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Processus de contact sur des graphes aléatoires

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Résumé

Le processus de contact est l'un des systèmes de particules en interaction les plus étudiés. Il peut s'interpréter comme un modèle pour la propagation d'un virus dans une population ou sur un réseau. L'objectif de cette thèse est d'étudier la relation entre la structure locale du réseau et le comportement global du processus sur le réseau tout entier.

Le cadre typique dans lequel on se place est celui d'une suite de graphes aléatoires (G_n) convergeant localement vers un graphe limite G. On étudie alors le comportement asymptotique du temps d'extinction τ_n du processus sur G_n; lorsqu'initialement tous les individus sont infectés. Nous montrons sur plusieurs exemples qu'il existe une transition de phase lorsque λ - le taux d'infection du processus - traverse une valeur critique $\lambda_c(G)$, qui ne dépend que de G. Plus précisément, pour certains modèles de graphes aléatoires comme le modèle de configuration, le graphe à attachement préférentiel, le graphe géométrique aléatoire, le graphe inhomogène, nous montrons que τ_n est d'ordre soit logarithmique soit exponentiel; selon que λ est soit inférieur ou supérieur à $\lambda_c(G)$.

De plus, dans certains cas, nous montrons des résultats de métastablité: en régime sur-critique, τ_n divisé par son espérance converge en loi vers une variable aléatoire exponentielle de moyenne 1, et la densité des sites infectés reste stable (et non nulle) sur une période de temps d'ordre typiquement τ_n.

Mots-clés: Processus de contact, Systèmes de particules en interaction, Graphes aléatoires, Metastabilité, Densité metastable, Temps de mort, Transition de phase.

Abstract

The contact process is one of the most studied interacting particle systems and is also often interpreted as a model for the spread of a virus in a population or a network. The aim of this thesis is to study the relationship between the local structure of the network and the global behavior of the contact process (the virus) on the whole network.

Let (G_n) be a sequence of random graphs converging weakly to a graph G. Then we study τ_n, the extinction time of the contact process on G_n starting from full occupancy. We prove in some examples that there is a phase transition of τ_n when λ - the infection rate of the contact process crosses a critical value $\lambda_c(G)$ depending only on G. More precisely, for some models of random graphs, such as the configuration model, preferential attachment graph, random geometric graph, inhomogeneous graph, we show that τ_n is of logarithmic (resp. exponential) order when $\lambda < \lambda_c(G)$ (resp. $\lambda < \lambda_c(G)$).

Moreover, in some cases we also prove metastable results: in the super-critical regime, τ_n divided by its expectation converges in law to an exponential random variable with mean 1, and the density of the infected sites is stable for a long time.

Keywords: Contact process, Interacting particle systems, Random graphs, Metastability, Metastable density, Extinction time, Phase transition.

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Chapter 1

Introduction

Le processus de contact a été introduit en 1974 par T. E. Harris dans [52]. C'est un système de spins simple décrivant la propagation d'un virus dans une population ou un réseau. Un individu (ou un sommet) dans la population (ou le réseau) est soit infecté soit sain. Un sommet infecté (resp. sain) devient sain (resp. infecté) au taux λ fois le nombre de ses voisins infectés (resp. au taux 1), où le paramètre λ est aussi appelé taux d'infection.

Initialement, le processus de contact a été considéré comme un modèle purement mathématique et il a été bien étudié sur des réseaux réguliers comme $\mathbb{Z}^d$ et $\mathbb{T}_d$, l'arbre homogène de degré $d+1$, dans les années $70, 80$ et 90. Plus récemment, avec le développement d'internet et l'apparition des virus, ce processus a connu un regain d'intérêt. Depuis les années 2000, le processus de contact a été étudié sur beaucoup de réseaux différents, spécialement sur des réseaux sociaux, aussi bien en informatique qu'en physique théorique, ou en mathématiques.

Dans cette thèse, on s'interesse à la relation entre la structure locale du réseau et le comportement global du processus sur le réseau tout entier. Le cadre typique dans lequel on se place est celui d'une suite de graphes aléatoires (G_n) convergeant localement vers un graphe limite G. On étudie alors le comportement asymptotique du temps d'extinction τ_n du processus sur G_n; lorsqu'initialement tous les individus sont infectés. Nous montrons sur plusieurs exemples qu'il existe une transition de phase lorsque λ traverse une valeur critique $\lambda_c(G)$, qui ne dépend que de G. Plus précisément, pour certains modèles de graphes aléatoires comme le modèle de configuration, le graphe à attachement préférentiel, le graphe géométrique aléatoire, le graphe inhomogène, nous montrons que τ_n est d'ordre soit logarithmique soit exponentiel en le nombre des sites de G_n; selon que λ est inférieur ou supérieur à $\lambda_c(G)$ respectivement.

De plus, dans certains cas, nous montrons des résultats de métastablité: en régime sur-critique, τ_n divisé par son espérance converge en loi vers une variable aléatoire exponentielle de moyenne 1, et la densité des sites infectés reste stable (et non nulle) sur une période de temps d'ordre typiquement τ_n.

Dans ce chapitre, nous donnons d'abord la définition du processus de contact et ses caractéristiques de base. Ensuite, nous résumons le développement de l'étude du processus et annonçons les résultats de la thèse.

1.1 Processus de contact

1.1.1 Définition du processus

Étant donné un graphe localement fini $G = (V, E)$ et un paramètre $\lambda > 0$, le processus de contact sur G est un processus de Markov $(\xi_t)_{t\geqslant 0}$ sur l'espace de configurations $\{0,1\}^V$ de générateur infinitésimal donné par

$$\Omega f(\xi) = \sum_{v\in V}(f(\phi_v\xi) - f(\xi)) + \lambda \sum_{v\sim w}(f(\phi_{(v,w)}\xi) - f(\xi)),$$

où $v \sim w$ signifie que $\{v,w\} \in E$, et

$$\phi_v\xi(w) = \begin{cases} \xi(w) & \text{si } w \neq v, \\ 0 & \text{si } w = v, \end{cases} \qquad \phi_{(v,w)}\xi(u) = \begin{cases} \xi(u) & \text{si } w \neq v, \\ 1(\max\{\xi(x),\xi(y)\} = 1) & \text{si } w = v. \end{cases}$$

Ici et dans toute la thèse, $1(E)$ désigne la fonction indicatrice de l'événement E. Étant donné $A \subset V$, on note (ξ_t^A) le processus de contact de configuration initiale A. Si $A = \{v\}$, on écrit simplement (ξ_t^v).

En considérant ξ_t comme un sous ensemble de V, via $\xi_t = \{v \in V : \xi_t(v) = 1\}$, les transitions du processus de contact sont données par

$$\begin{array}{lll} \xi_t \to \xi_t \setminus \{v\} & \text{au taux } 1 & \text{si } v \in \xi_t, \\ \xi_t \to \xi_t \cup \{v\} & \text{au taux } \lambda \deg_{\xi_t}(v) & \text{si } v \notin \xi_t, \end{array}$$

où $\deg_{\xi_t}(v)$ désigne le nombre de voisins de v qui sont infectés au temps t.

1.1.2 Construction graphique et propriétés du processus

Le processus de contact peut être vu comme un système de particules en interaction dans lequel

(a) une particule meurt au taux 1,

(b) une particule sur un site donne naissance à une autre particule sur chacun des sites voisins au taux λ,

(c) s'il y a deux particules sur un même site, elles se fondent en une immédiatement.

Ce point de vue donne deux avantages. Tout d'abord, il conduit à une comparaison utile avec un processus plus simple: la marche aléatoire branchante (i.e. sans la condition (c)), qui a été bien étudiée. Deuxièmement, à partir de ce point de vue, en donnant une représentation graphique du processus de contact, on obtient facilement des propriétés de base du processus. Nous verrons dans le prochain paragraphe une application du premier avantage. Nous nous concentrons maintenant sur le second.

Pour diriger la construction graphique, nous orientons le graphe G en associant à chaque arête deux arêtes orientées. Puis assignons des processus de Poisson indépendants $\mathcal{N}_v$ d'intensité 1 à chaque sommet $v \in V$ et $\mathcal{N}_e$ d'intensité λ pour chaque arête orientée e.

On dit qu'il y a un chemin d'infection de (v, s) à (w, t), et noté

$$(v, s) \leftrightarrow (w, t),$$

soit si $s = t$ et $v = w$, soit si $s < t$ et il y a une suite de temps $s = s_0 < s_1 < \ldots < s_\ell < s_{\ell+1} = t$, et une suite de sommets $v = v_0, v_1, \ldots, v_\ell = w$ de telle sorte que pour tout $i = 1, \ldots, \ell$

$$\begin{cases} s_i \in \mathcal{N}_{(v_{i-1}, v_i)} \quad \text{et} \\ \mathcal{N}_{v_i} \cap [s_i, s_{i+1}] = \varnothing. \end{cases}$$

Alors pour tout $A \subset V$, le processus de contact de configuration initiale A est défini par

$$\xi_t^A = \{w \in V : \exists\, v \in V \text{ tel que } (v, 0) \leftrightarrow (w, t)\}.$$

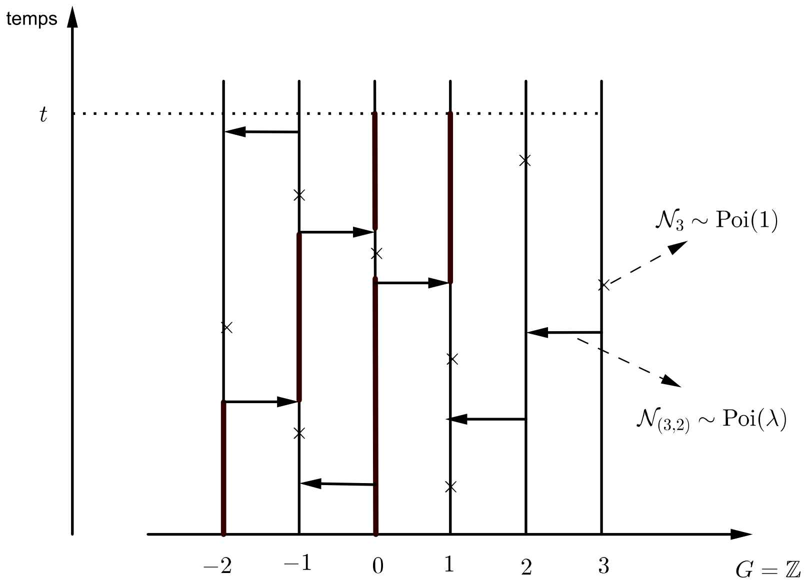

Figure 1.1: Construction graphique sur $\mathbb{Z}$.

Dans la figure 1.1, on désigne les processus de récupération par des croix et les processus d'infection par des flèches. Un chemin d'infection au temps t est un chemin orienté qui augmente au temps t, sans passer par une croix, et traverse les flèches d'infection. Dans cette figure, on dessine les chemins d'infection au temps t en gras.

Monotonie et additivité. La construction graphique nous permet de coupler des processus de contact avec des états initiaux et des taux différents. En effet, il découle directement de la construction que

$$\xi_t^{A\cup B} = \xi_t^A \cup \xi_t^B \qquad \text{pour tout } A, B \subset V. \tag{1.1}$$

On dit que le processus de contact est *additif*. En outre, le processus est aussi croissant en λ i.e. pour tout $\lambda_1 < \lambda_2$, il y a un couplage tel que pour toute configuration initiale A et temps $t \geqslant 0$,

$$\xi_t^{A,(\lambda_1)} \subset \xi_t^{A,(\lambda_2)}, \tag{1.2}$$

où $(\xi_t^{(\lambda)})$ est le processus de contact avec taux d'infection λ. En effet, nous considérons la construction graphique de $\xi_t^{(\lambda_2)}$ et pour chaque point d'un processus de Poisson $\mathcal{N}_e^{(\lambda_2)}$, nous l'effaçons avec probabilité λ_1/λ_2. Le processus obtenu par les nouveaux processus de points a la même distribution que $(\xi_t^{(\lambda_1)})$. Donc, nous avons fait un couplage satisfaisant (1.2).

Auto-dualité. Une propriété importante du processus de contact est l'auto-dualité i.e. pour tout $A, B \subset V$ et $t \geqslant 0$,

$$\mathbb{P}(\xi_t^A \cap B \neq \varnothing) = \mathbb{P}(\xi_t^B \cap A \neq \varnothing). \tag{1.3}$$

Suppose la représentation graphique du processus de contact. Pour $t \geqslant 0$ et $B \subset V$, on définit un processus dual $(\hat{\xi}_s^{B,t})_{s \leqslant t}$ en retournant l'axe du temps et les directions des processus d'infection (des flèches dans Figure 1.1). Plus formellement, pour $0 \leqslant s \leqslant t$

$$\hat{\xi}_s^{B,t} = \{v \in V : \exists\, w \in B \text{ tel que } (v, t-s) \leftrightarrow (w,t)\}.$$

Par la symétrie de la construction graphique,

$$(\hat{\xi}_s^{B,t})_{s \leqslant t} \overset{(\mathcal{L})}{=} (\xi_s^B)_{s \leqslant t}.$$

Donc,

$$\begin{aligned}
\mathbb{P}(\xi_t^B \cap A \neq \varnothing) &= \mathbb{P}(\hat{\xi}_t^{B,t} \cap A \neq \varnothing) \\
&= \mathbb{P}(\exists\, v \in A \text{ et } w \in B \text{ tel que } (v,0) \leftrightarrow (w,t)) \\
&= \mathbb{P}(\xi_t^A \cap B \neq \varnothing).
\end{aligned}$$

Mesures invariantes et convergence complète. On suppose maintenant que ξ_0 est aléatoire et est distribuée comme une mesure μ. Alors, on note la loi de ξ_t par μ_t. On appelle

$$\mathcal{I} = \{\mu : \mu_t = \mu \,\forall\, t \geqslant 0\},$$

l'ensemble des mesures invariantes. Ceci est un ensemble convexe et l'ensemble de ses points extrémaux est désigné par $\mathcal{I}_e$. Il est évident que $\delta_{\mathbf{0}} \in \mathcal{I}_e$, où $\delta_{\mathbf{0}}$ est la configuration nulle.

L'additivité du processus implique une propriété plus générale des systèmes de spins appelé *attractivité*. On munit l'espace d'état $\{0,1\}^V$ avec l'ordre partiel en écrivant $\eta \leqslant \zeta$ si $\eta(v) \leqslant \zeta(v)$ pour tout $v \in V$. On définit alors la *monotonie stochastique* pour les mesures de probabilité sur $\{0,1\}^V$. On dit que $\mu \preceq \nu$ si

$$\int f d\mu \leqslant \int f d\nu,$$

pour toute fonction continue croissante $f : \{0,1\}^V \to \mathbb{R}$. L'attractivité signifie que la monotonie stochastique est préservée par l'évolution du processus i.e.

$$\text{si} \quad \mu \preceq \nu \qquad \text{alors} \qquad \mu_t \preceq \nu_t \quad \text{pour tout } t \geqslant 0.$$

L'attractivité implique que $\delta_{\mathbf{1},t} \Rightarrow \bar{\nu}$, où $\delta_{\mathbf{1},t}$ dénote la loi du processus commençant par l'occupation complète et $\Rightarrow$ dénote la convergence en loi. Ici, $\bar{\nu} \in \mathcal{I}_e$ et est appelé la mesure invariante supérieure, tandis que $\delta_{\mathbf{0}}$ est appelé la mesure invariante inférieure. Si $\bar{\nu} = \delta_{\mathbf{0}}$, alors $\mu_t \Rightarrow \delta_{\mathbf{0}}$ pour toute μ et $\mathcal{I} = \{\delta_{\mathbf{0}}\}$.

On dit qu'il y a la *convergence complète* si pour toute configuration initiale A,

$$\xi_t^A \Rightarrow \alpha(A)\bar{\nu} + (1 - \alpha(A))\delta_{\mathbf{0}},$$

avec

$$\alpha(A) = \mathbb{P}(\xi_t^A \neq \varnothing \,\forall\, t \geqslant 0),$$

de manière équivalente (par l'auto-dualité), si pour tous A et B,

$$\mathbb{P}(\xi_t^A \cap B) \to \alpha(A)\alpha(B).$$

Dans le cadre des graphes finis, la convergence complète peut être comprise comme la stabilité du processus autour d'un certain état dans une très longue période (voir le théorème 1.2.3 (iii) et la transition de phase *(T3)*).

1.1.3 Valeurs critiques

La survie du processus de contact (un virus) est une question naturelle. En raison de la monotonie du processus de contact en λ, on peut définir une valeur critique

$$\lambda_1(G) = \inf\{\lambda > 0 : \mathbb{P}(\xi_t^v \neq \varnothing \;\forall\, t \geqslant 0) > 0\}.$$

Cette définition est indépendante du choix de v lorsque G est connexe. Cette valeur critique concerne la chance de *survie globale* du processus. On définit aussi une autre valeur critique pour la chance de *survie locale*, ou la chance de récurrence, du processus:

$$\lambda_2(G) = \inf\{\lambda > 0 : \mathbb{P}(\text{pour tout } T, \text{ il existe } t \geqslant T : v \in \xi_t^v) > 0\},$$

où v est un sommet arbitraire de G (cette définition ne dépend pas du choix de v si G est connexe). Il est évident que $\lambda_1(G) \leqslant \lambda_2(G)$. Les valeurs critiques ont été bien comprises pour les réseaux réguliers $\mathbb{Z}^d$ et $\mathbb{T}_d$, et elles seront rappelées dans la suite. En dominant le processus de contact par la marche aléatoire branchante, on obtient une borne inférieure sur les valeurs critiques pour les graphes de degré borné.

Lemme 1.1.1. *Soit G un graphe connexe de degré borné par K. Alors,*

$$\lambda_1(G) \geqslant 1/K,$$

et si G est infini, alors $\lambda_1(G) \leqslant \lambda_1(\mathbb{Z}) < \infty$.

Dans cette thèse, on étudie seulement la transition de phase du processus de contact sur des graphes finis lorsque le taux d'infection traverse la première valeur critique du graphe limite. Par conséquent, on note souvent

$$\lambda_c(G) = \lambda_1(G).$$

Le rôle de λ_2 dans le carde des graphes finis n'est pas encore compris à ce jour.

1.2 Processus de contact sur les réseaux réguliers

Dans la première période de l'histoire du processus de contact, à partir de la fin des années 1970 jusqu'au début des années 2000, le processus sur les réseaux réguliers comme $\mathbb{Z}^d$ et $\mathbb{T}_d$ a été soigneusement étudié. Beaucoup de beaux résultats ont été obtenus en développant des techniques importantes.

1.2.1 Processus de contact sur $\mathbb{Z}^d$ et $[\![0, n]\!]^d$

Sur $\mathbb{Z}^d$

Il est établi qu'il y a une transition de phase unique du processus de contact, i.e. la survie globale est équivalente à la survie locale.

Théorème 1.2.1. *Les assertions suivantes ont lieu.*

(i) $0 < \lambda_1(\mathbb{Z}^d) = \lambda_2(\mathbb{Z}^d) =: \lambda_c(\mathbb{Z}^d) < \infty$.

(ii) Si $\lambda \leqslant \lambda_c(\mathbb{Z}^d)$, le processus de contact meurt presque sûrement et $\mathcal{I} = \{\delta_{\mathbf{0}}\}$.

(iii) Si $\lambda > \lambda_c(\mathbb{Z}^d)$, la convergence complète a lieu et $\mathcal{I}_e = \{\delta_{\mathbf{0}}, \bar{\nu}\}$.

La partie (i) a été prouvée par Durrett dans [35] pour $d = 1$ en 1980. Plus tard, Bezuidenhout et Grimmett [16] ont généralisé (i) pour $d \geqslant 2$ et prouvé (ii). La partie (iii) a été établie par l'effort de nombreux auteurs, en particulier Durrett et Griffeath [40], Schonmann [79], Andjel [4]. Le résultat final a été prouvé par Bezuidenhout et Grimmett [16].

Bornes exponentielles. Bezuidenhout et Grimmett [17] et Durrett [37] ont montré la fluctuation du comportement du processus de contact lorsque le taux d'infection traverse la valeur critique. Pour tout $A \subset \mathbb{Z}^d$, on définit le temps de mort du processus partant de A par

$$\tau^A = \inf\{t \geqslant 0 : \xi_t^A = \varnothing\}.$$

Théorème 1.2.2. *On a*

(i) si $\lambda < \lambda_c(\mathbb{Z}^d)$, alors pour tout $v \in \mathbb{Z}^d$, la limite suivante existe et

$$\lim_{t\to\infty} \frac{-\log(\mathbb{P}(\tau^{\{v\}} \geqslant t))}{t} = \gamma_-(\lambda) > 0,$$

(ii) si $\lambda > \lambda_c(\mathbb{Z}^d)$, il existe des constantes positives C et ε, telles que

$$\begin{aligned} \mathbb{P}(\tau^A < \infty) &\leqslant \exp(-\varepsilon|A|), \\ \mathbb{P}(t < \tau^A < \infty) &\leqslant C\exp(-\varepsilon t). \end{aligned}$$

Ce théorème est un ingrédient important pour étudier la transition de phase du processus de contact sur les boîtes finies (voir le théorème 1.2.3).

Exposant critique. Une question classique est d'étudier la probabilité de survie lorsque λ est proche de la valeur critique. On définit

$$\rho(\lambda) = \mathbb{P}(\xi_t^0 \neq \varnothing \, \forall\, t \geqslant 0).$$

D'après le théorème 1.2.1 (ii), $\rho(\lambda_c) = 0$ et on dit que $\rho(\lambda)$ a un exposant critique γ si

$$\rho(\lambda) \sim (\lambda - \lambda_c)^\gamma \qquad \text{lorsque} \qquad \lambda \downarrow \lambda_c.$$

Il a été montré qu'il existe une constante positive $c = c(d)$, telle que pour $\lambda > \lambda_c$

$$\rho(\lambda) \geqslant c(\lambda - \lambda_c).$$

Ce résultat implique que l'exposant critique, s'il existe, vérifie $\gamma \leqslant 1$. La question de savoir si $\gamma = 1$ ou non est encore ouverte.

Sur $[\![0, n]\!]^d$

Il est naturel d'étudier le processus de contact sur les boîtes finies $[\![0, n]\!]^d$ et de comparer son comportement avec celui du processus sur le réseau infini. Le processus sur les graphes finis, en particulier sur $[\![0, n]\!]^d$, atteint finalement 0 qui est l'unique état absorbant. Une question intéressante est de savoir à quelle vitesse le processus atteint cet état absorbant. On définit le temps de mort du processus sur $[\![0, n]\!]^d$ à partir de la pleine occupation par

$$\tau_n = \inf\{t \geqslant 0 : \xi_t^{[\![0,n]\!]^d} = \varnothing\}.$$

Alors τ_n est fini presque sûrement et une transition de phase de τ_n a lieu à $\lambda_c(\mathbb{Z}^d)$. Le résultat suivant montre cette transition de phase et le comportement du processus sur-critique.

Théorème 1.2.3. *Les assertions suivantes ont lieu.*

(i) Si $\lambda > \lambda_c(\mathbb{Z}^d)$, il existe une constante positive $\gamma_+(\lambda)$, telle que

$$\frac{\log \tau_n}{n} \xrightarrow[n\to\infty]{(\mathbb{P})} \gamma_+(\lambda).$$

Si $\lambda < \lambda_c(\mathbb{Z}^d)$, alors

$$\frac{\tau_n}{\log n} \xrightarrow[n\to\infty]{(\mathbb{P})} \gamma_-(\lambda),$$

avec $\gamma_-(\lambda)$ comme dans le théorème 1.2.2.

(ii) Si $\lambda > \lambda_c(\mathbb{Z}^d)$, alors τ_n divisé par son espérance converge en loi vers une variable aléatoire exponentielle de moyenne 1.

(iii) Si $\lambda > \lambda_c(\mathbb{Z}^d)$, alors pour toute suite $(t_n) \nearrow \infty$ satisfaisant $t_n = o(\mathbb{E}(\tau_n))$,

$$\frac{|\xi_{t_n}^{[\![0,n]\!]^d}|}{n^d} \xrightarrow{(\mathbb{P})} \bar{\nu}\left(\xi \in \{0,1\}^{\mathbb{Z}^d} : \xi(0) = 1\right) = \mathbb{P}\left(\xi_t^0 \neq \varnothing \,\forall\, t \geqslant 0\right),$$

où (ξ_t^0) est le processus de contact sur $\mathbb{Z}^d$ partant de la configuration où seul 0 est infecté.

Les parties (i) et (ii) ont été prouvées par Durrett et Schonmann [43] et Durrett et Liu [41] pour $d = 1$. Mountford les a généralisé dans [62] et [63] pour les dimensions supérieures. Pour le cas critique en dimension 1, dans [44], les auteurs ont prouvé que si $\lambda = \lambda_c(\mathbb{Z})$

$$\mathbb{P}(n \leqslant \tau_n \leqslant n^4) \to 1.$$

La partie (iii) a été prouvée par Schonmann [78], puis généralisée par Simonis [82].

La partie (i) montre la variation de τ_n lorsque le taux d'infection traverse la valeur critique. La partie (ii) nous dit que dans le régime sur-critique, τ_n est proche d'une variable aléatoire exponentielle. De plus, par (iii), on a aussi que la densité des sites infectés est proche de celle en volume infini sur une très longue période de temps. Les parties (ii) et (iii) sont des résultats de métastablité pour le processus de contact.

1.2.2 Processus de contact sur $\mathbb{T}_d$ et $\mathbb{T}_d^n$

Sur $\mathbb{T}_d$

Soit $\mathbb{T}_d$ l'arbre connexe homogène dans lequel chaque sommet a $d+1$ voisins. Le comportement ergodique du processus sur $\mathbb{T}_d$ est plus riche que celui sur $\mathbb{Z}^d$.

Théorème 1.2.4. *Pour tout $d \geqslant 2$*

(i) $0 < \lambda_1 < \lambda_2 < \infty$,

(ii) si $\lambda \leqslant \lambda_1$, le processus meurt presque sûrement et $\mathcal{I}_e = \{\delta_{\mathbf{0}}\}$,

(iii) si $\lambda_1 < \lambda \leqslant \lambda_2$, le processus survit globalement mais pas localement et $|\mathcal{I}_e| = \infty$,

(iv) si $\lambda > \lambda_2$, la convergence complète a lieu et $\mathcal{I}_e = \{\delta_{\mathbf{0}}, \bar{\nu}\}$.

En trouvant une borne supérieure explicite pour λ_1 et une borne inférieure pour λ_2, Pemantle a prouvé la partie (i) dans [74] pour $d \geqslant 3$, puis Liggett l'a prouvé dans [57] pour $d = 2$. Dans [84], Stacey a donné une preuve élégante directe sans borner les valeurs critiques. La partie (ii) a été prouvée par Morrow, Schinazi et Zhang dans [61]. Pour (iii), Lalley a montré que le processus de contact avec $\lambda = \lambda_2$ ne survit pas localement. Durrett, Schinazi dans [39] et Liggett dans [58] ont donné deux façons de construire une infinité de mesures invariantes. La convergence complète a été prouvée par Zhang [88] puis une preuve simplifiée a été donnée par Salzano et Schonmann [77].

Barsky, Wu dans [11] et Schonmann dans [80] ont montré que l'exposant critique est 1 i.e.

$$\mathbb{P}(\xi_t^v \neq \varnothing \, \forall t \geqslant 0) \sim (\lambda - \lambda_1) \qquad \text{lorsque } \lambda \downarrow \lambda_1.$$

Sur $\mathbb{T}_d^n$

On note $\mathbb{T}_d^n$ l'abre homogène de hauteur n. Alors le volume de $\mathbb{T}_d^n$ est $(d+1)d^{n-1}$. Soit τ_n le temps de mort du processus de contact sur $\mathbb{T}_d^n$ partant de la pleine configuration. Comme sur les boîtes finies, on espère une transition de phase pour τ_n.

Théorème 1.2.5. *On a*

(i) si $\lambda < \lambda_2(\mathbb{T}_d)$ alors il existe une constante positive c_1, telle que

$$\frac{\tau_n}{n} \xrightarrow[n\to\infty]{(\mathbb{P})} c_1$$

(ii) si $\lambda > \lambda_2(\mathbb{T}_d)$ alors il existe une constante positive c_2, telle que

$$\lim_{n\to\infty} \frac{\log \mathbb{E}(\tau_n)}{d^n} = c_2,$$

de plus τ_n divisé par son espérance converge en loi vers une variable aléatoire exponentielle de moyenne 1.

Ce théorème a été prouvé par Cranston, Mountford, Mourrat et Valesin dans [22]. Avant eux, Stacey a prouvé dans [83] des résultats plus faibles. Plus précisément, il a montré que $\tau_n \sim n$ si $\lambda < \lambda_2$, et τ_n est d'ordre au moins $\exp(d^{n(1-\varepsilon)})$ pour tout $\varepsilon > 0$ si $\lambda > \lambda_2$.

Ce théorème semble montrer que la transition de phase a lieu au niveau de la deuxième valeur critique du graphe limite. Mais nous allons voir qu'en fait elle a lieu autour de la première valeur critique. Mais ici la bonne notion de limite de $\mathbb{T}_d^n$ n'est pas l'arbre $\mathbb{T}_d$, mais plutôt l'arbre de la canopée $\mathbb{CT}_d$, qui est tel que $\lambda_1(\mathbb{CT}_d) = \lambda_2(\mathbb{T}_d)$. La notion de convergence ici est celle de la convergence locale de graphes, introduite par Benjamini et Schramm que nous rappelons dans le prochain paragraphe.

1.2.3 Convergence faible de graphes et transition de phase pour le temps de mort

On donne ici la définition, introduite par Benjamini et Schramm [21], de la convergence locale d'une suite de graphes. Un graphe enraciné est un couple (G, v) composé d'un graphe connexe localement fini G et un sommet v dans G appelé la racine. Deux graphes enracinés (G, v) et (G', v') sont appelés *isomorphes* s'il existe un isomorphisme de G vers G' qui envoie v sur v'. Dans ce cas, on note $(G, v) \sim (G', v')$. On définit $\mathcal{G}$ l'ensemble des classes d'équivalence de graphes enracinés. On munit $\mathcal{G}$ de la topologie locale induite par la métrique suivante. Pour (G, v) et (G', v') dans $\mathcal{G}$,

$$d_{loc}((G, v), (G', v')) = 2^{-\max\{r \geqslant 0 :\ 1(B_G(v,r) \sim B_G(v',r))\}},$$

où $B_G(v, r)$ désigne la boule de rayon r autour de v dans G qui contient tous les sommets à distance inférieure à r de v et toutes les arêtes entre eux.

Alors, $\mathcal{E} = (\mathcal{G}, d_{loc})$ est un espace métrique complet séparable. Un graphe aléatoire enraciné est une variable aléatoire à valeur dans $\mathcal{E}$. On définit naturellement la notion de la convergence faible de graphes aléatoires enracinés comme suit. Une suite de graphes aléatoires enracinés (G_n, v_n) converge vers (G, v) si pour toute fonction continue bornée $F : \mathcal{G} \to \mathbb{R}$,

$$\mathbb{E}[F((G_n, v_n))] \to \mathbb{E}[F((G, v))] \qquad \text{lorsque} \quad n \to \infty. \tag{1.4}$$

On note dans ce cas

$$(G_n, v_n) \quad \overset{(w)}{\Longrightarrow} \quad (G, v).$$

Dans la proposition suivante, on donne une caractérisation très pratique de la convergence faible de graphes.

Proposition 1.2.6. *Une suite de graphes aléatoires enracinés (G_n, v_n) converge vers (G, v) si et seulement si pour tout (H, w) dans $\mathcal{G}$ et tout r fini,*

$$\mathbb{P}(B_{G_n}(v_n, r) \sim (H, w)) \longrightarrow \mathbb{P}(B_G(v, r) \sim (H, w)) \qquad \textit{lorsque } n \to \infty. \tag{1.5}$$

On dit qu'une suite de graphes aléatoires (G_n) converge faiblement vers un graphe enraciné (G, v), et note

$$(G_n) \quad \overset{(w)}{\Longrightarrow} \quad (G, v),$$

si

$$(G_n, u_n) \quad \overset{(w)}{\Longrightarrow} \quad (G, v),$$

où (G_n, u_n) est le graphe aléatoire G_n enraciné à un sommet choisi uniformément.

La notion de cette convergence faible (ou locale) a été largement étudiée pour de nombreuses classes de graphes aléatoires tels que les graphes d'Erdos-Rényi, modèles de configuration, graphes à attachement préférentiel, graphes inhomogènes aléatoires,...

Le graphe régulier aléatoire (défini dans la section suivante) par exemple converge faiblement vers l'arbre régulier $\mathbb{T}_d$. En outre, Mourrat et Valesin ont prouvé dans [66] qu'il y a une transition de phase de τ_n lorsque le taux d'infection traverse $\lambda_1(\mathbb{T}_d)$: τ_n est d'ordre exponentiel quand $\lambda > \lambda_1(\mathbb{T}_d)$, et d'ordre logarithmique quand $\lambda < \lambda_1(\mathbb{T}_d)$.

Nous observons donc que la transition de phase sur le temps de mort sur un graphe fini semble avoir toujours lieu autour de la première valeur critique du graphe limite. Autrement dit, dans tous les exemples précédents, on observe le phénomène remarquable suivant.

(T). Transition de phase (local-global). *Supposons que (G_n) converge faiblement vers un graphe enraciné (G, v) et soit τ_n le temps de mort du processus de contact sur G_n partant de la pleine occupation. Alors*

(T1) avec grande probabilité (i.e. avec probabilité qui tend vers 1), τ_n est d'ordre exponentiel (resp. logarithmique) en $|G_n|$ lorsque $\lambda > \lambda_c(G)$ (resp. $\lambda < \lambda_c(G)$).

(T2) Dans le régime sur-critique, τ_n divisé par son espérance converge en loi vers une variable aléatoire exponentielle de moyenne un.

Ce résultat a en fait été observé dans un certain nombre d'autres exemples: modèles de configuration avec distribution des degrés à queue lourde, graphes "small-world", graphes à attachement préférentiel, graphes inhomogènes aléatoires. Ces exemples seront revus en détail dans la section suivante. Il y a aussi quelques résultats généraux. Pour la phase sur-critique, dans [64], les auteurs ont montré que si $\lambda > \lambda_c(\mathbb{Z})$ et les degrés des graphes sont bornés, le temps de mort est d'ordre exponentiel. Récemment, Schapira et Valesin ont montré que pour tout graphe fini G (sans restriction sur les degrés) et $\lambda > \lambda_c(\mathbb{Z})$, le temps de mort est d'ordre au moins exponentiel en $|G|/\log(|G|)^{1+\varepsilon}$ pour tout $\varepsilon > 0$. Mais comme pour un graphe G infini, en général $\lambda_c(G) < \lambda_c(\mathbb{Z})$, la question se pose toujours de montrer la phase sur-critique dans *(T1)* pour $\lambda_c(G) < \lambda < \lambda_c(\mathbb{Z})$.

Notons par ailleurs que le résultat dans le régime sous-critique n'est pas vrai en général. En effet, nous allons construire dans le chapitre 5 un contre-exemple dans lequel τ_n est d'ordre exponentiel en $|G_n|^\alpha$, avec $\alpha > 0$, même si $\lambda < \lambda_c(G)$.

Dans le cas où $G_n = B_G(v, n)$ avec G un graphe transitif et $v \in G$, en appliquant un résultat de Aizenman et Jung [5], on peut montrer que τ_n est d'ordre logarithmique lorsque $\lambda < \lambda_c(G)$. En effet, ils ont montré que si $\lambda < \lambda_c(G)$ alors il existe des constantes positives c et C, telles que pour tout $t \geqslant 0$

$$\mathbb{P}(\tau^w \geqslant t) \leqslant Ce^{-ct},$$

où τ^w est le temps de mort du processus de contact sur G partant d'un sommet w dans G. Puisque G_n est un sous-graphe de G, pour tout $t \geqslant 0$

$$\mathbb{P}(\tau_n^w \geqslant t) \leqslant Ce^{-ct},$$

où τ_n^w est le temps de mort du processus de contact sur G_n partant d'un sommet w dans G_n. Alors

$$\mathbb{P}(\tau_n \geqslant t) = \mathbb{P}(\exists\, w \in G_n : \tau_n^w \geqslant t) \leqslant C|G_n|e^{-ct} \to 0 \quad \text{lorsque} \quad n \to \infty,$$

si $t = 2\log(|G_n|)/c$.

Cela étant, G_n ne converge nécessairement vers G (voir le cas des arbres homogènes finis). Cependant, si $|\partial B_G(v, n)| = o(|B_G(v, n)|)$, c'est bien le cas par exemple sur un réseau régulier de $\mathbb{R}^d$, et on a donc bien *(T1)*, au moins en régime sous-critique. Un exemple plus général est celui où G est un graphe transitif moyennable, (G_n) suite de Fölner de G (i.e. tq. $\frac{|\partial G_n|}{|G_n|} \to 0$).

Outre le temps de mort, on peut également étudier le comportement de la densité des sites infectés. Par exemple, le théorème 1.2.3 (iii) montre que pour le processus de contact sur les boîtes finies, la densité converge vers la probabilité de survie du processus de contact partant d'un seul sommet sur le graphe limite. Dans le cas général, nous pouvons expliquer ce résultat comme suit: par l'auto-dualité du processus de contact,

$$\frac{|\xi_t^{\mathbf{1}}|}{n} \overset{(\mathcal{L})}{=} \frac{1}{n} \sum_{v \in V_n} 1(\xi_t^v \neq \varnothing).$$

En conséquence, la densité des sommets infectés au temps t peut être interprétée comme la moyenne de la probabilité de survie. On peut donc s'attendre à ce que

(T3)

$$\rho_n(\lambda) = \frac{|\xi^{\mathbf{1}}_{t_n}|}{n} \xrightarrow[n\to\infty]{(\mathbb{P})} \rho_G(\lambda) = \mathbb{P}(\xi^v_t \neq \varnothing \, \forall t \geqslant 0),$$

pour toute suite $(t_n) \nearrow \infty$, au moins si $t_n = o(\mathbb{E}(\tau_n))$.

En outre, comme pour l'exposant critique de $\rho_G(\lambda)$, on s'intéresse également à l'ordre de grandeur de $\rho_n(\lambda)$ quand $\lambda \to \lambda_c(G)$ et $n \to \infty$. Cette question a été envisagée pour les boîtes finies, modèles de configuration avec la distribution de degrés en loi de puissance, graphes à attachement préférentiel et graphes réguliers aléatoires.

Dans la dernière section de ce chapitre, nous passerons en revue l'étude de toutes les trois questions *(T1)–(T3)* pour un certain nombre d'exemples de graphes aléatoires.

1.3 Processus de contact sur des graphes aléatoires et résultats de la thèse

1.3.1 Une brève histoire de graphes aléatoires

Le sujet des graphes aléatoires est né à la fin des années 1950, lorsque Erdos et Rényi ont essayé de décrire à quoi ressemble un choix aléatoire de graphe avec n sommets et m arêtes. Après, Gilbert a introduit un autre modèle plus simple, noté $G(n,p)$, dans lequel chacune des $n(n-1)/2$ arêtes est indépendamment présente avec probabilité p (si $p = 2m/n(n-1)$, alors les deux modèles d'Erdos- Rényi et de Gilbert sont presque équivalents). Erdos et Rényi ont découvert qu'il existe des transitions de phase pour la structure du graphe. Par exemple pour l'apparition d'une composante géante: si $p = c/n$ et $c < 1$, toutes les composantes connexes du graphe ont taille $\mathcal{O}(\log n)$, mais si $c > 1$, la composante la plus grande a $\varkappa n$ sommets avec $\varkappa = \varkappa(c) > 0$, et les autres composantes ont taille $\mathcal{O}(\log n)$. Les résultats d'Erdos et Rényi ont inspiré beaucoup de travaux dans la théorie des graphes aléatoires.

Une autre propriété intéressante du graphe d'Erdos-Rényi est sa compacité. Le diamètre ou la distance typique (la distance entre deux sommets choisis uniformément) est $\mathcal{O}(\log n)$, ce qui est très inférieur au nombre de sommets. On dit qu'un graphe à n sommets est *"small-world"* si le diamètre ou la distance typique est d'ordre $(\log n)^k$, avec $k > 0$.

Réseaux "small-world". Dans la pièce "six degrés de séparation" de John Guare, une mère dit à sa fille que le monde est vraiment petit : "Tout le monde sur la planète est séparé par seulement six personnes. Six degrés de séparation. Entre nous et tout le monde sur cette planète dont le président des États-Unis, un gondolier à Venise, ou un Eskimo".

Le phénomène de "small-wold" est observé dans divers réseaux. En 1967, Stanley Milgram a donné des lettres à ses amis. Les lettres devaient être envoyées vers une personne cible, mais les récipiendaires ne pouvaient qu'envoyer à quelqu'un qu'ils connaissaient. Trente cinq pour cent des lettres sont parvenues à destination et la médiane de nombres d'étapes que les lettres ont pris était d'environ 6.

Dans les graphes de la collaboration des acteurs (resp. des mathématiciens), deux personnes sont reliées par une arête si ils sont apparus dans un même film (resp. un même article). Les données empiriques montrent que le diamètre du graphe des acteurs est environ 6 et la distance typique dans le graphe des mathématiciens est environ 7.

Albert, Jeong et Barabási [2, 9] ont étudié le graphe du réseau internet dont les sommets sont les pages internets et les arrêts sont des liens. Ils ont estimé que la distance typique est environ $0.35 + 2.06 \log n$, où n est la taille du graphe.

Loi de puissance et deux modèles. Barabási et Albert [8] ont observé que dans les réseaux sociaux, les degrés des noeuds se répartissent selon une loi de puissance. Plus précisément, la proportion de sommets de degré k décroît asymptotiquement comme une puissance de k, simplement avec un exposant $a > 0$ qui dépend du réseau considéré. Par exemple pour l'internet, Barabási et Albert trouvent l'exposant $a = 2,1$, alors que pour le réseau des collaborations entre acteurs, ils trouvent $a = 2,3$.

Graphe à attachement préférentiel. Pour illustrer la loi de puissance dans les réseaux sociaux, Barabási et Albert [8] ont introduit un modèle dynamique, appelé le graphe à attachement préférentiel. On imagine que l'internet est constamment ajouté une nouvelle page qui envoie des liens aux pages existantes avec probabilité proportionnelle à leur popularité. Plus précisément, à l'instant 1, il y a deux sommets v_1, v_2 et m arêtes entre eux. Puis pour $t \geqslant 2$, un nouveau sommet v_t se connecte à m sommets existants avec la règle suivante. Chacune des m nouvelles arêtes est reliée à un sommet v_i avec probabilité proportionnelle à $\deg(v_i) + u$, avec $u > -1$. Alors, la fraction de sommets de degré k est d'ordre k^{-3-u}. Autrement dit, l'exposant du graphe est $3 + u$.

Modèle de configuration. Si on veut seulement un graphe avec distribution de degrés donné, il est simple d'utiliser le modèle de configuration. Étant donné une distribution (p_k) (ici, $p_k \sim k^{-a}$), nous construisons un graphe de n sommets $v_1, \dots, v_n$ comme suit. Soit $D_1, \dots, D_n$ une suite de variables aléatoires i.i.d. de même loi (p_k). Supposons que $D_1 + ... + D_n$ est pair, si non nous ajoutons simplement 1 à un des (D_i). Pour tout $1 \leqslant i \leqslant n$, nous allouons D_i demi-arêtes au sommet v_i. Puis nous collons uniformément au hasard toutes ces demi-arêtes par paires. Alors, dans le graphe obtenu, la fraction de sommets de degré k converge vers p_k lorsque n tends vers l'infini.

Nous remarquons que les graphes à attachement préférentiel et modèle de configuration sont des réseaux "small-world". De plus, les deux graphes convergent faiblement vers des arbres, nous allons le voir dans sous-sections suivantes.

Épidémie sur réseaux sociaux. Il y a deux modèles de base de l'épidémie: SIR et SIS. Dans le premier, un individu sensible (S) devient infecté (I) au taux λ fois le nombre de ses voisins infectés, et au taux 1 un sommet infecté entre la classe (R) des sommets immunisés contre l'infection. Le modèle SIR sur un graphe G a une relation étroite avec une percolation sur G construite comme suit. Pour chaque arête (x, y) de G, nous la supprimons et puis tirons indépendamment deux arrêts orientés $x \to y$ et $y \to x$ avec même probabilité $\lambda/(\lambda + 1)$, qui est la probabilité que le virus en x ou y infecte l'autre avant de mourir. Alors, le graphe des sites infectés d'une épidémie SIR partant de x est juste la composante connexe de x dans la percolation orientée. Pour de nombreux exemples, la percolation orientée (et donc le modèle SIR) est essentiellement équivalente aux processus branchants qui sont bien étudiés, voir par example [68].

D'un autre côté, dans le modèle SIS (ou le processus de contact), les individus peuvent être réinfectés à plusieurs reprises (puisqu'il n'y a pas d'état immunitaire (R)). Par conséquent, le processus de contact est en général plus compliqué à étudier et nous n'avons pas de résultats aussi précis que dans le cas du modèle SIR. Cependant, le modèle SIS peut présenter un état stationnaire, ou l'état endémique dans lequel la densité de sites infectés reste stable une très longue période (d'ordre typiquement du temps de mort du processus).

En utilisant des méthodes de champ moyen, Pastor-Satorras et Vesignani [76] ont étudié le processus de contact sur les graphes en loi de puissance. Leur calculs non-rigoureux suggèrent une conjecture sur λ_c-la valeur critique pour la persistance du virus (le processus de contact).

- Si a -l'exposant du graphe- est inférieur à 3, alors $\lambda_c = 0$ i.e. pour tout λ positif, τ_n -le temps de mort du virus- est d'ordre exponentiel en n, où n est le nombre des sommets du graphe.

- Si $3 < a \leqslant 4$, alors $\lambda_c > 0$, i.e. τ_n est d'ordre au plus polynomial (resp. exponentiel) en n si $\lambda < \lambda_c$ (resp. $\lambda < \lambda_c$). De plus, en régime sur-critique, la densité de sites infectés est $\rho(\lambda) \asymp (\lambda - \lambda_c)^{1/(a-3)}$, quand $\lambda \downarrow \lambda_c$.

- Si $a > 4$, alors $\lambda_c > 0$ et $\rho(\lambda) \asymp (\lambda - \lambda_c)$, quand $\lambda \downarrow \lambda_c$.

Le premier résultat rigoureux autour de cette conjecture est dû à Berger, Borgs, Chayes et Saberi [13]. Ils ont montré que pour le processus de contact sur le graphe à attachement préférentiel d'exposant 3, λ_c est nul et ils ont obtenu des estimations pour la densité $\rho(\lambda)$.

1.3.2 Arbres de Galton-Watson, graphe d'Erdos-Rényi et graphes inhomogène aléatoires

Le premier résultat sur le thème du processus de contact sur des graphes aléatoires est dû à Pemantle [74] sur la survie du processus sur des arbres de Galton-Watson. Soit $\mathbb{T}$ un arbre de Galton-Watson avec loi de reproduction $\mathbf{q} = (q_k)$. Il découle directement du Lemme 1.1.1 que si le support de $\mathbf{q}$ est borné alors $\lambda_c > 0$ i.e. le processus de contact sur $\mathbb{T}$ meurt presque sûrement lorsque λ est suffisamment petit. Pemantle a donné une condition suffisante sur $\mathbf{q}$ pour que la valeur critique soit zéro.

Théorème 1.3.1. *[74] Si*

$$\limsup_{k\to\infty} \frac{|\log q_k|}{k} = 0, \tag{P}$$

alors $\lambda_c(\mathbb{T}) = 0$.

Ce résultat n'a jusqu'à présent pas été amélioré, et en particulier la question suivante reste ouverte.

Problème ouvert. Est-ce que $\lambda_c(\mathbb{T}) > 0$ ou non, lorsque $|\log(q_k)| \asymp k$.

L'arbre de Galton-Watson est la limite faible de nombreux graphes aléatoires, tels que le graphe d'Erdos-Rényi, le graphe inhomogène, et le modèle de configuration. Nous introduisons maintenant notre résultat sur le processus de contact sur le graphe d'Erdos-Rényi et une généralisation.

Le graphe inhomogène aléatoire (GIA). Le graphe d'Erdos-Renyi est homogène en un certain sens: toutes les arêtes sont présentes avec la même probabilité, et les degrés des sommets se concentrent autour du degré moyen. Par contre, les données empiriques montrent que les réseaux réels sont hétérogènes. Alors, les probabilistes ont généralisé ce graphe comme suit. Soit (w_i) une suite de variables aléatoires i.i.d. de moyenne finie. Alors pour tout $1 \leqslant i, j \leqslant n$, v_i est connecté à v_j avec probabilité $p_{i,j} = p_{i,j}(w)$. Lorsque les $p_{i,j}$ sont égaux, le graphe obtenu est le graphe d'Erdos-Rényi. Il y a plusieurs versions différentes pour $(p_{i,j})$, telles que le modèle de Chung-Lu [1] où $p_{i,j} = \min\{1, w_i w_j/\ell_n\}$, ou le modèle de Britton, Deijfen, Martin-Lof [15] où $p_{i,j} = w_i w_j/(w_i w_j + \ell_n)$, avec $\ell_n = w_1 + \ldots + w_n$. Nous considérons ici le modèle du graphe Poisson introduit par Norros-Reitu [69], où

$$p_{i,j} = 1 - \exp(-w_i w_j/\ell_n),$$

car il conduit à un couplage entre le GIA et un arbre de Galton-Watson qui est très utile. Nous renvoyons à l'article de Bollobas, Janson et Riordan [19] pour de nombreux autres modèles et leurs propriétés. Le GIA converge faiblement, voir [19, 53], vers un arbre de Galton-Watson à deux types dans lequel la loi de reproduction de la racine est $\mathbf{p} = (p_k)$ et celle des autres sommets est $\mathbf{q} = (q_k)$, avec

$$p_k = \mathbb{P}(\mathrm{Poi}(w) = k),$$

et

$$q_k = \mathbb{P}(\mathrm{Poi}(w^*) = k),$$

où w^* est la distribution biaisée par la taille de w. On suppose que les hypothèses suivantes sont satisfaites:

(H1) l'arbre limite est sur-critique, i.e. $\mathbb{E}(\mathbf{q}) > 1$,

(H2) le support de $\mathbf{q}$ n'est pas borné et $|\log(q_k)| = o(k)$ quand $k \to \infty$.

Théorème 1.3.2. *Supposons (H1) et (H2). Soit τ_n le temps de mort du processus de contact sur le GIA (avec n sommets) partant de la pleine occupation. Alors pour tout $\lambda > 0$, avec grand probabilité*

$$\log \tau_n \asymp n.$$

De plus, $(T2)$ a lieu.

Notons que l'hypothèse (H1) est nécessaire, puisque si $\mathbb{E}(\mathbf{q})$ est inférieure à 1, toutes les composantes dans GIA ont taille $\mathcal{O}(\log n)$. D'autre part, l'hypothèse (H2) est essentielle dans notre preuve. De plus, ce théorème et le théorème 1.3.1 impliquent que la transition de phase **(T)** est vrai dans ce cas.

Nous prouvons aussi un résultat similaire pour le graphe d'Erdos-Rényi, $G(n,p)$. Notons que $G(n,p)$ converge faiblement vers un arbre de Galton-Watson avec loi de reproduction de Poisson, qui ne satisfait pas (H2). Donc, nous avons besoin d'une hypothèse plus forte sur le degré moyen np.

Théorème 1.3.3. *Soit τ_n le temps de mort du processus de contact sur $G(n,p)$ partant de tous les sites infectés. Il existe des constantes positives c, C dépendant de λ telles que pour tout $\lambda > 0$ et $np \geqslant C$,*

$$\mathbb{P}(\tau_n \geqslant \exp(cn)) \to 1.$$

De plus, (T2) a lieu.

De même à la situation des arbres de Galton-Watson, il reste un problème difficile sur le $G(n,p)$, où np est indépendant de λ.

Problème ouvert. Pour np fixé, est-ce que $\lambda_c(G(n,p)) > 0$ i.e. est-ce que pour λ assez petit, τ_n est d'ordre au plus polynomial en n?

1.3.3 Modèle de configuration

Nous considérons le modèle de configuration (définit dans la section 1.3.1) avec la distribution de degré en loi de puissance i.e. les degrés des sommets ont la même loi (p_k)

donnée par

$$p_k \sim k^{-a} \quad \text{pour} \quad k \geqslant 1,$$

avec $a > 1$. Nous notons le graphe par $MC_n(a)$.

On remarque que si $a > 2$ alors $MC_n(a)$ converge faiblement vers $\mathbb{T}(a)$-un arbre de Galton Watson à deux types. Dans cet arbre, la loi de reproduction de la racine est (p_k) et celle des autres sommets est (q_k)- la distribution biaisée par la taille de (p_k). Alors, par le théorème 1.3.1, on a $\lambda_c(\mathbb{T}(a)) = 0$. Donc, si la transition de phase **(T)** est vraie, la valeur critique sur $MC_n(a)$ devrait être aussi zéro. Par contre, la prédiction de Pastor-Satorras et Vesgnani dit que la valeur critique est nulle si et seulement si $a \leqslant 3$. Dans [23], Chatterjee et Durrett ont montré que lorsque $a \geqslant 2$, le temps de mort est d'ordre exponentiel, autrement dit la valeur critique est zéro, ce qui invalide la prédiction de Pastor-Satorras et Vesgnani et établit la transition de phase **(T)**. Plus précisément, Chatterjee et Durrett ont prouvé le théorème suivant.

Théorème 1.3.4. *[23] Soit (ξ_t) le processus de contact sur $MC_n(a)$ partant de la pleine configuration et soit τ_n le temps de mort du processus. Alors, pour tout $\lambda > 0$, les assertions suivantes ont lieu.*

(i) Si $a > 2$, alors pour tout $\varepsilon > 0$, $\mathbb{P}(\tau_n \geqslant \exp(n^{1-\varepsilon})) \to 1$, quand $n \to \infty$.

(ii) Si $a \geqslant 3$, alors il existe des constantes positives c et C (ne dépendants pas de λ), telles que

$$\mathbb{P}\left(c\lambda^{1+(a-2)(2-\delta)} \leqslant \frac{|\xi_{\exp(\sqrt{n})}|}{n} \leqslant C\lambda^{1+(a-2)(1-\delta)} \right) \to 1.$$

Ce résultat a été généralisé dans [64, 65] pour tout $a > 2$:

Théorème 1.3.5. *[64, 65] Soit (ξ_t) le processus de contact sur $CM_n(a)$ à partir de la pleine occupation, avec $a > 2$. Alors, pour tout $\lambda > 0$, il existe une constante positive $c = c(\lambda)$ telle que*

$$\frac{|\xi_{t_n}|}{n} \quad \xrightarrow[n\to\infty]{(\mathbb{P})} \quad \rho_a(\lambda),$$

pour toute suite (t_n) satisfaisant $t_n \to \infty$ et $t_n \leqslant \exp(cn)$, où $\rho_a(\lambda)$ est la probabilité de *survie du processus de contact sur l'arbre de Galton-Watson $\mathbb{T}(a)$. De plus, quand $\lambda \to 0$,*

$$\rho_a(\lambda) \asymp \begin{cases} \lambda^{\frac{1}{3-a}} & \textit{si } 2 < a \leqslant 5/2 \\ \frac{\lambda^{2a-3}}{|\log \lambda|^{a-2}} & \textit{si } 5/2 < a \leqslant 3 \\ \frac{\lambda^{2a-3}}{|\log \lambda|^{2a-4}} & \textit{si } a > 3. \end{cases}$$

Ce théorème est basé sur l'idée d'approximer le processus de contact sur $MC_n(a)$ (à partir d'un sommet typique) par le processus sur le graphe limite- $\mathbb{T}(a)$, avec une estimation très détaillée de la probabilité de survie $\rho_a(\lambda)$.

Dans le cas a est inférieure à 2, $MC_n(a)$ ne converge plus. En utilisant une analyse directe du processus de contact sur le modèle de configuration, nous obtenons le résultat suivant.

Théorème 1.3.6. *Soit (ξ_t) le processus de contact sur $MC_n(a)$ partant de la pleine configuration et soit τ_n le temps de mort du processus. Alors, pour tout $\lambda > 0$, les assertions suivantes ont lieu.*

(i) Pour tout $a > 1$, on a τ_n devisé par son espérance converges en loi vers une variable aléatoire de moyenne 1.

(ii) Si $1 < a \leqslant 2$, il existe une constante positive $c = c(\lambda)$ telle que

$$\frac{|\xi_{t_n}|}{n} \xrightarrow[n\to\infty]{(\mathbb{P})} \rho_a(\lambda),$$

pour toute séquence (t_n) satisfaisant $t_n \to \infty$ et $t_n \leqslant \exp(cn)$, où

$$\rho_a(\lambda) = \sum_{j=1}^{\infty} \frac{k\lambda}{k\lambda + 1} p_k.$$

De plus, lorsque $\lambda \to 0$,

$$\rho_a(\lambda) \asymp \begin{cases} \lambda^{a-1} & \text{si } 1 < a < 2 \\ \lambda|\log \lambda| & \text{si } a = 2, \end{cases}$$

(ce qui montre en particulier que la prédiction de Chatterjee et Durrett que $\rho(\lambda)$ devait être un $\mathcal{O}(\lambda)$ n'était pas correcte).

1.3.4 Graphe à attachement préférentiel

La première réponse mathématique à la prédiction de Pastor-Satorras et Vesignani était dans l'article "The spread of virus on the Internet" de Berger, Borgs, Chayes, Saberi [13]. Ils ont étudié le processus de contact sur le graphe à attachement préférentiel. Ils ont montré que si l'exposant du graphe, noté a, est 3 (dans ce cas la probabilité pour qu'un nouveau sommet soit relié à un sommet v_i est proportionnelle au degré de v_i), alors la valeur critique du processus de contact est égale à zéro. Plus précisément, ils ont prouvé que si $a = 3$, il existe des constantes positives c et C telles que pour tout $\lambda > 0$

$$\mathbb{P}\left(\lambda^c \leqslant \frac{|\xi_{\exp(\sqrt{n})}|}{n} \leqslant \lambda^C\right) \to 1,$$

où (ξ_t) est le processus de contact à partir de la pleine occupation. Dans notre article, nous généralisons ce résultat comme suit.

Théorème 1.3.7. *Soit (ξ_t) le processus de contact sur le graphe à attachement préférentiel d'exposant $a \geqslant 3$ commençant par tous les sommets infectés et soit τ_n le temps de mort du processus.*

(i) Il existe des constantes positives c and C telles que

$$\mathbb{P}\left(c\rho_a(\lambda) \leqslant \frac{|\xi_{t_n}|}{n} \leqslant C\rho_a(\lambda)\right) = 1 - o(1),$$

pour toute suite (t_n) satisfaisant $t_n \to \infty$ et $t_n \leqslant T_n = \exp\left(\frac{c\lambda^2 n}{(\log n)^{a-2}}\right)$ et

$$\rho_a(\lambda) = \frac{\lambda^{2a-3}}{|\log \lambda|^{a-2}}.$$

(ii) τ_n divisé par son espérance converge en loi vers une variable aléatoire de loi exponentielle de moyenne 1.

D'autre part, il a été montré dans [14] que le graphe à attachement préférentiel converge faiblement vers un arbre infini appelé le "*Pólya point graph*" (voir aussi le chapitre 3 pour sa définition). Notons cet arbre par $\mathbb{T}$ et sa racine par o. Nous prouvons que la valeur critique du processus de contact sur $\mathbb{T}$ est aussi zéro. De plus, la probabilité de survie du processus sur $\mathbb{T}$ est du même ordre que la densité métastable du processus sur le graphe à attachement préférentiel. Par conséquent, nous établissons la transition de phase **(T)** dans ce cas.

Théorème 1.3.8. *Soit* (ξ_t^o) *le processus de contact sur* $\mathbb{T}$ *tel que* $\xi_0^o = \{o\}$. *Alors il existe des constantes positives* c *et* C, *telles que*

$$c\rho_a(\lambda) \leqslant \mathbb{P}\left(\xi_t^o \neq \varnothing \ \forall t \geqslant 0\right) \leqslant C\rho_a(\lambda),$$

avec $\rho_a(\lambda)$ *comme dans le théorème 1.3.7.*

Pour les preuves des théorèmes 1.3.7 et 1.3.8, nous avons quelques remarques.

- Pour obtenir le temps T_n, nous reprenons une idée de [23]. Nous trouvons les sommets de degré plus grand que le diamètre du graphe. Ensuite, le virus se propage dans ces grands sommets un temps exponentiel en leur degré total. Dans le graphe à attachement préférentiel, le diamètre est d'ordre $\log n$. D'autre part, la distribution de degré est la loi de puissance d'exposant a. Donc le nombre de sommets de degré supérieure à $\log n$ est d'ordre $n/\log^{a-1} n$. Par conséquent, nous obtenons une borne inférieure sur le temps de mort de l'ordre $\exp(n/\log^{a-2} n)$.

  L'ordre optimal du temps de mort est exponentiel en n. Il nous semble qu'il est possible d'adapter la méthode de [86] pour obtenir une borne inférieure de l'ordre $\exp(n/\log^{1+\varepsilon} n)$ pour tout $\varepsilon > 0$. Cependant, cela ne semble ni facile ni optimal, et nous n'avons donc pas poursuivi dans cette direction.

