging to some P. Then again by Skorokhod's theorem, we can find a probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ together with some sequence of process $(\overline{\mathcal{X}}_t^{N,\eta(n),\varepsilon(n)}, \overline{\mathcal{V}}_t^{N,\eta(n),\varepsilon(n)}, \overline{B}_t)_n$ and a process $(\overline{\mathcal{X}}_t^N, \overline{\mathcal{V}}_t^N, W_t)_{t\in[0,T]}$ such that theirs law under $\overline{\mathbb{P}}$ are respectively $(P^{\eta(n),\varepsilon(n)})_n$ and P, and $(\overline{\mathcal{X}}_\cdot^{N,\eta(n),\varepsilon(n)}, \overline{\mathcal{V}}_\cdot^{N,\eta(n),\varepsilon(n)}, \overline{B}_\cdot)_n$ goes $\overline{\mathbb{P}}$-almost surely to $(\overline{\mathcal{X}}_t^N, \overline{\mathcal{V}}_t^N, W_t)_{t\in[0,T]}$. Note that the third marginal of $P^{\eta(n),\varepsilon(n)}$ is the law of a (dN)-Brownian motion so that $\overline{B}_\cdot$ is a Brownian motion under $\overline{\mathbb{P}}$. Thus $((\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}), (\overline{\mathcal{X}}_\cdot^{N,\eta(n),\varepsilon(n)}, \overline{\mathcal{V}}_\cdot^{N,\eta(n),\varepsilon(n)}, \overline{B}_\cdot))$ is a weak solution to (3.118) for each n since uniqueness in law holds for this equation. We now show that $((\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}), (\overline{\mathcal{X}}_t^N, \overline{\mathcal{V}}_t^N, W_t)_{t\in[0,T]})$ is a weak solution to stochastic integral inclusion system (2.114). Note that since $\overline{\mathcal{X}}_\cdot^{N,\eta(n),\varepsilon(n)}, \overline{\mathcal{V}}_\cdot^{N,\eta(n),\varepsilon(n)}, \overline{B}_\cdot$ is a weak solution to (3.118), we obtain that $\overline{\mathbb{P}}$-almost surely

(3.121)

$$\begin{aligned}&\overline{V}_t^i - \overline{V}_0^i - \frac{1}{N}\sum_{j=1}^N\int_0^t \mathbf{1}_{K(\overline{V}_s^i)\setminus\tilde{\partial}K(\overline{V}_s^i)}(\overline{X}_s^j - \overline{X}_s^i)(\overline{V}_s^j - \overline{V}_s^i)\,ds - \sqrt{2\sigma}\int_0^t \mathcal{R}(\overline{V}_s^i)\,d\overline{B}_s^i\\ &= \overline{V}_t^i - \overline{V}_t^{i,\eta,\varepsilon} - \sqrt{2\sigma}\int_0^t \big(\mathcal{R}(\overline{V}_s^i) - \mathcal{R}(\overline{V}_s^{i,\eta,\varepsilon})\big)d\overline{B}_s^i - \frac{1}{N}\sum_{j=1}^N\int_0^t I_s^{\eta,\varepsilon}\,ds\\ &= \overline{V}_t^i - \overline{V}_t^{i,\eta,\varepsilon} - \sqrt{2\sigma}\int_0^t \big(\mathcal{R}(\overline{V}_s^i) - \mathcal{R}(\overline{V}_s^{i,\eta,\varepsilon})\big)d\overline{B}_s^i - \frac{1}{N}\sum_{j=1}^N\int_0^t I_s^{\eta,\varepsilon}\mathbf{1}_{\{\overline{X}_s^j - \overline{X}_s^i \notin \tilde{\partial}K(V_s^i))\}}\,ds\\ &\quad - \frac{1}{N}\sum_{j=1}^N\int_0^t I_s^{\eta,\varepsilon}\mathbf{1}_{\{\overline{X}_s^j - \overline{X}_s^i \in \tilde{\partial}K(V_s^i))\}}\,ds,\end{aligned}$$

where

$$I_s^{\eta,\varepsilon} := \mathbf{1}_{K(\overline{V}_s^i)\setminus\tilde{\partial}K(\overline{V}_s^i)}(\overline{X}_s^j-\overline{X}_s^i)(\overline{V}_s^j-\overline{V}_s^i) - \mathbf{1}^{\eta,\varepsilon}_{K(\overline{V}_s^{i,\eta,\varepsilon})}(\overline{X}_s^{j,\eta,\varepsilon}-\overline{X}_s^{i,\eta,\varepsilon})(\overline{V}_s^{j,\eta,\varepsilon}-\overline{V}_s^{i,\eta,\varepsilon}).$$

We then estimate the third and fourth terms on the right hand side of the equality (3.121). By definition of the generalized boundary set $\tilde{\partial}K(\cdot)$, it is clear that if

$$\overline{X}_s^j-\overline{X}_s^i \notin \tilde{\partial}K(\overline{V}_s^i),$$

then

$$\mathbf{1}^{\eta,\varepsilon}_{K(\overline{V}_s^{i,\eta,\varepsilon})}(\overline{X}_s^{i,\eta,\varepsilon}-\overline{X}_s^{j,\eta,\varepsilon})(\overline{V}_s^{j,\eta,\varepsilon}-\overline{V}_s^{i,\eta,\varepsilon}) \to \mathbf{1}_{K(\overline{V}_s^i)\setminus\tilde{\partial}K(\overline{V}_s^i)}(\overline{X}_s^i-\overline{X}_s^j)(\overline{V}_s^j-\overline{V}_s^i),$$

weakly as $\varepsilon,\eta\to 0$. That is,

$$\frac{1}{N}\sum_{j=1}^N\int_0^t I_s^{\eta,\varepsilon}\mathbf{1}_{\{\overline{X}_s^j-\overline{X}_s^i\notin\tilde{\partial}K(V_s^i))\}}\,ds\to 0 \quad \text{weakly as } \varepsilon,\eta\to 0.$$

We also find

$$\begin{aligned}
&\frac{1}{N}\sum_{j=1}^N\int_0^t I_s^{\eta,\varepsilon}\mathbf{1}_{\{\overline{X}_s^j-\overline{X}_s^i\in\tilde{\partial}K(V_s^i))\}}\,ds\\
&\quad=\frac{1}{N}\sum_{j=1}^N\int_0^t\big(\mathbf{1}^{\eta,\varepsilon}_{K(\overline{V}_s^{i,\eta,\varepsilon})}(\overline{X}_s^{j,\eta,\varepsilon}-\overline{X}_s^{i,\eta,\varepsilon})(\overline{V}_s^{j,\eta,\varepsilon}-\overline{V}_s^{i,\eta,\varepsilon})\big)\mathbf{1}_{\{\overline{X}_s^j-\overline{X}_s^i\in\tilde{\partial}K(\overline{V}_s^i)\}}\,ds\\
&\quad=\frac{1}{N}\sum_{j=1}^N\int_0^t\big(\mathbf{1}^{\eta,\varepsilon}_{K(\overline{V}_s^{i,\eta,\varepsilon})}(\overline{X}_s^{j,\eta,\varepsilon}-\overline{X}_s^{i,\eta,\varepsilon})(\overline{V}_s^{j}+\overline{V}_s^{i}-\overline{V}_s^{i,\eta,\varepsilon})\big)\mathbf{1}_{\{\overline{X}_s^j-\overline{X}_s^i\in\tilde{\partial}K(\overline{V}_s^i)\}}\,ds\\
&\qquad+\frac{1}{N}\sum_{j=1}^N\int_0^t\big(\mathbf{1}^{\eta,\varepsilon}_{K(\overline{V}_s^{i,\eta,\varepsilon})}(\overline{X}_s^{j,\eta,\varepsilon}-\overline{X}_s^{i,\eta,\varepsilon})\big)(\overline{V}_s^j-\overline{V}_s^i)\mathbf{1}_{\{\overline{X}_s^j-\overline{X}_s^i\in\tilde{\partial}K(\overline{V}_s^i)\}}\,ds\\
&\quad=:\int_0^t J_s^{\eta,\varepsilon}ds+\frac{1}{N}\sum_{j=1}^N\int_0^t\alpha_s^{i,j,\eta,\varepsilon}(\overline{V}_s^j-\overline{V}_s^i)\mathbf{1}_{\{\overline{X}_s^j-\overline{X}_s^i\in\tilde{\partial}K(\overline{V}_s^i)\}}\,ds,
\end{aligned}$$

where $J_t^{\eta,\varepsilon}\to 0$ weakly as $\eta,\varepsilon\to 0$ for $t\in[0,T]$ due to the tightness. Thus, by combining the all of the above observations, we have that for any $h>0$, there exist $\eta,\varepsilon>0$ small enough such that (3.122)

$$\begin{aligned}
&\overline{V}_t^i-\overline{V}_0^i-\frac{1}{N}\sum_{j=1}^N\int_0^t\tilde{\mathbf{1}}_s^{i,j,\eta,\varepsilon}(\overline{V}_s^j-\overline{V}_s^i)\,ds-\sqrt{2\sigma}\int_0^t\mathcal{R}(\overline{V}_s^i)\,d\overline{B}_s^i=\\
&\overline{V}_t^i-\overline{V}_t^{i,\eta,\varepsilon}-\sqrt{2\sigma}\int_0^t\big(\mathcal{R}(\overline{V}_s^i)-\mathcal{R}(\overline{V}_s^{i,\eta,\varepsilon})\big)d\overline{B}_s^i-\frac{1}{N}\sum_{j=1}^N\int_0^t I_s^{\eta,\varepsilon}\mathbf{1}_{\{\overline{X}_s^j-\overline{X}_s^i\notin\tilde{\partial}K(V_s^i))\}}\,ds+\int_0^t J_s^{\eta,\varepsilon}ds,
\end{aligned}$$

where

$$\tilde{\mathbf{1}}_s^{i,j,\eta,\varepsilon}:=\mathbf{1}_{K(\overline{V}_s^i)\setminus\tilde{\partial}K(\overline{V}_s^i)}(\overline{X}_s^j-\overline{X}_s^i)+\alpha_s^{i,j,\eta,\varepsilon}\mathbf{1}_{\{\overline{X}_s^j-\overline{X}_s^i\in\tilde{\partial}K(\overline{V}_s^i)\}}.$$

Up to extracting a subsequence, since the function $s\in[0,T]\mapsto\alpha_s^{i,j,\eta,\varepsilon}\in[0,1]$ is bounded, we can find due to the fundamental Theorem for Young measures [**3**] for almost every $s\in[0,T]$ we can find a probability measure $\nu_s^{i,j}$ such that

$$\alpha_\cdot^{i,j,\eta,\varepsilon}\rightharpoonup\int_0^1 y\nu_\cdot^{i,j}(dy),$$

weakly in $L^\infty(]0,T[,[0,1])$. Therefore letting η,ε go to zero, we find that for almost $t\in[0,T]$

$$\overline{V}_t^i=\overline{V}_0^i+\frac{1}{N}\sum_{j=1}^N\int_0^t\alpha_s^{i,j}(\overline{V}_s^j-\overline{V}_s^i)\,ds+\sqrt{2\sigma}\int_0^t\mathcal{R}(\overline{V}_s^i)\,d\overline{B}_s^i,$$

with

$$\alpha_s^{i,j}:=\mathbf{1}_{K(\overline{V}_s^i)\setminus\tilde{\partial}K(\overline{V}_s^i)}(\overline{X}_s^j-\overline{X}_s^i)+\left(\int_0^1 y\nu_s^{i,j}(dy)\right)\mathbf{1}_{\{\overline{X}_s^j-\overline{X}_s^i\in\tilde{\partial}K(\overline{V}_s^i)\}}\in\mathcal{I}(X_s^j-X_s^i,V_s^i).$$

which concludes the proof.

□

4. Nonlinear stochastic integral system: Proof of Theorem 1.2

The purpose of this section is to show the existence and uniqueness of solutions to the following Nonlinear SIEs:

$$\begin{cases} Y_t = Y_0 + \displaystyle\int_0^t W_s\, ds, \\ W_t = W_0 + \displaystyle\int_0^t F[f_s](Y_s, W_s)\, ds + \sqrt{2\sigma} \int_0^t \mathcal{R}(W_s)\, dB_s, \\ f_t = \mathcal{L}(Y_t, W_t). \end{cases} \tag{4.123}$$

For this, we first give the so called weak-strong stability estimate under the assumptions (**H1**)-(**H2**) for the set-valued function $K(\cdot)$.

LEMMA 4.1. *Let (Y, W) and (Y', W') two random variables on $\mathbb{R}^{2d}$ and denote $f = \mathcal{L}(Y, W)$ and $f' = \mathcal{L}(Y', W')$. Assume that $f \in L^\infty(\mathbb{R}^{2d})$ and both f and f' are compactly supported in velocity in $\mathbb{B}^d_{V_{max}}$. Then there exists a constant depending only on V_m such that*

$$\mathbb{E}\left[|F[f](Y,W) - F[f'](Y',W')|\right] \le C\|f\|_{L^1\cap L^\infty}\mathbb{E}\left[|Y-Y'| + |W-W'|\right].$$

PROOF. Introducing $\pi := \mathcal{L}((Y,W),(Y',W'))$, we obtain

$$\begin{aligned}
&\mathbb{E}\left[|F[f](Y,W) - F[f'](Y',W')|\right] \\
&= \mathbb{E}\left[\left|\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \left(\mathbf{1}_{K(W)}(y-Y)(w-W) - \mathbf{1}_{K(W')}(y'-Y')(w'-W')\right)\pi(dy,dw,dy',dw')\right|\right] \\
&\le \mathbb{E}\left[\left|\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \left(\mathbf{1}_{K(W)}(y-Y)(w-W) - \mathbf{1}_{K(W)}(y-Y)(w'-W')\right)\pi(dy,dw,dy',dw')\right|\right] \\
&\quad + \mathbb{E}\left[\left|\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \left(\mathbf{1}_{K(W)}(y-Y) - \mathbf{1}_{K(W)}(y'-Y')\right)(w'-W')\,\pi(dy,dw,dy',dw')\right|\right] \\
&\quad + \mathbb{E}\left[\left|\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \left(\mathbf{1}_{K(W)}(y'-Y') - \mathbf{1}_{K(W')}(y'-Y')\right)(w'-W')\,\pi(dy,dw,dy',dw')\right|\right] \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

$\diamond$ Estimate I_1: First, we easily obtain

$$I_1 \le \mathbb{E}\left[|W-W'|\right] + \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} |w-w'|\pi(dy,dw,dy',dw') = 2\mathbb{E}\left[|W-W'|\right].$$

$\diamond$ Estimate I_2: Using the fact that f, f' are compactly supported in velocity together with Lemma 1.1, we find

$$\begin{aligned}
I_2 &\le 2V_m\mathbb{E}\left[\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \left|\mathbf{1}_{K(W)}(y-Y) - \mathbf{1}_{K(W)}(y'-Y')\right|\pi(dy,dw,dy',dw')\right] \\
&\le 2V_m\mathbb{E}\left[\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \left(\mathbf{1}_{\partial^{2|Y-Y'|}K(W)}(y-Y) + \mathbf{1}_{\partial^{2|y-y'|}K(W)}(y-Y)\right)\pi(dy,dw,dy',dw')\right] \\
&=: I_2^1 + I_2^2,
\end{aligned}$$

where I_2^1 can be estimated by

$$\begin{aligned}
I_2^1 &\le 2V_m\mathbb{E}\left[\|f\|_{L^\infty}\,|\Theta(W)^{2|Y-Y'|,+}|\mathbf{1}_{2|Y-Y'|\le 1}\right] + 2V_m\mathbb{E}\left[|Y-Y'|\mathbf{1}_{2|Y-Y'|>1}\right] \\
&\le C\|f\|_{L^1\cap L^\infty}\mathbb{E}\left[|Y-Y'|\right],
\end{aligned}$$

thanks to (**H2**) (i)-(ii). Then by using Fubini's theorem together with the fact $f = \mathcal{L}(Y,W)$, (**H2**) (i)-(ii) as the above, and compact support of f in velocity, we obtain

$$\begin{aligned}
I_2^2 &= 2V_m\mathbb{E}\left[\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \mathbf{1}_{\partial^{2|y-y'|}K(u)}(y-z)f(dz,du)\pi(dy,dw,dy',dw')\right] \\
&\le C(1+\|f\|_{L^\infty})\mathbb{E}\left[\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} |y-y'|\pi(dy,dw,dy',dw')\right] \\
&= C\|f\|_{L^1\cap L^\infty}\mathbb{E}\left[|Y-Y'|\right].
\end{aligned}$$

$\diamond$ Estimate of I_3: Using (**H2**) (iii) together with the fact that

$$\mathbf{1}_A(x) \le \mathbf{1}_{A^{|x-y|,+}}(y) \quad \text{for any set } A \subset \mathbb{R}^d \text{ and } x, y \in \mathbb{R}^d,$$

we split I_3 into two terms:

$$\begin{aligned}
I_3 &\le 2V_m\mathbb{E}\left[\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \mathbf{1}_{K(W)\Delta K(W')}(y'-Y')\pi(dy,dw,dy',dw')\right]\\
&\le 2V_m\mathbb{E}\left[\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \mathbf{1}_{\Theta(W)^{C}|W-W'|_+}(y'-Y')\pi(dy,dw,dy',dw')\right]\\
&\le 2V_m\mathbb{E}\left[\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \mathbf{1}_{\Theta(W)^{C}|W-W'|+|Y-Y'|+|y-y'|_+}(y-Y)\pi(dy,dw,dy',dw')\right]\\
&\le 2V_m\mathbb{E}\left[\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \left(\mathbf{1}_{\Theta(W)^{2C|W-W'|+2|Y-Y'|_+}}(Y-y)+\mathbf{1}_{\Theta(W)^{2|y-y'|_+}}(y-Y)\right)\pi(dy,dw,dy',dw')\right]\\
&=: I_3^1+I_3^2.
\end{aligned}$$

We first easily estimate I_3^1 as

$$I_3^1 \le 2V_m\|f\|_{L^1\cap L^\infty}\mathbb{E}\left[|W-W'|+|Y-Y'|\right],$$

due to (**H2**) (ii). For the estimate of I_3^2, we again use Fubini's theorem and compact support of f in velocity, we have similarly as the estimate of I_2^2 above that

$$\begin{aligned}
I_3^2 &= C\mathbb{E}\left[\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \mathbf{1}_{\Theta(u)^{2|y-y'|_+}}(y-z)f(dz,du)\pi(dy,dw,dy',dw')\right]\\
&\le C\|f\|_{L^1\cap L^\infty}\mathbb{E}\left[\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} |y-y'|\pi(dy,dw,dy',dw')\right]\\
&= C\|f\|_{L^1\cap L^\infty}\mathbb{E}\left[|Y-Y'|\right].
\end{aligned}$$

By combining all the above estimates, we conclude our desired result. □

We now consider the following regularized nonlinear SIEs:

$$\begin{cases}
Y_t^{\eta,\varepsilon} = Y_0 + \displaystyle\int_0^t W_s^{\eta,\varepsilon}\,ds,\\
W_t^{\eta,\varepsilon} = W_0 + \displaystyle\int_0^t F^{\eta,\varepsilon}[f_s^{\eta,\varepsilon}](Y_s^{\eta,\varepsilon},W_s^{\eta,\varepsilon})\,ds + \sqrt{2\sigma}\int_0^t \mathcal{R}(W_s^{\eta,\varepsilon})\,dB_s,\\
f_t^{\eta,\varepsilon} = \mathcal{L}(Y_t^{\eta,\varepsilon},W_t^{\eta,\varepsilon}),
\end{cases} \tag{4.124}$$

where $F^{\eta,\varepsilon}$ is defined as in (3.115). Due to the smoothness of the force fields and the diffusion coefficients, it is clear the global existence and uniqueness of solutions for the system (4.124).

In the lemma below, we provide the upper bound estimate of the solution $W_t^{\eta,\varepsilon}$ to the above system whose proof can be obtained by using the almost same argument as in Lemma 3.2.

LEMMA 4.2. *Let $T>0$, and suppose that there exists a solution on the time interval $[0,T]$ to the system* (4.123) *with the law of the initial data $f_0=\mathcal{L}(Y_0,W_0)$ which is compactly supported in velocity in $\mathbb{B}^d_{V_{max}}$. Then it holds*

$$|W_t^{\eta,\varepsilon}| \le V_m, \quad \mathbb{P}\text{- a.s.},$$

for $\eta,\varepsilon>0$ and $t\in[0,T]$.

In the proposition below, we show the existence of weak solutions to the corresponding Vlasov-Fokker-Planck type equation to (4.124).

PROPOSITION 4.1. *The family of time marginals of the solution $(f_t^{\eta,\varepsilon})$ to the system* (4.124) *is a global-in-time weak solution of the following kinetic equation:*

$$\partial_t f_t^{\eta,\varepsilon} + v\cdot\nabla_x f_t^{\eta,\varepsilon} + \nabla_v\cdot\left(F^{\eta,\varepsilon}[f_t^{\eta,\varepsilon}]f_t^{\eta,\varepsilon}\right) = \sigma\Delta_v(\mathcal{R}^2(v)f_t^{\eta,\varepsilon}), \quad (x,v)\in\mathbb{R}^d\times\mathbb{R}^d, \tag{4.125}$$

with the initial data $f_0=\mathcal{L}(Y_0,W_0)\in L^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ compactly supported in velocity in $\mathbb{B}^d_{V_{max}}$, where the velocity alignment force $F^{\eta,\varepsilon}[f_t^{\eta,\varepsilon}]$ is given by

$$F^{\eta,\varepsilon}[f_t^{\eta,\varepsilon}](x,v) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbf{1}^{\eta,\varepsilon}_{K(v)}(x-y)(w-v)f_t^{\eta,\varepsilon}(dy,dw).$$

Furthermore, we have

$$\sup_{0\le t\le T}\|f_t^{\eta,\varepsilon}\|_{L^\infty} \le C.$$

for some $T>0$ and positive constant $C>0$ independent of η,ε.

PROOF. For $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$, we apply Itô's formula to the system (4.124) to find

$$\begin{aligned}
&\phi(Y_t^{\eta,\varepsilon}, W_t^{\eta,\varepsilon}) \\
&= \phi(Y_0, W_0) + \int_0^t \langle \nabla_{x,v}\phi(Y_s^{\eta,\varepsilon}, W_s^{\eta,\varepsilon}), d(Y_s^{\eta,\varepsilon}, W_s^{\eta,\varepsilon})\rangle + \frac{1}{2}\int_0^t \Delta_{x,v}\phi(Y_s^{\eta,\varepsilon}, W_s^{\eta,\varepsilon}) d\,\langle (Y_s^{\eta,\varepsilon}, W_s^{\eta,\varepsilon})\rangle \\
&= \phi(Y_0, W_0) + \int_0^t W_s^{\eta,\varepsilon}\cdot\nabla_x\phi(Y_s^{\eta,\varepsilon}, W_s^{\eta,\varepsilon})ds + \int_0^t F^{\eta,\varepsilon}[f_s^{\eta,\varepsilon}](Y_s^{\eta,\varepsilon}, W_s^{\eta,\varepsilon})\cdot\nabla_v\phi(Y_s^{\eta,\varepsilon}, W_s^{\eta,\varepsilon})ds \\
&\quad + \sqrt{2\sigma}\int_0^t \langle \nabla_v\phi(Y_s^{\eta,\varepsilon}, W_s^{\eta,\varepsilon}), \mathcal{R}(W_s^{\eta,\varepsilon})dB_s\rangle + \sigma\int_0^t \Delta_v\phi(Y_s^{\eta,\varepsilon}, W_s^{\eta,\varepsilon})\mathcal{R}^2(W_s^{\eta,\varepsilon})\,ds.
\end{aligned}$$

We then take the expectation and use the fact that $f_t^{\eta,\varepsilon} = \mathcal{L}(Y_t^{\eta,\varepsilon}, W_t^{\eta,\varepsilon})$ to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}^d}\phi(x,v)f_t^{\eta,\varepsilon}(dx,dv) \\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d}\phi(x,v)f_0(dx,dv) + \int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d} v\cdot\nabla_x\phi(x,v)f_s^{\eta,\varepsilon}(dx,dv)\,ds \\
&\quad + \int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d} F^{\eta,\varepsilon}[f_s^{\eta,\varepsilon}](x,v)\cdot\nabla_v\phi(x,v)f_s^{\eta,\varepsilon}(dx,dv)\,ds \\
&\quad + \sigma\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d}\Delta_v\phi(x,v)\mathcal{R}^2(v)f_s^{\eta,\varepsilon}(dx,dv)\,ds.
\end{aligned}$$

This concludes that the family of time marginals of the process solutions $(Y_t^{\eta,\varepsilon}, W_t^{\eta,\varepsilon})$ to the system (4.124) is a weak solution for the kinetic equation (4.125).

