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Discipline: Mathématiques

**Angel PARDO**

# Comptage d'orbites périodiques dans le modèle de windtree

## Counting problem on wind-tree models

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Après avis des rapporteurs :

| | |
|---|---|
| Jayadev ATHREYA | University of Washington |
| Howard MASUR | University of Chicago |

Devant le jury composé de :

| | | |
|---|---|---|
| Marie-Claude ARNAUD | Université d'Avignon | |
| Nicolas BÉDARIDE | Aix-Marseille Université | |
| Pascal HUBERT | Aix-Marseille Université | *Directeur de thèse* |
| Erwan LANNEAU | Université Grenoble Alpes | |
| Alejandro MAASS | Universidad de Chile | |
| Carlos MATHEUS | Université Paris 13 | |
| Anton ZORICH | Université Paris 7 | |

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# Résumé

Le problème du cercle de Gauss consiste à compter le nombre de points entiers de longueur bornée dans le plan. Autrement dit, compter le nombre de géodésiques fermées de longueur bornée sur un tore plat bidimensionnel. De très nombreux problèmes de comptage en systèmes dynamiques se sont inspirés de ce problème. Depuis 30 ans, on cherche à comprendre l'asymptotique de géodésiques fermées dans les surfaces de translation. H. Masur a montré que ce nombre a une croissance quadratique. Calculer l'asymptotique quadratique (constante de Siegel–Veech) est un sujet de recherches très actif aujourd'hui. L'objet d'étude de cette thèse est le modèle de windtree, un modèle de billard non compact. Dans le cas classique, on place des obstacles rectangulaires identiques dans le plan en chaque point entier. On joue au billard sur le complémentaire. Nous montrons que le nombre de trajectoires périodiques a une croissance asymptotique quadratique et calculons la constante de Siegel–Veech pour le windtree classique ainsi que pour la généralisation de Delecroix–Zorich. Nous prouvons que, pour le windtree classique, cette constante ne dépend pas des tailles des obstacles (phénomène "non varying" analogue aux résultats de Chen–Möller). Enfin, lorsque la surface de translation compacte sous-jacente est une surface de Veech, nous donnons une version quantitative du comptage.



**Mots-clés** : Systèmes dynamiques, Géométrie, Modèle de windtree, Billards, Surfaces de translation, Problème de comptage.

# Abstract

The Gauss circle problem consists in counting the number of integer points of bounded length in the plane. In other words, counting the number of closed geodesics of bounded length on a flat two dimensional torus. Many counting problems in dynamical systems have been inspired by this problem. For 30 years, the experts try to understand the asymptotic behavior of closed geodesics in translation surfaces. H. Masur proved that this number has quadratic growth rate. Compute the quadratic asymptotic (Siegel–Veech constant) is a very active research domain these days. The object of study in this thesis is the wind-tree model, a non-compact billiard model. In the classical setting, we place identical rectangular obstacles in the plane at each integer point. We play billiard on the complement. We show that the number of periodic trajectories has quadratic asymptotic growth rate and we compute the Siegel–Veech constant for the classical wind-tree model as well as for the Delecroix–Zorich variant. We prove that, for the classical wind-tree model, this constant does not depend on the dimensions of the obstacles (non-varying phenomenon, analogous to results of Chen–Möller). Finally, when the underlying compact translation surface is a Veech surface, we give a quantitative version of the counting.



**Keywords**: Dynamical systems, Geometry, Wind-tree model, Billiards, Translation surfaces, Counting problem.

# Contents

# Introduction

This thesis is about asymptotic formulas for the number of (isotopy classes of) periodic billiard trajectories on wind-tree models, billiards in the plane endowed with $\mathbb{Z}^2$-periodic obstacles. This question has been widely studied in the context of finite rational billiards and compact flat surfaces, and it is related to many other questions such as the calculation of the volume of normalized strata [**EMZ**] or the sum of Lyapunov exponents of the geodesic Teichmüller flow [**EKZ**] on strata of flat surfaces (Abelian or quadratic differentials).

The body of this work is composed of three self-contained chapters, which corresponds to three papers of the author on the counting problem on wind-tree models [**Pa1, Pa2, Pa3**]. In the following, we present a brief summary of the motivations, main objects and related results. We end this introduction stating the main results contained in the body of this work, as well as a brief explanation of the key ideas and main difficulties.

**Polygonal billiards.** Consider a point particle bouncing around in a polygon. Away from the edges, the point moves inside a polygon at unit speed along a straight line until it reaches the boundary. At the edges, the point bounces changing direction instantaneously according to the usual law of geometrical optics: the angle of incidence equals the angle of reflection, and continues along the new line. If the point reaches a vertex, it stops moving: the orbit is not defined further. A path described in this way is called a billiard trajectory; see Figure 1.

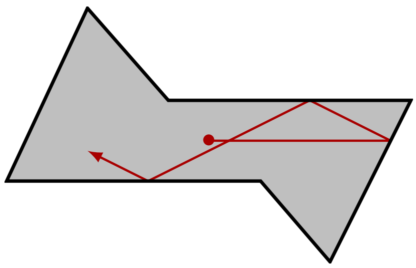


FIGURE 1. A billiard trajectory in a polygonal billiard.

The study of billiard trajectories is a basic problem in dynamical systems and arises naturally in physics. Consider, for example, two points of different masses moving on a segment. The points may elastically collide and reflect from the end points of the segment. This system can be modeled as billiard trajectories in a right angled triangle (see, for example, [**MT**, §1.2]).

For an introduction and general references to this subject, we refer the reader to the book of Tabachnikov [**Ta**].

## 1. Counting problem

In this work we are interested in the length spectrum of polygonal billiards, that is, on periodic billiard trajectories and their length. More precisely, for a polygonal billiard $B$, we would like to know as much as possible about the number $N(B, L)$ of (isotopy classes of) periodic billiard trajectories of length at most $L$ in $B$.

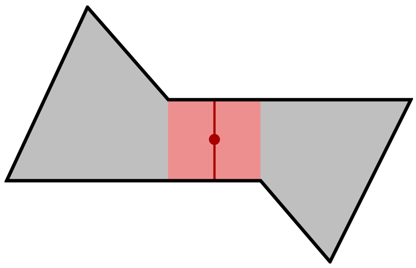


FIGURE 2. A family of isotopic periodic billiard trajectories.

A first natural question when tackling this problem is whether there is at least one periodic billiard orbit in every polygon, that is, whether for every polygonal billiard $B$, the number $N(B, L)$ is positive for $L$ large enough. This question is open even for non acute triangles.

**1.1. Triangular billiards.** In the case of an acute triangle, the simplest (and shortest) periodic trajectory in it is given by the orthic triangle: the triangle whose vertices are the bases of the altitudes in the original triangle form a periodic billiard trajectory in it (see Figure 3). The proof goes back to 1775 and is due to Fagnano [**Fa**]. For right triangles, the existence of periodic trajectories was proved independently by Holt [**Ho**] and Galperin–Stepin–Vorobets [**GSV91**]. Schwartz [**Sch**] proved the existence of periodic billiard orbits in every obtuse triangle with angle less than 100 degrees. However, it is unknown whether periodic trajectories exist in every obtuse triangle.

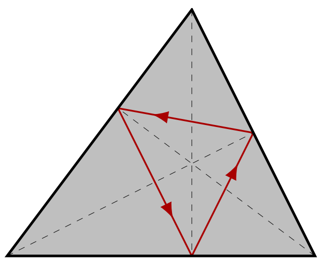
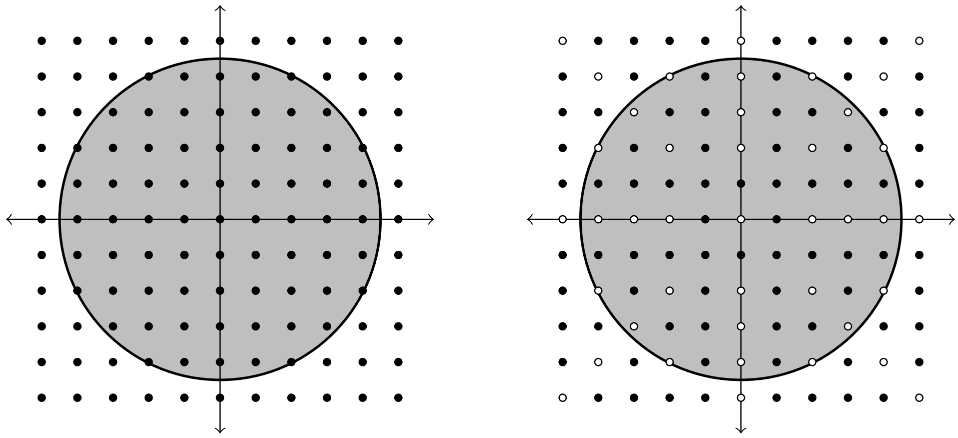


FIGURE 3. Orthic triangle: the shortest periodic billiard trajectory in any acute triangle.

A rational triangle, that is, a triangle whose angles are all rational multiples of $\pi$, has a dense set of periodic billiard trajectories [**BGKT**] (see also [**Ma86**]).

**1.2. Square billiard and the Gauss circle problem.** It is clear that in the square billiard $B$ there are periodic trajectories. Moreover, in this case, the exact value of $N(B, L)$ is equivalent to the (primitive) circle problem.

The Gauss circle problem is the problem of determining the number of integer lattice points inside the boundary of a circle of a given radius centered at the origin (see Figure 4a). This number is approximated by the area of the circle, so the real problem is to accurately bound the error term describing how the number of points differs from the area. The first progress on a solution to this problem was made by Gauss, hence its name. It is conjectured (see [**Gy**]) that the correct bound is $O(L^{1/2+\epsilon})$, the current known bound being $O(L^{131/208})$ (note that $131/208 \approx 0.6298$), proved by Huxley [**Hu03**].



(a) Gauss circle problem. (b) Primitive circle problem.

FIGURE 4. The counting problem on the square billiard is equivalent to the (primitive) circle problem.

To be precise, coming back to the square billiard, we do not want to count trajectories which go around a single periodic trajectory several times. Thus, our problem is equivalent not to the circle problem, but to the *primitive* circle problem, which is the problem of determining the number $N(L)$ of *primitive* integer lattice points in the circle of radius $L$, centered at the origin (see Figure 4b). By primitive integer lattice point we mean, as usual, coprime integer couples.

Using the same ideas as the usual Gauss circle problem and the fact that the probability that two integers are coprime is $1/\zeta(2)$, it is relatively straightforward to show that

$$N(L) = \frac{1}{\zeta(2)}\pi L^2 + O(L^{1+\epsilon}).$$

No bound on the error term of the form $O(L^{1-\delta})$ for any $\delta \geq 0$ is currently known without assuming the Riemann Hypothesis (see [**Wu**] for the best known error term assuming the Riemann Hypothesis).

The square billiard is a particular case of rational billiard, which are much more tractable than general polygons, without the rationality assumption.

**1.3. Rational billiards.** A rational billiard is a billiard in a rational polygon, that is, a polygon whose angles are all rational multiples of $\pi$. Thanks to the rationality assumption much more can be said.

Masur [**Ma86, Ma88, Ma90**] proved that for every rational billiard $B$, there is a dense set of directions such that there are periodic trajectories in those directions and that there are positive constants $c(B)$ and $C(B)$ such that

$$c(X)L^2 \leq N(B,L) \leq C(X)L^2$$

for large enough $L$. Boshernitzan–Galperin–Krüger–Troubetzkoy [**BGKT**] strengthened the density result to the whole tangent space.

Veech [**Ve89**] proved that there are in fact exact quadratic asymptotics for a special class of rational billiards now called Veech billiards, which includes for example all regular polygons (see [**Ve92**]).

It is still an open problem whether *all* rational billiards have exact quadratic asymptotics. Eskin–Mirzakhani–Mohammadi [**EMM**] showed that for *every* rational billiard we have *weak* quadratic asymptotic formulas,

$$\lim_{L\to\infty} \frac{1}{L}\int_0^L \frac{N(B,e^t)}{\pi e^{2t}}\mathrm{d}t = c(B),$$

for some $c(B) > 0$, and we write $N(B,L)$ "$\sim$" $c(B)\cdot \pi L^2$.

The rationality assumption leads to deep connections to algebraic geometry, Teichmüller theory, ergodic theory on homogenous spaces, and other areas of mathematics (see, for example, [**Ve89**] and references in [**MT**]). All these results relies on these connections.

**1.4. Unfolding rational billiards.** When a trajectory in a polygonal billiard $B$ is reflected in one of the edges of the polygon, it undergoes a transformation by the element of O(2) corresponding to (the derivative of) the reflection in that edge. The subgroup $\Gamma(B)$ of O(2) generated by all reflections in the edges of $B$ is either finite or dense (as any subgroup of O(2) containing reflections). A necessary condition for $\Gamma(B)$ to be finite is that the polygon is rational. It is also sufficient if the boundary is connected, that is, if the polygon is simply connected.

Thus, a given billiard trajectory in a rational billiard have only finitely many different directions. In particular, if instead of bouncing the trajectory we reflect (unfold) the billiard table, we obtain finitely many copies of the original polygon (up to translations). This unfolding construction is due to Fox–Kershner [**FK**] and it is often attributed to Katok–Zemlyakov [**KaZe**]. We illustrate the unfolding procedure in the case of a rectangular billiard in Figure 5.

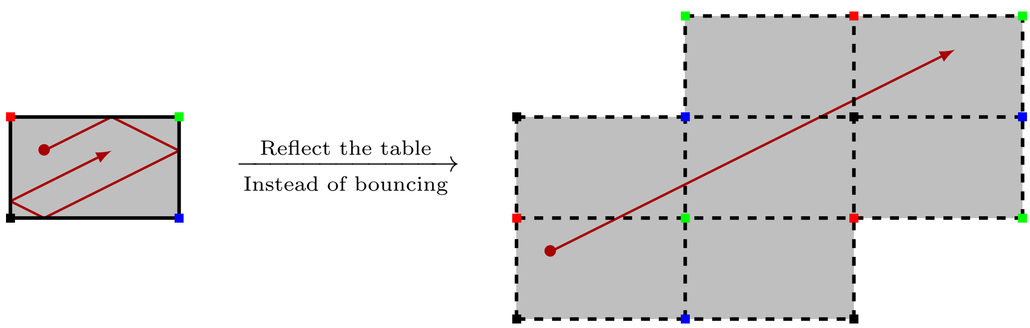



FIGURE 5. Unfolding a rectangular billiard table.

These polygons can be translated so that they are all disjoint in the plane and we identify the edges according to the reflections. This unfolding procedure gives raise to a closed surface with a natural flat metric (the one obtained from $\mathbb{R}^2$) with conical singularities (with angles multiples of $2\pi$). Thus, for example, in the case of a rectangular billiard we obtain a flat torus. In general, the result of the unfolding of a rational billiard is what is called a translation surface.

More details about billiards in polygons, specially rational polygons and their connection to translation surfaces, can be found in the surveys of Gutkin [**Gt**], Hubert–Schmidt [**HS**], Masur–Tabachnikov [**MT**] and Smillie [**Sm**].

## 2. Translation surfaces

For an introduction and general references to this subject, we refer the reader to the surveys of Zorich [**Zo06**], Forni–Matheus [**FM**], Wright [**Wr**].

Roughly, a translation surface is a surface which can be obtained by edge-to-edge gluing of polygons in $\mathbb{R}^2$ using translations only. More formally, a compact connected oriented surface $X$ is called a translation surface if it is equipped with a translation structure, that is, a complex atlas such that the transition functions are translations (the chart domains cover all the surface but finitely many singular points). The translation structure induces the structure of a smooth manifold, a flat Riemannian metric, and a Borel measure on the surface $X$ punctured at the singular points. We also require that the metric has a cone type singularity at each singular point or, equivalently, that the area of the surface is finite. The cone angles are integer multiples of $2\pi$. A conical singularity of angle $2\pi$ is removable: it is rather a marked point than a true singularity of the metric.

**2.1. Abelian differentials.** There is a one to one correspondence between translation surfaces and (non-zero) Abelian differentials, holomorphic 1-forms on (compact) Riemann surfaces. The holomorphic 1-form $\mathrm{d}z$ on $\mathbb{C}$ defines a holomorphic 1-form $\omega$ on $X$ which in local coordinates has the form $\omega = \mathrm{d}z$. Since the changes of local coordinates are defined by translations only, say $z' = z + c$, we see that $\mathrm{d}z = \mathrm{d}z'$. It is not difficult to verify that the complex structure extends to the conical singularities. On the other hand, given an Abelian differential $\omega$, the atlas consisting of all local coordinates $z$ with the property that $\omega = \mathrm{d}z$ defines a translation structure. The 1-form vanishes exactly at the (non-removable) conical singularities; conical singularities of angle $2\pi(n+1)$ corresponding to zeros of degree $n$ of the Abelian differential.

**2.2. Moduli space and strata.** The moduli space $\mathcal{H}_g$ of translation surfaces of genus $g$ is a vector bundle over the moduli space of Riemann surfaces of genus $g$. Moreover, this space is stratified according to the number and multiplicity of the zeros of the Abelian differentials. Let $g \geq 1$, $\mathbf{n} = \{n_1, ..., n_k\}$ be a partition of $2g-2$. Then, $\mathcal{H}(\mathbf{n})$ denotes the stratum of Abelian differentials, that is, holomorphic 1-forms on Riemann surfaces of genus $g$, with zeros of degrees $n_1, \ldots, n_k$.

Each stratum is a complex orbifold of dimension $d = 2g + k - 1$ which has a complex atlas with transition functions in $\mathrm{GL}(d, \mathbb{Z})$ (away from orbifold points or on an appropriate cover without orbifold points). Thus, a stratum looks locally like a complex affine space.

We also consider normalized strata; we denote by $\mathcal{H}_1(\mathbf{n})$, the (real) codimension 1 subspace of area 1 translation surfaces in $\mathcal{H}(\mathbf{n})$. Strata (normalized or not) are never compact and are not always connected. Their connected components have been classified by Kontsevich–Zorich [**KZ03**] and there are at most three connected components in each stratum.

**2.3.** $\mathrm{SL}(2,\mathbb{R})$**-action and Teichmüller flow.** There is a natural action of $\mathrm{SL}(2,\mathbb{R})$ on (connected components of) strata of translation surfaces, coming from the linear action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^2$, which generalizes the action of $\mathrm{SL}(2,\mathbb{R})$ on the space $\mathrm{GL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ of flat tori. Roughly, if $g \in \mathrm{SL}(2,\mathbb{R})$ and $X$ is a translation surface given as a collection of polygons, then $gS$ is the translation surface given by the collection of polygons obtained by acting linearly by $g$ on the polygons defining $X$, as in Figure 6.

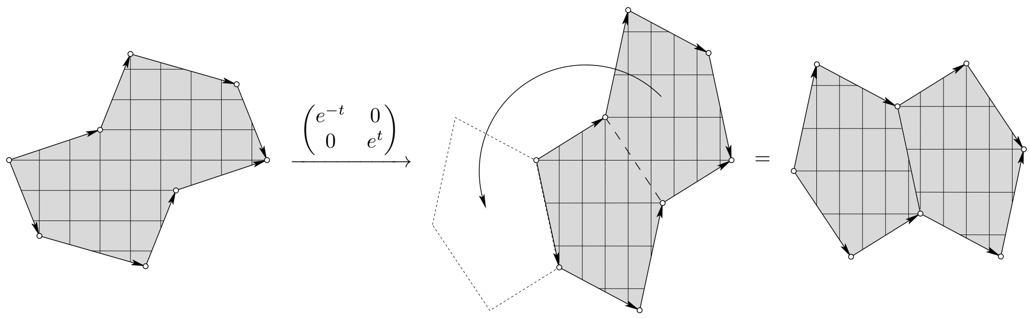



FIGURE 6. $\mathrm{SL}(2,\mathbb{R})$-action and *cut & paste* ([**Zo06**, Fig. 15]). The first modification of the polygon changes the translation structure while the second one just changes the way in which we unwrap the translation surface.

Let

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad \text{and} \quad r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

The element $r_\theta \in \mathrm{SL}(2,\mathbb{R})$ acts by $\omega \mapsto e^{i\theta}\omega$ and has the effect of rotating the translation surface by the angle $\theta \in [0, 2\pi)$. The action of $(g_t)_{t\in\mathbb{R}}$ is called the Teichmüller geodesic flow.

A deep result due to Eskin–Mirzakhani–Mohammadi [**EMM**] (see also [**EMi**]) says that the $\mathrm{SL}(2,\mathbb{R})$-orbit closure of any translation surface in $\mathcal{H}_1(\mathbf{n})$ is the space of unit area translation surfaces in some affine invariant submanifold, that is, a submanifold which locally looks like an affine subspace. Moreover, each such space is the support of an affine invariant measure, that is, an ergodic $\mathrm{SL}(2,\mathbb{R})$-invariant probability measure which locally is (up to normalization) the restriction of Lebesgue measure (see [**EMi**] for the precise definitions).

The most important case is a connected component of a normalized stratum $\mathcal{H}_1(\mathbf{n})$. The associated affine measure is known as the Masur–Veech measure: Masur [**Ma82**] and Veech [**Ve82**] independently proved that in this case, the total mass of this measure is finite and ergodic with respect to the Teichmüller geodesic flow.

**2.4. Veech surfaces.** Veech surfaces and billiards form an important subclass on the history of rational billiards and translation surfaces. They correspond to translation surfaces whose $\mathrm{SL}(2,\mathbb{R})$-orbit is closed and they have a large group of symmetries. For an introduction and general references to Veech surfaces, we refer the reader to the survey of Hubert–Schmidt [**HS**].

We denote the stabilizer of a translation surface $X$ under the action of $\mathrm{SL}(2,\mathbb{R})$ by $\mathrm{SL}(X)$. The group $\mathrm{SL}(X)$ is also the group of derivatives of affine orientation-preserving diffeomorphisms of $X$. Recall that $\mathrm{SL}(2,\mathbb{R})$ does not act faithfully on the upper half-plane $\mathbb{H}$; it is the projective group $\mathrm{PSL}(2,\mathbb{R})$ that does so. We define the Veech Group of $X$ to be the image of $\mathrm{SL}(X)$ in $\mathrm{PSL}(2,\mathbb{R})$ and we denote it by $\mathrm{PSL}(X)$.

A translation surface $X$ is called Veech surface if its Veech group $\mathrm{PSL}(X)$ is a lattice, that is, if $\mathbb{H}/\mathrm{PSL}(X)$ has finite volume. Veech surfaces correspond to closed $\mathrm{SL}(2,\mathbb{R})$-orbits. Such a closed orbits is called a Teichmüller curve.

The simplest examples of Veech surfaces are translation coverings of the torus, called square-tiled surfaces. They are those translation surfaces whose Veech group is arithmetic (commensurable with $\mathrm{PSL}(2,\mathbb{Z})$), after a theorem by Gutkin–Judge [**GJ96**]. Square-tiled surfaces were introduced by Thurston [**Th**]. See Figure 7 for an example of square-tiled surface. Square-tiled surfaces have been used for explicit computations of volumes of (normalized) strata by Zorich [**Zo02**] and Eskin–Okounkov [**EO**]: square-tiled surfaces correspond to integer points of strata of Abelian differentials; the volume of a stratum is computed from the asymptotic number of integer points in a large ball.

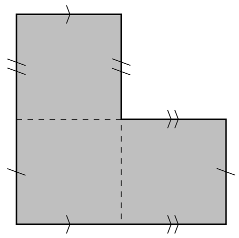


FIGURE 7. A square-tiled surface. It belongs to $\mathcal{H}(2)$.

**2.5. Counting problem on translation surfaces.** The unfolding construction provides a (one way) dictionary between rational billiards and translation surfaces. In the case of polygonal billiards, the counting problem we are concerned with is that of periodic billiards trajectories. This dictionary identifies billiards trajectories with geodesics (for the flat metric) on the corresponding translation surface. Thus, we are interested in the counting of closed geodesics on translation surfaces. This question has been widely studied and it is related to many other questions such as the calculation of the volume of normalized strata [**EMZ**] or the sum of Lyapunov exponents of the geodesic Teichmüller flow [**EKZ**] on strata of translation surfaces.

Together with every closed regular geodesic in a translation surface we have a bunch of parallel closed regular geodesics. A cylinder on a translation surface is a maximal open annulus filled by isotopic simple closed regular geodesics. A cylinder $C$ is isometric to the product of an open interval and a circle. Note that we do not count trajectories which go around a single closed trajectory several times, and we are counting unoriented trajectories.

The results on the counting problem on rational billiards above come in fact from analogous results in the more general case of translation surfaces. In this context, we have that: Masur [**Ma88, Ma90**] proved that for every translation surface $X$, there exist positive constants $c(X)$ and $C(X)$ such that the number $N(X,L)$ of (maximal) cylinders of closed geodesics of length at most $L$ satisfy

$$c(X)L^2 \leq N(X,L) \leq C(X)L^2$$

for large enough $L$. Veech [**Ve89**] proved that for Veech surfaces there are in fact exact quadratic asymptotics; Gutkin–Judge [**GJ00**] gave a different proof. Another proof for the upper quadratic bounds was given by Vorobets [**Vo97**]. Eskin–Masur [**EMa**] gave yet another one and proved that for each ergodic probability measure $\mu$ on strata of normalized (area 1) translation surfaces, there is a constant $c(\mu)$ such that for almost every surface, $N(X,L) \sim c(\mu) \cdot \pi L^2$, that is,

$$\lim_{L\to\infty} \frac{N(X,L)}{\pi L^2} = c(\mu).$$

The constant $c(\mu)$ is called the Siegel–Veech constant ([**EMa**]) of the counting problem; it is the constant in the Siegel–Veech formula ([**EMa**]), a Siegel-type formula introduced by W. Veech [**Ve98**], which can be translated into

$$c(\mu) = \frac{1}{\pi R^2} \int_{\mathcal{H}_1(\mathbf{n})} N(X,R)\mathrm{d}\mu(X).$$

It is still an open problem whether *all* translation surfaces have exact quadratic asymptotics. In particular, this result does not provides further information on the counting problem on rational billiards. In fact, the set of surfaces which are constructed from polygons has zero measure for every $\mathrm{SL}(2,\mathbb{R})$-invariant measure, but for those which are supported on $\mathrm{SL}(2,\mathbb{R})$-orbits of Veech surfaces and for which we already know the answer.

On the other hand, Eskin–Mirzakhani–Mohammadi [**EMM**] showed that for *every* (area 1) translation surfaces (in particular, for *every* rational billiard) we have *weak* quadratic asymptotic formulas,

$$\lim_{L\to\infty} \frac{1}{L} \int_0^L \frac{N(X,e^t)}{\pi e^{2t}} \mathrm{d}t = c(X),$$

which we write $N(X,L)$ "$\sim$" $c(X) \cdot \pi L^2$. The constant $c(X)$ being the Siegel–Veech constant associated to the affine invariant measure supported on the $\mathrm{SL}(2,\mathbb{R})$-orbit closure of the surface $X$, given by general invariant measure classification theorem of Eskin–Mirzakhani [**EMi**].

The particular constants for several Veech surfaces have been computed explicitly by Veech [**Ve89, Ve92**], Vorobets [**Vo97**], Gutkin–Judge [**GJ00**] and Schmoll [**Sc**]. Constants for some families of non-Veech surfaces were also given by Eskin–Masur–Schmoll [**EMS**] and Eskin–Marklof–Witte Morris [**EMW**].

Eskin–Masur–Zorich [**EMZ**] computed the Siegel–Veech constants for connected components of all strata of Abelian differentials, and also described all possible configurations of cylinders of closed geodesics which might be found on a generic translation surface. In general, the particular constants for Veech surfaces do not coincide with the Siegel–Veech constants of the strata where they live.

## 3. Wind-tree model

The classical wind-tree model corresponds to a billiard in the plane endowed with $\mathbb{Z}^2$-periodic obstacles of rectangular shape; the sides of the rectangles are aligned along the lattice, see Figure 8.

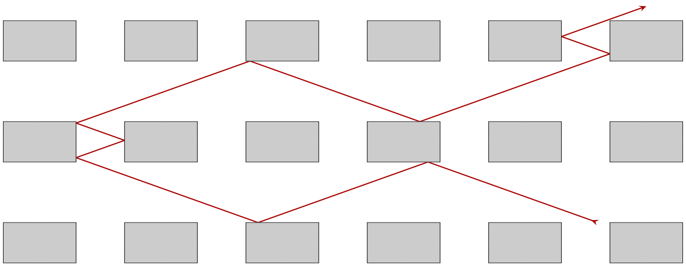


FIGURE 8. Original wind-tree model.

The wind-tree model (in a slightly different version) was introduced by Ehrenfest–Ehrenfest [**EE**] in 1912. Hardy–Weber [**HaWeb**] studied the periodic version. All these studies had physical motivations.

Several advances on the dynamical properties of the billiard flow in the wind-tree model were obtained recently using geometric and dynamical properties on moduli space of (compact) flat surfaces; billiard trajectories can be described by the linear flow on a flat surface.

Avila–Hubert [**AH**] showed that for all parameters of the obstacle and for almost all directions, the trajectories are recurrent. There are examples of divergent trajectories constructed by Delecroix [**De**]. The non-ergodicity was proved by Frączek–Ulcigrai [**FU**]. It was proved by Delecroix–Hubert–Lelièvre [**DHL**] that the diffusion rate is independent either on the concrete values of the parameters of the obstacle or on almost any direction and almost any starting point. It is equal to $2/3$. A generalization of this last result was shown by Delecroix–Zorich [**DZ**] for more complicated obstacles. In this work we study this last variant, corresponding to a billiard in the plane endowed with $\mathbb{Z}^2$-periodic obstacles of right-angled polygonal shape; the obstacles being horizontally and vertically symmetric and the sides of the rectangles are aligned along the lattice, see Figure 9 for an example.

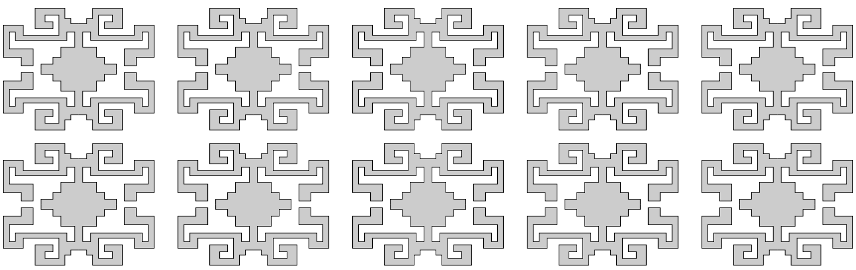


FIGURE 9. Delecroix–Zorich variant.

Recall that in the classical case of a billiard in a rectangle we can glue a flat torus out of four copies of the billiard table and unfold billiard trajectories to flat geodesics of the same length on the resulting flat torus. In the case of the wind-tree model we also start from gluing a translation surface out of four copies of the infinite billiard table $\Pi$. The resulting surface $X_\infty = X_\infty(\Pi)$ is $\mathbb{Z}^2$-periodic with respect to translations by vectors of the original lattice. Passing to the $\mathbb{Z}^2$-quotient we get a compact translation surface $X = X(\Pi)$. For the case of the original wind-tree billiard, with rectangular obstacles, the resulting translation surface is represented at Figure 10. It has genus 5 and belongs to the stratum $\mathcal{H}(2^4)$ (see [**DHL**, § 3] for more details).

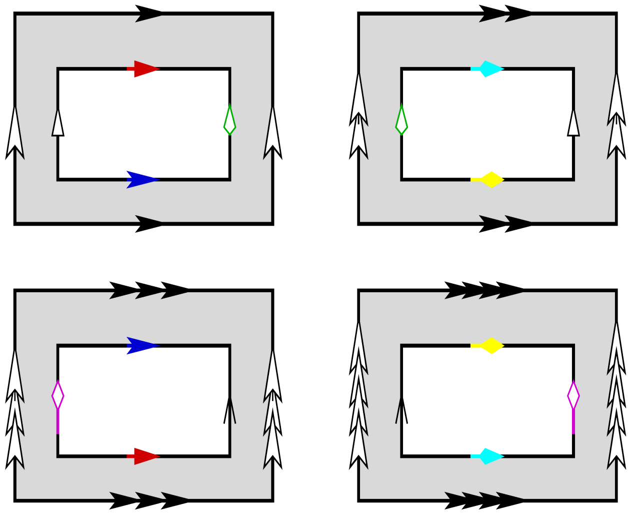


FIGURE 10. The translation surface $X$ obtained as quotient over $\mathbb{Z}^2$ of an unfolded wind-tree billiard table ([**DZ**, Figure 5]).

Similarly, when the obstacle has $4m$ corners with the angle $\pi/2$ —and $4(m-1)$ with angle $3\pi/2$—, the same construction gives a translation surface of genus $4m+1$ in $\mathcal{H}(2^{4m})$, consisting in four flat tori with holes (four copies of a $\mathbb{Z}^2$ fundamental domain of $\Pi$, the holes corresponding to the obstacles) with corresponding identifications, as in the classical setting ($m = 1$, see Figure 10).

## 4. Counting problem on wind-tree billiards

In this section we state and give a brief explanation of the main results on the body of this work. Each subsection is devoted to one chapter of this thesis. Note that each chapter has its own introduction were more details are given.

**4.1. Asymptotic formulas on generic wind-tree models.** In Chapter I, we prove asymptotic formulas for generic wind-tree models with respect to a natural Lebesgue-type measure (see [**AEZ, DZ**]) on the parameters of the wind-tree billiards, that is, the side lengths of the obstacles. Denote by $\mathcal{WT}(m)$ the family of wind-tree billiards such that the obstacle has $4m$ corners with the angle $\pi/2$. Say, all billiards from the original wind-tree family as in Figure 8 live in $\mathcal{WT}(1)$; the billiard in Figure 9 belongs to $\mathcal{WT}(17)$. We denote by $\text{Area}\,(\Pi/\mathbb{Z}^2)$ the area of a fundamental domain of the $\mathbb{Z}^2$-periodic billiard table $\Pi \in \mathcal{WT}(m)$.

THEOREM. *For almost every wind-tree billiard $\Pi \in \mathcal{WT}(m)$ the number $N(\Pi, L)$ of (isotopy classes of) periodic billiard trajectories of length at most $L$ in $\Pi$ has quadratic asymptotic growth rate*

$$N(\Pi, L) \sim c(m) \cdot \frac{\pi L^2}{\text{Area}\,(\Pi/\mathbb{Z}^2)},$$

*where*

$$c(m) = \left(20m^2 - 95m - 78 + 78 \cdot 4^m \frac{(m!)^2}{(2m)!}\right) \frac{1}{6\pi^2}.$$

The constant $c(m)$ is not a standard Siegel–Veech constant, but corresponds to Siegel–Veech constants of some particular configurations of cylinders on compact translation surfaces associated to generic wind-tree billiards.

**Strategy of the proof.** As described above, the unfolding construction on rational billiards can be exploited in this case as well. This allows us to reformulate the counting problem on a wind-tree billiard $\Pi$ in terms of a counting problem on a infinite $\mathbb{Z}^2$-periodic translation surface $X_\infty = X_\infty(\Pi)$: families of periodic trajectories in $\Pi$ are in one to one correspondence with families of (compact) closed geodesics in $X_\infty$.

It is clear that every closed geodesic in $X_\infty$ descends to its $\mathbb{Z}^2$-quotient $X = X(\Pi)$ (see Figure 10, for the resulting surface in the classical setting, $m = 1$). However, some closed geodesics in $X$ lifts to $X_\infty$ as a strip, isometric to the product of an open interval and a straight line (see Figure 11). Thus

$$N(\Pi, L) = N(X_\infty, L) < N(X, L)$$

for $L$ large enough. In particular, we obtain quadratic upper bounds. Recall that in $\Pi$ and $X_\infty$ we count periodic trajectories up to $\mathbb{Z}^2$-translations as every such trajectory can be translated to get a non-isotopic one of the same length.

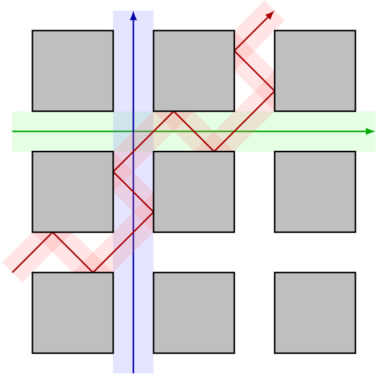


FIGURE 11. Some closed geodesics in $X$ lift to strips in $X_\infty$. They correspond to "unbounded periodic trajectories" in the wind-tree model.

Thus, the classical approach on (finite area) rational billiards and (compact) translation surfaces cannot be applied, at least not in the obvious way. Furthermore, one of the key tools in the classical approach is the $\mathrm{SL}(2,\mathbb{R})$-action on moduli space of translation surfaces and the collection $\mathcal{C}$ of cylinders in $X$ that lift to closed geodesics in $X_\infty$ has not the suitable dynamical properties.

In order to handle this lack of suitable dynamical properties we split the collection $\mathcal{C}$ into families of *good* and *bad* cylinders. The notion of good cylinders was first introduced by Avila–Hubert [**AH**] in order to give a geometric criterion for recurrence of $\mathbb{Z}^d$-periodic translation surfaces. The symmetries of the surface $X$ allows to split its homology group into $\mathrm{SL}(2,\mathbb{R})$-equivariant subspaces. One of these $\mathrm{SL}(2,\mathbb{R})$-equivariant subspace, say $E$, guarantees that every closed curve whose homology class belongs to $E$ lifts to a closed curve in $X_\infty$. Good cylinders are exactly those cylinders in $X$ whose core curve belongs to these subspace $E$ (for more details, see Chapter I). This allows us to tackle the counting of good cylinders using the classical approach. In particular, the counting function for good cylinders has quadratic lower bounds. In the case of bad cylinders, the classical approach is no longer possible. However, using technology for asymptotic formulas developed by Eskin–Masur [**EMa**], we prove the following result whose proof do not rely on ergodic theory.

Theorem. *Let $\Pi \in \mathcal{WT}(m)$ be a wind-tree billiard, $X = X(\Pi)$ the associated compact translation surface. Then, the number $N_{bad}(X, L)$ of bad cylinders in $X$ of length at most $L$, has* subquadratic *asymptotic growth rate, $N_{bad}(X, L) = o(L^2)$ or, which is the same,*

$$\lim_{L\to\infty} \frac{N_{bad}(X, L)}{\pi L^2} = 0.$$

As a consequence, the counting problem on the wind-tree billiard $\Pi$ is reduced to the counting problem of good cylinders in the compact translation surface $X$. The main difficulties on the counting of good cylinders are: (a) the classical approach only allows to conclude asymptotic formulas for *almost every* translation surface and the surfaces obtained from wind-tree billiards are of zero measure; and (b) the computation of Siegel–Veech constants is in general a difficult task.

Following ideas of Athreya–Eskin–Zorich [**AEZ**] and Delecroix–Zorich [**DZ**], and proving certain combinatorial identities for resulting hypergeometric sums, we are able to handle these difficulties and obtain the desired result.

**Side results.** As a by-product of our methods, we obtain several results:

*Area Siegel-Veech constant.* Following the same strategy, we are able to compute the area Siegel–Veech constant, associated to the counting of the area of maximal families of isotopy classes of periodic trajectories. More precisely, we have the following.

Theorem. *For almost every $\Pi \in \mathcal{WT}(m)$ the weighted number $N_{area}(\Pi, L)$ of maximal families of isotopic periodic billiard trajectories of length at most $L$ in $\Pi$, where the weight is the area covered by the family, has quadratic asymptotic growth rate*

$$N_{area}(\Pi, L) \sim c_{area}(m) \cdot \frac{\pi L^2}{\mathrm{Area}\,(\Pi/\mathbb{Z}^2)},$$

*where*

$$c_{area}(m) = \left(8m - 33 + 39 \cdot 4^m \frac{(m!)^2}{(2m+1)!}\right) \frac{4}{3\pi^2}.$$

*Polynomial diffusion.* Let $d(\cdot,\cdot)$ be the Euclidean distance on $\mathbb{R}^2$ and consider the wind-tree billiard table $\Pi \in \mathcal{WT}(m)$ as a subset of $\mathbb{R}^2$. Let $(\phi_t^\theta)_{t\in\mathbb{R}}$ be the billiard flow starting in direction $\theta \in [0, 2\pi)$ on $\Pi$, that is, $\phi_t^\theta(x)$ is the position of a particle after time $t$ starting from position $x \in \Pi$ in direction $\theta$.

Combining ideas of Delecroix–Hubert–Lelièvre [**DHL**], Delecroix–Zorich [**DZ**] and Forni [**Fo**], we obtain the following result on polynomial diffusion rates on wind-tree models.

Theorem. *For* every *wind-tree billiard* $\Pi \in \mathcal{WT}(m)$ *there exists* $\delta(\Pi) > 0$ *such that for almost every direction* $\theta \in [0, 2\pi)$ *and every starting point (with infinite forward orbit)*

$$\limsup_{t\to\infty} \frac{\log d(x, \phi_t^\theta(x))}{\log t} = \delta(\Pi).$$

Here, $\delta(\Pi)$ is the polynomial diffusion rate. Note that this result is already known for $m = 1$ and the diffusion rate is $2/3$ independently of the billiard table (see [**DHL**, Theorem 1]), and for almost every $\Pi \in \mathcal{WT}(m)$, for $m > 1$, with diffusion rate $\delta(m) = 4^m(m!)^2/(2m+1)!$, also independent of the billiard (see [**DZ**, Theorem 1]). Moreover, the value of $\delta(\Pi)$ depends only on $\mathrm{SL}(2,\mathbb{R})$-orbit closures (of the compact translation surface associated to the wind-tree billiard). Anyway, the interest of this result relies in the fact that the diffusion rate $\delta(\Pi)$ is *positive* for *every* $\Pi \in \mathcal{WT}(m)$.

*Recurrence.* Avila–Hubert [**AH**] gave a geometric criterion for the recurrence of a $\mathbb{Z}^d$-periodic translations surfaces in terms of good cylinders and proved the recurrence for the original wind-tree model. Using this criterion, our approach allows us to prove the recurrence for the Delecroix–Zorich variant. More precisely, we have the following.

Theorem. *For every wind-tree billiard* $\Pi \in \mathcal{WT}(m)$ *the billiard flow in* $\Pi$ *is recurrent for almost every direction* $\theta \in [0, 2\pi)$.

This result is already known for $m = 1$ ([**AH**, Theorem 1]). Moreover, as explained to us by V. Delecroix, a criterion of recurrence due to Chevallier–Conze [**CC**, Corollary 1.2] allows to conclude that the billiard flow $\phi_t^\theta$ is recurrent in $\Pi$ for almost every direction $\theta \in [0, 2\pi)$ when the polynomial diffusion rate (see above) $\delta(\Pi) < 1/2$. However, we only know that the polynomial diffusion rate is less than $1/2$ for *almost every* $\Pi \in \mathcal{WT}(m)$ and only for $m > 2$ ([**DZ**, Theorem 1]).

**4.2. Effective counting on Veech wind-tree models.** Veech, in his seminal work [**Ve89**], proved that for Veech surfaces (and billiards) there are exact quadratic asymptotics:

$$N(X, L) = c(X)L^2 + o(L^2).$$

Moreover, the methods used by Veech [**Ve89**] allows to conclude an effective version of the asymptotic formula above (see [**Ve92**, Remark 1.12]). Namely,

$$N(X, L) = c(X)L^2 + O(L^{2\delta(X)}) + O(L^{4/3})$$

as $L \to \infty$, for some $\delta(X) \in [1/2, 1)$. Furthermore, the number $\delta(X)$ has a specific interpretation in terms of spectral properties of the Veech group.

We say that $\Pi \in \mathcal{WT}(m)$ is a Veech wind-tree billiard if the underlying surface $X(\Pi)$ is a Veech surface. In Chapter II, we present an effective version of the counting problem, that is, the analogue of Veech's result, for Veech wind-tree billiards. More precisely, we prove the following.

THEOREM. *Let* $\Pi$ *be a Veech wind-tree billiard. Then, there exists* $c(\Pi) > 0$ *and* $\delta(\Pi) \in (1/2, 1)$ *such that*

$$N(\Pi, L) = c(\Pi)L^2 + O(L^{2\delta(\Pi)}) + O(L^{4/3})$$

*as* $L \to \infty$.

This result relies, on one hand, in the adaptation of Veech methods to our context, which allows to keep track trajectories corresponding to good cylinders. On the other hand, using tools from hyperbolic geometry we are able to handle trajectories corresponding to bad cylinders.

Furthermore, in the simplest case, when $\Pi$ is the wind-tree billiard with square obstacles of side length 1/2, the Veech group of $\Pi$ can be easily described and most of the involved objects can be explicitly computed, such as the contribution on the error term of the well behaved part of the periodic trajectories corresponding to good cylinders. Using results of Roblin–Tapie [**RT**], we explicitly estimate the contribution of the badly behaved family of periodic trajectories corresponding to bad cylinders. More precisely, we prove the following.

THEOREM 4.1. *Let* $\Pi$ *be the Veech wind-tree billiard with square obstacles of side length* 1/2*, and let* $\delta = \delta(\Pi) \in (1/2, 1)$ *be as in the conclusion on previous theorem. Then,*

$$\delta < 0.9992.$$

**Strategy of the proof.** Veech [**Ve89**] proved that for Veech surfaces, there are exact quadratic asymptotics and provided an effective version by means of spectral properties of the Veech group (see [**Ve92**, Remark 1.12]). Applying Veech's method to the counting problem on Veech wind-tree models, we are able to prove the analogous result in the case of good cylinders. We give the order of the error term by means of spectral properties of the Veech group of the underlying surface.

In the case of bad cylinders, this approach does not work anymore. However, bad cylinders can be described in terms of an intricate but well described subgroup $\Gamma_{bad}$ of the Veech group. Using tools from hyperbolic geometry, thanks to ideas of Dal'Bo [**Da**], we prove that the leading term on the counting of bad cylinders is related to the critical exponent of the group $\Gamma_{bad}$, and using results of Brooks [**Br**], we prove that this critical exponent is strictly less than 1.

The number $\delta(\Pi)$, giving the order of the error term, is completely defined by spectral properties of the involved groups.

In the case of the wind-tree billiard with square obstacles of side length 1/2, the underlying surface is a square-tiled surface whose Veech group is a congruence subgroup of level 2. Thanks to a result of Huxley [**Hu85**], we know that low level congruence groups satisfy the Selberg's 1/4 conjecture. This gives the contribution of good cylinders to the error term. The critical exponent of $\Gamma_{bad}$ requires much more attention and we are not able to give its exact value. Using results of Roblin–Tapie [**RT**], we estimate the critical exponent of $\Gamma_{bad}$. These estimates are far away from being optimal, but up to our knowledge, this is the only existing tool.

**4.3. Non-varying phenomenon.** The result about the polynomial diffusion rate on the classical wind-tree model due to Delecroix–Hubert–Lelièvre [**DHL**] *evince* a first *non-varying phenomenon*. Namely, the diffusion rate equals 2/3 independently of the size of the obstacles. In Chapter III we exhibit a non-varying phenomenon related to the counting problem we are interested in. More precisely, we prove the following.

THEOREM. *Let $\Pi(a,b)$ be a classical wind-tree model with rectangular obstacles of side lenghts $a, b \in \,]0,1[$. Denote by $N_{area}(\Pi(a,b),L)$ be the number of maximal families of isotopic periodic trajectories (up to $\mathbb{Z}^2$-translations) of length at most L in $\Pi(a,b)$, weighted by the area covered by the family.*

*(1) For Lebesgue-almost every $(a,b) \in \,]0,1[^2$ and, in particular, if $a,b$ are rational or can be written as $1/(1-a) = x + z\sqrt{D}$ and $1/(1-b) = y + z\sqrt{D}$ with $x,y,z \in \mathbb{Q}$ and $x+y=1$ and $D$ a positive square-free integer, then,*

$$N_{area}(\Pi(a,b),L) \sim \frac{4}{3\pi^2} \cdot \frac{\pi L^2}{1-ab}.$$

*(2) In any other case, we have the weak asymptotic formula*

$$N_{area}(\Pi(a,b),L) \text{“}\sim\text{”} \frac{4}{3\pi^2} \cdot \frac{\pi L^2}{1-ab}.$$

Evincing thus a new non-varying phenomenon: in terms of Siegel–Veech constants, this result means that the Siegel–Veech constant associated to this counting on wind-tree models equals $4/3\pi^2$ independently of the size of the obstacles.