- Nous suivons de près la preuve dans [65] pour calculer la densité métastable des sites infectés. Nous notons que cette quantité dans le cas du graphe à attachement préférentiel avec exposant $a \geqslant 3$ est légèrement plus grand que celle dans le cas du modèle de configuration avec le même exposant. La raison est que la distance entre les grands sommets (ayant degré plus grand que λ^{-2}) est inférieure à celle dans le modèle de configuration, et donc le virus se propage plus facilement.

1.3.5 Graphes géométriques aléatoires

Un graphe géométrique aléatoire infini, noté par $GGA(R)$, (resp. graphe fini $GGA(n, R)$) est construit comme suit. L'ensemble des sommets du graphe est composé des atomes d'un processus ponctuel de Poisson d'intensité 1 sur $\mathbb{R}^d$ (resp. sur $[0, \sqrt[d]{n}]^d$). Ensuite, deux sommets sont reliés si la distance euclidienne entre eux est inférieure à R.

Le graphe géométrique aléatoire n'est pas un bon modèle pour l'internet puisqu'il n'est ni un réseau de petit monde, ni un graphe présentant une distribution de degré à queue lourde. Toutefois, il est approprié pour la modélisation du réseau de capteurs ou de réseau sans fil. Par conséquent, il y a des recherches sur les processus qui se produisent sur ce graphe, y compris le processus de contact, dans les deux approches théoriques et pratiques.

Par le lemme 1.1.1, la valeur critique du processus de contact sur un graphe G est positive lorsque les degrés de G sont bornés. En revanche, il y a peu de connaissances de la phase sous-critique sur les graphes de degré non-borné. Dans un article récent, Ménard et Singh [67] ont introduit la notion de *partition fusionnée cumulativement* (PFC) qui est un outil pour regrouper des sommets du graphe en clusters et mesurer leur impact pour la diffusion du processus. Les auteurs fournissent une condition suffisante pour avoir $\lambda_c > 0$. Comme application, ils ont montré que $\lambda_c(GGA)$ est positif. À ma connaissance, ils sont les premiers qui donnent un graphe de degré non borné présentant une transition de phase non triviale.

Théorème 1.3.9. *[67] Pour tout R fixé, nous avons $\lambda_c(GGA(R)) > 0$.*

Nous allons présenter plus concrètement la notion de PFC et le théorème 1.3.9 aux chapitres 4 et 5. Nous considérons maintenant le processus de contact sur le graphe géométrique aléatoire finie. Il a été montré dans [48] qu'en dimension deux, si $R^2 = \mathcal{O}(\log n)$ et $R \to \infty$, alors τ_n - le temps de mort du processus de contact sur $GGA(R, n)$ partant de la pleine configuration- satisfait que $Cn \log n \geqslant \log \tau_n \geqslant cnR^2/\log n$ pour des constantes positives c et C.

Nous allons généraliser ce résultat pour toutes les dimensions plus grandes que 2, ainsi que donner des bornes sur $\lambda_c(GGA(R))$ en fonction de R.

Théorème 1.3.10. *Il existe des constantes positive c et C, telles que les assertions suivantes ont lieu.*

(i) Pour tout $\lambda, R > 0$,

$$\mathbb{P}(\log \tau_n \leqslant Cn \log(\lambda R^d)) \to 1,$$

et

$$\mathbb{P}(\tau_n \geqslant \sqrt{n}) \to 1.$$

(ii) On a

$$c/R^d \leqslant \lambda_c(GGA(R)) \leqslant C/R^d.$$

(iii) Si $\lambda \geqslant C/R^d$, alors

$$\mathbb{P}(\log \tau_n \geqslant cn \log(\lambda R^d)) \to 1,$$

et τ_n divisé par son espérance converge en loi vers une variable aléatoire de loi exponentielle de moyenne 1.

Voici quelques observations sur ce théorème.

- $\lambda_c(GGA(R)) \asymp 1/R^d$ quand $R \to \infty$. De plus, si R tend vers l'infini, alors pour tout λ positif, le temps de mort du processus de contact est sur-exponentiel en n. La raison en est que les graphes géométriques aléatoires contiennent de nombreux graphes complets sur lesquels le processus de contact survit un temps sur-exponentiel.

- La phase sous-critique de la transition de phase **(T)** est légèrement violée: le temps de mort n'est pas d'ordre logarithmique. Dans le chapitre 5, basé sur un résultat de [67], nous allons construire un contre-exemple plus fort dans lequel le temps de mort est d'ordre exponentiel "streched", même dans la phase sous-critique.

- Nous avons essayé de montrer que si λ est assez petit, $\tau_n \leqslant n^2$ en utilisant les résultats dans [67], afin d'établir la transition de phase du processus de contact sur $GGA(n, R)$. Cependant, ce travail exige une étude approfondie de la PFC sur GGA, qui est encore nouvelle et n'est pas suffisamment détaillée à ce jour.

1.3.6 Percolation long-range en dimension 1

La percolation long-range unidimensionnelle d'exposant $s > 1$, notée par G_s, est définie comme suit: indépendamment pour tous i et j dans $\mathbb{Z}$, il y a une arête reliant i et j avec probabilité $|i - j|^{-s}$.

Nous allons montrer que le graphe G_s avec grand exposant satisfait la condition dans [67] pour que le processus de contact présente une transition de phase non triviale.

Théorème 1.3.11. *Si $s > 102$, alors $\lambda_c(G_s) > 0$.*

Il y a une transition de phase dans la structure de la percolation long-range. Si $s < 2$, le graphe G_s présente le phénomène de petit monde. Plus précisément, la distance entre x et y est d'ordre $(\log|x-y|)^{\varkappa+o(1)}$ avec $\varkappa = \varkappa(s) > 1$, avec probabilité tendant vers 1 quand $|x-y| \to \infty$. En revanche, si $s > 2$, le graphe ressemble en quelque sorte à $\mathbb{Z}$ et la distance est d'ordre $|x-y|$. D'autre part, nous avons $\lambda_c(\mathbb{Z}) > 0$. Par conséquent, nous conjecturons que le théorème 1.3.11 a lieu pour tout $s > 2$.

1.3.7 Graphe régulier aléatoire et modèle de petit monde

Dans cette partie, nous résumons quelques résultats pour le processus de contact sur les graphes aléatoires avec des degrés bornés, pour lequel la transition de phase **(T)** a été établie.

Graphe régulier aléatoire de degré d, noté par $GRA(d)$ est le modèle de configuration avec la distribution de degré constante $D = d$. Ce graphe converge localement vers l'arbre régulier $\mathbb{T}_d$. Par les résultats de Mourrat et Valesin [66]; Lalley et Su [56], **(T)** a été prouvée dans ce cas. Plus précisément,

(i) si $\lambda \leqslant \lambda_c(\mathbb{T}_d)$, alors τ_n -le temps de mort du processus à partir de la pleine occupation est d'ordre $\log n$. En revanche, si $\lambda > \lambda_c(\mathbb{T}_d)$, alors τ_n est d'ordre exponentiel en n.

(ii) Si $\lambda > \lambda_c(\mathbb{T}_d)$, alors τ_n divisé par son espérance converges en loi vers une variable aléatoire de loi exponentielle de moyenne 1.

(iii) Si $\lambda > \lambda_c(\mathbb{T}_d)$, alors la densité métastable des sites infectés converge en probabilité vers la probabilité de survie du processus de contact sur l'arbre régulier infini.

La partie (i) a été prouvée indépendamment dans [66] et [56]. La partie (ii) a été prouvée en [85] et la partie (iii) a été prouvée dans [56]. L'idée de la preuve de (iii) est intéressante. Les auteurs ont fait une nouvelle construction du processus de contact à partir d'un seul sommet. Dans cette construction, le processus de contact coévolue avec le graphe. Pour plusieurs processus sur les modèles de configuration, l'idée de faire une construction dans laquelle le processus et le graphe évoluent à la même échelle de temps a été utilisée avec succès, par exemple dans [54] pour le modèle SIR.

Sur un graphe du petit monde. Dans [38], Durrett et Jung ont considéré une version discrète du processus de contact sur S_m^n, qui est construit comme suit. Nous commençons avec un tore $(\mathbb{Z} \mod n)^d$ de dimension d, et connectons tous les couples de sommets à distance inférieure ou égale à m. Par ailleurs, à chaque sommet est ajoutée une demi-arête. Puis, les demi-arêtes sont reliées uniformément comme dans le modèle de configuration. Ce graphe est souvent appelé BC "small-world", puisque dans [10], Bollobas et Chung ont introduit ce graphe pour $m = d = 1$.

Il a été montré dans [38] que S_m^n converge faiblement vers un graphe appelé le "Big-world", et noté par $\mathcal{B}_m$. Les degrés de $\mathcal{B}_m$ sont bornés. Par conséquent, la valeur critique du processus de contact sur $\mathcal{B}_m$ est positive. Durrett et Jung ont montré une partie de la transition de phase pour τ_n:

(i) Si $\lambda \leqslant \lambda_c(\mathcal{B}_m)$, alors $\tau_n \asymp \log n$.

(ii) Si $\lambda > \lambda_c(\mathcal{B}_m)$, ils n'ont pas réussi à montrer que τ_n est d'ordre exponentiel.

(iii) En revanche, dans le régime sur-critique $\lambda > \lambda_c(\mathcal{B}_m)$, si l'on fait une modification du processus où à chaque étape chaque site peut, selon un tire à pile ou face, infecter un sommet choisi uniformément, alors ils montrent que le temps d'extinction de cette modification du processus est d'ordre exponentiel en n^d. Ils conjecturent que la même résultat devrait avoir lieu pour le modèle d'origine.

Comme le graphe BC "small-world" et le graphe régulier aléatoire ont quelques similarités, nous nous attendons à ce que les techniques de [56, 66] puissent être adaptées pour prouver la conjecture de Durrett et Jung ainsi que les questions $(T2)$ et $(T3)$ dans **(T)**.

Chapter 2

Metastability for the contact process on the configuration model with infinite mean degree

Abstract. We study the contact process on the configuration model with a power law degree distribution, when the exponent is smaller than or equal to two. We prove that the extinction time grows exponentially fast with the size of the graph and prove two metastability results. First the extinction time divided by its mean converges in distribution toward an exponential random variable with mean one, when the size of the graph tends to infinity. Moreover, the density of infected sites taken at exponential times converges in probability to a constant. This extends previous results in the case of an exponent larger than 2 obtained in [23, 64, 65].

2.1 Introduction

In this chapter we will prove metastability results for the contact process on the configuration model with a power-law degree distribution, extending the main results of [23, 65, 64] to the case when the exponent of the power-law is smaller than or equal to 2.

The configuration model (a definition will be given later) is a popular model of random graphs with given degree distribution and it was thoroughly investigated by many authors, see [53] for a list of reference. The contact process on the configuration model was first studied by Chatterjee and Durret in [23]. They have shown that when the degree distribution has a power law (with exponent larger than two), the extinction time grows faster than any stretched exponential (in the number of vertices), which can be interpreted in saying that the critical value is zero for these graphs (invalidating thereby some physicists predictions). On the other hand, it is well known that when the degree distribution has finite mean, the sequence of graphs converges toward a Galton Watson tree as the number of vertices increases to infinity. Moreover, by Theorem 1.3.1 ([74, Theorem 3.2]), the critical value on the limiting Galton Watson tree is also zero (the process has always a positive probability to survive for any $\lambda > 0$). Therefore, the phase transition **(T)** mentioned in Chapter 1 is satisfied for this class of random graphs.

The configuration model is also interesting for another reason, highlighted in [23]: when the degree sequence has a power law, the contact process exhibits a metastable behaviour. This was first proved under a finite second moment hypothesis (equivalently for exponents larger than three) in [23], and the result has been later strengthened and extended to exponents larger than two in [65, 64]. To be more precise now, in [23] the authors proved that when the degree distribution has a power law with finite second moment, then

$$\mathbb{P}\left(c\lambda^{1+(a-2)(2-\delta)} \leqslant \frac{|\xi_{\exp(\sqrt{n})}|}{n} \leqslant C\lambda^{1+(a-2)(1-\delta)}\right) \to 1,$$

for some positive constants c and C (independent of λ), where ξ denotes the contact process starting from full occupancy. In [64] the authors have shown that when the degree distribution has finite mean (and a power law), the extinction time is w.h.p. exponential in the size of the graph (when starting from full occupancy), and combined with the results of [65], one obtains that

$$\mathbb{P}\left(c\rho_a(\lambda) \leqslant \frac{|\xi_{t_n}|}{n} \leqslant C\rho_a(\lambda)\right) \to 1,$$

for any sequence (t_n) satisfying $t_n \to \infty$ and $t_n \leqslant \exp(cn)$, where

$$\rho_a(\lambda) = \begin{cases} \lambda^{\frac{1}{3-a}} & \text{if } 2 < a \leqslant 5/2 \\ \frac{\lambda^{2a-3}}{|\log \lambda|^{a-2}} & \text{if } 5/2 < a \leqslant 3 \\ \frac{\lambda^{2a-3}}{|\log \lambda|^{2a-4}} & \text{if } a > 3. \end{cases}$$

In this paper we complete this picture by studying the case of power laws with exponents $a \in (1, 2]$. To simplify the discussion and some proofs we have chosen to consider mainly only two special choices of degree distribution. Namely we assume that it is given either by

$$p_{n,a}(j) = c_{n,a}\, j^{-a} \qquad \text{for } j = 1, \dots, n, \tag{2.1}$$

for graphs of size n, or by

$$p_a(j) = c_{\infty,a}\, j^{-a} \qquad \text{for } j \geqslant 1, \tag{2.2}$$

independently of the size of the graph, where $(c_{n,a})$ and $c_{\infty,a}$ are normalizing constants. However, at the end of the paper we also present straightforward extensions of our results to more general distributions, see Section 2.7 for more details. Our first main result in this setting is the following:

Theorem 2.1.1. *For each n, let G_n be the configuration model with n vertices and degree distribution given either by* (2.1) *or* (2.2) *with $a \in (1, 2]$. Consider the contact process $(\xi_t)_{t \geqslant 0}$ with infection rate $\lambda > 0$ starting from full occupancy on G_n. Then there is some positive constant $c = c(\lambda)$, such that the following convergence in probability holds:*

$$\frac{|\xi_{t_n}|}{n} \xrightarrow[n\to\infty]{(\mathbb{P})} \rho_a(\lambda), \tag{2.3}$$

for any sequence (t_n) sastifying $t_n \to \infty$ and $t_n \leqslant \exp(cn)$, where

$$\rho_a(\lambda) = \sum_{j=1}^{\infty} \frac{j\lambda}{j\lambda + 1} p_a(j). \tag{2.4}$$

Note that as $\lambda \to 0$,

$$\rho_a(\lambda) \asymp \begin{cases} \lambda^{a-1} & \text{if } 1 < a < 2 \\ \lambda |\log \lambda| & \text{if } a = 2, \end{cases}$$

which in particular shows that the guess of Chatterjee and Durrett [23] that $\rho_a(\lambda)$ should be $\mathcal{O}(\lambda)$ was not correct.

Now let us make some comments on the proof of this result. One first remark is that one of the main ingredients in the approach of [65] completely breaks down when the degree distribution has infinite mean (or when its mean is unbounded like in the case (2.1)), since in this case the sequence of graphs (G_n) does not locally converge anymore.

In particular we cannot transpose the analysis of the contact process on G_n (starting from a single vertex) into an analysis on an infinite limit graph. So instead we have to work directly on the graph G_n. In fact we will show that it contains w.h.p. a certain number of disjoint star graphs (i.e. graphs with one central vertex and all the others connected to the central vertex), which are all connected, and whose total size is of order n (the size of G_n). It is well known that the contact process on a star graph remains active w.h.p. for a time exponential in the size of the graph. So our main contribution here is to show that when we connect disjoint star graphs together, the process survives w.h.p. for a time which is exponential in the total size of these graphs. To this end we use the machinery introduced in [23], with their notion of lit stars. We refer to Proposition 2.4.1 and its proof for more details. Now it is interesting to notice that while this strategy works in all the cases we consider, the details of the arguments strongly depend on whether $a < 2$ or $a = 2$, and on the choice of the degree distribution. This explains why we found interesting to present the proof for the two examples (2.1) and (2.2) (note that these distributions were also considered in [46], where it was already proved that the distance between two randomly chosen vertices was a.s. equal either to two or three).

Then to obtain the asymptotic expression for the density (2.3), the point is to use the self-duality of the contact process. This allows to transpose the problem on the density of infected sites in terms of survival of the process starting from a single vertex. But starting from a single vertex, the process has a real chance to survive for a long time only if it infects one of its neighbors before extinction. Moreover, when it does, one can show that w.h.p. it immediately infects one of the star graphs mentioned above, and therefore the virus survives w.h.p. for a time at least t_n. The conclusion of the theorem follows once we observe that the probability to infect a neighbor before extinction starting from any vertex is exactly equal to $\rho_a(\lambda)$ in case (2.2) and to

$$\rho_{n,a}(\lambda) := \sum_{j=1}^{n} \frac{j\lambda}{j\lambda+1} p_{n,a}(j), \tag{2.5}$$

in case (2.1), which converges to $\rho_a(\lambda)$, as $n \to \infty$.

Our second result is often considered in the literature as another (weaker) expression of the metastability:

Theorem 2.1.2. *Assume that the degree distribution on G_n is given either by* (2.1) *or* (2.2) *with $a \in (1, 2]$, and let τ_n be the extinction time of the contact process with infection rate $\lambda > 0$ starting from full occupancy. Then*

(i) the following convergence in law holds

$$\frac{\tau_n}{\mathbb{E}(\tau_n)} \xrightarrow[n\to\infty]{(\mathcal{L})} \mathcal{E}(1),$$

with $\mathcal{E}(1)$ an exponential random variable with mean one,

(ii) there exists a constant $C > 0$, such that $\mathbb{E}(\tau_n) \leqslant \exp(Cn)$, for all $n \geqslant 1$.

In particular this result shows that Theorem 2.1.1 cannot be extended to sequences (t_n) growing faster than exponentially. In fact one can prove (see Remark 2.6.4) that Theorem 2.1.1 holds true for any constant c smaller than $\liminf(1/n)\log\mathbb{E}(\tau_n)$, and cannot be extended above this limit. This of course raises the question of knowing if the sequence $(1/n)\log\mathbb{E}(\tau_n)$ admits a limit or not. Such result has been obtained in a number of contexts, for instance in [22] or on finite boxes $[\![0, n]\!]^d$ (see [59] Section I.3), but we could not obtain it in our setting. One reason, which for instance prevents us to apply the strategy of [22], is that there does not seem to be a natural way to embed G_n into G_{n+1} (or another configuration model with larger size).

Our method for proving Theorem 2.1.2 (i) is rather general and only requires some simple hypothesis on the maximal degree and the diameter of the graph, which is satisfied in most scale-free random graphs models, like the configuration model with power law distribution having a finite mean (with the same hypothesis as in [23, 65]), or the preferential attachment graph (see Chapter 3). We refer to Proposition 2.6.2 and Remark 2.6.3 for more details.

Let us also stress the fact that (ii) would be well known if the graph had order n edges, as when the degrees have finite mean, but here it is not the case, so we have to use a more specific argument, see Section 6.

This chapter is organized as follows. In the next section, we recall the well-known and very useful graphical construction of the contact process. We also give a definition of the configuration model, fix some notation, and prove preliminary results on the graph structure. In Section 3, we prove that G_n contains w.h.p. a subgraph, called two-step star graph, which is made of several star graphs connected together, whose total size is comparable to the size of the whole graph. We refer to this section for a precise statement, which in fact depends on which case we consider ($a < 2$ or $a = 2$, and distribution (2.1) or (2.2)). In Section 4 we show that once a vertex (with high degree) of the two-step star graph is infected, the virus survives for an exponential time. Then we prove Theorem 2.1.1 and 2.1.2 in Sections 5 and 6 respectively. Finally in the last section we discuss several extenstions of our results to more general degree distributions.

2.2 Preliminaries

2.2.1 Graphical construction of the contact process.

We briefly recall here the graphical construction of the contact process (see more in Liggett's book [59]).

Fix $\lambda > 0$ and an oriented graph G (recall that a non-oriented graph can also be seen as oriented by associating to each edge two oriented edges). Then assign independent Poisson point processes $\mathcal{N}_v$ of rate 1 to each vertex $v \in V$ and $\mathcal{N}_e$ of rate λ to each oriented edge e. Set also $\mathcal{N}_{(v,w)} := \cup_{e:v\to w} \mathcal{N}_e$, for each ordered pair (v, w) of vertices, where the notation $e : v \to w$ means that the oriented edge e goes from v to w.

We say that there is an infection path from (v, s) to (w, t), and we denote it by

$$(v, s) \longleftrightarrow (w, t), \tag{2.6}$$

either if $s = t$ and $v = w$, or if $s < t$ and if there is a sequence of times $s = s_0 < s_1 < \ldots < s_l < s_{l+1} = t$, and a sequence of vertices $v = v_0, v_1, \ldots, v_l = w$ such that for every $i = 1, \ldots, l$

$$\begin{cases} s_i \in \mathcal{N}_{(v_{i-1},v_i)} \quad \text{and} \\ \mathcal{N}_{v_i} \cap [s_i, s_{i+1}] = \varnothing. \end{cases}$$

Furthermore, for any A, B two subsets of V_n and I, J two subsets of $[0, \infty)$, we write

$$A \times I \longleftrightarrow B \times J,$$

if there exists $v \in A$, $w \in B$, $s \in I$ and $t \in J$, such that (2.6) holds. Then for any $A \subset V_n$, the contact process with initial configuration A is defined by

$$\xi_t^A := \{v \in V_n : A \times \{0\} \longleftrightarrow (v,t)\},$$

for all $t \geqslant 0$. It is well known that $(\xi_t^A)_{t\geqslant 0}$ has the same distribution as the process defined in the introduction. Just note that in our definition, the Poisson processes associated to edges forming loops play no role (we could in particular remove them), but this definition will be convenient at one place of the proof (when we will use that the $Y_{n,v}$'s are i.i.d. in Subsection 2.5.1). We define next τ_n^A as the extinction time of the contact process starting from A. However, we will sometimes drop the superscript A from the notation when it will be clear from the context. We will also simply write ξ_t^v or τ_n^v when $A = \{v\}$.

Finally we introduce the following related notation:

$$\sigma(v) = \inf\{s \geqslant 0 : s \in \mathcal{N}_v\}, \tag{2.7}$$

and

$$\sigma(e) = \inf\{s \geqslant 0 : s \in \mathcal{N}_e\}, \tag{2.8}$$

for any vertex v and oriented edge e.

2.2.2 Configuration model and notation.

The configuration model is a well known model of random graph with prescribed degree distribution, see for instance [53]. In fact here we will consider a sequence (G_n) of such graphs. To define it, start for each n with a vertex set V_n of cardinality n and construct the edge set as follows. Consider a sequence of i.i.d. integer valued random variables $(D_v)_{v\in V_n}$ (whose law might depend on n) and assume that $L_n = \sum_v D_v$ is even (if not increase one of the D_v's by 1, which makes no difference in what follows). For each vertex v, start with D_v half-edges (sometimes called stubs) incident to v. Then match uniformly at random all these stubs by pairs. Once paired two stubs form an edge of the graph. Note that the random graph we obtain may contain multiple edges (i.e. edges between the same two vertices), or loops (edges whose two extremities are the same vertex).

In fact one can also define G_n by matching the stubs sequentially. This equivalent construction will be used in particular in Lemma 2.2.2, 2.2.3 and 2.2.4, so let us describe it now. As with the previous construction we start with a sequence of degrees $(D_v)_{v\in V_n}$, and for each $v \in V_n$, D_v half-edges emanating from v. Then we denote by $\mathcal{H}$ the set of all the half-edges. Select one of them h_1 arbitrarily and then choose a half-edge h_2 uniformly from $\mathcal{H} \setminus \{h_1\}$, and match h_1 and h_2 to form an edge. Next, select arbitrarily another half-edge h_3 from $\mathcal{H} \setminus \{h_1, h_2\}$ and match it to another h_4 uniformly chosen from $\mathcal{H} \setminus \{h_1, h_2, h_3\}$. Then continue this procedure until there are no more half-edges. It is possible to show that the two constructions of G_n have the same law.

Now we introduce some notation. We denote the indicator function of a set E by $\mathbf{1}(E)$. For any vertices v and w we write $v \sim w$ if there is an edge between them (in which case we say that they are neighbors or connected), and $v \not\sim w$ otherwise. We also denote by s_v the number of half-edges forming loops attached to a vertex v. We call size of a graph G the cardinality of its set of vertices, and we denote it by $|G|$.

A graph in which all vertices have degree one, except one which is connected to all the others is called a **star graph**. The only vertex with degree larger than one is called the center of the star graph, or central vertex. We call **two-step star graph** a graph formed by a family of disjoints star graphs, denoted by $S(v_i)_{1\leqslant i\leqslant k}$, centered respectively in vertices $(v_i)_{1\leqslant i\leqslant k}$, plus an additional vertex v_0 and edges between v_0 and all the v_i''s (or equivalently it is just a tree, which is of height 2 when rooted at v_0). The notation $\mathbf{S}(\mathbf{k}; \mathbf{d_1}, \dots, \mathbf{d_k})$ will refer to the two-step star graph where v_i has degree $d_i + 1$ for all i (which means that inside $S(v_i)$, v_i has degree d_i, or that $S(v_i)$ has size $d_i + 1$). These graphs will play a crucial role in our proof of Theorem 2.1.1.

Furthermore we denote by $\mathrm{Bin}(n, p)$ the binomial distribution with parameters n and p. If f and g are two real functions, we write $f = \mathcal{O}(g)$ if there exists a constant $C > 0$, such that $f(x) \leqslant Cg(x)$ for all x; $f \asymp g$ if $f = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$; $f = o(g)$ if $f(x)/g(x) \to 0$ as $x \to \infty$. Finally for a sequence of random variables (X_n) and a function $f : \mathbb{N} \to (0, \infty)$, we say that $X_n \asymp f(n)$ holds w.h.p. if there exist positive constants c and C, such that $\mathbb{P}(cf(n) \leqslant X_n \leqslant Cf(n)) \to 1$, as $n \to \infty$.

2.2.3 Preliminary estimates on the graph structure

We first recall a large deviations result which we will use throughout this paper (see for instance [32]): if $X \sim \mathrm{Bin}(n,p)$, then for all $c > 0$, there exists $\theta > 0$, such that

$$\mathbb{P}(|X - np| \geqslant cnp) \leqslant \exp(-\theta np) \quad \text{for all } n \in \mathbb{N} \text{ and } p \in [0,1]. \tag{2.9}$$

Now we present a series of lemmas deriving basic estimates on the degree sequence and the graph structure. The first one is very elementary and applies to all the cases we will consider in this paper.

Lemma 2.2.1. *Assume that the degree sequence is given either by (2.1) or (2.2), with $1 < a \leqslant 2$. For $j \geqslant 1$, let $A_j := \{v : D_v = j\}$ and $n_j = |A_j|$. Then there exist positive constants c and C, such that*

$$\mathbb{P}(n_j \in (cnj^{-a}, Cnj^{-a}) \text{ for all } j = 1, ..., n^{1/2a}) = 1 - o(1).$$

Proof. Observe that we always have $n_j \sim \mathrm{Bin}(n, p_j)$, for some $p_j \in (c_{\infty,a} j^{-a}, j^{-a})$, with $c_{\infty,a}$ as in (2.2). Thus the result directly follows from (2.9). □

Our next results depend more substantially on the value of a and the choice of the degree distribution.

Lemma 2.2.2. *Assume that the degree distribution is given by (2.1), with $a \in (1,2)$. Let $E := \{v : D_v \geqslant n/2\}$. Let also $\kappa > 2 - a$ and $\chi < 1$ be some constants. Then the following assertions hold*

(i) $L_n \asymp n^{3-a}$ *w.h.p.,*

(ii) $|E| \asymp n^{2-a}$ *w.h.p.,*

(iii) $\mathbb{P}(v \sim w \text{ for all } v \text{ and } w \text{ such that } D_v \geqslant n/2 \text{ and } D_w \geqslant n^\kappa) = 1 - o(1)$,

(iv) $\mathbb{P}(s_v \geqslant 1) = o(1)$, *for any* $v \in V_n$,

(v) $\mathbb{P}(\text{All neighbors of } v \text{ have degree larger than } n^\chi) = 1 - o(1)$, *for any* $v \in V_n$.

Proof. Let us start with Part (i). It follows from the definition (2.1) that

$$\mathbb{E}\left(D_v\right) \asymp n^{2-a} \quad \text{and} \quad \mathrm{Var}(D_v) \asymp n^{3-a}.$$

The result follows by using Chebyshev's inequality.

Part (ii) is similar to Lemma 2.2.1. For Part (iii), let v and w be two vertices such that $D_v \geqslant n/2$ and $D_w \geqslant n^\kappa$. Then conditionally on $(D_z)_{z\in V_n}$, the probability that the $n/8$ first stubs of v do not connect to w is smaller than $(1 - \frac{n^\kappa}{L_n - n/4})^{n/8}$. Hence,

$$\mathbb{P}\left(v \not\sim w \mid (D_z)_{z\in V_n},\, L_n \in (cn^{3-a}, Cn^{3-a})\right) \leqslant \left(1 - \frac{n^\kappa}{Cn^{3-a} - n/4}\right)^{n/8} = o(n^{-2}),$$

which proves (iii) by using (i) and a union bound.

We now prove (iv). To this end, notice that conditionally to D_v and L_n, s_v is stochastically dominated by a binomial random variable with parameters D_v and $D_v/(L_n - 2D_v + 2)$ (remark in particular that since $D_z \geqslant 1$ for all z, the denominator in the last term is always positive). Hence Markov's inequality shows that

$$\mathbb{P}(s_v \geqslant 1 \mid D_v, L_n) \leqslant \frac{D_v^2}{L_n - 2D_v + 2}.$$

The result follows by using (i) and that for any fixed $\varepsilon > 0$, $\mathbb{P}(D_v \geqslant n^\varepsilon) = o(1)$.

It remains to prove (v). Denote the degrees of the neighbors of v by $D_{v,i}$, $i = 1, \dots, D_v$. It follows from the definition of the configuration model that for any $i \leqslant D_v$ and $k \neq D_v$,

$$\mathbb{P}(D_{v,i} = k \mid (D_z)_{z\in V_n}) = \frac{kn_k}{L_n - 1},$$

where we recall that n_k is the number of vertices of degree k. Therefore,

$$\mathbb{P}(D_{v,i} \leqslant n^\chi \mid (D_z)_{z\in V_n}) \leqslant \frac{K_n}{L_n - 1},$$

where

$$K_n = \sum_{k\leqslant n^\chi} kn_k.$$

Summing over i, we get

$$\mathbb{P}(\exists i \leqslant D_v : D_{v,i} \leqslant n^\chi \mid (D_z)_{z\in V_n}) \leqslant \frac{K_n D_v}{L_n - 1}.$$

Moreover, similarly to the proof of (i), we can see that w.h.p.

$$K_n \asymp n^{1+\chi(2-a)}.$$

Together with (i), and using again that $D_v \leqslant n^\varepsilon$ w.h.p. for any fixed $\varepsilon > 0$, we get (v). □

Things drastically change when the degree distribution is given by (2.2). In this case L_n, as well as the k maximal degrees, for any fixed k, are all of order $n^{1/(a-1)}$ (for the comparison with the previous case note that $1/(a-1)$ is always larger than $a-3$ when $a \in (1,2)$, which is consistent with the fact that the distribution (2.2) stochastically dominates (2.1)):

Lemma 2.2.3. *Assume that the degree distribution is given by (2.2), with $a \in (1,2)$. Denote by $(D_i)_{1\leqslant i\leqslant n}$ the sequence of degrees ranged in decreasing order (in particular D_1 is the maximal degree). Let also $\kappa > (2-a)/(a-1)$ and $\chi < 1/(a-1)$ be some constants. Then the following assertions hold*

(i) there exist (a.s. positive and finite) random variables $(\gamma_i)_{i\geqslant 0}$, such that for any fixed $k \geqslant 1$,

$$\left(\frac{L_n}{n^{1/(a-1)}}, \frac{D_1}{n^{1/(a-1)}}, \ldots, \frac{D_k}{n^{1/(a-1)}}\right) \xrightarrow[n\to\infty]{(\mathcal{L})} (\gamma_0, \gamma_1, \ldots, \gamma_k).$$

(ii) For any $\varepsilon > 0$, there exists a positive constant $\eta = \eta(\varepsilon)$, such that for any fixed $k \geqslant 1$,

$$\liminf_{n\to\infty} \mathbb{P}\left(D_i/L_n \geqslant \eta\, i^{-1/(a-1)} \quad \textit{for all } 1 \leqslant i \leqslant k\right) \geqslant 1-\varepsilon,$$

and an integer $k = k(\varepsilon)$, such that

$$\liminf_{n\to\infty} \mathbb{P}(D_1 + \cdots + D_k \geqslant L_n/2) \geqslant 1-\varepsilon.$$

(iii) $\mathbb{P}(v \sim w$ for all v and w such that $D_v \geqslant n$ and $D_w \geqslant n^\kappa) = 1 - o(1)$,

(iv) $\mathbb{P}(s_v \geqslant 1) = o(1)$, for any $v \in V_n$,

(v) $\mathbb{P}\,($All neighbors of v have degree larger than $n^\chi) = 1 - o(1)$, for any $v \in V_n$.

Proof. Part (i) is standard, we refer for instance to Lemma 2.1 in [46]. More precisely let $(e_i)_{i\geqslant 1}$ be an i.i.d. sequence of exponential random variables with mean one and $\Gamma_i = e_1 + \ldots + e_i$, for all $i \geqslant 1$ (in particular Γ_i is a Gamma random variable with parameters i and 1). Then the result holds with

$$\gamma_i = ((a-1)\Gamma_i/c_{\infty,a})^{-1/(a-1)},$$

for all $i \geqslant 1$, and $\gamma_0 = \sum_i \gamma_i$ (which is well a.s. a convergent series).

For (ii) note that $\Gamma_i/i \to 1$ a.s. as $i \to \infty$. In particular for any ε, there exists $C > 0$, such that

$$\mathbb{P}(\Gamma_i \leqslant Ci \text{ for all } i \geqslant 1) \geqslant 1 - \varepsilon/2.$$

The first assertion follows with (i), using also that $\mathbb{P}(\gamma_0 \leqslant C) \geqslant 1 - \varepsilon/2$, for C large enough. The second one is an immediate corollary of (i) and the definition of γ_0 as the limit of the partial sum $\sum_{i \leqslant k} \gamma_i$, as $k \to \infty$.

Parts (iii)-(v) are similar to the previous case. $\square$

We now give an analogous result for the case $a = 2$, which we will not prove here since it is entirely similar to the case $a < 2$ (just for the case when the degree distribution is given by (2.2), one can use the elementary fact that w.h.p. all vertices have degree smaller than $n \log\log n$).

Lemma 2.2.4. *Assume that the degree distribution is given either by* (2.1) *or* (2.2), *with* $a = 2$. *Let* $E' := \{v : D_v \geqslant n^{3/4}\}$. *Then the following assertions hold*

(i) $L_n \asymp n \log n$ *w.h.p.,*

(ii) $|E'| \asymp n^{1/4}$ *and* $\sum_{v \in E'} D_v \asymp n \log n$ *w.h.p.,*

(iii) $\mathbb{P}(v \sim w$ *for all* v *and* w *such that* $D_v \geqslant n/\log n$ *and* $D_w \geqslant (\log n)^4) = 1 - o(1)$,

(iv) $\mathbb{P}(s_v \geqslant 1) = o(1)$, *for any* $v \in V_n$.

(v) $\mathbb{P}($*All neighbors of* v *have degree larger than* $(\log n)^4) = 1 - o(1)$, *for any* $v \in V_n$.

2.3 Existence of a large two-step star graph

In this section we will prove that the graph G_n contains w.h.p. a large two-step star graph $S(k; d_1, \dots, d_k)$, the term large meaning that $d_1 + \cdots + d_k$ will be of order n, and all the d_i's of order at least $\log n$. However, the precise values of k and the d_i's will depend on which case we consider (to be more precise, in the case of degree distribution given by (2.2) with $a \in (1, 2)$ we prove that for any $\varepsilon > 0$, G_n contains a large two-step star graph with probability at least $1 - \varepsilon$, with k and the d_i's depending on ε. Nevertheless, the rest of the proof works mutadis mutandis).

2.3.1 Case $1 < a < 2$

Bounded degree sequence

We assume here that the law of the degrees is given by (2.1). Recall that $E = \{v : D_v \geqslant n/2\}$ and $A_1 = \{v : D_v = 1\}$. In addition for any vertex v, let us denote by

$$d_1(v) := \sum_{w \in A_1} \mathbf{1}(\{w \sim v\}),$$

the number of neighbors of v in A_1.

Lemma 2.3.1. *There exist positive contants β and κ, such that*

$$\mathbb{P}\left(\#\{v \in E : d_1(v) \geqslant \beta n^{a-1}\} \geqslant \kappa\, n^{2-a}\right) = 1 - o(1). \tag{2.10}$$

Proof. It follows from the definition of the configuration model that for any $w \in A_1$ and $v \in E$,

$$\mathbb{P}(w \sim v \mid (D_z)_{z \in V_n}) = \frac{D_v}{L_n - 1}. \tag{2.11}$$

Similarly for any $v \in E$ and $w \neq w' \in A_1$,

$$\begin{aligned} |\mathrm{Cov}(w \sim v, w' \sim v \mid (D_z))| &= \left| \frac{D_v(D_v - 1)}{(L_n - 1)(L_n - 3)} - \left(\frac{D_v}{L_n - 1}\right)^2 \right| \\ &= \mathcal{O}\left(\frac{D_v}{L_n^2}\right) \end{aligned} \tag{2.12}$$

Define now the set

$$\mathcal{A}_n := \left\{cn^{3-a} \leqslant L_n \leqslant Cn^{3-a}\right\} \cap \{|A_1| \geqslant cn\},$$

with $0 < c \leqslant C$, such that

$$\mathbb{P}(\mathcal{A}_n) = 1 - o(1). \tag{2.13}$$

Note that the existence of c and C is guaranteed by Lemma 2.2.1 and 2.2.2. Set also $\beta = c/(4C)$. Then (2.11) and (2.12) show that on $\mathcal{A}_n$,

$$\sum_{w \in A_1} \mathbb{P}(w \sim v \mid (D_z)) \geqslant 2\beta n^{a-1},$$

and

$$\sum_{w \neq w' \in A_1} \mathrm{Cov}(w \sim v, w' \sim v \mid (D_z)) = o(n^{2a-2}).$$

Thus by using Chebyshev's inequality, we deduce that on $\mathcal{A}_n$,

$$\mathbb{P}(d_1(v) \geqslant \beta n^{a-1} \mid (D_z)) = \mathbb{P}\left(\sum_{w \in A_1} \mathbf{1}(\{w \sim v\}) \geqslant \beta n^{a-1} \;\Big|\; (D_z) \right) = 1 - o(1).$$

Hence for any $v \neq w \in E$,

$$\mathrm{Cov}(d_1(v) \geqslant \beta n^{a-1}, d_1(w) \geqslant \beta n^{a-1} \mid (D_z)_{z \in V_n}) = o(1).$$

Then by using Chebyshev's inequality again we obtain that on the event $\{|E| \geqslant 2\kappa n^{2-a}\}$,

$$\mathbb{P}\left(\#\{v \in E : d_1(v) \geqslant \beta n^{a-1}\} \geqslant \kappa\, n^{2-a} \mid (D_z)_{z \in V_n}\right) = 1 - o(1).$$

Then (2.10) follows by using (2.13), Lemma 2.2.2 (ii) and taking expectation. $\square$

As a corollary we get the following result:

Proposition 2.3.2. *Assume that the law of the degree sequence is given by* (2.1) *with* $a \in (1, 2)$. *There exist positive constants* β *and* κ, *such that w.h.p.* G_n *contains as a subgraph a copy of* $S(k; d_1, \dots, d_k)$, *with* $k = \kappa n^{2-a}$ *and* $d_i = \beta n^{a-1}$, *for all* $i \leqslant k$.

Proof. This is a direct consequence of Lemma 2.2.2 (iii) and Lemma 2.3.1. $\square$

Unbounded degree sequences

We assume here that the law of the degrees is given by (2.2). The proof of the next result is similar to the one of Lemma 2.3.1, so we omit it.

Lemma 2.3.3. *With the notation of Lemma 2.2.3, let* $(v_i)_{i \leqslant n}$ *be a reordering of the vertices of* G_n, *such that the degree of* v_i *is* D_i *for all* i *(in particular* v_1 *is a vertex with maximal degree). Then for any fixed* i,

$$\mathbb{P}(d_1(v_i) \geqslant D_i n_1 / (2L_n)) = 1 - o(1).$$

As a consequence we get

Proposition 2.3.4. *Assume that the degree distribution is given by* (2.2), *with* $a \in (1, 2)$. *There exists a constant* $c > 0$, *such that for any* $\varepsilon > 0$, *there exists* $\eta = \eta(\varepsilon) > 0$ *and an integer* $k = k(\varepsilon)$, *such that for* n *large enough, with probability at least* $1 - \varepsilon$, G_n *contains as a subgraph a copy of* $S(k; d_1, \dots, d_k)$, *with* $d_i \geqslant \eta i^{-1/(a-1)} n$ *for all* $i \geqslant 1$, *and* $d_1 + \cdots + d_k \geqslant cn$.

Proof. It follows from Lemma 2.3.3 that for any i,

$$\mathbb{P}(d_1(v_i) \geqslant D_i n_1/(2L_n)) = 1 - o(1).$$

Hence for any fixed k,

$$\mathbb{P}\left(d_1(v_1) + \ldots + d_1(v_k) \geqslant \frac{n_1(D_1 + \ldots + D_k)}{2L_n}\right) = 1 - o(1).$$

Moreover, by Lemma 2.2.1 we have $\mathbb{P}(n_1 \in (cn, Cn)) = 1 - o(1)$. On the other hand, for any $\varepsilon > 0$, by Lemma 2.2.3 (ii), there exist $\eta = \eta(\varepsilon)$ and $k = k(\varepsilon)$, such that

$$\mathbb{P}\left(\frac{D_1 + \ldots + D_k}{L_n} \geqslant \frac{1}{2}\right) \geqslant 1 - \varepsilon/4,$$
$$\mathbb{P}(D_i/L_n \geqslant \eta i^{-1/(a-1)} \quad \forall\, i \leqslant k+1) \geqslant 1 - \varepsilon/4.$$

Therefore with probability at least $1 - (3\varepsilon/4)$, for n large enough, η and k as above,

$$d_1(v_1) + \ldots + d_1(v_k) \geqslant cn/4,$$
$$d_1(v_i) \geqslant c\eta i^{-1/(a-1)} n/2 \;\forall\, i \leqslant k+1.$$

Then by using a similar argument as in the proof of Lemma 2.2.2 (iii), we can show that with probability larger than $1 - (\varepsilon/4)$, v_{k+1} and v_i are connected for all $i \leqslant k$. The result follows. $\square$

2.3.2 Case $a = 2$

In this case we can treat both distributions (2.1) and (2.2) in the same way. Recall that $E' = \{v : D_v \geqslant n^{3/4}\}$, and $d_1(v)$ denotes the number of neighbors in A_1 of a vertex v.

Lemma 2.3.5. *There exists a positive constant β, such that*

$$\mathbb{P}\left(d_1(v) \geqslant \beta D_v/\log n \quad \textit{for all } v \in E'\right) = 1 - o(1). \tag{2.14}$$

Proof. The proof is very close to the proof of Lemma 2.3.1. First, for any $v \in E'$ and $w \in A_1$, we have

$$\mathbb{P}(w \sim v \mid (D_z)) \asymp \frac{D_v}{L_n},$$

and furthermore for any $w \neq w' \in A_1$,

$$|\mathrm{Cov}(w \sim v, w' \sim v \mid (D_z))| = \mathcal{O}\left(\frac{D_v}{L_n^2}\right).$$

Then by using Chebyshev's inequality, we get that for any $v \in E'$,

$$\mathbb{P}\left(d_1(v) \leqslant \beta D_v n_1/L_n \mid (D_z)\right) = \mathcal{O}\left(\frac{L_n}{n_1 D_v}\right),$$

for some constant $\beta > 0$. The desired result follows by using a union bound and then Lemma 2.2.1 and 2.2.4 (i)-(ii). □

As a consequence we get

Proposition 2.3.6. *Assume that the law of the degree distribution is given either by* (2.1) *or* (2.2) *and that $a = 2$. There exists a positive constant β such that w.h.p. G_n contains as a subgraph a copy of $S(k; d_1, \dots, d_k)$, with $k \asymp n^{1/4}$, $d_i \geqslant \beta n^{3/4}/\log n$ for all $i \leqslant k$, and $d_1 + \dots + d_k \asymp n$.*

Proof. Just take for the v_i's the elements of E'. Then use Lemma 2.2.4 (ii)-(iii) and Lemma 2.3.5. □

2.4 Contact process on a two-step star graph

In this section we will study the contact process on a two-step star graph. Our main result is the following:

Proposition 2.4.1. *There exist positive constants c and C, such that for any two-step star graph $G = S(k; d_1, \dots, d_k)$, satisfying $d_i \geqslant C \log n/\lambda^2$, for all $i \leqslant k$, and $d_1 + \dots + d_k = n$,*

$$\mathbb{P}\left(\tau_n^{v_1} \geqslant \exp(c\lambda^2 n)\right) = 1 - o(1),$$

where $\tau_n^{v_1}$ is the extinction time of the contact process with infection parameter $\lambda \leqslant 1$ starting from v_1 on $S(k; d_1, \dots, d_k)$.

Note that since we are only concerned with the extinction time here, there is no restriction in assuming $\lambda \leqslant 1$, as the contact process is stochastically monotone in λ (see [59]). So when $\lambda > 1$ the same result holds; one just has to remove the λ everywhere in the statement of the proposition.

Now of course an important step in the proof is to understand the behavior of the process on a single star graph. This has already been studied for a long time, for instance it appears in Pemantle [74], and later in [13, 23, 65]. We will collect all the results we need in Lemma 2.4.2 below, but before that we give some new definition. We say that a vertex v is **lit** (the term is taken from [23]) at some time t if the proportion of its infected neighbors at time t is larger than $\lambda/(16e)$ (note that in [64] the authors also use the term *infested* for a similar notion).

Lemma 2.4.2. *There exists a constant $c \in (0,1)$, such that if (ξ_t) is the contact process with parameter $\lambda \leqslant 1$ on a star graph S with center v, satisfying $\lambda^2|S| \geqslant 64e^2$, then*

(i) $\mathbb{P}(\xi_{\exp(c\lambda^2|S|)} \neq \varnothing \mid v \textit{ is lit at time } 0) \geqslant 1 - \exp(-c\lambda^2|S|)$,

(ii) $\mathbb{P}(\exists t > 0 : v \textit{ is lit at time } t \mid \xi_0(v) = 1) \to 1 \qquad \textit{as } |S| \to \infty.$

(iii) $\mathbb{P}(v \textit{ is lit at time } 1 \mid \xi_0(v) = 1) \geqslant (1 - \exp(-c\lambda|S|))/e$,

(iv) $\mathbb{P}(v \textit{ is lit during } [\exp(c\lambda^2|S|), 2\exp(c\lambda^2|S|)] \mid v \textit{ is lit at time } 0) \geqslant 1-2\exp(-c\lambda^2|S|)$.

Proof. Parts (i), (ii) and (iii) are exactly Lemma 3.1 in [65], and (iv) can be proved similarly, see for instance Lemma 3.2.7 (iii) in Chapter 3 (similar results can be found in [13, 23, 34, 74]). □

Proof of Proposition 2.4.1. We first handle the easy case when there is some $1 \leqslant i \leqslant k$, such that $\deg(v_i) \geqslant n/2$. First by Lemma 2.4.2 we know that w.h.p. the virus survives inside $S(v_1)$ at least a time $\exp(c\lambda^2 d_1)$. Since by hypothesis d_1 diverges when n tends to infinity, and since v_1 and v_i are at distance at most two (both are connected to v_0), we deduce that w.h.p. v_i will be infected before the extinction of the virus. The proposition follows by another use of Lemma 2.4.2.

We now assume that $d_i \leqslant n/2$, for all i. First we need to introduce some more notation. For $s < t$ and $v, w \in S(v_i)$, we write

$$(v,s) \overset{(i)}{\longleftrightarrow} (w,t), \tag{2.15}$$

if there exists an infection path entirely inside $S(v_i)$ joining (v,s) and (w,t). Similarly if V and W are two subsets of G, we write

$$V \times \{s\} \overset{(i)}{\longleftrightarrow} W \times \{t\},$$

if there exists $v \in V \cap S(v_i)$ and $w \in W \cap S(v_i)$, such that (2.15) holds. Now for $\ell \geqslant 0$ and $1 \leqslant i \leqslant k$ define

$$E_{\ell,i} := \left\{ \xi_{\ell n^2} \times \{\ell n^2\} \overset{(i)}{\longleftrightarrow} S(v_i) \times \{(\ell+1)n^2\} \right\}.$$

We claim that for any $\ell \geqslant 0$ and $1 \leqslant i \leqslant k$, we have

$$\mathbb{P}\big(E_{\ell,i} \cap \big(\cap_{j\neq i} E^c_{\ell+1,j}\big)\big) \leqslant \exp(-c\lambda^2 n), \tag{2.16}$$

for some constant $c > 0$. To fix ideas we will prove the claim for $i = 1$ (clearly by symmetry there is no loss of generality in assuming this) and to simplify notation we also assume that $\ell = 0$ (the proof works the same for any ℓ). Furthermore, in the whole proof the notation c will stand for a positive constant independent of λ, whose value might change from line to line.

Now before we start the proof we give a new definition. We denote by $(\xi'_t)_{t\geqslant 0}$ the contact process on $\overline{S}(v_1) := S(v_1) \cup \{v_0\}$, which is defined by using the same Poisson processes as ξ, but only on this subgraph. In particular with ξ', the vertex v_0 can only be infected by v_1, and thus the restriction of ξ on $\overline{S}(v_1)$ dominates ξ'. We also assume that the starting configurations of ξ' and of the restriction of ξ on $\overline{S}(v_1)$ are the same. Now for any integer $m \leqslant n$, define

$$G_m = \{\xi'_t(v_0) = 1 \text{ for all } t \in [3m+2, 3m+3]\}.$$

Let also $\mathcal{F}_t = \sigma(\xi'_s, s \leqslant t)$ be the natural filtration of the process ξ'. Then observe that for any vertex $w \in S(v_1)$, conditionally on $\mathcal{F}_{3m}$, and on the event $\{\xi'_{3m}(w) = 1\}$, we have

$$\begin{aligned} G_m \quad \subset \quad & \{\mathcal{N}_w \cap [3m, 3m+1] = \varnothing, \mathcal{N}_{(w,v_1)} \cap [3m, 3m+1] \neq \varnothing, \mathcal{N}_{v_1} \cap [3m, 3m+2] = \varnothing, \\ & \mathcal{N}_{(v_1,v_0)} \cap [3m+1, 3m+2] \neq \varnothing, \mathcal{N}_{v_0} \cap [3m+1, 3m+3] = \varnothing\}, \end{aligned}$$

at least if $w \neq v_1$. Moreover, the event on the right hand side has probability equal to $(1-e^{-\lambda})^2 e^{-5}$, which is larger than $c\lambda^2$, for some $c > 0$, and a similar result holds if $w = v_1$. Therefore for any m and any nonempty subset $A \subset S(v_1)$,

$$\mathbb{P}(G^c_m \mid \mathcal{F}_{3m})\, \mathbf{1}(\xi'_{3m} = A) \leqslant (1 - c\lambda^2)\mathbf{1}(\xi'_{3m} = A).$$

In other words, if we define

$$H_m = \{\xi'_{3m} \cap S(v_1) \neq \varnothing\},$$

we get

$$\mathbb{P}(G_m^c \mid \mathcal{F}_{3m})\,\mathbf{1}(H_m) \leqslant 1 - c\lambda^2,$$

for all $m \leqslant n$. By using induction, it follows that

$$\mathbb{P}\left(\left(\bigcup_{m=0}^{n-1} G_m\right)^c \cap \left(\bigcap_{m=0}^{n-1} H_m\right)\right) \leqslant (1-c\lambda^2)^n.$$

But by construction

$$E_{0,1} \subset \bigcap_{m=0}^{n-1} H_m.$$

Therefore

$$\mathbb{P}\left(E_{0,1} \cap \{\exists m \in [0,3n-1] \,:\, \xi'_t(v_0) = 1 \text{ for all } t \in [m,m+1]\}^c\right) \leqslant \exp(-c\lambda^2 n).$$

Then by repeating the argument in each interval $[3Mn, 3(M+1)n]$, for every $M \leqslant n/3-1$, we get

$$\mathbb{P}\left(E_{0,1},\, |\mathcal{M}| < n/3\right) \leqslant \exp(-c\lambda^2 n), \tag{2.17}$$

where

$$\mathcal{M} := \left\{m \leqslant n^2 - 1 \,:\, \xi'_t(v_0) = 1 \text{ for all } t \in [m,m+1]\right\}.$$

Now for each $2 \leqslant j \leqslant k$ and $m \leqslant n^2 - 1$, define,

$$\begin{aligned} C_{m,j} :=& \left\{\mathcal{N}_{(v_0,v_j)} \cap [m,m+1] \neq \varnothing,\, \mathcal{N}_{v_j} \cap [m,m+2] = \varnothing\right\} \\ \cap & \left\{|\{w \in S(v_j) : \mathcal{N}_{(v_j,w)} \cap [m+1,m+2] \neq \varnothing \text{ and } \mathcal{N}_w \cap [m+1,m+2] = \varnothing\}| > \frac{\lambda d_j}{16e}\right\}. \end{aligned}$$

Note that these events are independent of $\mathcal{M}$ and $E_{0,1}$, as they depend on different Poisson processes. Note also that by using (2.9)

$$\begin{aligned} \mathbb{P}(C_{m,j}) &= (1-e^{-\lambda})e^{-2} \times \mathbb{P}(\mathcal{B}(d_j,(1-e^{-\lambda})/e) \geqslant \lambda d_j/(16e)) \\ &\geqslant c\lambda, \end{aligned} \tag{2.18}$$

and thus (since $C_{m,j}$ and $C_{m',j}$ are independent when $m - m' \geqslant 2$),

$$\mathbb{P}\left(\bigcap_{m\in\mathcal{M}} C_{m,j}^c \;\middle|\; |\mathcal{M}| \geqslant n/3\right) \leqslant \exp(-c\lambda n).$$

Moreover, by construction if $m \in \mathcal{M}$ and $C_{m,j}$ holds, then v_j is lit at some time $t \in [m+1,m+2]$. Therefore by using (2.17),

$$\mathbb{P}\left(E_{0,1} \cap \{\exists j \in \{2,\dots,k\} : v_j \text{ is never lit in } [0,n^2]\}\right) \leqslant \exp(-c\lambda n). \tag{2.19}$$

Finally define $U_j = \exp(c\lambda^2 d_j)$, for all $j \leqslant k$, with the constant c as in Lemma 3.2.8, and take C large enough, so that the hypothesis $d_j\lambda^2 \geqslant C\log n$ implies $U_j \geqslant 2n^2$. Then (2.19) together with Lemma 2.4.2 (i) imply that

$$\mathbb{P}\left(E_{0,1} \cap (\cap_{j\geqslant 2} E_{1,j}^c)\right) \leqslant \exp(-c\lambda^2 n) + \prod_{j\geqslant 2} U_j^{-1} \leqslant 2\exp(-c\lambda^2 n/2),$$

where for the last inequality we used that $d_2 + \cdots + d_k \geqslant n/2$. This concludes the proof of (2.16). The proposition immediately follows, since by using Lemma 2.4.2, we also know that $\mathbb{P}(E_{0,1}) = 1 - o(1)$, when v_1 is infected initially (observe that $\exp(c\lambda^2 d_1) \geqslant n^2$, if the constant C in the hypothesis is large enough). □

2.5 Proof of Theorem 2.1.1

The proof is the same in all the cases we considered, so to fix ideas we assume in all this section that the degree distribution is given by (2.1) with $a \in (1, 2)$. The other cases are left to the reader.

Let (t_n) be as in the statement of Theorem 2.1.1. Define for $v \in V_n$,

$$X_{n,v} = \mathbf{1}(\{\xi_{t_n}^v \neq \varnothing\}).$$

The self-duality of the contact process (see (1.7) p. 35 in [59]) implies that for any $\gamma > 0$,

$$\mathbb{P}\left(|\xi_{t_n}^{V_n}| > \gamma n\right) = \mathbb{P}\left(\sum_{v\in V_n} X_{n,v} > \gamma n\right)$$

and similarly for the reverse inequality. Hence, to prove that $|\xi_{t_n}^{V_n}|/n$ converges in probability to $\rho_a(\lambda)$, we have to show that

$$\mathbb{P}\left(\sum_{v\in V_n} X_{n,v} > (\rho_{n,a}(\lambda) + \varepsilon)n\right) \to 0 \quad \text{as } n \to \infty \tag{2.20}$$

and

$$\mathbb{P}\left(\sum_{v\in V_n} X_{n,v} < (\rho_{n,a}(\lambda) - \varepsilon)n\right) \to 0 \quad \text{as } n \to \infty \tag{2.21}$$

for all $\varepsilon > 0$ (recall that $\rho_{n,a}(\lambda)$ converges to $\rho_a(\lambda)$, as $n \to \infty$). We will prove these two statements in the next two subsections.

2.5.1 Proof of (2.20)

This part is quite elementary. The idea is to say that if the virus survives for a time t_n starting from some vertex v, then v has to infect one of its neighbors before $\sigma(v)$ (recall the definition (2.7)), unless $\sigma(v) \geqslant t_n$, but this last event has $o(1)$ probability so we can ignore it. Now the probability that v infects a neighbor before $\sigma(v)$, is bounded by the probability that one of the Poisson point processes associated to the edges emanating from v has a point before $\sigma(v)$ (actually it is exactly equal to this if there is no loop attached to v). Then having observed that the latter event has probability exactly equal to $\rho_{n,a}(\lambda)$, we get the desired upper bound, at least in expectation. The true upper bound will follow using Chebyshev's inequality and the domination of the $X_{n,v}$'s by suitable i.i.d. random variables.

Now let us write this proof more formally. Set $Y_{n,v} = \mathbf{1}(C_{n,v})$, with (recall (2.8))

$$C_{n,v} = \left\{ \min_{e:v\to\cdot} \sigma(e) < \sigma(v) \right\},$$

where the notation $e : v \to \cdot$ means that e is an (oriented) edge emanating from v (possibly forming a loop). By construction the $(Y_{n,v})_{v\in V_n}$ are i.i.d. random variables, and moreover, the above discussion shows that for all v,

$$X_{n,v} \leqslant Y_{n,v} + \mathbf{1}(\{\sigma(v) > t_n\}). \tag{2.22}$$

Now we have

$$\begin{aligned}\mathbb{E}(Y_{n,v}) = \mathbb{P}(C_{n,v}) &= \sum_{j=1}^{n} \mathbb{P}(C_{n,v} \mid D_v = j)\mathbb{P}(D_v = j) \\ &= \sum_{j=1}^{n} \frac{j\lambda}{j\lambda+1} p_{n,a}(j) = \rho_{n,a}(\lambda). \end{aligned} \tag{2.23}$$

Therefore it follows from Chebyshev's inequality that

$$\mathbb{P}\left(\sum_v Y_{n,v} > (\rho_{n,a}(\lambda) + \varepsilon/2)n \right) = o(1),$$

for any fixed $\varepsilon > 0$. On the other hand $\mathbb{P}(\sigma(v) > t_n) = e^{-t_n} = o(1)$. Thus by using Markov's inequality we get

$$\mathbb{P}\left(\sum_v \mathbf{1}(\{\sigma(v) > t_n\}) > \varepsilon n/2 \right) = o(1).$$

Then (2.20) follows with (2.22).

2.5.2 Proof of (2.21)

This part is more complicated and requires the results obtained so far in Sections 2, 3 and 4. First define $Z_{n,v} = \mathbf{1}(A_{n,v} \cap B_{n,v})$, for $v \in V_n$, where

$$A_{n,v} = \{v \text{ infects one of its neighbors before } \sigma(v)\},$$

and $B_{n,v} = \{\xi_{t_n}^v \neq \varnothing\}$. Remember that $X_{n,v} = \mathbf{1}(B_{n,v})$, which in particular gives $Z_{n,v} \leqslant X_{n,v}$. Therefore the desired lower bound follows from the next lemma and Chebyshev's inequality.

Lemma 2.5.1. *For any $v \neq w \in V_n$,*

(i) $\mathbb{E}(Z_{n,v}) \geqslant \rho_{n,a}(\lambda) - o(1)$.

(ii) $Cov(Z_{n,v}, Z_{n,w}) = o(1)$.

Proof. We claim that

$$\mathbb{P}(B_{n,v} \mid A_{n,v}) = 1 - o(1). \tag{2.24}$$

To see this first use that w.h.p. there is a large two-step star graph in G_n (given by Proposition 2.3.2). Then use Lemma 2.2.2 (iii) and (v) to see that w.h.p. all neighbors of v have large degree and are connected to all the v_i's of the two-step star graph (recall that by construction $D_{v_i} \geqslant n/2$, for all i). Note that in the case $a = 2$, this is not exactly true, but nevertheless the neighbors of v and the v_i's are still w.h.p. at distance at most two, since they are all connected to the set of vertices z satisfying $D_z \geqslant n/\log n$ (and w.h.p. this set is nonempty). Now if a neighbor, say w, of v is infected and has large degree, then Lemma 2.4.2 shows that w.h.p. the virus will survive in the star graph formed by w and its neighbors for a long time. But if in addition w and v_1 are connected (or more generally if they are at distance at most two), then v_1 will be infected as well w.h.p. before extinction of the process. Then Proposition 2.4.1 gives (2.24).

