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par

Hind AL BABA

Théorie des semi-groupes pour les équations de Stokes et de Navier-Stokes avec des conditions aux limites de type Navier

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Devant la commission d'examen composée de :

M.	Chérif Amrouche	Professeur à l'UPPA	*Directeur de thèse*
M.	Miguel Escobedo	Professeur à l'Universidad del País Vasco	*Directeur de thèse*
M.	Thierry Cazenave	Directeur de Recherche CNRS à l'UPMC	*Rapporteur*
M.	Carlos Conca	Professeur à l'Universidad de Chile	*Rapporteur*
M.	Luis Escauriaza	Professeur à l'Universidad del País Vasco	*Président*
M.	Jacques Giacomoni	Professeur à l'UPPA	*Examinateur*
M.	Jean-Pierre Raymond	Professeur à l'Université Paul Sabatier	*Examinateur*
M.	Franck Sueur	Professeur à l'université de Bordeaux	*Examinateur*

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Contents

Notations

Notations générales

$\mathbb{R}^3$	espace euclidien muni de sa norme usuelle notée $\lvert\cdot\rvert$
Ω	domaine borné de $\mathbb{R}^3$
$\partial\Omega$ ou Γ	frontière de Ω
$\boldsymbol{n}$	vecteur normal unitair à Γ
$\boldsymbol{x} = (x_1, x_2, x_3)$	élément de Ω
$\boldsymbol{v} = (v_1, v_2, v_3)$	champ de vecteurs
$p > 1$	exposant de Lebesgue
$p' > 1$	exposant conjugé de p vérifiant $\frac{1}{p} + \frac{1}{p'} = 1$
$\boldsymbol{x} \cdot \boldsymbol{y}$	produit scalaire de $\boldsymbol{x}$ et $\boldsymbol{y}$
$\boldsymbol{x} \times \boldsymbol{y}$	produit vectoriel de $\boldsymbol{x}$ et $\boldsymbol{y}$
$\boldsymbol{A} : \boldsymbol{B}$	produit matriciel de $\boldsymbol{A}$ et $\boldsymbol{B}$
$\boldsymbol{v}_\tau = \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}$	composante tangentielle de $\boldsymbol{v}$
$\nabla v = (\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, \frac{\partial v}{\partial x_3})$	gradient de v
$\operatorname{div} \boldsymbol{v}$	divergence de $\boldsymbol{v}$

$\mathbf{curl}\, \boldsymbol{v}$	rotationnel de $\boldsymbol{v}$
$(\mathbf{D}(\boldsymbol{v}))_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$	tenseur des déformations de $\boldsymbol{v}$
X'	espace dual de X
$\langle ., . \rangle_{X' \times X}$	produit de dualité X', X

Espaces fonctionnels

$\boldsymbol{\mathcal{D}}(\Omega)$	espace des fonctions indéfiniment différentiable
$\boldsymbol{\mathcal{D}}_\sigma(\Omega)$	espace des fonctions indéfiniment différentiable à divergence nulle
$\boldsymbol{\mathcal{D}}'(\Omega)$	espace des distributions sur Ω
$\boldsymbol{L}^p(\Omega)$	espace des fonctions $\boldsymbol{u}$ mesurables sur Ω telles que $\int_\Omega \lvert\boldsymbol{u}\rvert^p d\boldsymbol{x} < \infty$
$\boldsymbol{H}^p(\mathrm{div}, \Omega)$	espace des fonctions de $\boldsymbol{L}^p(\Omega)$ et à divergence dans $L^p(\Omega)$

Chapter 1

Introduction générale

Cette thèse est consacrée à l'étude théorique mathématique des équations de Stokes

$$\left\{\begin{array}{lll} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = 0 & \text{dans} & \Omega \times (0,T), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{dans} & \Omega \end{array}\right. \tag{1.0.1}$$

et de Navier-Stokes

$$\left\{\begin{array}{lll} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = 0 & \text{dans} & \Omega \times (0,T), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{dans} & \Omega. \end{array}\right. \tag{1.0.2}$$

Ces équations permettent de décrire le mouvement de certains fluides. Elles modélisent un fluide comme étant un milieu continu, caractérisé par des grandeurs physiques définies en tout point de l'espace et à tout instant (voir [67]). Les inconnues $\boldsymbol{u}$ et π représentent respectivement le vecteur vitesse et la pression du fluide occupant un domaine Ω. Les données intiales sont les forces extérieures $\boldsymbol{f}$ et la vitesse initiale $\boldsymbol{u}_0$.

L'étude du comportement des fluides remonte à l'Antiquité avec ***Archimède***. Dans son Traité "des corps flottants", il étudie avec rigueur l'immersion d'un corps solide ou fluide, dans un fluide de densité inférieure ou égale. Il découvre ainsi que tout corps plongé dans un fluide au repos, entièrement mouillé par celui-ci ou traversant sa surface libre, reçoit une poussée: une force verticale, dirigée de bas en haut et opposée au poids du volume de fluide déplacé. Cette force est appelée poussée d'Archimède.

Aprés une longue interruption, l'étude des fluides reprend un essor véritable au XV*ème* siècle avec ***Léonardo da Vinci***. Ce dernier propose de nombreuses descriptions d'écoulements (jets, tourbillons, ondes de surface) et formule ainsi le principe de conservation de la masse ([37]).

En 1687, ***Isaac Newton*** établit dans son oeuvre majeure "Principia mathematica", les trois lois universelles du mouvement. Cette oeuvre marque un tournant pour la physique. Il y avance le principe d'inertie, la proportionnalité des forces et des accélérations, l'égalité de l'action et de la réaction, les lois des collisions et montre le mouvement des fluides et la théorie de l'attraction universelle.

En 1738, ***Daniel Bernouilli*** expose dans son ouvrage “Hydrodynamica”, le théorème fondamental de la mécanique des fluides qui porte son nom : “le théorème de Bernoulli”. Dans cet ouvrage, Bernouilli montre l’importance du principe de la conservation de l’énergie dans l’étude des fluides non visqueux.

En 1749, ***Jean d’Alembert*** introduit de nouvelles notions mathématiques dans l’étude de la dynamique des fluides. Plus précisément les notions de dérivées partielles, le champ de vitesses et les pressions internes d’un fluide. Cette mathématisation de la physique avec l’introduction des outils du calcul différentiel va révolutionner la compréhension mathématique du mouvement des corps solides et liquides. On doit à ***d’Alembert*** la découverte des premières équations de la mécanique des fluides. Cependant, son analyse n’est pas complète et c’est à ***Euler*** qu’on doit l’écriture finale des équations de la mécanique des fluides incompressibles.

En appliquant la loi de Newton “$\sum \boldsymbol{F} = m \cdot a$” à un volume infinitésimal de fluide, ***Euler*** aboutit en 1755 dans son mémoire “Principes généraux du mouvement des fluides”, à un système d’équations aux dérivées partielles décrivant les fluides parfaits incompressibles. Il donna la première définition de la pression d’un fluide, une notion qui avait échappé à ***d’Alembert***.

Les équations d’Euler sont un sytème de trois équations à quatre fonctions inconnues de la position et du temps. Afin d’obtenir une solution unique, les équations d’Euler doivent être complétées par une équation complémentaire qui traduit l’incompressibilité et la conservation de la masse. Voici les équations d’***Euler***:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \qquad \operatorname{div} \boldsymbol{u} = 0 \qquad \text{dans} \quad \Omega \times (0, T).$$

La première équation ci-dessus n’est rien d’autre que la relation fondamentale de la dynamique de Newton, selon laquelle le produit de la masse par l’accélération est égale à la force appliquée. L’accélération (dérivée de la vitesse) correspond aux deux premiers termes du côté gauche de la première équation (voir [67]). En effet, l’accélération est égale à la dérivée partielle par rapport au temps, complétée par le terme $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ en raison du fait que la vitesse intervenant dans ces équations est celle caractérisant le fluide en un point donné de l’espace, et non la vitesse d’une particule fluide suivie dans son mouvement. Le terme $\nabla \pi$ représente les forces de pression. Ce sont des forces internes au fluide qui viennent du fait qu’un petit morceau de fluide se fait pousser par tout le reste du fluide qui l’entoure. Finalement $\boldsymbol{f}$ désigne les forces extérieures.

Cependant, contrairement à l’expérience physique, ***d’Alembert*** s’aperçoit que selon ***Euler***, un corps se déplaçant dans un fluide ne s’oppose à aucune résistance, c’est le “Paradoxe de d’Alembert”. Voici quelques exemples: Prenons l’exemple d’un oiseau qui se déplace dans un fluide: “l’air”. Si on calcule à partir de l’équation d’***Euler*** la pression exercée par l’air à ses ailes, on trouve zéro et donc l’oiseau tombe. Or, il ne tombe pas en réalité. Un autre exemple: Une barque qui se déplace dans l’eau. D’après l’équation d’***Euler***, elle ne devrait jamais ralentir. ***D’Alembert*** s’aperçoit que cette équation développée avec toute la rigueur possible, pose dans certains cas un problème qu'il laisse résoudre aux géomètre futurs.

Cent cinquante ans plus tard, l'ingénieur polytechnicien français ***Claude-Louis Navier*** observe qu'un fluide en évolution, va dissiper de l'énergie sous forme de chaleur et cela est simplement par le frottement d'une couche de fluide sur l'autre. Inclure un tel phénomène dans les équations ***d'Euler*** semble difficile ([37]), car les équations ***d'Euler*** formulent l'écoulement du fluide à l'échelle *macroscopique* alors que la dissipation d'énergie a lieu à un niveau microscopique. Ainsi, dans son oeuvre majeure "Mémoire sur les lois du mouvements du fluide" ([62]) de 1822, ***Navier*** réalisait une percée dans la prise en compte de la viscosité. Pour expliquer cette dernière, ***Navier*** postula l'existence d'une force d'attraction entre deux molécules de fluides. De ces forces résulte, une viscosité qui lisse les différences de vitesse entre un point du fluide et ces voisins. Dans l'équation, on verra donc apparaître un paramètre ν qui représente la viscosité du fluide. Voici le modèle proposé par ***Navier***

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{f}, \qquad \operatorname{div}\boldsymbol{u} = 0 \qquad \text{dans} \quad \Omega\times(0,T).$$

En 1845, ***George Gabriel Stokes*** publia les résultats de ses travaux de thèse "On the theories of the internal friction of fluids in motion" ([77]). Dans cette thèse consacrée à l'étude des fluides visqueux, ***Stokes*** démontre le modèle proposé par ***Navier*** en utilisant une approche mathématique. Pour cette raison, le nom de ***Stokes*** ainsi que celui de ***Navier*** ont été choisis pour désigner ces equations.

Bien qu'elles soient établies au XIXème siècle, les équations de Navier-Stokes continuent de fasciner les ingénieurs, les physiciens et même les mathématiciens. Ces équations sont censées décrire le mouvement des fluides qui peuvent être un liquide ou un gaz. Les équations de Navier-Stokes se révèlent donc cruciales pour décrire de nombreux phénmènes. Elles sont utilisées pour comprendre les mouvements des courants dans les océans, ainsi que ceux des grandes masses d'air dans l'atmosphère. Ces équations entrent également dans l'étude de la circulation du sang dans nos artères, dans la simulation des trajectoires d'air autour d'une aile d'un avion et dans la simulation des tourbillons. Ces équations sont même utilisées dans les jeux vidéos pour améliorer le réalisme de certaines scènes.

Malheureusement jusqu'à présent, on ne sait pas démontrer si pour toute condition initiale, il existe une solution régulière et globale en temps, pour $n \geq 3$. En dimension 2, dimension qui n'est pas très physique mais aide beaucoup mathématiquement, le problème est bien posé au sens ci-dessus grâce aux travaux fondamentaux de Leray en 1934 [53]. Mais en 2D le problème est beaucoup moins compliqué qu'en 3D. En dimension trois on sait (c.f. [40]) que si le champ de vitesse initial est suffisamment petit, il existe toujours une solution régulière globalement définie. Physiquement, cela correspond aux régimes où l'écoulement est laminaire et non pas turbulent. Le régime est dit "*Laminaire*" quand l'écoulement de fluide se fait d'une manière tranquille. C'est le cas des fluides, visqueux lents et plutôt confinés. Comme par exemple le mouvement de l'huile d'olive versée d'une bouteille. À l'opposé, le régime est dit "*turbulent*" quand les fluides sont peu visqueux, rapides et se déplacent sur de grandes distances. Dans ce cas là, les écoulements se produisent de manière chaotique et présentent de nombreux tourbillons, comme par exemple, le mouvement de l'air au passage d'un avion, c'est le chaos total. La turbulence est le pendant physique de la non-linéarité des équation de Navier-Stokes.

Outre les équations qui doivent être satisfaites en tout point du domaine, il sera nécessaire de former également les conditions qui doivent être satisfaites à la frontière. ***Stokes*** a formulé dans sa thèse [77] la condition de non glissement. Selon, ***Stokes***, un fluide en contact avec une paroi solide ne peut pas glisser le long de la paroi, au contraire sa vitesse au bord doit être nulle. C'est la condition de Dirichlet

$$\boldsymbol{u} = 0, \qquad \text{sur} \quad \Gamma \times (0, T),$$

où Γ est la frontière. ***Stokes*** a essayé de vérifier cette condition de non-glissement à travers des calculs expérimentaux. Il a remarqué, qu'à l'interface fluide-solide, la force d'attraction entre les particules de fluide et les particules solides (forces adhésives) sont supérieures à celles entre les particules de fluide (forces cohésives). Ce déséquilibre de forces fait baisser la vitesse du fluide à zéro. Cependant, en calculant selon les conditions de non-glissement, l'écoulement à travers de longs tuyaux circulaires droits et des canaux rectangulaires puis en comparant ces résultats avec l'expérience réelle, ***Stokes*** trouve que dans ce cas-là cette condition n'est pas compatible avec l'expérience (voir [77] pour plus de détails).

Lorsque le domaine occupé par le fluide est bordé par une surface solide, le fluide ne peut la traverser. Sa vitesse est donc forcément nulle dans la direction perpendiculaire à la surface. C'est la condition d'imperméabilité

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \qquad \text{sur} \quad \Gamma \times (0, T).$$

En revanche, elle n'est pas forcément nulle dans les directions tangentielles. En toute rigueur, il y a toujours un glissement, mais parfois il n'est appréciable qu'à des échelles spatiales microscopiques. Dans certains cas, le glissement est notable.

En 1823, ***Navier*** [62] a proposé une condition dite de glissement avec friction à la paroi. Cela permet de prendre en compte le glissement du fluide près du bord et de mesurer l'effet de friction, en considérant la composante tangentielle du tenseur des contraintes proportionnelles à la composante tangentielle du champ de vitesse :

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \qquad 2\,\nu\,[\mathbf{D}(\boldsymbol{u}) \cdot \boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\,\boldsymbol{u}_{\boldsymbol{\tau}} = 0 \qquad \text{sur} \;\; \Gamma \times (0, T), \tag{1.0.3}$$

oú ν est la viscosité, $\alpha \geq 0$ est le coefficient de friction et $\mathbf{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$ est le tenseur de déformation associé au vecteur vitesse $\boldsymbol{u}$.

La littérature scientifique récente montre un regain d'intérêt pour des choix différents de conditions aux limites, notamment en théorie mathématique des fluides. Pour le problème de Navier-Stokes, le choix de ces conditions n'est pas neutre, c'est même toujours un sujet de controverses. La plupart des travaux à ce jour considèrent des conditions de type Dirichlet pour les équations de Navier-Stokes. Cependant, ces conditions de non-glissement ne reflètent pas toujours la réalité et plusieurs études théoriques et expérimentales étaient faites pour justifier les conditions de non-glissement dans les équations de Navier-Stokes. En 1959, Serrin [68] a signalé que ces conditions ne sont pas toujours réalistes et conduisent en général à des phénomènes de couches limites près des parois. Serrin montre aussi que les conditions de non-glissement ne sont plus vraies quand une pression modérée est impliquée comme dans le cas de haute attitude de l'aérodynamiques. La nullité du champ de vitesse à la frontière est donc une contrainte. Une proposition alternative, physiquement et mathématiquement naturelle est celle de glissement de ***Navier*** (1.0.3). Le rôle de la viscosité et la notion de proportionnalité, c'est-à-dire la modélisation du frottement demeure un problème ouvert qui peut conduire à formuler de nouvelles conditions à la frontière. Un autre cas aussi intéressant est lorsque le coefficient de friction α est nul. Cela correspond à une condition de glissement de Navier, mais sans friction :

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \qquad 2\,\nu\,[\mathbf{D}(\boldsymbol{u}) \cdot \boldsymbol{n}]_{\boldsymbol{\tau}} = 0 \qquad \text{sur } \; \Gamma \times (0, T). \tag{1.0.4}$$

Parmi les premiers qui ont étudié le problème de Navier-Stokes avec les conditions de Navier, on peut citer Solonnikov et Ščadilov en 1973. Dans leurs article [76], les auteurs ont considéré le problème stationnaire de Navier-Stokes avec les conditions de Navier sur une partie de bord et une condition de Dirichlet sur le reste de bord et ils ont abouti à l'existence de solutions faibles dans $\boldsymbol{H}^1(\Omega) \times L^2(\Omega)$ et aussi de solutions fortes dans $\boldsymbol{H}^2(\Omega) \times H^1(\Omega)$.

Les conditions de Navier (1.0.4) sont souvent utilisées dans la modélisation d'écoulement en présence de parois rugueuses et perforées. L'effet de la rugosité du domaine sur l'écoulement du fluide a récemment été étudié par Amirat, Bresch, Lemoine et Simon dans [4] et par Bucur, Feireisl et Nečasová dans [21, 20], en considérant des conditions de Navier ce qui leur a permis d'obtenir un comportement asymptotique du fluide. On peut citer aussi les travaux de Jäger et Mikelic [46, 45] qui ont étudié l'influence de la rugosité des parois et donné une justification mathématique à l'utilisation des conditions de Navier. Dans le cas des parois perforées, on peut citer le travail de Beavers et al [47].

Les conditions de Navier sont aussi utilisées dans la simulation numérique des écoulements turbulents comme dans le travail de Berselli et al [16] ainsi que dans celui de Paès [63].

Casado et al [24, 23] donnent une justification mathématique de l'imposition des conditions de non-glissement dans certains cas et montrent l'équivalence entre la condition de glissement et celle d'adhérence dans le cas d'une frontière rugueuse de période ε et d'amplitude ε.

Un autre type de conditions de glissement sans friction de ***Navier*** est celui qui porte sur la composante tangentielle du tourbillon

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \qquad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \qquad \text{sur} \quad \Gamma \times (0, T). \tag{1.0.5}$$

On l'appellera condition de type-Navier. Dans le cas d'un bord plat et lorsque le coefficient de friction est nul, les deux conditions (1.0.4) et (1.0.5) sont équivalentes. Ce type de condition a récemment était étudié par Amrouche et Seloula dans [10, 11] qui ont développé des propriétés sur les potentiels vecteurs ainsi que quelques inégalités de Sobolev et des résultats d'existence de solutions faibles, fortes et très faibles pour le problème de Stokes stationnaire. On peut citer aussi le travail de Amrouche, Penel et Seloula [7].

Da Veiga [14] et Xiao et Xin [81] ont considéré le problème d'évolution de Navier-Stokes avec les conditions de type Navier (1.0.5) et ont étudié la convergence des solutions vers les solutions du problème d'Euler lorsque la viscosité tend vers zéro.

Les condition (1.0.5) ont aussi été étudiés par Mitrea et Monniaux dans [60, 59] qui ont démontré l'analyticité de semi-groupe de Stokes avec ces conditions ainsi qu'un résultat d'existence locale des solutions mild pour le problème non-linéaire. Miyakawa a étudié dans [61] le problème de Navier-Stokes avec les conditions (1.0.5) et abouti à un résultat d'existence locale des solutions.

Dans certaines situations, il est aussi naturel de prescrire la valeur de la pression, tout au moins sur une partie du bord, comme dans le cas de pipelines [57], de systèmes hydrauliques utilisant des pompes, réservoirs confinés. Notons qu'il n'y a aucune justification physique pour prescrire une condition limite de pression sur toute la paroi et celle-ci doit être complétée par une condition limite impliquant la vitesse. Dans la littérature, on peut trouver au moins deux types de conditions limites où la pression intervient. Une condition qui fait intervenir le tenseur de déformation

$$\boldsymbol{u}\cdot\boldsymbol{n}=0, \qquad [(-\pi\, I\, +\, 2\,\nu\mathbf{D}(\boldsymbol{u}))\cdot\boldsymbol{n}]_{\boldsymbol{\tau}}+\alpha\,\boldsymbol{u}_{\boldsymbol{\tau}}=0 \qquad \text{sur} \quad \Gamma\times(0,T)$$

et une seconde qui fait intervenir la composante tangentielle du champ de vitesse

$$\boldsymbol{u}\times\boldsymbol{n}=\mathbf{0}, \qquad \pi=\pi_0 \qquad \text{sur} \quad \Gamma\times(0,T). \tag{1.0.6}$$

Ce types de condition limites a fait l'objet de plusieurs travaux. On peut citer par exemple les travaux de Conca et al. [25, 26] qui ont considéreé le problème stationnaire de Stokes et de Navier-Stokes avec des conditions sur la pression et ont montré des résultats d'existence des solutions. Le problème de Navier-Stokes avec des conditions faisant intervenir la pression a aussi été étudié par Łukaszewicz dans [55], par Marušić [56] et récemment par Amrouche et Seloula dans [10, 11].

Signalons aussi les travaux de Saal [66], Shibata et Shimada [69] et la travail de Shimada [70], qui ont considéré les problèmes de Stokes et de Navier-Stokes avec des conditions de type Robin:

$$\boldsymbol{u}\times\boldsymbol{n}=\mathbf{0}, \qquad \lambda\boldsymbol{u}\cdot\boldsymbol{n}+\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}}\,=\,\mathbf{0}, \quad \text{sur} \quad \Gamma\times(0,T).$$

Le problème de Navier-Stokes peut aussi être étudié avec les conditions de Neumann

$$\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}}-\pi\boldsymbol{n}=\mathbf{0} \qquad \text{sur} \quad \Gamma\times(0,T),$$

voir Solonnikov [75] et Mitrea et Monniaux [58]. Physiquement, ces conditions expliquent l'absence de contrainte sur l'interface séparant deux milieux.

Cette thèse est consacrée à l'étude théorique mathématique du problème d'évolution de Stokes et de Navier-Stokes avec les trois types de conditions (1.0.4), (1.0.5) et (1.0.6) en utilisant la théorie des semi-groupes. Notre but est d'obtenir l'analyticité de semi-groupe de Stokes avec chacunes de ces conditions ainsi qu'une estimation sur les puissances imaginaires pures de l'opérateur de Stokes conduisant à un résultat de régularité $L^p - L^q$ maximale pour le problème de Stokes ainsi qu'à l'étude du problème non-linéaire. On s'intéresse également à la comparaison avec les résultats connu pour le problème de Stokes et de Navier-Stokes avec la condition de Dirichlet (cf. [38, 39]) et les conditions de Robin (cf. [66, 69, 70]).

Ce manuscrit est composé de six chapitres:

Le **Chapitre 2** est dédié aux notations, définitions et propriétés des espaces fonctionnels et aux résultats fondamentaux sur lesquels nous nous appuyons dans la suite. Ce chapitre comporte trois sections : La première section est consacrée à la définition du cadre fonctionnel essentiel à notre travail. Dans la deuxième section, on rappelle quelques résultats préliminaires et on démontre quelques propriétés fondamentales. Parmi ces propriétés, on démontre la formule suivante: Pour toute fonction $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ tel que $\Delta \boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ et pour tout $p \geq 2$, on a

$$\begin{aligned} - \int_\Omega |\boldsymbol{u}|^{p-2} \Delta \boldsymbol{u} \cdot \overline{\boldsymbol{u}} \,\mathrm{d}\,x \;=\; \int_\Omega |\boldsymbol{u}|^{p-2} |\nabla \boldsymbol{u}|^2 \,\mathrm{d}\,x \;+\; 4\,\frac{p-2}{p^2} \int_\Omega \Big| \nabla |\boldsymbol{u}|^{p/2} \Big|^2 \,\mathrm{d}\,x \\ +\; (p-2)\, i \sum_{k=1}^{3} \int_\Omega |\boldsymbol{u}|^{p-4} \,\mathrm{Re}\Big(\frac{\partial \boldsymbol{u}}{\partial x_k} \cdot \overline{\boldsymbol{u}} \Big) \mathrm{Im}\Big(\frac{\partial \boldsymbol{u}}{\partial x_k} \cdot \overline{\boldsymbol{u}} \Big) \,\mathrm{d}\,x \;-\; \Big\langle \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}},\, |\boldsymbol{u}|^{p-2} \boldsymbol{u} \Big\rangle_\Gamma, \quad (1.0.7) \end{aligned}$$

où $\langle .\,,\,. \rangle_\Gamma$ est l'anti-dualité entre $\boldsymbol{W}^{-1/p,p}(\Gamma)$ et $\boldsymbol{W}^{1/p,p'}(\Gamma)$. Cette dernière formule joue un rôle très important dans la démonstration de l'analyticité du semi-groupe de Stokes. Enfin dans la troisième section, on rappelle quelques propriétés des opérateurs sectoriels et des opérateurs fractionnaires. On donne aussi une condition nécessaire et suffisante à l'analyticité d'un semi-groupe engendré par un opérateur.

Dans le **Chapitre 3** on considère l'opérateur de Stokes avec les conditions (1.0.5), (1.0.4) et (1.0.6) respectivement et on démontre l'analyticité du semi-groupe de Stokes avec chacune de ces conditions. Ceci permet de résoudre le problème d'évolution de Stokes (1.0.1) avec chacune de ces trois conditions, respectivement, en utilisant la théorie des semi-groupes. Dans la suite on notera par A_p, (respectivement $\mathbb{A}_p$ et $\mathcal{A}_p$), l'opérateur de Stokes avec les conditions de type-Navier (1.0.5) (respectivement les conditions de Navier (1.0.4) et les conditions qui dépendent de la pression (1.0.6)).

Ce chapitre comporte trois sections : Dans la première section, on considère l'opérateur de Stokes avec les conditions de type Navier (1.0.5) et on démontre l'analyticité du semi-groupe de Stokes avec ces conditions sur les espaces $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, $[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\tau}$ et $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$

respectivement (voir (3.1.4), (3.1.64) et (3.1.71) pour la définition de ces espaces). Pour démontrer cette analyticité, on étudie la résolvante de l'opérateur de Stokes avec ces conditions. Une remarque très importante est que les conditions de type Navier (1.0.5) entraînent la disparition du gradient de pression dans la définition de l'opérateur de Stokes. Ainsi, notre problème se ramène à l'étude du problème :

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = 0 \qquad \text{dans } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \qquad \text{sur } \Gamma, \end{cases} \tag{1.0.8}$$

où $\lambda \in \Sigma_\varepsilon$,

$$\Sigma_\varepsilon = \{\lambda \in \mathbb{C}^*;\ |\arg \lambda| \leq \pi - \varepsilon\}$$

et $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ (respectivement $\boldsymbol{f} \in [\boldsymbol{H}^{p'}_0(\operatorname{div}, \Omega)]'_{\sigma,\tau}$ et $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$). Lorsque Ω est de classe $C^{2,1}$, on démontre l'existence d'une solution forte $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ (respectivement solution faible $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ et très faible $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$) au problème (1.0.8). De plus pour $\lambda \in \mathbb{C}^*$ tel que $\operatorname{Re} \lambda \geq 0$ la solution $\boldsymbol{u}$ satisfait l'estimation

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C(\Omega, p)}{|\lambda|} \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{1.0.9}$$

Pour les solutions faibles et très faibles on démontre aussi une estimation de type (1.0.9) pour les normes de $[\boldsymbol{H}^{p'}_0(\operatorname{div}, \Omega)]'$ et $[\boldsymbol{T}^{p'}(\Omega)]'$ respectivement.

L'existence de solution sur $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ ne pose pas de problème et s'obtient facilement en utilisant le théorème de Lax-Milgram et les injections de Sobolev. L'estimation sur la résolvante (1.0.9) demande plus de travail. Pour $p \geq 2$, on multiplie la première équation du problème (1.0.8) par $|\boldsymbol{u}|^{p-2}\overline{\boldsymbol{u}}$, on utilise la formule (1.0.7) et la relation :

$$\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \nabla_{\boldsymbol{\tau}}(\boldsymbol{u} \cdot \boldsymbol{n}) - \Big(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}\Big)_{\boldsymbol{\tau}} - \sum_{k=1}^{2} \big(\boldsymbol{u}_{\boldsymbol{\tau}} \cdot \frac{\partial \boldsymbol{n}}{\partial s_k}\big) \boldsymbol{\tau}_k, \qquad \text{sur} \quad \Gamma,$$

où ∇_τ est le gradient tangentiel. On obtient ainsi, après plusieurs calculs l'estimation :

$$\begin{aligned} &|\lambda|\, \|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega |\boldsymbol{u}|^{p-2}\, |\nabla \boldsymbol{u}|^2 \,\mathrm{d}\, x + 4\, \frac{p-2}{p^2} \int_\Omega |\nabla\, |\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\, x \\ &\quad \leq \frac{p-2}{2} \int_\Omega |\boldsymbol{u}|^{p-2}\, |\nabla \boldsymbol{u}|^2 \,\mathrm{d}\, x + 2\, C(\Omega) \int_\Gamma |\boldsymbol{u}|^p \,\mathrm{d}\, \sigma + 2\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}. \end{aligned} \tag{1.0.10}$$

Puis on discute par rapport aux valeurs de λ et de p.

L'analyticité sur les espaces duaux $[\boldsymbol{H}^{p'}_0(\operatorname{div}, \Omega)]'_{\sigma,\tau}$ et $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ se fait par dualité. Avant de démontrer cette analyticité, on définit la trace normale $\boldsymbol{f} \cdot \boldsymbol{n}_{|\Gamma}$ d'une distribution $\boldsymbol{f} \in [\boldsymbol{H}^{p'}_0(\operatorname{div}, \Omega)]'$ à divergence dans $L^p(\Omega)$ (respectivement $\boldsymbol{f} \in [\boldsymbol{T}^{p'}(\Omega)]'$ à divergence dans $L^p(\Omega)$). Ceci est essentiel à l'étude du problème (1.0.8) dans le cas où fonction $\boldsymbol{f} \in [\boldsymbol{H}^{p'}_0(\operatorname{div}, \Omega)]'_{\sigma,\tau}$ (respectivement $\boldsymbol{f} \in [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$). Notons que $[\boldsymbol{H}^{p'}_0(\operatorname{div}, \Omega)]' \hookrightarrow \boldsymbol{W}^{-1,p}(\Omega)$ et $[\boldsymbol{T}^{p'}(\Omega)]' \hookrightarrow \boldsymbol{W}^{-2,p}(\Omega)$.

Par ailleurs, On sait d'après les travaux de Amrouche et Seloula [10, 11] que dans le cas où le domaine Ω n'est pas simplement connexe, le noyau du problème du Stokes avec les conditions de type Navier n'est pas trivial et il est donné par l'espace

$$\boldsymbol{K}_{\tau}^{p}(\Omega) \;=\; \big\{\boldsymbol{u} \in \boldsymbol{X}_{\tau}^{p}(\Omega);\; \operatorname{div} \boldsymbol{u} = 0,\; \mathbf{curl}\, \boldsymbol{u} = \mathbf{0} \text{ dans } \Omega\big\}.$$

On peut considérer une autre variante du problème de Stokes avec les conditions (1.0.5) en prenant la restriction de l'opérateur de Stokes sur l'espace des fonctions $\boldsymbol{f} \in \boldsymbol{L}^{p}_{\sigma,\tau}(\Omega)$ satisfaisant la condition de compatibilité

$$\forall\ \boldsymbol{v} \in \boldsymbol{K}_{\tau}^{p'}(\Omega), \qquad \int_{\Omega} \boldsymbol{u} \cdot \overline{\boldsymbol{v}}\,\mathrm{d}\,x = 0. \tag{1.0.11}$$

On obtient ainsi un opérateur à inverse borné qui engendre un semi-groupe analytique et ce dernier décroit exponentiellement.

La deuxième section est consacrée à l'opérateur de Stokes avec des conditions de glissement sans friction de Navier (1.0.4). Dans cette section on démontre que l'opérateur de Stokes avec les conditions de Navier engendre un semi-groupe analytique borné sur les espaces $\boldsymbol{L}^{p}_{\sigma,\tau}(\Omega)$, $[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'_{\sigma,\tau}$ et $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$. Les techniques utilisées dans cette section ressemblent à celles de la section précédente. Cependant, dans ce cas, la pression n'est plus constante et ne disparaît donc pas du problème. Pour cela, on étudie la résolvante de l'opérateur de Stokes avec ces conditions :

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} \,+\, \nabla \pi = \boldsymbol{f}, \quad & \operatorname{div} \boldsymbol{u} = 0 \qquad & \text{dans } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0} & \text{sur } \Gamma, \end{cases} \tag{1.0.12}$$

où $\lambda \in \mathbb{C}^*$ est tel que $\operatorname{Re} \lambda \geq 0$. On démontre l'existence des solutions faibles, fortes et très faibles du problème (1.0.12) satisfaisant une estimation pour la résolvante de type (1.0.9) pour les normes des espaces $\boldsymbol{L}^{p}_{\sigma,\tau}(\Omega)$, $[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'_{\sigma,\tau}$ et $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ respectivement. Pour démontrer l'estimation (1.0.9), on procède de la même manière que la condition de type Navier (1.0.5). Pour $p \geq 2$, on multiplie la première équation du problème (1.0.12) par $|\boldsymbol{u}|^{p-2}\overline{\boldsymbol{u}}$, on utilise la formule (1.0.7) et la relation :

$$[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \nabla_{\boldsymbol{\tau}}(\boldsymbol{u} \cdot \boldsymbol{n}) + \Big(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}\Big)_{\boldsymbol{\tau}} - \sum_{k=1}^{2} \big(\boldsymbol{u}_{\boldsymbol{\tau}} \cdot \frac{\partial \boldsymbol{n}}{\partial s_k}\big) \boldsymbol{\tau}_k, \quad \text{sur} \quad \Gamma.$$

On obtient alors l'estimation :

$$\begin{aligned} &|\lambda| \|\boldsymbol{u}\|^{p}_{\boldsymbol{L}^p(\Omega)} + \int_{\Omega} |\boldsymbol{u}|^{p-2}\, |\nabla\, \boldsymbol{u}|^2 \,\mathrm{d}\,x \,+\, 2\,\frac{p-2}{p^2} \int_{\Omega} |\nabla\, |\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\,x \\ &\quad \leq\, C(\Omega,p)\, \|\boldsymbol{u}\|^{p}_{\boldsymbol{L}^p(\Omega)} \,+\, \frac{p-2}{2} \int_{\Omega} |\boldsymbol{u}|^{p-2} |\nabla \boldsymbol{u}|^2 \,\mathrm{d}\,x \,+\, 2\,\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \,+\, \|\nabla \pi\|_{\boldsymbol{L}^p(\Omega)}\big) \|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}. \end{aligned} \tag{1.0.13}$$

Grâce aux conditions limites (1.0.4) la pression peut être contrôlée par la norme $\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}$ et par la norme $\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)}$ pour $2 \leq p \leq 6$. La norme $\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)}$ est à son tour controlée par $\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}$ ce qui résoud le problème.

La troisième section de ce chapitre est consacrée à l'analyticité du semi-groupe engendré par l'opérateur de Stokes, avec les conditions qui dépendent de la pression et la composante tangentielle du champ de vitesse (1.0.6), sur les espaces $\boldsymbol{L}^p_\sigma(\Omega)$ et $[\boldsymbol{H}^p_0(\mathbf{curl},\Omega)]'_\sigma$ (voir (3.3.8) et (3.3.10) pour la définition de ces espaces). Grâce aux conditions limites (1.0.6), la pression peut être découplée du problème en résolvant directement un problème de Dirichlet. Ceci nous amène donc à étudier un problème pour le laplacien :

$$\begin{cases} \lambda\boldsymbol{u} - \Delta\boldsymbol{u} = \boldsymbol{f}, \quad \operatorname{div}\boldsymbol{u} = 0 & \text{dans} \quad \Omega, \\ \qquad\qquad\qquad \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0}, & \text{sur} \quad \Gamma. \end{cases} \tag{1.0.14}$$

On démontre l'existence de solutions faibles et de solutions fortes au problème (1.0.14) satisfaisant une estimation pour la résolvante de type (1.0.9) pour les normes des espaces $\boldsymbol{L}^p_\sigma(\Omega)$ et $[\boldsymbol{H}^p_0(\mathbf{curl},\Omega)]'_\sigma$ respectivement. On procède de la même manière comme précédemment. Pour $p \geq 2$, on multiplie la première équation du problème (1.0.14) par $|\boldsymbol{u}|^{p-2}\overline{\boldsymbol{u}}$, on utilise la formule (1.0.7) et la formule :

$$\operatorname{div}\boldsymbol{u} \,=\, \operatorname{div}_\Gamma \boldsymbol{u}_{\boldsymbol{\tau}} \,+\, 2\,K\,\boldsymbol{u}\cdot\boldsymbol{n} \,+\, \frac{\partial\,\boldsymbol{u}}{\partial\,\boldsymbol{n}}\cdot\boldsymbol{n} \qquad \text{sur } \Gamma,$$

où $K \,=\, \frac{1}{2}\sum_{j=1}^{2}\frac{\partial\,\boldsymbol{n}}{\partial\,s_j}\cdot\boldsymbol{\tau_j}$ est la courbure moyenne et $\operatorname{div}_\Gamma$ est la divergence surfacique.

Par ailleurs, on sait d'après les travaux de Amrouche et Seloula [10, 11] que dans le cas où la frontière Γ n'est pas connexe, le noyau du problème de Stokes avec les conditions (1.0.6) n'est pas trivial et il est égale à l'espace

$$\boldsymbol{K}^p_N(\Omega) \,=\, \big\{\boldsymbol{u}\in\boldsymbol{X}^p_N(\Omega);\ \operatorname{div}\boldsymbol{u} = 0,\ \mathbf{curl}\,\boldsymbol{u} = \boldsymbol{0} \text{ dans } \Omega\big\}.$$

En prenant la restriction de l'opérateur de Stokes sur l'espace des fonctions $\boldsymbol{f}\in\boldsymbol{L}^p_\sigma(\Omega)$ satisfaisant la condition de compatibilité:

$$\forall\ \boldsymbol{v}\in\boldsymbol{K}^{p'}_N(\Omega), \qquad \int_\Omega \boldsymbol{f}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x = 0,$$

on obtient un opérateur à inverse borné qui engendre un semi-groupe analytique et ce dernier décroît exponentiellement.

Dans la littérature, on trouve plusieurs résultats sur l'analyticité du semi-groupe Stokes. Tout d'abord, en 1977, Solonnikov [74] a démontré l'analyticité du semi-groupe de Stokes avec la condition de Dirichlet sur $\boldsymbol{L}^p(\Omega)$. Dans cet article, il établit une estimation de type (1.0.9) pour toute valeur de λ tel que $|\arg\lambda| \leq \delta + \pi/2$, où $\delta \geq 0$ est petit. Pour dériver cette estimation, Solonnikov utilise une idée de Sobolevskiĭ [73] (voir [74, Theorem 5.2])).

Quelques années plus tard Giga [38] démontre l'analyticité du semi-groupe de Stokes avec des conditions de Dirichlet en utilisant la théorie des opérateurs pseudo-différentiels. Son résultat étend celui de Solonnikov dans deux directions. Premièrement, il démontre l'estimation sur la résolvante (1.0.9) pour tout λ dans le secteur Σ_ε et pour tout $\varepsilon > 0$. Deuxièmement, dans [38] la résolvante est donnée explicitement, ce qui permet de décrire les domaines des opérateurs fractionnaires de l'opérateur de Stokes avec la condition de Dirichlet. En domaine extérieur, Giga et Zohr [42] approchent la résolvante de l'opérateur de Stokes avec les conditions de Dirichlet par la résolvante de l'opérateur de Stokes dans $\mathbb{R}^n$ pour démontrer l'analyticité. Plus tard, Farwig et Sohr [34] étudient la résolvante de l'opérateur de Stokes avec des conditions aux limites de Dirichlet dans le cas où $\operatorname{div} \boldsymbol{u} \neq 0$ sur Ω. Leurs résultats comprennent des domaines bornés et non bornés et les preuves reposent sur la technique de multiplicateur. Récemment, Abe et Giga [1] démontrent l'analyticité de semi-groupe de Stokes avec les conditions de Dirichlet dans les espaces des fonctions bornées. Leur approche est différente de l'approche usuelle. Pour établir leur résultat ils démontrent l'estimation

$$\|N(\boldsymbol{u},\pi)\|_{\boldsymbol{L}^\infty(\Omega\times]0,T_0[)} \leq C\,\|\boldsymbol{u}_0\|_{\boldsymbol{L}^\infty(\Omega)}$$

où

$$N(\boldsymbol{u},\pi)(x,t) = |\boldsymbol{u}(x,t)| \,+\, t^{1/2}\,|\nabla \boldsymbol{u}(x,t)| \,+\, t\,|\nabla^2 \boldsymbol{u}(x,t)| \,+\, t\,|\partial_t \boldsymbol{u}(x,t)| \,+\, |\nabla \pi(x,t)|.$$

La résolvante de l'opérateur de Stokes est aussi étudiée avec les conditions de Robin par Saal [66], Shibata et Shimada [69]. Saal [66] démontre que l'opérateur de Stokes avec les conditions de Robin est sectoriel et engendre un semi-groupe analytique borné sur $\boldsymbol{L}^p(\Omega)$. La stratégie pour prouver ces résultats est d'abord de construire une solution explicite pour le problème associé à la résolvante de l'opérateur de Stokes avec ces conditions. Dans des domaines bornés ou extérieurs, Shibata et Shimada ont démontré dans [69] une estimation géneralisée pour la résolvante de l'opérateur de Stokes, avec des conditions non-homogènes de Robin et quand la divergence ne s'annule pas sur Ω, dans un cadre L^p. Ils ont établi leur résultat en étendant le résultats de Farwig et Sohr [34]. Leur approche est donc différente de celui de Saal [66] et plutôt proche de celui de Farwig et Sohr [34].

Concernant les conditions de type-Navier, Mitrea et Monniaux [60] étudient la résolvante de l'opérateur de Stokes avec les conditions de type Navier dans des domaines Lipschitziens. Ils démontrent ainsi l'estimation (1.0.9) en utilisant la théorie des formes différentielles sur des sous-domaines Lipschitziens d'une variété riemannienne lisse et compacte. En outre, lorsque le domaine Ω est de classe C^∞, Miyakawa montre dans [61] que le laplacien avec les conditions de type Navier (1.0.5) laisse l'espace $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ invariant et donc engendre un semi-groupe analytique borné sur $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$.

Dans **le Chapitre 4**, on étudie les puissances complexes et fractionnaires de l'opérateur de Stokes avec les conditions de Navier (1.0.4), de type Navier (1.0.5) et les conditions qui dépendent de la pression (1.0.6) respectivement. Comme l'opérateur de Stokes avec ces conditions engendre un semi-groupe analytique borné, il est en particulier un opérateur positif. Il en résulte alors, d'après les résultats de Komatsu [52] et de Tribel [78], que les puissances complexes et fractionnaires de l'opérateur de Stokes avec chacune de ces conditions respectives sont des opérateurs bien définis, fermés et à domaines denses. Le but de ce chapitre est de montrer certaines propriétés et estimations de ces puissances. Les résultats sur les puissances imaginaires pures seront utilisés dans le chapitre 5 pour obtenir un résultat de régularité $L^p - L^q$ maximale pour le problème de Stokes inhomogène. Ceux sur les puissances fractionnaires seront utilisés dans le chapitre 5 pour obtenir des estimations de type $L^p - L^q$ pour le problème de Stokes homogène et dans le chapitre 6 pour obtenir des résultats d'existence et d'unicité locale pour le problème non-linéaire. Puisque l'opérateur de Stokes avec chacune de ces conditions n'est pas inversible, il n'est pas facile en général de calculer ces puissances. On peut éviter cette difficulté en suivant le même argument de Borchers et Miyakawa [17] ainsi que de celui de Giga et Sohr [43]. Nos résultats seront obtenus en utilisant la théorie d'interpolation et l'invariance d'un domaine étoilé par la transformation

$$\forall\, x \in \Omega, \qquad (S_\mu \boldsymbol{f})(x) = \boldsymbol{f}(x/\mu), \qquad \boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega), \tag{1.0.15}$$

où $\mu > 0$. On utilise aussi le fait que l'opérateur de Stokes avec ces conditions, satisfait la propriété

$$\mu^2 A_p \,=\, S_\mu\, A_p\, S_\mu^{-1}, \qquad I + \mu^2 A_p \,=\, S_\mu (I + A_p)\, S_\mu^{-1}. \tag{1.0.16}$$

Tout d'abord on démontre nos résultats pour l'opérateur $(1/\mu^2 I + A_p)^\alpha$, $\alpha \in \mathbb{C}$ puis on passe à la limite lorsque $\mu \to \infty$. On termine par le fait qu'un ouvert Lipschitzien continu est l'union d'un nombre fini d'ouverts Lipschitziens étoilés.

Ce chapitre comporte trois sections. Dans la première section, on étudie les puissances complexes et fractionnaires de l'opérateur de Stokes avec les conditions de type-Navier (1.0.5). On démontre que les puissances imaginaires pures de l'opérateur de Stokes avec les conditions de type-Navier sont bornées et satisfont l'estimation :

$$\forall\, s \in \mathbb{R}, \qquad \|(A_p)^{i\,s}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \,\leq\, K\, e^{|s|\theta_0}, \tag{1.0.17}$$

pour un certain $0 < \theta_0 < \pi/2$. Pour démontrer cette dernière on procède comme dans [43, Appendix A]. Tout d'abord, on prouve une estimation de type (1.0.17) pour l'opérateur $(I + A_p)^{i\,s}$, $s \in \mathbb{R}$, puis on déduit l'estimation (1.0.17) pour l'opérateur de Stokes en utilisant la transformation S_μ et en passant à la limite lorsque $\mu \to \infty$, exactement comme dans [43, Appendix A].

Après avoir établi l'estimation (1.0.17), on étudie les puissances fractionnaires de l'opérateur de Stokes avec les conditions de type-Navier ainsi que leurs domaines. On montre que

$$\mathbf{D}(A_p^{1/2}) = \boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega),$$

où

$$\boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega) = \{\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega);\ \operatorname{div} \boldsymbol{u} = 0 \ \text{ dans } \ \Omega \ \text{ et } \ \boldsymbol{u} \cdot \boldsymbol{n} = 0 \ \text{ sur } \ \Gamma\}. \tag{1.0.18}$$

On montre aussi l'équivalence des deux normes

$$\forall \boldsymbol{u} \in \mathbf{D}(A_p^{1/2}), \qquad \|A_p^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \simeq \|\mathbf{curl}\, \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

Notons que, puisque l'opérateur de Stokes avec les conditions de types-Navier n'est pas inversible, cette équivalence des normes n'est pas automatique. Cela vient du fait que le domaine $\mathbf{D}(A_p^{1/2})$ est muni de la norme du graphe $\|(I + A_p)^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$ et non de la norme de $\|A_p^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$.

Concernant les puissances fractionnaires de l'opérateur de Stokes, on montre une injection de type Sobolev. Plus précisément, pour tout $\alpha \in \mathbb{R}$, tel que $0 < \alpha < 3/2p$, on a

$$\mathbf{D}(A_p^{\alpha}) \hookrightarrow \boldsymbol{L}^q(\Omega), \qquad \frac{1}{q} = \frac{1}{p} - \frac{2\alpha}{3}. \tag{1.0.19}$$

De plus, pour tout $\boldsymbol{u} \in \mathbf{D}(A_p^{\alpha})$

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^q(\Omega)} \ \leq \ C(\Omega, p)\, \|A_p^{\alpha}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{1.0.20}$$

L'estimation (1.0.20) n'est pas immédiate puisque l'opérateur de Stokes avec les conditions de type Navier n'est pas inversible. Pour démontrer cette estimation, on procède comme dans [17]. L'injection (1.0.19) implique que pour tout $\boldsymbol{u} \in \mathbf{D}(A_p^{\alpha})$

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^q(\Omega)} \ \leq \ C(\Omega, p)\, \|(I + A_p)^{\alpha}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

En utilisant, la transformation S_μ (1.0.15), la relation (1.0.16) et en passant à la limite lorsque $\mu \longrightarrow \infty$ on obtient l'estimation (1.0.20).

Dans la deuxième section, on étudie les puissances complexes et fractionnaires de l'opérateur de Stokes avec les conditions de glissement de Navier (1.0.4), dont on a noté par $\mathbb{A}_p$. En procédant comme dans la première section de ce chapitre, on démontre que les puissances imaginaires pures de l'opérateur $\mathbb{A}_p$ sont bornées et satisfont une estimation de type (1.0.17). On démontre aussi que le domaine de $\mathbf{D}(\mathbb{A}_p^{1/2})$ est égal à l'espace $\boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega)$. Mais dans ce cas, on a l'équivalence des normes suivantes:

$$\forall \boldsymbol{u} \in \mathbf{D}(\mathbb{A}_p^{1/2}), \qquad \|\mathbb{A}_p^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \simeq \|\mathbf{D}(\boldsymbol{u})\|_{\boldsymbol{L}^p(\Omega)}.$$

On démontre également l'injection de Sobolev (1.0.19) et l'estimation (1.0.20) pour l'opérateur de Stokes $\mathbb{A}_p$.

La troisième section est consacrée à l'étude des puissances complexes et fractionnaires de l'opérateur $\mathcal{A}_p$ qui représente l'opérateur de Stokes avec les conditions (1.0.6) qui dépendent de la pression et de la composante tangentielle du champ de vitesses. Comme pour les autres conditions limites, les puissances imaginaires pures de l'opérateur de Stokes avec les conditions (1.0.6) sont bornées et satisfont l'estimation (1.0.17). On donne aussi dans cette section, une caractérisation du domaine $\mathbf{D}(\mathcal{A}_p^{1/2})$. Dans ce cas-là on a :

$$\mathbf{D}(\mathcal{A}_p^{1/2}) = \boldsymbol{W}^{1,p}_{\sigma,N}(\Omega),$$

où

$$\boldsymbol{W}^{1,p}_{\sigma,N}(\Omega) = \{\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega);\ \operatorname{div} \boldsymbol{u} = 0 \ \text{ dans } \ \Omega \ \text{ et } \ \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \ \text{ sur } \ \Gamma\}.$$

On a également l'équivalence des normes:

$$\forall \boldsymbol{u} \in \mathbf{D}(\mathcal{A}_p^{1/2}), \qquad \|\mathcal{A}_p^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \simeq \|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

Les domaines fractionnaires de l'opérateur de Stokes avec les conditions (1.0.6) satisfont aussi l'injection de Sobolev (1.0.19) et l'estimation (1.0.20).

Dans la littérature les puissances imaginaires pures de l'opérateur de Stokes avec les conditions de Dirichlet ont été étudiées par Giga et Sohr dans [39, 42, 43]. Dans le cas où le domaine Ω est borné, Giga établit dans [39] une estimation de type (1.0.17) pour l'opérateur de Stokes avec Dirichlet en utilisant la théorie des opérateurs pseudo-différentiels. Dans $\mathbb{R}^n$, Giga et Sohr [42] démontrent l'estimation (1.0.17) pour l'opérateur de Stokes en utilisant la transformation de Fourier et la technique des multiplicateurs. Dans le cas d'un domaine extérieur, Giga et Sohr obtiennent cette estimation en comparant les puissances imaginaires de l'opérateur de Stokes dans le domaine extérieur avec celles de l'opérateur de Stokes dans $\mathbb{R}^n$. Finalement dans le cas du demi-espace, Giga et Sohr démontrent l'estimation (1.0.17) pour l'opérateur de Stokes en suivant l'approche de Borchers et Miyakawa dans [17].

Concernant les puissances fractionnaires de l'opérateur de Stokes et leurs domaines avec les conditions de Dirichlet, celles-ci ont été étudiées par Borchers et Miyakawa dans [17, 18] et par Giga et Sohr dans [39, 42, 43] avec différents types de domaines (domaines bornés, domaines exterieurs, demi-espace). Ces auteurs ont démontré que :

$$\mathbf{D}(A_p^{1/2}) \ = \ \boldsymbol{W}^{1,p}_0(\Omega) \cap \boldsymbol{L}^p_\sigma(\Omega)$$

et

$$\forall \boldsymbol{u} \in \mathbf{D}(A_p^{1/2}), \qquad \|A_p^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \simeq \|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

Ils ont aussi démontré des injections de Sobolev de type (1.0.19) ainsi que l'estimation (1.0.20) pour les domaines fractionnaires de l'opérateur de Stokes.