These non-varying phenomena are expected to some extent to occur and arise from deep results on translation surfaces.

A connected component of a stratum of translation surfaces (or, more generally, an affine invariant submanifold) is said to be *non-varying* if for every Teichmüller curve in that component (resp. submanifold) the sum of positive Lyapunov exponents is the same. Such a non-varying phenomenon was observed numerically by Kontsevich–Zorich along with the initial observations on Lyapunov exponents for the Teichmüller geodesic flow [**Ko, KZ97**]. Nowadays, there are two types of non-varying results. One for low genus, due to Chen–Möller [**CM**], which uses a translation of the problem into algebraic geometry. The other one, for hyperelliptic loci, due to Eskin–Kontsevich–Zorich [**EKZ**], which is a consequence of their main result relating sum of Lyapunov exponents to Siegel–Veech constants and is thus related to the counting problems we are interested in. In particular, the non-varying phenomenon for the sum of Lyapunov exponents is equivalent to the non-varying of Siegel–Veech constants.

Motivated by the application to the wind-tree model, we study in Chapter III a related counting problem: that of cylinders whose core curve passes through two marked regular Weierstrass points on hyperelliptic surfaces in a hyperelliptic component (for the definition of the involved objects, see Chapter III); and we prove the following non-varying phenomenon analogous to the one described above.

THEOREM. *Let $\mu$ be the affine invariant measure supported on the* $\mathrm{SL}(2,\mathbb{R})$*-orbit closure of an hyperelliptic surface $X$ in a hyperelliptic component $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1,g-1)$, $g>1$. Then, the (area) Siegel–Veech constant associated to the counting problem of cylinders whose core curve passes through two marked regular Weierstrass points equals*

$$\begin{cases} \dfrac{1}{\pi^2}\cdot\dfrac{1}{2g-1}, & \text{if } X\in\mathcal{H}^{hyp}(2g-2),\\ \dfrac{1}{\pi^2}\cdot\dfrac{1}{2g}, & \text{if } X\in\mathcal{H}^{hyp}(g-1,g-1). \end{cases}$$

It is a natural question whether this non-varying phenomenon takes place in every hyperelliptic loci as well, as is the case for the counting problem of every cylinder (and not only those that pass through prescribed Weierstrass points). We prove that this is not true in general.

**Strategy of the proof.** From a hyperelliptic surface $X$ in a hyperelliptic component $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1,g-1)$, $g>1$, and given two fixed regular Weierstrass points, we build three different translation surfaces which are coverings of the original surface $X$. These coverings turn out to be hyperelliptic surfaces as well. We introduce some collections of cylinders associated to the monodromy of these coverings and describe the counting of cylinders whose core curve passes through the two Weierstrass points in terms of one of these collections. By elementary considerations on the Siegel–Veech formula, we can relate the Siegel–Veech constants of such collections of cylinders in $X$ to their liftings on the coverings.

Decomposing the Siegel–Veech constants of the involved surfaces in terms of these collections, we obtain a system of equations which allows us to describe the Siegel–Veech constants of each collection in terms of those of the surfaces. Since the surfaces are hyperelliptic, thanks to Eskin–Kontsevich–Zorich [**EKZ**], the result is non-varying. Describing the hyperelliptic loci where the surfaces lie and putting the values of the corresponding Siegel–Veech constants in the expression allows us to compute explicitly the value of the Siegel–Veech constant associated to the configurations and therefore, the one associated to the counting of cylinders whose core curve passes through the two Weierstrass points.

We present a family of counterexamples for hyperelliptic loci which are not hyperelliptic components. Using similar ideas, we exhibit hyperelliptic surfaces where the Siegel–Veech constant associated to the counting of cylinders whose core curve passes through two marked Weierstrass points does not coincide with the corresponding Siegel–Veech constant on the hyperelliptic loci where they lie.

Comming back to the wind-tree model, a simple description of good cylinders shows that, in the classical model, they coincide with the cylinders whose core curve passes through two specific regular Weierstrass points in a quotient of the surface $X=X(\Pi)$, which lies in the hyperelliptic component $\mathcal{H}^{hyp}(2)=\mathcal{H}(2)$. Thus, the previous result and some elementary considerations on the Siegel–Veech formula allows us then to conclude the result for wind-tree models.

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CHAPTER I

# Counting problem on wind-tree models

Abstract. We study periodic wind-tree models, billiards in the plane endowed with $\mathbb{Z}^2$-periodically located identical connected symmetric right-angled obstacles. We show asymptotic formulas for the number of (isotopy classes of) closed billiard trajectories (up to $\mathbb{Z}^2$-translations) on the wind-tree billiard. We also compute explicitly the associated Siegel-Veech constant for generic wind-tree billiards depending on the number of corners on the obstacle.

## 1. Introduction

The classical wind-tree model corresponds to a billiard in the plane endowed with $\mathbb{Z}^2$-periodic obstacles of rectangular shape; the sides of the rectangles are aligned along the lattice, see Figure 1.

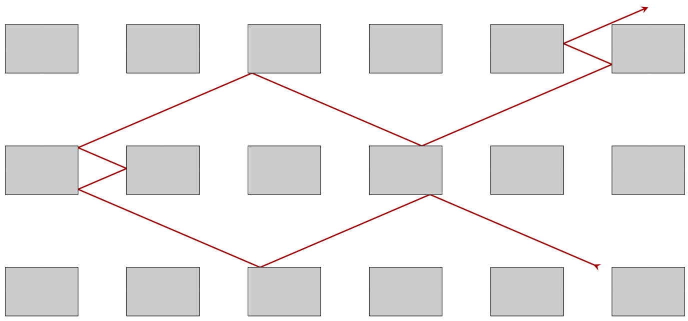


Figure 1. Original wind-tree model.

The wind-tree model (in a slightly different version) was introduced by P. Ehrenfest and T. Ehrenfest [**EE**] in 1912. J. Hardy and J. Weber [**HaWeb**] studied the periodic version. All these studies had physical motivations.

Several advances on the dynamical properties of the billiard flow in the wind-tree model were obtained recently using geometric and dynamical properties on moduli space of (compact) flat surfaces; billiard trajectories can be described by the linear flow on a flat surface.

A. Avila and P. Hubert [**AH**] showed that for all parameters of the obstacle and for almost all directions, the trajectories are recurrent. There are examples of divergent trajectories constructed by V. Delecroix [**De**]. The non-ergodicity was proved by K. Frączek and C. Ulcigrai [**FU**]. It was proved by V. Delecroix, P. Hubert and S. Lelièvre [**DHL**] that the diffusion rate is independent either on the concrete values of parameters of the obstacle or on almost any direction and almost any starting point and is equals to 2/3. A generalization of this last result was shown by V. Delecroix and A. Zorich [**DZ**] for more complicated obstacles. In this work we study this last variant, corresponding to a billiard in the plane endowed with $\mathbb{Z}^2$-periodic obstacles of right-angled polygonal shape; the obstacles being horizontally and vertically symmetric and the sides of the rectangles are aligned along the lattice, see Figure 2 for an example.

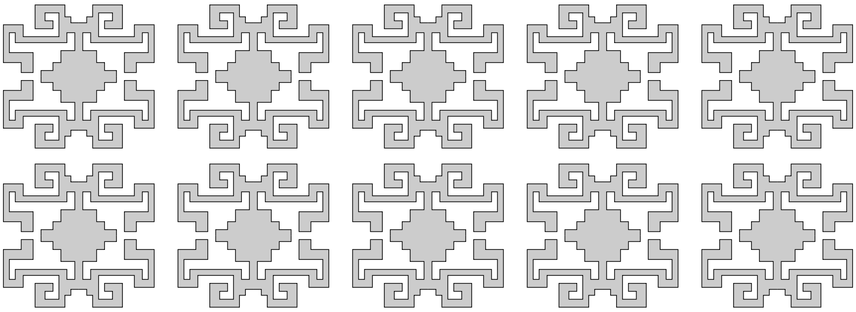


FIGURE 2. Delecroix–Zorich variant.

This work concerns asymptotic formulas for the number of (isotopy classes of) closed billiard trajectories (up to $\mathbb{Z}^2$-translations) on the wind-tree model. Note that we do not count trajectories which go around a single closed trajectory several times, and we are counting unoriented trajectories. This question has been widely studied in the context of (finite) rational billiards and compact flat surfaces, and it is related to many other questions such as the calculation of the volume of normalized strata [**EMZ**] or the sum of Lyapunov exponents of the geodesic Teichmüller flow [**EKZ**] on strata of flat surfaces (Abelian or quadratic differentials).

H. Masur [**Ma88, Ma90**] proved that for every flat surface $X$, there exist positive constants $c(X)$ and $C(X)$ such that the number $N(X,L)$ of (maximal) cylinders of closed geodesics of length at most $L$ satisfy

$$c(X)L^2 \leq N(X,L) \leq C(X)L^2$$

for large enough $L$. W. Veech [**Ve89**] proved that for Veech surfaces there are in fact exact quadratic asymptotics; E. Gutkin and C. Judge [**GJ00**] gave a different proof. Another proof for the upper quadratic bounds was given by Y. Vorobets [**Vo97**]. A. Eskin and H. Masur [**EMa**] gave yet another one and proved that for each ergodic probability measure $\mu$ on strata of normalized (area 1) flat surfaces, there is a constant $c(\mu)$ such that for almost every surface, $N(X,L) \sim c(\mu) \cdot \pi L^2$, that is,

$$\lim_{L\to\infty} \frac{N(X,L)}{\pi L^2} = c(\mu).$$

The constant $c(\mu)$ is called the Siegel–Veech constant ([**EMa**]) of the counting problem; it is the constant in the Siegel–Veech formula ([**EMa**]), a Siegel-type formula introduced by W. Veech [**Ve98**].

It is still an open problem whether *all* flat surfaces have exact quadratic asymptotics. The particular constants for several Veech surfaces have been computed explicitly by W. Veech [**Ve89, Ve92**], Y. Vorobets [**Vo97**], E. Gutkin and C. Judge [**GJ00**] and M. Schmoll [**Sc**]. Constants for some families of non-Veech surfaces were also given by A. Eskin, H. Masur and M. Schmoll [**EMS**] and A. Eskin, J. Marklof and D. Witte Morris [**EMW**]. A. Eskin, H. Masur and A. Zorich [**EMZ**] computed the Siegel–Veech constants for connected components of all strata of Abelian differentials, and also described all possible configurations of cylinders of closed geodesics which might be found on a generic flat surface. In general, the particular constants for Veech surfaces do not coincide with the Siegel–Veech constants of the strata where they live.

The case of quadratic differentials presents extra difficulties. However, J. Athreya, A. Eskin and A. Zorich [**AEZ**] gave explicit values for the Siegel–Veech constants on strata of quadratic differentials of genus zero surfaces. E. Goujard [**Gj**] generalized this approach to higher genera and obtained some exact values of Siegel–Veech constants for strata of quadratic differentials away from genus zero.

We prove asymptotic formulas for generic wind-tree models with respect to a natural Lebesgue-type measure (see [**AEZ, DZ**]) on the parameters of the wind-tree billiards, that is, the side lengths of the obstacles. Denote by $\mathcal{WT}(m)$ the family of wind-tree billiards such that the obstacle has $4m$ corners with the angle $\pi/2$. Say, all billiards from the original wind-tree family as in Figure 1 live in $\mathcal{WT}(1)$; the billiard in Figure 2 belongs to $\mathcal{WT}(17)$. We denote by $\mathrm{Area}\,(\Pi/\mathbb{Z}^2)$ the area of a fundamental domain of the $\mathbb{Z}^2$-periodic billiard table $\Pi \in \mathcal{WT}(m)$.

Theorem 1.1. *For almost every wind-tree billiard $\Pi \in \mathcal{WT}(m)$ the number $N(\Pi, L)$ of closed billiard trajectories of length at most $L$ in $\Pi$ (up to isotopy and $\mathbb{Z}^2$-translations) has quadratic asymptotic growth rate*

$$N(\Pi, L) \sim c(m) \cdot \frac{\pi L^2}{\mathrm{Area}\,(\Pi/\mathbb{Z}^2)},$$

*where*

$$c(m) = \left(20m^2 - 95m - 78 + 78 \cdot 4^m \frac{(m!)^2}{(2m)!}\right) \frac{1}{6\pi^2}.$$

The constant $c(m)$ is not the Siegel–Veech constant of one particular surface, but corresponds to Siegel–Veech constants of some particular configurations of cylinders on compact flat surfaces associated to generic wind-tree billiards.

On the other hand, A. Eskin, M. Mirzakhani and A. Mohammadi [**EMM**] showed that for *all* (area 1) flat surfaces we have *weak* quadratic asymptotic formulas,

$$\lim_{L\to\infty} \frac{1}{L} \int_0^L \frac{N(X, e^t)}{\pi e^{2t}} \mathrm{d}t = c(X),$$

which we write $N(X, L)$ "$\sim$" $c(X) \cdot \pi L^2$. The constant $c(X)$ being the Siegel–Veech constant associated to the affine invariant measure supported on the $\mathrm{SL}(2, \mathbb{R})$-orbit closure of the surface $X$ given by general invariant measure classification theorem of A. Eskin and M. Mirzakhani [**EMi**].

Using this technology, one can prove weak asymptotic formulas for individual wind-tree billiards. In particular, the following holds.

Theorem 1.2. *Let $\Pi \in \mathcal{WT}(m)$ be a wind tree billiard.*

*(1) Suppose that one of the following conditions holds*
 *(a) All the parameters of $\Pi$ are rational, or*
 *(b) $m = 1$ and there exists a square-free integer $D > 0$ such that the two parameters of $\Pi$, say $a, b \in (0, 1)$, can be written as $1/(1-a) = x + z\sqrt{D}$ and $1/(1-b) = y + z\sqrt{D}$ with $x, y, z \in \mathbb{Q}$ and $x + y = 1$.*

*Then,*

$$N(\Pi, L) \sim c(\Pi) \cdot \frac{\pi L^2}{\mathrm{Area}\,(\Pi/\mathbb{Z}^2)}.$$

*(2) In any other case, we have the weak asymptotic formula*

$$N(\Pi, L) \text{ ``}\!\sim\!\text{'' } c(\Pi) \cdot \frac{\pi L^2}{\mathrm{Area}\,(\Pi/\mathbb{Z}^2)}.$$

The case (1) corresponds to (particular cases of) Veech surfaces and formulas for the Siegel–Veech constants can be obtained following an approach similar to the one of E. Gutkin and C. Judge [**GJ00**, § 6]. In the case (a), when the parameters are rational, it corresponds to square-tiled surfaces and it is possible to obtain formulas similar to the obtained by A. Eskin, M. Kontsevich and A. Zorich [**EKZ**, Theorem 4]. In the other cases we do not know the Siegel–Veech constants for every wind-tree billiard. However, it depends only on $\mathrm{SL}(2,\mathbb{R})$-orbit closures (of a compact flat surface associated to the wind-tree billiard) and, in particular, it coincides with $c(m)$ for generic billiards.

**1.1. Strategy of the proof.** We reformulate the counting problem on wind-tree billiards in terms of a counting problem on a $\mathbb{Z}^2$-periodic flat surface. This is quite elementary and straightforward. For details on the reduction of the study of the billiard flow into the study of a $\mathbb{Z}^2$-cocycle over the linear flow of a finite flat surface, see [**DHL**, § 3].

In general, we can consider an infinite flat surface $X_\infty$ which is a ramified $\mathbb{Z}^d$-cover over a compact flat surface $X$, $d \geq 1$ ($d = 2$ in our case). Let $\Sigma$ be the finite set of singularity points of $X$. Since the intersection form $\langle \cdot, \cdot \rangle$ is non-degenerate between $H^1(X \setminus \Sigma, \mathbb{Z})$ and $H^1(X, \Sigma, \mathbb{Z})$, every such $\mathbb{Z}^d$-cover is defined by a $d$-tuple of independent elements $\mathbf{f} = (f_1, \ldots, f_d)$ in the group of relative cohomology $H^1(S, \Sigma, \mathbb{Z})$, but we restrict ourselves to the case when $\mathbf{f} \in H^1(X, \mathbb{Z}^d)$ —this is the case of the infinite $\mathbb{Z}^2$-periodic flat surface associated to a wind-tree model.

We are interested in counting (maximal) cylinders of closed geodesics in $X_\infty$ (up to $\mathbb{Z}^d$-translations, of course). Cylinders of closed geodesics in the cover $X_\infty$ clearly descends to cylinders in $X$, but not the other way around. In fact, by definition of the covering, cylinders in the cover $X_\infty$ are exactly the lift of those cylinders $C$ in $X$ such that $\gamma_C$, (the Poincaré dual of the homology class of) its core curve, verifies $\langle \gamma_C, f_i \rangle = 0$, for each $i = 1, \ldots, d$.

One of the main tools used in this kind of problems (and many others) is the $\mathrm{SL}(2,\mathbb{R})$-action on strata of flat surfaces (see, e.g., [**EMa, EMZ**]) and the associated cocycle over the Hodge bundle, the Kontsevich–Zorich cocycle. Let $\mathcal{M}$ be the $\mathrm{SL}(2,\mathbb{R})$-orbit closure of $X$, $F$ be a subbundle of the Hodge bundle over $\mathcal{M}$, invariant with respect to the Kontsevish–Zorich cocycle, and let $f \in F_X$.

Note that cylinders $C$ in $X$ such that $\langle \gamma_C, f \rangle = 0$ split naturally into two families: (a) the family of cylinders such that $\langle \gamma_C, h \rangle = 0$ for all $h \in F_X$, which we call *$F$-good cylinders*, and (b) the family of cylinders that are not $F$-good, but $\langle \gamma_C, f \rangle = 0$. These later are called *$(F, f)$-bad cylinders*. This notion of $F$-good cylinders was first introduced by A. Avila and P. Hubert [**AH**] in order to give a geometric criterion for recurrence of $\mathbb{Z}^d$-periodic flat surfaces.

Thus, counting cylinders in a $\mathbb{Z}^d$-periodic flat surface can be reduced to count separately cylinders which are $(\oplus_j F^{(j)})$-good cylinders and $(F^{(j_i)}, f_i)$-bad cylinders in the compact surface, for some appropriate subbundles $(F^{(j)})_j$.

In the case of the classical wind-tree model, that is, for $m = 1$, V. Delecroix, P. Hubert and S. Lelièvre [**DHL**] gave a complete description of the cocycles defining the surfaces and the corresponding decomposition of the Hodge bundle, which allows us to successfully apply this approach. This is extended naturally to the Delecroix–Zorich variant ($m > 1$). In fact, for every $\Pi \in \mathcal{WT}(m)$, there are two cocycles $h$ and $v$ in a compact flat surface $\mathrm{X} = \mathrm{X}(\Pi)$ defining the $\mathbb{Z}^2$-periodic flat surface $\mathrm{X}_\infty = \mathrm{X}_\infty(\Pi)$ associated to $\Pi$ and two 2-dimensional equivariant subbundles, which we denote by $F^{+-}$ and $F^{-+}$, such that $h \in F^{+-}$ and $v \in F^{-+}$.

Using the main result of A. Eskin and H. Masur in [**EMa**], it is a straightforward remark that we have asymptotic formulas for the number of $F$-good cylinders with an associated Siegel–Veech constant, for generic surfaces, for any $\mathrm{SL}(2,\mathbb{R})$-ergodic finite measure on any normalized strata. In the case of $(F, f)$-bad cylinders, this is no longer true. However, in the case of the wind-tree model, we prove the following.

THEOREM 1.3. *Let $\Pi \in \mathcal{WT}(m)$ be a wind-tree billiard, $\mathrm{X} = \mathrm{X}(\Pi)$ the associated compact flat surface and let $F$ be one of the associated subbundles $F^{+-}$ or $F^{-+}$. Then, for any $f \in F_\mathrm{X}$ the number $N_F(f, L)$, of $(F, f)$-bad cylinders in* X *of length at most $L$, has* subquadratic *asymptotic growth rate, that is, $N_F(f, L) = o(L^2)$ or, which is the same,*

$$\lim_{L\to\infty} \frac{N_F(f, L)}{\pi L^2} = 0.$$

We use technology for asymptotic formulas developed by A. Eskin and H. Masur [**EMa**] in order to prove (a slightly more general version of) Theorem 1.3. For this, we need in addition the condition of non-zero Lyapunov exponents for to the relevant subbundles $F^{+-}$ and $F^{-+}$. This is true for almost every wind-tree billiards thanks to one of the main results of V. Delecroix and A. Zorich in [**DZ**]. For the statement to be true for every wind-tree billiard, we use (a slightly more general version of) the so called Forni's criterion due to G. Forni [**Fo**], a geometric criterion for the positivity of Lyapunov exponents, applied to integer equivariant subbundles.

As a consequence of Theorem 1.3, the proof of Theorem 1.1 is reduced to compute the Siegel–Veech constant associated to configurations of $F^{+-} \oplus F^{-+}$-good cylinders. Furthermore, Theorem 1.2 becomes a compilation of several different results and we omit its proof here; it is almost identical to the proof of Theorem 1.7 in [**AEZ**], after the reduction given by Theorem 1.3, to the problem of counting only $F^{+-} \oplus F^{-+}$-good cylinders.

For the computation of the Siegel–Veech constant associated to configurations of $F^{+-} \oplus F^{-+}$-good cylinders, we make use of extra symmetries in the surface $\mathrm{X}(\Pi)$ to describe it as a cover of lower genus surfaces. In particular, configurations of $F^{+-} \oplus F^{-+}$-good cylinders are related to configurations of cylinders on some strata of genus zero surfaces, such that they lift to homologically trivial cylinders on some strata of genus one surfaces.

C. Boissy [**Bo**] described all possible configurations on generic surfaces in genus zero. Using this, we describe all possible configurations of cylinders satisfying the homological conditions ensuring they correspond to $F^{+-} \oplus F^{-+}$-good cylinders. Then, we relate Siegel–Veech constants of configurations in the genus zero surface with the constant for the higher genus surface and do the combinatorics. Finally, plugging in the resulting expression the explicit values of the Siegel–Veech constants for configurations on generic surfaces of genus zero obtained by J. Athreya, A. Eskin and A. Zorich [**AEZ**] and, proving certain combinatorial identities for resulting hypergeometric sums, we obtain the desired explicit value of $c(m)$.

**1.2. Side results.** As a by-product of our methods, we obtain several results as detailed below.

**Area Siegel-Veech constant.** Following the same strategy, we are able to compute the area Siegel–Veech constant, associated to the counting of the area of maximal families of isotopic compact trajectories. More precisely, we have the analogous of Theorem 1.1:

THEOREM 1.4. *For almost every $\Pi \in \mathcal{WT}(m)$ the weighted number $N_{area}(\Pi, L)$ of maximal families $C$ of isotopic closed billiard trajectories of length at most $L$ in $\Pi$ (up to $\mathbb{Z}^2$-translations), where the weight is the ratio* $\mathrm{Area}\,(C)\,/\,\mathrm{Area}\,(\Pi/\mathbb{Z}^2)$*, has quadratic asymptotic growth rate*

$$N_{area}(\Pi, L) \sim c_{area}(m) \cdot \frac{\pi L^2}{\mathrm{Area}\,(\Pi/\mathbb{Z}^2)},$$

*where*

$$c_{area}(m) = \left(8m - 33 + 39 \cdot 4^m \frac{(m!)^2}{(2m+1)!}\right) \frac{4}{3\pi^2}.$$

**Polynomial diffusion.** Let $d(\cdot, \cdot)$ be the Euclidean distance on $\mathbb{R}^2$ and consider the wind-tree billiard table $\Pi \in \mathcal{WT}(m)$ as a subset of $\mathbb{R}^2$. Let $(\phi_t^\theta)_{t\in\mathbb{R}}$ be the billiard flow in direction $\theta \in [0, 2\pi)$ on $\Pi$, that is, $\phi_t^\theta(x)$ is the position of a particle after time $t$ starting from position $x \in \Pi$ in direction $\theta$.

The application of the Forni's criterion to the relevant subbundles $F^{+-}$ and $F^{-+}$ allows us to show that they have non-zero Lyapunov exponents. Applying the result [**DZ**, Corollary 1] of V. Delecroix and A. Zorich, which is a generalization of the analogous result for the classical model due to V. Delecroix, P. Hubert and S. Lelièvre [**DHL**], we obtain the following.

THEOREM 1.5. *For every wind-tree billiard $\Pi \in \mathcal{WT}(m)$ there exists $\delta(\Pi) > 0$ such that for almost every direction $\theta \in [0, 2\pi)$ and every starting point (with infinite forward orbit)*

$$\limsup_{t\to\infty} \frac{\log d(x, \phi_t^\theta(x))}{\log t} = \delta(\Pi).$$

Here, $\delta(\Pi)$ is the polynomial diffusion rate and coincides with the Lyapunov exponent mentioned above. Note that this result is already known for $m = 1$ and the diffusion rate $\delta$ is $2/3$ independently of the billiard table (see [**DHL**, Theorem 1]), and for almost all $\Pi \in \mathcal{WT}(m)$, for $m > 1$, with $\delta(m) = 4^m(m!)^2/(2m+1)!$, also independent of the billiard (see [**DZ**, Theorem 1]). Moreover, the value of $\delta(\Pi)$ depends only on $\mathrm{SL}(2,\mathbb{R})$-orbit closures (of the compact flat surface associated to the wind-tree billiard). Anyway, the interest of this result relies in the fact that the diffusion rate $\delta(\Pi)$ is *positive* for *every* $\Pi \in \mathcal{WT}(m)$.

**Recurrence.** A. Avila and P. Hubert [**AH**] gave a geometric criterion for the recurrence of a $\mathbb{Z}^d$-periodic flat surfaces in terms of good cylinders and proved the recurrence for the original wind-tree model. Using this criterion, our approach allows us to prove the recurrence for the Delecroix–Zorich variant. More precisely, we have the following.

THEOREM 1.6. *For every wind-tree billiard* $\Pi \in \mathcal{WT}(m)$ *the billiard flow in* $\Pi$ *is recurrent for almost every direction* $\theta \in [0, 2\pi)$.

This result is already known for $m = 1$ (see [**AH**, Theorem 1]). Moreover, as explained to us by V. Delecroix, a criterion of recurrence due to N. Chevallier and J.-P. Conze [**CC**, Corollary 1.2] allows us to conclude that the billiard flow $\phi_t^\theta$ is recurrent in $\Pi$ for almost every direction $\theta \in [0, 2\pi)$ if the polynomial diffusion rate (see above) $\delta(\Pi) < 1/2$. However, we only know that the polynomial diffusion rate is less than $1/2$ for *almost every* $\Pi \in \mathcal{WT}(m)$ and only for $m > 2$.

**1.3. Structure of the paper.** In § 2 we briefly recall all the background necessary to formulate and prove the results. In § 3 we do the reduction of the counting problem on general $\mathbb{Z}^d$-periodic flat surfaces to the counting of $(\oplus_j F^{(j)})$-good cylinders and $(F^{(j_i)}, f_i)$-bad cylinders in the compact surface, for some appropriate subbundles $(F^{(j)})_j$ of the Hodge bundle. In § 4 we prove Theorem 4.1, a slightly more general version of Theorem 1.3, but with the extra condition that some particular Lyapunov exponent is positive. In § 5 we show that the relevant Lyapunov exponent is positive applying the Forni's criterion to integer equivariant subbundles, which ends the proof of Theorem 1.3 and allows us to reduce the problem to the counting of $F^{+-} \oplus F^{-+}$-good cylinders. In § 6 we study configurations of cylinders on generic genus zero surfaces in order to describe $F^{+-} \oplus F^{-+}$-good cylinders. In § 6.1 we show which configurations of cylinders on generic genus zero surfaces lift to $F^{+-} \oplus F^{-+}$-good cylinders in the higher genus surface by means of topological considerations. Then, in § 6.2, we describe how these cylinders lift to the higher genus surface, that is, the number of cylinders we obtain and their length. With this, we are able to relate in § 6.3 the Siegel–Veech constants of the genus zero and the higher genus surfaces.

Finally, in § 7 we compute the Siegel–Veech constant of $F^{+-} \oplus F^{-+}$-good cylinders: we count the possible configurations taking part in the computations and plug in the explicit values of the Siegel–Veech constants obtained by J. Athreya, A. Eskin and A. Zorich [**AEZ**]. This allows us to conclude the computations by means of a combinatorial identity for certain hypergeometric sums proved separately in an appendix.

Side results mentioned above are proved in § 8.

**Acknowledgments.** The author is greatly indebted to Pascal Hubert for his guide and invaluable help at every stage of this work, to Anton Zorich for introducing me in the theory of flat surfaces, to both of them for their constant encouragement, kind explanations and useful discussions. The author is grateful to Vincent Delecroix for his kind explanation of his work with A. Zorich about the polynomial diffusion rate of generalized wind-tree models and for pointing out the recurrence in these models when there is a low polynomial diffusion rate. The author is grateful to Carlos Matheus for corroborate and give some details about the validity of Forni's criterion for an integer equivariant subbundle.

## 2. Background

**2.1. Flat surfaces.** For an introduction and general references to this subject, we refer the reader to the surveys of Zorich [**Zo06**], Forni–Matheus [**FM**], Wright [**Wr**].

**Flat surfaces and strata.** Let $g > 1$, $\alpha = \{n_1, \dots, n_k\} \subset \mathbb{N}$ be a partition of $2g-2$ and $\mathcal{H}(\alpha)$ be a stratum of Abelian differentials, that is, the space of pairs $X = (S, \omega)$ where $\omega$ is a holomorphic 1-form on a Riemann surface $S$ of genus $g$, with zeros of degrees $n_1, \dots, n_k \in \mathbb{N}$. Let $\Sigma = \Sigma(\omega)$ be the set of singularities of $X$, the zeros of $\omega$. The form $\omega$ defines a canonical flat metric on $S$ with conical singularities of angle $2\pi(n+1)$ at zeros of degree $n$ of $\omega$.

We also consider strata $\mathcal{Q}(d_1, \dots, d_k)$ of meromorphic quadratic differentials with at most simple poles, the spaces of pairs $(S, q)$ where $q$ is a meromorphic quadratic differential on a Riemann surface $S$ of genus $g$ with zeros of order $d_1, \dots, d_k$, $d_i \in \{-1\} \cup \mathbb{N}$ for $i = 1, \dots, k$ (in a slight abuse of vocabulary, we are considering poles as zeros of order $-1$) and $\sum_{i=1}^k d_i = 4g-4$. The quadratic differential $q$ also defines a canonical flat metric with conical singularities of angle $\pi(d+2)$ at zeros of order $d$.

In this paper, a quadratic differential is not the square of an Abelian differential and a flat surface is the Riemann surface with the flat metric corresponding to an Abelian or quadratic differential.

The area of a flat surface is the one obtained from the flat metric. Let $\mathcal{H}_1(\alpha)$ denote the codimension 1 subspace of area 1 on $\mathcal{H}(\alpha)$ denote the codimension 1 subspace of (flat) area 1.

$\mathrm{SL}(2,\mathbb{R})$**-action and the Teichmüller geodesic flow.** There is a natural action of $\mathrm{SL}(2,\mathbb{R})$ on strata of Abelian differentials, which generalizes the action of $\mathrm{SL}(2,\mathbb{R})$ on the space $\mathrm{GL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ of flat tori. Let

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad \text{and} \quad r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

The element $r_\theta \in \mathrm{SL}(2,\mathbb{R})$ acts by $(S,\omega) \mapsto (S, e^{i\theta}\omega)$. This has the effect of rotating the flat surface by the angle $\theta \in [0, 2\pi)$. The action of $(g_t)_{t\in\mathbb{R}}$ is called the Teichmüller geodesic flow.

**Affine invariant measures and manifolds.** Each stratum carries a natural Lebesgue measure, invariant under the action of $\mathrm{SL}(2,\mathbb{R})$, which is given by the pullback of the Lebesgue measure on $H^1(S,\Sigma,\mathbb{C}) \cong \mathbb{C}^{2g+k-1}$.

An affine invariant manifold is an $\mathrm{SL}(2,\mathbb{R})$-invariant closed subset of $\mathcal{H}_1(\alpha)$, which looks like an affine subspace in period coordinates (see, e.g., [**Zo06**, § 3]). Each affine invariant manifold $\mathcal{M}$ is the support of an ergodic $\mathrm{SL}(2,\mathbb{R})$-invariant probability measure $\nu_{\mathcal{M}}$. Locally, in period coordinates, this measure is (up to normalization) the restriction of Lebesgue measure to the subspace $\mathcal{M}$ (see [**EMi**] for the precise definitions). Eskin–Mirzakhani–Mohammadi [**EMM**] proved that any $\mathrm{SL}(2,\mathbb{R})$-orbit closure is an affine invariant manifold. The most important case of an affine invariant manifold is a connected component of a stratum $\mathcal{H}_1(\alpha)$. Masur [**Ma82**] and Veech [**Ve82**] independently proved that in this case, the total mass of this measure is finite and ergodic with respect to the Teichmüller geodesic flow. The associated affine measure is known as the Masur–Veech measure.

**Hodge bundle and the Kontsevich–Zorich cocycle.** The (real) Hodge bundle $H^1$ is the real vector bundle of dimension $2g$ over an affine invariant manifold $\mathcal{M}$, where the fiber over $X = (S,\omega)$ is the real cohomology $H^1_X = H^1(S,\mathbb{R})$. Each fiber $H^1_X$ has a natural lattice $H^1_X(\mathbb{Z}) = H^1(S,\mathbb{Z})$ which allows identification of nearby fibers and definition of the Gauss–Manin (flat) connection. The monodromy of the Gauss–Manin connection restricted to $\mathrm{SL}(2,\mathbb{R})$-orbits provides a cocycle called the Kontsevich–Zorich cocycle, which we denote by $A(g,X)$, for $g \in \mathrm{SL}(2,\mathbb{R})$ and $X \in \mathcal{M}$. The Kontsevich–Zorich cocycle is a symplectic cocycle because it preserves the intersection form $\langle f_1, f_2\rangle = \int_S f_1 \wedge f_2$ on $H^1(S,\mathbb{R})$, which is a symplectic form on the $2g$-dimensional real vector space $H^1(S,\mathbb{R})$. Let $\|\cdot\|_\omega$ be the Hodge norm (for precise definition see, e.g., [**FM**, § 3.4]). The Hodge norm depends continuously on $(S,\omega)$, but is not preserved by the Kontsevich–Zorich cocycle in general.

**Lyapunov exponents.** Given any affine invariant manifold $\mathcal{M}$, we know from Oseledets theorem that there are real numbers $\lambda_1(\mathcal{M}) \geq \cdots \geq \lambda_{2g}(\mathcal{M})$, the Lyapunov exponents, and a measurable $g_t$-equivariant filtration of the Hodge bundle $H^1(S,\mathbb{R}) = V_1(X) \supset \cdots \supset V_{2g}(X) = \{0\}$ at $\nu_{\mathcal{M}}$-almost every $X = (S,\omega) \in \mathcal{M}$ and

$$\lim_{t\to\infty} \frac{1}{t} \log \|A(g_t, X)f\|_{g_t\omega} = \lambda_i$$

for every $f \in V_i \setminus V_{i+1}$.

THEOREM 2.1 (Chaika-Eskin [**CE**]). *Let $X$ be a flat surface and $\mathcal{M}$ be the $\mathrm{SL}(2,\mathbb{R})$-orbit closure of $X$. Then, for almost every $\theta \in [0, 2\pi)$ we have the $g_t$-equivariant filtration $H^1(S,\mathbb{R}) = V_1(r_\theta X) \supset \cdots \supset V_{2g}(r_\theta X) = \{0\}$ and, for every $f \in V_i \setminus V_{i+1}$,*

$$\lim_{t\to\infty} \frac{1}{t} \log \|A(g_t, r_\theta X)f\|_{g_t r_\theta \omega} = \lambda_i(\mathcal{M}).$$

The set $\Lambda(\mathcal{M})$ of Lyapunov exponents is called Lyapunov spectrum (of the Kontsevich–Zorich cocycle over the Teichmüller flow on $\mathcal{M}$). The fact that the Kontsevich–Zorich cocycle is symplectic means that the Lyapunov spectrum is always symmetric, $\Lambda(\mathcal{M}) = -\Lambda(\mathcal{M})$.

**Equivariant subbundles of the Hodge bundle.** Let $\mathcal{M}$ be an affine invariant submanifold and $F$ a subbundle of the Hodge bundle over $\mathcal{M}$. We say that $F$ is equivariant if it is invariant under the Kontsevich–Zorich cocycle, and we say that $F$ is irreducible if it has no proper equivariant subbundles. Since $\mathcal{M}$ is $\mathrm{SL}(2,\mathbb{R})$-invariant, by the definition of the Kontsevich–Zorich cocycle, a flat (locally constant) subbundle is always equivariant.

Previous discussion about Lyapunov exponents applies in this context as well and we have that, as before, for every $X = (S, \omega) \in \mathcal{M}$ such that $\mathcal{M}$ is the $\mathrm{SL}(2, \mathbb{R})$-orbit closure of $X$ and almost every $\theta \in [0, 2\pi)$, there is a $g_t$-equivariant filtration $F_{r_\theta X} = U_1(r_\theta X) \supset \cdots \supset U_r(r_\theta X) = \{0\}$, where $r = \operatorname{rank} F = \dim F_X$ and, for every $f \in U_i \setminus U_{i+1}$,

$$\lim_{t\to\infty} \frac{1}{t} \log \|A(g_t, r_\theta X) f\|_{g_t r_\theta \omega} = \lambda_i(\mathcal{M}, F).$$

The Lyapunov spectrum restricted to $F$ is $\Lambda(\mathcal{M}, F) = \{\lambda_i(\mathcal{M}, F)\}_{i=1}^r \subset \Lambda(\mathcal{M})$.

Remark 2.2. If $F$ is irreducible and admits a non-zero Lyapunov exponent in its Lyapunov spectrum, then $F$ is symplectic with respect to the intersection form, that is, the symplectic intersection form is non-degenerate on $F$ (this is a nontrivial fact that can be deduced from [**EMi**, Theorem A.9], which in turn is deduced from [**FMZ**]). In particular, $F$ is an even-dimensional subbundle and, as before, the associated Lyapunov spectrum is symmetric, $\Lambda(\mathcal{M}, F) = -\Lambda(\mathcal{M}, F)$.

We denote by $F^\dagger$ the symplectic complement of $F$ and, when $F$ is symplectic, define $F_X^{\mathrm{pr}}(\mathbb{Z}) = \mathrm{pr}_{F_X} H_X^1(\mathbb{Z})$, where $\mathrm{pr}_{F_X} : H_X^1 \to F_X$ is the symplectic projection, that is, the first component of the decomposition $H_X^1 = F_X \oplus F_X^\dagger$.

We denote by $F_X(\mathbb{Z}) = F_X \cap H_X^1(\mathbb{Z})$ the set of integer cocycles in $F_X$. We say that $F$ is defined over $\mathbb{Z}$ if it is generated by integer cocycles, that is, if $F_X = \langle F_X(\mathbb{Z})\rangle_\mathbb{R}$. When $F$ is defined over $\mathbb{Z}$, $F_X(\mathbb{Z})$ is a lattice in $F_X$. If, in addition, $F$ is symplectic, we have that $F_X^{\mathrm{pr}}(\mathbb{Z})$ is also a lattice and $F_X(\mathbb{Z}) \subset F_X^{\mathrm{pr}}(\mathbb{Z})$.

**2.2. Counting problem.** We are interested in the counting of closed geodesics of bounded length on flat surfaces.

**Cylinders of closed geodesics and saddle connections.** Together with every closed regular geodesic in a flat surface $X = (S, \omega)$ (resp. $(S, q)$) we have a bunch of parallel closed regular geodesics. A cylinder on a flat surface is a maximal open annulus filled by isotopic simple closed regular geodesics. A cylinder $C$ is isometric to the product of an open interval and a circle, and its core curve $\gamma_C$ is the geodesic projecting to the middle of the interval. A saddle connection is a geodesic joining two different singularities or a singularity to itself, with no singularities in its interior. Cylinders are always bounded by parallel saddle connections.

**Holonomy.** Integrating $\omega$ (resp. a locally defined square-root of $q$) along the core curve of a cylinder, a saddle connection or, more generally, any homology class $\gamma \in H_1(S, \Sigma, \mathbb{Z})$, we get a complex number. Considered as a planar vector, this complex number represents the affine holonomy along $\gamma$ and we denote this holonomy vector by $\mathrm{hol}_\omega(\gamma)$. In particular, in the case of a cylinder or saddle connection, its euclidean length corresponds to the modulus of its holonomy vector.

**Systole.** Let $\mathrm{sys}(X)$ be the systole of the flat surface $X$, that is, the length of its shortest saddle connection, and let $K_\epsilon = \{X : \mathrm{sys}(X) \geq \epsilon\}$. $K_\epsilon$ form a compact exhaustion on any affine invariant manifold (which are never compact).

**Counting problem and Siegel–Veech constants.** Consider the set of all cylinders on a flat surface $X$ and consider its image $V(X) \in \mathbb{R}^2 \cong \mathbb{C}$ under the holonomy map, $V(X) = \{\mathrm{hol}\gamma_C : C \text{ is a cylinder in } X\}$. This is a discret set of $\mathbb{R}^2$. We are concerned with the asymptotic behavior of the number $N(X, L) = \#V(X) \cap B(L)$ of cylinders in $X$ of length at most $L$, when $L \to \infty$.

THEOREM 2.3 (Eskin–Masur [**EMa**]). *Let $\mathcal{M}$ be an affine invariant manifold. Then, there is a constant $c(\mathcal{M})$ such that for $\nu_{\mathcal{M}}$-almost all $X \in \mathcal{M}$*

$$\lim_{L\to\infty} \frac{N(X, L)}{\pi L^2} = c(\mathcal{M}), \tag{1}$$

*where $c(\mathcal{M})$ is the Siegel–Veech constant given by the Siegel–Veech formula*

$$c(\mathcal{M}) = \frac{1}{\pi\rho^2} \int_{\mathcal{M}} N(Y, \rho)\mathrm{d}\nu_{\mathcal{M}}(Y). \tag{2}$$

We use some of the tools developed by Eskin–Masur when proving this theorem. In particular, the following are of special utility to us.

THEOREM 2.4 ([**EMa**, Theorem 5.1(b)]). *For any $X \in \mathcal{H}(\alpha)$ and all $\delta, \rho > 0$,*

$$N(X, \rho) \leq \frac{c(\rho, \delta)}{\mathrm{sys}(X)^{1+\delta}}.$$

THEOREM 2.5 ([**EMa**, Theorem 5.2]). *For any $X \in \mathcal{H}(\alpha)$, any $\beta < 2$ and all $t > 0$,*

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{\mathrm{sys}(g_t r_\theta X)^\beta} \leq c(X, \beta).$$

We remark that these two results are true for *every* flat surface, in contrast to Theorem 2.3, which holds for *almost every* flat surface.

**Configurations of cylinders.** A collection $\mathbf{C} = \{C_1, \ldots, C_n\}$ of cylinders determines the data on combinatorial geometry of the decomposition of $S \setminus \mathbf{C}$. It determines the number of components, their boundary structure, the singularity data for each component and how the components are glued to each other. These data are referred to as configuration of cylinders (see [**EMZ**]). The multiplicity of a configuration is the number of cylinders it defines. Remark that we reserve the notion of configuration for geometric types of possible collections of cylinders, and not for the collections themselves.

In this work, we are only concerned with multiplicity one configurations, that is, those defining a single cylinder. We are also concerned with some homological conditions —and not only the geometric combinatorics— when considering configurations (see § 3). However, this information is also carried by configurations because of topological considerations.

REMARK 2.6. Let $\mathcal{C}$ be a configuration of cylinders and consider now $N_{\mathcal{C}}(X, L)$, the number of cylinders in $X$ of length at most $L$ forming a configuration of type $\mathcal{C}$. Then, the analogous of Theorem 2.3 is also true in this context (see [**EMa, EMZ**]), with the Siegel-Veech constant associated to this counting problem depending also on the configuration, $c_{\mathcal{C}}(\mathcal{M}) = c(\mathcal{C}, \mathcal{M})$.

**2.3. Configuration of cylinders in genus zero and associated Siegel–Veech constants.** Boissy [**Bo**] described all generic configurations of cylinders for flat surfaces in genus zero and Athreya–Eskin–Zorich [**AEZ**] provided the values of the corresponding Siegel–Veech constants. In this section we recall briefly this results (cf. [**DZ**, Section 4.2]).

According to [**Bo**] and [**MZ**], for almost any flat surface in any stratum of meromorphic quadratic differentials with at most simple poles on the sphere , different from $\mathcal{Q}(-1^4)$, every single regular closed geodesic corresponds to one of the two configurations described below.

**Pocket configurations.** These configurations are defined by single cylinders bounded by a saddle connection joining a fixed pair of poles $P_{j_1}, P_{j_2}$ and by a saddle connection joining a fixed zero $P_i$ of order $d_i \geq 1$ to itself (see Figure 3). By convention, the holonomy associated to these configurations corresponds to closed geodesics and not to the saddle connection joining the two poles, which is twice as short as the closed geodesic.

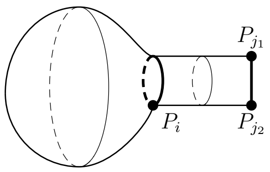



FIGURE 3. A pocket configuration formed by cylinders bounded by a saddle connection joining two fixed poles on one side and by a saddle connection joining a fixed zero to itself on the other.

The Siegel–Veech constant $c^{\text{pocket}}_{j_1,j_2;i}$ corresponding to these configurations has the form ([**AEZ**, Theorem 4.5])

$$c^{\text{pocket}}_{j_1,j_2;i} = \frac{d_i+1}{k-4}\frac{1}{2\pi^2}.$$

If we consider the union of several pocket configurations, fixing the poles $P_{j_1}, P_{j_2}$ and considering any zero $P_i$ on the boundary of the cylinder, then the resulting Siegel–Veech constant $c^{\text{pocket}}_{j_1,j_2}$ corresponding to this configuration has the form ([**AEZ**, Corollary 4.7])

$$c^{\text{pocket}}_{j_1,j_2} = \frac{1}{2\pi^2}. \tag{3}$$

**Dumbbell configurations.** In this case, we still have a single cylinder, which is now bounded by saddle connections joining a fixed zero to itself on each side. Say, a saddle connection joining the fixed zero $P_{i_1}$ of order $d_{i_1} \geq 1$ to itself and the other, joining the fixed zero $P_{i_2}$ of order $d_{i_2} \geq 1$ to itself (see Figure 4). Such a cylinder separates the original surface W in two flat spheres. Let $P_{i_{11}}, \ldots, P_{i_{1k_1}}$ be the singularities (zeros and poles) on one part and $P_{i_{21}}, \ldots, P_{i_{2k_2}}$, the rest. In particular, we have $i_1 \in \{i_{11}, \ldots, i_{1k_1}\}$ and $i_2 \in \{i_{21}, \ldots, i_{2k_2}\}$. All this information is carried by the configuration.

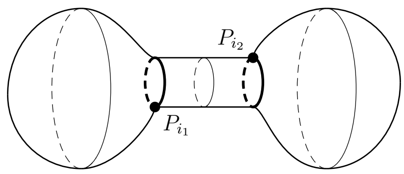



FIGURE 4. A dumbbell configuration, consisting of two flat spheres joined by a cylinder whose boundary components are saddle connections joining a zero to itself.