On the other hand observe that

$$\{s_v = 0\} \cap C_{n,v} \subset A_{n,v}.$$

Therefore (2.23) and Lemma 2.2.2 (iv) give Part (i) of the lemma. The second part follows easily by using that we also have $A_{n,v} \subset C_{n,v}$, and that the $C_{n,v}$'s are independent. $\square$

2.6 Proof of Theorem 2.1.2

We first prove a lower bound on the probability that the extinction time is smaller than n^2. Together with the following lemma, we will get the assertion (ii) of the theorem:

Lemma 2.6.1. *For every $s > 0$, we have*

$$\mathbb{P}(\tau_n \leqslant s) \leqslant \frac{s}{\mathbb{E}(\tau_n)}.$$

This lemma is a direct consequence of the Markov property and the attractiveness of the contact process, see for instance Lemma 4.5 in [64].

For simplicity we assume that $\lambda \leqslant 1$, and for the other case $\lambda > 1$, we just need to slightly modify the values of some constants. We also assume first that the degree distribution is given by (2.1).

Let $\bar{n}_a$ be the number of vertices having degree larger than $n^{1/2a}$. Then $\bar{n}_a \sim \mathrm{Bin}(n, \bar{p}_a)$, where $\bar{p}_a = \sum_{j>n^{1/2a}} p_{n,a}(j) \asymp n^{(1-a)/2a}$. Hence, as for Lemma 2.2.1, there exists a constant $K > 0$, such that

$$\mathbb{P}\left(\bar{n}_a \leqslant K n^{(1+a)/2a}\right) = 1 - o(1).$$

In fact thanks to Lemma 2.2.1, we can even assume that

$$\mathbb{P}(\mathcal{E}_n) = 1 - o(1), \tag{2.25}$$

where

$$\mathcal{E}_n := \left\{n_j \leqslant Knj^{-a} \text{ for all } j \leqslant n^{1/2a}\right\} \cap \left\{\bar{n}_a \leqslant K n^{(1+a)/2a}\right\}.$$

Now if a vertex has degree j, the probability that it becomes healthy before spreading infection to another vertex is at least equal to $1/(1 + j\lambda)$ (it is in fact exactly equal to this if there is no loop attached to this vertex). Since this happens independently for all vertices, we have that a.s. for n large enough, on $\mathcal{E}_n$,

$$\begin{aligned}
\mathbb{P}(\tau_n \leqslant \min_v \sigma(v) \mid (D_v)_{v\in V_n}) &\geqslant (1/(1+\lambda n))^{\bar{n}_a} \prod_{j=1}^{n^{1/2a}} (1/(1+\lambda j))^{n_j} \\
&\geqslant (2\lambda n)^{-\bar{n}_a} \prod_{j=1}^{1/\lambda} 2^{-n_j} \prod_{1/\lambda}^{n^{1/2a}} (2\lambda j)^{-n_j} \\
&\geqslant (2\lambda)^{-n} n^{-\bar{n}_a} \prod_{1/\lambda}^{n^{1/2a}} j^{-n_j} \\
&\geqslant \exp\left(-n\left(\log(2\lambda) + K\sum_{1/\lambda}^{n^{1/2a}} j^{-a}\log j\right) - \bar{n}_a \log n\right) \\
&\geqslant \exp(-Cn/4),
\end{aligned}$$

for some constant $C = C(\lambda) > 0$.

Now for each vertex v, $\sigma(v)$ is an exponential random variable with mean 1. Hence, a.s. for n large enough and on $\mathcal{E}_n$,

$$\mathbb{P}(\tau_n \leqslant n^2 \mid (D_v)_{v\in V_n}) \geqslant e^{-Cn/4} - \mathbb{P}(\exists v : \sigma(v) \geqslant n^2) \geqslant e^{-Cn/2}.$$

The same can be proved in the case when the degree distribution is given by (2.2). One just has to use that w.h.p. all the degrees are bounded by $n^{2/(a-1)}$, but this does not seriously affect the proof.

Together with (2.25), it follows that

$$\mathbb{P}(\tau_n \leqslant n^2) \geqslant \exp(-Cn)(1-o(1)),$$

and as we already mentioned above, with Lemma 2.6.1 we get the assertion (ii) of the theorem.

We now prove (i). This will be a consequence of a more general result:

Proposition 2.6.2. *Let (G_n^0) be a sequence of connected graphs, such that $|G_n^0| \leqslant n$, for all n. Let τ_n denote the extinction time of the contact process on G_n^0 starting from full occupancy. Assume that*

$$\frac{D_{n,\max}}{d_n \vee \log n} \to \infty, \tag{2.26}$$

with $D_{n,\max}$ the maximum degree and d_n the diameter of G_n^0. Then

$$\frac{\tau_n}{\mathbb{E}(\tau_n)} \xrightarrow[n\to\infty]{(\mathcal{L})} \mathcal{E}(1).$$

Proof. According to Proposition 1.2 in [62] and Lemma 2.6.1 above it suffices to show that there exists a sequence (a_n), such that $a_n = o(\mathbb{E}(\tau_n))$ and

$$\sup_{v\in V_n} \mathbb{P}(\xi_{a_n}^v \neq \xi_{a_n}, \xi_{a_n}^v \neq \varnothing) = o(1), \tag{2.27}$$

where $(\xi_t)_{t\geqslant 0}$ denotes the process starting from full occupancy.

Set $\bar\lambda = \lambda \wedge 1$. Using Lemma 2.4.2, we get

$$\mathbb{E}(\tau_n) \geqslant \exp(c\bar\lambda^2 D_{n,\max}), \tag{2.28}$$

with c as in this lemma. Using next (2.26), we can find a sequence (φ_n) tending to infinity, such that

$$\frac{D_{n,\max}}{(\log n \vee d_n)\varphi_n} \to \infty. \tag{2.29}$$

Now define

$$b_n = \exp(c\bar\lambda^2(\log n \vee d_n)\varphi_n) \quad \text{and} \quad a_n = 4b_n + 1.$$

Then (2.28) and (2.29) show that $a_n = o(\mathbb{E}(\tau_n))$, so it amounts now to prove (2.27) for this choice of (a_n). To this end it is convenient to introduce the dual contact process. Given some positive real t and A a subset of the vertex set V_n of G_n, the dual process $(\hat\xi_s^{A,t})_{s\leqslant t}$ is defined by

$$\hat\xi_s^{A,t} = \{v \in V_n : (v, t-s) \longleftrightarrow A \times \{t\}\},$$

for all $s \leqslant t$. It follows from the graphical construction that for any v,

$$\begin{aligned}
&\mathbb{P}(\xi_{a_n}^v \neq \xi_{a_n}, \xi_{a_n}^v \neq \varnothing)\\
=\ &\mathbb{P}(\exists w \in V_n : \xi_{a_n}^v(w) = 0,\ \xi_{a_n}^v \neq \varnothing,\ \hat\xi_{a_n}^{w,a_n} \neq \varnothing)\\
\leqslant\ &\sum_{w\in V_n} \mathbb{P}\left(\xi_{a_n}^v \neq \varnothing,\ \hat\xi_{a_n}^{w,a_n} \neq \varnothing,\ \text{and } \hat\xi_{a_n-t}^{w,a_n} \cap \xi_t^v = \varnothing \text{ for all } t \leqslant a_n\right),
\end{aligned} \tag{2.30}$$

So let us prove now that the last sum above tends to 0 when $n \to \infty$. Set

$$\beta_n = [\varphi_n(d_n \vee \log n)],$$

and let u be a vertex with degree larger than β_n. Let then $S(u)$ be a star graph of size β_n centered at u. Now we slightly change the definition of a lit vertex, and say that u is lit if the number of its infected neighbors *in* $S(u)$ is larger than $\bar{\lambda}\beta_n/(16e)$. We first claim that

$$\mathbb{P}(\xi^v_{b_n} \neq \varnothing, u \text{ is not lit before } b_n) = o(1/n). \tag{2.31}$$

To see this, define $K_n = [b_n/d_n]$ and for any $0 \leqslant k \leqslant K_n - 1$

$$A_k := \{\xi^v_{kd_n} \neq \varnothing\},$$

and

$$B_k := \left\{\xi^v_{kd_n} \times \{kd_n\} \longleftrightarrow (u, (k+1)d_n - 1)\right\} \cap \{u \text{ is lit at time } (k+1)d_n\}.$$

Note that

$$\{\xi^v_{b_n} \neq \varnothing, u \text{ is not lit before } b_n\} \ \subset \ \bigcap_{k=0}^{K_n-1} A_k \cap B_k^c. \tag{2.32}$$

Moreover, by using a similar argument as for (2.18), we obtain

$$\mathbb{P}\left((z,t) \longleftrightarrow (z', t + d_n - 1)\right) \geqslant \exp(-Cd_n) \quad \text{for any } z, z' \in V_n \text{ and } t \geqslant 0,$$

for some constant $C > 0$ (in fact this is not true if $d_n = 1$; but in this case one can just consider time intervals of length $d_n + 1$ instead of d_n). On the other hand, Lemma 2.4.2 (iii) implies that if u is infected at time t then it is lit at time $t+1$ with probability larger than 1/3, if n is large enough. Therefore for any $k \leqslant K_n - 1$,

$$\mathbb{P}(B_k^c \mid \mathcal{G}_k)\mathbf{1}(A_k) \leqslant 1 - \exp(-Cd_n)/3,$$

with $\mathcal{G}_k$ the sigma-field generated by all the Poisson processes introduced in the graphical construction in the time interval $[0, kd_n]$. Iterating this, we get

$$\mathbb{P}\left(\bigcap_{k=0}^{K_n-1} A_k \cap B_k^c\right) \ \leqslant \ (1 - \exp(-Cd_n)/3)^{K_n-1} = o(1/n),$$

where the last equality follows from the definition of b_n. Together with (2.32) this proves our claim (2.31). Then by using Lemma 2.4.2 (iv) we get

$$\mathbb{P}(\xi^v_{b_n} \neq \varnothing, u \text{ is not lit at time } 2b_n) = o(1/n). \tag{2.33}$$

Therefore, if we define

$$\mathcal{A}(v) = \{\xi^v_{b_n} \neq \varnothing, u \text{ is lit at time } 2b_n\},$$

we get

$$\mathbb{P}(\mathcal{A}(v)^c, \xi^v_{b_n} \neq \varnothing) = o(1/n).$$

Likewise if

$$\hat{\mathcal{A}}(w) = \{\hat{\xi}^{w,4b_n+1}_{b_n} \neq \varnothing, \exists\, U \subset S(u) : |U| \geqslant \frac{\bar{\lambda}}{16e}\beta_n \text{ and } (x, 2b_n+1) \leftrightarrow (w, 4b_n+1)\,\forall\, x \in U\}.$$

then

$$\mathbb{P}(\hat{\mathcal{A}}(w)^c, \hat{\xi}^{w,4b_n+1}_{b_n} \neq \varnothing) = o(1/n).$$

Moreover, $\mathcal{A}(v)$ and $\hat{\mathcal{A}}(w)$ are independent for all v, w. Now the result will follow if we can show that for any $A, B \subset S(u)$ with $|A|, |B|$ larger than $\bar{\lambda}\beta_n/(16e)$

$$\mathbb{P}(A \times \{2b_n\} \overset{S(u)}{\longleftrightarrow} B \times \{2b_n+1\}) = 1 - o(1/n), \tag{2.34}$$

where the notation

$$A \times \{2b_n\} \overset{S(u)}{\longleftrightarrow} B \times \{2b_n+1\}$$

means that there is an infection path inside $S(u)$ from a vertex in A at time $2b_n$ to a vertex in B at time $2b_n + 1$. To prove (2.34), define

$$\bar{A} = \{x \in A \setminus \{u\} : \mathcal{N}_x \cap [2b_n, 2b_n+1] = \varnothing\},$$
$$\bar{B} = \{y \in B \setminus \{u\} : \mathcal{N}_y \cap [2b_n, 2b_n+1] = \varnothing\}.$$

Since for any x,

$$\mathbb{P}(\mathcal{N}_x \cap [2b_n, 2b_n+1] = \varnothing) = 1 - e^{-1},$$

standard large deviations results show that $|\bar{A}|$ and $|\bar{B}|$ are larger than $(1-e^{-1})\bar{\lambda}\beta_n/(32e)$, with probability at least $1 - o(1/n)$. Now let

$$\mathcal{E} = \{|\bar{A}| \geqslant (1-e^{-1})\bar{\lambda}\beta_n/(32e)\} \cap \{|\bar{B}| \geqslant (1-e^{-1})\bar{\lambda}\beta_n/(32e)\}.$$

Set

$$\varepsilon_n = \frac{1}{(\log n)\sqrt{\varphi_n}} \quad \text{and} \quad J_n = \left[\frac{(\log n)\sqrt{\varphi_n}}{2}\right],$$

and define for $0 \leqslant j \leqslant J_n - 1$

$$\begin{aligned} C_j &= \{\mathcal{N}_u \cap [2b_n + 2j\varepsilon_n, 2b_n + (2j+2)\varepsilon_n] = \varnothing\} \\ &\cap \{\exists x \in \bar{A} : \mathcal{N}_{(x,u)} \cap [2b_n + 2j\varepsilon_n, 2b_n + (2j+1)\varepsilon_n] \neq \varnothing\} \\ &\cap \{\exists y \in \bar{B} : \mathcal{N}_{(u,y)} \cap [2b_n + (2j+1)\varepsilon_n, 2b_n + (2j+2)\varepsilon_n] \neq \varnothing\}. \end{aligned}$$

Observe that

$$\bigcup_{j=0}^{J_n-1} C_j \subset \left\{ A \times \{2b_n\} \overset{S(u)}{\longleftrightarrow} B \times \{2b_n + 1\} \right\}. \tag{2.35}$$

Moreover, conditionally on $\bar{A}$ and $\bar{B}$, the events (C_j) are independent, and

$$\begin{aligned} \mathbb{P}(C_j \mid \bar{A}, \bar{B}) &= e^{-2\varepsilon_n} \mathbb{P}(\mathrm{Bin}(|\bar{A}|, 1 - e^{-\varepsilon_n}) \geqslant 1) \times \mathbb{P}(\mathrm{Bin}(|\bar{B}|, 1 - e^{-\varepsilon_n}) \geqslant 1) \\ &\geqslant 1/2, \end{aligned}$$

on the event $\mathcal{E}$, if n is large enough. Therefore

$$\mathbb{P}\left(\mathcal{E}, \bigcap_{j=0}^{J_n-1} C_j^c\right) \leqslant 2^{-J_n} = o(1/n).$$

This together with (2.35) imply (2.34), and concludes the proof of the proposition. □

Remark 2.6.3. This proposition can be used in various examples, for instance to the case of the configuration model with degree distribution satisfying $p(1) = p(2) = 0$, and

$$p(k) \sim ck^{-a} \qquad \text{as } k \to \infty,$$

for some constants $c > 0$ and $a > 2$. This is the degree distribution considered in [23, 64]. In this case it is known that w.h.p. the graph is connected and has diameter $\mathcal{O}(\log n)$, see [23, Lemma 1.2], and since the maximal degree is at least polynomial, the proposition applies well here. It also applies to the preferential attachment graph model considered by Berger et al [13], see Chapter 3.

Remark 2.6.4. Assume that on a sequence of graphs (G_n), one can prove that w.h.p. $\tau_n \geqslant \varphi(n)$, for some function $\varphi(n)$, and that in the mean time we can prove (2.27) for some $a_n \leqslant \varphi(n)$. Then observe that if (2.3) holds with $t_n = a_n$, then by using the self-duality, we can see that the same holds as well with $t_n = \varphi(n)$. In particular, in our setting, by using Theorem 2.1.2, we deduce that (2.3) holds with $t_n = \exp(cn)$, for any $c < c_{\text{crit}} := \liminf(1/n)\log\mathbb{E}(\tau_n)$, but (using again Theorem 2.1.2) it does not when $c > c_{\text{crit}}$. This argument also explains why the combination of the results in [65] and [64] give the statement that was mentioned in the introduction for the case $a > 2$.

Now to complete the proof of Theorem 2.1.2 (i), it remains to show that the hypothesis of the proposition is well satisfied in our case, namely for the maximal connected component – call it G_n^0 – of the configuration model G_n. It amounts to show first that the size of all the other connected components is much smaller, to ensure that w.h.p. the extinction time on G_n and on G_n^0 coincide. Remember that with Theorem 2.1.1 we know that on G_n it is w.h.p. larger than $\exp(cn)$. In the mean time we will show that the diameter of G_n^0 is $o(n)$. Since we could not find a reference, we provide a short proof here (in fact much more is true, see below).

For $v \in V_n$, we denote by $\mathcal{C}(v)$ the connected component of G_n containing v, and by $||\mathcal{C}(v)||$ its number of edges. We also define

$$d_n' := \max_{v \notin G_n^0} ||\mathcal{C}(v)||.$$

Lemma 2.6.5. *Let G_n be the configuration model with n vertices and degree distribution given either by* (2.1) *or* (2.2)*, with $a \in (1,2]$. Let $d_n = diam(G_n^0)$ be the maximal distance between pair of vertices in G_n^0. Then there exists a positive constant C, such that w.h.p.*

$$\max(d_n, d_n') \leqslant \begin{cases} C & \text{when } 1 < a < 2 \\ 4\log n/\log\log n & \text{when } a = 2. \end{cases}$$

Proof. We only prove the result for $a = 2$ here, the case $a < 2$ being entirely similar. To fix ideas we also assume that the degree distribution is given by (2.1), but the proof works as well with (2.2). Set

$$F = \left\{ v : D_v \geqslant (\log n)^4 \right\}.$$

Lemma 2.2.4 (iii) shows that w.h.p. all the elements of F are in the same connected component, and Lemma 2.2.4 (v) then shows that w.h.p. this component has size $n(1 - o(1)$, in particular it is the maximal connected component. In conclusion we get

$$\mathbb{P}(F \subset G_n^0) = 1 - o(1). \tag{2.36}$$

Now let

$$R_n := \sum_{v \in F} D_v.$$

By construction, the probability that any stub incident to some vertex $v \notin F$ is matched with a stub incident to a vertex lying in F is equal to $R_n/(L_n - 1)$. By iterating this argument, we get

$$\mathbb{P}\left(d(v,F) > k \text{ and } ||\mathcal{C}(v)|| > k \mid (D_w)_{w \in V_n}\right) \leqslant \frac{R_n}{L_n - 1} \frac{R_n}{L_n - 3} \cdots \frac{R_n}{L_n - 2k + 1},$$

for any k, where $d(v,F)$ denotes the graph distance between v and F (which by convention we take infinite when there is no element of F in $\mathcal{C}(v)$). Then it follows from Lemma 2.2.4 (i) and the fact that $R_n \asymp n \log\log n$, that

$$\mathbb{P}\left(d(v,F) > k_n \text{ and } ||\mathcal{C}(v)|| > k_n\right) \leqslant \left(\frac{C \log\log n}{\log n}\right)^{2\log n/\log\log n - 1} = o(n^{-1}),$$

for some constant $C > 0$, with $k_n = 2\log n/(\log\log n) - 1$. This proves the lemma, using a union bound and (2.36). □

To complete the proof of Part (i) of the theorem, we just need to remember that on any graph with k edges, and for any $t \geqslant 1$, the extinction time is bounded by $2t$ with probability at least $1 - (1 - \exp(-Ck))^t$ (since on each time interval of length 1 it has probability at least $\exp(-Ck)$ to die out, for some constant $C > 0$, independently of the past). Therefore the previous lemma shows that w.h.p. the extinction time on G_n^0 and on G_n are equal, as was announced just above the previous lemma. Then Part (i) of the theorem follows with Proposition 2.6.2.

2.7 Extension to more general degree distributions

We present here some rather straightforward extensions of our results to more general degree distributions.

A first one, which was also considered in [46], is to take distributions which interpolate between (2.1) and (2.2): for any fixed $\alpha \in [1, \infty]$, define

$$p_{n,a,\alpha}(j) := c_{n,a,\alpha}\, j^{-a} \qquad \text{for all } 1 \leqslant j \leqslant n^{\alpha},$$

where $(c_{n,a,\alpha})$ are normalizing constants, and with the convention that the case $\alpha = \infty$ corresponds to the distribution given by (2.2).

It turns out that if $a < 2$ and $\alpha < 1/(a-1)$, one can use exactly the same proof as in the case $\alpha = 1$. When $\alpha > 1/(a-1)$, using that w.h.p. all vertices have degree smaller than $n^{1/(a-1)} \log\log n$, one can use the same proof as in the case $\alpha = \infty$. The case $\alpha = 1/(a-1)$ is more complicated, and as in [46], a proof would require a more careful look at it.

When $a = 2$, using that w.h.p. all vertices have degree smaller than $n \log\log n$, one can see that the same proof applies for any $\alpha > 1$.

Another extension is to assume that there exist positive constants c and C, and some fixed $m \geqslant 1$, such that for any vertex v,

$$cj^{-a} \leqslant \mathbb{P}(D_v = j) \leqslant Cj^{-a} \qquad \text{for } m \leqslant j \leqslant n^{\alpha},$$

say with $\alpha = 1$, but it would work with $\alpha = \infty$ as well. The only minor change in this case is in the proof of Lemma 2.3.1. But one can argue as follows: just replace the set A_1 by the set of vertices in A_m whose first $m-1$ stubs are not connected to any of the vertices in E. By definition these vertices have at most one neighbor in E and moreover, it is not difficult to see that this set also has w.h.p. a size of order n. Then the rest of the proof applies, mutadis mutandis. All other arguments in the proof of Theorem 2.1.1 remain unchanged. Therefore in this case we obtain that:

$$\frac{|\xi_{t_n}^{V_n}|}{n} - \rho_{n,a}(\lambda) \xrightarrow{(\mathbb{P})} 0,$$

with $\rho_{n,a}(\lambda)$ as in (2.5). Theorem 2.1.2 remains also valid in this setting.

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Chapter 3

Metastability for the contact process on the preferential attachment graph

Abstract. We consider the contact process on the preferential attachment graph. The work of Berger, Borgs, Chayes and Saberi [13] confirmed physicists' predictions that the contact process starting from a typical vertex becomes epidemic for an arbitrarily small infection rate λ with positive probability. More precisely, they showed that with probability $\lambda^{\Theta(1)}$, it survives for a time exponential in the largest degree. Here we obtain sharp bounds for the density of infected sites at a time close to exponential in the number of vertices (up to some logarithmic factor).

3.1 Introduction

The aim of this chapter is to prove a metastability result for the contact process on the preferential attachment random graph, improving [13]'s result in two aspects: obtaining a better bound on the extinction time, and estimating more accurately the density of the infected sites.

The preferential attachment graph (a definition will be given later) is well-known as a pattern of scale-free or social networks. Indeed, it not only explains the power-law degree sequence of a host in real world networks, but also reflects a wisdom that the rich get richer - the newbies are more likely to get acquainted with more famous people rather than a relatively unknown person. Therefore there has been great interest in this random graph as well as the processes occurring on it, including the contact process. In [13], the authors proved a remarkable result which validated physicists' predictions that the phase transition of the contact process occurs at $\lambda = 0$. More precisely, they showed that there are positive constants θ, c and C, such that for all $\lambda > 0$

$$\lambda^c \leqslant \mathbb{P}\left(\xi^u_{\exp(\theta\lambda^2\sqrt{n})} \neq \varnothing\right) \leqslant \lambda^C, \tag{3.1}$$

where (ξ^u_t) is the contact process starting from a uniformly chosen vertex.

We improve (3.1) as follows.

Theorem 3.1.1. *Let (G_n) be the sequential model of the preferential attachment graph with parameters $m \geqslant 1$ and $\alpha \in [0,1)$. Consider the contact process (ξ_t) with infection rate $\lambda > 0$ starting from full occupancy on G_n. Then there exist positive constants c and C, such that*

$$\mathbb{P}\left(c\lambda^{1+\frac{2}{\psi}}|\log\lambda|^{-\frac{1}{\psi}} \leqslant \frac{|\xi_{t_n}|}{n} \leqslant C\lambda^{1+\frac{2}{\psi}}|\log\lambda|^{-\frac{1}{\psi}}\right) = 1 - o(1). \tag{3.2}$$

where $\psi = \frac{1-\alpha}{1+\alpha}$ and (t_n) is any sequence satisfying $t_n \to \infty$ and $t_n \leqslant T_n = \exp\left(\frac{c\lambda^2 n}{(\log n)^{1/\psi}}\right)$.

By a well-known property of the contact process called self-duality (see [59], Section I.1) for any $t \geqslant 0$ we have

$$\sum_{v \in V_n} 1(\{\xi^v_t \neq \varnothing\}) \overset{(\mathcal{L})}{=} |\xi_t|. \tag{3.3}$$

Therefore the survival probability as in (3.1) is just the expected value of the density of infected sites as in Theorem 3.1.1, so that our result is a stronger form of the one in [13]. Additionally we get a more precise estimate of the density and we allow (t_n) to be larger.

Since the empty configuration is the unique absorbing state of the contact process on finite graphs, the contact process on G_n always dies out. Theorem 3.1.1 shows that before dying, the contact process remains for a long time in a stationary situation in the sense that the density of infected sites is stable for a time streched exponential in the number of vertices. This metastable behavior for the contact process has been observed in some examples, such as the finite boxes (see Chapter 1), the configuration models (see Chapter 2), the random regular graphs (see [56]).

We also prove another metastable result for the extinction time of the contact process.

Proposition 3.1.2. *Let τ_n be the extinction time of the contact process on the sequential preferential attachment graph with infection rate $\lambda > 0$ starting from full occupancy. Then*

(i) the following convergence in law holds

$$\frac{\tau_n}{\mathbb{E}(\tau_n)} \xrightarrow[n\to\infty]{(\mathcal{L})} \mathcal{E}(1),$$

with $\mathcal{E}(1)$ an exponential random variable with mean one,

(ii) there exists a constant $C > 0$, such that $\mathbb{E}(\tau_n) \leqslant \exp(Cn)$, for all $n \geqslant 1$.

This metatstable result has been proved for the contact process on finite boxes (see [24, 62]), finite homogeneous trees (see [22]), configuration models (see [26]), random regular graphs (see [85]). The proof of Proposition 3.1.2 is immediate, since it is consequence of Proposition 2.1.2 in Chapter 2 and Lemma 5.3.1 in Chapter 5.

Now let us make some comments on the proof of Theorem 3.1.1.

First, to obtain the time T_n we will use the maintenance mechanism as in [23] instead of the one in [13]. In the latter the authors used that in the preferential attachment graph the maximal degree is of order $\sqrt{n}$, plus the well-known fact that for any vertex v, the process survives a time exponential in the degree of v, once it is infected, yielding (3.1). In the former, on the other hand, when considering the contact process on the configuration model, Chatterjee and Durrett employed many vertices with total degree of order $n^{1-\varepsilon}$, for any $\varepsilon > 0$, and derived a much better bound on the extinction time. Here, our strategy is to find vertices with degree larger than $Cd(G_n)$, where $C = C(\lambda) > 0$ is a constant and $d(G_n)$ is the diameter of G_n, which is of order $\log n$. Thanks to Proposition 1 in [23], we can deduce that the virus propagates along these vertices for a time exponential in their total degree. Moreover, the degree distribution of the graph, denoted by $\mathbf{p}$, has a power-law with exponent $\nu = 2 + 1/\psi$. Thus the number of these vertices is of order $n(\log n)^{1-\nu}$ and their total degree is of order $n(\log n)^{-1/\psi}$, which explains the bound on t_n in Theorem 3.1.1.

Secondly, to gain the precise estimate on the density, we use ideas from [74, 13, 23, 65]: if the virus starting at a typical vertex wants to survive a long time, it has to infect a big vertex of degree significantly larger than λ^{-2}. Then the virus is likely to survive in the neighborhood of this vertex for a time which is long enough to infect another big vertex, and so on. We can see that the time required for a virus to spread from one big vertex to another is at least $\lambda^{-\Theta(1)}$ (corresponding to the case when the distance between them is constant). Besides, it was shown that if $\deg(v) \geqslant K/\lambda^2$, then the survival time of the contact process on the star graph formed by v and it neighbors is about $\exp(cK)$. Hence the degree of big vertices should be larger than $C\lambda^{-2}|\log\lambda|$.

Using the idea in [23, 65] for the configuration model, we study the spread of infection starting from potential vertices which are the vertices having a big neighbor. A potential vertex infects its big neighbor with probability of order λ. Moreover, we will show in Section 4 that any big vertex has a positive probability to make the virus survive up to time T_n. Therefore the probability for the virus starting from a potential vertex to be active at time T_n is of order λ. Hence the density of vertices from where the virus survives up to T_n is of order at least λ times the density of potential vertices. This is of order $\lambda \times \mathbf{p}\big([\lambda^{-2}|\log\lambda|, \infty)\big)\lambda^{-2}|\log\lambda| \asymp \lambda^{1+\frac{2}{\psi}}|\log\lambda|^{-\frac{1}{\psi}}$ yielding the desired lower bound.

For the upper bound, we adapt the proof in [65] for Galton-Watson trees, see the appendix.

It is interesting to note also that if we consider the contact process on the configuration model with the same power-law degree distribution, the density is of order $\lambda^{1+\frac{2}{\psi}}|\log\lambda|^{-\frac{2}{\psi}}$ (see [65, Theorem 1.1]), which is slightly smaller than the one in (3.2). This difference is due to the fact that the distance between big vertices in the preferential attachment is less than the one in the configuration model.

In [14] the authors show that the preferential attachment graph converges weakly to an infinite graph limit, called the *Pólya-point graph*. We also estimate the survival probability of the contact process on this infinite graph.

Proposition 3.1.3. *Let (ξ_t^o) be the contact process on the Pólya-point graph with infection rate $\lambda > 0$ starting from the root o. There exist positive constants c and C, such that*

$$c\lambda^{1+2/\psi}|\log\lambda|^{-1/\psi} \leqslant \mathbb{P}(\xi_t^o \neq \varnothing \ \forall t) \leqslant C\lambda^{1+2/\psi}|\log\lambda|^{-1/\psi}.$$

Theorem 3.1.1 (resp. Proposition 3.1.3) implies that for all $\lambda > 0$, the contact process becomes endemic (resp. survives forever) with positive probability. In other words, the critical values of the contact process on the preferential attachment graph and its weak limit are all zero. This provides a new example of the phase transition **(T)** mentioned in Chapter 1.

Chapter 3 is organized as follows. In the next section, based on [14], we give the definition of the sequential model of the preferential attachment graph as well as its weak local limit, the Pólya-point graph. We also prove preliminary results on the graph structure and fix some notation. We prove Proposition 3.1.3 is proved in Sections 3. The main theorem 3.1.1 is proved in Section 4 and Proposition 3.1.2 is proved in Section 5.

3.2 Preliminaries

3.2.1 Construction of the random graph and notation.

Let us give a definition following [14] of the sequential model of the preferential attachment graph with parameters $m \geqslant 1$ and $\alpha \in [0,1)$. We construct a sequence of graphs (G_n) with vertex set $V_n = \{v_1, \ldots, v_n\}$ as follows.

First G_1 contains one vertex v_1 and no edge, and G_2 contains 2 vertices v_1, v_2 and m edges connecting them. Given G_{n-1}, we define G_n the following way. Add the vertex v_n to the graph, and draw edges between v_n and m vertices $w_{n,1}, \ldots, w_{n,m}$ (possibly with repetitions) from G_{n-1} as follows: with probability $\alpha_n^{(i)}$, the vertex $w_{n,i}$ is chosen uniformly at random from V_{n-1} where

$$\alpha_n^{(i)} = \begin{cases} \alpha & \text{if } i = 1, \\ \alpha \frac{2m(n-1)}{2m(n-2)+2m\alpha+(1-\alpha)(i-1)} = \alpha + \mathcal{O}(n^{-1}) & \text{if } i \geqslant 2. \end{cases}$$

Otherwise, $w_{n,i} = v_k$ with probability

$$\frac{\deg_{n-1}^{(i)}(v_k)}{Z_{n-1}^{(i)}},$$

where

$$\deg_{n-1}^{(i)}(v_k) = \deg_{n-1}(v_k) + \#\{1 \leqslant j \leqslant i-1 : w_{n,j} = v_k\},$$

is the degree of v_k before choosing $w_{n,i}$, and

$$Z_{n-1}^{(i)} = \sum_{k=1}^{n-1} \deg_{n-1}^{(i)}(v_k) = \sum_{k=1}^{n-1} \deg_{n-1}(v_k) + i - 1 = 2m(n-2) + i - 1,$$

with $\deg_{n-1}(v_k)$ the degree of v_k in G_{n-1}.

This construction might seem less natural than in the independent model where with probability α we choose $w_{n,i}$ uniformly from V_{n-1} and with probability $1-\alpha$ it is chosen according to a simpler rule: $w_{n,i} = v_k$ with probability $\deg_{n-1}(v_k)/2m(n-2)$. However the sequential model constructed above is easier to analyze because it is exchangeable, and as a consequence it admits an alternative representation which contains more independence. In [14], the authors called it the Pólya urn representation which we now recall in the following theorem. To this end, we denote by $\beta(a,b)$ the Beta distribution, whose density is proportional to $x^{a-1}(1-x)^{b-1}$ on $[0,1]$, and by $\Gamma(a,b)$ the Gamma distribution, whose density is proportional to $x^{a-1}e^{-bx}$ on $[0,\infty)$. For any $a<b$, $\mathcal{U}([a,b])$ stands for the uniform distribution on $[a,b]$.

Theorem 3.2.1. *[14, Theorem 2.1] Fix $m \geqslant 1$, $\alpha \in [0,1)$ and $n \geqslant 1$. Set $r = \alpha/(1-\alpha), \psi_1 = 1$, and let $\psi_2, \dots, \psi_n$ be independent random variables with law*

$$\psi_j \sim \beta(m + 2mr, (2j-3)m + 2mr(j-1)).$$

Define

$$\varphi_j = \psi_j \prod_{t=j+1}^{n} (1-\psi_t), \qquad S_k = \sum_{j=1}^{k} \varphi_j, \quad \text{and} \qquad I_k = [S_{k-1}, S_k).$$

Conditionally on $\psi_1, \dots, \psi_n$, let $\{U_{k,i}\}_{k=1,\dots,n, i=1,\dots,m}$ be a sequence of independent random variables, with $U_{k,i} \sim \mathcal{U}([0, S_{k-1}])$. Start with the vertex set $V_n = \{v_1, \dots, v_n\}$. For $j<k$, join v_j and v_k by as many edges as the number of indices $i \in \{1,\dots,m\}$, such that $U_{k,i} \in I_j$. Denote the resulting random graph by G_n.

Then G_n has the same distribution as the sequential model of the preferential attachment graph.

We remark that in [14], the authors state the theorem for $m \geqslant 2$. However, their proof also works properly when $m = 1$.

From now on, we always consider the random multi-graph G_n constructed as in this theorem.

We now look at the local structure of G_n. It was shown in [14] that G_n is locally tree-like, with some subtle degree distribution that we now recall. First we fix some constants:

$$\chi = \frac{1+2r}{2+2r} \qquad \text{and} \qquad \psi = \frac{1-\chi}{\chi} = \frac{1}{1+2r}.$$

Note that $1/2 \leqslant \chi < 1$ and $0 < \psi \leqslant 1$. Let

$$F \sim \Gamma(m+2mr, 1) \quad \text{and} \quad F' \sim \Gamma(m+2mr+1, 1).$$

We will construct inductively a random rooted tree (T, o) with vertices identified with elements of $\cup_{\ell \geqslant 1} \mathbb{N}^\ell$ (where vertices at generation ℓ are elements of $\mathbb{N}^\ell$) and a map which associates to each vertex v a position x_v in $[0,1]$. Additionally each vertex (except the root) will be assigned a type, either R or L.

- The root $o = (0)$ has position $x_o = U_0^\chi$, where $U_0 \sim \mathcal{U}([0,1])$.
- Given $v \in T$ and its position x_v, define

  $$m_v = \begin{cases} m & \text{if } v \text{ is the root or of type L,} \\ m-1 & \text{if } v \text{ is of type R.} \end{cases}$$

  and

  $$\gamma_v \sim \begin{cases} F & \text{if } v \text{ is the root or of type R,} \\ F' & \text{if } v \text{ is of type L.} \end{cases}$$

  The children of v are $(v,1)$,…,(v,m_v), (v,m_v+1),…,(v,m_v+q_v), the first m_v's are of type L and the remaining ones are of type R. Conditionally on x_v, $x_{(v,1)}, \ldots, x_{(v,m_v)}$ are i.i.d. uniform random variable in $[0, x_v]$, and $x_{(v,m_v+1)}, \ldots, x_{(v,m_v+q_v)}$ are the points of the Poisson point process on $[x_v, 1]$ with intensity

  $$\rho_v(x)dx = \gamma_v \frac{\psi x^{\psi-1}}{x_v^\psi} dx.$$

  This procedure defines inductively an infinite rooted tree (T, o), which is called the Pólya-point graph and $(x_v)_{v \in T}$ is called the Pólya-point process.

Theorem 3.2.2. *[14, Theorem 2.2] The random graph G_n constructed as in Theorem 3.2.1 converges weakly to the Pólya-point graph.*

Note that in [14] the authors prove this theorem for $m \geqslant 2$. For $m = 1$, recently, in [20] the authors show that the local limit of preferential attachment graph is always the Pólya-point graph regardless the initial (seed) graph (in our case, the initial graph contains two vertices v_1, v_2 and one edge connecting them).

Now we introduce some notation. We denote the indicator function of a set A by $\mathbf{1}(A)$. For any vertices v and w we write $v \sim w$ if there is an edge between them (in which case we say that they are neighbors or connected), and $v \not\sim w$ otherwise. We call size of G the cardinality of its set of vertices, and we denote it by $|G|$.

A graph in which all vertices have degree one, except one which is connected to all the others is called a **star graph**. The only vertex with degree larger than one is called the center of the star graph, or central vertex.

If f and g are two real functions, we write $f = \mathcal{O}(g)$ if there exists a constant $C > 0$, such that $f(x) \leqslant Cg(x)$ for all x; $f \gtrsim g$ (or equivalently $g \lesssim f$) if $g = \mathcal{O}(f)$; $f \asymp g$ if $f = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$; $f = o(g)$ if $f(x)/g(x) \to 0$ as $x \to \infty$. Finally for a sequence of r.v.s (X_n) and a function $f : \mathbb{N} \to (0, \infty)$, we say that $X_n \asymp f(n)$ holds w.h.p. if there exist positive constants c and C, such that $\mathbb{P}(cf(n) \leqslant X_n \leqslant Cf(n)) \to 1$.

3.2.2 Preliminary results on the random graph.

We first recall a version of the Azuma-Hoeffding inequality for martingales which we will use throughout this paper (see for instance [25]).

Lemma 3.2.3. *Let $(X_i)_{i\geqslant 0}$ be a martingale satisfying $|X_i - X_{i-1}| \leqslant 1$ for all $i \geqslant 1$. Then for any n and $t > 0$, we have*

$$\mathbb{P}\left(|X_n - X_0| \geqslant t\right) \leqslant 2\exp(-t^2/2n).$$

From this inequality we can deduce a large deviations result. Let $(X_i)_{i\geqslant 1}$ be a sequence of independent Bernoulli random variables. Assume that $0 < 2p \leqslant \mathbb{E}(X_i) \leqslant Mp$ for all i. Then there exists $c = c(M) > 0$, such that for all n

$$\mathbb{P}\left(np \leqslant \sum_{i=1}^{n} X_i \leqslant 2Mnp\right) \geqslant 1 - \exp(-cnp). \tag{3.4}$$

Now we present some estimates on the sequences (φ_i), (ψ_j) and (S_k).

Lemma 3.2.4. *Let $(\varphi_i)_i$, $(\psi_j)_j$ and $(S_k)_k$ be sequences of random variables as in Theorem* 3.2.1. *Then there exist positive constants μ and θ_0, such that for all $\theta \leqslant \theta_0$, the following assertions hold.*

(i) $\mathbb{E}(\psi_j) = \frac{\chi}{j} + \mathcal{O}(\frac{1}{j^2})$, $\quad \mathbb{E}(\psi_j^2) \asymp \frac{1}{j^2}$.

(ii) For any $\varepsilon > 0$, *there exists* $K = K(\varepsilon) < \infty$, *such that*

$$\mathbb{P}(\mathcal{E}_\varepsilon) \geqslant 1 - \varepsilon,$$

where

$$\mathcal{E}_\varepsilon = \Big\{ \Big| \frac{S_j}{S_k} - \Big(\frac{j}{k}\Big)^\chi \Big| \leqslant \varepsilon (j/k)^\chi \quad \forall\, K(\varepsilon) \leqslant j \leqslant k \Big\}.$$

(iii) As n *tends to infinity,*

$$\mathbb{P}\left(i\psi_i \leqslant 2\log n \; \forall\, i = 1, \dots, n \right) = 1 - o(1).$$

(iv) $\mathbb{P}\left(\mu/j \geqslant \psi_j \geqslant \theta/j\right) \geqslant 2\theta$.

(v) $\mathbb{E}\left[\varphi_j 1(\psi_j \geqslant \theta/j)\right] \geqslant \theta \mathbb{E}(\varphi_j)$.

Proof. Let us start with Part (i). Observe that if $\psi \sim \beta(a, b)$, then

$$\mathbb{E}(\psi) = \frac{a}{a+b} \quad \text{and} \quad \mathbb{E}(\psi^2) = \frac{a(a+1)}{(a+b)(a+b+1)}.$$

Hence the result follows from the fact that $\psi_j \sim \beta(m + 2mr, (2j-3)m + 2mr(j-1))$.
Part (ii) is a direct consequence of Lemma 3.1 in [14].
To prove (iii), we observe that $\psi_i \sim \beta(a, b_i)$ with $a = m + 2mr$ and $b_i = (2m + 2mr)i - (3m + 2mr)$. Hence, for all $i \geqslant 2\log n$

$$\begin{aligned} \mathbb{P}\left(\psi_i > \frac{2\log n}{i}\right) &= \frac{1}{\beta(a, b_i)} \int_{2\log n/i}^{1} x^{a-1}(1-x)^{b_i - 1} dx \\ &\lesssim b_i^a \left(1 - \frac{2\log n}{i}\right)^{b_i} \lesssim n^{-2}. \end{aligned}$$

Here, we have used that $\beta(a, b) \asymp \mathcal{O}(b^{-a})$ when a is fixed. On the other hand, when $i < 2\log n$, this probability is zero. Therefore

$$\mathbb{P}(i\psi_i \leqslant 2\log n \; \forall\, i = 1, \dots, n) = 1 - o(1).$$

For Part (iv), Markov's inequality gives that for any $\delta \in (0, 1)$

$$\mathbb{P}\big(|\psi_j - \mathbb{E}(\psi_j)| > (1-\delta)\mathbb{E}(\psi_j)\big) \leqslant \frac{\mathrm{Var}_n(\psi_j)}{(1-\delta)^2 \mathbb{E}(\psi_j)^2}.$$

Moreover, if $\psi \sim \beta(a,b)$, then

$$\frac{\mathrm{Var}(\psi)}{\mathbb{E}(\psi)^2} = \frac{\mathbb{E}(\psi^2)}{\mathbb{E}(\psi)^2} - 1 = \frac{(a+1)(a+b)}{a(a+b+1)} - 1 = \frac{b}{a(a+b+1)} \leqslant \frac{1}{a}.$$

Therefore for any j

$$\mathbb{P}\big[\delta\mathbb{E}(\psi_j) \leqslant \psi_j \leqslant (2-\delta)\mathbb{E}(\psi_j)\big] \geqslant 1 - \frac{1}{(1-\delta)^2(m+2mr)}.$$

Hence thanks to (i) we can choose positive constants μ and θ, such that for all j

$$\mathbb{P}\left(\psi_j \in \left(\frac{\theta}{j}, \frac{\mu}{j}\right)\right) \geqslant 2\theta. \tag{3.5}$$

For (v), we notice that

$$\begin{aligned}
\mathbb{E}\big[\varphi_j 1(\psi_j \geqslant \theta/j)\big] &= \mathbb{E}\big[\psi_j 1(\psi_j \geqslant \theta/j)\big]\mathbb{E}\left(\prod_{t=j+1}^{n}(1-\psi_t)\right) \\
&\geqslant c\mathbb{E}(\psi_j)\mathbb{E}\left(\prod_{t=j+1}^{n}(1-\psi_t)\right) \\
&= c\mathbb{E}(\varphi_j),
\end{aligned}$$

for some $c > 0$, independent of j. Thus the result follows by taking θ small enough. □

The preferential attachment graph is known as a prototype of small world networks whose diameter and typical distance (the distance between two randomly chosen vertices) are of logarithmic order. In fact, these quantities in the independent model were well-studied, see for instance [33] or [53]. In the following two lemmas, we prove similar estimates for the sequential model. These estimates are in fact weaker but sufficient for our purpose.

Lemma 3.2.5. *Let $d(G_n)$ be the diameter of the random graph G_n, i.e. the maximal distance between pair of vertices in G_n. Then there exists a positive constant b_1, such that*

$$\mathbb{P}(d(G_n) \leqslant b_1 \log n) = 1 - o(1).$$

Proof. Let $\varepsilon \in (0, 1/2)$ be given, and recall the definitions of $K(\varepsilon)$ and $\mathcal{E}_\varepsilon$ given in Lemma 3.2.4 (ii). We first bound $d(v_1, v_n)$. Define a decreasing random sequence $(n_i)_{i \geqslant 0}$ as follows

$n_0 = n$ and for $i \geqslant 1$, n_{i+1} is arbitrarily chosen from indices such that $v_{n_{i+1}}$ receives an edge emanating from v_{n_i}. Define

$$X_i = 1(\{n_i \leqslant n_{i-1}/2\}) \text{ and } \mathcal{F}_i = \sigma(n_j : j \leqslant i) \vee \sigma((\varphi_t)).$$

By the construction of the graph in Theorem 3.2.1, we have

$$\mathbb{E}(X_{i+1} \mid \mathcal{F}_i) = \mathbb{P}(n_{i+1} \leqslant n_i/2 \mid \mathcal{F}_i) = \frac{S_{[n_i/2]}}{S_{n_i-1}},$$

with (S_i) as in Theorem 3.2.1. We now define

$$\sigma_n = \inf\{i : n_{i+1} \leqslant \log n\}.$$

Observe that if $\sum_1^k X_i \geqslant (\log_2 n - \log_2 \log n)$, then

$$n_k \leqslant n_0 \times 2^{-\sum_1^k X_i} \leqslant \log n,$$

which implies that $\sigma_n \leqslant k$. Hence for all k

$$\mathbb{P}\big[\sigma_n \geqslant k \mid (\varphi_t), \mathcal{E}_\varepsilon\big] \leqslant \mathbb{P}_n \left(\sum_{i=1}^k X_i \leqslant \log_2 n - \log_2 \log n \mid (\varphi_t), \mathcal{E}_\varepsilon \right). \tag{3.6}$$

On the other hand, if $i \leqslant \sigma_n$, then $n_i > \log n > 2K(\varepsilon)$ for n large enough. Therefore by Lemma 3.2.4 (ii), we have on $\mathcal{E}_\varepsilon$ for $i \leqslant \sigma_n$

$$\frac{S_{[n_i/2]}}{S_{n_i-1}} \geqslant (1-\varepsilon) \left(\frac{[n_i/2]}{n_i - 1} \right)^{\chi} \geqslant \frac{1}{2^{\chi+1}} =: p.$$

In other words, we have for $i \leqslant \sigma_n$

$$\mathbb{E}(X_{i+1} \mid \mathcal{F}_i, \mathcal{E}_\varepsilon) \geqslant p. \tag{3.7}$$

Let us define $Y_0 = 0$ and for $k \geqslant 1$,

$$Y_k = \sum_{i=1}^k \left(X_i - \mathbb{E}(X_i \mid \mathcal{F}_{i-1}, \mathcal{E}_\varepsilon) \right).$$

Then (Y_k) is a martingale with respect to the filtration $(\mathcal{F}_k)$ and $|Y_k - Y_{k-1}| \leqslant 1$. By using Lemma 3.2.3 we get that

$$\mathbb{P}\big[Y_k \leqslant -kp/2 \mid (\varphi_t)\big] \leqslant 2\exp(-kp^2/8). \tag{3.8}$$

Then (3.7) and (3.8) imply that

$$\mathbb{P}\left(\sum_{i=1}^{k} X_i \leqslant kp/2 \mid (\varphi_t), \mathcal{E}_\varepsilon\right) \leqslant 2\exp(-kp^2/8). \tag{3.9}$$

Combining (3.6) and (3.9) gives that

$$\mathbb{P}(\sigma_n \geqslant C\log n \mid (\varphi_t), \mathcal{E}_\varepsilon) = \mathcal{O}(n^{-2}), \tag{3.10}$$

for some $C = C(p) > 0$. On the other hand,

$$\{\sigma_n \leqslant C\log n\} \subset \{\exists\, k \leqslant \log n : d(v_n, v_k) \leqslant C\log n\},$$

and $d(v_1, v_k) \leqslant \log n$ for all $k \leqslant \log n$. Therefore,

$$\{\sigma_n \leqslant C\log n\} \subset \{d(v_n, v_1) \leqslant (C+1)\log n\}.$$

Hence by (3.10), we obtain that

$$\mathbb{P}\big[d(v_1, v_n) \geqslant (C+1)\log n \mid (\varphi_t), \mathcal{E}_\varepsilon\big] = \mathcal{O}(n^{-2}). \tag{3.11}$$

Let $d_{G_k}(v_i, v_j)$ be the distance between v_i and v_j in G_k for $i, j \leqslant k \leqslant n$. Note that

$$d_{G_k}(v_i, v_j) \geqslant d(v_i, v_j) = d_{G_n}(v_i, v_j).$$

Similarly to (3.11), we deduce that on $\mathcal{E}_\varepsilon$, for all $i \geqslant C\log n$,

$$\mathbb{P}\big[d(v_1, v_i) \geqslant (C+1)\log i \mid (\varphi_t)\big] \leqslant \mathbb{P}\big[d_{G_i}(v_1, v_i) \geqslant (C+1)\log i \mid (\varphi_t)\big] = \mathcal{O}(i^{-2}).$$

Hence on $\mathcal{E}_\varepsilon$

$$\mathbb{P}\big[d(v_1, v_i) \leqslant (C+1)\log n \;\forall\, i \geqslant C\log n \mid (\varphi_t)\big] = 1 - o(1).$$

Therefore by taking expectation with respect to (φ_t) and using Lemma 3.2.4 (ii), we get

$$\mathbb{P}\big[d(G_n) \leqslant 2(C+1)\log n\big] \geqslant 1 - 2\varepsilon,$$

which proves the result by letting ε tend to 0. □

Before proving the lower bound on the typical distance, we make a remark which will be used frequently in this chapter. It follows from the definition of G_n that for all $i < j$,

$$\mathbb{P}(v_i \not\sim v_j \mid (\varphi_t)) = \left(1 - \frac{\varphi_i}{S_{j-1}}\right)^m.$$

Hence

$$\frac{\varphi_i}{S_{j-1}} \leqslant \mathbb{P}(v_i \sim v_j \mid (\varphi_t)) \leqslant \frac{m\varphi_i}{S_{j-1}}.$$

Then by using the following identities

$$S_{j-1} = \sum_{t=1}^{j-1} \varphi_t = \prod_{t=j}^{n}(1 - \psi_t) \quad \text{and} \quad \varphi_i = \psi_i \prod_{t=i+1}^{n} (1 - \psi_t),$$

we obtain that

$$\frac{\psi_i S_i}{S_{j-1}} \leqslant \mathbb{P}(v_i \sim v_j \mid (\varphi_t)) \leqslant m\frac{\psi_i S_i}{S_{j-1}}. \tag{3.12}$$

Lemma 3.2.6. *Let w_1 and w_2 be two uniformly chosen vertices from V_n. Then there is a positive constant b_2, such that w.h.p.*

$$d(w_1, w_2) \geqslant \frac{b_2 \log n}{\log \log n}.$$

Proof. Fix $\varepsilon \in (0, 1/2)$ a positive constant. Then we define

$$\mathcal{I}_\varepsilon = \mathcal{E}_\varepsilon \cap \{i\psi_i \leqslant 2 \log n \;\forall\, i = 1, \ldots, n\},$$

with $\mathcal{E}_\varepsilon$ as in Lemma 3.2.4. It follows from 3.2.4 (ii) and (iii) that for all n large enough

$$\mathbb{P}(\mathcal{I}_\varepsilon) \geqslant 1 - 2\varepsilon. \tag{3.13}$$

We now use an argument from [53, Vol II, Lemma 7.16] to bound the typical distance. We call a sequence of distinct vertices $\pi = (\pi_1, \ldots, \pi_k)$ a self-avoiding path. We write $\pi \subset G_n$ if π_i and π_{i+1} are neighbors for all $1 \leqslant i \leqslant k - 1$. Let $\Pi(i, j, k)$ be the set of all self-avoiding paths of length k starting from v_i and finishing at v_j. We then claim that for all $i, j, k \geqslant 1$,

(i) $\mathbb{P}\big[d(v_i, v_j) = k \mid (\varphi_t), \mathcal{I}_\varepsilon\big] \leqslant g_k(i, j) := \sum_{\pi \in \Pi(i,j,k)} \mathbb{P}\big[\pi \subset G_n \mid (\varphi_t), \mathcal{I}_\varepsilon\big]$,

(ii) $g_{k+1}(i, j) \leqslant \sum_{s \neq i,j} g_1(i, s) g_k(s, j)$.

The first claim is clear, because if $d(v_i, v_j) = k$ then there exists a self-avoiding path in $\Pi(i, j, k)$ which is in G_n. For the second one, we note that for any self-avoiding path $\pi = (\pi_1, \ldots, \pi_k)$,

$$\mathbb{P}(\pi \subset G_n \mid (\varphi_t), \mathcal{I}_\varepsilon) = \mathbb{P}(\pi_1 \sim \pi_2 \mid (\varphi_t), \mathcal{I}_\varepsilon) \times \mathbb{P}(\bar{\pi} \subset G_n \mid (\varphi_t), \mathcal{I}_\varepsilon),$$

where $\bar{\pi} = (\pi_2, \ldots, \pi_k)$. Indeed, if $j < k$, then the event that $v_j \sim v_k$ depends only on the $(U_{k,i})_{i \leqslant m}$. Hence this result follows from the facts that the vertices in π are distinct and that the $\{(U_{k,i})_{i \leqslant m}\}_k$ are independent. We are now in position to prove (ii):

$$\begin{aligned} g_{k+1}(i,j) &= \sum_{s \neq i,j} \sum_{\substack{v_i \notin \bar{\pi} \\ \bar{\pi} \in \Pi(s,j,k)}} \mathbb{P}(v_i \sim v_s, \bar{\pi} \subset G_n \mid (\varphi_t), \mathcal{I}_\varepsilon) \\ &\leqslant \sum_{s \neq i,j} \sum_{\bar{\pi} \in \Pi(s,j,k)} \mathbb{P}(v_i \sim v_s \mid (\varphi_t), \mathcal{I}_\varepsilon) \times \mathbb{P}(\bar{\pi} \subset G_n \mid (\varphi_t), \mathcal{I}_\varepsilon) \\ &= \sum_{s \neq i,j} g_1(i,s) g_k(s,j). \end{aligned}$$

We prove by induction on k that there is a positive constant C, such that

$$g_k(i,j) \leqslant \frac{(C \log n)^{2k-1}}{\sqrt{ij}}. \tag{3.14}$$

For $k = 1$, it follows from (3.12) that for all $i < j$,

$$g_1(i,j) = \mathbb{P}(v_i \sim v_j \mid (\varphi_t), \mathcal{I}_\varepsilon) \leqslant m \mathbb{E}\left(\frac{\psi_i S_i}{S_{j-1}} \mid (\varphi_t), \mathcal{I}_\varepsilon \right). \tag{3.15}$$

We now claim that on $\mathcal{I}_\varepsilon$

$$\frac{\psi_i S_i}{S_{j-1}} \leqslant \frac{4 \log n}{\sqrt{ij}}. \tag{3.16}$$

Indeed, we recall that on $\mathcal{I}_\varepsilon$,

$$\psi_i \leqslant \frac{2 \log n}{i} \; \forall \, i = 1, \ldots, n \quad \text{and} \quad \left| \frac{S_i}{S_j} - \left(\frac{i}{j} \right)^\chi \right| \leqslant \varepsilon (i/j)^\chi \quad \text{for all } j \geqslant i \geqslant K(\varepsilon),$$

with $K(\varepsilon)$ as in Lemma 3.2.4 (ii).

If $i < j \leqslant K(\varepsilon)$ then (3.16) holds for all n large enough (the left hand-side is bounded by 1 and the right-hand side tends to infinity). If $j > i \geqslant K(\varepsilon)$ then

$$\psi_i \frac{S_i}{S_{j-1}} \leqslant \frac{2 \log n}{i} \times (1 + \varepsilon) \left(\frac{i}{j-1} \right)^\chi \leqslant \frac{4 \log n}{\sqrt{ij}}, \tag{3.17}$$

since $i < j$ and $\chi \geqslant 1/2$. If $i \leqslant K(\varepsilon) < j$ then using $\psi_i \leqslant 1$ and the fact that the sequence (S_i) is increasing, we obtain

$$\psi_i \frac{S_i}{S_{j-1}} \leqslant \frac{S_{K(\varepsilon)}}{S_{j-1}} \leqslant (1+\varepsilon)\left(\frac{K(\varepsilon)}{j-1}\right)^{\varkappa} \leqslant \frac{4\log n}{\sqrt{ij}},$$

for all n large enough. In any case, (3.16) holds. It follows from (3.15) and (3.16) that (3.14) holds for $k = 1$.

Assume now that (3.14) is true for some k, and let us prove it for $k+1$. By using the induction hypothesis and (ii), we get that

$$g_{k+1}(i,j) \leqslant \sum_{s \neq i,j} g_1(i,s) g_k(s,j) \leqslant \sum_{s \neq i,j} \frac{C \log n}{\sqrt{is}} \frac{(C\log n)^{2k-1}}{\sqrt{sj}} \leqslant \frac{(C\log n)^{2k+1}}{\sqrt{ij}},$$

which proves the induction step. Now it follows from (i) and (3.14) that

$$\begin{aligned} \mathbb{P}(d(w_1, w_2) \leqslant K \mid (\varphi_t), \mathcal{I}_\varepsilon) &\leqslant \frac{1}{n^2} \sum_{k=1}^{K} \sum_{1 \leqslant i,j \leqslant n} g_k(i,j) \\ &\leqslant \frac{1}{n^2} \sum_{k=1}^{K} \sum_{1 \leqslant i,j \leqslant n} \frac{(C\log n)^{2k-1}}{\sqrt{ij}} \leqslant \frac{(C\log n)^{2K}}{n}. \end{aligned}$$

Therefore if $K = \log n/(3C \log\log n)$, then $\mathbb{P}(d(w_1, w_2 \mid (\varphi_t), \mathcal{I}_\varepsilon) \leqslant K) = o(1)$. Combining this with (3.13) gives the desired lower bound with $b_2 = 1/(3C)$. □

3.2.3 Contact process on star graphs

As in Chapter 2, we say that a vertex v is **lit** (the term is taken from [23]) at some time t if the proportion of its infected neighbors at time t is larger than $\lambda/(16e)$.

Lemma 3.2.7. *Let (ξ_t) be the contact process on a star graph S with center v. There exists a positive constant c^*, such that the following assertions hold.*

(i) $\mathbb{P}\,(v$ *is lit at time* $1 \mid \xi_0(v) = 1) \geqslant c^*(1 - \exp(-c^*\lambda|S|))$.

(ii) $\mathbb{P}\,(\exists t > 0 : v$ *is lit at time* $t \mid \xi_0(v) = 1) \to 1 \quad$ *as* $|S| \to \infty$.

(iii) If $\lambda^2|S| \geqslant 64e^2$, *and v is lit at time 0, then v is lit during the time interval* $[\exp(c^*\lambda^2|S|), 2\exp(c^*\lambda^2|S|)]$ *with probability larger than* $1 - 2\exp(-c^*\lambda^2|S|)$.

Proof. Parts (i) and (ii) are exactly Lemma 3.1 (i), (iii) in [65]. For (iii) we need an additional definition: a vertex v is said to be *hot* at some time t if the proportion of its infected neighbors at time t is larger than $\lambda/(8e)$. Then in [23] the authors proved (with different constants in the definition of lit and hot vertices, but this does not effect the proof) that

- if v is lit at some time t, then it becomes hot before $t + \exp(c^*\lambda^2 \deg(v))$,
- if v is hot at some time t, then it remains lit until $t + 2\exp(c^*\lambda^2 \deg(v))$,

with probability larger than $1 - \exp(-c^*\lambda^2 \deg(v))$. Now (iii) follows from these claims. □

The following result is Lemma 3.2 in [65].