We next show the uniform bound estimate of $\|f^{\eta,\varepsilon}\|_{L^\infty}$ in $\eta,\varepsilon > 0$. Due to the diffusion term, we first estimate L^p-norm of the solution $f^{\eta,\varepsilon}$ and send $p\to\infty$ to have the desired L^∞-estimate. For $p \geq 1$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}^d}(f_t^{\eta,\varepsilon})^p\,dxdv \\
&= p\int_{\mathbb{R}^d\times\mathbb{R}^d}\partial_t f_t^{\eta,\varepsilon}(f_t^{\eta,\varepsilon})^{p-1}dxdv \\
&= -p\int_{\mathbb{R}^d\times\mathbb{R}^d}\nabla_v\cdot(F^{\eta,\varepsilon}[f_t^{\eta,\varepsilon}]f_t^{\eta,\varepsilon})(f_t^{\eta,\varepsilon})^{p-1}dxdv + \sigma p\int_{\mathbb{R}^d\times\mathbb{R}^d}\Delta_v(\mathcal{R}^2(v)f_t^{\eta,\varepsilon})(f_t^{\eta,\varepsilon})^{p-1}dxdv \\
&=: J_1 + J_2,
\end{aligned}$$

where $J_i, i = 1,2$ are estimated as follows.

$$\begin{aligned}
J_1 &= -p\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\nabla_v\cdot F^{\eta,\varepsilon}[f_t^{\eta,\varepsilon}]\right)(f_t^{\eta,\varepsilon})^p\,dxdv - \int_{\mathbb{R}^d\times\mathbb{R}^d}F^{\eta,\varepsilon}[f_t^{\eta,\varepsilon}]\cdot\nabla_v(f_t^{\eta,\varepsilon})^p\,dxdv \\
&= -(p-1)\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\nabla_v\cdot F^{\eta,\varepsilon}[f_t^{\eta,\varepsilon}]\right)(f_t^{\eta,\varepsilon})^p\,dxdv \\
J_2 &= -\sigma p\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\nabla_v(\mathcal{R}^2(v))f_t^{\eta,\varepsilon} + \mathcal{R}^2(v)\nabla_v f_t^{\eta,\varepsilon}\right)\cdot\nabla_v((f_t^{\eta,\varepsilon})^{p-1})\,dxdv \\
&= \sigma(p-1)\int_{\mathbb{R}^d\times\mathbb{R}^d}\Delta_v(\mathcal{R}^2(v))(f_t^{\eta,\varepsilon})^p\,dxdv - \sigma p(p-1)\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathcal{R}^2(v)(f_t^{\eta,\varepsilon})^{p-2}|\nabla_v f_t^{\eta,\varepsilon}|^2\,dxdv.
\end{aligned}$$

This yields

$$\frac{d}{dt}\|f_t^{\eta,\varepsilon}\|_{L^p}^p \leq (p-1)\left(\|\nabla_v\cdot F^{\eta,\varepsilon}[f_t^{\eta,\varepsilon}]\|_{L^\infty} + \sigma\|\Delta_v\mathcal{R}^2(v)\|_{L^\infty}\right)\|f_t^{\eta,\varepsilon}\|_{L^p}^p.$$

On the other hand, it follows from [**6**, Proposition 4.1] together with Lemma 4.2 that

$$\|\nabla_v\cdot F^{\eta,\varepsilon}[f_t^{\eta,\varepsilon}]\|_{L^\infty} \leq C(1 + \|f_t^{\eta,\varepsilon}\|_{L^\infty}),$$

where $C > 0$ is independent of η, ε, and p. Thus we obtain

$$\frac{d}{dt}\left\|f_t^{\eta,\varepsilon}\right\|_{L^p}^p \leq Cp(1 + \|f_t^{\eta,\varepsilon}\|_{L^\infty})\left\|f_t^{\eta,\varepsilon}\right\|_{L^p}^p.$$

By applying Gronwall's inequality, we get

$$\|f_t^{\eta,\varepsilon}\|_{L^p} \leq \|f_0\|_{L^p}\exp\left(C\int_0^t(1 + \|f_s^{\eta,\varepsilon}\|_{L^\infty})ds\right).$$

We now send $p \to \infty$ to find

$$\|f_t^{\eta,\varepsilon}\|_{L^\infty} \le \|f_0\|_{L^\infty} \exp\left(C \int_0^t (1 + \|f_s^{\eta,\varepsilon}\|_{L^\infty})\, ds \right).$$

Set

$$g(t) := \|f_0\|_{L^\infty} \exp\left(C \int_0^t (1 + \|f_s^{\eta,\varepsilon}\|_{L^\infty}) ds \right) \ge 0.$$

Then $g(0) = \|f_0\|_{L^\infty}$ and g satisfies

$$g'(t) = Cg(t)(1 + \|f_t^{\eta,\varepsilon}\|_{L^\infty}) \le Cg(t)(1 + g(t)).$$

This yields

$$(g^{-1}(t))' + Cg^{-1}(t) \ge -C \quad \text{and} \quad g(t) \le \frac{1}{(1 + g_0^{-1})e^{-Ct} - 1}.$$

Hence we have

$$\|f_t^{\eta,\varepsilon}\|_{L^\infty} \le g(t) \le \frac{1}{(1 + g_0^{-1})e^{-Ct} - 1} = \frac{\|f_0\|_{L^\infty}}{(1 + \|f_0\|_{L^\infty})e^{-Ct} - \|f_0\|_{L^\infty}}.$$

□

We then provide the existence and stability of strong solutions to the nonlinear SIEs (4.123) in the proposition below.

PROPOSITION 4.2. *There exists only one process on the time interval $[0,T]$ for some $T > 0$ solving the system (4.123) in the strong sense starting from (Y_0, W_0) with law $f_0 \in L^\infty(\mathbb{R}^{2d})$, independent of $(B_t)_{t\in[0,T]}$ such that its time marginal satisfies $f_t \in L^\infty(0,T; L^\infty(\mathbb{R}^{2d}))$ and solves the kinetic equation (1.111). Moreover if $(Y_t, W_t)_{t\in[0,T]}$ and $(Y_t', W_t')_{t\in[0,T]}$ are two solutions to the system (4.123) with the initial data (Y_0, W_0) and (Y_0', W_0'), respectively, such that $f_t = \mathcal{L}(Y_t, W_t)$ with $f_t \in L^\infty(0,T; L^\infty(\mathbb{R}^{2d}))$. Then we have*

$$\mathbb{E}\big[|Y_t - Y_t'| + |W_t - W_t'|\big] \le e^{C\int_0^t \|f_s\|_{L^1\cap L^\infty} ds} \mathbb{E}\big[|Y_0 - Y_0'| + |W_0 - W_0'|\big],$$

for $t \in [0,T]$.

PROOF. As mentioned in Introduction, we cannot directly use the 1-Wasserstein distance. Set $\varphi_\gamma(x) := \sqrt{\gamma^2 + |x|^2}$ with $\gamma > 0$. Note that

$$\nabla\varphi_\gamma(x) = \frac{x}{\sqrt{\gamma^2 + |x|^2}} \quad \text{and} \quad \Delta\varphi_\gamma(x) = \frac{d\gamma^2 + (d-1)|x|^2}{(\gamma^2 + |x|^2)^{3/2}}.$$

We also find from the above that

$$|\nabla\varphi_\gamma(x)| \le 1 \quad \text{and} \quad |\Delta\varphi_\gamma(x)| \le \frac{d}{\sqrt{\gamma^2 + |x|^2}}. \tag{4.126}$$

⋄ **Step A (Cauchy estimates).-** Let $(Y_t^{\eta,\varepsilon}, W_t^{\eta,\varepsilon})$ and $(Y_t^{\eta',\varepsilon'}, W_t^{\eta',\varepsilon'})$ be solutions to the regularized system (4.124). Then, by applying Itô's formula, we obtain

(4.127)

$$\begin{aligned}
&\sqrt{\gamma^2 + |W_t^{\eta,\varepsilon} - W_t^{\eta',\varepsilon'}|^2} \\
&= \int_0^t \left(\frac{W_s^{\eta,\varepsilon} - W_s^{\eta',\varepsilon'}}{\sqrt{\gamma^2 + |W_s^{\eta,\varepsilon} - W_s^{\eta',\varepsilon'}|^2}} \right) \cdot \Big(F^{\eta,\varepsilon}[f_s^{\eta,\varepsilon}](Y_s^{\eta,\varepsilon}, W_s^{\eta,\varepsilon}) - F^{\eta',\varepsilon'}[f_s^{\eta',\varepsilon'}](Y_s^{\eta',\varepsilon'}, W_s^{\eta',\varepsilon'}) \Big)\, ds \\
&\quad + \sqrt{2\sigma} \int_0^t \Big(\mathcal{R}(W_s^{\eta,\varepsilon}) - \mathcal{R}(W_s^{\eta',\varepsilon'}) \Big) \left(\frac{W_s^{\eta,\varepsilon} - W_s^{\eta',\varepsilon'}}{\sqrt{\gamma^2 + |W_s^{\eta,\varepsilon} - W_s^{\eta',\varepsilon'}|^2}} \right) \cdot dB_s \\
&\quad + \sigma \int_0^t \left(\frac{d\gamma^2 + (d-1)|W_s^{\eta,\varepsilon} - W_s^{\eta',\varepsilon'}|^2}{(\gamma^2 + |W_s^{\eta,\varepsilon} - W_s^{\eta',\varepsilon'}|^2)^{3/2}} \right) \Big(\mathcal{R}(W_s^{\eta,\varepsilon}) - \mathcal{R}(W_s^{\eta',\varepsilon'}) \Big)^2\, ds,
\end{aligned}$$

due to $W_0^{\eta,\varepsilon} = W_0^{\eta',\varepsilon'} = W_0$. We next take the expectation to the above equality to find

$$\begin{aligned}
&\mathbb{E}\left[\sqrt{\gamma^2 + |W_t^{\eta,\varepsilon} - W_t^{\eta',\varepsilon'}|^2}\right] \\
&\quad\le \int_0^t \mathbb{E}\left[\left|F^{\eta,\varepsilon}[f_s^{\eta,\varepsilon}](Y_s^{\eta,\varepsilon}, W_s^{\eta,\varepsilon}) - F^{\eta',\varepsilon'}[f_s^{\eta',\varepsilon'}](Y_s^{\eta',\varepsilon'}, W_s^{\eta',\varepsilon'})\right|\right] ds \\
&\qquad + \sigma d \int_0^t \mathbb{E}\left[\frac{1}{\sqrt{\gamma^2 + |W_s^{\eta,\varepsilon} - W_s^{\eta',\varepsilon'}|^2}} \left(\mathcal{R}(W_s^{\eta,\varepsilon}) - \mathcal{R}(W_s^{\eta',\varepsilon'})\right)^2\right] ds \\
&\quad=: \int_0^t K_1\, ds + \int_0^t K_2\, ds,
\end{aligned}$$

where we used (4.126). Here K_2 can be easily estimated as

$$\begin{aligned}
\int_0^t K_2\, ds &\le \sigma d \|\mathcal{R}\|_{Lip}^2 \int_0^t \mathbb{E}\left[\frac{|W_s^{\eta,\varepsilon} - W_s^{\eta',\varepsilon'}|^2}{\sqrt{\gamma^2 + |W_s^{\eta,\varepsilon} - W_s^{\eta',\varepsilon'}|^2}}\right] ds \\
&\le \sigma d \|\mathcal{R}\|_{Lip}^2 \int_0^t \left(\gamma + \mathbb{E}[|W_s^{\eta,\varepsilon} - W_s^{\eta',\varepsilon'}|]\right) ds.
\end{aligned}$$

For the estimate of K_1, we decompose it as

$$\begin{aligned}
K_1 &\le \mathbb{E}\left[|F^{\eta,\varepsilon}[f_s^{\eta,\varepsilon}](Y_s^{\eta,\varepsilon}, W_s^{\eta,\varepsilon}) - F[f_s^{\eta,\varepsilon}](Y_s^{\eta,\varepsilon}, W_s^{\eta,\varepsilon})|\right] \\
&\quad + \mathbb{E}\left[\left|F[f_s^{\eta,\varepsilon}](Y_s^{\eta,\varepsilon}, W_s^{\eta,\varepsilon}) - F[f_s^{\eta',\varepsilon'}](Y_s^{\eta',\varepsilon'}, W_s^{\eta',\varepsilon'})\right|\right] \\
&\quad + \mathbb{E}\left[\left|F[f_s^{\eta',\varepsilon'}](Y_s^{\eta',\varepsilon'}, W_s^{\eta',\varepsilon'}) - F^{\eta',\varepsilon'}[f_s^{\eta',\varepsilon'}](Y_s^{\eta',\varepsilon'}, W_s^{\eta',\varepsilon'})\right|\right] \\
&=: K_1^1 + K_1^2 + K_1^3.
\end{aligned}$$

$\diamond$ Estimates of K_1^1 and K_1^3: We again split K_1^1 into two terms:

$$\begin{aligned}
K_1^1 &= \mathbb{E}\left[\left|\int_{\mathbb{R}^d\times\mathbb{R}^d} \left(\mathbf{1}^{\eta,\varepsilon}_{K(W_s^{\eta,\varepsilon})}(Y_s^{\eta,\varepsilon} - y) - \mathbf{1}_{K(W_s^{\eta,\varepsilon})}(Y_s^{\eta,\varepsilon} - y)\right)(w - W_s^{\eta,\varepsilon}) f_s^{\eta,\varepsilon}(dy, dw)\right|\right] \\
&\le \mathbb{E}\left[\int_{\mathbb{R}^d\times\mathbb{R}^d} \left|\mathbf{1}^{\eta,\varepsilon}_{K(W_s^{\eta,\varepsilon})}(Y_s^{\eta,\varepsilon} - y) - \mathbf{1}^{\varepsilon}_{K(W_s^{\eta,\varepsilon})}(Y_s^{\eta,\varepsilon} - y)\right| |w - W_s^{\eta,\varepsilon}| f_s^{\eta,\varepsilon}(dy, dw)\right] \\
&\quad + \mathbb{E}\left[\int_{\mathbb{R}^d\times\mathbb{R}^d} \left|\mathbf{1}^{\varepsilon}_{K(W_s^{\eta,\varepsilon})}(Y_s^{\eta,\varepsilon} - y) - \mathbf{1}_{K(W_s^{\eta,\varepsilon})}(Y_s^{\eta,\varepsilon} - y)\right| |w - W_s^{\eta,\varepsilon}| f_s^{\eta,\varepsilon}(dy, dw)\right] \\
&=: K_1^{1,1} + K_1^{1,2}.
\end{aligned}$$

Using the estimate of compact support of $f_s^{\eta,\varepsilon}$ in velocity and (3.140), we find for $\eta \le 1$

$$\begin{aligned}
K_1^{1,1} &\le 2V_m |\mathbb{B}^d_{V_{max}}| \|f_s^{\eta,\varepsilon}\|_{L^\infty} \mathbb{E}\left[\int_{\mathbb{R}^d} \left|\mathbf{1}^{\eta,\varepsilon}_{K(W_s^{\eta,\varepsilon})}(Y_s^{\eta,\varepsilon} - y) - \mathbf{1}^{\varepsilon}_{K(W_s^{\eta,\varepsilon})}(Y_s^{\eta,\varepsilon} - y)\right| dy\right] \\
&\le C\|f_s^{\eta,\varepsilon}\|_{L^\infty}\eta.
\end{aligned}$$

Similarly, by using (3.138) and (**H2**) (i)-(ii), we get for $\varepsilon < 1/2$

$$\begin{aligned}
K_1^{1,2} &\le 2V_m |\mathbb{B}^d_{V_{max}}| \|f_s^{\eta,\varepsilon}\|_{L^\infty} \mathbb{E}\left[\int_{\mathbb{R}^d} \left|\mathbf{1}^{\varepsilon}_{K(W_s^{\eta,\varepsilon})}(Y_s^{\eta,\varepsilon} - y) - \mathbf{1}_{K(W_s^{\eta,\varepsilon})}(Y_s^{\eta,\varepsilon} - y)\right| dy\right] \\
&\le C\|f_s^{\eta,\varepsilon}\|_{L^\infty} \sup_{v\in\mathbb{R}^d} |\Theta(v)^{2\varepsilon,+}| \\
&\le C\|f_s^{\eta,\varepsilon}\|_{L^\infty}\varepsilon.
\end{aligned}$$

Employing the almost same argument as above, we estimate K_1^3 as

$$K_1^3 \le C\|f_s^{\eta,\varepsilon}\|_{L^\infty}\left(\eta' + \varepsilon'\right) \quad \text{for} \quad \eta', \varepsilon' < 1/2.$$

$\diamond$ Estimate of K_1^2: It follows from Lemma 1.2 that

$$K_1^2 \le C(1 + \|f_s^{\eta,\varepsilon}\|_{L^\infty})\mathbb{E}\left[|Y_s^{\eta,\varepsilon} - Y_s^{\eta',\varepsilon'}| + |W_s^{\eta,\varepsilon} - W_s^{\eta',\varepsilon'}|\right].$$

Combining the all of above estimates, we have

$$\begin{aligned}
&\mathbb{E}\left[|Y_t^{\eta,\varepsilon}-Y_t^{\eta',\varepsilon'}|+\sqrt{\gamma^2+|W_t^{\eta,\varepsilon}-W_t^{\eta',\varepsilon'}|^2}\right]\\
&\quad\le \gamma + C\int_0^t (1+\|f_s^{\eta,\varepsilon}\|_{L^\infty})\,\mathbb{E}\left[|Y_s^{\eta,\varepsilon}-Y_s^{\eta',\varepsilon'}|+\sqrt{\gamma^2+|W_s^{\eta,\varepsilon}-W_s^{\eta',\varepsilon'}|^2}\right]ds\\
&\qquad + C\int_0^t \|f_s^{\eta,\varepsilon}\|_{L^\infty}(\eta+\varepsilon+\eta'+\varepsilon')\,ds,
\end{aligned}$$

where $C>0$ is independent of $\eta,\varepsilon,\eta',\varepsilon'$ and γ. We finally let $\gamma\to 0$ and use Gronwall's inequality together with the uniform bound estimate in Proposition 4.1, we have

$$\sup_{t\in[0,T]}\mathbb{E}\left[|Y_t^{\eta,\varepsilon}-Y_t^{\eta',\varepsilon'}|+|W_t^{\eta,\varepsilon}-W_t^{\eta',\varepsilon'}|\right]\le C(\eta+\varepsilon+\eta'+\varepsilon')\quad\text{for}\quad \eta,\varepsilon,\eta',\varepsilon'<1/2,$$

where the constant C depends only on $V_m, d, \|\mathcal{R}\|_{Lip}$, and T.

◇ **Step B (Uniform-in-time estimate).-** Before taking the expectation to the equality (4.127), we first take the supremum over the time interval $[0,T]$. Then we can estimate similarly as in **Step A** to estimate

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left(|Y_t^{\eta,\varepsilon}-Y_t^{\eta',\varepsilon'}|+|W_t^{\eta,\varepsilon}-W_t^{\eta',\varepsilon'}|\right)\right]$$

except the following term

$$\int_0^t\left(\mathcal{R}(W_s^{\eta,\varepsilon})-\mathcal{R}(W_s^{\eta',\varepsilon'})\right)\left(\frac{W_s^{\eta,\varepsilon}-W_s^{\eta',\varepsilon'}}{\sqrt{\gamma^2+|W_s^{\eta,\varepsilon}-W_s^{\eta',\varepsilon'}|^2}}\right)\cdot dB_s$$

since this term will not vanish if we take the expectation after the supremum. For this, we use Doob's inequality to estimate it as

$$\begin{aligned}
&\mathbb{E}\left[\sup_{\tau\le t}\left|\int_0^\tau\left(\mathcal{R}(W_s^{\eta,\varepsilon})-\mathcal{R}(W_s^{\eta',\varepsilon'})\right)\left(\frac{W_s^{\eta,\varepsilon}-W_s^{\eta',\varepsilon'}}{\sqrt{\gamma^2+|W_s^{\eta,\varepsilon}-W_s^{\eta',\varepsilon'}|^2}}\right)\cdot dB_s\right|\right]\\
&\le \mathbb{E}\left[\sup_{\tau\le t}\left|\int_0^\tau\left(\mathcal{R}(W_s^{\eta,\varepsilon})-\mathcal{R}(W_s^{\eta',\varepsilon'})\right)\left(\frac{W_s^{\eta,\varepsilon}-W_s^{\eta',\varepsilon'}}{\sqrt{\gamma^2+|W_s^{\eta,\varepsilon}-W_s^{\eta',\varepsilon'}|^2}}\right)\cdot dB_s\right|^2\right]^{1/2}\\
&\le 2\|\mathcal{R}\|_{Lip}\mathbb{E}\left[\int_0^t|W_s^{\eta,\varepsilon}-W_s^{\eta',\varepsilon'}|^2ds\right]^{1/2}.
\end{aligned}$$

Then this together with the similar estimates as in **Step A** for other terms gives

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left(|Y_t^{\eta,\varepsilon}-Y_t^{\eta',\varepsilon'}|+|W_t^{\eta,\varepsilon}-W_t^{\eta',\varepsilon'}|\right)\right]\le C\left(\eta+\varepsilon+\eta'+\varepsilon'\right),$$

for $\eta,\varepsilon,\eta',\varepsilon'<1/2$, where $C>0$ is independent of the regularization parameters, η,ε,η', and ε'.

◇ **Step C (Passing to the limit).-** It follows from **Step B** that there exists a limit process $(Y_t,W_t)_{t\in[0,T]}$ of the $(Y_t^{\eta,\varepsilon},W_t^{\eta,\varepsilon})_{t\in[0,T]}$ as $\eta,\varepsilon\to 0$ in $L^1(\Omega\times[0,T])$. Using the fact that

$$\mathcal{W}_1(f_t^{\eta,\varepsilon},f_t^{\eta',\varepsilon'})\le\mathbb{E}\left[|Y_t^{\eta,\varepsilon}-Y_t^{\eta',\varepsilon'}|+|W_t^{\eta,\varepsilon}-W_t^{\eta',\varepsilon'}|\right],$$

we also deduce that $(f_t^{\eta,\varepsilon})_{t\in[0,T]}$ is a Cauchy sequence in $\mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}^d\times\mathbb{R}^d))$. Then, by completeness of this space, we define $f\in\mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}^d\times\mathbb{R}^d))$ by $f_t:=\lim_{\eta,\varepsilon\to0}f_t^{\eta,\varepsilon}$ for $t\in[0,T]$. We next show that the limiting process $(Y_t,W_t)_{t\in[0,T]}$ obtained before is the solution to the nonlinear SIEs (4.123) and its time marginal f_t is the weak solution to the kinetic equation (1.111). For this, it is enough to show that

$$K_3^{\eta,\varepsilon}:=\mathbb{E}\left[\left|\int_0^t(F^{\eta,\varepsilon}[f_s^{\eta,\varepsilon}](Y_s^{\eta,\varepsilon},W_s^{\eta,\varepsilon})-F[f_s](Y_s,W_s))\,ds\right|\right]\to0$$

and

$$K_4^{\eta,\varepsilon}:=\mathbb{E}\left[\left|\int_0^t(\mathcal{R}(W_s^{\eta,\varepsilon})-\mathcal{R}(W_s))\cdot dB_s\right|\right]\to0$$

as $\eta, \varepsilon \to 0$. Using similar arguments for the term K_1 in **Step A**, we easily estimate $K_3^{\eta,\varepsilon}$ as

$$K_3^{\eta,\varepsilon} \le C(\eta+\varepsilon) + C(1+\|f_s\|_{L^\infty})\mathbb{E}\left[|Y_s^{\eta,\varepsilon} - Y_s| + |W_s^{\eta,\varepsilon} - W_s|\right] \to 0 \quad \text{as} \quad \eta, \varepsilon \to 0.$$

We next use the Lipschitz regularity of $\mathcal{R}$ together with Itô isometry to get

$$\begin{aligned} K_4^{\eta,\varepsilon} &\le \mathbb{E}\left[\left|\int_0^t (\mathcal{R}(W_s^{\eta,\varepsilon}) - \mathcal{R}(W_s)) \cdot dB_s\right|^2\right]^{1/2} \\ &= \mathbb{E}\left[\int_0^t (\mathcal{R}(W_s^{\eta,\varepsilon}) - \mathcal{R}(W_s))^2\, ds\right]^{1/2} \\ &\le \|\mathcal{R}\|_{Lip}\, \mathbb{E}\left[\int_0^t |W_s^{\eta,\varepsilon} - W_s|^2\, ds\right]^{1/2} \to 0 \quad \text{as} \quad \eta, \varepsilon \to 0. \end{aligned}$$

This concludes that the liming process $(Y_t, W_t)_{t\in[0,T]}$ is the solution to the nonlinear SIEs (4.123). For the weak solutions to the kinetic equation (1.111), we take a test function $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$ and apply Itô's formula to that solution of the system (4.123) as in the proof of Proposition 4.1. Then we can find that its time marginals f_t solves that kinetic equation (1.111) in the distributional sense. We also easily check that solution $f \in L^\infty(\mathbb{R}^{2d} \times [0,T])$.