Denoting by $d_i$ the order of the singularity $P_i$, we can represent the sets (with multiplicities) of orders of all zeros and poles $\alpha := \{d_1, \dots, d_k\}$ as a disjoint union of the two subsets

$$\alpha = \{d_{i_{11}}, \dots, d_{i_{1k_1}}\} \sqcup \{d_{i_{21}}, \dots, d_{i_{2k_2}}\} =: \alpha_1 \sqcup \alpha_2.$$

The corresponding Siegel–Veech constant $c^{\text{dumbbell}}_{i_1,i_2;\alpha_1,\alpha_2}$ is given by ([**AEZ**, Theorem 4.8],

$$c^{\text{dumbbell}}_{i_1,i_2;\alpha_1,\alpha_2} = (d_{i_1}+1)(d_{i_2}+1)\frac{(k_1-3)!(k_2-3)!}{(k-4)!}\frac{1}{2\pi^2} \tag{4}$$

**2.4. From billiards to flat surfaces.** Recall that in the classical case of a billiard in a rectangle we can glue a flat torus out of four copies of the billiard table and unfold billiard trajectories to flat geodesics of the same length on the resulting flat torus.

**Wind-tree model.** The wind-tree model corresponds to a billiard $\Pi$ in the plane endowed with $\mathbb{Z}^2$-periodic horizontally and vertically symmetric right-angled obstacles, where the sides of the obstacles are aligned along the lattice as in Figure 1 and Figure 2.

In the case of the wind-tree model we also start from gluing a flat surface out of four copies of the infinite billiard table $\Pi$. The resulting surface $\mathrm{X}_\infty = \mathrm{X}_\infty(\Pi)$ is $\mathbb{Z}^2$-periodic with respect to translations by vectors of the original lattice. Passing to the $\mathbb{Z}^2$-quotient we get a compact flat surface $\mathrm{X} = \mathrm{X}(\Pi)$. For the case of the original wind-tree billiard, with rectangular obstacles, the resulting flat surface is represented at Figure 5. It has genus 5 and belongs to the stratum $\mathcal{H}(2^4)$ (see [**DHL**, § 3] for details).

Similarly, when the obstacle has $4m$ corners with the angle $\pi/2$ —and $4(m-1)$ with angle $3\pi/2$—, the same construction gives a flat surface of genus $4m+1$ in $\mathcal{H}(2^{4m})$, consisting in four flat tori with holes (four copies of a $\mathbb{Z}^2$ fundamental domain of $\Pi$, the holes corresponding to the obstacles) with corresponding identifications, as in the classical setting ($m = 1$, see Figure 5). Let $\mathcal{WT}(m)$ denote the set of wind-tree billiards $\Pi$ whose obstacles have $4m$ corners with angle $\pi/2$. The space $\mathcal{WT}(m)$ has a natural Lebesgue measure coming from the consideration of lengths and position of the sides of the obstacle. The construction $\Pi \mapsto \mathrm{X}(\Pi)$ defines a map $\mathcal{WT}(m) \to \mathcal{H}(2^{4m})$ and we define $\mathcal{B}(m)$ to be the image of this map, that is, the set of all compact surfaces $\mathrm{X}(\Pi)$ such that $\Pi \in \mathcal{WT}(m)$, and we consider in $\mathcal{B}(m)$ the pushforward of the measure on $\mathcal{WT}(m)$.

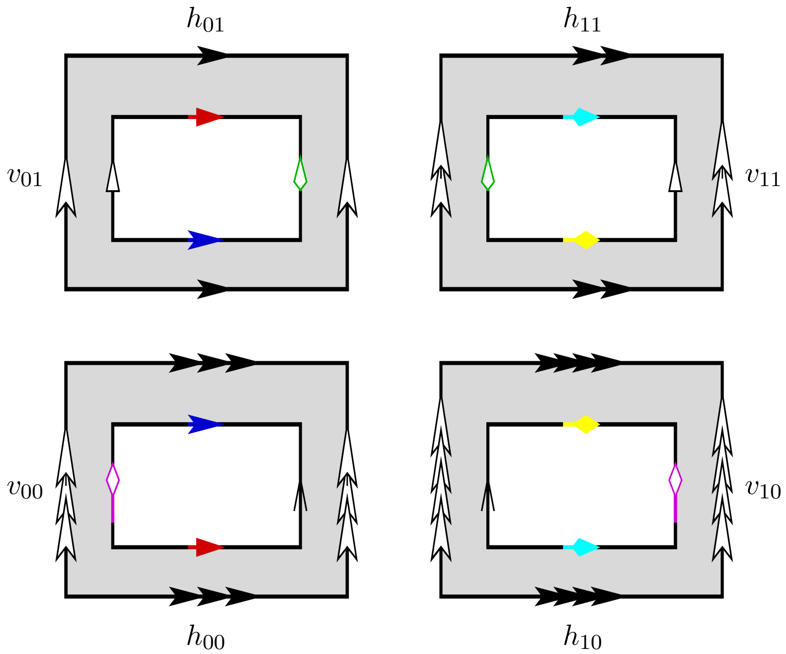



FIGURE 5. The flat surface X obtained as quotient over $\mathbb{Z}^2$ of an unfolded wind-tree billiard table ([**DZ**, Figure 5]).

Note that any resulting flat surface $X \in \mathcal{B}(m)$ has (at least) the group $(\mathbb{Z}_2)^3$ as a group of isometries. We have the isometry $\tau_h$, interchanging the pairs of flat tori with holes in the same rows by parallel translations, the isometry $\tau_v$, interchanging columns, and $\iota$, the isometry acting on each of the four tori with holes as the central symmetry with the center in the center of the hole (rotation by $\pi$).

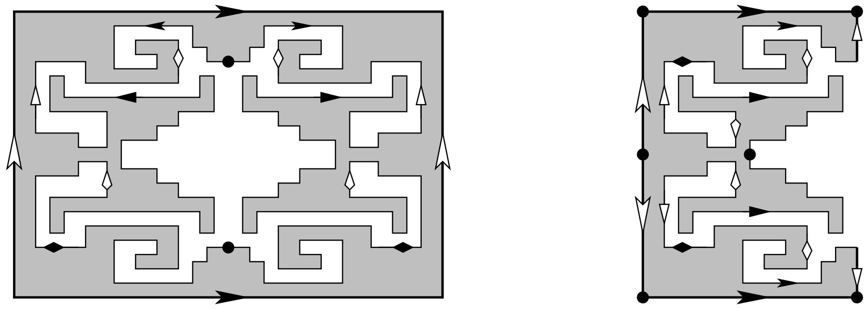
Consider the quotient $W_h$ of the flat surface X over the subgroup $(\mathbb{Z}_2)^2$ of isometries spanned by $\tau_h$ and $\iota \circ \tau_v$. The resulting surface $W_h$ (see Figure 6a) belongs to the stratum $\mathcal{Q}(1^{2m}, -1^{2m})$. In particular, it has genus 1, $W_h = (E_h, q_h)$. Similarly, $W_v = X/\langle \tau_v, \iota \circ \tau_h \rangle = (E_v, q_v) \in \mathcal{Q}(1^{2m}, -1^{2m})$. The surface W obtained as the quotient of the original flat surface X over the entire group $(\mathbb{Z}_2)^3$ (see Figure 6b) belongs to the stratum $\mathcal{Q}(1^m, -1^{m+4})$. In particular, it has genus zero, $W = (\mathbb{CP}^1, q)$. Clearly, $W_h$ and $W_v$ are ramified double covers over W with ramification points at four (out of $m+4$) simple poles of the flat surface W (see [**DZ**, § 3.1, 3.2] for details). Moreover, $W_h$ and $W_v$ share three out of their four ramified simple poles.

Furthermore, the isometries $\tau_h$ and $\tau_v$ decompose the Hodge bundle over $\mathcal{M}$. In fact, we have that

$$H^1_X = E^{++} \oplus E^{+-} \oplus E^{-+} \oplus E^{--},$$

where $E^{++}$ is the vector space invariant by $\tau_h$ and $\tau_v$, $E^{+-}$ the vector space invariant by $\tau_h$ and anti-invariant by $\tau_v$, etc. This decomposition is flat, defined over $\mathbb{Z}$ and symplectic; each subbundle is symplectic and the sum is orthogonal with respect to the intersection form.

Consider now the cohomology classes $h, v \in H^1(X, \mathbb{Z})$ Poincaré-dual to the cycles $h_{00} - h_{01} + h_{10} - h_{11}$ and $v_{00} - v_{10} + v_{01} - v_{11}$ respectively (see Figure 5) as elements of the fiber over the point X of the (real) Hodge bundle $H^1$ over the $SL(2, \mathbb{R})$-orbit



(a) $\mathrm{W}_h = (E_h, q_h) \in \mathcal{Q}(1^{2m}, -1^{2m})$. (b) $\mathrm{W} = (\mathbb{CP}^1, q) \in \mathcal{Q}(1^m, -1^{m+4})$.

FIGURE 6. The flat surface $\mathrm{W}_h$ is a double cover over the underlying surface W branched at the four simple poles represented by bold dots ([**DZ**, Figure 7]).

closure of $\mathrm{X} \in \mathcal{B}(m)$. The pair $(h, v) \in H^1(\mathrm{X}, \mathbb{Z}^2)$ defines the $\mathbb{Z}^2$-covering $\mathrm{X}_\infty$ of X and the coordinates of this $\mathbb{Z}^2$-cocycle defining $\mathrm{X}_\infty$ belong to $E^{+-} \oplus E^{-+}$, more precisely, we have that $h \in E^{+-}$ and $v \in E^{-+}$.

We further consider $F^{+-} \subset E^{+-}$, the vector space invariant by $\tau_h$ and $\iota \circ \tau_v$, which is naturally isomorphic to the Hodge bundle over the genus one flat surface $\mathrm{W}_h = (E_h, q_h)$. Then, $F^{+-}$ is a two dimensional, defined over $\mathbb{Z}$, flat —it is locally defined by two cocycles in $H^1(\mathrm{X}, \mathbb{Z})$ and the Gauss–Manin connection— and symplectic subbundle of the Hodge bundle. In particular, it is continuous and equivariant (invariant with respect to the Kontsevich–Zorich cocycle). Analogously, we consider $F^{-+} \subset E^{-+}$, the vector space invariant by $\tau_v$ and $\iota \circ \tau_h$, with the analogous properties. We have that $h \in F^{+-}$ and $v \in F^{-+}$ (see [**DZ**, Lemma 3.1]).

THEOREM 2.7 (Delecroix–Zorich [**DZ**]). *For almost every billiard* $\Pi \in \mathcal{WT}(m)$, *the* $\mathrm{GL}(2,\mathbb{R})$*-orbit closure of* $\mathrm{W}(\Pi)$ *coincides with the whole stratum* $\mathcal{Q}(1^m, -1^{m+4})$ *and the Lyapunov exponents on the* $\mathrm{SL}(2,\mathbb{R})$*-orbit closure of* $\mathrm{X}(\Pi)$ *over the subbundles* $F^{+-}$ *and* $F^{-+}$ *are* $\pm\delta(m)$*, where*

$$\delta(m) = \frac{(2m)!!}{(2m+1)!!} = 4^m \frac{(m!)^2}{(2m+1)!} > 0.$$

Here, the double factorial means the product of all even (correspondingly odd) natural numbers from 2 to $2m$ (correspondingly from 1 to $2m+1$). For the original wind-tree model, that is, when $m = 1$, this was first shown by Delecroix–Hubert–Lelièvre [**DHL**]. In this case we have, in particular, that $F^{+-} = E^{+-}$, $F^{-+} = E^{-+}$ and $\delta(1) = 2/3$.

Since the subbundles $F^{+-}$ and $F^{-+}$ have non-zero Lyapunov exponents and are 2-dimensional, they are irreducible and then, symplectic (see Remark 2.2).

In this work, we are concerned with counting closed trajectories in the wind-tree billiard. Obviously, any closed trajectory can be translated by an element in $\mathbb{Z}^2$ to obtain a new closed trajectory. Then, we shall count (isotopy classes of) closed trajectories of bounded length in the wind-tree billiard up to $\mathbb{Z}^2$-translations. There is a one to one correspondence between billiard trajectories in $\Pi$ and geodesics in $\mathrm{X}_\infty$. But $\mathrm{X}_\infty$ is the $\mathbb{Z}^2$-covering of X given by $h, v \in H^1(\mathrm{X}, \mathbb{Z})$, which means that closed curves $\gamma$ in X lift to closed curves in $X_\infty$ if and only if $\langle \gamma, h\rangle = \langle \gamma, v\rangle = 0$. This is a general fact about $\mathbb{Z}^d$-periodic flat surfaces.

## 3. Counting problem in $\mathbb{Z}^d$-periodic flat surfaces

We consider an infinite $\mathbb{Z}^d$-periodic flat surface $X_\infty$ which is a ramified cover over a compact flat surface $X = (S, \omega)$, the covering group being $\mathbb{Z}^d$, $d \geq 1$. Let $\Sigma$ be the finite set of singularity points of $X$. Since the intersection form $\langle \cdot, \cdot\rangle$ is non-degenerate between $H^1(S\backslash\Sigma, \mathbb{Z})$ and $H^1(S, \Sigma, \mathbb{Z})$, every such $\mathbb{Z}^d$-cover is defined by a $d$-tuple of independent elements $\mathbf{f} = (f_1, \dots, f_d)$ in the group of relative cohomology $H^1(S, \Sigma, \mathbb{Z})$.

We are interested in counting cylinders in $X_\infty$ modulo $\mathbb{Z}^d$-translations. Cylinders in the cover $X_\infty$ clearly descends to cylinders in $X$, but not the other way around. In fact, by definition of the covering, the monodromy of a closed curve $\gamma$ is translation by $(\langle \gamma, f_i\rangle)_{i=1}^d \in \mathbb{Z}^d$. It follows that cylinders in the cover $X_\infty$ are exactly the lift of those cylinders $C$ in $X$ such that its core curve $\gamma_C$ verifies $\langle \gamma_C, f_i\rangle = 0$, for each $i = 1, \dots, d$. Note that, in this case, the monodromy is always trivial and cylinders in $X_\infty$ are always isometric to their projection on $X$. When a cylinder $C$ does not satisfy this condition, it lifts to $X_\infty$ as a strip, isometric to the product of an open interval and a straight line.

We restrict ourselves to the case when $\mathbf{f}$ is an absolute covector, that is, it is a $d$-tuple of independent elements in the group of *absolute* cohomology $H^1(S, \mathbb{Z})$. Let $\mathcal{M}$ be the $\mathrm{SL}(2, \mathbb{R})$-orbit closure of $X$, $F$ be an equivariant subbundle of the Hodge bundle over $\mathcal{M}$ and $f \in F_X$.

Note that cylinders $C$ in $X$ such that $\langle \gamma_C, f\rangle = 0$, split naturally into two families: (a) the family of cylinders such that $\langle \gamma_C, h\rangle = 0$ for all $h \in F_X$, which we call *$F$-good cylinders*, and (b) the family of cylinders that are not $F$-good, but $\langle \gamma_C, f\rangle = 0$. These later are called *$(F, f)$-bad cylinders*. The notion of $F$-good cylinders was first introduced by Avila–Hubert [**AH**] in order to give a geometric criterion for recurrence of $\mathbb{Z}^d$-periodic flat surfaces.

Thus, counting cylinders in the $\mathbb{Z}^d$-periodic flat surface can be reduced to counting separately cylinders which are $(\oplus_j F^{(j)})$-good cylinders and $(F^{(j_i)}, f_i)$-bad cylinders in the compact surface, for some appropriate subbundles $(F^{(j)})_j$.

REMARK 3.1. When $F$ is symplectic, in particular, if $\Lambda(F) \neq \{0\}$ (see Remark 2.2), $F$-good cylinders are exactly those that $\mathrm{pr}_{F_X}\gamma_C = 0$. If, in addition, $F$ is 2-dimensional (in particular, irreducible if $\Lambda(F) \neq \{0\}$), $C$ is an $(F, f)$-bad cylinder if and only if $\mathrm{pr}_{F_X}\gamma_C \neq 0$ is colinear to $f$.

Since the Kontsevich–Zorich cocycle preserves the intersection form and $F$ is equivariant, it is clear that the set of $F$-good cylinders is $\mathrm{SL}(2, \mathbb{R})$-equivariant. Then, classical results can be applied. In particular, applying the main result of [**EMa**], if there is at least one $F$-good cylinder in $X$, then we can deduce that $F$-good cylinders have quadratic asymptotic growth rate (with positive Siegel–Veech constant) for $\nu_{\mathcal{M}}$-almost every flat surface in $\mathcal{M}$, the $\mathrm{SL}(2, \mathbb{R})$-orbit closure of $X$. However, this is no longer true in the case of $(F, f)$-bad cylinders.

For $f \in F_X$ define the set $V_F(f)$ of holonomy vectors of $(F, f)$-bad cylinders in $X$. We have that $V_F(A(g, X)f) = gV_F(f)$, since $F$ is equivariant and the Kontsevich-Zorich cocycle respects the intersection form. Finally, let

$$N_F(f, L) = \#V_F(f) \cap B(L)$$

be the number of $(F, f)$-bad cylinders in $X$ of length bounded by $L$.

## 4. Bad cylinders have subquadratic asymptotic growth rate

In this section, we prove the following general result about bad cylinders which applies to *some* $\mathbb{Z}^d$-periodic flat surfaces and, in particular, to the family of wind-tree models we are interested in.

THEOREM 4.1. *Let $X$ be a flat surface and $F$ a 2-dimensional equivariant continuous subbundle of the Hodge bundle on $\mathcal{M}$, the $\mathrm{SL}(2, \mathbb{R})$-orbit closure of $X$. Suppose that $F$ is defined over $\mathbb{Z}$ and has non-zero Lyapunov exponents. Then, for all $f \in F_X$ the number $N_F(f, L)$, of $(F, f)$-bad cylinders in $X$ of length at most $L$, has subquadratic asymptotic growth rate, that is, $N_F(f, L) = o(L^2)$ or, which is the same,*

$$\lim_{L \to \infty} \frac{N_F(f, L)}{\pi L^2} = 0.$$

REMARK 4.2. When $F$ is 2-dimensional, symplectic (in particular, when it has non-zero Lyapunov exponents) and defined over $\mathbb{Z}$, if $f \in F_X$ is not colinear to an integer cocycle, then, there are no $(F, f)$-bad cylinders, since $\mathrm{pr}_{F_X} \gamma_C$ is always a rational multiple of an integer cocycle. Since the notion of bad cylinder is clearly projective, the proof of Theorem 4.1 is then reduced to prove the conclusion only for $f \in F_X(\mathbb{Z})$, instead that for all $f \in F_X$.

To prove Theorem 4.1 we use technology for asymptotic formulas for counting closed geodesics developed by Eskin–Masur [**EMa**]. In particular, the following proposition, which is a restatement of Proposition 3.5 and Lemma 8.1 in [**EMa**], is a key step in the proof.

PROPOSITION 4.3 (Eskin–Masur). *Let $\mathcal{V} \subset \mathbb{R}^2 \setminus \{0\}$, define $\mathcal{N}(\mathcal{V}, T) := \#\mathcal{V} \cap B(T)$ and suppose that $\mathcal{N}(\mathcal{V}, T) < \infty$ for all $T > 0$. Then, for all $\rho, t > 0$*

$$\mathcal{N}(\mathcal{V}, 2\rho e^t) - \mathcal{N}(\mathcal{V}, \rho e^t) \leq c(\rho) e^{2t} \int_0^{2\pi} \mathcal{N}(g_t r_\theta \mathcal{V}, 4\rho) \mathrm{d}\theta.$$

Hence, the proof of Theorem 4.1 is reduced to show the following.

THEOREM 4.4. *Under the hypothesis of Theorem 4.1, for every $f \in F_X(\mathbb{Z})$ and all $\rho > 0$,*

$$\lim_{t \to \infty} \int_0^{2\pi} N_F(A(g_t r_\theta, X)f, \rho) \mathrm{d}\theta = 0.$$

PROOF OF THEOREM 4.1. It is clear that $V_F(\cdot) \subset \mathbb{R}^2 \setminus \{0\}$ is $\mathrm{SL}(2, \mathbb{R})$-equivariant and $N_F(f, L)$ is finite, since it is bounded by $N(X, L)$, the number of all cylinders of length bounded by $L$ and $N(X,L) \leq c(X)L^2$ ([**Ma90**]). Then, by Proposition 4.3, we have that, for all $f \in F_X(\mathbb{Z})$, all $\rho > 0$ and all $t > 0$,

$$N_F(f, 2\rho e^t) - N_F(f, \rho e^t) \leq c(\rho)e^{2t} \int_0^{2\pi} N_F(A(g_t r_\theta, X)f, 4\rho)\mathrm{d}\theta.$$

But then, by Theorem 4.4,

$$\limsup_{t\to\infty} \frac{N_F(f, 2\rho e^t) - N_F(f, \rho e^t)}{\rho^2 e^{2t}} \leq \frac{c(\rho)}{\rho^2} \lim_{t\to\infty} \int_0^{2\pi} N_F(A(g_t r_\theta, X)f, 4\rho)\mathrm{d}\theta = 0.$$

That is

$$\limsup_{T\to\infty} \frac{N_F(f, 2T) - N_F(f,T)}{T^2} = 0.$$

It follows that

$$\begin{aligned}
\bar{c}_F(f) \coloneqq \limsup_{L\to\infty} \frac{N_F(f,L)}{\pi L^2} &= \limsup_{T\to\infty} \frac{1}{4\pi} \frac{N_F(f,2T)}{T^2} \\
&= \frac{1}{4\pi} \limsup_{T\to\infty} \left( \frac{N_F(f,2T) - N_F(f,T)}{T^2} + \frac{N_F(f,T)}{T^2} \right) \\
&\leq \frac{1}{4\pi} \left( \limsup_{T\to\infty} \frac{N_F(f,2T) - N_F(f,T)}{T^2} + \limsup_{T\to\infty} \frac{N_F(f,T)}{T^2} \right) \\
&= \frac{1}{4\pi} \left(0 + \bar{c}_F(f)\right) = \frac{1}{4\pi} \bar{c}_F(f)
\end{aligned}$$

and then, $\bar{c}_F(f) = 0$. We conclude that

$$\lim_{L\to\infty} \frac{N_F(f,L)}{\pi L^2} = 0.$$

□

**4.1. Proof of Theorem 4.4.** In order to show that

$$\lim_{t\to\infty} \int_0^{2\pi} N_F(A(g_t r_\theta, X)f, \rho)\mathrm{d}\theta = 0,$$

we split the integral in whether $g_t r_\theta X \in K_\epsilon = \{\mathrm{sys} \geq \epsilon\}$ or not, and show that both parts tend to zero as $t$ tends to infinity and $\epsilon$, to zero.

When $g_t r_\theta X \in K_\epsilon$, the corresponding part of the integral tends to zero as a consequence of the following proposition, whose proof is postponed to § 4.2.

PROPOSITION 4.5. *Under the hypothesis of Theorem 4.4, for all $f \in F_X(\mathbb{Z})$, all $\rho, \epsilon > 0$ and almost every $\theta$*

$$N_F(A(g_t r_\theta, X)f, \rho) \cdot \mathbf{1}_{K_\epsilon}(g_t r_\theta X) = 0$$

*for sufficiently large $t$, $t \geq t_0(x, \rho, \epsilon, \theta)$.*

REMARK 4.6. The intuition behind this apparently technical proposition is the following. By hypothesis, the Lyapunov exponent of $f \in F_X(\mathbb{Z})$ is positive and then, for almost every $\theta$, $A(g_t r_\theta, X)f$ becomes very long for large $t$. Without loss of generality, we can suppose that $f$ is primitive. Therefore, no short cycle (of length bounded by $\rho$) can have projection on $F_X$ colinear to $A(g_t r_\theta, X)f$, because this latter is primitive and longer. We formalize this idea in § 4.2.

Recall that $N_F(f,L) \leq N(X,L)$. Furthermore, $N(\cdot,\rho)$ is bounded in $K_\epsilon$. Indeed, by Theorem 2.4, for $\delta = 1$,

$$\mathbf{1}_{K_\epsilon} N(\cdot,\rho) \leq \mathbf{1}_{K_\epsilon} \frac{c(\rho,1)}{\mathrm{sys}^2} \leq \frac{c(\rho,1)}{\epsilon^2} = c(\rho,\epsilon).$$

Then, for fixed $\rho, \epsilon > 0$,

$$\int_0^{2\pi} N_F(A(g_t r_\theta, X)f,\rho) \cdot \mathbf{1}_{K_\epsilon}(g_t r_\theta X) \mathrm{d}\theta \leq c(\rho,\epsilon) \cdot |\{\theta \in [0,2\pi) : N_F(A(g_t r_\theta, X)f,\rho) \cdot \mathbf{1}_{K_\epsilon}(g_t r_\theta X) \neq 0\}|,$$

where $|\cdot|$ is the Lebesgue measure on $[0,2\pi)$. Finally, by Proposition 4.5, the right side of the inequality tends to zero as $t$ tends to infinity. That is,

$$\lim_{t\to\infty} \int_0^{2\pi} N_F(A(g_t r_\theta, X)f,\rho) \cdot \mathbf{1}_{K_\epsilon}(g_t r_\theta X) \mathrm{d}\theta = 0. \tag{5}$$

For the rest of the integral we use the following.

Lemma 4.7. *For any flat surface $X$, any $\beta < 2$ and all $\epsilon > 0$,*

$$|\{\theta \in [0,2\pi) : \mathrm{sys}(g_t r_\theta X) < \epsilon\}| < c(X,\beta)\epsilon^\beta$$

*for all $t > 0$.*

Proof.

$$\begin{aligned} |\{\theta \in [0,2\pi) : \mathrm{sys}(g_t r_\theta X) < \epsilon\}| &= \int_0^{2\pi} \mathbf{1}_{\mathrm{sys}<\epsilon}(g_t r_\theta X) \mathrm{d}\theta \\ &\leq \int_0^{2\pi} \mathbf{1}_{\mathrm{sys}<\epsilon}(g_t r_\theta X) \cdot \frac{\epsilon^\beta}{\mathrm{sys}(g_t r_\theta X)^\beta} \mathrm{d}\theta \\ &\leq \epsilon^\beta \int_0^{2\pi} \frac{\mathrm{d}\theta}{\mathrm{sys}(g_t r_\theta X)^\beta} \end{aligned}$$

Then, by Theorem 2.5, we conclude that

$$|\ \{\theta \in [0,2\pi) : \mathrm{sys}(g_t r_\theta X) < \epsilon\}| \leq c(X,\beta)\epsilon^\beta.$$

□

Moreover, since $N_F(f,\rho) \leq N(X,\rho)$ and, by Theorem 2.4, for any $\delta > 0$

$$N(X,\rho) \leq \frac{c(\delta,\rho)}{\mathrm{sys}(X)^{1+\delta}},$$

it follows that

$$\begin{aligned}
&\int_0^{2\pi} N_F(A(g_t r_\theta, X)f, \rho)\cdot \mathbf{1}_{\mathrm{sys}<\epsilon}(g_t r_\theta X)\mathrm{d}\theta \\
&\qquad\le \sum_{n=0}^{\infty}\int_0^{2\pi} N(g_t r_\theta X, \rho)\cdot \mathbf{1}_{\mathrm{sys}\in\left[\frac{\epsilon}{2^{n+1}},\frac{\epsilon}{2^n}\right)}(g_t r_\theta X)\mathrm{d}\theta \\
&\qquad\le c(\delta,\rho)\sum_{n=0}^{\infty}\int_0^{2\pi} \frac{1}{\mathrm{sys}(g_t r_\theta X)^{1+\delta}}\cdot \mathbf{1}_{\mathrm{sys}\in\left[\frac{\epsilon}{2^{n+1}},\frac{\epsilon}{2^n}\right)}(g_t r_\theta X)\mathrm{d}\theta \\
&\qquad\le c(\delta,\rho)\sum_{n=0}^{\infty}\int_0^{2\pi} \frac{1}{\left(\frac{\epsilon}{2^{n+1}}\right)^{1+\delta}}\cdot \mathbf{1}_{\mathrm{sys}\in\left[\frac{\epsilon}{2^{n+1}},\frac{\epsilon}{2^n}\right)}(g_t r_\theta X)\mathrm{d}\theta \\
&\qquad\le c(\delta,\rho)\sum_{n=0}^{\infty}\frac{2^{(n+1)(1+\delta)}}{\epsilon^{1+\delta}}\int_0^{2\pi} \mathbf{1}_{\mathrm{sys}<\frac{\epsilon}{2^n}}(g_t r_\theta X)\mathrm{d}\theta \\
&\qquad\le c(\delta,\rho)\sum_{n=0}^{\infty}\frac{2^{(n+1)(1+\delta)}}{\epsilon^{1+\delta}}|\{\theta\in[0,2\pi) : \mathrm{sys}(g_t r_\theta X) < \frac{\epsilon}{2^n}\}|.
\end{aligned}$$

Then, by Lemma 4.7, for $1+\delta<\beta<2$,

$$\begin{aligned}
\lim_{t\to\infty}\int_0^{2\pi} N_F(A(g_t r_\theta, X)f, \rho)\cdot \mathbf{1}_{\mathrm{sys}<\epsilon}(g_t r_\theta X)\mathrm{d}\theta &\le c(\delta,\rho)\sum_{n=0}^{\infty}\frac{2^{(n+1)(1+\delta)}}{\epsilon^{1+\delta}}c(X,\beta)\frac{\epsilon^\beta}{2^{n\beta}} \\
&\le c(\delta,\rho,X,\beta)\epsilon^{\beta-(1+\delta)}. \qquad (6)
\end{aligned}$$

Joining both parts of the integral, (5) and (6), we obtain that, for every $\epsilon,\delta,\rho>0$, $f\in F_X(\mathbb{Z})$ and $1+\delta<\beta<2$,

$$\lim_{t\to\infty}\int_0^{2\pi} N_F(A(g_t r_\theta, X)f, \rho)\mathrm{d}\theta \le 0 + c(\delta,\rho,X,\beta)\epsilon^{\beta-(1+\delta)}.$$

Then, fixing $\rho>0$, $0<\delta<1$ and $1+\delta<\beta<2$, and letting $\epsilon\to 0$, we conclude that

$$\lim_{t\to\infty}\int_0^{2\pi} N_F(A(g_t r_\theta, X)f, \rho)\mathrm{d}\theta = 0.$$

□

**4.2. Proof of Proposition 4.5.** The first step is to show that, for a cylinder, being bounded in length implies having bounded projection in $F_X$.

LEMMA 4.8. *Let $\rho>0$ and $K\subset\mathcal{M}$ be a compact subset. Then, for all $X'\in K$ and all cylinder $C$ on $X'$ such that $|\mathrm{hol}_{\omega'}\gamma_C|\le\rho$ we have that*

$$\|\mathrm{pr}_{F_{X'}}[\gamma_C]\|_{\omega'}\le c(\rho,K,F).$$

PROOF. Let $\mathbf{C}(\rho,X')$ be the finite set of cylinders on $X'$ of length at most $\rho$. Then, $c_0(\rho,X',F)=\max\{\|\mathrm{pr}_{F_{X'}}[\gamma]\|_{\omega'} : C\in\mathbf{C}(\rho,X')\}$ is finite.

Define $\Gamma(\rho,X')=\{\gamma_C : C\in\mathbf{C}(\rho,X')\}$. Then, since $F$ is continuous, $\mathrm{pr}_{F_{(\cdot)}}(\cdot)$ is continuous and since the Hodge norm $\|\cdot\|_{(\cdot)}$ is continuous, there is a neighborhood $U(X')$ of $X'$ in $\mathcal{M}$ such that, for all $\bar{X}=(\bar{S},\bar{\omega})\in U(X')$,

- $\Gamma(\rho,\bar{X})\subset\Gamma(2\rho,X')$ (after local identification), and

- $\|\mathrm{pr}_{F_{\bar{X}}} \cdot \|_{\bar{\omega}} \leq 2\|\mathrm{pr}_{F_{X'}} \cdot \|_{\omega'}$.

Therefore, if $\bar{C}$ is a cylinder in $\bar{X} \in U(X')$ with $|\mathrm{hol}_{\bar{\omega}} \gamma_{\bar{C}}| \leq \rho$, then

$$\|\mathrm{pr}_{F_{\bar{X}}}[\gamma_{\bar{C}}]\|_{\bar{\omega}} \leq 2\|\mathrm{pr}_{F_{X'}}[\gamma_{\bar{C}}]\|_{\omega'} \leq 2c_0(2\rho, X', F) =: c(\rho, X', F).$$

Since $U(X')$ is open and $K$ is compact, there is a finite set $A \subset K$ such that $K \subset \cup_{X' \in A} U(X')$. We conclude, taking $\max_{X' \in A} c(\rho, X', F)$ to be $c(\rho, K, F)$. □

Since $F$ is 2-dimensional and has non-zero Lyapunov exponents, it is symplectic and its Lyapunov spectrum is symmetric (see Remark 2.2), say $\Lambda(\mathcal{M}, F) = \{\pm\lambda\}$, $\lambda > 0$. Moreover, since $f \in F_X(\mathbb{Z})$ is an integer covector, its associated Lyapunov exponent has to be positive. Then, for almost every $\theta$, we have that

$$\lim_{t \to \infty} \frac{\log \|A(g_t r_\theta, X) f\|_{g_t r_\theta \omega}}{t} = \lambda > 0,$$

in particular, for almost every $\theta$ and sufficiently large $t$, $t \geq t_0(r_\theta X, f)$,

$$\|A(g_t r_\theta, X) f\|_{g_t r_\theta \omega} \geq e^{\frac{\lambda}{2} t}. \tag{7}$$

Recall that, since $F$ is defined over $\mathbb{Z}$, $F_X^{\mathrm{pr}}(\mathbb{Z}) = \mathrm{pr}_{F_X} H^1_X(\mathbb{Z})$ is a lattice and $F_X(\mathbb{Z}) \subset F_X^{\mathrm{pr}}(\mathbb{Z})$. Let $m = m(f)$ be a positive integer such that $\frac{1}{m} f$ is a primitive element in the lattice $F_X^{\mathrm{pr}}(\mathbb{Z})$, and let $c(\rho, \epsilon, F)$ be the constant given by Lemma 4.8 for $K = K_\epsilon$. Then, for large $t$, $t \geq t_0(\epsilon, \rho, f)$,

$$e^{\frac{\lambda}{2} t} > m(f) c(\rho, \epsilon, F). \tag{8}$$

Therefore, putting (7) and (8) together, for almost every $\theta$ and all $t$ sufficiently large, $t \geq t_0(\epsilon, \rho, \theta, X, f)$, we have that

$$\|A(g_t r_\theta, X) f\|_{g_t r_\theta \omega} \geq e^{\frac{\lambda}{2} t} > m(f) c(\rho, \epsilon, F).$$

Fix $\theta$ and $t$ as before, consider $X_t = g_t r_\theta X$, $\omega_t = g_t r_\theta \omega$ and $f_t = A(g_t r_\theta, X) f$, and suppose that $X_t \in K_\epsilon$. Now, if $\gamma$ is the core curve of a cylinder in $X_t$ such that $|\mathrm{hol}_{\omega_t} \gamma| \leq \rho$, then

$$\|\mathrm{pr}_{F_{X_t}}[\gamma]\|_{\omega_t} \leq c(\rho, \epsilon, F) < \frac{1}{m} \|f_t\|_{\omega_t},$$

where the first inequality is given by Lemma 4.8, for $X' = X_t$ and $K = K_\epsilon$.

Recall that under our hypothesis, an $(F, f_t)$-bad cylinder $C$ in $X_t$ has to verify that $\mathrm{pr}_{F_{X_t}}[\gamma_C] \neq 0$ is colinear to $f_t$ (see Remark 3.1). But no element in $F_{X_t}^{\mathrm{pr}}(\mathbb{Z})$ colinear to $f_t$ can be shorter than $\frac{1}{m} f_t$, since this last is primitive in the lattice $F_{X_t}^{\mathrm{pr}}(\mathbb{Z})$, by definition of $m$ and, evidently, $\mathrm{pr}_{F_{X_t}}[\gamma]$ belongs to $F_{X_t}^{\mathrm{pr}}(\mathbb{Z})$

Then $\gamma$, as before, cannot be the core curve of an $(F, f_t)$-bad cylinder in $X_t$. And thus, $N_F(A(g_t r_\theta, X) f, \rho) = N_F(f_t, \rho) = 0$, for $\theta$ and $t$ as before. That is, for all $f \in F_X(\mathbb{Z})$, all $\rho, \epsilon > 0$ and almost every $\theta$

$$N_F(A(g_t r_\theta, X) f, \rho) \cdot \mathbf{1}_{K_\epsilon}(X_t) = 0$$

for sufficiently large $t$, $t \geq t_0(x, \rho, \epsilon, \theta)$. □

## 5. Application to wind-tree models

In this section we apply previous discussion to wind-tree models. As we have seen, there is an identification between cylinders (up to $\mathbb{Z}^2$-translations) in the infinite billiard $\Pi \in \mathcal{WT}(m)$ and the union of $(F^{+-} \oplus F^{-+})$-good cylinders, $(F^{+-}, h)$-bad cylinders and $(F^{-+}, v)$-bad cylinders in $\mathrm{X} = \mathrm{X}(\Pi) \in \mathcal{B}(m)$. Moreover, the subbundles $F^{+-}$ and $F^{-+}$ are always 2 dimensional flat subbundles defined over $\mathbb{Z}$ and, by Theorem 2.7, we know that for almost every $\mathrm{X} \in \mathcal{B}(m)$, $\Lambda(\mathcal{M}, F^{+-}) = \Lambda(\mathcal{M}, F^{-+}) = \{\pm\delta(m)\}$, where $\mathcal{M}$ is the $\mathrm{SL}(2,\mathbb{R})$-orbit closure of X and $\delta(m) > 0$. In particular, for almost every $\mathrm{X} \in \mathcal{B}(m)$, $F^{+-}$ and $F^{-+}$ satisfy the hypothesis of Theorem 4.1.

This suffices for the almost everywhere statement of Theorem 1.1, but it does not for the everywhere statement of Theorem 1.2. However, an adaptation of Forni's criterion [**Fo**] allows us to prove that the top Lyapunov exponents of $F^{+-}$ and $F^{-+}$ are in fact positive.

THEOREM 5.1 (Forni's criterion for integer equivariant subbundles). *Let $\mathcal{M}$ be an affine invariant manifold and $F$ be an equivariant subbundle of the Hodge bundle on $\mathcal{M}$ defined over $\mathbb{Z}$. Suppose that there exists a flat surface $X \in \mathcal{M}$ and a family of parallel closed geodesics in $X$ such that the space generated by the (Poincaré dual of the) homology classes of these closed geodesics is a subspace of $F_X$ of dimension $d \geq 1$. Then, the top $d$ Lyapunov exponents on $F$ are strictly positive, that is,*

$$\lambda_1(\mathcal{M}, F) \geq \cdots \geq \lambda_d(\mathcal{M}, F) > 0.$$

PROOF. The proof follows as the original proof of [**Fo**, Theorem 1.6]. In fact, as communicated to as by C. Matheus, the main steps of the proof are:

(1) [**Fo**, § 3]: The unstable bundle of the Kontsevich–Zorich cocycle is $\nu_{\mathcal{M}}$-almost everywhere transverse to all integral isotropic subspaces (see [**Fo**, Lemma 3.1]). In our case, we can restrict the unstable bundle to the equivariant subbundle $F$ and this statement remains true since the subbundle $F$ is defined over $\mathbb{Z}$.
(2) [**Fo**, § 4]: $d \times d$-block of the second fundamental form converges to $-\mathrm{Id}$ along an isotropic subspace transverse to the (Poincaré dual of the) $d$-dimensional subspace generated by the closed geodesics (see [**Fo**, Lemma 4.4]). This remains true when restricting to the subbudle $F$; the proof relies only on classical formulas for the period matrix near the boundary of the Deligne–Mumford compactification of the moduli space of abelian differentials (see [**Fo**, Lemma 4.1]).
(3) [**Fo**, § 5]: Finally, the proof of [**Fo**, Theorem 1.6] remains valid since the argument combines the two previous points with a hypothesis of local product structure, which is always true after Eskin–Mirzakhani [**EMi**].

□

COROLLARY 5.2. *For every* $\mathrm{X} \in \mathcal{B}(m)$*, the subbundles $F^{+-}$ and $F^{-+}$ defined on the* $\mathrm{SL}(2,\mathbb{R})$*-orbit closure of* X*, satisfy the hypothesis of Theorem 4.1.*

PROOF. We already know that the subbundles $F^{+-}$ and $F^{-+}$ are 2 dimensional flat subbundles defined over $\mathbb{Z}$. Then, it remains to prove that they have non-zero Lyapunov exponents.

Let $F_{\mathrm{X}}$ be the (Poincaré dual of the) symplectic subspace generated by cycles $h_{00}, h_{10}, h_{01}, h_{11}, v_{00}, v_{10}, v_{01}, v_{11}$ (see Figure 5). This defines a flat (that is, a locally constant) subbundle of the Hodge bundle, which is clearly defined over $\mathbb{Z}$. Moreover, $F$ has rank 8 and is symplectic. In particular, its Lyapunov spectrum is symmetric. Taking the closed geodesics given by $h_{00}, h_{10}, h_{01}, h_{11}$, which are horizontal and homologically independent, and applying Theorem 5.1, we conclude that $F$ has 4 positive Lyapunov exponents and therefore all eight Lyapunov exponents are non-zero. Finally, we note that $F^{+-}$ and $F^{-+}$ are subbundles of $F$ and, in particular, their Lyapunov spectra are contained in the one of $F$. Thus, they have non-zero Lyapunov exponents. □

Thus, by Theorem 4.1, $(F^{+-}, h)$-bad cylinders and $(F^{-+}, v)$-bad cylinders in X have subquadratic asymptotic growth rate, proving Theorem 1.3. Thus, asymptotic formulas for the wind-tree model correspond to those of $(F^{+-} \oplus F^{-+})$-good cylinders. In particular, this justifies why we can conclude Theorem 1.2, so we have weak asymptotic formulas for *every* wind-tree model.

For simplicity, henceforth, we will call simply good cylinders the $(F^{+-} \oplus F^{-+})$-good cylinders, and by bad cylinders we will refer to $(F^{+-}, h)$ and $(F^{-+}, v)$-bad cylinders.

As a direct consequence of Theorem 1.3 and an adapted version of Theorem 2.3 (see Remark 2.6), we have the following.

Corollary 5.3. *For almost every wind-tree billiard $\Pi \in \mathcal{WT}(m)$, the number $N(\Pi, L)$ of closed billiard trajectories of length bounded by $L$ in $\Pi$ (up to isotopy and $\mathbb{Z}^2$-translations) has quadratic asymptotic growth rate,*

$$N(\Pi, L) \sim \frac{1}{4} c_{good}(\mathcal{M}) \frac{\pi L^2}{\mathrm{Area}\,(\Pi/\mathbb{Z}^2)},$$

*where $c_{good}(\mathcal{M})$ is the Siegel-Veech constant associated to the counting problem of good cylinders in $\mathcal{M}$, the* $\mathrm{SL}(2,\mathbb{R})$*-orbit closure of* $\mathrm{X}(\Pi)$.

The factor 1/4 coming from the fact that $\mathrm{Area}\,(\mathrm{X}(\Pi)) = 4 \cdot \mathrm{Area}\,(\Pi/\mathbb{Z}^2)$.

In addition, a cylinder in X is a good cylinder if (and only if) the homology class of its core curve projects trivially to $F^{+-}$ and to $F^{-+}$ (see Remark 3.1). We have also the following useful characterization of good cylinders (see Figure 7 for notation).

Lemma 5.4. *Let $C$ be a cylinder in* X. *Then $C$ is a good cylinder in* X *if and only if the core curve of $C$ projects to homologically trivial curves in* $\mathrm{W}_h$ *and* $\mathrm{W}_v$.

Proof. Let $\gamma$ be the core curve of $C$. Then $C$ is an $F^{+-}$-good cylinder in X if and only if $\mathrm{pr}_{F^{+-}}[\gamma] = 0$. But $F^{+-}$ is naturally isomorphic to $H^1(\mathrm{W}_h)$ by the pushforward of the covering map $\mathbf{p}_h$. Then $\mathrm{pr}_{F^{+-}}[\gamma] = 0$ if and only if $\mathbf{p}_{h*}[\gamma] = [\mathbf{p}_h\gamma] = 0$. Analogously, the same holds for $F^{-+}$ and $\mathrm{W}_v$. And good cylinders are exactly those which are $F^{+-}$ and $F^{-+}$-good cylinders. □

Then, good cylinders in X are exactly those which project to homologically trivial cylinders in the flat surfaces $\mathrm{W}_h$ and $\mathrm{W}_v$. Cylinders in X also project to the flat surface W, of genus zero. The $\mathrm{SL}(2,\mathbb{R})$-orbit closure $\mathcal{M}$ of X projects to the $\mathrm{SL}(2,\mathbb{R})$-orbit closure $\mathcal{L}$ of W, and for almost every $\mathrm{X} \in \mathcal{B}(m)$, $\mathbb{R}\mathcal{L}$ coincides

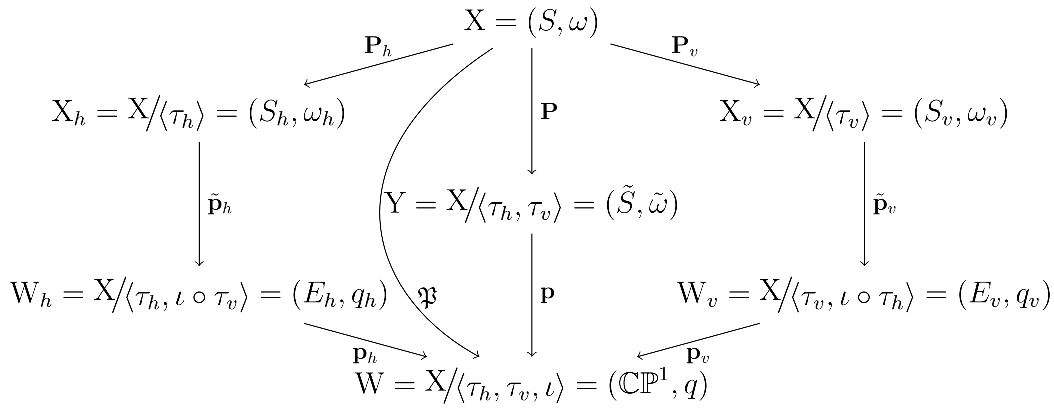



FIGURE 7. Surfaces and covering maps notation

with the whole stratum $\mathcal{Q}(1^m, -1^{m+4})$ ([**DZ**, Proposition 2]). Moreover, we have seen in § 2.3 that generic flat surfaces in $\mathcal{Q}(1^m, -1^{m+4})$ have only two types of configurations of cylinders, the so called pocket and dumbbell configurations. But generic flat surfaces are not pertinent to our study. In fact, the set of flat surfaces $\mathrm{W} \in \mathcal{Q}(1^m, -1^{m+4})$ coming from wind-tree billiards is negligible. However, we have the following.

PROPOSITION 5.5. *For almost any wind-tree billiard* $\Pi \in \mathcal{WT}(m)$ *the following property holds. Consider a cylinder in* $\mathrm{W}(\Pi) = \mathrm{X}(\Pi)/\langle \iota, \tau_h, \tau_v \rangle$ *and suppose it is not horizontal nor vertical. Then, the cylinder make part of one of the configurations described in § 2.3, that is, a pocket or a dumbbell configuration.*

PROOF. See [**AEZ**, Proposition 2.2] (the proof of which mimics the proof of [**EMZ**, Theorem 7.4]). □

COROLLARY 5.6. *For almost every wind-tree billiard* $\Pi \in \mathcal{WT}(m)$,

$$c_{good}(\mathcal{M}) = c_{good}^{pocket}(\mathcal{M}) + c_{good}^{dumbbell}(\mathcal{M}),$$

*where* $c_{good}^{pocket}(\mathcal{M})$ *(resp.* $c_{good}^{dumbbell}(\mathcal{M})$*) corresponds to the Siegel–Veech constant associated to the counting problem of configurations of good cylinders in* $\mathcal{M}$*, the* $\mathrm{SL}(2,\mathbb{R})$*-orbit closure of* $\mathrm{X}(\Pi)$*, such that those configurations project to pocket (resp. dumbbell) configurations in* $\mathcal{Q}(1^m, -1^{m+4})$.