Lemma 3.2.8. *Let us consider the contact process on a graph $G = (V, E)$. There exist positive constants c^* and λ_0, such that if $0 < \lambda < \lambda_0$, the following holds. Let v and w be two vertices satisfying $\deg(v) \geqslant \frac{7}{c^*}\frac{1}{\lambda^2}\log\left(\frac{1}{\lambda}\right) d(v, w)$. Assume that v is lit at time 0. Then w is lit before $\exp(c^*\lambda^2 \deg(v))$ with probability larger than $1 - 2\exp(-c^*\lambda^2 \deg(v))$.*

3.3 Proof of Proposition 3.1.3

In this section we study the contact process on the Pólya-point graph (T, o). To prove Proposition 1.2, we have to show that

$$\mathbb{P}(\xi_t^o \neq \varnothing \ \forall t \geqslant 0) \lesssim \lambda^{1+\frac{2}{\psi}} |\log \lambda|^{\frac{-1}{\psi}}, \tag{3.18}$$

and

$$\mathbb{P}(\xi_t^o \neq \varnothing \ \forall t \geqslant 0) \gtrsim \lambda^{1+\frac{2}{\psi}} |\log \lambda|^{\frac{-1}{\psi}}. \tag{3.19}$$

The proof of (3.18) is based on the proof of the upper bound in Proposition 1.4 in [65] for the case of the contact process on Galton-Watson trees, and we put it in Appendix.

3.3.1 Proof of (3.19)

In this part, we first estimate the probability that there is an infinite sequence of vertices, including a neighbor of the root, with larger and larger degree and a small enough distance between any two consecutive elements of the sequence. We then repeatedly apply Lemma 3.2.7 and 3.2.8 to bound from below the probability that the virus propagates along these vertices, and like this survives forever. To this end, we denote by

$$\varphi(\lambda) = \frac{7}{c^*}\frac{1}{\lambda^2}\log\left(\frac{1}{\lambda}\right), \tag{3.20}$$

with c^* as in Lemma 3.2.7 and 3.2.8.

We denote by $w_0 = (0)$, and $x_0 = x_{w_0}$. For any $i \geqslant 1$, let

$$w_i = (0, 1, ..., 1) \text{ and } x_i = x_{w_i}.$$

Then w_i's degree conditioned on x_i is $m+1$ plus a Poisson random variable with parameter

$$\frac{\gamma}{x_i^\psi}\int_{x_i}^{1} \psi x^{\psi-1}dx = \gamma\frac{1-x_i^\psi}{x_i^\psi},$$

where γ is a Gamma random variable with parameters $a = m + 2mr + 1$ and 1. Therefore letting $\kappa = (1 - x_i^\psi)/x_i^\psi$, we have

$$\begin{aligned}
\mathbb{P}(\deg(w_i) = m+1+k \mid x_i) = \mathbb{E}\left(\frac{e^{-\gamma\kappa}}{k!}(\gamma\kappa)^k \,\middle|\, x_i\right) &= \frac{\kappa^k}{k!}\mathbb{E}(e^{-\gamma\kappa}\gamma^k \mid x_i) \\
= \frac{\kappa^k}{\Gamma(a)k!}\int_0^\infty e^{-(\kappa+1)x}x^{k+a-1}dx &= \frac{\kappa^k}{\Gamma(a)k!(\kappa+1)^{k+a}}\int_0^\infty e^{-y}y^{k+a-1}dy \\
= \frac{\Gamma(k+a)\kappa^k}{\Gamma(a)k!(\kappa+1)^{k+a}} &= \frac{\Gamma(k+a)}{\Gamma(a)k!}(1-x_i^\psi)^k x_i^{a\psi},
\end{aligned} \tag{3.21}$$

where $\Gamma(b) = \int_0^\infty x^{b-1}e^{-x}dx$.

Lemma 3.3.1. *There is a positive constant c, such that all $\lambda > 0$,*

$$\mathbb{P}(\mathcal{N}) \geqslant c\varphi(\lambda)^{-1/\psi},$$

where

$$\mathcal{N} = \{\exists (j_\ell)_{\ell\geqslant 1} : j_1 = 1, \deg(w_{j_\ell}) \geqslant 2^{\ell+1}\varphi(\lambda)/\psi \geqslant \varphi(\lambda)d(w_{j_\ell}, w_{j_{\ell+1}})\ \forall \ell \geqslant 1\}.$$

Proof. It follows from Markov's inequality that for any $k \geqslant 1$,

$$\mathbb{P}\left(\prod_{i=1}^{k} U_i > 2^{-(k+1)/2}\right) \leqslant 2^{-(k-1)/2}, \tag{3.22}$$

where (U_i) is a sequence of i.i.d. uniform random variables in $[0,1]$.

Since $\Gamma(k+a)/k! \asymp k^{a-1}$, there is a positive constant C, such that for all $k \geqslant 1$

$$\frac{\Gamma(k+a)}{\Gamma(a)k!} \leqslant Ck^{a-1}.$$

Then it follows from (3.21) that

$$\mathbb{P}\big(\deg(w_i) \leqslant m+1+(c/x_i^{\psi}) \mid x_i\big) \leqslant \sum_{k=0}^{[c/x_i^{\psi}]} Ck^{a-1}x_i^{a\psi} \leqslant Cc^a, \tag{3.23}$$

for any $c>0$. Set $j_1 = 1$ and $j_\ell = [4\ell/\psi]$ for $\ell \geqslant 2$. Then define

$$\mathcal{N}_\ell = \left\{x_{j_\ell} \leqslant \left(4^\ell \varphi(\lambda)/c\psi\right)^{-1/\psi}, \deg(w_{j_\ell}) \geqslant 2^{\ell+1}\varphi(\lambda)/\psi\right\}$$

for all $\ell \geqslant 1$, where c is a positive constant to be chosen later.

Since $2^{\ell+1}\varphi(\lambda)/\psi \geqslant 4\varphi(\lambda)/\psi \geqslant \varphi(\lambda)d(w_{j_\ell}, w_{j_{\ell+1}})$, we have

$$\mathcal{N} \supset \bigcap_{\ell=1}^{\infty} \mathcal{N}_\ell. \tag{3.24}$$

Since x_{j_ℓ} is distributed as $x_1 \times U_1 \times \ldots \times U_{j_\ell - 1}$, applying (3.22) gives that

$$\mathbb{P}(x_{j_\ell} \leqslant x_1 4^{-\ell/\psi}) \geqslant 1 - 2(4^{-\ell/\psi}).$$

Therefore

$$\mathbb{P}\big[x_{j_\ell} \leqslant (4^\ell \varphi(\lambda)/c\psi)^{-1/\psi} \mid \mathcal{N}_1\big] \geqslant 1 - 2(4^{-\ell/\psi}). \tag{3.25}$$

By using (3.23) with $(2^{1-\ell}c)$ instead of c we obtain that

$$\mathbb{P}\big[\deg(w_{j_\ell}) \geqslant 2^{\ell+1}\varphi(\lambda)/\psi \mid x_{j_\ell} \leqslant (4^\ell \varphi(\lambda)/c\psi)^{-1/\psi}\big] \geqslant 1 - C(2^{1-\ell}c)^a. \tag{3.26}$$

Then it follows from (3.25) and (3.26) that

$$\mathbb{P}\left(\bigcap_{\ell=2}^{\infty} \mathcal{N}_\ell \mid \mathcal{N}_1\right) \geqslant 1 - 2\sum_{\ell=2}^{\infty} 4^{-\ell/\psi} - C\sum_{\ell=2}^{\infty}(2^{1-\ell}c)^a \geqslant 1/4, \tag{3.27}$$

provided c is small enough. We now estimate $\mathbb{P}(\mathcal{N}_1)$. Let

$$\bar{\varphi}(\lambda) = (4\varphi(\lambda)/(c\psi))^{-1/\psi}.$$

Recall that x_1 is uniformly distributed on $[0, x_0]$, with $x_0 \sim U_0^{\chi}$ and $U_0 \sim \mathcal{U}([0,1])$. Therefore

$$\begin{aligned} \mathbb{P}(x_1 \leqslant \bar{\varphi}(\lambda)) = \mathbb{E}\left(\frac{\min\{\bar{\varphi}(\lambda), x_0\}}{x_0}\right) &\geqslant \bar{\varphi}(\lambda)\mathbb{P}(x_0 \geqslant \bar{\varphi}(\lambda)) \\ &= \bar{\varphi}(\lambda)(1 - \bar{\varphi}(\lambda)^{1/\chi}) \geqslant \bar{\varphi}(\lambda)/2, \end{aligned} \tag{3.28}$$

for λ small enough. On the other hand, (3.23) gives that for c small enough

$$\mathbb{P}(\mathcal{N}_1 \mid x_1 \leqslant \bar{\varphi}(\lambda)) \geqslant 1 - Cc^a \geqslant 1/2. \tag{3.29}$$

We thus can choose c such that the two inequalities in (3.27) and (3.29) are satisfied. Now it follows from (3.24), (3.27), (3.28) and (3.29) that

$$\mathbb{P}(\mathcal{N}) \gtrsim \bar{\varphi}(\lambda),$$

which implies the result. □

Proof of (3.19). By repeatedly applying Lemma 3.2.8 to the pairs of vertices $(w_{i_\ell}, w_{i_{\ell+1}})$, we obtain that

$$\begin{aligned} \mathbb{P}(\xi_t \neq \varnothing \, \forall t \geqslant T \mid \mathcal{N}, w_1 \text{ is lit at some time } T) &\geqslant 1 - 2\sum_{\ell=1}^{\infty} \exp(-c^*\lambda^2 2^{\ell+1}\varphi(\lambda)/\psi) \\ &\geqslant 1 - 2\sum_{\ell=1}^{\infty} \exp(-7(2^{\ell+1})|\log\lambda|/\psi) \\ &\geqslant 1/2, \end{aligned} \tag{3.30}$$

for λ small enough. On the other hand, by using Lemma 3.2.7 (i), we have

$$\begin{aligned} &\mathbb{P}(w_1 \text{ is lit at some time } T \mid \mathcal{N}, o \text{ is infected at time } 0) \\ &\geqslant c\lambda\mathbb{E}(1 - \exp(-c^*\lambda\deg(w_1)) \mid \mathcal{N}) \geqslant c\lambda/2, \end{aligned} \tag{3.31}$$

for some $c > 0$ (note that on $\mathcal{N}$, we have $c^*\lambda\deg(w_1) \geqslant 7$). Now it follows from (3.30), (3.31) and Lemma 3.3.1 that

$$\mathbb{P}(\xi_t \neq \varnothing \, \forall t \geqslant 0) \geqslant (1/2) \times (c\lambda/2) \times \mathbb{P}(\mathcal{N}) \gtrsim \lambda^{1+2/\psi}|\log\lambda|^{-1/\psi},$$

which proves (3.19) □

3.4 Proof of Theorem 3.1.1.

By using the self-duality of the contact process (3.3), we see that to prove (3.2), it is sufficient to show that

$$\mathbb{P}\left(\frac{1}{n}\sum_{v\in V_n} 1(\{\xi^v_{t_n} \neq \varnothing\}) \leqslant C\lambda^{1+2/\psi}|\log\lambda|^{-1/\psi}\right) = 1 - o(1), \tag{3.32}$$

and

$$\mathbb{P}\left(\frac{1}{n}\sum_{v\in V_n} 1(\{\xi^v_{t_n} \neq \varnothing\}) \geqslant c\lambda^{1+2/\psi}|\log\lambda|^{-1/\psi}\right) = 1 - o(1), \tag{3.33}$$

for some positive constants c and C. We will prove these two statements in the next two subsections.

3.4.1 Proof of (3.32)

For $r \geqslant 1$, we define

$$\mathcal{L}_T(o,r) = \{(o,0) \leftrightarrow B_T(o,r)^c \times \mathbb{R}_+\},$$

the event that the contact process on the Pólya-point graph starting from the root infects vertices outside $B_T(o,r)$. Then we have

$$\{\xi^o_t \neq \varnothing\, \forall t\} = \cap_{r=1}^{\infty}\mathcal{L}_T(o,r).$$

Hence, it follows from (3.18) that there are positive constants C and $R = R(\lambda)$, such that

$$\mathbb{P}(\mathcal{L}_T(o,R)) \leqslant C\lambda^{1+\frac{2}{\psi}}|\log\lambda|^{\frac{-1}{\psi}}. \tag{3.34}$$

For any $v \in V_n$ and R as in (3.34), we define

$$\mathcal{L}_n(v,R) = \{(v,0) \leftrightarrow B_{G_n}(v,R)^c \times \mathbb{R}_+\}$$

and

$$X_v = 1(\mathcal{L}_n(v,R)).$$

Theorem 3.2.2 yields that

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{L}_n(u,R)) = \mathbb{P}(\mathcal{L}_T(o,R)), \tag{3.35}$$

where u is a uniformly chosen vertex from V_n. By combing this with (3.34) we obtain that

$$\lim_{n\to\infty} \mathbb{P}(X_u = 1) \leqslant C\lambda^{1+2/\psi}|\log\lambda|^{-1/\psi},$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{n}\sum_{v\in V_n} \mathbb{P}(X_v = 1) \leqslant C\lambda^{1+2/\psi}|\log\lambda|^{-1/\psi}. \tag{3.36}$$

Now, let us consider

$$W_n = \{(v, v') \in V_n \times V_n : d(v, v') \geqslant 2R + 3\},$$

with R as in (3.34). Since $R + 1 \leqslant b_2 \log n/(\log\log n)$ for n large enough, Lemma 3.2.6 implies that

$$\sum_{v,v'\in V_n} \mathbb{P}((v, v') \notin W_n) = o(n^2).$$

On the other hand, if $(v, v') \in W_n$ then X_v and $X_{v'}$ are independent. Therefore

$$\sum_{v,v'\in V_n} \mathrm{Cov}(X_v, X_{v'}) = o(n^2). \tag{3.37}$$

Thanks to (3.36) and (3.37) by using Chebyshev's inequality we get that

$$\mathbb{P}\left(\frac{1}{n}\sum_{v\in V_n} X_v \leqslant 2C\lambda^{1+2/\psi}|\log\lambda|^{-1/\psi}\right) = 1 - o(1). \tag{3.38}$$

Since the contact process on a finite ball in the Pólya-point graph a.s. dies out,

$$\lim_{t\to\infty} \mathbb{P}(\mathcal{L}_T(o, R)^c \cap \{\xi_t^o \neq \varnothing\}) = 0.$$

Hence for any $\varepsilon > 0$, there exists t_ε, such that

$$\mathbb{P}(\mathcal{L}_T(o, R)^c \cap \{\xi_{t_\varepsilon}^o \neq \varnothing\}) \leqslant \varepsilon. \tag{3.39}$$

For any $v \in V_n$, define

$$X_{v,\varepsilon} = 1(\mathcal{L}_n(v, R)^c \cap \{\xi_{t_\varepsilon}^v \neq \varnothing\}).$$

Then for n large enough such that $t_n \geqslant t_\varepsilon$, we have

$$1(\{\xi_{t_n}^v \neq \varnothing\}) \leqslant X_v + X_{v,\varepsilon}. \tag{3.40}$$

It follows from Theorem 3.2.2 and (3.39) that

$$\lim_{n\to\infty} \mathbb{E}(X_{u,\varepsilon}) = \mathbb{P}(\mathcal{L}_T(o,R)^c \cap \{\xi^o_{t_\varepsilon} \neq \varnothing\}) \leqslant \varepsilon.$$

By using this and Markov's inequality we get that for n large enough, and for any $\eta > 0$,

$$\mathbb{P}\left(\frac{1}{n}\sum_{v\in V_n} X_{v,\varepsilon} > \eta\right) \leqslant \frac{\sum_{v\in V_n}\mathbb{E}(X_{v,\varepsilon})}{n\eta} = \frac{\mathbb{E}(X_{u,\varepsilon})}{\eta} \leqslant \frac{2\varepsilon}{\eta}. \tag{3.41}$$

By combining (3.38), (3.40) and (3.41), then letting ε tend to 0, we obtain that

$$\mathbb{P}\left(\frac{1}{n}\sum_{v\in V_n} 1(\{\xi^v_{t_n} \neq \varnothing\}) \leqslant 3C\lambda^{1+2/\psi}|\log\lambda|^{-1/\psi}\right) = 1 - o(1),$$

which proves (3.32). □

3.4.2 Proof of (3.33)

This subsection is divided into three parts. In the first one, we will show that w.h.p. there are many vertices with large degree (larger than $\varkappa^* \log n$). By using on the other hand that the diameter of the graph is smaller than $b_1 \log n$, we can deduce that if one of these large degree vertices is infected, then the virus survives w.h.p. for a time $\exp(cn/(\log n)^{1/\psi})$, see Proposition 3.4.2. In the second part, we measure the density of *potential* vertices which are promising for spreading the virus to some of these large degree vertices. In the last part, we estimate the proportion of potential vertices which really send the virus to large degree vertices, getting this way (3.33).

Lower bound on the extinction time

Our aim in this part is to find large degree vertices as mentioned above. We then prove that if one of them is infected, the virus is likely to survive a long time.

Lemma 3.4.1. *Let $\varkappa > 0$ be given. Then there exists a positive constant $\bar{c} = \bar{c}(\varkappa)$, such that $\mathcal{A}_n$ holds w.h.p. with*

$$\mathcal{A}_n = \{G_n \textit{ contains } [\bar{c}n/(\log n)^{1/(1-\chi)}] \textit{ disjoint star graphs of size larger than } \varkappa \log n\}.$$

Proof. Let $\varepsilon \in (0, 1/3)$ be given, and let $K = K(\varepsilon)$ and $\mathcal{E}_\varepsilon$ be as in Lemma 3.2.4. Set $a_n = (M \log n)^{1/(1-\chi)}$, with M to be chosen later. Denote by

$$A = \{v_i : i \in [n/a_n, 2n/a_n] \text{ and } \psi_i \in (\theta/i, \mu/i)\}$$

and

$$\mathcal{J}_0 = \mathcal{E}_\varepsilon \cap \{|A| \geqslant \theta n/a_n\},$$

with θ, μ as in Lemma 3.2.4. Recall that the events $\{\psi_i \in (\theta/i, \mu/i)\}$ are independent and have probability larger than 2θ. Therefore (3.4) implies that w.h.p. $|A| \geqslant \theta n/a_n$. Hence by Lemma 3.2.4 (ii), we have for n large enough

$$\mathbb{P}(\mathcal{J}_0) \geqslant 1 - 2\varepsilon.$$

We now suppose that $\mathcal{J}_0$ happens. Then, the elements of A can be written as $\{v_{j_1}, ..., v_{j_\ell}\}$ with $\ell \in [\theta n/a_n, n/a_n]$. Then define

$$A_1 = \{v_j : n/2 \leqslant j \leqslant n\}.$$

We will show that all vertices in A have a large number of neighbors in A_1. Indeed, it follows from (3.12) and Lemma 3.2.4 (ii) that for $K(\varepsilon) \leqslant j < k$,

$$\mathbb{P}(v_j \sim v_k \mid (\varphi_t), \mathcal{E}_\varepsilon) \asymp \frac{\psi_j S_j}{S_{k-1}} \asymp \psi_j \left(\frac{j}{k-1}\right)^\chi.$$

Hence, for all $v_j \in A$ and $v_k \in A_1$,

$$\mathbb{P}(v_j \sim v_k \mid \mathcal{J}_0) \asymp \frac{j^{\chi-1}}{k^\chi} \asymp \frac{a_n^{1-\chi}}{n}. \tag{3.42}$$

Conditionally on (ψ_j), the events $\{\{v_{j_1} \sim v_k\}\}_{k \in A_1}$ are independent. Hence thanks to (3.4) we get that there are positive constants θ_1, c, and C, such that

$$\mathbb{P}\left(c a_n^{1-\chi} \leqslant \sum_{v_k \in A_1} 1(v_{j_1} \sim v_k) \leqslant C a_n^{1-\chi} \mid \mathcal{E}\right) \geqslant 1 - \exp(-\theta_1 a_n^{1-\chi}),$$

or equivalently

$$\mathbb{P}(\mathcal{J}_1 \mid \mathcal{J}_0) \geqslant 1 - \exp(-\theta_1 a_n^{1-\chi}), \tag{3.43}$$

where

$$\mathcal{J}_1 = \{c a_n^{1-\chi} \leqslant |B_1| \leqslant C a_n^{1-\chi}\}$$

and

$$B_1 = \{v_k \in A_1 : v_{j_1} \sim v_k\}.$$

Note that in this proof, the value of the constant θ_1 may change from line to line. Now let us consider

$$A_2 = A_1 \setminus B_1 \quad \text{and} \quad B_2 = \{v_k \in A_2 : v_{j_2} \sim v_k\}.$$

We notice that on $\mathcal{J}_1 \cap \mathcal{E}$, the cardinality of A_2 is larger than $n/2 - Ca_n^{1-\chi} \geqslant n/4$. Thus, similarly to (3.43), there exist positive constants c_1 and C_1, such that

$$\mathbb{P}(\mathcal{J}_2 \mid \mathcal{J}_1 \cap \mathcal{J}_0) \geqslant 1 - \exp(-\theta_1 a_n^{1-\chi}),$$

where

$$\mathcal{J}_2 = \{c_1 a_n^{1-\chi} \leqslant |B_2| \leqslant C_1 a_n^{1-\chi}\}.$$

Here we can also assume that $c_1 \leqslant c$ and $C_1 \geqslant C$. Likewise for all $2 \leqslant s \leqslant \ell$, define recursively

$$A_s = A_{s-1} \setminus B_{s-1}, \quad B_s = \{v_k \in A_s : v_{j_s} \sim v_k\},$$

$$\mathcal{J}_s = \{c_1 a_n^{1-\chi} \leqslant |B_s| \leqslant C_1 a_n^{1-\chi}\}.$$

On $\overset{s-1}{\underset{i=0}{\cap}} \mathcal{J}_i$, we have $|A_s| \geqslant n/2 - sC_1 a_n^{1-\chi} \geqslant n/4$. Therefore, similarly to (3.43)

$$\mathbb{P}\left(\mathcal{J}_s \mid \overset{s-1}{\underset{i=0}{\cap}} \mathcal{J}_i\right) \geqslant 1 - \exp -(\theta_1 a_n^{1-\chi}).$$

Hence

$$\mathbb{P}\left(\overset{\ell}{\underset{i=1}{\cap}} \mathcal{J}_i \mid \mathcal{J}_0\right) \geqslant 1 - n\exp(-\theta_1 a_n^{1-\chi})/a_n.$$

Taking M large enough such that $c_1 a_n^{1-\chi} \geqslant \varkappa \log n$ and $n\exp(-\theta_1 a_n^{1-\chi}) \leqslant 1$ yields that

$$\mathbb{P}(|B_s| \geqslant \varkappa \log n \;\forall\, 1 \leqslant s \leqslant \ell \mid \mathcal{J}_0) \geqslant 1 - a_n^{-1}. \tag{3.44}$$

Moreover, by definition $B_s \cap B_t = \varnothing$ for all $s \neq t$. Hence, all vertices in A have more than $\varkappa \log n$ distinct neighbors. Finally, take $\bar{c}$ such that $\bar{c}n/(\log n)^{1/(1-\chi)} \leqslant \theta n/a_n$, for instance $\bar{c} \leqslant \theta M^{-1/(1-\chi)}$. In conclusion, we have shown that for any given $\varepsilon \in (0, 1/3)$,

$$\mathbb{P}(\mathcal{A}_n) \geqslant 1 - 2\varepsilon - a_n^{-1} \geqslant 1 - 3\varepsilon,$$

for n large enough. Since this holds for any $\varepsilon > 0$, the result follows. □

To determine the constant $\varkappa$ in the definition of $\mathcal{A}_n$, we first recall that

$$\mathbb{P}(\mathcal{B}_n) = 1 - o(1),$$

where

$$\mathcal{B}_n = \{d(G_n) \leqslant b_1 \log n\}.$$

Hence to apply Lemma 3.2.8 to the large degree vertices exhibited in the previous lemma, we need

$$\varkappa \log n \geqslant \frac{7}{c^*}\frac{1}{\lambda^2}\log\left(\frac{1}{\lambda}\right) b_1 \log n.$$

Moreover, in (3.46), we will use that $\varkappa \geqslant 3/(c^*\lambda^2)$. So we let

$$\varkappa^* = \max\left\{\frac{7}{c^*}\frac{1}{\lambda^2}\log\left(\frac{1}{\lambda}\right) b_1, \frac{3}{c^*\lambda^2}\right\}. \tag{3.45}$$

Then we let $\bar{c}^* = \bar{c}^*(\varkappa^*)$ and $\mathcal{A}_n$ be defined accordingly as in Lemma 3.4.1.

A set of vertices $V = \{w_1, \ldots, w_k\} \subset V_n$ is called **good** if $|S(w_i)\backslash \cup_{j\neq i} S(w_j)| \geqslant \varkappa^* \log n$ for all $1 \leqslant i \leqslant k$, where $S(v)$ denotes the star graph formed by v and its neighbors.

Let V_n^* be a maximal good set i.e. $|V_n^*| = \max\{|V| : V \subset V_n \text{ is good}\}$.

Proposition 3.4.2. *There exists a positive constant c, such that*

$$\mathbb{P}\left(\xi_{T_n} \neq \varnothing \mid \xi_0 \cap V_n^* \neq \varnothing\right) = 1 - o(1),$$

where $T_n = \exp(c\lambda^2 n/(\log n)^{1/\psi})$.

Proof. Thanks to Lemma 3.2.5 and 3.4.1, we can assume that $d(G_n) \leqslant b_1 \log n$ and $|V_n^*| \geqslant \bar{c}^* n/(\log n)^{1/(1-\chi)}$. Assume also that at time 0 a vertex in V_n^*, say v, is infected.

Due to the definiton of V_n^*, for any $w \in V_n^*$, we can select from the set of w's neighbors a subset $D(w)$ of size $\varkappa^* \log n$, such that $D(w) \cap D(w') = \varnothing$ for all $w \neq w'$.
We say that a vertex w in V_n^* is *infested* at some time t if the proportion of infected sites in $D(w)$ at time t is larger than $\lambda/(16e)$ (the term is taken from [64]).

It follows from Lemma 3.2.7 (ii) that v becomes infested with probability tending to 1, as $n \to \infty$. Using Lemma 3.2.7 (iii) and 3.2.8 (note that $|D(w)| \geqslant (7/c^*\lambda^2)|\log \lambda| d(w, w')$), we deduce that for any $t \geqslant 0$ and $w \in V_n^*$,

$$\begin{aligned}&\mathbb{P}(w \text{ is infested at } t + 2\exp(c^*\lambda^2\varkappa^* \log n) \mid v \text{ is infested at } t)\\ &\geqslant 1 - 4\exp(-c^*\lambda^2\varkappa^* \log n).\end{aligned}$$

Therefore

$$
\begin{aligned}
&\mathbb{P}(\text{All vertices in } V_n^* \text{ are infested at } t + 2\exp(c^*\lambda^2\varkappa^* \log n) \mid v \text{ is infested at } t) \\
&\geqslant 1 - 4n\exp(-c^*\lambda^2\varkappa^* \log n) \\
&\geqslant 1 - n^{-1},
\end{aligned} \tag{3.46}
$$

where we have used that $c^*\lambda^2\varkappa^* \geqslant 3$. Now if all vertices in V_n^* are infested at the same time, then the proof of Proposition 1 in [23] shows that the virus survives a time exponential in $\sum_{v\in V_n^*} \deg(v)$. More precisely, let $I_{n,t}$ be the number of infested vertices in V_n^* at time t. Then there is a positive constant η, such that for all $k \leqslant |V_n^*|$,

$$
\mathbb{P}\left(I_{n,s_k} \geqslant k/2 \mid I_{n,0} \geqslant k\right) \geqslant 1 - s_k^{-1},
$$

where $s_k = \exp(\eta\lambda^2 k\varkappa^* \log n)$. The result follows by taking $k = \left[\bar{c}^* n/(\log n)^{1/(1-\chi)}\right]$. $\square$

Density of potential vertices

In this part we will estimate the proportion of *potential* sites from where the virus can be sent with positive probability to a vertex at distance quite small (of order $(\log\log n)^2$) and with large degree (larger than $\varkappa^* \log n$).

This proportion approximates the probability that there is an infection path from the uniformly chosen vertex u to a vertex with degree larger than $\varkappa^* \log n$. To bound from below this probability, we use the same ideas as in Lemma 3.3.1. Indeed, we will find a sequence of vertices starting from a neighbor of u and ending at a large degree vertex, satisfying the hypothesis of Lemma 3.2.8 for spreading the virus from u to the ending vertex (see Lemma 3.4.4).

Here are just some comments on the order of magnitude above. First, if a vertex with degree larger than $\varkappa^* \log n$ is infected, then w.h.p. it will infect a site in V_n^* , and then we can conclude with Proposition 3.4.2. Secondly, $(\log\log n)^2$ is the distance from a potential vertex to a large degree vertex and is much smaller than the typical distance between two different potential vertices. Hence the propagation of the virus from these potential vertices to their closest large degree vertex are approximately independent events.

Set

$$
R_n = \left[(\log\log n)^2\right].
$$

For $w \in V_n$, define $k_0(w)$ by $w = v_{k_0(w)}$, and for $i \geqslant 1$ let $k_i(w)$ be chosen arbitrarily from the indices such that $v_{k_i(w)}$ receives an edge emanating from $v_{k_{i-1}(w)}$. We define also

$$\mathcal{H}_n(w) = \{k_0(w) \geqslant n/\log n\} \cap \{k_{i+1}(w) \geqslant k_i(w)/\log k_i(w) \geqslant n^{1/2} \ \forall\, 0 \leqslant i \leqslant R_n\}.$$

Lemma 3.4.3. *There is a positive constant θ_0, such that for all $\theta \leqslant \theta_0$, for all $\varepsilon \in (0,1/2)$, and for any vertex w, we have*

(i) for all $i \leqslant R_n$

$$\mathbb{P}\left(\max_{v \in B_{G_n}(w,i)} \deg(v) \geqslant \theta e^{\theta i}(n/k_0(w))^{1-\chi} \mid \mathcal{E}_\varepsilon \cap \mathcal{H}_n(w)\right) \geqslant 1 - e^{-\theta i},$$

(ii) $\mathbb{P}\big[\mathcal{H}_n(w) \mid k_0(w) \geqslant n/\log n, \mathcal{E}_\varepsilon\big] = 1 - o(1/\log\log n)$,

with $\mathcal{E}_\varepsilon$ as in Lemma 3.2.4 (ii).

Proof. We first prove (ii). It follows from the construction of G_n and Lemma 3.2.4 (ii) that if $k_i(w)/\log k_i(w) \geqslant K(\varepsilon)$, then

$$\mathbb{P}\Big[k_{i+1}(w) \leqslant k_i(w)/\log k_i(w) \mid \mathcal{E}_\varepsilon, k_i(w), (\varphi_t)\Big] = \frac{S_{[k_i(w)/\log k_i(w)]}}{S_{k_i(w)-1}} \leqslant (1+\varepsilon)\left(\frac{1}{\log k_i(w)}\right)^{\varkappa}.$$

Hence for all $i \leqslant R_n$, we have

$$\begin{aligned}
& \mathbb{P}\Big[k_{i+1}(w) \geqslant n/(\log n)^{i+2} \mid \mathcal{E}_\varepsilon, k_i(w) \geqslant n/(\log n)^{i+1}\Big] \\
\geqslant\ & \mathbb{P}\Big[k_{i+1}(w) \geqslant k_i(w)/\log k_i(w) \mid \mathcal{E}_\varepsilon, k_i(w) \geqslant n/(\log n)^{i+1}\Big] \\
=\ & 1 - o((\log n)^{-\chi/2}).
\end{aligned}$$

Therefore

$$\begin{aligned}
& \mathbb{P}\Big[k_{i+1}(w) \geqslant k_i(w)/\log k_i(w) \geqslant n/(\log n)^{i+2} \ \forall\, i \leqslant R_n \mid \mathcal{E}_\varepsilon, k_0(w) \geqslant n/\log n\Big] \\
=\ & 1 - o(R_n(\log n)^{-\chi/2}) = 1 - o(1).
\end{aligned}$$

This implies (ii), since $n/(\log n)^{i+2} \geqslant \sqrt{n}$ for all $i \leqslant R_n$.

We now prove (i). First, we claim that there is a positive constant c_0, such that for any $c < c_0$, there exists $c' = c'(c) > 0$, such that for all $i \leqslant R_n$

(a) $\mathbb{P}\Big[k_{[i/2]}(w) \leqslant e^{-ci} k_0(w) \mid \mathcal{E}_\varepsilon \cap \mathcal{H}_n(w)\Big] \geqslant 1 - e^{-c'i}$,

(b) $\mathbb{P}\Big[\exists j \in (i/2, i) : \psi_{k_j(w)} \geqslant c/k_j(w) \mid \mathcal{E}_\varepsilon \cap \mathcal{H}_n(w)\Big] \geqslant 1 - e^{-c'i}$,

(c) $\mathbb{P}\Big[\deg(v_k) \geqslant c'(n/k)^{1-\chi} \mid \psi_k \geqslant c/k, \mathcal{E}_\varepsilon\Big] \geqslant 1 - \exp(-c'(n/k)^{1-\chi})$, for any $v_k \in V_n$.

From these claims we can deduce the result. Indeed, (a) and (b) imply that with probability larger than $1 - 2\exp(-c'i)$ there is an integer $j \in (i/2, i)$ such that $k_j(w) \leqslant e^{-ci}k_0(w)$ and $\psi_{k_j(w)} \geqslant c/k_j(w)$. Then (c) gives that $\deg(v_{k_j(w)}) \geqslant c'e^{c(1-\chi)i}(n/k_0(w))^{1-\chi}$ with probability larger than $1 - \exp(-c'e^{c(1-\chi)i})$. Hence (i) follows by taking θ small enough.

To prove (a), similarly to Lemma 3.2.6, we consider

$$X_j(w) = 1(\{k_j(w) \leqslant k_{j-1}(w)/2\}) \text{ and } \mathcal{F}_j(w) = \sigma(k_t(w) : t \leqslant j) \vee \sigma((\varphi_t)).$$

On $\mathcal{H}_n(w)$, we have $K(\varepsilon) \leqslant \sqrt{n} \leqslant k_j(w)$ for all $j \leqslant R_n$. Then by using the same argument as in Lemma 3.2.6 we obtain that on $\mathcal{H}_n(w) \cap \mathcal{E}_\varepsilon$,

$$\mathbb{E}\Big[X_j(w) \mid \mathcal{F}_{j-1}(w)\Big] \geqslant \frac{S_{[k_j(w)/2]} - S_{[k_j(w)/\log k_j(w)]}}{S_{k_j(w)-1}} \geqslant p \tag{3.47}$$

and

$$\mathbb{P}\left(\sum_{j=1}^{[i/2]} X_j(w) \geqslant ip/4\right) \geqslant 1 - 2\exp(-ip^2/16), \tag{3.48}$$

for some constant $p > 0$. Since $k_{[i/2]}(w) \leqslant 2^{-ip/4}k_0(w)$ as soon as $\sum_{j=1}^{[i/2]} X_j(w) \geqslant ip/4$, the claim (a) follows from (3.48).

We now prove (b). Let θ be the constant as in Lemma 3.2.4 (v). Fix some $j \in (i/2, i)$ and set $k = k_j(w) - 1$ and $\ell = [k_j(w)/\log k_j(w)]$. Then we have

$$\begin{aligned}\mathbb{P}\Big[\psi_{k_{j+1}(w)} \geqslant \theta/k_{j+1}(w) \mid k, \ell\Big] &= \mathbb{E}\left(\frac{1}{S_k - S_\ell}\sum_{t=\ell+1}^{k} \varphi_t 1(\psi_t \geqslant \theta/t) \mid k, \ell\right) \\ &\geqslant \mathbb{E}\left(\frac{1}{S_k}\sum_{t=\ell+1}^{k} \varphi_t 1(\psi_t \geqslant \theta/t) \mid k, \ell\right)\end{aligned} \tag{3.49}$$

On $\mathcal{H}_n(w) \cap \mathcal{E}_\varepsilon$, we have $k \geqslant k_{j+1}(w) \geqslant \ell \geqslant \sqrt{n} \geqslant K(\varepsilon)$. Hence, using Lemma 3.2.4 (ii) and the fact that $S_n = 1$, we get on $\mathcal{H}_n(w) \cap \mathcal{E}_\varepsilon$,

$$|S_k - (k/n)^\chi| = |S_k/S_n - (k/n)^\chi| \leqslant \varepsilon(k/n)^\chi \quad \text{and} \quad |S_\ell - (\ell/n)^\chi| \leqslant \varepsilon(\ell/n)^\chi. \tag{3.50}$$

Moreover, by Lemma 3.2.4 (v),

$$\mathbb{E}[\varphi_t 1(\psi_t \geqslant \theta/t)] \geqslant \theta. \tag{3.51}$$

Now using (3.50) and (3.51), we obtain on $\mathcal{H}_n(w) \cap \mathcal{E}_\varepsilon$

$$
\begin{aligned}
\mathbb{E}\left(\frac{1}{S_k}\sum_{t=\ell+1}^{k}\varphi_t 1(\psi_t \geqslant \theta/t) \mid k,\ell\right) &\geqslant \frac{n^\chi}{(1+\varepsilon)k^\chi}\mathbb{E}\left(\theta\sum_{t=\ell+1}^{k}\varphi_t \mid k,\ell\right) \\
&\geqslant \frac{\theta n^\chi}{(1+\varepsilon)k^\chi}\mathbb{E}\left(S_k - S_\ell \mid k,\ell\right) \\
&\geqslant \frac{\theta n^\chi}{(1+\varepsilon)k^\chi}\left[(1-\varepsilon)(k/n)^\chi - (1+\varepsilon)(\ell/n)^\chi\right] \\
&\geqslant \theta/4, \qquad (3.52)
\end{aligned}
$$

since $\varepsilon \in (0,1/2)$ and $\ell = [(k+1)/\log(k+1)]$. It follows from (3.49) and (3.52) that

$$
\mathbb{P}\left(\psi_{k_{j+1}(w)} \geqslant \theta/k_{j+1}(w) \mid k_j(w), \mathcal{H}_n(w)\cap\mathcal{E}_\varepsilon\right) \geqslant \theta/4. \qquad (3.53)
$$

Now it follows from (3.53) that

$$
\mathbb{P}\left(\nexists j \in (i/2, i) : \psi_{k_{j+1}(w)} \geqslant \theta/k_{j+1}(w) \mid \mathcal{H}_n(w)\cap\mathcal{E}_\varepsilon\right) \leqslant (1-\theta/4)^{[i/2]},
$$

which implies (b). Finally, (c) can be proved similarly as for (3.43). □

Lemma 3.4.4. *Let u be a uniformly chosen vertex from V_n. Then*

$$
\mathbb{P}(\mathcal{M}) \gtrsim \lambda\varphi(\lambda)^{-1/\psi},
$$

where

$$
\mathcal{M} = \{\exists w \in B_{G_n}(u, R_n) : \deg(w) \geqslant \varkappa^* \log n\} \cap \{(\xi_t^u) \text{ makes } w \text{ lit inside } B_{G_n}(u, R_n)\}.
$$

Proof. Define k_0 by $v_{k_0} = u$ and for $i \geqslant 1$ let k_i be chosen arbitrarily from the indices that v_{k_i} receives an edge emanating from $v_{k_{i-1}}$. Let us denote $u_1 = v_{k_1}$ and define also

$$
\mathcal{H}_n := \mathcal{H}_n(u_1) = \{k_1 \geqslant n/\log n\} \cap \{k_{i+1} \geqslant k_i/\log k_i \geqslant \sqrt{n}\ \forall 1 \leqslant i \leqslant R_n + 1\}.
$$

In this proof, we assume that $\varepsilon = o(\lambda\varphi(\lambda)^{-1/\psi})$ as $\lambda \to 0$. Similarly to Lemma 3.4.3 by using that k_0 is chosen uniformly from $\{1, \ldots, n\}$, we have $\mathbb{P}(\mathcal{H}_n \mid \mathcal{E}_\varepsilon) = 1 - o(1/\log\log n)$ and hence $\mathbb{P}(\mathcal{E}_\varepsilon \cap \mathcal{H}_n) = 1 - o(\lambda\varphi(\lambda)^{-1/\psi})$. We assume now that these two events happen.

We recall the claim (c) in the proof of Lemma 3.4.3: there is a positive constant c_0, such that for any $c < c_0$, there exists $c' = c'(c) > 0$, such that

$$
\mathbb{P}(\deg(v_k) \geqslant c'(n/k)^{1-\chi} \mid \psi_k \geqslant c/k) \geqslant 1 - \exp(-c'(n/k)^{1-\chi}), \qquad (3.54)
$$

for any $v_k \in V_n$. Let us consider

$$\mathcal{M}_1 = \big\{k_1 \leqslant n/\tilde{\varphi}(\lambda)\big\},$$

where

$$\tilde{\varphi}(\lambda) = (4\varphi(\lambda)/c'\theta^2)^{1/1-\chi},$$

with θ a small enough constant (smaller than θ_0 as in Lemma 3.2.4 and 3.4.3 and smaller than c_0), and $c' = c'(\theta)$. Define

$$\mathcal{M}_2 = \mathcal{M}_1 \cap \big\{\forall\, 1 \leqslant \ell \leqslant R'_n \ \ \exists w_\ell : d(u_1, w_\ell) \leqslant r_\ell, \deg(w_\ell) \geqslant \varphi(\lambda)\exp(\theta r_\ell)\big\},$$

where

$$r_\ell = [4\ell/\theta^2] \quad \text{for} \quad 1 \leqslant \ell \leqslant R'_n := [\theta^2 R_n/8].$$

By using Lemma 3.4.3 for u_1 we get that for any $\ell \leqslant R'_n$

$$\mathbb{P}\left(\max_{v \in B_{G_n}(u_1, r_\ell)} \deg(v) \geqslant \theta e^{\theta r_\ell}(n/k_1)^{1-\chi}\right) \geqslant 1 - e^{-\theta r_\ell}.$$

If $k_1 \leqslant n/\tilde{\varphi}(\lambda)$, then

$$\theta\exp(\theta r_\ell)(n/k_1)^{1-\chi} \geqslant \theta\exp(\theta r_\ell)\tilde{\varphi}(\lambda)^{1-\chi} \geqslant \varphi(\lambda)\exp(\theta r_\ell).$$

Thus

$$\mathbb{P}\left(\exists v : d(v, u_1) \leqslant r_\ell, \deg(v) \geqslant \varphi(\lambda)\exp(\theta r_\ell) \mid \mathcal{M}_1\right) \geqslant 1 - e^{-4\ell/\theta}.$$

Hence

$$\mathbb{P}\left(\mathcal{M}_2 \mid \mathcal{M}_1\right) \geqslant 1 - \sum_{\ell=1}^{R'_n} \exp(-4\ell/\theta) \geqslant 1 - 2\exp(-4/\theta). \tag{3.55}$$

Define

$$\mathcal{M}_3 = \mathcal{M}_1 \cap \{\deg(u_1) \geqslant 4\varphi(\lambda)/\theta^2\}.$$

Similarly to (3.52), we can show that

$$\mathbb{P}\left(\psi_{k_1} \geqslant \theta/k_1 \mid k_1 \leqslant n/\tilde{\varphi}(\lambda)\right) \geqslant \theta/4.$$

It follows from (3.54) and the fact that $c'\tilde{\varphi}(\lambda)^{1-\chi} = 4\varphi(\lambda)/\theta^2$, that

$$\mathbb{P}\left(\deg(u_1) \geqslant c'(n/k_1)^{1-\chi} \mid k_1 \leqslant n/\tilde{\varphi}(\lambda), \psi_{k_1} \geqslant \theta/k_1\right) \geqslant 1 - \exp(-4\varphi(\lambda)/\theta^2) \geqslant 1/2.$$

From the last two inequalities we deduce that

$$\mathbb{P}(\mathcal{M}_3 \mid \mathcal{M}_1) \geqslant \theta/8.$$

Combining this with (3.55) we obtain that

$$\mathbb{P}(\mathcal{M}_2 \cap \mathcal{M}_3 \mid \mathcal{M}_1) \geqslant \theta/8 - 2\exp(-4/\theta) \geqslant \theta/16. \tag{3.56}$$

We now bound from below $\mathbb{P}(\mathcal{M}_1)$. Observe that

$$\mathbb{P}\left(k_1 \leqslant n/\tilde{\varphi}(\lambda) \,\middle|\, k_0, (\varphi_j)\right) \geqslant \frac{S_{[n/\tilde{\varphi}(\lambda)]} 1(k_0 > n/\tilde{\varphi}(\lambda))}{S_{k_0 - 1}} \gtrsim \tilde{\varphi}(\lambda)^{-\chi} \left(\frac{k_0}{n}\right)^{\chi} 1(k_0 > n/\tilde{\varphi}(\lambda)).$$

Since k_0 is distributed uniformly on $\{1, \dots, n\}$, we get

$$\mathbb{E}[(k_0/n)^{\chi} 1(k_0 > n/\tilde{\varphi}(\lambda))] \asymp 1.$$

Therefore

$$\mathbb{P}\left(\mathcal{M}_1\right) \gtrsim \tilde{\varphi}(\lambda)^{-\chi} \gtrsim \varphi(\lambda)^{-1/\psi}.$$

This and (3.56) give that

$$\mathbb{P}\left(\mathcal{M}_2 \cap \mathcal{M}_3\right) \gtrsim \varphi(\lambda)^{-1/\psi}. \tag{3.57}$$

Observe that on $\mathcal{M}_2 \cap \mathcal{M}_3$, we have $\deg(u_1) \geqslant \varphi(\lambda) r_1 \geqslant \varphi(\lambda) d(u_1, w_1)$ and

$$\deg(w_\ell) \geqslant \varphi(\lambda) \exp(\theta r_\ell) \geqslant 2\varphi(\lambda) r_{\ell+1} \geqslant \varphi(\lambda) d(w_\ell, w_{\ell+1})$$

for all $1 \leqslant \ell \leqslant R'_n$. In other words, u_1 and the vertices (w_ℓ) satisfy the condition in Lemma 3.2.8, and thus applying this lemma inductively yields that

$$\begin{aligned} & \mathbb{P}(w_{R'_n} \text{ is lit inside } B_{G_n}(u_1, R_n) \mid \mathcal{M}_2 \cap \mathcal{M}_3, u_1 \text{ is lit}) \\ \geqslant \;& 1 - \sum_{\ell=1}^{R'_n} \exp(-c^* \lambda^2 \varphi(\lambda) e^{\theta r_\ell}) \gtrsim 1. \end{aligned} \tag{3.58}$$

Similarly to (3.31), the probability that (ξ_t^u) makes u_1 lit is of order λ. It follows from this and (3.58) that

$$\mathbb{P}((\xi_t^u) \text{ makes } w_{R'_n} \text{ lit inside } B_{G_n}(u, R_n) \mid \mathcal{M}_2 \cap \mathcal{M}_3) \gtrsim \lambda. \tag{3.59}$$

In addition, $\deg(w_{R'_n}) \geqslant \varkappa^* \log n$. Therefore

$$\mathcal{M} \supset \mathcal{M}_2 \cap \mathcal{M}_3 \cap \{(\xi_t^u) \text{ makes } w_{R'_n} \text{ lit inside } B_{G_n}(u, R_n)\}.$$

Combining this with (3.57) and (3.59) gives the result. $\square$

Proof of (3.33)

For any $v \in V_n$, we define

$$Y_v = 1(\{\exists w \in B_{G_n}(v, R_n) : \deg(w) \geqslant \varkappa^* \log n\} \cap \{(\xi_t^v) \text{ makes } w \text{ lit inside } B_{G_n}(v, R_n)\})$$

and

$$Z_v = Y_v 1(\{\xi_{T_n}^v \neq \varnothing\}),$$

where T_n is as in Proposition 3.4.2. Then

$$\sum_{v \in V_n} Z_v \leqslant \sum_{v \in V_n} 1(\{\xi_{T_n}^v \neq \varnothing\}). \tag{3.60}$$

Lemma 3.4.5. *The following assertions hold.*

(i) $\mathbb{P}\left(\frac{1}{n} \sum_{v \in V_n} Y_v \geqslant c\lambda\varphi(\lambda)^{-1/\psi}\right) = 1 - o(1)$, *for some* $c > 0$, *independent of* λ.

(ii) $\mathbb{P}(Z_v = 1 \mid Y_v = 1) \to 1$, *as* $n \to \infty$ *uniformly in* $v \in V_n$.

Proof. For (i), let $\varepsilon \in (0, 1/2)$ be given. We have to show that the probability in the left-hand side is larger than $1 - 2\varepsilon$ for n large enough. First, Lemma 3.4.4 implies that

$$\mathbb{P}(Y_u = 1) \gtrsim \lambda\varphi(\lambda)^{-1/\psi},$$

with u the uniformly chosen vertex, or equivalently

$$\frac{1}{n} \sum_{v \in V_n} \mathbb{P}(Y_v = 1) \gtrsim \lambda\varphi(\lambda)^{-1/\psi}.$$

Using Chebyshev's inequality, Part (i) follows from this and the following claim: on $\mathcal{E}_\varepsilon$

$$\sum_{v, v' \in V_n} \operatorname{Cov}(Y_v, Y_{v'}) = o(n^2). \tag{3.61}$$

To prove (3.61), we consider

$$\mathcal{V}_n = \{(v_i, v_j) : i, j \geqslant n/\log n, d(v_i, v_j) \geqslant 2R_n + 3\}.$$

Since $R_n + 1 \leqslant b_2 \log n/(\log\log n)$ for n large enough, it follows from Lemma 3.2.6 that

$$\sum_{v, v' \in V_n} \mathbb{P}((v, v') \notin \mathcal{V}_n) = o(n^2). \tag{3.62}$$

On the other hand, Lemma 3.4.3 gives that if $i \geqslant n/\log n$, then on $\mathcal{E}_\varepsilon$

$$\mathbb{P}\left(\exists w \in B_{G_n}(v_i, R_n) : \deg(w) \geqslant \varkappa^* \log n\right) = 1 - o(1/\log\log n).$$

Moreover, given the graph G_n, Y_v and $Y_{v'}$ only depend on the Poisson processes defined on the vertices and edges on the balls $B_{G_n}(v, R_n)$ and $B_{G_n}(v', R_n)$ respectively. Hence on $\mathcal{E}_\varepsilon$ for all $(v, v') \in \mathcal{V}_n$,

$$\operatorname{Cov}(Y_v, Y_{v'}) = o(1/\log\log n). \tag{3.63}$$

Now (3.61) follows from (3.62) and (3.63).

We now prove (ii). If $Y_v = 1$, then there exists a vertex w such that $\deg(w) \geqslant \varkappa^* \log n$ and w is lit at some time. Besides, on $\mathcal{B}_n$ the diameter of the graph is bounded by $b_1 \log n$ w.h.p. Hence similarly to Lemma 3.2.8, we can show that on $\mathcal{B}_n$

$$\mathbb{P}(w \text{ infects a vertex in } V_n^* \,) \geqslant 1 - \exp(-c^* \varkappa^* \lambda^2 \log n).$$

If one of the vertices in V_n^* is infected, it follows from Proposition 3.4.2 that w.h.p. the virus survives up to time T_n. Hence we obtain (ii) by using that $\mathcal{B}_n$ holds w.h.p. □

It follows from (3.60), Lemma 3.4.5 and Markov's inequality that w.h.p.

$$\frac{1}{n} \sum_{v \in V_n} 1(\{\xi^v_{T_n} \neq \varnothing\}) \geqslant \frac{1}{n} \sum_{v \in V_n} Z_v \gtrsim \lambda \varphi(\lambda)^{-1/\psi},$$

which proves (3.33). □

3.5 Proof of Proposition 3.1.2

By applying Proposition 2.1.2, we observe that to prove the convergence in law of $\tau_n/\mathbb{E}(\tau_n)$, it suffices to show that

$$\frac{D_{n,\max}}{d(G_n) \vee \log n} \to \infty, \tag{3.64}$$

with $D_{n,\max}$ the maximum degree and $d(G_n)$ the diameter of G_n. By Lemma 3.2.5, w.h.p.

$$d(G_n) = \mathcal{O}(\log n). \tag{3.65}$$

Using the same argument for (3.43), we have for any sequence (a_n) tending to infinity

$$\mathbb{P}\Big(\exists \ell \in [cn/a_n, n/a_n] : \deg(v_\ell) \geqslant c a_n^{1-\chi}\Big) = 1 - o(1),$$

for some $c > 0$. By taking $a_n = \sqrt{n}$, we obtain that

$$\mathbb{P}(D_{n,\max} \geqslant cn^{(1-\chi)/2}) = 1 - o(1). \tag{3.66}$$

Therefore (3.64) follows from (3.65) and (3.66) and Proposition 2.1.2 (i) has been proved.

The exponential upper bound of τ_n follows from the sparsity of the preferential attachment graph, by applying Lemma 5.3.1.

3.6 Appendix: Proof of (3.18)

We closely follow the proof given in [65] for Galton-Watson trees. However, we have to take care that in our situation the degrees of the vertices are not independent as in Galton-Watson trees. This leads to some complications.

To simplify the computation, we consider a modified version of the Pólya-point graph defined as follows: we use the same construction except that now $m_v = m$ and $\gamma_v \sim F'$ for all vertices. Then, the new tree stochastically dominates the original tree (note that $F \preceq F'$) and thus, it is sufficient to prove the upper bound for the contact process on this new tree. In this appendix, we always consider this modified graph, and for simplicity, we use the same notation as for the Pólya-point graph.

To describe more precisely the distribution of the Pólya-point graph, we recall a basic fact of Poisson processes (see for example [45, Chapter 2]).

Claim. For any $a < b$, let $\Lambda(a, b)$ be the set of arrivals of the Poisson process on $[a, b]$ with intensity $f(x)$. Then conditional on $|\Lambda(a, b)| = k$, these k arrivals are independently distributed on $[a, b]$ with the same density $f(x)/\int_a^b f(t)dt$.

For any vertex v, conditioned on its position x_v and its number of descendants $m+k$, $x_{(v,1)}, \ldots, x_{(v,m)}$ are i.i.d. uniform random variables on $[0, x_v]$, and $x_{(v,m+1)}, \ldots, x_{(v,m+k)}$ are arrivals of a Poisson process on $[x_v, 1]$ with intensity $\gamma_v \frac{\psi x^{\psi-1}}{1-x_v^\psi} dx$ (conditional on having k arrivals), and thus $x_{(v,m+1)}, \ldots, x_{(v,m+k)}$ are distributed on $[x_v, 1]$ with density $\frac{\psi x^{\psi-1}}{1-x_v^\psi} dx$.

On the other hand, similarly to (3.21), for all v with $a = m + 2mr + 1$,

$$\begin{aligned} p(k,x) &= \mathbb{P}(\text{number of descendants of } v = m+k \mid x_v = x) \\ &= \frac{\Gamma(k+a)}{\Gamma(a)k!}(1-x^{\psi})^k x^{a\psi} \\ &\asymp k^{a-1}(1-x^{\psi})^k x^{a\psi}, \end{aligned} \tag{3.67}$$

since $\Gamma(k+a)/k! \asymp k^{a-1}$. Moreover,

$$\begin{aligned} \sum_{k\leqslant M} k^{a-1}(1-x^{\psi})^k &\asymp \sum_{k=0}^{M\wedge[x^{-\psi}]} k^{a-1} + \int\limits_{M\wedge x^{-\psi}}^{M} \exp(-tx^{\psi})t^{a-1}dt \\ &\asymp (M\wedge x^{-\psi})^a + x^{-a\psi} \int\limits_{Mx^{\psi}\wedge 1}^{Mx^{\psi}} \exp(-u)u^{a-1}du \\ &\asymp (M\wedge x^{-\psi})^a. \end{aligned}$$

Therefore

$$\sum_{k\leqslant M} p(k,x) \asymp (M\wedge x^{-\psi})^a x^{a\psi}. \tag{3.68}$$

Similarly,

$$\sum_{k\leqslant M} kp(k,x) \asymp (M\wedge x^{-\psi})^{a+1} x^{a\psi}, \tag{3.69}$$

$$\sum_{k\leqslant M} k^2 p(k,x) \asymp (M\wedge x^{-\psi})^{a+2} x^{a\psi}, \tag{3.70}$$

$$\sum_{k\geqslant 0} kp(k,x) = \mathcal{O}(x^{-\psi}). \tag{3.71}$$

Let $r > 0$ and $M \in \mathbb{N}$ be given. As in [65], we define for any vertex v the truncated tree starting from v as

$$\begin{aligned} T^v_{r,M} = \{v\} \cup \{&w \text{ descendant of } v : d(v,w) \leqslant r, \deg(y) \leqslant M \\ &\text{for all } y \notin \{v,w\} \text{ in the geodesic from } v \text{ to } w\}, \end{aligned}$$

and for $1 \leqslant i \leqslant r$, set

$$\begin{aligned} T^v_{i,r,M} &= \{w \in T^v_{r,M} : d(v,w) = i\}. \\ S^v_{i,r,M} &= \{w : d(v,w) = i \text{ and } w \text{ is a leaf of } T^v_{r,M}\}. \end{aligned}$$

If $v = o$, we simply write $T_{r,m}$, $T_{i,r,m}$, and $S_{i,r,m}$. By definition, if $v \neq o$, then

$$|T^v_{1,r,M}| = \deg(v) - 1, \tag{3.72}$$

$$|S^v_{1,r,M}| = |\{w : w \text{ is a child of } v \text{ with } \deg(w) > M\}|, \tag{3.73}$$

$$|T^v_{i+1,r,M}| = \sum_{j=1}^{\deg(v)-1} |T^{(v,j)}_{i,r,M}| 1(\deg((v,j)) \leqslant M) \quad \text{for } 1 \leqslant i \leqslant r-1, \tag{3.74}$$

$$|S^v_{i+1,r,M}| = \sum_{j=1}^{\deg(v)-1} |S^{(v,j)}_{i,r,M}| 1(\deg((v,j)) \leqslant M) \quad \text{for } 1 \leqslant i \leqslant r-2, \tag{3.75}$$

$$|S^v_{r,r,M}| = |T^v_{r,r,M}|. \tag{3.76}$$

If $v = o$, we just replace $\deg(o) - 1$ by $\deg(o)$ in these equations.

As in [65] (more precisely, Sections 6.1, 6.2 and 6.3), we will prove the upper bound for the contact process on the modified tree by using the four following lemmas.

Lemma 3.6.1. *There is a positive constant C, such that*

(i) for all $1 \leqslant i \leqslant r$ and $M \geqslant m+1$,

$$\mathbb{E}(|T_{i,r,M}|) \leqslant C^i (\log M)^{i-1},$$

(ii) for all $1 \leqslant i \leqslant r-1$ and $M \geqslant m+1$,

$$\mathbb{E}(|S_{i,r,M}|) \leqslant C^i (\log M)^{i-1} M^{-1/\psi}.$$

In fact, the bound for $\mathbb{E}(|T_{i,r,M}|)$ plays the same role as the estimate (6.2) in [65], and the bound for $\mathbb{E}(|S_{i,r,M}|)$ replaces to (6.1).

Lemma 3.6.2. *For all $r > 0$ and $M \geqslant m+1$, we have*

$$\mathbb{E}(|S_{1,r,M}| 1(|S_{1,r,M}| \geqslant 2) \mid \deg(o) \leqslant M) = \mathcal{O}(M^{-1-1/\psi} \log M).$$

This result is used in the estimate in (6.7) in [65].

Lemma 3.6.3. *For all $r > 0$ and $M' \geqslant M \geqslant m+1$, we have*

(i) $\mathbb{P}(\deg(o) \leqslant M, |S_{1,r,M}| = 1) = \mathcal{O}(M^{-1/\psi})$,

(ii) $\mathbb{P}\Big(\deg(o^*) \geqslant M' \mid \deg(o) \leqslant M, S_{1,r,M} = \{o^*\}\Big) = \mathcal{O}\left((M/M')^{1/\psi}\right).$

The first estimate is used for the event B_4^4 in [65], and the second one is an analogue of the bound for $q[M', \infty)/q[M, \infty)$ in Proposition 6.3 (at the first line of page 27).

For any $r > 0$ and $M' \geqslant M \geqslant m + 1$, we define the conditional probability measures

$$\mathbb{Q}_1(\cdot) = \mathbb{P}(\cdot \mid \deg(o) \leqslant M, |S_{1,r,M}| = 1),$$
$$\mathbb{Q}_2(\cdot) = \mathbb{P}(\cdot \mid \deg(o) \leqslant M, S_{1,r,M} = \{o^*\}, \deg(o^*) = M').$$

We call T^* the tree T rooted at o^*. Similarly as for T, we also define $T^*_{i,r,M}$, $S^*_{i,r,M}$.

Lemma 3.6.4. *There is a positive constant C, such that*

(i) for all $1 \leqslant i \leqslant r$ and $M' \geqslant M \geqslant m + 1$

$$\mathbb{E}_{\mathbb{Q}_1}(|T_{i,r,M}|) \leqslant (C \log M)^i,$$

(ii) for all $1 \leqslant i \leqslant r$ and $M' \geqslant M \geqslant m + 1$

$$\mathbb{E}_{\mathbb{Q}_2}(|T^*_{i,r,M}|) \leqslant C^i (\log M)^{i-1} M',$$

(iii) for all $1 \leqslant i \leqslant r - 1$ and $M \geqslant m + 1$,

$$\mathbb{E}_{\mathbb{Q}_2}(|S^*_{i,r,M}|) \leqslant C^i (\log M)^{i-1} M' M^{-1}.$$

The bound for $\mathbb{E}_{\mathbb{Q}_1}(|T_{i,r,M}|)$ is used in (6.12) in [65], the bounds for $\mathbb{E}_{\mathbb{Q}_2}(|T^*_{i,r,M}|)$ and $\mathbb{E}_{\mathbb{Q}_2}(|S^*_{i,r,M}|)$ are used in their Section 6.3.