$\diamond$ **Step D (Stability estimate).-** If $(Y_t^i, W_t^i), i = 1, 2$ are two processes obtained above with the initial data $(Y_0^i, W_0^i), i = 1, 2$, respectively, and $\mathcal{L}(Y_t^i, W_t^i) = f_t, i = 1, 2$, $t \in [0, T]$, then by employing almost same argument as in **Step A**, we can easily find

$$\begin{aligned} &\mathbb{E}\left[|Y_t^1 - Y_t^2| + \sqrt{\gamma^2 + |W_t^1 - W_t^2|^2}\right] \\ &\le \mathbb{E}\left[|Y_0^1 - Y_0^2| + \sqrt{\gamma^2 + |W_0^1 - W_0^2|^2}\right] \\ &\quad + C\int_0^t \left(1 + \|f_s^1\|_{L^\infty}\right)\mathbb{E}\left[|Y_s^1 - Y_s^2| + \sqrt{\gamma^2 + |W_s^1 - W_s^2|^2}\right] ds. \end{aligned}$$

Applying Gronwall's inequality and letting $\gamma \to 0$ conclude the desired stability estimate. □

Proof of Theorem 1.2. We already discussed the existence and uniqueness of strong solutions to the nonlinear SIEs (1.110) and the existence of weak solutions to the kinetic equation (1.111) in Proposition 3.2. For the uniqueness of weak solutions to (1.111), we use the fact that for any solutions to (1.111) can be represented as the time marginals of some solutions to (4.123). More precisely, let $(\tilde{f}_t)_{t\ge0}$ be a weak solution to the equation (1.111) with the initial data $\tilde{f}_0 \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$ and $(f_t)_{t\ge0}$ be another weak solution to (1.111) with the initial data $f_0 \in (L^1 \cap L^\infty)(\mathbb{R}^d \times \mathbb{R}^d) \cap \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$ such that $f \in L^\infty(0, T; (L^1 \cap L^\infty)(\mathbb{R}^d \times \mathbb{R}^d)) \cap \mathcal{C}([0,T]; \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d))$. Then we can find a probability space $(\Omega, \mathbb{P}, (\mathcal{F}_t)_{t\ge0}, \mathcal{F})$, a Brownian motion $(B_t)_{t\ge0}$ on that basis and a process $(X_t, V_t)_{t\ge0}$ solution to (4.123), which has the time marginal $\tilde{f}_t$ at any time $t \ge 0$. We refer to [**15**] for more details on that. On that probability space, let (Y_0, W_0) be a random variable on $\mathbb{R}^d \times \mathbb{R}^d$ with the law f_0 independent of $(B_t)_{t\ge0}$ such that

$$\mathcal{W}_1(f_0, \tilde{f}_0) = \mathbb{E}\left[|X_0 - Y_0| + |V_0 - W_0|\right].$$

Since the force fields has a kind of Lipschitz property for the given f_t, we can construct some stochastic process $(Y_t, W_t)_{t\ge0}$ which is a solution to (4.123) with the initial condition (Y_0, W_0), and same Brownian motion as before such that its time marginal is f_t. Then, by using the stability estimate for SIEs (4.123) in Proposition 3.2, we have

$$\begin{aligned} \mathcal{W}_1(f_t, \tilde{f}_t) &\le \mathbb{E}\left[|Y_t - X_t| + |V_t - W_t|\right] \\ &\le \mathbb{E}\left[|Y_0 - X_0| + |V_0 - W_0|\right] e^{\int_0^t \|f_s\|_{L^1\cap L^\infty} ds} \\ &= \mathcal{W}_1(f_0, \tilde{f}_0) e^{\int_0^t \|f_s\|_{L^1\cap L^\infty} ds}, \end{aligned}$$

and this concludes the uniqueness of solutions to the kinetic equation (1.111). □

5. Propagation of chaos: Proof of Theorem 1.3

We first show law of large numbers-like estimates whose proofs rely on the nice property of our communication weights in Lemma 1.1. Even though similar results observed in [**15, 18**], for the sake of completeness we give the details of it in Appendix 6.

LEMMA 5.1. *Let $(Y_i, W_i)_{i=1,\cdots,N}$ be i.i.d. random variables with law $f \in (\mathcal{P}_1 \cap L^\infty)(\mathbb{R}^{2d})$, and let μ_N be the associated empirical measure $\mu_N = \frac{1}{N}\sum_{i=1}^N \delta_{(Y_i,W_i)}$ and a pair (Y,W) be independent of $(Y_i,W_i)_{i=1,\cdots,N}$. Let us also denote by ρ the first marginal of f and $\rho_N = \frac{1}{N}\sum_{i=1}^N \delta_{Y_i}$. Then we have*

$$\mathbb{E}\left[\sup_{u\geq 0}\left|\int_{\mathbb{R}^d} \mathbf{1}_{\Theta(W)^{u,+}}(y-Y)(\rho_N-\rho)(dy)\right|\right] \leq \frac{C}{\sqrt{N}}$$

and

$$\mathbb{E}\left[\sup_{u\geq 0}\left|\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbf{1}_{\Theta(W)^{u,+}}(y-Y)(\mu_N-f)(dy,dw)\right|\right] \leq \frac{C}{\sqrt{N}}.$$

LEMMA 5.2. *Let $(Y_i)_{i=1,\cdots,N}$ be i.i.d. random variables with law $\rho \in \mathcal{P}(\mathbb{R}^d)$, and set h a measurable function in $\mathbb{R}^d$ such that $h(Y_1 - Y_2)$ is almost surely bounded from above by some constant c_0 and $h(0) = 0$. Then we have*

$$\mathbb{E}\left[\left|\int_{\mathbb{R}^d} h(Y_1, Y_1 - y)(\rho_N - \rho)(dy)\right|\right] \leq \frac{C}{\sqrt{N}},$$

where $\rho_N = \frac{1}{N}\sum_{i=1}^N \delta_{Y_i}$.

We next provide a quantitative estimate between velocity fields of the stochastic integral inclusion system (2.114) and the nonlinear SIEs (4.123).

LEMMA 5.3. *Let $(X_i, V_i)_{i=1,\cdots,N}$ be N random variables on $\mathbb{R}^{2d}$ and $(Y_i, W_i)_{i=1,\cdots,N}$ be N i.i.d random variables on $\mathbb{R}^{2d}$ with law $f \in L^\infty(\mathbb{R}^{2d})$ compactly supported in velocity in $\mathbb{B}^d_{V_{max}}$. Then there exists a constant C depending only on V_m, d such that*

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N \left|\tilde{F}[\mu_N](X_i,V_i) - F[f](Y_i,W_i)\right|\right] \leq C\|f\|_{L^1\cap L^\infty}\frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[|X_i - Y_i| + |V_i - W_i|\right] + \frac{C}{\sqrt{N}},$$

where $\mu_N = \frac{1}{N}\sum_{i=1}^N \delta_{(X_i,V_i)}$ and we abused the notation $|\cdot|$ for the distance between a set and a vector.

PROOF. We first introduce an empirical measure $\nu_N = \frac{1}{N}\sum_{i=1}^N \delta_{(Y_i,W_i)}$. For any $z_i \in \tilde{F}[\mu^N](X_i,V_i)$ with $i \in \{1\cdots,N\}$, there exists some (random variables) $(\alpha_{i,j})_{j=1,\cdots,N} \in [0,1]^N$ such that

$$z_i = \frac{1}{N}\sum_{j=1}^N \alpha_{i,j}(V_j - V_i).$$

Then, for $i \in \{1,\cdots,N\}$, we get

$$\begin{aligned}
&\mathbb{E}\left[|z - F[f](Y_i,W_i)|\right] \\
&\quad\leq \mathbb{E}\left[\frac{1}{N}\sum_{j=1}^N \left(\alpha_{i,j} - \mathbf{1}_{K(V_i)}(X_i - X_j)\right)(V_j - V_i)\right] \\
&\qquad + \mathbb{E}\left[|F[\nu_N](Y_i,W_i) - F[\mu_N](X_i,V_i)|\right] + \mathbb{E}\left[|F[\nu_N](Y_i,W_i) - F[f](Y_i,W_i)|\right] \\
&\quad=: I_1^i + I_2^i + I_3^i.
\end{aligned}$$

$\diamond$ Estimate of I_1^i: Note that $\alpha_{i,j} - \mathbf{1}_{K(V_i)}(X_j - X_i) \neq 0$ only if $X_j - X_i \in \tilde{\partial}K(V_i)$. This together with the assumption (**H2**) (iii) yields

$$\begin{aligned}
|\alpha_{i,j} - \mathbf{1}_{K(V_i)}(X_j - X_i)| &\leq C\mathbf{1}_{\{X_j - X_i \in \tilde{\partial}K(V_i)\}} \leq C\mathbf{1}_{\left\{Y_j - Y_i \in \Theta(W_i)^{C|Y_j - X_j| + |Y_i - X_i| + |V_i - W_i|,+}\right\}} \\
&\leq C\mathbf{1}_{\left\{Y_j - Y_i \in \Theta(W_i)^{C(|Y_i - X_i| + |V_i - W_i|),+}\right\}} + C\mathbf{1}_{\left\{Y_i - Y_j \in \Theta(W_i)^{C|Y_j - X_j|,+}\right\}}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
I_1^i &\le CV_m\mathbb{E}\left[\frac{1}{N}\sum_{j=1}^N \mathbf{1}_{\Theta(W_i)^{|Y_i-X_i|+|V_i-W_i|,+}}(Y_i-Y_j)+\mathbf{1}_{Y_i-Y_j\in\Theta(W_i)^{C|Y_j-X_j|,+}}\right]\\
&\le CV_m\mathbb{E}\left[\int_{\mathbb{R}^d}\mathbf{1}_{\left\{y-Y_i\in\Theta(W_i)^{C(|Y_i-X_i|+|V_i-W_i|),+}\right\}}\rho(dy)\right]\\
&\quad + CV_m\mathbb{E}\left[\frac{1}{N}\sum_{j=1}^N \mathbf{1}_{\left\{Y_i-Y_j\in\Theta(W_i)^{C|Y_j-X_j|,+}\right\}}\right]\\
&\quad + CV_m\mathbb{E}\left[\sup_{u\ge 0}\left|\int_{\mathbb{R}^d}\mathbf{1}_{\left\{y-Y_i\in\Theta(W_i)^{u,+}\right\}}(\rho-\rho^N)(dy)\right|\right].
\end{aligned}$$

Summing I_1^i over $i=1,\cdots,N$ and dividing it by N lead

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N I_1^i\right] &\le C\|\rho\|_{L^1\cap L^\infty}\frac{1}{N}\sum_{i=1}^N(|X_i-Y_i|+|V_i-W_i|)\\
&\quad + C\mathbb{E}\left[\sup_{u\ge 0}\left|\int_{\mathbb{R}^d}\mathbf{1}_{\left\{y-Y_i\in\Theta(W_i)^{u,+}\right\}}(\rho-\rho^N)(dy)\right|\right]\\
&\quad + C\mathbb{E}\left[\sup_{u\ge 0}\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}_{\left\{y-Y_i\in\Theta(w)^{u,+}\right\}}(f-\nu^N)(dy,dw)\right|\right].
\end{aligned}$$

$\diamond$ Estimate of I_2^i: We divide it into three terms:

$$\begin{aligned}
I_2^i &= \mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N\Big(\mathbf{1}_{K(V_i)}(X_j-X_i)(V_i-V_j)-\mathbf{1}_{K(W_i)}(Y_j-Y_i)(W_i-W_j)\Big)\right|\right]\\
&= \mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N\mathbf{1}_{K(V_i)}(X_j-X_i)\Big((V_i-V_j)-(W_i-W_j)\Big)\right|\right]\\
&\quad + \mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N\Big(\mathbf{1}_{K(V_i)}(X_j-X_i)-\mathbf{1}_{K(W_i)}(X_j-X_i)\Big)(W_i-W_j)\right|\right]\\
&\quad + \mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N\Big(\mathbf{1}_{K(W_i)}(X_j-X_i)-\mathbf{1}_{K(W_i)}(Y_j-Y_i)\Big)(W_i-W_j)\right|\right]\\
&=: I_{2,1}^i+I_{2,2}^i+I_{2,3}^i.
\end{aligned}$$

• Estimate of $I_{2,1}^i$: We easily find

$$I_{2,1}^i \le \mathbb{E}[|V_i-W_i|]+\frac{1}{N}\sum_{j=1}^N\mathbb{E}[|V_j-W_j|]\quad\text{for}\quad i=1,\cdots,N.$$

• Estimate of $I_{2,2}^i$: Using the assumptions (**H2**) (ii)-(iii) and the fact that $\mathbf{1}_A(x)\le\mathbf{1}_{A^{|x-y|,+}}(y)$ for $A\subset\mathbb{R}^d$ and $x,y\in\mathbb{R}^d$, we get

$$\begin{aligned}
&\big|\mathbf{1}_{K(V_i)}(X_j-X_i)-\mathbf{1}_{K(W_i)}(X_j-X_i)\big|\\
&\quad\le \mathbf{1}_{K(V_i)\Delta K(W_i)}(X_j-X_i)\\
&\quad\le \mathbf{1}_{\Theta(W_i)^{C|V_i-W_i|,+}}(X_j-X_i)\\
&\quad\le \mathbf{1}_{\Theta(W_i)^{C|V_i-W_i|+|X_i-Y_i|+|X_j-Y_j|,+}}(Y_j-Y_i)\\
&\quad\le \mathbf{1}_{\Theta(W_i)^{2C|V_i-W_i|+2|X_i-Y_i|,+}}(Y_j-Y_i)+\mathbf{1}_{\Theta(W_i)^{2|X_j-Y_j|,+}}(Y_j-Y_i).
\end{aligned}$$

Regarding the first term on the right hand side of the above last inequality, we obtain

$$\begin{aligned}
&\frac{1}{N}\sum_{j=1}^{N}\mathbf{1}_{\Theta(W_i)^{2C|V_i-W_i|+2|X_i-Y_i|,+}}(Y_j-Y_i)\\
&\quad=\int_{\mathbb{R}^d}\mathbf{1}_{\Theta(W_i)^{2C|V_i-W_i|+2|X_i-Y_i|,+}}(y-Y_i)\,\rho_N(dy)\\
&\quad\le\int_{\mathbb{R}^d}\mathbf{1}_{\Theta(W_i)^{2C|V_i-W_i|+2|X_i-Y_i|,+}}(y-Y_i)\,\rho(dy)\\
&\qquad+\left|\int_{\mathbb{R}^d}\mathbf{1}_{\Theta(W_i)^{2C|V_i-W_i|+2|X_i-Y_i|,+}}(y-Y_i)\,(\rho_N-\rho)(dy)\right|\\
&\quad\le C(\|\rho\|_{L^1\cap L^\infty})\big(|V_i-W_i|+|X_i-Y_i|\big)+\sup_{u\ge0}\left|\int_{\mathbb{R}^d}\mathbf{1}_{\Theta(W_i)^{u,+}}(y-Y_i)\,(\rho_N-\rho)(dy)\right|,
\end{aligned}$$

for each $i=1,\cdots,N$, where we used the assumption (**H2**) (ii). Similarly, for each $j=1,\cdots,N$, we find

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}_{\Theta(W_i)^{2|X_j-Y_j|,+}}(Y_j-Y_i)\\
&\quad=\int_{\mathbb{R}^d}\mathbf{1}_{\Theta(w)^{2|X_j-Y_j|,+}}(Y_j-y)\,\nu_N(dy)\\
&\quad\le\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}_{\Theta(w)^{2|X_j-Y_j|,+}}(Y_j-y)\,f(dy,dw)+\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}_{\Theta(w)^{2|X_j-Y_j|,+}}(Y_j-y)(\nu_N-f)(dy,dw)\right|\\
&\quad\le C\|f\|_{L^1\cap L^\infty}|X_j-Y_j|+\sup_{u\ge0}\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}_{\Theta(w)^{u,+}}(Y_j-y)(\nu_N-f)(dy,dw)\right|.
\end{aligned}$$

Combining the above estimates together with Lemma 4.1, we have

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[I^i_{2,2}]\le C\|f\|_{L^1\cap L^\infty}\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[|X_i-Y_i|+|V_i-W_i|\right]+\frac{C}{\sqrt{N}},$$

due to $\|\rho\|_{L^\infty}\le C\|f\|_{L^\infty}$ for some $C>0$ depending on V_m and d.

• Estimate of $I^i_{2,3}$: It follows from Lemma 1.1 that

$$\big|\mathbf{1}_{K(W_i)}(X_j-X_i)-\mathbf{1}_{K(W_i)}(Y_j-Y_i)\big|\le\mathbf{1}_{\partial^{|X_i-Y_i|}K(W_i)}(Y_j-Y_i)+\mathbf{1}_{\partial^{|X_j-Y_j|}K(W_i)}(Y_j-Y_i).$$

We then define an empirical measure $\rho_N:=\frac{1}{N}\sum_{j=1}^{N}\delta_{Y_i}$ to estimate that for each $i=1,\cdots,N$

$$\begin{aligned}
&\frac{1}{N}\sum_{j=1}^{N}\mathbf{1}_{\partial^{2|X_i-Y_i|}K(W_i)}(Y_j-Y_i)\\
&\quad=\int_{\mathbb{R}^d}\mathbf{1}_{\partial^{2|X_i-Y_i|}K(W_i)}(y-Y_i)\,\rho_N(dy)\\
&\quad\le\int_{\mathbb{R}^d}\mathbf{1}_{\partial^{2|X_i-Y_i|}K(W_i)}(y-Y_i)\,\rho(dy)+\left|\int_{\mathbb{R}^d}\mathbf{1}_{\partial^{2|X_i-Y_i|}K(W_i)}(y-Y_i)\,(\rho_N-\rho)(dy)\right|\\
&\quad\le C\,\|\rho\|_{L^1\cap L^\infty}\,|X_i-Y_i|+\sup_{u\ge0}\left|\int_{\mathbb{R}^d}\mathbf{1}_{\partial^u K(W_i)}(y-Y_i)\,(\rho_N-\rho)(dy)\right|.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}_{\partial^{2|Y_j-X_j|}K(W_i)}(Y_j-Y_i)\\
&\quad=\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}_{\partial^{2|Y_j-X_j|}K(w)}(Y_j-y)\,\nu_N(dy,dw)\\
&\quad\le\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}_{\partial^{2|X_j-Y_j|}K(w)}(Y_i-y)\,f(dy,dw)+\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}_{\partial^{2|X_j-Y_j|}K(w)}(Y_j-y)\,(\nu_N-f)(dy,dw)\right|\\
&\quad\le C\,\|f\|_{L^1\cap L^\infty}\,|X_j-Y_j|+\sup_{u\ge0}\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}_{\partial^u K(w)}(Y_j-y)\,(\nu_N-f)(dy,dw)\right|,
\end{aligned}$$

for each $j = 1, \cdots, N$. From the above estimates together with Lemma 4.1, we obtain

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[I_{2,3}^i] \le C\|f\|_{L^1\cap L^\infty}\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[|X_i - Y_i|\right] + \frac{C}{\sqrt{N}}.$$

Collecting the all of above estimates, we have

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[I_2^i] \le C\|f\|_{L^1\cap L^\infty}\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[|X_i - Y_i| + |V_i - W_i|\right] + \frac{C}{\sqrt{N}},$$

where $C > 0$ is independent of N.

$\diamond$ Estimate of I_3^i: For each $i = 1, \cdots, N$, we define $h_i(y, w) = -\mathbf{1}_{K(W_i)}(y)w$ and rewrite I_3^i as

$$I_3^i = \mathbb{E}\left[\left|\int_{\mathbb{R}^{2d}} h_i(Y_i - y, W_i - w)(\nu_N - f)(dy, dw)\right|\right].$$

Then since $|h_i(Y_i - Y_j)(W_j - W_i)| \le 2V_m$ for any $j \neq i$, it follows from Lemma 4.2 that

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[I_3^i] \le \frac{C}{\sqrt{N}}.$$

Finally, we combine the above estimates to conclude our desired result. □

Proof of Theorem 1.3. Define μ_t^N, ν_t^N by empirical measures associated to systems (2.114) and (1.110):

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^{N}\delta_{(X_t^i, V_t^i)} \quad \text{and} \quad \nu_t^N = \frac{1}{N}\sum_{i=1}^{N}\delta_{(Y_t^i, W_t^i)}.$$

Then, for $i = 1, \cdots, N$ and $t \in [0, T]$, by using Itô's formula, we find

$$\begin{aligned}\sqrt{\gamma^2 + |V_t^i - W_t^i|^2} = \gamma &+ \int_0^t \left\langle \frac{V_s^i - W_s^i}{\sqrt{\gamma^2 + |V_s^i - W_s^i|^2}}, \tilde{F}[\mu_s^N](X_s^i, V_s^i) - F[f_s](Y_s^i, W_s^i)\right\rangle ds \\ &+ \sqrt{2\sigma}\int_0^t \frac{\mathcal{R}(V_s^i) - \mathcal{R}(W_s^i)}{\sqrt{\gamma^2 + |V_s^i - W_s^i|^2}}\left\langle V_s^i - W_s^i,\, dB_s^i\right\rangle \\ &+ \sigma\int_0^t \frac{d\gamma^2 + (d-1)|V_s^i - W_s^i|^2}{\sqrt{\gamma^2 + |V_s^i - W_s^i|^2}^3}\left(\mathcal{R}(V_s^i) - \mathcal{R}(W_s^i)\right)^2 ds,\end{aligned}$$

due to $V_0^j = W_0^j$ for all $j \in \{1, \cdots, N\}$. We then take the averaged value of $\sqrt{\gamma^2 + |V_t^i - W_t^i|^2}$ and the expectation to that to obtain

$$\begin{aligned}\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\sqrt{\gamma^2 + |V_t^i - W_t^i|^2}\right] \le C\gamma &+ \int_0^t \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N}\tilde{F}[\mu_s^N](X_s^i, V_s^i) - F[f_s](Y_s^i, W_s^i)\right|\right] ds \\ &+ C\int_0^t \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}|V_s^i - W_s^i|\right] ds,\end{aligned}$$

due to the Lipschitz regularity of $\mathcal{R}$. Using Lemma 4.3 yields

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[\sqrt{\gamma^2 + |V_t^i - W_t^i|^2}\right] \le C\gamma + C\|f_s\|_{L^1\cap L^\infty}\frac{1}{N}\sum_{i=1}^{N}\int_0^t \mathbb{E}\left[\sqrt{\gamma^2 + |V_s^i - W_s^i|^2}\right] ds + \frac{C}{\sqrt{N}},$$

where C is a positive constant depending only on $\mathcal{R}, T$, and V_m. Moreover, by letting $\gamma \to 0$ and using the fact

$$\mathbb{E}\left[|X_t^i - Y_t^i|\right] \le \int_0^t \mathbb{E}\left[|V_s^i - W_s^i|\right] ds,$$

we obtain

$$\begin{aligned}&\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\left(|X_t^i - Y_t^i| + |V_t^i - W_t^i|\right)\right] \\ &\le C\int_0^t \|f_s\|_{L^1\cap L^\infty}\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}(|X_s^i - Y_s^i| + |V_s^i - W_s^i|)\right] ds + \frac{C}{\sqrt{N}}.\end{aligned}$$

Applying Gronwall's inequality to the above inequality gives

$$\mathbb{E}\left[\mathcal{W}_1(\mu_t^N,\nu_t^N)\right] \le \frac{C}{\sqrt{N}}\exp\left(\int_0^t \|f_s\|_{L^1\cap L^\infty}\,ds\right).$$

Finally, we use the convergence estimate stated in Proposition 2.1 together with the moment estimate in Remark 1.2 to find that for all $t\in[0,T]$

$$\mathbb{E}\left[\mathcal{W}_1(\mu_t^N,f_t)\right] \le C\begin{cases} N^{-1/2}+N^{-(q-1)/q} & \text{if } 2>d \text{ and } q\neq 2,\\ N^{-1/2}\log(1+N)+N^{-(q-1)/q} & \text{if } 2=d \text{ and } q\neq 2,\\ N^{-1/d}+N^{-(q-1)/q} & \text{if } 2<d \text{ and } q\neq d/(d-1).\end{cases}$$

This completes the proof. □

6. Appendix : Proofs of Lemmas 4.1 and 4.2

Proof of Lemma 4.1. Let $(Y_n,W_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables with law $f\in\mathcal{P}(\mathbb{R}^{2d})$ and independent of the (Y,W) and K_N be a Poisson random variable of parameter N independent of $(Y_n,W_n)_{n\in\mathbb{N}}$. Define ϱ_N the following random measure

$$\varrho_N=\sum_{i=1}^{K_N}\delta_{Y_i},$$

where $\delta_x(A)$ is a degenerate measure located in x, i.e., $\delta_x(A)=\mathbf{1}_A(x)$. Then ϱ_N is a Poisson random measure of intensity $N\rho$. Then it is straightforward to get $\|\varrho_N-N\rho_N\|_{TV}\le|K_N-N|$, where $\|\cdot\|_{TV}$ represents the total variation distance of probability measures. Then we get

$$\begin{aligned}
&\sup_{u\ge0}\left|\int_{\mathbb{R}^d}\mathbf{1}_{\Theta(v)^{u,+}}(y-a)(\rho_N-\rho)(dy)\right|\\
&\le\frac{1}{N}\sup_{u\ge0}\left|\int_{\mathbb{R}^d}\mathbf{1}_{\Theta(v)^{u,+}}(y-a)(\varrho_N-N\rho)(dy)\right|+\frac{1}{N}\sup_{u\ge0}\left|\int_{\mathbb{R}^d}\mathbf{1}_{\Theta(v)^{u,+}}(y-a)(\varrho_N-N\rho_N)(dy)\right|\\
&\le\frac{1}{N}\sup_{u\ge0}\left|\int_{\mathbb{R}^d}\mathbf{1}_{\Theta(v)^{u,+}}(y-a)\overline{\varrho}_N(dy)\right|+\frac{1}{N}\|\varrho_N-N\rho_N\|_{TV}\\
&\le\frac{1}{N}\sup_{u\ge0}\left|\mathcal{M}_u^N\right|+\frac{|K_N-N|}{N},\quad \mathbb{P}-a.s.,
\end{aligned}$$

for all (a,v), where

$$\mathcal{M}_u^N=\int_{\mathbb{R}^d}\mathbf{1}_{\Theta(v)^{u,+}}(y-a)\overline{\varrho}_N(dy)\quad\text{and}\quad\overline{\varrho}_N=\varrho_N-N\rho.$$