It follows that the study of configurations of cylinders on generic flat surfaces in $\mathcal{Q}(1^m, -1^{m+4})$ suffices for our purposes.

## 6. Configurations of good cylinders

Here we show which conditions a cylinder in $\mathrm{W} = (\mathbb{CP}^1, q) \in \mathcal{L} = \mathcal{Q}(1^m, -1^{m+4})$ has to satisfy so that it lifts to a good cylinder in $\mathrm{X} = (S, \omega) \in \mathcal{M}$, and then we interpret this in terms of configurations of generic surfaces of genus zero, that is, pocket and dumbbell configurations (see § 2.3).

Recall that, by Lemma 5.4, a cylinder in X is good if it projects to a homologically trivial cylinder in the surfaces $\mathrm{W}_h$ and $\mathrm{W}_v$, of genus 1. Then, our classification will consist in finding the configurations on W which lift to homologically trivial closed geodesics in $W_h$ and $W_v$.

Since there are clear analogies between objects with subindex $h$ and subindex $v$ (see Figure 7), in this section we will use the label $\circ$ for both labels $h$ and $v$. Thus, any result in terms of labels $\circ$ will give the corresponding result for $h$ and $v$.

**6.1. Cylinders in W who lift to good cylinders in X.** Let $C$ be a cylinder in the genus zero surface W. Then, since all curves are homologically trivial on W, the core curve of $C$, say $\gamma$, cuts the surface in two components, say $W_1$ and $W_2$.

For our purposes here, the only relevant information about $C$ we need, is the number $q_l$ of cone singularities of angle $3\pi$ and the number $r_l$ of ramified poles in $W_l$ for the double cover $\mathbf{p}_\circ : W_\circ \to W$, $l = 1, 2$. The number $p_l$, of unramified poles for $\mathbf{p}_\circ$ in $W_l$ is also relevant, but since $W_l$ is a genus zero surface with only simple zeros and poles, and a single boundary component, then

$$4g(W_l) - 4 = -4 = q_l - p_l - r_l - 2,$$

and $p_l$ can be written in terms of $q_l$ and $r_l$ as $p_l = q_l - r_l + 2$, $l = 1, 2$. Also, $q_2 = m - q_1$ and $r_2 = 4 - r_1$, so we will only consider $r = r_1$ and $q = q_1$.

Remark that the number $r$ depends on the configuration as well as on the double cover $\mathbf{p}_\circ$ (of which there are two, $\mathbf{p}_h$ and $\mathbf{p}_v$), while $q$ does not depend on the double cover. Call then, the former number $r_\circ = r(C, \mathbf{p}_\circ)$. Furthermore, since $W_1$ and $W_2$ were arbitrarily chosen, we can fix them such that $r_\circ = r_1 \leq r_2$. Note that $|r_h - r_v| \leq 1$, since three out of four ramified poles are shared by both covering maps. In particular, we can always choose $W_1$ and $W_2$ coherently such that $r_\circ = r_{\circ 1} \leq r_{\circ 2}$, for both coverings. Furthermore, there is only one way to do this unless $r_h = r_v = 2$. Note that with this setting, $r_h, r_v \in \{0, 1, 2\}$. Call $W' = W_2$ and $W'_\circ = \mathbf{p}_\circ^{-1} W'$, and recall that $\mathbf{p}_{\circ *} : \pi_1(W_\circ) \to \pi_1(W)$ is the pushforward of the projection $\mathbf{p}_\circ : W_\circ \to W$, which sends closed curves in $W_\circ$ to closed curves in W. In particular, $b_\circ = \#\mathbf{p}_{\circ *}^{-1}(\gamma)$ is the number of curves (connected components) in $\mathbf{p}_\circ^{-1}(\gamma)$, and $b_\circ \in \{1, 2\}$, since $\mathbf{p}_\circ$ is a double cover.

Remark 6.1. In particular, the number $b_\circ$ corresponds to the number of boundary components of the surface $W'_\circ$. This number also defines the monodromy of the core curve of $C$, $\gamma$, for $\mathbf{p}_\circ$. In fact, $b_\circ = 2$ means that $\gamma$ has two $\mathbf{p}_{\circ *}$-preimages and, since $\mathbf{p}_\circ$ is a double cover, this gives trivial monodromy. While non trivial monodromy, and equals to $\mathbb{Z}_2$, arises when $b_\circ = 1$.

Lemma 6.2. *Let $C$ be a cylinder in* W*, $\gamma$ its core curve and consider $b_\circ = \#\mathbf{p}_{\circ *}^{-1}(\gamma)$. Then, $b_\circ = 4 - r_\circ - 2g(W'_\circ)$. In particular, $b_\circ \equiv r_\circ \mod 2$.*

Proof. Clearly, $W'$ has one boundary component, which is equal $\gamma$. Note that $b_\circ$ is the number of boundary components of $W'_\circ$, $b_\circ = \#\mathbf{p}_{\circ *}^{-1}(\gamma) \in \{1, 2\}$.

In $W'$, there are $4 - r_\circ$ ramified and $m - (q - r_\circ + 2)$ unramified poles for $\mathbf{p}_\circ$, and $m - q$ simple zeros. Thus, we have $2(m - q + 2 - r_h)$ poles and $2(m - q)$ simple zeros in $W'_\circ$. But then,

$$4g(W'_\circ) - 4 = 2(m - q) - 2(m - q + 2 - r_\circ) - 2b_\circ,$$

That is, $b_\circ = 4 - r_\circ - 2g(W'_\circ)$ and, in particular, $b_\circ \equiv r_\circ \mod 2$. □

Proposition 6.3. *Let $C$ be a cylinder in* W*. Then $C$ lifts to good cylinders in* X *if and only if $r_h, r_v \in \{0,1\}$.*

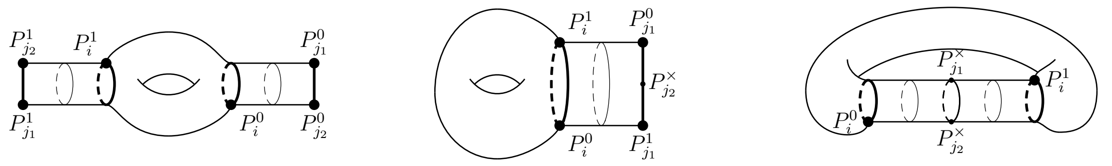
Proof. Let $\gamma$ be the core curve of $C$. Then, we want to show that if $\gamma_\circ \in \mathbf{p_\circ}_*^{-1}(\gamma)$, $[\gamma_\circ] = 0$ if and only if $r_\circ \neq 2$. Note that, since $g(\mathrm{W}_\circ) = 1$, a homologically trivial curve always cut the surface into a genus zero surface and a genus one surface.

As before, let $\mathrm{W}' = \mathrm{W}_2$ and $\mathrm{W}'_\circ = \mathbf{p}_\circ{}^{-1}\mathrm{W}'$. By the previous lemma, we know that $\#\mathbf{p_\circ}_*^{-1}(\gamma) = b_\circ = 4 - r_\circ - 2g(\mathrm{W}'_\circ)$, $b_\circ \equiv r_\circ \mod 2$. Then,

- If $r_\circ = 0$, then $b_\circ = 2$ and $g(\mathrm{W}'_\circ) = 1$. That is, $\gamma$ has two $\mathbf{p_\circ}_*$-preimages ($b_\circ = 2$) bounding a genus one surface ($g(\mathrm{W}'_\circ) = 1$) in $\mathrm{W}_\circ$. But $g(\mathrm{W}_\circ) = 1$, and therefore both $\mathbf{p_\circ}_*$-preimages of $\gamma$ are homologically trivial (see, e.g., Figure 8a and Figure 9a).
- When $r_\circ = 1$, we have $b_\circ = 1$ and $g(\mathrm{W}'_\circ) = 1$. It follows that $\gamma$ has one $\mathbf{p_\circ}_*$-preimage which is homologically trivial (see, e.g., Figure 8b and Figure 9b).
- Finally, if $r_\circ = 2$, then $b_\circ = 2$ and $g(\mathrm{W}'_\circ) = 0$. Therefore, $\gamma$ has two $\mathbf{p_\circ}_*$-preimages and both together bounds each of two genus zero surfaces which form the whole surface $\mathrm{W}_h$ of genus one (see, e.g., Figure 8c and Figure 9c). Then, both preimages of $\gamma$ are not homologically trivial.

□

Thus, we know which cylinders in W lift to good cylinders in X. It remains to see how these cylinders lift, that is, the number of cylinders in X we obtain and their length.




(a) $r_\circ = 0$. A torus with two "pockets".

(b) $r_\circ = 1$. A torus with a "pocket" twice longer.

(c) $r_\circ = 2$. A torus with a non homologically trivial cylinder.

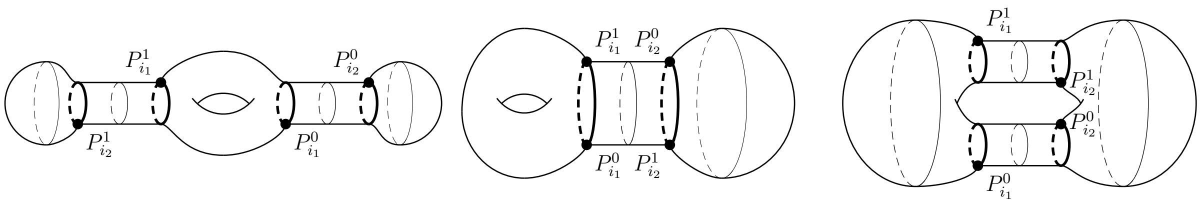
Figure 8. Possible liftings for $\mathbf{p}_\circ$ of a pocket configuration.

**6.2. How cylinders in W lift to good cylinders in X.** Here we show how lift to X those cylinders in W who lift to good cylinders in X. More precisely, we determine the number of cylinders in X we obtain and their length. To do this, we will lift one by one the covering maps $\mathbf{p}_\circ : \mathrm{W}_\circ \to \mathrm{W}$, then $\tilde{\mathbf{p}}_\circ : \mathrm{X}_\circ \to \mathrm{W}_\circ$ and finally $\mathbf{P}_\circ : \mathrm{X} \to \mathrm{X}_\circ$ (see Figure 7). Recall we are using the label $^\circ$ instead of $h$ and $v$.

The following is a direct consequence of Remark 6.1 and Lemma 6.2.

Lemma 6.4. *Let $C$ be a cylinder in* W*. Then, the core curve $\gamma$ of $C$ has trivial monodromy for $\mathbf{p}_\circ$ if $r_\circ \neq 1$, and equals to $\mathbb{Z}_2$, if $r_\circ = 1$.*

Proof. From Remark 6.1, we know that the number $b_\circ$ defines the monodromy of $\gamma$, being trivial for $b_\circ = 2$ and equals to $\mathbb{Z}_2$ when $b_\circ = 1$. But, by Lemma 6.2, we also know that $b_\circ \equiv r_\circ \mod 2$, and $r_\circ \in \{0,1,2\}$. □




(a) $r_\circ = 0$. A torus joined to two flat spheres by homologically trivial cylinders.

(b) $r_\circ = 1$. A torus joined to a flat spheres by a homologically trivial cylinder twice longer.

(c) $r_\circ = 2$. Two flat spheres joined by two non homologically trivial cylinders.

FIGURE 9. Possible liftings for $\mathbf{p}_\circ$ of a dumbbell configuration.

The meaning of previous lemma can be noticed in Figure 8 and Figure 9.

LEMMA 6.5. *Let $C_\circ$ be a cylinder in $\mathrm{W}_\circ$ such that $r_\circ(\mathbf{p}_\circ(C_\circ)) \neq 2$. Then, the core curve of $C_\circ$ has trivial monodromy for $\tilde{\mathbf{p}}_\circ : \mathrm{X}_\circ \to \mathrm{W}_\circ$.*

PROOF. Let $\gamma_\circ$ be the core curve of $C_\circ$. Since $r_\circ(\mathbf{p}_\circ(C_\circ)) \neq 2$, by Proposition 6.3 and Lemma 5.4, $\gamma_\circ$ is homologically trivial. Then, it cuts the surface $\mathrm{W}_\circ$ in two components. Let $\mathrm{W}^{\backprime}_\circ$ be one of these two components and consider $\mathrm{X}^{\backprime}_\circ = \tilde{\mathbf{p}}_\circ^{-1}\mathrm{W}^{\backprime}_\circ$.

Let $q^{\backprime}$ be the number of double zeros and $b^{\backprime}$, the number of boundary components, on $\mathrm{X}^{\backprime}_\circ$. Then, $4g(\mathrm{X}^{\backprime}_\circ) - 4 = 4q^{\backprime} - 2b^{\backprime}$, and $b^{\backprime} \equiv 0 \mod 2$. That is, $b^{\backprime} = 2$ and $\gamma_\circ$ has two $\tilde{\mathbf{p}}_{\circ *}$-preimages. Since $\tilde{\mathbf{p}}_\circ$ is a double cover, then $\gamma_\circ$ has trivial monodromy. □

Thus, the possible $\tilde{\mathbf{p}}_\circ$-liftings in the surface $\mathrm{X}_\circ$ of a cylinder $C_\circ$ in the surface $\mathrm{W}_\circ$ (with $r_\circ(\mathbf{p}_\circ(C_\circ)) \neq 2$) looks like as in Figure 10 or Figure 11.

(a) $r_\circ = 0$.

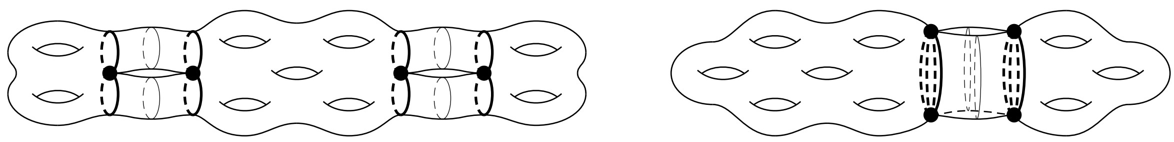
(b) $r_\circ = 1$.

FIGURE 10. Possible $\tilde{\mathbf{p}}_\circ$-liftings in $\mathrm{X}_\circ$ of cylinders in $\mathrm{W}_\circ$ coming from a pocket configuration in W.

Finally, we can describe how cylinders in W lift to good cylinders in X. Recall that $\mathfrak{P} : \mathrm{X} \to \mathrm{W}$ is a covering of degree 8.

LEMMA 6.6. *Let $C$ be a cylinder in* W *and $\gamma$ be its core curve. Suppose that $r_h, r_v \in \{0, 1\}$. Then,*

*(1) If $r_h = r_v = 0$, then $\gamma$ has trivial monodromy for $\mathfrak{P}$. In particular, $\gamma$ has eight $\mathfrak{P}_*$-preimages of the same length than $\gamma$.*



(a) $r_\circ = 0$. (b) $r_\circ = 1$.

FIGURE 11. Possible $\tilde{\mathbf{p}}_\circ$-liftings in $\mathrm{X}_\circ$ of cylinders in $\mathrm{W}_\circ$ coming from a dumbbell configuration in W.

*(2) In any other case, $\gamma$ has monodromy $\mathbb{Z}_2$ for $\mathfrak{P}$. In particular, $\gamma$ has four $\mathfrak{P}_*$-preimages twice longer than $\gamma$.*

PROOF. Recall first that $\mathfrak{P} : \mathrm{X} \to \mathrm{W}$ is a covering of degree 8, $\mathfrak{P} = \mathbf{p}_\circ \circ \tilde{\mathbf{p}}_\circ \circ \mathbf{P}_\circ$ and also $\mathfrak{P} = \mathbf{p} \circ \mathbf{P}$, where $\mathbf{P} : \mathrm{X} \to \mathrm{Y}$ and $\mathbf{p} : \mathrm{Y} \to \mathrm{W}$ (see the diagram in Figure 7 for a recall in notation).

(1) Suppose $r_h = r_v = 0$. By Lemma 6.4, we know that $\gamma$ has trivial monodromy for both $\mathbf{p}_h$ and $\mathbf{p}_v$. Then, by Lemma 6.5, we deduce that $\gamma$ has trivial monodromy for $\mathbf{p}_h \circ \tilde{\mathbf{p}}_h$ and for $\mathbf{p}_v \circ \tilde{\mathbf{p}}_v$. Then, the monodromy of $\gamma$ for $\mathfrak{P} = \mathbf{p}_\circ \circ \tilde{\mathbf{p}}_\circ \circ \mathbf{P}_\circ$ can be at most $\mathbb{Z}_2$, since $\mathbf{P}_\circ : \mathrm{X} \to \mathrm{X}_\circ$ is a double cover.

Suppose it is $\mathbb{Z}_2$. Then, the monodromy for $\mathbf{P}_\circ$ of the corresponding curves $\bar{\gamma}_{\circ i}$, $i = 1, \ldots, 4$, in $\mathrm{X}_\circ$ is $\mathbb{Z}_2$. This means, in particular, that $\tau_h$ and $\tau_v$ fix the corresponding curves $\bar{\gamma}_i$, $i = 1, \ldots, 4$, in X. Consider $D = \mathbf{P}_*(\{\bar{\gamma}_i\}_{i=1}^4)$ and note that $D = \mathbf{p}_*^{-1}(\gamma)$. Then, since $\tau_h$ and $\tau_v$ fix each $\bar{\gamma}_i$, $i = 1, \ldots, 4$, we have that $\#D = 4$, but $\mathbf{p}$ is a double cover, so this is impossible. Thus, assuming that the monodromy for $\mathfrak{P}$ of $\gamma$ is $\mathbb{Z}_2$, we get a contradiction. Therefore, the monodromy is trivial (see Figure 12a and Figure 13a).

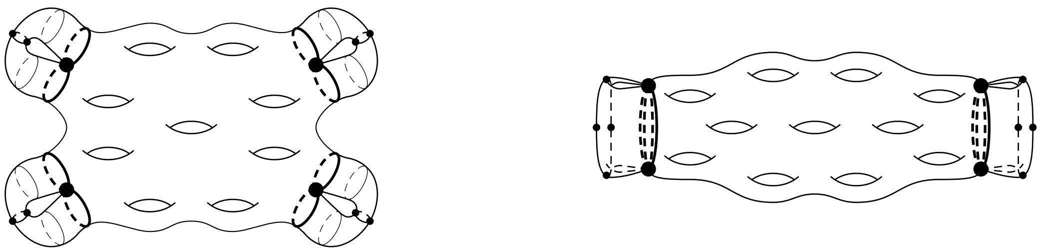
(2) For the other cases, we will prove that $\gamma$ has monodromy $\mathbb{Z}_2$. Remember we are assuming that $r_h, r_v \neq 2$.

(a) Suppose $r_h = r_v = 1$. From Lemma 6.4 we know that $\gamma$ has monodromy $\mathbb{Z}_2$ for both $\mathbf{p}_h$ and $\mathbf{p}_v$. Then, by Lemma 6.5, we deduce that $\gamma$ has monodromy $\mathbb{Z}_2$ for $\mathbf{p}_h \circ \tilde{\mathbf{p}}_h$ and for $\mathbf{p}_v \circ \tilde{\mathbf{p}}_v$. Then, the monodromy of $\gamma$ for $\mathfrak{P} = \mathbf{p}_\circ \circ \tilde{\mathbf{p}}_\circ \circ \mathbf{P}_\circ$ can be $\mathbb{Z}_2$ or $\mathbb{Z}_4$, since $\mathbf{P}_\circ$ is a double cover.

Suppose it is $\mathbb{Z}_4$. Then, the monodromy for $\mathbf{P}_\circ$ of the corresponding curves $\bar{\gamma}_{\circ i}$, $i = 1, 2$, in $\mathrm{X}_\circ$ is $\mathbb{Z}_2$, and $\tau_h$ and $\tau_v$ fix each $\bar{\gamma}_i$, $i = 1, 2$ in X. To continue with the argument, we need to remark first that $\tau_h$ and $\tau_v$ are orientation preserving isometric involutions. Then, when they fix a cylinder, the only way to do this is, either being the identity or a rotation by half the length of the cylinder, when restricted to the cylinder. In particular, $\check{\gamma}_i := \mathfrak{P}(\bar{\gamma}_i) = \bar{\gamma}_i / \langle \tau_h, \tau_v \rangle$ has at least half the length of $\bar{\gamma}_i$, $i = 1, 2$, that is, at least twice the length of $\gamma$. But $\check{\gamma}_i \in \mathbf{p}_*^{-1}(\gamma)$, $i = 1, 2$, and $\mathbf{p}$ is a double cover, so it is impossible to have two $\mathbf{p}$-preimages of at least twice the length. Thus, assuming that the monodromy of $\gamma$ for $\mathfrak{P}$ is $\mathbb{Z}_4$, we get a contradiction. Therefore, the monodromy is $\mathbb{Z}_2$ (see Figure 12b and Figure 13b).

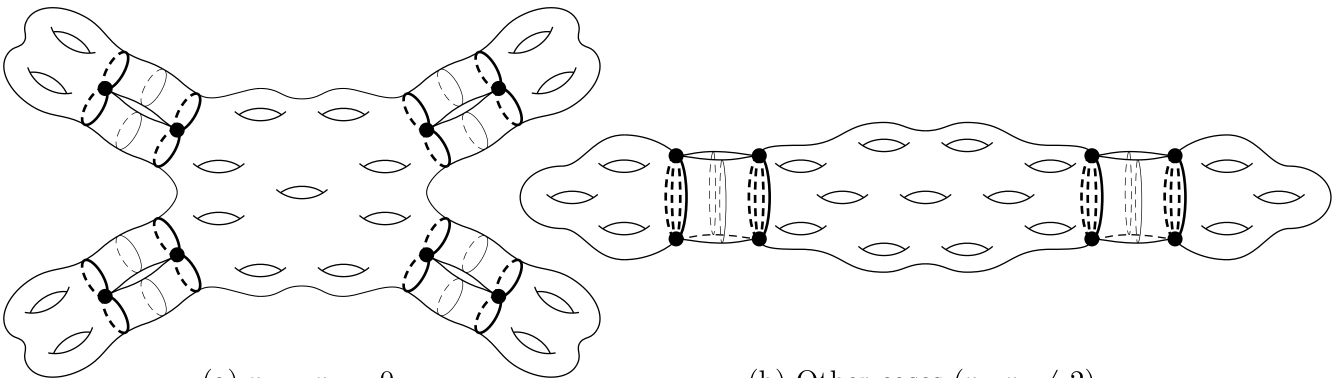
(b) Suppose that $r_h = 0$ and $r_v = 1$. Then, as before, we find that $\gamma$ has trivial monodromy for $\mathbf{p}_h \circ \tilde{\mathbf{p}}_h$, and monodromy $\mathbb{Z}_2$ for $\mathbf{p}_v \circ \tilde{\mathbf{p}}_v$. Then, since $\mathbf{P}_h$ and $\mathbf{P}_v$ are double covers, $\gamma$ has trivial or $\mathbb{Z}_2$ monodromy for $\mathbf{p}_h \circ \tilde{\mathbf{p}}_h \circ \mathbf{P}_h$ and monodromy $\mathbb{Z}_2$ or $\mathbb{Z}_4$ for $\mathbf{p}_v \circ \tilde{\mathbf{p}}_v \circ \mathbf{P}_v$. But $\mathbf{p}_h \circ \tilde{\mathbf{p}}_h \circ \mathbf{P}_h = \mathbf{p}_v \circ \tilde{\mathbf{p}}_v \circ \mathbf{P}_v = \mathfrak{P}$, and therefore, the only alternative is to have monodromy equals to $\mathbb{Z}_2$ (see Figure 12a and Figure 13a). Analogously, we have monodromy $\mathbb{Z}_2$ for $r_h = 1$ and $r_v = 0$.

□



(a) $r_h = r_v = 0$.  (b) Other cases $(r_h, r_v \neq 2)$.

FIGURE 12. Lifting of a pocket configuration in W to X.



(a) $r_h = r_v = 0$.  (b) Other cases $(r_h, r_v \neq 2)$.

FIGURE 13. Lifting of a dumbbell configuration in W to X.

**6.3. Relation between Siegel-Veech constants in $\mathcal{Q}(1^m, -1^{m+4})$ and its lifting to $\mathcal{M}$.** We conclude the study of which and how cylinders in W lift to good cylinders in X by relating the Siegel-Veech constants of configurations in W and its liftings to X.

Let $\mathcal{L}$ be an invariant affine submanifold in $\mathcal{Q}(1^m, -1^{m+4})$ and let $\mu$ be the associated affine invariant measure on $\mathcal{L}$. Consider the locus $\mathcal{M}$ of all possible $\mathfrak{P}$-covers surfaces from $\mathcal{L}$. Note that, by construction, this gives an $\mathrm{SL}(2, \mathbb{R})$-equivariant one-to-one correspondence between $\mathcal{L}$ and $\mathcal{M}$. In particular, $\mathcal{M}$ is an affine invariant submanifold on $\mathcal{H}(2^{4m})$. Let $\nu$ be the affine invariant measure on $\mathcal{M}$. Note that that $\mu$ is the direct image of $\nu$ with respect to the projection $\mathcal{M} \to \mathcal{L}$.

Let $c = c_{\mathcal{C}}(\mathcal{L})$ be the Siegel-Veech constant associated to the counting of a multiplicity one configuration $\mathcal{C}$ of cylinders in $\mathcal{L}$ (see § 2.2 for the definitions). Then, the configuration $\mathcal{C}$ induces a cylinder configuration $\bar{\mathcal{C}}$ on the covering space $\mathcal{M}$, defined by the covering maps $\mathfrak{P}$. Let $\bar{c} = c_{\bar{\mathcal{C}}}(\mathcal{M})$ be the associated Siegel-Veech constant. The lemma below relates $c$ and $\bar{c}$. It is the analogous of Lemma 1.1 in [**EKZ**] and Lemma 4.1 in [**DZ**], adapted for our purposes.

We say that $\mathcal{C}$ is a *pocket-like* configuration, if the singularities in one of the boundary components of the cylinder are only poles. Note that, in particular, there are exactly two poles in that boundary component. Denote by $r_h(\mathcal{C})$ and $r_v(\mathcal{C})$ the values of $r_h$ and $r_v$ in the cylinders defined by configuration $\mathcal{C}$. These values are well defined, since a configuration defines all that data. Call the pair $(r_h, r_v)$ the *profile of the configuration* $\mathcal{C}$. We say that $\mathcal{C}$ is a *good configuration* if it is a multiplicity one configuration of cylinders in $\mathcal{L}$ such that $r_h(\mathcal{C}), r_v(\mathcal{C}) \in \{0, 1\}$.

LEMMA 6.7. *Let $\mathcal{C}$ be a good configuration.*

*(1) If $\mathcal{C}$ is pocket-like, then*
   *(a) If $\mathcal{C}$ has profile $(0,0)$, then $\bar{c} = 32c$.*
   *(b) In any other case, $\bar{c} = 4c$.*
*(2) If $\mathcal{C}$ is not pocket-like, then*
   *(a) If $\mathcal{C}$ has profile $(0,0)$, then $\bar{c} = 64c$.*
   *(b) In any other case, $\bar{c} = 8c$.*

PROOF. First of all, suppose we know the exact number and the relative length of cylinders in X we obtain by lifting a cylinder from configuration $\mathcal{C}$ in W. Say, a cylinder from $\mathcal{C}$ in W is lifted to $n$ cylinders in X and their lengths are $s$ times the length of $\gamma$. Then,

$$N_{\bar{\mathcal{C}}}(\mathrm{X}, L) = nN_{\mathcal{C}}\left(\mathrm{W}, s^{-1}L\right)$$

and therefore,

$$\bar{c} = \frac{n}{s^2}\frac{\mathrm{Area(X)}}{\mathrm{Area(W)}}c = 8\frac{n}{s^2}c,$$

where we used the fact that Area(X) = 8Area(W), since X is a metric 8-fold covering of W. But we know, by Lemma 6.6, the exact number of $\mathfrak{P}_*$-preimages of the core curve of $C$, $\gamma$, and the relative length of these, depending on $r_h$ and $r_v$.

If $\mathcal{C}$ is not a pocket-like configuration, then, there is at least one singularity in each boundary of the cylinder in W which is not a pole. Then, for each $\mathfrak{P}_*$-preimage, $\bar{\gamma}$, of its core curve $\gamma$, there is a cylinder in X with core curve $\bar{\gamma}$ (see Figure 13). Thus, the values of $n$ and $s$ are given by Lemma 6.6. That is, $n = 8$ and $s = 1$ for profile $(0,0)$, and $n = 4$, $s = 2$, for all other profiles of good configurations.

In the case of pocket-like configurations, the poles defining the pocket-like configuration become regular points in the interior of the corresponding cylinders in X (see Figure 12) and, therefore, each cylinder in X has two $\mathfrak{P}_*$-preimages of $\gamma$ in its interior, instead of one, as in the case of non pocket-like configurations. Hence, the number $n$, of cylinders in X obtained by lifting a cylinder in W is half the number of $\mathfrak{P}_*$-preimages of $\gamma$, which is given by Lemma 6.6. That is, in the case of pocket-like configurations, we have that $n = 4$ and $s = 1$ for profile $(0, 0)$, and $n = 2$, $s = 2$, for all other profiles of good configurations. $\square$

Remark 6.8. If we were working with the area Siegel-Veech constant, instead of the classical Siegel-Veech constant, there would be no difference for pocket-like or not pocket-like configurations in the previous result, since area Siegel-Veech constant depends only on monodromy.

## 7. Siegel-Veech constants of good configurations for generic surfaces

In this section we use the results of the previous section to compute the exact value of the Siegel-Veech constant of good configurations for generic surfaces in $\mathcal{Q}(1^m, -1^{m+4})$ with respect to the Masur–Veech measure.

Recall that for almost every surface in $\mathcal{L} = \mathcal{Q}(1^m, -1^{m+4})$, the only possible configurations are pocket and dumbbell configurations. Note that both configurations are multiplicity one configurations, that is, they define a single cylinder.

By Proposition 6.3, a multiplicity one configuration is a good configuration if and only if $r_h, r_v \in \{0, 1\}$, where $r_h$ and $r_v$ are the number of ramified poles for $\mathbf{p}_h$ and $\mathbf{p}_v$, respectively, in a component of the surface W after cutting along the core curve of the cylinder defined by the configuration. Lastly, recall that $\mathbf{p}_h$ and $\mathbf{p}_v$ have four ramified poles each, from which they share three. In particular, there are five "special" poles, the three shared ramified poles and one more for each one of $\mathbf{p}_h$ and $\mathbf{p}_v$.

**Good pocket configurations.** Recall that in a pocket configuration, we have a single cylinder bounded by a saddle connection joining a fixed pair of poles $P_{j_1}, P_{j_2}$ on one side and by a separatrix loop emitted from a fixed zero $P_i$ of order $d_i \geq 1$, on the other side (see Figure 3). Then, $r_h$ and $r_v$, as defined in the previous section, is the number of ramified poles among the poles $P_{j_1}$ and $P_{j_2}$ of the configuration for the double cover $\mathbf{p}_h$ and $\mathbf{p}_v$, respectively. By Proposition 6.3, the configuration is good if and only if $r_h, r_v \in \{0, 1\}$. Recall that the profile of the configuration is the pair $(r_h, r_v)$.

Profile $(0, 0)$ means that none of the ramified poles, for $\mathbf{p}_h$ and $\mathbf{p}_v$, is one of the poles defining the pocket configuration, $P_{j_1}$ or $P_{j_2}$. Then, since there are $m - 1 = (m + 4) - 5$ poles which are unramified poles for both $\mathbf{p}_h$ and $\mathbf{p}_v$, there are exactly $\binom{m-1}{2} = (m - 1)(m - 2)/2$ pocket configurations of profile $(0, 0)$.

In order to have profile $(1, 1)$, we should have one ramified and one unramified pole for both $\mathbf{p}_h$ and $\mathbf{p}_v$, or one which is ramified for $\mathbf{p}_h$ but unramified for $\mathbf{p}_v$ and vice versa. This latter case occurs once, because $\mathbf{p}_h$ and $\mathbf{p}_v$ share three out of four of their ramified poles. The former case happens exactly $\binom{3}{1}\binom{m-1}{1} = 3m - 3$ times. Therefore, we have $3m - 2$ pocket configurations of profile $(1, 1)$.

Profile $(1, 0)$ occurs when one of the poles is ramified for $\mathbf{p}_h$ but unramified for $\mathbf{p}_v$ and the other is unramified for both $\mathbf{p}_h$ and $\mathbf{p}_v$. Then, there are $\binom{1}{1}\binom{m-1}{1} = m-1$ pocket configurations of profile $(1, 0)$. Similarly, we have $m-1$ pocket configurations of profile $(0, 1)$.

Summarizing good profiles and applying Lemma 6.7, we get that good pocket configurations contribute to the Siegel-Veech constant of good cylinders in $\mathcal{M}$ by

$c_{good}^{\text{pocket}}(\mathcal{M})$, which is $16(m-1)(m-2)+4((3m-2)+2(m-1))$ times the Siegel-Veech constant for pocket configurations in $\mathcal{L}$. Thus, by formula (3),

$$c_{good}^{\text{pocket}}(\mathcal{M}) = \left(4m^2 - 7m + 4\right) \frac{2}{\pi^2}.$$

**Good dumbbell configurations.** Recall that in this configuration, we have a single cylinder, bounded by a saddle connection joining a zero to itself on each side (see Figure 4). Such a cylinder separates the original surface W in two parts. This yields a partition of $\alpha = \{1^m, -1^{m+4}\}$ (where superindices stand for the multiplicities) into two subsets $\alpha = \alpha_1 \sqcup \alpha_2$, which is also considered to be part of the configuration, and we consider $\alpha_1$ to contain the $r_h$ ramified poles for $\mathbf{p}_h$ and the $r_v$ ramified poles for $\mathbf{p}_v$. We stress in the fact that, even if there are several singularities with the same degree, we differentiate them, so they are named and, by a slight abuse of notation, we consider this information is also carried by the partition.

Let $k_l = \#\alpha_l$, counting multiplicities, $l = 1, 2$, and note that $k = k_1+k_2 = 2m+4$. Let $q$ be the number of simple zeros in $\alpha_1$. Then, there are $k_1 - q$ poles in $\alpha_1$, but also, by topological considerations, we have that this number is equal to $q+2$, since we are restricted to a genus zero surface with one boundary component. Therefore, we will always have that $\alpha_1 = \{1^q, -1^{q+2}\}$ and $\alpha_2 = \{1^{m-q}, -1^{m-q+2}\}$ (up to the names of the singularities). In particular, $k_1 = 2q + 2$ and $k_2 = 2m - 2q + 2$. Thus, in this context, formula (4) becomes

$$c_{i_1,i_2;\alpha_1,\alpha_2}^{\text{dumbbell}} = \frac{(2q-1)!(2m-2q-1)!}{(2m)!} \frac{2}{\pi^2}. \tag{9}$$

Since this value depends only on $q$, it is natural to try to group configurations sharing this number $q$ and study the corresponding combinatorics. But, by Lemma 6.7, different profiles give different weights when lifted to $\mathcal{M}$. Hence, we have to consider different profiles separately.

For dumbbell configurations, profile $(0, 0)$ means that there are only unramified poles in $\alpha_1$, that is, all the five ramified poles for $\mathbf{p}_h$ and $\mathbf{p}_v$, are in $\alpha_2$. Then, the combinatorics are given by the remaining $m - 1$ poles and the $m$ simple zeros.

Hence, to compute the number of these configurations, that is, dumbbell configurations of profile $(0, 0)$ with $q$ simple zeros in $\alpha_1$, we remark that we have to choose $q$ of the $m$ (named) simple zeros and $q + 2$ of the remaining $m - 1$ (named) poles, to have in total $q + 2$ poles in $\alpha_1$, as required by the topology. Finally, note that we have to choose one of $q$ zeros to be located at the boundary of the cylinder on one side and one of $m - q$ zeros to be located at the boundary of the cylinder on the other side. For any given $q$, where $1 \leq q \leq m - 1$, the count gives

$$\binom{m}{q}\binom{m-1}{q+2} q(m-q)$$

dumbbell configurations of profile $(0, 0)$.

In order to have profile $(1, 1)$, there are two possibilities. The first one is to have one simple pole in $\alpha_1$ which is ramified for $\mathbf{p}_h$ but unramified for $\mathbf{p}_v$ and vice versa. In this case, there is only one choice for this two ramified poles, because $\mathbf{p}_h$ and $\mathbf{p}_v$ share three out of four of their ramified poles. The three ramified poles shared by $\mathbf{p}_h$ and $\mathbf{p}_v$ are then in $\alpha_2$. As before, we have to choose $q$ of the $m$ simple zeros to be in $\alpha_1$, one of them to be in a boundary component of the cylinder and one of the remaining $m-q$ simple zeros to be in the other boundary component. For poles, since we have already taken two poles to be in $\alpha_1$, we have to choose $q$ poles among the $m-1$ unramified poles, to have $q+2$ poles in total, as required by the topology. Then, this case of profile $(1,1)$ occurs $\binom{m}{q}\binom{m-1}{q}q(m-q)$ times.

The other case which gives profile $(1,1)$ is when there is only one ramified pole for both $\mathbf{p}_h$ and $\mathbf{p}_v$ in $\alpha_1$ and all the remaining ramified poles (for $\mathbf{p}_h$ or $\mathbf{p}_v$) are in $\alpha_2$. Thus, there are 3 possibilities in choosing the common ramified pole and therefore, by an analogous computation, this case happens $\binom{m}{q}\binom{3}{1}\binom{m-1}{q+1}q(m-q)$ times. Then, for fixed $q$, $1 \le q \le m-1$, we have

$$\binom{m}{q}\left[3\binom{m-1}{q+1}+\binom{m-1}{q}\right]q(m-q)$$

dumbbell configurations of profile $(1,1)$.

Profile $(1,0)$ occurs when only one of the poles in $\alpha_1$ is ramified for $\mathbf{p}_h$ but unramified for $\mathbf{p}_v$ and all others are unramified for both $\mathbf{p}_h$ and $\mathbf{p}_v$. Then, by an analogous computation, there are $\binom{m}{q}\binom{1}{1}\binom{m-1}{q+1}q(m-q)$ dumbbell configurations of profile $(1,0)$. Similarly, we have

$$\binom{m}{q}\binom{m-1}{q+1}q(m-q)$$

dumbbell configurations of profile $(0,1)$.

In summary, by Lemma 6.7, good dumbbell configurations contribute to the Siegel-Veech constant of good cylinders in $\mathcal{M}$ by

$$\tbinom{m}{q}\left[64\tbinom{m-1}{q+2}+8\left(3\tbinom{m-1}{q+1}+\tbinom{m-1}{q}+2\tbinom{m-1}{q+1}\right)\right]q(m-q)$$

times the Siegel-Veech constant for a dumbbell configurations in $\mathcal{L}$ with $q$ simple zeros in $\alpha_1$, that is,

$$c_{q,good}^{\text{dumbbell}}(\mathcal{M})=8\binom{m}{q}\left[8\binom{m-1}{q+2}+5\binom{m-1}{q+1}+\binom{m-1}{q}\right]q(m-q)c_q^{\text{dumbbell}},$$

where $c_q^{\text{dumbbell}}$ is given by formula (9). Finally, summing up all the contribution of good dumbbell configurations and plugging in formula (9), we obtain that

$$\begin{aligned}
c_{good}^{\text{dumbbell}}(\mathcal{M}) &= 8\sum_{q=1}^{m-1}\tbinom{m}{q}\left[8\tbinom{m-1}{q+2}+5\tbinom{m-1}{q+1}+\tbinom{m-1}{q}\right]q(m-q)\tfrac{(2q-1)!(2m-2q-1)!}{(2m)!}\tfrac{2}{\pi^2}\\
&= 8\sum_{q=1}^{m-1}\tbinom{m}{q}\left[8\tbinom{m-1}{q+2}+5\tbinom{m-1}{q+1}+\tbinom{m-1}{q}\right]\tfrac{1}{4}\tfrac{(2q)!(2m-2q)!}{(2m)!}\tfrac{2}{\pi^2}\\
&= \frac{4}{\pi^2}\sum_{q=1}^{m-1}\frac{\binom{m}{q}}{\binom{2m}{2q}}\left[8\tbinom{m-1}{q+2}+5\tbinom{m-1}{q+1}+\tbinom{m-1}{q}\right]. \qquad (10)
\end{aligned}$$

But, by Proposition A.1, formula (10) can be written as

$$\begin{aligned}
c^{\text{dumbbell}}_{good}(\mathcal{M}) &= \frac{4}{\pi^2}\sum_{q=1}^{m-1}\frac{\binom{m}{q}}{\binom{2m}{2q}}\left[8\tbinom{m-1}{q+2}+5\tbinom{m-1}{q+1}+\tbinom{m-1}{q}\right]\\
&= \frac{4}{\pi^2}\left[8\left(\tfrac{1}{6}m^2-\tfrac{13}{6}m-3+\tfrac{5}{2}4^m\tfrac{(m!)^2}{(2m)!}\right)\right.\\
&\qquad \left.+5\left(m+2-\tfrac{3}{2}4^m\tfrac{(m!)^2}{(2m)!}\right)+\left(-1+\tfrac{1}{2}4^m\tfrac{(m!)^2}{(2m)!}\right)\right]\\
&= \frac{2}{3\pi^2}\left[8\left(m^2-13m-18+15\cdot 4^m\tfrac{(m!)^2}{(2m)!}\right)\right.\\
&\qquad \left.+5\left(6m+12-9\cdot 4^m\tfrac{(m!)^2}{(2m)!}\right)+\left(-6+3\cdot 4^m\tfrac{(m!)^2}{(2m)!}\right)\right]\\
&= \frac{2}{3\pi^2}\left(8m^2-74m-90+78\cdot 4^m\tfrac{(m!)^2}{(2m)!}\right).
\end{aligned}$$

We conclude the computation of the Siegel-Veech constant for good cylinders in $\mathcal{M}$, for generic surfaces, summing up the contribution of pocket and dumbbell good configurations

$$\begin{aligned}
c_{good}(\mathcal{M}) &= c^{\text{pocket}}_{good}(\mathcal{M})+c^{\text{dumbbell}}_{good}(\mathcal{M})\\
&= \left(4m^2-7m+4\right)\frac{2}{\pi^2}+\left(8m^2-74m-90+78\cdot 4^m\frac{(m!)^2}{(2m)!}\right)\frac{2}{3\pi^2}\\
&= \left(20m^2-95m-78+78\cdot 4^m\frac{(m!)^2}{(2m)!}\right)\frac{2}{3\pi^2}. \qquad (11)
\end{aligned}$$

## 8. Side results

**8.1. Area Siegel-Veech constant.** Following the same treatment, we can deduce that for almost every wind-tree billiard $\Pi\in\mathcal{WT}(m)$, the number $N_{area}(\Pi, L)$ has quadratic asymptotic growth rate and

$$N_{area}(\Pi, L)\sim c_{a,good}(\mathcal{M})\frac{\pi L^2}{\text{Area}\,(\Pi/\mathbb{Z}^2)},$$

where $c_{a,good}(\mathcal{M})$ is the area Siegel-Veech constant associated to the counting problem of the area of good cylinders in $\mathcal{M}$, the $\text{SL}(2,\mathbb{R})$-orbit closure of $\text{X}(\Pi)$.

Note that, unlike the case of the classical (non-weighted) counting, in this case we do not have the factor $1/4$ (see Corollary 5.3). This is because, in the weighted counting, the area is already taken into consideration.

Moreover, for almost every wind-tree billiard $\Pi\in\mathcal{WT}(m)$,

$$c_{a,good}(\mathcal{M}) = c^{\text{pocket}}_{a,good}(\mathcal{M})+c^{\text{dumbbell}}_{a,good}(\mathcal{M}),$$

where $c^{\text{pocket}}_{a,good}(\mathcal{M})$ (resp. $c^{\text{dumbbell}}_{a,good}(\mathcal{M})$) corresponds to the area Siegel–Veech constant associated to configurations of good cylinders in $\mathcal{M}$ which project to pocket (resp. dumbbell) configurations in $\mathcal{Q}(1^m,-1^{m+4})$.

Furthermore, there exist a relation between classical Siegel–Veech constants and area Siegel-Veech constants for configurations $\mathcal{C}$ of cylinders in $\mathcal{L} = \mathcal{Q}(1^m, -1^{m+4})$:

$$c_{a,\mathcal{C}}(\mathcal{L}) = \frac{1}{2m+1} c_{\mathcal{C}}(\mathcal{L}).$$

This is a consequence of a generalization of Vorobets formula [**Vo05**, Theorem 1.6(b)], proved by Athreya–Eskin–Zorich [**AEZ**, Proposition 4.9] for any configuration of cylinders on any strata $\mathcal{Q}(d_1, \ldots, d_k)$ of quadratic differentials on $\mathbb{CP}^1$.

Then, we can relate the Siegel–Veech constant on $\mathcal{M}$ with that of $\mathcal{L}$, using the analogous of Lemma 6.7 (keeping in mind Remark 6.8).

Finally, we have

$$\begin{aligned} c_{a,good}(\mathcal{M}) &= c_{a,good}^{\text{pocket}}(\mathcal{M}) + c_{a,good}^{\text{dumbbell}}(\mathcal{M}) \\ &= \frac{1}{2m+1}\left(4m^2 - 7m + 4\right)\frac{4}{\pi^2} \\ &\quad + \frac{1}{2m+1}\left(8m^2 - 74m - 90 + 78 \cdot 4^m \frac{(m!)^2}{(2m)!}\right)\frac{2}{3\pi^2} \\ &= \frac{1}{2m+1}\left(16m^2 - 58m - 33 + 39 \cdot 4^m \frac{(m!)^2}{(2m)!}\right)\frac{4}{3\pi^2} \\ &= \left(8m - 33 + 39 \cdot 4^m \frac{(m!)^2}{(2m+1)!}\right)\frac{4}{3\pi^2} \end{aligned}$$

**8.2. Polynomial diffusion rate.** The main result of Delecroix–Hubert–Lelièvre in [**DHL**] relates the polynomial diffusion rate on the classical model to the Lyapunov exponents of the subbundles $F^{+-}$ and $F^{-+}$. In this case, the polynomial diffusion rate is 2/3 for *every* wind-tree billiard in $\mathcal{WT}(1)$. This result was generalized by Delecroix–Zorich [**DZ**] for $m \geq 2$. However, in the general case, the value of the diffusion rate is also explicitly known but only for *almost every* wind-tree billiard in $\mathcal{WT}(m)$ and numerically for some explicit examples (see [**DZ**, Remark 2]).

The explicit values of the polynomial diffusion rate for *all* wind-tree billiards in $\mathcal{WT}(m)$, $m \geq 2$, is still an open problem. However, an application of Forni's criterion for integer equivariant subbundles (Theorem 5.1) allows us to show that the relevant Lyapunov exponents is always positive, for every wind-tree billiard in $\mathcal{WT}(m)$, for all $m \geq 1$ (Corollary 5.2).

Thus, we can conclude that we have always positive polynomial diffusion rate.

**8.3. Recurrence.** A geometric criterion for the recurrence of the directional linear flow on $\mathbb{Z}^d$-periodic flat surfaces in terms of good cylinders by Avila–Hubert [**AH**] says that if the positive $g_t$-orbit of the compact surface accumulates on a flat surfaces with a vertical good cylinder, then the vertical linear flow on the $\mathbb{Z}^d$-periodic flat surface is recurrent ([**AH**, Proposition 2]).