Assume that Lemmas 3.6.1–3.6.4 hold for a moment, we now prove the upper bound of the survival probability of the contact process on T.

Proof of (3.18). Using the same notation in [65], we set

$$M = \left\lceil \frac{1}{8\lambda^2} \right\rceil \text{ and } R = \left\lceil \frac{2/\psi + 5}{2/\psi - 1} \right\rceil + 1.$$

and define

$$\begin{aligned}
B_1^4 &= \{\deg(o) > M\}, \\
B_2^4 &= \left\{ \deg(o) \leqslant M, (o,0) \leftrightarrow \left(\bigcup_{i=2}^{R} S_{i,R,M} \right) \times \mathbb{R}_+ \text{ in } T_{R,M} \right\}, \\
B_3^4 &= \{\deg(o) \leqslant M, |S_{1,R,M}| \geqslant 2, (o,0) \leftrightarrow S_{1,R,M} \times \mathbb{R}_+ \text{ in } T_{R,M}\}.
\end{aligned}$$

When $S_{1,R,M} = \{o^*\}$, let $0 < t^* < t^{**}$ be the first two arrival times of the process $D_{(o,o^*)}$, the Poisson point process (with intensity λ) of transmissions from o to o^*. Then, we define

$$B_4^4 = \{\deg(o) \leqslant M, |S_{1,R,M}| = 1, t^{**} < \inf D_o\},$$

where D_o is the Poisson point process (with intensity 1) of recoveries at o.

We say that o^* becomes infected *directly* if $t^* < \inf D_o$. We say that it becomes infected *indirectly* if there are infection paths from o to o^* but all these paths must visit at least one vertex different from o and o^*. Define

$$B_5^4 = \{\deg(o) \leqslant M, |S_{1,R,M}| = 1, \exists\, y \in T_{R,M}, 0 < s < t : (o,0) \leftrightarrow (y,s) \leftrightarrow (o,t) \text{ inside } T_{R,M}\}.$$

Note that if o^* becomes infected indirectly, then B_5^4 must occur. Let us define

$$B_6^4 = \{\deg(o) \leqslant M, S_{1,R,M} = \{o^*\}, t^* < \inf D_o, (o^*, t^*) \leftrightarrow B(o,R)^c \times [t^*, \infty)\}.$$

Then it was shown in [65] that

$$\{(o,0) \leftrightarrow B(o,R)^c \times \mathbb{R}_+\} \subset \bigcup_{i=1}^{6} B_i^4. \tag{3.77}$$

Event B_1^4. We observe that by (3.67)

$$\begin{aligned}
\mathbb{P}(\deg(o) > M \mid x_o = x) &\asymp x^{a\psi} \sum_{k>M} k^{a-1}(1 - x^{\psi})^k \\
&\lesssim x^{a\psi} \int_M^{\infty} t^{a-1} \exp(-t x^{\psi}) dt \\
&\asymp \int_{Mx^{\psi}}^{\infty} t^{a-1} \exp(-t) dt \\
&\lesssim \exp(-M x^{\psi}/2).
\end{aligned}$$

We recall that $x_o \sim \mathcal{U}([0,1])^{\chi}$, with $\chi = 1/(\psi+1)$. Thus x_o is distributed on $[0,1]$ with density $(\psi+1)x^{\psi}dx$. Hence

$$\begin{aligned}
\mathbb{P}(B_1^4) &= \int_0^1 \mathbb{P}(\deg(o) > M \mid x_o = x)(\psi+1)x^{\psi} dx \\
&\lesssim \int_0^1 \exp(-M x^{\psi}/2) x^{\psi} dx \\
&\lesssim M^{-1-1/\psi} = \mathcal{O}(\lambda^{2+2/\psi}).
\end{aligned}$$

Event B_2^4. As in [65], we have

$$\begin{aligned}\mathbb{P}(B_2^4) &\leqslant \sum_{i=2}^{R}(2\lambda)^i\mathbb{E}(|S_{i,R,M}|)\\ &\leqslant M^{-1/\psi}\sum_{i=2}^{R-1}(C\lambda\log M)^i+(C\lambda\log M)^R\\ &\lesssim \lambda^{2/\psi}\sum_{i=2}^{R-1}(\lambda|\log\lambda|)^i+(\lambda|\log\lambda|)^R\\ &\lesssim \lambda^{3/2+2/\psi}.\end{aligned}$$

Here, for the first inequality we have used Lemma 3.6.1 and (3.76).

Event B_3^4. As in [65], we have

$$\begin{aligned}\mathbb{P}(B_3^4) &\leqslant (2\lambda)\mathbb{E}(|S_{1,R,M}|1(|S_{1,R,M}|\geqslant 2)\mid \deg(o)\leqslant M)\\ &= \mathcal{O}(\lambda M^{-1-1/\psi}\log M)\\ &= \mathcal{O}(\lambda^{2+2/\psi}),\end{aligned}$$

where we have used Lemma 3.6.2 in the second line.

Event B_4^4. The number of transmissions from o to o^* before time t has Poisson distribution with parameter λt. Thus,

$$\begin{aligned}\mathbb{P}(B_4^4) &= \mathbb{P}(\deg(o)\leqslant M, |S_{1,R,M}|=1)\int_0^\infty \mathbb{P}(\mathrm{Poi}(\lambda t)\geqslant 2)e^{-t}dt\\ &= \mathcal{O}(\lambda^{2/\psi})\int_0^\infty \lambda^2t^2e^{-t}dt\\ &= \mathcal{O}(\lambda^{2+2/\psi}).\end{aligned}$$

Note that in the second line, we have used Lemma 3.6.3 (i) to estimate the first term and the fact that $\mathbb{P}(\mathrm{Poi}(u)\geqslant 2)\leqslant u^2$ to bound the second one.

Event B_5^4. As in [65],

$$\mathbb{P}(B_5^4)\leqslant \mathbb{P}(\deg(o)\leqslant M, |S_{1,R,M}|=1)\mathbb{P}\left(\begin{array}{c|c}\exists\, y\in T_{R,M}, 0<s<t: & \deg(o)\leqslant M,\\ (o,0)\leftrightarrow(y,s)\leftrightarrow(o,t)\text{ inside }T_{R,M} & |S_{1,R,M}|=1\end{array}\right).$$

By Lemma 3.6.3 (i), the first term is $\mathcal{O}(\lambda^{2/\psi})$. Using the same argument in [65], the second one is bounded by

$$\begin{aligned}
& \sum_{i=1}^{R} \lambda^{2i} \times \mathbb{E}(|\{x \in T_{R,M} : d(o,x) = i\}| \mid \deg(o) \leqslant M, |S_{1,R,M}| = 1) \\
= & \sum_{i=1}^{R} \lambda^{2i} \times \mathbb{E}_{\mathbb{Q}_1}(|T_{i,R,M}|) \\
\leqslant & \sum_{i=1}^{R} (C\lambda^2 |\log \lambda|)^i \\
= & \ \mathcal{O}(\lambda^{3/2}).
\end{aligned}$$

Thus we have

$$\mathbb{P}(B_5^4) = \mathcal{O}(\lambda^{3/2+2/\psi}).$$

Event B_6^4. We have

$$\begin{aligned}
\mathbb{P}(B_6^4) \ \leqslant & \ \mathbb{P}(\deg(o) \leqslant M, |S^1_{R,M}| = 1) \times \mathbb{P}(t^* < \inf D_o) \\
& \times \mathbb{P}((o^*, t^*) \leftrightarrow B(o,R)^c \times [t^*, \infty) \mid \deg(o) \leqslant M, |S^1_{R,M}| = 1, t^* < \inf D_o) \\
\lesssim & \ \lambda^{2/\psi} \times \lambda \times \mathbb{P}((o^*, 0) \leftrightarrow B(o,R)^c \times \mathbb{R}_+ \mid \deg(o) \leqslant M, |S^1_{R,M}| = 1).
\end{aligned} \tag{3.78}$$

Note that we have used Lemma 3.6.3 (i) to bound the first probability. Now, it remains to bound the third term. We observe that for any $M' > M$,

$$\begin{aligned}
& \mathbb{P}((o^*, 0) \leftrightarrow B(o,R)^c \times \mathbb{R}_+ \mid \deg(o) \leqslant M, |S_{1,R,M}| = 1) \\
= & \sum_{k=M+1}^{\infty} \mathbb{P}((o^*, 0) \leftrightarrow B(o,R)^c \times \mathbb{R}_+ \mid \deg(o^*) = k, \deg(o) \leqslant M, S_{1,R,M} = \{o^*\}) \\
& \qquad \times \mathbb{P}(\deg(o^*) = k \mid \deg(o) \leqslant M, S_{1,R,M} = \{o^*\}) \\
\leqslant & \ \mathbb{P}((o^*, 0) \leftrightarrow B(o,R)^c \times \mathbb{R}_+ \mid \deg(o^*) = M', \deg(o) \leqslant M, S_{1,R,M} = \{o^*\}) \\
& + \mathbb{P}(\deg(o^*) \geqslant M' \mid \deg(o) \leqslant M, S_{1,R,M} = \{o^*\}) \\
\lesssim & \ \mathbb{Q}_2((o^*, 0) \leftrightarrow B(o,R)^c \times \mathbb{R}_+) + \mathcal{O}((M/M')^{1/\psi}).
\end{aligned} \tag{3.79}$$

Here, we used Lemma 3.6.3 (ii) to bound the second term and $\mathbb{Q}_2$ is the conditional probability depending on M' which was defined in Lemma 3.6.4.

As in [65], we take

$$M' = \lceil \varepsilon_1 \lambda^{-2} |\log \lambda| \rceil,$$

with

$$\varepsilon_1 = \varepsilon_1'/64 \qquad \text{and} \qquad \varepsilon_1' = \min\{(2/\psi - 1), 2\}/4.$$

Then the second term is of order

$$(M/M')^{1/\psi} \asymp |\log \lambda|^{-1/\psi}. \tag{3.80}$$

To bound the first term, we notice that

$$\mathbb{Q}_2((o^*, 0) \leftrightarrow B(o, R)^c \times \mathbb{R}_+) \leqslant \mathbb{Q}_2((o^*, 0) \leftrightarrow B(o^*, R-1)^c \times \mathbb{R}_+).$$

Lemma 3.6.5. *There exists $\delta > 0$, such that*

$$\mathbb{Q}_2((o^*, 0) \leftrightarrow B(o^*, R-1)^c \times \mathbb{R}_+) < \lambda^\delta.$$

Proof. We follow the proof and notation in [65, Section 6.3], let $R' = R - 1$, and $L_1 = \lceil \lambda^{-\varepsilon_1'/2} \rceil$. Then we define

$$\begin{aligned}
\phi(T^*) &= \sum_{i=1}^{R'} (2\lambda)^i |S^*_{i,R',M}(T)|, \\
\psi(T^*) &= \sum_{i=2}^{R'} (2\lambda)^{2i} |T^*_{i,R',M}|,
\end{aligned}$$

where T^* is the tree T rooted at o^* and $T^*_{i,r,M}, S^*_{i,r,M}$ are defined in Lemma 3.6.4. We now define

$$\begin{aligned}
B_1^5 &= \{\phi(T^*) > \lambda^{\varepsilon_1'}\}, \qquad B_2^5 = \{\psi(T^*) > \lambda^{\varepsilon_1'}\}, \\
B_3^5 &= (B_1^5 \cup B_2^5)^c \cap \left\{ \{o^*\} \times [0, L_1] \leftrightarrow \left(\bigcup_{i=1}^{R'} S^*_{i,R',M} \right) \times \mathbb{R}_+ \right\}, \\
B_4^5 &= (B_1^5 \cup B_2^5)^c \cap \{\exists z : d(o^*, z) \geqslant 2, \{o^*\} \times [0, L_1] \leftrightarrow (z, s) \leftrightarrow \{o^*\} \times [s, \infty) \\
&\qquad\qquad\qquad\qquad \text{inside } T(z) \cap T_{R',M}\}, \\
B_5^5 &= \{B(o^*, 1) \times \{0\} \leftrightarrow B(o^*, 1) \times \{L_1\} \text{ inside } B(o^*, 1)\}.
\end{aligned}$$

It was explained in [65] that

$$\{(o^*, 0) \leftrightarrow B(o^*, R')^c \times \mathbb{R}_+\} \subset \bigcup_{i=1}^5 B_i^5.$$

Event B_1^5. Similarly to B_1^4, using Lemma 3.6.4, we have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_2}(\phi(T^*)) &\lesssim \sum_{i=1}^{R'-1}(2\lambda)^i(\log M)^{i-1}M'M^{-1} + (2\lambda)^{R'}M'M^{-1}(\log M)^{R'} \\ &\lesssim \lambda^{3/4}.\end{aligned}$$

Then using Markov's inequality we get

$$\mathbb{Q}_2(B_1^5) = \mathcal{O}(\lambda^{3/4-\varepsilon_1'}) = \mathcal{O}(\lambda^{1/4}),$$

since $\varepsilon_1' \leqslant 1/2$.

Event B_2^5. We have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_2}(\psi(T^*)) &\lesssim M'\sum_{i=2}^{R'}(2\lambda)^{2i}(\log M)^{(i-1)} \\ &\lesssim \lambda^{3/2}.\end{aligned}$$

Then it follows from Markov's inequality that

$$\mathbb{Q}_2(B_2^5) = \mathcal{O}(\lambda^{3/2-\varepsilon_1'}) = \mathcal{O}(\lambda).$$

The events B_3^5, B_4^5, B_5^5 can be estimated by the same way in [65] to get the bound as in Lemma 3.6.5. $\square$

Conclusion of the proof of (3.18). By (3.78), (3.79), (3.80) and Lemma 3.6.5, we have

$$\mathbb{P}(B_6^4) \lesssim \lambda^{1+2/\psi}(\lambda^{\delta} + |\log\lambda|^{-1/\psi}) = \mathcal{O}(\lambda^{1+2/\psi}|\log\lambda|^{-1/\psi}).$$

Now, it follows from (3.77) and the estimates of events $(B_i^4)_{i\leqslant 6}$ that

$$\mathbb{P}(\xi_t^o \neq \varnothing\, \forall t \geqslant 0) \leqslant \mathbb{P}((o,0) \leftrightarrow B(o,R)^c \times \mathbb{R}_+) \lesssim \mathbb{P}(B_6^4) = \mathcal{O}(\lambda^{1+2/\psi}|\log\lambda|^{-1/\psi}),$$

which proves the desired result. $\square$

We now turn to prove the series of lemmas 3.6.1–3.6.4.

Proof of Lemma 3.6.1. Since we fix r and M, we omit it in the notation. Let us define for $1 \leqslant i \leqslant r$

$$f_i(x) = \mathbb{E}(|T_i^v| 1(\deg(v) \leqslant M) \mid x_v = x),$$

where v is any vertex different from the root o. For $1 \leqslant i \leqslant r-1$,

$$\begin{aligned} f_{i+1}(x) &= \mathbb{E}(|T^v_{i+1}| 1(\deg(v) \leqslant M) \mid x_v = x) \\ &= \sum_{k \leqslant M-m-1} \mathbb{E}(|T^v_{i+1}| 1(\deg(v) = m+1+k) \mid x_v = x) \\ &= \sum_{k \leqslant M-m-1} \mathbb{E}(|T^v_{i+1}| \mid x_v = x, \deg(v) = m+1+k) p(k,x) \\ &= \sum_{k \leqslant M-m-1} \sum_{j=1}^{m+k} \mathbb{E}\Big(|T^{(v,j)}_i| 1(\deg((v,j)) \leqslant M) \mid x_v = x, \deg(v) = m+1+k\Big) p(k,x), \end{aligned}$$

where for the last line, we used (3.74).

On the event $\{x_v = x, \deg(v) = m+1+k\}$, $x_{(v,1)}, \dots, x_{(v,m)}$ are uniformly distributed on $[0,x]$ and $x_{(v,m+1)}, \dots, x_{(v,m+k)}$ are distributed on $[x,1]$ with density $\frac{\psi y^{\psi-1}}{1-x^\psi} dy$. Therefore

$$\begin{aligned} f_{i+1}(x) &= \sum_{k \leqslant M-m-1} \left(\frac{m}{x} \int_0^x f_i(y) dy + \frac{k}{1-x^\psi} \int_x^1 \psi y^{\psi-1} f_i(y) dy \right) p(k,x) \\ &\leqslant \frac{m}{x} \int_0^x f_i(y) dy + \frac{F(M,x)}{1-x^\psi} \int_x^1 \psi y^{\psi-1} f_i(y) dy, \end{aligned} \tag{3.81}$$

where

$$F(M,x) = \sum_{k \leqslant M-m-1} k p(k,x).$$

Moreover, it follows from (3.72) that

$$f_1(x) = \mathbb{E}\Big((\deg(v) - 1) 1(\deg(v) \leqslant M) \mid x_v = x\Big) = m + F(M,x). \tag{3.82}$$

Hence by (3.69),

$$f_1(x) \lesssim F^*(M,x) := (M \wedge x^{-\psi})^{a+1} x^{a\psi} + 1. \tag{3.83}$$

After some simple computations, we have

$$\frac{1}{x} \int_0^x F^*(M,y) dy \lesssim F^*(M,x) \log M. \tag{3.84}$$

and

$$\frac{1}{1-x^\psi} \int_x^1 \psi y^{\psi-1} F^*(M,y) dy \lesssim \log M. \tag{3.85}$$

From (3.81), (3.82), (3.83), (3.84) and (3.85), we can prove by induction that for $1 \leqslant i \leqslant r$,

$$f_i(x) \leqslant C^i (\log M)^{i-1} F^*(M,x). \tag{3.86}$$

for some constant $C > 0$. Similarly to (3.81), we also have

$$\mathbb{E}(|T_i| \mid x_o = x) \leqslant \frac{m}{x} \int_0^x f_{i-1}(y)dy + \sum_{k\geqslant 0} \frac{kp(k,x)}{1-x^\psi} \int_x^1 \psi y^{\psi-1} f_{i-1}(y)dy.$$

It follows from this estimate, (3.71) and (3.86) that

$$\mathbb{E}(|T_i| \mid x_o = x) \leqslant C^i (\log M)^{i-1} (F^*(M,x) + x^{-\psi}).$$

Hence using that $x_o \sim \mathcal{U}([0,1])^\chi$ with $\chi = 1/(\psi+1)$, we get that for $i \leqslant r$

$$\begin{aligned} \mathbb{E}(|T_i|) &= (\psi+1) \int_0^1 \mathbb{E}(|T_i| \mid x_o = x) x^\psi dx \\ &\leqslant C^i (\log M)^{i-1}. \end{aligned}$$

This proves Lemma 3.6.1 (i). We now prove (ii). To estimate $\mathbb{E}(|S_i|)$, we define for $1 \leqslant i \leqslant r-1$

$$g_i(x) = \mathbb{E}(|S_i^v| 1(\deg(v) \leqslant M) \mid x_v = x).$$

As for $f_i(x)$, we also have for $1 \leqslant i \leqslant r-2$,

$$g_{i+1}(x) \leqslant \frac{m}{x} \int_0^x g_i(y)dy + \frac{F(M,x)}{1-x^\psi} \int_x^1 \psi y^{\psi-1} g_i(y)dy, \tag{3.87}$$

and by (3.73),

$$\begin{aligned} g_1(x) &= \mathbb{E}(|\{w : w \text{ is a child of } v \text{ with } \deg(w) > M\}| 1(\deg(v) \leqslant M) \mid x_v = x) \\ &\leqslant \frac{m}{x} \int_0^x g_0(y)dy + \frac{F(M,x)}{1-x^\psi} \int_x^1 \psi y^{\psi-1} g_0(y)dy, \end{aligned} \tag{3.88}$$

where

$$g_0(y) = \mathbb{P}(\deg(w) > M \mid x_w = y).$$

It follows from (3.67) that

$$\begin{aligned}
g_0(y) &\asymp y^{a\psi} \sum_{k>M} k^{a-1}(1-y^{\psi})^k \\
&\lesssim y^{a\psi} \int\limits_M^{\infty} t^{a-1} \exp(-ty^{\psi}) dt \\
&\asymp \int\limits_{My^{\psi}}^{\infty} t^{a-1} \exp(-t) dt \\
&\lesssim \exp(-My^{\psi}/2).
\end{aligned}$$

On the other hand, if $1 > My^{\psi} > 1/2$ then

$$\begin{aligned}
g_0(y) &\gtrsim y^{a\psi} \sum_{k=M+1}^{2M} k^{a-1}(1-y^{\psi})^k \\
&\gtrsim 1.
\end{aligned}$$

Therefore

$$1(1/2 < My^{\psi} < 1) \lesssim g_0(y) \lesssim \exp(-My^{\psi}/2). \tag{3.89}$$

Let us define for $M \geqslant 2$

$$\alpha = p_{L,M}(x) = \mathbb{P}(\deg((v,1)) > M \mid x_v = x) = \frac{1}{x} \int\limits_0^x g_0(y) dy \tag{3.90}$$

$$\beta = p_{R,M}(x) = \mathbb{P}(\deg((v,m+1)) > M \mid x_v = x) = \frac{1}{1-x^{\psi}} \int\limits_x^1 \psi y^{\psi-1} g_0(y) dy. \tag{3.91}$$

Then using (3.89) we obtain that

$$(Mx^{\psi})^{-1/\psi} 1(Mx^{\psi} \geqslant 1) \lesssim \alpha \lesssim 1(Mx^{\psi} < 1) + (Mx^{\psi})^{-1/\psi} 1(Mx^{\psi} \geqslant 1) \tag{3.92}$$

$$\beta \lesssim M^{-1} \exp(-Mx^{\psi}/2). \tag{3.93}$$

Define

$$G^*(M,x) = 1(Mx^{\psi} < 1) + (Mx^{\psi})^{-1/\psi} 1(Mx^{\psi} \geqslant 1).$$

Then

$$\beta F(M,x) = \mathcal{O}(G^*(M,x)).$$

Therefore using this, (3.88), (3.92) and (3.93) we get

$$g_1(x) = \mathcal{O}(G^*(M,x)). \tag{3.94}$$

Furthermore,

$$\frac{1}{x}\int_0^x G^*(M,y)dy = \mathcal{O}((\log M)G^*(M,x)), \tag{3.95}$$

$$\frac{1}{1-x^\psi}\int_x^1 \psi y^{\psi-1}G^*(M,y)dy = \mathcal{O}(M^{-1}1(Mx^\psi < 1) + M^{-1/\psi}1(Mx^\psi \geqslant 1)). \tag{3.96}$$

Hence

$$\frac{m}{x}\int_0^x G^*(M,y)dy + \frac{F(M,x)}{1-x}\int_x^1 G^*(M,y)dy = \mathcal{O}((\log M)G^*(M,x)).$$

From this estimate, (3.87) and (3.94), we can prove by induction that for $1 \leqslant i \leqslant r-1$

$$g_i(x) = \mathcal{O}((\log M)^{i-1}G^*(M,x)). \tag{3.97}$$

We now have

$$\begin{aligned}
\mathbb{E}(|S_i| \mid x_o) &\leqslant \frac{m}{x_o}\int_0^{x_o} g_{i-1}(y)dy + \sum_{k\geqslant 0}\frac{kp(k,x_o)}{1-x_o^\psi}\int_{x_o}^1 \psi y^{\psi-1}g_{i-1}(y)dy. \\
&= \mathcal{O}\left(\frac{1}{x_o}\int_0^{x_o} g_{i-1}(y)dy + \frac{x_o^{-\psi}}{(1-x_o^\psi)}\int_{x_o}^1 \psi y^{\psi-1}g_{i-1}(y)dy\right) \\
&= \mathcal{O}\left((\log M)^{i-1}[G^*(M,x_o) + x_o^{-\psi}(M^{-1}1(Mx^\psi < 1) + M^{-1/\psi}1(Mx^\psi \geqslant 1))]\right).
\end{aligned}$$

Finally,

$$\mathbb{E}[G^*(M,x_o) + x_o^{-\psi}(M^{-1}1(Mx^\psi < 1) + M^{-1/\psi}1(Mx^\psi \geqslant 1))] = \mathcal{O}(M^{-1/\psi}).$$

Then the result follows from the last two estimates. □

Proof of Lemma 3.6.2. We also omit here r and M in the notation. Then

$$\begin{aligned}
&\mathbb{E}(|S_1|1(|S_1| \geqslant 2) \mid \deg(o) \leqslant M, x_o = x) \\
&\leqslant \sum_{k\leqslant M}\mathbb{E}(|S_1|1(|S_1| \geqslant 2) \mid x_o = x, \deg(o) = m+k)p(k,x).
\end{aligned}$$

Conditionally on the event $\{x_o = x, \deg(o) = m + k\}$, $|S_1|$ has the same distribution as

$$X = X_1 + \ldots + X_m + Y_1 + \ldots + Y_k,$$

where (X_i) and (Y_j) are independent Bernoulli random variables with mean $\alpha = p_{L,M}(x)$ and $\beta = p_{R,M}(x)$ respectively, as defined in (3.90) and (3.91). Then

$$\begin{aligned}
\mathbb{E}(X1(X \geqslant 2)) &= \mathbb{E}(X) - \mathbb{P}(X = 1) \\
&= m\alpha + k\beta - m\alpha(1-\alpha)^{m-1}(1-\beta)^k - k\beta(1-\alpha)^m(1-\beta)^{k-1} \\
&\leqslant (m\alpha + k\beta)^2 \\
&\leqslant 2(m^2\alpha^2 + k^2\beta^2).
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbb{E}(|S_1|1(|S_1| \geqslant 2) \mid \deg(o) \leqslant M, x_o = x) &\leqslant \sum_{k \leqslant M} (2m^2\alpha^2 + 2k^2\beta^2) p(k, x) \\
&\leqslant 2m^2\alpha^2 + 2\beta^2 \sum_{k \leqslant M} k^2 p(k, x).
\end{aligned}$$

We now take the expectation with respect to x_o. Since it has density $(\psi + 1)x^\psi dx$ on $[0, 1]$, the expectation of the first term is of order

$$\int_0^1 \alpha^2 x^\psi dx = \mathcal{O}(M^{-1-1/\psi} \log M).$$

By (3.70) and (3.93), the expectation of the second term is equivalent to

$$\int_0^1 \beta^2 (M \wedge x^{-\psi})^{a+2} x^{(a+1)\psi} dx = \mathcal{O}(M^{-1-1/\psi} \log M).$$

Combining the last two estimates gives that

$$\begin{aligned}
\mathbb{E}(|S_1|1(|S_1| \geqslant 2) \mid \deg(o) \leqslant M) &= \int_0^1 \mathbb{E}(|S_1|1(|S_1| \geqslant 2) \mid \deg(o) \leqslant M, x_o = x)(\psi + 1)x^\psi dx \\
&= \mathcal{O}(M^{-1-1/\psi} \log M),
\end{aligned}$$

which proves the lemma. $\square$

Proof of Lemma 3.6.3. With the same α and β as in the previous lemma, we have

$$
\begin{aligned}
& \mathbb{P}(\deg(o) \leqslant M, |S_1| = 1 \mid x_o = x) \\
= & \sum_{k \leqslant M-m} \left[m\alpha(1-\alpha)^{m-1}(1-\beta)^k + k\beta(1-\alpha)^m(1-\beta)^{k-1} \right] p(k,x) \\
\leqslant & \sum_{k \leqslant M} \left[m\alpha + k\beta \right] p(k,x) \\
\lesssim & \ \alpha + \beta(M \wedge x^{-\psi})^{a+1} x^{a\psi}.
\end{aligned}
$$

Then we take expectation in x_o and get

$$
\mathbb{P}(\deg(o) \leqslant M, |S_1| = 1) = \mathcal{O}(M^{-1/\psi}),
$$

which proves Lemma 3.6.3 (i). For (ii), we note that $(1-\beta)^k \asymp 1$ for $1 \leqslant k \leqslant M$ since $\beta = \mathcal{O}(M^{-1})$. Hence using (3.68), we get

$$
\mathbb{P}(\deg(o) \leqslant M, |S_1| = 1 \mid x_o = x) \gtrsim \alpha(1-\alpha)^{m-1}(M \wedge x^{-\psi})^a x^{a\psi}.
$$

By (3.92), there is a positive constant $c < 1$, such that

$$
(1-\alpha)^{m-1} \geqslant c1(Mx^{\psi} \geqslant 1/c). \tag{3.98}
$$

Therefore

$$
\begin{aligned}
\mathbb{P}(\deg(o) \leqslant M, |S_1| = 1) &= (\psi+1) \int_0^1 \mathbb{P}(\deg(o) \leqslant M, |S_1| = 1 \mid x_o = x) x^{\psi} dx \\
&\gtrsim \int_0^1 \alpha(1-\alpha)^{m-1}(M \wedge x^{-\psi})^a x^{a\psi} x^{\psi} dx \\
&\gtrsim \int_{(cM)^{-1/\psi}}^1 (Mx^{\psi})^{-1/\psi} x^{\psi} dx \\
&\gtrsim M^{-1/\psi}.
\end{aligned} \tag{3.99}
$$

On the other hand,

$$
\begin{aligned}
\mathbb{P}(\deg(o) \leqslant M, S_1 = \{o^*\}, \deg(o^*) \geqslant M' \mid x_o = x) &\leqslant \sum_{k \leqslant M-m} [m\alpha' + k\beta'] p(k,x). \\
&\leqslant m\alpha' + \beta' F(M,x),
\end{aligned}
$$

with

$$\alpha' = p_{L,M'}(x) \qquad \text{and} \qquad \beta' = p_{R,M'}(x).$$

Similarly to the calculus for α and β, we get

$$\int_0^1 \alpha' x^{\psi} dx = \mathcal{O}((M')^{-1/\psi}),$$

and

$$\begin{aligned}\int_0^1 \beta' F(M,x) x^{\psi} &\lesssim \frac{1}{M'} \int_0^1 e^{-M'x^{\psi}/2} (M \wedge x^{-\psi})^{a+1} x^{(a+1)\psi} dx \\ &\lesssim (M')^{-1-1/\psi}.\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{P}(\deg(o) \leqslant M, S_1 = \{o^*\}, \deg(o^*) \geqslant M') &= \mathcal{O}\left(\int_0^1 (m\alpha' + \beta' F(M,x)) x^{\psi} dx\right) \\ &= \mathcal{O}\left((M')^{-1/\psi}\right). \qquad (3.100)\end{aligned}$$

Now Lemma 3.6.3 (ii) follows from (3.99) and (3.100). $\square$

Proof of Lemma 3.6.4. We start with (ii), the estimate for $|T^*_{i,r,M}|$. Let us define

$$A = \{\deg(o) \leqslant M\} \quad \text{and} \quad B = \{S_{1,r,M} = \{o^*\}, \deg(o^*) = M'\}.$$

Similarly to the previous lemma,

$$\begin{aligned}&\mathbb{P}(\deg(o) \leqslant M, S_{1,r,M} = \{o^*\}, \deg(o^*) = M' \mid x_o = x) \\ &= \sum_{k \leqslant M-m} \left[m\alpha''(1-\alpha)^{m-1}(1-\beta)^k + k\beta''(1-\alpha)^m(1-\beta)^{k-1}\right] p(k,x) \qquad (3.101) \\ &\gtrsim \alpha''(1-\alpha)^{m-1}(M \wedge x^{-\psi})^a x^{a\psi}, \qquad (3.102)\end{aligned}$$

where

$$\alpha'' := \mathbb{P}(\deg((0,1)) = M' \mid x_o = x) = \frac{1}{x}\int_0^x p(M',y) dy$$

and

$$\beta'' := \mathbb{P}(\deg((0,m+1)) = M' \mid x_o = x) = \frac{1}{1-x^{\psi}} \int_x^1 p(M',y)\psi y^{\psi-1} dy.$$

Similarly to the calculus for α and β, we have

$$\begin{aligned}(M')^{-1-1/\psi}x^{-1}1(M'x^{\psi}\geqslant 1)\lesssim \alpha'' \lesssim & x^{\psi}(M'x^{\psi})^{a-1}1(M'x^{\psi}<1) \qquad (3.103)\\ & +(M')^{-1-1/\psi}x^{-1}1(M'x^{\psi}\geqslant 1),\\ \beta'' \lesssim & (M')^{-2}e^{-M'x^{\psi}/2}. \qquad (3.104)\end{aligned}$$

Hence, it follows from (3.98), (3.102) and (3.103) that

$$\mathbb{P}(A\cap B\mid x_o=x)\gtrsim (M')^{-1-1/\psi}x^{-1}1(Mx^{\psi}\geqslant 1/c).$$

Therefore

$$\mathbb{P}(A\cap B)\gtrsim (M')^{-1-1/\psi}. \qquad (3.105)$$

We now prove that

$$\mathbb{E}(|T^{*}_{i,r,M}|1(A)1(B))\leqslant C^{i}(\log M)^{i-1}M'(M')^{-1-1/\psi}.$$

For $i=1$, observe that on $A\cap B$, $|T^{*}_{1,r,,M}|=\deg(o^{*})=M'$. It follows from (3.101) that

$$\mathbb{P}(A\cap B\mid x_o=x)\leqslant m\alpha''+\beta''F(M,x).$$

Then using that

$$\int_0^1 \alpha'' x^{\psi}dx=\mathcal{O}((M')^{-1-1/\psi})$$

$$\int_0^1 \beta'' F(M,x)x^{\psi}dx=\mathcal{O}((M')^{-2-1/\psi}),$$

we obtain

$$\mathbb{P}(A\cap B)=\mathcal{O}((M')^{-1-1/\psi}). \qquad (3.106)$$

Therefore

$$\mathbb{E}(|T^{*}_{1,r,M}|1(A)1(B))=\mathcal{O}(M'(M')^{-1-1/\psi}).$$

For $i\geqslant 2$, we notice that

$$|T^{*}_{i,r,M}|=|T^{o^{*}}_{i,r,M}|+\sum_{(0,j)\neq o^{*}}|T^{(0,j)}_{i-2,r-2,M}|, \qquad (3.107)$$

with the convention $|T_0| = 1$. We have

$$\mathbb{E}(|T^*_{1,r,M}|1(A)1(B) \mid x_o = x) = h_{2,i}(x) + h_{1,i-2}(x), \tag{3.108}$$

where for $1 \leqslant i \leqslant r$,

$$\begin{aligned} h_{1,i}(x) &= \mathbb{E}\left(\sum_{(0,j)\neq o^*} |T^{(0,j)}_{i,r,M}|1(A)1(B) \mid x_o = x\right) \\ h_{2,i}(x) &= \mathbb{E}\Big(|T^{o^*}_{i,r,M}|1(A)1(B) \mid x_o = x\Big). \end{aligned}$$

We first study $h_{1,i}(x)$.

$$\begin{aligned} h_{1,i}(x) &= \mathbb{E}\left(\sum_{(0,j)\neq o^*} |T^{(0,j)}_{i,r,M}|1(A)1(B) \mid x_o = x\right) \\ &= \sum_{k=0}^{M-m} \mathbb{E}\left(\sum_{(0,j)\neq o^*} |T^{(0,j)}_{i,r,M}|1(B) \mid x_o = x, \deg(o) = m+k\right) p(k,x) \\ &= \sum_{k=0}^{M-m} \mathbb{E}\left(\sum_{1\leqslant j\neq s\leqslant m+k} |T^{(0,j)}_{i,r,M}|1(B_{s,k}) \mid x_o = x, \deg(o) = m+k\right) p(k,x), \end{aligned} \tag{3.109}$$

where

$$B_{s,k} = \{\deg(o) = m+k\} \cap \{\deg((0,s)) = M', \deg((0,j)) \leqslant M \ \forall j \neq s\}.$$

If $j \neq s$, then

$$\begin{aligned} &\mathbb{E}\left(|T^{(0,j)}_{i,r,M}|1(B_{s,k}) \mid x_{(0,1)}, \ldots, x_{(0,m+k)}\right) \\ &\leqslant \mathbb{E}\left(|T^{(0,j)}_{i,r,M}|1(\deg((0,j)) \leqslant M) \mid x_{(0,j)}\right) \mathbb{P}(\deg((0,s)) = M' \mid x_{(0,s)}) \\ &= f_i(x_{(0,j)}) p(M', x_{(0,s)}). \end{aligned} \tag{3.110}$$

Now using this estimate and the facts that $x_{(0,1)}, \ldots, x_{(0,m)}$ are uniformly distributed on $[x_o, 1]$ and that $x_{(0,m+1)}, \ldots, x_{(0,m+k)}$ are distributed on $[x_o, 1]$ with density $\frac{\psi y^{\psi-1} dy}{1-x_o^{\psi}}$, we deduce from (3.109) and (3.110) that

$$
\begin{aligned}
h_{1,i}(x) \leqslant & \left(\frac{1}{x}\int_0^x f_i(y)dy\right)\alpha'' \sum_{k=0}^{M-m} m^2 p(k,x) \\
&+ \left(\frac{1}{1-x^{\psi}}\int_x^1 \psi y^{\psi-1} f_i(y)dy\right)\beta'' \sum_{k=0}^{M-m} k^2 p(k,x) \\
&+ \left[\left(\frac{1}{x}\int_0^x f_i(y)dy\right)\beta'' + \left(\frac{1}{1-x^{\psi}}\int_x^1 \psi y^{\psi-1} f_i(y)dy\right)\alpha''\right] \sum_{k=0}^{M-m} (mk)p(k,x).
\end{aligned}
\tag{3.111}
$$

Then using these estimates, (3.84), (3.85), (3.86), we obtain

$$
\begin{aligned}
h_{1,i}(x) &\lesssim (\log M)^i \left(\alpha'' F^*(M,x)\sum_{k\leqslant M} p(k,x) + \beta''\sum_{k\leqslant M} k^2 p(k,x) + (\beta'' F^*(M,x) + \alpha'')\right) \\
&= (\log M)^i (H_1(x) + H_2(x) + H_3(x)).
\end{aligned}
$$

After some computations, we get

$$\int_0^1 (H_1(x) + H_2(x) + H_3(x))x^{\psi}dx = \mathcal{O}((M')^{-1-1/\psi}\log M)$$

Hence

$$\int_0^1 h_{1,i}(x)x^{\psi}dx \leqslant (C\log M)^{i+1}(M')^{-1-1/\psi}.$$

Note that in (3.107), we need an estimate for $h_{1,i-2}(x)$:

$$\int_0^1 h_{1,i-2}(x)x^{\psi}dx \leqslant (C\log M)^{i-1}(M')^{-1-1/\psi}. \tag{3.112}$$

We now study $h_{2,i}(x)$. Similarly to (3.107),

$$
\begin{aligned}
h_{2,i}(x) &= \mathbb{E}\left(|T^{o^*}_{i,r,M}|1(A)1(B) \mid x_o = x\right) \\
&= \sum_{k=0}^{M-m} \mathbb{E}\left(\sum_{j=1}^{m+k} |T^{(0,j)}_{i,r,M}|1(B_{j,k}) \mid x_o = x, \deg(o) = m+k\right) p(k,x),
\end{aligned}
\tag{3.113}
$$

with $B_{j,k}$ as in (3.107). Using the same computations in (3.110) and (3.111), we get

$$\begin{aligned}
& \mathbb{E}\left(|T^{(0,j)}_{i,r,M}|1(B_{j,k}) \mid x_{(0,1)},\ldots,x_{(0,m+k)}\right) \\
\leqslant\ & \mathbb{E}\left(|T^{(0,j)}_{i,r,M}|1(\deg((v,j))=M') \mid x_{(0,j)}\right) \\
\leqslant\ & \left(\frac{m}{x_{(0,j)}}\int_0^{x_{(0,j)}} f_{i-1}(y)dy + \frac{M'-m}{1-x^{\psi}_{(0,j)}}\int_{x_{(0,j)}}^1 \psi x^{\psi-1} f_{i-1}(y)dy\right) p(M',x_{(0,j)}) \\
\leqslant\ & C^i(\log M)^{i-1}M'p(M',x_{(0,j)}). \qquad (3.114)
\end{aligned}$$

Combining (3.113) and (3.114) gives that

$$h_{2,i}(x)\leqslant C^i(\log M)^{i-1}M'\sum_{k=0}^{M-m}(m\alpha''+k\beta'')\,p(k,x).$$

Therefore using the same estimate as (3.106), we get

$$\int_0^1 h_{2,i}(x)x^{\psi}dx \leqslant C^i(\log M)^{i-1}M'(M')^{-1-1/\psi}. \qquad (3.115)$$

It follows from (3.108), (3.112) and (3.115) that

$$\begin{aligned}
\mathbb{E}(|T^*_{i,r,M}|1(A\cap B)) &= \int_0^1 \mathbb{E}(|T^*_{i,r,M}|1(A\cap B)\mid x_o=x)(\psi+1)x^{\psi}dx \\
&= \int_0^1 (h_{2,i}(x)+h_{1,i-1}(x))(\psi+1)x^{\psi}dx \\
&\leqslant C^i(\log M)^{i-1}M'(M')^{-1-1/\psi}. \qquad (3.116)
\end{aligned}$$

From (3.105), (3.116), we deduce that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_2}(|T^*_{i,r,M}|)=\mathbb{E}(|T^*_{i,r,M}|\mid A\cap B) &= \frac{1}{\mathbb{P}(A\cap B)}\mathbb{E}(|T^*_{i,r,M}|1(A\cap B)) \\
&\leqslant C^i(\log M)^{i-1}M', \qquad (3.117)
\end{aligned}$$

which proves Lemma 3.6.4 (ii). We now prove (iii). Similarly to (3.116) and (3.117),

$$\mathbb{E}_{\mathbb{Q}_2}(|S^*_{i,r,M}|) = \frac{1}{\mathbb{P}(A\cap B)}\int_0^1 (l_{2,i}(x)+l_{1,i-2}(x))(\psi+1)x^{\psi}dx, \qquad (3.118)$$

where

$$\begin{aligned}
l_{1,i}(x) &= \mathbb{E}\left(\sum_{(0,j)\neq o^*}|S^{(0,j)}_{i,r,M}|1(A)1(B)\mid x_o=x\right) \\
l_{2,i}(x) &= \mathbb{E}\left(|S^{(o^*}_{i,r,M}|1(A)1(B)\mid x_o=x\right).
\end{aligned}$$

We first study $l_{1,i}(x)$. Using the same computation for $h_{1,i}(x)$, we have

$$\begin{aligned}
l_{1,i}(x) &= \mathbb{E}\left(\sum_{(0,j)\neq o^*} |S_{i,r,M}^{(0,j)}| 1(A)1(B) \mid x_o = x\right) \\
&\leqslant \left(\frac{1}{x}\int_0^x g_i(y)dy\right)\alpha'' \sum_{k=0}^{M-m} m^2 p(k,x) \\
&+ \left(\frac{1}{1-x^\psi}\int_x^1 \psi y^{\psi-1} g_i(y)dy\right)\beta'' \sum_{k=0}^{M-m} k^2 p(k,x) \\
&+ \left[\left(\frac{1}{x}\int_0^x g_i(y)dy\right)\beta'' + \left(\frac{1}{1-x^\psi}\int_x^1 \psi y^{\psi-1} g_i(y)dy\right)\alpha''\right] \sum_{k=0}^{M-m} (mk)p(k,x).
\end{aligned}$$

Then using (3.95), (3.96), (3.97) and some calculus, we get

$$\int_0^1 l_{1,i-2}(x)x^\psi dx = \mathcal{O}((\log M)^{i-1} M' M^{-1} (M')^{-1-1/\psi}). \tag{3.119}$$

For $l_{2,i}(x)$, we observe that

$$\begin{aligned}
l_{2,i}(x) &= \mathbb{E}\left(|S_{i,r,M}^{o^*} 1(A)1(B)| \mid x_o = x\right) \\
&= \sum_{k=0}^{M-m} \mathbb{E}\left(\sum_{j=1}^{m+k} |S_{i,r,M}^{(0,j)}| 1(B_{j,k}) \mid x_o = x, \deg(o) = m+k\right) p(k,x).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathbb{E}\left(|S_{i,r,M}^{(0,j)}| 1(B_{j,k}) \mid x_{(0,1)}, \ldots, x_{(0,m+k)}\right) \\
\leqslant &\left(\frac{m}{x_{(0,j)}}\int_0^{x_{(0,j)}} g_{i-1}(y)dy + \frac{M'-m}{1-x_{(0,j)}^\psi}\int_{x_{(0,j)}}^1 g_{i-1}(y)\psi y^{\psi-1}dy\right) p(M', x_{(0,j)}) \\
= &\ \mathcal{O}((\log M)^{i-1} M' M^{-1} p(M', x_{(0,j)})).
\end{aligned}$$

Here, we used (3.96) to estimate the second term. Therefore

$$\int_0^1 l_{2,i}(x)x^\psi dx = \mathcal{O}((\log M)^{i-1} M' M^{-1} (M')^{-1-1/\psi}). \tag{3.120}$$

Now, it follows from (3.105), (3.118), (3.119) and (3.120) that

$$\mathbb{E}_{\mathbb{Q}_2}(|S_{i,r,M}^*|) = \mathcal{O}((\log M)^{i-1} M' M^{-1}),$$

which proves Lemma 3.6.4 (iii). We now prove (i). To estimate $\mathbb{E}_{\mathbb{Q}_1}(|T_{i,r,M}|)$, we use (3.99) and the same argument as for $\mathbb{E}_{\mathbb{Q}_2}(|T^*_{i,r,M}|)$. More precisely, we replace B by $\tilde{B} = \{|S_{1,r,M}| = 1\}$, replace $B_{s,k}$ by

$$\tilde{B}_{s,k} = \{\deg(o) = m+k\} \cap \{\deg((0,s)) > M, \deg((0,j)) \leqslant M \ \forall\ j \neq s\},$$

replace α'' and β'' by α and β respectively. We now have

$$\begin{aligned}
&\mathbb{E}(|T_{i,r,M}|1(A)1(\tilde{B}) \mid x_o = x) \\
&= \mathbb{E}\left(\sum_{j=1}^{\deg(o)} |T^{(0,j)}_{i-1,r-1,M}|1(A)1(\tilde{B})1(\deg((0,j)) \leqslant M) \mid x_o = x\right) \\
&= \sum_{k=0}^{M-m} \mathbb{E}\left(\sum_{1 \leqslant j \neq s \leqslant m+k} |T^{(0,j)}_{i-1,r-1,M}|1(\tilde{B}_{s,k}) \mid x_o = x, \deg(o) = m+k\right) p(k,x) \\
&:= \tilde{h}_{i-1}(x).
\end{aligned}$$

Finally, $\int_0^1 \tilde{h}_{i-1}(x)x^{\psi}dx$ can be estimated by the same way for $h_{1,i-2}(x)$, and then we get the desired result. $\square$

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Chapter 4

Contact process on one-dimensional long range percolation

Abstract. Recently, by introducing the notion of cumulatively merged partition, Ménard and Singh provide in [67] a sufficient condition on graphs ensuring that the critical value of the contact process is positive. In this chapter, we show that the one-dimensional long range percolation with high exponent satisfies their condition and thus the contact process exhibits a non-trivial phase transition.

4.1 Introduction

We study the contact process on G_s, the one-dimensional long range percolation graph with exponent $s > 1$, defined as follows: independently for any i and j in $\mathbb{Z}$ there is an edge connecting them with probability $|i-j|^{-s}$. In particular, G_s contains $\mathbb{Z}$ so it is connected.

As we have shown in Lemma 1.1.1, the contact process on graphs with bounded degrees exhibits a non-trivial phase transition, i.e. λ_c is positive. In contrast, there is a little knowledge about the sub-critical phase on unbounded degree graphs. For Galton-Watson trees, Pemantle proved in [74] that if the reproduction law B asymptotically satisfies that $\mathbb{P}(B \geqslant x) \geqslant \exp(-x^{1-\varepsilon})$, for some $\varepsilon > 0$, then $\lambda_c = 0$. Recently, in [67], by introducing the notion of cumulatively merged partition (abbr. CMP) (see Section 4.2.2), the authors provided a sufficient condition on graphs ensuring that $\lambda_c > 0$. As an application, they show that the contact process on random geometric graphs and Delaunay triangulations exhibits a non-trivial phase transition.

The long range percolation graph was first introduced in [81, 89]. Then it gained interest in some contexts such as the graph distance, diameter, random walk, see [31] for a list of reference. The long range percolation is locally finite if and only if $s > 1$, so we only consider the contact process on such graphs. Moreover, it follows from the ergodicity of G_s that there is a non negative constant $\lambda_c(s)$, such that

$$\lambda_c(G_s) = \lambda_c(s) \text{ for almost all graphs } G_s. \tag{4.1}$$

It is clear that the sequence of graphs (G_s) is stochastically decreasing in s in the sense that G_{s_1} can be coupled as a subgraph of G_{s_2} if $s_1 \geqslant s_2$. Therefore $\lambda_c(s_1) \geqslant \lambda_c(s_2)$. Hence, we can define

$$s_c = \inf\{s : \lambda_c(s) > 0\}. \tag{4.2}$$

We will apply the method in [67] to show that $s_c < +\infty$. Here is our main result.

Theorem 4.1.1. *We have*

$$s_c \leqslant 102.$$

There is a phase transition in the structure of the long range percolation. If $s < 2$, the graph G_s exhibits the small-world phenomenon. More precisely, the distance between x and y is of order $(\log|x-y|)^{\varkappa + o(1)}$ with $\varkappa = \varkappa(s) > 1$, with probability tending to 1 as $|x-y| \to \infty$, see for instance [7]. In contrast, if $s > 2$, the graph somehow looks like $\mathbb{Z}$ (see Section 4.2.1) and the distance now is of order $|x-y|$, see [6]. On the other hand, as mentioned above, we know that $\lambda_c(\mathbb{Z}) > 0$. Hence, we conjecture that

$$s_c \leqslant 2.$$

The results in [67] can be slightly improved and thus we could get a better bound on s_c, but it would still be far from the critical value 2.

This chapter is organized as follows. In Section 2, we first describe the structure of the graph and show that G_s can be seen as the gluing of i.i.d. finite subgraphs. Then we recall the definitions and results of [67] on the CMP. By studying the moment of the total weight of a subgraph, we are able to apply the results from [67] and prove our main theorem. In Section 3, we consider the CMP on G_s with s close to 1.

4.2 Proof of Theorem 4.1.1

4.2.1 Structure of the graph

We fix $s > 2$. For any $k \in \mathbb{Z}$, we say that k is a *cut-point* if there is no edge (i, j) with $i < k$ and $j > k$.

Lemma 4.2.1. *The following statements hold.*

(i) For all $k \in \mathbb{Z}$

$$\mathbb{P}(k \text{ is a cut-point}) = \mathbb{P}(0 \text{ is a cut-point}) > 0.$$

As a consequence, almost surely there exist infinitely many cut-points.

(ii) The subgraphs induced in the intervals between consecutive cut-points are i.i.d. In particular, the distances between consecutive cut-points form a sequence of i.i.d. random variables.

Proof. We first prove (i). Observe that

$$\begin{aligned}
\mathbb{P}(k \text{ is a cut-point}) &= \mathbb{P}(0 \text{ is a cut-point}) \\
&= \prod_{i<0<j} \left(1 - |i-j|^{-s}\right) \\
&\geqslant \exp\left(-2 \sum_{i<0<j} |i-j|^{-s}\right) \\
&\geqslant e^{2/(2-s)},
\end{aligned}$$

where we used that $1 - x \geqslant \exp(-2x)$ for $0 \leqslant x \leqslant 1/2$ and

$$\begin{aligned}
\sum_{i<0<j} |i-j|^{-s} = \sum_{i,j\geqslant 1} (i+j)^{-s} &\leqslant \frac{1}{s-1} \sum_{i\geqslant 1} i^{1-s} \\
&\leqslant \frac{1}{s-1}\left(1 + \frac{1}{s-2}\right) = \frac{1}{s-2},
\end{aligned}$$

using series integral comparison.

Then the ergodic theorem implies that there are infinitely many cut-points a.s.

Part (ii) is immediate, since there are no edges between different intervals between consecutive cut-points. $\square$

We now study some properties of the distance between two consecutive cut-points.

Proposition 4.2.2. *Let D be the distance between two consecutive cut-points. Then there exists a sequence of integer-valued random variables $(\varepsilon_i)_{i\geqslant 0}$ with $\varepsilon_0 = 1$, such that*

(i) $D = \sum_{i=0}^{T} \varepsilon_i$ *with* $T = \inf\{i \geqslant 1 : \varepsilon_i = 0\}$,

(ii) T *is stochastically dominated by a geometric random variable with mean* $e^{2/(2-s)}$,

(iii) for all $i, \ell \geqslant 1$

$$\mathbb{P}(\varepsilon_i > \ell \mid T \geqslant i) \leqslant \ell^{2-s}/(s-2).$$

Proof. To simplify notation, we assume that 0 is a cut-point. Set $X_{-1} = 0$ and $X_0 = 1$, then we define for $i \geqslant 1$

$$X_i = \max\{k : \exists\, X_{i-2} \leqslant j \leqslant X_{i-1} - 1,\ j \sim k\},$$
$$\varepsilon_i = X_i - X_{i-1}.$$

Then $\varepsilon_i \geqslant 0$ and we define

$$T = \inf\{i \geqslant 1 : X_i = X_{i-1}\} = \inf\{i \geqslant 1 : \varepsilon_i = 0\}.$$

We have $X_i = X_{i-1}$ for all $i \geqslant T$, or equivalently $\varepsilon_i = 0$ for all $i \geqslant T$.

Note that X_T is the closest cut-point on the right of 0, so it has the same law as D, by definition. Moreover

$$X_T = \sum_{i=0}^{T} \varepsilon_i, \tag{4.3}$$

which implies (i). Observe that for $i \geqslant 1$ we have $\{T \geqslant i\} = \{X_{i-2} < X_{i-1}\}$ and

$$\begin{aligned}
\mathbb{P}(T = i \mid T \geqslant i) &= \mathbb{P}(X_i = X_{i-1} \mid X_{i-2} < X_{i-1}) \\
&= \mathbb{P}(\nexists\, X_{i-2} \leqslant j < X_{i-1} < k :\ j \sim k \mid X_{i-2} < X_{i-1}) \\
&\geqslant \prod_{j<0<k} \left(1 - |j-k|^{-s}\right) \\
&\geqslant e^{2/(2-s)}.
\end{aligned}$$

This implies (ii). For (iii), we note that for $i, \ell \geqslant 1$,

$$\begin{aligned}\mathbb{P}(X_i \leqslant X_{i-1} + \ell \mid X_{i-2} < X_{i-1}) &\geqslant \prod_{\substack{j<0\\k>\ell}} \left(1 - |j-k|^{-s}\right)\\ &\geqslant 1 - \sum_{\substack{j<0\\k>\ell}} |j-k|^{-s}.\end{aligned}$$

We have

$$\begin{aligned}\sum_{\substack{j<0\\k>\ell}} |j-k|^{-s} &= \sum_{j=1}^{\infty} \sum_{k=\ell+1}^{\infty} (k+j)^{-s}\\ &\leqslant \frac{1}{s-1} \sum_{j=1}^{\infty} (j+\ell)^{1-s}\\ &\leqslant \ell^{2-s}/(s-2).\end{aligned}$$

Therefore,

$$\mathbb{P}(\varepsilon_i > \ell \mid T \geqslant i) \leqslant \ell^{2-s}/(s-2),$$

which proves (iii). □

Since the definition of λ_c is independent of the starting vertex, we can assume that the initially infected vertex is a cut-point.

It will be convenient to assume that 0 is a cut-point. Suppose that conditioned on 0 being a cut-point and infected at the beginning, we can prove that $\lambda_c > 0$. Since the distribution is invariant under translations, we have $\lambda_c > 0$ for the contact process starting from any cut point.

Hence, from now on we condition on the event 0 is a cut-point. Set $K_0 = 0$, for $i \geqslant 1$, we call K_i (resp. K_{-i}) the i^{th} cut point from the right (resp. left) of 0. By Lemma 4.2.1 (ii), the graphs induced in the intervals $[K_i, K_{i+1})$ are i.i.d. Therefore, G_s is isomorphic to the graph $\tilde{G}_s$ obtained by gluing an i.i.d. sequence of graphs with distribution of the graph $[0, K_1)$. We have to prove that the contact process on $\tilde{G}_s$ exhibits a non-trivial phase transition.

4.2.2 Cumulatively merged partition

We recall here the definitions and results introduced in [67]. Given a locally finite graph $G = (V, E)$, an expansion exponent $\alpha \geqslant 1$, and a sequence of non-negative weights defined on the vertices

$$(r(x), x \in V) \in [0, \infty)^V,$$

a partition $\mathcal{C}$ of the vertex set V is said to be (r, α)-admissible if it satisfies

$$\forall\, C, C' \in \mathcal{C}, \quad C \neq C' \qquad \Longrightarrow \qquad d(C, C') > \min\{r(C), r(C')\}^{\alpha},$$

with

$$r(C) = \sum_{x \in C} r(x).$$

We call *cumulatively merged partition* (CMP) of the graph G with respect to r and α the finest (r, α)- admissible partition and denote it by $\mathcal{C}(G, r, \alpha)$. It is the intersection of all (r, α)-admissible partitions of the graph, where the intersection is defined as follows: for any sequence of partitions $(\mathcal{C}_i)_{i \in I}$,

$$x \sim y \text{ in } \cap_{i \in I} \mathcal{C}_i \quad \text{if} \quad x \sim y \text{ in } \mathcal{C}_i \text{ for all } i \in I.$$

As for Bernoulli percolation on $\mathbb{Z}^d$, the question we are interested in is the existence of an infinite cluster (here an infinite partition). For the CMP on $\mathbb{Z}^d$ with i.i.d. weights, we have the following result.

Proposition 4.2.3. *[67, Proposition 3.7] For any $\alpha \geqslant 1$, there exists a positive constant $\beta_c = \beta_c(\alpha)$, such that for any positive random variable Z satisfying $\mathbb{E}(Z^{\gamma}) \leqslant 1$ with $\gamma = (4\alpha d)^2$ and any $\beta < \beta_c$, almost surely $\mathcal{C}(\mathbb{Z}^d, \beta Z, \alpha)$-the CMP on $\mathbb{Z}^d$ with expansion exponent α and i.i.d. weights distributed as βZ-has no infinite cluster.*

We note that in [67, Proposition 3.7], the authors only assume that $\mathbb{E}(Z^{\gamma}) < \infty$ and they do not precise the dependence of β_c with $\mathbb{E}(Z^{\gamma})$. However, we can deduce from their proof a lower bound on β_c depending only on $\mathbb{E}(Z^{\gamma})$ (and only on α, γ, d if we suppose $\mathbb{E}(Z^{\gamma}) \leqslant 1$), see Appendix for more details. Finally, our $\beta_c(\alpha)$ is a lower bound of the critical parameter $\lambda_c(\alpha)$ introduced by Ménard and Singh.

Using the notion of CMP, they give a sufficient condition on a graph G ensuring that the critical value of the contact process is positive.

Theorem 4.2.4. *[67, Theorem 4.1] Let $G = (V, E)$ be a locally finite connected graph. Consider $\mathcal{C}(G, r_{\triangle}, \alpha)$ the CMP on G with expansion exponent α and degree weights*

$$r_{\triangle}(x) = \deg(x) 1(\deg(x) \geqslant \triangle).$$

Suppose that for some $\alpha \geqslant 5/2$ and $\triangle \geqslant 0$, the partition $\mathcal{C}(G, r_\triangle, \alpha)$ has no infinite cluster. Then

$$\lambda_c(G) > 0.$$

Thanks to this result, Theorem 4.1.1 will follow from the following proposition.

Proposition 4.2.5. *Fix $s > 102$. There exists a positive constant $\triangle$, such that the partition $\mathcal{C}(\tilde{G}_s, r_\triangle, 5/2)$ has no infinite cluster a.s.*

4.2.3 Proof of Proposition 4.2.5

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two CMPs. We write $\mathcal{C}_1 \preceq \mathcal{C}_2$, if there is a coupling such that $\mathcal{C}_1$ has an infinite cluster only if $\mathcal{C}_2$ has an infinite cluster.

Lemma 4.2.6. *We have*

$$\mathcal{C}(\tilde{G}_s, r_\triangle, 5/2) \preceq \mathcal{C}(\mathbb{Z}, Z_\triangle, 5/2), \tag{4.4}$$

with

$$Z_\triangle = \sum_{0 \leqslant x < K_1} \deg(x) 1(\deg(x) \geqslant \triangle).$$

Proof. For $i \in \mathbb{Z}$, we define

$$Z_i = \sum_{K_i \leqslant x < K_{i+1}} \deg(x) 1(\deg(x) \geqslant \triangle).$$

Then $(Z_i)_{i \in \mathbb{Z}}$ is a sequence of i.i.d. random variables with the same distribution as $Z_\triangle$, since the graph $\tilde{G}_s$ is composed of i.i.d. subgraphs $[K_i, K_{i+1})$. Therefore, $\mathcal{C}(\mathbb{Z}, (Z_i), 5/2)$ has the same law as $\mathcal{C}(\mathbb{Z}, Z_\triangle, 5/2)$. Thus to prove Lemma 4.2.6, it remains to show that

$$\mathcal{C}(\tilde{G}_s, r_\triangle, 5/2) \preceq \mathcal{C}(\mathbb{Z}, (Z_i), 5/2). \tag{4.5}$$

For any subset A of the vertices of $\tilde{G}_s$, we define its projection

$$p(A) = \{i \in \mathbb{Z} : A \cap [K_i, K_{i+1}) \neq \varnothing\}.$$

Since all intervals $[K_i, K_{i+1})$ have finite mean, if $|A| = \infty$ then $|p(A)| = \infty$. Therefore, to prove (4.5), it suffices to show that

$$x \sim y \text{ in } \mathcal{C}(\tilde{G}_s, r_\triangle, 5/2) \quad \text{implies} \quad p(x) \sim p(y) \text{ in } \mathcal{C}(\mathbb{Z}, (Z_i), 5/2). \tag{4.6}$$

We prove (4.6) by contradiction. Suppose that there exist x_0 and y_0 such that $x_0 \sim y_0$ in $\mathfrak{C}(\tilde{G}_s, r_\vartriangle, 5/2)$ and $p(x_0) \not\sim p(y_0)$ in $\mathfrak{C}(\mathbb{Z}, (Z_i), 5/2)$. Then by definition there exists $\mathcal{C}$, a $((Z_i), 5/2)$-admissible partition of $\mathbb{Z}$, such that $p(x_0) \not\sim p(y_0)$ in $\mathcal{C}$.