Note that $(\mathcal{M}_u^N)_{u\ge0}$ is a martingale with respect to the filtration $(\mathcal{F}_u)_{u\ge0}$ defined by:

$$\mathcal{F}_u=\sigma\left\{\int_{\mathbb{R}^d}h(y)\varrho_N(dy)\mid supp\ h\in\Theta(v)^{u,+}+a\right\},$$

since ϱ_N is the Poisson random measure. We then use Doob's inequality to obtain

$$\mathbb{E}\left[\sup_{u\ge0}\left|\mathcal{M}_u^N\right|\right]\le\left(\mathbb{E}\left[\sup_{u\ge0}\left|\mathcal{M}_u^N\right|^2\right]\right)^{1/2}\le\sqrt{2}\left(\mathbb{E}\left[\left|\mathcal{M}_\infty^N\right|^2\right]\right)^{1/2}=\sqrt{2}N^{1/2},$$

where we also used the fact that $\mathcal{M}_\infty^{N,a}=M_N(\mathbb{R}^d)-N$ and $M_N(\mathbb{R}^d)$ is a Poisson random variable of parameter $N\rho(\mathbb{R}^d)=N$ so that $\mathbb{V}(\mathcal{M}_\infty^{N,a})=N$. Moreover, by the property of Poisson random variable, we get

$$\mathbb{E}\left[|K_N-N|\right]\le(\mathbb{V}(K_N))^{1/2}=N^{1/2}.$$

Combining the above estimates, we find

$$\mathbb{E}\left[\sup_{u\ge0}\left|\int_{\mathbb{R}^d}\mathbf{1}_{\Theta(v)^{u,+}}(y-a)(\rho_N-\rho)(dy)\right|\right]\le\frac{3}{\sqrt{N}}\quad\text{for all}\quad a\in\mathbb{R}^d. \tag{6.128}$$

We then use the classical property of conditional expectation to get

$$\mathbb{E}\left[\sup_{u\ge0}\left|\int_{\mathbb{R}^d}\mathbf{1}_{\Theta(v)^{u,+}}(y-Y)(\rho_N-\rho)(dy)\right|\right]=\mathbb{E}\left[\mathbb{E}\left[\sup_{u\ge0}\left|\int_{\mathbb{R}^d}\mathbf{1}_{\Theta(v)^{u,+}}(y-Y)(\rho_N-\rho)(dy)\right|\mid(Y,W)\right]\right].$$

This and together with (6.128) and the fact that (Y, W) is independent of the $(Y_i, W_i)_{i=1,\cdots,N}$ yields

$$\mathbb{E}\left[\sup_{u\geq 0}\left|\int_{\mathbb{R}^d}\mathbf{1}_{\Theta(v)^{u,+}}(y-Y)(\rho_N-\rho)(dy)\right| \mid (Y,W)\right] \leq \frac{3}{\sqrt{N}}.$$

Similarly, we define

$$M_N = \sum_{i=1}^{K_N}\delta_{(Y_i,W_i)},$$

then we obtain

$$\begin{aligned}
&\sup_{u\geq 0}\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}_{\Theta(w)^{u,+}}(y-a)(\mu_N-f)(dy,dw)\right| \\
&\quad\leq \frac{1}{N}\sup_{u\geq 0}\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}_{\Theta(w)^{u,+}}(y-a)(M_N-Nf)(dy,dw)\right| \\
&\qquad+\frac{1}{N}\sup_{u\geq 0}\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}_{\Theta(w)^{u,+}}(y-a)(M_N-N\mu_N)(dy,dw)\right| \\
&\quad\leq \frac{1}{N}\sup_{u\geq 0}\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}_{\Theta(w)^{u,+}}(y-a)\overline{M}_N(dy,dw)\right|+\frac{1}{N}\left\|M_N-N\mu_N\right\|_{TV} \\
&\quad\leq \frac{1}{N}\sup_{u\geq 0}\left|\widetilde{\mathcal{M}}_u^N\right|+\frac{|K_N-N|}{N},
\end{aligned}$$

with

$$\widetilde{\mathcal{M}}_u^N := \int_{\mathbb{R}^d\times\mathbb{R}^d}\mathbf{1}_{\Theta(w)^{u,+}}(y-a)\overline{M}_N(dy,dw) = \overline{M}_N(A_{a,u}), \quad A_{a,u} := \{(y,w)\in\mathbb{R}^{2d} \mid y\in\Theta(w)^{u,+}+a\}.$$

Next we introduce a filtration $\widetilde{\mathcal{F}}_u = \sigma\{r\leq u,\ M_N(A_{a,r})\}$ and use the similar arguments as the above to deduce that $(\widetilde{\mathcal{M}}_u^N)_{u\geq 0}$ is a martingale. Then the rest of proof is almost same as before. □

Proof of Lemma 4.2. Note that the random variables $Y_2,\cdots,Y_N$ conditioned to Y_1 are still i.i.d. Then, by standard property on the conditional expectation, we estimate

$$\begin{aligned}
\mathbb{E}\left[\left|\int_{\mathbb{R}^d}h(Y_1-y)(\rho_N-\rho)(dy)\right|\right] &= \mathbb{E}\left[\mathbb{E}\left[\left|\int_{\mathbb{R}^d}h(Y_1-y)(\rho_N-\rho)(dy)\right| \mid Y_1\right]\right] \\
&\leq \mathbb{E}\left[\left(\mathbb{E}\left[\frac{1}{N^2}\sum_{i=2}^{N}|h(Y_1-Y_i)-h*\rho(Y_1)|^2 \mid Y_1\right]\right)^{1/2}\right] \\
&\leq \mathbb{E}\left[\sqrt{\frac{N-1}{N^2}}\left(\mathbb{E}\left[|h(Y_1-Y_2)-h*\rho(Y_1)|^2 \mid Y_1\right]\right)^{1/2}\right].
\end{aligned}$$

On the other hand, since X_2 is independent of X_1, we get

$$\mathbb{E}\big[|h(Y_1-Y_2)-h*\rho(Y_1)|^2 \mid Y_1\big] = \int_{\mathbb{R}^d}|h(Y_1-y)-h*\rho(Y_1)|^2\rho(dy) \leq 4c_0^2.$$

Here we also used the facts that $h(Y_1-Y_2)$ is almost surely bounded from above by c_0 and Y_1 is independent of Y_2 to obtain $|h*\rho(Y_1)| = \mathbb{E}\left[|h(Y_1-Y_2)| \mid Y_1\right] \leq c_0$. □

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CHAPTER 4

Agregation equations on bounded domain and sharp sensing zones

1. Introduction

Mathematical modelling of collective behaviors, such as flocks of birds, schools of fish, or aggregation of bacteria, etc, has received a bulk of attention because of its possible applications in the field of engineering, biology, industry, and sociology [**2, 19, 23, 30, 31**]. These models are usually based on incorporating different mechanisms of interactions between individuals, for instance, a short-range repulsion, a long-range attraction, and an alignment in certain spatial regions. We refer the reader to [**5, 7, 9**] and the references therein for recent surveys of collective behavior models. In this current work, we consider the continuity-type model for collective behavior in the presence of diffusion. More precisely, we are interested in the propagation of chaos for interacting diffusing particles with reflecting boundary conditions describing collective behavior of individuals with vision geometrical constraints. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$. On that stochastic basis let $\{B_t^i\}_{i=1}^N$ be N independent d-dimensional Brownian motions and $\mathcal{O}$ be a bounded open convex set of $\mathbb{R}^d$ with Lipschitz boundary $\partial\mathcal{O}$. In this setting, our main stochastic integral equations(in short, SIEs) are given by

$$\begin{cases} X_t^i = X_0^i + \displaystyle\int_0^t V[\mu_s^N](X_s^i)\, ds + \sqrt{2\sigma} B_t^i - K_t^i, \quad i = 1, \cdots, N, \quad t > 0, \\ K_t^i = \displaystyle\int_0^t n(X_s^i)\, d|K^i|_s, \quad |K^i|_t = \int_0^t \mathbf{1}_{\partial\mathcal{O}}(X_s^i)\, d|K^i|_s, \end{cases} \tag{1.129}$$

where $\mu_s^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_s^i}$ and $n(x)$ denotes the outward normal to $\partial\mathcal{O}$ at the point $x \in \partial\mathcal{O}$. Here X_t^i is the position of i-th particle at time $t \geq 0$, K_t^i is a $\mathbb{R}^d$-bounded total variation process called the reflecting force, and $V[\mu]$ represents the velocity field non-locally computed in terms of the density:

$$V[\mu](x) = \int_{\mathbb{R}^d} \nabla\varphi(x-y)\mathbf{1}_{K(w(x))}(y-x)\mu(dy) \quad \text{for} \quad \mu \in \mathcal{P}(\mathbb{R}^d), \tag{1.130}$$

where $\mathbf{1}_{K(w(x))}$ is the indicator function on the set $K(w(x)) \subset \mathbb{R}^d$ and w is an orientational field, and $\nabla\varphi$ is a bounded Lipschitz interaction field.

As the total number of individuals gets large, the particle system leads to a macroscopic description based on the evolution of the probability density by means of mean-field limit. The rigorous derivation of the mean-field equation is well studied for sufficiently regular forces [**4, 12**], and it is extended to the equations with non-Lipschitz forces and noises in [**3**] under some uniform moment bounds conditions. For the deterministic particle system with singular kernels, the rigorous derivation of continuum descriptions is studied in [**5, 17**]. For the system (1.129) without noises and reflecting forces, the rigorous derivation of mean-field limit model can be obtained by employing the similar strategy as in [**6**], in which the second order collective behavior models with sharp sensitivity regions are considered.

Solving the SIEs (1.129) is known as Skorokhod problem [**26**]. This kind of problem is studied in [**29**] where a convex domain is considered, and then it is extended to a general domain satisfying some admissible conditions in [**20**]. Here, the admissibility roughly means that the domain can be approximated by smooth domains in a certain sense, see [**20**, p. 521]. Later, those conditions on the domain are removed in [**25**] by employing the strategy used in [**29**] and approximating the Skorokhod equation. It is worth mentioning that so far Skorokhod problems related to the propagation of chaos are only studied when that system has drift or force terms regular enough,

see [**21**], i.e., those are bounded and Lipschitz continuous, to the best of knowledge of the authors. Moreover, the rigorous derivation of mean-field limit of stochastic differential equations(in short, SDEs) with reflecting boundary conditions are only studied in [**27**], see [**18**] for the propagation of chaos of one dimensional Vlasov-Poisson-Fokker-Planck system.

The main purpose of this paper is to extend the result in [**27**], where stochastic differential equations with reflecting boundary conditions and bounded Lipschitz velocity fields are taken into account, to the case of discontinuous velocity fields. To be more precise, we will show that the N interacting processes $(X_t^i)_{i=1}^N$ of the system (1.129) well approximates as $N \to \infty$ the processes $(Y_t^i)_{i=1}^N$ to the following nonlinear SIEs:

(1.131)
$$\begin{cases} Y_t^i = Y_0^i + \displaystyle\int_0^t V[\rho_s](Y_s^i)\,ds + \sqrt{2\sigma}B_t^i - \widetilde{K}_t^i, \quad \mathcal{L}(Y_t^i) = \rho_t, \quad i = 1, \cdots, N, \quad t \geq 0, \\ \widetilde{K}_t^i = \displaystyle\int_0^t n(Y_s^i)\,d|\widetilde{K}^i|_s, \quad |\widetilde{K}^i|_t = \int_0^t \mathbf{1}_{\partial\mathcal{O}}(Y_s^i)\,d|\widetilde{K}^i|_s, \\ Y_0^i = X_0^i, \quad i = 1, \cdots, N. \end{cases}$$

By a straightforward application of Itô's formula, we find that the probability density function ρ_t is determined by a continuity type equation of the form:

(1.132)
$$\begin{cases} \partial_t \rho_t + \nabla \cdot (\rho_t V[\rho_t]) = \sigma \Delta \rho_t, \quad x \in \mathcal{O}, \quad t > 0, \\ V[\rho_t](x) = \displaystyle\int_{\mathcal{O}} \nabla\varphi(x-y)\mathbf{1}_{K(w(x))}(y-x)\rho_t(dy), \end{cases}$$

with the following initial data and boundary conditions:

$$\rho_t(x)|_{t=0} =: \rho_0(x), \quad x \in \mathcal{O} \quad \text{and} \quad \langle \sigma\nabla\rho - \rho V[\rho], n\rangle = 0 \quad \text{on} \quad \partial\mathcal{O}.$$

The equation (1.132) with the set $K \equiv \mathbb{R}^d$, i.e., without vision geometrical constrains, is of the classical form, usually called the *aggregation equation* [**5, 30**]. On the other hand, due to the presence of vision geometrical constraints in (1.132) which comes from the cutoff interaction function $\mathbf{1}_K$ in the velocity field V, the individuals at position x are only interacting with others inside the region $K(w(x))$. Considering the vision geometrical constraints is quite natural in the modelling of animal and human behavior, and realistic modelling of collective behaviors should deal with that, see [**1, 6, 24**]. In order to show the convergence of some probability measure, as well as for stability estimate for nonlinear PDEs, we use the Wasserstein distance which is defined by

$$\mathcal{W}_p^p(\mu,\nu) := \inf_{\xi\in\Gamma(\mu,\nu)} \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p \xi(dx,dy) \right) = \inf_{(X\sim\mu, Y\sim\nu)} \mathbb{E}[|X-Y|^p],$$

for $p \geq 1$ and $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$, where $\Gamma(\mu,\nu)$ is the set of all probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with first and second marginals μ and ν, respectively, and (X,Y) are all possible couples of random variables with μ and ν as respective laws. Given the types of diffusions we are taking into account in this current work, quadratic Wasserstein distances, for instance $\mathcal{W}_{2n}$ with $n \in \mathbb{N}$, could seem more convenient. Notice that the propagation of chaos for a system of interacting diffusing particles with normal reflection boundary condition is proved in [**27**] by making use of Wasserstein distance of order 4. However it has already been pointed out in [**6**, Remark 3.1] by the authors and their collaborators that the strategy used to deal with those discontinuous kernels does not work in Wasserstein distance of order p with $p \in (1,\infty)$. Thus, the use of either $\mathcal{W}_1$ or $\mathcal{W}_\infty$ becomes essential in our framework due to the form of velocity fields. Compared to the work in [**6**], in which the sensitivity region is independent of the position, the singularity of the interaction function is somehow stronger due to the position dependency of the set K, and this constrains us to use the infinite Wasserstein distance $\mathcal{W}_\infty$ defined as

$$\mathcal{W}_\infty(\mu,\nu) := \inf_{\xi\in\Gamma(\mu,\nu)} \xi\text{-}\operatorname*{ess\,sup}_{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d} |x-y| = \inf_{(X\sim\mu,Y\sim\nu)} \mathbb{P}\text{-}\operatorname{ess\,sup} |X-Y|.$$

Here we introduce several notations used throughout the paper. $|\cdot|$ and $\langle\cdot,\cdot\rangle$ denote the Euclidean distance and the standard inner product on $\mathbb{R}^d$, respectively. We also use the notation $|\cdot|$ for the Lebesgue measure of some set or the cardinal of finite index sets when there is no confusion. $\mathcal{P}(\mathcal{O})$ and $\mathcal{P}_p(\mathcal{O})$ stand for the sets of all probability measures and probability measures with finite moments of order $p \in [1,\infty)$ on $\mathcal{O}$, respectively. The notation for a probability measure and its probability density is often abused for notational simplicity. For a function $f(x)$, $\|f\|_{L^p}$ represents the usual $L^p(\mathcal{O})$-norm, and $\|f\|_{L^p\cap L^q} := \|f\|_{L^p} + \|f\|_{L^q}$. For $p \in [1,\infty]$ and $T > 0$,

$L^p(0,T;E)$ is the set of the L^p functions from an interval $(0,T)$ to a Banach space E. We also denote by C a generic positive constant. For a set $A \subset \mathbb{R}^d$, $\overline{A}$ represents the closure of A.

The rest of this paper is organized as follows. In the next subsection, we will give precise statements of our main results on the existence of solutions to the SIEs and the partial differential equations(in short, PDEs), and the propagation of chaos under suitable assumptions on the sensitivity regions. In Section 2, we present a global existence of solutions to the particle systems (1.129). Section 3 is devoted to provide the existence and uniqueness of solution to the PDE and its associated nonlinear SIEs. In Section 4, we give the details of proof for the propagation of chaos for the systems (1.129) with the aid of law of large number like estimates. In Appendix 5.1, we provide two Gronwall's type inequalities to be used in the proof. Finally, Appendix 5.2 is devoted to study a representation for solutions of the equation (1.132) giving some relations between uniqueness of solutions to the SDEs and the PDE, which complements the proof of Theorem 1.2 below.

1.1. Main results. We first introduce several notations for the set valued function $x \in \mathbb{R}^d \mapsto K(x) \subseteq \mathbb{R}^d$.

DEFINITION 1.1. *Let $K \subset \mathbb{R}^d$ be a non-empty compact set and $\varepsilon > 0$. We define the ε-boundary of K by:*

$$\partial^\varepsilon K := \{x + y \mid x \in \partial K, |y| \leq \varepsilon\},$$

and also the ε-enlargement(resp. ε-reduction) $K^{\varepsilon,+}$ (resp. $K^{\varepsilon,-}$) by

$$K^{\varepsilon,+} := K \cup \partial^\varepsilon K \quad \text{and} \quad K^{\varepsilon,-} := K \setminus \partial^\varepsilon K.$$

Note that $\partial^\varepsilon K = K^{\varepsilon,+} \setminus K^{\varepsilon,-}$ and $(\partial^\varepsilon K)^{\delta,+} \subset \partial^{\varepsilon+\delta} K$ for $\varepsilon > 0$ and $\delta > 0$.

Using those notations for K together with the so called *rope argument* used in [**16**] for the propagation of chaos of Vlasov-Poission system in one dimension, we present a useful estimate for the cut off interaction function. We refer to [**6**, Lemma 2.2] for details of the proof.

LEMMA 1.1. *For $K \subset \mathbb{R}^d$ For $x_1, y_1, x_2, y_2 \in \mathbb{R}^d$, we have*

$$|\mathbf{1}_K(y_1 - x_1) - \mathbf{1}_K(y_2 - x_2)| \leq \mathbf{1}_{\partial^{2|x_1-x_2|}K}(y_1 - x_1) + \mathbf{1}_{\partial^{2|y_1-y_2|}K}(y_1 - x_1).$$

In this paper, we consider that the set valued function K satisfying the following conditions:

(**H1**) K is globally compact, i.e., $K(x)$ is compact and there exists a compact set $\mathcal{K}$ such that $K(x) \subseteq \mathcal{K}, \forall\, x \in \mathbb{R}^d$.

(**H2**) There exists a family of closed sets $x \mapsto \Theta(x)$ and a constant C independent $\varepsilon > 0$ such that:

- (i) $\partial K(x) \subseteq \Theta(x)$, for all $x \in \mathbb{R}^d$,
- (ii) $\sup_{x \in \mathbb{R}^d} |\Theta(x)^{\varepsilon,+}| \leq C\varepsilon$, for all $\varepsilon \in (0,1)$,
- (iii) $K(x) \Delta K(x') \subseteq \Theta(x)^{C|x-x'|,+}$ for $x, x' \in \mathbb{R}^d$,
- (iv) $\Theta(x) \subseteq \Theta(x')^{C|x-x'|,+}$ for $x, x' \in \mathbb{R}^d$.

The set-valued function Θ in the above is a kind of generalized boundary of the set K. It is introduced in [**6**], where the sensitivity set K depends on the velocity variable, in order to give a sense to the time-derivative of the particle trajectories when they cross the boundary of K. It is also used to consider the vision cone with varying angles with respect to the speed. For more details, we refer to [**6**, Section 2].

In the next subsection, we provide several examples of sensitivity sets satisfying the above assumptions.

REMARK 1.1. *If w is Lipschitz, then it follows from* (**H1**)-(**H2**) *that*

$$\begin{aligned}
|K(w(x))\Delta K(w(x'))| &\leq |K(w(x))\Delta K(w(x'))| \left(\mathbf{1}_{C\|w\|_{Lip}|x-x'|<1} + \mathbf{1}_{C\|w\|_{Lip}|x-x'|\geq 1}\right) \\
&\leq \left|\Theta(w(x))^{C|w(x)-w(x')|,+}\right| \mathbf{1}_{C\|w\|_{Lip}|x-x'|<1} + 2|\mathcal{K}|\mathbf{1}_{C\|w\|_{Lip}|x-x'|\geq 1} \\
&\leq C\|w\|_{Lip}|x-x'|,
\end{aligned}$$

for $x, x' \in \mathbb{R}^d$.

We now present the main results of this paper. First we are concerned with the global existence of weak solutions to the SIEs (1.129). For this, we recall the definition of weak solutions for the SIEs (1.129).

DEFINITION 1.2. [**22**] *A weak solution of the stochastic integral equation* (1.129) *with initial data* $\mathcal{X}_0^N := (X_0^1, \cdots, X_0^N)$ *is a couple* $(\mathcal{X}_t^N, \mathcal{B}_t^N, \mathcal{K}_t^N)_{t\geq 0}$, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *such that*

(i) $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *is some stochastic basis,*
(ii) $\mathcal{B}_t^N = (B_t^1, \cdots, B_t^N)$ *is a* dN *dimensional* $(\mathcal{F}_t)$*-Brownian motion under* $\mathbb{P}$, $(\mathcal{X}_t^N, \mathcal{K}_t^N)$ *are some* $(\mathcal{F}_t)$*-adapted processes,*
(iii) $\mathbb{P}$*-almost surely* $\mathcal{X}_t^N, \mathcal{B}_t^N, \mathcal{K}_t^N$ *satisfy* (1.129).

THEOREM 1.1. *Let* $N \geq 2$. *For any initial data* $\mathcal{X}_0^N \in \overline{\mathcal{O}}^N$ *and for any* $T > 0$, *there exists at least one weak existence for the system* (1.129) *on the time interval* $[0, T]$.

We next state the theorem on the existence of solutions to the nonlinear SIEs and its associated PDEs.

THEOREM 1.2. *Let* ρ_0 *be a probability measure on* $\mathcal{O} \subset \mathbb{R}^d$ *satisfying* $\rho_0 \in L^\infty(\mathcal{O})$ *and let* $(X_0^i)_{i=1,\cdots,N}$ *be* N *independent variables with the law* ρ_0. *Suppose that the set-valued functions* K *satisfies* (**H1**)-(**H2**), w *is Lipschitz, and* $\nabla\varphi \in W^{1,\infty}(\mathbb{R}^d)$. *Then, for some* $T > 0$, *there exist a unique pathwise solution* $(Y_t^i)_{i=1,\cdots,N}$ *to the nonlinear SIEs* (1.131) *and a unique weak solution* $\rho \in L^\infty(0, T; (L^1_+ \cap L^\infty)(\mathcal{O})) \cap \mathcal{C}([0, T]; \mathcal{P}_1(\mathcal{O}))$ *to* (1.132) *for the initial condition* ρ_0, *which is the law of solution for* (1.131) *up to time* $T > 0$. *Moreover if* $\tilde{\rho}$ *is another weak solution to* (1.132) *up to time* $T > 0$ *with initial condition* $\tilde{\rho}_0 \in \mathcal{P}(\mathcal{O})$ *then for any* $t \in [0, T]$ *we have*

$$\mathcal{W}_\infty(\rho_t, \tilde{\rho}_t) \leq e^{\int_0^t \|\rho_s\|_{L^1\cap L^\infty} ds}\, \mathcal{W}_\infty(\rho_0, \tilde{\rho}_0).$$

REMARK 1.2. *Since* $\mathcal{O}$ *is bounded and the mass is conserved, we easily get*

$$\int_{\mathcal{O}} |x|^q \rho_t\, dx \leq (diam(\mathcal{O}))^q \int_{\mathcal{O}} \rho_t\, dx = (diam(\mathcal{O}))^q \int_{\mathcal{O}} \rho_0\, dx \quad for \quad q \geq 1, \quad t \geq 0,$$

i.e., $\rho \in L^\infty(0, T; \mathcal{P}_q(\mathcal{O}))$ *for any* $q \geq 1$.