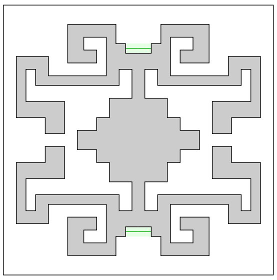
A result of Chaika–Eskin [**CE**] allows us to extend this criterion. In fact, we have the following.

THEOREM 8.1. *Let $X$ be a flat surface, $\mathcal{M}$ its* $\mathrm{SL}(2,\mathbb{R})$*-orbit closure and $F$ a continuous equivariant subbundle. Let $\mathbf{f}$ be a $d$-tuple of elements in $F_X(\mathbb{Z})$ and consider $X_\infty$, the $\mathbb{Z}^d$-periodic flat surface defined by $X$ and $\mathbf{f}$. Suppose that there* *exists $Y \in \mathcal{M}$ with an $F$-good cylinder. Then, for almost every $\theta \in [0, 2\pi)$, the linear flow in direction $\theta$ is recurrent on $X_\infty$.*

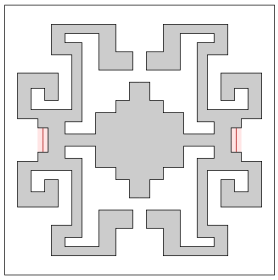
PROOF. By [**CE**, Theorem 1.1], for almost every $\theta \in [0, 2\pi)$, $r_{-\theta}X$ is Birkhoff generic for the $g_t$-flow with respect to $\nu_{\mathcal{M}}$. Since $Y \in \mathcal{M}$ has a $F$-good cylinder, then $Y' = r_\phi Y$ has a vertical cylinder for some $\phi \in [0, 2\pi)$. Obviously $Y' \in \mathcal{M}$ and, since $r_{-\theta}X$ is Birkhoff generic, its positive $g_t$-orbit accumulates on $Y'$. Then, by [**AH**, Proposition 2], the linear flow in direction $\theta$ is recurrent in $X_\infty$. □

Thus, to prove the recurrence of every wind-tree billiard $\Pi \in \mathcal{WT}(m)$, we shall show that we can find good cylinders in the compact surface X(Π).

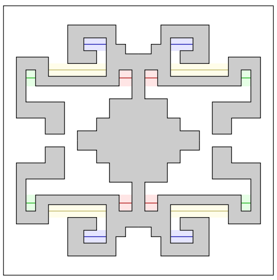
For $m = 1$ this was first proved by Avila–Hubert [**AH**, Lemma 4]. Consider $m \geq 2$ and recall that the obstacles of a wind-tree billiard $\Pi \in \mathcal{WT}(m)$ are horizontal and vertically symmetric right-angled polygons with $4m$ corners with the angle $\pi/2$ and $4(m-1)$, with the angle $3\pi/2$.



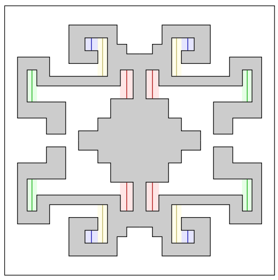
(a) Horizontal good cylinder of profile $(1, 0)$.



(b) Vertical good cylinder of profile $(0, 1)$.



(c) Horizontal good cylinder of profile $(0, 0)$.



(d) Vertical good cylinder of profile $(0, 0)$.

FIGURE 14. Good cylinders for obstacles with two consecutive corners with angle $3\pi/2$.

If the obstacle has two consecutive angles $3\pi/2$, then we have (horizontal or vertical) good cylinders of profile $(1, 0)$, $(0, 1)$ or $(0, 0)$. In fact, if the two consecutive angles are symmetric with respect to the vertical reflection, then we obtain horizontal good cylinders of profile $(1, 0)$ as in Figure 14a. Similarly, if the angles are symmetric with respect to the horizontal reflection, then we have vertical good cylinders of profile $(0, 1)$ as in Figure 14b. In other case, we obtain horizontal or vertical good cylinders of profile $(0, 0)$ as in Figure 14c and Figure 14d.

If there are no consecutive corners of angles $3\pi/2$, then there are good cylinders of profile $(1, 1)$ as in Figure 15.

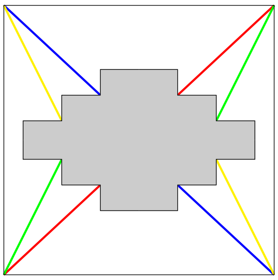


FIGURE 15. Core curves of good cylinders of profile $(1,1)$ for obstacles with no consecutive corners with angle $3\pi/2$.

Thus, for every $\Pi \in \mathcal{WT}(m)$ we can exhibit good cylinders in $\mathrm{X}(\Pi)$ and then, by Theorem 8.1, we conclude that the billiard flow in direction $\theta$ is recurrent for almost every $\theta \in [0, 2\pi)$.

## Appendix A. Combinatorial identities

In this appendix we prove the following identities.

PROPOSITION A.1. *For any $m \in \mathbb{N}$ the following identities hold*

$$\sum_{q=1}^{m-1} \frac{\binom{m}{q}\binom{m-1}{q+2}}{\binom{2m}{2q}} = \frac{1}{6}m^2 - \frac{13}{6}m - 3 + \frac{5}{2}4^m \frac{(m!)^2}{(2m)!} \tag{12}$$

$$\sum_{q=1}^{m-1} \frac{\binom{m}{q}\binom{m-1}{q+1}}{\binom{2m}{2q}} = m + 2 - \frac{3}{2}4^m \frac{(m!)^2}{(2m)!} \tag{13}$$

$$\sum_{q=1}^{m-1} \frac{\binom{m}{q}\binom{m-1}{q}}{\binom{2m}{2q}} = -1 + \frac{1}{2}4^m \frac{(m!)^2}{(2m)!} \tag{14}$$

PROOF. Define

$$B(m,s) := \sum_{q=1}^{m-1} \frac{\binom{m}{q}\binom{m-1}{q+s}}{\binom{2m}{2q}}$$

and note that

$$\frac{\binom{m}{q}\binom{m-1}{q+s}}{\binom{2m}{2q}} = \frac{m!(m-1)!}{(2m)!}\binom{2q}{q}\binom{2m-2q}{m-q}\frac{q!}{(q+s)!}\frac{(m-q)!}{(m-1-q-s)!}.$$

Consider

$$A(m,s) = \sum_{q=0}^{m} \binom{2q}{q}\binom{2m-2q}{m-q}\frac{q!}{(q+s)!}\frac{(m-q)!}{(m-1-q-s)!}.$$

Then

$$B(m,s) = \frac{m!(m-1)!}{(2m)!}A(m,s) - \binom{m-1}{s}. \tag{15}$$

Note now than we can write

$$\frac{(m-q)!}{(m-1-q-s)!} = \prod_{i=0}^{s}(m-q-i) =: P^{(m,s)}(q),$$

where $P^{(m,s)}$ is a computable polynomial of degree $s+1$, and suppose

$$P^{(m,s)}(q) = \sum_{j=0}^{s+1} p_j^{(m,s)} q^j.$$

Then, we can write

$$A(m,s) = \sum_{j=0}^{s+1} p_j^{(m,s)} \sum_{q=0}^{m} \binom{2q}{q}\binom{2m-2q}{m-q}\frac{q!}{(q+s)!}q^j$$

and define

$$D(m,s,j) = \sum_{q=0}^{m} \binom{2q}{q}\binom{2m-2q}{m-q}\frac{q!}{(q+s)!}q^j,$$

so that

$$A(m,s) = \sum_{j=0}^{s+1} p_j^{(m,s)} D(m,s,j). \tag{16}$$

Note that

$$\begin{aligned}
D(m,s,j) &= \sum_{q=0}^{m} \binom{2q}{q}\binom{2m-2q}{m-q}\frac{q!}{(q+s)!}q^j \\
&= \sum_{q=0}^{m} \binom{2q}{q}\binom{2m-2q}{m-q}\frac{q!}{(q+s)!}q^j \frac{q+s+1}{q+s+1} \\
&= \sum_{q=0}^{m} \binom{2q}{q}\binom{2m-2q}{m-q}\frac{q!}{(q+s+1)!}q^j (q+s+1) \\
&= D(m,s+1,j+1) + (s+1)\, D(m,s+1,j).
\end{aligned}$$

Then, $D$ satisfies the following recurrence relation,

$$D(m,s,j) = D(m,s-1,j-1) - s\, D(m,s,j-1) \tag{17}$$

and, in particular, we can deduce that $D(m,s,j)$ can be written as a linear combination of $D(m,i,0)$, $i = 1,\dots,s$, and $D(m,0,l)$, $0 \le l \le j-s$. But, since $j$ takes values in $\{0,\dots,s+1\}$, for the $D(m,0,l)$ terms, we are interested only in $D(m,0,1)$ and $D(m,0,0)$. The value of $D(m,0,0)$ is given in [**Gl**, (3.90)],

$$D(m,0,0) = \sum_{q=0}^{m} \binom{2q}{q}\binom{2m-2q}{m-q} = 4^m. \tag{18}$$

On the other hand,

$$\begin{aligned} D(m,0,1) &= \sum_{q=0}^{m} \binom{2q}{q}\binom{2m-2q}{m-q} q \\ &= \sum_{r=0}^{m} \binom{2m-2r}{m-r}\binom{2r}{r}(m-r) \\ &= m\, D(m,0,0) - D(m,0,1). \end{aligned}$$

Then, $2\, D(m,0,1) = m\, D(m,0,0)$ and, by the identity (18),

$$D(m,0,1) = \frac{m}{2}4^m. \tag{19}$$

REMARK A.2. In fact, it is not difficult to show that $D(m,0,l) = (m/2)^l\ 4^m$, $l \geq 0$.

For the other terms, of the form $D(m,i,0)$, we use the following identity ([**Gl**, (3.95)])

$$\mathcal{X}(m,i) := \sum_{q=0}^{m} \binom{2q}{q}\binom{2m-2q}{m-q}\frac{i}{q+i} = \frac{\binom{2m+2i-1}{m+i}}{\binom{2i-1}{i}}. \tag{20}$$

But, a simple partial fraction decomposition gives

$$\frac{q!}{(q+i)!} = \prod_{j=1}^{i}\frac{1}{q+j} = \sum_{j=1}^{i}\frac{(-1)^{j-1}}{(j-1)!(i-j)!}\frac{1}{q+j} = \sum_{j=1}^{i}\frac{(-1)^{j-1}}{j!(i-j)!}\frac{j}{q+j}$$

and thus,

$$D(m,i,0) = \sum_{j=1}^{i}\frac{(-1)^{j-1}}{(j)!(i-j)!}\mathcal{X}(m,j). \tag{21}$$

**Proof of identity (14).** Following previous discution, $P^{(m,0)}(q) = m - q$ and then, by (16), we have that,

$$A(m,0) = m\, D(m,0,0) - D(m,0,1) = \frac{m}{2}4^m,$$

where last equality comes from (18) and (19). Finally, from (15), we have that

$$B(m,0) = \frac{m!(m-1)!}{(2m)!}A(m,0) - \binom{m-1}{0} = \frac{1}{2}4^m\frac{(m!)^2}{(2m)!} - 1,$$

which is (14).

**Proof of identity (13).** Note that $P^{(m,1)}(q) = m^2 - m - (2m-1)q + q^2$. Then, by (16), we have that,

$$A(m,1) = (m^2 - m)\, D(m,1,0) - (2m-1)\, D(m,1,1) + D(m,1,2).$$

Using the recurrence rule (17), we have that

$$\begin{aligned} &D(m,1,1) = D(m,0,0) - D(m,1,0), \text{ and} \\ &D(m,1,2) = D(m,0,1) - D(m,1,1) = D(m,0,1) - D(m,0,0) + D(m,1,0). \end{aligned}$$

It follows that

$$\begin{aligned} A(m,1) &= (m^2 - m + (2m-1) + 1)\ D(m,1,0) - (2m-1+1)\ D(m,0,0) + D(m,0,1) \\ &= (m^2+m)\ D(m,1,0) - 2m\ D(m,0,0) + D(m,0,1). \end{aligned}$$

By identity (21) for $i = 1$, $D(m,1,0) = \mathcal{X}(1)$, and from (20),

$$D(m,1,0) = \mathcal{X}(1) = \binom{2m+1}{m+1} = \frac{(2m+1)!}{m!(m+1)!}.$$

Therefore,

$$\begin{aligned} A(m,1) &= (m^2+m)\frac{(2m+1)!}{m!(m+1)!} - 2m\ 4^m + \frac{m}{2}4^m \\ &= \frac{(2m+1)!}{m!(m-1)!} - \frac{3m}{2}4^m, \end{aligned}$$

where we have also used (18) and (19). Thus, from (15),

$$\begin{aligned} B(m,1) &= \frac{m!(m-1)!}{(2m)!}A(m,1) - \binom{m-1}{1} \\ &= \frac{m!(m-1)!}{(2m)!}\left(\frac{(2m+1)!}{m!(m-1)!} - \frac{3m}{2}4^m\right) - (m-1) \\ &= 2m+1 - \frac{3}{2}4^m\frac{(m!)^2}{(2m)!} - (m-1) \\ &= m+2 - \frac{3}{2}4^m\frac{(m!)^2}{(2m)!}, \end{aligned}$$

which is (13).

**Proof of identity (12).** (For the sake of readability, we will omit $m$ from notation in this part.) From (16), we have that

$$A(2) = p_0^{(2)}D(2,0) + p_1^{(2)}D(2,1) + p_2^{(2)}D(2,2) + p_3^{(2)}D(2,3),$$

where

$$P^{(2)}(q) = \sum_{j=0}^{3} p_j^{(2)}q^j = (m^3 - 3m^2 + 2m) - (3m^2 - 6m + 2)q + (3m-3)q^2 - q^3.$$

Using the recurrence rule (17), we have that

$$\begin{aligned} D(2,1) &= D(1,0) - 2\ D(2,0), \\ D(2,2) &= D(1,1) - 2\ D(2,1) \\ &= D(0,0) - D(1,0) - 2\ (D(1,0) - 2\ D(2,0)) \\ &= D(0,0) - 3\ D(1,0) + 4\ D(2,0), \text{ and} \\ D(2,3) &= D(1,2) - 2\ D(2,2) \\ &= D(0,1) - D(1,1) - 2\ (D(0,0) - 3\ D(1,0) + 4\ D(2,0)) \\ &= D(0,1) - D(0,0) + D(1,0) - 2\ D(0,0) + 6\ D(1,0) - 8\ D(2,0) \\ &= D(0,1) - 3\ D(0,0) + 7\ D(1,0) - 8\ D(2,0). \end{aligned}$$

It follows that,

$$\begin{aligned}
A(2) &= p_0^{(2)} D(2,0) + p_1^{(2)} D(2,1) + p_2^{(2)} D(2,2) + p_3^{(2)} D(2,3) \\
&= p_3^{(2)} D(0,1) + (p_2^{(2)} - 3\, p_3^{(2)}) D(0,0) + (p_1^{(2)} - 3\, p_2^{(2)} + 7\, p_3^{(2)}) D(1,0) \\
&\qquad + (p_0^{(2)} - 2\, p_1^{(2)} + 4\, p_2^{(2)} - 8\, p_3^{(2)}) D(2,0) \\
&= -D(0,1) + 3m\, D(0,0) + q_1^{(2)} D(1,0) + q_2^{(2)} D(2,0) \\
&= \frac{5m}{2} 4^m + q_1^{(2)} D(1,0) + q_2^{(2)} D(2,0),
\end{aligned}$$

where we have used (18), (19) and the values of $p_3^{(2)} = -1$ and $p_2^{(2)} = 3m - 3$. We have also defined $q_1^{(2)} = p_1^{(2)} - 3p_2^{(2)} + 7p_3^{(2)}$ and $q_2^{(2)} = p_0^{(2)} - 2p_1^{(2)} + 4p_2^{(2)} - 8p_3^{(2)}$.

Thus, by identity (21),

$$\begin{aligned}
A(2) &= \frac{5m}{2} 4^m + q_1^{(2)} \mathcal{X}(1) + q_2^{(2)} \left( \mathcal{X}(1) - \frac{1}{2} \mathcal{X}(2) \right) \\
&= \frac{5m}{2} 4^m + (q_1^{(2)} + q_2^{(2)}) \mathcal{X}(1) - \frac{1}{2} q_2^{(2)} \mathcal{X}(2) \\
&= \frac{5m}{2} 4^m + (p_0^{(2)} - p_1^{(2)} + p_2^{(2)} - p_3^{(2)}) \mathcal{X}(1) - \frac{1}{2} (p_0^{(2)} - 2p_1^{(2)} + 4p_2^{(2)} - 8p_3^{(2)}) \mathcal{X}(2) \\
&= \frac{5m}{2} 4^m + (m^3 - m) \mathcal{X}(1) - \frac{1}{2} (m^3 + 3m^2 + 2m) \mathcal{X}(2).
\end{aligned}$$

Plugging in identity (20), we obtain

$$\begin{aligned}
A(2) &= \frac{5m}{2} 4^m + (m^3 - m) \binom{2m+1}{m+1} - \frac{1}{2} (m^3 + 3m^2 + 2m) \frac{\binom{2m+3}{m+2}}{\binom{3}{2}} \\
&= \frac{5m}{2} 4^m + (m-1)m(m+1) \frac{(2m+1)!}{m!(m+1)!} - \frac{1}{6} m(m+1)(m+2) \frac{(2m+3)!}{(m+1)!(m+2)!} \\
&= \frac{5m}{2} 4^m + \left( (m-1) - \frac{1}{3}(2m+3) \right) \frac{(2m+1)!}{m!(m-1)!} = \frac{5m}{2} 4^m + \frac{1}{3}(m-6) \frac{(2m+1)!}{m!(m-1)!}.
\end{aligned}$$

Finally, by (15),

$$\begin{aligned}
B(2) &= \frac{m!(m-1)!}{(2m)!} A(2) - \binom{m-1}{2} \\
&= \frac{m!(m-1)!}{(2m)!} \left( \frac{5m}{2} 4^m + \frac{1}{3}(m-6) \frac{(2m+1)!}{m!(m-1)!} \right) - \frac{1}{2}(m-1)(m-2) \\
&= \frac{5}{2} 4^m \frac{(m!)^2}{(2m)!} + \frac{1}{3}(2m^2 - 11m - 6) - \frac{1}{2}(m^2 - 3m + 2) \\
&= \frac{5}{2} 4^m \frac{(m!)^2}{(2m)!} + \frac{1}{6}(m^2 - 13m - 18),
\end{aligned}$$

which is (12).

□

Remark A.3. Note that the proof of Theorem A.1 states a procedure or algorithm in order to compute $A(m,s)$ and $B(m,s)$ for all $s \geq 0$. Anyway, an algorithm is not a formula, and evidently, the complexity increase enormously when $s$ becomes larger. However, with this method, it is possible to show that $B(m,s)$ has the form

$$(2m+1)\ \mathcal{P}_s(m) + (-1)^s\ \tfrac{2s+1}{2}\ 4^m \frac{(m!)^2}{(2m)!} - \tbinom{m-1}{s},$$

where $\mathcal{P}_s$ is a polynomial of degree $s-1$ (in particular, $\mathcal{P}_0 = 0$), which can also be explicitly computed. Moreover, $\mathcal{P}_s$ can be deduced from the fact that $B(m,s) = 0$ for $m = 1, \ldots, s+1$. In particular,

$$\mathcal{P}_s(m) = (-1)^{s+1}\ \tfrac{2s+1}{2}\ 4^m \frac{(m!)^2}{(2m+1)!}$$

for $m = 1, \ldots, s$. Anyway, we do not perform the computations here.

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CHAPTER II

# Quantitative error term in the counting problem on Veech wind-tree models

Abstract. We study periodic wind-tree models, billiards in the plane endowed with $\mathbb{Z}^2$-periodically located identical connected symmetric right-angled obstacles. We exhibit effective asymptotic formulas for the number of (isotopy classes of) periodic billiard trajectories (up to $\mathbb{Z}^2$-translations) on Veech wind-tree billiards, that is, wind-tree billiards whose underlying compact translation surfaces are Veech surfaces. We show that the error term depends on spectral properties of the Veech group and give explicit estimates in the case when obstacles are squares of side length 1/2.

## 1. Introduction

The classical wind-tree model corresponds to a billiard in the plane endowed with $\mathbb{Z}^2$-periodic obstacles of rectangular shape aligned along the lattice, as in Figure 1.

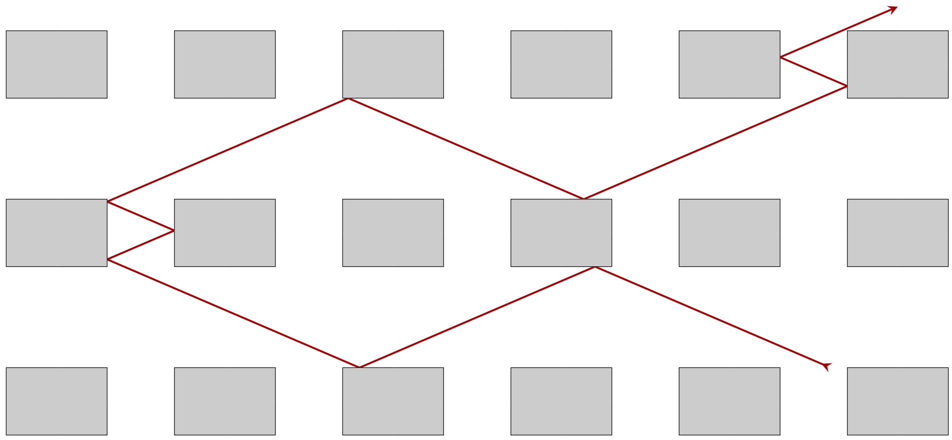

Figure 1. Original wind-tree model.

The wind-tree model (in a slightly different version) was introduced by P. Ehrenfest and T. Ehrenfest [**EE**] in 1912. J. Hardy and J. Weber [**HaWeb**] studied the periodic version. All these studies had physical motivations.

Several advances on the dynamical properties of the billiard flow in the wind-tree model were obtained recently using geometric and dynamical properties on moduli space of (compact) flat surfaces; billiard trajectories can be described by the linear flow on a flat surface.

A. Avila and P. Hubert [**AH**] showed that for all parameters of the obstacle and for almost all directions, the trajectories are recurrent. There are examples of divergent trajectories constructed by V. Delecroix [**De**]. The non-ergodicity was proved by K. Frączek and C. Ulcigrai [**FU**]. It was proved by V. Delecroix, P. Hubert and S. Lelièvre [**DHL**] that the diffusion rate is independent either on the concrete values of parameters of the obstacle or on almost any direction and almost any starting point and is equals to 2/3. A generalization of this last result was shown by V. Delecroix and A. Zorich [**DZ**] for more complicated obstacles. In the present work we study this last variant, corresponding to a billiard in the plane endowed with $\mathbb{Z}^2$-periodic obstacles of right-angled polygonal shape, aligned along the lattice and horizontally and vertically symmetric. See Figure 2 for an example.

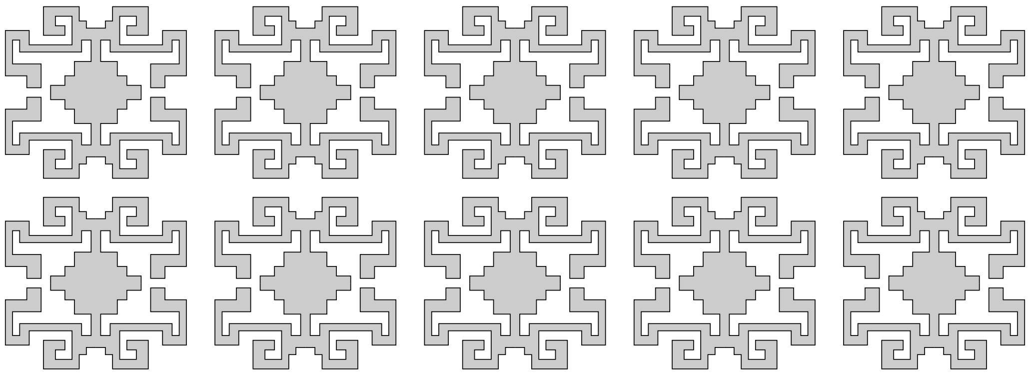

FIGURE 2. Delecroix–Zorich variant.

We are concerned with asymptotic formulas for the number of (isotopy classes of) periodic trajectories on the wind-tree model. This question has been widely studied in the context of (finite area) rational billiards and compact flat surfaces, and it is related to many other questions such as the calculation of the volume of normalized strata [**EMZ**] or the sum of Lyapunov exponents of the geodesic Teichmüller flow [**EKZ**] on strata of flat surfaces (Abelian or quadratic differentials).

H. Masur [**Ma88, Ma90**] proved that for every flat surface (resp. rational billiard) $X$, there exist positive constants $c(X)$ and $C(X)$ such that the number $N(X,L)$ of maximal cylinders of closed geodesics (resp. isotopy classes of periodic trajectories) of length at most $L$ satisfies

$$c(X)L^2 \leq N(X,L) \leq C(X)L^2$$

for $L$ large enough. W. Veech, in his seminal work [**Ve89**], proved that for Veech surfaces (resp. billiards) there are in fact exact quadratic asymptotics:

$$N(X,L) = c(X)L^2 + o(L^2).$$

In this work study the error term in this kind of asymptotic formulas. In the compact case, the methods used by W. Veech [**Ve89**] give the following result (see [**Ve92**, Remark 1.12]).

THEOREM (Veech). *Let $X$ be a Veech surface. Then, there exists $c(X) > 0$ and $\delta(X) \in [1/2, 1)$ such that*

$$N(X,L) = c(X)L^2 + O(L^{2\delta(X)}) + O(L^{4/3})$$

*as $L \to \infty$.*

Furthermore, the number $\delta(X)$ has a specific interpretation in terms of spectral properties of the Veech group.

**Asymptotic formulas for wind-tree models.** In [**Pa1**], we proved asymptotic formulas for generic wind-tree models with respect to a natural Lebesgue-type measure on the parameters of the wind-tree billiards, that is, the side lengths of the obstacles (cf. [**AEZ, DZ**]) and gave the exact value of the quadratic coefficient, which depends only in the number of corners of the obstacle (see [**Pa1**] for more details on the counting problem on wind-tree models). Asymptotic formulas were also given in the case of Veech wind-tree billiards, that is, wind-tree billiards such that the underlying compact translation surface is a Veech surface[1] (see §2.3 for precise definitions).

In the present work, we present an effective version of this result, that is, the analogue of Veech's Theorem, for Veech wind-tree billiards. More precisely, we prove the following.

THEOREM 1.1. *Let* $\Pi$ *be a Veech wind-tree billiard. Then, there exists* $c(\Pi) > 0$ *and* $\delta(\Pi) \in (1/2, 1)$ *such that*

$$N(\Pi, L) = c(\Pi)L^2 + O(L^{2\delta(\Pi)}) + O(L^{4/3})$$

*as* $L \to \infty$.

This result relies, on one hand, in the adaptation of Veech methods to our context, which allows to keep track one well behaved part of periodic trajectories on wind-tree billiards (*good cylinders*, see §2.4). On the other hand, there is a family of badly behaved trajectories (*bad cylinders*, see §2.4) which we attack using tools from hyperbolic geometry. Thanks to ideas of F. Dal'Bo [**Da**], we are able to relate the error term for this family with the Poincaré critical exponent of an associated subgroup of the Veech group. We prove then that this critical exponents is strictly less than 1 using results of R. Brooks [**Br**] (see also [**RT**]).

**Explicit estimates.** In the simplest case, when $\Pi$ is the wind-tree billiard with square obstacles of side length 1/2, the Veech group of $\Pi$ can be easily described and most of the involved objects can be explicitly computed, such as the contribution on the error term of the well behaved part of the periodic trajectories. Using results of T. Roblin and S. Tapie [**RT**], we explicitly estimate the contribution of the badly behaved family of periodic trajectories. More precisely, we prove the following.

THEOREM 1.2. *Let* $\Pi$ *be the Veech wind-tree billiard with square obstacles of side length* 1/2*, and let* $\delta = \delta(\Pi) \in (1/2, 1)$ *be as in the conclusion of Theorem 1.1. Then,*

$$\delta < 0.9992.$$

**Strategy of the proof.** W. Veech [**Ve89**] proved that for Veech surfaces, there are exact quadratic asymptotics relating the Dirichlet series of their length spectrum to Eisenstein series associated to the cusps of their (lattice) Veech group. An application of Ikehara's tauberian theorem allows then him to conclude. An effective version of this last tool allows to quantify the error term in terms of spectral properties of the Veech group (see [**Ve92**, Remark 1.12]).

In [**Pa1**], we showed that the counting problem on wind-tree models can be reduced to the study of two families of cylinders in the associated translation surface,

[1]We stress that this definition of "Veech wind-tree billiard" is not standard.

these are called *good* and *bad* cylinders (see §2.4.1, for the precise definition). The notion of good cylinders was first introduced by A. Avila and P. Hubert [**AH**] in order to give a geometric criterion for recurrence of $\mathbb{Z}^d$-periodic translation surfaces.

Applying Veech's method to the counting problem on Veech wind-tree models, we are able to prove the analogous result in the case of good cylinders, that is, to give the order of the error term in terms of ad-hoc spectral properties of the Veech group of the underlying surface. This is possible because the collection of good cylinders is $\mathrm{SL}(2,\mathbb{R})$-equivariant and then, there is a simple description of good cylinders in terms of some particular cusps of the Veech group, which allows to connect the counting problem to the corresponding Eisenstein series as Veech did.

In the case of bad cylinders, this approach does not work anymore since this family is not $\mathrm{SL}(2,\mathbb{R})$-equivariant and there is no simple description of bad cylinders in terms of (cusps of) the Veech group of the underlying surface. However, bad cylinders can be described in terms of some intricate but well described subgroup $\Gamma_{bad}$ of the Veech group. Using tools from hyperbolic geometry, thanks to ideas of F. Dal'Bo [**Da**], we prove that the leading term on the counting of bad cylinders is related to the critical exponent of this subgroup $\Gamma_{bad}$.

Using results of R. Brooks [**Br**], we prove that this critical exponent is strictly less than 1. We use the representation of the Veech group given by the restriction of the Kontsevich–Zorich cocycle to a corresponding equivariant subbundle of the real Hodge bundle. The kernel of this representation is a subgroup of $\Gamma_{bad}$. One first application of Brooks results allows us to show that the critical exponents of these two groups coincide. A second application shows that the critical exponent of the kernel of the representation is strictly less than that of the Veech group, which equals 1.

The number $\delta(\Pi)$ in the statement of Theorem 1.1, giving the order of the error term, is completely defined by spectral properties of the involved groups. More precisely, it is the maximum between the critical exponent of the group $\Gamma_{bad}$, associated to bad cylinders, and the second largest pole of the meromorphic continuation of (linear combination of) Eisenstein series, associated to good cylinders. The 4/3 in the conclusion of Theorem 1.1 appears because of technicalities in the effective version of the tauberian theorem for Eisenstein series ([**Ve92**, Remark 1.12]).

In the case when $\Pi$ is the wind-tree billiard with square obstacles of side length 1/2, the Veech group of $\Pi$ is a congruence subgroup of level 2. Thanks to a result of M. Huxley [**Hu85**], we know that low level congruence groups satisfies the Selberg's 1/4 conjecture. To our purposes, this means that the Eisenstein series has no poles in $(1/2, 1)$. The critical exponent of $\Gamma_{bad}$ requires much more attention and we are not able to give the exact value. Using results of T. Roblin and S. Tapie [**RT**], we estimate the critical exponent of $\Gamma_{bad}$. These estimates are far away from being optimal, but up to our knowledge, this is the only existing tool.

In order to apply this method to estimate the critical exponent of $\Gamma_{bad}$, we have first to give energy estimates on a Dirichlet fundamental domain of the Veech group and to estimate the bottom of the spectrum of the combinatorial Laplace operator associated to the quotient of the Veech group by $\Gamma_{bad}$, which turn out to be isomorphic to $\mathrm{PSL}(2,\mathbb{Z})$.

It is very likely that estimates (or even, the exact value) of the bottom of the combinatorial spectrum of $\mathrm{PSL}(2,\mathbb{Z})$ are known by the experts. However, we could not find any clue of this and we give our own estimates.

**Structure of the paper.** In §2 we briefly recall all the background necessary to formulate and prove the results. In §3 we study the counting problem on Veech surfaces associated to collections of cylinders described by a subgroup of the Veech group. We restate Veech's theorem in the case when the subgroup is a lattice and we relate the growth rate to the critical exponent for general subgroups of the Veech group. In §4 we apply this results to the counting problem on Veech wind-tree billiards. Veech's theorem is applied to good cylinders, giving the quadratic asymptotic growth rate with the error term depending in the spectrum of the Veech group. We show that bad cylinders are described by an infinitely generated Fuchsian group of the first kind and prove that its critical exponent is strictly less than one, showing thus the subquadratic asymptotic growth rate of bad cylinders in an effective way.

Finally, in §5 we study the case of the wind-tree billiard with square obstacles of side length 1/2. We estimate the critical exponent of the group associated to bad cylinders. In order to perform this, we give energy estimates in Appendix A and we estimate the combinatorial specrum of $\mathrm{PSL}(2,\mathbb{Z})$ in Appendix B. Both appendices are self contained and can be read independently of the rest of the paper.

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## 2. Background

**2.1. Rational billiards and translation surfaces.** For an introduction and general references to this subject, we refer the reader to the surveys of Masur–Tabachnikov [**MT**], Zorich [**Zo06**], Forni–Matheus [**FM**], Wright [**Wr**].

2.1.1. *Rational billiards.* Given a polygon whose angles are rational multiples of $\pi$, consider the trajectories of an ideal point mass, that moves at a constant speed without friction in the interior of the polygon and enjoys elastic collisions with the boundary (angles of incidence and reflection are equal). Such an object is called a rational billiard. There is a classical construction of a translation surface from a rational billiard (see [**FK, KZ**]).

2.1.2. *Translation surfaces.* Let $g \geq 1$, $\mathbf{n} = \{n_1, \dots, n_k\}$ be a partition of $2g-2$ and $\mathcal{H}(\mathbf{n})$ denote a stratum of Abelian differentials, that is, holomorphic 1-forms on Riemann surfaces of genus $g$, with zeros of degrees $n_1, \dots, n_k \in \mathbb{N}$. There is a one to one correspondence between Abelian differentials and translation surfaces, surfaces which can be obtained by edge-to-edge gluing of polygons in $\mathbb{R}^2$ using translations only. Thus, we refer to elements of $\mathcal{H}(\mathbf{n})$ as translation surfaces.

A translation surface has a canonical flat metric, the one obtained form $\mathbb{R}^2$, with conical singularities of angle $2\pi(n+1)$ at zeros of degree $n$ of the Abelian differential.

2.1.3. $\mathrm{SL}(2,\mathbb{R})$-*action.* There is a natural action of $\mathrm{SL}(2,\mathbb{R})$ on strata of translation surfaces, coming from the linear action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^2$, which generalizes the action of $\mathrm{SL}(2,\mathbb{R})$ on the space $\mathrm{GL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ of flat tori. Let $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$; the action of $(g_t)_{t\in\mathbb{R}}$ is called the Teichmüller geodesic flow.

2.1.4. *Hodge bundle and the Kontsevich–Zorich cocycle.* The (real) Hodge bundle $H^1$ is the real vector bundle of dimension $2g$ over an affine invariant manifold $\mathcal{M}$ (see [**EMi, EMM**] for the precise definition), where the fiber over $X$ is the real cohomology $H^1_X = H^1(X,\mathbb{R})$. Each fiber $H^1_X$ has a natural lattice $H^1_X(\mathbb{Z}) = H^1(X,\mathbb{Z})$ which allows identification of nearby fibers and definition of the Gauss–Manin (flat) connection. The monodromy of the Gauss–Manin connection restricted to $\mathrm{SL}(2,\mathbb{R})$-orbits provides a cocycle called the Kontsevich–Zorich cocycle, which we denote by $\mathrm{KZ}(A,X)$, for $A \in \mathrm{SL}(2,\mathbb{R})$ and $X \in \mathcal{M}$. The Kontsevich–Zorich cocycle is a symplectic cocycle preserving the symplectic intersection form $\langle f_1, f_2\rangle = \int_S f_1 \wedge f_2$ on $H^1(X,\mathbb{R})$.

2.1.5. *Lyapunov exponents.* Given any affine invariant manifold $\mathcal{M}$, we know from Oseledets theorem that there are real numbers $\lambda_1(\mathcal{M}) \geq \cdots \geq \lambda_{2g}(\mathcal{M})$, the Lyapunov exponents of the Kontsevich–Zorich cocycle over the Teichmüller flow on $\mathcal{M}$ and a measurable $g_t$-equivariant filtration of the Hodge bundle $H^1(X,\mathbb{R}) = V_1(X) \supset \cdots \supset V_{2g}(X) = \{0\}$ at $\nu_{\mathcal{M}}$-almost every $X \in \mathcal{M}$ such that

$$\lim_{t\to\infty} \frac{1}{t} \log \|\mathrm{KZ}(g_t, X) f\|_{g_t\omega} = \lambda_i$$

for every $f \in V_i \setminus V_{i+1}$.

The fact that the Kontsevich–Zorich cocycle is symplectic implies that the Lyapunov spectrum is symmetric, $\lambda_j = -\lambda_{2g-j}$, $j = 0, \dots, g$.

2.1.6. *Equivariant subbundles of the Hodge bundle.* Let $\mathcal{M}$ be an affine invariant submanifold and $F$ a subbundle of the Hodge bundle over $\mathcal{M}$. We say that $F$ is equivariant if it is invariant under the Kontsevich–Zorich cocycle. Since $\mathcal{M}$ is $\mathrm{SL}(2,\mathbb{R})$-invariant, by the definition of the Kontsevich–Zorich cocycle, a flat (locally constant) subbundle is always equivariant.

We say that $F$ admit an almost invariant splitting, if there exists $n \geq 1$ and for $\nu_{\mathcal{M}}$-almost every $X \in \mathcal{M}$ there exist proper subspaces $W_1(X), \dots, W_n(X) \subset F_X$ such that $W_i(X) \cap W_j(X) = \{0\}$ for $1 \leq i < j \leq n$, such that, for every $i \in \{1, \dots, n\}$ and almost every $A \in \mathrm{SL}(2,\mathbb{R})$, $\mathrm{KZ}(A,X)W_i(X) = W_j(AX)$ for some $j \in \{1, \dots, n\}$, and such that the map $X \mapsto \{W_1(X), \dots, W_n(X)\}$ is $\nu_{\mathcal{M}}$-measurable. We say that $F$ is strongly irreducible if is does not admit an almost invariant splitting.

Previous discussion about Lyapunov exponents applies in this context as well and we have that, as before, for almost every $X \in \mathcal{M}$, there is a measurable $g_t$-equivariant filtration $F_X = U_1(X) \supset \cdots \supset U_r(X) = \{0\}$, where $r = \operatorname{rank} F = \dim F_X$ and, for every $f \in U_i \setminus U_{i+1}$,

$$\lim_{t\to\infty} \frac{1}{t} \log \|\mathrm{KZ}(g_t, X)f\|_{g_t r_\theta \omega} = \lambda_i(\mathcal{M}, F).$$

We denote by $F_X(\mathbb{Z}) = F_X \cap H^1_X(\mathbb{Z})$ the set of integer cocycles in $F_X$. We say that $F$ is defined over $\mathbb{Z}$ if it is generated by integer cocycles, that is, if $F_X = \langle F_X(\mathbb{Z})\rangle_{\mathbb{R}}$. When $F$ is defined over $\mathbb{Z}$, $F_X(\mathbb{Z})$ is a lattice in $F_X$.

2.1.7. *Veech group and Veech surfaces.* We denote the stabilizer of a translation surface $X$ under the action of $\mathrm{SL}(2,\mathbb{R})$ by $\mathrm{SL}(X)$. The group $\mathrm{SL}(X)$ is also the group of derivatives of affine orientation-preserving diffeomorphisms of $X$.

Recall that $\mathrm{SL}(2,\mathbb{R})$ does not act faithfully on the upper half-plane $\mathbb{H}$; it is the projective group $\mathrm{PSL}(2,\mathbb{R})$ that does so. If $G$ is a subgroup of $\mathrm{SL}(2,\mathbb{R})$, we denote by $\mathrm{P}G$ its image in $\mathrm{PSL}(2,\mathbb{R})$. In a slight abuse of notation we sometimes shall omit P whenever it is clear from the context that we see $G$ as a subgroup of $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{PSL}(2,\mathbb{R})$. We define the *Veech Group* of $X$ to be $\mathrm{PSL}(X)$, that is, the image of $\mathrm{SL}(X)$ in $\mathrm{PSL}(2,\mathbb{R})$.

A translation surface $X$ is called *Veech surface* if its Veech group $\mathrm{PSL}(X)$ is a lattice, that is, if $\mathbb{H}/\mathrm{PSL}(X)$ has finite volume. Veech surfaces correspond to closed $\mathrm{SL}(2,\mathbb{R})$-orbits. Such a closed orbits is called a Teichmüller curve. In this work we are devoted to Veech surfaces. For an introduction and general references to Veech surfaces, we refer the reader to the survey of Hubert–Shcmidt [**HS**].

2.1.8. *Veech group representation.* When $A \in \mathrm{SL}(X)$, the Kontsevich–Zorich cocycle defines a symplectic map $\mathrm{KZ}(A,X) : H^1_X \to H^1_X$ which preserves $H^1_X(\mathbb{Z})$. This defines thus a representation $\rho_{H^1}$ of $\mathrm{SL}(X)$ on the symplectic group $\mathrm{Sp}(H^1_X,\mathbb{Z})$,

$$\begin{array}{rccc} \rho_{H^1}: & \mathrm{SL}(X) & \to & \mathrm{Sp}(H^1_X,\mathbb{Z}), \\ & A & \mapsto & \mathrm{KZ}(A,X). \end{array}$$

If $F$ is an equivariant subbundle, then the restriction of the Kontsevich–Zorich cocycle to $F$ gives another representation which, is not faithful in general and, we denote by $\rho_F : \mathrm{SL}(X) \to \mathrm{SL}(F_X)$. Note that this representation is not symplectic nor defined over $\mathbb{Z}$ in general. However, if the subbundle is symplectic or defined over $\mathbb{Z}$, so is the representation.

If $-id \notin \mathrm{SL}(X)$, every representation $\rho_F$ descends to a representation of $\mathrm{PSL}(X)$ on $\mathrm{PSL}(F_X)$. Nevertheless, if $-id \in \mathrm{SL}(X)$, this is not true in general. In fact, for a symplectic equivariant subbundles $F$ defined over $\mathbb{Z}$, $\rho_F(-id) \in \mathrm{Sp}(F_X,\mathbb{Z})$ is a symplectic permutation matrix which might not be in the center of $\mathrm{Sp}(F_X)$, so we cannot descend to the corresponding projective groups. Anyway, if $F_X$ is two dimensional, the only symplectic permutation matrices are $\pm id_{F_X}$ and therefore, $\rho_F$ descends to a representation of $\mathrm{PSL}(X)$ on $\mathrm{PSL}(F_X)$.

**2.2. Counting problem.** We are interested in the counting of closed geodesics of bounded length on translation surfaces.

2.2.1. *Cylinders.* Together with every closed regular geodesic in a translation surface $X$ we have a bunch of parallel closed regular geodesics. A cylinder on a translation surface is a maximal open annulus filled by isotopic simple closed regular geodesics. A cylinder $C$ is isometric to the product of an open interval and a circle, and its core curve $\gamma_C$ is the geodesic projecting to the middle of the interval.

2.2.2. *Holonomy.* Integrating the corresponding Abelian differential along the core curve of a cylinder or, more generally, any homology class $\gamma \in H_1(X, \mathbb{Z})$, we get a complex number. Considered as a planar vector, it represents the affine holonomy along $\gamma$ and we denote this holonomy vector by $\mathrm{hol}(\gamma)$. In particular, the euclidean length of a cylinder corresponds to the modulus of its holonomy vector.

A relevant equivariant subbundle is given by $\ker \mathrm{hol}$ which in turn is the symplectic complement of the so called tautological (sub)bundle.

2.2.3. *Counting problem.* Consider the collection of all cylinders on a translation surface $X$ and consider its image $V(X) \subset \mathbb{R}^2$ under the holonomy map, $V(X) = \{\mathrm{hol}\, \gamma_C : C \text{ is a cylinder in } X\}$. This is a discrete set of $\mathbb{R}^2$. We are concerned with the asymptotic behavior of the number $N(X, L) = \# V(X) \cap B(L)$ of cylinders in $X$ of length at most $L$, when $L \to \infty$.

More generally, we can consider any collection of cylinders $\mathcal{C} \subset \mathcal{A}$, and study the asymptotic behavior of the number of cylinders in $\mathcal{C}$ of length at most $L$, $N_{\mathcal{C}}(X, L) = \# V_{\mathcal{C}}(X) \cap B(L)$, as $L \to \infty$, where $V_{\mathcal{C}}(X) = \{\mathrm{hol}\, \gamma_C : C \in \mathcal{C}\}$.

**2.3. Wind-tree model.** The wind-tree model corresponds to a billiard $\Pi$ in the plane endowed with $\mathbb{Z}^2$-periodic horizontally and vertically symmetric right-angled obstacles, where the sides of the obstacles are aligned along the lattice as in Figure 1 and Figure 2.

Recall that in the classical case of a billiard in a rectangle we can glue a flat torus out of four copies of the billiard table and unfold billiard trajectories to flat geodesics of the same length on the resulting flat torus. In the case of the wind-tree model we also start from gluing a translation surface out of four copies of the infinite billiard table $\Pi$. The resulting surface $X_\infty = X_\infty(\Pi)$ is $\mathbb{Z}^2$-periodic with respect to translations by vectors of the original lattice. Passing to the $\mathbb{Z}^2$-quotient we get a compact translation surface $X = X(\Pi)$. For the case of the original wind-tree billiard, with rectangular obstacles, the resulting translation surface is represented at Figure 3 (see [**DHL**, § 3] for more details).

Similarly, when the obstacle has $4m$ corners with the angle $\pi/2$ (and therefore, $4m-4$ with angle $3\pi/2$), the same construction gives a translation surface consisting in four flat tori with holes —four copies of a $\mathbb{Z}^2$-fundamental domain of $\Pi$, the holes corresponding to the obstacles— with corresponding identifications, as in the classical setting ($m = 1$, see Figure 3).

2.3.1. *Description of the $\mathbb{Z}^2$-covering and relevant subbundles.* There are two cohomology classes $h, v \in H^1(X, \mathbb{Z})$ defining the $\mathbb{Z}^2$-covering $X_\infty$ of $X$. Let $\mathcal{M}$ be the $\mathrm{SL}(2, \mathbb{R})$-orbit closure of $X$. Then, thanks to the symmetries of $X$, there are two equivariant subbundles $F^{(h)}$ and $F^{(v)}$ of $H^1$ defined over $\mathcal{M}$, such that $h \in F^{(h)}$ and $v \in F^{(v)}$ (see [**Pa1**] for more details). Furthermore, we have the following (see [**Pa1**, Corollary 5]).

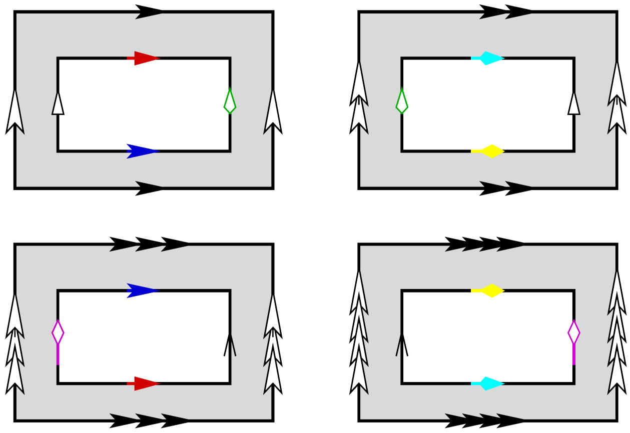

FIGURE 3. The translation surface $X$ obtained as quotient over $\mathbb{Z}^2$ of an unfolded wind-tree billiard table ([**DZ**, Figure 5]).