Le Chapitre 5 est dédié au problème d'évolution de Stokes avec chacune des conditions traitées dans les chapitres précédents. Pour le problème de Stokes homogène (*i.e.* $\boldsymbol{f} = \mathbf{0}$), l'analyticité du semi-groupe de Stokes établie dans le chapitre 3 nous donne une unique solution satisfaisant toute la régularité souhaitée:

$$\forall\ k \in \mathbb{N},\ \ \ell \in \mathbb{N}^*, \qquad \boldsymbol{u} \in C^k(]0\,,\,\infty[\,,\,\mathbf{D}(A_p^\ell)).$$

Cependant, pour le problème non homogène ($\boldsymbol{f} \neq 0$), l'analyticité du semi-groupe n'est pas suffisante pour obtenir une unique solution $\boldsymbol{u}$ satisfaisant la régularité maximale et normalement il faut imposer plus de régularité sur $\boldsymbol{f}$. On résout ce problème en utilisant les puissances imaginaires pures de l'opérateur de Stokes.

Ce chapitre est composé de trois sections : Dans la première section, on considère le problème d'évolution (1.0.1) avec les conditions de type Navier (1.0.5). Pour le problème de Stokes homogène, grâce aux conditions limites (1.0.5), le gradient de pression disparaît du problème et le problème de Stokes homogène avec les conditions de type-Navier (1.0.5) est équivalent au problème:

$$\left\{\begin{array}{rclc} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} = \boldsymbol{0}, & \operatorname{div} \boldsymbol{u} = 0 & \text{dans} & \Omega \times (0,T), \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, & \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} & \text{sur} & \Gamma \times (0,T), \\ & \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{dans} & \Omega. \end{array}\right. \tag{1.0.21}$$

L'analyticité du semi-groupe de Stokes avec les conditions de type-Navier sur les espaces $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$ et $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ respectivement, nous donne les solutions faibles, fortes et très faibles du problème (1.0.21).
Pour une donnée initiale $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, le problème (1.0.21) admet une unique solution qui outre que $\boldsymbol{u} \in C^k(]0\,,\,\infty[\,,\,\mathbf{D}(A_p^\ell))$, satisfait en particulier les propriétés :

$$\forall\, 1 \leq q < 2,\ \forall\ T < \infty, \quad \boldsymbol{u} \in L^q(0,T;\, \boldsymbol{W}^{1,p}(\Omega)) \ \text{ et } \ \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\, [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]').$$

De plus, pour une donnée initiale $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, on démontre que l'unique solution $\boldsymbol{u}(t)$ du problème de Stokes homogène (1.0.21) est dans $\boldsymbol{L}^q(\Omega)$, pour tout $t > 0$ et pour tout $1 < p \leq q < \infty$ et satisfait l'estimation de type $L^p - L^q$

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \leq C\, t^{-3/2(1/p-1/q)} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}, \tag{1.0.22}$$

qui joue un rôle très important dans le problème non-linéaire.

Dans la littérature, on trouve différents résultats sur les estimations $L^p - L^q$ pour le semi-groupe de Stokes. Dans le cas où $\Omega = \mathbb{R}^3$, Kato montre dans [50] que les estimations $L^p - L^q$ pour le semi-groupe de Stokes découlent directement des estimations correspondantes pour le semi-groupe de la chaleur. Dans le demi-espace, Borchers et Miyakawa [18] obtiennent ces estimations pour le semi-groupe de Stokes avec les conditions de Dirichlet de la formule d'Ukai [79]. Dans le cas d'un domaine borné, Giga établit dans [40] l'estimation (1.0.22) pour le semi-groupe de Stokes avec les conditions de Dirichlet grâce à des injections de Sobolev pour les domaines fractionnaires de l'opérateur de Stokes. En procédant de la même manière, Giga et Sohr établissent dans [42] l'estimation (1.0.22) pour le semi-groupe de Stokes avec Dirichlet dans le cas des domaines extérieurs avec des limitations sur les indices p et q. Récemment Coulhon et Lamberton [27] montrent que l'estimation $L^p - L^q$ du semi-groupe de Stokes avec les conditions de Dirichlet dans le cas d'un domaine borné ou d'un domaine extérieur ou bien encore du demi-espace, s'obtiennent par un simple transfert de propriétés du semi-groupe de la chaleur.

On considère également dans ce chapitre le cas où la donnée initiale $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ et satisfait la condition de compatibilité:

$$\forall\, \boldsymbol{v} \in \boldsymbol{K}^{p'}_{\tau}(\Omega), \qquad \int_{\Omega} \boldsymbol{u}_0 \cdot \overline{\boldsymbol{v}}\, \mathrm{d}\, x = 0.$$

Dans ce cas, l'unique solution du problème (1.0.21) décroît exponentiellement et satisfait en particulier la même condition que celle vérifiée par $\boldsymbol{u}_0$ ci-dessus. Par conséquent, dans ce cas, les estimations $L^p - L^q$ sont de la forme :

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \;\leq\; C\, e^{-\delta\, t}\, t^{-3/2(1/p-1/q)} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)},$$

pour un certain $\delta > 0$.

Lorsque le domaine Ω est de classe $C^{2,1}$ et lorsque la donnée initiale est plus régulière et appartient à l'espace $\boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega)$ (1.0.18), on obtient une solution forte du problème (1.0.21) qui satisfait

$$\forall\, 1 \leq q < 2,\;\; \forall\, T < \infty, \quad \boldsymbol{u} \in L^q(0, T;\, \boldsymbol{W}^{2,p}(\Omega)) \;\text{ et }\; \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0, T;\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega)).$$

On démontre aussi l'existence des solutions très faibles au problème (1.0.21) quand Ω est de classe $C^{2,1}$ et quand $\boldsymbol{u}_0 \in [\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]'_{\sigma,\tau}$. Dans ce cas là, la solution $\boldsymbol{u}$ du problème (1.0.21) satisfait

$$\forall\, 1 \leq q < 2,\;\; \forall\, T < \infty, \quad \boldsymbol{u} \in L^q(0, T;\, \boldsymbol{L}^p(\Omega)) \;\text{ et }\; \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0, T;\, [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}).$$

Le concept des *"solutions très faibles"* a été d'abord introduit par Lions et Magenes dans [54]. Ensuite, Amann considère ce type de solutions dans [3], dans le cadre des espaces de Besov. Récemment, ce concept a été modifié par Galdi et Sohr dans [31, 32, 30], Farwig et Kozono dans [33] et Galdi, Simader et Sohr dans [36] à un cadre des espaces L^p classiques. Le concept de solutions très faibles est fortement basé sur des arguments de dualité des solutions fortes. Pour cela, la régularité du domaine pour les solutions très faibles est la même que pour les solutions fortes.

On traite aussi dans cette section, le problème de Stokes non-homogène où $\boldsymbol{u}_0 = \boldsymbol{0}$ et $\boldsymbol{f} \in L^q(0, T;\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$, $1 < p, q < \infty$. Dans ce cas, la pression est aussi une constante. Pour une telle fonction $\boldsymbol{f}$ l'unique solution de problème (1.0.1) satisfait $\boldsymbol{u} \in C([0, T]\,;\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$ pour tout $T < \infty$. Cependant, pour une fonction $\boldsymbol{f} \in L^q(0, T;\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$ l'analyticité du semi-groupe de Stokes ne suffit pas pour obtenir une unique solution $\boldsymbol{u}(t)$ du problème satisfaisant la régularité maximale:

$$\boldsymbol{u} \in L^q(0, T;\, \boldsymbol{W}^{2,p}(\Omega)), \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0, T;\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega)). \tag{1.0.23}$$

Normalement, il faut imposer plus de régularité sur $\boldsymbol{f}$ (par exemple $\boldsymbol{f}$ est localement continue Hölderienne [64]).
Dans cette section, on démontre que lorsque la force extérieure est dans $L^q(0, T;\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$ et $\boldsymbol{u}_0 = \mathbf{0}$, l'unique solution du problème (1.0.1)-(1.0.5) satisfait la régularité $L^p - L^q$ maximale (1.0.23). Pour démontrer ce résultat on suit l'approche de Giga et Sohr [43].

La régularité maximale pour le problème de Stokes a été d'abord étudiée par Solonnikov dans [74] avec les conditions de Dirichlet pour $0 < T < \infty$. Celui-ci construit une unique solution $(\boldsymbol{u}, \pi)$ du problème (1.0.1) dans $\Omega \times [0, T)$ satisfaisant la régularité:

$$\int_0^T \left\| \frac{\partial \boldsymbol{u}}{\partial t} \right\|^p_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\, t + \int_0^T \|\nabla^2 \boldsymbol{u}(t)\|^p_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\, t + \int_0^T \|\nabla \pi(t)\|^p_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\, t \\ \leq C(T, \Omega, p) \int_0^T \|\boldsymbol{f}(t)\|^p_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\, t,$$

où $\nabla^2 \boldsymbol{u} = (\partial_i \partial_j \boldsymbol{u})_{i,j=1,2,3}$ est la matrice des dérivées de second ordre de $\boldsymbol{u}$. Lorsque Ω est non borné l' estimation de Solonnikov n'est pas globale en temps car $C(T, \Omega, p)$ peut tendre vers l'infini lorsque $T \to \infty$. Son approche est basée sur les méthodes de la théorie des potentiels. Quelques années plus tard, Giga et Sohr [43] étendent le résultat de Solonnikov [74] dans deux directions. Leur estimation est globale en temps et peut contenir différents indices p et q. Pour obtenir cette régularité $L^p - L^q$ maximale et globale en temps pour le problème de Stokes avec Dirichlet, ils utilisent la théorie de perturbation et les puissances imaginaires pures d'un opérateur non-négatif. En effet, Giga et Sohr établissent dans [43] un résultat pour un problème de Cauchy abstrait qui étend le résultat de Dore et Venni [28].

Concernant le problème de Stokes avec les conditions de Robin, Saal établit dans [66] un résultat de régularité $L^p - L^q$ maximale pour le problème de Stokes avec des conditions de Robin homogène en utilisant les puissances imaginaires pures de l'opérateur de Stokes avec ces conditions. Cependant, cette approche ne s'applique pas dans le cas où des conditions de Robin non-homogènes sont considérées. Pour cela Shimada [70] dérive cette régularité $L^p - L^q$ maximale pour le problème de Stokes avec des conditions de Robin non-homogènes en appliquant l'opérateur de Weis et le multiplicateur de Fourier à la formule explicite donnant la solution du problème de Stokes avec ces conditions.

Comme il est indiqué ci-dessus, lorsque la force extérieure f est à divergence nulle et sa composante normale est nulle à la frontière, le gradient de pression disparaît du problème. Par contre, pour une fonction $\boldsymbol{f} \in L^q(0,T;\, \boldsymbol{L}^p(\Omega))$, le gradient de pression n'ai pas nul et peut être découplé du problème en résolvant un problème de Newmann faible (Voir [71])

$$\operatorname{div}(\mathbf{grad}\,\pi - \boldsymbol{f}) = 0 \qquad \text{dans } \Omega \times (0,T), \qquad (\mathbf{grad}\,\pi - \boldsymbol{f}) \cdot \boldsymbol{n} = 0 \qquad \text{sur} \quad \Gamma \times (0,T).$$

Notre résultat est le suivant: lorsque Ω est de classe $C^{2,1}$, $\boldsymbol{f} \in L^q(0,T;\, \boldsymbol{L}^p(\Omega))$, $1 < p, q < \infty$, $T \leq \infty$ et $\boldsymbol{u}_0 = \mathbf{0}$, le problème de Stokes (1.0.1)-(1.0.5) admet une unique solution $(\boldsymbol{u}, \pi)$ telle que:

$$\boldsymbol{u} \in L^q(0,T_0;\, \boldsymbol{W}^{2,p}(\Omega)), \quad T_0 \leq T \text{ si } T < \infty \text{ et } T_0 < T \text{ si } T = \infty,$$

$$\pi \in L^q(0,T;\, W^{1,p}(\Omega)/\mathbb{R}), \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\, \boldsymbol{L}^p(\Omega)).$$

On a aussi l'estimation suivante

$$\int_0^T \left\| \frac{\partial \boldsymbol{u}}{\partial t} \right\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\,t + \int_0^T \|\Delta \boldsymbol{u}(t)\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\,t + \int_0^T \|\pi(t)\|^q_{W^{1,p}(\Omega)/\mathbb{R}} \mathrm{d}\,t$$
$$\leq C(p,q,\Omega) \int_0^T \|\boldsymbol{f}(t)\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\,t.$$

On montre aussi dans cette section l'existence des solutions faible et très faible au problème de Stokes non-homogène avec les conditions de type-Navier. Pour une donnée initiale $\boldsymbol{u}_0 = 0$ et une force $\boldsymbol{f} \in L^q(0,T;\, [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]')$, $1 < p, q < \infty$ et $0 < T \leq \infty$ le problème (1.0.1)-(1.0.5) admet une unique solution $(\boldsymbol{u}, \pi)$ satisfaisant

$$\boldsymbol{u} \in L^q(0,T_0;\, \boldsymbol{W}^{1,p}(\Omega)), \quad T_0 \leq T \text{ si } T < \infty \text{ et } T_0 < T \text{ si } T = \infty,$$

$$\pi \in L^q(0,T;\, L^p(\Omega)/\mathbb{R}), \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T; \in [\boldsymbol{H}_0^{p'}(\mathrm{div}\Omega)]'_{\sigma,T})$$

avec l'estimation suivante

$$\int_0^T \left\| \frac{\partial \boldsymbol{u}}{\partial t} \right\|^q_{[\boldsymbol{H}_0^{p'}(\mathrm{div}\Omega)]'} \mathrm{d}\,t + \int_0^T \|\Delta \boldsymbol{u}(t)\|^q_{[\boldsymbol{H}_0^{p'}(\mathrm{div}\Omega)]'} \mathrm{d}\,t + \int_0^T \|\pi(t)\|^q_{L^p(\Omega)/\mathbb{R}} \mathrm{d}\,t$$
$$\leq C(p,q,\Omega) \int_0^T \|\boldsymbol{f}(t)\|^q_{[\boldsymbol{H}_0^{p'}(\mathrm{div}\Omega)]'} \mathrm{d}\,t.$$

Lorsque Ω est de classe $C^{2,1}$, $0 < T \leq \infty$, $1 < p, q < \infty$, $\boldsymbol{u}_0 = 0$ et $\boldsymbol{f} \in L^q(0,T;\, [\boldsymbol{T}^{p'}(\Omega)]')$ le problème (1.0.1)-(1.0.5) admet une unique solution $(\boldsymbol{u}, \pi)$ qui satisfait

$$\boldsymbol{u} \in L^q(0,T_0;\, \boldsymbol{L}^p(\Omega)), \quad T_0 \leq T \text{ si } T < \infty \text{ et } T_0 < T \text{ si } T = \infty,$$

$$\pi \in L^q(0,T;\, W^{-1,p}(\Omega)/\mathbb{R}), \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T; \in [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau})$$

et

$$\int_0^T \left\| \frac{\partial \boldsymbol{u}}{\partial t} \right\|^q_{[\boldsymbol{T}^{p'}(\Omega)]'} \mathrm{d}\,t + \int_0^T \|\Delta \boldsymbol{u}(t)\|^q_{[\boldsymbol{T}^{p'}(\Omega)]'} \mathrm{d}\,t + \int_0^T \|\pi(t)\|^q_{W^{-1,p}(\Omega)/\mathbb{R}} \mathrm{d}\,t$$
$$\leq C(p,q,\Omega) \int_0^T \|\boldsymbol{f}(t)\|^q_{[\boldsymbol{T}^{p'}(\Omega)]'} \mathrm{d}\,t.$$

Dans la deuxième section, on considère le problème d'évolution de Stokes (1.0.1) avec les conditions de glissement de Navier (1.0.4). Dans ce cas pour une fonction $\boldsymbol{f}$ à divergence nulle et dont la composante normale est nulle au bord, la pression n'est pas constante et ne disparaît pas du problème. Comme dans la première section de ce chapitre, on donne l'existence des solutions fortes, faibles et très faibles pour le problème (1.0.1)-(1.0.4) satisfaisant la régularité $L^p - L^q$ maximale. Pour le problème homogène cela découle directement de l'analyticité du semi-groupe de Stokes avec les conditions de glissement de Navier. Pour le problème non-homogène, la régularité $L^p - L^q$ maximale est obtenue en utilisant les puissances imaginaires pures de l'opérateur de Stokes avec ces conditions.

Dans la troisième section, on considère le problème d'evolution de Stokes (1.0.1) avec les conditions qui dépendent de la pression (1.0.6). Grâce aux conditions limites (1.0.6), la pression peut être découplée du problème (1.0.1) en utilisant le problème de Dirichlet

$$\Delta\pi = \operatorname{div}\boldsymbol{f} \qquad \text{dans } \ \Omega \times (0,T), \qquad \pi = 0 \qquad \text{sur } \ \Gamma \times (0,T). \tag{1.0.24}$$

Donc, pour une fonction $\boldsymbol{f}$ tel que $\operatorname{div}\boldsymbol{f} = 0$ dans Ω, la pression est nulle et notre problème revient à étudier le problème pour le laplacien suivant:

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta\boldsymbol{u} = \boldsymbol{f}, \quad \operatorname{div}\boldsymbol{u} = 0 & \text{dans} \quad \Omega \times (0,T), \\ \qquad\qquad\qquad\quad \boldsymbol{u} \times \boldsymbol{n} = 0 & \text{sur} \quad \Gamma \times (0,T), \\ \qquad\qquad\qquad\quad \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{dans} \quad \Omega. \end{cases} \tag{1.0.25}$$

On montre l'existence d'une unique solution forte (respectivement faible) pour le problème (1.0.25) pour tout $\boldsymbol{f} \in L^q(0,T;\, \boldsymbol{L}^p_\sigma(\Omega))$ (respectivement $\boldsymbol{f} \in L^q(0,T;\, [\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'_\sigma)$) et $\boldsymbol{u}_0 \in \boldsymbol{W}^{1,p}_{\sigma,N}(\Omega)$, où

$$\boldsymbol{W}^{1,p}_{\sigma,N}(\Omega) = \{\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega);\ \operatorname{div}\boldsymbol{u} = 0 \text{ dans } \Omega,\ \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \text{ sur } \Gamma\}$$

(respectivement $\boldsymbol{u}_0 \in \boldsymbol{L}^p_\sigma(\Omega)$). Maintenant pour une fonction $\boldsymbol{f}$ qui n'est pas à divergence nulle on récupère une pression non nulle en utilisant le problème (1.0.24).

Dans le **Chapitre 6** on considère le problème de Navier-Stokes (1.0.2) avec chacune des conditions considérées dans les chapitres précédents. Pour une donnée initiale $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, $p \geq 3$ et lorsque le second membre $\boldsymbol{f} = \mathbf{0}$, on démontre l'existence locale d'une unique solution $\boldsymbol{u} \in C((0,T_*];\, \mathbf{D}(A_p)) \cap C^1((0,T_*];\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$. Tout d'abord, en utilisant le résultat de Giga [40, Theorem 1, Theorem 2] on démontre l'existence d'une unique solution locale $\boldsymbol{u}(t)$ qui vérifie

$$\boldsymbol{u} \in BC([0,T_0);\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega)) \cap L^q(0,T_0;\, \boldsymbol{L}^r_{\sigma,\tau}(\Omega)),$$

$$q,\, r > p, \qquad \frac{2}{q} + \frac{3}{r} = \frac{3}{p}.$$

De plus, pour une donnée initiale petite (*i.e.* $\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \leq \varepsilon$), on obtient l'existence globale des solutions. En effet, Giga considère dans [40] un problème semi-linéaire parabolique abstrait

$$\frac{\partial \boldsymbol{u}}{\partial t} + \mathcal{A}\boldsymbol{u} = \boldsymbol{F}\boldsymbol{u} \qquad u(0) = \boldsymbol{a}, \qquad \text{dans} \quad \Omega \times (0, T], \tag{1.0.26}$$

où Ω est un domaine quelconque de $\mathbb{R}^n$, $\boldsymbol{F}\boldsymbol{u}$ représente le terme non-linéaire et $\mathcal{A}$ est un opérateur elliptique tel que $-\mathcal{A}$ engendre un C_0 semi-groupe $e^{-t\mathcal{A}}$ sur un sous-espace fermé $\boldsymbol{E}^p$ de $\boldsymbol{L}^p(\Omega)$. Il construit une unique *solution mild* au problème (1.0.26) lorsque le semi-groupe $e^{-t\mathcal{A}}$ satisfait les estimations $L^p - L^q$

$$\|e^{-t\mathcal{A}}\boldsymbol{f}\|_{\boldsymbol{L}^q(\Omega)} \leq \, M\, t^{-\frac{n}{m}(\frac{1}{p}-\frac{1}{q})}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \qquad \boldsymbol{f} \in \boldsymbol{E}^p, \quad 0 < t < T, \tag{1.0.27}$$

et sous certaines hypothèses sur le terme non-linéaire $\boldsymbol{F}\boldsymbol{u}$. Giga suppose que le terme non-linéaire peut s'écrire sous la forme:

$$\boldsymbol{F}\boldsymbol{u} = \boldsymbol{L}\,\boldsymbol{G}\boldsymbol{u},$$

où $\boldsymbol{\Gamma}$ est un opérateur linéaire fermé à domaine dense de $\boldsymbol{L}^p(\Omega)$ dans $\boldsymbol{E}^q$ pour un certain $q > 1$ et $\boldsymbol{G}$ est un opérateur non-linéaire de $\boldsymbol{E}^p$ dans $\boldsymbol{L}^h(\Omega)$, $h > 1$. Il suppose de plus qu'il existe $0 \leq \gamma < m$, tel que

$$\|e^{-t\mathcal{A}}\boldsymbol{\Gamma}\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \leq N_1 t^{-\gamma/m}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \qquad \boldsymbol{f} \in \boldsymbol{E}^p, \quad 0 < t < T, \tag{1.0.28}$$

où $N_1 = N_1(p, T)$ est une constante qui dépend de p et T. Concernant la fonction $\boldsymbol{G}$, Giga suppose qu'il existe une constante $\alpha > 0$ telle que:

$$\|\boldsymbol{G}\boldsymbol{v} - \boldsymbol{G}\boldsymbol{w}\|_{\boldsymbol{L}^h(\Omega)} \leq N_2\|\boldsymbol{v} - \boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)}(\|\boldsymbol{v}\|^\alpha_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{w}\|^\alpha_{\boldsymbol{L}^p(\Omega)}), \qquad \boldsymbol{G}\boldsymbol{0} = \boldsymbol{0}, \tag{1.0.29}$$

avec $1 \leq h = p/(1+\alpha)$ et la constante $N_2 = N_2(p)$ dépend seulement de p, $1 < p < \infty$.

En supposant (1.0.27),(1.0.28) et (1.0.29), Giga démontre l'existence d'une unique solution $\boldsymbol{u}$ qui vérifie

$$\boldsymbol{u} \in BC([0, T_0);\, \boldsymbol{E}^p) \cap L^q(0, T_0;\, \boldsymbol{E}^r),$$

avec $\frac{1}{q} = \frac{n}{m}(\frac{1}{p} - \frac{1}{r})$ et $\boldsymbol{a} \in \boldsymbol{E}^p$. Il montre aussi que son résultat s'applique au problème de Navier-Stokes ainsi qu'au problème de la *chaleur semi-linéaire.*

Après avoir étabi notre résultat d'existence et d'unicité de *solution mild* pour le problème de Navier-Stokes avec les conditions de type-Navier, on démontre que cette solution est régulière. En effet, maintenant on sait que la solution existe et peut s'écrire sous la forme:

$$\boldsymbol{u}(t) = \boldsymbol{u}_0(t) + \boldsymbol{S}\boldsymbol{u}(t)$$

où

$$\boldsymbol{u}_0(t) = e^{-tA_p}\boldsymbol{u}_0 \qquad \text{et} \qquad \boldsymbol{S}\boldsymbol{u}(t) = \int_0^t e^{-(t-s)A_p}\boldsymbol{F}\boldsymbol{u}(s)\,\mathrm{d}\,s$$

et $\boldsymbol{Fu} = -P(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$. En procédant comme dans [41] on démontre que $u \in C((0, T_*]; \mathbf{D}(A_p))$. Tout d'abord, on démontre que pour tout $0 < t \leq T_*$, $u(t) \in \mathbf{D}(A_p^{\alpha})$, pour tout $0 < \alpha < 1-\delta$, pour un certain $0 \leq \delta < 1$ et satisfait:

$$\|(I + A_p)^{\alpha}\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq K_{\alpha} t^{-\alpha},$$

pour une certaine constante $K_{\alpha} > 0$. Pour établir ce résultat, on estime le terme non-linéaire $\boldsymbol{Fu}$ en fonction des puissances fractionnaires de l'opérateur $I + A_p$. En procédant comme dans [41, Lemme 2.2], on montre que pour un certain $0 \leq \delta \leq \frac{1}{2} + \frac{3}{2}(1 - \frac{1}{p})$ fixé et pour $1 < p < \infty$ on a:

$$\forall\ \boldsymbol{u},\ \boldsymbol{v} \in \boldsymbol{\mathcal{D}}_{\sigma}(\Omega), \qquad \|(I + A_p)^{-\delta} P(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \ \leq\ M\, \|(I + A_p)^{\theta}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \|(I + A_p)^{\rho}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}, \tag{1.0.30}$$

où la constant $M = M(\delta, \theta, \rho, p)$, pourvu que

$$\delta + \theta + \rho \geq 1/2 + 3/2p, \qquad \theta > 0, \quad \rho > 0, \qquad \rho + \delta > 1/2.$$

Ensuite, on démontre que pour tout $0 < \alpha < 1 - \delta$, $(I + A_p)^{\alpha}\boldsymbol{u}$ est continue hölderienne sur tout intervalle $[\varepsilon, T_*]$, $0 < \varepsilon < T_*$. Par conséquent, en utilisant l'estimation (1.0.30) pour $\delta = 0$, on déduit que $\boldsymbol{Fu}$ est continue hölderienne sur tout intervalle $[\varepsilon, T_*]$, $0 < \varepsilon < T_*$. En utilisant le résultat de Pazy [64, Chapitre 4, Corollaire 3.3], on déduit que $\boldsymbol{Su}(t) \in \mathbf{D}(A_p)$ pour tout $0 < t \leq T_*$. Comme $\boldsymbol{u}_0(t) = e^{-tA_p}\boldsymbol{u}_0 \in \mathbf{D}(A_p)$, la solution $\boldsymbol{u}(t) \in \mathbf{D}(A_p)$ pour tout $0 < t \leq T_*$.

Chapter 2

Basic properties of the functional framework

2.1 Functional framework

In this section we review some basic notations, definitions and functional framework which are essential in our work.

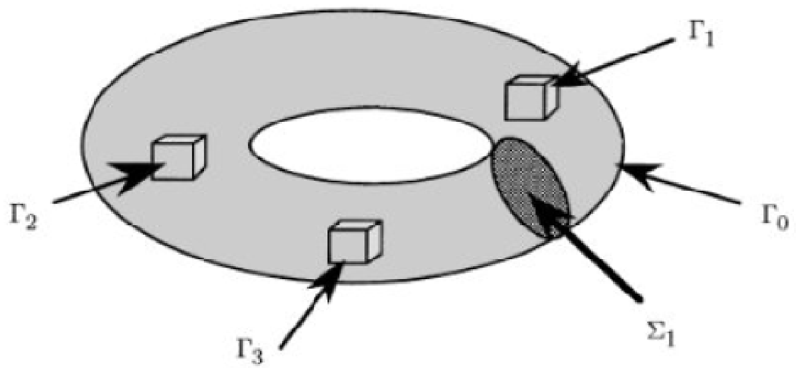
In what follows, if we do not state otherwise, Ω will be considered as an open bounded connected domain of $\mathbb{R}^3$ of class at least $C^{1,1}$ and sometimes of class $C^{2,1}$. Then a unit exterior normal vector to the boundary can be defined almost everywhere it will be denoted by $\boldsymbol{n}$. The generic point in Ω is denoted by $\boldsymbol{x} = (x_1, x_2, x_3)$.

We do not assume that the boundary Γ is connected and in the case where Γ is not connected we denote by Γ_i, $0 \leq i \leq I$, the connected component of Γ, Γ_0 being the boundary of the only unbounded connected component of $\mathbb{R}^3 \backslash \overline{\Omega}$. We also fix a smooth open set ϑ with a connected boundary (a ball, for instance), such that $\overline{\Omega}$ is contained in ϑ, and we denote by Ω_i, $0 \leq i \leq I$, the connected component of $\vartheta \backslash \overline{\Omega}$ with boundary Γ_i ($\Gamma_0 \cup \partial\vartheta$ for $i = 0$).

We do not assume that Ω is simply-connected and in the case where Ω is not simply connected we suppose that there exist J connected open surfaces Σ_j, $1 \leq j \leq J$, called "cuts", contained in Ω, such that each surface Σ_j is an open subset of a smooth manifold, the boundary of Σ_j is contained in Γ. The intersection $\overline{\Sigma}_i \cap \overline{\Sigma}_j$ is empty for $i \neq j$ and finally the open set $\Omega^\circ = \Omega \backslash \cup_{j=1}^{J} \Sigma_j$ is simply connected and pseudo-$C^{1,1}$ (see [5] for instance).

We denote by $[\cdot]_j$ the jump of a function over Σ_j, *i.e.* the difference of the traces for $1 \leq j \leq J$. In addition, for any function q in $W^{1,p}(\Omega^\circ)$, $\mathbf{grad}\, q$ is the gradient of q in the sense of distribution in $\boldsymbol{\mathcal{D}}'(\Omega^\circ)$, it belongs to $\boldsymbol{L}^p(\Omega^\circ)$ and therefore can be extended to $\boldsymbol{L}^p(\Omega)$. In order to distinguish this extension from the gradient of q in $\boldsymbol{\mathcal{D}}'(\Omega^\circ)$ we denote it by $\widetilde{\mathbf{grad}}\, q$.

Finally, vector fields, matrix fields and their corresponding spaces defined on Ω will be



denoted by bold character. The functions treated here are complex valued functions. We will use also the symbol σ to represent a set of divergence free functions. In other words If $\boldsymbol{E}$ is Banach space, then

$$\boldsymbol{E}_\sigma = \{\boldsymbol{v} \in \boldsymbol{E};\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\}.$$

Now, we introduce some functional spaces. Let $\boldsymbol{L}^p(\Omega)$ denotes the usual vector valued $\boldsymbol{L}^p$-space over Ω. Let us define the spaces:

$$\boldsymbol{H}^p(\mathbf{curl}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\},$$

$$\boldsymbol{H}^p(\mathrm{div}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \operatorname{div} \boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\},$$

$$\boldsymbol{X}^p(\Omega) = \boldsymbol{H}^p(\mathbf{curl}, \Omega) \cap \boldsymbol{H}^p(\mathrm{div}, \Omega),$$

equipped with the graph norm. Thanks to [11] we know that $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$, $\boldsymbol{H}^p(\mathrm{div}, \Omega)$ and $\boldsymbol{X}^p(\Omega)$.
We also define the subspaces:

$$\boldsymbol{H}^p_0(\mathbf{curl}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^p(\mathbf{curl}, \Omega);\ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\},$$

$$\boldsymbol{H}^p_0(\mathrm{div}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^p(\mathrm{div}, \Omega);\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\},$$

$$\boldsymbol{X}^p_N(\Omega) = \{\boldsymbol{v} \in \boldsymbol{X}^p(\Omega);\ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\},$$

$$\boldsymbol{X}^p_\tau(\Omega) = \{\boldsymbol{v} \in \boldsymbol{X}^p(\Omega);\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}$$

and

$$\boldsymbol{X}^p_0(\Omega) = \boldsymbol{X}^p_N(\Omega) \cap \boldsymbol{X}^p_\tau(\Omega).$$

We have denoted by $\boldsymbol{v} \times \boldsymbol{n}$ (respectively by $\boldsymbol{v} \cdot \boldsymbol{n}$) the tangential (respectively normal) boundary value of $\boldsymbol{v}$ defined in $\boldsymbol{W}^{-1/p,p}(\Gamma)$ (respectively in $W^{-1/p,p}(\Gamma)$) as soon as $\boldsymbol{v}$ belongs to $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$ (respectively to $\boldsymbol{H}^p(\mathrm{div}, \Omega)$). More precisely, any function $\boldsymbol{v}$ in $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$ (respectively in $\boldsymbol{H}^p(\mathrm{div}, \Omega)$) has a tangential (respectively normal) trace $\boldsymbol{v} \times \boldsymbol{n}$ (respectively $\boldsymbol{v} \cdot \boldsymbol{n}$) in $\boldsymbol{W}^{-1/p,p}(\Gamma)$ (respectively in $W^{-1/p,p}(\Gamma)$) defined by:

$$\forall\, \boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega),\ \ \langle \boldsymbol{v} \times \boldsymbol{n},\, \boldsymbol{\varphi} \rangle_\Gamma = \int_\Omega \mathbf{curl}\, \boldsymbol{v} \cdot \overline{\boldsymbol{\varphi}}\, \mathrm{d}\, x - \int_\Omega \boldsymbol{v} \cdot \mathbf{curl}\, \overline{\boldsymbol{\varphi}}\, \mathrm{d}\, x \tag{2.1.1}$$

and

$$\forall\, \varphi \in W^{1,p'}(\Omega),\ \ \langle \boldsymbol{v}\cdot\boldsymbol{n},\, \varphi\rangle_\Gamma \;=\; \int_\Omega \boldsymbol{v}\cdot\mathbf{grad}\,\overline{\varphi}\,\mathrm{d}\,x \;+\; \int_\Omega \mathrm{div}\,\boldsymbol{v}\,\overline{\varphi}\,\mathrm{d}\,x, \tag{2.1.2}$$

where $\langle .,.\rangle_\Gamma$ is the anti-duality between $\boldsymbol{W}^{-1/p,\,p}(\Gamma)$ and $\boldsymbol{W}^{1/p,\,p'}(\Gamma)$ in (2.1.1) and between $W^{-1/p,\,p}(\Gamma)$ and $W^{1/p,\,p'}(\Gamma)$ in (2.1.2). Thanks to [11] we know that $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{H}^p_0(\mathbf{curl},\Omega)$ and in $\boldsymbol{H}^p_0(\mathrm{div},\Omega)$.

We denote by $[\boldsymbol{H}^p_0(\mathbf{curl},\Omega)]'$ and $[\boldsymbol{H}^p_0(\mathrm{div},\Omega)]'$ the dual spaces of $\boldsymbol{H}^p_0(\mathbf{curl},\Omega)$ and $\boldsymbol{H}^p_0(\mathrm{div},\Omega)$ respectively.

Notice that we can characterize these dual spaces as follows: A distribution $\boldsymbol{f}$ belongs to $[\boldsymbol{H}^p_0(\mathbf{curl},\Omega)]'$ if and only if there exist functions functions $\boldsymbol{\psi}\in\boldsymbol{L}^{p'}(\Omega)$ and $\boldsymbol{\xi}\in\boldsymbol{L}^{p'}(\Omega)$, such that $\boldsymbol{f} \,=\, \boldsymbol{\psi} \,+\, \mathbf{curl}\,\boldsymbol{\xi}$. Moreover one has

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}^p_0(\mathbf{curl},\Omega)]'} \;=\; \inf_{\boldsymbol{f}\,=\,\boldsymbol{\psi}\,+\,\mathbf{curl}\,\boldsymbol{\xi}} \max\,(\|\boldsymbol{\psi}\|_{\boldsymbol{L}^{p'}(\Omega)},\, \|\boldsymbol{\xi}\|_{\boldsymbol{L}^{p'}(\Omega)}).$$

Similarly, a distribution $\boldsymbol{f}$ belongs to $[\boldsymbol{H}^p_0(\mathrm{div},\Omega)]'$ if and only if there exist $\boldsymbol{\psi}\in\boldsymbol{L}^{p'}(\Omega)$ and $\chi\in L^{p'}(\Omega)$ such that $\boldsymbol{f} \,=\, \boldsymbol{\psi} \,+\, \mathbf{grad}\,\chi$ and

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}^p_0(\mathrm{div},\Omega)]'} \;=\; \inf_{\boldsymbol{f}\,=\,\boldsymbol{\psi}\,+\,\mathbf{grad}\,\chi} \max\,(\|\boldsymbol{\psi}\|_{\boldsymbol{L}^{p'}(\Omega)}\,,\, \|\chi\|_{\boldsymbol{L}^{p'}(\Omega)}).$$

Finally we consider the space

$$\boldsymbol{T}^p(\Omega) = \big\{\boldsymbol{v}\in\boldsymbol{H}^p_0(\mathrm{div},\Omega);\ \ \mathrm{div}\,\boldsymbol{v}\in W^{1,p}_0(\Omega)\big\}, \tag{2.1.3}$$

equipped with the graph norm. Thanks to [10, Lemma 4.11, Lemma 4.12] we know that $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{T}^p(\Omega)$ and a distribution $\boldsymbol{f}\in(\boldsymbol{T}^p(\Omega))'$ if and only if there exists a function $\boldsymbol{\psi}\in\boldsymbol{L}^{p'}(\Omega)$ and a function $\chi\in W^{-1,p'}(\Omega)$ such that $\boldsymbol{f} \,=\, \boldsymbol{\psi} \,+\, \nabla\chi$. Moreover we have the estimate

$$\|\boldsymbol{\psi}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|\chi\|_{W^{-1,p'}}(\Omega) \;\leq\; \|\boldsymbol{f}\|_{(\boldsymbol{T}^p(\Omega))'}.$$

2.2 Preliminary results

In this section, we review some known results which are essential in our work. First, We recall that the vector-valued Laplace operator of a vector field $\boldsymbol{v} = (v_1, v_2, v_3)$ is equivalently defined by

$$\Delta\,\boldsymbol{v} = \mathbf{grad}\,(\mathrm{div}\,\boldsymbol{v}) - \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v}$$

or by

$$\Delta\,\boldsymbol{v} = 2\mathrm{div}\,\mathbf{D}(\boldsymbol{v}) - \mathbf{grad}\,(\mathrm{div}\,\boldsymbol{v}).$$

Next, we review some Sobolev embeddings (see [11]):

Lemma 2.2.1. *The spaces $\boldsymbol{X}^p_N(\Omega)$ and $\boldsymbol{X}^p_\tau(\Omega)$ defined above are continuously embedded in $\boldsymbol{W}^{1,p}(\Omega)$.*

Consider now the spaces

$$\boldsymbol{X}^{2,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \operatorname{div} \boldsymbol{v} \in W^{1,p}(\Omega),\ \mathbf{curl}\, \boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega) \text{ and } \boldsymbol{v}\cdot\boldsymbol{n} \in W^{1-1/p,p}(\Gamma)\} \tag{2.2.1}$$

and

$$\boldsymbol{Y}^{2,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \operatorname{div} \boldsymbol{v} \in W^{1,p}(\Omega),\ \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega) \text{ and } \boldsymbol{v}\times\boldsymbol{n} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)\}.$$

Lemma 2.2.2. *Assume that Ω is of class $C^{2,1}$, then the spaces $\boldsymbol{X}^{2,p}(\Omega)$ and $\boldsymbol{Y}^{2,p}(\Omega)$ are continuously embedded in $\boldsymbol{W}^{2,p}(\Omega)$.*

Consider now the space

$$\boldsymbol{E}^{\,p}(\Omega) = \{\boldsymbol{v} \in\ \boldsymbol{W}^{\,1,p}(\Omega);\ \Delta\boldsymbol{v} \in [\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'\},$$

which is a Banach space for the norm:

$$\|\ \boldsymbol{v}\ \|_{\boldsymbol{E}^{\,p}(\Omega)} = \|\ \boldsymbol{v}\ \|_{\boldsymbol{W}^{\,1,p}(\Omega)} + \|\ \Delta\boldsymbol{v}\ \|_{[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'}\ .$$

Thanks to [10, Lemma 4.1] we know that $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{E}^{\,p}(\Omega)$. Moreover we have the following Lemma (see [10, Corollary 4.2]):

Lemma 2.2.3. *The linear mapping $\gamma : \boldsymbol{v} \longrightarrow \mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n}$ defined on $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended to a linear and continuous mapping*

$$\gamma : \boldsymbol{E}^p(\Omega) \longrightarrow \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma).$$

Moreover, we have the Green formula: for any $\boldsymbol{v} \in \boldsymbol{E}^p(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{X}_{\tau}^{p'}(\Omega)$ such that $\operatorname{div}\boldsymbol{\varphi} = 0$ *in Ω.*

$$-\langle\Delta\boldsymbol{v},\boldsymbol{\varphi}\rangle_{\Omega} = \int_{\Omega} \mathbf{curl}\,\boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} - \langle\mathbf{curl}\boldsymbol{v}\times\boldsymbol{n},\boldsymbol{\varphi}\rangle_{\Gamma}.$$

where $\langle .,.\rangle_{\Gamma}$ denotes the anti-duality between $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)$ and $\langle .,.\rangle_{\Omega}$ denotes the anti-duality between $[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'$ and $\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)$.

In addition we have the following lemma (see [8, Lemma 2.4]).

Lemma 2.2.4. *Suppose that Ω is of class $C^{1,1}$. The linear mapping $\Theta : \boldsymbol{v} \longrightarrow [\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}|\Gamma}$ defined on $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping*

$$\Theta : \boldsymbol{E}^p(\Omega) \longrightarrow \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma).$$

Moreover, we have the Green formula: for any $\boldsymbol{v} \in \boldsymbol{E}^p(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{W}^{1\,p'}(\Omega)$ such that $\operatorname{div}\boldsymbol{\varphi} = 0$ *in Ω and $\boldsymbol{\varphi}\cdot\boldsymbol{n} = 0$ on Γ,*

$$-\langle\Delta\boldsymbol{v},\boldsymbol{\varphi}\rangle_{\Omega} = 2\int_{\Omega} \mathbf{D}(\boldsymbol{v}) : \mathbf{D}(\boldsymbol{\varphi})\,d\boldsymbol{x} - 2\langle[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}},\boldsymbol{\varphi}\rangle_{\Gamma}, \tag{2.2.2}$$

where $\langle .,.\rangle_{\Gamma}$ denotes the anti-duality between $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)$ and $\langle .,.\rangle_{\Omega}$ denotes the anti-duality between $[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'$ and $\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)$.

Next we consider the space

$$\boldsymbol{H}^p(\Delta,\Omega) = \big\{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\;\; \Delta\boldsymbol{v} \in (\boldsymbol{T}^{p'}(\Omega))'\big\},$$

which is a Banach space for the graph norm. Thanks to [10, Lemma 4.13, Lemma 4.14] we know that

Proposition 2.2.5. *The space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{H}^p(\Delta,\Omega)$. Moreover for every $\boldsymbol{v}$ in $\boldsymbol{H}^p(\Delta,\Omega)$ the trace $\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n}$ exists and belongs to $\boldsymbol{W}^{-1-1/p,p}(\Gamma)$. In addition we have the Green formula: for all $\boldsymbol{v}\in\boldsymbol{H}^p(\Delta,\Omega)$ and for all $\boldsymbol{\varphi}\in\boldsymbol{W}^{2,p}(\Omega)$ such that $\operatorname{div}\boldsymbol{\varphi} = \boldsymbol{\varphi}\cdot\boldsymbol{n} = 0$ on Γ and $\mathbf{curl}\,\boldsymbol{\varphi}\times\boldsymbol{n} = \mathbf{0}$ on Γ:*

$$\langle \Delta\boldsymbol{v}\,,\,\boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)} = \int_\Omega \boldsymbol{v}\cdot\Delta\overline{\boldsymbol{\varphi}}\,dx \;+\; \langle \mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n}\,,\,\boldsymbol{\varphi}\rangle_\Gamma, \tag{2.2.3}$$

where $\langle .\,,\,.\rangle_\Gamma = \langle .\,,\,.\rangle_{\boldsymbol{W}^{-1-1/p,p}(\Gamma)\times\boldsymbol{W}^{1+1/p,p'}(\Gamma)}$.

Moreover, thanks to [8, Lemma 5.4] we have the following proposition

Proposition 2.2.6. *For every $\boldsymbol{v}$ in $\boldsymbol{H}^p(\Delta,\Omega)$ the trace $[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}$ exists and belongs to $\boldsymbol{W}^{-1-1/p,p}(\Gamma)$. Moreover we have the Green formula: for any $\boldsymbol{u}\in\boldsymbol{H}_p(\Delta;\Omega)$ and for all $\boldsymbol{\varphi}\in\;\boldsymbol{W}^{2,p'}(\Omega)$; $\boldsymbol{\varphi}\cdot\boldsymbol{n} =$ 0, $\operatorname{div}\boldsymbol{\varphi} = 0$, $[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0}$ on Γ,*

$$\langle \Delta\boldsymbol{u},\boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)} = \int_\Omega \boldsymbol{u}\cdot\Delta\overline{\boldsymbol{\varphi}}d\boldsymbol{x} + \langle 2[\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}},\boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)\times\boldsymbol{W}^{1+\frac{1}{p},p'}(\Gamma)}. \tag{2.2.4}$$

We recall also the following De Rham result:

Lemma 2.2.7. *Let $\boldsymbol{F}\in\boldsymbol{W}^{-2,p}(\Omega)$ verifying*

$$\forall\,\boldsymbol{v}\in\boldsymbol{\mathcal{D}}_\sigma(\Omega), \qquad \langle \boldsymbol{F}\,,\,\boldsymbol{v}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega)\times\boldsymbol{\mathcal{D}}(\Omega)} = 0.$$

Then there exists $\chi\in W^{-1,p}(\Omega)$ such that $\boldsymbol{F} = \nabla\,\chi$.

Next, we recall also the following lemma (see [11, Lemma 3.2]) that gives the normal trace of a function $\boldsymbol{\psi}\in\boldsymbol{H}^p_0(\operatorname{div},\Omega)$ on Σ_j, $1\le j\le J$.

Lemma 2.2.8. *Let $\boldsymbol{\psi}\in\boldsymbol{H}^p_0(\operatorname{div},\Omega)$, the restriction of $\boldsymbol{\psi}\cdot\boldsymbol{n}$ to any Σ_j, $1\le j\le J$ belongs to the dual space $\big(\boldsymbol{W}^{1-1/p',p'}(\Sigma_j)\big)'$ and the following Green's formula holds:*

$$\forall\,\chi\in W^{1,p'}(\Omega^\circ),\;\; \sum_{j=1}^{J}\langle \boldsymbol{\psi}\cdot\boldsymbol{n}\,,\,[\chi]_j\rangle_{\Sigma_j} \;=\; \int_{\Omega^\circ}\boldsymbol{\psi}\cdot\mathbf{grad}\,\overline{\chi}\,dx \;+\; \int_{\Omega^\circ}\chi\operatorname{div}\overline{\boldsymbol{\psi}}\,dx \tag{2.2.5}$$

Moreover we have the estimate

$$\|\boldsymbol{\psi}\cdot\boldsymbol{n}\|_{\left(\boldsymbol{W}^{1-1/p',p'}(\Sigma_j)\right)'} \;\le\; C\,\|\boldsymbol{\psi}\|_{\boldsymbol{H}^p(\operatorname{div},\Omega)}. \tag{2.2.6}$$

Next we consider the problem:

$$\operatorname{div}(\mathbf{grad}\,\pi - \boldsymbol{f}) = 0 \qquad \text{in } \Omega, \qquad (\mathbf{grad}\,\pi - \boldsymbol{f})\cdot\boldsymbol{n} = 0 \qquad \text{on } \Gamma. \tag{2.2.7}$$

We recall the following lemma concerning the weak Neumann problem (see [71] for the first point and [6] for the two other points)

Lemma 2.2.9. ***(i)*** *Let $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ the Problem (2.2.7) has a unique solution $\pi \in W^{1,p}(\Omega)/\mathbb{R}$ satisfying the estimate*

$$\|\mathbf{grad}\,\pi\|_{\boldsymbol{L}^p(\Omega)} \;\leq\; C_1(\Omega)\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)},$$

for some constant $C_1(\Omega) > 0$.

(ii) *Let $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'$, the Problem (2.2.7) has a unique solution $\pi \in L^p(\Omega)/\mathbb{R}$ satisfying the estimate*

$$\|\pi\|_{L^p(\Omega)/\mathbb{R}} \;\leq\; C_2(\Omega,p)\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'}.$$

(iii) *Let $\boldsymbol{f} \in (\boldsymbol{T}^{p'}(\Omega))'$, where $\boldsymbol{T}^p(\Omega)$ is given by (2.1.3). The Problem (2.2.7) has a unique solution $\pi \in W^{-1,p}(\Omega)/\mathbb{R}$ satisfying the estimate*

$$\|\pi\|_{W^{-1,p}(\Omega)} \;\leq\; C(\Omega,p)\,\|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'}.$$

The following lemma plays an important role in the proof of the resolvent estimate:

Lemma 2.2.10. *Let $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ such that $\Delta\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$. Then for all $p \geq 2$ one has:*

$$\begin{aligned} -\int_\Omega |\boldsymbol{u}|^{p-2}\Delta\boldsymbol{u}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\,x \;=\; &\int_\Omega |\boldsymbol{u}|^{p-2}|\nabla\boldsymbol{u}|^2\,\mathrm{d}\,x \;+\; 4\,\frac{p-2}{p^2}\int_\Omega \Big|\nabla|\boldsymbol{u}|^{p/2}\Big|^2\,\mathrm{d}\,x \\ &+\;(p-2)\,i\sum_{k=1}^{3}\int_\Omega |\boldsymbol{u}|^{p-4}\operatorname{Re}\Big(\frac{\partial\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\operatorname{Im}\Big(\frac{\partial\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\,\mathrm{d}\,x \;-\;\Big\langle\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}},\,|\boldsymbol{u}|^{p-2}\boldsymbol{u}\Big\rangle_\Gamma, \end{aligned} \tag{2.2.8}$$

where $\langle .\,,.\rangle_\Gamma$ is the anti-duality between $\boldsymbol{W}^{-1/p,p}(\Gamma)$ and $\boldsymbol{W}^{1/p,p'}(\Gamma)$.

Proof. Let $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ such that $\Delta\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$. We recall that $\boldsymbol{u} = (\ u_1,\ \ u_2,\ \ u_3\ )$ is a vector complex valued function. We recall also that the vectors $\overline{\boldsymbol{u}}$ and $\operatorname{Re}\boldsymbol{u}$ given by

$$\overline{\boldsymbol{u}} = (\ \overline{u_1},\ \ \overline{u_2},\ \ \overline{u_3}\ ), \qquad \operatorname{Re}\boldsymbol{u} = (\ \operatorname{Re}u_1,\ \ \operatorname{Re}u_2,\ \ \operatorname{Re}u_3\ )$$

are the conjugate and the real part of the vector $\boldsymbol{u}$ respectively. We can easily verify that for any $1 \leq k \leq 3$ one has

$$\frac{\partial\,|\boldsymbol{u}|^2}{\partial x_k} = \sum_{j=1}^{3}\Big[\frac{\partial\,u_j}{\partial x_k}\overline{u_j} + u_j\frac{\partial\,\overline{u_j}}{\partial x_k}\Big] = 2\operatorname{Re}\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big).$$

As a result

$$\frac{\partial\,|\boldsymbol{u}|^{p-2}}{\partial x_k} = (p-2)\,|\boldsymbol{u}|^{p-4}\operatorname{Re}\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big) \;\text{ and }\; \Big|\frac{\partial\,|\boldsymbol{u}|^{p/2}}{\partial x_k}\Big|^2 \;=\; \frac{p^2}{4}\,|\boldsymbol{u}|^{p-4}\Big[\operatorname{Re}\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\Big]^2. \tag{2.2.9}$$

Now, using (2.2.9) we have

$$\sum_{k=1}^{3} \frac{\partial\,|\boldsymbol{u}|^{p-2}}{\partial\, x_k} \frac{\partial\, \boldsymbol{u}}{\partial\, x_k} \cdot \overline{\boldsymbol{u}}\, \mathrm{d}\, x \;=\; 4\, \frac{p-2}{p^2} \int_{\Omega} \left| \nabla |\boldsymbol{u}|^{p/2} \right|^2 \mathrm{d}\, x \;+$$
$$(p-2)\, i \sum_{k=1}^{3} \int_{\Omega} |\boldsymbol{u}|^{p-4}\, \mathrm{Re}\left(\frac{\partial\, \boldsymbol{u}}{\partial x_k} \cdot \overline{\boldsymbol{u}} \right) \mathrm{Im}\left(\frac{\partial\, \boldsymbol{u}}{\partial x_k} \cdot \overline{\boldsymbol{u}} \right) \mathrm{d}\, x.$$

Finally applying the Green-Formula one gets (2.2.8). □

To give explicitly the boundary condition on Γ, we recall some properties concerning some traces of vector fields. Let us consider any point P on Γ and choose an open neighbourhood W of P in Γ small enough to allow the existence of 2 families of C^2 curves on W with these properties: a curve of each family passes through every point of W and the unit tangent vectors to these curves form an orthonormal system (which we assume to have the direct orientation) at every point of W. The lengths s_1, s_2 along each family of curves, respectively, are a possible system of coordinates in W. We denote by $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$ the unit tangent vectors to each family of curves, respectively. With this notation, we have $\boldsymbol{v} = \sum_{k=1}^{2} v_k \boldsymbol{\tau}_k + (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}$ where $\boldsymbol{\tau}_k^T = (\tau_{k1}, \tau_{k2}, \tau_{k3})$ and $v_k = \boldsymbol{v}\cdot\boldsymbol{\tau}_k$. As a result for any $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$ the following formulas holds (see [8]):

$$[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = \nabla_{\boldsymbol{\tau}}(\boldsymbol{v}\cdot\boldsymbol{n}) + \big(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}\big)_{\boldsymbol{\tau}} - \boldsymbol{\Lambda}\boldsymbol{v} \qquad \text{on} \quad \Gamma \tag{2.2.10}$$

and

$$\mathbf{curl}\,\boldsymbol{v} \times \boldsymbol{n} = \nabla_{\boldsymbol{\tau}}(\boldsymbol{v}\cdot\boldsymbol{n}) - \big(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}\big)_{\boldsymbol{\tau}} - \boldsymbol{\Lambda}\boldsymbol{v} \qquad \text{on} \quad \Gamma, \tag{2.2.11}$$

where

$$\boldsymbol{\Lambda}\boldsymbol{w} = \sum_{k=1}^{2} \big(\boldsymbol{w}_{\boldsymbol{\tau}} \cdot \frac{\partial \boldsymbol{n}}{\partial s_k}\big)\boldsymbol{\tau}_k. \tag{2.2.12}$$

In particular, observe that, if $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ and $[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0}$ we have $\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n} = -2\boldsymbol{\Lambda}\boldsymbol{v}$. As consequence

$$\Delta\boldsymbol{v}\cdot\boldsymbol{n} = \mathrm{div}_{\Gamma}\boldsymbol{\Lambda}\boldsymbol{v} \quad \text{on} \quad \Gamma, \tag{2.2.13}$$

where div_{Γ} denote the surface divergence on Γ.