REMARK 1.3. *It is worth emphasizing that our strategy is directly applicable to the whole space case when* $\sigma = 0$ *under additional assumptions on the initial moment bounds, i.e., there is no diffusion, this is why we specify the regularity of solutions even though there is the inclusion between* L^p *spaces. Note that it is impossible to define the infinite Wasserstein distance between two solutions to* (1.132) *in the whole space.*

Before stating our result on propagation of chaos, we recall the definition of a chaotic sequence. We refer to [**28**] for details of the proof of the equivalence relation in the definition below.

DEFINITION 1.3. *Let* ρ *be a probability on some polish space* E *and* $\big((X_i^N)_{i\leq N}\big)_{N\in\mathbb{N}}$ *be a sequence of exchangeable random variables. Let us denote by* $(u^N)_{N\in\mathbb{N}}$ *the sequence of their laws. Then* $\big((X_i^N)_{i\leq N}\big)_{N\in\mathbb{N}}$ *is* ρ*-chaotic if for any* $k \geq 2$ *and test functions* $\phi_1, \cdots, \phi_k \in \mathcal{C}^\infty(E)$ *it holds*

$$\int_{E^N} \phi_1 \otimes \cdots \otimes \phi_k \otimes 1 \cdots \otimes 1\, u^N(dx_1, \cdots, dx_N) \to \prod_{i=1}^k \int_E \phi_i(x)\, \rho(dx) \quad as \quad N \to \infty.$$

Or equivalently if

$$\mu^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_i^N} \xrightarrow{\mathcal{L}} \rho \quad as \quad N \to \infty,$$

where the convergence is in a law, in the space of probability measure on E.

REMARK 1.4. *Assume that* ρ *lies in* $\mathcal{P}_p(E)$ *endowed with the* $\mathcal{W}_p$ *distance. Then a sufficient condition for the sequence* $\big((X_i^N)_{i\leq N}\big)_{N\in\mathbb{N}}$ *to be* ρ*-chaotic is the following:*

$$\mathbb{E}\Big[\mathcal{W}_p(\mu^N, \rho)\Big] \to 0 \quad as \quad N \to \infty.$$

Then starting from an ρ_0-chaotic initial condition on the dynamics of (1.129) (in fact, we assume that the random variables $(X_0^i)_{i=1,\cdots,N}$ are independent identically distributed with the law ρ_0 which is stronger than being ρ_0-chaotic), we show that this chaotic character is preserved on time, more precisely, that solutions at time $t > 0$ to (1.129) are ρ_t-chaotic where ρ_t is solution at time t to (1.132). Moreover, we provide some quantitative estimate in the theorem below.

THEOREM 1.3. *Suppose that the set-valued function K satisfies* (**H1**)-(**H2**), *w is Lipschitz, and $\nabla\varphi \in W^{1,\infty}(\mathbb{R}^d)$. Let ρ be a solution to the equation* (1.132) *up to time $T > 0$, such that $\rho \in L^\infty(0,T;(L^1_+ \cap L^\infty)(\mathcal{O})) \cap \mathcal{C}([0,T];\mathcal{P}_1(\mathcal{O}))$ with initial data $\rho_0 \in (L^1_+ \cap L^\infty)(\mathcal{O}) \cap \mathcal{P}_1(\mathcal{O})$. Furthermore, assume that $(X_0^i)_{i=1,\cdots,N}$ are N independent variables with the law ρ_0. Then, for any $1 \le p < q < \infty$ and integer m such that $N \ge (2m)^2$, there exists a constant $C > 0$ depending only on ρ_0, q, $\|w\|_{Lip}$, $\|\nabla_x\varphi\|_{W^{1,\infty}}$, and T such that*

$$\sup_{t\in[0,T]} \mathbb{E}\left[\mathcal{W}_p(\mu_t^N,\rho_t)\right] \le C c_m N^{-\frac{1}{2}+\frac{1}{2m}} + \begin{cases} N^{-1/2p} + N^{-(q-p)/qp} & \text{if } 2p > d \text{ and } q \neq 2p, \\ N^{-1/2p}\log(1+N)^{1/p} + N^{-(q-p)/qp} & \text{if } 2p = d \text{ and } q \neq 2p, \\ N^{-1/d} + N^{-(q-p)/qp} & \text{if } 2p < d \text{ and } q \neq \dfrac{d}{d-p}, \end{cases}$$

with $c_m = (2m!)^{\frac{1}{2m}}\sqrt{8m}+8e^{2m}$ and where $\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}$ is the empirical measure associated to the particle system (1.129) *with initial condition $(X_0^i)_{i=1,\cdots,N}$.*

1.2. Examples of sensitivity sets. In this part, we list several sensitivity sets satisfying our main assumptions (**H1**)-(**H2**). It is worth mentioning that, in the majority of cases, we do not need to introduce the generalized boundary set Θ.

A fixed closed ball.- If we choose $K = B(0,r) := \{x \in \mathbb{R}^d : |x| \le r\}$ with $r > 0$, then it is clear that $B(0,r)$ satisfies the assumptions (**H1**)-(**H2**) with $\Theta = \partial B(0,r)$.

A closed ball with varying radius.- Let $\bar{r} : \mathbb{R}_+ \to \mathbb{R}_+$ be bounded and Lipschitz function, and take into account the case $K(x) = B(0,\bar{r}(|x|))$. In this case, it is easy to check the conditions (**H1**), and (**H2**) with $\Theta(x) = \partial B(0,\bar{r}(|x|))$ since the symmetric difference is always included in a form of torus which can be expressed by the enlargement of $B(0,\bar{r}(|x|))$.

A vision cone with a fixed angle.- Let us consider the vision cone with a fixed angle $\theta \in (0,\pi)$ and a radius $r > 0$, and a direction $w(x)$, so that the set valued function K is defined as

$$K(x) = C(r,w(x),\theta) := \left\{ y \in \mathbb{R}^d : |y| \le r\,, \cos^{-1}\left(\frac{\langle y, w(x)\rangle}{|y||w(x)|}\right) \in [-\theta,\theta] \right\} \quad \text{with} \quad d = 2,3.$$

Suppose that the direction function w is Lipschitz and bounded from the both above and below by some positive constant, i.e., $w^* \ge |w| \ge w_* > 0$. Then it is clear that $K \circ w$ satisfies (**H1**), and it is not hard to check that satisfies (**H2**) with $\Theta = \partial K$ due to the boundedenss of w. For this, similar estimates in [**6**] can be used. Note that such cutoff interaction function is considered in [**11**] for the dynamics of pedestrians.

A vision cone with varying angles.- We now consider the vision cone with varying angles with respect to the speed. For this, we first define the angle function $0 < \theta(z) \in \mathcal{C}^\infty(\mathbb{R}_+)$ by $\theta(z) = \pi$ for $0 \le z \le 1$, $\theta(z)$ is decreasing for $z \ge 1$, and $\theta(z) \to \theta_* > 0$ as $z \to +\infty$. Using this θ function, we set

$$K(x) = C(r,w(x),\theta(|w(x)|)) := \left\{ y \in \mathbb{R}^d : |y| \le r\,, \cos^{-1}\left(\frac{\langle y, w(x)\rangle}{|y||w(x)|}\right) \in [-\theta(|w(x)|),\theta(|w(x)|)] \right\},$$

with $d = 2,3$. Very similar consideration is studied in [**6**] for second-order collective behavior models. In this case, it is required to use the following generalized boundary set Θ:

$$\Theta(x) := \begin{cases} \partial C(r,w(x),\theta(|w(x)|)) \cup R(w(x)) & \text{if } |w(x)| \in (1/2,1), \\ \partial C(r,w(x),\theta(|w(x)|)) & \text{else,} \end{cases}$$

where $R(w(x)) = [a(w(x)),b(w(x))]$ with

$$a(w(x)) = -r\frac{w(x)}{|w(x)|} \quad , \quad b(w(x)) = 2r(|w(x)|-1)\frac{w(x)}{|w(x)|}.$$

Then, by assuming the Lipschitz continuity for the direction function w and using similar arguments as in [**6**, Section 5.3], we can check that the above vision cone satisfies the assumptions (**H1**)-(**H2**).

1.3. Weak-strong Lipschitz estimate. In order to give a main idea of the proof, we provide a crucial weak-strong Lipschitz estimate for the velocity fields generated by two probability measures under the assumptions (**H1**)-(**H2**) on the set valued function $K(\cdot)$.

LEMMA 1.2. *Let Y and Y' be two random variables on $\mathcal{O}$ and denote $\rho = \mathcal{L}(Y)$ and $\rho' = \mathcal{L}(Y')$. Assume that $\rho \in (L^1 \cap L^\infty)(\mathcal{O})$. Then there exists a constant depending only on $\|\nabla\varphi\|_{W^{1,\infty}}, \|w\|_{Lip}$ such that*

$$|V[\rho](Y) - V[\rho'](Y')| \le C\|\rho\|_{L^1\cap L^\infty}\mathbb{P}\text{-}\operatorname{ess\,sup}|Y - Y'|, \tag{1.133}$$

where V is given in (1.130).

PROOF. Introducing $\pi := \mathcal{L}(Y, Y')$, we first decompose the left hand side of (1.133) into three terms:

$$\begin{aligned}
&|V[\rho](Y) - V[\rho'](Y')| \\
&= \left|\int \left(\mathbf{1}_{K(w(Y))}(y-Y)\nabla\varphi(Y-y) - \mathbf{1}_{K(w(Y'))}(y'-Y')\nabla\varphi(Y'-y')\right)\pi(dy,dy')\right| \\
&\le \int \left(\mathbf{1}_{K(w(Y))}(y-Y)\nabla\varphi(Y-y) - \mathbf{1}_{K(w(Y))}(y-Y)\nabla\varphi(Y'-y')\right)\pi(dy,dy') \\
&\quad + \int \left(\mathbf{1}_{K(w(Y))}(y-Y) - \mathbf{1}_{K(w(Y'))}(y-Y)\right)\nabla\varphi(Y'-y')\pi(dy,dy') \\
&\quad + \int \left(\mathbf{1}_{K(w(Y'))}(y-Y) - \mathbf{1}_{K(w(Y'))}(y'-Y')\right)\nabla\varphi(Y'-y')\pi(dy,dy') \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

Here $I_i, i = 1, 2, 3$ can be estimated as follows.
$\diamond$ Estimate I_1: Due to the regularity of $\nabla\varphi$, we easily obtain

$$I_1 \le \|\nabla\varphi\|_{Lip}\left(\mathbb{P}\text{-}\operatorname{ess\,sup}|Y-Y'| + \pi\text{-}\operatorname{ess\,sup}|y-y'|\right) = 2\|\nabla\varphi\|_{Lip}\mathbb{P}\text{-}\operatorname{ess\,sup}|Y-Y'|.$$

$\diamond$ Estimate I_2: Using our main assumptions together with Remark 1.1 yields

$$\begin{aligned}
I_2 &\le \|\nabla\varphi\|_{L^\infty}\int \left|\mathbf{1}_{K(w(Y))}(y-Y) - \mathbf{1}_{K(w(Y'))}(y-Y)\right|\pi(dy,dy') \\
&\le \|\nabla\varphi\|_{L^\infty}\int \mathbf{1}_{K(w(Y))\Delta K(w(Y'))}(y-Y)\rho(dy) \\
&\le C\|\nabla\varphi\|_{L^\infty}\|\rho\|_{L^1\cap L^\infty}|K(w(Y))\Delta K(w(Y'))| \\
&\le C\|\nabla\varphi\|_{L^\infty}\|\rho\|_{L^1\cap L^\infty}\|w\|_{Lip}|Y-Y'| \\
&\le C\|\nabla\varphi\|_{L^\infty}\|\rho\|_{L^1\cap L^\infty}\mathbb{P}\text{-}\operatorname{ess\,sup}|Y-Y'|.
\end{aligned}$$

$\diamond$ Estimate I_3: It follows from Lemma 1.1, and (**H2**) that

$$\begin{aligned}
I_3 &\le \|\nabla\varphi\|_{L^\infty}\int \left|\mathbf{1}_{K(w(Y'))}(y-Y) - \mathbf{1}_{K(w(Y'))}(y'-Y')\right|\pi(dy,dy') \\
&\le \|\nabla\varphi\|_{L^\infty}\int \left(\mathbf{1}_{\partial^{2|Y-Y'|}K(w(Y'))}(y-Y) + \mathbf{1}_{\partial^{2|y-y'|}K(w(Y'))}(y-Y)\right)\pi(dy,dy') \\
&\le \|\nabla\varphi\|_{L^\infty}\int \left(\mathbf{1}_{\partial^{2\mathbb{P}\text{-}\operatorname{ess\,sup}|Y-Y'|}K(w(Y'))}(y-Y) + \mathbf{1}_{\partial^{2\pi\text{-}\operatorname{ess\,sup}|y-y'|}K(w(Y'))}(y-Y)\right)\rho(dy) \\
&\le C\|\nabla\varphi\|_{L^\infty}\|\rho\|_{L^1\cap L^\infty}\left(\mathbb{P}\text{-}\operatorname{ess\,sup}|Y-Y'| + \pi\text{-}\operatorname{ess\,sup}|y-y'|\right) \\
&\le C\|\nabla\varphi\|_{L^\infty}\|\rho\|_{L^1\cap L^\infty}\mathbb{P}\text{-}\operatorname{ess\,sup}|Y-Y'|.
\end{aligned}$$

By combining all the above estimates, we conclude our desired result. □

REMARK 1.5. *As mentioned in Introduction, we are imposed to use the infinite Wasserstein distance to have the above weak-strong Lipschitz estimate due to the stronger singularity in the velocity fields than the one in* [**6**]. *Note that in* [**6**, Proposition 2.3] *the similar estimate is obtained in the Wasserstein distance of order 1.*

2. Global existence of weak solutions for the SIEs

In this section, we provide the details of proof of Theorem 1.1 on the global existence of weak solutions to the stochastic particle system (1.129). The proof relies on an adapted use of Girsanov's Theorem which is useful for SDEs with non-smooth drift but additive noise.

2.1. Proof of Theorem 1.1. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis and $(\mathcal{B}_t^N)_{t\geq 0}$ be a dN dimensional $(\mathcal{F}_t)$-Brownian motion on this basis. We define a $\mathbb{R}^{dN}$-valued function $\mathcal{A}^N := (\mathcal{A}_1^N, \cdots, \mathcal{A}_N^N)$ by

$$\mathcal{A}_i^N(x_1, \cdots, x_N) := V[\mu^N](x_i),$$

where $\mu^N := \frac{1}{N}\sum_{j=1}^N \delta_{x_j}$. Next we define

$$\mathcal{X}_t^N := \mathcal{X}_0^N + \sqrt{2\sigma}\mathcal{B}_t^N - \mathcal{K}_t^N, \quad \mathcal{K}_t^N := (K_t^1, \cdots, K_t^N), \quad t > 0,$$
$$K_t^i := \int_0^t n(X_s^i)\, d|K^i|_s \ , \ |K^i|_t := \int_0^t \mathbf{1}_{\partial\mathcal{O}}(X_s^i)\, d|K^i|_s, \quad \text{for } i = 1, \cdots, N,$$

for which the well-posedness, together with the fact $\mathcal{X}_t^N \in \mathcal{O}^N$ $\mathbb{P}$-a.s. for all $t \geq 0$, is ensured by [**20**, Theorem 3.1] or [**29**]. Then we define

$$\mathcal{Y}_t^N := \mathcal{B}_t^N - \frac{1}{\sqrt{2\sigma}}\int_0^t \mathcal{A}^N\left(\mathcal{X}_s^N\right) ds. \tag{2.134}$$

This implies that $(\mathcal{X}_t^N)_{t\geq 0}$ and $(\mathcal{Y}_t^N)_{t\geq 0}$ satisfy

$$\mathcal{X}_t^N = \mathcal{X}_0^N + \int_0^t \mathcal{A}^N\left(\mathcal{X}_s^N\right) ds + \sqrt{2\sigma}\mathcal{Y}_t^N - \mathcal{K}_t^N,$$
$$K_t^i = \int_0^t n(X_s^i)\, d|K^i|_s \ , \ |K^i|_t = \int_0^t \mathbf{1}_{\partial\mathcal{O}}(X_s^i)\, d|K^i|_s, \quad \text{for } i = 1, \cdots, N.$$

Note that the above stochastic integral equation has the same form with (1.129). We now look for a proper stochastic basis under which $(\mathcal{Y}_t^N)_{t\geq 0}$ is a dN dimensional Brownian motion. Fix $T > 0$ and define $(Z_t^N)_{t\geq 0}$ by

$$Z_t^N := \exp\left(\int_0^t \left\langle \mathcal{A}^N\left(\mathcal{X}_s^N\right), dB_s^N\right\rangle - \frac{1}{2}\int_0^t |\mathcal{A}^N\left(\mathcal{X}_s^N\right)|^2\, ds\right).$$

Since

$$\left|\mathcal{A}^N\left(\mathcal{X}_s^N\right)\right| \leq \|\nabla\varphi\|_{L^\infty} \quad \text{a.s.},$$

we obtain

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \left|\mathcal{A}^N\left(\mathcal{X}_s^N\right)\right|^2 ds\right)\right] \leq e^{\|\nabla\varphi\|_{L^\infty}^2 T/2}.$$

This together with the classical exponential martingale theory yields that the process $(Z_t^N)_{t\in[0,T]}$ is a positive martingale with respect to $(\mathcal{F}_t)_{t\in[0,T]}$ and $\mathbb{E}[Z_t^N] = 1$ for $t \in [0, T]$. Then, by Girsanov's Theorem(see for instance [**22**, Theorem 2.51]), the stochastic process $\mathcal{Y}_t^N$ defined in (2.134) is a dN dimensional Brownian motion under the probability measure $\mathbb{Q}$ defined by

$$\mathbb{Q}(A) = \int_A Z_t^N d\mathbb{P}, \quad \text{for} \quad A \in \mathcal{F}_t.$$

This concludes that the following couple

$$\left((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{Q}), (\left(\mathcal{X}_t^N\right)_{t\in[0,T]}, \mathcal{Y}_t^N)_{t\geq 0}, \mathcal{K}_t^N)_{t\in[0,T]}\right),$$

is a weak solution to (1.129).

3. Existence and stability of the nonlinear SIEs and PDEs

In this section we study the existence and uniqueness of solutions to the nonlinear SDEs (1.131) which process solutions have time marginals solutions to the continuity equation (1.132). As mentioned in Introduction, the existence of such process solutions are studied in [**29**] where $\mathcal{O}$ is an open convex, and later it is refined in [**20, 27**] for the case where $\mathcal{O}$ is an open domain satisfying the uniform exterior sphere condition which reads

$$\exists r_0 > 0 \quad , \forall\, x \in \partial\mathcal{O}, \quad \exists y_x \in \mathbb{R}^d \quad \text{such that} \quad B(y_x, r_0) \cap \overline{\mathcal{O}} = \{x\}. \tag{3.135}$$

We here set ourselves in the case where $\mathcal{O}$ is convex. Note that it implies that

$$n(w) \cdot (w - w') \geq 0 \qquad \text{for any } w \in \partial\mathcal{O} \text{ and } w' \in \overline{\mathcal{O}}, \tag{3.136}$$

and if for some $w \in \partial\mathcal{O}$ some vector $k \in \mathbb{R}^d$ satisfies

$$k \cdot (w - w') \geq 0 \qquad \text{for any } w' \in \overline{\mathcal{O}},$$

then it holds $k = \theta n(w)$ for some $\theta > 0$. Later, we will use these observations for the existence of strong solutions to the system (1.131), see the proof of Proposition 3.2 below.

3.1. Regularized system. In this part, we introduce a regularized system, and show the uniform boundedness of solutions to that regularized system in regularization parameters. Consider a mollified interaction function $\mathbf{1}_K^{\varepsilon,\eta}$ defined by

$$\mathbf{1}_{K(x')}^{\varepsilon,\eta}(x) = \int_{\mathbb{R}^d} \mathbf{1}_{K(x'-y')}(y-x)\phi_\varepsilon(y)\phi_\eta(y')\,dydy',$$

and consistently with the notation introduced before we set

$$V^{\varepsilon,\eta}[\mu](x) := \int_{\mathcal{O}} \mathbf{1}_{K(w(x))}^{\varepsilon,\eta}(y-x)\nabla\phi(x-y)\mu(dy).$$

LEMMA 3.1. *Assume $\rho_0 \in (L^1_+ \cap L^\infty)(\mathcal{O}) \cap \mathcal{P}_1(\mathcal{O})$, let Y_0 be with the law ρ_0 and $(B_t)_{t\geq 0}$ be a Brownian motion independent of Y_0. Then for any $T > 0$, there exists a unique process solving the following nonlinear SIEs up to time $T > 0$ in the strong sense:*

(3.137)
$$\begin{cases} Y_t^{\varepsilon,\eta} = Y_0 + \displaystyle\int_0^t \left(\int \mathbf{1}_{K(w(Y_s^{\varepsilon,\eta}))}^{\varepsilon,\eta}(y - Y_s^{\eta,\varepsilon})\nabla\varphi(Y_s^{\eta,\varepsilon} - y)\rho_s^{\varepsilon,\eta}(dy)\right) ds + \sqrt{2\sigma}B_t - K_t^{\varepsilon,\eta}, \\ K_t^{\varepsilon,\eta} = \displaystyle\int_0^t n(Y_s^{\varepsilon,\eta})\, d|K^{\varepsilon,\eta}|_s, \quad |K^{\varepsilon,\eta}|_t = \int_0^t \mathbf{1}_{\partial\mathcal{O}}(Y_s^{\varepsilon,\eta})\, d|K^{\varepsilon,\eta}|_s, \\ \mathcal{L}(Y_t^{\varepsilon,\eta}) = \rho_t^{\varepsilon,\eta}. \end{cases}$$

PROOF. Since both drift and diffusion terms in the above regularized SIEs are smooth, we deduce from [**27**] the strong existence and uniqueness of the process $(Y_t^{\varepsilon,\eta})$ to the system (3.137). □

We next provide some basic properties of the mollified indicator function in the lemma below. The proof of that can be found in [**6**, Lemma 4.2].

LEMMA 3.2. *Denote*

$$\mathbf{1}_{K(x')}^{\varepsilon}(x) = \int_{\mathbb{R}^d} \mathbf{1}_{K(x')}(y-x)\phi_\varepsilon(y)\,dy.$$

(i) For all $\varepsilon > 0$, it holds

$$\int |\mathbf{1}_K^\varepsilon(x) - \mathbf{1}_K(x)|\,dx \leq |\partial^{2\varepsilon}K|. \tag{3.138}$$

(ii) For all $\varepsilon > 0$ and $x_1, y_1, x_2, y_2, v \in \mathbb{R}^d$ we have

$$|\mathbf{1}_K^\varepsilon(y_1 - x_1) - \mathbf{1}_K^\varepsilon(y_2 - x_2)| \leq \mathbf{1}_{\partial^{2|x_1-x_2|}K}^\varepsilon(y_1 - x_1) + \mathbf{1}_{\partial^{2|y_1-y_2|}K}^\varepsilon(y_1 - x_1). \tag{3.139}$$

(iii) For all $x \in \mathcal{O}$ and $0 < \eta \leq 1$, it holds

$$\int_{\mathbb{R}^d} \left|\mathbf{1}_{K(w(x))}^{\eta,\varepsilon}(y-x) - \mathbf{1}_{K(w(x))}^{\varepsilon}(y-x)\right| dy \leq C\eta, \tag{3.140}$$

where C is a positive constant independent of ε and η.

In the proposition below, we show the existence of weak solutions to the corresponding continuity equation to (3.137) with no-flux boundary condition. We also provide a uniform bound estimate of the solution in regularization parameters.

PROPOSITION 3.1. *The family of time marginals $(\rho_t^{\varepsilon,\eta})_{t\geq 0}$ of the solution to* (3.137)*, is a global-in-time weak solution of*

$$\begin{cases} \partial_t\rho_t^{\varepsilon,\eta} + \nabla\cdot\left(V^{\varepsilon,\eta}[\rho_t^{\varepsilon,\eta}]\rho_t^{\varepsilon,\eta}\right) = \sigma\Delta\rho_t^{\varepsilon,\eta}, \quad x \in \mathcal{O}, \quad t > 0, \\ V^{\varepsilon,\eta}[\rho_t^{\varepsilon,\eta}](x) = \displaystyle\int_{\mathcal{O}} \mathbf{1}_{K(w(x))}^{\varepsilon,\eta}(y-x)\nabla\varphi(x-y)\rho_t^{\varepsilon,\eta}(dy), \\ \left\langle \sigma\nabla\rho_t^{\varepsilon,\eta} - \rho_t^{\varepsilon,\eta}V^{\varepsilon,\eta}[\rho_t^{\varepsilon,\eta}], n\right\rangle = 0 \quad \text{on} \quad \partial\mathcal{O}, \end{cases} \tag{3.141}$$

with $\rho_0 = \mathcal{L}(Y_0) \in (L^1 \cap L^\infty)(\mathcal{O})$. Furthermore, there exist a time $T > 0$ and a constant $C > 0$ such that

$$\sup_{0\le t\le T} \|\rho_t^{\varepsilon,\eta}\|_{L^1\cap L^\infty} \le C,$$

where $C > 0$ is independent of $\varepsilon, \eta > 0$.