THEOREM 2.1. *Let* $\Pi$ *be a wind-tree billiard,* $X = X(\Pi)$. *Then, the subbundles* $F^{(h)}$ *and* $F^{(v)}$ *defined over the* $\mathrm{SL}(2,\mathbb{R})$*-orbit closure of* $X$ *are* 2*-dimensional flat subbundles defined over* $\mathbb{Z}$ *and have non-zero Lyapunov exponents.*

As consequence, these subbundles are strongly irreducible and symplectic. Indeed, by [**AEM**, Theorem 1.4] and [**EMi**, Theorem A.9], any measurable equivariant subbundle with at least one non-zero Lyapunov exponent is symplectic and, in particular, even dimensional. Thus, a two-dimensional subbundle is automatically strongly irreducible provided it has non-zero Lyapunov exponents. Furthermore, these subbundles are subbundles of $\ker \mathrm{hol}$.

2.3.2. *The* $(1/2, 1/2)$ *wind-tree model.* We give a little more details in the case of the wind-tree billiard with square obstacles of side length $1/2$, $\Pi = \Pi(1/2, 1/2)$.

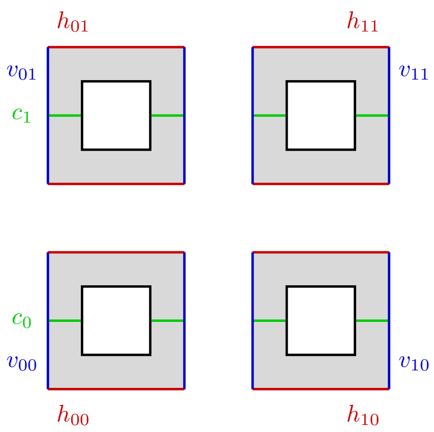


FIGURE 4. The surface $X = X(\Pi(1/2, 1/2))$ and the cycles $h_{ij}$, $v_{ij}$ and $c_j$, $i, j \in \{0, 1\}$ (cf. [**DHL**, Figure 4]).

The surface $X = X(\Pi)$ is a covering of a genus 2 surface $L$ which is a so called L-shaped surface that belongs to the stratum $\mathcal{H}(2)$ (see for example [**DHL**]). In particular, $\mathrm{SL}(X) \subset \mathrm{SL}(L)$. In this case, $L$ is a square-tiled surface, tiled by 3 squares, see Figure 5.

It is elementary to see that $\mathrm{SL}(L) = \langle r, u^2 \rangle$, where $r = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $u = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ (see for example [**Zo06**, §9.5]). Moreover, it is not difficult to verify that $\mathrm{SL}(X) = \mathrm{SL}(L)$.

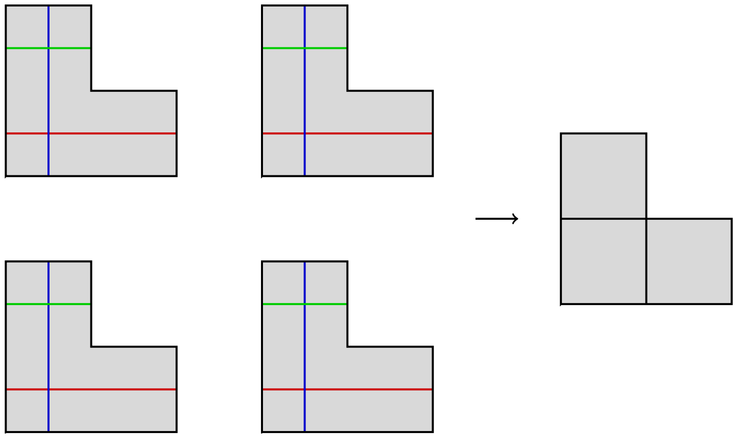

FIGURE 5. The surface $X = X(\Pi(1/2, 1/2))$ seen as a cover of the square-tiled L-shaped surface $L$.

For $i, j \in \{0, 1\}$, let $h_{ij}$, $v_{ij}$ and $c_j$ be as in Figure 4. Let $E^{+-}$ be the subspace of $H^1(X, \mathbb{R})$ with symplectic integer basis $\{h^{+-}, v^{+-}\}$, where $h^{+-}$ is the Poincaré dual of the cycle $h_{00}+h_{01}-h_{10}-h_{11}$ and $v^{+-}$, of $v_{00}+v_{01}-v_{10}-v_{11}$. Similarly, define $E^{-+}$, with basis $\{h^{-+}, v^{-+}\}$, $h^{-+} = (h_{00}-h_{01}+h_{10}-h_{11})^*$ and $v^{-+} = (v_{00}-v_{01}+v_{10}-v_{11})^*$.

We have that $F_X^{(h)} = E^{+-}$, $h = h^{+-}$, $F_X^{(v)} = E^{-+}$ and $v = v^{-+}$.

The action of $u^2 \in \mathrm{SL}(X)$ on the $h_{ij}, v_{ij}$, $i, j \in \{0, 1\}$ is shown in Figure 6 and is described by

$$\rho_{H^1}(u^2) : \quad \begin{aligned} h_{ij}^* &\mapsto h_{ij}^* \\ v_{ij}^* &\mapsto v_{ij}^* + h_{ij}^* + c_j^*, \end{aligned}$$

and therefore, for $\sigma \in \{+-, -+\}$, we have that

$$\rho_{E^\sigma}(u^2) : \quad \begin{aligned} h^\sigma &\mapsto h^\sigma \\ v^\sigma &\mapsto v^\sigma + h^\sigma. \end{aligned}$$

Thus, with the right choice of basis for $F = F^{(h)}$ or $F^{(v)}$, we get $\rho_F(u^2) = u$. Similarly, we can find that $\rho_F(r) = r$. In particular, $\rho_F(\mathrm{SL}(X)) \cong \mathrm{SL}(2, \mathbb{Z})$, for $F = F^{(h)}, F^{(v)}$.

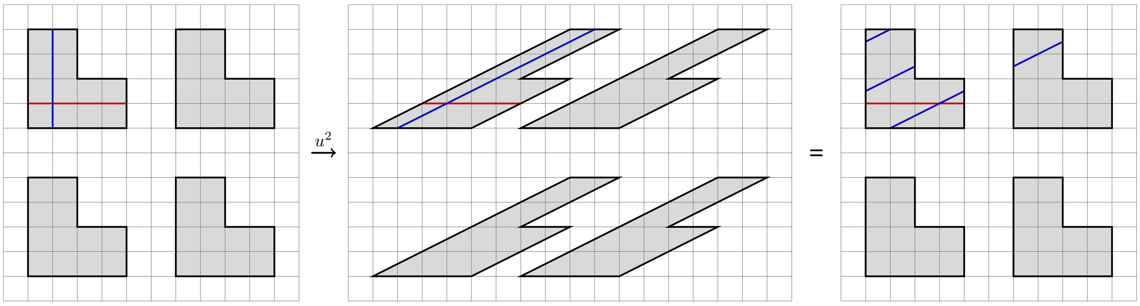


FIGURE 6. The action of $u^2$ on $h_{ij}, v_{ij}$, $i, j \in \{0, 1\}$.

**2.4. Counting problem on wind-tree models.** In this work, we are concerned with counting periodic trajectories in the wind-tree billiard. Obviously, any periodic trajectory can be translated by an element in $\mathbb{Z}^2$ to obtain a new (non-isotopic) periodic trajectory. Then, we shall count (isotopy classes of) periodic trajectories of bounded length in the wind-tree billiard, up to $\mathbb{Z}^2$-translations.

There is a one to one correspondence between billiard trajectories in $\Pi$ and geodesics in $X_\infty$. But $X_\infty$ is the $\mathbb{Z}^2$-covering of $X$ given by $h, v \in H^1(X, \mathbb{Z})$, which means that closed curves $\gamma$ in $X$ lift to closed curves in $X_\infty$ if and only if $h(\gamma_C) = v(\gamma_C) = 0$. In fact, by definition of the covering, the monodromy of a closed curve $\gamma$ in $X$ is the translation by $(h(\gamma), v(\gamma)) \in \mathbb{Z}^2$. The cylinders in the cover $X_\infty$ are exactly the lift of those cylinders $C$ in $X$ whose core curve $\gamma_C$ has trivial monodromy. In particular, cylinders in $X_\infty$ are always isometric to their projection on $X$. When a cylinder $C$ does not satisfy this condition, it lifts to $X_\infty$ as a strip, isometric to the product of an open interval and a straight line.

2.4.1. *Good and bad cylinders.* Let $f = h$ or $v$, and $F = F^{(f)}$. Note that cylinders $C$ in $X$ such that $f(\gamma_C) = 0$, split naturally into two families: (a) the family of cylinders such that $\hat{f}(\gamma_C) = 0$ for all $\hat{f} \in F_X$, that is, $\gamma_C \in \mathrm{Ann}(F_X)$, which we call $F$-*good cylinders*, and (b) the family of cylinders that are not $F$-good, but $f(\gamma_C) = 0$. These later are called $(F, f)$-*bad cylinders*. The notion of $F$-good cylinders was first introduced by Avila–Hubert [**AH**] in order to give a geometric criterion for recurrence of $\mathbb{Z}^d$-periodic flat surfaces. Good cylinders are favorable to our purposes. In fact, since the Kontsevich–Zorich cocycle preserves the intersection form and $F$ is equivariant, they define an $\mathrm{SL}(2, \mathbb{R})$-equivariant family of cylinders, which is much more tractable than arbitrary collections of cylinders.

For a wind-tree billiard $\Pi$, we denote by $N(\Pi, L)$, the number of (isotopy classes of) periodic trajectories (up to $\mathbb{Z}^2$-translations) of length at most $L$, by $N_{good}(X, L)$ the number of $F^{(h)} \oplus F^{(v)}$-good cylinders in $X = X(\Pi)$ of length at most $L$ and $N_{f-bad}(X, L)$, of $(F, f)$-bad cylinders in $X$ of length at most $L$, for $f = h$ or $v$ and $F = F^{(f)}$. Note that

$$N_{good}(X, L) \leq N(\Pi, L) \leq N_{good}(X, L) + N_{h-bad}(X, L) + N_{v-bad}(X, L).$$

Thus, it is enough to understand the asymptotic behavior of $N_{good}(X, L)$, $N_{h-bad}(X, L)$ and $N_{v-bad}(X, L)$ separately.

The author [**Pa1**] used this to reduce the counting problem on wind-tree models to the counting of good cylinders. In fact, we have the following.

THEOREM 2.2 ([**Pa1**, Theorem 1.3]). *Let $\Pi$ be a wind-tree billiard, $X = X(\Pi)$ the associated compact flat surface, let $f = h$ or $v$ and $F = F^{(f)}$ be one of the associated subbundles $F^{(h)}$ or $F^{(v)}$. Then, the number $N_{f-bad}(X, L)$, of $(F, f)$-bad cylinders in $X$ of length at most $L$, has* subquadratic *asymptotic growth rate, that is, $N_{f-bad}(X, L) = o(L^2)$.*

Thus, the counting problem on wind-tree models may be reduced to count $F^{(h)} \oplus F^{(v)}$-good cylinders, which has quadratic asymptotic growth rate thanks to a result of Eskin–Masur [**EMa**]. However, in this work, we are interested in an effective version and therefore, bad cylinders have to be taken into account.

REMARK 2.3. An useful characterization of bad cylinders in our case is the following. A cylinder $C$ is $(F, f)$-bad if and only if $\mathrm{pr}_{F_X}\gamma_C = \pm f$. In fact, since $F$ is symplectic and two dimensional, $C$ is an $(F, f)$-bad cylinder if and only if $\mathrm{pr}_{F_X}\gamma_C \neq 0$ is colinear to $f$ (see [**Pa1**, Remark 3.1]). Moreover, the action of $\mathrm{SL}(2, \mathbb{R})$ on homology (that is, the Kontsevich–Zorich cocycle) is by integer matrices, then, this is equivalent to say that $\mathrm{pr}_{F_X}\gamma_C = \pm f$.

2.4.2. *Veech wind-tree billiards.* Let $\Pi$ be a wind-tree billiard. We define the *Veech group of* $\Pi$ to be $\mathrm{PSL}(\Pi) = \mathrm{PSL}(X(\Pi))$ and we say that $\Pi$ is a *Veech wind-tree billiard* if $\mathrm{PSL}(\Pi)$ is a lattice. We stress that these definitions are not standard as it does not correspond to the (projection to $\mathrm{PSL}(2, \mathbb{R})$ of the) derivatives of affine orientation-preserving diffeomorphisms of the unfolded billiard $X_\infty(\Pi)$, but to those of $X(\Pi)$, the $\mathbb{Z}^2$-quotient of the unfolded billiard.

In the classical case, of rectangular obstacles, we denote $\Pi(a, b)$ the wind-tree billiard with rectangular obstacles of side lengths $a, b \in \,]0, 1[$. Thank to results of Calta [**Ca**] and McMullen [**McM03, McM05**], it is possible to classify completely Veech wind-tree models in the classical case (see [**DHL**, Theorem 3]).

THEOREM 2.4. *The wind-tree model* $\Pi(a, b)$ *is a Veech wind-tree billiard if and only if either* $a, b \in \mathbb{Q}$ *or there exist* $x, y \in \mathbb{Q}$ *and a square-free integer* $D > 1$ *such that* $1/(1 - a) = x + y\sqrt{D}$ *and* $1/(1 - b) = (1 - x) + y\sqrt{D}$.

In this work we are concerned only with Veech wind-tree billiards. Most of the tools we use to deal with bad cylinders comes from geometric considerations of the action (on the upper half-plane $\mathbb{H}$) of the lattice Veech group $\mathrm{PSL}(\Pi)$ and, more precisely, of some particular subgroups of $\mathrm{PSL}(\Pi)$. These groups are Fuchsian groups. In the following, we present a brief recall of the objects we need and some of their properties.

**2.5. Fuchsian groups.** A Fuchsian group is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$. A Fuchsian group $\Gamma$ acts properly discontinuously on $\mathbb{H}$. In particular, the orbit $\Gamma z$ of any point $z \in \mathbb{H}$ under the action of $\Gamma$ has no accumulation points in $\mathbb{H}$. There may, however, be limit points on the real axis. Let $\Lambda(\Gamma)$ be the limit set of $\Gamma$, that is, the set of limits points for the action of $\Gamma$ on $\overline{\mathbb{H}}$, $\Lambda(\Gamma) \subset \overline{\mathbb{R}}$. The limit set may be empty, or may contain one or two points, or may contain an infinite number. A Fuchsian group is of the first type if its limit set is the closed real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. This happens in the case of lattices, but there are Fuchsian groups of the first kind of infinite covolume. These latter are always infinitely generated.

When the limit set is finite, we say that $\Gamma$ is elementary. In such case, $\Gamma$ is cyclic.

In this work we shall mainly handle two type of Fuchsian groups. The first are Veech groups of Veech surfaces, which are lattices by definition and the other are the subgroups of the Veech group given by $\mathrm{Pker}\,\rho_F$, for equivariant subbundles $F \subset H^1$. Recall that $\rho_F : \mathrm{SL}(X) \to \mathrm{SL}(F_X)$. Thus, $\ker \rho_F$ is a subgroup of $\mathrm{SL}(X)$, $\mathrm{Pker}\,\rho_F$ is the image of $\ker \rho_F$ in $\mathrm{PSL}(X)$.

The following result allows us to better understand these groups when $F$ is a 2-dimensional subbundle of $\ker \mathrm{hol}$.

THEOREM 2.5 ([**HoWei**, Theorem 5.6]). *Let $X$ be a Veech surface and $F$ an integer (defined over $\mathbb{Z}$) 2-dimensional subbundle of* $\ker \mathrm{hol}$ *over the* $\mathrm{SL}(2,\mathbb{R})$*-orbit of $X$. Then,* $\mathrm{Pker}\, \rho_F$ *is a Fuchsian group of the first kind.*

In particular, in the case of Veech wind-tree billiards, the hypothesis are satisfied by the subbundles $F^{(h)}$ and $F^{(v)}$ and therefore, $\mathrm{Pker}\, \rho_F$ is a Fuchsian group of the first kind for $F = F^{(h)}, F^{(v)}$.

2.5.1. *Critical exponent.* Another concept which is of major relevance in this work is that of the critical exponent of a Fuchsian group. For an introduction to the subject, we refer the reader to Peigné [**Pe**].

Let $\Gamma$ be a Fuchsian group. The orbital function $n_\Gamma : \mathbb{R}_+ \to \mathbb{N}$ is defined by $n_\Gamma(R) = \#\{g \in \Gamma : \ d_{\mathbb{H}}(i, gi) \leq R\}$. The exponent

$$\delta(\Gamma) := \limsup_{R\to\infty} \frac{1}{R} \ln n_\Gamma(R)$$

is the critical exponent of $\Gamma$. It corresponds to the critical exponent (the abscissa of convergence in $\mathbb{R}_+$) of the Poincaré series defined by

$$P_\Gamma(s) := \sum_{g\in\Gamma} e^{-s d_{\mathbb{H}}(i,gi)}.$$

That is, $P_\Gamma(s)$ diverges for $s < \delta(\Gamma)$ and converges for $s > \delta(\Gamma)$.

Note that in the definition of the critical exponent $\delta(\Gamma)$ it is innocuous if we change $d_{\mathbb{H}}(i, gi)$ for $d_{\mathbb{H}}(x, gy)$, for some $x, y \in \mathbb{H}$ or, in particular, if we change $\Gamma$ for some conjugate of $\Gamma$, either in the definition of the orbital function $n_\Gamma$ or in the Poincaré series $P_\Gamma$.

A result of Roblin [**Ro99, Ro02**] relates in a sharper way the asymptotic behavior of the orbital function and the critical exponent.

THEOREM 2.6 (Roblin). *Let $\Gamma$ be a non-elementary Fuchsian group. Then*

$$n_\Gamma(r) = O(e^{\delta r}),$$

*as $L \to \infty$.*

Consider now the following sungroups of $\mathrm{PSL}(2,\mathbb{R})$:

- $K = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \ \theta \in [0,\pi) \right\}$,
- $A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : \ t \in \mathbb{R} \right\}$, and
- $N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : \ t \in \mathbb{R} \right\}$.

Every element $g \in \mathrm{PSL}(2,\mathbb{R}) \setminus \{1\}$ is conjugated to some element in $K$, $A$ or $N$. In fact, we have the following:

- $|\operatorname{tr}(g)| < 2$ if and only if $g$ is conjugated to some element of $K$. In this case $g$ is called elliptic and it fixes exactly one point in $\overline{\mathbb{H}}$, which belongs to $\mathbb{H}$;
- $|\operatorname{tr}(g)| > 2$ if and only if $g$ is conjugated to some element of $A$. In this case $g$ is called hyperbolic and it fixes exactly two point in $\overline{\mathbb{H}}$, which belongs to $\partial\overline{\mathbb{H}} = \overline{\mathbb{R}}$; and

- $|\operatorname{tr}(g)| = 2$ if and only if $g$ is conjugated to some (and therefore, to every) element of $N$. In this case $g$ is called parabolic and it fixes exactly one point in $\overline{\mathbb{H}}$, which belongs to $\partial\mathbb{H}$.

If $\Gamma$ is a non-elementary Fuchsian group, it has positive critical exponent $\delta(\Gamma) > 0$ and if it contains a parabolic element, then $\delta(\Gamma) > 1/2$.

One of the main ingredients we use to prove our results is the following result of Brooks [**Br**] (see also [**RT**]).

THEOREM 2.7 (Brooks). *Let $\Gamma_0$ be a Fuchsian group and $\Gamma$ be a non-elementary normal subgroup of $\Gamma_0$ such that $\delta(\Gamma) > 1/2$.*

*(1) If $\Gamma_0/\Gamma$ is amenable, then $\delta(\Gamma) = \delta(\Gamma_0)$.*
*(2) If $\Gamma_0$ is a lattice and $\Gamma_0/\Gamma$ is non-amenable, then $\delta(\Gamma) < \delta(\Gamma_0) = 1$.*

This last result is based on the fact that the critical exponent $\delta(\Gamma)$ is related to $\lambda_0(\Gamma)$, the bottom of the spectrum of the Laplace operator on $\mathbb{H}/\Gamma$. In fact, when $\delta(\Gamma) \geq 1/2$, we have that $\lambda_0(\Gamma) = \delta(\Gamma)(1 - \delta(\Gamma))$ (see for example [**RT**]).

## 3. Counting problems on Veech surfaces

Let $X$ be a Veech surface, that is, $X$ is a translation surface whose Veech group $\mathrm{PSL}(X)$ is a (non-uniform) lattice. In particular, $\mathbb{H}/\mathrm{PSL}(X)$ has a finite number of cusps. It is well known (since Veech [**Ve89**]) that, for Veech surfaces, cylinders correspond to the cusps of the Veech group and, in particular, the family of all cylinders can be written as the union of a finite number of $\mathrm{SL}(X)$-orbit of cylinders. That is, there are finitely many cylinders $A_1, \ldots, A_n$ in $X$ such that

$$\mathcal{A} := \{\text{all cylinders in } X\} = \mathrm{SL}(X) \cdot \{A_j\}_{j=1}^n.$$

In particular, any collection $\mathcal{C} \subset \mathcal{A}$ of cylinders is contained in a finite union of cusps, in the sense that it satisfies $\mathcal{C} \subset \mathrm{SL}(X) \cdot \mathbf{C}$, for some *finite* collection $\mathbf{C} \subset \mathcal{C}$.

**3.1. Finitely saturated collections of cylinders.** Let $\Gamma$ be a subgroup of $\mathrm{SL}(X)$. A collection $\mathcal{C}$ of cylinders in $X$ is said to be *finitely saturated by* $\Gamma$ (or $\Gamma$-*finitely saturated*) if it can be expressed as a finite union of $\Gamma$-orbits of cylinders and $\Gamma$ contains every cusp. More precisely, $\mathcal{C}$ is finitely saturated by $\Gamma$ if $\mathcal{C} = \Gamma \cdot \mathbf{C}$, for some finite collection $\mathbf{C} \subset \mathcal{C}$ and $stab_{\mathrm{SL}(X)}(C) \subset \Gamma$ for every $C \in \mathcal{C}$. Equivalently, we can ask $stab_{\mathrm{SL}(X)}(C) \subset \Gamma$ only for $C \in \mathbf{C}$.

Thus, as already said in different terms, the collection $\mathcal{A}$ of all cylinders in $X$ is $\mathrm{SL}(X)$-finitely saturated.

REMARK 3.1. In the definition of finitely saturated collections of cylinders, the *finite* part is fundamental. Consider, for example, the group $\Gamma$ generated by all parabolics in $\mathrm{SL}(X)$. Then, when the Teichmüller curve defined by $X$ has positive genus[2], $\mathcal{A}$ is saturated by $\Gamma$, but it is not $\Gamma$-finitely saturated.

In general, any $\mathrm{SL}(2,\mathbb{R})$-equivariant collection of cylinders (defined in the $\mathrm{SL}(2,\mathbb{R})$-orbit of $X$) is $\mathrm{SL}(X)$-finitely saturated. In particular, configurations of cylinders, in the sense of Eskin–Masur–Zorich [**EMZ**], define $\mathrm{SL}(X)$-finitely saturated collections of cylinders.

[2]See [**HL**] for examples of Teichmüller curves with arbitrary large genus in a fixed stratum.

However, in this work, we have to deal with collections of cylinders which are finitely saturated by groups which are not lattices as $\mathrm{SL}(X)$ is. In fact, we have to deal with groups which are not even finitely generated.

Remark 3.2. If $\Gamma$ is a Fuchsian group such that a (non-empty) collection of cylinders $\mathcal{C}$ is finitely saturated by $\Gamma$, then, by definition, $stab_{\mathrm{SL}(X)}(C) \subset \Gamma$ for every $C \in \mathcal{C}$. But $\mathrm{P}stab_{\mathrm{SL}(X)}(C)$ is cyclic parabolic. Thus, $\Gamma$ contains parabolics and therefore $\delta(\Gamma) \geq 1/2$, with equality if and only if $\Gamma$ is elementary (and $\mathcal{C}$ is a finite collection of parallel cylinders).

**3.2. Counting problem.** We are interested in counting cylinders in some particular collections. Let $\mathcal{C}$ be a collection of cylinders in $X$ and let $N_{\mathcal{C}}(X, L)$ be the number of cylinders in $C$ of length at most $L$. We are able to study the asymptotic behavior in the case of finitely saturated collections.

In the case of $\mathcal{A}$, the collection of all cylinders in $X$, Veech proved the quadratic asymptotic behavior in [**Ve89**] and gave then an effective version in [**Ve92**, Remark 1.12]. In the case of collections of cylinders saturated by lattice groups, Veech's approach can be applied exactly the same. In fact, we have the following.

Theorem 3.3 (Veech). *Let $X$ be a Veech surface and let $\mathcal{C}$ be a $\Gamma$-finitely saturated collection of cylinders on $X$ with $\Gamma$ being a lattice. Then*

$$N_{\mathcal{C}}(X, L) = c(\mathcal{C})L^2 + \sum_{j=1}^{k} c_j(\mathcal{C})L^{2\delta_j} + O(L^{4/3}),$$

*as $L \to \infty$, for some $c(\mathcal{C}), c_1(\mathcal{C}), \ldots, c_k(\mathcal{C}) > 0$, where $\{\delta_j(1-\delta_j)\}_{j=1}^{k}$ is the discrete spectrum of the Laplace operator on $\mathbb{H}/\Gamma$ on $(0, 1/4)$. In particular, $\delta_j \in (1/2, 1)$, for $j = 1, \ldots, k$. Possibly $k = 0$.*

Proof. For $\mathcal{C} = \mathcal{A}$, the collection of all cylinders in $X$ (which is finitely saturated by $\Gamma = \mathrm{SL}(X)$), Veech proved in [**Ve89**] the principal term $c(\mathcal{C})L^2$. The remainder was observed in [**Ve92**, Remark 1.12], by an application of [**Gd**, Theorem 4]. The proof relies only in the fact that $\mathcal{A}$ is finitely saturated by a lattice group, namely $\mathrm{SL}(X)$. Thus, in the case of collections finitely saturated by a lattice group, the proof follows exactly the same. □

In the case of infinite covolume groups this method cannot be adapted properly. However, following ideas of Dal'Bo [**Da**], we are able to prove the following.

Theorem 3.4. *Let $X$ be a Veech surface and $\mathcal{C}$, a $\Gamma$-finitely saturated collection of cylinders on $X$ with $\Gamma$ non-elementary. Let $\delta = \delta(\Gamma)$ be the critical exponent of $\Gamma$. In particular, $\delta > 1/2$. Then,*

$$N_{\mathcal{C}}(X, L) = O(L^{2\delta}),$$

*as $L \to \infty$.*

Proof. Without loss of generality, we can assume that $\mathcal{C} = \Gamma \cdot C$, for some cylinder $C$ in $X$. Let $p = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $P = \langle p \rangle$ and $x = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Up to conjugation, we can suppose that $\mathrm{hol}(\gamma_C) = x$ and $stab_{\mathrm{SL}(X)}(C) = P$. Note that $\delta$ is invariant by conjugation, so there is no loss of generality. Denote $N_\Gamma(L) := N_{\Gamma \cdot C}(X, L)$. The idea is to relate $N_\Gamma$ to $n_\Gamma$ in order to apply Theorem 2.6.

It is clear that

$$\begin{aligned} N_\Gamma(L) &= \#\{gx : |gx| \leq L,\, g \in \Gamma\} \\ &= \#\{gP \in \Gamma\big/P : |gx| \leq L\} \\ &= \#\{Pg \in P\big\backslash\Gamma : |g^{-1}x| \leq L\}. \end{aligned}$$

A simple computation shows that $|g^{-1}x| = \operatorname{Im}(gi)^{-1/2}$. In addition, for each coset in $P\big\backslash\Gamma$, there is exactly one representative $g \in \Gamma$ such that $\operatorname{Re}(gi) \in [0,1)$. Thus,

$$\begin{aligned} N_\Gamma(L) &= \#\{Pg \in P\big\backslash\Gamma : |g^{-1}x| \leq L\} \\ &= \#\{g \in \Gamma : \operatorname{Re}(gi) \in [0,1)\,,\ \operatorname{Im}(gi)^{-1/2} \leq L\}. \end{aligned}$$

Moreover, there exists $c(\Gamma) > 0$ such that if $g \in \Gamma$ satisfies $\operatorname{Re}(gi) \in [0,1)$, then $d_{\mathbb{H}}(i, gi) \leq -\ln \operatorname{Im}(gi) + c(\Gamma)$. In fact, let $g \in \Gamma$. Note first that $\operatorname{Im}(gi)$ is bounded above, since $P$ is a subgroup of $\Gamma$ (we have a cusp at infinity). In addition, we have that

$$d_{\mathbb{H}}(i, gi) = \operatorname{acosh}\left(1 + \frac{\operatorname{Re}(gi)^2 + (1 - \operatorname{Im}(gi))^2}{2\operatorname{Im}(gi)}\right)$$

and therefore, if $g \in \Gamma$ and $\operatorname{Re}(gi) \in [0,1)$, then

$$d_{\mathbb{H}}(i, gi) \leq \operatorname{acosh}\left(1 + \frac{\tilde{c}(\Gamma)}{\operatorname{Im}(gi)}\right),$$

for some $\tilde{c}(\Gamma) > 0$. Once again, since $\operatorname{Im}(gi)$ is bounded above, we get that

$$d_{\mathbb{H}}(i, gi) \leq \ln\left(\frac{1}{\operatorname{Im}(gi)}\right) + c(\Gamma),$$

for some $c(\Gamma) > 0$.

It follows that

$$\begin{aligned} N_\Gamma(L) &= \#\{g \in \Gamma : \operatorname{Re}(gi) \in [0,1)\,,\ \operatorname{Im}(gi)^{-1/2} \leq L\} \\ &\leq \#\{g \in \Gamma : d_{\mathbb{H}}(i, gi) \leq 2\ln L + c(\Gamma)\} \\ &= n_\Gamma(2\ln L + c(\Gamma)). \end{aligned}$$

Finally, by Theorem 2.6, $n_\Gamma(r) = O(e^{\delta(\Gamma)r})$ and thus

$$N_\Gamma(L) \leq n_\Gamma(2\ln L + c(\Gamma)) = O(e^{\delta(\Gamma)(2\ln L + c(\Gamma))}) = O(L^{2\delta(\Gamma)}).$$

□

## 4. Veech wind-tree billiards

In [**Pa1**], we proved asymptotic formulas for generic wind-tree models. To prove such result, we had to split the associated collection of cylinders into two. The collection of *good cylinders* and the collection of *bad cylinders* (see §2.4.1). We proved then that good cylinders have quadratic asymptotic growth rate (and gave the associated coefficient in the generic case) and that bad cylinders have sub-quadratic asymptotic growth rate.

In this work we exhibit a quantitative version of these results in the case of Veech wind-tree billiards.

**4.1. Good cylinders.** Being a good cylinder is a $\mathrm{SL}(2,\mathbb{R})$-invariant condition, then, in particular, for Veech wind-tree billiards $\Pi$, with Veech group $\mathrm{PSL}(\Pi)$ (see §2.4.2), the collection of good cylinders is $\mathrm{SL}(\Pi)$-finitely saturated (see §3.1) and thus, as a corollary of Veech's theorem (Theorem 3.3), we obtain the following.

Corollary 4.1. *Let $\Pi$ be a Veech wind-tree billiard. Then, there exists $c(\Pi) > 0$ and $\delta_{good}(\Pi) \in [1/2, 1)$ such that*

$$N_{good}(\Pi, L) = c(\Pi) \cdot \frac{\pi L^2}{\mathrm{Area}\,(\Pi/\mathbb{Z}^2)} + O(L^{2\delta_{good}(\Pi)}) + O(L^{4/3})$$

*as $L \to \infty$, where $\delta = \delta_{good}(\Pi)$ is such that $\delta(1-\delta)$ is the second smallest eigenvalue of the Laplace operator on $\mathbb{H}/\mathrm{PSL}(\Pi)$, $\delta(1-\delta) \in (0, 1/4]$.*

**4.2. Bad cylinders.** In the case of bad cylinders, Veech's approach is no longer possible since collection of bad cylinders is not $\mathrm{SL}(2,\mathbb{R})$-equivariant and, in particular, bad cylinders are not $\mathrm{SL}(\Pi)$-finitely saturated. However, it is finitely saturated by a subgroup $\Gamma_{bad}$ of $\mathrm{SL}(\Pi)$, so we can use the approach on Theorem 3.4.

Remark 4.2. We shall see that $\Gamma_{bad}$ is quite intricate. It is a not normal subgroup of $\mathrm{SL}(\Pi)$ and it is an infinitely generated Fuchsian group of the first kind.

By this means, we prove that bad cylinders have sub-quadratic asymptotic growth rate in an effective way. More precisely, we prove the following.

Theorem 4.3. *Let $\Pi$ be a Veech wind-tree billiard. Then, there exists $\delta_{bad}(\Pi) \in (1/2, 1)$ such that*

$$N_{bad}(\Pi, L) = O(L^{2\delta_{bad}(\Pi)})$$

*as $L \to \infty$.*

Proof. Let $f = h, v$ and $F = F^{(f)}$. Henceforth, by bad cylinder we mean $(F, f)$-bad cylinder. Recall that a cylinder $C$ in $X = X(\Pi)$ is a bad cylinder if and only if $\mathrm{pr}_F \gamma_C = \pm f$ (see Remark 2.3).

Let $\mathcal{B}$ be the collection of all bad cylinders in $X$. Then, since the collection of all cylinders can be written as a finite union of $\mathrm{SL}(X)$-orbits of cylinders, then there is a finite collection of bad cylinders $\mathbf{B}$ such that $\mathcal{B} \subset \mathrm{SL}(X) \cdot \mathbf{B}$.

Now, given a bad cylinder $B$ in $X$, define

$$\Gamma_{bad}(B) \coloneqq \{g \in \mathrm{SL}(X) : g \cdot B \text{ is a bad cylinder}\},$$

so that

$$\mathcal{B} = \bigcup_{B \in \mathbf{B}} \Gamma_{bad}(B) \cdot B.$$

Since $B$ is a bad cylinder if and only if $\mathrm{pr}_F \gamma_C = \pm f$, then $g \in \Gamma_{bad}(B)$ if and only if $\mathrm{pr}_F \gamma_{g \cdot B} = \pm f$. But $\mathrm{pr}_F \gamma_{g \cdot B} = \mathrm{pr}_F \rho_{H^1}(g) \gamma_B = \rho_F(g) \mathrm{pr}_F \gamma_B = \rho_F(g)(\pm f)$, where $\rho_F$ denotes the representation of $\mathrm{SL}(X)$ on $\mathrm{Sp}(F_X, \mathbb{Z})$ (see §2.1.8 and §2.3.1). It follows then that

$$\Gamma_{bad}(B) = \Gamma_{bad} \coloneqq \{g \in \mathrm{SL}(X) : \rho_F(g) f = \pm f\},$$

which is a group and does not depend on $B \in \mathcal{B}$. Thus, $\mathcal{B} = \Gamma_{bad} \cdot \mathbf{B}$. Moreover, if $B \in \mathcal{B}$ and $p \in stab_{\mathrm{SL}(X)}(B)$, then $p \cdot B = B$, which is a bad cylinder. Therefore, $p \in \Gamma_{bad}(B) = \Gamma_{bad}$ and $\mathcal{B}$ is finitely saturated by $\Gamma_{bad}$ (see §3.1).

We can apply then Theorem 3.4 to obtain $\delta_{bad}(\Pi) = \delta(\Gamma_{bad})$. To conclude, we have to prove that $\delta(\Gamma_{bad}) < 1$.

Proposition 4.4. *The critical exponent of* $\Gamma_{bad}$ *is strictly less than one.*

It follows then, by Proposition 4.4 and Theorem 3.4, that $N_{bad}(\Pi, L) = O(L^{2\delta_{bad}(\Pi)})$ as $L \to \infty$, where $\delta_{bad}(\Pi) = \delta(\Gamma_{bad}) \in (1/2, 1)$. This proves Theorem 4.3. □

To conclude, we have to prove now Proposition 4.4.

Proof of Proposition 4.4. Consider the normal subgroup of $\mathrm{SL}(X)$ given by $\ker \rho_F$ and note that it is also a subgroup of $\Gamma_{bad}$.

Since the action on homology is via (symplectic) integer matrices, then

$$\rho_F(\Gamma_{bad}) \subset stab(\pm f) := \{\hat{g} \in \mathrm{Sp}(F_X, \mathbb{Z}) : \hat{g} f = \pm f\}.$$

Since $F_X$ is two-dimensional, $\mathrm{Sp}(F_X, \mathbb{Z}) \cong \mathrm{SL}(2, \mathbb{Z})$ and $stab(\pm f) \cong stab_{\mathrm{SL}(2,\mathbb{Z})}(\pm\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right))$, which is virtually cyclic parabolic. Thus, the quotient group $\Gamma_{bad}/\ker \rho_F \cong \rho_F(\Gamma_{bad})$ is amenable (as it is isomorphic to a subgroup of an amenable group).

In a slight abuse of notation we will refer in the following to (discrete) subgroups of $\mathrm{SL}(2, \mathbb{R})$ as if they were Fuchsian groups (discrete subgroups of $\mathrm{PSL}(2, \mathbb{R})$).

By Theorem 2.5, $\ker \rho_F$ is of the first kind and, in particular, non-elementary. Thus, we can apply Theorem 2.7 to obtain that $\delta(\Gamma_{bad}) = \delta(\ker \rho_F)$.

Consider now the quotient group $\mathrm{SL}(X)/\ker \rho_F \cong \rho_F(\mathrm{SL}(X))$. The aim is to prove that $\rho_F(\mathrm{SL}(X))$ is not amenable. We first note that, since $F$ has positive Lyapunov exponents (Theorem 2.1), $\rho_F(\mathrm{SL}(X))$ has at least one hyperbolic element and then, a maximal cyclic hyperbolic subgroup $H$. Suppose $\rho_F(\mathrm{SL}(X))$ is elementary and, in particular, virtually $H$. But then, $F$ would admit an almost invariant splitting (see §2.1.6). But $F$ is two dimensional and has no zero Lyapunov exponents, in particular, it is strongly irreducible and do not admit almost invariant splittings. Thus $\rho_F(\mathrm{SL}(X))$ is non-elementary and it contains a Schottky group as subgroup.

Since Schottky groups are free and, in particular, non-amenable, it follows that $\rho_F(\mathrm{SL}(X))$ is not amenable. That is, $\mathrm{SL}(X)/\ker \rho_F$ is not amenable, and then, by Theorem 2.7, we have that $\delta(\ker \rho_F) < \delta(\mathrm{SL}(X))$. Thus, we conclude that

$$\delta(\Gamma_{bad}) = \delta(\ker \rho_F) < \delta(\mathrm{SL}(X)) = 1.$$

□

Proof of Remark 4.2. We have to show that $\Gamma_{bad}$ is an infinitely generated group of the first kind. Since $\ker \rho_F$ is of the first kind and $\ker \rho_F \subset \Gamma_{bad}$, so is $\Gamma_{bad}$. Moreover, $\delta(\Gamma_{bad}) < 1$, so it cannot be a lattice and therefore, it has to be infinitely generated, since finitely generated groups of the first kind are always lattices. □

## 5. Explicit estimates for the $(1/2, 1/2)$ wind-tree model

In the case of the wind-tree billiard with square obstacles of side length $1/2$, $\Pi = \Pi(1/2, 1/2)$, the Veech group can be easily computed (see §2.3.2). Indeed, $\mathrm{SL}(\Pi) = \langle r, u^2 \rangle$, where $r = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $u = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. In particular, $\mathrm{PSL}(\Pi)$ is a congruence subgroup of level 2.

**5.1. Good cylinders.** A result of Huxley [**Hu85**] shows that congruence groups $\Gamma$ of low level satisfies the Selberg's 1/4 conjecture, that is, that the bottom of the spectrum of the Laplace operator on $\mathbb{H}/\Gamma$ equals 1/4. That means (see §2.5.1) that we have $\delta_{good}(\Pi) = 1/2$ in Corollary 4.1.

**5.2. Bad cylinders.** We have now to estimate $\delta_{bad}(\Pi)$ from Theorem 4.3. For this, we use a version of Brook's theorem (Theorem 2.7) by Roblin–Tapie [**RT**], formulated in a much more general context, which we adapt to ours.

THEOREM 5.1 (Roblin–Tapie). *Let $\Gamma_0$ be a lattice and $\Gamma$ be a non-elementary normal subgroup of $\Gamma_0$ such that $\delta(\Gamma) > 1/2$. Let $\mathcal{D}$ be a Dirichlet domain for $\Gamma_0$ and $S_0$ the associated symmetric system of generators (see §5.2.2). Consider $G = \Gamma_0/\Gamma$ and $S = S_0/\Gamma$ the corresponding systems of generators of $G$. Then,*

$$\lambda_0(\Gamma) \geq \frac{\eta(\Gamma_0) E_{\mathcal{D}} \mu_0(G, S)}{\eta(\Gamma_0) + E_{\mathcal{D}} \mu_0(G, S)},$$

*where $\eta(\Gamma_0)$ is the spectral gap associated to $\Gamma_0$ (see §5.2.1), $E_{\mathcal{D}}$ is any lower bound for the energy on $\mathcal{D}$ (see §5.2.3) and $\mu_0(G, S)$ is the bottom of the combinatorial spectrum of $G$ associated to $S$ (see §5.2.4), as defined below.*

5.2.1. *Critical exponent and spectrum of the Laplace operator.* Let $\Gamma$ be a non-elementary Fuchsian group with critical exponent $\delta(\Gamma) > 1/2$. Then, the critical exponent $\delta(\Gamma)$ is related to $\lambda_0(\Gamma)$, the bottom of the spectrum of the Laplace operator on $\mathbb{H}/\Gamma$, by $\lambda_0(\Gamma) = \delta(\Gamma)(1 - \delta(\Gamma)) \in (0, 1/4)$.

If moreover $\Gamma$ is finitely generated, then the bottom of the spectrum $\lambda_0(\Gamma)$ is an isolated eigenvalue. We consider then the *spectral gap of the Laplace operator on* $\mathbb{H}/\Gamma$, $\eta(\Gamma) := \lambda_1(\Gamma) - \lambda_0(\Gamma) > 0$, where $\lambda_1(\Gamma)$ is the second smallest eigenvalue of the Laplace operator on $\mathbb{H}/\Gamma$.

5.2.2. *Dirichlet domains and transition zones.* Let $\Gamma$ be a finitely generated Fuchsian group and consider a Dirichlet domain $\mathcal{D} \subset \mathbb{H}$ for the action of $\Gamma$. Its boundary $\partial\mathcal{D}$ is piecewise geodesic, with finitely many pieces. To $\mathcal{D}$, we can associate a finite symmetric system of generators $S$ of $\Gamma$. Each such generator $s \in S$ is associated to one geodesic piece of $\partial\mathcal{D}$. Namely, $\beta_s = \mathcal{D} \cap s\mathcal{D}$. And every geodesic piece of $\partial\mathcal{D}$ has an associated generator in this way.

We say that $L, R > 0$ are *admisible (for $\mathcal{D}$)* if for each $s \in S$, there exists a geodesic segment $\alpha_s \subset \beta_s$ of length $L$ such that $\alpha_s = s\alpha_{s^{-1}}$ and such that $\alpha_s$ admits a tubular neighborhood of radius $R$ which are pairwise disjoint (see Appendix A for more details). These tubular neighborhoods are *transition zones* of length $L$ and radius $R$ (cf. [**RT**, p. 72]).

5.2.3. *Energy on transition zones.* Roblin–Tapie [**RT**] introduced the volume and capacity of transition zones (in a much more general context). In our context, for a transition zone of length $L$ and radius $R$, its *area* is $\mathrm{A}(L, R) := L \cdot \sinh(R)$ and its *capacity* is $\mathrm{C}(L, R) := L/\arctan(\sinh(R))$. We say that $E_{\mathcal{D}} \in \mathbb{R}_+$ is a *lower bound for the energy on $\mathcal{D}$* if there are admissible $L, R > 0$ such that

$$E_{\mathcal{D}} = \frac{1}{2\,\mathrm{Area}(\mathcal{D})} \cdot \frac{\eta(\Gamma) \cdot \mathrm{A}(L, R) \cdot \mathrm{C}(L, R)}{\left(\sqrt{\eta(\Gamma) \cdot \mathrm{A}(L, R)} + \sqrt{\mathrm{C}(L, R)}\right)^2}.$$

In Appendix A we estimate $E_{\mathcal{D}}$ in the case of the Dirichlet domain of the Veech group of $\Pi$, $\mathcal{D} = \{z \in \mathbb{H} : |z| \geq 1, \, |\operatorname{Re}(z)| \leq 1\}$, with associated system of generators $S_0 = \{r, u^2\}$.

5.2.4. *Combinatorial spectrum.* Let $G$ be a finitely generated group and $S \subset G$ be a symmetric finite system of generators of $G$.

Let $\ell^2(G)$ be the space of square-summable sequences on $G$ with the inner product

$$\langle h, h' \rangle := \sum_{g \in G} h_g h'_g,$$

for $h, h' \in \ell^2(G)$, and define $\Delta_S : \ell^2(G) \to \ell^2(G)$, the combinatorial Laplace operator associated to $S$ on $\ell^2(G)$, by

$$(\Delta_S h)_g := \sum_{s \in S} (h_g - h_{gs}).$$

Then, we define $\mu_0(G, S)$, *the bottom of the combinatorial spectrum of* $G$ *associated to* $S$ to be the bottom of the spectrum of $\Delta_S$, that is,

$$\mu_0(G, S) := \inf \left\{ \frac{\langle \Delta_S h, h \rangle}{\langle h, h \rangle}, \; h \in \ell^2(G) \right\}.$$

We estimate $\mu_0(G, S)$ in the case of $G = \mathrm{PSL}(2, \mathbb{Z})$ and $S = \{r, u\}$ in Appendix B.

**Estimates for $\delta_{bad}(\Pi)$.** An application of Theorem 5.1 allows us to estimate $\delta_{bad}(\Pi)$ in the present case. More precisely, we have the following.

THEOREM 5.2. *Let $\Pi$ be the Veech wind-tree billiard with square obstacles of side length $1/2$, and let $\delta = \delta_{bad}(\Pi) \in (1/2, 1)$ be as in the conclusion of Theorem 4.3. Then,*

$$\delta < 0.9992.$$

PROOF. Following §4.2, we have that $\delta = \delta_{bad}(\Pi)$ corresponds to the critical exponent of the group $\Gamma_{bad}$[3]. Moreover, $\delta(\Gamma_{bad}) = \delta(\ker \rho_F)$. Let then $\delta = \delta(\ker \rho_F)$. As we have already seen, $\delta(1 - \delta) = \lambda_0(\ker \rho_F)$.

The idea is to apply Theorem 5.1 to $\Gamma_0 = \mathrm{PSL}(\Pi)$ and $\Gamma = \mathrm{P}\ker \rho_F$. Thus, it is enough to estimate

$$\frac{\eta(\Gamma_0) E_{\mathcal{D}} \mu_0(G, S)}{\eta(\Gamma_0) + E_{\mathcal{D}} \mu_0(G, S)}$$

from below.

Note that the function $x/(1+x)$ is an increasing function in $(0, \infty)$ and therefore, the problem can be reduced to find lower bounds for $\eta(\Gamma_0)$, $E_{\mathcal{D}}$ and $\mu_0(G, S)$.