We have also the following result:

Lemma 2.2.11. *Assume that Ω is of class $C^{1,1}$. Let $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ such that $\Delta\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$. Then $[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}$ and $\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n}$ belongs to $\boldsymbol{W}^{-1/p,p}(\Gamma)$ and satisfy respectively formulas* (2.2.10) *and* (2.2.11).

Proof. Let us prove formula (2.2.11) for a function $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ such that $\Delta\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$. We can easily verify that $\mathbf{grad}\,\boldsymbol{v} \in \boldsymbol{H}^p(\mathrm{div},\Omega)$ and then $\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}$ exists and belongs to $\boldsymbol{W}^{-1/p,p}(\Gamma)$. On the other hand, we consider the space

$$\boldsymbol{E}^p(\Omega) \;=\; \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\, \Delta\,\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\},$$

which is a Banach space for the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{E}^p(\Omega)} \,=\, \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \,+\, \|\Delta\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}.$$

using the density of $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ in the space $\boldsymbol{E}^p(\Omega)$, there exists a sequence $(\boldsymbol{v}_k)_k$ in $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ such that $(\boldsymbol{v}_k)_k$ converges to $\boldsymbol{v}$ in $\boldsymbol{E}^p(\Omega)$. As a result the sequence $(\boldsymbol{v}_k)_k$, $(\Delta \boldsymbol{v}_k)_k$ and $(\mathbf{grad}\, \boldsymbol{v}_k)_k$ converge respectively to $\boldsymbol{v}$, $\Delta \boldsymbol{v}$ and $\mathbf{grad}\, \boldsymbol{v}$ in the respective spaces $\boldsymbol{W}^{1,p}(\Omega)$, $\boldsymbol{L}^p(\Omega)$ and $\boldsymbol{H}^p(\mathrm{div}, \Omega)$. As a consequence the sequence $(\boldsymbol{v}_k \cdot \boldsymbol{n})_k$ converges to $\boldsymbol{v}\cdot\boldsymbol{n}$ in $W^{1-1/p,p}(\Gamma)$, then $(\nabla_{\boldsymbol{\tau}}(\boldsymbol{v}_k \cdot \boldsymbol{n}))_k$ converges to $\nabla_{\boldsymbol{\tau}}(\boldsymbol{v}\cdot\boldsymbol{n})$ in $\boldsymbol{W}^{-1/p,p}(\Gamma)$. Moreover the sequence $(\boldsymbol{v}_{k_{\boldsymbol{\tau}}})_k$ converges to $\boldsymbol{v}_{\boldsymbol{\tau}}$ in $\boldsymbol{W}^{1-1/p,p}(\Gamma)$. Now since Ω is of class $C^{1,1}$ then $\boldsymbol{n} \in \boldsymbol{W}^{1,\infty}(\Gamma)$, this means that $\frac{\partial \boldsymbol{n}}{\partial s_j} \in \boldsymbol{L}^\infty(\Gamma)$ and the sequence $(\boldsymbol{v}_{k_{\boldsymbol{\tau}}} \cdot \frac{\partial \boldsymbol{n}}{\partial s_j})_k$ converges to $\boldsymbol{v}_{\boldsymbol{\tau}} \cdot \frac{\partial \boldsymbol{n}}{\partial s_j}$ in $W^{1-1/p,p}(\Gamma)$ and then in $W^{-1/p,p}(\Gamma)$. In addition the sequences $(\frac{\partial \boldsymbol{v}_k}{\partial \boldsymbol{n}})_k$ and $(\mathbf{curl}\, \boldsymbol{v}_k \times \boldsymbol{n})_k$ converge respectively to $\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}$ and $\mathbf{curl}\, \boldsymbol{v}\times\boldsymbol{n}$ in $\boldsymbol{W}^{-1/p,p}(\Gamma)$. Moreover, one has for all $k \in \mathbb{N}$

$$\mathbf{curl}\, \boldsymbol{v}_k \times \boldsymbol{n} \,=\, \nabla_{\boldsymbol{\tau}}(\boldsymbol{v}_k\cdot\boldsymbol{n}) \,-\, \Big(\frac{\partial\, \boldsymbol{v}_k}{\partial\, \boldsymbol{n}}\Big)_{\boldsymbol{\tau}} \,-\, \sum_{j=1}^{2}\Big(\frac{\partial\, \boldsymbol{n}}{\partial\, s_j}\cdot \boldsymbol{v}_{k_{\boldsymbol{\tau}}}\Big)\boldsymbol{\tau}_j \qquad \text{on } \Gamma. \tag{2.2.14}$$

Finally passing to the limit as $k\to+\infty$ in (2.2.14) one gets formula (2.2.11) holds for any vector $\boldsymbol{v}\in\boldsymbol{E}^p(\Omega)$. Which ends the proof. □

Moreover, we know that for any vector $\boldsymbol{v}$ of $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ one has:

$$\mathrm{div}\, \boldsymbol{v} \,=\, \mathrm{div}_\Gamma\, \boldsymbol{v}_{\boldsymbol{\tau}} \,+\, 2\, K\, \boldsymbol{v}\cdot\boldsymbol{n} \,+\, \frac{\partial\, \boldsymbol{v}}{\partial\, \boldsymbol{n}}\cdot\boldsymbol{n} \qquad \text{on} \quad \Gamma, \tag{2.2.15}$$

where $K \,=\, \frac{1}{2}\sum_{j=1}^{2} \frac{\partial\, \boldsymbol{n}}{\partial\, s_j}\cdot\boldsymbol{\tau_j}$ denotes the mean curvature and div_Γ is the surface divergence.

Consider the space

$$\boldsymbol{Z}^p(\Omega) \,=\, \Big\{\boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega);\ \ \mathrm{div}\, \boldsymbol{v}\in W^{1,p}(\Omega)\Big\},$$

which is a Banach space for the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{Z}^p(\Omega)} \,=\, \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \,+\, \|\mathrm{div}\, \boldsymbol{v}\|_{W^{1,p}(\Omega)}.$$

Proposition 2.2.12. *The space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{Z}^p(\Omega)$.*

Proof. Let us consider the linear continuous mapping $P\,:\,W^{1,p}(\Omega)\longrightarrow W^{1,p}(\mathbb{R}^3)$ and the vector $\boldsymbol{P}\,:\,\boldsymbol{W}^{1,p}(\Omega)\longrightarrow\boldsymbol{W}^{1,p}(\mathbb{R}^3)$ such that $P\, v_{|\Omega} \,=\, v$ for all $v\in W^{1,p}(\Omega)$ and $\boldsymbol{P}\, \boldsymbol{v}_{|\Omega} \,=\, \boldsymbol{v}$ for all $\boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega)$. Moreover for all $\boldsymbol{v}\in\boldsymbol{Z}^p(\Omega)$ we define $\mathrm{div}(\boldsymbol{P}\boldsymbol{v})$ by

$$\mathrm{div}(\boldsymbol{P}\boldsymbol{v}) \,=\, P(\mathrm{div}\, \boldsymbol{v}).$$

Now let $\boldsymbol{v}\in\boldsymbol{Z}^p(\Omega)$, then $\boldsymbol{P}\, \boldsymbol{v}\in\boldsymbol{W}^{1,p}(\mathbb{R}^3)$ and $\mathrm{div}(\boldsymbol{P}\boldsymbol{v}) \,=\, P(\mathrm{div}\, \boldsymbol{v})\in W^{1,p}(\mathbb{R}^3)$. This means that $\boldsymbol{P}\, \boldsymbol{v}\in\boldsymbol{Z}^p(\mathbb{R}^3)$. Thanks to [9, Lemma 8] we know that $\boldsymbol{\mathcal{D}}(\mathbb{R}^3)$ is dense in $\boldsymbol{Z}^p(\mathbb{R}^3)$, then there exists a sequence $(\boldsymbol{\varphi}_k)_k$ in $\boldsymbol{\mathcal{D}}(\mathbb{R}^3)$ such that

$$\boldsymbol{\varphi}_k \longrightarrow \boldsymbol{P}\boldsymbol{v} \ \text{ in } \boldsymbol{Z}^p(\mathbb{R}^3), \qquad \text{as } \ k\to+\infty.$$

i.e.

$$\boldsymbol{\varphi}_k \longrightarrow \boldsymbol{P}\boldsymbol{v} \text{ in } \boldsymbol{W}^{1,p}(\mathbb{R}^3), \qquad \text{as } k \to +\infty$$

and

$$\operatorname{div} \boldsymbol{\varphi}_k \longrightarrow \operatorname{div} \boldsymbol{P}\boldsymbol{v} = P\operatorname{div}\boldsymbol{v} \text{ in } W^{1,p}(\mathbb{R}^3), \qquad \text{as } k \to +\infty.$$

As a result

$$\boldsymbol{\varphi}_{k_{|\Omega}} \longrightarrow \boldsymbol{v} \text{ in } \boldsymbol{W}^{1,p}(\Omega), \qquad \text{as } k \to +\infty$$

and

$$\operatorname{div}(\boldsymbol{\varphi}_{k_{|}\Omega}) = (\operatorname{div} \boldsymbol{\varphi}_k)_{|\Omega} \longrightarrow \operatorname{div}\boldsymbol{v} \text{ in } W^{1,p}(\Omega), \qquad \text{as } k \to +\infty.$$

Finally we conclude that

$$\boldsymbol{\varphi}_{k_{|\Omega}} \longrightarrow \boldsymbol{v} \text{ in } \boldsymbol{Z}^p(\Omega), \qquad \text{as } k \to +\infty,$$

which ends the proofs. □

As a consequence we have the following corollary:

Corollary 2.2.13. *Suppose that Ω is of class $C^{1,1}$, then for all $\boldsymbol{v} \in \boldsymbol{Z}^p(\Omega)$ formula (2.2.15) holds in $W^{-1/p,p}(\Gamma)$.*

Proof. Let $\boldsymbol{v} \in \boldsymbol{Z}^p(\Omega)$ and let $(\boldsymbol{v}_k)_k$ a sequence in $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ such that $(\boldsymbol{v}_k)_k$ converges to $\boldsymbol{v}$ in $\boldsymbol{Z}^p(\Omega)$. *i.e.*

$$\boldsymbol{v}_k \longrightarrow \boldsymbol{v} \text{ in } \boldsymbol{W}^{1,p}(\Omega), \qquad \text{as } k \to +\infty$$

and

$$\operatorname{div} \boldsymbol{v}_k \longrightarrow \operatorname{div} \boldsymbol{v} \text{ in } W^{1,p}(\Omega), \qquad \text{as } k \to +\infty.$$

In addition we know that for all $k \in \mathbb{N}$ relation (2.2.15) is satisfied. On the other hand the sequence $(\boldsymbol{v}_{k_{\boldsymbol{\tau}}})_k$ converges to $\boldsymbol{v}_{\boldsymbol{\tau}}$ in $\boldsymbol{W}^{1-1/p,p}(\Gamma)$, then

$$\operatorname{div}_\Gamma \boldsymbol{v}_{k_{\boldsymbol{\tau}}} \longrightarrow \operatorname{div}_\Gamma \boldsymbol{v}_{\boldsymbol{\tau}} \text{ as } k \to +\infty \qquad \text{in } W^{-1/p,p}(\Gamma).$$

Moreover the sequences $(\boldsymbol{v}_k \cdot \boldsymbol{n})_k$ and $(\operatorname{div} \boldsymbol{v}_{k_{|\Gamma}})_k$ converge respectively to $\boldsymbol{v} \cdot \boldsymbol{n}$ and $\operatorname{div} \boldsymbol{v}_{|\Gamma}$ in $W^{1-1/p,p}(\Gamma)$ and then in $W^{-1/p,p}(\Gamma)$. Moreover since Ω is of class $C^{1,1}$ then $\boldsymbol{n} \in \boldsymbol{W}^{1,\infty}(\Gamma)$ and $K \in L^\infty(\Gamma)$, then $K\, \boldsymbol{v}_k \cdot \boldsymbol{n} \longrightarrow K\, \boldsymbol{v} \cdot \boldsymbol{n}$, in $W^{-1/p,p}(\Gamma)$. Then the sequence $(\frac{\partial \boldsymbol{v}_k}{\partial \boldsymbol{n}} \cdot \boldsymbol{n})_k$ converges in $W^{-1/p,p}(\Gamma)$ to an element which we shall denote it by $\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \cdot n$. As a consequence for all $\boldsymbol{v} \in \boldsymbol{Z}^p(\Omega)$ one has

$$\operatorname{div} \boldsymbol{v} = \operatorname{div}_\Gamma \boldsymbol{v}_{\boldsymbol{\tau}} + 2\, K\, \boldsymbol{v} \cdot \boldsymbol{n} + \frac{\partial\, \boldsymbol{v}}{\partial\, \boldsymbol{n}} \cdot \boldsymbol{n} \qquad \text{in } W^{-1/p,p}(\Gamma)$$

and the result is proved. □

2.3 Some Properties of sectorial and non-negative operators

This section is devoted to the definitions and some relevant properties of sectorial and non-negative operators very useful in our work. In all this section X denotes a Banach space and $\mathcal{A} : D(\mathcal{A}) \subset X \mapsto X$ is a closed linear operator. $D(\mathcal{A})$ is equipped with the graph norm and form a Banach space with the graph norm.

We start by the definition of a sectorial operator (see [29, Chapter 2, page 96]). Let $0 \leq \theta < \pi/2$ and let Σ_θ be the sector

$$\Sigma_\theta \; = \; \Big\{ \lambda \in \mathbb{C}^*; \;\; |\arg \lambda| < \pi - \theta \Big\}.$$

Definition 2.3.1. *We say that a linear densely defined operator $\mathcal{A} : D(\mathcal{A}) \subseteq X \longmapsto X$ is sectorial if there exists a constant $M > 0$ and an angle $0 \leq \theta < \pi/2$ such that*

$$\forall\, \lambda \in \Sigma_\theta, \qquad \|R(\lambda, \mathcal{A})\|_{\mathcal{L}(X)} \; \leq \; \frac{M}{|\lambda|}, \tag{2.3.1}$$

where $R(\lambda, \mathcal{A}) \; = \; (\lambda I \; - \; \mathcal{A})^{-1}$.

This means that the resolvent of a sectorial operator contains a sector Σ_θ for some $0 \leq \theta < \pi/2$ and for every $\lambda \in \Sigma_\theta$ one has estimate (2.3.1).

The following theorem gives the necessary and sufficient condition for an operator to generates a bounded analytic semi-group (see [29, Chapter 2, Theorem 4.6, page 101]).

Theorem 2.3.2. *An operator $\mathcal{A}$ generates a bounded analytic semi-group if and only if $\mathcal{A}$ is sectorial.*

Nevertheless, it is not always easy to prove that an operator is sectorial in the sense of Definition 2.3.1. Although, Yosida [82] has proved that it is suffices to prove (2.3.1) in the half plane $\{\lambda \in \mathbb{C}^*; \operatorname{Re} \lambda \geq w\}$, for some $w \geq 0$. This result is stated in [13, Chapter 1, Theorem 3.2, page 30] and proved by K. Yosida.

Proposition 2.3.3. *Let $\mathcal{A} : \mathrm{D}(\mathcal{A}) \subseteq X \longmapsto X$ be a linear densely defined operator, let $w \geq 0$ and $M > 0$ such that*

$$\forall\, \lambda \in \mathbb{C}^*, \;\; \operatorname{Re} \lambda \geq w, \qquad \|R(\lambda, \mathcal{A})\|_{\mathcal{L}(X)} \; \leq \; \frac{M}{|\lambda|}.$$

Then $\mathcal{A}$ is sectorial.

Proof. For simplicity we will suppose that $w = 0$. Thanks to Yosida [82, Chapter VIII, Theorem 1, page 211] we know that $\rho(\mathcal{A})$ is an open subset of $\mathbb{C}$ and for all $\lambda_0 \in \rho(\mathcal{A})$, the disc of centre λ_0 and radius $|\lambda_0|/M$ is contained in $\rho(\mathcal{A})$. In particular, for every $r > 0$, the open disks with center $\pm\, i\, r$ and radius $|r|/M$ is contained in $\rho(\mathcal{A})$. The union of such disks and of the half plane $\{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq 0\}$ contains the sector

$$\Big\{ \lambda \in \mathbb{C}; \;\; \lambda \neq 0, \; |\arg \lambda| < \pi \; - \; \arctan(M) \Big\},$$

hence it contains the sector

$$S = \Big\{\lambda \in \mathbb{C};\ \lambda \neq 0,\ |\arg \lambda| < \pi - \arctan(2\,M)\Big\}.$$

If $\lambda \in S$ and $\operatorname{Re}\lambda < 0$, we write λ in the form $\lambda = \pm\, i\, r - (\theta\, r)/(2\,M)$ for some $\theta \in (0,1)$. Thanks to [29, Chapter 4, formula 1.2, page 239] we know that

$$R(\lambda, \mathcal{A}) = R(\pm\, i\, r, \mathcal{A})\big[I + (\lambda \mp i\, r)R(\pm\, i\, r, \mathcal{A})\big]^{-1}.$$

We can easily verify that $\big\|\big[I + (\lambda \mp i\, r)R(\pm\, i\, r, \mathcal{A})\big]^{-1}\big\|_{\mathcal{L}(X)} \leq 2$.

Next, observe that $|\lambda| = \sqrt{r^2 + \frac{\theta^2\, r^2}{4\,M^2}} = r\,\frac{\sqrt{4\,M^2 + \theta^2}}{2\,M}$. Then

$$\|R(\lambda, \mathcal{A})\| \leq \frac{2\,M}{r} \leq \frac{2\,M\,\frac{\sqrt{4\,M^2+\theta^2}}{2\,M}}{r\,\frac{\sqrt{4\,M^2+\theta^2}}{2\,M}} \leq \frac{\sqrt{4\,M^2 + 1}}{|\lambda|}.$$

Now if $\lambda \in S$ such that $\operatorname{Re}\lambda \geq 0$ then thanks to our assumption one has

$$\|R(\lambda, \mathcal{A})\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda|} \tag{2.3.2}$$

which ends the proof. $\square$

Remark 2.3.4. Proposition 2.3.3 means that there exists an angle $0 < \theta_0 < \pi/2$ such that the resolvent set of the operator $\mathcal{A}$ contains the sector

$$\Sigma_{\theta_0} = \big\{\lambda \in \mathbb{C};\ \ |\arg\lambda| \leq \pi - \theta_0\big\}$$

where estimate (2.3.2) is satisfied.

Next we recall some definitions and properties concerning the fractional powers of a non-negative operator. We start by the following definition.

Definition 2.3.5. *An operator $\mathcal{A}$ is said to be a non-negative operator if its resolvent set contains all negative real numbers and*

$$\sup_{t>0}\, t\,\|(t\,I + \mathcal{A})^{-1}\|_{\mathcal{L}(X)} < \infty.$$

It is clear that if an operator $\mathcal{B}$ is sectorial then $-\mathcal{B}$ is in particular a non-negative operator.

For a non-negative operator $\mathcal{A}$ it is possible to define its complex power $\mathcal{A}^z$ for every $z \in \mathbb{C}$ as a densely defined closed linear operator in the closed subspace $X_{\mathcal{A}} = \overline{D(\mathcal{A})} \cap \overline{R(\mathcal{A})}$ in X. Here $D(\mathcal{A})$ and $R(\mathcal{A})$ denote, respectively, the domain and the range of $\mathcal{A}$. Observe that, if both $D(\mathcal{A})$ and $R(\mathcal{A})$ are dense in X, then $X_{\mathcal{A}} = X$. We refer to [52, 78] for the definition and some relevant properties of the complex power of a non-negative operator.

In addition, for a non-negative bounded operator whose inverse $\mathcal{A}^{-1}$ exists and it is bounded (*i.e.* $0 \in \rho(\mathcal{A})$), the complex power $\mathcal{A}^z$ can be defined for all $z \in \mathbb{C}$ by the means of the Dunford integral ([82]):

$$\mathcal{A}^z f = \frac{1}{2\pi i} \int_{\Gamma_\theta} (-\lambda)^z (\lambda I + \mathcal{A})^{-1} f \, \mathrm{d}\lambda, \tag{2.3.3}$$

where Γ_θ runs in the resolvent set of $-\mathcal{A}$ from $\infty \, e^{i(\theta-\pi)}$ to zero and from zero to $\infty \, e^{i(\pi-\theta)}$, $0 < \theta < \pi/2$ in $\mathbb{C}$ avoiding the non negative real axis. The branch of $(-\lambda)^z$ is taken so that $\mathrm{Re}((-\lambda)^z) > 0$ for $\lambda < 0$. It is proved by Triebel [78] that when the operator $\mathcal{A}$ is of bounded inverse, the complex powers $\mathcal{A}^z$ for $\mathrm{Re}\, z > 0$ are isomorphisms from $D(\mathcal{A}^z)$ to $X_{\mathcal{A}}$.

The following property plays an important role in the study of the abstract inhomogeneous Cauchy-Problem and give us more regularity for the solutions (see [43]).

Definition 2.3.6. *Let $\theta \geq 0$ and $K \geq 1$. A non-negative operator $\mathcal{A}$ belongs to $E_K^\theta(X)$ if $\mathcal{A}^{is} \in \mathcal{L}(X_{\mathcal{A}})$ for every $s \in \mathbb{R}$ and its norm in $\mathcal{L}(X_{\mathcal{A}})$ satisfies the estimate*

$$\|\mathcal{A}^{is}\|_{\mathcal{L}(X_{\mathcal{A}})} \leq K \, e^{\theta \, |s|}. \tag{2.3.4}$$

If in addition $D(\mathcal{A})$ and $R(\mathcal{A})$ are dense in X, we say that $\mathcal{A} \in \mathcal{E}_K^\theta(X)$.

We note that, these spaces $E_K^\theta(X)$ and $\mathcal{E}_K^\theta(X)$ were introduced by Dore and Venni [28], Giga and Zohr [43] in the abstract perturbation theory.

We recall some properties of the pure imaginary powers of a non-negative operator very useful in our work (see [43, Appendix,Lemma A2] for the proof).

Lemma 2.3.7. *Let $\mathcal{A}$ be a non-negative operator in a Banach space X. It holds*
(i) *For all $a > 0$ and for all $s \in \mathbb{R}$, $(a\mathcal{A})^{i\,s} = a^{i\,s}\mathcal{A}^{i\,s}$.*
(ii) *For all $\delta > 0$ and for all $f \in D(\mathcal{A}) \cap R(\mathcal{A})$,*

$$(\delta I + \mathcal{A})^{i\,s} f \to \mathcal{A}^{i\,s} f, \qquad \text{as} \quad \delta \to 0.$$

The following lemma is proved by Komatsu (see [52]) and plays an important role in the study of the domains of fractional powers of the Stokes operator.

Lemma 2.3.8. *Let $\mathcal{A}$ be a non-negative closed linear operator. If $\mathrm{Re}\alpha > 0$ the domain $D((\nu I + \mathcal{A})^\alpha)$ doesn't depend on $\nu \geq 0$ and coincides with $D((\mu I + \mathcal{A})^\alpha)$ for $\mu \geq 0$. In other words*

$$\forall \, \mu, \nu > 0, \qquad D(\mathcal{A}^\alpha) = D((\mu I + \mathcal{A})^\alpha) = D((\nu I + \mathcal{A})^\alpha).$$

For a non-negative operator $\mathcal{A}$ such that $0 \in \rho(\mathcal{A})$, the boundedness of $\mathcal{A}^{is}$, $s \in \mathbb{R}$ allows us to determine the domain of definition of $D(\mathcal{A}^\alpha)$, for complex number α satisfying $\mathrm{Re}\, \alpha > 0$ using complex interpolation. The following result is due to [78]

Theorem 2.3.9. *Let $\mathcal{A}$ be a non-negative operator with bounded inverse. We suppose that there exist two positive numbers ε and C such that $\mathcal{A}^{is}$ is bounded for $-\varepsilon \leq s \leq \varepsilon$ and $\|\mathcal{A}^{is}\|_{\mathcal{L}(X_A)} \leq C$. If α is a complex number such that $0 < \operatorname{Re}\alpha < \infty$ and $0 < \theta < 1$ then*

$$[X\,,\,D(\mathcal{A}^{\alpha})]_{\theta} \;=\; D(\mathcal{A}^{\alpha\theta}).$$

Remark 2.3.10. Let $\mathcal{A}$ be a non-negative operator of bounded inverse (*i.e.* $0 \in \rho(\mathcal{A})$). We know, thanks to [78, Theorem 1.15.2, part (e)] that for every complex number α such that $\operatorname{Re}\alpha > 0$, the fractional powers $\mathcal{A}^{\alpha}$ is an isomorphism from $\mathbf{D}(\mathcal{A}^{\alpha})$ to X.

When $-\mathcal{A}$ is the infinitesimal generator of a bounded analytic semi-group $e^{-t\mathcal{A}}$, the following proposition is proved by Komatsu (see [52, Theorem 12.1] for instance)

Proposition 2.3.11. *Let $-\mathcal{A}$ be the infinitesimal generator of a bounded analytic semi-group $e^{-t\mathcal{A}}$. For any complex number α such that $\operatorname{Re}\alpha > 0$ one has*

$$\forall t > 0, \qquad \|\mathcal{A}^{\alpha} e^{-t\mathcal{A}}\|_{\mathcal{L}(X)} \leq C\, t^{-\operatorname{Re}\alpha}. \tag{2.3.5}$$

Moreover, due to [52] we have the following property

Proposition 2.3.12. *Let $-\mathcal{A}$ be the infinitesimal generator of a bounded analytic semi-group $e^{-t\mathcal{A}}$. Then for all, $0 < \alpha < 1$, the fraction power $-\mathcal{A}^{\alpha}$ generates a bounded analytic semi-group. Moreover,*

$$\forall x \in D(\mathcal{A}^{\alpha}), \qquad \mathcal{A}^{\alpha} e^{-t\mathcal{A}} x \;=\; e^{-t\mathcal{A}} \mathcal{A}^{\alpha} x. \tag{2.3.6}$$

Now, when the infinitesimal generator of a bounded analytic semi-group is of bounded inverse and when $0 < \alpha < 1$, the fractional power $\mathcal{A}^{\alpha}$ of $\mathcal{A}$ can be defined as follows (see [64, Chapter 2, Theorem 6.9] for instance).

Theorem 2.3.13. *Let $-\mathcal{A}$ be the infinitesimal generator of an analytic semi-group such that $0 \in \rho(\mathcal{A})$. For $0 < \alpha < 1$ and for $x \in D(\mathcal{A})$ we have the following explicit formula for $\mathcal{A}^{\alpha}x$,*

$$\mathcal{A}^{\alpha} x \;=\; \frac{\sin \pi\alpha}{\pi} \int_0^{\infty} t^{\alpha-1} \mathcal{A}(t\,I \,+ \mathcal{A})^{-1} x \,\mathrm{d}\,t. \tag{2.3.7}$$

Remark 2.3.14. We can easily check using formula (2.3.7) that in the case where $\mathcal{A}$ is a self adjoint operator, it is the same for $\mathcal{A}^{\alpha}$, $0 < \alpha < 1$.

Using the explicit formula (2.3.7) we can verify that Lemma 2.3.7 proved in [43, Appendix,Lemma A2] holds for the fractional powers of an infinitesimal generator of an analytic semi-group.

Lemma 2.3.15. *Let $-\mathcal{A}$ be the infinitesimal generator of an analytic semi-group. Then*
(i) *For all $\beta > 0$ and for all $0 < \alpha < 1$, $(\beta\mathcal{A})^{\alpha} \;=\; \beta^{\alpha}\mathcal{A}^{\alpha}$.*
(ii) *For all $\delta > 0$ and for all $x \in D(\mathcal{A})$,*

$$(\delta\, I \,+\, \mathcal{A})^{\alpha} x \to \mathcal{A}^{\alpha} x, \qquad \text{as} \quad \delta \to 0.$$

Lemma 2.3.16. *Let $-\mathcal{A}$ be the infinitesimal generator of a bounded analytic semi-group $e^{-t\mathcal{A}}$ on a Banach space X such that $0 \in \rho(\mathcal{A})$. Then for all $0 < \alpha < 1$*

$$\|(I - e^{-h\mathcal{A}})\mathcal{A}^{-\alpha}\|_X \leq \frac{C}{\alpha} h^{\alpha}. \tag{2.3.8}$$

Remark 2.3.17. Lemma 2.3.16 has been proved by Kato and Yosida in [35, Lemma 2.11] in the case where the operator $\mathcal{A}$ is a strictly positive self-adjoint operator an a Hilbert space H and $-\mathcal{A}$ generates a strongly continuous semi-group on H. In Lemma 2.3.16 we generalize their result to the case of a sectorial operator $\mathcal{A}$ of bounded inverse on a Banach space.

Proof of Lemma 2.3.16. First recall that since $-\mathcal{A}$ is the infinitesimal generator of a bounded analytic semi-group on X then for all $x \in X$, $e^{-t\mathcal{A}}x \in D(\mathcal{A})$ and

$$\frac{\mathrm{d}e^{-t\mathcal{A}}x}{\mathrm{d}t} = -\mathcal{A}e^{-t\mathcal{A}}x \qquad \text{and} \qquad \frac{\mathrm{d}(I - e^{-t\mathcal{A}})x}{\mathrm{d}t} = -\mathcal{A}e^{-t\mathcal{A}}x.$$

As a result

$$\forall x \in X, \qquad (I - e^{-h\mathcal{A}})x = -\int_0^h \mathcal{A}e^{-t\mathcal{A}}x \,\mathrm{d}\, t.$$

Moreover in the particular case where $x \in D(\mathcal{A}^{\alpha})$ one has

$$(I - e^{-h\mathcal{A}})x = -\int_0^h \mathcal{A}e^{-t\mathcal{A}}x \,\mathrm{d}\, t = -\int_0^h \mathcal{A}^{1-\alpha}e^{-t\mathcal{A}}\mathcal{A}^{\alpha}x \,\mathrm{d}\, t.$$

Thus

$$\begin{aligned} \|(I - e^{-h\mathcal{A}})x\|_X &\leq \int_0^h \|\mathcal{A}^{1-\alpha}e^{-t\mathcal{A}}\|_{\mathcal{L}(X)}\|\mathcal{A}^{\alpha}x\|_X \,\mathrm{d}\, t \\ &\leq \int_0^h t^{\alpha-1}\|\mathcal{A}^{\alpha}x\|_X \,\mathrm{d}\, t \\ &\leq \frac{C}{\alpha}h^{\alpha}\|\mathcal{A}^{\alpha}x\|_X. \end{aligned}$$

Finally observe that for all $x \in X$, $\mathcal{A}^{-\alpha}x \in D(\mathcal{A}^{\alpha})$ and

$$\|(I - e^{-h\mathcal{A}})\mathcal{A}^{-\alpha}x\|_X \leq \frac{C}{\alpha}h^{\alpha}\|\mathcal{A}^{\alpha}\mathcal{A}^{-\alpha}x\|_X \leq \frac{C}{\alpha}h^{\alpha}\|x\|_X$$

and the result is proved. □

Next, we recall the definition of a Hölder continuous function

Definition 2.3.18. *Let I be an interval and let X be a Banach space. A function $f : \longmapsto X$ is Hölder continuous with exponent ϑ, $0 < \vartheta < 1$ on I if there is a constant L such that*

$$\|f(t) - f(s)\|_X \leq L|t - s|^{\vartheta}.$$

We denote the family of all Hölder continuous functions with exponent ϑ on I by $C^{\vartheta}(I; X)$.

Proposition 2.3.19. *Let $-\mathcal{A}$ be the infinitesimal generator of a bounded analytic semi-group $e^{-t\mathcal{A}}$ on a Banach space X such that $0 \in \rho(\mathcal{A})$. Then for all $0 < \alpha \leq 1$, and for all $x \in X$ $A^{\alpha}e^{-t\mathcal{A}}x$ is Hölder continuous on every interval $[\varepsilon, T]$ for all $\varepsilon > 0$.*

Proof. Let $x \in X$, then

$$\begin{aligned}
\|\mathcal{A}^{\alpha}e^{-(t+h)\mathcal{A}}x - \mathcal{A}^{\alpha}e^{-t\mathcal{A}}x\|_X &= \|\mathcal{A}^{\alpha}e^{-t\mathcal{A}}(e^{-h\mathcal{A}} - I)x\|_X \\
&= \|\mathcal{A}^{2\alpha}e^{-t\mathcal{A}}(e^{-h\mathcal{A}} - I)\mathcal{A}^{-\alpha}x\|_X \\
&\leq \frac{C}{t^{2\alpha}}\|(e^{-h\mathcal{A}} - I)\mathcal{A}^{-\alpha}x\|_X \\
&\leq C_{\alpha}h^{\alpha}\|x\|_x.
\end{aligned}$$

The last inequality is obtained using Lemma 2.3.16. □

Consider now the function

$$v(t) = \int_0^t e^{-(t-s)\mathcal{A}}f(s)\,\mathrm{d}\,s, \qquad 0 < t < T, \quad T > 0. \tag{2.3.9}$$

The following theorem is due to Pazy [64, Chapter 4, Corollary 3.3]

Theorem 2.3.20. *Let $f \in L^1(0,T;\,X)$ and $e^{-t\mathcal{A}}$ is a bounded analytic semi-group generated by $-\mathcal{A}$ on a Banach space X. If the function f is locally Hölder continuous then $v(t) \in D(\mathcal{A})$ for all $0 < t < T$.*

Chapter 3

Analyticity of the Stokes semi-group

This chapter is devoted to the proof of the analyticity of the Stokes semi-group with three different types of boundary conditions respectively. The Navier-type boundary conditions (1.0.5), the Navier slip boundary conditions (1.0.4) and the boundary condition involving the pressure (1.0.6). This is a key tool to solve the time dependent Stokes problem using semi-group theory. The proof is based on the study of the complex resolvent of the stokes operator with each of these boundary conditions.

This chapter is organised as follows :

In section 1 we prove the analyticity of the Stokes semi-group with Navier-type boundary conditions (1.0.5) in some spaces to be determined. We also impose further conditions on the Stokes operator, the flux through the cuts Σ_j, $1 \leq j \leq J$ is equal to zero. This assumption allows us to obtain an operator of bounded inverse and thus an exponential decay of the Stokes semi-group.

Section 2 is devoted to the analyticity of the Stokes semi-group with the Navier slip boundary condition (1.0.4). The proofs in this section are similar to those in the previous section, for this reason many details will be omitted.

In section 3, we prove that the Stokes operator with the boundary condition involving the pressure and the tangential component of the velocity (1.0.6) is sectorial and generates a bounded analytic semi-group on some spaces to be determined. We will also define the Stokes operator with flux through the connected components of Γ. These boundary conditions enable the Stokes operator to be invertible and thus Stokes semi-group decays exponentially.

3.1 Stokes operator with Navier-type boundary conditions

In this section we consider the Stokes operator with Navier-type boundary

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \qquad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \ \text{ on } \qquad \Gamma. \tag{3.1.1}$$

We prove the analyticity of the semi-group generated by the Stokes operator with boundary conditions on the spaces $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, $[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\tau}$ and $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ respectively (see (3.1.4), (3.1.64) and (3.1.71) for the definition of these spaces). This allows us to obtain weak, strong and very weak solutions for the time dependent Stokes problem with this Navier-type boundary condition. We define also the Stokes operator With flux boundary conditions, this enables us to obtain an operator of bounded inverse.

Consider the space

$$\boldsymbol{V}^p_\tau(\Omega) \;=\; \Big\{\boldsymbol{u}\in \boldsymbol{X}^p_\tau(\Omega);\;\mathrm{div}\,\boldsymbol{u}=0\;\text{ in }\Omega\Big\}, \tag{3.1.2}$$

which is a Banach space for the norm $\boldsymbol{X}^p(\Omega)$ and it is equivalent to the space

$$\boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega)=\{\boldsymbol{u}\in \boldsymbol{W}^{1,p}(\Omega);\,\mathrm{div}\,\boldsymbol{u}=0\text{ in }\Omega\text{ and }\boldsymbol{u}\cdot\boldsymbol{n}=0\text{ on }\Gamma\} \tag{3.1.3}$$

with equivalent norms. We also consider the space

$$\boldsymbol{L}^p_{\sigma,\tau}(\Omega) \;=\; \Big\{\boldsymbol{f}\in \boldsymbol{L}^p(\Omega);\;\mathrm{div}\,\boldsymbol{f}=0\;\text{ in }\;\Omega,\;\;\boldsymbol{f}\cdot\boldsymbol{n}=0\;\text{ on }\;\Gamma\Big\}. \tag{3.1.4}$$

It is clear that for $p=2$, $\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ is a closed subspace of $\boldsymbol{L}^2(\Omega)$ and it is an Hilbert space for the inner product of $\boldsymbol{L}^2(\Omega)$.

The Stokes operator with Navier-type boundary conditions is defined by

$$\forall\,\boldsymbol{u}\in \boldsymbol{V}^p_\tau(\Omega),\;\;\forall\,\boldsymbol{v}\in \boldsymbol{V}^{p'}_\tau(\Omega),\qquad \langle A_p\,\boldsymbol{u}\,,\,\boldsymbol{v}\rangle_{(\boldsymbol{V}^{p'}_\tau(\Omega))'\times \boldsymbol{V}^{p'}_\tau(\Omega)} \;=\; \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}\,x. \tag{3.1.5}$$

On other words, the Stokes operator with Navier-type boundary conditions is the linear mapping

$$A_p\;:\;\mathbf{D}(A_p)\subset \boldsymbol{L}^p_{\sigma,\tau}(\Omega)\longmapsto \boldsymbol{L}^p_{\sigma,\tau}(\Omega),$$

where

$$\begin{aligned}\mathbf{D}(A_p)=\big\{\boldsymbol{u}\in \boldsymbol{W}^{1,p}(\Omega);\;\Delta\boldsymbol{u}\in \boldsymbol{L}^p(\Omega),\;\mathrm{div}\,\boldsymbol{u}=0\text{ in }\Omega,&\\ \boldsymbol{u}\cdot\boldsymbol{n}\;=\;0,\;\;\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}\;=\;\mathbf{0}\text{ on }\Gamma\big\}&\end{aligned} \tag{3.1.6}$$

and

$$\forall\,\boldsymbol{u}\in \mathbf{D}(A_p),\qquad A_p\,\boldsymbol{u}\;=\;-P\Delta\boldsymbol{u}\qquad\text{in}\quad\Omega.$$

Here $P:\boldsymbol{L}^p(\Omega)\longmapsto \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ is the Helmholtz projection defined by:

$$\forall\,\boldsymbol{f}\in \boldsymbol{L}^p(\Omega),\qquad P\boldsymbol{f}\;=\;\boldsymbol{f}\;-\;\mathbf{grad}\,\pi, \tag{3.1.7}$$

where π is the unique solution of Problem (2.2.7).

Notice that for all $\boldsymbol{u}\in \mathbf{D}(A_p)\cap\mathbf{D}(A_q)$ and for all $1<p,q<\infty$, $A_p\boldsymbol{u}=A_q\boldsymbol{u}$ in Ω.

The following proposition gives a basic property of the Stokes operator with Navier-type boundary conditions (3.1.1).

Proposition 3.1.1. *For all $\boldsymbol{u} \in \mathbf{D}(A_p)$, $A_p\, \boldsymbol{u} = -\Delta \boldsymbol{u}$ in Ω.*

Proof. Let $\boldsymbol{u} \in \mathbf{D}(A_p)$, it is clear that $\Delta \boldsymbol{u} \in \boldsymbol{H}^p(\mathrm{div}, \Omega)$. Moreover since $\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0}$ on Γ then we can easily verify that $\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ. This means that $\Delta \boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ. As a consequence, $\Delta \boldsymbol{u} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and $A_p \boldsymbol{u} = -P\Delta \boldsymbol{u} = -\Delta \boldsymbol{u}$ in Ω. Notice that here the pressure π is a solution of the problem

$$\Delta \pi = 0 \qquad \text{in } \Omega, \qquad \frac{\partial \pi}{\partial \boldsymbol{n}} = \Delta\, \boldsymbol{u} \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma.$$

Thus $\pi =$ Constant and $\mathbf{grad}\, \pi = 0$ in Ω. □

Remark 3.1.2. One of our main goals is to see the difference between the Stokes operator with Navier-type boundary conditions and with the Dirichlet boundary condition. For this reason we consider the space

$$\boldsymbol{V}^p_0(\Omega) \;=\; \big\{ \boldsymbol{v} \in \boldsymbol{W}^{1,p}_0(\Omega);\, \mathrm{div}\, \boldsymbol{v} = 0 \;\text{ in } \Omega \big\}$$

which is a Banach space for the norm of $\boldsymbol{W}^{1,p}(\Omega)$. For every $\boldsymbol{u} \in \boldsymbol{V}^p_0(\Omega)$, we define the Stokes operator with Dirichlet boundary condition by

$$\forall\, \boldsymbol{v} \in \boldsymbol{V}^{p'}_0(\Omega), \qquad \langle A\boldsymbol{u}\,,\, \boldsymbol{v}\rangle_{(\boldsymbol{V}^{p'}_0(\Omega))' \times \boldsymbol{V}^{p'}_0(\Omega)} \;=\; \int_\Omega \nabla \boldsymbol{u} : \nabla \overline{\boldsymbol{v}}\, \mathrm{d}\, x.$$

Equivalently, the Stokes operator with Dirichlet boundary conditions is defined by :

$$\boldsymbol{u} \in \mathbf{D}(A), \qquad A\boldsymbol{u} \;=\; -P\Delta \boldsymbol{u} \;=\; -\Delta \boldsymbol{u} + \mathbf{grad}\, \pi,$$

where $\mathbf{D}(A) \;=\; \boldsymbol{W}^{2,p}(\Omega) \cap \boldsymbol{W}^{1,p}_0(\Omega) \cap \boldsymbol{L}^p_\sigma(\Omega)$ and π is the unique solution up to an additive constant of the problem

$$\mathrm{div}(\mathbf{grad}\, \pi \,-\, \Delta \boldsymbol{u}) = 0 \qquad \text{in } \Omega, \qquad (\mathbf{grad}\, \pi \,-\, \Delta \boldsymbol{u}) \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma.$$

Unlike the Stokes operator with Navier-type boundary condition, the pressure here cannot be constant since $\Delta \boldsymbol{u} \cdot \boldsymbol{n}$ doesn't vanish on Γ.

The following two propositions give the density and a regularity property of the domain of the Stokes operator.

Proposition 3.1.3. *The space $\mathbf{D}(A_p)$ is dense in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$.*

Proof. It is clear that $\boldsymbol{\mathcal{D}}_\sigma(\Omega) \subset \mathbf{D}(A_p) \subset \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. Now, since $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ is dense in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, then $\mathbf{D}(A_p)$ is dense in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. □

Proposition 3.1.4. *Suppose that Ω is of class $C^{2,1}$, then*

$$\mathbf{D}(A_p) = \Big\{ \boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega);\; \mathrm{div}\, \boldsymbol{u} = 0 \text{ in } \Omega,\; \boldsymbol{u} \cdot \boldsymbol{n} = 0,\; \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma \Big\}. \tag{3.1.8}$$

Proof. Let $\boldsymbol{u} \in \mathbf{D}(A_p)$ and set $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{u}$. Then $\boldsymbol{z} \in \boldsymbol{L}^p(\Omega)$, $\operatorname{div} \boldsymbol{z} = 0$ in Ω, $\mathbf{curl}\,\boldsymbol{z} = -\Delta \boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{z} \times \boldsymbol{n} = \mathbf{0}$ on Γ. Thus $\boldsymbol{z} \in \boldsymbol{X}^p_N(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$. Finally observe that $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$, $\mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$, $\operatorname{div} \boldsymbol{u} = 0$ in Ω and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ. Thanks to Lemma 2.2.2, we conclude that $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$, which ends the proof. $\square$

Remark 3.1.5. (i) Notice that thanks to Lemmas 2.2.1 and 2.2.2, when Ω is of class $C^{2,1}$ we have

$$\forall\, \boldsymbol{u} \in \mathbf{D}(A_p), \qquad \|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \simeq \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\Delta \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

(ii) Thanks to [10, Proposition 4.7], when Ω is of class $C^{2,1}$ for all $\boldsymbol{u} \in \mathbf{D}(A_p)$ such that $\langle \boldsymbol{u} \cdot \boldsymbol{n}\,,\, 1\rangle_{\Sigma_j} = 0$, $1 \leq j \leq J$ we have

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \simeq \|\Delta \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

3.1.1 Analyticity on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$

In this subsection we prove that the Stokes operator with Navier-type boundary conditions generates a bounded analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ for all $1 < p < \infty$. Since the Hilbertian case is different from the general $\boldsymbol{L}^p$-theory we will treat each case separately.

The Hilbertian case

Before we state our theorem we give a basic property essential in the proof of the Hilbertian case.

For all $\varepsilon \in \,]0, \pi[$, let Σ_ε be the sector

$$\Sigma_\varepsilon = \big\{\lambda \in \mathbb{C}^*;\ |\arg \lambda| \leq \pi - \varepsilon\big\}. \tag{3.1.9}$$

Lemma 3.1.6. *Let $\varepsilon \in \,]0, \pi[$ be fixed. There exists a constant $C_\varepsilon > 0$ such that for every positive real numbers a and b one has:*

$$\forall\, \lambda \in \Sigma_\varepsilon, \qquad |\lambda\, a \,+\, b| \geq C_\varepsilon(|\lambda| a \,+\, b). \tag{3.1.10}$$

Proof. For $\lambda \in \mathbb{R}^+$ the inequality (3.1.10) is obvious. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and let $\varepsilon \in \,]0, \pi[$ such that $\varepsilon < |\arg \lambda| < \pi - \varepsilon$. We write λ in the exponential form $\lambda = r\, e^{i\,\theta}$ where $\theta = \arg \lambda$. Next, we search a constant $C > 0$ such that $|\lambda\, a \,+ b|^2 \,\geq\, C^2(|\lambda|\, a + b)^2$. This is equivalent to find a constant $C > 0$ such that polynomial $(1 - C^2)\, b^2 \,+\, 2\, a\, r\, (\cos\theta \,-\, C^2)\, b \,+\, (1 \,-\, C^2)\, a^2\, r^2$ is positive. An easy computation shows that the constant C depends on ε and satisfies $C_\varepsilon \in \Big]0,\, \sqrt{\frac{\sin^2 \varepsilon}{2\,(1 + \cos \varepsilon)}}\Big[$. $\square$

Now we want to study the resolvent of the Stokes operator. For that we consider the problem

$$\left\{\begin{aligned} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} = \boldsymbol{f}, &\qquad \operatorname{div} \boldsymbol{u} = 0 &&\text{in} \quad \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, &\qquad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} &&\text{on} \quad \Gamma, \end{aligned}\right. \tag{3.1.11}$$

where $\boldsymbol{f} \in \boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ and $\lambda \in \Sigma_\varepsilon$.

Remark 3.1.7. Observe that, Problem (3.1.11) is equivalent to the problem

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} = \boldsymbol{f}, & \text{in} \quad \Omega, \\ \quad \boldsymbol{u} \cdot \boldsymbol{n} = 0, \qquad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} & \text{on} \quad \Gamma. \end{cases} \tag{3.1.12}$$

In fact, let $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ be the unique solution of Problem (3.1.12) and set $\operatorname{div} \boldsymbol{u} = \chi$. It is clear that $\lambda\chi - \Delta\chi = 0$ in Ω. Moreover, since $\boldsymbol{f} \cdot \boldsymbol{n} = 0$ and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ then $\Delta \boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ. Notice also that the condition $\mathbf{curl} \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0}$ on Γ implies that $\mathbf{curl}\,\mathbf{curl} \boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ. Finally since $\Delta \boldsymbol{u} = \mathbf{grad}(\operatorname{div} \boldsymbol{u}) - \mathbf{curl}\,\mathbf{curl} \boldsymbol{u}$ one gets $\frac{\partial \chi}{\partial \boldsymbol{n}} = 0$ on Γ. Thus $\chi = 0$ in Ω and the result is proved.

In what follows we use the following formula for the Laplacian operator

$$\Delta\, \boldsymbol{v} = \mathbf{grad}\,(\operatorname{div} \boldsymbol{v}) - \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{v} \qquad \text{in} \quad \Omega.$$

The following theorem gives the solution of the resolvent of the Stokes operator A_2 as well as a resolvent estimate.

Theorem 3.1.8. *Let $\varepsilon \in]0, \pi[$ be fixed, $\boldsymbol{f} \in \boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ and $\lambda \in \Sigma_\varepsilon$.*
(i) *The Problem (3.1.11) has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$.*
(ii) *There exist a constant $C'_\varepsilon > 0$ independent of $\boldsymbol{f}$ and λ such that the solution $\boldsymbol{u}$ satisfies the estimates*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C'_\varepsilon}{|\lambda|} \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \tag{3.1.13}$$

and

$$\|\mathbf{curl}\, \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C'_\varepsilon}{\sqrt{|\lambda|}} \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}. \tag{3.1.14}$$

($C'_\varepsilon = 1/C_\varepsilon$, where C_ε is the constant in (3.1.10)).
(iii) *If Ω is of class $C^{2,1}$ then $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$ and satisfies the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} \leq \frac{C(\Omega, \lambda, \varepsilon)}{|\lambda|} \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}, \tag{3.1.15}$$

where $C(\Omega, \lambda, \varepsilon) = C(\Omega)(C'_\varepsilon + 1)(|\lambda| + 1)$.

Remark 3.1.9. We note that for $\lambda > 0$ the constant C'_ε is equal to 1 and we recover the m-accretiveness property of the Stokes operator on $\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$.

Proof. **(i) Variational formulation**: Consider the space $\boldsymbol{V}^2_\tau(\Omega)$ given by (3.1.2) (for $p = 2$). It is clear that $\boldsymbol{V}^2_\tau(\Omega)$ is a closed subspace of $\boldsymbol{X}^2_\tau(\Omega)$ and it is an Hilbert space for the inner product of $\boldsymbol{X}^2(\Omega)$. We also recall that on $\boldsymbol{V}^2_\tau(\Omega)$ the norm of $\boldsymbol{X}^2_\tau(\Omega)$ is equivalent to the norm of $\boldsymbol{H}^1(\Omega)$.

Now, consider the variational problem: find $\boldsymbol{u} \in \boldsymbol{V}^2_\tau(\Omega)$ such that for any $\boldsymbol{v} \in \boldsymbol{V}^2_\tau(\Omega)$

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}\, x, \tag{3.1.16}$$

where

$$a(\boldsymbol{u},\boldsymbol{v}) = \lambda\int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x.$$

We can easily verify that a is a continuous sesqui-linear form on $\boldsymbol{V}^2_\tau(\Omega)$. In fact

$$\begin{aligned}
|a(\boldsymbol{u},\boldsymbol{v})| &\le |\lambda|\,\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}\\
&\le (|\lambda|+1)\|\boldsymbol{u}\|_{\boldsymbol{X}^2_T(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{X}^2_\tau(\Omega)}.
\end{aligned}$$

For the coercivity, observe that since $\lambda\in\Sigma_\varepsilon$, thanks to Lemma 3.1.6 there exists a constant C_ε such that

$$\begin{aligned}
|\,a(\boldsymbol{v},\boldsymbol{v})| &= \left|\lambda\,\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)}\right|\\
&\ge C_\varepsilon\left(|\lambda|\,\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)}\right)\\
&\ge C_\varepsilon\,\min(|\lambda|\,,1)\|\boldsymbol{v}\|^2_{\boldsymbol{X}^2_\tau(\Omega)}.
\end{aligned}$$

Then for all $\lambda\in\Sigma_\varepsilon$ a is a sesqui-linear continuous coercive form on $\boldsymbol{V}^2_\tau(\Omega)$. Due to Lax-Milgram Lemma, Problem (3.1.16) has a unique solution $\boldsymbol{u}\in\boldsymbol{V}^2_\tau(\Omega)$ since the right-hand side belongs to the anti-dual $(\boldsymbol{V}^2_\tau(\Omega))'$.