PROOF. • (*Existence of weak solutions*): For $\phi \in \mathcal{C}^\infty(\mathcal{O})$ satisfying $\langle \nabla\phi(x), n(x)\rangle = 0$ for $x \in \partial\mathcal{O}$, by applying Itô's formula, we get

$$\begin{aligned}\phi(Y_t^{\varepsilon,\eta}) &= \phi(Y_0) + \int_0^t \langle \nabla\phi(Y_s^{\varepsilon,\eta}), dY_s^{\varepsilon,\eta}\rangle + \frac{1}{2}\int_0^t (dY_s^{\varepsilon,\eta})^t\, \nabla^2\phi(Y_s^{\varepsilon,\eta})\, dY_s^{\varepsilon,\eta}\\ &= \phi(Y_0) + \int_0^t \langle V^{\varepsilon,\eta}[\rho_s^{\varepsilon,\eta}](Y_s^{\varepsilon,\eta}), \nabla\phi(Y_s^{\varepsilon,\eta})\rangle ds + \sqrt{2\sigma}\int_0^t \langle \nabla\phi(Y_s^{\varepsilon,\eta}), dB_s\rangle\\ &\quad - \int_0^t \langle \nabla\phi(Y_s^{\varepsilon,\eta}), dK_s^{\eta,\varepsilon}\rangle + \sigma\int_0^t \Delta\phi(Y_s^{\varepsilon,\eta})\, ds.\end{aligned}$$

Since

$$\int_0^t \langle \nabla\phi(Y_s^{\varepsilon,\eta}), dK_s^{\varepsilon,\eta}\rangle\, ds = \int_0^t \langle \nabla\phi(Y_s^{\varepsilon,\eta}), n(Y_s^{\varepsilon,\eta})\rangle\, d|K^{\varepsilon,\eta}|_s = 0,$$

by taking the expectation and using the fact that $\rho_t^{\varepsilon,\eta} = \mathcal{L}(Y_t^{\varepsilon,\eta})$, we obtain

$$\int_{\mathcal{O}} \phi(x)\rho_t^{\varepsilon,\eta}(dx) = \int_{\mathcal{O}} \phi(x)\rho_0(dx) + \int_0^t\int_{\mathcal{O}} V^{\varepsilon,\eta}[\rho_s^{\varepsilon,\eta}](x)\cdot\nabla\phi(x)\,\rho_s^{\varepsilon,\eta}(dx)\,ds + \sigma\int_0^t\int_{\mathcal{O}} \Delta\phi(x)\rho_s^{\varepsilon,\eta}(dx)\,ds.$$

This implies that the family of time marginals of the process solutions to (3.137) is a weak solution for the equation (3.141).

• (*Uniform bound estimate*): It is straightforward to get that for $p \ge 1$

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{O}} (\rho_t^{\varepsilon,\eta})^p\, dx &= p\int_{\mathcal{O}} \partial_t\rho_t^{\varepsilon,\eta}(\rho_t^{\varepsilon,\eta})^{p-1}dx\\ &= -p\int_{\mathcal{O}} \nabla\cdot(V^{\varepsilon,\eta}[\rho_t^{\varepsilon,\eta}]\rho_t^{\varepsilon,\eta})(\rho_t^{\varepsilon,\eta})^{p-1}dx + \sigma p\int_{\mathcal{O}} \Delta\rho_t^{\varepsilon,\eta}(\rho_t^{\varepsilon,\eta})^{p-1}dx\\ &=: I_1 + I_2,\end{aligned}$$

where $I_i, i = 1, 2$ are estimated as follows.

$$\begin{aligned}I_1 &= p\int_{\mathcal{O}} V^{\varepsilon,\eta}[\rho_t^{\varepsilon,\eta}]\rho_t^{\varepsilon,\eta}\cdot\nabla\left((\rho_t^{\varepsilon,\eta})^{p-1}\right) - p\int_{\partial\mathcal{O}} V^{\varepsilon,\eta}[\rho_t^{\varepsilon,\eta}]\rho_t^{\varepsilon,\eta}\cdot n(x)(\rho_t^{\varepsilon,\eta})^{p-1}\, dS(x)\\ &= -(p-1)\int_{\mathcal{O}} \left(\nabla\cdot V^{\varepsilon,\eta}[\rho_t^{\varepsilon,\eta}]\right)(\rho_t^{\varepsilon,\eta})^p\, dx - p\int_{\partial\mathcal{O}} V^{\varepsilon}[\rho_t^{\varepsilon,\eta}]\rho_t^{\varepsilon,\eta}\cdot n(x)(\rho_t^{\varepsilon,\eta})^{p-1}\, dS(x),\\ I_2 &= -\sigma p\int_{\mathcal{O}} \nabla\rho_t^{\varepsilon,\eta}\cdot\nabla\left((\rho_t^{\varepsilon,\eta})^{p-1}\right)dx + \sigma p\int_{\partial\mathcal{O}} \nabla\rho_t^{\varepsilon,\eta}\cdot n(x)(\rho_t^{\varepsilon,\eta})^{p-1}\, dS(x)\\ &= -\sigma p(p-1)\int_{\mathcal{O}} (\rho_t^{\varepsilon,\eta})^{p-2}|\nabla\rho_t^{\varepsilon,\eta}|^2dx + \sigma p\int_{\partial\mathcal{O}} \nabla\rho_t^{\varepsilon,\eta}\cdot n(x)(\rho_t^{\varepsilon,\eta})^{p-1}\, dS(x).\end{aligned}$$

Thus we have

$$\begin{aligned}\frac{d}{dt}\|\rho_t^{\varepsilon,\eta}\|_{L^p}^p &= -(p-1)\int_{\mathcal{O}} \left(\nabla\cdot V^{\varepsilon,\eta}[\rho_t^{\varepsilon,\eta}]\right)(\rho_t^{\varepsilon,\eta})^p\, dx - \sigma p(p-1)\int_{\mathcal{O}} (\rho_t^{\varepsilon,\eta})^{p-2}|\nabla\rho_t^{\varepsilon,\eta}|^2dx\\ &\le (p-1)\|\nabla\cdot V^{\varepsilon,\eta}[\rho_t^{\varepsilon,\eta}]\|_{L^\infty}\|\rho_t^{\varepsilon,\eta}\|_{L^p}^p,\end{aligned}$$

due to the boundary condition. On the other hand, we can estimate

$$\|\nabla\cdot V^{\varepsilon,\eta}[\rho_t^{\varepsilon,\eta}]\|_{L^\infty} \le C\|\rho_t^{\varepsilon,\eta}\|_{L^\infty}, \tag{3.142}$$

where $C > 0$ is independent of ε, η, and p. Indeed, for $i = 1, \cdots, d$, and $|h| \le 1$, we get

$$\begin{aligned}&\int_{\mathcal{O}} \left(\partial_i\varphi(x + he_i - y)\mathbf{1}^{\varepsilon,\eta}_{K(w(x+he_i))}(y - x - he_i) - \partial_i\varphi(x-y)\mathbf{1}^{\varepsilon,\eta}_{K(w(x))}(y-x)\right)\rho_t^{\varepsilon,\eta}(y)\, dy\\ &= \int_{\mathcal{O}} \left(\partial_i\varphi(x + he_i - y) - \partial_i\varphi(x-y)\right)\mathbf{1}^{\varepsilon,\eta}_{K(w(x+he_i))}(y - x - he_i)\rho_t^{\varepsilon,\eta}(y)\, dy\\ &\quad + \int_{\mathcal{O}} \partial_i\varphi(x-y)\left(\mathbf{1}^{\varepsilon,\eta}_{K(w(x+he_i))}(y - x - he_i) - \mathbf{1}^{\varepsilon,\eta}_{K(w(x))}(y-x)\right)\rho_t^{\varepsilon,\eta}(y)\, dy,\end{aligned}$$

where the first term on the right hand side of the above equality can be easily estimated as

$$\left|\int_{\mathcal{O}} \left(\partial_i\varphi(x+he_i-y)-\partial_i\varphi(x-y)\right)\mathbf{1}^{\varepsilon,\eta}_{K(w(x+he_i))}(y-x-he_i)\rho_t^{\varepsilon,\eta}(y)\,dy\right| \le h\|\nabla\varphi\|_{Lip}\|\rho_t^{\varepsilon,\eta}\|_{L^1}.$$

For the estimate of second term, we use Lemma 1.1, Remark 1.1, (3.139), together with the assumption (**H2**) to find

$$\begin{aligned}
&\left|\int_{\mathcal{O}} \partial_i\varphi(x-y)\left(\mathbf{1}^{\varepsilon,\eta}_{K(w(x+he_i))}(y-x-he_i)-\mathbf{1}^{\varepsilon,\eta}_{K(w(x))}(y-x)\right)\rho_t^{\varepsilon,\eta}(y)\,dy\right| \\
&\quad= \left|\int_{\mathcal{O}\times\mathbb{R}^d} \partial_i\varphi(x-y)\left(\mathbf{1}^{\varepsilon}_{K(w(x+he_i)-y')}(y-x-he_i)-\mathbf{1}^{\varepsilon}_{K(w(x+he_i)-y')}(y-x)\right)\rho_t^{\varepsilon,\eta}(y)\phi_\eta(y')\,dydy'\right. \\
&\qquad\left.+\int_{\mathcal{O}\times\mathbb{R}^d} \partial_i\varphi(x-y)\left(\mathbf{1}^{\varepsilon}_{K(w(x+he_i)-y')}(y-x)-\mathbf{1}^{\varepsilon}_{K(w(x)-y')}(y-x)\right)\rho_t^{\varepsilon,\eta}(y)\phi_\eta(y')\,dydy'\right| \\
&\quad\le \|\nabla\varphi\|_{L^\infty}\int_{\mathcal{O}\times\mathbb{R}^d}\mathbf{1}^{\varepsilon}_{\partial^{2h}K(w(x+he_i)-y')}(y-x-he_i)\rho_t^{\varepsilon,\eta}(y)\phi_\eta(y')\,dydy' \\
&\qquad+\|\nabla\varphi\|_{L^\infty}\int_{\mathcal{O}\times\mathbb{R}^d}\mathbf{1}^{\varepsilon}_{K(w(x+he_i)-y')\Delta K(w(x)-y')}(y-x)\rho_t^{\varepsilon,\eta}(y)\phi_\eta(y')\,dydy' \\
&\quad\le \|\nabla\varphi\|_{L^\infty}\|\rho_t^{\varepsilon,\eta}\|_{L^\infty}\left(\sup_{x\in\mathbb{R}^d}|\partial^{2h}K(x)|+\int_{\mathbb{R}^d}|K(w(x+he_i)-y')\Delta K(w(x)-y')|\phi_\eta(y')\,dy'\right) \\
&\quad\le Ch\|\nabla\varphi\|_{L^\infty}\|\rho_t^{\varepsilon,\eta}\|_{L^\infty}\left(1+\|w\|_{Lip}\right).
\end{aligned}$$

This proves that inequality (3.142) holds. Thus we obtain

$$\frac{d}{dt}\left\|\rho_t^{\varepsilon,\eta}\right\|_{L^p}^p \le Cp\|\rho_t^{\varepsilon,\eta}\|_{L^\infty}\left\|\rho_t^{\varepsilon,\eta}\right\|_{L^p}^p.$$

By applying Gronwall's inequality, we get

$$\|\rho_t^{\varepsilon,\eta}\|_{L^p}^p \le \|\rho_0\|_{L^p}^p\exp\left(Cp\int_0^t\|\rho_s^{\varepsilon,\eta}\|_{L^\infty}ds\right) \quad \text{i.e.,} \quad \|\rho_t^{\varepsilon,\eta}\|_{L^p}\le\|\rho_0\|_{L^p}\exp\left(C\int_0^t\|\rho_s^{\varepsilon,\eta}\|_{L^\infty}ds\right).$$

We then send $p\to\infty$ to find

$$\|\rho_t^{\varepsilon,\eta}\|_{L^\infty}\le\|\rho_0\|_{L^\infty}\exp\left(C\int_0^t\|\rho_s^{\varepsilon,\eta}\|_{L^\infty}ds\right).$$

We finally use Lemma 5.1 (i) with $f(t)=\|\rho_t^{\varepsilon,\eta}\|_{L^\infty}$ to have

$$\|\rho_t^{\varepsilon,\eta}\|_{L^\infty}\le\frac{\|\rho_0\|_{L^\infty}}{1-C\|\rho_0\|_{L^\infty}t},$$

which concludes the proof. □

3.2. Existence of solutions for the nonlinear SIEs. In this part, we show the existence of strong solutions to (1.131) by obtaining the weak-strong stability estimate.

PROPOSITION 3.2. *There exists only one strong solution* $[0,T]$ *to the following nonlinear SIEs:*

(3.143)

$$\begin{cases} Y_t = Y_0+\displaystyle\int_0^t\left(\int_{\mathcal{O}}\mathbf{1}_{K(w(Y_s))}(y-Y_s)\nabla\varphi(Y_s-y)\rho_s(dy)\right)ds+\sqrt{2\sigma}B_t-K_t, \quad \mathcal{L}(Y_t)=\rho_t, \quad t>0, \\ K_t=\displaystyle\int_0^t n(Y_s)\,d|K|_s, \quad |K|_t=\int_0^t\mathbf{1}_{\partial\mathcal{O}}(Y_s)\,d|K|_s, \\ \mathcal{L}(Y_0)=\rho_0. \end{cases}$$

Moreover, if $(Y_t^i)_{t\in(0,T]}, i=1,2$ *are two solutions to* (3.143) *with the initial data* (Y_0^i)*, respectively, and* $\mathcal{L}(Y_\cdot^1)=\rho_\cdot\in L^\infty(\mathcal{O}\times[0,T])$ *then we have the following stability estimate:*

$$\mathbb{P}\text{-}\operatorname{ess\,sup}|Y_t^1-Y_t^2|\le e^{C\int_0^t\|\rho_s\|_{L^1\cap L^\infty}ds}\,\mathbb{P}\text{-}\operatorname{ess\,sup}|Y_0^1-Y_0^2| \quad \textit{for} \quad t\in[0,T].$$

PROOF. For the proof, we split it into three steps:

• **Step A** (*Cauchy estimate*): Let $\varepsilon,\varepsilon',\eta,\eta'>0$, and consider the solutions $Y_\cdot^{\varepsilon,\eta}$ and $Y_\cdot^{\varepsilon',\eta'}$ to the regularized nonlinear SIEs (3.137). For notational simplicity, we set

$$V_t^{\varepsilon,\eta,\varepsilon',\eta'}(Y_t) := V^{\varepsilon,\eta}[\rho_t^{\varepsilon,\eta}](Y_t^{\varepsilon,\eta}) - V^{\varepsilon',\eta'}[\rho_t^{\varepsilon',\eta'}](Y_t^{\varepsilon',\eta'}).$$

Applying Ito's formula yields

$$\begin{aligned}\left|Y_t^{\varepsilon,\eta} - Y_t^{\varepsilon',\eta'}\right|^2 &= 2\int_0^t \left\langle Y_s^{\varepsilon,\eta} - Y_s^{\varepsilon',\eta'}, V_s^{\varepsilon,\eta,\varepsilon',\eta'}(Y_s)\right\rangle ds \\ &\quad - 2\int_0^t \left\langle Y_s^{\varepsilon,\eta} - Y_s^{\varepsilon',\eta'}, n(Y_s^{\varepsilon,\eta})\right\rangle d|K^{\varepsilon,\eta}|_s - 2\int_0^t \left\langle Y_s^{\varepsilon',\eta'} - Y_s^{\varepsilon,\eta}, n(Y_s^{\varepsilon',\eta'})\right\rangle d|K^{\varepsilon',\eta'}|_s \\ &\le 2\int_0^t \left\langle Y_s^{\varepsilon,\eta} - Y_s^{\varepsilon',\eta'}, V_s^{\varepsilon,\eta,\varepsilon',\eta'}(Y_s)\right\rangle ds,\end{aligned}$$

since $(Y_s^{\varepsilon,\eta} - Y_s^{\varepsilon',\eta'}) \cdot n(Y_s^{\varepsilon,\eta}) \ge 0$, $d|K^{\varepsilon,\eta}|_s$ almost surely, due to the convexity of the domain $\mathcal{O}$, see (3.136). Thus it only remains to estimate the following term:

$$J := \left|V^{\varepsilon,\eta}[\rho_s^{\varepsilon,\eta}](Y_s^{\varepsilon,\eta}) - V^{\varepsilon',\eta'}[\rho_s^{\varepsilon',\eta'}](Y_s^{\varepsilon',\eta'})\right|.$$

We decompose J as

$$\begin{aligned}J &\le |V^{\varepsilon,\eta}[\rho_s^{\varepsilon,\eta}](Y_s^{\varepsilon,\eta}) - V[\rho_s^{\varepsilon,\eta}](Y_s^{\varepsilon,\eta})| + \left|V[\rho_s^{\varepsilon,\eta}](Y_s^{\varepsilon,\eta}) - V[\rho_s^{\varepsilon',\eta'}](Y_s^{\varepsilon',\eta'})\right| \\ &\quad + \left|V[\rho_s^{\varepsilon',\eta'}](Y_s^{\varepsilon',\eta'}) - V^{\varepsilon',\eta'}[\rho_s^{\varepsilon',\eta'}](Y_s^{\varepsilon',\eta'})\right| \\ &=: J_1 + J_2 + J_3.\end{aligned}$$

$\diamond$ Estimates of J_1 and J_3: Using (3.138) and (3.140), we easily get for $\varepsilon, \eta \le 1/2$

$$\begin{aligned}J_1 &= \left|\int \left(\mathbf{1}^{\varepsilon,\eta}_{K(w(Y_s^{\varepsilon,\eta}))}(y - Y_s^{\varepsilon,\eta}) - \mathbf{1}_{K(w(Y_s^{\varepsilon,\eta}))}(y - Y_s^{\varepsilon,\eta})\right) \nabla\varphi(Y_s^{\varepsilon,\eta} - y)\rho_s^{\varepsilon,\eta}(dy)\right| \\ &\le \left|\int \left(\mathbf{1}^{\varepsilon,\eta}_{K(w(Y_s^{\varepsilon,\eta}))}(y - Y_s^{\varepsilon,\eta}) - \mathbf{1}^{\varepsilon}_{K(w(Y_s^{\varepsilon,\eta}))}(y - Y_s^{\varepsilon,\eta})\right) \nabla\varphi(Y_s^{\varepsilon,\eta} - y)\rho_s^{\varepsilon,\eta}(dy)\right| \\ &\quad + \left|\int \left(\mathbf{1}^{\varepsilon}_{K(w(Y_s^{\varepsilon,\eta}))}(y - Y_s^{\varepsilon,\eta}) - \mathbf{1}_{K(w(Y_s^{\varepsilon,\eta}))}(y - Y_s^{\varepsilon,\eta})\right) \nabla\varphi(Y_s^{\varepsilon,\eta} - y)\rho_s^{\varepsilon,\eta}(dy)\right| \\ &\le C\|\nabla\varphi\|_{L^1\cap L^\infty}\|\rho_s^{\varepsilon,\eta}\|_{L^\infty}(\varepsilon + \eta),\end{aligned}$$

where $C > 0$ is independent of $\varepsilon, \eta > 0$. Employing the same argument as above, we estimate J_3 as

$$J_3 \le C\|\nabla\varphi\|_{L^\infty}\|\rho_s^{\varepsilon',\eta'}\|_{L^\infty}(\varepsilon' + \eta') \quad \text{for} \quad \varepsilon', \eta' < 1/2.$$

$\diamond$ Estimate of J_2: It follows from Lemma 1.2 that

$$J_2 \le C\|\rho_s^{\varepsilon,\eta}\|_{L^1\cap L^\infty}\,\mathbb{P}\text{-}\operatorname{ess\,sup}|Y_s^{\varepsilon,\eta} - Y_s^{\varepsilon',\eta'}|.$$

Combining all the above estimates, we find

$$\begin{aligned}|Y_t^{\varepsilon,\eta} - Y_t^{\varepsilon',\eta'}|^2 &\le C\int_0^t \|\rho_s^{\varepsilon,\eta}\|_{L^1\cap L^\infty}\left(\mathbb{P}\text{-}\operatorname{ess\,sup}|Y_s^{\varepsilon,\eta} - Y_s^{\varepsilon',\eta'}|\right)|Y_s^{\varepsilon,\eta} - Y_s^{\varepsilon',\eta'}|\,ds \\ &\quad + C\int_0^t (\varepsilon + \eta + \varepsilon' + \eta')\left(\|\rho_s^{\varepsilon,\eta}\|_{L^1\cap L^\infty} + \|\rho_s^{\varepsilon',\eta'}\|_{L^1\cap L^\infty}\right)|Y_s^{\varepsilon,\eta} - Y_s^{\varepsilon',\eta'}|\,ds,\end{aligned}$$

where $C > 0$ is independent of $\varepsilon, \varepsilon', \eta, \eta' > 0$. Then using Lemma 5.1 (ii) with $f(t) = |Y_t^{\varepsilon,\eta} - Y_t^{\varepsilon',\eta'}|$ and $p = 2$ yields

$$\sup_{0\le s\le t}\mathbb{P}\text{-}\operatorname{ess\,sup}|Y_s^{\varepsilon,\eta} - Y_s^{\varepsilon',\eta'}| \le C(\varepsilon + \varepsilon' + \eta + \eta')\exp\left(C\int_0^t \left(\|\rho_s^{\varepsilon,\eta}\|_{L^1\cap L^\infty} + \|\rho_s^{\varepsilon',\eta'}\|_{L^1\cap L^\infty}\right)ds\right)$$

for $t \in [0, T]$, where the constant C is independent of $\varepsilon, \varepsilon', \eta$ and η'.

• **Step B** (*Passing to the limit*): It follows from **Step A** that there exists a limit process $(Y_t)_{t\in[0,T]}$ of the $(Y_t^{\varepsilon,\eta})_{t\in[0,T]}$ as $\varepsilon, \eta \to 0$ in $L^1(\Omega, \mathcal{O}\times[0,T])$. Using the fact that

$$\sup_{t\in[0,T]}\mathcal{W}_\infty(\rho_t^{\varepsilon,\eta}, \rho_t^{\varepsilon',\eta'}) \le \sup_{t\in[0,T]}\mathbb{P}\text{-}\operatorname{ess\,sup}|Y_t^{\varepsilon,\eta} - Y_t^{\varepsilon',\eta'}|,$$

we also deduce that $(\rho_t^{\varepsilon,\eta})_{t\in[0,T]}$ is a Cauchy sequence in $\mathcal{C}([0,T];\mathcal{P}_1(\mathcal{O}))$. Then, by completeness of this space, we define $\rho \in \mathcal{C}([0,T];\mathcal{P}_1(\mathcal{O}))$ by $\rho_t := \lim_{\varepsilon,\eta\to 0}\rho_t^{\varepsilon,\eta}$ for $t \in [0,T]$.

We now check $(3.143)_2$. For this, we define K_t as

$$K_t := Y_t - Y_0 - \int_0^t V[\rho_s](Y_s)\,ds + \sqrt{2\sigma}B_t$$

and recall

$$K_t^{\varepsilon,\eta} = Y_t^{\varepsilon,\eta} - Y_0 - \int_0^t V^{\varepsilon,\eta}[\rho_s^{\varepsilon,\eta}](Y_s^{\varepsilon,\eta})\,ds + \sqrt{2\sigma}B_t.$$

We then claim that

$$\mathbb{E}\left[\left|\int_0^t (V[\rho_s](Y_s) - V^{\varepsilon,\eta}[\rho_s^{\varepsilon,\eta}](Y_s^{\varepsilon,\eta}))\,ds\right|\right] \to 0, \quad \text{as} \quad \varepsilon,\eta \to 0. \tag{3.144}$$

Let us assume that (3.144) holds at the moment. We will give the proof of that in the last part of the step. We also notice that if (3.144) holds, then the rest of the proof can be obtained by using the almost same argument as in [**20**, Lemma 1.2]. However, we provide the details of the proof for the reader's convenience and the completeness.