- $\Gamma_0 = \langle r, u^2 \rangle$ is a level two congruence group and, as already seen in §5.1, its spectral gap is

$$\eta(\Gamma_0) = 1/4.$$

[3]Here, in a slight abuse of notation, we are referring to (discrete) subgroups of $\mathrm{SL}(2, \mathbb{R})$ as if they were Fuchsian groups (discrete subgroups of $\mathrm{PSL}(2, \mathbb{R})$).

- We consider the Dirichlet domain $\mathcal{D} = \{z \in \mathbb{H} : |z| \geq 1,\ |\operatorname{Re}(z)| \leq 1\}$ for $\Gamma_0$. We estimate $E_{\mathcal{D}}$ in Appendix A. By Theorem A.1, we have that

$$E_{\mathcal{D}} > 0.012.$$

- Recall that $\mathrm{SL}(\Pi) = \langle r, u^2 \rangle$ and that, with the right choice of basis for $F_X$, $\rho_F(r) = r$ and $\rho_F(u^2) = u$ (see §2.3.2).

  Moreover since $F$ is a 2-dimensional symplectic equivariant subbundle defined over $\mathbb{Z}$, $\rho_F$ descends to a representation $\tilde{\rho}_F$ of $\mathrm{PSL}(X)$ on $\mathrm{PSL}(F_X, \mathbb{Z})$ (see §2.1.8), where $X = X(\Pi)$. Furthermore, by definition, the kernel of this latter representation coincides with $\mathrm{Pker}\,\rho_F$, the image of $\ker \rho_F$ in $\mathrm{PSL}(2, \mathbb{R})$. Analogously, for the image of the representation we have $\tilde{\rho}_F(\mathrm{PSL}(X)) = \mathrm{P}\rho_F(\mathrm{SL}(X))$. In summary, we have
  - $\Gamma_0 = \mathrm{PSL}(\Pi) = \langle r, u^2 \rangle$,
  - $\Gamma = \mathrm{Pker}\,\rho_F = \ker \tilde{\rho}_F$,
  - $\Gamma_0 / \Gamma = \mathrm{PSL}(\Pi) / \ker \tilde{\rho}_F \cong \tilde{\rho}_F(\mathrm{PSL}(X)) = \mathrm{P}\rho_F(\mathrm{SL}(X))$, and
  - $\rho_F(\mathrm{SL}(X)) = \langle \rho_F(r), \rho_F(u^2) \rangle = \langle r, u \rangle = \mathrm{SL}(2, \mathbb{Z})$.

  The combinatorial spectrum is invariant under isomorphisms of groups (with generators). But $\Gamma_0 / \Gamma$ is isomorphic to $\mathrm{P}\rho_F(\mathrm{SL}(X))$ which in turn is isomorphic to $\mathrm{PSL}(2, \mathbb{Z})$. In addition, the system of generators associated to the Dirichlet domain $\mathcal{D}$ is $S_0 = \{r, u^2, u^{-2}\}$, and the corresponding image into $G = \mathrm{PSL}(2, \mathbb{Z})$ is $S = \{r, u, u^{-1}\}$.

  We estimate $\mu_0(G, S)$ in Appendix B. By Theorem B.2, we have that

$$\mu_0(G, S) > 0.07.$$

Putting all together, we get that

$$\lambda_0(\Gamma) \geq \frac{\eta(\Gamma_0) E_{\mathcal{D}} \mu_0(G, S)}{\eta(\Gamma_0) + E_{\mathcal{D}} \mu_0(G, S)} > 0.0008,$$

and we conclude that

$$\delta(\Gamma) = \frac{1 + \sqrt{1 - 4\lambda_0(\Gamma)}}{2} < 0.9992.$$

□

## Appendix A. Energy estimates

In this appendix we give lower bounds for the energy (see §A.2 for precise definition) on the Dirichlet domain $\mathcal{D} = \{z \in \mathbb{H} : |z| \geq 1,\ |\operatorname{Re}(z)| \leq 1\}$ of the Fuchsian group $\Gamma = \langle r, u^2 \rangle$, where $r = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $u = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. More precisely, we prove the following.

Theorem A.1. *Let $\mathcal{D} = \{z \in \mathbb{H} : |z| \geq 1,\ |\operatorname{Re}(z)| \leq 1\}$ be the Dirichlet domain of $\Gamma = \langle r, u^2 \rangle$. Then, there is a lower bound for the energy on $\mathcal{D}$ which satisfies*

$$E_{\mathcal{D}} > 0.012.$$

In the following we recall the definition of the involved objects (see [**RT**] for a much more general and detailed discussion).

**A.1. Dirichlet domains and transition zones.** Let $\Gamma$ be a finitely generated Fuchsian group and consider a Dirichlet domain $\mathcal{D} \subset \mathbb{H}$ for the action of $\Gamma$. Its boundary $\partial\mathcal{D}$ is piecewise geodesic, with finitely many pieces. To $\mathcal{D}$, we can associate a finite symmetric system of generators $S$ of $\Gamma$. To each such generator $s \in S$ we can associate one geodesic piece of $\partial\mathcal{D}$. Namely, $\beta_s = \mathcal{D} \cap s\mathcal{D}$. And every such piece has an associated generator in this way. Moreover, it is clear from the definition that $\beta_s = s\beta_{s^{-1}}$. In Figure A.1, we show the case of the elementary group $\langle u \rangle$.

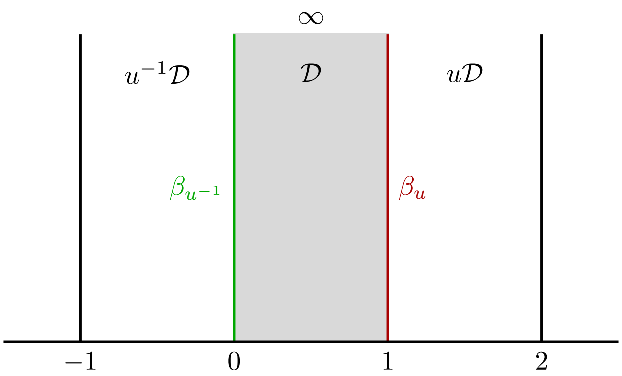


FIGURE A.1. Dirichlet domain for the elementary (cyclic parabolic) group $\langle u \rangle$, $\mathcal{D} = \{0 \leq \operatorname{Re} z \leq 1\}$. The associated symmetric systems of generators is $S = \{u, u^{-1}\}$ and the corresponding geodesic boundaries $\beta_u = \{\operatorname{Re} z = 1\}$, $\beta_{u^{-1}} = \{\operatorname{Re} z = 0\}$.

Let $z \in \mathring{\beta}_s$, for some $s \in S$, and let $\rho > 0$ sufficiently small such that there is a point $b_s(z, \rho) \in \mathcal{D}$ satisfying $d_{\mathbb{H}}(b_s(z, \rho), \beta_s) = d_{\mathbb{H}}(b_s(z, \rho), z) = \rho$. In particular, such point $b_s(z, \rho)$ is unique. See Figure A.2 for an example of $b_s(z, \rho)$, in the case of $\langle u \rangle$, for $s = u^{-1}$.

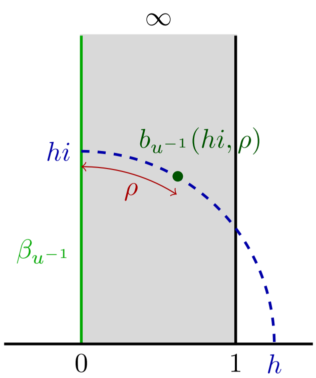


FIGURE A.2. The point $b_s(z, \rho)$. It corresponds to the point in $\mathcal{D}$ which lie on the geodesic passing through $hi$ perpendicularly to $\beta_{u^{-1}}$, in the case of the elementary group $\langle u \rangle$, for $s = u^{-1}$, $z = hi$

We say that $L, R > 0$ are *admisible (for $\mathcal{D}$)* if for each $s \in S$, there exists a geodesic segment $\alpha_s \subset \beta_s$ of length $L$ such that $\alpha_s = s\alpha_{s^{-1}}$, $b_s(z, R)$ is well defined and the sets

$$A_s := \{b_s(z,\rho) \in \mathcal{D} : \; z \in \alpha_s, \; 0 \le \rho < R\}$$

are pairwise disjoint. (see Figure A.3). We call these sets, *transition zones* of length $L$ and radius $R$ (cf. [**RT**, p. 72]).

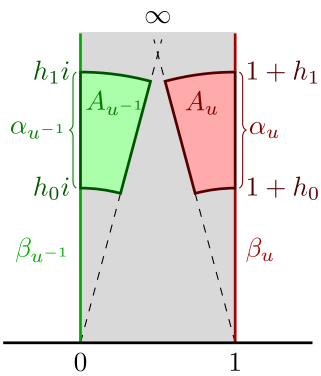


FIGURE A.3. Transition zones, in the case of the elementary group $\langle u \rangle$.

**A.2. Energy on transition zones.** Roblin–Tapie [**RT**] introduced the volume and capacity of transition zones (in a much more general context). In our context, for a transition zone of length $L$ and radius $R$, its *area* is $\mathrm{A}(L,R) := L \cdot \sinh(R)$ and its *capacity* is $\mathrm{C}(L,R) := L/\arctan(\sinh(R))$.

We say that $E_{\mathcal{D}} \in \mathbb{R}_+$ is a *lower bound for the energy on* $\mathcal{D}$ if there are admissible $L, R > 0$ such that

$$E_{\mathcal{D}} = \frac{1}{2\,\mathrm{Area}(\mathcal{D})} \cdot \frac{\eta(\Gamma)\cdot \mathrm{A}(L,R)\cdot \mathrm{C}(L,R)}{\left(\sqrt{\eta(\Gamma)\cdot \mathrm{A}(L,R)} + \sqrt{\mathrm{C}(L,R)}\right)^2},$$

where $\eta(\Gamma)$ is the spectral gap of the Laplace operator on $\mathbb{H}/\Gamma$, which is well defined and positive, since $\Gamma$ is finitely generated.

We can now start the discussion in the case of $\mathcal{D} = \{z \in \mathbb{H} : |z| \ge 1, \; |\mathrm{Re}(z)| \le 1\}$, the Dirichlet domain of $\Gamma = \langle r, u^2 \rangle$.

**A.3. Proof of Theorem A.1.** The following result, whose proof is postponed to §A.4, provides a sufficient condition for $L, R > 0$ to be admissible (see §A.1).

PROPOSITION A.2. *Let* $L, R > 0$. *If* $4e^L \tanh^2(R) \le 1$, *then* $L, R$ *are admissible.*

We want now to estimate $E_{\mathcal{D}}$ (see §A.2).

We first note that $\Gamma = \langle r, u^2 \rangle$ is a congruence group of level two and therefore, by a result of Huxley [**Hu85**], we have that $\eta(\Gamma) = 1/2$. Moreover, the Dirichlet domain $\mathcal{D}$ is an ideal triangle, with vertices $1$, $-1$ and $\infty$ (see Figure A.4). In particular, $\mathrm{Area}(\mathcal{D}) = \pi$.

By Proposition A.2, $L, R > 0$ are admissible if $4e^L \tanh^2(R) \le 1$. It suffices then to find the largest possible lower bound for the energy in this region. That is, we want to find $E^* = \max\{E_{\mathcal{D}}(L,R) : \; 4e^L \tanh^2 R \le 1\}$. This can be done numerically: we get $L^* \approx 2.286$, $R^* \approx 0.1608$ and

$$E^* = E_{\mathcal{D}}(L^*, R^*) \approx 0.01258 > 0.012.$$

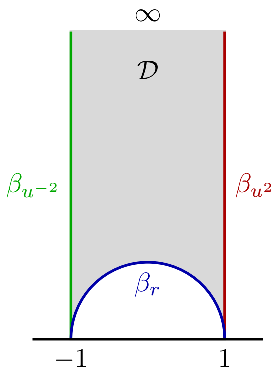


FIGURE A.4. Dirichlet domain $\mathcal{D} = \{z \in \mathbb{H} : |z| \geq 1, |\operatorname{Re}(z)| \leq 1\}$ for $\Gamma = \langle r, u^2 \rangle$. The associated symmetric systems of generators is $S = \{r, u, u^{-1}\}$ and the corresponding geodesic boundaries are $\beta_{u^{\pm 2}} = \{\operatorname{Re} z = \pm 1\}$, $\beta_r = \{|z| = 1\}$.

□

**A.4. Proof of Proposition A.2.** In this section we prove Proposition A.2, thus providing a sufficient condition for $L, R > 0$ to be admissible.

For $a, b \in \overline{\mathbb{R}}$, let $\gamma(a,b)$ denote the (bi-infinite) geodesic in $\mathbb{H}$ which goes from $a$ to $b$. And for $x, y, z \in \overline{\mathbb{H}}$, let $T(x,y,z)$ denote the geodesic triangle with vertices $x, y, z$. Thus, the Dirichlet domain $\mathcal{D} = \{z \in \mathbb{H} : |z| \geq 1, |\operatorname{Re}(z)| \leq 1\}$ coincides with $T(-1, 1, \infty)$ and $\partial\mathcal{D} = \gamma(-1,1) \cup \gamma(1,\infty) \cup \gamma(\infty,-1)$ (see Figure A.4). Note that the symmetric system of generators associated to $\mathcal{D}$ is $S = \{r, u^2, u^{-2}\}$ and, following the notation on §A.1, we have

$$\beta_r = \{z \in \mathbb{H} : \ |z| = 1\} = \gamma(-1, 1)$$
$$\beta_{u^{\pm 2}} = \{z \in \mathbb{H} : \ \operatorname{Re}(z) = \pm 1\} = \gamma(\pm 1, \infty).$$

It follows that, in particular, any geodesic segment $\alpha_{u^{\pm 2}} \subset \beta_{u^{\pm 2}}$ (see §A.1) is of the form $\{z \in \mathbb{H} : \ \operatorname{Re}(z) = \pm 1, \ h_0 < \operatorname{Im}(z) < h_1\}$, for some $h_1 > h_0 > 0$, with the same $h_1$ and $h_0$ for both $\alpha_{u^2}$ and $\alpha_{u^{-2}}$ since $\alpha_{u^2} = u^2 \alpha_{u^{-2}}$. And any geodesic segment $\alpha_r \in \beta_r$ is of the form $\{z \in \mathbb{H} : \ |z| = 1, \ |\operatorname{Re}(z)| < c\}$, for some $c > 0$, since $\alpha_r = r\alpha_r$.

For simplicity, we shall consider a "symmetric" partition of $\mathcal{D}$ as in Figure A.5, given by a homography $g$, defined by the matrix $\left(\begin{smallmatrix} 1 & 3 \\ -1 & 1 \end{smallmatrix}\right)$, which is an isometry of order 3 fixing $i\sqrt{3}$ and such that permutes cyclically $-1$, $1$ and $\infty$. In particular,

$$\begin{aligned} g: \quad \gamma(-1,1) &\mapsto \gamma(1,\infty), \\ \gamma(1,\infty) &\mapsto \gamma(\infty,-1), \\ \gamma(\infty,-1) &\mapsto \gamma(-1,1). \end{aligned}$$

Note that it corresponds to the elliptic element $g = \begin{pmatrix} 1/2 & 3/2 \\ -1/2 & 1/2 \end{pmatrix} \in \operatorname{PSL}(2,\mathbb{R})$. Thus, $g$ divides $\mathcal{D}$ in three isometric triangular regions. Namely, $T(-1, i\sqrt{3}, 1)$,

$T(1, i\sqrt{3}, \infty)$, $T(\infty, i\sqrt{3}, -1)$. Moreover, it is clear that

$$\begin{aligned} g: \quad T(-1, i\sqrt{3}, 1) &\mapsto T(1, i\sqrt{3}, \infty), \\ T(1, i\sqrt{3}, \infty) &\mapsto T(\infty, i\sqrt{3}, -1), \\ T(\infty, i\sqrt{3}, -1) &\mapsto T(-1, i\sqrt{3}, 1). \end{aligned}$$

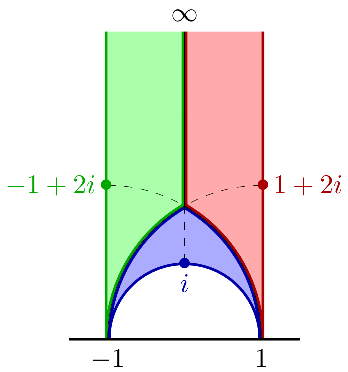


FIGURE A.5. Symmetric partition of $\mathcal{D}$ given by the homography defined by the elliptic element $g = \begin{pmatrix} 1/2 & 3/2 \\ -1/2 & 1/2 \end{pmatrix}$ of order 3.

In particular, if we consider the transition zones to be contained in these triangular regions, it is direct that they are pairwise disjoint. And since these regions are isometric, we can consider the transition zones to be isometric and interchanged by the isometry $g$. That is, we impose

$$\begin{aligned} g: \quad \alpha_r &\mapsto \alpha_{u^2}, \\ \alpha_{u^2} &\mapsto \alpha_{u^{-2}}, \\ \alpha_{u^{-2}} &\mapsto \alpha_r. \end{aligned}$$

Recall that $\alpha_r = r\alpha_r$, so $\alpha_r$ is "centered" at $i \in \mathbb{H}$. Then, by the imposed symmetry $\alpha_{u^{\pm 2}} = g^{\pm 1}(\alpha_r)$, we have that $\alpha_{u^{\pm 2}}$ has to be "centered" at $g^{\pm 1}(i) = 1 \pm 2i$. That means that $h_1 = 2e^{L/2}$ and $h_0 = 2e^{-L/2}$ in the definition of $\alpha_{u^{\pm 2}}$ (see above).

We have now to study the points $b_s(z, R)$, $s \in S$, $z \in \alpha_s$, in order to give conditions to $L, R$ to be admissible (see §A.1). Moreover, by the imposed symmetries, it is enough to find conditions for $b_0(h, R) := b_{u^2}(1 + 2hi, R)$ to be contained in $T_0 := T(1, i\sqrt{3}, \infty)$, for $h \in \left[1, e^{L/2}\right)$.

Now, by definition, $b_0(h, R)$ is the only point in $\mathcal{D}$ such that

$$d_{\mathbb{H}}(b_0(h, R), \beta_{u^2}) = d_{\mathbb{H}}(b_0(h, R), 1 + 2hi) = R,$$

for $R > 0$ small enough. By the leftmost equality, such points correspond to points in $\mathcal{D}$ which lie on the geodesic passing through $1 + 2hi$ perpendicularly to $\beta_{u^2}$ (see Figure A.6, cf. Figure A.2). That is, $b_0(h, R) = 1 + 2he^{i\theta(R)}i$, for some $\theta(R) > 0$. Moreover,

$$d_{\mathbb{H}}(1 + 2he^{i\theta}i, 1 + 2hi) = d_{\mathbb{H}}(e^{i\theta}i, i) = \operatorname{acosh}(\sec(\theta)).$$

Thus, $\cos(\theta(R)) = \operatorname{sech}(R)$ and therefore, $\sin(\theta(R)) = \tanh(R)$. It follows that

$$b_{u^2}(1 + 2hi, R) = b_0(h, R) = 1 + 2he^{i\theta(R)}i = 1 - 2h\tanh(R) + i2h\operatorname{sech}(R).$$

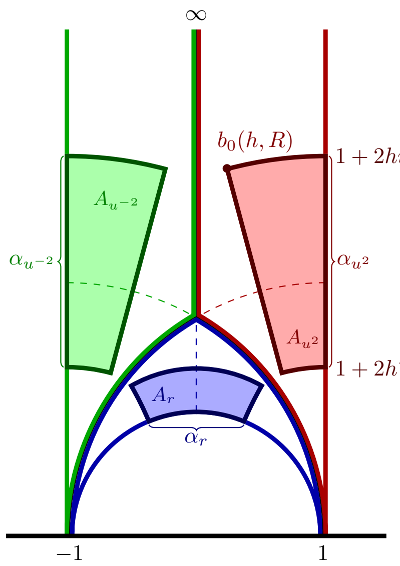


FIGURE A.6. Symmetric transition zones and $b_0(z,R) \in T_0$.

Then, for $h \geq 1$, the condition $b_0(h,R) \in T_0$ is equivalent to $2h\tanh(R) \leq 1$. Since $2h\tanh(R)$ is increasing, the condition $b_0(h,R) \in T_0$ for every $h \in \left[1, e^{L/2}\right)$ is equivalent to $2e^{L/2}\tanh(R) \leq 1$.

Thus, $L, R$ are admissible if $2e^{L/2}\tanh(R) \leq 1$ or, equivalently, if

$$4e^L \tanh^2(R) \leq 1.$$

□

## Appendix B. Estimates for the combinatorial spectrum

In this appendix we estimate from below the bottom of the combinatorial spectrum $\mu_0(G,S)$, for $G = \mathrm{PSL}(2,\mathbb{Z})$ associated to the system of generators $S = \{r,u\}$, where $r = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $u = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. By combinatorial spectrum, we refer to the spectrum of the combinatorial Laplace operator on the Cayley graph.

REMARK B.1. It is very likely that in the present case, estimates (or even, the exact value) of $\mu_0(G,S)$ are known by the experts. However, we could not find any clue of this and therefore, we give our own estimates.

We estimate $\mu_0(G,S)$ from below following ideas of Nagnibeda [**Na**] and prove the following.

THEOREM B.2. *Let $G = \mathrm{PSL}(2,\mathbb{Z})$ and $S = \{r,u\}$. Then, the bottom of the combinatorial spectrum associated to $S$ satisfies*

$$\mu_0(G,S) > 0.07.$$

REMARK B.3. It can be proved that the bottom of the combinatorial spectrum associated to a symmetric finite system of $k > 1$ generators, is bounded from above by $k - 2\sqrt{k-1}$ (which corresponds to the bottom of the combinatorial spectrum of a regular tree of degree $k$). In our case, this means that $\mu_0(G,S) < 3 - 2\sqrt{2}$ or, numerically, $\mu_0(G,S) < 0.1716$. In particular, this shows that our estimate is less than 2.5 times worse than the actual value.

In the following, we recall some aspects of combinatorial group theory we need and, in particular, we recall the definition of the bottom of the combinatorial spectrum $\mu_0(G,S)$. The following discussion is completely general.

**B.1. Combinatorial group theory.** Let $G$ be any group, and let $S$ be a subset of $G$. A *word* in $S$ is any expression of the form

$$w = s_1^{\sigma_1} s_2^{\sigma_2} \cdots s_n^{\sigma_n}$$

where $s_1, \dots, s_n \in S$ and $\sigma_i \in \{+1, -1\}$, $i = 1, \dots, n$. The number $l(w) = n$ is the *length* of the word.

Each word in $S$ represents an element of $G$, namely the product of the expression. The identity element can be represented by the empty word, which is the unique word of length zero.

*Notation.* We use an overline to denote inverses, thus $\bar{s}$ stands for $s^{-1}$.

In these terms, a subset $S$ of a group $G$ is a system of generators if and only if every element of $G$ can be represented by a word in $S$. Henceforth, let $S$ be a fixed system of generators of $G$ and a word is assumed to be a word in $S$. A *relator* is a non-empty word that represent the identity element of $G$.

Any word in which a generator appears next to its own inverse ($s\bar{s}$ or $\bar{s}s$) can be simplified by omitting the redundant pair. We say that a word is *reduced* if it contains no such redundant pairs.

Let $v, w$ be two words. We say that $v$ is a *subword* of $w$ if $w = v'vv''$, for some words $v', v''$. If $v'$ is the empty word we say that $v$ is a *prefix* of $w$. If $v''$ is the empty word we say that $v$ is a *suffix* of $w$.

We say that a word is *reduced in $G$* if it has no non-empty relators as subword. In particular, if a word is reduced in $G$, any of its subwords is also reduced in $G$.

For an element $g \in G$, we consider the *word norm* $|g|$ to be the least length of a word which is equals to $g$ when considered as a product in $G$, and every such word is called a *path*, that is, if its length coincides with its word norm when considered as a product in $G$. In particular, a path is always reduced in $G$. Moreover, a subword of a path is also a path. We say that two words are *equivalent* if they represent the same element in $G$.

For a relator, we call a subword that is a relator, a *subrelator*. We say that a relator is *primitive* if every proper subword is reduced in $G$, that is, if it does not contain proper subrelators. In particular, a word is reduced in $G$ if and only if it contains no primitive relators as subword. Note that, if $P$ is the set of all primitive relators, then $\langle S \mid P \rangle$ is a presentation of $G$.

The following elementary results (see Figure B.1) will be useful in §B.3.

LEMMA B.4. *Let $v, w$ be two different equivalent paths. Then, there are paths $v_0, v_1$, $w_0, w_1$ and $x$ such that $v = v_0v_1x$ and $w = w_0w_1x$, and $v_1\bar{w}_1$ is a primitive relator (of even length).*

PROOF. Let $x$ be the largest common suffix of $v$ and $w$ (possibly $x$ is empty). Write $v = v'x$ and $w = w'x$. Let $w_1$ and $v_1$ be the smallest non-empty suffixes of $w'$

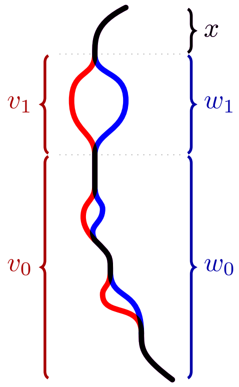


FIGURE B.1. Decomposition of two equivalent paths

and $v'$ respectively such that $v_1$ and $w_1$ are equivalent. Such $v_1$ and $w_1$ exist since $v$ and $w$ are different words. Moreover, they have the same length since they are equivalent, that is, they are paths that evaluate to the same element in $G$. Write $v' = v_0 v_1$ and $w' = w_0 w_1$ (possibly $v_0$ and $w_0$ are empty). In particular $v_0$ and $w_0$ are equivalent, since the same holds for $v', w'$ and $v_1, w_1$.

It remains to prove that $v_1 \bar{w}_1$ is primitive. Suppose $z$ is a subrelator of $v_1 \bar{w}_1$. Since $v_1$ and $w_1$ are paths, they are in particular reduced in $G$ and also their subwords. Then $z = v_2 \bar{w}_2$ for some non-empty suffixes $v_2$ and $w_2$ of $v_1$ and $w_1$ respectively. In particular, $v_2$ and $w_2$ are non-empty suffixes of $w'$ and $v'$ respectively and $v_2, w_2$ are equivalent. But, by definition, $v_1$ and $w_1$ are the smallest such suffixes and therefore $v_2 = v_1$ and $w_2 = w_1$. Thus, $v_1 \bar{w}_1$ has no proper subrelators and therefore, $v_1 \bar{w}_1$ is primitive. □

As a direct consequence of the previous lemma, we have the following.

COROLLARY B.5. *Let $v = v'yx$ and $w = w'zx$ be two equivalent paths such that $y\bar{z}$ is reduced in $G$. Then, $y\bar{z}$ is a subword of some primitive relator (of even length).*

PROOF. Consider the decomposition given by the previous lemma. It is clear that $y$ is a subword of $v_1$ and $z$, of $w_1$. Then $y\bar{z}$ is a subword of the primitive relator $v_1 \bar{w}_1$. □

**B.2. Combinatorial spectrum.** Let $G$ be a finitely generated group and $S \subset G$ be a finite system of generators of $G$. Let $\ell^2(G)$ be the space of square-summable sequences on $G$ with the inner product

$$\langle h, h' \rangle := \sum_{g \in G} h_g h'_g,$$

for $h, h' \in \ell^2(G)$, and define $\Delta_S : \ell^2(G) \to \ell^2(G)$, the combinatorial Laplace operator on $G$ associated to $S$, by

$$(\Delta_S h)_g := \sum_{s \in S \cup \bar{S}} (h_g - h_{gs}),$$

for $h \in \ell^2(G)$. Then, we define $\mu_0(G,S)$, *the bottom of the combinatorial spectrum of G associated to S* to be the bottom of the spectrum of $\Delta_S$, that is,

$$\mu_0(G,S) := \inf \left\{ \frac{\langle \Delta_S h, h\rangle}{\langle h,h\rangle},\ h \in \ell^2(G) \right\}.$$

Remark B.6. The subjacent object in this discussion is the Laplace operator on the Cayley graph of $G$ associated to $S$. However we do not explain this here.

B.2.1. *Nagnibeda's ideas.* In order to give estimates from below to the combinatorial spectrum we follow ideas of Nagnibeda [**Na**], which are based in the following result, whose proof is elementary (see, for example, [**Co**, §7.1]).

Proposition B.7 (Gabber–Galil's lemma). *Let $G$ be a finitely generated group and $S$ a finite symmetric system of generators of $G$. Suppose there exists a function $L : G \times S \to \mathbb{R}_+$ such that, for every $g \in G$ and $s \in S$,*

$$L(g,s) = \frac{1}{L(gs, s^{-1})} \qquad \text{and} \qquad \sum_{s\in S} L(g,s) \le k,$$

*for some $k > 0$. Then,*

$$\mu_0(G,S) \ge \#S - k.$$

Let $S$ be a symmetric finite system of generators of $G$. For $g \in G$, denote by $|g|$ the word norm with respect to $S$ and define $S^{\pm}(g) := \{s \in S : |gs| = |g| \pm 1\}$. For $g \in G$ and $s \in S$, we say that $gs$ is a *successor* of $g$ if $s \in S^+(g)$ and that $gs$ is a *predecessor* of $g$ if $s \in S^-(g)$. Henceforth we assume $S^+(g) \cup S^-(g) = S$, for every $g \in G$. Note that this is equivalent to say that every relator has even length.

A function $t : G \to \mathbb{N}$ is called a *type function on G* and its value $t(g)$ at $g \in G$ is called the *type of g*. We say that a type function $t$ is *compatible with S*, or simply that $t$ is a *compatible type function*, if the following two conditions are equivalent:

(1) $t(g) = t(g')$;
(2) $\#\{s \in S^+(g) : t(gs) = k\} = \#\{s' \in S^+(g') : t(g's') = k\}$, for every $k \in \mathbb{N}$.

Equivalently, $t$ is a compatible type function if the (multiset of) types of successors of an element $g \in G$ (is/)are completely defined by its type $t(g)$.

For any type function $t : G \to \mathbb{N}$ and positive valuation $c : \mathbb{N} \to \mathbb{R}_+$, we can consider a function $L_c : G \times S \to \mathbb{R}_+$ defined by

$$L_c(g,s) = \begin{cases} c_k, & \text{if } s \in S^+(g),\ k = t(gs), \\ 1/c_k, & \text{if } s \in S^-(g),\ k = t(g). \end{cases}$$

It is clear then, by the definition, that any $L_c : G \times S \to \mathbb{R}_+$ defined as above satisfies $L_c(g,s) = 1/L_c(gs,s^{-1})$, since $s \in S^+(g)$ if and only if $s^{-1} \in S^-(gs)$, and $S = S^+(g) \cup S^-(g)$, for every $g \in G$.

Moreover, for a compatible type function $t$, we define for $k = t(g) \in \mathbb{N}$, $g \in G$,

$$f_k(c) := \sum_{s\in S} L_c(g,s) = \sum_{s \in S^+(g)} c_{t(gs)} + \frac{\#S^-(g)}{c_k}.$$

Note that this is well defined since $t$ is compatible with $S$ and therefore the sum depends only on $k$, the type of $g$.

As a direct consequence of Gabber–Galil's lemma (Proposition B.7), we get the following.

Corollary B.8. *Let $t : G \to \{0, \dots, K\}$ be a compatible type function. Then,*

$$\mu_0(G, S) \geq \#S - \max_{k=0,\dots,K} f_k(c),$$

*for every $c : \{0, \dots, K\} \to \mathbb{R}_+$, where $f_k$ is defined as above.*

Then, every compatible (finite) type function gives lower bounds for the combinatorial spectrum.

**B.3. Compatible type functions for $G = \mathrm{PSL}(2, \mathbb{Z})$, $S = \{r, u\}$.** Until now, the discussion is completely general. We now specialize to the case of $G = \mathrm{PSL}(2, \mathbb{Z})$ with generators $r = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $u = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. The aim in the following is to give a compatible finite type function in this case, in order to give estimates for the bottom of the combinatorial spectrum with the aid of Corollary B.8. For this, we define a *suffix type function* and prove that it is compatible with $S = \{r, u\}$.

It is classical that $\langle r, u \mid r^2, (ru)^3 \rangle$ is a presentation of $G$. Since we have the relator $r^2$, for the sake of simplicity, we omit henceforth $\bar{r}$, since as element in $G$ it coincides with $r$. The set of primitive relators is then (up to include the variants with $\bar{r}$ instead of $r$) given by

$$\{r^2, (ru)^3, (r\bar{u})^3, (ur)^3, (\bar{u}r)^3\}.$$

In particular, every relator has even length and we can apply previous discussion.

Let $\mathbf{S}(g)$ be the set of all suffixes of paths for $g \in G$. Then, by the description of the primitive relators, as a direct consequence of Corollary B.5, we have the following.

Corollary B.9. *The following cases cannot happen:*

- *$u, \bar{u} \in \mathbf{S}(g)$;*
- *$ur, \bar{u}r \in \mathbf{S}(g)$;*
- *$ar, \bar{a}^2 \in \mathbf{S}(g)$, for $a = u$ or $\bar{u}$; or*
- *$ar, a \in \mathbf{S}(g)$, for $a = u$ or $\bar{u}$.*

Proof. Neither $u^2$, $ur\bar{u}$ nor $\bar{u}ru$ are subwords of a primitive relator. □

Let $\mathbf{S}_n(g)$ be the set of all suffixes of length $n \in \mathbb{N}$ of paths for $g \in G$ and define, by recurrence, $\mathbf{S}_1^*(g) = \mathbf{S}_1(g)$ and

$$\mathbf{S}_{n+1}^*(g) = \begin{cases} \mathbf{S}_{n+1}(g) & \text{if } S_{n+1}(g) \neq \emptyset, \\ \mathbf{S}_n^*(g) & \text{if } S_{n+1}(g) = \emptyset. \end{cases}$$

Note that any injective function $j : \mathbf{S}_n^*(G) \to \mathbb{N}$ defines a (finite) type function $t = j \circ \mathbf{S}_n^* : G \to \mathbb{N}$, which we call *suffix type function of level $n$*.

Lemma B.10. *Let $t : G \to \mathbb{N}$ be a suffix type function of level 2. Then, it is compatible with $S$.*

Proof. Being compatible with $S$ means that the type $t(g)$ of $g \in G$ completely defines the types of its successors. Then, it is enough to show that $\mathbf{S}_2^*(g)$ defines completely the multiset $\{\mathbf{S}_2^*(gs) : s \in S^+(g)\}$.

By the previous corollary, we have that it cannot happen that $u, \bar{u} \in \mathbf{S}_1(g)$, that $\#\mathbf{S}_2(g) \leq 2$ and that $\mathbf{S}_2(g) = 2$ if and only if $\mathbf{S}_2(g) = \{ra, a^2\}$ or $\{ra, \bar{a}r\}$, for $a = u$ or $\bar{u}$. It follows then, that $\mathbf{S}_1^*(g) \in \{\emptyset, \{r\}, \{u\}, \{\bar{u}\}, \{u, r\}, \{\bar{u}r\}\}$ and

$$\mathbf{S}_2^*(g) \in \{\emptyset, \{r\}, \{ar\}, \{a\}, \{ra\}, \{a^2\}, \{ra, a^2\}, \{ra, \bar{a}r\}\}_{a=u,\bar{u}}.$$

Moreover, it is clear that $s \in S^+(g)$ if and only if $\bar{s} \notin \mathbf{S}_1(g)$.

Let $a = u$ or $\bar{u}$.

- If $\mathbf{S}_2^*(g) = \emptyset$, $g = 1$ and evidently $\mathbf{S}_2^*(gs) = \{s\}$, for $s \in S = S^+(g)$.
- If $\mathbf{S}_2^*(g) = \{r\}$, $g = r$ and $\mathbf{S}_2^*(gb) = \{rb\}$, for $b \in \{u, \bar{u}\} = S^+(r)$.
- If $\mathbf{S}_2^*(g) = \{ar\}$, $S^+(g) = \{u, \bar{u}\}$, $\mathbf{S}_2^*(ga) = \{ra, \bar{a}r\}$ and $\mathbf{S}_2^*(g\bar{a}) = \{r\bar{a}\}$.
- If $\mathbf{S}_2^*(g) = \{a\}$, $g = a$ and $S^+(g) = \{r, a\}$. Moreover, $\mathbf{S}_2^*(gr) = \{ar\}$ and $\mathbf{S}_2^*(ga) = \{a^2\}$.
- If $\mathbf{S}_2^*(g) = \{ra\}$, then $S^+(g) = \{r, a\}$, $\mathbf{S}_2^*(gr) = \{ar, r\bar{a}\}$ and $\mathbf{S}_2^*(ga) = \{a^2\}$.
- If $\mathbf{S}_2^*(g) = \{a^2\}$, then $S^+(g) = \{r, a\}$, $\mathbf{S}_2^*(gr) = \{ar\}$ and $\mathbf{S}_2^*(ga) = \{a^2\}$.
- If $\mathbf{S}_2^*(g) = \{ra, a^2\}$, $S^+(g) = \{r, a\}$, $\mathbf{S}_2^*(gr) = \{ar, r\bar{a}\}$ and $\mathbf{S}_2^*(ga) = \{a^2\}$.
- If $\mathbf{S}_2^*(g) = \{ra, \bar{a}r\}$, then $S^+(g) = \{a\}$ and $\mathbf{S}_2^*(ga) = \{ra, a^2\}$.

Thus, given only the value of $\mathbf{S}_2^*(g)$ we can tell $\mathbf{S}_2^*(gs)$, $s \in S^+(g)$ and therefore, suffix type functions are compatible with $S$. □

We summarize the proof of the previous lemma by the following diagram which shows each possible $\mathbf{S}_2^*(g)$, $g \in G$ with its respective multiset of $\mathbf{S}_2^*(gs)$, $s \in S^+(g)$:

$$\mathbf{S}_2^*(g) \to \mathbf{S}_2^*(gs), s \in S^+(g)$$

$$\begin{aligned}
\emptyset &\to \{r\}, \{u\}, \{\bar{u}\} \\
\{r\} &\to \{ru\}, \{r\bar{u}\} \\
\{ar\} &\to \{ra, \bar{a}r\}, \{r\bar{a}\} \\
\{a\} &\to \{ar\}, \{a^2\} \\
\{ra\} &\to \{ar, r\bar{a}\}, \{a^2\} \\
\{a^2\} &\to \{ar\}, \{a^2\} \\
\{ra, a^2\} &\to \{ar, r\bar{a}\}, \{a^2\} \\
\{ra, \bar{a}r\} &\to \{ra, a^2\},
\end{aligned}$$

where $a = u$ or $\bar{u}$.

It is not difficult to see in the previous diagram that there are different suffix types which share the types of the successors. Namely $\{a\}, \{a^2\}$ and $\{ra\}, \{ra, a^2\}$. This allows us to reduce the number of types. Furthermore, it is clear that distinguishing $u$ and $\bar{u}$ in the previous description has no major benefit. This motivates the definition of the following type function. Let $T : G \to \{0, \ldots, 5\}$ be the type function defined as follows:

$$T(g) = \begin{cases} 0 & \text{if } \mathbf{S}_2^*(g) = \emptyset, \\ 1 & \text{if } \mathbf{S}_2^*(g) = \{r\}, \\ 2 & \text{if } \mathbf{S}_2^*(g) = \{ar\},\ a = u \text{ or } \bar{u}, \\ 3 & \text{if } \mathbf{S}_2^*(g) = \{a\} \text{ or } \{a^2\},\ a = u \text{ or } \bar{u}, \\ 4 & \text{if } \mathbf{S}_2^*(g) = \{ra\} \text{ or } \{ra, a^2\},\ a = u \text{ or } \bar{u}, \\ 5 & \text{if } \mathbf{S}_2^*(g) = \{ar, r\bar{a}\},\ a = u \text{ or } \bar{u}. \end{cases}$$

From the previous discussion, we deduce the following.

THEOREM B.11. *The type function $T : G \to \{0, \ldots, 5\}$ is compatible with $S$. Moreover,*

- *Type 0 elements have one type 1 and two type 3 successors;*
- *Type 1 elements have two type 4 successors;*
- *Type 2 elements have one type 4 and one type 5 successor;*
- *Type 3 elements have one type 2 and one type 3 successor;*
- *Type 4 elements have one type 3 and one type 5 successor; and*
- *Type 5 elements have one type 4 successor;*

Thus, we have a compatible type function with a full description of the types of the successors for each type. We can then finally apply Nagnibeda's ideas (Corollary B.8) to give estimates for the bottom of the combinatorial spectrum.

**B.4. Estimates for the bottom of the combinatorial spectrum.** By Theorem B.11, the $f_k$ of Corollary B.8 are given by:

- $f_0(c) = c_1 + 2c_3$;
- $f_1(c) = 2c_4 + 1/c_1$;
- $f_2(c) = c_4 + c_5 + 1/c_2$;
- $f_3(c) = c_2 + c_3 + 1/c_3$;
- $f_4(c) = c_3 + c_5 + 1/c_4$; and
- $f_5(c) = c_4 + 2/c_5$.

It follows that $\mu_0(G, S) \geq \#S - \max_k f_k(c)$, for every $c = (c_1, \ldots, c_5) \in \mathbb{R}_+^5$. Thus, the problem can be reduced to find the optimal such bound. This can be solved numerically: we get that $\bar{c} \in \mathbb{R}_+^5$ with

$$\bar{c}_1 = 1;\ \bar{c}_2 \approx 0.8323;\ \bar{c}_3 \approx 0.7326;\ \bar{c}_4 \approx 0.7927;\ \bar{c}_5 \approx 0.9358;$$

is a (local) minimun for $\max_k f_k(c)$, and $\max_k f_k(\bar{c}) \approx 2.9299 < 2.93$.

Finally, since $\#S = 3$, it follows that

$$\mu_0(G, S) > 0.07.$$

This concludes the proof of Theorem B.2 □

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CHAPTER III

# A non-varying phenomenon with an application to the wind-tree model

ABSTRACT. We exhibit a non-varying phenomenon for the counting problem of cylinders, weighted by their area, passing through two marked (regular) Weierstrass points of a translation surface in a hyperelliptic connected component $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1,g-1)$, $g>1$. As an application, we obtain the non-varying phenomenon for the counting problem of (weighted) periodic trajectories on the classical wind-tree model, a billiard in the plane endowed with $\mathbb{Z}^2$-periodically located identical rectangular obstacles.

## 1. Introduction

A connected component of a stratum of Abelian differentials is said to be *non-varying* if for every Teichmüller curve in that component the sum of (positive) Lyapunov exponents is the same. Such a non-varying phenomenon was observed numerically by M. Kontsevich and A. Zorich along with the initial observations on Lyapunov exponents for the Teichmüller geodesic flow [**Ko, KZ97**]. Today, there are two types of non-varying results. One for low genus, due to D. Chen and M. Möller [**CM**], which uses a translation of the problem into algebraic geometry. The other one, for hyperelliptic loci, due to A. Eskin, M. Kontsevich and A. Zorich [**EKZ**], which is a consequence of their main result relating sum of Lyapunov exponents to Siegel–Veech constants, which, roughly speaking, measure the growth rate of the number of cylinders of bounded length on translation surfaces. In particular, the non-varying phenomenon for the sum of Lyapunov exponents is equivalent to the non-varying of Siegel–Veech constants.

The related counting problem has been widely studied and it is related to many other questions such as the calculation of the volume of strata of normalized translation surfaces [**EMZ**]. H. Masur [**Ma88, Ma90**] proved that for every translation surface $X$, there exist positive constants $c(X)$ and $C(X)$ such that the number $N(X,L)$ of (maximal) cylinders of closed geodesics of length at most $L$ satisfy

$$c(X)L^2 \leq N(X,L) \leq C(X)L^2$$

for large enough $L$. W. Veech [**Ve89**] proved that for Veech surfaces there are in fact exact quadratic asymptotics. A. Eskin and H. Masur [**EMa**] proved that for each ergodic probability measure $\mu$ on strata of normalized (area 1) translation surfaces, there is a constant $c(\mu)$ such that for almost every surface $X$, $N(X,L) \sim c(\mu)\cdot \pi L^2$.

This constant $c(\mu)$ is the Siegel–Veech constant ([**EMa**]); it is the constant in the Siegel–Veech formula ([**EMa**]), a Siegel-type formula introduced by W. Veech [**Ve98**], which can be translated into

$$c(\mu) = \frac{1}{\pi R^2}\int N(X,R)\mathrm{d}\mu(X).$$

The first explicit computations where made by W. Veech [**Ve89, Ve92**]. A. Eskin, H. Masur and A. Zorich [**EMZ**] computed the Siegel–Veech constants for connected components of all strata of Abelian differentials, and also described all possible configurations of cylinders of closed geodesics which might be found on a generic flat surface. In general, the particular constants for Veech surfaces do not coincide with the Siegel–Veech constants of the strata where they live. Unless, of course, we face a non-varying phenomenon, as is for example the case of the hyperelliptic components $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1,g-1)$, $g>1$.

In this work we study a different but related counting problem: that of cylinders whose core curve passes through two marked regular Weierstrass points on hyperelliptic surfaces in a hyperelliptic component; and we prove the following non-varying phenomenon analogous to the one described above.

Theorem 1. *Let $\mu$ be the affine invariant measure supported on the $\mathrm{SL}(2,\mathbb{R})$-orbit closure of an hyperelliptic surface $X$ in a hyperelliptic component $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1,g-1)$, $g>1$. Then, the (area) Siegel–Veech constant associated to the counting problem of cylinders whose core curve passes through two marked regular Weierstrass points equals*

$$\begin{cases} \dfrac{1}{\pi^2}\cdot\dfrac{1}{2g-1}, & \text{if } X\in\mathcal{H}^{hyp}(2g-2),\\ \dfrac{1}{\pi^2}\cdot\dfrac{1}{2g}, & \text{if } X\in\mathcal{H}^{hyp}(g-1,g-1).\end{cases}$$

It is a natural question whether this non-varying phenomenon takes place in every hyperelliptic loci as well, as is the case for the counting problem of every cylinder (and not only those that pass through prescribed Weierstrass points). We shall see that this is not true in general.

The main motivation of this result, is an application to the wind-tree model.

**Wind-tree model.** The wind-tree model corresponds to a billiard in the plane endowed with $\mathbb{Z}^2$-periodic obstacles of rectangular shape aligned along the lattice, as in Figure 1. Denote by $\Pi(a,b)$ the wind-tree model whose obstacles have dimensions $(a,b)\in\,]0,1[$.

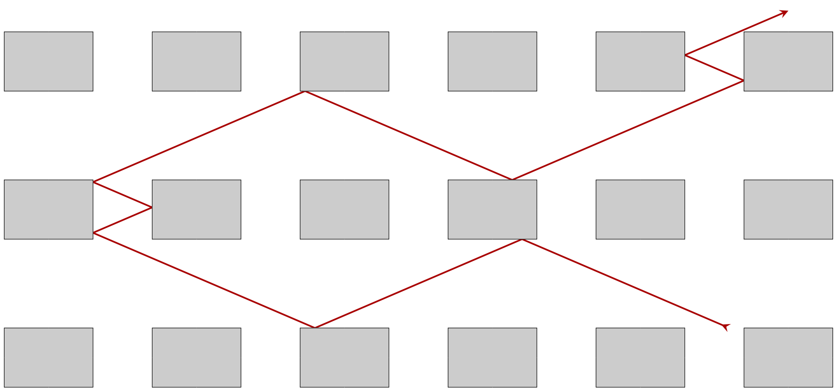

Figure 1. The wind-tree model.

The wind-tree model (in a slightly different version) was introduced by P. Ehrenfest and T. Ehrenfest [**EE**] in 1912. J. Hardy and J. Weber [**HaWeb**] studied the periodic version. All these studies had physical motivations.