(ii) Equivalent Problem: Now we want to extend (3.1.16) to any test function $\boldsymbol{v}\in\boldsymbol{X}^2_\tau(\Omega)$. In fact, we proceed exactly in the same way as in [10, Proposition 4.3]. Let $\boldsymbol{v}\in\boldsymbol{X}^2_\tau(\Omega)$ and consider the solution $\chi\in H^1(\Omega)$ unique up to an additive constant of the problem

$$\Delta\chi = \operatorname{div}\boldsymbol{v} \text{ in } \Omega, \qquad \frac{\partial\chi}{\partial\boldsymbol{n}} = 0 \text{ on } \Gamma.$$

Setting

$$\boldsymbol{\varphi} = \boldsymbol{v} - \mathbf{grad}\,\chi$$

we can easily verify that $\boldsymbol{\varphi}\in\boldsymbol{L}^2(\Omega)$, $\operatorname{div}\boldsymbol{\varphi}=0$ in Ω, $\mathbf{curl}\,\boldsymbol{v} = \mathbf{curl}\,\boldsymbol{\varphi}\in\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{\varphi}\cdot\boldsymbol{n}=0$ on Γ (*i.e.* $\boldsymbol{\varphi}\in\boldsymbol{V}^2_\tau(\Omega)$).

Moreover, it is clear that $\mathbf{grad}\,\chi$ belongs to $\boldsymbol{H}^2_0(\operatorname{div},\Omega)$ and since $\boldsymbol{f}$ and $\boldsymbol{u}$ are divergence free then we can easily verify that

$$\int_\Omega \boldsymbol{f}\cdot\mathbf{grad}\,\overline{\chi}\,\mathrm{d}\,x = 0 \qquad\text{and}\qquad \int_\Omega \boldsymbol{u}\cdot\mathbf{grad}\,\overline{\chi}\,\mathrm{d}\,x = 0.$$

As a consequence we obtain $a(\boldsymbol{u},\boldsymbol{v}) = a(\boldsymbol{u},\boldsymbol{\varphi})$ and then Problem (3.1.16) is equivalent to the problem: find $\boldsymbol{u}\in\boldsymbol{V}^2_\tau(\Omega)$ such that for all $\boldsymbol{v}\in\boldsymbol{X}^2_\tau(\Omega)$

$$\lambda\int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x. \tag{3.1.17}$$

Finally we prove that Problem (3.1.11) is equivalent to Problem (3.1.17). It is clear that if $\boldsymbol{u}\in\boldsymbol{H}^1(\Omega)$ is a solution of (3.1.11) then $\boldsymbol{u}$ solves (3.1.17). Conversely, let $\boldsymbol{u}$ be a solution of Problem (3.1.17). Then for all $\boldsymbol{\varphi}\in\boldsymbol{X}^2_\tau(\Omega)$ one has

$$\lambda\int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{\varphi}}\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{\varphi}}\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\overline{\boldsymbol{\varphi}}\,\mathrm{d}\,x. \tag{3.1.18}$$

In particular, for all $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega)$ (3.1.18) is satisfied. As a result one gets

$$\forall\, \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega), \qquad \langle \lambda\, \boldsymbol{u} - \Delta \boldsymbol{u} - \boldsymbol{f}\,,\, \boldsymbol{\varphi} \rangle_{\boldsymbol{\mathcal{D}}'(\Omega) \times \boldsymbol{\mathcal{D}}(\Omega)} \;=\; 0.$$

This means that $\lambda\, \boldsymbol{u} - \Delta \boldsymbol{u} = \boldsymbol{f}$ in Ω. Moreover, since $\boldsymbol{u} \in \boldsymbol{V}^2_{\tau}(\Omega)$ then $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, $\operatorname{div} \boldsymbol{u} = 0$ in Ω and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ. It remains to prove the boundary condition $\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = 0$ on Γ. In fact, for all $\boldsymbol{v} \in \boldsymbol{X}^2_{\tau}(\Omega)$ one has

$$\langle \lambda\, \boldsymbol{u} - \Delta \boldsymbol{u}\,,\, \boldsymbol{v} \rangle_{[\boldsymbol{H}^2_0(\mathrm{div},\Omega)]' \times \boldsymbol{H}^2_0(\mathrm{div},\Omega)} \;=\; \langle \boldsymbol{f}\,,\, \boldsymbol{v} \rangle_{[\boldsymbol{H}^2_0(\mathrm{div},\Omega)]' \times \boldsymbol{H}^2_0(\mathrm{div},\Omega)}.$$

Then

$$\lambda \int_{\Omega} \boldsymbol{u} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x + \int_{\Omega} \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x \;-\; \langle \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n}\,,\, \boldsymbol{v} \rangle_{\Gamma} = \int_{\Omega} \boldsymbol{f} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}\, x,$$

where $\langle .\,,\, . \rangle_{\Gamma} \;=\; \langle .\,,\, . \rangle_{\boldsymbol{H}^{-1/2}(\Gamma) \times \boldsymbol{H}^{1/2}(\Gamma)}$. On the other hand, since $\boldsymbol{u}$ is a solution of (3.1.17) then one gets

$$\forall\; \boldsymbol{v} \in \boldsymbol{X}^2_{\tau}(\Omega), \qquad \langle \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n}\,,\, \boldsymbol{v} \rangle_{\Gamma} \;=\; 0.$$

Let $\boldsymbol{\mu} \in \boldsymbol{H}^{1/2}(\Gamma)$, we know that $\boldsymbol{\mu_{\tau}} = \boldsymbol{\mu} - (\boldsymbol{\mu} \cdot \boldsymbol{n})\boldsymbol{n}$ on Γ. Since Ω is of class $C^{1,1}$ then $\boldsymbol{\mu_{\tau}} \in \boldsymbol{H}^{1/2}(\Gamma)$. As a result, there exists a function $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{\mu_{\tau}} = \boldsymbol{v}$ on Γ. Clearly $\boldsymbol{v} \in \boldsymbol{X}^2_{\tau}(\Omega)$ and

$$\langle \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n}\,,\, \boldsymbol{\mu} \rangle_{\Gamma} \;=\; \langle \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n}\,,\, \boldsymbol{\mu_{\tau}} \rangle_{\Gamma} \;=\; \langle \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n}\,,\, \boldsymbol{v} \rangle_{\Gamma} \;=\; 0.$$

This means that $\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0}$ on Γ and $\boldsymbol{u}$ is a solution of (3.1.11).

As a result Problem (3.1.11) and (3.1.17) are equivalent and Problem (3.1.11) has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$.

(ii) Estimates: Multiplying the first equation of System (3.1.11) by $\overline{\boldsymbol{u}}$ and integrating both sides one gets

$$\lambda \int_{\Omega} |\boldsymbol{u}|^2\, \mathrm{d}\, x \;+\; \int_{\Omega} |\mathbf{curl}\, \boldsymbol{u}|^2\, \mathrm{d}\, x \;=\; \int_{\Omega} \boldsymbol{f} \cdot \overline{\boldsymbol{u}}\, \mathrm{d}\, x.$$

Now as described above, since $\lambda \in \Sigma_{\varepsilon}$, there exists a constant $C'_{\varepsilon} = 1/C_{\varepsilon}$ such that

$$\begin{aligned} |\lambda| \|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\, \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \;&\leq\; C'_{\varepsilon} \big| \lambda \|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\, \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \big| \\ &=\; C'_{\varepsilon} \Big| \int_{\Omega} \boldsymbol{f} \cdot \overline{\boldsymbol{u}}\, \mathrm{d}\, x \Big| \\ &\leq\; C'_{\varepsilon}\, \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}. \end{aligned}$$

As a result

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C'_{\varepsilon}}{|\lambda|} \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)},$$

which is estimate (3.1.13). In addition, it is clear that

$$\begin{aligned} \|\mathbf{curl}\, \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \;&\leq\; C'_{\varepsilon}\, \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \\ &\leq\; \frac{C'^2_{\varepsilon}}{|\lambda|}\, \|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)}, \end{aligned}$$

which is estimate (3.1.14). We recall that C_ε is the constant in (3.1.10).

(iii) Regularity: The regularity of the solution is a direct application of Proposition 3.1.4. Let us prove estimate (3.1.15). Thanks to (3.1.13) it is clear that

$$\|\Delta \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \,\leq\, \|\boldsymbol{f} - \lambda\, \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \,\leq\, (C'_\varepsilon + 1)\, \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}.$$

Now, since $\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} \,\simeq\, \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \,+\, \|\Delta \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}$ one has estimate (3.1.15). □

The following theorem gives us the analyticity of the semi-group generated by the Stokes operator on $\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$.

Theorem 3.1.10. *The operator $-A_2$ generates a bounded analytic semi-group on $\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$.*

Proof. Thanks to Theorem 2.3.2 it suffices to prove that $-A_2$ is sectorial which is a direct application of Theorem 3.1.8. We recall that, with the Navier-type boundary conditions (3.1.1) the Stokes operator coincides with the $-\Delta$ operator. □

Remark 3.1.11. We recall that the restriction of an analytic semi-group to the non negative real axis is C_0 semi-group. Thanks to Remark 3.1.9 the restriction of our analytic semi-group to the real axis gives a C_0 semi-group of contraction.

Remark 3.1.12. Consider the sesqui-linear form (see [5]):

$$\forall\, \boldsymbol{u},\, \boldsymbol{v} \in \boldsymbol{V}^2_\tau(\Omega), \qquad a(\boldsymbol{u}, \boldsymbol{v}) \,=\, \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}\, x. \tag{3.1.19}$$

If Ω is simply connected, we know that for all $\boldsymbol{v} \in \boldsymbol{V}^2_\tau(\Omega)$ one has

$$\|\boldsymbol{v}\|_{\boldsymbol{X}^2(\Omega)} \leq C\, \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}. \tag{3.1.20}$$

As a result, the sesqui-linear form a is coercive and we can apply Lax-Milgram Lemma to find solution to the problem: find $\boldsymbol{u} \in \boldsymbol{V}^2_\tau(\Omega)$ such that for all $\boldsymbol{v} \in \boldsymbol{V}^2_\tau(\Omega)$

$$a(\boldsymbol{u}, \boldsymbol{v}) \,=\, \int_\Omega \boldsymbol{f} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}\, x,$$

where $\boldsymbol{f} \in \boldsymbol{L}^2_{\sigma,\tau}(\Omega)$. This means that the operator $A_2 \,:\, \mathbf{D}(A_2) \subset \boldsymbol{L}^2_{\sigma,\tau}(\Omega) \longmapsto \boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ is bijective.

Now, if Ω is multiply-connected, the inequality (3.1.20) is false. Indeed we introduce the Kernel $\boldsymbol{K}^2_\tau(\Omega)$:

$$\boldsymbol{K}^2_\tau(\Omega) \,=\, \big\{\boldsymbol{v} \in \boldsymbol{X}^2_\tau(\Omega);\ \mathrm{div}\, \boldsymbol{v} = 0,\ \mathbf{curl}\, \boldsymbol{v} = \mathbf{0} \text{ in } \Omega\big\}. \tag{3.1.21}$$

Thanks to [5, Proposition 3.14] we know that this kernel is not trivial, it is of finite dimension and it is spanned by the functions $\widetilde{\mathbf{grad}}\, q^\tau_j$, $1 \leq j \leq J$, where q^τ_j is the unique solution up to an additive constant of the problem:

$$\begin{cases} -\Delta q_j^\tau = 0 & \text{in } \Omega^\circ, \\ \partial_n q_j^\tau = 0 & \text{on } \Gamma, \\ \left[q_j^\tau\right]_k = \text{constant}, & 1 \leq k \leq J, \\ \left[\partial_n q_j^\tau\right]_k = 0; & 1 \leq k \leq J, \\ \langle \partial_n q_j^\tau \,,\, 1\rangle_{\Sigma_k} = \delta_{jk}, & 1 \leq k \leq J. \end{cases} \tag{3.1.22}$$

Moreover, thanks to [5, Corollary 3.16], for all $\boldsymbol{v} \in \boldsymbol{X}_\tau^2(\Omega)$ we have the following Poincaré-type inequality:

$$\|\boldsymbol{v}\|_{\boldsymbol{X}_\tau^2(\Omega)} \leq C_2(\Omega)(\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \sum_{j=1}^{J} |\langle \boldsymbol{v}\cdot\boldsymbol{n}\,,\, 1\rangle_{\Sigma_j}|). \tag{3.1.23}$$

The following proposition gives the eigenvalues of the Stokes operator. We will see later that the following proposition allows us to obtain an explicit form for the unique solution of the homogeneous Stokes Problem as a linear combination of the eigenfunctions of the Stokes operator.

Proposition 3.1.13. *There exists a sequence of functions $(\boldsymbol{z}_k)_k \subset \mathbf{D}(A_2)$ and an increasing sequence of real numbers $(\lambda_k)_k$ such that $\lambda_k \geq 0$, $\lambda_k \to +\infty$ as $k \to +\infty$ and*

$$\forall\, \boldsymbol{v} \in \boldsymbol{X}_\tau^2(\Omega), \qquad \int_\Omega \mathbf{curl}\,\boldsymbol{z}_k \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}\,x \;=\; \lambda_k \int_\Omega \boldsymbol{z}_k \cdot \overline{\boldsymbol{v}}\,\mathrm{d}\,x.$$

In other words, $(\lambda_k)_k$ are the eigenvalues of the Stokes operator and $(\boldsymbol{z}_k)_k$ are the associated eigenfunctions.

Proof. Consider the operator

$$\Lambda \,:\, \boldsymbol{L}^2_{\sigma,\tau}(\Omega) \longmapsto \mathbf{D}(A_2) \longmapsto \boldsymbol{L}^2_{\sigma,\tau}(\Omega)$$

$$\boldsymbol{f} \longmapsto \boldsymbol{u} \longmapsto \boldsymbol{u},$$

where $\boldsymbol{u}$ is the unique solution of the problem

$$\begin{cases} \boldsymbol{u} + A_2\boldsymbol{u} = \boldsymbol{f}, & \mathrm{div}\,\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, & \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} & \text{on } \Gamma. \end{cases}$$

Thanks to Theorem 3.1.8, we know that Λ is a bounded linear operator from $\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ into itself. Moreover, thanks to Lemma 2.2.1 and the compact embedding of $\boldsymbol{H}^1(\Omega)$ in $\boldsymbol{L}^2(\Omega)$, the canonical embedding $\mathbf{D}(A_2) \hookrightarrow \boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ is compact. Equivalently, the operator Λ is compact from $\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ into itself. Moreover we can easily verify that this operator is also a self adjoint operator. Thus $\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ has a Hilbertian basis formed from the eigenvectors of the operator Λ. Then, there exists a sequence of real numbers $(\mu_k)_{k\geqslant 0}$ and eigenfunctions $(\boldsymbol{z}_k)_{k\geqslant 0}$ such that

$\Lambda \boldsymbol{z}_k = \mu_k \boldsymbol{z}_k$ and $\mu_k \longrightarrow 0$ as $k \to +\infty$. This means that $-\mu_k \Delta \boldsymbol{z}_k + \mu_k \boldsymbol{z}_k = \boldsymbol{z}_k$. Note that $0 < \mu_k \leq 1$. As a result $A_2 \boldsymbol{z}_k = \lambda_k \boldsymbol{z}_k$, where $\lambda_k = \frac{1}{\mu_k} - 1$ and $\lambda_k \longrightarrow +\infty$ as $k \to +\infty$. In conclusion $(\boldsymbol{z}_k)_k$ is a sequence of eigenfunctions of the Stokes operator associated to the eigenvalues $(\lambda_k)_k$. $\square$

Remark 3.1.14. As a consequence of Proposition 3.1.13, $\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ can be written in the form

$$\boldsymbol{L}^2_{\sigma,\tau}(\Omega) = \mathbf{Ker} A_2 \bigoplus_{k=1}^{+\infty} \mathbf{Ker}(\lambda_k I - A_2).$$

In other words, any vector $\boldsymbol{v} \in \boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ can be written in the form

$$\boldsymbol{v} = \sum_{k=1}^{J} \alpha_k \widetilde{\mathbf{grad}}\, q_k^\tau + \sum_{k=1}^{+\infty} \beta_k \boldsymbol{z}_k,$$

where $(\widetilde{\mathbf{grad}}\, q_k^\tau)_{1\leq k\leq J}$ is a basis for $\mathbf{ker}\, A_2 = \boldsymbol{K}^2_\tau(\Omega)$ and $\forall\, k \in \mathbb{N}$, $\boldsymbol{z}_k \in \mathbf{ker}\,(\lambda_k I - A_2)$. We recall that J is the dimension of $\mathbf{ker}\, A_2 = \boldsymbol{K}^2_\tau(\Omega)$, (see [5]).

As described above, when Ω is simply-connected, $\boldsymbol{K}^2_\tau(\Omega) = \{\mathbf{0}\}$, $\lambda_0 = 0$ is not an eigenvalue and the Stokes operator is bijective from $\mathbf{D}(A_2)$ into $\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ with bounded and compact inverse. In this case,

$$\boldsymbol{L}^2_{\sigma,\tau}(\Omega) = \bigoplus_{k=1}^{+\infty} \mathbf{Ker}(\lambda_k I - A_2),$$

where $(\lambda_k)_{k\geq 1}$ are the eigenvalues of the Stokes operator and $(\boldsymbol{z}_{\boldsymbol{k}})_k$ are the eigenfunctions associated to eigenvalues $(\lambda_k)_{k\geq 1}$. Moreover, the sequence $(\lambda_k)_{k\geq 1}$ is an increasing sequence of positive real numbers and the first eigenvalue λ_1 is equal to $\frac{1}{C_2(\Omega)}$ where $C_2(\Omega)$ is the constant that comes from the Poincaré-type inequality (3.1.23).

L^p-theory

We have seen that the Hilbert case can be obtained easily using Lax-Milgram Lemma. However the general case $p \neq 2$ is not as easy as the particular case $p = 2$ and demand extra work. In this section we extend Theorem 3.1.8 to every $1 < p < \infty$. We start by the existence theorem:

Theorem 3.1.15. *Let $\lambda \in \mathbb{C} \in \Sigma_\varepsilon$ and let $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. The Problem (3.1.11) has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$. Moreover, if Ω is of class $C^{2,1}$ then $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$.*

Proof. As in the proof of Theorem 3.1.8, we can easily verify that Problem (3.1.11) is equivalent to the variational problem: Find $\boldsymbol{u} \in \boldsymbol{V}^p_\tau(\Omega)$ such that for all $\boldsymbol{v} \in \boldsymbol{X}^{p'}_\tau(\Omega)$

$$\lambda \int_\Omega \boldsymbol{u} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x + \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}\, x,$$

where $\boldsymbol{V}^p_\tau(\Omega)$ is given by (3.1.2). The proof is done in three steps:
(i) Case $2 \leq p \leq 6$. Let $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ be the unique solution of Problem (3.1.11). We write Problem (3.1.11) in the form:

$$\begin{cases} -\Delta \boldsymbol{u} = \boldsymbol{F}, & \operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, & \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \qquad \text{on } \Gamma, \end{cases} \tag{3.1.24}$$

where $\boldsymbol{F} = \boldsymbol{f} - \lambda\,\boldsymbol{u}$. Thans to the embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$ one has $\boldsymbol{F} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$.

Thanks to [10, Proposition 4.3] we know that Problem (3.1.24) has a solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ when $\boldsymbol{F}$ satisfies the compatibility condition

$$\forall\, \boldsymbol{v} \in \boldsymbol{K}^{p'}_\tau(\Omega), \qquad \int_\Omega \boldsymbol{F}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x = 0, \tag{3.1.25}$$

where

$$\boldsymbol{K}^{p'}_\tau(\Omega) \;=\; \big\{\boldsymbol{v} \in \boldsymbol{X}^{p'}_\tau(\Omega);\ \operatorname{div}\boldsymbol{v} = 0,\ \mathbf{curl}\,\boldsymbol{v} = \mathbf{0} \text{ in } \Omega\big\}.$$

To this end let $\boldsymbol{v} \in \boldsymbol{K}^{p'}_\tau(\Omega)$, thanks to Lemma 2.2.3 one has:

$$\int_\Omega \boldsymbol{F}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x \;=\; -\int_\Omega \Delta\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x \;=\; \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}\,x - \langle \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}\,,\,\boldsymbol{v}\rangle_\Gamma \;=\; 0.$$

Now applying [10, Proposition 4.3], our solution $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$.

(ii) Case $p \geq 6$. Since $\boldsymbol{f} \in \boldsymbol{L}^6(\Omega)$, Problem (3.1.11) has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,6}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$. Now proceeding in the same way as above one gets that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$.

(iii) Case $p \leq 2$. As described above, for $p \geq 2$ the operator $\lambda\,I + A_p$ is an isomorphism from $\boldsymbol{V}^p_\tau(\Omega)$ to $(\boldsymbol{V}^{p'}_\tau(\Omega))'$. Then the adjoint operator which is equal to $\lambda\,I + A_p$ is an isomorphism from $\boldsymbol{V}^{p'}_\tau(\Omega)$ to $(\boldsymbol{V}^p_\tau(\Omega))'$ for $p' \leq 2$. This means that, the operator $\lambda\,I + A_p$ is an isomorphism for $p \leq 2$, which ends the proof. Notice that the operator $\lambda\,I + A_p \in \mathcal{L}(\boldsymbol{V}^p_\tau(\Omega), (\boldsymbol{V}^{p'}_\tau(\Omega))')$ is defined by: for all $\boldsymbol{\varphi} \in \boldsymbol{V}^p_\tau(\Omega)$, for all $\boldsymbol{\xi} \in \boldsymbol{V}^{p'}_\tau(\Omega)$

$$\langle(\lambda\,I + A_p)\boldsymbol{\varphi}\,,\,\boldsymbol{\xi}\rangle_{(\boldsymbol{V}^{p'}_\tau(\Omega))'\times\boldsymbol{V}^{p'}_\tau(\Omega)} \;=\; \lambda\int_\Omega \boldsymbol{\varphi}\cdot\overline{\boldsymbol{\xi}}\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{curl}\,\overline{\boldsymbol{\xi}}\,\mathrm{d}x.$$

□

Now, we want to prove a resolvent estimate similar to the estimate (3.1.13) for all $1 < p < \infty$. But this case is not as obvious as the case $p = 2$ and the proof will be done in several steps.

Proposition 3.1.16. *Let $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re}\lambda \geq 0$ and $|\lambda| \geq \lambda_0$, where $\lambda_0 = \lambda_0(\Omega, p)$ is defined in (3.1.35). Moreover, let $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, where $1 < p < \infty$ and let $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ be the unique solution of Problem (3.1.11). Then $\boldsymbol{u}$ satisfies the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \;\leq\; \frac{\kappa_1(\Omega,\,p)}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{3.1.26}$$

where the constant $\kappa_1(\Omega,\,p)$ is independent of λ and $\boldsymbol{f}$. Moreover, for $\frac{4}{3} \leq p \leq 4$ the constant κ_1 is independent of Ω and p.

Proof. Suppose that $p \geq 2$, multiplying the first equation of Problem (3.1.11) by $|\boldsymbol{u}|^{p-2}\,\overline{\boldsymbol{u}}$ and integrating both sides one gets thanks to Lemma 2.2.10

$$\begin{aligned}\lambda \int_\Omega |\boldsymbol{u}|^p \,\mathrm{d}\,x + \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2 \,\mathrm{d}\,x + 4\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\,x \\ + (p-2)\,i\sum_{k=1}^{3}\int_\Omega |\boldsymbol{u}|^{p-4}\,\mathrm{Re}\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\mathrm{Im}\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\,\mathrm{d}\,x \\ = \int_\Gamma |\boldsymbol{u}|^{p-2}\Big(\frac{\partial\,\boldsymbol{u}}{\partial\,\boldsymbol{n}}\Big)_{\boldsymbol{\tau}}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\,\sigma + \int_\Omega |\boldsymbol{u}|^{p-2}\,\boldsymbol{f}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\,x. \end{aligned} \tag{3.1.27}$$

Notice that the integral on Γ is well defined. In fact, thanks to Lemma 2.2.11 and to the boundary conditions satisfied by $\boldsymbol{u}$ we know that $\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}$ belongs to $\boldsymbol{W}^{-1/p,p}(\Gamma)$ and satisfies formulas (2.2.11). As a result, $\Big(\frac{\partial\,\boldsymbol{u}}{\partial\,\boldsymbol{n}}\Big)_{\boldsymbol{\tau}} = -\sum_{j=1}^{2}\Big(\frac{\partial\,\boldsymbol{n}}{\partial\,\boldsymbol{s}_j}\cdot\boldsymbol{u}_{\boldsymbol{\tau}}\Big)\boldsymbol{\tau}_j$. Moreover, since Ω is of class $C^{1,1}$ then $\boldsymbol{n}\in\boldsymbol{W}^{1,\infty}(\Gamma)$ and since $\boldsymbol{u}_{\boldsymbol{\tau}}$ belongs to $\boldsymbol{W}^{1-1/p,p}(\Gamma)\hookrightarrow\boldsymbol{L}^p(\Gamma)$. As a result $\Big(\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}}\Big)_{\boldsymbol{\tau}}$ belongs to $\boldsymbol{L}^p(\Gamma)$. In addition, it is clear that $|\boldsymbol{u}|^{p-2}\overline{\boldsymbol{u}}\in\boldsymbol{W}^{1,p'}(\Omega)$ and then its trace belongs to $\boldsymbol{W}^{1-1/p',p'}(\Gamma)\hookrightarrow\boldsymbol{L}^{p'}(\Gamma)$. Which justify the integral on Γ.

Now observe that

$$\begin{aligned}\Big(\frac{\partial\,\boldsymbol{u}}{\partial\,\boldsymbol{n}}\Big)_{\boldsymbol{\tau}}\cdot\overline{\boldsymbol{u}}_{\boldsymbol{\tau}} &= -\sum_{j=1}^{2}\Big(\frac{\partial\,\boldsymbol{n}}{\partial s_j}\cdot\boldsymbol{u}_{\boldsymbol{\tau}}\Big)\boldsymbol{\tau}_j\cdot\sum_{k=1}^{2}\overline{u}_k\,\boldsymbol{\tau}_k \\ &= \boldsymbol{n}\cdot\sum_{j,k=1}^{2}\overline{u}_j\,u_k\,\frac{\partial\,\boldsymbol{\tau}_k}{\partial s_j}. \end{aligned} \tag{3.1.28}$$

Next we put together the two formulas (3.1.27) and (3.1.28), we study separately the real and the imaginary parts of formula (3.1.27) and using the fact that Ω is of class $C^{1,1}$ one gets

$$\begin{aligned}\mathrm{Re}\,\lambda\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2\,\mathrm{d}\,x + 4\,\frac{p-2}{p^2}\int_\Omega|\nabla\,|\boldsymbol{u}|^{p/2}|^2\,\mathrm{d}\,x \\ \leq C_1(\Omega)\int_\Gamma|\boldsymbol{u}|^p\,\mathrm{d}\,\sigma + \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)} \end{aligned} \tag{3.1.29}$$

and

$$\begin{aligned}|\mathrm{Im}\,\lambda|\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} \leq \frac{p-2}{2}\int_\Omega|\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2\,\mathrm{d}\,x + C_1(\Omega)\int_\Gamma|\boldsymbol{u}|^p\,\mathrm{d}\,\sigma + \\ + \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}, \end{aligned} \tag{3.1.30}$$

for some constant $C_1(\Omega) > 0$. Now putting together (3.1.29) and (3.1.30) one has

$$\begin{aligned}|\lambda|\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega|\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2\,\mathrm{d}\,x + 4\,\frac{p-2}{p^2}\int_\Omega|\nabla\,|\boldsymbol{u}|^{p/2}|^2\,\mathrm{d}\,x \\ \leq \frac{p-2}{2}\int_\Omega|\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2\,\mathrm{d}\,x + 2\,C_1(\Omega)\int_\Gamma|\boldsymbol{u}|^p\,\mathrm{d}\,\sigma + 2\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}. \end{aligned} \tag{3.1.31}$$

Moreover, thanks to [44, Chapter 1, Theorem 1.5.1.10, page 41] we know that:

$$\int_\Gamma |w|^2 \,\mathrm{d}\,\sigma \,\le\, \varepsilon \int_\Omega |\nabla\, w|^2 \,\mathrm{d}\,x \,+\, C_\varepsilon \int_\Omega |w|^2 \,\mathrm{d}\,x, \tag{3.1.32}$$

for all $w \in H^1(\Omega)$ and for all $\varepsilon \in\,]0,1[$. Applying formula (3.1.32) to $w = |\boldsymbol{u}|^{p/2}$ and substituting in (3.1.31) one gets

$$\begin{aligned} &|\lambda|\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} \,+\, \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2 \,\mathrm{d}\,x \,+\, 4\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\,x \\ &\quad\le\, \frac{p-2}{2}\int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2 \,\mathrm{d}\,x \,+\, 2\,C_1(\Omega)\Big[\,\varepsilon\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\,x \,+\, C_\varepsilon\int_\Omega |\boldsymbol{u}|^p \,\mathrm{d}\,x\Big] \\ &\quad+\, 2\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}. \end{aligned} \tag{3.1.33}$$

We chose $\varepsilon > 0$ such that $\varepsilon\, C_1(\Omega) = \frac{p-2}{p^2}$. As a result the constant C_ε in (3.1.33) depends on p and Ω. Then by setting $C_\varepsilon = C_2(\Omega, p)$ one has

$$\begin{aligned} &|\lambda|\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2 \,\mathrm{d}\,x \,+\, 2\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\,x \\ &\qquad\le\, C_3(\Omega,p)\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} \,+\, \frac{p-2}{2}\int_\Omega |\boldsymbol{u}|^{p-2}|\nabla\boldsymbol{u}|^2 \,\mathrm{d}\,x \,+\, 2\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}, \end{aligned}$$

where

$$C_3(\Omega, p) = 2\,C_1(\Omega)C_2(\Omega, p). \tag{3.1.34}$$

We define

$$\lambda_0 = 2\,C_3(\Omega, p). \tag{3.1.35}$$

Now, for $|\lambda| \ge \lambda_0$ one has

$$\begin{aligned} &\frac{|\lambda|}{2}\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2 \,\mathrm{d}\,x \,+\, 2\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\,x \\ &\qquad\le\, \frac{p-2}{2}\int_\Omega |\boldsymbol{u}|^{p-2}|\nabla\boldsymbol{u}|^2 \,\mathrm{d}\,x \,+\, 2\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}. \end{aligned}$$

In fact we have two different cases.

(i) Case $2 \le p \le 4$. One has

$$\begin{aligned} &\frac{|\lambda|}{2}\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \frac{4-p}{2}\int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2 \,\mathrm{d}\,x \,+\, 2\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\,x \le \\ &\qquad 2\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}. \end{aligned}$$

Thus

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \,\le\, \frac{4}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{3.1.36}$$

which is the required estimate.

(ii) Case $p > 4$. We write Problem (3.1.11) in the form (3.1.24). Thanks to [10, Proposition 4.3] we have

$$\big\|\boldsymbol{u} - \sum_{j=1}^{J} \langle \boldsymbol{u}\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j} \widetilde{\mathbf{grad}}\, q_j^\tau \big\|_{\boldsymbol{W}^{1,4}(\Omega)} \leq C_4(\Omega)\, \|\boldsymbol{f} - \lambda\, \boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)}.$$

Thus

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,4}(\Omega)} \leq \big\| \sum_{j=1}^{J} \langle \boldsymbol{u}\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j} \widetilde{\mathbf{grad}}\, q_j^\tau \big\|_{\boldsymbol{W}^{1,4}(\Omega)} + C_4(\Omega)\, \|\boldsymbol{f}\|_{\boldsymbol{L}^4(\Omega)} + \\ + C_4(\Omega)\, |\lambda|\, \|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)}. \quad (3.1.37)$$

On the other hand, thanks to Lemma 2.2.8 and (3.1.36) we have

$$|\langle \boldsymbol{u}\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j}| \leq C_5(\Omega)\, \|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)} \leq \frac{C_5(\Omega)}{|\lambda|}\, \|\boldsymbol{f}\|_{\boldsymbol{L}^4(\Omega)} \leq \frac{C_5(\Omega)}{\lambda_0}\, \|\boldsymbol{f}\|_{\boldsymbol{L}^4(\Omega)}.$$

As a result, using (3.1.36) with $p = 4$ and substituting in (3.1.37) one gets

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,4}(\Omega)} \leq C_7(\Omega)\, \|\boldsymbol{f}\|_{\boldsymbol{L}^4(\Omega)},$$

where $C_7(\Omega) = C_6(\Omega)\, \frac{C_5(\Omega)}{\lambda_0} + 5\, C_4(\Omega)$ and $\|\widetilde{\mathbf{grad}}\, q_j^\tau\|_{\boldsymbol{W}^{1,4}(\Omega)} \leq C_6(\Omega)$.

Now since $\boldsymbol{W}^{1,4}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$, then

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(\Omega)} &\leq C_8(\Omega)\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,4}(\Omega)} \leq C_8(\Omega)\, C_7(\Omega)\|\boldsymbol{f}\|_{\boldsymbol{L}^4(\Omega)} \\ &\leq C_8(\Omega)\, C_7(\Omega)(\mathrm{mes}\,\Omega)^{(p-4)/4p}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \end{aligned}$$

Consequently

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C_9(\Omega)\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \quad (3.1.38)$$

where

$$C_9(\Omega) = C_8(\Omega) C_7(\Omega) (\mathrm{mes}\,\Omega)^{1/4}. \quad (3.1.39)$$

Notice that

$$\begin{aligned} \|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} &= \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)} \\ &\leq C_9(\Omega)\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}. \end{aligned} \quad (3.1.40)$$

Thus proceeding exactly as above and putting together (3.1.29), (3.1.32) and (3.1.40) one has

$$\mathrm{Re}\,\lambda\, \|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega |\boldsymbol{u}|^{p-2}\, |\nabla\, \boldsymbol{u}|^2\, \mathrm{d}\,x + 2\,\frac{p-2}{p^2} \int_\Omega |\nabla\, |\boldsymbol{u}|^{p/2}|^2\, \mathrm{d}\,x \\ \leq (C_3(\Omega, p) C_9(\Omega) + 1)\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}.$$

As a result one has

$$\mathrm{Re}\,\lambda\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C_{10}(\Omega, p)\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \quad (3.1.41)$$

$$\int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla \boldsymbol{u}|^2 \,\mathrm{d}\,x \;\leq\; C_{10}(\Omega, p)\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}, \tag{3.1.42}$$

and

$$2\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\,x \;\leq\; C_{10}(\Omega, p)\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}, \tag{3.1.43}$$

where

$$C_{10}(\Omega, p) = 1 \,+\, C_3(\Omega, p)C_9(\Omega). \tag{3.1.44}$$

In addition, using (3.1.30), (3.1.42) and (3.1.43) one has

$$|\mathrm{Im}\,\lambda|\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \;\leq\; C_{11}(\Omega, p)\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.1.45}$$

Thus putting together (3.1.41) and (3.1.45) one gets for $p > 4$

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \;\leq\; \frac{C_{12}(\Omega, p)}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{3.1.46}$$

which ends the case $p > 4$.

Finally putting together (3.1.36) and (3.1.46) we conclude that for $p \geq 2$ we have

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \;\leq\; \frac{\kappa_1(\Omega, p)}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{3.1.47}$$

with

$$\kappa_1(\Omega, p) = \max(4, C_{12}(\Omega, p)). \tag{3.1.48}$$

By duality we obtain estimate (3.1.47) for all $1 < p < \infty$. □

Proposition 3.1.17. *Let $\lambda \in \mathbb{C}^*$ such that $\mathrm{Re}\,\lambda \geq 0$ and $0 < |\lambda| \leq \lambda_0$, with λ_0 as in Proposition 3.1.16. Moreover, let $1 < p < \infty$, $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and let $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ be the unique solution of Problem 3.1.11. Then $\boldsymbol{u}$ satisfies the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \;\leq\; \frac{\kappa_2(\Omega, p)}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.1.49}$$

For some constant $\kappa_2(\Omega, p)$ independent of λ and $\boldsymbol{f}$.

Proof. Thanks to (3.1.13) with $\varepsilon = \frac{\pi}{2}$ we have

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \;\leq\; \frac{C_{13}}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}$$

and

$$\|\mathbf{curl}\,\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \;\leq\; \frac{C_{13}^2}{|\lambda|}\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)}.$$

Moreover we know that

$$\begin{aligned}\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} &\leq C_{14}(\Omega)(\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} \,+\, \|\mathbf{curl}\,\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)})\\ &\leq C_{14}(\Omega)C_{13}^2\frac{1+|\lambda|}{|\lambda|^2}\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)}.\end{aligned}$$

Now because $|\lambda| \leq \lambda_0$ we deduce that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq \frac{C_{15}(\Omega)}{|\lambda|} \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)},$$

where

$$C_{15}(\Omega) = C_{13}\sqrt{C_{14}(\Omega)(1+\lambda_0)}. \tag{3.1.50}$$

In fact we have two different cases.

(i) Case $2 \leq p \leq 6$. Because $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$ we have

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} &\leq C_{16}(\Omega,p)\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \\ &\leq \frac{C_{16}(\Omega,p)C_{15}(\Omega)}{|\lambda|}\|\boldsymbol{f}\|_{L^2(\Omega)} \leq \frac{C_{17}(\Omega,p)}{|\lambda|}\|\boldsymbol{f}\|_{L^p(\Omega)}, \end{aligned} \tag{3.1.51}$$

where

$$C_{17}(\Omega,p) = (\operatorname{mes}\Omega)^{(p-2)/2p}\, C_{15}(\Omega,p)C_{16}(\Omega). \tag{3.1.52}$$

(ii) Case $p \geq 6$. Proceeding in a similar way as in Proposition 3.1.16 (case $p > 4$), we obtain

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C_{18}(\Omega,p)}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.1.53}$$

Finally putting together (3.1.51) and (3.1.53), we deduce the estimate (3.1.49) with

$$\kappa_2(\Omega,p) = \max(C_{17}(\Omega,p)\,, C_{18}(\Omega,p)). \tag{3.1.54}$$

□

As a conclusion of Propositions 3.1.16 and 3.1.17 we have the following theorem:

Theorem 3.1.18. *Let $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re}\lambda \geq 0$, let $1 < p < \infty$, $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and let $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ be the unique solution of Problem (3.1.11). Then $\boldsymbol{u}$ satisfies the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{\kappa_3(\Omega,p)}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{3.1.55}$$

where $\kappa_3(\Omega,p) = \max(\kappa_1(\Omega,p), \kappa_2(\Omega,p))$.

In addition, if Ω is of class $C^{2,1}$ we have the following estimate

$$\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{\kappa_4(\Omega,p)}{\sqrt{|\lambda|}}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \tag{3.1.56}$$

and

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq \kappa_5(\Omega,p)\,\frac{1+|\lambda|}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.1.57}$$

Proof. The proof of estimate (3.1.55) is a conclusion of Propositions 3.1.16 and 3.1.17. Let us prove estimate (3.1.56). The proof is done in two steps.
(i) Case $\langle \boldsymbol{u}\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j} = 0$, $1 \leq j \leq J$. Thanks to [10, Proposition 4.7] we know that $\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \simeq \|\Delta\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$. Now, using the Gagliardo-Nirenberg inequality (see [2, Chapter IV, Theorem 4.14, Theorem 4.17] for instance) we have

$$\begin{aligned}\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} &\leq C(\Omega,p)\,\|\Delta\boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}\\ &= C(\Omega,p)\,\|\boldsymbol{f}-\lambda\,\boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}\\ &\leq \frac{C(\Omega,p)}{\sqrt{|\lambda|}}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.\end{aligned}$$

(ii) General case. Let $\boldsymbol{u}\in\mathbf{D}(A_p)$ be the unique solution of Problem (3.1.11) and set

$$\widetilde{\boldsymbol{u}} = \boldsymbol{u} - \sum_{j=1}^{J}\langle \boldsymbol{u}\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j}\widetilde{\mathbf{grad}}\,q_j^{\tau}.$$

As a result, thanks to the previous case we have

$$\|\mathbf{curl}\,\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\,\|\Delta\widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}\|\widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}.$$

Thus

$$\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} = \|\mathbf{curl}\,\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^p(\Omega)} \leq \|\Delta\widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}\|\widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)} = \|\Delta\boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}\|\widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}.$$

Moreover, thanks to Lemma 2.2.8 we know that

$$\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\,\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

As a consequence we deduce estimate (3.1.56).

Finally, when Ω is of class $C^{2,1}$, on $\mathbf{D}(A_p)$ the norm of $\boldsymbol{W}^{2,p}(\Omega)$ is equivalent to the graph norm of the Stokes operator with Navier-type boundary conditions (3.1.1). As a result one has estimate (3.1.57). □

As in the Hilbertian case, Proposition 3.1.3 and Theorems 3.1.15 allow us to deduce the analyticity of the semi-group generated by the Stokes operator with Navier-type boundary conditions on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$.

Theorem 3.1.19. *The operator $-A_p$ generates a bounded analytic semigroup on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ for all $1 < p < \infty$.*

Proof. The proof is a direct application of Proposition 2.3.3 with $w = 0$. In fact, thanks to Proposition 3.1.3 and Theorems 3.1.15 and 3.1.18 the operator $-A_p$ satisfies the assumptions of Proposition 2.3.3. This justify the analyticity of the semi-group generated by the operator $-A_p$ on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ for all $1 < p \leq \infty$. □

Remark 3.1.20. Notice that, unlike the Hilbertian case, we can not use the result of [29, Chapter II, Theorem 4.6, page 101] to prove the analyticity of the semi-group generated by the Stokes operator in the $\boldsymbol{L}^p$-space where we have supposed that $\mathrm{Re}\,\lambda \geq 0$.

3.1.2 Analyticity on $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$

This subsection is devoted to the analyticity of the semi-group generated by the Stokes operator with Navier-type boundary conditions (3.1.1) on $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$ (given by (3.1.64)). This analyticity allows us to obtain the weak solution to the time dependent problem.

First consider the space:

$$\boldsymbol{E} = \{\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]';\ \mathrm{div}\,\boldsymbol{f} \in L^p(\Omega)\},$$

which is a Banach space with the norm

$$\|\boldsymbol{f}\|_{\boldsymbol{E}} = \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} + \|\mathrm{div}\,\boldsymbol{f}\|_{L^p(\Omega)}. \tag{3.1.58}$$

Lemma 3.1.21. *The space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{E}$.*

Proof. Let $\boldsymbol{\ell} \in \boldsymbol{E}'$ such that $\langle \boldsymbol{\ell}\,,\,\boldsymbol{v}\rangle_{\boldsymbol{E}'\times\boldsymbol{E}} = 0$ for all $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$ and let us show that $\boldsymbol{\ell}$ is null in $\boldsymbol{E}$. We know that there exists a function $\boldsymbol{u}$ in $\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)$ and a function χ in $L^{p'}(\Omega)$ such that for all $\boldsymbol{f}$ in $\boldsymbol{E}$ one has:

$$\langle \boldsymbol{\ell}\,,\,\boldsymbol{f}\rangle_{\boldsymbol{E}'\times\boldsymbol{E}} \;=\; \langle \boldsymbol{f}\,,\,\boldsymbol{u}\rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'\times\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)} + \int_\Omega \mathrm{div}\,\boldsymbol{f}\,\overline{\chi}\,\mathrm{d}\,x. \tag{3.1.59}$$

We denote by $\widetilde{\boldsymbol{u}}$ and $\widetilde{\chi}$ the extension of $\boldsymbol{u}$ and χ by zero to $\mathbb{R}^3$. As a result for every $\boldsymbol{f} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3)$ one has

$$\langle \boldsymbol{f}\,,\,\widetilde{\boldsymbol{u}}\rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\mathbb{R}^3)]'\times\boldsymbol{H}_0^{p'}(\mathrm{div},\mathbb{R}^3)} + \int_{\mathbb{R}^3} \mathrm{div}\,\boldsymbol{f}\,\overline{\widetilde{\chi}}\mathrm{d}\,x \;=\; 0.$$

Then $\widetilde{\boldsymbol{u}} = \nabla\,\widetilde{\chi}$ and $\boldsymbol{u} = \nabla\,\chi$. This means that $\widetilde{\chi} \in L^{p'}(\mathbb{R}^3)$ and $\nabla\,\widetilde{\chi} \in \boldsymbol{H}_0^{p'}(\mathrm{div},\,\mathbb{R}^3)$. Then $\widetilde{\chi} \in W^{2,p'}(\mathbb{R}^3)$ and $\chi \in W_0^{2,p'}(\Omega)$. Now since $\mathcal{D}(\Omega)$ dense in $W_0^{2,p'}(\Omega)$ there exists a sequence $(\chi_k)_k$ in $\mathcal{D}(\Omega)$ that converges to χ in $W^{2,p'}(\Omega)$. Finally for all $\boldsymbol{f} \in \boldsymbol{E}$ one has:

$$\begin{aligned}
\langle \boldsymbol{\ell}\,,\,\boldsymbol{f}\rangle_{\boldsymbol{E}'\times\boldsymbol{E}} &= \langle \boldsymbol{f}\,,\,\boldsymbol{u}\rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'\times\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)} + \int_\Omega \mathrm{div}\,\boldsymbol{f}\,\overline{\chi}\,\mathrm{d}\,x.\\
&= \lim_{k\to+\infty}\Big[\langle \boldsymbol{f}\,,\,\nabla\,\chi_k\rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'\times\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)} + \int_\Omega \mathrm{div}\,\boldsymbol{f}\,\overline{\chi_k}\,\mathrm{d}\,x.\Big]\\
&= 0.
\end{aligned}$$

□

The following corollary gives us the normal trace of a function $\boldsymbol{f}$ in $\boldsymbol{E}$.

Corollary 3.1.22. *The linear mapping $\gamma\,:\,\boldsymbol{f} \longmapsto \boldsymbol{f}\cdot\boldsymbol{n}$ defined on $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended to a linear continuous mapping still denoted by $\gamma\,:\,\boldsymbol{E} \longmapsto W^{-1-1/p,p}(\Gamma)$. Moreover we have the following Green formula: for all $\boldsymbol{f} \in \boldsymbol{E}$ and for all $\chi \in W^{2,p'}(\Omega)$ such that $\frac{\partial\chi}{\partial\boldsymbol{n}} = 0$ on Γ,*

$$\int_\Omega (\mathrm{div}\,\boldsymbol{f})\,\overline{\chi}\,d\,x \;=\; -\,\langle \boldsymbol{f}\,,\,\nabla\,\chi\rangle_\Omega + \langle \boldsymbol{f}\cdot\boldsymbol{n}\,,\,\chi\rangle_\Gamma, \tag{3.1.60}$$

where $\langle\cdot\,,\,\cdot\rangle_\Omega = \langle\cdot\,,\,\cdot\rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'\times\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}$ and $\langle\cdot\,,\,\cdot\rangle_\Gamma \;=\; \langle\cdot\,,\,\cdot\rangle_{W^{-1-1/p,p}(\Gamma)\times W^{1+1/p,p'}(\Gamma)}$.

Proof. Let $\boldsymbol{f} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$ and let $\chi \in W^{2,p'}(\Omega)$ such that $\frac{\partial \chi}{\partial \boldsymbol{n}} = 0$ on Γ, then the Green Formula (3.1.60) obviously holds. Moreover we can easily verify that

$$|\langle \boldsymbol{f} \cdot \boldsymbol{n}\,,\, \chi \rangle_\Gamma| \leq C\, \|\boldsymbol{f}\|_{\boldsymbol{E}} \|\chi\|_{W^{2,p'}(\Omega)}.$$

On the other hand, for every $\mu \in W^{1+1/p,p'}(\Gamma)$, there exists a function $\chi \in W^{2,p'}(\Omega)$ such that $\chi = \mu$ on Γ and $\frac{\partial \chi}{\partial \boldsymbol{n}} = 0$ on Γ with the estimate

$$\|\chi\|_{W^{2,p'}(\Omega)} \leq C\, \|\mu\|_{W^{1+1/p,p'}(\Gamma)}.$$

As a result,

$$|\langle \boldsymbol{f} \cdot \boldsymbol{n}\,,\, \mu \rangle_\Gamma| = |\langle \boldsymbol{f} \cdot \boldsymbol{n}\,,\, \chi \rangle_\Gamma| \leq C\, \|\boldsymbol{f}\|_{\boldsymbol{E}} \|\chi\|_{W^{2,p'}(\Omega)} \leq C\, \|\boldsymbol{f}\|_{\boldsymbol{E}} \|\mu\|_{W^{1+1/p,p'}(\Gamma)}$$

and

$$\|\boldsymbol{f} \cdot \boldsymbol{n}\|_{W^{-1-1/p,p}(\Gamma)} \leq C\, \|\boldsymbol{f}\|_{\boldsymbol{E}}.$$

Thus the linear mapping $\gamma : \boldsymbol{\mathcal{D}}(\overline{\Omega}) \mapsto W^{-1-1/p,p}(\Gamma)$ is continuous for the norm of $\boldsymbol{E}$. Since $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{E}$, γ can be extended by continuity to a linear continuous mapping from $\boldsymbol{E}$ to $W^{-1-1/p,p}(\Gamma)$ and the Green Formula (3.1.60) holds for all $\boldsymbol{f} \in \boldsymbol{E}$ and for all $\chi \in W^{2,p'}(\Omega)$ such that $\frac{\partial \chi}{\partial \boldsymbol{n}} = 0$ on Γ. $\square$

Now consider the problem:

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = 0 \qquad \text{in} \quad \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \qquad \text{on} \quad \Gamma, \end{cases} \tag{3.1.61}$$

where $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re} \lambda \geq 0$ and $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\operatorname{div}, \Omega)]'$. The following theorem gives us the existence and uniqueness of solution to Problem (3.1.61):

Theorem 3.1.23. *Let $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re} \lambda \geq 0$ and let $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\operatorname{div}, \Omega)]'$. The Problem (3.1.61) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ satisfying*

$$\|\boldsymbol{u}\|_{[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'} \leq \frac{C(\Omega, p)}{|\lambda|}\, \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\operatorname{div},\Omega)]'} \tag{3.1.62}$$

for some constant $C(\Omega, p) > 0$ independent of λ and $\boldsymbol{f}$.

Proof. **(i)** For the existence of solutions for Problem (3.1.61) we proceed in the same way as in [10, Theorem 4.4], Theorem 3.1.8 and Theorem 3.1.15.
(ii) To prove estimate (3.1.62) we proceed as follows: Consider the problem:

$$\begin{cases} \lambda \boldsymbol{v} - \Delta \boldsymbol{v} + \nabla \theta = \boldsymbol{F}, & \operatorname{div} \boldsymbol{v} = 0 \qquad \text{in} \quad \Omega, \\ \boldsymbol{v} \cdot \boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \qquad \text{on} \quad \Gamma, \end{cases} \tag{3.1.63}$$

where $\boldsymbol{F} \in \boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)$ and $\lambda \in \mathbb{C}^*$ such that $\mathrm{Re}\,\lambda \geq 0$. Thanks to Lemma 2.2.9 there exists a unique up to an additive function $\theta \in W^{1,p'}(\Omega)/\mathbb{R}$ solution of

$$\mathrm{div}(\nabla\theta - \boldsymbol{F}) = 0 \qquad \text{in } \Omega \qquad (\nabla\theta - \boldsymbol{F}) \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma.$$

Moreover the function θ satisfies the estimate

$$\|\nabla\theta\|_{\boldsymbol{L}^{p'}(\Omega)} \leq C(\Omega, p')\, \|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

As a result, thanks to Theorem 3.1.15 and Theorem 3.1.18, Problem (3.1.63) has a unique solution $(\boldsymbol{v}, \theta) \in \boldsymbol{W}^{1,p'}(\Omega) \times W^{1,p'}(\Omega)/\mathbb{R}$ that satisfies the estimate

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)} \leq \frac{C(\Omega, p')}{|\lambda|}\, \|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Thus

$$\|\boldsymbol{v}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)} \leq \frac{C(\Omega, p')}{|\lambda|}\, \|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}.$$

Now let $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{L}^p(\Omega)/\mathbb{R}$ be the solution of Problem (3.1.61), then by using (3.1.60) we have:

$$\begin{aligned}
\|\boldsymbol{u}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} &= \sup_{\boldsymbol{F}\in \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega), \boldsymbol{F}\neq 0} \frac{|\langle \boldsymbol{u}\,,\, \boldsymbol{F}\rangle_\Omega|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}} \\
&= \sup_{\boldsymbol{F}\in \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega), \boldsymbol{F}\neq 0} \frac{|\langle \boldsymbol{u}\,,\, \lambda\,\boldsymbol{v} - \Delta\boldsymbol{v} + \nabla\theta\rangle_\Omega|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}} \\
&= \sup_{\boldsymbol{F}\in \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega), \boldsymbol{F}\neq 0} \frac{|\langle \lambda\,\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla\pi\,,\, \boldsymbol{v}\rangle_\Omega|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}} \\
&= \sup_{\boldsymbol{F}\in \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega), \boldsymbol{F}\neq 0} \frac{|\langle \boldsymbol{f}\,,\, \boldsymbol{v}\rangle_\Omega|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}} \\
&\leq \frac{C(\Omega, p')}{|\lambda|}\, \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'},
\end{aligned}$$

which is estimate (3.1.62). □

As consequence of Theorem 3.1.23 we have the following corollary

Corollary 3.1.24. *Let $\lambda \in \mathbb{C}^*$ such that $\mathrm{Re}\,\lambda \geq 0$ and let $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'$ such that $\mathrm{div}\,\boldsymbol{f} = 0$ in Ω and $\boldsymbol{f} \cdot \boldsymbol{n} = 0$ on Γ. The Problem (3.1.11) has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ satisfying the estimate (3.1.62).*

Now we consider the space

$$[\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'_{\sigma,\tau} = \big\{\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]';\ \mathrm{div}\,\boldsymbol{f} = 0 \text{ in } \Omega,\ \boldsymbol{f} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\big\}. \tag{3.1.64}$$

We define the operator

$$B_p\,:\, \mathbf{D}(B_p) \subset [\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'_{\sigma,\tau} \longmapsto [\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'_{\sigma,\tau},$$

by

$$\forall\, \boldsymbol{u} \in \mathbf{D}(B_p), \qquad B_p\, \boldsymbol{u} = -\Delta \boldsymbol{u} \qquad \text{in } \Omega. \tag{3.1.65}$$

The domain of B_p is given by

$$\mathbf{D}(B_p) = \big\{ \boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega);\ \Delta \boldsymbol{u} \in [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'\ \mathrm{div}\, \boldsymbol{u} = 0 \text{ in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0,\ \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma \big\}. \tag{3.1.66}$$

Remark 3.1.25. The operator B_p is the extension of the Stokes operator to $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$.