The convergence (3.144) implies that $(K_t^{\varepsilon,\eta})_{t\in[0,T]}$ converges to $(K_t)_{t\in[0,T]}$ in $L^1(\Omega, \mathcal{O}\times[0,T])$ as ε and η go to 0, and subsequently, for any $\phi \in \mathcal{C}_c^\infty(\mathcal{O})$ with $0 \le \phi \le 1$, it deduces

$$\int_0^t \phi(Y_s)\,d|K|_s \le \liminf_{\varepsilon,\eta\to 0} \int_0^t \phi(Y_s^{\varepsilon,\eta})\,d|K^{\varepsilon,\eta}|_s = 0.$$

Taking an increasing sequence converging to $\mathbf{1}_{\mathcal{O}}$, we find

$$\int_0^t \mathbf{1}_{\mathcal{O}}(Y_s)\,d|K|_s = 0.$$

This yields

$$|K|_t = \int_0^t d|K|_s = \int_0^t (\mathbf{1}_{\mathcal{O}}(Y_s) + \mathbf{1}_{\partial\mathcal{O}}(Y_s))\,d|K|_s = \int_0^t \mathbf{1}_{\partial\mathcal{O}}(Y_s)\,d|K|_s.$$

We now again use the convexity of the domain $\mathcal{O}$ to get that for any $w \in \mathcal{O}$ and $0 \le \phi \in C([0,T])$

$$\int_0^t \langle Y_s^{\varepsilon,\eta} - w, n(Y_s^{\varepsilon,\eta})\rangle\,\phi(s)\,d|K^{\varepsilon,\eta}|_s \ge 0,$$

which can be rewritten as

$$\int_0^t \phi(s)\,\langle Y_s^{\varepsilon,\eta} - w, dK_s^{\varepsilon,\eta}\rangle \ge 0.$$

We then let $\varepsilon,\eta \to 0$ to find that $d|K^{\varepsilon,\eta}|_s$ converges weakly (up to subsequence) to some measure dm_s with $d|K|_s \le dm_s$ and deduce

$$\int_0^t \phi(s)\,\langle Y_s - w, k_s\rangle\; dm_s \ge 0,$$

where we denoted k_s a nonzero vector valued function such that $dK_s = k_s dm_s$ with $d|K|_s = |k_s|dm_s$ we obtain that

$$\langle Y_s - w, k_s\rangle \ge 0 \quad \text{for all} \quad w \in \overline{\mathcal{O}} \quad m_s - \text{a.s.}.$$

Then we find $k_s = |k_s|n(Y_s)$ due to the convexity of $\mathcal{O}$. Hence we have

$$K_t = \int_0^t n(Y_s)|k_s|\,dm_s = \int_0^t n(Y_s)\,d|K|_s.$$

Proof of Claim (3.144).- We first split (3.144) into two parts:

$$\begin{aligned}
&\left|\int_0^t (V[\rho_s](Y_s) - V^{\varepsilon,\eta}[\rho_s^{\varepsilon,\eta}](Y_s^{\varepsilon,\eta}))\,ds\right| \\
&\quad\le \left|\int_0^t (V[\rho_s](Y_s) - V[\rho_s^{\varepsilon,\eta}](Y_s^{\varepsilon,\eta}))\,ds\right| + \left|\int_0^t (V[\rho_s^{\varepsilon,\eta}](Y_s^{\varepsilon,\eta}) - V^{\varepsilon,\eta}[\rho_s^{\varepsilon,\eta}](Y_s^{\varepsilon,\eta}))\,ds\right| \\
&\quad=: L_1^{\varepsilon,\eta} + L_2^{\varepsilon,\eta}.
\end{aligned}$$

It follows from Lemma 1.2 that

$$L_1^{\varepsilon,\eta} \le C\|\nabla\varphi\|_{W^{1,\infty}}\|w\|_{Lip}\int_0^t \|\rho_s\|_{L^1\cap L^\infty}\,(\mathbb{P}\text{-}\operatorname{ess\,sup}|Y_s - Y_s^{\varepsilon,\eta}|)\;ds \to 0,$$

as $\varepsilon,\eta \to 0$. Using similar arguments for the term J_1 in **Step A**, L_2 can be estimated as

$$L_2^{\varepsilon,\eta} \le C(\varepsilon+\eta)\|\nabla\varphi\|_{L^\infty}\int_0^t \|\rho_s^{\varepsilon,\eta}\|_{L^1\cap L^\infty}\,ds \quad \text{for} \quad \varepsilon,\eta \le 1/2.$$

Thus $L_2^{\varepsilon,\eta} \to 0$ as $\varepsilon, \eta \to 0$, and this concludes the proof of claim.

• **Step C** (*Stability estimate*): Using similar arguments as in **Step A**, if $Y_t^i, i = 1, 2$ are two processes obtained as the above with the initial data Y_0^i, respectively, and $\mathcal{L}(Y_t^i) = \rho_t^i, t \geq 0$ for $i = 1, 2$, using Lemma 1.2, we easily find

$$|Y_t^1 - Y_t^2|^2 \leq C \int_0^t 2\|\rho_s\|_{L^1 \cap L^\infty} \left(\mathbb{P}\text{-ess sup}\, |Y_s^1 - Y_s^2|\right) |Y_s^1 - Y_s^2|\, ds$$

Similarly as in **Step A**, we apply Lemma 5.1 (ii) with $p = 2$ to conclude the stability estimate. □

3.3. Proof of Theorem 1.2. The existence and uniqueness of strong solutions to the nonlinear SIEs (1.131) just follows from Proposition 3.2. For the existence of weak solutions for the equation (1.132), just take any test function $\phi \in \mathcal{C}^\infty(\mathcal{O})$ with $\langle \nabla\phi(x), n(x)\rangle = 0$ on $\partial\mathcal{O}$ and apply Ito's formula to the solution to (1.131), then find that its time marginals $(\rho_t)_{t\in[0,T]}$ solves the equation (1.132) in the distributional sense. For the uniqueness of solutions, we move the stability estimate of solutions for SIEs obtained in Proposition 3.2 on to some stability estimate for the corresponding PDE. In order to do so, we use the fact that for any solutions to (1.132) can be seen as the time marginals of some solutions to (3.143). Let $(\tilde{\rho}_t)_{t\geq 0}$ be a weak solution to (1.132) with the initial data $\tilde{\rho}_0 \in \mathcal{P}_1(\mathcal{O})$ and $(\rho_t)_{t\geq 0}$ be another weak solution to (1.132) with the initial data $\rho_0 \in (L^1_+ \cap L^\infty)(\mathcal{O}) \cap \mathcal{P}_1(\mathcal{O})$ such that $\rho \in L^\infty(0, T; (L^1_+ \cap L^\infty)(\mathcal{O})) \cap \mathcal{C}([0, T]; \mathcal{P}_1(\mathcal{O}))$. Then, by Lemma 5.2, we can find a probability space $(\Omega, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F})$, a Brownian motion $(B_t)_{t\geq 0}$ on that basis and a process $(X_t)_{t\geq 0}$ solution to (3.143), which has the time marginal $\tilde{\rho}_t$ at any time $t \geq 0$. On that probability space, let Y_0 be a random variable on $\mathcal{O}$ with the law ρ_0 independent of $(B_t)_{t\geq 0}$ such that

$$\mathcal{W}_\infty(\rho_0, \tilde{\rho}_0) = \mathbb{P}\text{-ess sup}\, |Y_0 - X_0|.$$

Note that it is known that such an optimal coupling exists when ρ_0 is absolutely continuous with respect to the Lebesgue measure, see [**8**]. On the other hand, since ρ has a sufficient regularity for the velocity field to be Lipschitz (see the proof of Proposition 3.1), the standard theory on linear SDEs allows to build some stochastic process $(Y_t)_{t\geq 0}$ which is a solution to (3.143) with the initial condition Y_0, and same Brownian motion as exhibited in the beginning of this step, such that its marginal at time t is ρ_t. Hence, by definition of $\mathcal{W}_\infty$ distance and Proposition 3.2, it is straightforward to deduce that

$$\mathcal{W}_\infty(\rho_t, \tilde{\rho}_t) \leq \mathbb{P}\text{-ess sup}\, |Y_t - X_t| \leq \mathbb{P}\text{-ess sup}\, |Y_0 - X_0| e^{\int_0^t \|\rho_s\|_{L^1\cap L^\infty} ds} = \mathcal{W}_\infty(\rho_0, \tilde{\rho}_0) e^{\int_0^t \|\rho_s\|_{L^1\cap L^\infty} ds},$$

from which the uniqueness of solutions to (1.132) follows. This completes the proof.

Remark 3.1. *It is worth noticing that we are not able at this point to extend this result to the case where $\mathcal{O}$ is not convex but only satisfies the exterior sphere condition* (3.135)*. This is due to the fact that we can only obtain the weak-strong stability estimate in the $\mathcal{W}_\infty$ metric. That is why we have to estimate the $\mathbb{P}$ essential supremum of the distance between two regularized solutions. If the domain $\mathcal{O}$ only satisfies the condition* (3.135)*, then we need to use the similar strategy as in* [**20**]*, together with approximating the $\mathbb{P}$-essential supremum by $\mathbb{E}[|\cdot|^p]^{1/p}$ with $p \geq 1$. However, this gives a p-dependent constant in the estimates and it cannot be removed. Thus our arguments fail to the case in which the domain $\mathcal{O}$ only satisfies the condition* (3.135)*.*

4. Propagation of chaos

4.1. Law of large numbers like estimates. In this subsection, we provide types of the law of large numbers estimates which relies on the nice property of our communication function observed in Lemma 1.1.

Lemma 4.1. *Let $m, N \in \mathbb{N}$ with $N \geq (2m)^2$ and $Y_1, \cdots, Y_N$ be N i.i.d. random variables with the law $\rho \in \mathcal{P}(\mathcal{O})$, Y be independent of $Y_1, \cdots, Y_N$, and let ρ^N be the associated empirical measure $\rho^N = \frac{1}{N}\sum_{i=1}^N \delta_{Y_i}$. Then we have*

$$\mathbb{E}\left[\sup_{i=1,\cdots,N} \sup_{u\geq 0} \left| \int_{\mathcal{O}} \mathbf{1}_{\Theta(w(Y_i))^{u,+}}(y - Y_i)(\rho^N - \rho)(dy) \right|^{2m}\right]^{\frac{1}{2m}} \leq 8e^{2m} N^{-\frac{1}{2}+\frac{1}{2m}}.$$

PROOF. Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables with the law $\rho \in \mathcal{P}(\mathcal{O})$, and K_N be a Poisson random variable of parameter N independent of $(Y_n)_{n\in\mathbb{N}}$. Define ϱ^N by the following random measure

$$\varrho^N := \sum_{i=1}^{K_N} \delta_{Y_i}.$$

Then ϱ^N is a Poisson random measure of the intensity measure $N\rho$. It is straightforward to get $\left\|\varrho^N - N\rho^N\right\|_{TV} = |K_N - N|$, where $\|\cdot\|_{TV}$ represents the total variation of signed measures. This yields

$$\begin{aligned}
&\mathbb{E}\left[\sup_{i=1,\cdots,N}\sup_{u\geq 0}\left|\int_{\mathcal{O}} \mathbf{1}_{\Theta(w(Y_i))^{u,+}}(y-Y_i)(\rho^N-\rho)(dy)\right|^{2m}\right]^{\frac{1}{2m}}\\
&\quad\leq \mathbb{E}\left[\sum_{i=1}^{N}\left(\sup_{u\geq 0}\left|\int_{\mathcal{O}} \mathbf{1}_{\Theta(w(Y_i))^{u,+}}(y-Y_i)(\rho^N-\rho)(dy)\right|\right)^{2m}\right]^{\frac{1}{2m}}\\
&\quad\leq \mathbb{E}\left[N^{-2m}\sum_{i=1}^{N}\sup_{u\geq 0}\left(\left|\int_{\mathcal{O}} \mathbf{1}_{\Theta(w(Y_i))^{u,+}}(y-Y_i)(N\rho^N-\varrho^N)(dy)+\mathcal{M}_u^{N,Y_i}\right|\right)^{2m}\right]^{\frac{1}{2m}}\\
&\quad\leq 2\mathbb{E}\left[N^{-2m}\sum_{i=1}^{N}\left(\sup_{u\geq 0}\left|\int_{\mathcal{O}} \mathbf{1}_{\Theta(w(Y_i))^{u,+}}(y-Y_i)(N\rho^N-\varrho^N)(dy)\right|^{2m}+\sup_{u\geq 0}\left|\mathcal{M}_u^{N,Y_i}\right|^{2m}\right)\right]^{\frac{1}{2m}}\\
&\quad\leq 2\mathbb{E}\left[N^{-2m}\sum_{i=1}^{N}\left(\left\|\varrho^N-N\rho^N\right\|_{TV}^{2m}+\sup_{u\geq 0}\left|\mathcal{M}_u^{N,Y_i}\right|^{2m}\right)\right]^{\frac{1}{2m}},
\end{aligned}$$

where

$$\mathcal{M}_u^{N,Y_i} := \int_{\mathcal{O}} \mathbf{1}_{\Theta(w(Y_i))^{u,+}}(y-Y_i)\overline{\varrho}^N(dy) \quad \text{with} \quad \overline{\varrho}^N := \varrho^N - N\rho.$$

Since the $(Y_i)_{i=1,\cdots,N}$ are i.i.d, we find

$$\begin{aligned}
&\mathbb{E}\left[\sup_{i=1,\cdots,N}\sup_{u\geq 0}\left|\int_{\mathcal{O}} \mathbf{1}_{\Theta(w(Y_i))^{u,+}}(y-Y_i)(\rho^N-\rho)(dy)\right|^{2m}\right]^{\frac{1}{2m}}\\
&\qquad\leq 2N^{\frac{-2m+1}{2m}}\left(\mathbb{E}\left[|K_N-N|^{2m}\right]^{\frac{1}{2m}}+\mathbb{E}\left[\sup_{u\geq 0}\left|\mathcal{M}_u^{N,Y_1}\right|^{2m}\right]^{\frac{1}{2m}}\right).
\end{aligned} \tag{4.145}$$

We next observe that $(\mathcal{M}_u^{N,Y_1})_{u\geq 0}$ conditioned to Y_1 is a martingale. Indeed, for $a\in\mathcal{O}$, we define the filtration $(\mathcal{F}_u^a)_{u\geq 0}$ as

$$\mathcal{F}_u^a = \sigma\left\{\int_{\mathcal{O}} h(y)\varrho^N(dy) \mid supp\ h \subseteq \Theta(w(a))^{r,+}+a,\ r\leq u\right\},$$

Then, for $s>u$, we find

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathcal{O}} \mathbf{1}_{\Theta(w(a))^{s,+)}}(y-a)\overline{\varrho}^N(dy)\,\middle|\,\mathcal{F}_u^a\right]\\
&\quad=\mathbb{E}\left[\int_{\mathcal{O}} \mathbf{1}_{\Theta(w(a))^{s,+}\setminus\Theta(w(a))^{u,+}}(y-a)\overline{\varrho}^N(dy)\,\middle|\,\mathcal{F}_u^a\right]+\mathbb{E}\left[\int_{\mathcal{O}} \mathbf{1}_{\Theta(w(a))^{u,+}}(y-a)\overline{\varrho}^N(dy)\,\middle|\,\mathcal{F}_u^a\right]
\end{aligned}$$

due to the linearity of conditional expectation and the following property of indicator function

$$\mathbf{1}_A = \mathbf{1}_{A\setminus B}+\mathbf{1}_B \quad \text{for} \quad A,B\subset\mathbb{R}^d \text{ and } B\subseteq A.$$

Since $\varrho^N(A)$ is independent of $\varrho^N(B)$ for disjoint sets $A,B\subset\mathbb{R}^d$, by the definition of Poisson random measure (see [**10**]),

$$\int_{\mathcal{O}} \mathbf{1}_{\Theta(w(a))^{s,+}\setminus\Theta(w(a))^{u,+}}(y-a)\,\overline{\varrho}^N(dy) = \overline{\varrho}^N\left(\left(\Theta(w(a))^{s,+}\setminus\Theta(w(a))^{u,+}\right)+a\right),$$

is independent of all $\varrho^N(\Theta(w(a))^{s,+} \setminus \Theta(w(a))^{r,+})$ for $r \le u$, thus it is also independent of $\mathcal{F}_u^a$. This yields

$$\mathbb{E}\left[\int_{\mathcal{O}} \mathbf{1}_{\Theta(w(a))^{s,+}\setminus\Theta(w(a))^{u,+}}(y-a)\,\overline{\varrho}^N(dy) \mid \mathcal{F}_u^a\right] = \mathbb{E}\left[\int_{\mathcal{O}} \mathbf{1}_{\Theta(w(a))^{s,+}\setminus\Theta(w(a))^{u,+}}(y-a)\,\overline{\varrho}^N(dy)\right] = 0.$$

On the other hand, since $\int_{\mathcal{O}} \mathbf{1}_{\Theta(w(a))^{u,+}}(y-a)\overline{\varrho}^N(dy)$ is $\mathcal{F}_u^a$-measurable, we deduce

$$\mathbb{E}\left[\int_{\mathcal{O}} \mathbf{1}_{\Theta(w(a))^{u,+}}(y-a)\,\overline{\varrho}^N(dy) \mid \mathcal{F}_u^a\right] = \int_{\mathcal{O}} \mathbf{1}_{\Theta(w(a))^{u,+}}(y-a)\,\overline{\varrho}^N(dy),$$

and $(\mathcal{M}_u^{N,a})_{u\ge 0}$ is a martingale. We now use Doob's inequality to obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{u\ge 0}\left|\mathcal{M}_u^{N,a}\right|^{2m}\right] &\le \left(\frac{2m}{2m-1}\right)^{2m}\mathbb{E}\left[\left|\mathcal{M}_\infty^{N,a}\right|^{2m}\right]\\ &= \left(\frac{2m}{2m-1}\right)^{2m}\mathbb{E}\left[\left|\varrho_N(\mathcal{O}) - N\rho(\mathcal{O})\right|^{2m}\right]\\ &= \left(\frac{2m}{2m-1}\right)^{2m}\mathbb{E}\left[\left|K_N - N\right|^{2m}\right],\end{aligned}$$

We next use a standard property of the conditional expectation to get

$$\begin{aligned}\mathbb{E}\left[\sup_{u\ge 0}\left|\mathcal{M}_u^{N,1}\right|^{2m}\right] &= \mathbb{E}\left[\mathbb{E}\left[\sup_{u\ge 0}\left|\int_{\mathcal{O}} \mathbf{1}_{\Theta(w(Y_1))^{u,+}}(y-Y_1)(\rho^N-\rho)(dy)\right|^{2m} \mid Y_1\right]\right]\\ &\le \left(\frac{2m}{2m-1}\right)^{2m}\mathbb{E}\left[\left|K_N - N\right|^{2m}\right].\end{aligned}$$

Coming back to (4.145), we then find

$$\begin{aligned}&\mathbb{E}\left[\sup_{i=1,\cdots,N}\sup_{u\ge 0}\left|\int_{\mathcal{O}} \mathbf{1}_{\Theta(w(Y_i))^{u,+}}(y-Y_i)(\rho^N-\rho)(dy)\right|^{2m}\right]^{\frac{1}{2m}}\\ &\quad\le 2\left(1+\frac{2m}{2m-1}\right)N^{-1+\frac{1}{2m}}\mathbb{E}\left[\left|K_N-N\right|^{2m}\right]^{\frac{1}{2m}}.\end{aligned}$$

Note that K_N is the Poisson(N)-distributed random variable, thus it holds

$$\mathbb{E}\left[\exp\left(2m\frac{|K_N-N|}{\sqrt{N}}\right)\right] \le 2\exp\left(N\left(e^{\frac{2m}{\sqrt{N}}} - 1 - \frac{2m}{\sqrt{N}}\right)\right) \le 2e^{(2m)^2},$$

due to

$$N\left(e^{\frac{2m}{\sqrt{N}}} - 1 - \frac{2m}{\sqrt{N}}\right) = (2m)^2\int_0^1 (1-t)e^{\frac{2m}{\sqrt{N}}t}\,dt \le (2m)^2\frac{\sqrt{N}}{2m}\left(e^{\frac{2m}{\sqrt{N}}}-1\right) \le (2m)^2,$$

where we used the fact that the function $(1-t)e^{\frac{2m}{\sqrt{N}}t}$ on $[0,1]$ is bounded by $\frac{\sqrt{N}}{2m}(e^{\frac{2m}{\sqrt{N}}}-1)$ and $e^{x-1} \le x$ for $0 \le x \le 1$. This yields

$$\frac{1}{N^m}\mathbb{E}\big[|K_N-N|^{2m}\big] \le \mathbb{E}\left[e^{2m\frac{|K_N-N|}{\sqrt{N}}}\right] \le 2e^{(2m)^2}.$$

□

LEMMA 4.2. *Let $m, N \in \mathbb{N}$ with $N \ge (2m)^2$ and $Y_1, \cdots, Y_N$ be N i.i.d. random variables with the law $\rho \in \mathcal{P}(\mathcal{O})$. Then we have*

$$\mathbb{E}\left[\sup_{i=1,\cdots,N}\left|\int_{\mathcal{O}} \mathbf{1}_{K(w(Y_i))}(y-Y_i)\nabla\varphi(Y_i-y)(\rho^N(dy)-\rho(dy))\right|^{2m}\right]^{\frac{1}{2m}} \le \|\nabla\varphi\|_{L^\infty}(2m!)^{\frac{1}{2m}}\sqrt{8m}N^{-\frac{1}{2}+\frac{1}{2m}},$$

where $\rho^N = \frac{1}{N}\sum_{i=1}^N \delta_{Y_i}$.

PROOF. Let $\mathcal{M}$ be a set of N-dimensional multi-index of order $2m$, i.e.,

$$\mathcal{M} := \left\{\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{N}^N : |\alpha| = \sum_{i=1}^N \alpha_i = 2m\right\},$$

and define $\mathcal{M}_1 := \{(\alpha_j)_j \in \mathcal{M} : \alpha_j \neq 1, \forall\, j = 1, \cdots, N\}$. Let us denote by $\ell_\alpha := |\{i = 1, \cdots, N : \alpha_i > 0\}|$ with $\alpha \in \mathcal{M}_1$. It is clear that $\ell_\alpha \leq m$. We now consider two functions $\phi_\alpha : \{1, \cdots, \ell_\alpha\} \to \{1, \cdots, N\}$ and $\psi_\alpha : \{1, \cdots, \ell_\alpha\} \to \{2, \cdots, 2m\}$ with $\alpha \in \mathcal{M}_1$ defined by

$$\phi_\alpha(i) = \text{“the index of the } i\text{-th nonzero component of } \alpha\text{”}$$

and

$$\psi_\alpha(i) = \text{“the value of the } i\text{-th nonzero component of } \alpha\text{”},$$

respectively. Then we get that

$$\Phi : \mathcal{M}_1 \ni \alpha \mapsto (\phi_\alpha, \psi_\alpha) \in \bigcup_{k=1}^{m} \{1, \cdots, N\}^{\{1, \cdots, k\}} \times \{1, \cdots, 2m\}^{\{1, \cdots, k\}},$$

is injective and thus

$$|\mathcal{M}_1| \leq \sum_{k=1}^{m} N^k (2m)^k \leq 2(2m)^m N^m.$$

For notational simplicity, we now set for a fixed $i \in \{1, \cdots, N\}$

$$h_k^i := \nabla\varphi(Y_i - Y_k)\mathbf{1}_{K(w(Y_i))}(Y_k - Y_i) - \int_{\mathcal{O}} \nabla\varphi(Y_i - y)\mathbf{1}_{K(w(Y_i))}(y - Y_i)\rho(dy) \quad \text{and} \quad \mathbf{h}^i := (h_1^i, \cdots, h_N^i).$$

A straightforward computation yields

$$\begin{aligned}
&\mathbb{E}\left[\sup_{i=1,\cdots,N}\left|\int_{\mathcal{O}} \mathbf{1}_{K(w(Y_i))}(y - Y_i)\nabla\varphi(Y_i - y)(\rho^N(dy) - \rho(dy))\right|^{2m}\right]^{\frac{1}{2m}} \\
&\quad \leq \mathbb{E}\left[\left(\sum_{i=1}^{N}\left(\int_{\mathcal{O}} \nabla\varphi(Y_i - y)\mathbf{1}_{K(w(Y_i))}(y - Y_i)(\rho^N(dy) - \rho(dy))\right)^{2m}\right)\right]^{\frac{1}{2m}} \\
&\quad = \mathbb{E}\left[\left(\sum_{i=1}^{N} N^{-2m}\left(\sum_{j=1}^{N} h_j^i\right)^{2m}\right)\right]^{\frac{1}{2m}} \\
&\quad = \left(\sum_{i=1}^{N} N^{-2m}\mathbb{E}\left[\left(\sum_{j=1}^{N} h_j^i\right)^{2m}\right]\right)^{\frac{1}{2m}}.
\end{aligned}$$

Then it follows from the law of total expectation that

$$\mathbb{E}\left[\left(\sum_{j=1}^{N} h_j^i\right)^{2m}\right] = \mathbb{E}\left[\mathbb{E}\left[\left(\sum_{j=1}^{N} h_j^i\right)^{2m} \mid Y_i\right]\right] = \mathbb{E}\left[\mathbb{E}\left[\sum_{\alpha\in\mathcal{M}} \binom{2m}{\alpha}(\mathbf{h}^i)^\alpha \mid Y_i\right]\right].$$

Note that $(h_k^i)_{k=1,\cdots,N}^{\alpha_k}$ conditioned on Y_i are independent since $(Y_k)_{k=1,\cdots,N}$ are independent. Thus we find

$$\mathbb{E}\left[\left(\sum_{j=1}^{N} h_j^i\right)^{2m}\right] = \sum_{\alpha\in\mathcal{M}} \binom{2m}{\alpha}\mathbb{E}\left[\prod_{j=1}^{N} \mathbb{E}\left[(h_j^i)^{\alpha_j} \mid Y_i\right]\right].$$

We also notice that

(4.146)

$$\mathbb{E}\left[\nabla\varphi(Y_i - Y_j)\mathbf{1}_{K(w(Y_i))}(Y_j - Y_i) \mid Y_i\right] = \int_{\mathcal{O}} \nabla\varphi(Y_i - y)\mathbf{1}_{K(w(Y_i))}(y - Y_i)\,\rho(dy), \quad \text{i.e.,} \quad \mathbb{E}\left[h_j^i \mid Y_i\right] = 0.$$

This implies

$$\mathbb{E}\left[(\mathbf{h}^i)^\alpha \,|\, Y_i\right] = 0 \quad \text{for} \quad \alpha \notin \mathcal{M}_1.$$

On the other hand, we obtain

$$\mathbb{E}\left[(\mathbf{h}^i)^\alpha \,|\, Y_i\right] \leq \left(2\|\nabla\varphi\|_{W^{1,\infty}}\right)^{2m},$$

due to $|h_k^i| \leq 2$. This and together with (4.146) yields

$$\mathbb{E}\left[\left(\sum_{j=1}^{N} h_j^i\right)^{2m}\right] \leq \sup_{\alpha\in\mathcal{M}} \binom{2m}{\alpha}\left(2\|\nabla\varphi\|_{W^{1,\infty}}\right)^{2m} |\mathcal{M}_1| \leq \sup_{\alpha\in\mathcal{M}} \binom{2m}{\alpha}(8\|\nabla\varphi\|_{W^{1,\infty}}^2 m)^m N^m,$$

and the result follows.