Several advances on the dynamical properties of the billiard flow in the wind-tree model were obtained using geometric and dynamical properties on moduli space of (compact) translation surfaces. A. Avila and P. Hubert [**AH**] showed that for all parameters of the obstacle and for almost all directions, the trajectories are recurrent. There are examples of divergent trajectories constructed by V. Delecroix [**De**]. The non-ergodicity was proved by K. Frączek and C. Ulcigrai [**FU**]. It was proved by V. Delecroix, P. Hubert and S. Lelièvre [**DHL**] that the diffusion rate is independent either on the concrete values of the parameters of the obstacle or on almost any direction and almost any starting point and is equals to $2/3$. A generalization of this last result was shown by V. Delecroix and A. Zorich [**DZ**] for more complicated obstacles.

The result of V. Delecroix, P. Hubert and S. Lelièvre about the diffusion rate *evince* a first non-varying phenomenon in the case of the classical wind-tree model, which corresponds to the 'sum of Lyapunov exponents' counterpart. In this work we describe the 'Siegel–Veech constant' counterpart of the non-varying phenomenon.

The author [**Pa1**] studied the counting problem on wind-tree models proving that the number of periodic trajectories has quadratic asymptotic growth rate and computed, in the generic case, the Siegel–Veech constants for the classical wind-tree model as well as for the Delecroix–Zorich variant. In this work we prove that, for the classical wind-tree model, this constant does not depend on the dimensions of the obstacles, exhibiting a non-varying phenomenon analogous to the one described above. More precisely, as a direct consequence of Theorem 1, we have the following.

THEOREM 2. Denote by $N_{area}(\Pi(a,b),L)$ be the number of maximal families of isotopic periodic trajectories (up to $\mathbb{Z}^2$-translations) of length at most $L$ in $\Pi(a,b)$, weighted by the area covered by the family.

(1) For Lebesgue-almost every $(a,b) \in ]0,1[$ and, in particular, if $a, b$ are rational or can be written as $1/(1-a) = x + z\sqrt{D}$ and $1/(1-b) = y + z\sqrt{D}$ with $x, y, z \in \mathbb{Q}$ and $x + y = 1$ and $D$ a positive square-free integer, then,

$$N_{area}(\Pi(a,b),L) \sim \frac{4}{3\pi^2} \cdot \frac{\pi L^2}{1-ab}.$$

(2) In any other case, we have the weak asymptotic formula

$$N_{area}(\Pi(a,b),L) \text{ “}\sim\text{” } \frac{4}{3\pi^2} \cdot \frac{\pi L^2}{1-ab}.$$

PROOF. The statement is a compilation of several different results and is equivalent to say that $c_{area}(\Pi(a,b)) = 4/3\pi^2$ (cf. [**AEZ**, Theorem 1.7] and [**Pa1**, Theorem 1.2]). By [**Pa1**, Corollary 5.6], the counting problem on the wind-tree model coincides with the counting problem of cylinders whose core curve passes through two marked regular Weierstrass points on a surface $L(a,b) \in \tilde{\mathcal{Q}}(1,-1^5) = \mathcal{H}(2)$.

By elementary considerations on the Siegel–Veech formula (cf. [**EKZ**, Lemma 1.1]) combined with the lifting properties of cylinders in $L(a,b)$ (see for example [**AH**, Lemma 3]), we have that $c_{area}(\Pi(a,b))$ is four times the Siegel–Veech constant associated to the corresponding counting on $L(a,b)$.

Thus, by Theorem 1, we conclude that $c_{area}(\Pi(a,b)) = 4/3\pi^2$. □

**Strategy of the proof.** From a hyperelliptic surface $X$ in a hyperelliptic component $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1,g-1)$, $g>1$, and given two fixed regular Weierstrass points, we build three different translation surfaces which are covering of the original surface $X$. These coverings turn out to be hyperelliptic surfaces as well. We introduce some configurations of cylinders associated to the monodromy of these coverings and describe the counting of cylinders whose core curve passes through the two Weierstrass points in terms of one of these configurations. We relate the Siegel–Veech constants of the configurations on $X$ to their liftings on the coverings. Decomposing the Siegel–Veech constants of the involved surfaces in terms of these configurations, we obtain a system of equations which allows us to describe the Siegel–Veech constants of the configurations in terms of those of the surfaces. Since the surfaces are hyperelliptic, thanks to Eskin–Kontsevich–Zorich [**EKZ**], the result is non-varying. Describing the hyperelliptic loci where the surfaces lie and putting the values of the corresponding Siegel–Veech constants in the expression allows us to compute explicitly the value of the Siegel–Veech constant associated to the configurations and therefore, the one associated to the counting of cylinders whose core curve passes through the two Weierstrass points.

We present a family of counterexamples for hyperelliptic loci which are not hyperelliptic components. We exhibit hyperelliptic surfaces where the Siegel–Veech constant associated to the counting of cylinders whose core curve passes through two marked Weierstrass points does not coincide with the corresponding Siegel–Veech constant on the hyperelliptic loci where they lie. For this, we use one of the covers defined above, which lies in a hyperelliptic locus which is not a hyperelliptic component. We relate the configuration of cylinders whose core curve passes through (any) two Weierstrass points to one of the configurations mentioned above and we compute the value of the corresponding Siegel–Veech constant analogously. Using a result of Athreya–Eskin–Zorich [**AEZ**], we show the corresponding generic value for the hyperelliptic locus, which does not coincide with the one obtained for the constructed surface, showing that the relevant Siegel–Veech constant vary along the hyerelliptic locus.

**Structure of the paper.** In §2 we briefly recall all the background necessary to formulate and prove the results. In §3 we prove the result in the case of the hyperelliptic component $\mathcal{H}^{hyp}(2g-2)$, $g>1$. We describe the covering construction in §3.1 and prove that they are hyperelliptic surfaces, giving also the corresponding hyperelliptic loci where they lie. In §3.2 we introduce the associated configurations of cylinders and relate the counting of cylinders whose core curve passes through the two Weierstrass points in terms of one of these configurations. We describe the system of equations they satisfy and find the value of the desired Siegel–Veech constant. In §4 we prove the result in the case of the hyperelliptic component $\mathcal{H}^{hyp}(2g-2)$, $g>1$, following the same outline.

We present in §5 the family of counterexamples, providing the values of (the sum of) the pertinent Siegel–Veech constants for the counterexamples as well as for the generic case.

## 2. Background

**2.1. Flat surfaces.** For an introduction and general references to this subject, we refer the reader to the surveys of Zorich [**Zo06**], Forni–Matheus [**FM**], Wright [**Wr**].

2.1.1. *Flat surfaces and strata.* Let $g \geq 1$, $\{n_1, \ldots, n_k\}$ be a partition of $2g-2$ and $\mathcal{H}(n_1, \ldots, n_k)$ denote a stratum of Abelian differentials, that is, the space holomorphic 1-forms on Riemann surfaces of genus $g$, with zeros of degree $n_1, \ldots, n_k \in \mathbb{N}$. There is a one to one correspondence between Abelian differentials and translation surfaces, surfaces which can be obtained by edge-to-edge gluing of polygons in $\mathbb{R}^2$ using translations only. Thus, we refer to elements of $\mathcal{H}(n_1, \ldots, n_k)$ as translation surfaces. A translation surface has a canonical flat metric, the one obtained form $\mathbb{R}^2$, with conical singularities of angle $2\pi(n+1)$ at zeros of degree $n$ of the Abelian differential.

We also consider strata $\mathcal{Q}(d_1, \ldots, d_k)$ of meromorphic quadratic differentials with at most simple poles on Riemann surfaces with zeros of order $d_1, \ldots, d_k$, $d_i \in \{-1\} \cup \mathbb{N}$ for $i = 1, \ldots, k$ (in a slight abuse of vocabulary, we are considering poles as zeros of order $-1$) and $\sum_{i=1}^{k} d_i = 4g-4$. A quadratic differential also defines a canonical flat metric with conical singularities of angle $\pi(d+2)$ at zeros of order $d$.

In this paper, a quadratic differential is not the square of an Abelian differential. This condition is automatically satisfied if at least one of parameters $d_j$ is odd.

*Notation.* As usual, we use "exponential" notation to denote multiple zeroes (or simple poles) of the same degree, for example $\mathcal{Q}(1, -1^5) = \mathcal{Q}(1, -1, -1, -1, -1, -1)$.

A flat surface is a Riemann surface with the flat metric corresponding to an Abelian or quadratic differential.

2.1.2. *Canonical orientation double cover.* One can canonically associate with every meromorphic quadratic differential $q$ on a Riemann surface $S$ another connected curve with an Abelian differential on it. It is the unique double covering of $S$ (possibly ramified at singularities of $q$) such that the pullback of $q$ is the square of an Abelian differential.

*Notation.* We denote by $\tilde{\mathcal{Q}}(d_1, \ldots, \mathrm{d}_n)$ the locus of translation surfaces consisting on the canonical orientating double cover of surfaces in the strata of half-translation surfaces $\mathcal{Q}(d_1, \ldots, \mathrm{d}_n)$.

2.1.3. *Hyperelliptic surfaces, loci and components.* We say that a translation surface $X$ is a hyperelliptic surface if it corresponds to the canonical orientation double cover of a quadratic differential on a Riemann surface of genus zero. Equivalently, if $X \in \tilde{\mathcal{Q}}(d_1, \ldots, d_n)$ with $\sum_{j=1}^{n} d_j = -4$ and, in this case, we say that $\tilde{\mathcal{Q}}(d_1, \ldots, d_n)$ is an hyperelliptic locus.

There are two series of hyperelliptic loci which play a special role: for $g > 1$,

$$\tilde{\mathcal{Q}}(2g-3, -1^{2g+1}) \subset \mathcal{H}(2g-2), \text{ and}$$
$$\tilde{\mathcal{Q}}(2g-2, -1^{2g+2}) \subset \mathcal{H}(g-1, g-1),$$

In these cases, the hyperelliptic loci coincides with a connected component of the corresponding stratum (see [**KZ03**, §2.1]), the hyperelliptic compontent, which is denoted by

$$\mathcal{H}^{hyp}(2g-2) = \tilde{\mathcal{Q}}(2g-3, -1^{2g+1}), \text{ and}$$
$$\mathcal{H}^{hyp}(g-1, g-1) = \tilde{\mathcal{Q}}(2g-2, -1^{2g+2}).$$

2.1.4. *Hyperelliptic involution and Weierstrass points.* Every translation surface obtained as an orientation covering comes with an involution. In the case of hyperelliptic surfaces, we call it the hyperelliptic involution. The hyperelliptic involution of a hyperelliptic surface of genus $g$ has exactly $2g+2$ fixed points. These fixed points are called Weierstrass points. We say that a Weierstrass point is regular if it is regular for the flat metric, that is, if it is not a conical singularity. Note that regular Weierstrass points are exactly those points who projects to poles in the corresponding quadratic differential on the sphere.

Moreover, a translation surface of genus $g$ is a hyperelliptic surface if and only if it has an involution which fixes $2g+2$ points.

**2.2. Counting problem.** We are interested in the counting of closed geodesics of bounded length on translation surfaces. Together with every closed regular geodesic in a translation surface $X$ we have a bunch of parallel closed regular geodesics. A cylinder on a flat surface is a maximal open annulus filled by isotopic simple closed regular geodesics. A cylinder $C$ is isometric to the product of an open interval and a circle, its core curve $\gamma_C$ is the geodesic projecting to the middle of the interval and its length $l(C)$ is the circumference of the circle. A saddle connection is a geodesic joining two different singularities or a singularity to itself, with no singularities in its interior. Cylinders are always bounded by parallel saddle connections.

The number of cylinders of bounded length is finite. Thus, for any $L > 0$ the following quantity is well-defined:

$$N_{area}(X, L) = \frac{1}{\text{Area}(X)} \sum_{\substack{C \subset X \\ l(C) \leq L}} \text{Area}(C),$$

where the sum is over all cylinders $C$ in $X$ of length bounded by $L$.

The following theorem is a special case of a fundamental result of Veech [**Ve98**], considered by Vorobets in [**Vo05**].

THEOREM (Veech). *Let $\nu$ be an ergodic* $\mathrm{SL}(2,\mathbb{R})$*-invariant probability measure on a stratum $\mathcal{H}_1(n_1, \ldots, n_k)$ of Abelian differentials of area one. Then, the following ratio is constant (i.e. does not depend on the value of a positive parameter $R$):*

$$c_{area}(\nu) = \frac{1}{\pi R^2} \int N_{area}(X, R) \mathrm{d}\nu.$$

This formula is the Siegel–Veech formula, and the corresponding constant $c_{area}(\nu)$ is the Siegel–Veech constant.

A fundamental result of Eskin–Mirzhakani–Mohammadi [**EMM**] says that every $\mathrm{SL}(2,\mathbb{R})$-orbit closure $\mathcal{M}$ is an affine invariant manifold and, in paricular, it is the support of an affine invariant measure $\nu_{\mathcal{M}}$ (see [**EMM, EMi**] for the precise definitions). For simplicity, we denote $c_{area}(\mathcal{M}) = c_{area}(\nu_{\mathcal{M}})$.

We call a configuration of cylinders on an affine invariant manifold $\mathcal{M}$, a continuous $\mathrm{SL}(2,\mathbb{R})$-equivariant application $\mathcal{C}$ which associates to $X \in \mathcal{M}$ (or any finite cover of $\mathcal{M}$) a collection of cylinders in $X$ (cf. [**EMZ**]). The previous discussion on the counting problem and Siegel–Veech constants applies as well in the case of configurations of cylinders and we denote by $c_{area}(\mathcal{M}, \mathcal{C})$ the corresponding Siegel–Veech constant.

*Notation.* For a translation surface $X$, we denote by $c_{area}(X)$, the Siegel–Veech constant associated to the affine invariant measure $\nu_{\mathcal{M}}$ supported on its $\mathrm{SL}(2,\mathbb{R})$-orbit closure $\mathcal{M} = \overline{\mathrm{SL}(2,\mathbb{R})X}$. That is

$$c_{area}(X) := c_{area}(\mathcal{M}) = \frac{1}{\pi R^2} \int_{\mathcal{M}} N_{area}(Y, R) \mathrm{d}\nu_{\mathcal{M}}(Y).$$

Similarly, for a configuration of cylinders $\mathcal{C}$ defined on $\mathcal{M}$, we denote by $c_{area}(X, \mathcal{C})$, the corresponding Siegel–Veech constant, $c_{area}(X, \mathcal{C}) = c_{area}(\mathcal{M}, \mathcal{C})$.

**2.3. Non-varying phenomenon.** The following result summarize the non-varying phenomenon for Siegel–Veech constants observed on hyperelliptic loci by Eskin–Kontsevich–Zorich [**EKZ**, Theorem 3 and Lemma 1.1].

THEOREM 2.1 (Eskin–Kontsevich–Zorich). *Let $X$ be a hyperelliptic surface such that the quotient sphere belongs to $\mathcal{Q}(d_1, \dots, d_n)$. That is, $X \in \tilde{\mathcal{Q}}(d_1, \dots, d_n)$ with $\sum_{j=1}^{n} d_j = -4$. Then*

$$c_{area}(X) = -\frac{1}{4\pi^2} \sum_{j=1}^{n} d_j \frac{d_j + 4}{d_j + 2}.$$

## 3. The case of $\mathcal{H}^{hyp}(2g-2)$

In this section we prove the statement of Theorem 1 in the case of the hyperelliptic component $\mathcal{H}^{hyp}(2g-2)$, $g > 1$.

**3.1. Hyperelliptic coverings.** Let $X \in \mathcal{H}^{hyp}(2g-2)$, $w_0, w_1, w_2 \in X$ be three different regular Weierstrass points[1] and $z \in X$ be the zero of order $2g-2$ on $X$. Consider two saddle connections $s_1, s_2$ passing through $w_1, w_2$ respectively (and joining $z$ to itself). In particular, $s_1$ and $s_2$ are $h$-invariant, where $h : X \to X$ is the hyperelliptic involution.

For $(i, j) \in \{0,1\}^2 \setminus \{(0,0)\}$, consider the covering $X_{ij}$ over $X$ defined by the subgroup of $\pi_1(X, w_0)$

$$\Gamma_{ij} = \{\gamma \in \pi_1(X, w_0) : \ \iota(\gamma, is_1 + js_2) \equiv_2 0\}. \tag{22}$$

Note that, since $s_1, s_2$ are closed loops, these coverings are unramified.

LEMMA 3.1. *$X_{ij}$ is hyperelliptic, for $(i, j) \in \{0,1\}^2 \setminus \{(0,0)\}$.*

PROOF. From general covering space theory, we know that points in $X_{ij}$ can be taken to be equivalence classes of pairs $(x, \rho)$, which we denote by $[x, \rho]_{ij}$, where $\rho$ is a path joining $w_0$ to $x \in X$ and $(x_1, \rho_1)$ is equivalent to $(x_2, \rho_2)$ provided $x_1 = x_2$ and $\rho_1 \rho_2^{-1} \in \Gamma_{ij}$, where $\Gamma_{ij}$ is the subgroup of $\pi_1(X, w_0)$ defining the covering $X_{ij}$, described by (22) above.

[1] Every hyperelliptic surface has at least four different regular Weierstrass points.

We define now $h_{ij} : X_{ij} \to X_{ij}$ by $h_{ij}([x, \rho]_{ij}) = [h(x), h \circ \rho]_{ij}$, where $h : X \to X$ is the hyperelliptic involution on $X$. Note that $h_{ij}$ is a well defined involution which is a lift of $h$. It is clear from the definition that the only possible fixed points of $h_{ij}$ are the points lying above the Weierstrass points of $X$. Moreover, in $X_{ij}$, if one of the two points above a given Weierstrass point in $X$ is fixed by $h_{ij}$, then both are.

Let $w \in X$ be a Weierstrass point and $\rho : [0, 1] \to X$ be a path from $w_0$ to $X_{11}$. Consider now $\hat{\rho} := h(\rho)\rho^{-1} \in \pi_1(X, w_0)$. Then, by definition of $X_{ij}$ (see (22) above) $[w, \rho]_{ij} \in X_{ij}$ is fixed by $h_{ij}$ if and only if $\iota(\hat{\rho}, is_1 + js_2) \equiv_2 0$. Moreover, up to homotopy, we can suppose that $\rho|_{]0,1[}$ avoids $z, w_1, w_2 \in X$, and that $\rho$ intersects transversally $s_1, s_2$ and avoids tangencies and self-intersections over $s_1, s_2$. It follows that $\iota(\hat{\rho}, s_k) \equiv_2 \#\hat{\rho} \cap s_k$, $k = 1, 2$. Moreover, $\hat{\rho}$ and $s_k$ are $h$-invariant and therefore, so is $\hat{\rho} \cap s_k$. Since $h$ is an involution, the parity of $\#\hat{\rho} \cap s_k$ equals (the parity of) the number of its fixed points.

Now, since $w_0 \notin \{z, w_1, w_2\}$, we have that, for $k = 1, 2$,

(1) if $w \notin \{z, w_k\}$, then $\hat{\rho} \cap s_k$ is an $h$-invariant set with no fixed points and therefore, $\iota(\hat{\rho}, s_k) \equiv_2 \#\hat{\rho} \cap s_k \equiv_2 0$;
(2) if $w \in \{z, w_k\}$, then $\hat{\rho} \cap s_k$ is an $h$-invariant set with exactly one fixed point, namely $X_{11}$, and therefore, $\iota(\hat{\rho}, s_k) \equiv_2 \#\hat{\rho} \cap s_k \equiv_2 1$.

Thus,

(1) the two points on $X_{10}$ above $X_{11}$ are fixed by $h_{10}$ if and only if $w \notin \{z, w_1\}$;
(2) the two points on $X_{01}$ above $X_{11}$ are fixed by $h_{01}$ if and only if $w \notin \{z, w_2\}$; and
(3) the two points on $X_{11}$ above $X_{11}$ are fixed by $h_{11}$ if and only if $w \notin \{w_1, w_2\}$.

It follows that, for each $(i, j) \in \{0, 1\}^2 \setminus \{(0, 0)\}$, the number of points in $X_{ij}$, fixed by $h_{ij}$ is twice the number of Weierstrass points in $X$ but two. Since $X$ is hyperelliptic of genus $g$, it has $2g + 2$ Weierstrass points and thus, the number of fixed points for $h_{ij}$ is $2(2g + 2 - 2) = 4g$.

Moreover, since the coverings are regular double covers, $\chi(X_{ij}) = 2\chi(X)$. That is, $g(X_{ij}) = (2g - 2) + 1 = 2g - 1$ and $h_{ij}$ fixes $4g = 2g(X_{ij}) + 2$ points. We conclude that $X_{ij}$ is hyperelliptic. □

Since $X_{ij}$ is a regular double cover of $X \in \mathcal{H}(2g - 2)$, we have that $X_{ij} \in \mathcal{H}(2g - 2, 2g - 2)$. Furthermore, we have the following, which shall be needed latter.

Lemma 3.2. *The surfaces $X_{10}$ and $X_{01}$ belong to the hyperelliptic connected component $\mathcal{H}^{hyp}(2g - 2, 2g - 2) = \tilde{\mathcal{Q}}(4g - 4, -1^{4g})$, while the surface $X_{11}$ belongs to the hyperelliptic locus $\tilde{\mathcal{Q}}(2g - 3, 2g - 3, -1^{4g-2}) \subset \mathcal{H}^{odd}(2g - 2, 2g - 2)$.*

Proof. Following the proof of Lemma 3.1, we know that the hyperelliptic involution $h_{10}$ fixes the points in $X_{10}$ above the Weierstrass points in $X$ but $z$ and $w_1$. In particular, the conical singularities of $X_{10}$ are not fixed by $h_{10}$ and therefore $X_{10} \in \tilde{\mathcal{Q}}(4g - 4, -1^{4g}) = \mathcal{H}^{hyp}(2g - 2, 2g - 2)$. The proof for $X_{01}$ is analogous.

Similarly, the hyperelliptic involution $h_{11}$ fixes the points in $X_{11}$ above the Weierstrass points in $X$ but $w_1$ and $w_2$. Then, the conical singularities of $X_{11}$ are fixed by $h_{11}$ and therefore $X_{11} \in \tilde{\mathcal{Q}}(2g - 3, 2g - 3, -1^{4g-2})$. The parity of the spin structure of surfaces in $\tilde{\mathcal{Q}}(2g - 3, 2g - 3, -1^{4g-2})$ is deduced from [**KZ03**, Proposition 7]. □

**3.2. Configurations and Siegel–Veech constants.** We are concerned with the counting of cylinders whose core curve passes through two fixed Weierstrass points $w_1, w_2 \in X$. For a cylinder $C$ in $X$, we define the profile of $C$ to be the couple $(\iota(\gamma_C, s_k) \mod 2)_{k=1,2} \in \{0,1\}^2$, where $\gamma_C$ is the core curve of $C$. We also consider $\mathcal{C}_{pq}$ to be the configuration of cylinders in $X$ of profile $(p,q) \in \{0,1\}^2$.

LEMMA 3.3. *The configuration $\mathcal{C}_{11}$ coincides with the configuration of cylinders whose core curve passes through $w_1$ and $w_2$.*

PROOF. Since $X \in \mathcal{H}^{hyp}(2g-2)$, the core curve of every cylinder $C$ in $X$ is $h$-invariant and passes through exactly two different Weierstrass points[2]. Denote by $\mathcal{W}(C)$ the set of this two Weierstrass points lying on the core curve of $C$. We claim that $\iota(\gamma_C, s_k) \equiv_2 \mathbf{1}_{\mathcal{W}(C)}(w_k)$. In fact, $\gamma_C$ can be written as $h(\rho)\rho^{-1}$, where $\rho$ is a geodesic path from one Weierstrass point in $\mathcal{W}(C)$ to the other. Moreover, $\rho|_{]0,1[}$ avoids every Weierstrass point, in particular $z$ and $w_k$. Furthermore, since $\rho$ and $s_k$ are geodesics, $\rho$ intersects transversally $s_j$ and avoids tangencies and self-intersections over $s_k$. Thus, as in the proof of Lemma 3.1, it follows that $\iota(\gamma, s_k) \equiv_2 \#\gamma_C \cap s_k$ and therefore, $\iota(\gamma, s_k) \equiv_2 1$ if and only if $w_k \in \mathcal{W}(C)$, proving the claim. We conclude thus, that the core curve of $C$ pass through $w_1$ and $w_2$, that is, $\mathcal{W}(C) = \{w_1, w_2\}$, if and only if $\iota(\gamma, s_1) \equiv_2 \iota(\gamma, s_2) \equiv_2 1$, that is, if and only if $C$ has profile $(1,1)$. □

Denote by $c_{pq} = c_{area}(X, \mathcal{C}_{pq})$. Then, it is clear that

$$c_{area}(X) = c_{00} + c_{10} + c_{01} + c_{11}.$$

Now, consider the configuration $\mathcal{C}_{pq}^{ij}$ of cylinders $C$ in $X_{ij}$ such that they project to $X$ to cylinders in $\mathcal{C}_{pq}$. Again, it is clear that $c_{area}(X_{ij})$ decomposes into the sum of the Siegel–Veech constants $c_{area}(X_{ij}, \mathcal{C}_{pq}^{ij})$, $(p,q) \in \{0,1\}^2$.

The following general result relates the Siegel–Veech constants of configurations of cylinders on a double covering to the constant on the base space (see [**DZ**, Lemma 4.1], which is a slight generalization of [**EKZ**, Lemma 1.1]).

LEMMA 3.4. *Let $\hat{X} \to X$ be a double covering, $\mathcal{C}$ be a configuration of cylinders on $C$ and $\hat{\mathcal{C}}$ be the lift of this configuration to $\hat{X}$. If the core curve of every cylinder in $\mathcal{C}$ has non-trivial monodromy, then $c_{area}(\hat{X}, \hat{\mathcal{C}}) = c_{area}(X, \mathcal{C})/2$. If the core curve of every cylinder in $\mathcal{C}$ has trivial monodromy, then $c_{area}(\hat{X}, \hat{\mathcal{C}}) = 2c_{area}(X, \mathcal{C})$.*

Note that, in our case, the monodromy of a cylinder $C$ in $X$ for the covering $X_{ij}$ is, by definition, given by the intersection number $\iota(\gamma_C, is_1 + js_2) \mod 2$. But then, profiles define the monodromy on each covering and, in particular, the relation between the Siegel–Veech constants given in Lemma 3.4 above. Thus, we have that cylinders in $\mathcal{C}_{pq}^{ij}$ have monodromy equals to $ip + jq \mod 2$. Using this, it is easy to

[2]This claim is true only for $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1, g-1)$, and this is the main reason this argument does not work on other hyperelliptic loci.

verify that, by the previous lemma, we have the following:

$$\begin{aligned}
c_{area}(X) &= \ c_{00} + \ c_{10} + \ c_{01} + \ c_{11}, \\
c_{area}(X_{10}) &= 2c_{00} + 2c_{10} + \frac{1}{2}c_{01} + \frac{1}{2}c_{11}, \\
c_{area}(X_{01}) &= 2c_{00} + \frac{1}{2}c_{10} + 2c_{01} + \frac{1}{2}c_{11}, \\
c_{area}(X_{11}) &= 2c_{00} + \frac{1}{2}c_{10} + \frac{1}{2}c_{01} + 2c_{11}.
\end{aligned}$$

Moreover

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 1/2 & 1/2 \\ 2 & 1/2 & 2 & 1/2 \\ 2 & 1/2 & 1/2 & 2 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} -3 & 1 & 1 & 1 \\ 2 & 1 & -1 & -1 \\ 2 & -1 & 1 & -1 \\ 2 & -1 & -1 & 1 \end{pmatrix}$$

and therefore,

$$c_{11} = \frac{1}{3}\left[2c_{area}(X) - c_{area}(X_{10}) - c_{area}(X_{01}) + c_{area}(X_{11})\right]. \tag{23}$$

Thus, it is enough to compute the Siegel–Veech constants for each surface and, since all of them are hyperelliptic, by Theorem 2.1, it is enough to find the strata of quadratic differentials where the quotient spheres belong. But, by Lemma 3.2, we already know this and hence,

$$c_{area}(X) = -\frac{1}{4\pi^2}\left((2g-3)\frac{2g+1}{2g-1} - 3(2g-1)\right) = \frac{g}{\pi^2}\cdot\frac{2g+1}{2g-1},$$

$$c_{area}(X_{10}) = c_{area}(X_{01}) = -\frac{1}{4\pi^2}\left((4g-4)\frac{4g}{4g-2} - 3(4g)\right) = \frac{g}{\pi^2}\cdot\frac{4g-1}{2g-1}$$

and

$$c_{area}(X_{11}) = -\frac{1}{4\pi^2}\left((2g-3)\frac{2g+1}{2g-1}2 - 3(4g-2)\right) = \frac{1}{\pi^2}\cdot\frac{4g^2-4g+3}{2g-1}.$$

Finally, plugging this in formula (23), we obtain

$$c_{11} = \frac{1}{3\pi^2}\cdot\frac{1}{2g-1}\left[2g(2g+1) - 2g(4g-1) + (4g^2-4g+3)\right] = \frac{1}{\pi^2}\cdot\frac{1}{2g-1}.$$

That is, by Lemma 3.3, the Siegel–Veech constant associated to the counting of cylinders whose core curve passes through the two regular Weierstrass points $w_1$ and $w_2$ equals

$$\frac{1}{\pi^2}\cdot\frac{1}{2g-1}$$

for surfaces in $\mathcal{H}^{hyp}(2g-2)$. □

## 4. The case of $\mathcal{H}^{hyp}(g-1, g-1)$

In this section we prove Theorem 1 for the hyperelliptic connected components $\mathcal{H}^{hyp}(g-1, g-1)$, $g > 1$. The proof follows almost in the same way than in the case of $\mathcal{H}^{hyp}(2g-2)$ but some small details that we present below.

**4.1. Hyperelliptic coverings.** The construction of the hyperelliptic coverings is slightly different, in particular, the saddle connections we use to define them are no longer closed curves.

Let $X \in \mathcal{H}^{hyp}(g-1,g-1)$, $w_0, w_1, w_2 \in X$ be three different regular Weierstrass points and $z \in X$ be one of the zeros of order $g-1$ on $X$. Consider two saddle connections $s_1, s_2$ passing through $w_1, w_2$ respectively. In particular, $s_1$ and $s_2$ are $h$-invariant and goes from $z$ to $h(z)$, where $h : X \to X$ is the hyperelliptic involution and $h(z)$ is the other zero of order $g-1$ on $X$.

As above, for $(i,j) \in \{0,1\}^2 \setminus \{(0,0)\}$, we consider the covering $X_{ij}$ over $X$ defined by the subgroup of $\pi_1(X,w_0)$

$$\Gamma_{ij} = \{\gamma \in \pi_1(X,w_0) : \ \iota(\gamma, is_1 + js_2) \equiv_2 0\}.$$

Note that, since $s_1, s_2$ are no longer closed loops, these coverings might be branched coverings. In fact, it is not hard to see that $X_{10}$ and $X_{01}$ are ramified over $z$ and $h(z)$, while $X_{11}$ is unramified. Anyway, they are still hyperelliptic coverings:

LEMMA 4.1. *$X_{ij}$ is hyperelliptic, for $(i,j) \in \{0,1\}^2 \setminus \{(0,0)\}$.*

PROOF. The proof follows as in the proof of Lemma 3.1. For a Weierstrass point $w \in X$ and $\rho : [0,1] \to X$ a path from $w_0$ to $X_{11}$, we get that $\iota(\hat{\rho}, s_k) \equiv_2 \#\hat{\rho} \cap s_k$, $k = 1,2$, where $\hat{\rho} = h(\rho)\rho^{-1} \in \pi_1(X,w_0)$. And, for $k = 1,2$, we have that,

(1) if $w \neq w_k$, then $\hat{\rho} \cap s_k$ is an $h$-invariant set with no fixed points and therefore, $\iota(\hat{\rho}, s_k) \equiv_2 \#\hat{\rho} \cap s_k \equiv_2 0$;
(2) if $w = w_k$, then $\hat{\rho} \cap s_k$ is an $h$-invariant set with exactly one fixed point, namely $X_{11}$, and therefore, $\iota(\hat{\rho}, s_k) \equiv_2 \#\hat{\rho} \cap s_k \equiv_2 1$.

Thus,

(1) the two points on $X_{10}$ above $X_{11}$ are fixed by $h_{10}$ if and only if $w \neq w_1$;
(2) the two points on $X_{01}$ above $X_{11}$ are fixed by $h_{01}$ if and only if $w \neq w_2$; and
(3) the two points on $X_{11}$ above $X_{11}$ are fixed by $h_{11}$ if and only if $w \notin \{w_1, w_2\}$.

It follows that there are exactly $2(2g+2-1) = 4g+2$ fixed points for $h_{10}$ and $h_{01}$. Moreover, since the coverings $X_{10}$ and $X_{01}$ are branched over two points, by Riemann–Hurwitz formula, $\chi(X_{10}) = \chi(X_{01}) = 2\chi(X) + 2$. That is, for $ij = 10, 01$, $g(X_{ij}) = 2g - 2 + 1 + 1 = 2g$, so $h_{ij}$ fixes $4g+2 = 2g(X_{ij}) + 2$ points and therefore $X_{ij}$ is hyperelliptic. Similarly, $h_{11}$ has $2(2g+2-2) = 4g$ fixed points and since the coverings $X_{11}$ is a regular double cover, $\chi(X_{11}) = 2\chi(X)$. That is, $g(X_{11}) = (2g-2) + 1 = 2g-1$. Thus $h_{11}$ fixes $4g = 2g(X_{11}) + 2$ points and $X_{11}$ is hyperelliptic. □

And we have the following.

LEMMA 4.2. *The surfaces $X_{10}$ and $X_{01}$ belong to the hyperelliptic connected component $\mathcal{H}^{hyp}(2g-1, 2g-1) = \tilde{\mathcal{Q}}(4g-2, -1^{4g+2})$, while the surface $X_{11}$ belongs to the hyperelliptic locus $\tilde{\mathcal{Q}}(2g-2, 2g-2, -1^{4g}) \subset \mathcal{H}(g-1,g-1,g-1,g-1)$, which is connected.*

PROOF. The proof is analogous to the proof of Lemma 3.2. □

**4.2. Configurations and Siegel–Veech constants.** As before, we are concerned with the counting of cylinders whose core curve passes through two fixed Weierstrass points $w_1, w_2 \in X$. For a cylinders $C$ in $X$, we define the profile of $C$ as before, that is, the couple $(\iota(\gamma_C, s_k) \mod 2)_{k=1,2} \in \{0,1\}^2$, where $\gamma_C$ is the core curve of $C$; and we consider $\mathcal{C}_{pq}$ to be the configuration of cylinders in $X$ of profile $(p,q) \in \{0,1\}^2$.

Lemma 4.3. *The configuration $\mathcal{C}_{11}$ coincides with the configuration of cylinders whose core curve passes through $w_1$ and $w_2$.*

Proof. The proof is the same as for Lemma 3.3. □

Denote by $c_{pq} = c_{area}(X, \mathcal{C}_{pq})$, so

$$c_{area}(X) = c_{00} + c_{10} + c_{01} + c_{11}.$$

We consider the configurations $\mathcal{C}_{pq}^{ij}$ as in §3.2 and, applying Lemma 3.4 and the definition of profile, we obtain the same system as in the previous case. Thus, it follows that $c_{11} = \left[2c_{area}(X) - c_{area}(X_{10}) - c_{area}(X_{01}) + c_{area}(X_{11})\right]/3$.

It suffices now to compute the Siegel–Veech constants for each surface. By Theorem 2.1 and Lemma 4.2, we have that

$$c_{area}(X) = -\frac{1}{4\pi^2}\left((2g-2)\frac{2g+2}{2g} - 3(2g+1)\right) = \frac{g+1}{\pi^2}\cdot\frac{2g+1}{2g},$$

$$c_{area}(X_{10}) = c_{area}(X_{01}) = -\frac{1}{4\pi^2}\left((4g-2)\frac{4g+2}{4g} - 3(4g+2)\right) = \frac{2g+1}{\pi^2}\cdot\frac{4g+1}{4g}$$

and

$$c_{area}(X_{11}) = -\frac{1}{4\pi^2}\left((2g-2)\frac{2g+2}{2g}2 - 3(4g)\right) = \frac{1}{\pi^2}\cdot\frac{2g^2+1}{g}.$$

Finally, we obtain

$$c_{11} = \frac{1}{3\pi^2}\cdot\frac{1}{4g}\left[4(g+1)(2g+1) - 2(2g+1)(4g+1) + 4(2g^2+1)\right] = \frac{1}{\pi^2}\cdot\frac{1}{2g-1}.$$

That is, by Lemma 4.3, the Siegel–Veech constant associated to the counting of cylinders whose core curve passes through the two regular Weierstrass points $w_1$ and $w_2$ equals

$$\frac{1}{\pi^2}\cdot\frac{1}{2g}$$

for surfaces in $\mathcal{H}^{hyp}(g-1, g-1)$. □

## 5. Counterexamples

In this section we present a family of counterexamples: we exhibit hyperelliptic surfaces where the Siegel–Veech constant associated to the counting of cylinders whose core curve passes through two marked Weierstrass points does not coincide with the corresponding Siegel–Veech constant on the hyperelliptic loci where they lie.

Let $X$ be a hyperelliptic surface in a hyperelliptic component, that is, $X$ belongs to $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1, g-1)$, $g \geq 1$. We consider the surface $X_{11}$ from the previous sections. By Lemma 3.2 and 4.2, we know that $X_{11}$ belongs to the hyperelliptic locus $\tilde{\mathcal{Q}}(2g-3, 2g-3, -1^{4g-2})$ or $\tilde{\mathcal{Q}}(2g-2, 2g-2, -1^{4g})$, respectively.

Recall that in $X_{11}$, regular Weierstrass points are exaclty those that project to regular Weierstrass points in $X$ but $w_1$ and $w_2$ (see proof of Lemma 3.1/4.1, point (3)). For a cylinder $C$ in $X_{11}$, denote by $\mathcal{W}(C)$ the set of (regular) Weierstrass points on its core curve. Thus, a cylinder $C$ in $X_{11}$ whose core curve passes through two regular Weierstrass points (that is, $\#\mathcal{W}(C) = 2$) projects to a cylinder in $X$ whose core curve passes through two regular Weierstrass points different from $w_1$ and $w_2$.

Let $\mathcal{C}_{\mathcal{W}}$ be the configuration of cylinders in $X_{11}$ whose core curve passes through (any) two regular Weierstrass points[3], and recall that $\mathcal{C}_{00}^{11}$ denotes the configuration of cylinders in $X_{11}$ such that their projection in $X$ have profile $(0,0)$ (see §3.2/4.2).

Lemma 5.1. *The configurations $\mathcal{C}_{\mathcal{W}}$ and $\mathcal{C}_{00}^{11}$ coincide.*

Proof. The proof follows as in Lemma 3.3/4.3. In fact, let $\bar{\mathcal{C}}_{\mathcal{W}}$ denote the configuration of cylinders in $X$ who lift to cylinders on $\mathcal{C}_{\mathcal{W}}$. By the previous discussion, $\bar{\mathcal{C}}_{\mathcal{W}}$ coincides with the configuration of cylinders in $X$ whose core curve passes through two regular Weierstrass points different from $w_1$ and $w_2$.

In the proof of Lemma 3.3, we showed that for any cylinder $C$ in $X$ we have that $\iota(\gamma_C, s_k) \equiv_2 \mathbf{1}_{\mathcal{W}(C)}(w_k)$, $k = 1, 2$. Then, since the profile of a cylinder $C$ in $X$ is defined as $(\iota(\gamma_C, s_k) \mod 2)_{k=1,2}$, $C$ has profile $(0,0)$ if and only if $\mathbf{1}_{\mathcal{W}(C)}(w_1) = \mathbf{1}_{\mathcal{W}(C)}(w_2) = 0$. That is, if and only if its core curve passes through two regular Weierstrass points different from $w_1$ and $w_2$ (recall that the core curve of every cylinder in $X$ passes through two regular Weierstrass points). Thus, $\bar{\mathcal{C}}_{\mathcal{W}} = \mathcal{C}_{00}$ and therefore, $\mathcal{C}_{\mathcal{W}} = \mathcal{C}_{00}^{11}$. ☐

It follows from the previous lemma and Lemma 3.4 that $c_{area}(X_{11}, \mathcal{C}_{\mathcal{W}}) = 2c_{00}$. But we know, by the equation system it satisfies (see §3.2), that

$$c_{00} = \frac{1}{3}\left[-3c_{area}(X) + c_{area}(X_{10}) + c_{area}(X_{01}) + c_{area}(X_{11})\right].$$

We have already computed the Siegel–Veech constants for each surface. Putting all together, we obtain that

$$c_{area}(X_{11}, \mathcal{C}_{\mathcal{W}}) = \begin{cases} \dfrac{2g-2}{\pi^2}, & \text{if } X \in \mathcal{H}^{hyp}(2g-2), \\ \dfrac{2g-1}{\pi^2}, & \text{if } X \in \mathcal{H}^{hyp}(g-1, g-1). \end{cases}$$

Thus, we have computed the sum of the Siegel–Veech constants corresponding to the counting of cylinders in $X_{11}$ whose core curve passes through two marked regular Weierstrass points, for any such marking, for a particular choice of a surfaces in the hyperelliptic loci $\tilde{\mathcal{Q}}(2g-3, 2g-3, -1^{4g-2})$ and $\tilde{\mathcal{Q}}(2g-2, 2g-2, -1^{4g})$. We shall see that these do not coincide with the generic case in such hyperelliptic loci. In fact, we have the following.

[3]Unlike the case of surfaces in hyperelliptic components, for surfaces in other hyperelliptic loci, not every core curve of a cylinder passes through Weierstrass points. However, if it passes through a Weierstrass point, it passes through exactly two of them.

LEMMA 5.2. *In the generic case, the value of the Siegel–Veech constants is*

$$c_{area}(\mathcal{M},\mathcal{C}_{\mathcal{W}}) = \begin{cases} \dfrac{2g-1}{\pi^2}, & \text{if } \mathcal{M} = \tilde{\mathcal{Q}}(2g-3,2g-3,-1^{4g-2}),\\ \dfrac{2g}{\pi^2}, & \text{if } \mathcal{M} = \tilde{\mathcal{Q}}(2g-2,2g-2,-1^{4g}).\end{cases}$$

PROOF. By a result of Athreya–Eskin–Zorich ([**AEZ**, Corollary 4.7]), the generic *classical* Siegel–Veech constant associated to the counting of cylinders in $\mathbb{CP}^1$ (with a meromorphic quadratic differential with at most simple poles) bounded by a saddle connection joining two marked poles equals $1/2\pi^2$ (whichever stratum of quadratic differentials in genus zero). Note that, these cylinders correspond exactly to the projection of cylinders in the orientation double cover whose core curve passes through two marked regular Weierstrass points.

However, we are interested in the *area* Siegel–Veech constant. For configurations $\mathcal{C}$ of cylinders in strata $\mathcal{L} = \mathcal{Q}(d_1,\dots,d_k)$ of quadratic differentials on $\mathbb{CP}^1$ (that is, $\sum_{j=1}^k d_k = -4$), there exist a relation between the classical Siegel–Veech constant $c(\mathcal{L},\mathcal{C})$ and the area Siegel-Veech constant $c_{area}(\mathcal{L},\mathcal{C})$, namely

$$c_{area}(\mathcal{L},\mathcal{C}) = \frac{1}{k-3}c(\mathcal{L},\mathcal{C}).$$

This is a consequence of a generalization of Vorobets formula [**Vo05**, Theorem 1.6(b)], proved by Athreya–Eskin–Zorich [**AEZ**, Proposition 4.9] for any configuration of cylinders on any strata of quadratic differentials on $\mathbb{CP}^1$. Moreover, by [**EKZ**, Lemma 1.1] (cf. Lemma 3.4), we have that the corresponding Siegel–Veech constant in the hyperelliptic locus, say $c_{area}(\tilde{\mathcal{L}},\tilde{\mathcal{C}})$, is twice this value,

$$c_{area}(\tilde{\mathcal{L}},\tilde{\mathcal{C}}) = 2c_{area}(\mathcal{C}) = \frac{2}{k-3}c(\mathcal{L},\mathcal{C}).$$

Putting all together, in the case of $\mathcal{M}_1 = \tilde{\mathcal{Q}}(2g-3,2g-3,-1^{4g-2})$ we get

$$c_{area}(\mathcal{M}_1,\mathcal{C}_{\mathcal{W}}) = \frac{2}{4g-3}\binom{4g-2}{2}\frac{1}{2\pi^2} = \frac{2g-1}{\pi^2},$$

where the binomial coefficient stands for all possible choices of two regular Weierstrass points, which correspond to the poles of the quadratic differential on $\mathbb{CP}^1$.

Similarly, in the case of $\mathcal{M}_2 = \tilde{\mathcal{Q}}(2g-2,2g-2,-1^{4g})$, we get

$$c_{area}(\mathcal{M}_2,\mathcal{C}_{\mathcal{W}}) = \frac{2}{4g-1}\binom{4g}{2}\frac{1}{2\pi^2} = \frac{2g}{\pi^2}.$$

□

Thus, the Siegel–Veech constant for the counting of cylinders whose core curve passes through two marked regular Weierstrass points cannot be non-varying in $\tilde{\mathcal{Q}}(2g-3,2g-3,-1^{4g-2})$ or $\tilde{\mathcal{Q}}(2g-2,2g-2,-1^{4g})$, as the sum of all of them is not.

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**Résumé**

Le problème du cercle de Gauss consiste à compter le nombre de points entiers de longueur bornée dans le plan. Autrement dit, compter le nombre de géodésiques fermées de longueur bornée sur un tore plat bidimensionnel. De très nombreux problèmes de comptage en systèmes dynamiques se sont inspirés de ce problème. Depuis 30 ans, on cherche à comprendre l'asymptotique de géodésiques fermées dans les surfaces de translation. H. Masur a montré que ce nombre a une croissance quadratique. Calculer l'asymptotique quadratique (constante de Siegel–Veech) est un sujet de recherches très actif aujourd'hui. L'objet d'étude de cette thèse est le modèle de windtree, un modèle de billard non compact. Dans le cas classique, on place des obstacles rectangulaires identiques dans le plan en chaque point entier. On joue au billard sur le complémentaire. Nous montrons que le nombre de trajectoires périodiques a une croissance asymptotique quadratique et calculons la constante de Siegel–Veech pour le windtree classique ainsi que pour la généralisation de Delecroix–Zorich. Nous prouvons que, pour le windtree classique, cette constante ne dépend pas des tailles des obstacles (phénomène "non varying" analogue aux résultats de Chen–Möller). Enfin, lorsque la surface de translation compacte sous-jacente est une surface de Veech, nous donnons une version quantitative du comptage.

**Mots-clés** : Systèmes dynamiques, Géométrie, Modèle de windtree, Billards, Surfaces de translation, Problème de comptage.

**Abstract**

The Gauss circle problem consists in counting the number of integer points of bounded length in the plane. In other words, counting the number of closed geodesics of bounded length on a flat two dimensional torus. Many counting problems in dynamical systems have been inspired by this problem. For 30 years, the experts try to understand the asymptotic behavior of closed geodesics in translation surfaces. H. Masur proved that this number has quadratic growth rate. Compute the quadratic asymptotic (Siegel–Veech constant) is a very active research domain these days. The object of study in this thesis is the wind-tree model, a non-compact billiard model. In the classical setting, we place identical rectangular obstacles in the plane at each integer point. We play billiard on the complement. We show that the number of periodic trajectories has quadratic asymptotic growth rate and we compute the Siegel–Veech constant for the classical wind-tree model as well as for the Delecroix–Zorich variant. We prove that, for the classical wind-tree model, this constant does not depend on the dimensions of the obstacles (non-varying phenomenon, analogous to results of Chen–Möller). Finally, when the underlying compact translation surface is a Veech surface, we give a quantitative version of the counting.

**Keywords**: Dynamical systems, Geometry, Wind-tree model, Billiards, Translation surfaces, Counting problem.