Proposition 3.1.26. *The space $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ is dense in $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$.*

Proof. Let $\boldsymbol{\ell}$ be a linear form on $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$ such that $\boldsymbol{\ell}$ vanishes on $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ and let us show that $\boldsymbol{\ell}$ is null on $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$. Thanks to the Hahn-Banach theorem, $\boldsymbol{\ell}$ can be extended to a linear continuous form on $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'$ denoted by $\widetilde{\boldsymbol{\ell}}$. Moreover

$$\forall\, \boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}, \qquad \boldsymbol{\ell}(\boldsymbol{f}) = \langle \widetilde{\boldsymbol{\ell}}\,,\, \boldsymbol{f} \rangle_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)\times[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}.$$

Since $\boldsymbol{\ell}$ vanishes on $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ then thanks to De-Rham lemma there exists a function $\pi \in W^{2,p'}(\Omega)$ such that $\frac{\partial \pi}{\partial \boldsymbol{n}} = 0$ on Γ and $\widetilde{\boldsymbol{\ell}} = \nabla \pi$ in Ω. Now let $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$ then by Corollary 3.1.22 we have

$$\begin{aligned} \boldsymbol{\ell}(\boldsymbol{f}) &= \langle \boldsymbol{f}\,,\, \nabla\pi \rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'\times \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)} \\ &= -\int_\Omega (\mathrm{div}\, \boldsymbol{f})\, \overline{\pi}\, \mathrm{d}\, x + \langle \boldsymbol{f}\cdot\boldsymbol{n}\,,\, \pi \rangle_\Gamma \\ &= 0. \end{aligned}$$

$\square$

As a result of Proposition 3.1.26 we deduce the density of the domain of the operator B_p.

Corollary 3.1.27. *The operator B_p is a densely defined operator.*

Next, using Proposition 2.3.3 with $w = 0$, one gets the analyticity of the semi-group generated by the operator B_p:

Theorem 3.1.28. *The operator $-B_p$ generates a bounded analytic semi-group on $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$.*

3.1.3 Analyticity on $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$

In this subsection we suppose that Ω is of class $C^{2,1}$. We give the analyticity of the semi-group generated by the Stokes operator with Navier-type boundary conditions on $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$. This gives us a way to get very weak solutions to the time dependent Stokes problem with the Navier-type boundary condition (3.1.1). We proceed in a very similar way to the previous section.

For this reason, we consider the space

$$\boldsymbol{G} = \big\{ \boldsymbol{f} \in (\boldsymbol{T}^{p'}(\Omega))'; \; \operatorname{div} \boldsymbol{f} \in L^p(\Omega) \big\},$$

equipped with the graph norm. We skip the proof of the following lemma because it is similar to the proof of Lemma 3.1.21:

Lemma 3.1.29. *The space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{G}$.*

As in the previous subsection The following corollary gives the normal trace of functions in $\boldsymbol{G}$.

Corollary 3.1.30. *The linear mapping $\gamma \, : \, \boldsymbol{f} \longmapsto \boldsymbol{f} \cdot \boldsymbol{n}$ defined on $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended to a linear continuous mapping still denoted by $\gamma \, : \, \boldsymbol{G} \longmapsto W^{-2-1/p,p}(\Gamma)$. Moreover we have the following Green formula: for all $\boldsymbol{f} \in \boldsymbol{G}$ and for all $\chi \in W^{3,p'}(\Omega)$ such that $\frac{\partial \chi}{\partial \boldsymbol{n}} = 0$ on Γ and $\Delta \chi = 0$ on Γ,*

$$\int_\Omega (\operatorname{div} \boldsymbol{f}) \, \overline{\chi} \, dx \; = \; - \langle \boldsymbol{f} \, , \, \nabla \, \chi \rangle_{(\boldsymbol{T}^{p'}(\Omega))' \times \boldsymbol{T}^{p'}(\Omega)} \; + \; \langle \boldsymbol{f} \cdot \boldsymbol{n} \, , \, \chi \rangle_\Gamma. \tag{3.1.67}$$

We recall that $\langle . \, , \, . \rangle_\Gamma \; = \; \langle . \, , \, . \rangle_{W^{-2-1/p,p}(\Gamma) \times W^{2+1/p,p'}(\Gamma)}$.

The following theorem gives the very weak solution to Problem (3.1.61).

Theorem 3.1.31. *Let $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re} \lambda \geq 0$ and let $\boldsymbol{f} \in (\boldsymbol{T}^{p'}(\Omega))'$ then the Problem (3.1.61) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$. Moreover we have the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \; \leq \; \frac{C(\Omega, p)}{|\lambda|} \, \|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'}, \tag{3.1.68}$$

for some constant $C(\Omega, p) > 0$ independent of λ and $\boldsymbol{f}$.

Proof. **(i)** Thanks to the Green formula (2.2.3) and to [10, Theorem 4.15] we can easily verify that Problem (3.1.61) is equivalent to the problem: Find $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ such that for all $\boldsymbol{\varphi} \in \mathbf{D}(A_{p'})$ (given by (3.1.8)) and for all $q \in W^{1,p'}(\Omega)$

$$\begin{aligned} \lambda \textstyle\int_\Omega \boldsymbol{u} \cdot \overline{\boldsymbol{\varphi}} \, \mathrm{d}\, x \; - \; \int_\Omega \boldsymbol{u} \cdot \Delta \overline{\boldsymbol{\varphi}} \, \mathrm{d}\, x &= \langle \boldsymbol{f}, \, \boldsymbol{\varphi} \rangle_{(\boldsymbol{T}^{p'}(\Omega))' \times \boldsymbol{T}^{p'}(\Omega)} \\ \textstyle\int_\Omega \boldsymbol{u} \cdot \nabla \overline{q} \, \mathrm{d}\, x &= 0. \end{aligned} \tag{3.1.69}$$

Notice that we recuperate the pressure using the De-Rham argument: if $\boldsymbol{F} \in \boldsymbol{W}^{-2,p}(\Omega)$ verifying $\langle \boldsymbol{F} \, , \, \boldsymbol{v} \rangle_{\boldsymbol{\mathcal{D}'}(\Omega) \times \boldsymbol{\mathcal{D}}(\boldsymbol{\Omega})} = 0$, for all $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega)$ then there exists $\chi \in W^{-1,p}(\Omega)$ such that $\boldsymbol{F} = \nabla \, \chi$.

(ii) Let us now solve (3.1.69). As in the proof of Theorem 3.1.23, we know that for all $\boldsymbol{F} \in \boldsymbol{L}^{p'}(\Omega)$ the problem:

$$\begin{cases} \lambda \boldsymbol{\varphi} - \Delta \boldsymbol{\varphi} - \nabla \theta = \boldsymbol{F}, & \operatorname{div} \boldsymbol{\varphi} = 0 \qquad \text{in} \quad \Omega, \\ \boldsymbol{\varphi} \cdot \boldsymbol{n} = 0, & \mathbf{curl} \, \boldsymbol{\varphi} \times \boldsymbol{n} = \boldsymbol{0} \qquad \text{on} \quad \Gamma, \end{cases} \tag{3.1.70}$$

has a unique solution $(\boldsymbol{\varphi}, \theta) \in \mathbf{D}(A_{p'}) \times W^{1,p'}(\Omega)/\mathbb{R}$ that satisfies the estimate

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\Omega)} \leq \frac{C(\Omega, p')}{|\lambda|} \|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Consider now the linear mapping:

$$\begin{aligned} L \ : \ \boldsymbol{L}^{p'}(\Omega) &\longmapsto \mathbb{C} \\ \boldsymbol{F} &\longmapsto \langle \boldsymbol{f} \,,\, \boldsymbol{\varphi} \rangle_{(\boldsymbol{T}^{p'}(\Omega))' \times \boldsymbol{T}^{p'}(\Omega)}, \end{aligned}$$

where $\boldsymbol{\varphi}$ is the unique solution of Problem (3.1.70). We have :

$$|L(\boldsymbol{F})| \ \leq \ \|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'} \|\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\Omega)} \ \leq \ \frac{C(\Omega, p')}{|\lambda|} \|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'} \|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Then there exists a unique $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ such that

$$L(\boldsymbol{F}) \ = \ \int_\Omega \boldsymbol{u} \cdot \overline{\boldsymbol{F}} \,\mathrm{d}\,x \ = \ \langle \boldsymbol{f} \,,\, \boldsymbol{\varphi} \rangle_{(\boldsymbol{T}^{p'}(\Omega))' \times \boldsymbol{T}^{p'}(\Omega)}.$$

Moreover $\boldsymbol{u}$ satisfies the estimate (3.1.68). On other worlds $\boldsymbol{u}$ is the unique solution of Problem (3.1.69). □

As a consequence of Theorem 3.1.31 we deduce the very weak solutions to Problem (3.1.11).

Corollary 3.1.32. *Let $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re} \lambda \geq 0$ and let $\boldsymbol{f} \in (\boldsymbol{T}^{p'}(\Omega))'$ such that $\operatorname{div} \boldsymbol{f} = 0$ in Ω and $\boldsymbol{f} \cdot \boldsymbol{n} = 0$ on Γ. The Problem (3.1.11) has a unique solution $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ that satisfies the estimate (3.1.68).*

Next we consider the space

$$[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau} \ = \ \big\{ \boldsymbol{f} \in (\boldsymbol{T}^{p'}(\Omega))'; \ \operatorname{div} \boldsymbol{f} = 0 \text{ in } \Omega, \ \boldsymbol{f} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma \big\}. \tag{3.1.71}$$

We define the extension of the Stokes operator to the space $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ by

$$C_p \,:\, \mathbf{D}(C_p) \subset [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau} \longmapsto [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$$

with

$$\forall\, \boldsymbol{u} \in \mathbf{D}(C_p), \qquad C_p\, \boldsymbol{u} = -\Delta \boldsymbol{u} \qquad \text{in} \Omega. \tag{3.1.72}$$

The domain of C_p is given by

$$\mathbf{D}(C_p) \ = \big\{ \boldsymbol{u} \in \boldsymbol{L}^p(\Omega); \ \Delta \boldsymbol{u} \in (\boldsymbol{T}^{p'}(\Omega))', \ \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega, \ \boldsymbol{u} \cdot \boldsymbol{n} \ = \ 0, \ \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} \ = \ \mathbf{0} \text{ on } \Gamma \big\}. \tag{3.1.73}$$

We skip the proof of the following proposition because it is similar to the proof of Proposition 3.1.26:

Proposition 3.1.33. *The space $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ is dense in $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$.*

As described above, using Proposition 2.3.3 with $w = 0$, we have the analyticity of the semi-group generated by the Stokes operator on $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$:

Theorem 3.1.34. *The operator $-C_p$ is a densely defined operator and generates a bounded analytic semi-group on $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$.*

3.1.4 Stokes operator with flux boundary conditions

In this subsection we also consider the Stokes operator associated to Problem (3.1.11) but with adding an extra boundary condition which is the flux through the cuts Σ_j, $1 \leq j \leq J$. This last condition enables the Stokes operator to be invertible with bounded and compact inverse.

Consider the space

$$\boldsymbol{X}_p = \left\{ \boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega); \int_\Omega \boldsymbol{f} \cdot \overline{\boldsymbol{v}} \, \mathrm{d}\, x = 0, \ \forall \ \boldsymbol{v} \in \boldsymbol{K}^{p'}_\tau(\Omega) \right\} \tag{3.1.74}$$

(do not confuse between this space and the space $\boldsymbol{X}^p(\Omega)$ defined in the Chapter I). The space $\boldsymbol{K}^{p'}_\tau(\Omega)$ is given by:

$$\boldsymbol{K}^{p'}_\tau(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{X}^{p'}_\tau(\Omega); \ \operatorname{div} \boldsymbol{v} = 0, \ \mathbf{curl}\, \boldsymbol{v} = \mathbf{0} \text{ in } \Omega \right\}. \tag{3.1.75}$$

Next, we define the operator $A'_p : \mathbf{D}(A'_p) \subset \boldsymbol{X}_p \longmapsto \boldsymbol{X}_p$ by:

$$\mathbf{D}(A'_p) = \left\{ \boldsymbol{u} \in \mathbf{D}(A_p); \ \langle \boldsymbol{u} \cdot \boldsymbol{n} \,,\, 1 \rangle_{\Sigma_j} = 0, \ 1 \leq j \leq J \right\} \tag{3.1.76}$$

and

$$\forall\, \boldsymbol{u} \in \mathbf{D}(A'_p), \qquad A'_p \boldsymbol{u} = A_p \boldsymbol{u}. \tag{3.1.77}$$

The operator A'_p is the restriction of the Stokes operator to the space $\boldsymbol{X}_p$. It is clear that when Ω is simply connected the Stokes operator A_p coincides with the operator A'_p.

Remark 3.1.35. Let $\boldsymbol{u} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, we note that the condition

$$\forall \ \boldsymbol{v} \in \boldsymbol{K}^{p'}_\tau(\Omega), \qquad \int_\Omega \boldsymbol{u} \cdot \overline{\boldsymbol{v}} \, \mathrm{d}\, x = 0, \tag{3.1.78}$$

is equivalent to the condition:

$$\langle \boldsymbol{u} \cdot \boldsymbol{n} \,,\, 1 \rangle_{\Sigma_j} = 0, \ 1 \leq j \leq J. \tag{3.1.79}$$

In fact, if $\boldsymbol{u} \in \boldsymbol{L}^p_\sigma(\Omega)$ satisfying the condition (3.1.79) then thanks to [11, Theorem 3.14] there exists a vector potential $\boldsymbol{\psi} \in \boldsymbol{W}^{1,p}(\Omega)$ such that $\boldsymbol{u} = \mathbf{curl}\, \boldsymbol{\psi}$, $\operatorname{div} \boldsymbol{\psi} = 0$ in Ω, $\boldsymbol{\psi} \times \boldsymbol{n} = \mathbf{0}$ on Γ and $\langle \boldsymbol{\psi} \cdot \boldsymbol{n} \,,\, 1 \rangle_{\Gamma_i} = 0, \ 1 \leq i \leq I$. As a result, for all $\boldsymbol{v} \in \boldsymbol{K}^{p'}_\tau(\Omega)$ one has

$$\int_\Omega \boldsymbol{u} \cdot \overline{\boldsymbol{v}} \, \mathrm{d}\, x = \int_\Omega \mathbf{curl}\, \boldsymbol{\psi} \cdot \overline{\boldsymbol{v}} \, \mathrm{d}\, x = \int_\Omega \boldsymbol{\psi} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \, \mathrm{d}\, x - \langle \boldsymbol{\psi} \times \boldsymbol{n} \,;\, \boldsymbol{v} \rangle_\Gamma = 0,$$

where the duality on Γ is $\langle .\,,\,. \rangle_\Gamma = \langle .\,,\,. \rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma) \times \boldsymbol{W}^{1/p,p'}(\Gamma)}$.

Conversely, if $\boldsymbol{u} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ satisfying the compatibility condition (3.1.78), then $\boldsymbol{u}$ can be written in the form

$$\boldsymbol{u} = \boldsymbol{\psi} - \sum_{j=1}^{J} \langle \boldsymbol{\psi} \cdot \boldsymbol{n} \,;\, 1 \rangle_{\Sigma_j} \widetilde{\mathbf{grad}} q_j^\tau,$$

for some function $\boldsymbol{\psi} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. As a result using Lemma 2.2.8 and the properties of the functions q_j^τ (3.1.22) one has $\langle \boldsymbol{u} \cdot \boldsymbol{n} \,,\, 1 \rangle_{\Sigma_j} = 0, \ 1 \leq j \leq J$.

Proposition 3.1.36. *The operator A'_p is a well defined operator of dense domain.*

Proof. Thanks to Remark 3.1.35 it is clear that $\mathbf{D}(A'_p) \subset \boldsymbol{X}_p$. Moreover, using Lemma 2.2.3 we can easily verify that for all $\boldsymbol{v} \in \boldsymbol{K}^{p'}_{\tau}(\Omega)$, $\int_\Omega \Delta \boldsymbol{u} \cdot \overline{\boldsymbol{v}} \,\mathrm{d}\, x = 0$. As a result $A'_p\boldsymbol{u} \in \boldsymbol{X}_p$ and A'_p is a well defined operator.

Now, for the density, let $\boldsymbol{w} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ such that $\langle \boldsymbol{w} \cdot \boldsymbol{n}\,,\, 1\rangle_{\Sigma_j} = 0$ for all $1 \leqslant j \leqslant J$. We know that there exists a sequence $(\boldsymbol{w}_k)_k$ in $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ such that $\boldsymbol{w}_k \longrightarrow \boldsymbol{w}$ in $\boldsymbol{L}^p(\Omega)$. As a consequence for all $1 \leqslant j \leqslant J$, $\langle \boldsymbol{w}_k \cdot \boldsymbol{n}\,,\, 1\rangle_{\Sigma_j} \longrightarrow \langle \boldsymbol{w} \cdot \boldsymbol{n}\,,\, 1\rangle_{\Sigma_j} = 0$, as $k \to +\infty$.
Now for all $k \in \mathbb{N}$, setting $\widetilde{\boldsymbol{w}}_k = \boldsymbol{w}_k - \sum_{j=1}^J \langle \boldsymbol{w}_k \cdot \boldsymbol{n}\,,\, 1\rangle_{\Sigma_j} \,\mathbf{grad}\, q^\tau_j$. We can easily verify that $(\widetilde{\boldsymbol{w}}_k)_k$ is in $\mathbf{D}(A'_p)$ and converges to $\boldsymbol{w}$ in $\boldsymbol{L}^p(\Omega)$. $\square$

Next we study the resolvent of the operator A'_p. For this reason we consider the problem

$$\begin{cases} \lambda\, \boldsymbol{u} - \Delta \boldsymbol{u} = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega,\\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} & \text{on } \Gamma,\\ \langle \boldsymbol{u} \cdot \boldsymbol{n}\,,\, 1\rangle_{\Sigma_j} = 0, & 1 \le j \le J, & \end{cases} \tag{3.1.80}$$

where $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re}\lambda \geq 0$ and $\boldsymbol{f} \in \boldsymbol{X}_p$. We skip the proof of the following theorem because it is similar to the proof of [10, Proposition 4.3], Theorem 3.1.15 and 3.1.18.

Theorem 3.1.37. *Let $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re}\lambda \geq 0$ and $\boldsymbol{f} \in \boldsymbol{X}_p$. The Problem (3.1.80) has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ that satisfies the estimates (3.1.55)-(3.1.56). In addition, when Ω is of class $C^{2,1}$ the solution $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$ and satisfies the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C(\Omega, p)\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{3.1.81}$$

where $C(\Omega, p)$ is independent of λ and $\boldsymbol{f}$.

Remark 3.1.38. Consider the Problem:

$$\begin{cases} -\Delta \boldsymbol{u} = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega,\\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} & \text{on } \Gamma,\\ \langle \boldsymbol{u} \cdot \boldsymbol{n}\,,\, 1\rangle_{\Sigma_j} = 0, & 1 \le j \le J, & \end{cases} \tag{3.1.82}$$

Thanks to [10, Proposition 4.3] we know that this problem has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and when Ω is of class $C^{2,1}$ the solution $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$. This means that the operator A'_p is invertible of bounded inverse $(0 \in \rho(A'_p))$.

We also Know due to [10], when Ω is not simply connected and when the flux through the cuts Σ_j, $1 \leq j \leq J$ is not equal to zero (*i.e.* if we consider Problem (3.1.82) without the last condition $\langle \boldsymbol{u} \cdot \boldsymbol{n}\,,\, 1\rangle_{\Sigma_j} = 0$, $1 \leq j \leq J$), Problem (3.1.82) has a solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ that is unique up to an additive element in $\boldsymbol{K}^p_\tau(\Omega)$ (given by (3.1.75) with p' replaced by p). On other words the Stokes operator $A_p : \mathbf{D}(A_p) \subset \boldsymbol{L}^p_{\sigma,\tau}(\Omega) \mapsto \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ is not invertible, its kernel is not trivial and it is equal to the space $\boldsymbol{K}^p_\tau(\Omega)$.

As a result of Theorem 3.1.37 the following theorem holds.

Theorem 3.1.39. *The operator $-A'_p$ generates a bounded analytic semi-group on $\boldsymbol{X}_p$ for all $1 < p < \infty$.*

Remark 3.1.40. **(i)** Let $e^{-tA'_p}$ be the semi-group generated by $-A'_p$ on $\boldsymbol{X}_p$. We notice that $e^{-tA'_p}$ is the restriction of e^{-tA_p} to $\boldsymbol{X}_p$, where e^{-tA_p} is the analytic semi-group generated by the operator $-A_p$ on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$.
(ii) The Stokes semi-group $e^{-tA'_p}$ decays exponentially on $\boldsymbol{X}_p$. This is due to the compatibility condition (3.1.78) that makes the Stokes operator with Navier-boundary condition of bounded inverse.

Remark 3.1.41. Thanks to the proof of Proposition 3.1.13 we can conclude that The space $\boldsymbol{X}_2$ has a Hilbertian basis formed from the eigenfunctions of the operator A'_p. Moreover $\sigma(A_p) = \sigma(A'_p) \cup \{0\}$ and $\boldsymbol{X}_2 = \bigoplus_{k=1}^{+\infty} \mathbf{Ker}(\lambda_k I - A_2)$.

The following remark gives the definition of the Stokes operator with flux boundary conditions on some subspaces of $[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\tau}$ and $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ satisfying some compatibility conditions

Remark 3.1.42. **(i)** Consider the space

$$\boldsymbol{Y}_p = \Big\{ \boldsymbol{f} \in [\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\tau};\ \forall\, \boldsymbol{v} \in \boldsymbol{K}^{p'}_\tau(\Omega),\ \ \langle \boldsymbol{f},\, \boldsymbol{v}\rangle_\Omega = 0 \Big\}, \tag{3.1.83}$$

where $\langle .\,,\,.\rangle_\Omega = \langle .\,,\,.\rangle_{[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]' \times \boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)}$.

We define the operator $B'_p : \mathbf{D}(B'_p) \subset \boldsymbol{Y}_p \longmapsto \boldsymbol{Y}_p$ by:

$$\mathbf{D}(B'_p) = \big\{ \boldsymbol{u} \in \mathbf{D}(B_p);\ \langle \boldsymbol{u}\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j} = 0,\ 1 \le j \le J \big\}$$

and

$$\forall\, \boldsymbol{u} \in \mathbf{D}(B'_p), \qquad B'_p\boldsymbol{u} = B_p\boldsymbol{u} = -\Delta\boldsymbol{u} \qquad \text{in} \quad \Omega.$$

We recall that $\mathbf{D}(B_p)$ is given by (3.1.66). Observe that, the operator B'_p is the restriction of the Stokes operator to the space $\boldsymbol{Y}_p$. It is clear that when Ω is simply connected the Stokes operator B_p coincides with the operator B'_p. We can easily verify that $\boldsymbol{f} \in \boldsymbol{Y}_p$ and for all $\lambda \in \mathbb{C}^*$ such that $\mathrm{Re}\lambda \ge 0$ the Problem (3.1.80) has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ satisfying the estimate (3.1.62). In other words, the operator B'_p is a well densely defined operator and $-B'_p$ generates a bounded analytic semi-group on $\boldsymbol{Y}_p$.
(ii) Consider the space

$$\boldsymbol{Z}_p = \Big\{ \boldsymbol{f} \in [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau};\ \forall\, \boldsymbol{v} \in \boldsymbol{K}^{p'}_\tau(\Omega),\ \ \langle \boldsymbol{f},\, \boldsymbol{v}\rangle_\Omega = 0 \Big\}, \tag{3.1.84}$$

where $\langle .\,,\,.\rangle_\Omega = \langle .\,,\,.\rangle_{[\boldsymbol{T}^{p'}(\Omega)]' \times \boldsymbol{T}^{p'}(\Omega)}$.

We define the operator $C'_p : \mathbf{D}(C'_p) \subset \boldsymbol{Z}_p \longmapsto \boldsymbol{Z}_p$ by:

$$\mathbf{D}(C'_p) = \big\{\boldsymbol{u} \in \mathbf{D}(C_p);\ \langle \boldsymbol{u}\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j} = 0,\ 1 \leq j \leq J\big\}$$

and

$$\forall\, \boldsymbol{u} \in \mathbf{D}(C'_p), \qquad C'_p\boldsymbol{u} = C_p\boldsymbol{u} = -\Delta\boldsymbol{u} \qquad \text{in} \quad \Omega.$$

We recall that $\mathbf{D}(C_p)$ is given by (3.1.73). Notice that, the operator C'_p is the restriction of the Stokes operator to the space $\boldsymbol{Z}_p$. Similarly, when Ω is simply connected the Stokes operator C_p coincides with the operator C'_p. We can easily verify that $\boldsymbol{f} \in \boldsymbol{Z}_p$ and for all $\lambda \in \mathbb{C}^*$ such that $\mathrm{Re}\lambda \geq 0$ the Problem (3.1.80) has a unique solution $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ satisfying the estimate (3.1.68). In other words, the operator C'_p is a well densely defined operator and $-C'_p$ generates a bounded analytic semi-group on $\boldsymbol{Z}_p$.

3.2 Stokes operator with Navier slip boundary conditions

This section deals with the Stokes operator with the Navier slip boundary conditions

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0, \qquad [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0} \text{ on } \Gamma. \tag{3.2.1}$$

We prove the analyticity of the Stokes semi-group with Navier boundary conditions on the spaces $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, $[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\tau}$ and $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ respectively. This analyticity allows us to solve the evolutionary Stokes problem and to obtain weak, strong and very weak solutions for this Problem. Since the proof of the analyticity is similar to the analyticity of the Stokes semi-group with Navier-type boundary conditions we will skip some details and we will give only a sketch of the proof.

As in the previous section, to prove the analyticity we use a classical approach. We study the resolvent of the Stokes operator:

$$\begin{cases} \lambda\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{f}, & \mathrm{div}\,\boldsymbol{u} = 0 & \text{in } \Omega, \\ \qquad\qquad \boldsymbol{u}\cdot\boldsymbol{n} = 0, & [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0} & \text{on } \Gamma, \end{cases} \tag{3.2.2}$$

where $\lambda \in \mathbb{C}^*$ such that $\mathrm{Re}\,\lambda \geq 0$. We prove the existence of weak, strong and very weak solutions to Problem (3.2.2) satisfying a resolvent estimate of type

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C(\Omega,p)}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \tag{3.2.3}$$

for the case of strong solution. We also prove an estimate of type (3.2.3) for the norm of $[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'$ (respectively the norm $[\boldsymbol{T}^{p'}(\Omega)]'$) in the case of weak (respectively very weak solution). As in the previous section, for $p=2$ one has estimate (3.2.3) in a sector $\lambda \in \Sigma_\varepsilon$ for a fixed $\varepsilon \in\,]0,\pi[$, where Σ_ε is given by (3.1.9).

Before giving our main results in this section we define the Stokes operator with Navier boundary conditions (3.2.1). Let $\boldsymbol{u} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and consider the mapping

$$\begin{aligned} \mathbb{A}_p \boldsymbol{u} \quad : \quad & \boldsymbol{W} \longrightarrow \mathbb{C} \\ & \boldsymbol{v} \longrightarrow -\int_\Omega \boldsymbol{u} \cdot \Delta \overline{\boldsymbol{v}} \, \mathrm{d}\, x, \end{aligned}$$

where

$$\boldsymbol{W} = \boldsymbol{V}^{p'}_\tau(\Omega) \cap \boldsymbol{W}^{2,p'}(\Omega)$$

with $\boldsymbol{V}^p_\tau(\Omega)$ is given by (3.1.2). It is clear that $\mathbb{A}_p \in \mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega),\, \boldsymbol{W}')$ and thanks to de Rham's Lemma there exists $\pi \in W^{-1,p}(\Omega)$ such that

$$\mathbb{A}_p \boldsymbol{u} + \Delta \boldsymbol{u} = \nabla \pi, \qquad \text{in} \quad \Omega.$$

Now suppose that $\boldsymbol{u} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and $\mathbb{A}_p \boldsymbol{u} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. Since $\Delta \boldsymbol{u} = -\mathbb{A}_p \boldsymbol{u} + \nabla \pi$ then using [8, Lemma 5.4] one has $[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau \in \boldsymbol{W}^{-1-1/p,p}(\Gamma)$. Moreover if we suppose that $[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau = \mathbf{0}$ on Γ then $(\boldsymbol{u},\, \pi) \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega) \times W^{-1,p}(\Omega)$ is a solution of the problem

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \mathbb{A}_p \boldsymbol{u}, & \operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega, \\ \qquad\quad \boldsymbol{u} \cdot \boldsymbol{n} = 0, & [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0} \qquad \text{on } \Gamma. \end{cases}$$

As a result using the regularity of the Stokes Problem [8, Theorem 4.1] one has $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ when Ω is of class $C^{2,1}$.

The Stokes operator with Navier slip boundary conditions is a densely defined closed operator

$$\mathbb{A}_p \,:\, \mathbf{D}(\mathbb{A}_p) \subset \boldsymbol{L}^p_{\sigma,\tau}(\Omega) \longmapsto \boldsymbol{L}^p_{\sigma,\tau}(\Omega),$$

where

$$\mathbf{D}(\mathbb{A}_p) = \Big\{\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega);\ \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega,\ \boldsymbol{u} \cdot \boldsymbol{n} = 0,\ [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0} \text{ on } \Gamma\Big\}, \tag{3.2.4}$$

provided that Ω is of class $C^{2,1}$. Moreover

$$\forall\, \boldsymbol{u} \in \mathbf{D}(\mathbb{A}_p), \qquad \mathbb{A}_p \boldsymbol{u} = -\Delta \boldsymbol{u} + \mathbf{grad}\, \pi, \tag{3.2.5}$$

where π is the unique solution up to an additive constant of the problem

$$\operatorname{div}(\mathbf{grad}\, \pi - \Delta \boldsymbol{u}) = 0 \qquad \text{in } \Omega, \qquad (\mathbf{grad}\, \pi - \Delta \boldsymbol{u}) \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma.$$

Remark 3.2.1. Unlike the Stokes operator with Navier-type boundary condition (3.1.1), the pressure here cannot be a constant since $\Delta \boldsymbol{u} \cdot \boldsymbol{n}$ doesn't vanishes on Γ.

3.2.1 Analyticity in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$

The main goal of this subsection is to prove that the Stokes operator with Navier slip boundary conditions (3.2.1) generates a bounded analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ for all $1 < p < \infty$. Our study will be done in two steps. In the first steps we treat the Hilbertian case and in the second step we treat the general $\boldsymbol{L}^p$-theory.

In what follows we use the following formula of the Laplacian operator

$$\Delta\, \boldsymbol{v} = 2\mathrm{div}\, \mathbf{D}(\boldsymbol{v}) - \mathbf{grad}\,(\mathrm{div}\, \boldsymbol{v}) \qquad \text{in} \quad \Omega.$$

The following theorem study the resolvent Problem (3.2.2) on $\boldsymbol{L}^2(\Omega)$.

Theorem 3.2.2. *Let $\varepsilon \in \,]0,\pi[$ be fixed, where Σ_ε is given by (3.1.9), let $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\lambda \in \Sigma_\varepsilon$.*
(i) *The problem (3.2.2) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$.*
(ii) *There exists a constant $C'_\varepsilon > 0$ independent of $\boldsymbol{f}$ and λ such that the solution $\boldsymbol{u}$ satisfies the estimates*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C'_\varepsilon}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \tag{3.2.6}$$

and

$$\|\mathbf{D}(\boldsymbol{u})\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C'_\varepsilon}{\sqrt{2|\lambda|}}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}. \tag{3.2.7}$$

($C'_\varepsilon = 1/C_\varepsilon$, where C_ε is the constant in (3.1.10)).
(iii) *If Ω is of class $C^{2,1}$ then $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^2(\Omega) \times H^1(\Omega)$ and satisfies the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} \leq \frac{C(\Omega, \lambda, \varepsilon)}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}, \tag{3.2.8}$$

where $C(\Omega, \lambda, \varepsilon) = C(\Omega)(C'_\varepsilon + 1)(|\lambda| + 1)$.

Proof. The proof is similar to the proof of Theorem 3.1.8 and it is done in two parts. We start by the proof of existence and uniqueness.
(i) Existence and uniqueness: We consider the variational problem: find $\boldsymbol{u} \in \boldsymbol{V}^2_\tau(\Omega)$ such that for any $\boldsymbol{v} \in \boldsymbol{V}^2_\tau(\Omega)$

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \overline{\boldsymbol{v}}\,\mathrm{d}\,x, \tag{3.2.9}$$

where

$$a(\boldsymbol{u}, \boldsymbol{v}) = \lambda \int_\Omega \boldsymbol{u} \cdot \overline{\boldsymbol{v}}\,\mathrm{d}x + 2\int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\overline{\boldsymbol{v}})\,\mathrm{d}x.$$

As in the proof of Theorem 3.1.8 and using Korn's inequality we can easily verify that the sesqui-linear form a is a continuous coercive form on $\boldsymbol{V}^2_\tau(\Omega)$. As consequence, due to Lax-Milgram Lemma, Problem (3.2.9) has a unique solution $\boldsymbol{u} \in \boldsymbol{V}^2_\tau(\Omega)$ since the right-hand side belongs to the anti-dual $(\boldsymbol{V}^2_\tau(\Omega))'$.

Using Green formula (2.2.2), we deduce that every solution of (3.2.2) also solves (3.2.9). Conversely, let $\boldsymbol{u}$ a solution of the problem (3.2.9) and let $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega)$ such that $\operatorname{div} \boldsymbol{v} = 0$ in Ω. As a result one has

$$\lambda \int_{\Omega} \boldsymbol{u} \cdot \overline{\boldsymbol{v}} \,\mathrm{d}\, x + 2 \int_{\Omega} \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\overline{\boldsymbol{v}}) \,\mathrm{d}\, x = \langle \lambda \boldsymbol{u} - \Delta \boldsymbol{u}, \boldsymbol{v} \rangle_{\boldsymbol{\mathcal{D}}'(\Omega) \times \boldsymbol{\mathcal{D}}(\Omega)}.$$

As a consequence,

$$\forall \boldsymbol{v} \in \boldsymbol{\mathcal{D}}_{\sigma}(\Omega), \qquad \langle \lambda \boldsymbol{u} - \Delta \boldsymbol{u} - \boldsymbol{f}, \boldsymbol{v} \rangle_{\boldsymbol{\mathcal{D}}'(\Omega) \times \boldsymbol{\mathcal{D}}(\Omega)} = 0.$$

As a result, by De Rham's Theorem, there exists a distribution $\pi \in L^2(\Omega)$ defined uniquely up to an additive constant such that

$$\lambda \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \qquad \text{in} \quad \Omega. \tag{3.2.10}$$

It remains to prove the Navier boundary condition $[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0}$ on Γ. We multiply the equation (3.2.10) by $\boldsymbol{v} \in \boldsymbol{V}^2_{\tau}(\Omega)$ and we integrate both sides, we have :

$$\lambda \int_{\Omega} \boldsymbol{u} \cdot \overline{\boldsymbol{v}} \,\mathrm{d}\, x + 2 \int_{\Omega} \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\overline{\boldsymbol{v}}) \,\mathrm{d}\, x - 2\langle [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{v} \rangle_{\Gamma} = \int_{\Omega} \boldsymbol{f} \cdot \overline{\boldsymbol{v}} \,\mathrm{d}\, x. \tag{3.2.11}$$

Using (3.2.9) and (3.2.11), we deduce that

$$\forall \boldsymbol{v} \in \boldsymbol{V}^2_{\tau}(\Omega), \qquad \langle [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{v} \rangle_{\Gamma} = 0.$$

Let now $\boldsymbol{\mu}$ any element of the space $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$. We know that there exists $\boldsymbol{v} \in \boldsymbol{H}^{1}(\Omega)$ such that $\operatorname{div} \boldsymbol{v} = 0$ in Ω and $\boldsymbol{v} = \boldsymbol{\mu}_{\boldsymbol{\tau}}$ on Γ. Its clear that $\boldsymbol{v} \in \boldsymbol{V}^2_{\tau}(\Omega)$ and

$$\langle [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\mu} \rangle_{\Gamma} = \langle [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{\mu}_{\boldsymbol{\tau}} \rangle_{\Gamma} = \langle [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}, \boldsymbol{v} \rangle_{\Gamma} = 0.$$

This implies that

$$[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \boldsymbol{0} \quad \text{on} \quad \Gamma.$$

This prove that the two problems (3.2.2) and (3.2.9) are equivalent. Thus we obtain the existence and the uniqueness of solution to problem (3.2.2).

(ii) Estimates: To prove the resolvent estimate (3.2.6) and (3.2.7) we proceed as in the proof of Theorem 3.1.8. We multiply the first equation of System (3.2.2) by $\overline{\boldsymbol{u}}$, integrate both sides and we use Lemma 3.1.6. These two estimates follow directly. □

We observe the following remark concerning the Stokes operator with Navier bounday condition:

Remark 3.2.3. Consider the sesqui-linear form

$$\forall\, \boldsymbol{u},\, \boldsymbol{v} \in \boldsymbol{V}^2_{\tau}(\Omega), \qquad a(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega} \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\overline{\boldsymbol{v}}) \,\mathrm{d}\, x. \tag{3.2.12}$$

Thanks to [8] we know that in general this sesqui-linear form is not coercive. In fact we introduce the kernel $\boldsymbol{\mathcal{T}}(\Omega)$ for any $1 < p < \infty$:

$$\boldsymbol{\mathcal{T}}^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega);\, \mathbf{D}(\boldsymbol{v}) = \mathbf{0} \,\text{in}\, \Omega,\, \operatorname{div} \boldsymbol{v} = 0 \,\text{in}\ \Omega \ \text{and}\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \ \text{on}\ \Gamma\}. \tag{3.2.13}$$

If Ω is obtained by rotation around a such vector $\boldsymbol{b}$ belongs to $\mathbb{R}^3$, then

$$\boldsymbol{\mathcal{T}}^p(\Omega) = \operatorname{Span}\{\boldsymbol{\beta}\}, \qquad \boldsymbol{\beta}(\boldsymbol{x}) = \boldsymbol{b}\times\boldsymbol{x}, \quad \text{for}\, \boldsymbol{x}\in\,\Omega.$$

Else, the kernel $\boldsymbol{\mathcal{T}}^p(\Omega)$ is equal to zero (see [80] for more details).

Thanks to [8, Lemma 3.3], we know that on $\boldsymbol{V}^2_\tau(\Omega)$ the semi-norm $\|\mathbf{D}(\boldsymbol{u})\|_{\boldsymbol{L}^2(\Omega)}$ is a norm equivalent to the norm $\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}$ if $\displaystyle\int_\Omega \boldsymbol{u}\cdot\boldsymbol{\beta} d\boldsymbol{x} = 0$. In this case the sesqui-linear form (3.2.12) is coercive and we can apply Lax-Milgram Lemma to find solution to the problem: find $\boldsymbol{u}\in\boldsymbol{V}^2_\tau(\Omega)$ such that for all $\boldsymbol{v}\in\boldsymbol{V}^2_\tau(\Omega)$

$$a(\boldsymbol{u},\boldsymbol{v}) \;=\; \int_\Omega \boldsymbol{f}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x,$$

where $\boldsymbol{f}\in\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$. This means that the Stokes operator with Navier slip boundary conditions (3.2.1) $\mathbb{A}_2 \,:\, \mathbf{D}(\mathbb{A}_2) \subset \boldsymbol{L}^2_{\sigma,\tau}(\Omega) \longmapsto \boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ is bijective and of bounded inverse. In the general case the sesqui-linear form (3.2.12) is not coercive and the Stokes operator with Navier boundary conditions is not bijective of bounded inverse.

Now we extend Theorem 3.2.2 to every $1 < p < \infty$. The proof of the following theorem is similar to the proof of Theorem 3.2.4

Theorem 3.2.4. *Let $\lambda\in\mathbb{C}\in\Sigma_\varepsilon$ and let $\boldsymbol{f}\in\boldsymbol{L}^p(\Omega)$. The Problem* (3.2.2) *has a unique solution $(\boldsymbol{u},\pi)\in\boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)/\mathbb{R}$. Moreover, if Ω is of class $C^{2,1}$ then $(\boldsymbol{u},\pi)\in\boldsymbol{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)$.*

Proof. **Existence:** As in proof of Theorem 3.2.2, the resolvent problem (3.2.2) is equivalent to the following variational problem: Find $\boldsymbol{u}\in\boldsymbol{V}^p_\tau(\Omega)$ such that for all $\boldsymbol{v}\in\boldsymbol{V}^{p'}_\tau(\Omega)$

$$\lambda\int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x + 2\int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\overline{\boldsymbol{v}})\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x.$$

To prove existence, we distinguish three cases:
(i) Case $2\le p\le 6$. Let $(\boldsymbol{u},\pi)\in\boldsymbol{H}^1(\Omega)\times L^2(\Omega)/\mathbb{R}$ be the unique solution of problem (3.2.2). We write our problem in the form:

$$\begin{cases} -\Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{F}, & \operatorname{div}\boldsymbol{u} = 0 \qquad \text{in}\ \Omega,\\ \qquad\quad \boldsymbol{u}\cdot\boldsymbol{n} = 0, & [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0} \qquad \text{on}\ \Gamma, \end{cases} \tag{3.2.14}$$

where $\boldsymbol{F} = \boldsymbol{f} - \lambda\,\boldsymbol{u}$. Using the embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$ one has $\boldsymbol{F} \in \boldsymbol{L}^p(\Omega)$. It remains to verify (see [8, Theorem 3.7]) that $\boldsymbol{F}$ satisfies the following compatibility condition

$$\int_\Omega \boldsymbol{F} \cdot \overline{\boldsymbol{\beta}}\,\mathrm{d}\,x = 0. \tag{3.2.15}$$

Thanks to Lemma 2.2.4, one has

$$\int_\Omega \boldsymbol{F} \cdot \overline{\boldsymbol{\beta}}\,\mathrm{d}\,x \;=\; \int_\Omega (-\Delta\boldsymbol{u} + \nabla\pi) \cdot \overline{\boldsymbol{\beta}}\,\mathrm{d}\,x \;=\; 2\int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\overline{\boldsymbol{\beta}})\,\mathrm{d}\,x - 2\langle[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}\,, \boldsymbol{\beta}\rangle_\Gamma \;=\; 0.$$

As consequence, applying [8, Theorem 3.7], we deduce that $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$.

(ii) Case $p \geq 6$. Since $\boldsymbol{f} \in \boldsymbol{L}^6(\Omega)$, according to the point **(i)**, problem (3.2.2) has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,6}(\Omega \hookrightarrow \boldsymbol{L}^\infty(\Omega)$. Again, applying [8, Theorem 3.7]), the solution $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$.

(iii) The case $p \leq 2$ is obtained by a duality argument.

Regularity: If Ω is of class $C^{2,1}$, we consider the problem (3.2.14) and we apply the Stokes regularity ([8, Theorem 4.1]), which ends the proof. □

Remark 3.2.5. 1. The pressure π given by Theorem 3.2.4, satisfies also the following problem:

$$\mathrm{div}(\nabla\pi - \boldsymbol{f}) = 0 \qquad \text{in } \Omega, \qquad (\nabla\pi - \boldsymbol{f}) \cdot \boldsymbol{n} = \Delta\,\boldsymbol{u} \cdot \boldsymbol{n} = 2\,\mathrm{div}_\Gamma(\boldsymbol{\Lambda}\boldsymbol{u}) \qquad \text{on } \Gamma.$$

Moreover, we have the estimate:

$$\|\nabla\pi\|_{\boldsymbol{L}^p(\Omega)} \leq C(\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}). \tag{3.2.16}$$

2. The result of the previous theorem holds in particular for $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. In this case, notice that the pressure π is a solution of the problem

$$\Delta\pi = 0 \qquad \text{in } \Omega, \qquad \frac{\partial\pi}{\partial\boldsymbol{n}} = \Delta\,\boldsymbol{u} \cdot \boldsymbol{n} = 2\,\mathrm{div}_\Gamma(\boldsymbol{\Lambda}\boldsymbol{u}) \qquad \text{on } \Gamma.$$

Moreover, we have the estimate:

$$\|\nabla\pi\|_{\boldsymbol{L}^p(\Omega)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}. \tag{3.2.17}$$

The latter estimates will be useful in the sequel to show the resolvent estimates.

In the following, we want to prove the resolvent estimate (3.2.3) for all $1 < p < \infty$. But this case will not be easy to show as in the case p $= 2$ and the proof will be done in several steps. The proof is similar to Proposition 3.1.16 and Proposition 3.1.17.

Proposition 3.2.6. *Assume that* Ω *is of class* $C^{2,1}$*. Let* $\lambda \in \mathbb{C}^*$ *such that* $\mathrm{Re}\,\lambda \geq 0$ *and* $|\lambda| \geq \lambda_0$*, where* $\lambda_0 = \lambda_0(\Omega, p)$ *is defined in (3.2.24). Moreover, let* $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$*, where* $1 < p < \infty$ *and let* $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ *be the unique solution of problem (3.2.2). Then* $\boldsymbol{u}$ *satisfies the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{\kappa_1(\Omega,\, p)}{|\lambda|}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{3.2.18}$$

where the constant $\kappa_1(\Omega,\, p)$ *is independent of* λ *and* $\boldsymbol{f}$*.*

Proof. As in the proof of Proposition 3.1.16, for $p \geq 2$ we multiply the first equation of problem (3.2.2) by $|\boldsymbol{u}|^{p-2}\,\overline{\boldsymbol{u}}$ and we integrate both sides. As a result, one gets thanks to Lemma 2.2.10

$$\begin{aligned}
\lambda \int_\Omega |\boldsymbol{u}|^p \,\mathrm{d}\,x &+ \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2 \,\mathrm{d}\,x \,+\, 4\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\,x \\
&+\,(p-2)\,i\sum_{k=1}^{3}\int_\Omega |\boldsymbol{u}|^{p-4}\,\mathrm{Re}\,\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\mathrm{Im}\,\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\,\mathrm{d}\,x \\
&= \int_\Gamma |\boldsymbol{u}|^{p-2}\big(\frac{\partial\,\boldsymbol{u}}{\partial\,\boldsymbol{n}}\big)_{\boldsymbol{\tau}}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\,\sigma \,+\, \int_\Omega |\boldsymbol{u}|^{p-2}\,\boldsymbol{f}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\,x - \int_\Omega |\boldsymbol{u}|^{p-2}\,\nabla\pi\cdot\overline{\boldsymbol{u}}.
\end{aligned} \tag{3.2.19}$$

Notice that the integral on Γ is well defined. In fact, since Ω is of class $C^{2,1}$ then $\boldsymbol{n} \in \boldsymbol{W}^{2,\infty}(\Gamma)$. Moreover since $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$, $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ and $[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0}$ on Γ, then using Lemma 2.2.11 and formula (2.2.10) we deduce that $\big(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}\big)_{\boldsymbol{\tau}} = \boldsymbol{\Lambda}\boldsymbol{u}$ belongs to $\boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma) \hookrightarrow \boldsymbol{L}^p(\Gamma)$ with $\Lambda\boldsymbol{u}$ is given by (2.2.12). In addition, we have $|\boldsymbol{u}|^{p-2}\overline{\boldsymbol{u}} \in \boldsymbol{W}^{1,p'}(\Omega)$ and then its trace belongs to $\boldsymbol{W}^{1-1/p',p'}(\Gamma) \hookrightarrow \boldsymbol{L}^{p'}(\Gamma)$. As consequence, the integral $\int_\Gamma |\boldsymbol{u}|^{p-2}\big(\frac{\partial\,\boldsymbol{u}}{\partial\,\boldsymbol{n}}\big)_{\boldsymbol{\tau}}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\,\sigma$ is well defined and we can replace in (2.2.8) the term $\left\langle \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}},\, |\boldsymbol{u}|^{p-2}\boldsymbol{u}\right\rangle_\Gamma$ by $\int_\Gamma |\boldsymbol{u}|^{p-2}\big(\frac{\partial\,\boldsymbol{u}}{\partial\,\boldsymbol{n}}\big)_{\boldsymbol{\tau}}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\,\sigma$. As a result, we have

$$\Big(\frac{\partial\,\boldsymbol{u}}{\partial\,\boldsymbol{n}}\Big)_{\boldsymbol{\tau}}\cdot\overline{\boldsymbol{u}}_{\boldsymbol{\tau}} = -\boldsymbol{n}\cdot\sum_{j,k=1}^{2}\overline{u}_j\,u_k\,\frac{\partial\,\boldsymbol{\tau}_k}{\partial s_j}. \tag{3.2.20}$$

Putting together (3.2.19) and (3.2.20) one gets

$$\begin{aligned}
\lambda \int_\Omega |\boldsymbol{u}|^p \,\mathrm{d}\,x &+ \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2 \,\mathrm{d}\,x \,+\, 4\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\,x \\
&+\,(p-2)\,i\sum_{k=1}^{3}\int_\Omega |\boldsymbol{u}|^{p-4}\,\mathrm{Re}\,\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\mathrm{Im}\,\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\,\mathrm{d}\,x \\
= \int_\Gamma |\boldsymbol{u}|^{p-2}\sum_{j=1}^{2}\sum_{k=1}^{2}\overline{u}_j\,u_k\,\boldsymbol{\tau}_k\cdot\frac{\partial\,\boldsymbol{n}}{\partial s_j}\,\mathrm{d}\,\sigma &+ \int_\Omega |\boldsymbol{u}|^{p-2}\,\boldsymbol{f}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\,x - \int_\Omega |\boldsymbol{u}|^{p-2}\,\nabla\pi\cdot\overline{\boldsymbol{u}}.
\end{aligned} \tag{3.2.21}$$

Next, we split formula (3.2.21) into two parts, the real and the imaginary parts and we treat each part separately. As a result, using (3.2.20), we have

$$\begin{aligned}
\mathrm{Re}\,\lambda\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} &+ \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2 \,\mathrm{d}\,x \,+\, 4\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\,x \\
&\leq C_1(\Omega)\int_\Gamma |\boldsymbol{u}|^p \,\mathrm{d}\,\sigma \,+\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)} \,+\, \|\nabla\pi\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}
\end{aligned} \tag{3.2.22}$$

and

$$|\mathrm{Im}\,\lambda|\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} \leq \frac{p-2}{2}\int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2\,\mathrm{d}\,x \,+\, C_1(\Omega)\int_\Gamma |\boldsymbol{u}|^p\,\mathrm{d}\,\sigma + \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)} \,+\, \|\nabla\pi\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}, \tag{3.2.23}$$

for some constant $C_1(\Omega) > 0$.

Proceeding exactly in the same way as in the proof of Proposition 3.1.16 one has

$$|\lambda|\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2\,\mathrm{d}\,x \,+\, 2\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2\,\mathrm{d}\,x$$
$$\leq\, C_3(\Omega,p)\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} \,+\, \frac{p-2}{2}\int_\Omega |\boldsymbol{u}|^{p-2}|\nabla\boldsymbol{u}|^2\,\mathrm{d}\,x \,+\, 2\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \,+\, \|\nabla\pi\|_{\boldsymbol{L}^p(\Omega)}\right)\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)},$$

where $C_3(\Omega,p) = 2\,C_1(\Omega)C_2(\Omega,p)$.