We define a partition $\tilde{\mathcal{C}}$ of $\tilde{G}_s$ as follows:

$$x \sim y \text{ in } \tilde{\mathcal{C}} \quad \text{ if and only if } \quad p(x) \sim p(y) \text{ in } \mathcal{C}.$$

In other words, an element in $\tilde{\mathcal{C}}$ is $\cup_{i \in C}[K_i, K_{i+1})$ with C a set in $\mathcal{C}$. We now claim that $\tilde{\mathcal{C}}$ is $(r_\vartriangle, 5/2)$-admissible. Indeed, let $\tilde{C}$ and $\tilde{C}'$ be two different sets in $\tilde{\mathcal{C}}$. Then by the definition of $\tilde{\mathcal{C}}$, we have $p(\tilde{C})$ and $p(\tilde{C}')$ are two different sets in $\mathcal{C}$ and

$$Z(p(\tilde{C})) := \sum_{i \in p(\tilde{C})} Z_i = \sum_{x \in \tilde{C}} \deg(x) 1(\deg(x) \geqslant \vartriangle) = r_\vartriangle(\tilde{C}).$$

Moreover, since these intervals $[K_i, K_{i+1})$ are disjoint,

$$d(\tilde{C}, \tilde{C}') \geqslant d(p(\tilde{C}), p(\tilde{C}')).$$

On the other hand, as $\mathcal{C}$ is $((Z_i), 5/2)$-admissible,

$$d(p(\tilde{C}), p(\tilde{C}')) > \min\{Z(p(\tilde{C})), Z(p(\tilde{C}'))\}^{5/2}.$$

It follows from the last three inequalities that

$$d(\tilde{C}, \tilde{C}') > \min\{r_\vartriangle(\tilde{C}), r_\vartriangle(\tilde{C}')\}^{5/2},$$

which implies that $\tilde{\mathcal{C}}$ is $(r_\vartriangle, 5/2)$-admissible.

Let C_0 and C_0' be the two sets in the partition $\mathcal{C}$ containing $p(x_0)$ and $p(y_0)$ respectively. Then by assumption $C_0 \neq C_0'$. We define

$$\tilde{C}_0 = \bigcup_{i \in C_0} [K_i, K_{i+1}) \qquad \text{and} \qquad \tilde{C}_0' = \bigcup_{i \in C_0'} [K_i, K_{i+1}).$$

Then both $\tilde{C}_0$ and $\tilde{C}_0'$ are in $\tilde{\mathcal{C}}$, and $\tilde{C}_0 \neq \tilde{C}_0'$. Moreover $\tilde{C}_0$ contains x_0 and $\tilde{C}_0'$ contains y_0. Hence $x_0 \not\sim y_0$ in $\tilde{\mathcal{C}}$ which is a $(r_\vartriangle, 5/2)$-admissible partition. Therefore, $x_0 \not\sim y_0$ in $\mathfrak{C}(\tilde{G}_s, r_\vartriangle, 5/2)$, which leads to a contradiction. Thus (4.6) has been proved. $\square$

We now apply Proposition 4.2.3 and Lemma 4.2.6 to prove Proposition 4.2.5. To do that, we fix a positive constant $\beta < \beta_c(5/2)$ with $\beta_c(5/2)$ as in Proposition 4.2.3 with $d = 1$ and rewrite

$$Z_\vartriangle = \beta \frac{Z_\vartriangle}{\beta}.$$

If we can show that there is $\vartriangle = \vartriangle(\beta, s)$, such that

$$\mathbb{E}\left(\left(\frac{Z_\vartriangle}{\beta}\right)^{100}\right) \leqslant 1, \tag{4.7}$$

then Proposition 4.2.3 implies that a.s. $\mathcal{C}(\mathbb{Z}, Z_\vartriangle, 5/2)$ has no infinite cluster. Therefore, by Lemma 4.2.6, there is no infinite cluster in $\mathcal{C}(\tilde{G}_s, r_\vartriangle, 5/2)$ and thus Proposition 4.2.5 follows. Now it remains to prove (4.7).

It follows from Proposition 4.2.2 (i) that

$$\mathbb{E}(K_1^{100}) = \mathbb{E}(D^{100}) = \mathbb{E}\left(\left(\sum_{i=0}^{T} \varepsilon_i\right)^{100}\right), \tag{4.8}$$

where T and (ε_i) are as in Proposition 4.2.2.

Applying the inequality $(x_1 + \ldots + x_n)^{100} \leqslant n^{99}(x_1^{100} + \ldots + x_n^{100})$ for any $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathbb{R}$, we get

$$\begin{aligned} \mathbb{E}\left(\left(\sum_{i=0}^{T} \varepsilon_i\right)^{100}\right) &\leqslant \mathbb{E}\left[(T+1)^{99} \sum_{i=0}^{T} \varepsilon_i^{100}\right] \\ &= \sum_{i=0}^{\infty} \mathbb{E}\left[(T+1)^{99} \varepsilon_i^{100} 1(T \geqslant i)\right]. \end{aligned} \tag{4.9}$$

Let $p = 1 + (s - 102)/200 > 1$ and q be its conjugate, i.e. $p^{-1} + q^{-1} = 1$. Then applying Hölder's inequality, we obtain

$$\mathbb{E}\left[(T+1)^{99} \varepsilon_i^{100} 1(T \geqslant i)\right] \leqslant \mathbb{E}\left((T+1)^{99q}\right)^{1/q} \mathbb{E}\left(\varepsilon_i^{100p} 1(T \geqslant i)\right)^{1/p}. \tag{4.10}$$

On the other hand,

$$\mathbb{E}\left(\varepsilon_i^{100p} 1(T \geqslant i)\right) = \mathbb{E}\left(\varepsilon_i^{100p} \mid T \geqslant i\right) \mathbb{P}(T \geqslant i). \tag{4.11}$$

Using Proposition 4.2.2 (iii) we have for $i \geqslant 1$

$$\begin{aligned}
\mathbb{E}\left(\varepsilon_i^{100p} \mid T \geqslant i\right) &\leqslant 100p \sum_{\ell=0}^{\infty} \mathbb{P}(\varepsilon_i > \ell \mid T \geqslant i)(\ell+1)^{100p-1} \\
&\leqslant 100p \left[1 + \sum_{\ell \geqslant 1} \ell^{2-s}(1+\ell)^{100p-1}/(s-2)\right] \\
&\leqslant C_1 = C_1(s) < \infty,
\end{aligned}$$

since by definition

$$2 - s + 100p - 1 = -1 - (s-102)/2 < -1.$$

Hence for all $i \geqslant 1$

$$\mathbb{E}\left(\varepsilon_i^{100p} 1(T \geqslant i)\right) \leqslant C_1 \mathbb{P}(T \geqslant i). \tag{4.12}$$

It follows from (4.8), (4.9), (4.10) and (4.12) that

$$\begin{aligned}
\mathbb{E}(K_1^{100}) &\leqslant \mathbb{E}\left[(T+1)^{99q}\right]^{1/q} \left[1 + \sum_{i=1}^{\infty} \left(C_1 \mathbb{P}(T \geqslant i)\right)^{1/p}\right] \\
&= M < \infty,
\end{aligned} \tag{4.13}$$

since T is stochastically dominated by a geometric random variable.

For any $j \in \mathbb{Z}$ and any interval I, we denote by $\deg_I(j)$ the number of neighbors of j in I when we consider the original graph (without conditioning on 0 being a cut-point).

Now for any non decreasing sequence $(x_k)_{k\geqslant 1}$ with $x_1 \geqslant 1$, conditionally on $\varepsilon_1 = x_1 - 1, \varepsilon_2 = x_2 - x_1, \ldots$, we have for all $j \in (x_{k-1}, x_k)$,

$$\deg(j) \prec 1 + \deg_{[x_{k-2}, x_{k+1})}(j),$$

where $\prec$ means stochastic domination.

Indeed, the conditioning implies that j is only connected to vertices in $[x_{k-2}, x_{k+1}]$ and that there is a vertex in $[x_{k-1}, x_k)$ connected to x_{k+1}.

Similarly, if $j = x_k$, it is only connected to vertices in $[x_{k-2}, x_{k+2}]$. Moreover, j is connected to at least one vertex in $[x_{k-2}, x_{k-1})$ and there is a vertex in $[x_k, x_{k+1})$ connected to x_{k+2}. Therefore,

$$\deg(x_k) \prec 2 + \deg_{[x_{k-2}, x_{k+2})}(x_k).$$

In conclusion, conditionally on $j \in [0, K_1)$,

$$\deg(j) \prec 2 + Y,$$

where

$$Y = \deg_{(-\infty,+\infty)}(j).$$

Hence,

$$\mathbb{E}\left(\deg(j)^{100} 1(\deg(j) \geqslant \triangle) \mid j \in [0, K_1)\right) \leqslant \mathbb{E}\left((2+Y)^{100} 1(Y \geqslant \triangle - 2)\right). \tag{4.14}$$

On the other hand,

$$\begin{aligned} \mathbb{P}(Y = k) &= \mathbb{P}(\deg_{(-\infty,+\infty)}(0) = k) \\ &\leqslant \mathbb{P}(\deg_{(-\infty,+\infty)}(0) \geqslant k) \\ &\leqslant \sum_{i_1 < i_2 < \ldots < i_k} |i_1|^{-s} |i_2|^{-s} \ldots |i_k|^{-s} \\ &\leqslant \frac{1}{k!} \sum_{i_1, i_2, \ldots, i_k} |i_1|^{-s} |i_2|^{-s} \ldots |i_k|^{-s} \\ &= \frac{1}{k!} \left(2 \sum_{i \geqslant 1} i^{-s} \right)^k = \frac{C^k}{k!}, \end{aligned}$$

with $C = 2 \sum_{i \geqslant 1} i^{-s}$. Therefore,

$$\begin{aligned} \mathbb{E}\left((2+Y)^{100} 1(Y \geqslant \triangle - 2)\right) &\leqslant \sum_{k \geqslant \triangle - 2} \frac{C^k (k+2)^{100}}{k!} \\ &:= f(\triangle). \end{aligned} \tag{4.15}$$

It follows from (4.13), (4.14) and (4.15) that

$$\begin{aligned} \mathbb{E}(Z_\triangle^{100}) &= \mathbb{E}\left[\left(\sum_{0 \leqslant j < K_1} \deg(j) 1(\deg(j) \geqslant \triangle) \right)^{100} \right] \\ &\leqslant \mathbb{E}\left[K_1^{99} \sum_{0 \leqslant j < K_1} \deg(j)^{100} 1(\deg(j) \geqslant \triangle) \right] \\ &\leqslant \mathbb{E}(K_1^{100}) f(\triangle) \\ &\leqslant M f(\triangle). \end{aligned}$$

Since $f(\triangle) \to 0$ as $\triangle \to \infty$, there exists $\triangle_0 \in (0, \infty)$, such that $M f(\triangle_0) \leqslant \beta^{100}$ and thus (4.7) is satisfied. $\square$

4.3 CMP on G_s with s close to 1

In contrast to Proposition 4.2.5, if the exponent s is small enough (close to 1), then the cumulatively merged partition on G_s has an infinite cluster.

Proposition 4.3.1. *For any $\alpha > 1$, there exists a positive constant $s_0 = s_0(\alpha)$, such that $\mathcal{C}(G_s, r_\triangle, \alpha)$ has an infinite cluster a.s. for any $\triangle \geqslant 0$ and $1 < s \leqslant s_0$.*

Proof. Fix $\alpha = 1 + \delta$ with $\delta > 0$ and $\triangle \geqslant 0$. To prove the proposition, we will prove the following: there exists $s_0 = s_0(\alpha)$, such that if $1 < s \leqslant s_0$, then for all N large enough

$$\mathbb{P}\left(d_{[0,N]}(0, N) \leqslant (\log N)^{1+\delta/2}\right) \geqslant 1 - 1/\log N, \tag{4.16}$$

with $d_{[0,N]}(x, y)$ the distance between x and y using only edges with extremities in $[0, N]$.

Assume that (4.16) holds for a moment. We now prove the existence of an infinite cluster in the CMP $\mathcal{C}(G_s, r_\triangle, \alpha)$. For $N \in \mathbb{N}$, we define

$$d_N = \left[\frac{\log N}{(\log\log N)^2}\right].$$

For any $1 \leqslant \ell \leqslant [N/d_N]$, we have

$$\mathbb{P}\left(N - \ell d_N \sim k \;\forall\, k \in \{N - \ell d_N + 1, \ldots, N - (\ell - 1)d_N\}\right) = (d_N!)^{-s}.$$

Let us define

$$\sigma = \inf\Big\{\ell \geqslant 1 : N - \ell d_N \sim k \;\forall\, k \in \{N - \ell d_N + 1, \ldots, N - (\ell - 1)d_N\}\Big\}.$$

As these events are independent, we have for all N large enough

$$\begin{aligned}
\mathbb{P}(\sigma \leqslant [N/(2d_N)]) &\geqslant 1 - (1 - (d_N!)^{-s})^{[N/(2d_N)]} \\
&\geqslant 1 - \exp(-N/(2d_N(d_N!)^s)) \\
&\geqslant 1 - \exp(-\sqrt{N}),
\end{aligned} \tag{4.17}$$

since

$$d_N! \leqslant \exp(d_N \log d_N) \leqslant \exp(\log N/\log\log N) = N^{o(1)}.$$

Let $j_N = N - \sigma d_N$. Then on the event $\sigma \leqslant [N/(2d_N)]$, we have $j_N \in [N/2, N]$ and $\deg_{[0,N]}(j_N) \geqslant d_N$. Moreover, applying (4.16) we get for all N large enough

$$\begin{aligned}
& \mathbb{P}(d_{[0,j_N]}(0, j_N) \leqslant (\log N)^{1+\delta/2} \mid \sigma \leqslant [N/(2d_N)]) \\
\geqslant\ & \mathbb{P}(d_{[0,j_N]}(0, j_N) \leqslant (\log j_N)^{1+\delta/2} \mid \sigma \leqslant [N/(2d_N)]) \\
\geqslant\ & 1 - 1/\log j_N \\
\geqslant\ & 1 - 1/\log(N/2).
\end{aligned}$$

Combining this with (4.17), we obtain that for all N large enough

$$\mathbb{P}\left(\exists\, j \in (0, N] : d_{[0,N]}(0, j) \leqslant (\log N)^{1+\delta/2}, \deg_{[0,N]}(j) \geqslant d_N\right) \geqslant 1 - 2/\log N. \qquad (4.18)$$

We take N_0 large enough such that (4.18) holds and $d_{N_0} \geqslant \triangle$. Then with probability larger than $1 - 2/\log N_0$, there exists $j_0 \in (0, N_0]$, such that $\deg(j_0) \geqslant d_{N_0}$. Applying (4.18) for $N_1 = [N_0^{1+\delta/2}]$, we get

$$\mathbb{P}(\exists j_1 \in (j_0, j_0 + N_1] : d(j_0, j_1) \leqslant (\log N_1)^{1+\delta/2}, \deg(j_1) \geqslant d_{N_1}) \geqslant 1 - 2/\log N_1.$$

Applying consecutively (4.18), we obtain

$$\begin{aligned}
& \mathbb{P}\left(\exists (j_i)_{i\geqslant 0} : j_i < j_{i+1}, d(j_i, j_{i+1}) \leqslant \log(N_{i+1})^{1+\delta/2}, \deg(j_i) \geqslant d_{N_i} \text{ for all } i \geqslant 0\right) \\
& \geqslant 1 - \sum_{i\geqslant 0} 2/\log N_i, \qquad (4.19)
\end{aligned}$$

with

$$N_i = \left[N_0^{(1+\delta/2)^i}\right].$$

We note that for all $i \geqslant 0$,

$$\deg(j_i) \geqslant d_{N_i} \geqslant d_{N_0} \geqslant \triangle \qquad \text{and}$$

$$\min\{\deg(j_i)^\alpha, \deg(j_{i+1})^\alpha\} \geqslant d_{N_i}^\alpha \geqslant (\log N_{i+1})^{1+\delta/2} \geqslant d(j_i, j_{i+1}).$$

Therefore, j_i and j_{i+1} are in the same cluster of $\mathcal{C}(G_s, r_\triangle, \alpha)$ for all $i \geqslant 0$. Hence, by (4.19)

$$\begin{aligned}
\mathbb{P}(\text{there is an infinite cluster in } \mathcal{C}(G_s, r_\triangle, \alpha)) &\geqslant 1 - \sum_{i\geqslant 0} 2/\log N_i \\
&\geqslant 1 - \sum_{i\geqslant 0} \frac{2}{(1+\delta/2)^i \log N_0} \\
&\to 1,
\end{aligned}$$

as $N_0 \to \infty$. Hence, a.s. $\mathcal{C}(G_s, r_\triangle, \alpha)$ has an infinite cluster.

Now it remains to prove (4.16). Let us define for $u \in (1,2)$

$$\varkappa(u) = \frac{\log 2}{\log(2/u)}.$$

Then

$$\varkappa(u) \searrow 1 \quad \text{as} \quad u \searrow 1. \tag{4.20}$$

Therefore (4.16) follows from the following. For any $s \in (1,2)$ and $\varepsilon > 0$, we will show that for all N large enough

$$\mathbb{P}\left(d_{[0,N]}(0,N) \leqslant (\log N)^{\varkappa(s)+\varepsilon}\right) \geqslant 1 - 1/\log N. \tag{4.21}$$

In [7], Biksup proves that the upper bound of $d_{[0,N]}(0,N)$ holds with high probability. Here based on his proof, we give a lower bound for this probability.

Fix $s \in (1,2)$ and $\varepsilon > 0$. Since the function $\varkappa(u)$ is continuous, there exists a constant $\varepsilon_1 > 0$, such that $s + \varepsilon_1 < 2$ and $\varkappa(s+\varepsilon_1) \leqslant \varkappa(s) + \varepsilon/2$. Now, let us define

$$\nu = \frac{s+\varepsilon_1}{2} \in (0,1).$$

For any $x \in \mathbb{Z}$ and $m \in \mathbb{N}$, we define

$$B_m^+(x) = [x, x+m] \quad \text{and} \quad B_m^-(x) = [x-m, x].$$

Then for all $x \leqslant y \leqslant x + M$ and $M_1 = [M^\nu] = [M^{(s+\varepsilon_1)/2}]$ large enough, we have

$$\begin{aligned}\mathbb{P}(B_{M_1}^+(x) \sim B_{M_1}^-(y)) &\geqslant \mathbb{P}(\mathrm{Bin}(M_1^2, M^{-s}) \geqslant 1) \\ &\geqslant 1 - \exp(-M^{\varepsilon_1}),\end{aligned} \tag{4.22}$$

where $A \sim B$ means that there in an edge between A and B. Applying this inequality for $x = z_0 = 0$ and $y = z_1 = N$ and $M = N$, we get

$$\mathbb{P}(B_{N_1}^+(z_0) \sim B_{N_1}^-(z_1)) \geqslant 1 - \exp(-N^{\varepsilon_1}),$$

with $N_1 = [N^\nu]$. Conditionally on this event, there are $z_{01} \in B_{N_1}^+(z_0)$ and $z_{10} \in B_{N_1}^-(z_1)$, such that $z_{01} \sim z_{10}$. Therefore

$$d_{[0,N]}(0,N) \leqslant d(z_0, z_{01}) + d(z_{01}, z_{10}) + d(z_{10}, z_1) \leqslant 2N_1 + 1. \tag{4.23}$$

Hence,

$$\mathbb{P}(d_{[0,N]}(0,N) \leqslant 2N_1+1) \geqslant 1-\exp(-N^{\varepsilon_1}). \tag{4.24}$$

Similarly to (4.24), since $|z_0 - z_{01}| \leqslant N_1$ and $|z_1 - z_{10}| \leqslant N_1$, we have

$$\mathbb{P}(d_{[z_0,z_{01}]}(z_0,z_{01}) \leqslant 2N_2+1, d_{[z_{10},z_1]}(z_{10},z_1) \leqslant 2N_2+1) \geqslant 1-2\exp(-N_1^{\varepsilon_1}),$$

with $N_2 = \left[N^{\nu^2}\right]$. Therefore by (4.23),

$$\mathbb{P}(d_{[0,N]}(0,N) \leqslant 2(2N_2+1)+1) \geqslant 1-\exp(-N^{\varepsilon_1})-2\exp(-N_1^{\varepsilon_1}).$$

For any L, by continuing this procedure to the L^{th} step, we have

$$\mathbb{P}(d_{[0,N]}(0,N) \leqslant 2^L(N_L+1)) \geqslant 1-\sum_{i=0}^{L-1} 2^i \exp(-N_i^{\varepsilon_1}), \tag{4.25}$$

where $N_i = \left[N^{\nu^i}\right]$ for $i \geqslant 0$. We set

$$L = [\varphi(N)],$$

with

$$\varphi(N) = \frac{\log\log N - \psi(N)}{\log(1/\nu)} \quad \text{ and } \quad \psi(N) = \log(2/\varepsilon_1) + \log\log\log\log N.$$

Then

$$\exp(\exp(\psi(N))) = (\log\log N)^{2/\varepsilon_1},$$

and

$$\nu^{\varphi(N)} = \exp(\varphi(N)\log\nu) = \exp(\psi(N) - \log\log N) = \exp(\psi(N))/\log N.$$

Thus

$$N^{\nu^{\varphi(N)}} = \exp(\nu^{\varphi(N)}\log N) = \exp(\exp(\psi(N))) = (\log\log N)^{2/\varepsilon_1}.$$

Since $\varphi(N)-1 \leqslant L \leqslant \varphi(N)$ and $\nu<1$, we have

$$(\log\log N)^{2/\varepsilon_1} \leqslant N_L = \left[N^{\nu^L}\right] \leqslant (\log\log N)^{2/(\nu\varepsilon_1)}. \tag{4.26}$$

On the other hand

$$\begin{aligned}2^L = \exp(L\log 2) &\leqslant \exp(\varphi(N)\log 2)\\ &\leqslant \exp(\log\log N \log 2/\log(1/\nu))\\ &= \exp(\varkappa(2\nu)\log\log N)\\ &= (\log N)^{\varkappa(2\nu)}. \end{aligned}\tag{4.27}$$

It follows from (4.26) and (4.27) that for all N large enough

$$2^L(N_L+1) \leqslant (\log N)^{\varkappa(2\nu)}(\log\log N)^{2/(\nu\varepsilon_1)} \leqslant (\log N)^{\varkappa(s)+\varepsilon}, \tag{4.28}$$

since by the definition of ν, we have $\varkappa(2\nu) \leqslant \varkappa(s) + \varepsilon/2$.

On the other hand, for any $i \leqslant L$

$$\begin{aligned}2^i\exp(-N_i^{\varepsilon_1}) &\leqslant 2^L\exp(-N_L^{\varepsilon_1})\\ &\leqslant (\log N)^{\varkappa(2\nu)}\exp(-(\log\log N)^2)\\ &\leqslant (\log N)^{-2}, \end{aligned}\tag{4.29}$$

for N large enough. Then combining (4.25), (4.28) and (4.29), we obtain (4.21). □

Appendix: a lower bound on β_c

In [67], Proposition 3.7 (our Proposition 4.2.3) follows from Lemmas 3.9, 3.10, 3.11 and a conclusion argument. Let us find in their proof a lower bound on β_c.

At first, they define a constant $c = 2\alpha d + 1$ and some sequences

$$L_n = 2^{c^n} \quad \text{and} \quad R_n = L_1 \ldots L_n \quad \text{and} \quad \varepsilon_n = 2^{-2dc^{n+1}}.$$

In Lemma 3.9, the authors do not use any information on Z and β. They set a constant $k_0 = [2^{d+1}(c+1)]$.

In Lemma 3.10, they suppose that $\beta \leqslant 1$ and the information concerning Z is as follows. There exists n_0, such that for all $n \geqslant n_0$, we have

$$2^d\mathbb{E}(Z^\gamma)L_{n+1}^{-\mu} \leqslant 1/2,$$

with

$$\mu = \frac{\gamma - 1}{2\alpha} - 3d - 4\alpha d^2 > 0.$$

In fact, under the assumption $\mathbb{E}(Z^\gamma) \leqslant 1$, we can take

$$n_0 = \left\lceil \frac{\log\left(\frac{d+1}{\mu}\right)}{\log c} \right\rceil. \tag{4.30}$$

In Lemma 3.10, they also assume that $\beta \leqslant 1$ and define a constant n_1, such that $n_1 \geqslant n_0$ and for all $n \geqslant n_1$

$$3k_0^{\alpha+1} L_{n+1} \leqslant \frac{R_{n+1}}{20},$$

or equivalently,

$$60k_0^{\alpha+1} \leqslant R_n. \tag{4.31}$$

In the conclusion leading to the proof of [67, Proposition 3.7], a lower bound on β_c is implicit. Indeed, with Lemmas 3.9, 3.10, 3.11 in hand, the authors only require that

$$\mathbb{P}(\mathcal{E}(R_{n_1})) \geqslant 1 - \varepsilon_{n_1}, \tag{4.32}$$

where for any $N \geqslant 1$

$$\mathcal{E}(N) = \{\text{there exits a stable set } S \text{ such that } [\![N/5, 4N/5]\!]^d \subset S \subset [\![1, N]\!]^d\}.$$

We do not recall the definition of stable sets here. However, we notice that by the first part of Proposition 2.5 and Corollary 2.13 in [67], the event $\mathcal{E}(N)$ occurs when the weights of all vertices in $[\![1, N]\!]^d$ are less than $1/2$. Therefore

$$\begin{aligned}
\mathbb{P}(\mathcal{E}(N)) &\geqslant \mathbb{P}\left(r(x) \leqslant 1/2 \text{ for all } x \in [\![1, N]\!]^d\right) \\
&= \mathbb{P}(\beta Z \leqslant 1/2)^{N^d} \\
&= (1 - \mathbb{P}(\beta Z > 1/2))^{N^d} \\
&= \left(1 - \mathbb{P}(Z^\gamma > (2\beta)^{-\gamma})\right)^{N^d} \\
&\geqslant (1 - (2\beta)^\gamma \mathbb{E}(Z^\gamma))^{N^d}.
\end{aligned}$$

Hence (4.32) is satisfied if

$$(1 - (2\beta)^\gamma \mathbb{E}(Z^\gamma))^{R_{n_1}^d} \geqslant (1 - \varepsilon_{n_1}),$$

or equivalently

$$(2\beta)^\gamma \mathbb{E}(Z^\gamma) \leqslant 1 - (1-\varepsilon_{n_1})^{R_{n_1}^{-d}}.$$

Hence, under the assumption $\mathbb{E}(Z^\gamma) \leqslant 1$, we can take

$$\beta_c = \frac{1}{2}\left(1 - (1-\varepsilon_{n_1})^{R_{n_1}^{-d}}\right)^{1/\gamma},$$

with n_1 as in (4.31).

Chapter 5

Contact process on random geometric graphs

Abstract. In [67], the authors show that the critical value of the contact process on infinite random geometric graphs is positive. We will give an asymptotic behavior of this critical value when R -the connecting radius of the graphs tends to infinity. Moreover, when R is large enough, the contact process on random geometric graphs restricted to finite boxes survives a time super-exponential in the number of vertices.

5.1 Introduction

An infinite random geometric graph with connecting radius R, noted by $RGG(R)$, (resp. finite graph $RGG(n,R)$) is constructed as follows. The vertex set is composed of the atoms of a Poisson point process with intensity 1 on $\mathbb{R}^d$ (resp. on $[1, \sqrt[d]{n}]^d$). Then for any two vertices $v \neq w$, we draw an edge between them if $\|v-w\| \leqslant R$, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$.

Random geometric graphs have been extensively studied for a long time by many authors, see in particular Penrose's book [72]. Recently, these graphs have also been considered as models of wireless networks (see e.g. [51]). Therefore, there has been interest in processes occurring on it, including the contact process in both theoretical and practical approaches, see for example [47, 48, 73].

In [67], by introducing the notion *cumulatively merged partition* Ménard and Singh provide a sufficient condition on graphs, say G, ensuring that $\lambda_c(G)$ -the critical value of the contact process on G is positive. As an application, they prove that $\lambda_c(RGG(R)) > 0$ for all R finite, see Theorem 5.3.6. This is the first example of graphs with unbounded degree on which the contact process exhibits a non-trivial phase transition.

We will study the asymptotic behavior of $\lambda_c(RGG(R))$ when R tends to infinity and the extinction time of the contact process restricted on $RGG(n, R)$.

Theorem 5.1.1. *Let $d \geqslant 2$ and τ_n be the extinction time of the contact process on the graph $RGG(n, R)$ starting from full occupancy. Then there exist positive constants ε, c, C and K depending only on d, such that the following statements hold.*

(i) For all $\lambda, R > 0$

$$\mathbb{P}(\tau_n \leqslant \exp(Cn \log(\lambda R^d))) = 1 - o(1),$$

and

$$\mathbb{P}\big(\tau_n \geqslant n^{c \log \log n}\big) = 1 - o(1).$$

(ii) If $R^d \geqslant K/(\lambda \wedge 1)$, then

$$\mathbb{P}(\tau_n \geqslant \exp(cn \log(\lambda R^d))) = 1 - o(1)$$

and

$$\frac{\tau_n}{\mathbb{E}(\tau_n)} \xrightarrow[n\to\infty]{(\mathcal{L})} \mathcal{E}(1),$$

with $\mathcal{E}(1)$ an exponential random variable with mean one.

(iii) We have

$$\varepsilon/R^d \leqslant \lambda_c(RGG(R)) \leqslant K/R^d.$$

Theorem 5.1.1 shows that $\lambda_c(RGG(R)) \asymp 1/R^d$ as $R \to \infty$. Moreover, when R tends to infinity, for any positive infection rate λ, the extinction time of the contact process on $RGG(n, R)$ is super-exponential in n. On the other hand, (i) implies that the sub-critical regime in the phase transition **(T)** is slightly violated: the extinction time is not of logarithmic order when $\lambda < \lambda_c(RGG(R))$. In fact, we can construct a stronger counter-example.

Proposition 5.1.2. *There exists a sequence of random graphs G_n converging locally to an infinite graph G satisfying $\lambda_c(G) > 0$, such that for all $\lambda > 0$, the extinction time of the contact process on G_n is at least $\exp(|G_n|^c)$, for some constant $c > 0$.*

We now make some comments on the proof of Theorem 5.1.1. To obtain the asymptotic behavior of $\lambda_c(RGG(R))$, we improve some details in the proof of [67]. To prove the lower bound of the extinction time in (ii), we will find in $RGG(n, R)$ a subgraph composed of many adjacent complete graphs, see Lemma 5.3.2. Then we can compare the contact process with a super-critical oriented percolation. The upper bound in (i) follows from a quite general argument: the extinction of the contact process on a graph $G = (V, E)$ is at most $\exp(C|V|\log(|E|/|V|))$, for some positive constant C. The lower bound in (i) is a simple consequence of the appearance of a clique of size larger than $\log(\sqrt{n})$ in $RGG(n, R)$ and Lemma 5.2.2.

This chapter is organized as follows. In Section 2, we recall the main result in [67], then we prove some preliminary results on the contact process on complete graphs, the oriented percolation in two dimensions and long paths in a site percolation. The main theorem is proved in Section 3. In the last section, we prove the Proposition 5.1.2.

5.2 Preliminairies

5.2.1 The phase transition of the contact process on $RRG(R)$

The notion of cumulatively partition percolation (abbr. CMP) and some useful results applied to the contact process have been presented in Chapter 4, so we do not recall here. Using Theorem 4.2.4 ([67, Theorem 4.1]) and the following result, the authors prove that the critical value of the contact process on random geometric graphs is positive.

Theorem 5.2.1. *[67, Proposition 3.12] Consider the CMP on $RGG(R)$ with expansion exponent $\alpha \geqslant 1$ and degree weights*

$$r_\triangle(x) = \deg(x)1(\deg(x) \geqslant \triangle).$$

There exists positive constant $\triangle{=}\triangle\ (R)$, such that $\mathcal{C}(RGG(R), r_\triangle, \alpha)$ has no infinite clus-

ter. As a consequence,

$$\lambda_c(RGG(R)) > 0.$$

5.2.2 Contact process on complete graphs

We denote by K_m the complete graph of size m. Similarly to the results for the contact process on star graphs in [23, 65], we prove the following.

Lemma 5.2.2. *Assume that $\lambda \leqslant 1$ and $m\lambda \geqslant 640$. Then the following assertions hold.*

(i) Let (ξ_t) be the contact process on K_m. Then

$$\mathbb{P}\left(\inf_{T_m/2 \leqslant t \leqslant T_m} |\xi_t| \geqslant m/4 \mid |\xi_0| \geqslant m/4\right) \geqslant 1 - 2T_m^{-1},$$

with $T_m = \exp(m \log(\lambda m)/16)$.

(ii) Let K_m^1 and K_m^2 be two disjoint complete graphs of size m, and $K_{m,m}$ be the graph formed by adding an edge between these two graphs. Let (ξ_t) be the contact process on $K_{m,m}$. Then

$$\mathbb{P}(|\xi_{T_m} \cap K_m^2| \geqslant m/4 \mid |\xi_0 \cap K_m^1| \geqslant m/4) \geqslant 1 - 5T_m^{-1}.$$

Proof. Part (i) follows from the following claims

$$\mathbb{P}\left(\inf_{0 \leqslant t \leqslant T_m} |\xi_t| \geqslant m/4 \mid |\xi_0| \geqslant m/2\right) \geqslant 1 - T_m^{-1}, \tag{5.1}$$

$$\mathbb{P}\left(\exists\, t \leqslant T_m/2 : |\xi_t| \geqslant m/2 \mid |\xi_0| \geqslant m/4\right) \geqslant 1 - T_m^{-1}. \tag{5.2}$$

First, we observe that $|\xi_t|$ increases by 1 with rate $\lambda|\xi_t|(m - |\xi_t|)$ and decreases by 1 with rate $|\xi_t|$. Therefore, the skeleton of $(|\xi_t|)$ is a random walk (U_r) trapped at 0, which satisfies $U_0 = |\xi_0|$ and

$$U_{r+1} = U_r + 1 \quad \text{with probability } p_1 = \frac{\lambda(m - U_r)}{\lambda(m - U_r) + 1},$$

$$U_{r+1} = U_r + 1 \quad \text{with probability } 1 - p_1.$$

We now prove (5.1). Assume that $|\xi_0| \geqslant m/2$. Then $U_0 \geqslant m/2$. Moreover, if $U_r \in (m/4, 3m/4)$ then $p_1 \geqslant \lambda m/(\lambda m + 4)$. Hence, when $U_r \in (m/4, 3m/4)$, it stochastically dominates a random walk (X_r) satisfying $X_0 = m/2$ and

$$X_{r+1} = X_r + 1 \quad \text{with probability } \frac{\lambda m}{\lambda m + 4},$$
$$X_{r+1} = X_r - 1 \quad \text{with probability } \frac{4}{\lambda m + 4}.$$

Then θ^{X_r} is a martingale, where

$$\theta = \frac{4}{\lambda m}.$$

Let q be the probability that X_r goes below $m/4$ before hitting $3m/4$. It follows from the optional stopping theorem that

$$q\theta^{m/4} + (1-q)\theta^{3m/4} \leqslant \theta^{m/2}.$$

Therefore using $\lambda m \geqslant 640$, we get

$$q \leqslant \theta^{m/4} = (4/\lambda m)^{m/4} \leqslant T_m^{-3}/(2m^2). \tag{5.3}$$

Hence, the random walk (X_r) (and thus $(|\xi_t|)$) makes at least $[m^2 T_m]$ upcrossings between $m/2$ and $3m/4$ before hitting $m/4$ with probability larger than

$$1 - [m^2 T_m] T_m^{-3}/(2m^2) \geqslant 1 - T_m^{-1}/2. \tag{5.4}$$

The law of the waiting time between two upcrossings of $(|\xi_t|)$ stochastically dominates $\mathcal{E}(L)$, with $L = \lambda[m/2](m - [m/2]) + [m/2]$, the waiting time when $|\xi_t| = [m/2]$.

Suppose that $(|\xi_t|)$ makes more than $[m^2 T_m]$ consecutive upcrossings. Then the time that $(|\xi_t|)$ stays above $m/4$ stochastically dominates S, the sum of $[m^2 T_m]$ i.i.d. exponential random variables with mean $1/L$. By applying Chebyshev's inequality, we get

$$\mathbb{P}(S < [m^2 T_m]/2L) \leqslant 4/([m^2 T_m]) \leqslant T_m^{-1}/2. \tag{5.5}$$

Since $L \leqslant m^2/2$, we deduce (5.1) from (5.4) and (5.5).

We now prove (5.2). Assume that $|\xi_0| \geqslant m/4$. Using a similar argument as for (X_r), we get that when $U_r \in (m/8, m/2)$, it stochastically dominates a random walk (Y_r) satisfying $Y_0 = m/4$ and

$$Y_{r+1} = Y_r + 1 \text{ with probability } p_2 = \frac{\lambda m}{\lambda m + 2},$$
$$Y_{r+1} = Y_r - 1 \text{ with probability } 1 - p_2.$$

Let us define

$$\sigma_Y = \inf\{r : Y_r \geqslant m/2\} \qquad \text{and} \qquad \tilde{\sigma}_Y = \inf\{r : Y_r \leqslant m/8\}.$$

Then similarly to (5.3), we have

$$\mathbb{P}(\tilde{\sigma}_Y < \sigma_Y) \leqslant (2/\lambda m)^{m/8} \leqslant T_m^{-1}/3. \tag{5.6}$$

Since $Y_r - (2p_2 - 1)r$ is a martingale, it follows from the optional stopping theorem that

$$m/4 = \mathbb{E}(Y_{\sigma_Y \wedge r}) - (2p_2 - 1)\mathbb{E}(\sigma_Y \wedge r) \leqslant m/2 - (2p_2 - 1)\mathbb{E}(\sigma_Y \wedge r).$$

Therefore using $m\lambda \geqslant 640$, we get

$$\mathbb{E}(\sigma_Y \wedge r) \leqslant \frac{m}{4(2p_2 - 1)} \leqslant m/3.$$

Letting r go to infinity, we obtain

$$\mathbb{E}(\sigma_Y) \leqslant m/3.$$

Thus using Markov inequality, we have

$$\mathbb{P}(\sigma_Y \geqslant mT_m) \leqslant \mathbb{E}(\sigma_Y)/mT_m \leqslant T_m^{-1}/3. \tag{5.7}$$

Now, let us define

$$\sigma = \inf\{t : |\xi_t| \geqslant m/2\} \qquad \text{and} \qquad \tilde{\sigma} = \inf\{t : |\xi_t| \leqslant m/8\}.$$

Then by (5.6),

$$\mathbb{P}(\tilde{\sigma} < \sigma) \leqslant \mathbb{P}(\tilde{\sigma}_Y < \sigma_Y) \leqslant T_m^{-1}/3. \tag{5.8}$$

On the other hand, when $|\xi_t| \in (m/8, m/2)$ the waiting time at each stage is an exponential random variable with mean less than $1/M$, with $M = \lambda[m/8](m - [m/8]) + [m/8]$, the mean of the waiting time when $|\xi_t| = [m/8]$. Therefore

$$\sigma 1(\sigma < \tilde{\sigma}) \preceq \sum_{i=1}^{\sigma_Y} E_i,$$

where (E_i) is a sequence of i.i.d. exponential random variables with mean $1/M$ and independent of σ_Y. Hence

$$\begin{aligned}\mathbb{P}(T_m/2 \leqslant \sigma < \tilde{\sigma}) &\leqslant \mathbb{P}(\sigma_Y \geqslant mT_m) + \mathbb{P}\left(\sum_{i=1}^{[mT_m]} E_i \geqslant T_m/2\right) \\ &\leqslant 2T_m^{-1}/3. \end{aligned} \tag{5.9}$$

Here, we have used (5.7) to bound the first term, and for the second one we note that

$$\mathbb{E}(E_i) = 1/M \leqslant 64/(7\lambda m^2) \leqslant 1/(70m),$$

thus using a standard large deviation result, we get a bound for this term. Now, it follows from (5.8) and (5.9) that

$$\mathbb{P}\left(\sigma \geqslant T_m/2\right) \leqslant T_m^{-1},$$

which proves (5.2).

For (ii), let v and w be two vertices in K_m^1 and K_m^2 respectively, such that there is an edge between v and w. Let (ξ'_t) (resp. (ξ''_t)) be the contact process on K_m^1 (resp. K_m^2). By (i), we have

$$\mathbb{P}(\xi'_{T_m} \neq \varnothing \mid |\xi'_0| \geqslant m/4) \geqslant 1 - 2T_m^{-1}. \tag{5.10}$$

We now claim that

$$\mathbb{P}(\exists\, t \leqslant m^2/2 : |\xi''_t| \geqslant m/4 \mid \xi'_{m^2/2} \neq \varnothing) \geqslant e^{-m/4}. \tag{5.11}$$

To prove (5.11), it amounts to show that

$$\mathbb{P}(\exists\, t \leqslant m/2 : |\xi''_t| \geqslant m/4 \mid |\xi''_0| = 1) \geqslant 2e^{-m/4}, \tag{5.12}$$

and

$$\mathbb{P}(w \text{ gets infected before } m^2/4 \mid \xi'_{m^2/4} \neq \varnothing) \geqslant 1/2. \tag{5.13}$$

For (5.12), observe that when $|\xi''_t| \leqslant m/4$, it increases by 1 in the next stage with probability

$$\frac{\lambda(m - |\xi''_t|)}{\lambda(m - |\xi''_t|) + 1} \geqslant \frac{3\lambda m}{3\lambda m + 4} > 0.9,$$

as $\lambda m \geqslant 640$. Moreover, the waiting time to the next stage is an exponential random variable with mean less than 1. Therefore, the probability that in all the $[m/4]+1$ first stages, $|\xi''_t|$ increases and the waiting time is less than 1, is larger than

$$\big(0.9(1-e^{-1})\big)^{[m/4]+1} \geqslant 2e^{-m/4},$$

which implies (5.12). For (5.13), we note that

$$\{\xi'_{m^2/4} \neq \varnothing\} \subset \bigcap_{i=0}^{[m^2/8]-1} \mathcal{E}_i,$$

where

$$\mathcal{E}_i = \{\exists\, v_i \in K_m^1 : \xi'_{2i}(v_i) = 1\}.$$

We define

$\mathcal{I}_i = \mathcal{E}_i \cap \{$there is no recovery at v_i in $[2i, 2i+1]$ and there is an infection spread from v_i to v in $[2i, 2i+1]$, there is no recovery at v in $[2i, 2i+2]$ and there is an infection spread from v to w in $[2i+1, 2i+2]\}$.

If $v_i \equiv v$, we only consider the recovery in v and the infection spread from v to w. We see that if one of $(\mathcal{I}_i)$ occurs then w gets infected before $m^2/4$ and for any $i = 0, \ldots, [m^2/8]-1$

$$\mathbb{P}(\mathcal{I}_i \mid \cap_{j=0}^{i}\mathcal{E}_j) \geqslant e^{-3}(1-e^{-\lambda})^2 \geqslant \lambda^2/(4e^3), \tag{5.14}$$

as $\lambda \leqslant 1$. Therefore, by using induction we have

$$\begin{aligned}
&\mathbb{P}\left(w \text{ is not infected before } m^2/4\right) \\
&\leqslant \mathbb{P}\left(\left(\bigcup_{i=0}^{[m^2/8]-1} \mathcal{I}_i\right)^c \cap \left(\bigcap_{i=0}^{[m^2/8]-1} \mathcal{E}_i\right)\right) \\
&\leqslant (1-\lambda^2/(4e^3))^{[m^2/8]} \\
&\leqslant 1/2,
\end{aligned} \tag{5.15}$$

since $\lambda m \geqslant 640$. Thus (5.13) follows.

We now prove (ii) by using (5.11). Suppose that $\xi'_{T_m} \neq \varnothing$. We divide the time interval $[0, T_m/2]$ into $[T_m/m^2]$ small intervals of length $m^2/2$. Then (5.11) implies that in each interval with probability larger than $e^{-m/4}$, there is a time s, such that that $|\xi''_s| \geqslant m/4$. Hence, similarly to (5.15) we have

$$\mathbb{P}(\exists\, s \leqslant T_m/2 : |\xi''_s| \geqslant m/4 \mid \xi'_{T_m/2} \neq \varnothing) \geqslant 1 - (1 - e^{-m/4})^{[T_m/m^2]} \geqslant 1 - T_m^{-1}. \qquad (5.16)$$

Suppose that $|\xi''_s| \geqslant m/4$ with $s \leqslant T_m/2$. Then (i) implies that $|\xi''_{T_m}| \geqslant m/4$ with probability larger than $1 - 2T_m^{-1}$. Combining this with (5.10) and (5.16), we get (ii). $\square$

5.2.3 Oriented percolation on finite sets

For any positive integer ℓ, we consider an oriented percolation process on $[0, \ell]$ with parameter q defined as follows. Let

$$\Gamma = \{(i, k) \in [0, \ell] \times \mathbb{N} : i + k \text{ is even}\}.$$

We then independently draw an arrow from (i, k) to $(j, k + 1)$ with probability q, where $j = i - 1$ or $j = i + 1$. Given the initial configuration $A \subset [0, \ell]$, the oriented percolation $(\eta_t)_{t \geqslant 0}$ is defined by

$$\eta_t^A = \{i \in [0, \ell] : \exists\, j \in A \text{ s.t. } (j, 0) \leftrightarrow (i, t)\} \text{ for } t \in \mathbb{N},$$

where the notation $(j, 0) \leftrightarrow (i, t)$ means that there is an oriented path from $(j, 0)$ to (i, t). If $A = \{x\}$, we simply write (η_t^x). We call (η_t) a Bernoulli oriented percolation with parameter q.

The oriented percolation on $\mathbb{Z}$, denoted by $(\bar{\eta}_t)$, was investigated by Durrett in [36]. Using his results and techniques, we will prove the following.

Lemma 5.2.3. *Let (η_t) be the oriented percolation on $[0, \ell]$ with parameter q. Then there exist positive constants δ and c independent of q and ℓ, such that if $q \geqslant 1 - \delta$ then the following statements hold.*

(i) For any ℓ,

$$\mathbb{P}(\sigma_\ell \leqslant 2\ell) \geqslant c,$$

where

$$\sigma_\ell = \inf\{t : \eta_t^0(\ell) = 1\}.$$

(ii) For any ℓ,

$$\mathbb{P}(\eta^{\mathbf{1}}_{t_\ell} \neq \varnothing) \geqslant 1 - 1/t_\ell,$$

where $t_\ell = \left[(1-q)^{-c\ell}\right]$ and $(\eta^{\mathbf{1}}_t)$ is the oriented percolation starting with $\eta^{\mathbf{1}}_0 = [0,\ell]$.

(iii) There exist a positive constant $\beta \in (0,1)$ and an integer $s_\ell \in [\exp(c\ell), 2\exp(c\ell)]$, such that

$$\mathbb{P}(|\eta^{\mathbf{1}}_{s_\ell} \cap [(1-\beta)\ell/2, (1+\beta)\ell/2]| \geqslant 3\beta\ell/4) \geqslant 1 - \exp(-c\ell).$$

Proof. Part (i) is Theorem B.24 (a) in [59] and (ii) can be proved using a contour argument as in Section 10 in [36].

We now prove (iii). Let $(\bar{\eta}_t)$ be the oriented percolation on $\mathbb{Z}$. Then

$$\alpha = \mathbb{P}(\bar{\eta}_t{}^0 \neq \varnothing \,\forall\, t) \to 1 \quad \text{as} \quad q \to 1.$$

Hence we can assume that $\alpha > 3/4$. Now we define

$$\ell_1 = [(8-\alpha)\ell/16], \qquad \ell_2 = [(8+\alpha)\ell/16] \quad \text{and} \quad \ell_3 = [\ell/4].$$

Then for all $t \leqslant \ell_3$,

$$\bar{\eta}^{[\ell_1,\ell_2]}_t \subset [\ell_1 - t, \ell_2 + t] \subset [\ell_1 - \ell_3, \ell_2 + \ell_3] \subset [0,\ell].$$

Therefore, for any $A \subset [\ell_1, \ell_2]$

$$\left(\bar{\eta}^A_t\right)_{0 \leqslant t \leqslant \ell_3} \equiv \left(\eta^A_t\right)_{0 \leqslant t \leqslant \ell_3}.$$

Hence, to simplify notation, we use (η_t) for the both processes in the interval $[0,\ell_3]$. To prove (iii), our goal is to show that there exists a positive constant c, such that for any $A \subset [\ell_1, \ell_2]$ with $|A| \geqslant 3\alpha\ell/32$,

$$\mathbb{P}(|\eta^A_{\ell_3} \cap [\ell_1, \ell_2]| \geqslant 3\alpha\ell/32) \geqslant 1 - \exp(-c\ell). \tag{5.17}$$

Then repeatedly applying (5.17) implies (iii) with $\beta = \alpha/8$. (Note that $\ell_1 = [(1-\beta)\ell/2]$ and $\ell_2 = [(1+\beta)\ell/2]$). To prove (5.17), it suffices to show that there exists a positive constant c, such that

$$\mathbb{P}(|\eta^x_{\ell_3} \cap [\ell_1, \ell_2]| \geqslant 3\alpha\ell/32 \mid \eta^x_{\ell_3} \neq \varnothing) \geqslant 1 - \exp(-c\ell) \text{ for all } x \in [\ell_1, \ell_2], \tag{5.18}$$

$$\mathbb{P}(\eta^A_{\ell_3} \neq \varnothing) \geqslant 1 - \exp(-c\ell) \text{ for all } A \subset [\ell_1, \ell_2] \text{ with } |A| \geqslant 3\alpha\ell/32. \tag{5.19}$$

To prove (5.18), we define for any $A \subset \mathbb{Z}$ and $t \geqslant 0$

$$r^A_t := \sup\{x : \exists y \in A, (y,0) \leftrightarrow (x,t)\}$$
$$l^A_t := \inf\{x : \exists y \in A, (y,0) \leftrightarrow (x,t)\}.$$

Then (5.18) is a consequence of the following.

- It is not hard to see that if $[\ell_1, \ell_2] \subset [l^x_{\ell_3}, r^x_{\ell_3}]$, then

$$\eta^{\mathbf{1}}_{\ell_3} \cap [\ell_1, \ell_2] \equiv \eta^x_{\ell_3} \cap [\ell_1, \ell_2],$$

- $\mathbb{P}(|\eta^{\mathbf{1}}_{\ell_3} \cap [\ell_1, \ell_2]| \geqslant 3(\ell_2 - \ell_1)/4) \geqslant 1 - \exp(-c\ell)$, as $3/4 < \alpha$,
- $\mathbb{P}\left([l^x_{\ell_3}, r^x_{\ell_3}] \supset [\ell_1, \ell_2] \mid \eta^x_{\ell_3} \neq \varnothing\right) \geqslant 1 - \exp(-c\ell)$.

The second claim is a consequence of Theorem 1 in [42]. (Note that in [42] the result is proved for the contact process, but as mentioned by the author the proof works as well for oriented percolation). In fact, it still holds if we replace 3/4 by any positive constant strictly less than α. To prove the third one, we observe that if $\eta^x_{\ell_3} \neq \varnothing$ then

$$r^x_{\ell_3} = r^{(-\infty, x]}_{\ell_3}.$$

Moreover, by the main result of Section 11 in [36], there is a positive constant c, such that for all integer x,

$$\mathbb{P}\left(r^{(-\infty,x]}_{\ell_3} \leqslant x + \alpha\ell_3/2\right) \leqslant \exp(-c\ell).$$

Therefore if $x \in [\ell_1, \ell_2]$, then

$$\mathbb{P}\left(r^x_{\ell_3} \geqslant \ell_2 \mid \eta^x_{\ell_3} \neq \varnothing\right) \geqslant 1 - \exp(-c\ell),$$

since $x + \alpha\ell_3/2 \geqslant \ell_1 + \alpha\ell_3/2 \geqslant \ell_2$. Similarly

$$\mathbb{P}\left(l^x_{\ell_3} \leqslant \ell_1 \mid \eta^x_{\ell_3} \neq \varnothing\right) \geqslant 1 - \exp(-c\ell).$$

Then the third claim follows from the last two estimates.

To prove (5.19), we use the same argument as in Section 10 in [36]. We say that A is more spread out than B (and write $A \succ B$) if there is an increasing function φ from B into A such that $|\varphi(x) - \varphi(y)| \geqslant |x - y|$ for all $x, y \in B$. (Note that this implies $|A| \geqslant |B|$). In [36], Durrett proves that there is a coupling such that if $A \succ B$ then

$$\eta_t^A \succ \eta_t^B \text{ for all } t \geqslant 0,$$

and as a consequence $|\eta_t^A| \geqslant |\eta_t^B|$ for all t. Hence

$$\mathbb{P}(\eta_t^A = \varnothing) \leqslant \mathbb{P}(\eta_t^B = \varnothing).$$

On the other hand, by (ii)

$$\mathbb{P}\left(\eta_{\ell_3}^{[\ell_1, \ell_1 + \ell_4]} = \varnothing\right) \leqslant \exp(-c\ell),$$

with $\ell_4 = [3\alpha\ell/32] - 1$. We observe that $A \succ [\ell_1, \ell_1 + \ell_4]$ for any A with $|A| \geqslant 3\alpha\ell/32$. Thus (5.19) follows from the last two inequalities. $\square$

5.2.4 Long path in a site percolation.

The Bernoulli site percolation on $\mathbb{Z}^d$ with parameter p is defined as usual: designate each vertex in $\mathbb{Z}^d$ to be open independently with probability p and closed otherwise. A path in $\mathbb{Z}^d$ is called open if all its sites are open. Then there is a critical value $p_c^s(d) \in (0, 1)$, such that if $p > p_c^s(d)$, then a.s. there exists an infinite open path (cluster), whereas if $p < p_c^s(d)$, a.s. there is no infinite cluster.

Lemma 5.2.4. *Consider the Bernoulli site percolation on $[0, n]^d$ with $d \geqslant 2$ and $p > p_c^s(2)$. Then there exists a positive constant $\rho = \rho(p, d)$, such that w.h.p. there is an open path whose length is larger than ρn^d.*

Proof. We set $m = [n^{1/4}]$. For $n, d \geqslant 2$, we say that the box $[0, n]^d$ is ρ-**good** if the site percolation cluster on it satisfies:

there exist two vertices x in $\{m\} \times [0, n]^{d-1}$ and y in $\{n - m\} \times [0, n]^{d-1}$ and an open path composed of three parts: the first one included in $[0, m] \times [0, n]^{d-1}$ has length larger than m and ends at x; the second one included in $[m, n - m] \times [0, n]^{d-1}$ has length larger than ρn^d, starts at x and ends at y; the third one included in $[n-m,n]\times[0,n]^{d-1}$ starts at y and has length larger than m.

We now prove by induction on d that if $p > p_c^s(2)$, there is a positive constant $\rho_d = \rho(p,d)$, such that w.h.p. the box $[0,n]^d$ is ρ_d-good. Then Lemma 5.2.4 immediately follows.

When $d=2$, the statement follows from the main result in [50]. We will prove it for $d=3$, the proof for $d \geqslant 4$ is exactly the same and will not be reproduced here.

For $1 \leqslant i \leqslant n$, let $\Lambda_i = \{i\}\times[2m, n-2m]^2$. We define

$$n_1 = n - 4m \quad \text{and} \quad m_1 = [n_1^{1/4}].$$

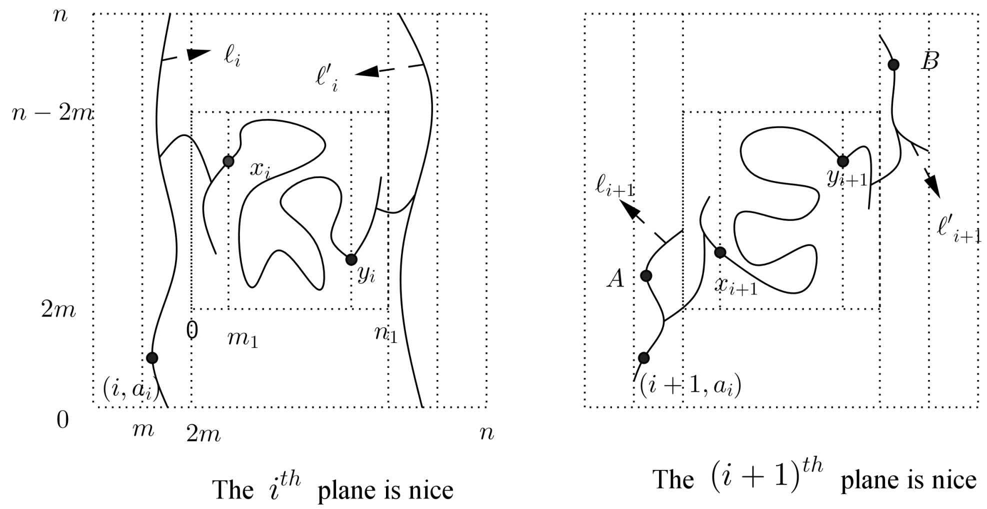
We say that the i^{th} plane is **nice** (or Λ_i is nice) if the site percolation on this plane satisfies: Λ_i is ρ_2-good (we consider Λ_i as a box in $\mathbb{Z}^2$); in each of the rectangles $\{i\}\times[m, 2m+m_1]\times[0,n]$ and $\{i\}\times[n-2m-m_1, n-m]\times[0,n]$, there is a unique connected component of size larger than m_1, see Figure 5.1 for a sample of a nice plane.

The result for $d=2$ implies that w.h.p. Λ_i is ρ_2-good. On the other hand, we know that w.h.p. in the percolation on a box of size n there is a unique open cluster having diameter larger than $C\log n$ for some C large enough (see for example Theorem 7.61 in [49]). Thus w.h.p. there is a unique open cluster of size larger $(C\log n)^d$. Hence Λ_i is nice w.h.p. for all $i=1,\ldots,n$. Moreover, the events $\{\Lambda_i \text{ is nice}\}$ are independent since the planes are disjoint. Therefore $\mathcal{A}_n$ holds w.h.p. with

$$\mathcal{A}_n = \{\#\{i : m \leqslant i \leqslant n-m, \Lambda_i \text{ is nice}\} \geqslant n/2\}.$$

On $\mathcal{A}_n$, there are more than $n/2$ disjoint open paths (they are in disjoint planes), each of which has length larger than $\rho_2 n_1^2$. Thus, to obtain an open path of length of order n^3, we will glue these long paths using shorter paths in good boxes of nice planes. To do that, we define

$\mathcal{B}_n = \{$ for all $1 \leqslant i \leqslant [n/2]$, there exist open paths: $\ell_{2i-1} \subset \{2i-1\}\times[m,2m]\times[0,n]$ whose end vertices are u and v with $u_3=0, v_3=n$; $\ell'_{2i-1} \subset \{2i-1\}\times[n-2m,n-m]\times[0,n]$ whose end vertices are u' and v' with $u'_3=0, v'_3=n$; $\ell_{2i} \subset \{2i\}\times[m,2m]\times[0,n]$ whose end vertices are z and t with $z_2=m, t_2=2m$; $\ell'_{2i} \subset \{2i\}\times[n-2m,n-m]\times[0,n]$ whose end vertices are z' and t' with $z'_2=n-2m, t'_2=n-m\}$.



Legend : $A = (i+1, a_{i+1}), B = (i+1, b_{i+1})$

Figure 5.1: Gluing two long paths.

We observe that ℓ_{2i-1} is a bottom-top crossing and ℓ_{2i} is a left-right crossing in two consecutive rectangles. Then they intersect when we consider only the last two coordinates, and the same holds for ℓ'_{2i-1} and ℓ'_{2i}. Hence on $\mathcal{B}_n$, for all $1 \leqslant i \leqslant n-2$, there exist $a_i \in [m, 2m] \times [0, n]$ and $b_i \in [n-2m, n-m] \times [0, n]$, such that

$$(i, a_i) \in \ell_i \text{ and } (i+1, a_i) \in \ell_{i+1},$$
$$(i, b_i) \in \ell'_i \text{ and } (i+1, b_i) \in \ell'_{i+1}.$$

In other word, we can jump from the i^{th} plane to the next one in two ways. Moreover, on $\mathcal{A}_n$ for all i such that the i^{th} plane is nice, the first part of the long open path in Λ_i is connected to ℓ_i (as these paths are in the same rectangle $\{i\} \times [m, 2m + m_1] \times [0, n]$ and have length larger than m_1), and similarly the third part is connected to ℓ'_i, see Figure 5.1.