□

We also provide a kind of weak-strong Lipschitz estimate for the velocity fields in the lemma below.

LEMMA 4.3. *Let $N \in \mathbb{N}$ and $X_1, \cdots, X_N$ be N exchangeable random variables on $\mathcal{O}$. Define $\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_i}$ the empirical measure associated. Let $Y_1, \cdots, Y_N$ be N i.i.d random variables on $\mathcal{O}$ with the law $\rho \in (L^1_+ \cap L^\infty)(\mathcal{O}) \cap \mathcal{P}_1(\mathcal{O})$ and $\rho^N = \frac{1}{N}\sum_{i=1}^N \delta_{Y_i}$. Then there exists a constant $C > 0$ which is independent of N and a random variable H_N such that*

$$\sup_{i=1,\cdots,N} \Big| V[\mu^N](X_i) - V[\rho](Y_i) \Big| \le C \, \|\rho\|_{L^1\cap L^\infty} \sup_{i=1,\cdots,N} |X_i - Y_i| + H_N,$$

where V is given in (1.130) *and the set K in V satisfies the assumption* (**H1**)-(**H2**). *Here H_N is given by*

$$H_N := \|\nabla\varphi\|_{L^\infty} \sup_{i=1,\cdots,N} \sup_{u\ge 0} \left| \int_{\mathcal{O}} \mathbf{1}_{\Theta(w(Y_i))^{u,+}}(y - Y_i)(\rho^N - \rho)(dy) \right| + \sup_{i=1,\cdots,N} \Big| V[\rho^N](Y_i) - V[\rho](Y_i) \Big|$$

and satisfies

$$\mathbb{E}\big[H_N^{2m}\big]^{\frac{1}{2m}} \le C_m N^{-\frac{1}{2}+\frac{1}{2m}},$$

for $m \in \mathbb{N}$ such that $(2m)^2 \le N$, where C_m is a positive constant, specified in the proof, depending on m, but not N.

PROOF. For notational simplicity, we denote by $A_N = \sup_{j=1,\cdots,N} |X_j - Y_j|$. For any $i = 1, \cdots, N$, we have

$$\begin{aligned}
&\Big| V[\mu^N](X_i) - V[\rho](Y_i) \Big| \\
&\quad \le \Big| V[\rho^N](Y_i) - V[\mu^N](X_i) \Big| + \Big| V[\rho^N](Y_i) - V[\rho](Y_i) \Big| \\
&\quad \le \left| \frac{1}{N}\sum_{j=1}^N \Big(\mathbf{1}_{K(w(X_i))}(X_j - X_i)\nabla\varphi(X_i - X_j) - \mathbf{1}_{K(w(Y_i))}(Y_j - Y_i)\nabla\varphi(Y_i - Y_j) \Big) \right| \\
&\qquad + \left| \frac{1}{N}\sum_{j=1}^N \mathbf{1}_{K(w(Y_i))}(Y_j - Y_i)\nabla\varphi(Y_i - Y_j) - V[\rho](Y_i) \right| \\
&\quad \le \left| \frac{1}{N}\sum_{j=1}^N \mathbf{1}_{K(w(X_i))}(X_j - X_i)\Big(\nabla\varphi(X_i - X_j) - \nabla\varphi(Y_i - Y_j) \Big) \right| \\
&\qquad + \left| \frac{1}{N}\sum_{j=1}^N \Big(\mathbf{1}_{K(w(X_i))}(X_j - X_i) - \mathbf{1}_{K(w(X_i))}(Y_j - Y_i) \Big)\nabla\varphi(Y_i - Y_j) \right| \\
&\qquad + \left| \frac{1}{N}\sum_{j=1}^N \Big(\mathbf{1}_{K(w(X_i))}(Y_j - Y_i) - \mathbf{1}_{K(w(Y_i))}(Y_j - Y_i) \Big)\nabla\varphi(Y_i - Y_j) \right| + \Big| V[\rho^N](Y_i) - V[\rho](Y_i) \Big| \\
&\quad =: I_1^i + I_2^i + I_3^i + I_4^i.
\end{aligned}$$

◇ Estimate of I_1^i: We easily find

$$I_1^i \le \|\nabla\varphi\|_{W^{1,\infty}} \left(|X_i - Y_i| + \frac{1}{N}\sum_{j=1}^N |X_j - Y_j| \right) \le 2\|\nabla\varphi\|_{W^{1,\infty}} A_N \le C\|\rho\|_{L^1\cap L^\infty} A_N.$$

◇ Estimate of I_2^i: It follows from Lemma 1.1, the assumptions (**H2**) (i) and (iv) that

$$\begin{aligned}
\big| \mathbf{1}_{K(w(X_i))}(X_j - X_i) - \mathbf{1}_{K(w(X_i))}(Y_j - Y_i) \big| &\le \mathbf{1}_{\partial^{2|X_i - Y_i|} K(w(X_i))}(Y_j - Y_i) + \mathbf{1}_{\partial^{2|X_j - Y_j|} K(w(X_i))}(Y_j - Y_i) \\
&\le \mathbf{1}_{\partial^{2A_N} K(w(X_i))}(Y_j - Y_i) \\
&\le \mathbf{1}_{\Theta(w(Y_i))^{2A_N + 2\|w\|_{Lip} A_N, +}}(Y_j - Y_i) \\
&\le \mathbf{1}_{\Theta(w(Y_i))^{CA_N, +}}(Y_j - Y_i).
\end{aligned}$$

Then, thanks to (**H2**) (ii), we obtain

$$\begin{aligned}
&\frac{1}{N}\sum_{j=1}^{N}\mathbf{1}_{\Theta(w(Y_i))^{CA_N,+}}(Y_j-Y_i)\\
&\quad=\int_{\mathcal{O}}\mathbf{1}_{\Theta(w(Y_i))^{CA_N,+}}(y-Y_i)\rho^N(dy)\\
&\quad\le\int_{\mathcal{O}}\mathbf{1}_{\Theta(w(Y_i))^{CA_N,+}}(y-Y_i)\rho(dy)+\left|\int_{\mathcal{O}}\mathbf{1}_{\Theta(w(Y_i))^{CA_N,+}}(y-Y_i)(\rho^N-\rho)(dy)\right|\\
&\quad\le C\left\|\rho\right\|_{L^1\cap L^\infty}A_N+\sup_{u\ge 0}\left|\int_{\mathcal{O}}\mathbf{1}_{\Theta(w(Y_i))^{u,+}}(y-Y_i)(\rho^N-\rho)(dy)\right|.
\end{aligned}$$

$\diamond$ Estimate of I_3^i: Using the assumptions (**H2**) (ii) and (iii) together with similar estimates as the above, we get

$$\begin{aligned}
I_3^i&\le\|\nabla\varphi\|_{L^\infty}\left|\int_{\mathcal{O}}\mathbf{1}_{K(w(X_i))\Delta K(w(Y_i))}(y-Y_i)\rho^N(dy)\right|\\
&\le\|\nabla\varphi\|_{L^\infty}\left|\int_{\mathcal{O}}\mathbf{1}_{\Theta(w(Y_i))^{C\|w\|_{Lip}|X_i-Y_i|,+}}(y-Y_i)\rho^N(dy)\right|\\
&\le\|\nabla\varphi\|_{L^\infty}\int_{\mathcal{O}}\mathbf{1}_{\Theta(w(Y_i))^{C\|w\|_{Lip}A_N,+}}(y-Y_i)\rho(dy)\\
&\quad+\|\nabla\varphi\|_{L^\infty}\left|\int_{\mathcal{O}}\mathbf{1}_{\Theta(w(Y_i))^{C\|w\|_{Lip}A_N,+}}(y-Y_i)(\rho^N-\rho)(dy)\right|\\
&\le\|\nabla\varphi\|_{L^\infty}\left(C\|w\|_{Lip}\|\rho\|_{L^1\cap L^\infty}A_N+\sup_{u\ge 0}\left|\int_{\mathcal{O}}\mathbf{1}_{\Theta(w(Y_i))^{u,+}}(y-Y_i)(\rho^N-\rho)(dy)\right|\right).
\end{aligned}$$

$\diamond$ Estimate of I_4^i: It just follows from Lemma 4.2 that

$$\mathbb{E}\left[\sup_{i=1,\cdots,N}\left(I_4^i\right)^{2m}\right]^{\frac{1}{2m}}\le\|\nabla\varphi\|_{L^\infty}(2m!)^{\frac{1}{2m}}\sqrt{8m}N^{-\frac{1}{2}+\frac{1}{2m}}.$$

We then combine the above estimates and take the supremum over $i=1,\cdots,N$ to find

$$\sup_{i=1,\cdots,N}\left|V[\mu^N](X_i)-V[\rho](Y_i)\right|\le C\left\|\rho\right\|_{L^1\cap L^\infty}A_N+H_N.$$

Finally, we use Lemma 4.1 for the second term on the right hand side of the above inequality to conclude the desired result. □

4.2. Proof of Theorem 1.3. It follows from Theorems 1.1 and 1.2 that there exist a weak solution to (1.129) and a unique pathwise solution to (1.131) on the time interval $[0,T]$ for some $T>0$. This implies that we are able to define solutions for those two equations on the same probability space with the same initial condition and Brownian motion. On that probability space, we define μ_t^N,ρ_t^N the empirical measures associated to system (1.129) and (1.131) by

$$\mu_t^N=\frac{1}{N}\sum_{i=1}^{N}\delta_{X_t^i}\quad\text{and}\quad\rho_t^N=\frac{1}{N}\sum_{i=1}^{N}\delta_{Y_t^i},$$

respectively. Using the similar argument as before, we get

$$\begin{aligned}
|X_i^t-Y_i^t|^2&\le 2\int_0^t\left\langle X_s^i-Y_s^i,V[\mu_s^N](X_s^i)-V[\rho_s](Y_s^i)\right\rangle ds\\
&\qquad-2\int_0^t\left\langle X_s^i-Y_s^i,n(X_s^i)\right\rangle d|K^i|_s-2\int_0^t\left\langle Y_s^i-X_s^i,n(Y_s^i)\right\rangle d|\tilde{K}^i|_s\\
&\le 2\int_0^t|X_s^i-Y_s^i|\sup_{k=1,\cdots,N}|V[\mu_s^N](X_s^k)-V[\rho_s](Y_s^k)|\,ds,
\end{aligned}$$

due to the convexity of the domain $\mathcal{O}$. Using Lemmas 5.1 and 4.3, we find

$$\sup_{i=1,\cdots,N}|X_t^i-Y_t^i|\le C\int_0^t\left(\|\rho_s\|_{L^1\cap L^\infty}\sup_{i=1,\cdots,N}|X_s^i-Y_s^i|+H_N\right)ds.$$

Applying Gronwall's inequality, taking the expectation, and using Holder's inequality lead to (4.147)

$$\mathbb{E}\left[\mathcal{W}_\infty(\mu_t^N,\rho_t^N)\right] \le \mathbb{E}\left[\sup_{i=1,\cdots,N}|X_t^i - Y_t^i|\right] \le t\exp\left(\int_0^t \|\rho_s\|_{L^1\cap L^\infty}\,ds\right)\mathbb{E}\left[(H_N)^{2m}\right]^{\frac{1}{2m}}.$$

Finally, we use the convergence estimate obtained in [**14**, Theorem 1] together with the moment estimate in Remark 1.2 to find that for all $t\in[0,T]$

$$\mathbb{E}\left[\mathcal{W}_p(\rho_t^N,\rho_t)\right] \le C_m N^{-\frac{1}{2}+\frac{1}{2m}} + C\begin{cases} N^{-1/2p}+N^{-(q-p)/qp} & \text{if } 2p>d \text{ and } q\neq 2p,\\ N^{-1/2p}\log(1+N)^{1/p}+N^{-(q-p)/qp} & \text{if } 2p=d \text{ and } q\neq 2p,\\ N^{-1/d}+N^{-(q-p)/qp} & \text{if } 2p<d \text{ and } q\neq d/(d-p).\end{cases}$$

Combining the above inequality and (4.147) concludes the proof.

5. Appendix

5.1. Gronwall-type inequalities. In this appendix, we present several Gronwall-type inequalities which used for the estimates of uniform bound and stability for solutions in the current work.

LEMMA 5.1. *Let f,g be nonnegative scalar functions.*

(i) *If f satisfies*

$$f(t)\le f_0\exp\left(C\int_0^t f(s)\,ds\right),\quad t\ge 0,$$

then we have

$$f(t)\le \frac{f_0}{1-Cf_0t},\quad t\ge 0,$$

where C is a positive constant.

(ii) *If f and g satisfy*

$$f^p(t)\le f_0^p + Cp\int_0^t \left(g(s)f^{p-1}(s)+f^p(s)\right)ds,\quad t\ge 0,$$

with $p\ge 1$, then we have

$$f(t)\le f_0e^t + Ce^t\int_0^t g(s)e^{-s}\,ds,\quad t\ge 0,$$

where C is a positive constant.

PROOF. (i) Set

$$h(t) := f_0\exp\left(C\int_0^t f(s)\,ds\right),$$

then we get

$$h'(t) = Ch(t)f(t)\le Ch(t)^2 \quad\text{with}\quad h_0=f_0.$$

This yields

$$(h^{-1}(t))'\ge -C \quad\text{and}\quad h(t)\le \frac{1}{h_0^{-1}-Ct} = \frac{h_0}{1-Ch_0t}.$$

Thus we have

$$f(t)\le h(t)\le \frac{f_0}{1-Cf_0t}.$$

(ii) Set

$$F(t) := f_0^p + Cp\int_0^t \left(g(s)f^{p-1}(s)+f^p(s)\right)ds.$$

Then we find that F satisfies

$$F'(t) = Cp\left(g(t)f^{p-1}(t)+f^p(t)\right)\le Cp\left(g(t)F^{(p-1)/p}(t)+F(t)\right)\quad\text{with}\quad F_0=f_0^p.$$

Dividing both sides of the above inequality by $pF^{(1-p)/p}$ implies

$$\left(F^{1/p}(t)\right)'\le Cg(t)+CF^{1/p}(t),$$

and this gives the following inequality for f:

$$f(t) \le F^{1/p}(t) \le F_0^{1/p} e^t + Ce^t \int_0^t g(s)e^{-s}\,ds = f_0 e^t + Ce^t \int_0^t g(s)e^{-s}\,ds.$$

This completes the proof. □

5.2. A representation for solutions to the PDE (1.132)**.** In this part, we show that any weak solution $\rho \in L^\infty(0,T;\mathcal{P}(\mathcal{O}))$ to (1.132) can be represented as the family of time marginals of a solution to (3.143).

LEMMA 5.2. *For any $\rho \in L^\infty(0,T;\mathcal{P}(\mathcal{O}))$ solution to* (1.132) *with initial condition $\rho_0 \in \mathcal{P}(\mathcal{O})$, there exists a stochastic basis, and on that basis a random variable Y_0 with the law ρ_0 and an independent Brownian motion $(B_t)_{t\in[0,T]}$ such that the solution to* (3.143) *generated by the initial condition Y_0 and Brownian motion $(B_t)_{t\in[0,T]}$ has time marginals ρ_t.*

PROOF. **Step A** (*Regularization*): Let $(\mu_\varepsilon)_{\varepsilon>0}$ be a sequence of mollifier and define $\rho_t^\varepsilon = \rho_t * \mu_\varepsilon$. Then we can easily find that ρ_t^ε satisfies the following Cauchy problem:

$$\begin{cases} \partial_t \rho_t^\varepsilon + \nabla\cdot(\rho_t^\varepsilon \mathcal{V}_t^\varepsilon) = \sigma\Delta\rho_t^\varepsilon, \quad x\in\mathcal{O}, \quad t>0, \\ \langle \sigma\nabla\rho^\varepsilon - \rho^\varepsilon \mathcal{V}^\varepsilon, n\rangle = 0 \quad \text{on} \quad \partial\mathcal{O}, \end{cases}$$

where

$$\mathcal{V}_t^\varepsilon := \frac{(\rho_t V[\rho_t]) * \mu_\varepsilon}{\rho_t^\varepsilon}.$$

Since the vector fields $\mathcal{V}_t^\varepsilon$ are smooth and satisfy $\|\mathcal{V}_t^\varepsilon\|_{L^\infty} \le \|V[\rho_t]\|_{L^\infty} \le \|\nabla\varphi\|_{L^\infty}$, we can define the solution to the following SDE

$$\begin{cases} Y_t^\varepsilon = Y_0^\varepsilon + \displaystyle\int_0^t \mathcal{V}_s^\varepsilon(Y_s^\varepsilon)\,ds + \sqrt{2\sigma}B_t - K_t^\varepsilon, \\ K_t^\varepsilon = \displaystyle\int_0^t n(Y_s^\varepsilon)\,d|K^\varepsilon|_s, \quad |K^\varepsilon|_t = \int_0^t \mathbf{1}_{\partial\mathcal{O}}(Y_s^\varepsilon)\,d|K^\varepsilon|_s. \end{cases} \tag{5.148}$$

By applying Itô's formula, similarly as in proof of Proposition 3.1, we can also find the time marginals of solution to (5.148) is the solution to

$$\begin{cases} \partial_t \overline{\rho}_t^\varepsilon + \nabla\cdot(\overline{\rho}_t^\varepsilon \mathcal{V}_t^\varepsilon) = \sigma\Delta\overline{\rho}_t^\varepsilon, \quad x\in\mathcal{O}, \quad t>0, \\ \langle \sigma\nabla\overline{\rho}^\varepsilon - \overline{\rho}^\varepsilon \mathcal{V}_t^\varepsilon, n\rangle = 0 \quad \text{on} \quad \partial\mathcal{O}. \end{cases} \tag{5.149}$$

It is clear that there exists a unique weak solution (5.149) with the initial condition ρ_0^ε due to the linearity together with the smoothness of the vector fields. Thus we have $\mathcal{L}(Y_t^\varepsilon) = \rho_t^\varepsilon$.
Step B (*Tightness*): We first show that the family $\{(Y_t^\varepsilon + K_t^\varepsilon)_{t\in[0,T]}, \varepsilon > 0\}$ is tight. Define $K(R,A)$ as

$$K(R,A) := \left\{ (f_t)_{t\in[0,T]} \,:\, |f(0)| \le A \quad \text{and} \quad \sup_{0\le s<t\le T} \frac{|f(t)-f(s)|}{|t-s|^{1/3}} \le R \right\}.$$

Note that $K(R,A)$ is compact subset of $C([0,T],\mathbb{R}^d)$ due to Ascoli's Theorem. On the other hand, a straightforward computation gives

$$\left| \int_s^t \mathcal{V}_u^\varepsilon(Y_u^\varepsilon)\,du + \sqrt{2\sigma}(B_t - B_s) \right| \le \left(T^{2/3}\|\nabla\varphi\|_{L^\infty} + \sup_{0\le s<t\le T} \frac{|B_t - B_s|}{|t-s|^{1/3}} \right) |t-s|^{1/3} =: U_T |t-s|^{1/3}.$$

This implies

$$\mathbb{P}\left((Y_t^\varepsilon + K_t^\varepsilon)_{t\in[0,T]} \notin K(R,A) \right) \le \mathbb{P}\left(|Y_0^\varepsilon| \ge A \right) + \mathbb{P}\left(U_T \ge R \right).$$

Note that U_T is almost surely bounded since the trajectories of the Brownian motion are almost surely $\frac{1}{3}$-Hölder. This together with the tightness of ρ_0^ε yields that for any $\eta > 0$ we can find some $R, A > 0$ such that

$$\sup_{\varepsilon>0} \mathbb{P}\left((Y_t^\varepsilon + K_t^\varepsilon)_{t\in[0,T]} \notin K(R,A) \right) \le \eta.$$

Thus the family of the law of the $(Y_t^\varepsilon + K_t^\varepsilon)_{t\in[0,T]}$ is tight. On the other hand, it follows from [**20**, Theorem 1.1] that the mapping $(w_t)_{t\in[0,T]}$ to $(x_t)_{t\in[0,T]}$ solution to the following Skorokhod problem:

$$\begin{cases} x_t + k_t = w_t, \\ k_t = \displaystyle\int_0^t h_s n(x_s)\, d|k|_s,\ |k|_t = \int_0^t \mathbf{1}_{\partial\mathcal{O}}(x_s)\, d|k|_s \end{cases},$$

is continuous. This concludes that the family of the law of $(Y_\cdot^\varepsilon)_{\varepsilon>0}$ is also tight.

Step C (*Passing to the limit*): Since the family of the law of $\{(Y_t^\varepsilon, B_t)_{t\in[0,T]}, \varepsilon > 0\}$ is tight, it is possible to find a subsequence $\{(Y_t^{\varepsilon_n}, B_t)_{t\in[0,T]}, \varepsilon > 0\}$ converging in law. Furthermore, up to changing the probability space, we can assume that this convergence holds almost surely due to Skorokhod representation Theorem. For the sake of shortness of notation, we assume the probability space is unchanged. Then it is sufficient to check

$$\mathbb{E}\left[\left|\int_0^t (\mathcal{V}^\varepsilon(Y_s^\varepsilon) - V[\rho_s](Y_s))\, ds\right|\right] \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

For this, inspired by the strategy used in [**13**, Theorem 2.6] and [**15**, Theorem B.1], we introduce a sequence of continuous function $V_\cdot^k : [0,T]\times\mathcal{O} \mapsto \mathbb{R}^d$ converging to $V[\rho_\cdot]$ in $L^1([0,T]\times\mathcal{O}, \rho_s(dy)ds)$ as $k \to \infty$. This is possible since the measure $\rho_s(dy)ds$ is a Radon measure. Then we define $\mathcal{V}_\cdot^{\varepsilon,k}$ by

$$\mathcal{V}_s^{\varepsilon,k} = \frac{(\rho_s V_s^k) * \mu_\varepsilon}{\rho_s^\varepsilon}.$$

Using this newly defined vector fields, we estimate as

$$\begin{aligned} &\int_0^t \mathbb{E}\left[|(\mathcal{V}^\varepsilon(Y_s^\varepsilon)) - V[\rho_s](Y_s)|\right] ds \\ &\quad\le \int_0^t\int_{\mathcal{O}} |\mathcal{V}_s^\varepsilon(y) - \mathcal{V}_s^{\varepsilon,k}(y)|\rho_s^\varepsilon(y)\, dyds + \int_0^t\int_{\mathcal{O}} |\mathcal{V}_s^{\varepsilon,k}(y) - V_s^k(y)|\rho_s^\varepsilon(y)\, dyds \\ &\qquad + \int_0^t\int_{\mathcal{O}} |V_s^k(y_1) - V_s^k(y_2)|\, \pi_s^\varepsilon(dy_1, dy_2)ds + \int_0^t\int_{\mathcal{O}} |V_s^k(y) - V[\rho_s](y)|\rho_s(dy)\, ds \\ &\quad=: I_1 + I_2 + I_3 + I_4, \end{aligned}$$

Here we can easily find that the terms I_2 and I_4 can be arbitrarily close to zero due to the definition of the sequence $(V_\cdot^k)_{k\ge0}$. For the estimate of I_1, we get

$$I_1 = \int_0^t\int_{\mathcal{O}} \left|\frac{\left[(V[\rho_s] - V_s^k)\rho_s\right] * \mu_\varepsilon}{\rho_s^\varepsilon}\right| \rho_s^\varepsilon\, dyds = \int_0^t\int_{\mathcal{O}} \left|\left[(V[\rho_s] - V_s^k)\rho_s\right] * \mu_\varepsilon\right| dyds.$$

Then we can again use the choice of the sequence $(V_\cdot^k)_{k\ge0}$ to have that I_1 can be arbitrarily small as $k \to \infty$. Finally, we notice that I_3 can be rewritten as

$$I_3 = \int_0^t \mathbb{E}\left[\left|(V_s^k(Y_s^\varepsilon)) - V_s^k(Y_s)\right|\right] ds.$$

Then it is clear that $I_3 \to 0$ as $\varepsilon \to 0$ since $V_\cdot^k$ is continuous and $(Y_t^\varepsilon)_{t\in[0,T]}$ converges to $(Y_t)_{t\in[0,T]}$ $\mathbb{P}$-a.s. This completes the proof. □

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