Setting

$$\lambda_0 = 2\,C_3(\Omega,p), \tag{3.2.24}$$

one has for $|\lambda| \geq \lambda_0$ the following inequality

$$\frac{|\lambda|}{2}\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2\,\mathrm{d}\,x \,+\, 2\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2\,\mathrm{d}\,x$$
$$\leq\, \frac{p-2}{2}\int_\Omega |\boldsymbol{u}|^{p-2}|\nabla\boldsymbol{u}|^2\,\mathrm{d}\,x \,+\, 2\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \,+\, \|\nabla\pi\|_{\boldsymbol{L}^p(\Omega)}\right)\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}.$$

(i) Case $2 \leq p \leq 4$. We can write

$$\frac{|\lambda|}{2}\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} \,+\, \frac{4-p}{2}\int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2\,\mathrm{d}\,x \,+\, 2\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2\,\mathrm{d}\,x$$
$$\leq\, 2\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \,+\, \|\nabla\pi\|_{\boldsymbol{L}^p(\Omega)}\right)\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}.$$

To estimate the pressure term, we proceed as follows: Since Ω is of class $C^{2,1}$ then $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$, for all $p \leq 6$. Using (3.2.17) one has :

$$\|\nabla\pi\|_{\boldsymbol{L}^p(\Omega)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)}.$$

In addition, using the regularity estimates of Stokes problem (see [8]) one has

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} \leq C(\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{f} - \lambda\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}).$$

Therefore

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} \leq C(1 + \frac{1}{|\lambda|} + C'_\varepsilon)\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \leq C(1 + \frac{1}{\lambda_0} + C'_\varepsilon)\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}.$$

As consequence one has

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C(\Omega,p)}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{3.2.25}$$

for some constant $C(\Omega, p) > 0$. In the sequel, if we do not state otherwise, the letter C denotes a constant that may change from one time to another.
(ii) Case $p > 4$**.** We have

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,4}(\Omega)} \leq C(\|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)} + \|\boldsymbol{f} - \lambda \boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)}) \leq C\,\|\boldsymbol{f}\|_{\boldsymbol{L}^4(\Omega)}.$$

Using the fact that $\boldsymbol{W}^{2,4}(\Omega) \hookrightarrow \boldsymbol{W}^{1,\infty}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$ one has

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.$$

As consequence

$$\|\nabla \pi\|_{\boldsymbol{L}^p(\Omega)} \leq C\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.2.26}$$

Notice that

$$\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} = \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)} \leq C\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}.$$

Now proceeding in the same way as in the proof of Proposition 3.1.16 case $p > 4$ one gets estimate (3.2.18) which ends the case $p > 4$.

By duality we obtain estimate (3.2.18) for all $1 < p < 2$. □

The resolvent estimate (3.2.18) holds also for $0 < |\lambda| \leq \lambda_0$. For the proof, we proceed as in the proof of Proposition 3.1.17. As a result, we have the following theorem:

Theorem 3.2.7. *Assume that Ω is of class $C^{2,1}$. Let $\lambda \in \mathbb{C}^*$ such that $\mathrm{Re}\,\lambda \geq 0$, let $1 < p < \infty$, $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and let $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ be the unique solution of problem* (3.2.2)*. Then $\boldsymbol{u}$ satisfies the estimates*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \;\leq\; \frac{\kappa_2(\Omega, p)}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{3.2.27}$$

$$\|\mathbf{D}(\boldsymbol{u})\|_{\boldsymbol{L}^p(\Omega)} \;\leq\; \frac{\kappa_3(\Omega, p)}{\sqrt{|\lambda|}}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \tag{3.2.28}$$

and

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \;\leq\; \kappa_4(\Omega, p)\,\frac{1 + |\lambda|}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.2.29}$$

Proof. Let us prove estimate (3.2.28). The proof is done in two steps.
(i) Case $\displaystyle\int_\Omega \boldsymbol{u} \cdot \boldsymbol{\beta}\,\mathrm{d}\,x = 0$. Thanks to the regularity estimates of the Stokes problem (see [8]) we know that

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \simeq \|-\Delta \boldsymbol{u} + \nabla \pi\|_{\boldsymbol{L}^p(\Omega)}$$

and using the Gagliardo-Nirenberg inequality (see [2, Chapter IV, Theorem 4.14, Theorem 4.17] for instance) we have

$$\begin{aligned}
\|\mathbf{D}(\boldsymbol{u})\|_{\boldsymbol{L}^p(\Omega)} &\leq C(\Omega, p)\,\|\boldsymbol{u}\|^{1/2}_{\boldsymbol{W}^{2,p}(\Omega)}\|\boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)} \\
&\leq C'(\Omega, p)\,\|\boldsymbol{f} - \lambda\,\boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)} \\
&\leq \frac{C(\Omega, p)}{\sqrt{|\lambda|}}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.
\end{aligned}$$

(ii) General case. Let $\boldsymbol{u} \in \mathbf{D}(\mathbb{A}_p)$ be the unique solution of Problem (3.2.2) and set

$$\widetilde{\boldsymbol{u}} = \boldsymbol{u} - \frac{\displaystyle\int_\Omega \boldsymbol{u} \cdot \boldsymbol{\beta} \,\mathrm{d}\,x}{\|\boldsymbol{\beta}\|^2_{\boldsymbol{L}^2(\Omega)}} \boldsymbol{\beta}.$$

As a result, thanks to the previous case we have

$$\|\mathbf{D}(\widetilde{\boldsymbol{u}})\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega, p)\, \|\mathbb{A}_p \widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)} \|\widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}.$$

Thus

$$\|\mathbf{D}(\boldsymbol{u})\|_{\boldsymbol{L}^p(\Omega)} = \|\mathbf{D}(\widetilde{\boldsymbol{u}})\|_{\boldsymbol{L}^p(\Omega)} \leq \|\mathbb{A}_p \widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)} \|\widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)} = \|\mathbb{A}_p \boldsymbol{u}\|^{1/2}_{\boldsymbol{L}^p(\Omega)} \|\widetilde{\boldsymbol{u}}\|^{1/2}_{\boldsymbol{L}^p(\Omega)}.$$

Moreover, it is clear that

$$\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega, p)\, \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

As a consequence we deduce estimate (3.2.28) and estimate (3.2.29) follows directly. □

As a result we recover the analyticity of the semi-group generated by the Stokes operator with Navier slip boundary condition (3.2.1) in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$.

Theorem 3.2.8. *The operator $-\mathbb{A}_p$ generated a bounded analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ for all $1 < p < \infty$.*

Remark 3.2.9. (i) As described in Remark 3.2.3 if the domain Ω is not obtained by a rotation around a vector $\boldsymbol{b}$, the Stokes operator $\mathbb{A}_p$ with Navier slip boundary conditions (3.2.1) is bijective of bounded inverse. In other words $0 \in \rho(\mathbb{A}_p)$ and the Stokes semi-group with Navier boundary conditions decays exponentially.
(ii) In the general case the Stokes operator with Navier-slip boundary condition is not invertible. However, the operator $I + \mathbb{A}_p$ is an isomorphism from $\mathbf{D}(\mathbb{A}_p)$ (given by (3.2.4)) to $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. As a result, when Ω is of class $C^{2,1}$ one has

$$\forall \boldsymbol{u} \in \mathbf{D}(\mathbb{A}_p), \qquad \|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \simeq \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbb{A}_p \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$$

3.2.2 Analyticity in $[\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]'_{\sigma,\tau}$

In this subsection we study the resolvent Problem (3.2.2) when $\boldsymbol{f} \in [\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]'_{\sigma,\tau}$ (given by (3.1.64)). We obtain weak solution to Problem (3.2.2) as well as a resolvent estimate for the norm of $[\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]'$. This gives us the analyticity of the Stokes semi-group with Navier slip boundary condition in $[\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]'_{\sigma,\tau}$.

Theorem 3.2.10. *Let $\lambda \in \mathbb{C}^*$ such that $\mathrm{Re}\,\lambda \geq 0$ and let $\boldsymbol{f} \in [\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]'$. The Problem (3.2.2) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$. Moreover, we have the following estimate:*

$$\|\boldsymbol{u}\|_{[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'} \leq \frac{C(\Omega, p)}{|\lambda|} \|\boldsymbol{f}\|_{[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'} \tag{3.2.30}$$

for some constant $C(\Omega, p) > 0$ independent of λ and $\boldsymbol{f}$.

Proof. For the existence of solutions, we proceed in the same way as in the proof of Theorem 3.2.4. Let us prove the resolvent estimate (3.2.30). We know that according to Theorem 3.2.4, for $\boldsymbol{F} \in \boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)$ and $\lambda \in \mathbb{C}^*$ such that $\mathrm{Re}\,\lambda \geq 0$ the following problem

$$\begin{cases} \lambda \boldsymbol{v} - \Delta \boldsymbol{v} + \nabla \theta = \boldsymbol{F}, & \mathrm{div}\, \boldsymbol{v} = 0 \qquad \text{in} \quad \Omega, \\ \boldsymbol{v} \cdot \boldsymbol{n} = 0, & [\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0} \qquad \text{on} \quad \Gamma, \end{cases} \tag{3.2.31}$$

has a unique solution $(\boldsymbol{v}, \theta) \in \boldsymbol{W}^{1,p'}(\Omega) \times L^{p'}(\Omega)/\mathbb{R}$ satisfying

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)} \leq \frac{C(\Omega, p')}{|\lambda|} \|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Thus

$$\|\boldsymbol{v}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)} \leq \frac{C(\Omega, p')}{|\lambda|} \|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}.$$

Now let $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{L}^p(\Omega)/\mathbb{R}$ be the solution of Problem (3.2.2). Observe that

$$\begin{aligned} \|\boldsymbol{u}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} &= \sup_{\boldsymbol{F} \in \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega), \boldsymbol{F} \neq 0} \frac{|\langle \boldsymbol{u}\,,\, \boldsymbol{F} \rangle_\Omega|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}} \\ &= \sup_{\boldsymbol{F} \in \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega), \boldsymbol{F} \neq 0} \frac{|\langle \boldsymbol{u}\,,\, \lambda\, \boldsymbol{v} - \Delta \boldsymbol{v} - \nabla \theta \rangle_\Omega|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}} \\ &= \sup_{\boldsymbol{F} \in \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega), \boldsymbol{F} \neq 0} \frac{|\langle \lambda\, \boldsymbol{u} - \Delta \boldsymbol{u} - \nabla \pi\,,\, \boldsymbol{v} \rangle_\Omega|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}} \\ &= \sup_{\boldsymbol{F} \in \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega), \boldsymbol{F} \neq 0} \frac{|\langle \boldsymbol{f}\,,\, \boldsymbol{v} \rangle_\Omega|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}} \\ &\leq \frac{C(\Omega, p')}{|\lambda|} \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}. \end{aligned}$$

□

We define the extension of the Stokes operator with Navier slip boundary condition (3.2.1) to the space $[\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'_{\sigma,\tau}$ by

$$\mathbb{B}_p \,:\, \mathbf{D}(\mathbb{B}_p) \subset [\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'_{\sigma,\tau} \longmapsto [\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'_{\sigma,\tau},$$

where

$$\mathbf{D}(\mathbb{B}_p) = \left\{ \boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega);\; \mathrm{div}\, \boldsymbol{u} = 0 \text{ in } \Omega,\; \boldsymbol{u} \cdot \boldsymbol{n} = 0,\; [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0} \text{ on } \Gamma \right\} \tag{3.2.32}$$

and

$$\forall\, \boldsymbol{u} \in \mathbf{D}(\mathbb{B}_p), \qquad \mathbb{B}_p\, \boldsymbol{u} = -\Delta \boldsymbol{u} + \mathbf{grad}\, \pi \qquad \text{in } \Omega, \tag{3.2.33}$$

where π is the unique solution up to an additive constant of the problem

$$\mathrm{div}(\mathbf{grad}\, \pi - \Delta \boldsymbol{u}) = 0 \qquad \text{in } \Omega, \qquad (\mathbf{grad}\, \pi - \Delta \boldsymbol{u}) \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma.$$

Using the density of $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ in $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$(see Proposition 3.1.26), we know that the operator $\mathbb{B}_p$ is a densely defined operator. Moreover, due to Theorem 3.2.10 we have the following theorem:

Theorem 3.2.11. *The operator $-\mathbb{B}_p$ generates a bounded analytic semi-group on $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$ for all $1<p<\infty$.*

We end this section by the following remark, very useful in the sequel

Remark 3.2.12. Since the operator $I+\mathbb{B}_p$ is an isomorphism from $\mathbf{D}(\mathbb{B}_p)$ to $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$, then

$$\forall \boldsymbol{u}\in\mathbf{D}(\mathbb{B}_p),\qquad \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}\simeq\|\boldsymbol{u}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}+\|\mathbb{B}_p\boldsymbol{u}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}. \tag{3.2.34}$$

3.2.3 Analyticity in $(\boldsymbol{T}^{p'}(\Omega))'_{\sigma,\tau}$

In this subsection we give the analyticity of the Stokes operator with Navier slip boundary conditions (3.2.1) in $(\boldsymbol{T}^{p'}(\Omega))'_{\sigma,\tau}$. This gives us very weak solution to the time dependent Stokes problem with the Navier slip boundary conditions.

For this reason we consider the Problem (3.2.2). Using the Green formula (2.2.4), the De Rham Lemma 2.2.7 and a duality argument, we can prove the existence of very weak solution to Problem (3.2.2) and we established the desired resolvent estimate. The proof of the following theorem because is similar to [8, Theorem 5.5] and Theorem 3.1.31.

Theorem 3.2.13. *Let $\lambda\in\mathbb{C}^*$ such that $\mathrm{Re}\,\lambda\geq 0$ and let $\boldsymbol{f}\in(\boldsymbol{T}^{p'}(\Omega))'$ then the Problem (3.2.2) has a unique solution $(\boldsymbol{u},\pi)\in\boldsymbol{L}^p(\Omega)\times W^{-1,p}(\Omega)/\mathbb{R}$. Moreover we have the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\;\leq\;\frac{C(\Omega,p)}{|\lambda|}\,\|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'},$$

for some constant $C(\Omega,p)>0$ independent of λ and $\boldsymbol{f}$.

Proof. Using the Green formula (2.2.4), we can easily check as in the proof of [8, Theorem 5.5] that the problem: Find $\boldsymbol{u}\in\boldsymbol{L}^p(\Omega)$ solution of problem (3.2.2), is equivalent to the problem: Find $\boldsymbol{u}\in\boldsymbol{L}^p(\Omega)$ such that for any function $\boldsymbol{\varphi}\in\boldsymbol{W}^{2,p'}(\Omega)$, such that $\boldsymbol{\varphi}\cdot\boldsymbol{n}=0$, $\mathrm{div}\,\boldsymbol{\varphi}=0$, $[\mathbf{D}(\boldsymbol{\varphi})\boldsymbol{n}]_{\boldsymbol{\tau}}=\mathbf{0}$ on Γ, and for all $q\in W^{1,p'}(\Omega)$

$$\begin{aligned}\lambda\int_\Omega\boldsymbol{u}\cdot\overline{\boldsymbol{\varphi}}\,\mathrm{d}\,x-\int_\Omega\boldsymbol{u}\cdot\Delta\overline{\boldsymbol{\varphi}}\,\mathrm{d}\,x&=\langle\boldsymbol{f},\,\boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)}\\ \int_\Omega\boldsymbol{u}\cdot\nabla\overline{q}\,\mathrm{d}\,x&=0.\end{aligned} \tag{3.2.35}$$

Note that the pressure can be recovered by De Rham argument given by Lemma 2.2.7.

Usinf Theorem 3.2.4 we know that for any $\boldsymbol{F}\in\boldsymbol{L}^{p'}(\Omega)$ there exists a unique solution $(\boldsymbol{\varphi},q)\in\boldsymbol{W}^{2,p'}(\Omega)\times W^{1,p'}(\Omega)/\mathbb{R}$ such that

$$\lambda\boldsymbol{\varphi}-\Delta\boldsymbol{\varphi}+\nabla q=\boldsymbol{F}\quad\text{and}\quad\mathrm{div}\,\boldsymbol{\varphi}=0\quad\text{in}\,\Omega,\quad\boldsymbol{\varphi}\cdot\boldsymbol{n}=0,\quad\text{and}\quad[\mathbf{D}(\boldsymbol{\varphi})\boldsymbol{n}]_{\boldsymbol{\tau}}=\mathbf{0}\quad\text{on}\,\Gamma. \tag{3.2.36}$$

In addition, thanks to Theorem 3.2.7 we have

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\Omega)} \leq \frac{C(\Omega, p')}{|\lambda|}\,\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Let T be the linear form from $\boldsymbol{L}^{p'}(\Omega)$ onto $\mathbb{C}$ defined by:

$$T : \boldsymbol{F} \longmapsto \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times \boldsymbol{T}^{p'}(\Omega)}.$$

where $\boldsymbol{\varphi}$ is the unique solution of (3.2.36). Notice that

$$|L(\boldsymbol{F})| \leq \|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'}\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\Omega)} \leq \frac{C(\Omega, p')}{|\lambda|}\,\|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'}\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Thus, the linear form T is continuous on $\boldsymbol{L}^{p'}(\Omega)$ and there exists a unique $\boldsymbol{u}$ in $\boldsymbol{L}^{p}(\Omega)$ such that

$$L(\boldsymbol{F}) = \int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{F}}\,\mathrm{d}\,x = \langle \boldsymbol{f}\,,\,\boldsymbol{\varphi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)}$$

and satisfying the estimate (3.2.13). On other worlds $\boldsymbol{u}$ is the unique solution of problem (3.2.35). Which completes the proof of the theorem. □

Now we define the extension of the Stokes operator with Navier slip boundary condition (3.2.1) to the space $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ (see (3.1.71) for the definition of the space $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$), as the linear operator

$$\mathbb{C}_p\,:\,\mathbf{D}(C_p)\subset[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}\longmapsto[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau},$$

where

$$\mathbf{D}(\mathbb{C}_p) = \big\{\boldsymbol{u}\in\boldsymbol{L}^p(\Omega);\ \operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega,\ \boldsymbol{u}\cdot\boldsymbol{n} = 0,\ [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0} \text{ on } \Gamma\big\} \tag{3.2.37}$$

and

$$\forall\,\boldsymbol{u}\in\mathbf{D}(\mathbb{C}_p),\qquad \mathbb{C}_p\,\boldsymbol{u} = -\Delta\boldsymbol{u} + \mathbf{grad}\,\pi \qquad \text{in } \Omega, \tag{3.2.38}$$

where π is the unique solution up to an additive constant of the problem

$$\operatorname{div}(\mathbf{grad}\,\pi - \Delta\boldsymbol{u}) = 0 \qquad \text{in } \Omega, \qquad (\mathbf{grad}\,\pi - \Delta\boldsymbol{u})\cdot\boldsymbol{n} = 0 \qquad \text{on } \Gamma.$$

Using the density of of $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ in $(\boldsymbol{T}^{p'}(\Omega))'_{\sigma,\tau}$ (see Proposition 3.1.26), we know that the operator $\mathbb{C}_p$ is a densely defined operator.

Finally, as a consequence of Theorem 3.2.13 we have the following result concerning the analyticity of the Stokes semi-group with Navier boundary conditions on $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$.

Theorem 3.2.14. *The operator $-\mathbb{C}_p$ generates a bounded analytic semi-group on $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$.*

We end this section by the following remark, very useful in the sequel

Remark 3.2.15. Since the operator $I + \mathbb{C}_p$ is an isomorphism from $\mathbf{D}(\mathbb{C}_p)$ to $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$, then

$$\forall\boldsymbol{u}\in\mathbf{D}(\mathbb{C}_p),\qquad \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \simeq \|\boldsymbol{u}\|_{[\boldsymbol{T}^{p'}(\Omega)]'} + \|\mathbb{C}_p\boldsymbol{u}\|_{[\boldsymbol{T}^{p'}(\Omega)]'}. \tag{3.2.39}$$

3.3 Stokes operator with Normal and pressure boundary conditions

In this section we consider the Stokes operator with Normal and pressure boundary conditions

$$\boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{0},\qquad \pi=0\qquad \text{on}\quad \Gamma. \tag{3.3.1}$$

We prove the analyticity of the semi-group generated by the Stokes operator with the boundary condition (3.3.1) in some spaces to be determined. Since the proof of the analyticity is similar to the analyticity of the Stokes semi-group with Navier-type boundary conditions we will skip some details and we will give only a sketch of the proof.

The proofs are based on the study of the following complex resolvent of the Stokes operator :

$$\begin{cases} \lambda\boldsymbol{u}-\Delta\boldsymbol{u}\,+\,\nabla\pi=\boldsymbol{f}, & \operatorname{div}\boldsymbol{u}=0 \qquad \text{in}\quad \Omega,\\ \qquad \boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{0}, & \qquad \pi=0 \qquad\quad \text{on}\quad \Gamma. \end{cases} \tag{3.3.2}$$

Due to boundary conditions (3.3.1) we will see in this section that the pressure can be decoupled from the Problem (3.3.2) using a Dirichlet problem. For this reason we are reduced to study the following Laplacian problem:

$$\begin{cases} \lambda\boldsymbol{u}-\Delta\boldsymbol{u}=\boldsymbol{f}, & \operatorname{div}\boldsymbol{u}=0 \qquad \text{in}\quad \Omega,\\ & \boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{0}, \qquad \text{on}\quad \Gamma. \end{cases} \tag{3.3.3}$$

Now we move to define the Stokes operator with the boundary conditions (3.3.1). For this reason we consider the space

$$\boldsymbol{V}^p_N(\Omega)\;=\;\Big\{\boldsymbol{u}\in\boldsymbol{X}^p_N(\Omega);\;\operatorname{div}\boldsymbol{u}=0\ \text{ in }\Omega\Big\}, \tag{3.3.4}$$

equipped with the norm of $\boldsymbol{X}^p(\Omega)$. Thanks to Lemma 2.2.1, we know that $\boldsymbol{V}^p_N(\Omega)$ is equal to the space

$$\boldsymbol{W}^{1,p}_{\sigma,N}(\Omega)=\{\boldsymbol{u}\in\boldsymbol{W}^{1,p}(\Omega);\;\operatorname{div}\boldsymbol{u}=0\text{ in }\Omega,\;\boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{0}\text{ on }\Gamma\} \tag{3.3.5}$$

with equivalent norm.

The Stokes operator with normal and pressure boundary conditions is a linear mapping

$$\mathcal{A}_p\;:\;\mathbf{D}(\mathcal{A}_p)\subset\boldsymbol{L}^p_\sigma(\Omega)\longmapsto\boldsymbol{L}^p_\sigma(\Omega),$$

where

$$\mathbf{D}(\mathcal{A}_p)=\big\{\boldsymbol{u}\in\boldsymbol{W}^{1,p}(\Omega);\;\Delta\boldsymbol{u}\in\boldsymbol{L}^p(\Omega),\;\operatorname{div}\boldsymbol{u}=0\text{ in }\Omega,\boldsymbol{u}\times\boldsymbol{n}\;=\;\boldsymbol{0}\text{ on }\Gamma\big\} \tag{3.3.6}$$

and

$$\forall\,\boldsymbol{u}\in\boldsymbol{V}^p_N(\Omega),\quad\forall\,\boldsymbol{v}\in\boldsymbol{V}^{p'}_N(\Omega),\qquad \langle\mathcal{A}_p\,\boldsymbol{u}\,,\,\boldsymbol{v}\rangle_{(\boldsymbol{V}^{p'}_N(\Omega))'\times\boldsymbol{V}^{p'}_N(\Omega)}\;=\;\int_\Omega\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}\,x. \tag{3.3.7}$$

The space $\boldsymbol{L}^p_\sigma(\Omega)$ is given by :

$$\boldsymbol{L}^p_\sigma(\Omega) = \{\boldsymbol{f} \in \boldsymbol{L}^p(\Omega);\ \operatorname{div} \boldsymbol{f} = 0 \text{ in } \Omega\} \tag{3.3.8}$$

which is a closed subspace of $\boldsymbol{L}^p(\Omega)$.

A key observation is that

Proposition 3.3.1. *For all $\boldsymbol{u} \in \mathbf{D}(\mathcal{A}_p)$, $\mathcal{A}_p \boldsymbol{u} = -\Delta \boldsymbol{u}$ in Ω.*

Proof. The result follows immediately since the pressure π is harmonic and equal to 0 on Γ. □

The following proposition shows the density of the domain of the Stokes operator

Proposition 3.3.2. *The Stokes operator with the boundary condition* (3.3.1) *is a densely defined operator.*

Proof. Since $\boldsymbol{\mathcal{D}}(\boldsymbol{\Omega})$ is dense in $\boldsymbol{L}^p(\Omega)$, then the space $\big\{\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega);\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\big\}$ is dense in $\boldsymbol{L}^p(\Omega)$. Let $\boldsymbol{u} \in \boldsymbol{L}^p_\sigma(\Omega)$ and $(\boldsymbol{u}_k)_{k\in\mathbb{N}}$ a sequence in $\boldsymbol{W}^{2,p}(\Omega)$ such that $\boldsymbol{u}_k \times \boldsymbol{n} = \boldsymbol{0}$ on Γ for any k in $\mathbb{N}$ and

$$\boldsymbol{u}_k \longrightarrow \boldsymbol{u} \text{ in } \boldsymbol{L}^p(\Omega) \qquad \text{as } k \to +\infty.$$

Now, for any k in $\mathbb{N}$, we consider the unique solution $\chi_k \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ of the Problem

$$\Delta \chi_k = \operatorname{div} \boldsymbol{u}_k \text{ in } \Omega \qquad \text{and} \qquad \chi_k = 0 \qquad \text{on } \Gamma.$$

For all $k \in \mathbb{N}$, χ_k satisfies the estimate

$$\|\chi_k\|_{W^{1,p}(\Omega)} \le C\, \|\operatorname{div} \boldsymbol{u}_k\|_{W^{-1,p}(\Omega)},$$

for some constant $C > 0$. Observe that since $\operatorname{div} \boldsymbol{u}_k \longrightarrow 0$ in $W^{-1,p}(\Omega)$, then $\chi_k \longrightarrow 0$ in $W^{1,p}(\Omega)$. Finally, by setting $\boldsymbol{\varphi}_k = \boldsymbol{u}_k - \mathbf{grad}\chi_k$, we can easily show that

$$\boldsymbol{\varphi}_k \in \boldsymbol{W}^{1,p}(\Omega),\ \Delta\boldsymbol{\varphi}_k \in \boldsymbol{L}^p(\Omega),\ \operatorname{div} \boldsymbol{\varphi}_k = 0 \text{ in } \Omega,\ \boldsymbol{\varphi}_k \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma \text{ and } \boldsymbol{\varphi}_k \longrightarrow \boldsymbol{u} \text{ in } \boldsymbol{L}^p(\Omega),$$

this ends the proof. □

When Ω is of class $C^{2,1}$ we have the following regularity for the domain $\mathbf{D}(\mathcal{A}_p)$ given by (3.3.6).

Lemma 3.3.3. *Suppose that Ω is of class $C^{2,1}$, then*

$$\mathbf{D}(\mathcal{A}_p) = \Big\{\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega);\ \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega,\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\Big\}. \tag{3.3.9}$$

Proof. Let $\boldsymbol{u} \in \mathbf{D}(\mathcal{A}_p)$ and set $\boldsymbol{z} = \mathbf{curl}\, \boldsymbol{u}$. It is clear that $\boldsymbol{z} \in \boldsymbol{L}^p(\Omega)$, $\operatorname{div} \boldsymbol{z} = 0$ in Ω, $\mathbf{curl}\, \boldsymbol{z} = -\Delta \boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{z} \cdot \boldsymbol{n} = 0$ on Γ. Thus $\boldsymbol{z} \in \boldsymbol{X}^p_\tau(\Omega)$ and thanks to Lemma 2.2.1 $\boldsymbol{z} \in \boldsymbol{W}^{1,p}(\Omega)$. Next observe that $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$, $\mathbf{curl}\, \boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$, $\operatorname{div} \boldsymbol{u} = 0$ in Ω and $\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}$ in Γ. Thanks to Lemma 2.2.2, we deduce that that $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$, which ends the proof. □

3.3.1 Analyticity on $\boldsymbol{L}^p_\sigma(\Omega)$

In this subsection we prove the analyticity of the Stokes operator with the boundary condition (3.3.1). As described in Proposition 3.3.1, on $\boldsymbol{L}^p_\sigma(\Omega)$ the Stokes operator coincides with the $-\Delta$ operator. For this reason our work is reduced to study the resolvent problem (3.3.3).

In what follows we use the following formula for the Laplacian operator

$$\Delta\, \boldsymbol{v} = \mathbf{grad}\,(\mathrm{div}\, \boldsymbol{v}) - \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{v} \qquad \text{in} \quad \Omega.$$

The following theorem gives weak and strong solutions for the resolvent Problem (3.3.3). Before we state our theorem we consider the following space

$$[\boldsymbol{H}^p_0(\mathbf{curl},\Omega)]'_\sigma = \{\boldsymbol{f} \in [\boldsymbol{H}^p_0(\mathbf{curl},\Omega)]';\ \mathrm{div}\, \boldsymbol{f} = 0,\ \text{in}\ \Omega\}. \tag{3.3.10}$$

Theorem 3.3.4. *Let $\varepsilon \in\,]0,\pi[$ be fixed and $\lambda \in \Sigma_\varepsilon$, where Σ_ε is given by* (3.1.9).
(i) *If $\boldsymbol{f} \in [\boldsymbol{H}^2_0(\mathbf{curl},\Omega)]'_\sigma$, then Problem (3.3.3) has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$satisfying the following estimate:*

$$\|\boldsymbol{u}\|_{[\boldsymbol{H}^2_0(\mathbf{curl},\Omega)]'} \leq \frac{C_\varepsilon}{|\lambda|}\, \|\boldsymbol{f}\|_{[\boldsymbol{H}^2_0(\mathbf{curl},\Omega)]'}. \tag{3.3.11}$$

(ii) *Moreover, if $\boldsymbol{f} \in \boldsymbol{L}^2_\sigma(\Omega)$ and Ω is of class $C^{2,1}$, then $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$ and satisfies the estimates:*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C_\varepsilon}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}, \tag{3.3.12}$$

$$\|\mathbf{curl}\, \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq \frac{C_\varepsilon}{\sqrt{|\lambda|}}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}. \tag{3.3.13}$$

and

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} \leq \frac{C(\Omega,\lambda,\varepsilon)}{|\lambda|}\, \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}, \tag{3.3.14}$$

for some constant $C_\varepsilon > 0$ and $C(\Omega,\lambda,\varepsilon) = C(\Omega)(C_\epsilon + 1)(1 + |\lambda|)$.

Proof. The proof of the existence is similar to the proof of Theorem 3.1.8 and it is is done in two steps:
(i) Variational formulation: Consider the variational problem: Find $\boldsymbol{u} \in \boldsymbol{V}^2_N(\Omega)$ such that for any $\boldsymbol{v} \in \boldsymbol{V}^2_N(\Omega)$

$$a(\boldsymbol{u},\boldsymbol{v}) = \langle \boldsymbol{f}\,,\, \boldsymbol{v}\rangle_\Omega, \tag{3.3.15}$$

where

$$a(\boldsymbol{u},\boldsymbol{v}) = \lambda \int_\Omega \boldsymbol{u}\cdot \overline{\boldsymbol{v}}\, \mathrm{d}x + \int_\Omega \mathbf{curl}\, \boldsymbol{u}\cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x$$

and

$$\langle .\,,\,.\rangle_\Omega = \langle .\,,\,.\rangle_{[\boldsymbol{H}^2_0(\mathbf{curl},\Omega)]'\times \boldsymbol{H}^2_0(\mathbf{curl},\Omega)}.$$

As in the proof of Theorem 3.1.8, we can easily verify that a is a continuous coercive sesquilinear form on $\boldsymbol{V}^2_N(\Omega)$. Thus due to Lax-Milgram Lemma, Problem (3.3.15) has a unique solution $\boldsymbol{u} \in \boldsymbol{V}^2_N(\Omega)$ since the right-hand side belongs to the anti-dual $(\boldsymbol{V}^2_N(\Omega))'$.

(ii) Equivalent problem: Now we want to extend (3.3.15) to any test function $\boldsymbol{v} \in \boldsymbol{X}^2_N(\Omega)$. In fact, we proceed exactly in the same way as in [11, Proposition 4.2]. Let $\boldsymbol{v} \in \boldsymbol{X}^2_N(\Omega)$ and consider the unique solution $\chi \in H^2(\Omega) \cap H^1_0(\Omega)$ of the problem

$$\Delta\chi = \operatorname{div}\boldsymbol{v} \;\text{ in }\; \Omega, \qquad \chi = 0 \;\text{ on }\; \Gamma.$$

Setting

$$\boldsymbol{\varphi} \,=\, \boldsymbol{v} \,-\, \mathbf{grad}\,\chi$$

we can easily verify that $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\Omega)$, $\operatorname{div}\boldsymbol{\varphi} = 0$ in Ω, $\mathbf{curl}\,\boldsymbol{v} \,=\, \mathbf{curl}\,\boldsymbol{\varphi} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{\varphi}\times\boldsymbol{n} = \boldsymbol{0}$ on Γ (*i.e.* $\boldsymbol{\varphi} \in \boldsymbol{V}^2_N(\Omega)$). It is clear that $\mathbf{grad}\,\chi$ belongs to $\boldsymbol{H}^2_0(\mathbf{curl},\Omega)$ and using the density of $\mathcal{D}(\Omega)$ in $H^1_0(\Omega)$ we can easily verify that

$$\langle \boldsymbol{f}\,,\, \mathbf{grad}\,\chi\rangle_\Omega \,=\, 0.$$

As a result

$$\langle \boldsymbol{f}\,,\, \boldsymbol{v}\rangle_\Omega \,=\, \langle \boldsymbol{f}\,,\, \boldsymbol{\varphi}\rangle_\Omega.$$

We recall that $\langle .\,,\,.\rangle_\Omega = \langle .\,,\,.\rangle_{[\boldsymbol{H}^2_0(\mathbf{curl},\Omega)]'\times\boldsymbol{H}^2_0(\mathbf{curl},\Omega)}$.
On the other hand since $\operatorname{div}\boldsymbol{u} \,=\, 0$ in Ω, we can easily verify that

$$\int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x \,=\, \int_\Omega \boldsymbol{u}\cdot\bar{\boldsymbol{\varphi}}\,\mathrm{d}\,x.$$

As a consequence we obtain $a(\boldsymbol{u},\boldsymbol{v}) = a(\boldsymbol{u},\boldsymbol{\varphi})$ and then Problem (3.3.15) is equivalent to the problem: Find $\boldsymbol{u} \in \boldsymbol{V}^2_N(\Omega)$ such that for all $\boldsymbol{v} \in \boldsymbol{X}^2_N(\Omega)$

$$\lambda\int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{v}}\;\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\;\mathrm{d}x \;\;=\;\; \langle \boldsymbol{f},\boldsymbol{v}\rangle_\Omega. \tag{3.3.16}$$

Next, we can easily check that Problem (3.3.3) is equivalent to Problem (3.3.16) and thus Problem (3.3.3) has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$.

Now suppose that $\boldsymbol{f} \in \boldsymbol{L}^2_\sigma(\Omega)$ and that Ω is of class $C^{2,1}$. Proceeding exactly in the same way as in Lemma 3.3.3 we deduce that that $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$.

The proof of the estimate (3.3.11)-(3.3.14) is done in the same way as in the proof Theorem 3.1.8. □

As a consequence, we have the following result

Corollary 3.3.5. *The operator $-\mathcal{A}_2$ generates a bounded analytic semi-group on $\boldsymbol{L}^2_\sigma(\Omega)$.*

Remark 3.3.6. Consider the sesqui-linear form (see [5]):

$$\forall\, \boldsymbol{u},\, \boldsymbol{v} \in \boldsymbol{V}^2_N(\Omega), \qquad a(\boldsymbol{u}, \boldsymbol{v}) \;=\; \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}\, x. \tag{3.3.17}$$

If the boundary Γ is connected, we know that for all $\boldsymbol{v} \in \boldsymbol{V}^2_N(\Omega)$ one has

$$\|\boldsymbol{v}\|_{\boldsymbol{X}^2(\Omega)} \leq C\, \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}. \tag{3.3.18}$$

As a result, the sesqui-linear form a is coercive and we can apply Lax-Milgram Lemma to find solution to the problem: Find $\boldsymbol{u} \in \boldsymbol{V}^2_N(\Omega)$ such that for all $\boldsymbol{v} \in \boldsymbol{V}^2_N(\Omega)$

$$a(\boldsymbol{u}, \boldsymbol{v}) \;=\; \int_\Omega \boldsymbol{f} \cdot \overline{\boldsymbol{v}} \,\mathrm{d}\, x,$$

where $\boldsymbol{f} \in \boldsymbol{L}^2_\sigma(\Omega)$. This means that the Stokes operator $\mathcal{A}_2 \,:\, \mathbf{D}(\mathcal{A}_2) \subset \boldsymbol{L}^2_\sigma(\Omega) \longmapsto \boldsymbol{L}^2_\sigma(\Omega)$ is bijective.

Now, if the boundary Γ is not connected, the inequality (3.3.18) is false. Indeed we introduce the Kernel $\boldsymbol{K}^2_N(\Omega)$:

$$\boldsymbol{K}^2_N(\Omega) \;=\; \big\{\boldsymbol{v} \in \boldsymbol{X}^2_N(\Omega);\; \mathrm{div}\, \boldsymbol{v} = 0,\; \mathbf{curl}\, \boldsymbol{v} = \mathbf{0} \text{ in } \Omega\big\}. \tag{3.3.19}$$

Thanks to [5, Proposition 3.18] we know that this kernel is not trivial, is of finite dimension and is spanned by the functions ∇q^N_i, $i = 1\ldots, I$, where q^N_i is the unique solution in $H^2(\Omega)$ of the problem

$$\begin{cases} -\Delta q^N_i = 0 \qquad \text{in } \Omega, \\ q^N_i|_{\Gamma_0} = 0 \quad \text{and} \quad q^N_i|_{\Gamma_k} = \text{constant}, \;\; 1 \leq k \leq I, \\ \big\langle \partial_n\, q^N_i,\, 1\big\rangle_{\Gamma_k} = \delta_{i\,k},\;\; 1 \leq k \leq I, \quad \text{and} \quad \big\langle \partial_n\, q^N_i,\, 1\big\rangle_{\Gamma_0} = -1. \end{cases} \tag{3.3.20}$$

Moreover, thanks to [5, Corollary 3.19], for all $\boldsymbol{v} \in \boldsymbol{X}^2_N(\Omega)$ we have the following Poincaré-type inequality:

$$\|\boldsymbol{v}\|_{\boldsymbol{X}^2_N(\Omega)} \leq C(\Omega)(\|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \,+\, \|\mathrm{div}\, \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \,+\, \sum_{i=1}^{I} |\langle \boldsymbol{v} \cdot \boldsymbol{n}\,,\, 1\rangle_{\Gamma_i}|). \tag{3.3.21}$$

The following theorem extends Theorem 3.3.4 to every $1 < p < \infty$.

Theorem 3.3.7. *Let $\lambda \in \Sigma_\varepsilon$ and let $\boldsymbol{f} \in [\boldsymbol{H}^{p'}_0(\mathbf{curl}, \Omega)]'_\sigma$. The Problem (3.3.3) has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$. Moreover if Ω is of class $C^{2,1}$ and $\boldsymbol{f} \in \boldsymbol{L}^p_\sigma(\Omega)$, then $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$.*

Proof. As in the proof of Theorem 3.3.4 we can easily verify that Problem (3.3.3) is equivalent to the variational problem: Find $\boldsymbol{u} \in \boldsymbol{V}^p_N(\Omega)$ such that for all $\boldsymbol{v} \in \boldsymbol{X}^{p'}_N(\Omega)$

$$\lambda \int_\Omega \boldsymbol{u} \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x + \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\; \mathrm{d}x \;=\; \int_\Omega \boldsymbol{f} \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x.$$

The proof is done in three steps and it is similar to the proof of Theorem 3.1.15:
(i) Case $2 \le p \le 6$. Since $[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]' \hookrightarrow [\boldsymbol{H}_0^2(\mathbf{curl},\Omega)]'$, Problem (3.3.3) has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$. We write Problem (3.3.3) in the form:

$$\begin{cases} -\Delta \boldsymbol{u} = \boldsymbol{F} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma, \end{cases} \tag{3.3.22}$$

where $\boldsymbol{F} = \boldsymbol{f} - \lambda\, \boldsymbol{u}$. Observe that for $p \le 6$ we have:

$$\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega) \hookrightarrow \boldsymbol{L}^{p'}(\Omega) \hookrightarrow \boldsymbol{L}^{6/5}(\Omega),$$

thanks to the embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$ one has $\boldsymbol{F} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'_\sigma$. Next, thanks to [11] we can easily verify that $\boldsymbol{F}$ satisfies the compatibility condition:

$$\forall\, \boldsymbol{v} \in \boldsymbol{K}_N^{p'}(\Omega), \quad \langle \boldsymbol{F}, \boldsymbol{v}\rangle_\Omega = 0, \tag{3.3.23}$$

where

$$\boldsymbol{K}_N^{p'}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^{p'}(\Omega),\ \operatorname{div} \boldsymbol{v} = 0,\ \mathbf{curl}\, \boldsymbol{v} = \boldsymbol{0} \text{ in } \Omega \quad \text{and} \quad \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\}.$$

Thanks to [11, Proposition 4.2], the solution $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$.

(ii) Case $p \ge 6$. Since $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]' \hookrightarrow [\boldsymbol{H}_0^{6/5}(\mathbf{curl},\Omega)]'$,, Problem (3.3.3) has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,6}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$. Now proceeding in the same way as above one gets that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$.

(iii) Case $p \le 2$. The existence of solutions is obtained by a duality argument.

Finally, if Ω is of class $C^{2,1}(\Omega)$ and $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, due to Lemma 3.3.3, the solution $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$. □

Now, we want to prove a resolvent estimate similar to the estimate (3.3.12) for any $1 < p < \infty$. We begin with the case where the mean curvature K defined in (2.2.15) is positive.

Proposition 3.3.8. *Let $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re}\lambda \ge 0$, $\boldsymbol{f} \in \boldsymbol{L}^p_\sigma(\Omega)$ and let $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ be the unique solution of Problem* (3.3.3). *Suppose moreover that the mean curvature K is positive. Then $\boldsymbol{u}$ satisfies the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \le \frac{C_p}{|\lambda|}\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{3.3.24}$$

where $C_p = \sqrt{\frac{p^2}{4} + 1}$.

Proof. Suppose that $p \geq 2$. Thanks to Lemma 2.2.10, multiplying the first equation of Problem (3.3.3) by $|\boldsymbol{u}|^{p-2}\,\overline{\boldsymbol{u}}$, and integrating both sides one gets

$$\lambda \int_\Omega |\boldsymbol{u}|^p \,\mathrm{d}\,x + \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2 \,\mathrm{d}\,x + 4\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\,x$$
$$+ (p-2)\,i\sum_{k=1}^{3}\int_\Omega |\boldsymbol{u}|^{p-4}\,\mathrm{Re}\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\mathrm{Im}\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\,\mathrm{d}\,x - \langle\frac{\partial\,\boldsymbol{u}}{\partial\,\boldsymbol{n}}\,,\,|\boldsymbol{u}|^{p-2}\boldsymbol{u}\rangle_\Gamma$$
$$= \int_\Omega |\boldsymbol{u}|^{p-2}\,\boldsymbol{f}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\,x, \quad (3.3.25)$$

where

$$\langle\,.\,,\,.\rangle_\Gamma = \langle\,.\,,\,.\rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma)\times\boldsymbol{W}^{1/p,p'}(\Gamma)}.$$

Since $\operatorname{div}\boldsymbol{u} = 0$ and $\boldsymbol{u}\times\boldsymbol{n} = 0$ on Γ then using Formula (2.2.15) and Corollary 2.2.13, we have

$$\frac{\partial\,\boldsymbol{u}}{\partial\,\boldsymbol{n}}\cdot\boldsymbol{n} = -\,2\,K\,\boldsymbol{u}\cdot\boldsymbol{n}, \qquad \text{on} \quad \Gamma. \quad (3.3.26)$$

Using the fact that $\boldsymbol{u}\cdot\boldsymbol{n}$ belongs to $W^{1-1/p,p}(\Gamma)$, we can write:

$$\langle\frac{\partial\,\boldsymbol{u}}{\partial\,\boldsymbol{n}}\,,\,|\boldsymbol{u}|^{p-2}\boldsymbol{u}\rangle_\Gamma = \langle\Big(\frac{\partial\,\boldsymbol{u}}{\partial\,\boldsymbol{n}}\cdot\boldsymbol{n}\Big)\boldsymbol{n}\,,\,|\boldsymbol{u}|^{p-2}\,(\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}\rangle_\Gamma = \int_\Gamma |\boldsymbol{u}|^{p-2}\Big(\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}}\cdot\boldsymbol{n}\Big)\overline{(\boldsymbol{u}\cdot\boldsymbol{n})}\,\mathrm{d}\,\sigma, \quad (3.3.27)$$

and

$$\Big(\frac{\partial\,\boldsymbol{u}}{\partial\,\boldsymbol{n}}\cdot\boldsymbol{n}\Big)\overline{(\boldsymbol{u}\cdot\boldsymbol{n})} = -\,2\,K\,|\boldsymbol{u}\cdot\boldsymbol{n}|^2 = -\,2\,K\,|\boldsymbol{u}|^2. \quad (3.3.28)$$

Putting together (3.3.25), (3.3.27) and (3.3.28) one gets

$$\lambda \int_\Omega |\boldsymbol{u}|^p \,\mathrm{d}\,x + \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2 \,\mathrm{d}\,x + 4\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\,x$$
$$+ (p-2)\,i\sum_{k=1}^{3}\int_\Omega |\boldsymbol{u}|^{p-4}\,\mathrm{Re}\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\mathrm{Im}\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\,\mathrm{d}\,x + 2\int_\Gamma K\,|\boldsymbol{u}|^p\,\mathrm{d}\,\sigma$$
$$= \int_\Omega |\boldsymbol{u}|^{p-2}\,\boldsymbol{f}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\,x.$$

As a result

$$(\mathrm{Re}\,\lambda)\int_\Omega |\boldsymbol{u}|^p \,\mathrm{d}\,x + \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2 \,\mathrm{d}\,x + 4\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\,x + 2\int_\Gamma K\,|\boldsymbol{u}|^p\,\mathrm{d}\,\sigma$$
$$= \mathrm{Re}\int_\Omega |\boldsymbol{u}|^{p-2}\,\boldsymbol{f}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\,x. \quad (3.3.29)$$

Observe that since $K > 0$ then

$$\mathrm{Re}\,\lambda\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \quad (3.3.30)$$

and

$$\int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2\,\mathrm{d}\,x \leq \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}. \quad (3.3.31)$$

Moreover

$$\operatorname{Im}\lambda\int_\Omega |\boldsymbol{u}|^p \,\mathrm{d}\,x \;+\; (p-2)\sum_{k=1}^{3}\int_\Omega |\boldsymbol{u}|^{p-4}\operatorname{Re}\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\operatorname{Im}\Big(\frac{\partial\,\boldsymbol{u}}{\partial x_k}\cdot\overline{\boldsymbol{u}}\Big)\,\mathrm{d}\,x \;= \\ \operatorname{Im}\int_\Omega |\boldsymbol{u}|^{p-2}\,\boldsymbol{f}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\,x. \quad (3.3.32)$$

Then putting together (3.3.31) and (3.3.32) one has

$$|\operatorname{Im}\lambda|\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \le \frac{p}{2}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \quad (3.3.33)$$

Finally putting together (3.3.30) and (3.3.33) one gets

$$|\lambda|^2\|\boldsymbol{u}\|^2_{\boldsymbol{L}^p(\Omega)} \;\le\; (\frac{p^2}{4}\;+\;1)\,\|\boldsymbol{f}\|^2_{\boldsymbol{L}^p(\Omega)}.$$

which is estimate (3.3.24). By duality, we obtain this estimate for $1 < p < \infty$. □

In the case where the mean curvature is arbitrary we have the following result :

Theorem 3.3.9. *Assume that Ω is of class $C^{2,1}$. Let $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re}\lambda \ge 0$, let $1 < p < \infty$, $\boldsymbol{f} \in \boldsymbol{L}^p_\sigma(\Omega)$ and let $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ be the unique solution of Problem (3.3.3). Then $\boldsymbol{u}$ satisfies the estimates*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \;\le\; \frac{\kappa_1(\Omega,p)}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \quad (3.3.34)$$

$$\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \;\le\; \frac{\kappa_2(\Omega,p)}{\sqrt{|\lambda|}}\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \quad (3.3.35)$$

and

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \;\le\; \kappa_3(\Omega,p)\,\frac{1+|\lambda|}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \quad (3.3.36)$$

Proof. Proceeding exactly in the same way as in Proposition 3.3.8 and using (3.3.29) one has

$$\operatorname{Re}\lambda\int_\Omega |\boldsymbol{u}|^p\,\mathrm{d}\,x \;+\; \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2\,\mathrm{d}\,x \;+\; 4\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2\,\mathrm{d}\,x \\ = \;-2\int_\Gamma K\,|\boldsymbol{u}|^p\,\mathrm{d}\,\sigma + \operatorname{Re}\int_\Omega |\boldsymbol{u}|^{p-2}\,\boldsymbol{f}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\,x. \quad (3.3.37)$$

Observe that because Ω is of class $C^{2,1}$ we have

$$\|K\|_{\boldsymbol{L}^\infty(\Omega)} \le C_1(\Omega). \quad (3.3.38)$$

Next as in the proof of Proposition 3.1.16 one has

$$|\lambda|\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega |\boldsymbol{u}|^{p-2}\,|\nabla\,\boldsymbol{u}|^2\,\mathrm{d}\,x \;+\; 2\,\frac{p-2}{p^2}\int_\Omega |\nabla\,|\boldsymbol{u}|^{p/2}|^2\,\mathrm{d}\,x \\ \le\; C_2(\Omega,p)\,\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} \;+\; \frac{p-2}{2}\int_\Omega |\boldsymbol{u}|^{p-2}|\nabla\boldsymbol{u}|^2\,\mathrm{d}\,x \;+\; 2\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}.$$

We set

$$\lambda_0 = 2\, C_2(\Omega, p), \tag{3.3.39}$$

the proof is done in two steps.

Step 1. We suppose that $|\lambda| \geq \lambda_0$, then

$$\begin{aligned} \frac{|\lambda|}{2}\, \|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \int_\Omega |\boldsymbol{u}|^{p-2}\, |\nabla\, \boldsymbol{u}|^2 \,\mathrm{d}\, x \,+\, 2\,\frac{p-2}{p^2} \int_\Omega |\nabla\, |\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\, x \\ \leq\, \frac{p-2}{2} \int_\Omega |\boldsymbol{u}|^{p-2} |\nabla \boldsymbol{u}|^2 \,\mathrm{d}\, x \,+\, 2\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}. \end{aligned}$$

We distinguish two different cases
(i) **case** $2 \leq p \leq 4$. In this case, we can write:

$$\begin{aligned} \frac{|\lambda|}{2}\, \|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} + \,\frac{4-p}{2} \int_\Omega |\boldsymbol{u}|^{p-2}\, |\nabla\, \boldsymbol{u}|^2 \,\mathrm{d}\, x \,+\, 2\,\frac{p-2}{p^2} \int_\Omega |\nabla\, |\boldsymbol{u}|^{p/2}|^2 \,\mathrm{d}\, x \leq \\ 2\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}. \end{aligned}$$

Thus

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \,\leq\, \frac{4}{|\lambda|}\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{3.3.40}$$

which is the required estimation.
(ii) **case** $p > 4$. We write our problem in the form (3.3.22). Thanks to [11, Proposition 4.2], we know that

$$\|\boldsymbol{u} - \sum_{i=1}^{I} \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i} \nabla\, q_i^N\|_{\boldsymbol{W}^{1,4}(\Omega)} \leq C_3(\Omega)\, \|\boldsymbol{f} \,-\, \lambda\, \boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)}$$

Thus, using (3.3.40) with $p = 4$ and substituting in the last inequality, we obtain

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,4}(\Omega)} \leq \|\sum_{i=1}^{I} \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i} \nabla\, q_i^N\|_{\boldsymbol{W}^{1,4}(\Omega)} + 5\, C_3(\Omega) \|\boldsymbol{f}\|_{\boldsymbol{L}^4(\Omega)} \tag{3.3.41}$$

Moreover, thanks to [11] we have:

$$|\langle \boldsymbol{u} \cdot \boldsymbol{n}\,,\, 1\rangle_{\Gamma_i}| \,\leq\, C_4(\Omega)\, \|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)} \,\leq\, \frac{C_4(\Omega)}{|\lambda|}\, \|\boldsymbol{f}\|_{\boldsymbol{L}^4(\Omega)} \leq \frac{C_4(\Omega)}{\lambda_0}\, \|\boldsymbol{f}\|_{\boldsymbol{L}^4(\Omega)}.$$

As a result, substituting this in (3.3.41) one gets

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,4}(\Omega)} \leq\, C_5(\Omega)\, \|\boldsymbol{f}\|_{\boldsymbol{L}^4(\Omega)}.$$

Now, since $\boldsymbol{W}^{1,4}(\Omega) \,\hookrightarrow\, \boldsymbol{L}^\infty(\Omega)$ one has directly

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \,\leq\, C_6(\Omega, p)\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.$$

Thus

$$\|\boldsymbol{u}\|^p_{\boldsymbol{L}^p(\Omega)} = \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)} \leq C_7(\Omega,p)\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{u}\|^{p-1}_{\boldsymbol{L}^p(\Omega)}.$$

Proceeding in the same way as in Proposition 3.1.16 case $p > 4$ one gets

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C_7(\Omega,p)}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{3.3.42}$$

with some constant $C_7(\Omega,p) > 0$ independent of λ and $\boldsymbol{f}$, which ends the case $p > 4$.

Finally putting together (3.3.40) and (3.3.42) we obtain estimate (3.3.34) for all $p \geq 2$ with

$$\kappa_1(\Omega,p) = \max(4, C_7(\Omega,p)).$$

By duality argument estimate (3.3.34) holds for all $p < 2$.