On $\mathcal{A}_n \cap \mathcal{B}_n$, we can find in $[m, n-m] \times [0, n]^2$ a path of length larger than $\rho_2 n^3/3$. Indeed, let i be the first index, such that $i \geqslant m$ and Λ_i is nice. We start at an end point, from the right for example, of the long path in Λ_i, then go along this long path towards the other end point. Then we can go to ℓ_i and arrive at (i, a_i). Now we jump to $(i+1, a_i)$ (recall that it is a neighbor of (i, a_i)). If the $(i+1)^{th}$ plane is not nice, we go to $(i+1, a_{i+1})$ to jump to the next plane (note that both $(i+1, a_i)$ and $(i+1, a_{i+1})$ are in ℓ_{i+1}). If the $(i+1)^{th}$ plane is nice, we now can touch and then go along to the long path in this plane and arrive at $(i+1, b_{i+1})$ to jump to the next plane. By continuing this procedure, we can go through all the long paths of nice planes in the definition of $\mathcal{A}_n$. The resulting path is in $[m, n-m] \times [0, n]^2$ and has length larger than $\rho_2 n^3/3$.

Moreover, in the slabs $[0, m] \times [0, n]^2$ and $[n-m, n] \times [0, n]^2$, w.h.p. we can find two paths of length larger than m which are connected to the long path we have just found above. These paths form the required three-parts long path. Therefore on $\mathcal{A}_n \cap \mathcal{B}_n$, w.h.p. the box $[0, n]^3$ is ρ_3-good with $\rho_3 = \rho_2/3$.

Now it remains to show that $\mathcal{B}_n$ holds w.h.p. We observe that the probability of the existence of such a path ℓ_i is larger than $1 - \exp(-cm)$ for some $c > 0$ (see for instance (7.70) in [49]). Thus $\mathcal{B}_n$ holds w.h.p.

We summary here the change of proving the induction from $d-1$ to d when $d \geqslant 4$. First, in the definition of a *nice* box, we consider

$$\Lambda_i = \{i\} \times [2m, n-2m]^{d-1},$$

and the uniqueness of the connected component of size larger than m_1 in the slabs $\{i\} \times [m, 2m+m_1] \times [0, n]^{d-2}$ and $\{i\} \times [n-2m-m_1, n-m] \times [0, n]^{d-2}$. Secondly, in the definition of $\mathcal{B}_n$, we consider $\ell_i \subset \{i\} \times \{m\}^{d-3} \times [0, n]^2$ and $\ell_i' \subset \{i\} \times \{n-m\}^{d-3} \times [0, n]^2$, two bottom-top (resp. left-right) crossings in the last two coordinates when i is odd (resp. even). $\square$

5.3 Proof of Theorem 5.1.1

5.3.1 Proof of Theorem 5.1.1 (i)

Proof of the lower bound $\tau_n \geqslant n^{c \log \log n}$ w.h.p.

Let $x_n = [\sqrt[d]{n}/(R/\sqrt{d})]$ and we divide the box $[0, \sqrt[d]{n}]^d$ into x_n^d separated boxes with the same volume $(R/\sqrt{d})^d$. Then the numbers of vertices in these boxes form a sequence of independent Poisson random variables with the same mean $(R/\sqrt{d})^d$. We have

$$n\mathbb{P}(\mathrm{Poi}((R/\sqrt{d})^d) \geqslant \log(\sqrt{n})) \to \infty.$$

Therefore,

$$x_n^d \times \mathbb{P}(\text{There are more than } \log(\sqrt{n}) \text{ vertices in a box}) \to \infty.$$

Thus

$$\mathbb{P}(\text{There is at least a box having more than } \log(\sqrt{n}) \text{ vertices}) \to 1.$$

On the other hand, all vertices in a small box (of length $R/\sqrt{d}$) are connected. Hence,

$$\mathbb{P}(\text{There is a clique of size } [\log(\sqrt{n})] \text{ in } RGG(n,R)) \to 1.$$

Hence, by Lemma 5.2.2 (i), w.h.p. the extinction time is larger than $n^{c \log\log n}$, for some positive constant c. $\square$

Proof the upper bound $\log \tau_n \leqslant Cn \log(\lambda R^d)$ **w.h.p.**

We prove an upper bound on the extinction time of the contact process on an arbitrary graph.

Lemma 5.3.1. *Let τ_G be the extinction time of the contact process on a graph $G = (V, E)$ starting from full occupancy. Then*

(a) $\mathbb{P}(\tau_G \leqslant F(|V|, |E|)) \geqslant 1 - \exp(-|V|)$,

(b) $\mathbb{E}(\tau_G) \leqslant 2F(|V|, |E|)$

with

$$F(|V|, |E|) = |V| \left(2 + \frac{4\lambda|E|}{|V|}\right)^{|V|}.$$

Proof. Observe that (b) is a consequence of (a) and the following. For any $s > 0$

$$\mathbb{E}(\tau_G) \leqslant \frac{s}{\mathbb{P}(\tau_G \leqslant s)}.$$

This result is Lemma 4.5 in [64].

We now prove (a). Let us denote by (ξ_t) the contact process on G starting with full occupancy. By using Markov's property and the monotonicity of the contact process, it suffices to show that

$$\mathbb{P}(\xi_1 = \varnothing) \geqslant \exp(-|V| \log(2 + 4\lambda|E|/|V|)). \tag{5.20}$$

Observe that the process dies at time 1 if for any vertex v, it heals before 1 and does not infect any neighbor. Let σ_v be the time of the first recovery at v, then $\sigma_v \sim \mathcal{E}(1)$. Let $\sigma_{v\to}$ be the time of the first infection spread from v to one of its neighbors. Then it is the minimum of $\deg(v)$ i.i.d. exponential random variables with mean λ and thus $\sigma_{v\to} \sim \mathcal{E}(\lambda \deg(v))$. Moreover σ_v and $\sigma_{v\to}$ are independent. Therefore

$$\mathbb{P}(\sigma_v < \min\{\sigma_{v\to}, 1\}) = \frac{1 - e^{-(1+\lambda \deg(v))}}{1 + \lambda \deg(v)} \geqslant \frac{1}{2(1 + \lambda \deg(v))}.$$

On the other hand, these events $\{\sigma_v < \min\{\sigma_{v\to}, 1\}\}_v$ are independent. Then using Cauchy's inequality, we get that

$$\begin{aligned} \mathbb{P}(\xi_1 = \varnothing) \geqslant \prod_{v \in V} (2 + 2\lambda \deg(v))^{-1} &\geqslant \left(\frac{2|V| + 2\lambda \sum_{v\in V} \deg(v)}{|V|} \right)^{-|V|} \\ &= \left(2 + \frac{4\lambda|E|}{|V|} \right)^{-|V|}, \end{aligned}$$

which implies (5.20). □

The upper bound on τ_n follows from Lemma 5.3.1 and the following: w.h.p. $G(n, R) = (V_n, E_n)$ with

- $|V_n| \leqslant 2n$
- $|E_n| \leqslant CnR^d$, for some $C = C(d)$.

The first claim is clear, since $|V_n|$ is a Poisson random variable with mean n. For the second one, let $y_n = 1 + [\sqrt[d]{n}/R]$. Then we cover $[0, \sqrt[d]{n}]^d$ using translations by $R/2$ for each coordinate, accounting for $(2y_n - 1)^d$ boxes of volume R^d. We observe that points at distance larger than R are not connected. Hence, $|E_n|$ is less than the sum of the number of edges in the covering small boxes.

These small boxes are partitioned into 2^d groups such that each group contains at most y_n^d disjoint boxes with the same volume R^d.

The number of vertices in each box has the same distribution as W, a Poisson random variable with mean R^d. Hence, the number of edges in a box is stochastically dominated by W^2.

Moreover, in each group the numbers of edges are independent, as the boxes are disjoint. Therefore, using Chebyshev's inequality, the total number of edges in a group is w.h.p. smaller than

$$2y_n^d \mathbb{E}(W^2) = 2y_n^d R^d(R^d + 1).$$

Hence, $|E_n|$ is w.h.p. less than

$$2^{d+1} y_n^d R^d(R^d + 1) \leqslant CnR^d,$$

for some $C = C(d)$ large enough. □

5.3.2 Proof of Theorem 5.1.1 (ii)

We denote by $\mathcal{C}(\ell, m)$ (resp. $\mathcal{C}(\mathbb{Z}, m)$) the graph obtained by attaching a complete graph of size m to each vertex in $[\![0, \ell]\!]$ (resp. $\mathbb{Z}$).

Lemma 5.3.2. *There exist positive constants c and K, such that if $R^d \geqslant K$ then*

(a) w.h.p. $RGG(n, R)$ contains as a subgraph a copy of $\mathcal{C}([cnR^{-d}], [cR^d])$,

(b) almost surely $RGG(R)$ contains a copy of $\mathcal{C}(\mathbb{Z}, [cR^d])$.

Proof. We first prove (a). If n/R^d is bounded from above, then w.h.p. $RGG(n, R)$ contains a clique of size of order n and thus the result follows. Indeed, by definition the vertices in $A = [0, R/\sqrt{d}]^d$ form a complete graph. Moreover the number of vertices in A is a Poisson random variable with mean $R^d \asymp n$, and hence w.h.p. it is of order n.

We now assume that n/R^d tends to infinity. Let $z_n = [\sqrt[d]{n}/(R/2\sqrt{d})]$, we divide the box $[0, \sqrt[d]{n}]^d$ into z_n^d smaller boxes of equal size, numerated by $(E_a)_{a \in [\![1,\ell]\!]^d}$, whose side length is $R/(2\sqrt{d})$. We see that if v and w are in the same small box or in adjacent ones, then $\|v - w\| \leqslant R$, hence these two vertices are connected. This implies that the vertices on a small box form a clique and two adjacent cliques are connected.

For any $a \in [\![1, \ell]\!]^d$, let us denote by

$$X_a = \#\{v : v \in E_a\}$$

the number of vertices located in E_a. Then (X_a) are independent and X_a is a Poisson random variable with mean

$$\mu := \left(\frac{R}{2\sqrt{d}}\right)^d. \tag{5.21}$$

For any a, we define

$$Y_a = 1(\{X_a \geqslant \mu/2\}).$$

Since $\mathbb{P}(\mathrm{Poi}(\mu) \geqslant \mu/2) \to 1$ as $\mu \to \infty$, it follows from (5.21) that

$$\mathbb{P}(Y_a = 1) \to 1 \qquad \text{as} \qquad R \to \infty.$$

Therefore there is a positive constant K, such that if $R^d \geqslant K$, then

$$\mathbb{P}(Y_a = 1) \geqslant p := (1 + p_c^s(2))/2.$$

We note that the Bernoulli random variables (Y_a) are independent. Hence if we say the small box E_a open when $Y_a = 1$ and closed otherwise, then we get a site percolation on $[\![1, \ell]\!]^d$ which stochastically dominates the Bernoulli site percolation on $[\![1, \ell]\!]^d$ with parameter $p > p_c^s(2)$. Then Lemma 5.2.4 gives that w.h.p. there is an open path of length $\rho z_n^d \asymp nR^{-d}$. On the other hand, in each open box, there is a clique of size $\mu/2 \asymp R^d$ and these cliques in adjacent open boxes are connected. Hence, (a) has been proved.

The proof of (b) is similar. We divide the whole space $\mathbb{R}^d$ into small boxes of length $R/2\sqrt{d}$. Then the boxes having more than cR^d vertices (for some c small enough) form a percolation which stochastically dominate a supercritical site percolation on $\mathbb{Z}^d$. Thus almost surely there is an infinite path of adjacent boxes, each of which contains more than cR^d vertices. Therefore, almost surely there exists a copy of $\mathcal{C}(\mathbb{Z}, [cR^d])$ in $RGG(R)$. $\square$

Lemma 5.3.3. *Let $\tau_{\ell,M}$ be the extinction time of the contact process on $C(\ell, M)$ starting from full occupancy. Then there exist positive constants c and K independent of λ, such that if $\bar{\lambda} M \geqslant K$, then*

$$\mathbb{P}(\tau_{\ell,M} \geqslant \exp(c\ell M \log(\bar{\lambda} M))) \to 1 \quad \textit{as} \quad \ell \to \infty, \tag{5.22}$$

with $\bar{\lambda} = \lambda \wedge 1$.

Proof. Let (ξ_t) be the contact process on $\mathcal{C}(\ell, M)$ with parameter $\lambda > 0$. It is sufficient to consider the case $\lambda \leqslant 1$, since the contact process is monotone in λ. We assume also that $M\lambda \geqslant 640$.

For $i \in [\![0, \ell]\!]$, we say that it is **lit** at time t (the term is taken from [23]) if the number of infected vertices in its attached complete graph at time t is larger than $M/4$.

Let $T = \exp(M \log(\lambda M)/16)$. For $r \geqslant 0$ and $i, j \in [\![0, \ell]\!]$ s.t. $|i - j| = 1$ and $i + r$ is even, we define

$$\begin{aligned} Z_{i,j}^r =& 1(\{i \text{ is not lit at time } rT\}) \\ &+ 1(\{i \text{ is lit at time } rT \text{ and } i \text{ makes } j \text{ lit at time } (r+1)T\}), \end{aligned}$$

where "i makes j lit at time $(r+1)T$" means that

$$\begin{aligned} &|\{y \in C(j) : \exists\, x \in C(i) \cap \xi_{rT} \text{ s.t. } (x, rT) \longleftrightarrow (y, (r+1)T) \text{ inside } C(i) \cup C(j) \cup \{i, j\}\}| \\ &\geqslant M/4, \end{aligned}$$

with $C(i)$ the complete graph attached at i. Then $(Z_{i,j}^r)$ naturally define an oriented percolation by identifying

$$\{(i, r) \leftrightarrow (j, r+1)\} \Leftrightarrow \{Z_{i,j}^r = 1\}.$$

It follows from Lemma 5.2.2 (ii) that

$$\mathbb{P}\left(Z_{i,j}^r = 1 \mid \mathcal{F}_{rT}\right) \geqslant 1 - 5T^{-1} \quad \forall\, r \geqslant 0 \text{ and } |i - j| = 1,$$

where $\mathcal{F}_t$ denotes the sigma-field generated by the contact process up to time t.

Moreover if $x \neq i$ and $y \neq j$, then $Z_{x,y}^r$ is independent of $Z_{i,j}^r$. Hence by a result of Liggett, Schonmann and Stacey [60] (see also Theorem B26 in [59]) the distribution of the family $(Z_{i,j}^r)$ stochastically dominates the measure of a Bernoulli oriented percolation with parameter

$$q \geqslant 1 - T^{-\gamma},$$

with $\gamma \in (0, 1)$. Moreover, if λM is large enough, then $1 - T^{-\gamma} > 1 - \delta$, with δ as in Lemma 5.2.3.

In summary, when λM is large enough, the distribution of $(Z_{i,j}^r)$ stochastically dominates the one of an oriented percolation on $[\![0, \ell]\!]$ with density close to 1. On the other hand, it follows from Lemma 5.2.3 (ii) that w.h.p. the oriented percolation process survives up to the step

$$\big[(1-q)^{-c\ell}\big] \geqslant [T^{c\gamma\ell}] \geqslant \exp(c\gamma\ell M\log(\lambda M)),$$

for some constant $c > 0$. Hence the result has been proved. $\square$

We now prove a metastablity result for connected graphs containing a copy of $\mathcal{C}(\ell, M)$.

Lemma 5.3.4. *Let (G_n^0) be a sequence of connected graphs, such that $|G_n^0| \leqslant n$, for all n. Let τ_n denote the extinction time of the contact process on G_n^0 starting from full occupancy. Assume that G_n^0 contains as a subgraph a copy of $\mathcal{C}(\ell_n, M)$. Then there exists a positive constant K, such that if $M \geqslant K/(\lambda\wedge 1)$ and*

$$\frac{\ell_n}{d_n \vee \log n} \to \infty, \tag{5.23}$$

with d_n the diameter of G_n^0, then

$$\frac{\tau_n}{\mathbb{E}(\tau_n)} \quad \underset{n\to\infty}{\overset{(\mathcal{L})}{\longrightarrow}} \quad \mathcal{E}(1).$$

Proof. According to Proposition 1.2 in [62], it suffices to show that there exists a sequence (a_n), such that $a_n = o(\mathbb{E}(\tau_n))$ and

$$\sup_{v\in V_n} \mathbb{P}(\xi^v_{a_n} \neq \xi_{a_n}, \xi^v_{a_n} \neq \varnothing) = o(1), \tag{5.24}$$

where $(\xi_t)_{t\geqslant 0}$ denotes the process starting from full occupancy.

Set $\bar\lambda = \lambda \wedge 1$. By Lemma 5.3.3, we get that if $\bar\lambda M$ is large enough, then

$$\mathbb{E}(\tau_n) \geqslant \exp(c\ell_n M \log(\bar\lambda M)), \tag{5.25}$$

with c as in this lemma. By (5.23), there is a sequence (φ_n) tending to infinity, such that

$$\frac{\ell_n}{k_n} \to \infty, \tag{5.26}$$

with

$$k_n = [(\log n \vee d_n)\varphi_n].$$

Now define

$$b_n = s_{k_n} T \quad \text{and} \quad a_n = 2b_n + 1,$$

with s_{k_n} as in Lemma 5.2.3 (iii) and $T = \exp(M \log(\bar{\lambda} M)/16)$.

Then (5.25) and (5.26) show that $a_n = o(\mathbb{E}(\tau_n))$, so it remains to prove (5.24) for this choice of (a_n). We recall the definition of the dual contact process. Given some positive real t and A a subset of the vertex set V_n of G_n, the dual process $(\hat{\xi}_s^{A,t})_{s \leqslant t}$ is defined by

$$\hat{\xi}_s^{A,t} = \{v \in V_n : (v, t-s) \longleftrightarrow A \times \{t\}\},$$

for all $s \leqslant t$. For any v,

$$\begin{aligned} & \mathbb{P}(\xi_{a_n}^v \neq \xi_{a_n}, \xi_{a_n}^v \neq \varnothing) \\ = \; & \mathbb{P}(\exists w \in V_n : \xi_{a_n}^v(w) = 0,\, \xi_{a_n}^v \neq \varnothing,\, \hat{\xi}_{a_n}^{w,a_n} \neq \varnothing) \\ \leqslant \; & \sum_{w \in V_n} \mathbb{P}\left(\xi_{a_n}^v \neq \varnothing,\, \hat{\xi}_{a_n}^{w,a_n} \neq \varnothing, \text{ and } \hat{\xi}_{a_n - t}^{w,a_n} \cap \xi_t^v = \varnothing \text{ for all } t \leqslant a_n\right). \end{aligned} \tag{5.27}$$

So let us prove now that the last sum above tends to 0 when $n \to \infty$.

By the hypothesis, there are vertices $x_0, \dots, x_{k_n}$ in G_n^0 together with complete graphs of size M, $C(x_0), \dots, C(x_{k_n})$, which form a graph isomorphic to $\mathcal{C}(k_n, M)$. Now we slightly change the definition of a lit vertex, and say that x_i is lit if the number of its infected neighbors *in* $C(x_i)$ is larger than $M/4$ for $i = 0, \dots, k_n$.

We first claim that for any v

$$\mathbb{P}(\mathcal{A}(v)^c, \xi_{b_n}^v \neq \varnothing) = o(1/n), \tag{5.28}$$

where

$$\mathcal{A}(v) = \left\{\xi_{b_n}^v \neq \varnothing,\, |\{i \in [(1-\beta)k_n/2, (1+\beta)k_n/2] : x_i \text{ is lit at time } b_n\}| \geqslant 3\beta k_n/4\right\},$$

with β as in Lemma 5.2.3.

Suppose for a moment that (5.28) holds. On the other hand, the dual process has the same law as the original process. Therefore it also holds that for any w

$$\mathbb{P}(\hat{\mathcal{A}}(w)^c, \hat{\xi}_{b_n}^{w, 2b_n+1} \neq \varnothing) = o(1/n), \tag{5.29}$$

with

$$\hat{\mathcal{A}}(w) = \Big\{\hat{\xi}^{w,2b_n+1}_{b_n} \neq \varnothing,\ \exists\, S \subset [(1-\beta)k_n/2, (1+\beta)k_n/2] \text{ with } |S| \geqslant 3\beta k_n/4 \text{ and}$$
$$W_i \subset C(x_i) \text{ with } |W_i| \geqslant M/4\,\forall\, i \in S : (x, b_n+1) \longleftrightarrow (w, 2b_n+1)\,\forall\, x \in \cup_{i\in S} W_i\Big\}.$$

Note that $\mathcal{A}(v)$ and $\hat{\mathcal{A}}(w)$ are independent for all v and w. Moreover, on $\mathcal{A}(v)\cap\hat{\mathcal{A}}(w)$, there are more than $\beta k_n/2$ vertices which are lit in both the original and the dual processes. More precisely, there is a set $S \subset [(1-\beta)k_n/2, (1+\beta)k_n/2]$ with $|S| \geqslant \beta k_n/2$ and sets $U_i, W_i \subset C(x_i)$ with $|U_i|, |W_i| \geqslant M/4$ for all $i \in S$, such that

$$(v, 0) \longleftrightarrow (x, b_n) \quad \text{for all} \quad x \in \cup_{i\in S} U_i$$
$$(y, b_n+1) \longleftrightarrow (w, 2b_n+1) \quad \text{for all} \quad y \in \cup_{i\in S} W_i.$$

It is not difficult to show that there is a positive constant c, such that for any non-empty sets $U_i, W_i \subset C(x_i)$,

$$\mathbb{P}(U_i \times \{b_n\} \overset{C(x_i)}{\longleftrightarrow} W_i \times \{b_n+1\}) \geqslant c,$$

where the notation

$$U_i \times \{b_n\} \overset{C(x_i)}{\longleftrightarrow} W_i \times \{b_n+1\}$$

means that there is an infection path inside $C(x_i)$ from a vertex in U_i at time b_n to a vertex in W_i at time b_n+1.

Moreover, conditionally on the sets U_i, W_i, these events are independent. Therefore,

$$\mathbb{P}(\exists i : U_i \times \{b_n\} \overset{C(x_i)}{\longleftrightarrow} W_i \times \{b_n+1\} \mid U_i, W_i) \geqslant 1-(1-c)^{\beta k_n/2} = 1 - o(1/n),$$

by our choice of k_n. This implies that

$$\mathbb{P}(\mathcal{A}(v), \hat{\mathcal{A}}(w), \hat{\xi}^{w,a_n}_{a_n-t} \cap \xi^v_t = \varnothing \text{ for all } t \leqslant a_n) = o(1/n). \tag{5.30}$$

Combining (5.28), (5.29) and (5.30) we obtain (5.27). Now it remains to prove (5.28).

To see this, we define an oriented percolation $(\tilde{\eta}_r)_{r\geqslant 0}$ on $[0, k_n]$ similarly as in Lemma 5.3.3. For $0 \leqslant i, j \leqslant k_n$ and $r \geqslant 0$, such that $|i-j| = 1$ and $i + r$ is even, we let $Z^r_{i,j} = 1$ (or equivalently $(i, r) \leftrightarrow (j, r+1)$) if either x_i is not lit at time rT or x_i is lit at time rT and x_i makes x_j lit at time $(r+1)T$.

As in Lemma 5.3.3, there exists a positive constant K, such that if $\bar{\lambda}M \geqslant K$, then $(\tilde{\eta}_r)$ stochastically dominates a Bernoulli oriented percolation with parameter $1-\delta$, with δ as in Lemma 5.2.3.

Assume that d_n is even, if not we just take the smallest even integer larger than d_n. Then we set

$$\tilde{d}_n = d_n + 2k_n.$$

Now define for $k \geqslant 0$,

$$\tilde{\sigma}_{k_n}(k) = \inf\{r \geqslant k\tilde{d}_n + d_n : \tilde{\eta}_r^{0,k\tilde{d}_n+d_n}(k_n) = 1\},$$

where for any $A \subset [0, k_n]$ and $t \geqslant s \geqslant 0$,

$$\tilde{\eta}_t^{A,s} = \{x \in [0, k_n] : \exists y \in A, (y, s) \leftrightarrow (x, t)\}.$$

Note that $k\tilde{d}_n + d_n + 2k_n = (k+1)\tilde{d}_n$. Then using Lemma 5.2.3 (i), we get

$$\mathbb{P}(\tilde{\sigma}_{k_n}(k) \leqslant (k+1)\tilde{d}_n \mid \mathcal{F}_{k\tilde{d}_n+d_n}) \geqslant c. \tag{5.31}$$

We observe that if $\tilde{\sigma}_{k_n}(k) \leqslant (k+1)\tilde{d}_n$, then there is a horizontal crossing before $(k+1)\tilde{d}_n$. Hence,

$$\tilde{\eta}_r^{\mathbf{1}} = \tilde{\eta}_r^{0,k\tilde{d}_n+d_n} \qquad \text{for all } r \geqslant (k+1)\tilde{d}_n.$$

Define

$$\mathcal{E} = \Big\{|\tilde{\eta}_{s_{k_n}}^{\mathbf{1}} \cap [(1-\beta)k_n/2, (1+\beta)k_n/2]| \geqslant 3\beta k_n/4\Big\}.$$

On $\{\tilde{\sigma}_{k_n}(k) \leqslant (k+1)\tilde{d}_n\} \cap \mathcal{E}$, if $(k+1)\tilde{d}_n \leqslant s_{k_n}$ then

$$|\tilde{\eta}_{s_{k_n}}^{0,k\tilde{d}_n+d_n} \cap [(1-\beta)k_n/2, (1+\beta)k_n/2]| \geqslant 3\beta k_n/4. \tag{5.32}$$

Let $K_n = [s_{k_n}/\tilde{d}_n]$ and for any $0 \leqslant k \leqslant K_n - 1$

$$A_k = \{\xi_{k\tilde{d}_n}^v \neq \varnothing\},$$

and

$$\begin{aligned} B_k = &\Big\{\xi_{k\tilde{d}_n}^v \times \{k\tilde{d}_n\} \longleftrightarrow (x_0, (k\tilde{d}_n + d_n - 1)T)\Big\} \cap \{x_0 \text{ is lit at time } (k\tilde{d}_n + d_n)T\} \\ &\cap \{\tilde{\sigma}_{k_n}(k) \leqslant (k+1)\tilde{d}_n\}. \end{aligned}$$

We have

$$\{\xi^v_{b_n} \neq \varnothing\} \subset \bigcap_{k=0}^{K_n-1} A_k. \tag{5.33}$$

On the other hand, if x_0 is lit at time rT and $\tilde{\eta}^{0,r}_s(i) = 1$ for $s > r$, then x_i is lit at time sT. Hence by (5.32) on $\mathcal{E}$, if one of the events $(A_k \cap B_k)$ happens then $\mathcal{A}(v)$ occurs. Combing this with (5.33), we get

$$\{\xi^v_{b_n} \neq \varnothing\} \cap \mathcal{A}(v)^c \subset \mathcal{E}^c \cup \bigcap_{k=0}^{K_n-1} A_k \cap B_k^c. \tag{5.34}$$

Using Lemma 5.2.3 (iii), we obtain a bound for the first term

$$\mathbb{P}(\mathcal{E}^c) \leqslant \exp(-ck_n) = o(1/n), \tag{5.35}$$

by the choice of k_n. For the second term, by using a similar argument as for (5.14), we have

$$\mathbb{P}\left((v,t) \longleftrightarrow (x_0, t+(d_n-1)T)\right) \geqslant \exp(-C(d_n-1)T) \quad \text{for any } t \geqslant 0,$$

for some constant $C > 0$. On the other hand, if x_0 is infected at time t then it is lit at time $t+T$ with probability larger than $\exp(-CT)$. Therefore combing with (5.31), we get that for any $k \leqslant K_n - 1$,

$$\mathbb{P}(B_k^c \mid \mathcal{G}_k)\mathbf{1}(A_k) \leqslant 1 - c\exp(-Cd_nT),$$

where $\mathcal{G}_k = \mathcal{F}_{k\tilde{d}_n}$. Iterating this, we get

$$\mathbb{P}\left(\bigcap_{k=0}^{K_n-1} A_k \cap B_k^c\right) \leqslant (1 - c\exp(-Cd_nT))^{K_n-1} = o(1/n), \tag{5.36}$$

where the last equality follows from the definition of s_{k_n}. Combining (5.34), (5.35) and (5.36) we get (5.28) and finish the proof. □

Proof of Theorem 5.1.1 (i). Suppose that $\lambda \geqslant K/R^d$, with K as in Lemma 5.3.2. Then it follows from Lemma 5.3.2 that w.h.p. $RGG(n,R)$ contains a copy of $\mathcal{C}(\ell_n, M)$, with $\ell_n = [cnR^{-d}]$ and $M = [cR^d]$.

If ℓ_n is bounded (or $R^d \asymp n$), then $\mathcal{C}(\ell_n, M)$ contains a complete graph of size of order n. Then Lemma 5.2.2 (i) implies that w.h.p. the extinction time is larger than $\exp(cn\log(\lambda n))$, for some $c > 0$.

If ℓ_n tends to infinity, then the result follows from Lemma 5.3.3.

To prove the convergence in law of $\tau_n/\mathbb{E}(\tau_n)$, we recall some results about the diameter of the giant component and the size of small components in RGGs. There is a positive constant R_0, such that if $R > R_0$, then w.h.p.

(a) the diameter of the largest component is $d_n = \mathcal{O}(n^{1/d}/R)$,

(b) the size of the second largest component is $\mathcal{O}((\log n)^{d/(d-1)})$.

The first claim is proved in [47] (Corollary 6) and the second one is proved in [72] (Theorem 10.18).

The second claim together with Lemma 5.3.1 above shows that w.h.p. the extinction time of the contact process on $RGG(n, R)$ and on its largest component are equal. We are now in a position to complete the proof of (i).

- If $R^d = o(n/\log n)$, then by (a)
$$\frac{\ell_n}{d_n \vee \log n} \to \infty.$$
Therefore, Lemmas 5.3.2 and 5.3.4 imply the convergence in law of $\tau_n/\mathbb{E}(\tau_n)$.

- If $n/\log n = \mathcal{O}(R^d)$, then
$$\frac{D_{n,\max}}{d_n \vee \log n} \to \infty,$$
since $R^d = \mathcal{O}(D_{n,\max})$, with $D_{n,\max}$ the maximum degree in the largest component. Thus the result follows from Proposition 2.6.2 in Chapter 2.

□

5.3.3 Proof of Theorem 5.1.1 (iii)

Proof of the upper bound $\lambda_c \leqslant K/R^d$

Lemma 5.3.2 (b) shows that when $R^d \geqslant C$, almost surely $RGG(R)$ contains $\mathcal{C}(\mathbb{Z}, [cR^d])$, for some positive constants c and C. On the other hand, similarly to Lemma 5.3.3, by using the comparison the contact process with the oriented percolation, we have the following.

Lemma 5.3.5. *There exists a positive constant K independent of λ, such that if $\lambda M \geqslant K$, then*

$$\mathbb{P}(\xi_t \neq \varnothing \, \forall \, t \geqslant 0) > 0,$$

where (ξ_t) is the contact process on $\mathcal{C}(\mathbb{Z}, M)$ starting from a single vertex.

Hence, we get that if λR^d is large enough, then the contact process on $RGG(R)$ starting from a single vertex survives forever with positive probability. In other words,

$$\lambda_c(RGG(R)) \leqslant K/R^d,$$

with K large enough.

Proof of the lower bound $\lambda_c \geqslant \varepsilon/R^d$

The lower bound on λ_c follows from the two following propositions, which are improved versions of Theorem 4.1 and Proposition 3.2 in [67].

Proposition 5.3.6. *Let $G = (V, E)$ be a locally finite connected graph. Consider $\mathcal{C}(G, r_\triangle, \alpha)$ the CMP on G with expansion exponent α and degree weights*

$$r_\triangle(x) = \deg(x) 1(\deg(x) \geqslant \triangle).$$

Suppose that for some $\alpha \geqslant 5/2$ and $\triangle \geqslant 0$, the partition $\mathcal{C}(G, r_\triangle, \alpha)$ has no infinite cluster. Then there exists a positive constant ε, such that for any infection rate $\lambda \leqslant \varepsilon/\triangle$ the contact process starting from a single vertex dies out almost surely. In other words

$$\lambda_c(G) \geqslant \varepsilon/\triangle.$$

Proposition 5.3.7. *Consider the CMP on $RGG(R)$ with expansion exponent $\alpha \geqslant 1$ and degree weights*

$$r_\triangle(x) = \deg(x) 1(\deg(x) \geqslant \triangle).$$

Then there exists positive constant K, such that for $\triangle = K(R^d \vee 1)$, the $\mathcal{C}(RGG(R), r_\triangle, \alpha)$ has no infinite cluster.

Proof of Proposition 5.3.6. Suppose that $\mathcal{C} = \mathcal{C}(G, r_\triangle, \alpha)$ has no infinite cluster. The keys to prove Theorem 4.1 in [67] are the main estimates in Proposition 4.8 in [67]. More precisely, in [67] the authors show that the contact process on G with infection rate λ dies out almost surely if the following holds: For any cluster $C \in \mathcal{C}$ satisfying $r_{\triangle}(C) \leqslant K\,\triangle$, with K some large constant we have

$$\mathbb{E}\left[\begin{array}{c}\text{Total number of infections existing } \mathcal{S}_C^\eta \\ \text{for the contact process } \xi^C_{\mathcal{S}_C^\eta}\end{array}\right] \leqslant \exp(-(r_{\triangle}(C)+2)^{1.01})/2 \qquad \text{(I1)}$$

and

$$\mathbb{E}\left[\text{Extinction time of the contact process } \xi^C_{\mathcal{S}_C^\eta}\right] \leqslant \exp(3r_{\triangle}(C)), \qquad \text{(I2)}$$

where $\eta = 0.1$ and $\mathcal{S}_C^\eta$ is the η - stabiliser of the cluster C, and $\xi^C_{\mathcal{S}_C^\eta}$ is the contact process restricted to $\mathcal{S}_C^\eta$ starting from C.

We do not recall the definition of the η - stabiliser here. However, we note that this is a subgraph of G with degree (considering in G) bounded by $r_{\triangle}(C) \vee \triangle$. Thus by the assumption $r_{\triangle}(C) \leqslant K\,\triangle$, all the degrees in $\mathcal{S}_C^\eta$ are smaller than $K\,\triangle$.

Observe that the contact process is stochastically dominated by the branching random walk with the birth rate λ and death rate 1. Therefore, the number of infections existing $\mathcal{S}_C^\eta$ for the contact process $\xi^C_{\mathcal{S}_C^\eta}$ is stochastically dominated by the number of particles in $\partial\bar{\mathcal{S}}_C^\eta$ for $\zeta^C_{\bar{\mathcal{S}}_C^\eta}$ - the branching random walk on $\bar{\mathcal{S}}_C^\eta$ starting from C, where

$$\bar{\mathcal{S}}_C^\eta = \mathcal{S}_C^\eta \cup \{y : d(\mathcal{S}_C^\eta, y) = 1\},$$

and

$$\partial\bar{\mathcal{S}}_C^\eta = \{y : d(\mathcal{S}_C^\eta, y) = 1\}.$$

For $x \in C$ and $\ell \geqslant 0$, we define I_ℓ^x the number of particles at distance larger than ℓ from x of the branching random walk on $\bar{\mathcal{S}}_C^\eta$ starting from a particle at x. Then by using the fact that all vertices in $\bar{\mathcal{S}}_C^\eta$ have degree smaller than $K\,\triangle$, we get

$$\begin{aligned}\mathbb{E}(I_\ell^x) &\leqslant \sum_{k \geqslant \ell} \lambda^k |\{\text{walks of length } k \text{ in } \bar{\mathcal{S}}_C^\eta \text{ starting from } x\}| \\ &\leqslant \sum_{k\geqslant \ell} (\lambda K\,\triangle)^k \leqslant 2(\lambda K\,\triangle)^{-\ell},\end{aligned}$$

provided that $\lambda \leqslant 1/(2K\,\triangle)$. On the other hand, by Proposition 4.4 in [67], we have if $x \in C$ and $y \in \partial\mathcal{S}_C^\eta$ then

$$d(x,y) \geqslant \left\lceil \eta r_{\triangle}(C)^{5/2} \right\rceil := \ell(C). \qquad (5.37)$$

Therefore

$$\begin{aligned}
&\mathbb{E}\left[\text{Total number of infections existing } \mathcal{S}_C^\eta \text{ for the contact process } \xi^C_{\mathcal{S}_C^\eta}\right]\\
\leqslant\ &\mathbb{E}\left[\text{Total number of particles in } \partial\mathcal{S}_C^\eta \text{ for the branching random walk } \zeta^C_{\bar{\mathcal{S}}_C^\eta}\right]\\
\leqslant\ &\sum_{x\in C}\mathbb{E}(I^x_{\ell(C)}) \leqslant 2|C|(\lambda K\ \triangle)^{-\ell(C)}.
\end{aligned}$$

We observe that $r_\triangle(C)+1 \geqslant |C|$. Hence with $(\lambda K\ \triangle)$ small enough,

$$2|C|(\lambda K\ \triangle)^{-\ell(C)} \leqslant 2(r_\triangle(C)+1)(\lambda K\ \triangle)^{-\ell(C)} \leqslant \exp(-(r_\triangle(C)+2)^{1.01})/2$$

for all $r_\triangle(C)$. Hence, (I1) has been proved.

To prove (I2), we need the following estimate. Let τ^x be the extinction time of the contact process on a graph G with degree bounded by D. Then for any $\lambda \leqslant 1/2D$ there exists a positive constant C_λ satisfying $C_\lambda \to 0$ as $\lambda \to 0$, such that for all $t > 0$

$$\mathbb{P}(\tau^x > t) \leqslant C_\lambda e^{-c_0 t},$$

for some positive constant c_0 (independent of λ). This is consequence of Proposition 3.1 and Lemma 3.2 in [66]. Therefore

$$\mathbb{E}(\tau^x) \leqslant C_\lambda \int_0^\infty e^{-c_0 t}dt = C_\lambda/c_0.$$

Then using the fact that all vertices in $\mathcal{S}_C^\eta$ have degree less than $K\ \triangle$, we obtain that for $\lambda K\ \triangle$ small enough

$$\mathbb{E}\left[\text{Extinction time of the contact process } \xi^C_{\mathcal{S}_C^\eta}\right] \leqslant |C|C_\lambda/c_0 \leqslant \exp(3r_\triangle(C)).$$

In conclusion, when $\lambda K\ \triangle$ is small enough (I1) and (I2) hold and thus the contact process dies out a.s. In other words,

$$\lambda_c(G) \geqslant \varepsilon/\ \triangle,$$

for ε small enough. □

Proof of Proposition 5.3.7. Using the same argument as in the proof of Appendix in Chapter 4 for the lower bound of the critical parameter of the Bernoulli CMP, we have $\mathcal{C}(RGG(R), r_\triangle, \alpha)$ has no infinite cluster when the following inequality holds.

$$\mathbb{P}(\text{All vertices in } [0, R_{n_1}]^d \text{ have weight zero}) \geqslant 1 - \varepsilon_{n_1},$$

where R_{n_1} and ε_{n_1} are explicit constants as in the Appendix. This is equivalent to

$$\mathbb{P}(\text{All vertices in } [0, R_{n_1}]^d \text{ have degree less than } \triangle) \geqslant 1 - \varepsilon_{n_1}. \tag{5.38}$$

We have for all $L \geqslant 1$

$$\begin{aligned} & \mathbb{P}(\exists\, x \in [0, R_{n_1}]^d : \deg(x) \geqslant \triangle) \\ \leqslant\ & \mathbb{P}(\text{there are more than } LR_{n_1}^d \text{ vertices in } [0, R_{n_1}]^d) + LR_{n_1}^d \mathbb{P}(\deg(x) \geqslant \triangle) \\ \leqslant\ & \mathbb{P}(\mathrm{Poi}(R_{n_1}^d) \geqslant LR_{n_1}^d) + LR_{n_1}^d \mathbb{P}(\mathrm{Poi}(R^d) \geqslant \triangle). \end{aligned}$$

We take L large enough, such that the first term is less than $\varepsilon_{n_1}/2$. Then we take K large enough, such that the second term is less than $\varepsilon_{n_1}/2$ when $\triangle = K(R^d \vee 1)$.

In conclusion, if $\triangle = K(R^d \vee 1)$, then (5.38) holds and thus $\mathcal{C}(RGG(R), r_\triangle, \alpha)$ has no infinite cluster. □

5.4 Proof of Proposition 5.1.2

Let Y be a random variable whose the law is given by

$$\mathbb{P}(Y = k) \sim k^{-102} \quad \text{for} \quad k \geqslant 1. \tag{5.39}$$

Then

$$\mathbb{E}(Y^{100}) < \infty.$$

Hence, Proposition 4.2.3 implies that $\mathcal{C}(\mathbb{Z}, \beta Y, 5/2)$ has no infinite cluster for all $\beta \leqslant \beta_c$, with $\beta_c = \beta_c(Y, 1, 5/2)$.

Let (Z_i) be a sequence of i.i.d. random variables with the same law as Z with

$$\mathbb{P}(Z = k) \sim k^{-104} \quad \text{for} \quad k \geqslant 1. \tag{5.40}$$

We construct a sequence of random graphs (G_n) as follows. For all n, we start with the one dimensional torus $(\mathbb{Z} \mod n)$. Then at each vertex i, we attach a star graph of size Z_i and denote the set of vertices attached to i by S_i.

Let u_n be a vertex chosen uniformly in G_n. Then

$$\begin{aligned} \mathbb{P}(u_n \text{ is in } (\mathbb{Z} \mod n) \mid (Z_i)) &= \frac{n}{n + Z_1 + \ldots + Z_n} \\ &\to p := \frac{1}{1 + \mathbb{E}(Z)} \quad a.s., \end{aligned}$$

and

$$\mathbb{P}(u_n \text{ is in one of } (S_i) \mid (Z_i)) \quad \to \quad 1-p \quad a.s.$$

We denote by $s(u_n)$ the size of the star graph containing u_n. Then

$$\mathbb{P}(s(u_n) = k \mid (Z_i)) = \frac{(k+1)n_k}{Z_1 + \ldots + Z_n + n},$$

where

$$n_k = |\{i : Z_i = k\}|.$$

We observe that

$$n_k/n \to \mathbb{P}(Z = k) \quad a.s.$$

Therefore

$$\mathbb{P}(s(u_n) = k \mid (Z_i)) \to \frac{\mathbb{P}(Z+1 = k+1)(k+1)}{\mathbb{E}(Z)+1} \quad a.s.$$

This means that $s(u_n)$ converges in law to $\bar{Z}$, where $\bar{Z}$ is a random variable whose the law is given by

$$\mathbb{P}(\bar{Z} = k) = \frac{\mathbb{P}(Z+1 = k+1)(k+1)}{\mathbb{E}(Z)+1}. \tag{5.41}$$

Moreover, the size of other star graphs at a finite distance from u_n converges in law to the distribution of Z. Therefore, the random rooted graph (G_n, u_n) converges locally to a graph (G, o) constructed as follows.

We start with $\mathbb{Z}$. Then for all $i \neq 0$, we attach on i a star graph of size Z_i. For the vertex 0, we attach on it a star graph of size $\bar{Z}$, where $\bar{Z}$ is an independent random variable with law as (5.41). The root o is the vertex 0 with probability p and otherwise o is a vertex in the star graph attached to 0 (when $\bar{Z} = 0$, the root o is always 0).

We observe that in G, $\deg(i) = Z_i + 2$ for $i \neq 0$ and $\deg(0) = \bar{Z} + 2$, and the other vertices have degree 1. Hence for all $\triangle \geqslant 2$, in $\mathcal{C}(G, r_\triangle, 5/2)$ - the CMP on G with degree weights with parameter $\triangle$ - the total weight of a star graph at a vertex i is

$$r_\triangle(\{i\} \cup S_i) = r_\triangle(i) + \sum_{v \in S_i} r_\triangle(v) = r_\triangle(i) = \deg(i) 1(\deg(i) \geqslant \triangle). \tag{5.42}$$

Moreover, as $k \to \infty$,

$$\mathbb{P}(Z = k) \asymp k^{-104} \qquad \text{and} \qquad \mathbb{P}(\bar{Z} = k) \asymp k^{-103},$$

since Z is given in (5.40). Therefore, for $\triangle$ large enough,

$$(2+Z)1(Z \geqslant \triangle -2) \prec \beta_c Y \qquad \text{and} \qquad (2+\bar{Z})1(\bar{Z} \geqslant \triangle -2) \prec \beta_c Y, \tag{5.43}$$

since $\mathbb{P}(Y=k) \sim k^{-102}$ (where $\prec$ means stochastic domination).

Using (5.42) and (5.43), we get

$$r_{\triangle}(\{i\} \cup S_i) \prec \beta_c Y. \tag{5.44}$$

Therefore $\mathcal{C}(G, r_{\triangle}, 5/2)$ is stochastically dominated by the CMP $\mathcal{C}(\mathbb{Z}, \beta_c Y, 5/2)$, which almost surely has no infinite cluster by Proposition 4.2.3. Therefore, almost surely $\mathcal{C}(G, r_{\triangle}, 5/2)$ has no infinite cluster. Thus by Theorem 4.2.4, the critical value of the contact process on G is positive:

$$\lambda_c(G) > 0. \tag{5.45}$$

On the other hand,

$$\mathbb{P}(Z \geqslant k) \asymp k^{-103}.$$

Therefore,

$$\mathbb{P}(\exists 1 \leqslant i \leqslant n : Z_i \geqslant n^{1/104}) \to 1.$$

Thus

$$\mathbb{P}(G_n \text{ contains a star graph of size larger than } n^{1/104}) \to 1.$$

Combining this with Lemma 2.4.2 (i) implies that for all $\lambda > 0$,

$$\mathbb{P}(\text{The contact proces on } G_n \text{ survives up to time } \exp(n^{1/105})) \to 1. \tag{5.46}$$

It follows from (5.45) and (5.46) that there exits a sequence of random graphs (G_n) locally converges to a random rooted graph (G, o) satisfying $\lambda_c(G) > 0$, such that w.h.p. the extinction time of the contact process on G_n is at least $\exp(n^{1/105})$ for all $\lambda > 0$. $\square$

Chapter 6

Contact process on rank-one inhomogeneous random graphs

Abstract. We show that the contact process on the rank-one inhomogeneous random graphs and Erdos-Rényi graphs with mean degree large enough survives a time exponential in the size of these graphs for any positive infection rate. In addition, a metastable result for the extinction time is also proved.

6.1 Introduction

An inhomogeneous random graph (IRG), $G_n = (V_n, E_n)$, is defined as follows. Let $V_n = \{v_1, \ldots, v_n\}$ be the vertex set and let (w_i) be a sequence of i.i.d. positive random variables with the same law as w. Then for any $1 \leqslant i \neq j \leqslant n$, we independently draw an edge between v_i and v_j with probability

$$p_{i,j} = 1 - \exp(-w_i w_j / \ell_n),$$

where

$$\ell_n = \sum_{i=1}^{n} w_i.$$

It is shown in [53] that when $\mathbb{E}(w)$ is finite, G_n converges weakly to a two-stages Galton-Watson tree. In this tree, the reproduction law of the root is (p_k) and the one of other vertices is (g_k) with

$$p_k = \mathbb{P}(\mathrm{Poi}(w) = k) = \mathbb{E}\left(e^{-w}\frac{w^k}{k!}\right) \tag{6.1}$$

and

$$g_k = \mathbb{P}(\mathrm{Poi}(w^*) = k) = \frac{1}{\mathbb{E}(w)}\mathbb{E}\left(e^{-w}\frac{w^{k+1}}{k!}\right), \tag{6.2}$$

where w^* is the size-bias distribution of w. We also assume in addition that

$(H1)$ $w \geqslant 1$ a.s. and $\mathbb{E}(w) < \infty$,

$(H2)$ the limiting tree is super critical, or equivalently

$$\nu =: \mathbb{E}(g) = \frac{\mathbb{E}(w^2)}{\mathbb{E}(w)} > 1,$$

$(H3)$ there exits a function $\varphi(k)$ increasing to infinity, such that

$$\limsup_{k\to\infty} g_k e^{k/\varphi(k)} \geqslant 1.$$

Theorem 6.1.1. *Let τ_n be the extinction time of the contact process on inhomogeneous random graphs with the weight w satisfying the hypotheses $(H1)-(H3)$, starting from full occupancy. Then for any $\lambda > 0$, there exist positive constants c and C, such that*

$$\mathbb{P}(\exp(Cn) \geqslant \tau_n \geqslant \exp(cn)) \to 1 \quad \text{as } n \to \infty.$$

Moreover,

$$\frac{\tau_n}{\mathbb{E}(\tau_n)} \xrightarrow[n\to\infty]{(\mathcal{L})} \mathcal{E}(1),$$

with $\mathcal{E}(1)$ the exponential random variable with mean 1.

For simplicity, we will replace the hypothesis $(H3)$ by the following stronger version: there exits a function $\varphi(k)$ increasing to infinity, such that

$$g_k e^{k/\varphi(k)} \geqslant 1 \quad \text{for all } k \geqslant 1. \tag{H3$'$}$$

We will see in Section 3 that this assumption does not change anything to the proof.

We note that $(H1)$ is necessary for the weak convergence, $(H2)$ is essential since without it w.h.p. all components have size $o(n)$ and $(H3)$ is the key hypothesis in our proof.

We also remark that under $(H2)$ and $(H3)$, Pemantle proves in [74] that $\lambda_c(GW(g))$ is zero, with $GW(g)$ the Galton-Watson tree with reproduction law g.

It is worth noting that studying $\lambda_c(GW(g))$ when $g_k \asymp \exp(-ck)$ is still a challenge. An equivalently interesting problem is to study the extinction time of the contact process on a super-critical Erdos-Rényi graph, which is a special inhomogeneous graph and converges weakly to a Galton-Watson tree with Poisson reproduction law.

Let us make some comments on the proof of Theorem 6.1.1. The upper bound on τ_n follows from Lemma 5.3.1 in Chapter 5. To prove the lower bound, we will show that G_n contains a sequence of disjoint star graphs with large degree, whose total size is of order n. Moreover, the distance between two consecutive star graphs is not too large, so that the virus starting from a star graph can infect the other one with high probability. Then by comparing with an oriented percolation with density close to 1, we get the lower bound. The convergence in law can be proved similarly as in Chapter 5.

This chapter is organized as follows. In Section 2, we prove some preliminary results to describe the neighborhood of a vertex in the graph. In Section 3, by defining some exploration process of the vertices we prove the existence of the sequence of star graphs mentioned above. Then we prove our main theorem. In Section 4, we prove a similar result for the Erdos-Rényi graph: for any $\lambda > 0$, if the mean degree of the Erdos-Rényi graph is larger than some explicit function of λ, then the extinction time is also of exponential order.

Now we introduce some notation. We denote the indicator function of a set E by $\mathbf{1}(E)$. For any vertices v and w we write $v \sim w$ if there is an edge between them. We call size of a graph G the cardinality of its set of vertices, and we denote it by $|G|$. A graph in which all vertices have degree one, except one which is connected to all the others is called a *star graph*. The only vertex with degree larger than one is called the center of the star graph, or central vertex.

Furthermore we denote by $\mathrm{Bin}(n,p)$ the binomial distribution with parameters n and p and denote by $\mathrm{Poi}(\mu)$ the Poisson distribution with mean μ. Let X and Y be two random variables or two distributions, we write $X \preceq Y$ if X is stochastically dominated by Y. If f and g are two real functions, we write $f = \mathcal{O}(g)$ if there exists a constant $C > 0$, such that $f(x) \leqslant Cg(x)$ for all x; $f \asymp g$ if $f = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$; $f = o(g)$ if $f(x)/g(x) \to 0$ as $x \to \infty$. Finally for a sequence of random variables (X_n) and a function $f : \mathbb{N} \to (0,\infty)$, we say that $X_n \asymp f(n)$ holds w.h.p. if there exist positive constants c and C, such that $\mathbb{P}(cf(n) \leqslant X_n \leqslant Cf(n)) \to 1$, as $n \to \infty$.

6.2 Preliminaries

6.2.1 A preliminary result on the sequence of weights.

Lemma 6.2.1. *Let (w_i) be the sequence of i.i.d. weights as in the definition of IRGs. Then for any $\delta > 0$, there exists $\varkappa_1 = \varkappa_1(\delta) \in (0,1)$, such that*

$$\mathbb{P}\left(\sum_{i\in U} w_i \geqslant (1-\delta)\sum_{i=1}^{n} w_i \text{ for all } U \subset \{1,\ldots,n\} \text{ with } |U| \geqslant n(1-\varkappa_1)\right) \to 1.$$

Proof. Using the law of large numbers, we get

$$\mathbb{P}\left(\sum_{i=1}^{n} w_i = n(1+o(1))\mu\right) \to 1,$$

with

$$\mu = \mathbb{E}(w).$$

Hence, it is sufficient to show that for any $\delta > 0$, there exists $\varkappa_1 \in (0,1)$, such that

$$\mathbb{P}\left(\sum_{i\in U} w_i \geqslant n\mu(1-\delta) \text{ for all } U \subset \{1,\ldots,n\} \text{ with } |U| \geqslant n(1-\varkappa_1)\right) \to 1. \tag{6.3}$$

Let us define

$$\alpha = \sup\{t \geqslant 0 : \mathbb{E}(w1(w<t)) \leqslant \mu(1-\delta)\}.$$

Then $\alpha \in (0,+\infty)$ and $\mathbb{P}(w < \alpha) \in [0,1)$. We set

$$\beta = \mathbb{P}(w<\alpha) + \frac{\mu(1-\delta) - \mathbb{E}(w1(w<\alpha))}{\alpha}.$$

We now claim that

$$\beta < 1. \tag{6.4}$$

Indeed, it follows from the definition of α that for any $\varepsilon > 0$,

$$\mathbb{E}(w1(w < \alpha)) \leqslant \mu(1-\delta) \leqslant \mathbb{E}(w1(w < \alpha + \varepsilon)).$$

Therefore

$$\begin{aligned} \frac{\mu(1-\delta) - \mathbb{E}(w1(w<\alpha))}{\alpha} &\leqslant \frac{\mathbb{E}(w1(\alpha \leqslant w < \alpha+\varepsilon))}{\alpha} \\ &\leqslant \frac{\alpha+\varepsilon}{\alpha}\mathbb{P}(\alpha \leqslant w < \alpha + \varepsilon) \\ &\to \mathbb{P}(w = \alpha), \end{aligned}$$

as $\varepsilon \to 0$. Hence

$$\beta \leqslant \mathbb{P}(w \leqslant \alpha).$$

If $\mathbb{P}(w \leqslant \alpha) < 1$, then $\beta < 1$ and thus (6.4) is proved. Otherwise,

$$\mathbb{E}(w1(w < \alpha)) = \mathbb{E}(w) - \alpha\mathbb{P}(w = \alpha) = \mu - \alpha\mathbb{P}(w = \alpha).$$

Therefore

$$\begin{aligned} \beta &= \mathbb{P}(w < \alpha) + \frac{\mu(1-\delta) - \mathbb{E}(w1(w<\alpha))}{\alpha} \\ &= \mathbb{P}(w < \alpha) + \frac{\mu(1-\delta) - \mu + \alpha\mathbb{P}(w=\alpha)}{\alpha} \\ &= \mathbb{P}(w \leqslant \alpha) - \frac{\mu\delta}{\alpha} \\ &= 1 - \frac{\mu\delta}{\alpha} < 1, \end{aligned}$$

which proves (6.4). Now, we can define

$$\varkappa_1 = \frac{1-\beta}{2} \in (0,1).$$

Observe that to prove (6.3), it suffices to show that

$$\mathbb{P}\left(w_{(1)} + \ldots + w_{([(1-\varkappa_1)n])} \geqslant n\mu(1-\delta)\right) \to 1, \tag{6.5}$$

where $w_{(1)} \leqslant w_{(2)} \ldots \leqslant w_{(n)}$ is the order statistics of the sequence (w_i).

Using the law of large numbers, we have

$$\frac{|\Lambda|}{n} \to \mathbb{P}(w < \alpha) \quad a.s.$$
$$\frac{\sum_{i\in\Lambda} w_i}{n} \to \mathbb{E}(w1(w < \alpha)) \quad a.s.$$

where

$$\Lambda = \{i : w_i < \alpha\}.$$

Therefore, for any $\varepsilon > 0$, w.h.p.

$$\sum_{i\in\Lambda} w_i \geqslant n(\mathbb{E}(w1(w < \alpha)) - \varepsilon) \quad \text{and} \quad |\Lambda| \leqslant [\gamma n],$$

with

$$\gamma = \mathbb{P}(w < \alpha) + \varepsilon.$$

Hence for any $\varepsilon > 0$, w.h.p.

$$w_{(1)} + \ldots + w_{([\gamma n])} \geqslant n(\mathbb{E}(w1(w < \alpha)) - \varepsilon)$$

and

$$w_{(k)} \geqslant \alpha \qquad \text{for all} \quad k \geqslant [\gamma n] + 1.$$

Therefore for any $\varepsilon > 0$, w.h.p.

$$\begin{aligned} w_{(1)} + \ldots + w_{([\gamma n]+\ell)} &\geqslant n(\mathbb{E}(w1(w < \alpha)) - \varepsilon) + \ell\alpha \\ &\geqslant n\mu(1 - \delta), \end{aligned}$$

with

$$\ell = 1 + \left[n \left(\frac{\mu(1-\delta) - \mathbb{E}(w1(w < \alpha)) + \varepsilon}{\alpha} \right) \right].$$

On the other hand, by the definition of β, γ and ℓ, we have

$$[\gamma n] + \ell \leqslant 1 + n\,(\beta + \varepsilon + \varepsilon/\alpha) \leqslant [(1 - \varkappa_1)]n,$$

when $\varepsilon \leqslant \varkappa_1\alpha/2(1 + \alpha)$. Thus we get (6.5) and the result is proved. $\square$

6.2.2 Coupling of an IRG with a Galton-Watson tree

We describe here the neighborhood of a vertex in a set. Let $U \subset V_n$ and $v \in V_n \backslash U$, and let R be a positive integer. We denote by $B_R(v, U)$ the graph containing all vertices in U at distance less than or equal to R from v. We adapt the construction in [53, Vol. II, Section 3.4] to make a coupling between $B_R(v, U)$ and a *marked mixed-Poisson Galton-Watson tree.*

Conditionally on the weights (w_i), we define the *mark distribution* of U to be the random variable M_U with distribution

$$\mathbb{P}(M_U = m) = w_m / \ell_U \quad \text{for all indices } m \text{ such that } v_m \in U, \tag{6.6}$$

with $\ell_U = \sum w_i 1(v_i \in U)$. Note that $\ell_n = \ell_{V_n}$.

We define a random tree with root o as follows. We first define the mark of the root as $M_o = m$, with m the index such that $v_m = v$. Then o has X_o children, with

$$X_o \sim \mathrm{Poi}(w_{M_o} \ell_U / \ell_n).$$

Each child of the root, say x, is assigned an independent mark M_x, with the same distribution as M_U. Conditionally on M_x, the number of children of x has distribution $\mathrm{Poi}(w_{M_x} \ell_U / \ell_n)$.

Suppose that all the vertices at height smaller than or equal to i are defined. We determine the vertices at height $(i+1)$ as follows. Each vertex at height i, say y, has an independent mark M_y with the same distribution as M_U and it has X_y children, where X_y is a Poisson random variable with mean $w_{M_y} \ell_U / \ell_n$.

We denote the resulted tree by $\mathbb{T}(v, U)$ and call it the *marked mixed-Poisson Galton-Watson tree* associated to (v, U). In order to make a relation between $B_R(v, U)$ and $\mathbb{T}(v, U)$, we define a *thinning* procedure on $\mathbb{T}(v, U)$ as follows.

For a vertex y different from the root, we thin y when either one of the vertices on the unique path between the root and y has been thinned, or when $M_y = M_{y'}$, for some vertex y' on this path.

We denote by $\tilde{\mathbb{T}}(v, U)$ the tree resulting from the thinning on $\mathbb{T}(v, U)$.

Proposition 6.2.2. *[53, Vol. II, Proposition 3.10] Conditionally on (w_i), the set of vertices in U at distance k from v (considering the graph induced in $U \cup \{v\}$) has the same distribution as*

$$\Big(\{v_{M_x} : x \in \tilde{\mathbb{T}}(v, U) \text{ and } |x| = k\}\Big)_{k \geqslant 0},$$

with $|x|$ the height of x. Moreover, $B_R(v, U)$ contains a subgraph which has the same law as $\tilde{\mathbb{T}}_R(v, U)$- the graph containing all vertices in $\tilde{\mathbb{T}}(v, U)$ whose heights are smaller than or equal to R.

We note that in [53], the author only prove this proposition for $U = V \setminus \{v\}$. The proof for any subset of $V \setminus \{v\}$ is essentially the same, so we do not present here.