Step 2. We suppose that $0 < |\lambda| \leq \lambda_0$. We proceed in the same way as in Proposition 3.1.17 we obtain the desired resolvent estimate.

For the proof of estimate (3.3.35) and (3.3.36), we proceed exactly as Theorem 3.1.18. □

As a consequence, the above results allow us to deduce the analyticity of the semi-group generated by the Stokes operator with normal boundary conditions on $\boldsymbol{L}^p_\sigma(\Omega)$.

Theorem 3.3.10. *The operator $-\mathcal{A}_p$ generates a bounded analytic semigroup on $\boldsymbol{L}^p_\sigma(\Omega)$ for all $1 < p < \infty$.*

3.3.2 Analyticity on $[\boldsymbol{H}^{p'}_0(\mathbf{curl},\,\Omega)]'_\sigma$

In this subsection we give the analyticity of the Stokes semi-group with the boundary condition (3.3.1) on the dual space $[\boldsymbol{H}^{p'}_0(\mathbf{curl},\,\Omega)]'_\sigma$ (given by (3.3.10)). This allows us to obtain weak solution for the time dependent Stokes problem with the corresponding boundary conditions. To this end, we consider the Problem (3.3.2). The following theorem gives weak solutions to Problem (3.3.2) as well as a resolvent estimate with respect to the norm of $[\boldsymbol{H}^{p'}_0(\mathbf{curl},\,\Omega)]'$.

Theorem 3.3.11. *Let $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re}\lambda \geq 0$ and let $\boldsymbol{f} \in [\boldsymbol{H}^{p'}_0(\mathbf{curl},\Omega)]'$. The Problem (3.3.2) has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,p}(\Omega)$ satisfying*

$$\|\boldsymbol{u}\|_{[\boldsymbol{H}^{p'}_0(\mathbf{curl},\Omega)]'} \leq \frac{C(\Omega,p)}{|\lambda|}\|\boldsymbol{f}\|_{[\boldsymbol{H}^{p'}_0(\mathbf{curl},\Omega)]'} \tag{3.3.43}$$

for some constant $C(\Omega,p) > 0$ independent of λ and $\boldsymbol{f}$.

Proof. **(i)** For the existence of solution we proceed as follows: Consider the problem

$$\Delta\pi = \operatorname{div}\boldsymbol{f} \ \text{ in } \Omega, \qquad \pi = 0 \ \text{ on } \Gamma. \tag{3.3.44}$$

Since $\operatorname{div} \boldsymbol{f} \in W^{-1,p}(\Omega)$, Problem (3.3.44) has a unique solution $\pi \in W_0^{1,p}(\Omega)$. Set $\boldsymbol{F} = \boldsymbol{f} - \nabla\pi$, we can easily verify that $\boldsymbol{F} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'_\sigma$. Then thanks to Theorem 3.3.7, the problem

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} = \boldsymbol{F}, \qquad \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega, \\ \qquad\qquad\qquad \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}, & \text{on} \quad \Gamma. \end{cases}$$

has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$. As a result Problem (3.3.2) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$.

(ii) To prove estimate (3.3.43) we proceed as follows: Let $\boldsymbol{F} \in \boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)$ and consider the problem:

$$\begin{cases} \lambda \boldsymbol{v} - \Delta \boldsymbol{v} - \nabla\theta = \boldsymbol{F}, \quad \operatorname{div} \boldsymbol{v} = 0 & \text{in} \quad \Omega, \\ \qquad \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}, \qquad\qquad \theta = 0 & \text{on} \quad \Gamma, \end{cases} \tag{3.3.45}$$

where $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re} \lambda \geq 0$. It is clear that θ is solution of following Dirichlet problem

$$-\Delta\theta = \operatorname{div} \boldsymbol{F} \quad \text{in } \Omega \qquad \text{and} \qquad \theta = 0 \qquad \text{on } \Gamma.$$

Then, θ belongs to $W_0^{1,p'}(\Omega)$ and satisfies the estimate

$$\|\theta\|_{\boldsymbol{W}^{1,p'}(\Omega)} \leq C(\Omega, p') \, \|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Moreover, by moving $\nabla\theta$ in problem (3.3.45) to the right hand side and using Theorem 3.3.9, problem (3.3.45) has a unique solution $\boldsymbol{v} \in \boldsymbol{W}^{1,p'}(\Omega)$ satisfying the estimate:

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)} \leq \frac{C(\Omega, p')}{|\lambda|} \, \|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}. \tag{3.3.46}$$

Applying the curl operator to the first equation of (3.3.45), we obtain that $\boldsymbol{z} = \mathbf{curl}\, \boldsymbol{v}$ is solution of the following problem:

$$\begin{cases} \lambda \boldsymbol{z} - \Delta \boldsymbol{z} = \mathbf{curl}\, \boldsymbol{F}, \qquad \operatorname{div} \boldsymbol{z} = 0 & \text{in} \quad \Omega, \\ \qquad \boldsymbol{z} \cdot \boldsymbol{n} = 0, \qquad \mathbf{curl}\, \boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{0} & \text{on} \quad \Gamma. \end{cases} \tag{3.3.47}$$

The boundary conditions satisfied by $\boldsymbol{z}$ are obtained using the boundary conditions satisfied by $\boldsymbol{u}$ and $\boldsymbol{F}$. In fact, since $\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ then $\boldsymbol{z} \cdot \boldsymbol{n} = \mathbf{curl}\, \boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ. Moreover, since $\boldsymbol{F} \in \boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)$ (*i.e.* $\boldsymbol{F} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ) and $\Delta \boldsymbol{u} = \boldsymbol{F} - \lambda \boldsymbol{u}$ in Ω then $\Delta \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ and $\mathbf{curl}\, \boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ.

Thanks to Theorem 3.1.15 and Theorem 3.1.18, $\boldsymbol{z}$ belongs to $\boldsymbol{W}^{1,p'}(\Omega)$ and satisfies the estimate:

$$\|\boldsymbol{z}\|_{\boldsymbol{L}^{p'}(\Omega)} \leq \frac{C(\Omega, p')}{|\lambda|} \, \|\mathbf{curl} \boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}. \tag{3.3.48}$$

Putting together (3.3.46) and (3.3.48), we deduce that

$$\|\boldsymbol{v}\|_{\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)} \leq \frac{C(\Omega, p')}{|\lambda|} \, \|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)}.$$

Now let $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,p}(\Omega)$ be the solution of Problem (3.3.2), then

$$
\begin{aligned}
\|\boldsymbol{u}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'} &= \sup_{\boldsymbol{F}\in \boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega), \boldsymbol{F}\neq 0} \frac{|\langle \boldsymbol{u}\,,\, \boldsymbol{F}\rangle_\Omega|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)}} \\
&= \sup_{\boldsymbol{F}\in \boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega), \boldsymbol{F}\neq 0} \frac{|\langle \boldsymbol{u}\,,\, \lambda\,\boldsymbol{v} - \Delta \boldsymbol{v} - \nabla\theta\rangle_\Omega|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)}} \\
&= \sup_{\boldsymbol{F}\in \boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega), \boldsymbol{F}\neq 0} \frac{|\langle \lambda\,\boldsymbol{u} - \Delta \boldsymbol{u} - \nabla\pi\,,\, \boldsymbol{v}\rangle_\Omega|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)}} \\
&= \sup_{\boldsymbol{F}\in \boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega), \boldsymbol{F}\neq 0} \frac{|\langle \boldsymbol{f}\,,\, \boldsymbol{v}\rangle_\Omega|}{\|\boldsymbol{F}\|_{\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)}} \\
&\leq \frac{C(\Omega, p')}{|\lambda|}\, \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'},
\end{aligned}
$$

which is estimate (3.3.43). □

We can define the extension of the Stokes operator $\mathcal{A}_p$ to the space $[\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'_\sigma$ and it is a closed linear densely defined operator

$$
\mathcal{B}_p \,:\, \mathbf{D}(\mathcal{B}_p) \subset [\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'_\sigma \longmapsto [\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'_\sigma, \tag{3.3.49}
$$

and

$$
\forall\, \boldsymbol{u} \in \mathbf{D}(\mathcal{B}_p), \qquad \mathcal{B}_p\, \boldsymbol{u} = -\Delta \boldsymbol{u} \qquad \text{in } \Omega. \tag{3.3.50}
$$

The domain of $\mathcal{B}_p$ is given by

$$
\mathbf{D}(\mathcal{B}_p) \;=\; \big\{\boldsymbol{u} \,\in\, \boldsymbol{W}^{1,p}(\Omega);\; \Delta \boldsymbol{u} \,\in\, [\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]',\; \operatorname{div} \boldsymbol{u} \,=\, 0 \text{ in } \Omega,\;\; \boldsymbol{u} \times \boldsymbol{n} \;=\; \mathbf{0} \text{ on } \Gamma\big\}. \tag{3.3.51}
$$

The operator $\mathcal{B}_p$ is a densely defined operator, the proof is done in two steps.

Proposition 3.3.12. *The space* $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ *is dense in* $[\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'_\sigma$

Proof. Let $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'_\sigma$, and let $\boldsymbol{\psi} \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{\xi} \in \boldsymbol{L}^p(\Omega)$ such that $\boldsymbol{f} \;=\; \boldsymbol{\psi} + \mathbf{curl}\, \xi$. Since $\operatorname{div} \boldsymbol{f} = \operatorname{div} \boldsymbol{\psi} = 0$, then there exists a sequence $(\boldsymbol{\psi_k})_k$ in $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ such that

$$
\boldsymbol{\psi}_k \longrightarrow \boldsymbol{\psi} \qquad \text{in } \boldsymbol{L}^p(\Omega), \qquad \text{as } k \to +\infty.
$$

On the other hand there exists a sequence $(\boldsymbol{\xi_k})_k$ in $\boldsymbol{\mathcal{D}}(\Omega)$ such that

$$
\boldsymbol{\xi}_k \longrightarrow \boldsymbol{\xi} \qquad \text{in } \boldsymbol{L}^p(\Omega), \qquad \text{as } k \to +\infty.
$$

i.e.

$$
\mathbf{curl}\, \boldsymbol{\xi}_k \longrightarrow \mathbf{curl}\, \boldsymbol{\xi} \qquad \text{in } [\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]' \qquad \text{as } k \to +\infty.
$$

For all $k \in \mathbb{N}$ set

$$
\boldsymbol{f}_k = \boldsymbol{\psi}_k + \mathbf{curl}\, \boldsymbol{\xi}_k.
$$

It is clear that $(\boldsymbol{f}_k)_k$ is a sequence in $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ and $(\boldsymbol{f}_k)_k$ converges to $\boldsymbol{f}$ in $[\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'$, which ends the proof. □

Corollary 3.3.13. *The space* $\mathbf{D}(\mathcal{B}_p)$ *defined by (3.3.51) is dense in* $[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'_\sigma$

Proof. First observe that $\mathbf{D}(\mathcal{A}_p) \subset \mathbf{D}(\mathcal{B}_p) \subset \boldsymbol{L}^p_\sigma(\Omega)$, where $\mathbf{D}(\mathcal{A}_p)$ is given by (3.3.6). Since $\mathbf{D}(\mathcal{A}_p)$ is dense in $\boldsymbol{L}^p_\sigma(\Omega)$ (see Proposition 3.3.2), then $\mathbf{D}(\mathcal{B}_p)$ is dense in $\boldsymbol{L}^p_\sigma(\Omega)$. Second thanks to Proposition 3.3.12, we know that $\boldsymbol{L}^p_\sigma(\Omega)$ is dense in $[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'_\sigma$. As a result we recover the density of $\mathbf{D}(\mathcal{B}_p)$ in $[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'_\sigma$. ☐

Theorem 3.3.14. *The operator* $-\mathcal{B}_p$ *generates a bounded analytic semigroup on* $[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'_\sigma$ *for all* $1 < p < \infty$.

3.3.3 Flux through the connected components of Γ

Consider the problem

$$\begin{cases} -\Delta \boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} & \text{on } \Gamma, \end{cases}$$

where $\boldsymbol{f} \in \boldsymbol{L}^p_\sigma(\Omega)$. Thanks to [11, Proposition 4.2], we know that when the boundary Γ is not connected, this problem has a solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ which is unique up to an additive element of $\boldsymbol{K}^p_N(\Omega)$ given by

$$\boldsymbol{K}^p_N(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega),\ \operatorname{div}\boldsymbol{v} = 0,\ \mathbf{curl}\,\boldsymbol{v} = \mathbf{0} \text{ in } \Omega \quad \text{and} \quad \boldsymbol{v}\times\boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\}. \tag{3.3.52}$$

In other words, the Stokes operator $\mathcal{A}_p : \mathbf{D}(\mathcal{A}_p) \subset \boldsymbol{L}^p_\sigma(\Omega) \mapsto \boldsymbol{L}^p_\sigma(\Omega)$ is not injective, its kernel is not trivial and is equal to the space $\boldsymbol{K}^p_N(\Omega)$ (given by (3.3.52)).

Our goal in this section is to obtain an operator of bounded inverse. This allows us to obtain an exponential decay of the Stokes semi-group with the boundary conditions (3.3.1). For this reason we impose further conditions on the Stokes operator which is the flux through the connected components Γ_i, $0 \le i \le I$ of Γ :

$$\langle \boldsymbol{u}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0, \quad 1 \le i \le I. \tag{3.3.53}$$

Consider the space

$$\boldsymbol{\mathcal{X}}_p = \{\boldsymbol{f} \in \boldsymbol{L}^p_\sigma(\Omega);\ \forall\,\boldsymbol{v} \in \boldsymbol{K}^{p'}_N(\Omega),\ \int_\Omega \boldsymbol{f}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x = 0\}. \tag{3.3.54}$$

The restriction of $\mathcal{A}_p$ to the space $\boldsymbol{\mathcal{X}}_p$ is a well defined operator of dense domain

$$\mathbf{D}(\mathcal{A}'_p) = \big\{\boldsymbol{u} \in \mathbf{D}(\mathcal{A}_p);\ \langle \boldsymbol{u}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0, \quad 1 \le i \le I\big\} \tag{3.3.55}$$

and

$$\forall\,\boldsymbol{u} \in \mathbf{D}(\mathcal{A}'_p), \qquad \mathcal{A}'_p\boldsymbol{u} = \mathcal{A}_p\boldsymbol{u} \qquad \text{in} \quad \Omega. \tag{3.3.56}$$

It is clear that when the boundary Γ is connected, $\boldsymbol{K}^p_N(\Omega) = \{\mathbf{0}\}$, the Stokes operator $\mathcal{A}_p$ coincides with the operator $\mathcal{A}'_p$ and is an invertible operator of bounded inverse.

Proposition 3.3.15. *The operator $\mathcal{A}'_p$ is a well defined operator of dense domain.*

Proof. Thanks to [11, Lemma 3.1] we recall that a vector field $\boldsymbol{u}$ in $\boldsymbol{L}^p(\Omega)$ satisfies $\operatorname{div}\boldsymbol{u} = 0$ in Ω and $\langle \boldsymbol{u}\cdot\boldsymbol{n}\,,\,1\rangle_{\Gamma_i} = 0$, $0 \leq i \leq I$, if an only if there exists a vector potential $\boldsymbol{\psi}$ in $\boldsymbol{W}^{1,p}(\Omega)$ such that $\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi}$. With this lemma we can easily verify that $\mathbf{D}(\mathcal{A}'_p)$ is included in $\boldsymbol{\mathcal{X}}_p$. In fact, for all $\boldsymbol{v} \in \boldsymbol{K}^{p'}_N(\Omega)$ one has

$$\int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x = \int_\Omega \mathbf{curl}\,\boldsymbol{\psi}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x = \int_\Omega \boldsymbol{\psi}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}\,x + \langle \boldsymbol{\psi}\,;\,\boldsymbol{v}\times\boldsymbol{n}\rangle_\Gamma = 0,$$

where the duality on Γ is

$$\langle .\,,\,.\rangle_\Gamma = \langle .\,,\,.\rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma)\times\boldsymbol{W}^{1/p,p'}(\Gamma)}.$$

Moreover, for every $\boldsymbol{u} \in \mathbf{D}(\mathcal{A}'_p)$ and for every $\boldsymbol{v} \in \boldsymbol{K}^{p'}_N(\Omega)$ one has

$$\int_\Omega \Delta\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x = \int_\Omega \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x = \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}\,x + \langle \mathbf{curl}\,\boldsymbol{u}\,;\,\boldsymbol{v}\times\boldsymbol{n}\rangle_\Gamma = 0.$$

In other words, for every function $\boldsymbol{u} \in \mathbf{D}(\mathcal{A}'_p)$, $\mathcal{A}'_p\boldsymbol{u} \in \boldsymbol{\mathcal{X}}_p$ and $\mathcal{A}'_p$ is a well defined operator.

Now for the density, let $\boldsymbol{f} \in \boldsymbol{\mathcal{X}}_p$, thanks to [11, Theorem 3.22] we know that there exists $\boldsymbol{\xi}$ in $\boldsymbol{W}^{1,p}(\Omega)$ such that $\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{\xi}$ in Ω, $\operatorname{div}\boldsymbol{\xi} = 0$ in Ω and $\boldsymbol{\xi}\cdot\boldsymbol{n} = 0$ on Γ. Since $\boldsymbol{\mathcal{D}}_{\boldsymbol{\sigma}}(\Omega)$ is dense in the space

$$\boldsymbol{W}^{1,p}_{\sigma,N}(\Omega) = \big\{\boldsymbol{\xi} \in \boldsymbol{W}^{1,p}(\Omega);\ \operatorname{div}\boldsymbol{\xi} = 0\,\text{in}\,\Omega,\ \boldsymbol{\xi}\cdot\boldsymbol{n} = 0 \ \text{ on } \ \Gamma\big\},$$

there exists a sequence $(\boldsymbol{\xi}_k)_k$ in $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ such that $\boldsymbol{\xi}_k \longrightarrow \boldsymbol{\xi}$ in $\boldsymbol{W}^{1,p}(\Omega)$. Thus $\mathbf{curl}\,\boldsymbol{\xi}_k \longrightarrow \mathbf{curl}\,\boldsymbol{\xi}$ in $\boldsymbol{L}^p(\Omega)$. As a result, $\mathbf{curl}\,\boldsymbol{\xi}_k \longrightarrow \boldsymbol{f}$ in $\boldsymbol{\mathcal{X}}_p$ and $(\mathbf{curl}\,\boldsymbol{\xi}_k)_k \subset \mathbf{D}(\mathcal{A}'_p)$, this ends the proof. $\square$

Remark 3.3.16. Let $\boldsymbol{u} \in \boldsymbol{L}^p_\sigma(\Omega)$, we note that the condition

$$\forall\ \boldsymbol{v} \in \boldsymbol{K}^{p'}_N(\Omega), \qquad \int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x = 0, \tag{3.3.57}$$

is equivalent to the condition:

$$\langle \boldsymbol{u}\cdot\boldsymbol{n}\,,\,1\rangle_{\Gamma_i} = 0,\ \ 0 \leq i \leq I. \tag{3.3.58}$$

In fact, as in the proof of Proposition 3.3.15, we can easily verify that a function $\boldsymbol{u} \in \boldsymbol{L}^p_\sigma(\Omega)$ satisfying the condition (3.3.58), satifies the compatibility condition (3.3.57). Conversly, let $\boldsymbol{u} \in \boldsymbol{L}^p_\sigma(\Omega)$ satisfying the compatibility condition (3.3.57) then $\boldsymbol{u}$ can be written as

$$\boldsymbol{u} = \boldsymbol{\psi} - \sum_{i=1}^{I}\langle \boldsymbol{\psi}\cdot\boldsymbol{n}\,;\,1\rangle_{\Gamma_i}\nabla q_i^N,$$

for some function $\boldsymbol{\psi} \in \boldsymbol{L}^p_\sigma(\Omega)$. Since for all $0 \leq i \leq I$, ∇q^i_N satisfies (3.3.20) we can easily check that forl all $1 \leq k \leq I$

$$\langle \boldsymbol{u}\cdot\boldsymbol{n}\,;\,1\rangle_{\Gamma_k} = 0.$$

Now we will study the resolvent of the operator $\mathcal{A}'_p$. For this reason we consider the problem

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \Omega, \\ & \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \qquad \text{on } \Gamma, \\ & \langle \boldsymbol{u} \cdot \boldsymbol{n}\,, 1 \rangle_{\Gamma_i} = 0, \quad 1 \le i \le I, \end{cases} \tag{3.3.59}$$

where $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re} \lambda \geq 0$ and $\boldsymbol{f} \in \boldsymbol{\mathcal{X}}_p$. We skip the proof of the following theorem because it is similar to the proof of [11, Proposition 4.2], Theorem 3.3.7 and Theorem 3.3.9.

Theorem 3.3.17. *Let $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re} \lambda \geq 0$ and $\boldsymbol{f} \in \boldsymbol{\mathcal{X}}_p$. The Problem (3.3.59) has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ that satisfies the estimates (3.3.34)-(3.3.35). In addition, when Ω is of class $C^{2,1}$ the solution $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$ and satisfies the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C(\Omega, p)\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)},$$

where $C(\Omega, p)$ is independent of λ and $\boldsymbol{f}$.

As a result of Theorem 3.3.17 the following theorem holds.

Theorem 3.3.18. *The operator $-\mathcal{A}'_p$ generates a bounded analytic semi-group on $\boldsymbol{\mathcal{X}}_p$ for all $1 < p < \infty$. Moreover this semi-group decays exponentially.*

Remark 3.3.19. Consider the sapce

$$\boldsymbol{\mathcal{Y}}_p = \Big\{ \boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'_\sigma;\ \forall\, \boldsymbol{v} \in \boldsymbol{K}_N^{p'}(\Omega),\ \ \langle \boldsymbol{f},\, \boldsymbol{v} \rangle_\Omega = 0 \Big\}.$$

We can verify that the operator $\mathcal{B}'_p$ which is the restriction of $\mathcal{B}_p$ to the space $\boldsymbol{\mathcal{Y}}_p$ is a well defined operator of dense domain

$$\mathbf{D}(\mathcal{B}'_p) = \big\{ \boldsymbol{u} \in \mathbf{D}(\mathcal{B}_p);\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0, \quad 1 \le i \le I \big\},$$

and generates a bounded analytic semi-group on $\boldsymbol{\mathcal{Y}}_p$ for $1 < p < \infty$. Moreover this semi-group decays exponentially.

Chapter 4

Complex and fractional powers of the Stokes operator

In this chapter we will study the complex and the fractional powers of the Stokes operator with the boundary conditions considered in the previous chapter respectively.

Since the Stokes operator with Navier-type boundary conditions (3.1.1) in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ (respectively the Navier slip boundary conditions (3.2.1) in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and The Stokes operator with the pressure boundary condition (3.3.1) in $\boldsymbol{L}^p_{\sigma}(\Omega)$) generates a bounded analytic semi-group , it is in particular a non-negative operator. It then follows from the results in [52] and in [78] that the complex and fractional powers of the Stokes operator with these boundary conditions, are well, densely defined and closed linear operators.

The purpose of this Chapter is to prove some properties and estimates of these powers. These estimates and properties will be used to prove maximal regularity results for the linear inhomogeneous Stokes problem. Those on the fractional powers are used to obtain $L^p - L^q$ estimate for the solution of the homogeneous Stokes problem.

In bounded domains Giga [39] studied in details the fractional powers of the Stokes operator with Dirichlet boundary conditions. In this case the Stokes operator is bijective with bounded inverse. In our case the Stokes operator with each of these three types of boundary conditions respectively is not invertible and in such a case it is not easy in general to compute calculus inequalities involving the fractional powers. We can avoid this difficulty by following the same argument of [17]. The desired calculus inequalities involving the fractional powers will be obtained using the complex interpolation theory and the invariance of a strictly star shaped domain under the scaling transformation $S_\mu \boldsymbol{f}(x) = \boldsymbol{f}(x/\mu)$, $\mu > 0$. We will obtain the desired inequalities for the operator $(1/\mu^2\, I \,+\, A_p)^\alpha$, $0 < \alpha < 1$, then we will pass to the limit as $\mu \to \infty$. In the general case, we use the fact that a bounded Lipschitz continuous set is the union of a finite number of star-shaped, Lipschitz continuous open sets.

In this Chapter we will study in details the complex and fractional powers of the Stokes operator with Navier-type boundary condition (3.1.1). The same results hold for the Stokes operator with the two others boundary conditions using the same approach. For this reason the Chapter is organised as follows:

In section 1 we prove in details the boundedness of the pure imaginary powers of the Stokes operator with Navier-type boundary condition (3.1.1). We also give in details the characterization of the domains of fractional powers of this operator.

The same techniques can be applied to the Stokes operator with the Navier-slip boundary conditions (3.2.1) and the Stokes operator with the pressure boundary conditions (3.3.1).

Section 2 is devoted to the complex and fractional powers of the Stokes operator with Navier slip boundary condition. Many proofs will omitted because it is similar to the case of the Stokes operator with the Navier-type boundary conditions (3.1.1).

Section 3 deals with the Stokes operator with the pressure boundary condition (3.3.1). We will give our results concerning the boundedness of the pure imaginary powers of this operator and some characterisation for the domains of fractional powers. Some details will be omitted because it is similar to the proof in section 1 of this chapter.

4.1 Fractional Powers with Navier-type boundary conditions

In this section we study the complex and the fractional powers of the Stokes operator with Navier-type boundary conditions (3.1.1). As stated above, due to the analyticity of the Stokes semi-group, the complex and fractional powers A_p^α, $\alpha \in \mathbb{C}$, of the Stokes operator with Navier-type boundary conditions (3.1.1), are well densely defined and closed linear operators on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ with domain $\mathbf{D}(A_p^\alpha)$.

Moreover using the fact that

$$\forall \alpha \in \mathbb{C}, \qquad \boldsymbol{\mathcal{D}}_\sigma(\Omega) \hookrightarrow \mathbf{D}(A_p^\alpha) \hookrightarrow \boldsymbol{L}^p_{\sigma,\tau}(\Omega),$$

one has the density of $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ in $\mathbf{D}(A_p^\alpha)$ for all $\alpha \in \mathbb{C}$.

4.1.1 Pure imaginary powers of the Stokes operator

We are interested in this subsection in the pure imaginary powers of the Stokes operator A_p (defined by (3.1.5)). We recall that (see Proposition 3.1.1), due to the boundary conditions (3.1.1), for all $\boldsymbol{u} \in \mathbf{D}(A_p)$, $A_p\,\boldsymbol{u} = -\Delta \boldsymbol{u}$ in Ω .

Our main goal in this section is to prove estimate (2.3.4) for the Stokes operator with Navier-type boundary conditions on the space $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. This is equivalent to prove estimate (2.3.4) for the $-\Delta$ operator with Navier type boundary conditions. Since the Stokes operator with Navier-type boundary conditions A_p does not have bounded inverse we will proceed in a similar way as in [43, Appendix A]. First, we prove estimate (2.3.4) for the operator $(I+A_p)^{is}$, for all $s \in \mathbb{R}$. Next, we deduce an estimate of type (2.3.4) for the operator $(A_p)^{is}$ for all $s \in \mathbb{R}$ using a scaling transformation and then by passing to the limit, exactly as in the proof of [43, Theorem A1]. We start by the following proposition:

Proposition 4.1.1. ***(i)*** *There exists an angle $0 < \theta_0 < \pi/2$ such that the resolvent set of the operator $-(I + A_p)$ contains the sector*

$$\Sigma_{\theta_0} = \big\{\lambda \in \mathbb{C};\ \ |\arg\lambda| \leq \pi - \theta_0\big\}. \tag{4.1.1}$$

Moreover, we have the estimate

$$\forall\,\lambda \in \Sigma_{\theta_0},\ \ \lambda \neq 0, \qquad \|(\lambda\, I + I + A_p)^{-1}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \leq \frac{\kappa_3(\Omega, p)}{|\lambda|}, \tag{4.1.2}$$

where $\kappa_3(\Omega, p)$ is the constant in (3.1.55).
(ii) *Now, let $0 < \alpha < 1$ be fixed, then for $\lambda \in \Sigma_{\theta_0}$ such that $\lambda \neq 0$ and $|\lambda| \leq \frac{1}{2\,\kappa_3(\Omega,p)}$ one has*

$$\|(\lambda\, I + I + A_p)^{-1}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \leq 2^\alpha\, \kappa_1^\alpha(\Omega, p)\, |\lambda|^{\alpha-1}. \tag{4.1.3}$$

Proof. **(i)** Thanks to Theorem 3.1.15 we know that $I + A_p$ is an isomorphism from $\mathbf{D}(A_p) \subset \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. We recall that $\mathbf{D}(A_p)$ is given by (3.1.6). Next, Let $\lambda \in \mathbb{C}^*$ such that $\operatorname{Re}\lambda \geq 0$. It is clear that the operator $\lambda\, I + I + A_p$ is an isomorphism from $\mathbf{D}(A_p)$ to $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. Moreover, since $\operatorname{Re}\lambda \geq 0$ one has (thanks to Theorem 3.1.18)

$$\|(\lambda\, I + I + A_p)^{-1}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \leq \frac{\kappa_3(\Omega, p)}{|\lambda\, +\, 1|} \leq \frac{\kappa_3(\Omega, p)}{|\lambda|}.$$

This means that the resolvent set of the operator $-(I+A_p)$ contains the set $\{\lambda \in \mathbb{C}^*;\ \operatorname{Re}\lambda \geq 0\}$ where the estimate (4.1.2) is satisfied . As a result, thanks to Proposition 2.3.3 and Remark 2.3.4, there exists an angle $0 < \theta_0 < \pi/2$ such that the resolvent set of $-(I + A_p)$ contains the sector Σ_{θ_0} given by (4.1.1). In addition for every $\lambda \in \Sigma_{\theta_0}$ such that $\lambda \neq 0$ estimate (4.1.2) is satisfied.
(ii) Now, let $\lambda \in \Sigma_{\theta_0}$ such that $\lambda \neq 0$ and $|\lambda| \leq \frac{1}{2\,\kappa_3(\Omega,p)}$. Moreover, let $\boldsymbol{u} \in \mathbf{D}(A_p)$ and $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ such that $(\lambda\, I + I + A_p)^{-1}\boldsymbol{f} = \boldsymbol{u}$. This means that

$$\begin{cases} \boldsymbol{u} - \Delta\boldsymbol{u} = \boldsymbol{f} - \lambda\,\boldsymbol{u}, & \operatorname{div}\boldsymbol{u} = 0 \qquad \text{in} \quad \Omega, \\ \quad \boldsymbol{u}\cdot\boldsymbol{n} = 0, & \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0}, \qquad \text{on} \quad \Gamma. \end{cases}$$

As a result using (3.1.55) we have

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \kappa_3(\Omega, p)\, \|\boldsymbol{f} - \lambda\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

Finally, due to our assumption on λ and the fact that $2\,\kappa_3(\Omega, p)|\lambda| \leq 1$ we have

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq 2\,\kappa_3(\Omega, p)\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \leq 2^\alpha\, \kappa_3^\alpha(\Omega, p)\, |\lambda|^{\alpha-1}\, \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)},$$

which ends the proof. □

With this proposition in hand we can now prove the boundedness of the pure imaginary powers of the operator $I + A_p$ which is a key to prove the boundedness of the pure imaginary powers of the Stokes operator with Navier-type boundary conditions A_p.

Proposition 4.1.2. *Let θ_0 as in Proposition 4.1.1. For all $s \in \mathbb{R}$ we have*

$$\|(I + A_p)^{i\,s}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \;\le\; M\,\kappa_3(\Omega,p)\,e^{|s|\,\theta_0}, \tag{4.1.4}$$

for some constant $M > 0$.

Proof. The proof is done in two steps.
(i) Let $z \in \mathbb{C}$ such that $-1 < -\alpha < \operatorname{Re} z < 0$, where $0 < \alpha < 1$ is fixed. Thanks to Proposition 4.1.1 we know that the operator $I + A_p$ is a non-negative bounded operator with bounded inverse as a result using formula (2.3.3) we have

$$(I + A_p)^z \;=\; \frac{1}{2\,\pi\,i}\int_{\Gamma_{\theta_0}} (-\lambda)^z\,(\lambda\,I + I + A_p)^{-1}\,\mathrm{d}\,\lambda.$$

We recall that

$$\Gamma_{\theta_0} \;=\; \{\rho\,e^{i\,(\pi-\theta_0)};\; 0 \le \rho \le \infty\} \,\cup\, \{-\,\rho\,e^{i\,(\theta_0-\pi)};\; -\infty \le \rho \le 0\}.$$

This means that

$$\begin{aligned}(I + A_p)^z \;=\; \frac{1}{2\,\pi\,i}\Big[\int_0^{+\infty} &\big(-\,\rho\,e^{i\,(\pi-\theta_0)}\big)^z\big(\rho\,e^{i\,(\pi-\theta_0)}\,I \,+\, I + A_p\big)^{-1}\,e^{i\,(\pi-\theta_0)}\,\mathrm{d}\rho\\ &-\int_0^{+\infty}\big(-\,\rho\,e^{i\,(\theta_0-\pi)}\big)^z\big(\rho\,e^{i\,(\theta_0-\pi)}I \,+ I + A_p\big)^{-1}\,e^{i\,(\theta_0-\pi)}\mathrm{d}\rho\Big].\end{aligned}$$

In addition, we know that $(-\lambda)^z \;=\; e^{z(\ln|\lambda| + i\,\mathrm{Arg}(-\lambda))}$, where $\mathrm{Arg}(-\lambda)$ is the principal argument of $-\lambda$. An easy computation shows that

$$|(-\lambda)|^z \;\le\; \rho^{\operatorname{Re} z}\,e^{|\operatorname{Im} z|\,\theta_0}. \tag{4.1.5}$$

As a result we have

$$\|(I + A_p)^z\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \;\le\; \frac{e^{|\operatorname{Im} z|\,\theta_0}}{2\,\pi}\Big[\,I_1 \,+\, I_2\Big], \tag{4.1.6}$$

with

$$I_1 \;=\; \int_0^{+\infty}\rho^{\operatorname{Re} z}\,\|\big(\rho\,e^{i\,(\pi-\theta_0)}\,I \,+\, I + A_p\big)^{-1}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))}\,\mathrm{d}\rho$$

and

$$I_2 \;=\; \int_0^{+\infty}\rho^{\operatorname{Re} z}\,\|\big(\rho\,e^{i\,(\theta_0-\pi)}\,I \,+\, I + A_p\big)^{-1}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))}\,\mathrm{d}\rho.$$

Next, we write I_1 in the form

$$\begin{aligned}I_1 \;=\; \int_0^{1/2\,\kappa_3(\Omega,p)} &\rho^{\operatorname{Re} z}\,\|\big(\rho\,e^{i\,(\pi-\theta_0)}\,I \,+\, I + A_p\big)^{-1}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))}\,\mathrm{d}\rho\\ &+\int_{1/2\,\kappa_3(\Omega,p)}^{+\infty}\rho^{\operatorname{Re} z}\,\|\big(\rho\,e^{i\,(\pi-\theta_0)}\,I \,+\, I + A_p\big)^{-1}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))}\,\mathrm{d}\rho.\end{aligned}$$

As a consequence, thanks to Proposition 4.1.1 part (ii) we have

$$I_1 \;\leq\; 2^{\alpha}\, \kappa_3^{\alpha}(\Omega,p) \int_0^{1/2\,\kappa_3(\Omega,p)} \frac{\mathrm{d}\rho}{\rho^{1\,-\,\alpha\,-\,\mathrm{Re}\,z}} \;+\; \kappa_3(\Omega,p) \int_{1/2\,\kappa_3(\Omega,p)}^{+\infty} \frac{\mathrm{d}\rho}{\rho^{1\,-\,\mathrm{Re}\,z}}.$$

Now, thanks to our assumption on z we can easily verify that $I_1 < \infty$. Similarly we prove that $I_2 < \infty$. Finally substituting in (4.1.6) we have

$$\|(I + A_p)^z\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \;\leq\; M\, \kappa(\Omega,p)\, e^{|\mathrm{Im}\, z|\,\theta_0}, \tag{4.1.7}$$

for some constant $M\,\kappa(\Omega,p) > 0$.

(ii) Let $s \in \mathbb{R}$, to obtain estimate (4.1.4) we use the fact that for all $\boldsymbol{f} \in \mathbf{D}(A_p)$, $(I + A_p)^z \boldsymbol{f}$ is analytic in z for $-1 < \mathrm{Re}z < 1$ (see [52, Proposition 4.7, Proposition 4.10]). As a result for all $\boldsymbol{f} \in \mathbf{D}(A_p)$

$$\|(I + A_p)^{is}\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} = \lim_{\varepsilon\to 0} \|(I + A_p)^{-\varepsilon+is}\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \leq \lim_{\varepsilon\to 0} Ce^{|s|\theta_0}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \leq Ce^{|s|\theta_0}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \tag{4.1.8}$$

Finally using the density of $\boldsymbol{D}(A_p)$ in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ we obtain estimate (4.1.8) for all $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and thus one has estimate (4.1.4). $\square$

Before we prove our main result in this subsection we prove the following proposition.

Proposition 4.1.3. *Suppose that Ω is strictly star shaped with respect to one of its points and let θ_0 as in Proposition 4.1.1. For all $s \in \mathbb{R}$ we have*

$$\|(A_p)^{i\,s}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \;\leq\; M\, \kappa_3(\Omega,p)\, e^{|s|\,\theta_0}, \tag{4.1.9}$$

for some constant $M > 0$. In other words we have

$$\forall\, 1 < p < \infty,\;\; s \in \mathbb{R}, \qquad A_p \in \mathcal{E}^{\theta_0}_K(\boldsymbol{L}^p_{\sigma,\tau}(\Omega)),$$

where $K = M\,\kappa(\Omega,p)$.

Proof. Since Ω is strictly star shaped with respect to one of its points then after translation in $\mathbb{R}^3$, we can suppose that this point is 0. This amounts to say that

$$\forall\, \mu > 1, \qquad \mu\,\overline{\Omega} \subset \Omega,\;\; \forall\, 0 \leq \mu < 1 \qquad \text{and}\;\; \overline{\Omega} \subset \mu\,\Omega.$$

Here we take $\mu > 1$ and we set $\Omega_\mu = \mu\,\Omega$.

The proof is based on the scaling transformation

$$\forall\, x \in \Omega_\mu, \qquad (S_\mu \boldsymbol{f})(x) = \boldsymbol{f}(x/\mu), \qquad \boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega). \tag{4.1.10}$$

As in the proof of [43, Appendix, Theorem A1] we can easily verify that

$$\mu^2\, A_p \;=\; S_\mu\, A_p\, S_\mu^{-1}, \qquad I + \mu^2 A_p \;=\; S_\mu (I + A_p)\, S_\mu^{-1}$$

and for $-1 < -\alpha < \operatorname{Re} z < 0$ using (2.3.3) we can verify that,

$$(I + \mu^2 A_p)^z \;=\; S_\mu (I + A_p)^z S_\mu^{-1}.$$

Moreover we know that

$$\|(I + \mu^2 A_p)^z\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} = \|S_\mu (I + A_p)^z S_\mu^{-1}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \leq \|(I + A_p)^z\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))}$$

Now, thanks to Proposition 4.1.2 one has, for $-1 < -\alpha < \operatorname{Re} z < 0$,

$$\|(I + \mu^2 A_p)^z\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \;\leq\; M\,\kappa(\Omega, p)\, e^{|\operatorname{Im} z|\,\theta_0},$$

where the constant $M\,\kappa(\Omega, p)$ is independent of μ. Thus, for all $s \in \mathbb{R}$, one has

$$\|(I + \mu^2 A_p)^{i\,s}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \;\leq\; M\,\kappa(\Omega, p)\, e^{|s|\,\theta_0}. \tag{4.1.11}$$

On the other hand, using Lemma 2.3.7 (part (i)) one has

$$\Big(\frac{1}{\mu^2}\, I + A_p\Big)^{i\,s} \;=\; \frac{1}{\mu^{2\,i\,s}}\,(I + \mu^2 A_p)^{i\,s}.$$

As a result, using (4.1.11) we have

$$\Big\|\Big(\frac{1}{\mu^2}\, I + A_p\Big)^{i\,s}\Big\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \;\leq\; M\,\kappa(\Omega, p)\, e^{|s|\,\theta_0}. \tag{4.1.12}$$

On the other hand, thanks to Lemma 2.3.7 (part (ii)) one has for all $\boldsymbol{f} \in \mathbf{D}(A_p)$

$$\|(A_p)^{i\,s}\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \;=\; \lim_{\mu \to +\infty} \Big\|\Big(\frac{1}{\mu^2}\, I + A_p\Big)^{i\,s}\boldsymbol{f}\Big\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.1.13}$$

We deduce by (4.1.12) and (4.1.13) that (4.1.9) hold for all $\boldsymbol{f} \in \mathbf{D}(A_p)$. Using the density of $\mathbf{D}(A_p)$ in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ the estimate follows for all $\boldsymbol{f}$ in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. $\square$

In the general case, for a domain Ω of Class $C^{2,1}$, we use the fact that (see [12] for instance), a bounded Lipschitz-Continuous open set is the union of a finite number of star-shaped, Lipschitz-continuous open sets. Clearly, It suffices to apply the above argument to each of these sets to derive the desired result on the entire domain. However, the divergence-free condition of a function $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ is not preserved under the cut-off procedure and the proof of the general case is not obvious.

To prove our main result in this section we need the following remark

Remark 4.1.4. First consider the sesqui-linear form

$$\forall\, \boldsymbol{u} \in \boldsymbol{V}^p_\tau(\Omega),\;\; \forall\, \boldsymbol{v} \in \boldsymbol{V}^{p'}_\tau(\Omega), \qquad a(\boldsymbol{u},\, \boldsymbol{v}) \;=\; \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}\, x \tag{4.1.14}$$

with $\boldsymbol{V}^p_\tau(\Omega)$ defined in (3.1.2). Consider also the operator $\widetilde{A}_p \in \mathcal{L}(\boldsymbol{V}^p_\tau(\Omega), (\boldsymbol{V}^{p'}_\tau(\Omega))')$ associated to the sesqui-linear form (4.1.14). The operator $\widetilde{A}_p$ is defined by: For all $\boldsymbol{u} \in \boldsymbol{V}^p_\tau(\Omega)$ and for all $\boldsymbol{v} \in \boldsymbol{V}^{p'}_\tau(\Omega)$,

$$\langle \widetilde{A}_p \boldsymbol{u}\,,\, \boldsymbol{v}\rangle_{(\boldsymbol{V}^{p'}_\tau(\Omega))' \times \boldsymbol{V}^{p'}_\tau(\Omega)} \;=\; a(\boldsymbol{u}\,,\,\boldsymbol{v}) \;=\; \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}\,x. \tag{4.1.15}$$

If moreover $\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0}$ on Γ, then using the Green formula (**??**) one has for all $\boldsymbol{u} \in \boldsymbol{V}^p_\tau(\Omega)$ and for all $\boldsymbol{v} \in \boldsymbol{V}^{p'}_\tau(\Omega)$,

$$\langle \widetilde{A}_p \boldsymbol{u}\,,\, \boldsymbol{v}\rangle_{(\boldsymbol{V}^{p'}_\tau(\Omega))' \times \boldsymbol{V}^{p'}_\tau(\Omega)} \;=\; \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}\,x \;=\; \langle -\Delta\boldsymbol{u},\, \boldsymbol{v}\rangle_\Omega \tag{4.1.16}$$

with $\langle .,.\rangle_\Omega$ denotes the anti-duality between $[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'$ and $\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)$. In particular, for all $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega)$ one has

$$\langle \widetilde{A}_p \boldsymbol{u} - \Delta\boldsymbol{u}\,,\, \boldsymbol{v}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega)\times\boldsymbol{\mathcal{D}}(\Omega)} \;=\; 0.$$

Thus thanks to De Rham Lemma, there is a function $\pi \in L^p(\Omega)/\mathbb{R}$ such that

$$\widetilde{A}_p\boldsymbol{u} \;=\; -\Delta\boldsymbol{u} + \nabla\pi, \qquad \text{in} \quad \Omega.$$

Now if $\widetilde{A}_p\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$, then using the regularity of the Stokes Problem (see [10, Theorem 4.8]) we deduce that $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R}$.

Notice that, in the case where $\widetilde{A}_p\boldsymbol{u} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, π is constant and $\widetilde{A}_p\boldsymbol{u} = A_p\boldsymbol{u} = -\Delta\boldsymbol{u}$ in Ω.

The operator $\widetilde{A}_p : \mathbf{D}(\widetilde{A}_p) \subset \boldsymbol{L}^p(\Omega) \longmapsto \boldsymbol{L}^p(\Omega)$ is a linear operator with

$$\mathbf{D}(\widetilde{A}_p) = \Big\{\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega);\ \mathrm{div}\,\boldsymbol{u} = 0 \text{ in } \Omega,\ \boldsymbol{u}\cdot\boldsymbol{n} = 0,\ \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\Big\}$$

and for all $\boldsymbol{u} \in \mathbf{D}(\widetilde{A}_p)$, $\widetilde{A}_p\boldsymbol{u}$ is defined by (4.1.15).

Next, we consider the resolvent of the operator $\widetilde{A}_p$ on $\boldsymbol{L}^p(\Omega)$. We can easily verify that for a given function $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ and for a given $\lambda \in \mathbb{C}^*$ such that $\mathrm{Re}\,\lambda \geq 0$, there exists a unique function $\boldsymbol{u} \in \mathbf{D}(\widetilde{A}_p)$ such that for all $\boldsymbol{v} \in \boldsymbol{V}^{p'}_\tau(\Omega)$

$$\lambda\int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x + \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}\,x \;=\; \int_\Omega \boldsymbol{f}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x.$$

Moreover we have the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \;\leq\; \frac{C(\Omega,p)}{|\lambda|}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.1.17}$$

This means that, for all $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ and for all $\lambda \in \mathbb{C}^*$ such that $\mathrm{Re}\,\lambda \geq 0$, there exists a unique function $\boldsymbol{u} \in \mathbf{D}(\widetilde{A}_p)$ such that

$$\lambda\boldsymbol{u} + \widetilde{A}_p\boldsymbol{u} \;=\; \boldsymbol{f} \qquad \text{in} \quad \Omega$$

and satisfying estimate (4.1.17). Thus the resolvent set of the operator $\widetilde{A}_p$ contains the half-plane $\{\lambda \in \mathbb{C}^*;\ \mathrm{Re}\lambda \geq 0\}$. Using Remark 2.3.4, we deduce that there exists an angle $0 < \theta_0 < \pi/2$ such that the resolvent set of the operator $\widetilde{A}_p$ contains the sector

$$\Sigma_{\theta_0} = \big\{\lambda \in \mathbb{C};\ \ |\arg\lambda| \leq \pi - \theta_0\big\}$$

where estimate (4.1.17) is satisfied.

Proceeding in the same way as in the proof of Proposition 4.1.1, we can verify that Proposition 4.1.1 holds for the operator $-(I+\widetilde{A}_p)$. As a result, the operator $(I+\widetilde{A}_p)$ is a non-negative operator of bounded inverse and for any complex $z \in \mathbb{C}$ the complex power $(I+\widetilde{A}_p)^z$ is well defined by the mean of the Dunford integral and using formula (2.3.3) one has:

$$(I+\widetilde{A}_p)^z \;=\; \frac{1}{2\,\pi\,i}\int_{\Gamma_{\theta_0}} (-\lambda)^z\,(\lambda\, I + I + \widetilde{A}_p)^{-1}\,\mathrm{d}\,\lambda.$$

Proceeding in the same way as in the proof of Proposition 4.1.2 one can prove that for any complex $z \in \mathbb{C}$ such that $-1 < \mathrm{Re}\, z < 0$, one has

$$\|(I+\widetilde{A}_p)^z\|_{\mathcal{L}(\boldsymbol{L}^p(\Omega))} \;\leq\; M\, C(\Omega,p)\, e^{|\mathrm{Im}\, z|\,\theta_0},$$

for some constant $M\, C(\Omega,p) > 0$.

Next, we consider the case where Ω is strictly star shaped with respect to one of its point. As in the proof of Proposition 4.1.3, using the scaling transformation S_μ given by (4.1.10) and the fact that

$$\mu^2\,\widetilde{A}_p \;=\; S_\mu\,\widetilde{A}_p\,S_\mu^{-1}, \qquad I+\mu^2\widetilde{A}_p \;=\; S_\mu(I+\widetilde{A}_p)\,S_\mu^{-1},$$

one has for $-1 < \mathrm{Re}\, z < 0$,

$$\|(I+\mu^2\,\widetilde{A}_p)^z\|_{\mathcal{L}(\boldsymbol{L}^p(\Omega))} \;\leq\; M\, C(\Omega,p)\, e^{|\mathrm{Im}\, z|\,\theta_0}, \tag{4.1.18}$$

where the constant $M\, C(\Omega,p)$ is independent of μ.

We observe also the following remark

Remark 4.1.5. Since $\mathbf{D}(\widetilde{A}_p)$ is not dense in $\boldsymbol{L}^p(\Omega)$, estimate (4.1.4) doesn't necessarily hold for the operator $I+\widetilde{A}_p$ in $\boldsymbol{L}^p(\Omega)$.

Now we prove our main result in this section:

Theorem 4.1.6. *Suppose that Ω is of class $C^{2,1}$ and let θ_0 as in Proposition 4.1.1. For all $s \in \mathbb{R}$ we have*

$$\|(A_p)^{i\,s}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \;\leq\; M\, C(\Omega,p)\, e^{|s|\,\theta_0}, \tag{4.1.19}$$

for some constant $M > 0$. In other words we have

$$\forall\, 1 < p < \infty,\ \ s \in \mathbb{R}, \qquad A_p \in \mathcal{E}_K^{\theta_0}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega)),$$

where $K = M\, C(\Omega,p)$.

Proof. Let $(\Theta_j)_{j\in J}$ be an open covering of Ω by a finite number of star-shaped open sets and let us consider a partition of unity $(\varphi_j)_{j\in J}$ subordinated to the covering $(\Omega_j)_{j\in J}$ where for all $j\in J,\ \Omega_j=\Theta_j\cap\Omega$. This means that

$$\forall j\in J,\qquad \mathrm{Supp}\varphi_j\subset\Omega_j$$

and

$$\sum_{j\in J}\varphi_j=1,\qquad \varphi_j\in\mathcal{D}(\Omega_j).$$

Let $\boldsymbol{f}\in\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, $\boldsymbol{f}$ can be written as

$$\boldsymbol{f}=\sum_{j\in J}\boldsymbol{f}_j,\qquad \forall j\in J,\quad \boldsymbol{f}_j=\varphi_j\boldsymbol{f}.$$

Notice that for all $j\in J$, $\boldsymbol{f}_j$ is not necessarily a divergence free function.

Let $\mu>0$ and let $z\in\mathbb{C}$ such that $-1<\mathrm{Re}z<0$. Thanks to Remark 4.1.4 we know that

$$(I+\mu^2A_p)^z\boldsymbol{f}\ =\ (I+\mu^2\widetilde{A}_p)^z\boldsymbol{f}\ =\ \sum_{j\in J}(I+\mu^2\widetilde{A}_p)^z\boldsymbol{f}_j.$$

As a result, one has

$$\begin{aligned}\|(I+\mu^2A_p)^z\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\ &\le\ \sum_{j\in J}\|(I+\mu^2\widetilde{A}_p)^z\boldsymbol{f}_j\|_{\boldsymbol{L}^p(\Omega)}\\ &=\ \sum_{j\in J}\|(I+\mu^2\widetilde{A}_p)^z\boldsymbol{f}_j\|_{\boldsymbol{L}^p(\Omega_j)}\end{aligned}$$

Since for all $j\in J$, Ω_j is strictly star shaped with respect to one of its points, then using estimate (4.1.18) one has

$$\begin{aligned}\|(I+\mu^2A_p)^z\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\ &\le\ \sum_{j\in J}C_j\,e^{|\mathrm{Im}\,z|\,\theta_0}\|\boldsymbol{f}_j\|_{\boldsymbol{L}^p(\Omega_j)}\\ &\le\ C\,e^{|\mathrm{Im}\,z|\,\theta_0}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}\end{aligned}$$

with a constant C independent of μ and $\boldsymbol{f}$. As a result one has

$$\|(I+\mu^2\,A_p)^z\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))}\ \le\ M\,C(\Omega,p)\,e^{|\mathrm{Im}\,z|\,\theta_0},$$

with a constant C independent of μ.

As in the proof of Proposition 4.1.2, using the fact that $(I+\mu^2\,A_p)^z\boldsymbol{f}$ is analytic in z, $-1<\mathrm{Re}z<1$, for all $\boldsymbol{f}\in\mathbf{D}(A_p)$ and using the density of $\mathbf{D}(A_p)$ in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, one has for all $s\in\mathbb{R}$,

$$\|(I+\mu^2\,A_p)^{is}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))}\ \le C\,e^{|s|\,\theta_0},\tag{4.1.20}$$

with a constant C independent of μ.