The law of the marked-mixed Poisson Galton-Watson tree. The offspring distribution of the root is given by

$$p_k^U = \mathbb{P}(\mathrm{Poi}(w_{M_o}\ell_U/\ell_n) = k) \quad \text{for} \quad k \geqslant 0.$$

The individuals of the second and further generations have the same offspring distribution, denoted by (g_k^U). It is given as follows: for all $k \geqslant 0$

$$g_k^U = \mathbb{P}(\mathrm{Poi}(w_{M_U}\ell_U/\ell_n) = k) = \sum_{v_i \in U} \mathbb{P}(\mathrm{Poi}(w_i\ell_U/\ell_n) = k) w_i/\ell_U.$$

If $U = V_n$, we write $g_k^{(n)}$ for g_k^U. Hence

$$g_k^{(n)} = \sum_{i=1}^{n} \mathbb{P}(\mathrm{Poi}(w_i) = k) w_i/\ell_n = \mathbb{P}(\mathrm{Poi}(W_n^*) = k),$$

with W_n^* the size-bias distribution of the empirical mean weight $W_n = (w_1 + \ldots + w_n)/n$. It is shown in [53] that since the (w_i) are i.i.d. with the same law as w and $\mathbb{E}(w)$ is finite,

$$W_n \xrightarrow[n\to\infty]{(\mathcal{L})} w \quad \text{and} \quad W_n^* \xrightarrow[n\to\infty]{(\mathcal{L})} w^*,$$

with w^* the size-bias distribution of w. Therefore we have the following convergence.

Lemma 6.2.3. *[53, Vol. II, Lemma 3.12]. For all $k \geqslant 0$,*

$$\lim_{n\to\infty} g_k^{(n)} = g_k,$$

with (g_k) as in (6.2).

Using Lemmas 6.2.1 and 6.2.3, we will show that the distribution (g_k^U) approximates (g_k), provided $|U|$ is large enough.

Lemma 6.2.4. *For any $\varepsilon > 0$ and $K \in \mathbb{N}$, there exists a constant $\varkappa_2 = \varkappa_2(\varepsilon, K) \in (0, \varkappa_1(1/2))$, such that*

$$\mathbb{P}\left(g_k^U \geqslant (1-\varepsilon)g_k \text{ for all } 0 \leqslant k \leqslant K \text{ and } U \subset V_n \text{ with } |U| \geqslant (1-\varkappa_2)n\right) \to 1.$$

Proof. If $g_k = 0$, then $g_k^U \geqslant (1-\varepsilon)g_k$. Assume that $k \leqslant K$ and $g_k > 0$. We have

$$|g_k^U - g_k| \leqslant |g_k^U - g_k^{(n)}| + |g_k^{(n)} - g_k|.$$

Lemma 6.2.3 implies that for all n large enough

$$|g_k^{(n)} - g_k| \leqslant \varepsilon g_k/2. \tag{6.7}$$

On the other hand,

$$\begin{aligned}
|g_k^U - g_k^{(n)}| &= \left|\sum_{v_i \in U} \mathbb{P}\left(\mathrm{Poi}(w_i \ell_U/\ell_n) = k\right) \frac{w_i}{\ell_U} - \sum_{v_i \in V_n} \mathbb{P}\left(\mathrm{Poi}(w_i) = k\right) \frac{w_i}{\ell_n}\right| \\
&\leqslant \sum_{v_i \in U} \left|\mathbb{P}\left(\mathrm{Poi}(w_i \ell_U/\ell_n) = k\right) \frac{w_i}{\ell_U} - \mathbb{P}\left(\mathrm{Poi}(w_i) = k\right) \frac{w_i}{\ell_n}\right| + \sum_{v_i \notin U} \frac{w_i}{\ell_n} \\
&\leqslant \sum_{v_i \in U} \left|\mathbb{P}\left(\mathrm{Poi}(w_i \ell_U/\ell_n) = k\right) - \mathbb{P}\left(\mathrm{Poi}(w_i) = k\right)\right| \frac{w_i}{\ell_U} \\
&\quad + \sum_{v_i \in U} \mathbb{P}\left(\mathrm{Poi}(w_i) = k\right) \left(\frac{w_i}{\ell_U} - \frac{w_i}{\ell_n}\right) + \sum_{v_i \notin U} \frac{w_i}{\ell_n} \\
&= S_1 + S_2 + S_3.
\end{aligned}$$

Here, we have used that $|x_i y_i - a_i b_i| \leqslant |x_i - a_i| y_i + |y_i - b_i| a_i$ for all $x_i, y_i, a_i, b_i \geqslant 0$.

We now define

$$f_k(x) = \mathbb{P}(\mathrm{Poi}(x) = k) = \frac{e^{-x} x^k}{k!}.$$

By the mean value theorem, for any $x < y$

$$|f_k(x) - f_k(y)| \leqslant \max_{x \leqslant u \leqslant y} |f_k'(u)| |x - y|.$$

If $k = 0$, then $|f_0'(u)| = f_0(u) = e^{-u}$. Therefore $|f_0'(u)| \leqslant |f_0'(x)|$ for all $x \leqslant u \leqslant y$. Hence

$$|f_0(x) - f_0(y)| \leqslant f_0(x)(y - x).$$

If $k \geqslant 1$ then for $u > 0$,

$$
\begin{aligned}
f_k'(u) &= e^{-u}\left(\frac{u^{k-1}}{(k-1)!} - \frac{u^k}{k!}\right), \\
f_k''(u) &= e^{-u}(u^2 - 2ku + k(k-1))\frac{u^{k-2}}{k!} \\
&= e^{-u}(u - (k - \sqrt{k}))(u - (k + \sqrt{k}))\frac{u^{k-2}}{k!}.
\end{aligned}
$$

Thus $|f_k'(u)|$ is decreasing when $u \geqslant 2k$. Hence, for $2k \leqslant x \leqslant u$,

$$|f_k'(u)| \leqslant |f_k'(x)| \leqslant f_k(x).$$

On the other hand, $|f_k'(u)| \leqslant 1$ for all $u \geqslant 0$. Therefore

$$\max_{x \leqslant u \leqslant y} |f_k'(u)| \leqslant 1(x \leqslant 2k) + f_k(x).$$

In summary, for all k and $0 \leqslant x \leqslant y$

$$|f_k(x) - f_k(y)| \leqslant (1(x \leqslant 2k) + f_k(x))(y - x). \tag{6.8}$$

Applying (6.8), we get that if $\ell_U \geqslant \ell_n/2$ then

$$
\begin{aligned}
|f_k(w_i\ell_U/\ell_n) - f_k(w_i)| &\leqslant (1(w_i\ell_U/\ell_n \leqslant 2k) + f_k(w_i\ell_U/\ell_n))\,\frac{w_i(\ell_n - \ell_U)}{\ell_n} \\
&\leqslant (1(w_i \leqslant 4k) + e^{-w_i/2}w_i^k/k!)\frac{w_i(\ell_n - \ell_U)}{\ell_n}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
S_1 &= \sum_{v_i \in U} |f_k(w_i\ell_U/\ell_n) - f_k(w_i)|\frac{w_i}{\ell_U} \\
&\leqslant \sum_{i=1}^{n} \frac{w_i^2(\ell_n - \ell_U)}{\ell_U\ell_n} 1(w_i \leqslant 4k) + \sum_{i=1}^{n} \frac{e^{-w_i/2}w_i^{k+2}}{k!}\,\frac{(\ell_n - \ell_U)}{\ell_U\ell_n} \\
&\leqslant \frac{4k(\ell_n - \ell_U)}{\ell_U} + \left(\frac{1}{\ell_n}\sum_{i=1}^{n} \frac{e^{-w_i/2}w_i^{k+2}}{k!}\right)\left(\frac{\ell_n - \ell_U}{\ell_U}\right).
\end{aligned}
$$

Observe that

$$\frac{1}{n}\sum_{i=1}^{n} \frac{e^{-w_i/2}w_i^{k+2}}{k!} \quad \to \quad \mathbb{E}(e^{-w/2}w^{k+2}/k!) < \infty.$$

On the other hand, $\ell_n \asymp n$. Therefore

$$\frac{1}{\ell_n}\sum_{i=1}^{n} \frac{e^{-w_i/2}w_i^{k+2}}{k!} = \mathcal{O}(1).$$

Hence

$$S_1 = \mathcal{O}\left(\frac{k(\ell_n - \ell_U)}{\ell_U}\right).$$

Moreover,

$$\begin{aligned} S_2 &= \sum_{v_i \in U} \mathbb{P}(\mathrm{Poi}(w_i) = k)\left|\frac{w_i}{\ell_U} - \frac{w_i}{\ell_n}\right| \\ &\leqslant \sum_{i=1}^{n} \frac{w_i(\ell_n - \ell_U)}{\ell_U \ell_n} \\ &= \frac{\ell_n - \ell_U}{\ell_U}, \end{aligned}$$

and

$$S_3 = \sum_{v_i \notin U} \frac{w_i}{\ell_n} \leqslant \frac{\ell_n - \ell_U}{\ell_U}.$$

In conclusion, if $\ell_U \geqslant \ell_n/2$ and $\ell_n \asymp n$ then

$$|g_k^U - g_k^{(n)}| \leqslant S_1 + S_2 + S_3 = \mathcal{O}\left(\frac{k(\ell_n - \ell_U)}{\ell_U}\right).$$

Therefore, there exists $\delta_k = \delta_k(k, \varepsilon, g_k) > 0$, such that if $\ell_U \geqslant (1-\delta_k)\ell_n$ then

$$|g_k^U - g_k^{(n)}| \leqslant \varepsilon g_k/2. \tag{6.9}$$

Define $\delta = \min\{\delta_k : k \leqslant K \text{ and } g_k > 0\}$. Then we have $\delta > 0$. Now, by Lemma 6.2.1 there exists $\varkappa_2 = \varkappa_1(\delta) \wedge \varkappa_1(1/2) \in (0,1)$, such that w.h.p. for all $U \subset V_n$ with $|U| \geqslant (1-\varkappa_2)n$,

$$\ell_U \geqslant (1-\delta)\ell_n.$$

Thus by (6.7) and (6.9), w.h.p. for all $U \subset V_n$ with $|U| \geqslant (1-\varkappa_2)n$,

$$g_k^U \geqslant (1-\varepsilon)g_k \text{ for all } k \leqslant K,$$

which proves the result. □

For any $\varepsilon \in (0,1)$ and $K \in \mathbb{N}$, we define a distribution $(g_{\cdot}^{\varepsilon,K})$ as follow:

$$\begin{aligned} g_k^{\varepsilon,K} &= 0 && \text{if} \quad k \geqslant K+1, \\ g_k^{\varepsilon,K} &= (1-\varepsilon)g_k && \text{if} \quad 1 \leqslant k \leqslant K, \\ g_0^{\varepsilon,K} &= 1 - (1-\varepsilon)\sum_{k=1}^{K} g_k. \end{aligned}$$

The following result is a direct consequence of Lemma 6.2.4.

Lemma 6.2.5. *For any $\varepsilon > 0$ and $K \in \mathbb{N}$,*

$$\mathbb{P}\Big((g_\cdot^{\varepsilon,K}) \preceq (g_\cdot^U) \text{ for all } U \subset V_n \text{ with } |U| \geqslant (1-\varkappa_2)n\Big) \to 1,$$

with $\varkappa_2$ as in Lemma 6.2.4.

Observe that $(g_\cdot^{\varepsilon,K})$ stochastically increases (resp. decreases) in K (resp. ε). Moreover, it converges to $(g_\cdot)$ as $\varepsilon \to 0$ and $K \to \infty$. Therefore, by the hypothesis (H_2), there are positive constants ε_0 and K_0, such that for all $\varepsilon \leqslant \varepsilon_0$ and $K \geqslant K_0$,

$$\nu_{\varepsilon,K} := \sum_{k=0}^{\infty} k g_k^{(\varepsilon,K)} \geqslant \bar{\nu}, \tag{6.10}$$

where

$$\bar{\nu} = \frac{1+\nu}{2} \in (1,\nu).$$

Define for $K \geqslant K_0$,

$$\mathcal{E}(K) = \{(g_\cdot^{\varepsilon_0,K}) \preceq (g_\cdot^U) \text{ and } \ell_U \geqslant \ell_n/2 \text{ for all } U \subset V_n \text{ with } |U| \geqslant (1-\varkappa_2)n\},$$

with $\varkappa_2 = \varkappa_2(\varepsilon_0, K)$ as in Proposition 6.2.5. Using this proposition and Lemma 6.2.1 with the fact that $\varkappa_2 \leqslant \varkappa_1(1/2)$, we obtain

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{E}(K)) \to 1. \tag{6.11}$$

We call $T_{\varepsilon,K}$ the Galton-Watson tree with reproduction law $(g_\cdot^{\varepsilon,K})$. Then (6.10) implies that $T_{\varepsilon,K}$ is super critical when $\varepsilon \leqslant \varepsilon_0$ and $K \geqslant K_0$. From now on, we set

$$\psi_1(K) = [K/\sqrt{\varphi(3K)}],$$

with the function φ as in the hypothesis $(H3')$.

We prove here a key lemma saying that when U is large enough, with positive probability there exists a vertex in $\tilde{\mathbb{T}}(v,U)$ at distance less than $\psi_1(K)$ from the root having more than $3K$ children (which implies that there exists a vertex with degree larger than $3K$ in $B_{\psi_1(K)}(v,U)$ with positive probability).

Lemma 6.2.6. *There are positive constants θ_1 and K_1, such that for all $K \geqslant K_1$ and $U \subset V_n$ with $|U| \geqslant (1-\varkappa_2)n$ and n large enough,*

$$\mathbb{P}\Big(\exists\, x \in \tilde{\mathbb{T}}(v,U) : |x| \leqslant \psi_1(K), \deg(x) \geqslant 3K+1 \mid \mathcal{E}(K)\Big) \geqslant \theta_1,$$

with $\varkappa_2$ as in Lemma 6.2.5.

Proof. If w_v -the weight of v- is larger than $10K$, then $\deg(o)$ is larger than $3K$ with positive probability. Indeed, $\deg(o)$ is a Poisson random variable with parameter $w_v\ell_U/\ell_n$. Moreover, on $\mathcal{E}(K)$, we have $\ell_U \geqslant \ell_n/2$. Therefore

$$\begin{aligned} \mathbb{P}(\deg(o) \geqslant 3K \mid \mathcal{E}(K)) &= \mathbb{P}(\mathrm{Poi}(w_v\ell_U/\ell_n) \geqslant 3K \mid \mathcal{E}(K)) \\ &\geqslant \mathbb{P}(\mathrm{Poi}(5K) \geqslant 3K) > 0. \end{aligned}$$

Hence, the result follows. We now suppose that $w_v \leqslant 10K$. Then, in the proof of [53, Vol. II, Corollary 3.13], it is shown that for any ℓ

$$\mathbb{P}(\mathbb{T}_\ell(v,U) \equiv \tilde{\mathbb{T}}_\ell(v,U)) \to 1 \quad \text{as} \quad n \to \infty. \tag{6.12}$$

We denote by

$$Z_\ell^{v,U} = |\{x \in \mathbb{T}(v,U), |x| = \ell\}|.$$

By Lemma 6.2.5, conditionally on $\deg(o) \geqslant 1$,

$$Z_{\ell+1}^{v,U} \succeq Z_\ell^{\varepsilon_0,K}, \tag{6.13}$$

with $Z_\ell^{\varepsilon_0,K}$ the number of individuals at the ℓ^{th} generation of $T_{\varepsilon_0,K}$. We remark that

$$\mathbb{P}(Z_\ell \geqslant m^{\ell/2} \mid Z_\ell \geqslant 1) \to 1 \qquad \text{as } \ell \to \infty,$$

where Z_ℓ is the number of individuals at the ℓ^{th} generation of a Galton-Watson tree T with mean $m > 1$. Therefore, for all ℓ large enough

$$\mathbb{P}(Z_\ell \geqslant m^{\ell/2}) \geqslant \mathbb{P}(|T| = \infty)/2.$$

Hence, for $K \geqslant K_0$

$$\mathbb{P}(Z_\ell^{\varepsilon_0,K} \geqslant \bar{\nu}^{\ell/2}) \geqslant \mathbb{P}(Z_\ell^{\varepsilon_0,K_0} \geqslant \bar{\nu}^{\ell/2}) \geqslant \mathbb{P}(|T_{\varepsilon_0,K_0}| = \infty)/2. \tag{6.14}$$

It follows from (6.13) and (6.14) that for all ℓ large enough,

$$\begin{aligned} \mathbb{P}(Z_{\ell+1}^{v,U} \geqslant \bar{\nu}^{\ell/2} \mid \mathcal{E}(K)) &\geqslant \mathbb{P}(\deg(o) \geqslant 1 \mid \mathcal{E}(K)) \times \mathbb{P}(|T_{\varepsilon_0,K_0}| = \infty)/2 \\ &\geqslant (1 - e^{-w_v/2}) \times \mathbb{P}(|T_{\varepsilon_0,K_0}| = \infty)/2 \\ &\geqslant \mathbb{P}(|T_{\varepsilon_0,K_0}| = \infty)/8. \end{aligned} \tag{6.15}$$

Here, we have used that $w_v \geqslant 1$. Now, we set

$$\ell = \psi_1(K) - 1.$$

Then we have

$$\begin{aligned} & \mathbb{P}\left(\exists\, x \in \mathbb{T}(v,U) : |x| = \ell+1, \deg(x) \geqslant 3K+1 \mid Z_{\ell+1}^{v,U} \geqslant \bar{\nu}^{\ell/2}\right) \\ \geqslant\ & \mathbb{P}\left(\mathrm{Bin}\left([\bar{\nu}^{\ell/2}], g_{3K}\right) \geqslant 1\right) \to 1 \qquad \text{as} \qquad K \to \infty, \end{aligned} \tag{6.16}$$

since under $(H3')$,

$$\bar{\nu}^{\ell/2} g_{3K} \geqslant \bar{\nu}^{\ell/2} \exp(-3K/\varphi(3K)) \to \infty \quad \text{as} \quad K \to \infty.$$

Now, the result follows from (6.12), (6.15) and (6.16). □

6.3 Proof of Theorem 6.1.1

6.3.1 Structure of the proof

For $\ell, M \in \mathbb{N}$, we define the class $\mathcal{S}(\ell, M)$ as the set of all graphs containing a sequence of ℓ disjoint star graphs of size M with centers $(x_i)_{i\leqslant \ell}$, such that $d(x_i, x_{i+1}) \leqslant \psi_1(M)+1$ for all $i \leqslant \ell - 1$.

The proof of Theorem 6.1.1 relies on the following propositions.

Proposition 6.3.1. *For any positive integer M, there exist positive constants c and K, such that $K \geqslant M$ and w.h.p. G_n belongs to the class $\mathcal{S}([cn], K)$.*

Proposition 6.3.2. *Let $\tau_{\ell,M}$ be the extinction time of the contact process on a graph of the class $\mathcal{S}(\ell, M)$ starting from full occupancy. Then there exist positive constants c and C independent of λ, such that if $h(\lambda)M \geqslant C\psi_1(M)$, then*

$$\mathbb{P}(\tau_{\ell,M} \geqslant \exp(c\lambda^2 \ell M)) \to 1 \quad as \quad \ell \to \infty, \tag{6.17}$$

with $h(\lambda) = \bar{\lambda}^2/|\log \bar{\lambda}|$ and $\bar{\lambda} = \lambda \wedge 1/2$.

Proposition 6.3.3. *Let (G_n^0) be a sequence of connected graphs, such that $|G_n^0| \leqslant n$ and G_n^0 belongs to the class $\mathcal{S}(k_n, M)$, for some sequence (k_n). Let τ_n denote the extinction* *time of the contact process on G_n^0 starting from full occupancy. Then there exists a positive constant C, such that if $h(\lambda)M \geqslant C\psi_1(M)$ with $h(\lambda)$ as in Proposition 6.3.2 and*

$$\frac{k_n}{d_n \vee \log n} \to \infty, \tag{6.18}$$

with d_n the diameter of G_n^0, then

$$\frac{\tau_n}{\mathbb{E}(\tau_n)} \quad \xrightarrow[n\to\infty]{(\mathcal{L})} \quad \mathcal{E}(1).$$

Proof of Theorem 6.1.1. Observe that Proposition 6.3.1 and Lemma 6.3.2 imply the lower bound on τ_n. On the other hand, the upper bound follows from Lemma 5.3.1 in Chapter 5 and the fact that $|E_n| \asymp n$ w.h.p., see [53, Vol. I, Theorem 6.6]. Similarly to Theorem 2.1.2 (i), we can prove the convergence in law of $\tau_n/\mathbb{E}(\tau_n)$ by using Propositions 6.3.1, 6.3.3 and the following:

- w.h.p. $d_n = \mathcal{O}(\log n)$ with d_n the diameter of the largest component of the IRG,
- w.h.p. the size of the second largest component in the IRG is $\mathcal{O}(\log n)$.

These claims are proved in Theorems 3.12 and 3.16 in [19] for a general model of IRG. □

Proof of Proposition 6.3.2. Similarly to Lemma 5.3.3 in Chapter 5, we can prove (6.17) by using a comparison between the contact process and an oriented percolation on $[\![1, n]\!]$ with density close to 1. Note that here, we use a mechanism of infection between star graphs instead of complete graphs as it was the case in Chapter 5. The mechanism for star graphs is described in Lemmas 3.1 and 3.2 in [65] and the function $h(\lambda)$ is chosen appropriately to apply these results. □

Proof of Proposition 6.3.3. The proof is the same as for Lemma 5.3.4. □

6.3.2 Proof of Proposition 6.3.1

This subsection is divided into four parts. In the first part, we define a preliminary process, called an exploration, which uses Proposition 6.2.2 and Lemma 6.2.6 to discover the neighborhood of a vertex. In Parts two and three, we describe the two main tasks, and the last part gives the conclusion.

Exploration process

For $v \in V_n$ and $U \subset V_n \setminus \{v\}$, we will define an **exploration** of v in U of type K (and denote it by $E_K(v,U)$ and call U the *source set* of the exploration). The aim of this exploration is just to find a vertex x in U with degree larger than $3K$ at distance less than $\psi_1(K)$ from v.

First, we set $x_0 = v$, $U_0 = U$ and $W_0 = \{x_0\}$ and call it the *waiting set*. We define a sequence of trees $(T^k(v,U))_{k\geqslant 0}$ as the record of the exploration, starting with $T^0(v,U) = \{x_0\}$. Then we determine $\mathcal{N}(x_0,U_0)$ - the set of neighbors of x_0 in U_0.

- If $\mathcal{N}(x_0,U_0) = \varnothing$, we define $U_1 = U_0$ and $W_1 = W_0 \setminus \{x_0\}$ and $T^1(v,U) = T^0(v,U)$.
- If $|\mathcal{N}(x_0,U_0)| \geqslant 3K$, we arbitrarily choose $3K$ vertices in $\mathcal{N}(v,U)$ to form three *seed* sets of size K denoted by $F_{v,1}$, $F_{v,2}$ and $F_{v,3}$. Then we declare that $E_K(v,U)$ is *successful*; we stop the exploration and define
$$U_1 = U \setminus (F_{v,1} \cup F_{v,2} \cup F_{v,3}).$$
- If $1 \leqslant |\mathcal{N}(x_0,U_0)| < 3K$, we define
$$\begin{aligned} U_1 &= U_0 \setminus \mathcal{N}(x_0,U_0), \\ T^1(v,U) &= T^0(v,U) \cup \mathcal{N}(x_0,U_0) \text{ together with the edges} \\ &\qquad\qquad\qquad\qquad \text{between } x_0 \text{ and } \mathcal{N}(x_0,U_0), \\ W_1 &= (W_0 \setminus \{x_0\}) \cup \{x \in \mathcal{N}(x_0,U_0) : d_{T^1(v,U)}(x_0,x) \leqslant \psi_1(K)\}, \end{aligned}$$
with $d_T(x,y)$ the graph distance between x and y in a tree T.

  Then we chose an arbitrary vertex x_1 in W_1 and repeat this step with x_1 and U_1 in place of x_0 and U_0.

We continue like this until: the waiting set is empty, or we succeed at some step.

Note that after the k^{th} step, we define
$$T^{k+1}(v,U) = \begin{cases} T^k(v,U) \cup \mathcal{N}(x_k,U_k) \text{ together with the edges} & \\ \qquad\qquad\qquad \text{between } x_k \text{ and } \mathcal{N}(x_k,U_k) & \text{if } |\mathcal{N}(x_k,U_k)| < 3K \\ T^k(v,U) & \text{if } |\mathcal{N}(x_k,U_k)| \geqslant 3K, \end{cases}$$

and

$$U_{k+1} = \begin{cases} U_k \setminus \mathcal{N}(x_k, U_k) & \text{if } |\mathcal{N}(x_k, U_k)| < 3K \\ U_k \setminus (F_{x_k,1} \cup F_{x_k,2} \cup F_{x_k,3}) & \text{if } |\mathcal{N}(x_k, U_k)| \geqslant 3K, \end{cases}$$

and

$$W_{k+1} = (W_k \setminus \{x_k\}) \cup \{x \in \mathcal{N}(x_k, U_k) : d_{T^{k+1}(v,U)}(x_0, x) \leqslant \psi_1(K)\}.$$

If the process stop after k_0 step, we define the remaining source set

$$\tilde{U} = U_{k_0+1}.$$

When an exploration is successful, its outputs are the set $\tilde{U}$ and a vertex, say u, with three seed sets $F_{u,1}, F_{u,2}$ and $F_{u,3}$ of size K. Otherwise, the output is just $\tilde{U}$.

Lemma 6.3.4. *The following statements hold.*

(i) For all v and U,

$$|\tilde{U}| \geqslant |U| - \psi_2(K),$$

with $\psi_2(K) = (3K)^{\psi_1(K)+1}$.

(ii) For all $K \geqslant K_1$, we have

$$\lim_{n\to\infty} \mathbb{P}(E_K(v,U) \text{ is successful} \mid \mathcal{E}(K), |U| \geqslant (1-\varkappa_2)n) \geqslant \theta_1,$$

with $\varkappa_2$, θ_1 and K_1 as in Lemmas 6.2.5 and 6.2.6.

Proof. Part (i) follows from the facts that at each step we remove from the source set at most $3K$ vertices, and that we only explore the vertices at distance less than or equal to $\psi_1(K)$ from v.

We now prove (ii). Similarly to Lemma 6.2.6, if w_v -the weight of v- is larger than $10K$,

$$\mathbb{P}(\deg(v) \geqslant 3K) \geqslant \theta_1,$$

and thus (ii) follows. Suppose that $w_v \leqslant 10K$. Then using the same argument in Lemma 6.2.6, or [53, Vol. II, Corollary 3.13] (for showing that $\mathbb{T}_\ell(v,U) \equiv \tilde{\mathbb{T}}_\ell(v,U)$ w.h.p. when w_v is bounded), we get

$$\mathbb{P}(B_{\psi_1(K)}(v,U) \text{ has no cycle}) \to 1 \quad \text{as} \quad n \to \infty. \tag{6.19}$$

Suppose that $B_{\psi_1(k)}(v, U)$ is a tree (or has no cycle). Then the order of explorations of vertices in the waiting set does not effect to the outcome of $E_K(v, U)$. Thus

$$\{E_K(v,U) \text{ is successful}\} \supset \{\exists\, x \in B_{\psi_1(K)}(v,U) : \deg(x) \geqslant 3K+1\} \cap \{B_{\psi_1(K)}(v,U) \text{ is a tree}\}.$$

Therefore, the result follows from Proposition 6.2.2, Lemma 6.2.6 and (6.19). □

Task I

The goal of this task is to show that w.h.p. by discovering $o(\log n)$ vertices, we can find in G_n a subgraph belonging to the class $\mathcal{S}(L_n, 3K)$ with

$$L_n = [\log\log\log n].$$

For $v \in V_n$, $K \geqslant K_1$ and $U \subset V_n \setminus \{v\}$, with $|U| \geqslant (1 - \varkappa_2/2)n$ and $\varkappa_2 = \varkappa_2(\varepsilon_0, K)$ as in Lemma 6.2.5, we define a **trial** $Tr(v, U, L_n, K)$ as follows.

At *level* 0, we define $\tilde{W}_0 = \{v\}$ and call it the waiting set at level 0. Then we perform $E_K(v, U)$ and call $\tilde{U}_v$ the source set after this exploration. If it fails, we declare that $Tr(v, U, L_n, K)$ *fails*. Otherwise, we are now in *level* 1 and continue as follows:

Let x_1 be the vertex with degree larger than $3K + 1$ which makes $E_K(v, U)$ successful and let $F_{x_1,1}$, $F_{x_1,2}$ and $F_{x_1,3}$ be its three seed sets of size K. We denote $\tilde{W}_1 = F_{x_1,3}$ and call it the waiting set at level 1 ($F_{x_1,1}$ and $F_{x_1,2}$ are reserved for Task II). We sequentially perform explorations of $\tilde{W}_1$. More precisely, we choose arbitrarily a vertex, say y_1 in $\tilde{W}_1$ and operate $E_K(y_1, \tilde{U}_v)$ and get a new source set $\tilde{U}_{y_1}$. Then we operate $E_K(y_2, \tilde{U}_{y_1})$ with y_2 chosen arbitrarily from $\tilde{W}_1 \setminus \{y_1\}$, and so on.

If none of these explorations is successful, we declare that the trial *fails*.

If some of those are successful, we are in *level* 2 and get some triples of seed sets $F_{\cdot,1}, F_{\cdot,2}$ and $F_{\cdot,3}$. Denote by $\tilde{W}_2$ the waiting set at level 2, which is the union of all the seed sets of the third type $F_{\cdot,3}$. Then we sequentially perform the explorations of vertices in $\tilde{W}_2$.

We continue this process until either we explore all vertices in waiting sets, or when we exceed to the L_n-th level.

We declare that $Tr(x, W, L_n, K)$ is *successful* if we can access to the L_n-th level, or that it has failed otherwise.

Lemma 6.3.5. *The followings hold.*

(i) A trial discovers at most $K^{L_n+1}\psi_2(K)$ vertices.

(ii) There exist constants $\theta_2 \in (0,1)$ and $K_2 \geqslant K_1$, such that for any $K \geqslant K_2$ and $v \in V_n$ and $U \subset V_n \setminus \{v\}$ satisfying $|U| \geqslant (1-\varkappa_2/2)n$, we have

$$\lim_{n\to\infty} \mathbb{P}(Tr(v, U, L_n, K) \text{ is successful } \mid \mathcal{E}(K)) \geqslant \theta_2,$$

with $\varkappa_2$ and K_1 as in Lemmas 6.2.5 and 6.2.6.

Proof. For (i), we observe that an exploration creates at most one third type seed set of size K. Then in a trial, we operate at most

$$1 + K + K^2 + \ldots + K^{L_n} \leqslant K^{L_n+1} \quad \text{explorations.}$$

Moreover, an exploration uses at most $\psi_2(K)$ vertices. Therefore, a trial discovers at most

$$K^{L_n+1}\psi_2(K) = o(\log n) \text{ vertices.}$$

We now prove (ii). As a trial uses at most $o(\log n)$ vertices, and initially $|U| \geqslant (1-\varkappa_1/2)n$, during the trial $Tr(v, U, L_n, K)$ the source sets of explorations always have cardinality larger than $(1-\varkappa_2)n$.

Hence, by Lemma 6.3.4 (ii), for all n and K large enough, on $\mathcal{E}(K)$ each exploration in $Tr(v, U, L_n, K)$ is successful with probability larger than $\theta_1/2$.

On the other hand, each successful exploration creates K new vertices in the next level. Hence $(|\tilde{W}_i|)_{i\leqslant L_n}$- the numbers of vertices to explore up to the L_n-th level in the trial stochastically dominate a branching process $(\eta_i)_{i\leqslant L_n}$ starting from $\eta_0 = 1$ with reproduction law η given by

$$\mathbb{P}(\eta = K) = \theta_1/2$$

$$\mathbb{P}(\eta = 0) = 1 - \theta_1/2.$$

We choose K large enough, such that $K\theta_1 > 2$. Then (η_i) is super critical, and thus

$$\mathbb{P}(Tr(x, W, L_n, K) \text{ is successful } \mid \mathcal{E}(K)) \geqslant \mathbb{P}(\eta_{L_n} \geqslant 1) > 0,$$

which proves the result. $\square$

We can now define **Task I**, which consists in some trials as follows. We fix a constant $K \geqslant K_2$ and set

$$A = \{v_1, \ldots, v_{L_n}\} \quad \text{and} \quad U = V_n \setminus A.$$

We first operate $Tr(v_1, U_0, L_n, K)$ with $U_0 = U$.

- If $Tr(v_1, U_0, L_n, K)$ is successful, we declare that Task I is *successful*.
- Otherwise, we call $\bar{U}_1$ the source set after this trial. We then perform $Tr(v_2, \bar{U}_1, L_n, K)$.

We sequentially operate the trials with vertices in A until we get a successful trial or we use up the vertices of A.

We declare that Task I is *successful* if there is a successful trial and that it *fails* otherwise.

Lemma 6.3.6. *We have*

$$\lim_{n \to \infty} \mathbb{P}(\textit{Task I is successful} \mid \mathcal{E}(K)) = 1.$$

Proof. By Lemma 6.3.5 (i) in this task, we discover at most

$$L_n K^{L_n+1} \psi_2(K) = o(\log n) \text{ vertices.}$$

Hence, the cardinality of the source set is always larger than $n - o(\log n)$. Therefore by Lemma 6.3.5 (ii), each trial is successful with probability larger than $\theta_2/2$ for n large enough.

We define $\mathcal{I}$ the first index such that the trial of $v_{\mathcal{I}}$ in Task I is successful (if there is no such index, we let $\mathcal{I} = \infty$). Then conditionally on $\mathcal{I} \leqslant L_n$, it is stochastically dominated by a geometric random variable with parameter $\theta_2/2$. Therefore

$$\mathbb{P}(\mathcal{I} = \infty) \leqslant (1 - \theta_2/2)^{L_n} = o(1).$$

In other words, Task I is successful w.h.p. $\square$

Task II

Suppose that Task I is successful. Then there is a sequence of vertices $\{u_1, \ldots, u_{L_n}\}$, such that $d(u_{i-1}, u_i) \leqslant \psi_1(K)$ for all $2 \leqslant i \leqslant L_n$, together with L_n pairs of disjoint seed sets

$(F_{u_1,1}, F_{u_1,2}), \ldots, (F_{u_{L_n},1}, F_{u_{L_n},2})$ attached respectively to (u_i). Moreover, the cardinality of U^*-the source set after Task I- is larger than $n - o(\log n)$. Importantly, we have not yet discovered the vertices in

$$U^+ = U^* \cup \bigcup_{1 \leqslant i \leqslant L_n} (F_{u_i,1} \cup F_{u_i,2}).$$

For any set $F \subset U^+$ of size K and $S \subset U^* \setminus F$ we define an **experiment** $Ep(F, S, K)$ as follows.

We write F as $\{z_1, \ldots, z_K\}$. Then we sequentially operate explorations $E_K(z_1, S_{z_0})$, ..., $E_K(z_K, S_{z_{K-1}})$, where S_{z_i} is the source set after the exploration $E_K(z_i, S_{z_{i-1}})$ for $1 \leqslant i \leqslant K$ with $S_{z_0} = S$.

If none of these explorations is successful, we declare that $Ep(F, S, K)$ *fails*, otherwise we say that it is *successful*. In the latter case, there is a vertex u with $d(u, F) \leqslant \psi_1(K)$ together with two seed sets $F_{u,1}$ and $F_{u,2}$ of size K (in fact, we even have three sets, but we will only use two of them).

Lemma 6.3.7. *We have*

(i) the number of vertices used in an experiment is at most $\psi_3(K) = K\psi_2(K)$,

(ii) there exists a positive constant $K_3 \geqslant K_2$, such that for all $K \geqslant K_3$, and n large enough

$$\mathbb{P}(Ep(F, S, K) \textit{ is successful} \mid \mathcal{E}(K), |S| \geqslant (1 - \varkappa_2/2)n) \geqslant 2/3,$$

with $\varkappa_2$ as in Lemma 6.2.5.

Proof. Part (i) is immediate, since in an experiment, we perform K explorations and each exploration uses at most $\psi_2(K)$ vertices. For (ii), we note that by (i) and the assumption $|S| \geqslant (1 - \varkappa_2/2)n$, the source set S_{z_i} has more than $(1 - \varkappa_2)n$ vertices for all i. Hence, by Lemma 6.3.4 (ii), for all $1 \leqslant i \leqslant K$ and n large enough

$$\mathbb{P}(E_K(z_i, S_{z_{i-1}}) \text{ is successful} \mid \mathcal{E}(K)) \geqslant \theta_1/2.$$

Thus on $\mathcal{E}(K)$, the probability that the experiment fails is less than

$$(1 - \theta_1/2)^K \leqslant 1/3,$$

provided K is large enough. $\square$

We define **Task II** as follows. First, we fix a constant $K \geqslant K_3$ and let

$$\varepsilon_1 = \varkappa_2/(3\psi_3(K)).$$

We label the seed sets *active* and make an order as follows

$$F_{u_1,1} < F_{u_1,2} < F_{u_2,1} < \ldots < F_{u_{L_n},1} < F_{u_{L_n},2}.$$

We perform $Ep(F_{u_{L_n},2}, U^*, K)$ and let $F_{u_{L_n},2}$ be inactive. If the experiment is successful, we find a vertex u at distance smaller than $\psi_1(K) + 1$ from u_{L_n} and two seed sets $F_{u,1}$ and $F_{u,2}$ of size K. We now add u in the sequence: $u_{L_{n+1}} = u$, label these sets $F_{u_{L_{n+1}},1}$ and $F_{u_{L_{n+1}},2}$ active, and make an order

$$F_{u_1,1} < \ldots < F_{u_{L_n},1} < F_{u_{L_{n+1}},1} < F_{u_{L_{n+1}},2}.$$

We then perform the experiment of the newest active set i.e. the active set with the largest order. After an experiment of an active set, we let it be inactive and either get a new vertex with two active sets attached on it (if the experiment is successful), or get nothing (otherwise).

Continue this procedure until one of the three following conditions is satisfied.

- There is no more active set. We declare that Task II fails.
- We do more than $[\varepsilon_1 n]$ experiments. We declare that Task II fails.
- We have more than $[\varepsilon_1 n/4]$ active sets. We declare that Task II is successful.

Proposition 6.3.8. *For all $K \geqslant K_3$, we have*

$$\lim_{n\to\infty} \mathbb{P}(\textit{Task II is successful} \mid \mathcal{E}(K), \textit{Task I is successful}) = 1.$$

Proof. By Lemma 6.3.7 (i), the first $[\varepsilon_1 n]$ experiments use at most $[\varkappa_2 n/3]$ vertices. Therefore, during Task II, the source set always has cardinality larger than $(1 - \varkappa_2/2)n$. Thus by Lemma 6.3.7 (ii), on $\mathcal{E}(K)$ during the time to perform this task, each experiment is successful with probability larger than 2/3.

Therefore the number of active sets stochastically dominates a random walk (R_i) satisfying $R_0 = 2L_n$ and

$$R_{i+1} = R_i + 1 \text{ with prob. } 2/3$$

$$R_{i+1} = R_i - 1 \text{ with prob. } 1/3.$$

Define

$$T_0 = \inf\{i : R_i = 0\} \qquad \text{and} \qquad T_1 = \inf\{i : R_i \geqslant [\varepsilon_1 n/4]\}.$$

Then using the optional stopping time theorem, we get

$$\mathbb{P}(T_1 \leqslant T_0) = 1 - o(1).$$

On the other hand, the law of large numbers gives that

$$\mathbb{P}(T_1 \leqslant [\varepsilon_1 n]) \geqslant \mathbb{P}(R_{[\varepsilon_1 n]} \geqslant [\varepsilon_1 n/4]) = 1 - o(1).$$

It follows from the last two inequalities that

$$\begin{aligned} & \mathbb{P}(\text{Task II is successful} \mid \mathcal{E}(K), \text{Task I is successful}) \\ \geqslant \ & \mathbb{P}(T_1 \leqslant \min\{T_0, [\varepsilon_1 n]\}) = 1 - o(1), \end{aligned}$$

which proves the result. □

Proof of Proposition 6.3.1

By Lemmas 6.3.6, 6.3.8 and (6.11), we can assume that both Tasks I and II are successful. Then we have more than $[\varepsilon_1 n/4]$ active sets. Observe that a vertex is attached to at most two active sets. Then the number of vertices having at least one active set is larger than $[\varepsilon_1 n/8]$. Therefore w.h.p. G_n belongs to the class $\mathcal{S}([cn], 2K)$ with $c = \varepsilon_1/8$. Moreover, K can be chosen arbitrarily large, so Proposition 6.3.1 has been proved.

6.4 Contact process on Erdos-Rényi random graphs

We recall the definition of $G(n,p)$-the Erdos-Rényi graph with parameter p. Let $V_n = \{v_1, \ldots, v_n\}$ be the vertex set. Then for $1 \leqslant i \neq j \leqslant n$, we independently draw an edge between v_i and v_j with probability p.

Proposition 6.4.1. *Let τ_n be the extinction time of the contact process on $G(n,p)$ starting with all sites infected. There exists a positive constant C, such that for any $\lambda > 0$ and $np \geqslant [C/h(\lambda)]!$,*

$$\mathbb{P}(\tau_n \geqslant \exp(cn)) = 1 - o(1),$$

with $h(\lambda)$ as in Proposition 6.3.2 and $c = c(\lambda)$ a positive constant. Moreover, in this setting

$$\frac{\tau_n}{\mathbb{E}(\tau_n)} \xrightarrow[n\to\infty]{(\mathcal{L})} \mathcal{E}(1).$$

Let us denote by $S(\ell, M)$ the graph obtained by attaching to each vertex in $[\![1, n]\!]$ a star graph of size M. Similarly to Lemma 6.3.2, we get that if $h(\lambda)M$ is large enough, then the extinction time of the contact process on $S(\ell, M)$ is exponential in $\ell \times M$ w.h.p. Therefore, similarly to Theorem 6.1.1, Proposition 6.4.1 follows from the following lemma.

Lemma 6.4.2. *For any M, there exists a positive constant c, such that if $np \geqslant 16M!$ then w.h.p. $G(n, p)$ contains as a subgraph a copy of $S([cn], M)$.*

To prove Lemma 6.4.2, we will use the following.

Lemma 6.4.3. *[3, Theorem 2] If $np > c > 1$ then w.h.p. $G(n, p)$ contains a path of length $[\varkappa n]$ with some positive constant $\varkappa = \varkappa(c)$.*

Proof of Lemma 6.4.2. We first define

$$n_1 = [n/2], \qquad A = \{v_1, \ldots, v_{n_1 - 1}\} \quad \text{and} \quad A^c = V_n \setminus A.$$

For $v_i \in A$ and $v_j \in A^c$, define

$$Y_{i,j} = 1(v_i \sim v_j).$$

Then $(Y_{i,j})$ are i.i.d. Bernoulli random variables with mean p. We set $B_0 = \varnothing$ and

$$\sigma_1 = \inf\Big\{j \leqslant n : \sum_{k=n_1}^{j} Y_{1,k} \geqslant M\Big\},$$

with the convention $\inf \varnothing = \infty$. Define

$$B_1 = \begin{cases} \{k : k \leqslant \sigma_1, Y_{1,k} = 1\} & \text{if} \quad \sigma_1 < \infty, \\ B_0 & \text{if} \quad \sigma_1 = \infty. \end{cases}$$

Suppose that σ_i and B_i have been already defined. Then we set

$$\sigma_{i+1} = \inf\Big\{j \leqslant n : \sum_{k=n_1}^{j} Y_{i+1,k} 1(k \notin B_i) \geqslant M\Big\},$$

and

$$B_{i+1} = \begin{cases} B_i \cup \{k : k \leqslant \sigma_{i+1}, k \notin B_i, Y_{i+1,k} = 1\} & \text{if} \quad \sigma_{i+1} < \infty, \\ B_i & \text{if} \quad \sigma_{i+1} = \infty. \end{cases}$$

Then

$$|B_{i+1}| = \begin{cases} |B_i| + M & \text{if} \quad \sigma_{i+1} < \infty, \\ |B_i| & \text{if} \quad \sigma_{i+1} = \infty. \end{cases}$$

Hence, for all $i \leqslant [n/M]$,

$$|B_i| \leqslant iM. \tag{6.20}$$

We now define

$$Y_i = 1(\sigma_i < \infty).$$

Then

$$\begin{aligned} \mathbb{P}(Y_{i+1} = 1 \mid B_i) &= \mathbb{P}(\sigma_{i+1} < \infty \mid B_i) \\ &= \mathbb{P}\left(\sum_{k=n_1}^{n} Y_{i+1,k} 1(k \notin B_i) \geqslant M \mid B_i\right) \\ &= \mathbb{P}\left(\mathrm{Bin}(n - n_1 + 1 - |B_i|, p) \geqslant M \mid B_i\right). \end{aligned}$$

It follows from (6.20) that when $i \leqslant [n/4M]$,

$$n - n_1 + 1 - |B_i| \geqslant [n/4].$$

Thus

$$\begin{aligned} \mathbb{E}(Y_{i+1} \mid B_i) &= \mathbb{P}(Y_{i+1} = 1 \mid B_i) \\ &\geqslant \mathbb{P}(\mathrm{Bin}([n/4], p) \geqslant M) \\ &\geqslant \mathbb{P}(\mathrm{Poi}(np/4) \geqslant M)/2 \\ &\geqslant 1/(M-1)!, \end{aligned} \tag{6.21}$$

when $np \geqslant 8$. Now we set

$$\Gamma = \{i \leqslant [n/4M] : Y_i = 1\},$$

and let

$$Z_k = \sum_{i=1}^{k} (Y_i - \mathbb{E}(Y_i \mid B_{i-1})). \tag{6.22}$$

Then (Z_k) is a $(\sigma(B_k))$- martingale satisfying $|Z_k - Z_{k-1}| \leqslant 1$ for all $1 \leqslant k \leqslant [n/4M]$. Thus it follows from Doob's martingale inequality that

$$\mathbb{P}\left(\max_{k\leqslant \ell} |Z_k| \geqslant x\right) \leqslant \frac{\mathbb{E}(Z_\ell^2)}{x^2} \leqslant \frac{\ell}{x^2} \quad \text{for all} \quad \ell \leqslant [n/4M],\ x > 0.$$

In particular, w.h.p.

$$|Z_{[n/4M]}| = o(n). \tag{6.23}$$

It follows from (6.21), (6.22) and (6.23) that w.h.p.

$$\begin{aligned} |\Gamma| &= \sum_{i=1}^{[n/4M]} Y_i \\ &= Z_{[n/4M]} + \sum_{i=1}^{[n/4M]} \mathbb{E}(Y_i \mid B_{i-1}) \\ &\geqslant Z_{[n/4M]} + [n/4M]/(M-1)! \\ &\geqslant n/(8M!). \end{aligned}$$

Conditionally on $|\Gamma| \geqslant n/(8M!)$, the graph induced in Γ contains a Erdos-Rényi graph, say H_n, of size $[n/(8M!)]$ with probability of connection p. Observe that H_n is super critical when

$$np/(8M!) \geqslant 2.$$

Therefore by Lemma 6.4.3, w.h.p. H_n contains a path of length $[\varkappa n]$, with $\varkappa = \varkappa(M)$ if $np \geqslant 16M!$.

On the other hand, each vertex in Γ has at least M disjoint neighbors in A^c. Thus Lemma 6.4.2 follows. □

Remark 6.4.4. We can use Proposition 6.4.1 to prove the lower bound on the extinction time in Theorem 6.1.1 for a small class of weights. Indeed, for any p we define

$$A_p = \{v_i : w_i \geqslant \sqrt{4p\mathbb{E}(w)}\}.$$

Then

$$|A_p| = (\gamma_p + o(1))n,$$

with

$$\gamma_p = \mathbb{P}(w \geqslant \sqrt{4p\mathbb{E}(w)}).$$

On the other hand, for any v_i and v_j in A_p,

$$\begin{aligned}
\mathbb{P}(v_i \sim v_j) &= 1 - \exp(-w_i w_j/\ell_n) \\
&\geqslant \frac{w_i w_j}{2\ell_n} 1(w_i w_j \leqslant \ell_n) + \frac{1}{2} 1(w_i w_j > \ell_n) \\
&\geqslant \frac{4p\mathbb{E}(w)}{2\ell_n} \\
&\geqslant \frac{p}{n},
\end{aligned}$$

since $\ell_n = n(\mathbb{E}(w) + o(1))$. Therefore, the graph induced in A_p contains a Erdos-Rényi graph $G([\gamma_p n], p/n)$. By proposition 6.4.1, if $p\gamma_p \to \infty$, for all $\lambda > 0$, w.h.p. the extinction time of the contact process on $G([\gamma_p n], p/n)$ is exponential in n. Hence, when

$$\lim_{p \to \infty} p\mathbb{P}(w \geqslant \sqrt{4p\mathbb{E}(w)}) = \infty, \tag{6.24}$$

w.h.p. the extinction time on the IRG(w) is exponential in n for all $\lambda > 0$.

The condition (6.24) is satisfied for some weight w, for example the power-law distribution with exponent between 2 and 3.

Bibliography

[1] W. Aiello, F. Chung, L. Lu. *A random graph model for power law graphs*, Exp. Math. 10, 53–66 (2001).

[2] R. Albert, H. Jeong, A. L. Barabási. *Diameter of the world-wide web*, Nature. 401, 130–131 (1999).

[3] M. Ajtai, J. Komlós, E. Szemerédi. *The longest path in a random graph*, Combinatorica. 1, 1-12 (1981).

[4] A. D. Andjel. *The contact process in high dimensions*, Ann. Probab. 16, 1174–1183 (1988).

[5] M. Aizemann, P. Jung. *On the critical behavior of the lower phase transition of the contact process*, Alea. 3, 301–320 (2007).

[6] I. Benjamini, N. Berger. *The diameter for long range percolation cluster on finite cycles*, Random Struct. Algorithms, 19 (2001), no. 2, 102–111.

[7] M. Biskup. *On the scaling of the chemical distance in long range percolation models*, Ann. Probab. 32 (2004), no. 4, 2938–2977.

[8] A. Barabási, R. Albert. *Emergence of scaling in random network*, Science. 286, 509–512 (1999).

[9] A. L. Barabási, R. Albert, H. Jeong. *Scale-free characteristics of random networks: The topology of the world-wide web*, Physica A. 281, 69–77 (2000).

[10] B. Bollobás, F. Chung. *The diameter of a cycle plus a random matching*, SIAM J. Discrete Math. 1, 328–333 (1988).

[11] D. J. Barsky, C. C. Wu. *Critical exponent for the contact process under the triangle condition*, J. Statist. Phys. 91, 95–124 (1998).

[12] I. Benjamini. *Coarse geometry and randomness - École d'été de Probabilité de Saint-Flour 2011*, Lecture notes in mathematics 2100, Springer (2013).

[13] N. Berger, C. Borgs, J. T. Chayes, A. Saberi. *On the spread of viruses on the internet*, Proceedings of the sixteenth annual ACM-SIAM symposium on discrete algorithms, 301-310, (2005).

[14] N. Berger, C. Borgs, J.T. Chayes, A. Saberi. *Asymptotic behavior and distributional limits of preferential attachment graphs*, Ann. Probab. 42, 1–40 (2014).

[15] T. Britton, M. Deijfen, A. Martin-Lof. *Generating random graphs with prescribed degree distribution*, J. Stat. Phys. 124, 1377-1397 (2006).

[16] C. Bzuidenhout, G. Grimmett. *The critical contact process dies out*, Ann. Probab. 18, 1462–1482 (1990).

[17] C. Bzuidenhout, G. Grimmett. *Exponential decay for subcritical contact and percolation processes*, Ann. Probab. 19, 1984–1009 (1991).

[18] A. Bovier, F. Hollander. *Metastability, a potential-theoretic approach.* Grundlehren de Mathematischen Wissenschaften **351**, Springer (2015).

[19] B. Bollobás, S. Janson, O. Riordan. *The phase transition in inhomogeneous random graphs*, Random. Struct. Algorithms. 31, 3-122 (2007).

[20] S. Bubeck, E. Mossel, M.Z. Rácz. *On the influence of the seed graph in the preferential attachment model*, arXiv:1401.4849v3.

[21] I. Benjamini, O. Schramm. *Recurrence of distributional limits of finite planar graphs*, Electron. J. Probab. 6 (23), 13 pp. (2001).

[22] M. Cranston, T. Mountford, J.-C. Mourrat, D. Valesin. *The contact process on finite homogeneous trees revisited*, ALEA Lat. Am. J. Probab. Math. Stat. 11, 385–408, (2014).

[23] S. Chatterjee, R. Durrett. *Contact process on random graphs with degree power law distribution have critical value zero*, Ann. Probab. 37, 2332-2356 (2009).

[24] M. Cassandro, A. Galves, E. Oliveri, M. Vares. *Metastable behaviour of stochastic dynamics: a pathwise approach*, J. Stat. Phys. 35, 603–634 (1984).

[25] F. Chung, L. Lu. *Concentration inequalities and martingale inequalities: a survey*, Internet Math. 3, 79–127 (2006).

[26] V. H. Can, B. Schapira. *Metastability for the contact process on the configuration model with infinite mean degree*, Electron. J. Probab. 20, 1–22 (2015), arXiv:1410.3061v3.

[27] V. H. Can. *Metastability for the contact process on the preferential attachment graph*, arXiv:1502.05633.

[28] V. H. Can. *Super-exponential extinction time of the contact process on random geometric graphs*, arXiv:1506.02373.

[29] V. H. Can. *Contact process on one-dimensional long range percolation*, Electron. Commun. Probab. 20, 1-11 (2015).

[30] V. H. Can. *Exponential extinction time of the contact process on inhomogeneous random graphs*, arXiv:1602.02059.

[31] N. Crawford, A. Sly. *Simple random walk on long range percolation cluster II: Scaling limits*, Ann. probab. 41 (2013), 445–502.

[32] A. Dembo, O. Zeitouni. *Large deviations techniques and applications, 2nd edition.* Applications of Mathematics (NewYork). 38, Springer (1998).

[33] S. Dommers, R. van der Hofstad, G. Hooghiemstra. *Diameters in preferential attachment models*, J. Stat. Phys. 139, 72–107 (2010).

[34] R. Durrett. *Random Graph Dyamics.* Cambridge Univ. Press, Cambridge (2007).

[35] R. Durrett. *On the growth of one-dimensional contact process*, Ann. Probab. 8, 890–907 (1980).

[36] R. Durrett. *Oriented percolation in two dimensions*, Ann. Probab. **12**, 999-1040 (1984).

[37] R. Durrett. *The contact process*, Proceedings of the 1989 AMS Seminar on Random media, vol. 27, AMS Lectures in applied mathematics, 1–18 (1991).

[38] R. Durrett, P. Jung. *Two phases transitions of the contact process on small worlds*, Stochastic Process. Appl. 117 1910–1927 (2007).

[39] R. Durrett, R. Schinazi. *Intermediate phase for the contact process on a tree*, Ann. Probab. 23, 668–673 (1995).

[40] R. Durrett, D. Griffeath. *Contact process in several dimensions*, Z. Wahrsch. verw. Gebiete. 59, 535–552 (1982).

[41] R. Durrett, X. Liu. *The contact process on a finite set*, Ann. Probab. 16, 1158–1173 (1988).

[42] R. Durrett, R. H. Schonmann *Large deviation for the contact process and two dimensional percolation*, Probab. Th. Rel. Fields. 77, 583–603 (1988).

[43] R. Durrett, R. Schonmann. *The contact process on a finite set II*, Ann. Probab. 16, 1570–1583 (1988).

[44] R. Durrett, R. Schonmann, N. Tanaka. *The contact process on a finite set III*, Ann. Probab. 17, 1303–1321 (1989).

[45] D. J. Daley, D. Verre-Jones. *An introduction to the theory of point processes*, Vol I. Probability and Applications, Springer (2003).

[46] H. van den Esker, R. van der Hofstad, G. Hooghiemstra, and D. Znamenski. *Distances in random graph with infinite mean degrees*, Extremes **8**, 111–141 (2006).

[47] T. Friedrich, T. Sauerwald, A. Stauffer. *Diameter and broadcast time of random geometric graphs in arbitrary dimensions*, Algorithmica. **67**, no. 1, 65–88 (2013).

[48] G. Ganesan. *Infection spread in random geometric graphs*, Adv. in Appl. Probab. **47**, 164–181 (2015).

[49] G. Grimmett. *Percolation.* Grundlehren der mathematischen Wissenschaften, **324** Springer (1999).

[50] G. Grimmett. *Long paths an cycles in a random lattice.* Ann. Discrete Mathematics. 33, 69–76 (1985).

[51] P. Gupta, P. R. Kumar. *Critical power for asymptotic connectivity in wireless networks* , In Stochastic Analysis, Control, Optimization and Applications, Boston, MA, 547–566 (1999).

[52] T. E. Harris. *Contact interactions on a lattice*, Ann. probab. 2, 969–988 (1974).

[53] R. van der Hofstad. *Random graphs and complex networks.* Available at http://www.win.tue.nl/ rhofstad/NotesRGCN.html.

[54] S. Janson, M. Luczak, P. Windridge. *Law of large numbers for the SIR epidemic on a random graph with given degrees*, Random. Struct. Algorithms. 45, 724–761 (2014).

[55] S. Laley. *Growth profile and invariant measures for the weakly supercritical contact process on a homogeneous tree*, Ann. Probab. 27, 206–225 (1999).

[56] S. Laley, W. Su. *Contact process on random regular graphs*, arXiv:1502.07421

[57] T. M. Liggett. *Multiple transition points for the contact processes on the binary tree*, Ann. Probab. 24, 1675–1710 (1996).

[58] T. M. Liggett. *Branching random walks and contact processes on homogeneous trees*, Probab. Th. Rel. Fields. 106, 495–519 (1996).

[59] T. M. Liggett. *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes.* Grundlehren de Mathematischen Wissenschaften **324**, Springer (1999).

[60] T. Liggett, R. Schonmann, A. Stacey. *Domination by product measure,* Ann. Prob. 25, 71-95 (1997).

[61] G. Morrow, R. Schinazi, Y. Zhang. *The critical contact process on a homogeneous tree*, J. Appl. Probab. 31, 250–255 (1994).

[62] T. Mountford. *A metastable result for the finite multidimensional contact process*, Canad. Math. Bull. 36 (2), 216–226 (1993).

[63] T. Mountford. *Existence of constant for finite system extinction*, J. Statist. Phys. 96, 1331–1341 (1999).

[64] T. Mountford, J.C.-Mourrat, D. Valesin, Q. Yao. *Exponential extinction time of the contact process on finite graphs*, arXiv:1203.2972.

[65] T. Mountford, D. Valesin, Q. Yao. *Metastable densities for contact processes on random graphs*, Electron. J. Probab. 18, 1-36 (2013).

[66] J.-C. Mourrat, D. Valesin. *Phase transition of the contact process on random regular graphs*, arXiv:1405.0865.

[67] L. Ménard, A. Singh. *Percolation by cumulative merging and phase transition for the contact process on random graphs*, arXiv:1502.06982v1.

[68] M. Newman. *Spread of epidemic disease on networks*, Phys. Rev. E 66, paper 016128 (2002).

[69] I. Norros, H. Reittu. *On a conditionally Poissonian graph process*, Adv. in Appl. Prob. 38, 59–75 (2006).

[70] M. Newman, D. Watts. *Renormalization group analysis of the small-world network model*, Phys. Lett. A. 263, 341–346 (1999).

[71] E. Olivieri, M. E. Vares. *Large Deviations and Metastability.* Cambridge University Press (2005).

[72] M. Penrose. *Random geometric graph*, Oxford University Press (2003).

[73] V. M. Preciado, A. Jababaie. *Spectral analysis of virus spreading in random geometric networks*, IEEE Conference on decision and control, 4802–4807 (2009).

[74] R. Pemantle. *The contact process on trees*, Ann. Probab. 20, 2089–2116 (1992).

[75] R. Pemantle, A. Stacey. *The branching random walk and contact process on non-homogeneous and Galton-Watson trees*, Ann. Probab. 29, 1563–1590 (2000).

[76] R. Pastor-Satorras, A. Vespigniani, *Epidemic spreading in scale-free networks*, Phys. Rev. Lett. 86, 3200–3203 (2001).

[77] M. Salzano, R. Schonmann. *A new proof for the contact process on homogeneous trees local survival implies complete convergence*, Ann. Probab. 26, 1251–1258 (1998).

[78] R. Schonmann. *Metastability for the contact process*, J. Statist. Phys. 41, 445–464 (1985).

[79] R. Schonmann. *A new proof of the complete convergence for the contact process in several dimensions with large infection parameter*, Ann. Probab. 15, 382–387 (1987).

[80] R. Schonmann. *The triangle condition for the contact process on homogeneous tree*, J. Statist. Phys. 90, 1429–1440 (1998).

[81] L. S. Schulman. *Long range percolation in one dimension*, J. Phys. A. Lett. 16 (1983).

[82] A. Simonis. *Metastability for the d-dimensional contact process*, J. Statist. Phys. 83, 1225–1239 (1996).

[83] A. Stacey. *The contact process on finite homogeneous trees*, Probab. Th. Rel. Fields. 121, 551–576 (2001).

[84] A. Stacey. *The existence of an intermediate phase for the contact process on trees*, Ann. Probab. 24 (1996), no. 4, 1711–1726.

[85] W. Su. *A metastability result for the contact process on a random regular graph*, arXiv:1503.04895.

[86] B. Schapira, D. Valesin. *Extinction time for the contact process on general graphs*, arXiv:1509.04133.

[87] D. Watts, S. Strogatz. *Collective dynamics of 'small-world' networks*, Nature. 393, 440–442 (1998).

[88] Y. Zhang. *The complete convergence for the contact process on trees*, Ann. Probab. 24, 1408–1443 (1996).

[89] Z. Q. Zhang, F. C. Pu, B. Z. Li. *Long range percolation in one dimension*, J. Phys. A: Math. Gen. 16 (1983).