Furthermore, as in the proof of Proposition 4.1.3, we have

$$\Big(\frac{1}{\mu^2}\, I + A_p\Big)^{i\,s} \;=\; \frac{1}{\mu^{2\,i\,s}}\,(I + \mu^2 A_p)^{i\,s}.$$

As a result, using (4.1.20) we have

$$\Big\|\Big(\frac{1}{\mu^2}\, I + A_p\Big)^{i\,s}\Big\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \;\leq\; C\, e^{|s|\,\theta_0}. \tag{4.1.21}$$

Thus using Lemma 2.3.7 (part (ii)) we deduce that for all $\boldsymbol{f} \in \mathbf{D}(A_p)$

$$\|(A_p)^{i\,s}\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \;=\; \lim_{\mu\to+\infty}\Big\|\Big(\frac{1}{\mu^2}\, I + A_p\Big)^{i\,s}\boldsymbol{f}\Big\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.1.22}$$

This means that (4.1.19) hold for all $\boldsymbol{f} \in \mathbf{D}(A_p)$. Using the density of $\mathbf{D}(A_p)$ in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ the estimate follows for all $\boldsymbol{f}$ in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. $\square$

Remark 4.1.7. Similarly we can prove that there exists $\theta_0 < \pi/2$ such that for all $s \in \mathbb{R}$

$$\|(B_p)^{i\,s}\|_{\mathcal{L}([\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\tau})} = \|(-\Delta)^{i\,s}\|_{\mathcal{L}([\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\tau})} \;\leq\; C(\Omega,p)\, e^{|s|\,\theta_0} \tag{4.1.23}$$

and

$$\|(C_p)^{i\,s}\|_{\mathcal{L}([\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau})} = \|(-\Delta)^{i\,s}\|_{\mathcal{L}([\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau})} \;\leq\; C(\Omega,p)\, e^{|s|\,\theta_0}. \tag{4.1.24}$$

We recall that the operator B_p and C_p given by (3.1.65) and (3.1.72) respectively are the extension of the Stokes operator to the spaces $[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'_{\sigma,\tau}$ and $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ respectively and they are equal to the Laplacian operator with Navier-type boundary conditions.

We will see later that estimates (4.1.23) (respectively (4.1.24)) give us the weak (respectively the very weak) solutions for the inhomogeneous Stokes Problem with Navier-type boundary conditions with a maximal $L^p - L^q$ regularity.

4.1.2 Domains of fractional powers of the Stokes operator

This subsection is devoted to the study of the domains of fractional powers of the Stokes operator with Navier-type boundary conditions on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. Since the Stokes operator with Navier-type boundary conditions doesn't have bounded inverse the fractional powers A_p^α are not bounded for both $\mathrm{Re}\alpha > 0$ and $\mathrm{Re}\alpha < 0$ and attention should be paid in the calculus of the domain $\mathbf{D}(A_p^\alpha)$ of A_p^α and its norm. In our case, we will follow Borchers and Miyakawa [17]. We will use complex interpolation theory to calculate the domains of fractional powers of the Stokes operator. We will characterize the domain $\mathbf{D}(A_p^{1/2})$ and will prove the equivalence of the two norms $\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}$ and $\|A_p^{1/2}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}$ for all $\boldsymbol{v} \in \mathbf{D}(A_p^{1/2})$. We will also prove through an interpolation argument an embedding theorem of Sobolev type for the domains of fractional powers of the Stokes operator with Navier-type boundary conditions.

We start by the characterization of the domain of $\mathbf{A}_p^{1/2}$.

Theorem 4.1.8. *For all* $1 < p < \infty$, $\mathbf{D}(A_p^{1/2}) = \boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega)$ *(given by* (3.1.3)*) with equivalent norms.*

Proof. Thanks to Lemma 2.3.9 we know that $\mathbf{D}(A_p^{1/2}) = \mathbf{D}((I + A_p)^{1/2})$ with norm $\|(I + A_p)^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$ which is equivalent to the graph norm. Moreover, due to Proposition 4.1.1 we know that $I + A_p$ is a non-negative operator and due to Theorem 3.1.19 we know that $I + A_p$ is of bounded inverse. In addition thanks to Proposition 4.1.2 we know that that the pure imaginary powers of $I + A_p$ are bounded and satisfy estimate (4.1.4). As a result thanks to Theorem 2.3.9 we have

$$\mathbf{D}(A_p^{1/2}) = \mathbf{D}((I + A_p)^{1/2}) = \big[\mathbf{D}(I + A_p); \boldsymbol{L}^p_{\sigma,\tau}(\Omega)\big]_{1/2} = \big[\mathbf{D}(A_p); \boldsymbol{L}^p_{\sigma,\tau}(\Omega)\big]_{1/2}.$$

Consider now a function $\boldsymbol{u} \in \mathbf{D}(A_p)$, set $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{u}$ and $\boldsymbol{U} = (\boldsymbol{u}, \boldsymbol{z})$. It is clear that $\boldsymbol{z} \in \boldsymbol{H}^p_0(\mathbf{curl}, \Omega)$ and using Lemma 2.2.1 we deduce that $\boldsymbol{z} \in \boldsymbol{X}^p_N(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$ and $\boldsymbol{U} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega)$. On the other hand if $\boldsymbol{u} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ we know that, thanks to [10, 11], $\boldsymbol{U} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega) \times [\boldsymbol{H}^p_0(\mathbf{curl}, \Omega)]' \hookrightarrow \boldsymbol{L}^p_{\sigma,\tau}(\Omega) \times \boldsymbol{W}^{-1,p}(\Omega)$. Next let $\boldsymbol{u} \in \mathbf{D}(A_p^{1/2})$ then $\boldsymbol{U} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega) \times [\boldsymbol{W}^{1,p}(\Omega), \boldsymbol{W}^{-1,p}(\Omega)]_{1/2} = \boldsymbol{L}^p_{\sigma,\tau}(\Omega) \times \boldsymbol{L}^p(\Omega)$. Thus using Lemma 2.2.1 we deduce that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega)$. This amount to say that

$$\mathbf{D}(A_p^{1/2}) \hookrightarrow \boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega). \tag{4.1.25}$$

It remains to prove the second inclusion. Observe that thanks to [78, Theorem 1.15.2, part (e)], since $I + A_p$ has a bounded inverse, then for all $1 < p < \infty$, $(I + A_p)^{1/2}$ is an isomorphism from $\mathbf{D}((I + A_p)^{1/2})$ to $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. This means that for all $\boldsymbol{F} \in \boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega)$ there exists a unique $\boldsymbol{v} \in \mathbf{D}((I + A_{p'})^{1/2})$ solution of

$$(I + A_{p'})^{1/2}\boldsymbol{v} = \boldsymbol{F}. \tag{4.1.26}$$

As a result for all $\boldsymbol{u} \in \mathbf{D}(A_p)$ we have

$$\begin{aligned}
\|(I + A_p)^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} &= \sup_{\boldsymbol{F} \in \boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega),\, \boldsymbol{F} \neq \boldsymbol{0}} \frac{\left|\langle (I + A_p)^{1/2}\boldsymbol{u}\,,\, \boldsymbol{F}\rangle_{\boldsymbol{L}^p_{\sigma,\tau}(\Omega) \times \boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega)}\right|}{\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}} \\
&= \sup_{\boldsymbol{F} \in \boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega),\, \boldsymbol{F} \neq \boldsymbol{0}} \frac{\left|\langle (I + A_p)^{1/2}\boldsymbol{u}\,,\, (I + A_{p'})^{1/2}\boldsymbol{v}\rangle_{\boldsymbol{L}^p_{\sigma,\tau}(\Omega) \times \boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega)}\right|}{\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}},
\end{aligned}$$

where $\boldsymbol{v}$ is the unique solution of (4.1.26).

We recall that for all $0 < \alpha < 1$, the adjoint operator $(A_p^\alpha)^*$ is equal to $A_{p'}^\alpha$ and $[(I + A_p)^\alpha]^* = (I + A_p^*)^\alpha = (I + A_{p'})^\alpha$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

As a result,

$$\begin{aligned}\|(I\,+\,A_p)^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} &= \sup_{\boldsymbol{v}\in\mathbf{D}(A_{p'}^{1/2}),\,\boldsymbol{v}\neq\boldsymbol{0}} \frac{\Big|\langle (I\,+\,A_p)\boldsymbol{u}\,,\,\boldsymbol{v}\rangle_{\boldsymbol{L}^p_{\sigma,\tau}(\Omega)\times\boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega)}\Big|}{\|(I\,+\,A_{p'})^{1/2}\boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)}}\\ &= \sup_{\boldsymbol{v}\in\mathbf{D}(A_{p'}^{1/2}),\,\boldsymbol{v}\neq\boldsymbol{0}} \frac{\Big|\int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x\,+\,\int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}\,x\Big|}{\|(I\,+\,A_{p'})^{1/2}\boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)}}\\ &\leq C(\Omega,p)\,\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}. \qquad (4.1.27)\end{aligned}$$

Now since $\mathbf{D}(A_p)$ is dense in $\boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega)$ one gets inequality (4.1.27) for all $\boldsymbol{u}\in\boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega)$ and then

$$\boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega)\hookrightarrow\mathbf{D}(A_p^{1/2})$$

and the result is prove. □

The following proposition gives the equivalence in norm of $A_p^{1/2}\boldsymbol{u}$ and $\mathbf{curl}\,\boldsymbol{u}$ in $\boldsymbol{L}^p(\Omega)$ for every $\boldsymbol{u}\in\mathbf{D}(A_p^{1/2})$.

Proposition 4.1.9. *For every $\boldsymbol{u}\in\mathbf{D}(A_p^{1/2})$ one has the equivalence of the two norms $\|A_p^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$ and $\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$. In other words, there exists two constants C_1 and C_2 such that for all $\boldsymbol{u}\in\mathbf{D}(A_p^{1/2})$*

$$\|A_p^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\leq C_1\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\leq C_2\|A_p^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

Proof. We will proceed in a similar way as in the proof of [17, Theorem 3.6]. First we assume that the domain Ω is strictly star shaped with respect to one of its points. As a result the domain Ω is invariant under the scaling transformation S_μ (given by(4.1.10)) introduced in the proof of Theorem 4.1.6. As in the proof of [43, Appendix, Theorem A1] we can easily verify that

$$\mu^2A_p\,=\,S_\mu\,A_p\,S_\mu^{-1},\qquad I+\mu^2A_p\,=\,S_\mu(I+A_p)\,S_\mu^{-1}.$$

Moreover, for $0<\alpha<1$ and for all $\boldsymbol{v}\in\mathbf{D}(A_p)$ one has,

$$(I+\mu^2A_p)^\alpha\boldsymbol{v}(x)\,=\,S_\mu(I+A_p)^\alpha S_\mu^{-1}\boldsymbol{v}(x)$$

and

$$S_\mu^{-1}(I+\mu^2A_p)^\alpha\boldsymbol{v}(x)\,=\,(I+A_p)^\alpha S_\mu^{-1}\boldsymbol{v}(x). \qquad (4.1.28)$$

The proof is done in two steps:
(i) First we prove that

$$\forall\,\boldsymbol{u}\in\mathbf{D}(A_p^{1/2}),\qquad \|A_p^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\,\leq\,C(\Omega,p)\,\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \qquad (4.1.29)$$

Thanks to Theorem 4.1.8 one has for every $\boldsymbol{u}\in\mathbf{D}(A_p^{1/2})$

$$\|(I\,+\,A_p)^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\,\leq\,C(\Omega,p)\,(\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\,+\,\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}). \qquad (4.1.30)$$

Now, for $\boldsymbol{u} \in \mathbf{D}(A_p)$ we set $\boldsymbol{u} = S_\mu^{-1}\boldsymbol{v}$, for some $\boldsymbol{v} \in \mathbf{D}(A_p)$. This is true since S_μ^{-1} is an isomorphism from $\mathbf{D}(A_p)$ into itself. Now substituting in (4.1.30) one has

$$\|(I + A_p)S_\mu^{-1}\boldsymbol{v}(x)\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\,(\|\boldsymbol{v}(\mu\,x)\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}(\mu\,x)\|_{\boldsymbol{L}^p(\Omega)}).$$

As a result thanks to (4.1.28), since $\boldsymbol{u} \in \mathbf{D}(A_p)$ one has

$$\|S_\mu^{-1}(I+\mu^2A_p)^{1/2}\boldsymbol{v}(x)\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\,(\|\boldsymbol{v}(\mu\,x)\|_{\boldsymbol{L}^p(\Omega)} + \mu\,\|\mathbf{curl}\,\boldsymbol{v}(\mu\,x)\|_{\boldsymbol{L}^p(\Omega)}). \tag{4.1.31}$$

Next using the integral formula (2.3.3), an easy computation shows that for all $\boldsymbol{v} \in \mathbf{D}(A_p)$, $S_\mu^{-1}(I+\mu^2A_p)^{1/2}\boldsymbol{v}(x) = (I+\mu^2A_p)^{1/2}\boldsymbol{v}(\mu\,x)$ and $(I+\mu^2\,A_p)^{1/2}\boldsymbol{v}(x) = \mu(\frac{1}{\mu^2}\,I + A_p)^{1/2}\boldsymbol{v}(x)$. Thus using (4.1.31) one has

$$\mu\,\|(\frac{1}{\mu^2}\,I + A_p)^{1/2}\boldsymbol{v}(\mu\,x)\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\,(\|\boldsymbol{v}(\mu\,x)\|_{\boldsymbol{L}^p(\Omega)} + \mu\,\|\mathbf{curl}\,\boldsymbol{v}(\mu\,x)\|_{\boldsymbol{L}^p(\Omega)}).$$

As a result, using the change of variable $y = \mu\,x$ one has for every $\boldsymbol{v} \in \mathbf{D}(A_p)$

$$\|(\frac{1}{\mu^2}\,I + A_p)^{1/2}\boldsymbol{v}(y)\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\,(\frac{1}{\mu}\|\boldsymbol{v}(y)\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}(y)\|_{\boldsymbol{L}^p(\Omega)}).$$

Now using Lemma 2.3.15 (part (ii)) and passing to the limit as $\mu \longrightarrow +\infty$ one gets for every $\boldsymbol{v} \in \mathbf{D}(A_p)$

$$\|A_p^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\,\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

Finally using the density of $\mathbf{D}(A_p)$ in $\mathbf{D}(A_p^{1/2})$ one gets estimate (4.1.2).

(ii) Next we prove that

$$\forall\,\boldsymbol{u} \in \mathbf{D}(A_p^{1/2}), \qquad \|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\,\|A_p^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.1.32}$$

Thanks to Theorem 4.1.8 one has for every $\boldsymbol{u} \in \mathbf{D}(A_p^{1/2})$

$$\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\,\|(I + A_p)^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.1.33}$$

Let $\boldsymbol{u} \in \mathbf{D}(A_p)$ and set $\boldsymbol{u} = S_\mu^{-1}\boldsymbol{v}$ for some $\boldsymbol{v} \in \mathbf{D}(A_p)$. Substituting in (4.1.33) and proceeding in the same way as in the first step one has

$$\|\mathbf{curl}\,\boldsymbol{v}(\mu\,x)\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\,\|(\frac{1}{\mu^2}\,I + A_p)^{1/2}\boldsymbol{v}(\mu\,x)\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.1.34}$$

Now using the change of variable $y = \mu\,x$ in (4.1.34) and passing to the limit as $\mu \longrightarrow +\infty$ one has estimate (4.1.32) for every $\boldsymbol{u} \in \mathbf{D}(A_p)$. Finally using the density of $\mathbf{D}(A_p)$ in $\mathbf{D}(A_p^{1/2})$ one has estimate (4.1.32).

In the case where Ω is not star-shaped we have to recover Ω which is $C^{1,1}$ (in particular Lipschitz) by a finite number of star open sets. $\square$

The following proposition shows an embedding of Sobolev type for the domains of fractional powers of the Stokes operator. This embedding give us the $\boldsymbol{L}^p - \boldsymbol{L}^q$ estimates for the homogeneous Stokes problem.

Proposition 4.1.10. *For all $1 < p < \infty$ and for all $0 < \alpha \leq 1$ we define $\beta = \max(\alpha, 1-\alpha)$ then*

$$\mathbf{D}(A_p^\alpha) \hookrightarrow \boldsymbol{L}^q(\Omega) \tag{4.1.35}$$

for all q such that:
(i) *For $1 < p < \frac{3}{2\beta}$, $q \in \left[p,\, \frac{3p}{3-2\beta p}\right]$.*
(ii) *For $p = \frac{3}{2\beta}$, $q \in [1, +\infty[$.*
(iii) *For $p > \frac{3}{2\beta}$, $q = +\infty$.*

Moreover for such q, the following estimate holds

$$\forall \boldsymbol{u} \in \mathbf{D}(A_p^\alpha), \qquad \|\boldsymbol{u}\|_{\boldsymbol{L}^q(\Omega)} \leq C(\Omega, p)\, \|A_p^\alpha \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.1.36}$$

Proof. As described in the proof of Theorem 4.1.8 we know that

$$\mathbf{D}(A_p^\alpha) = \mathbf{D}((I + A_p)^\alpha) = \left[\mathbf{D}(I + A_p); \boldsymbol{L}^p_{\sigma,\tau}(\Omega)\right]_\alpha = \left[\mathbf{D}(A_p); \boldsymbol{L}^p_{\sigma,\tau}(\Omega)\right]_\alpha.$$

Moreover

$$\left[\mathbf{D}(A_p); \boldsymbol{L}^p_{\sigma,\tau}(\Omega)\right]_\alpha \hookrightarrow \left[\boldsymbol{W}^{2,p}(\Omega)\,,\, \boldsymbol{L}^p(\Omega)\right]_\alpha = \boldsymbol{W}^{2(1-\alpha),p}(\Omega). \tag{4.1.37}$$

It is clear that for $0 < \alpha < 1/2$, $1 - \alpha > \alpha$ and

$$\mathbf{D}(A_p^\alpha) \hookrightarrow \boldsymbol{W}^{2\alpha,p}(\Omega).$$

Similarly, for $1/2 \leq \alpha \leq 1$, observe that $\alpha \geq 1 - \alpha$ and

$$\mathbf{D}(A_p^\alpha) \hookrightarrow \mathbf{D}(A_p^{1-\alpha}) \hookrightarrow \boldsymbol{W}^{2\alpha,p}(\Omega).$$

Thus one has, for all $0 < \alpha \leq 1$

$$\mathbf{D}(A_p^\alpha) \hookrightarrow \mathbf{D}(A_p^{1-\alpha}) \hookrightarrow \boldsymbol{W}^{2\beta,p}(\Omega).$$

Now using the result of [2, Theorem 7.57] we deduce the Sobolev embedding (4.1.35) with p and q satisfying (i), (ii) and (iii).

To prove estimate (4.1.36) we proceed in a similar way as in the proof of Proposition 4.1.9. Thanks to the part (i) we know that for all $\boldsymbol{u} \in \mathbf{D}(A_p^\alpha)$

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^q(\Omega)} \leq C(\Omega, p)\, \|(I + A_p)^\alpha \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.1.38}$$

Suppose that Ω is strictly star shaped and let $\boldsymbol{u} \in \mathbf{D}(A_p) \hookrightarrow \mathbf{D}(A_p^\alpha)$. Consider again the operator S_μ defined by (4.1.10) above and set $\boldsymbol{u} = S_\mu^{-1} \boldsymbol{v}$ for some $\boldsymbol{v} \in \mathbf{D}(A_p)$, substituting in (4.1.38) and using the same argument as in the proof of proposition 4.1.9 one has

$$\|\boldsymbol{v}(\mu\, x)\|_{\boldsymbol{L}^q(\Omega)} \leq C(\Omega, p)\, \|(I + \mu^2 A_p)^\alpha \boldsymbol{v}(\mu\, x)\|_{\boldsymbol{L}^p(\Omega)}.$$

Next, using the change of variable $y = \mu\, x$ and Lemma 2.3.15 (part (i)) one has

$$\mu^{-3/q} \|\boldsymbol{v}\|_{\boldsymbol{L}^q(\Omega)} \leq C(\Omega, p)\, \mu^{2\alpha - 3/p} \|(\frac{1}{\mu^2}\, I + A_p)^\alpha \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}.$$

Thus, since $\frac{1}{q} = \frac{1}{p} - \frac{2\alpha}{3}$ one has

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^q(\Omega)} \leq C(\Omega, p)\, \|(\frac{1}{\mu^2} I + A_p)^\alpha \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)},$$

where the constant $C(\Omega, p)$ is independent of μ.

Finally using Lemma 2.3.15 (part (ii)), letting $\mu \longrightarrow \infty$ and using the density of $\mathbf{D}(A_p)$ in $\mathbf{D}(A_p^\alpha)$ one has estimate (4.1.36). □

The following Corollary extends Proposition 4.1.10 to any real α such that $0 < \alpha < 3/2p$. This result is similar to the result of Borchers and Miyakawa [18] who proved the same result for the Stokes operator with Dirichlet boundary conditions in exterior domains for $1 < p < 3$.

Corollary 4.1.11. *for all $1 < p < \infty$ and for all $\alpha \in \mathbb{R}$ such that $0 < \alpha < 3/2p$ the following Sobolev embedding holds*

$$\mathbf{D}(A_p^\alpha) \hookrightarrow \boldsymbol{L}^q(\Omega), \qquad \frac{1}{q} = \frac{1}{p} - \frac{2\alpha}{3}. \tag{4.1.39}$$

Moreover for all $\boldsymbol{u} \in \mathbf{D}(A_p^\alpha)$ the following estimate holds

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^q(\Omega)} \leq C(\Omega, p)\, \|A_p^\alpha \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.1.40}$$

Proof. First observe that for $0 < \alpha < \min(1, 3/2p)$ the Sobolev embedding (4.1.39) is a consequence of Proposition 4.1.10 part (i). Next, for any real α such that $0 < \alpha < 3/2p$ we write $\alpha = k + \theta$, where k is a non negative integer and $0 < \theta < 1$.

Next we set

$$\frac{1}{q_0} = \frac{1}{p} - \frac{2\theta}{3} \qquad \text{and} \qquad \frac{1}{q_j} = \frac{1}{q_0} - \frac{2j}{3}, \quad j = 0, 1, ..., k. \tag{4.1.41}$$

It is clear that $\frac{1}{q_j} = \frac{1}{q_{j-1}} - \frac{2}{3}$ and that $q_k = q$. Moreover, by assumptions on p and α we have for $j = 0, 1, ..., k$, $\theta + j < 3/2p$. As a consequence of Proposition 4.1.10 part (i) it follows that

$$\mathbf{D}(A_p^\theta) \hookrightarrow \boldsymbol{L}^{q_0}(\Omega)$$

and for all $1 \leq j \leq k$

$$\mathbf{D}(Aq_{j-1}) \hookrightarrow \boldsymbol{L}^{q_j}(\Omega).$$

It thus follows that for all $\boldsymbol{u} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega)$

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^q(\Omega)} \leq \|(I+A_{q_{k-1}})\boldsymbol{u}\|_{\boldsymbol{L}^{q_{k-1}}(\Omega)} \leq ... \leq \|(I+A_{q_0})^k \boldsymbol{u}\|_{\boldsymbol{L}^{q_0}(\Omega)} \leq \|(I+A_p)^\alpha \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.1.42}$$

By density of $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ in $\mathbf{D}(A_p^\alpha)$ on gets the Sobolev embeddings (4.1.39) and estimate (4.1.42). Finally using a scaling procedure as in the proof of estimate (4.1.36) one gets estimate (4.1.40) for all $\boldsymbol{u} \in \mathbf{D}(A_p^\alpha)$.

□

4.2 Factional powers with Navier-slip boundary conditions

In this section we collect some results on the complex and fractional powers of the Stokes operator $\mathbb{A}_p$ with Navier slip boundary condition (3.2.1). We recall that the operator $\mathbb{A}_p$ is defined by (3.2.5).

As stated above, the existence of complex and fractional powers of the operator $\mathbb{A}_p$ is justified by the analyticity of the Stokes semi-group with the Navier slip boundary conditions (3.2.1). These complex and fractional powers $\mathbb{A}_p^\alpha$, $\alpha \in \mathbb{C}$, of the Stokes operator with Navier-slip boundary conditions, are well densely defined and closed linear operators on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ with domain $\mathbf{D}(\mathbb{A}_p^\alpha)$.

Moreover using the fact that

$$\forall \alpha \in \mathbb{C}, \qquad \boldsymbol{\mathcal{D}}_\sigma(\Omega) \hookrightarrow \mathbf{D}(\mathbb{A}_p^\alpha) \hookrightarrow \boldsymbol{L}^p_{\sigma,\tau}(\Omega),$$

one has the density of $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ in $\mathbf{D}(\mathbb{A}_p^\alpha)$ for all $\alpha \in \mathbb{C}$.

We can easily check (using the same proof) that Proposition 4.1.1, Proposition 4.1.2 and Proposition 4.1.3 hold for the Stokes operator with the Navier slip boundary conditions $\mathbb{A}_p$. As a result one has the boundedness of the pure imaginary powers of the operator $\mathbb{A}_p$. We skip the proof of the following theorem because it is similar to the proof of Theorem 4.1.6.

Theorem 4.2.1. *There exist an angle $0 < \theta_0 < \pi/2$ such that for all $s \in \mathbb{R}$ we have*

$$\|(\mathbb{A}_p)^{i\,s}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \leq M\,\kappa(\Omega,p)\,e^{|s|\,\theta_0}, \tag{4.2.1}$$

for some constant $M, \kappa(\Omega,p) > 0$. In other words we have

$$\forall\, 1 < p < \infty, \;\; s \in \mathbb{R}, \qquad \mathbb{A}_p \in \mathcal{E}^{\theta_0}_K(\boldsymbol{L}^p_{\sigma,\tau}(\Omega)),$$

where $K = M\,\kappa(\Omega,p)$.

Remark 4.2.2. We also have an estimate of type (4.2.1) for the operators $\mathbb{B}_p$ and $\mathbb{C}_p$, which are the extension of the Stokes operator with Navier slip boundary conditions (3.2.1) to the spaces $[\boldsymbol{H}^{p'}_0(\mathrm{div}\Omega)]'_{\sigma,T}$ and $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ respectively. We recall that the operators $\mathbb{B}_p$ and $\mathbb{C}_p$ are defined by (3.2.33) and (3.2.38) respectively.

We will see in Chapter 5 that estimate (4.2.1) gives us a maximal $L^p - L^q$ regularity result for the inhomogeneous time dependent Stokes problem with the Navier slip boundary conditions.

As stated in Theorem 2.3.9, the boundedness of the pure imaginary powers of the Stokes operator $\mathbb{A}_p$ allows us to characterise the domains of fractional powers of the Stokes operator through an interpolation argument. As in the case of the Stokes operator with Navier-type boundary conditions (3.1.1) we have the following theorem.

Theorem 4.2.3. *For all* $1 < p < \infty$, $\mathbf{D}(\mathbb{A}_p^{1/2}) = \boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega)$ *(given by* (3.1.3)*) with equivalent norms.*

Proof. The proof is similar to the proof of Theorem 4.1.8. In fact since the pure imaginary powers of the Stokes operator with Navier slip boundary conditions are bounded and satisfy estimates (4.2.1), one has thanks to Theorem 2.3.9

$$\mathbf{D}(\mathbb{A}_p^{1/2}) = \mathbf{D}((I + \mathbb{A}_p)^{1/2}) = \big[\mathbf{D}(I + \mathbb{A}_p); \boldsymbol{L}^p_{\sigma,\tau}(\Omega)\big]_{1/2} = \big[\mathbf{D}(\mathbb{A}_p); \boldsymbol{L}^p_{\sigma,\tau}(\Omega)\big]_{1/2}.$$

Consider now a function $\boldsymbol{u} \in \mathbf{D}(\mathbb{A}_p)$ (see (3.2.4) for the definition of $\mathbf{D}(\mathbb{A}_p)$) and set $\boldsymbol{z} = \mathbf{D}(\boldsymbol{u})$ and $\boldsymbol{U} = (\boldsymbol{u}, \boldsymbol{z})$. It is clear that when $\boldsymbol{u} \in \mathbf{D}(\mathbb{A}_p)$ then $\boldsymbol{z} \in \boldsymbol{W}^{1,p}(\Omega)$ and $\boldsymbol{U} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega)$. In addition, when $\boldsymbol{u} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ then $\boldsymbol{z} \in \boldsymbol{W}^{-1,p}(\Omega)$ and $\boldsymbol{U} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega) \times \boldsymbol{W}^{-1,p}(\Omega)$. Now, let $\boldsymbol{u} \in \mathbf{D}(\mathbb{A}_p^{1/2})$, then

$$\boldsymbol{U} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega) \times \big[\boldsymbol{W}^{1,p}(\Omega)\,;\, \boldsymbol{W}^{-1,p}(\Omega)\big]_{1/2} = \boldsymbol{L}^p_{\sigma,\tau}(\Omega) \times \boldsymbol{L}^p(\Omega).$$

As a result, $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$, $\boldsymbol{z} = \mathbf{D}(\boldsymbol{u}) \in \boldsymbol{L}^p(\Omega)$, $\operatorname{div} \boldsymbol{u} = 0$ in Ω and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ. Thanks to [8] we know that on $\boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega)$, $\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}$ is equivalent to $\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{D}(\boldsymbol{u})\|_{\boldsymbol{L}^p(\Omega)}$. As a result $\boldsymbol{u} \in \boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega)$ and

$$\mathbf{D}(\mathbb{A}_p^{1/2}) \hookrightarrow \boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega).$$

To prove the second inclusion, observe that thanks to [78, Theorem 1.15.2, part (e)], since $I + \mathbb{A}_p$ has a bounded inverse, then for all $1 < p < \infty$, $(I + \mathbb{A}_p)^{1/2}$ is an isomorphism from $\mathbf{D}((I + \mathbb{A}_p)^{1/2})$ to $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. This means that for all $\boldsymbol{F} \in \boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega)$ there exists a unique $\boldsymbol{v} \in \mathbf{D}((I + \mathbb{A}_{p'})^{1/2})$ solution of

$$(I + \mathbb{A}_{p'})^{1/2}\boldsymbol{v} = \boldsymbol{F}. \tag{4.2.2}$$

Now let $\boldsymbol{u} \in \mathbf{D}(\mathbb{A}_p)$. Proceeding in the same way as in the proof of Theorem 4.1.8 one has

$$\begin{aligned}
\|(I + \mathbb{A}_p)^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} &= \sup_{\boldsymbol{F}\in\boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega),\, \boldsymbol{F}\neq\boldsymbol{0}} \frac{\Big|\langle(I + \mathbb{A}_p)^{1/2}\boldsymbol{u}\,,\, \boldsymbol{F}\rangle_{\boldsymbol{L}^p_{\sigma,\tau}(\Omega)\times\boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega)}\Big|}{\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)}} \\
&= \sup_{\boldsymbol{v}\in\mathbf{D}(\mathbb{A}_{p'}^{1/2}),\, \boldsymbol{v}\neq\boldsymbol{0}} \frac{\Big|\langle(I + \mathbb{A}_p)\boldsymbol{u}\,,\, \boldsymbol{v}\rangle_{\boldsymbol{L}^p_{\sigma,\tau}(\Omega)\times\boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega)}\Big|}{\|(I + \mathbb{A}_{p'})^{1/2}\boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)}} \\
&= \sup_{\boldsymbol{v}\in\mathbf{D}(\mathbb{A}_{p'}^{1/2}),\, \boldsymbol{v}\neq\boldsymbol{0}} \frac{\Big|\int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x + \int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\overline{\boldsymbol{v}})\,\mathrm{d}\,x\Big|}{\|(I + \mathbb{A}_{p'})^{1/2}\boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)}} \\
&\leq C(\Omega, p)\,\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}.
\end{aligned} \tag{4.2.3}$$

We recall that $\boldsymbol{v}$ is the unique solution of Problem (4.2.2).

Now since $\mathbf{D}(\mathbb{A}_p)$ is dense in $\boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega)$ one gets inequality (4.2.3) for all $\boldsymbol{u} \in \boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega)$ and then

$$\boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega) \hookrightarrow \mathbf{D}(\mathbb{A}_p^{1/2})$$

and the result is prove. □

The following proposition gives the equivalence in norm of $\mathbb{A}_p^{1/2}\boldsymbol{u}$ and $\mathbf{D}(\boldsymbol{u})$ in $\boldsymbol{L}^p(\Omega)$ for every $\boldsymbol{u} \in \mathbf{D}(\mathbb{A}_p^{1/2})$.

Proposition 4.2.4. *For every $\boldsymbol{u} \in \mathbf{D}(\mathbb{A}_p^{1/2})$ one has the equivalence of the two norms $\|\mathbb{A}_p^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$ and $\|\mathbb{D}(\boldsymbol{u})\|_{\boldsymbol{L}^p(\Omega)}$. In other words, there exists two constants C_1 and C_2 such that for all $\boldsymbol{u} \in \mathbf{D}(\mathbb{A}_p^{1/2})$*

$$\|\mathbb{A}_p^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C_1\|\mathbf{D}(\boldsymbol{u})\|_{\boldsymbol{L}^p(\Omega)} \leq C_2\|\mathbb{A}_p^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

Proof. We proceed in a similar way as in the proof of Proposition 4.1.9. First we assume that the domain Ω is strictly star shaped with respect to one of its points and we use the invariance of Ω under the scaling transformation S_μ (given by(4.1.10)). As in the proof of Proposition 4.1.9 we can easily verify that

$$\mu^2\mathbb{A}_p \,=\, S_\mu\,\mathbb{A}_p\,S_\mu^{-1}, \qquad I + \mu^2\mathbb{A}_p \,=\, S_\mu(I + \mathbb{A}_p)\,S_\mu^{-1}.$$

Moreover, for $0 < \alpha < 1$ and for all $\boldsymbol{v} \in \mathbf{D}(\mathbb{A}_p)$ one has,

$$(I + \mu^2\mathbb{A}_p)^\alpha\boldsymbol{v}(x) \,=\, S_\mu(I + \mathbb{A}_p)^\alpha S_\mu^{-1}\boldsymbol{v}(x)$$

and

$$S_\mu^{-1}(I + \mu^2\mathbb{A}_p)^\alpha\boldsymbol{v}(x) \,=\, (I + \mathbb{A}_p)^\alpha S_\mu^{-1}\boldsymbol{v}(x). \tag{4.2.4}$$

(i) First we prove that

$$\forall\,\boldsymbol{u} \in \mathbf{D}(\mathbb{A}_p^{1/2}), \qquad \|\mathbb{A}_p^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \,\leq\, C(\Omega, p)\,\|\mathbf{D}(\boldsymbol{u})\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.2.5}$$

In fact, thanks to Theorem 4.2.3 one has for every $\boldsymbol{u} \in \mathbf{D}(\mathbb{A}_p^{1/2})$

$$\|(I \,+\, \mathbb{A}_p)^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \,\leq\, C(\Omega, p)\,(\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \,+\, \|\mathbf{D}(\boldsymbol{u})\|_{\boldsymbol{L}^p(\Omega)}). \tag{4.2.6}$$

Now, for $\boldsymbol{u} \in \mathbf{D}(\mathbb{A}_p)$ we set $\boldsymbol{u} = S_\mu^{-1}\boldsymbol{v}$, for some $\boldsymbol{v} \in \mathbf{D}(\mathbb{A}_p)$. Substituting in (4.2.6) and proceeding in the same way as in the proof of Proposition 4.1.9 one has

$$\Big\|\Big(\frac{1}{\mu^2}\,I \,+\, \mathbb{A}_p\Big)^{1/2}\boldsymbol{v}(\mu\,x)\Big\|_{\boldsymbol{L}^p(\Omega)} \,\leq\, C(\Omega, p)\,\Big(\frac{1}{\mu}\|\boldsymbol{v}(\mu\,x)\|_{\boldsymbol{L}^p(\Omega)} \,+\, \mu\,\|\mathbf{D}(\boldsymbol{v}(\mu\,x))\|_{\boldsymbol{L}^p(\Omega)}\Big). \tag{4.2.7}$$

Now using the chane of variable $y = \mu\,x$ in (4.2.7) and passing to the limit as $\mu \to \infty$ one has estimate (4.2.5) for every $\boldsymbol{u} \in \mathbf{D}(\mathbb{A}_p)$. Finally using the density of $\mathbf{D}(\mathbb{A}_p)$ in $\mathbf{D}(\mathbb{A}_p^{1/2})$ one has estimate (4.2.5) for every $\boldsymbol{u} \in \mathbf{D}(\mathbb{A}_p^{1/2})$.

(ii) Next we prove that

$$\forall\,\boldsymbol{u} \in \mathbf{D}(\mathbb{A}_p^{1/2}), \qquad \|\mathbf{D}(\boldsymbol{u})\|_{\boldsymbol{L}^p(\Omega)} \,\leq\, C(\Omega, p)\,\|\mathbb{A}_p^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.2.8}$$

Thanks to Theorem 4.2.3 one has for every $\boldsymbol{u} \in \mathbf{D}(\mathbb{A}_p^{1/2})$

$$\|\mathbf{D}(\boldsymbol{u})\|_{\boldsymbol{L}^p(\Omega)} \,\leq\, C(\Omega, p)\,\|(I \,+\, \mathbb{A}_p)^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.2.9}$$

Let $\boldsymbol{u} \in \mathbf{D}(\mathbb{A}_p)$ and set $\boldsymbol{u} = S_\mu^{-1}\boldsymbol{v}$ for some $\boldsymbol{v} \in \mathbf{D}(\mathbb{A}_p)$. Substituting in (4.2.9) and proceeding in the same way as in the first step one has

$$\|\mathbf{D}(\boldsymbol{v}(\mu\, x))\|_{\boldsymbol{L}^p(\Omega)} \;\leq\; C(\Omega, p)\left\|\Big(\frac{1}{\mu^2}\, I \,+\, \mathbb{A}_p\Big)^{1/2}\boldsymbol{v}(\mu\, x)\right\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.2.10}$$

Now using the change of variable $y = \mu\, x$ in (4.2.10) and passing to the limit as $\mu \longrightarrow +\infty$ one has estimate (4.2.8) for every $\boldsymbol{u} \in \mathbf{D}(\mathbb{A}_p)$. Finally using the density of $\mathbf{D}(\mathbb{A}_p)$ in $\mathbf{D}(\mathbb{A}_p^{1/2})$ one has estimate (4.2.8).

In the case where Ω is not star-shaped we have to recover Ω which is in particular Lipschitz by a finite number of star open sets. $\square$

As in the previous section (see Proposition 4.1.10), the following proposition gives us an embeddings of Sobolev type for the domains of fractional powers of the Stokes operator $\mathbb{A}_p$ (defined by (3.2.5)). The proof of the following proposition will be omitted because it is similar to the proof of Proposition 4.1.10.

Proposition 4.2.5. *For all* $1 < p < \infty$ *and for all* $0 < \alpha \leq 1$ *we define* $\beta = \max(\alpha, 1-\alpha)$ *then*

$$\mathbf{D}(\mathbb{A}_p^\alpha) \hookrightarrow \boldsymbol{L}^q(\Omega)$$

for all q *such that:*
(i) *For* $1 < p < \frac{3}{2\beta}$, $q \in \left[p,\, \frac{3p}{3-2\beta p}\right]$.
(ii) *For* $p = \frac{3}{2\beta}$, $q \in [1,\, +\infty[$.
(iii) *For* $p > \frac{3}{2\beta}$, $q = +\infty$.
Moreover for such q*, the following estimate holds*

$$\forall \boldsymbol{u} \in \mathbf{D}(\mathbb{A}_p^\alpha), \qquad \|\boldsymbol{u}\|_{\boldsymbol{L}^q(\Omega)} \;\leq\; C(\Omega, p)\, \|\mathbb{A}_p^\alpha \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

As in the previous section (see Corollary 4.1.11), we can extend Proposition 4.2.5 to any real α such that $0 < \alpha < 3/2p$. We skip the proof of the following corollary because it is similar to the proof of Corollary 4.1.11

Corollary 4.2.6. *for all* $1 < p < \infty$ *and for all* $\alpha \in \mathbb{R}$ *such that* $0 < \alpha < 3/2p$ *the following Sobolev embedding holds*

$$\mathbf{D}(\mathbb{A}_p^\alpha) \hookrightarrow \boldsymbol{L}^q(\Omega), \qquad \frac{1}{q} = \frac{1}{p} - \frac{2\alpha}{3}. \tag{4.2.11}$$

Moreover for all $\boldsymbol{u} \in \mathbf{D}(\mathbb{A}_p^\alpha)$ *the following estimate holds*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^q(\Omega)} \;\leq\; C(\Omega, p)\, \|\mathbb{A}_p^\alpha \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.2.12}$$

4.3 Fractional powers with normal and pressure boundary conditions

In this section we give some results on the complex and the fractional powers of the Stokes operator with normal and pressure boundary conditions (3.3.1) boundary conditions $\mathcal{A}_p$ on $\boldsymbol{L}^p_\sigma(\Omega)$. We recall that the operator $\mathcal{A}_p$ is defined by (3.3.7).

As stated above, since the Stokes operator with normal and pressure boundary conditions in $\boldsymbol{L}^p_\sigma(\Omega)$ generates a bounded analytic semi-group, it is in particular a non-negative operator. It then follows that the complex and fractional powers $\mathcal{A}_p^\alpha$, $\alpha \in \mathbb{C}$, of the Stokes operator with Navier-type boundary conditions, are well, densely defined and closed linear operators on $\boldsymbol{L}^p_\sigma(\Omega)$ with domain $\mathbf{D}(\mathcal{A}_p^\alpha)$.

Proceeding in the same way as in Proposition 4.1.1, Proposition 4.1.2, Proposition 4.1.3 and Theorem 4.1.6, we can prove an estimate of type (4.1.19) for the operator $\mathcal{A}_p$ on $\boldsymbol{L}^p_\sigma(\Omega)$ and for the operator $\mathcal{B}_p$ on $[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'_\sigma$. We recall that the operator $\mathcal{B}_p$ is the extension of the Stokes operator $\mathcal{A}_p$ to $[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'_\sigma$ and it is defined by (3.3.49) and (3.3.50).

Theorem 4.3.1. *There exist an angle $0 < \theta_0 < \pi/2$ such that for all $s \in \mathbb{R}$ we have*

$$\|(\mathcal{A}_p)^{i\,s}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \leq M\,\kappa(\Omega,p)\,e^{|s|\,\theta_0} \tag{4.3.1}$$

and

$$\|(\mathcal{B}_p)^{i\,s}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \leq M\,\kappa(\Omega,p)\,e^{|s|\,\theta_0}, \tag{4.3.2}$$

for some constant $M, \kappa(\Omega,p) > 0$.

As stated in Theorem 2.3.9, the boundedness of the pure imaginary powers of the Stokes operator $\mathcal{A}_p$ allows us to characterise the domains of fractional powers of the Stokes operator through an interpolation argument and to obtain an embedding of Sobolev type for these domains.

We start by the characterization of the domain of $\mathcal{A}_p^{1/2}$.

Theorem 4.3.2. *For all $1 < p < \infty$, $\mathbf{D}(\mathcal{A}_p^{1/2}) = \boldsymbol{W}^{1,p}_{\sigma,N}(\Omega)$ (given by (3.3.5)) with equivalent norms.*

Proof. The proof is similar to the proof of Theorem 4.1.8 and Theorem 4.2.3. In fact since the pure imaginary powers of the Stokes operator with normal and pressure boundary conditions are bounded and satisfy estimates (4.3.1), one has, thanks to Theorem 2.3.9

$$\mathbf{D}(\mathcal{A}_p^{1/2}) = \mathbf{D}((I+\mathcal{A}_p)^{1/2}) = \big[\mathbf{D}(I+\mathcal{A}_p);\boldsymbol{L}^p_{\sigma,\tau}(\Omega)\big]_{1/2} \subset \big[\boldsymbol{W}^{2,p}(\Omega);\boldsymbol{L}^p(\Omega)\big]_{1/2} = \boldsymbol{W}^{1,p}(\Omega),$$

with continuous embedding.

Let $\boldsymbol{u} \in \mathbf{D}(\mathcal{A}_p^{1/2})$ and let $(\boldsymbol{u}_k)_{k\in\mathbb{N}}$ a sequence in $\mathbf{D}(\mathcal{A}_p)$ that converges to $\boldsymbol{u}$ in $\mathbf{D}(\mathcal{A}_p^{1/2})$. This is true, since $\mathbf{D}(\mathcal{A}_p)$ is dense in $\mathbf{D}(\mathcal{A}_p^{1/2})$. As a result, $(\boldsymbol{u}_k)_{k\in\mathbb{N}}$ converges to $\boldsymbol{u}$ in $\boldsymbol{W}^{1,p}(\Omega)$

and $\boldsymbol{u}_k \times \boldsymbol{n} \longrightarrow \boldsymbol{u} \times \boldsymbol{n}$ in $\boldsymbol{W}^{1-1/p,p}(\Gamma)$ as $k \to \infty$. Moreover, since $(\boldsymbol{u}_k)_{k\in\mathbb{N}} \subset \mathbf{D}(\mathcal{A}_p)$, then for all $k \in \mathbb{N}$, $\boldsymbol{u}_k \times \boldsymbol{n} = \boldsymbol{0}$ on Γ and $\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ. Thus

$$\mathbf{D}(\mathcal{A}_p^{1/2}) \hookrightarrow \boldsymbol{W}^{1,p}_{\sigma,N}(\Omega).$$

Proceeding in the same way as in the proof of Theorem 4.1.8 we prove that for all $\boldsymbol{u} \in \mathbf{D}(\mathcal{A}_p^{1/2})$

$$\|(I + \mathcal{A}_p)^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega, p)\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}$$

and then

$$\boldsymbol{W}^{1,p}_{\sigma,N}(\Omega) \hookrightarrow \mathbf{D}(\mathcal{A}_p^{1/2})$$

and the result is prove. □

The following proposition gives the equivalence in norm of $\mathcal{A}_p^{1/2}\boldsymbol{u}$ and $\mathbf{curl}\,\boldsymbol{u}$ in $\boldsymbol{L}^p(\Omega)$ for every $\boldsymbol{u} \in \mathbf{D}(\mathcal{A}_p^{1/2})$. We skip the proof of the following proposition because it is similar to the proof of Proposition 4.1.9.

Proposition 4.3.3. *For every $\boldsymbol{u} \in \mathbf{D}(\mathcal{A}_p^{1/2})$ one has the equivalence of the two norms $\|\mathcal{A}_p^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$ and $\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$. In other words, there exists two constants C_1 and C_2 such that for all $\boldsymbol{u} \in \mathbf{D}(\mathcal{A}_p^{1/2})$*

$$\|\mathcal{A}_p^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C_1\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C_2\|\mathcal{A}_p^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

As in the case of the Stokes operator with Navier or Navier-type boundary conditions, we have an embedding of Sobolev type for the domains of fractional powers of the operator $\mathcal{A}_p$. The proof of the following proposition will be omitted because it is similar to the proof of Proposition 4.1.10.

Proposition 4.3.4. *For all $1 < p < \infty$ and for all $0 < \alpha \leq 1$ we define $\beta = \max(\alpha, 1-\alpha)$ then*

$$\mathbf{D}(\mathcal{A}_p^{\alpha}) \hookrightarrow \boldsymbol{L}^q(\Omega)$$

for all q such that:
(i) *For $1 < p < \frac{3}{2\beta}$, $q \in \left[p, \, \frac{3p}{3-2\beta p}\right]$.*
(ii) *For $p = \frac{3}{2\beta}$, $q \in [1, +\infty[$.*
(iii) *For $p > \frac{3}{2\beta}$, $q = +\infty$.*

Moreover for such q, the following estimate holds

$$\forall \boldsymbol{u} \in \mathbf{D}(\mathcal{A}_p^{\alpha}), \qquad \|\boldsymbol{u}\|_{\boldsymbol{L}^q(\Omega)} \leq C(\Omega, p)\,\|\mathcal{A}_p^{\alpha}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

As in the previous section (Corollary 4.1.11), we can extend Proposition 4.3.4 to any real α such that $0 < \alpha < 3/2p$. We skip the proof of the following corollary because it is similar to the proof of Corollary 4.1.11

Corollary 4.3.5. *for all* $1 < p < \infty$ *and for all* $\alpha \in \mathbb{R}$ *such that* $0 < \alpha < 3/2p$ *the following Sobolev embedding holds*

$$\mathbf{D}(\mathcal{A}_p^\alpha) \hookrightarrow \boldsymbol{L}^q(\Omega), \qquad \frac{1}{q} = \frac{1}{p} - \frac{2\alpha}{3}. \tag{4.3.3}$$

Moreover for all $\boldsymbol{u} \in \mathbf{D}(\mathcal{A}_p^\alpha)$ *the following estimate holds*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^q(\Omega)} \leq C(\Omega, p)\, \|\mathcal{A}_p^\alpha \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.3.4}$$

Chapter 5

Time dependent Stokes Problem

In this Chapter we consider the linearised time dependent Stokes problem

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega \times (0, T), \\ \qquad\qquad\qquad\qquad\quad \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in} \quad \Omega. \end{cases} \tag{5.0.1}$$

As described in the introduction, Problem (5.0.1) describes the motion of a viscous incompressible fluid in Ω. The velocity of motion is denoted by $\boldsymbol{u}$ and the associated pressure by π. Given data are the external force $\boldsymbol{f}$ and the initial velocity $\boldsymbol{u}_0$.

Problem (5.0.1) has often been studied with Dirichlet boundary condition. In this Chapter we will study this problem with the boundary conditions (3.1.1), (3.2.1) and (3.3.1) respectively. We will prove the existence of weak, strong and very weak solutions to Problem (5.0.1) with each boundary condition using the semi-group theory. We will also prove the so called $L^p - L^q$ estimate for the homogeneous Stokes problem with these boundary conditions respectively. The proof is based on the use of the Sobolev embeddings for the domains of fractional powers of the Stokes operator proved in the Chapter 4 and an interpolation inequality. We also give a maximal $L^p - L^q$ regularity result for the inhomogeneous Stokes problem. The proof is based on the boundedness of the pure imaginary powers of the Stokes operator proved in Chapter 4.

5.1 Stokes Problem with Navier-type boundary conditions

In this section we solve the time dependent Stokes Problem (5.0.1) with the boundary condition (3.1.1) using the semi-group theory. As described in Chapter 3, Proposition 3.1.1, due to the boundary conditions (3.1.1) the Stokes operator coincides with the $-\Delta$ operator. We recall that the Stokes operator with Navier type boundary conditions A_p is defined by (3.1.5).

5.1.1 The homogeneous Stokes problem

Consider the problem:

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} = \boldsymbol{0}, & \operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \Omega \times (0,T), \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on} \quad \Gamma \times (0,T), \\ & \boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in} \quad \Omega. \end{cases} \tag{5.1.1}$$

Usually in the Problem (5.1.1) where figures the constraint $\operatorname{div} \boldsymbol{u} = 0$ in Ω, appears a gradient of pressure. However, it is clear that, thanks to our boundary conditions, the pressure is constant here. For this reason, the Problem (5.1.1) is equivalent to the homogeneous Stokes problem

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{0}, & \operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \Omega \times (0,T), \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on} \quad \Gamma \times (0,T), \\ & \boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in} \quad \Omega. \end{cases} \tag{5.1.2}$$

As in the case of the heat equation the following theorem shows that the solution of Problem (5.1.1) is regular for $t > 0$. Moreover it allows us to describe the decay in time of this solutions.

Theorem 5.1.1. *Let $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, then Problem (5.1.1) has a unique solution $\boldsymbol{u}(t)$ satisfying*

$$\boldsymbol{u} \in C([0, +\infty[, \boldsymbol{L}^p_{\sigma,\tau}(\Omega)) \cap C(]0, +\infty[, \mathbf{D}(A_p)) \cap C^1(]0, +\infty[, \boldsymbol{L}^p_{\sigma,\tau}(\Omega)), \tag{5.1.3}$$

$$\boldsymbol{u} \in C^k(]0, +\infty[, \mathbf{D}(A_p^\ell)), \qquad \forall\, k \in \mathbb{N},\ \forall\, \ell \in \mathbb{N}^*. \tag{5.1.4}$$

Moreover we have the estimates

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq C_1(\Omega, p)\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \tag{5.1.5}$$

and

$$\left\| \frac{\partial \boldsymbol{u}(t)}{\partial t} \right\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C_2(\Omega, p)}{t}\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}. \tag{5.1.6}$$

In addition, if Ω is of class $C^{2,1}$ the following estimates hold

$$\|\mathbf{curl}\, \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C_3(\Omega, p)}{\sqrt{t}}\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \tag{5.1.7}$$

and

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C_4(\Omega, p)\, (1 + \frac{1}{t})\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}. \tag{5.1.8}$$

Proof. Since the operator $-A_p$ generates a bounded analytic semi-group e^{-tA_p} on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, the Problem (5.1.1) has a unique solution $\boldsymbol{u}(t) = e^{-tA_p}\, \boldsymbol{u}_0$. Thanks to [29, Chapter 2, Proposition 4.3] we know that $\|e^{-tA_p}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \leq C_1(\Omega, p)$, where $C_1(\Omega, p) = M_1\, \kappa_3(\Omega, p)$ for some constant $M_1 > 0$. We recall that $\kappa_3(\Omega, p)$ is the constant in (3.1.55). As a result one has estimate (5.1.5). We also know thanks to [29, Chapter 2, Theorem 4.6] that this solution belongs to $\mathbf{D}(A_p)$ thus one has (5.1.3). We recall that $\mathbf{D}(A_p)$ is given by (3.1.6). Now using the fact that $e^{-tA_p}\,\boldsymbol{u}_0 \in \mathbf{D}(A_p^\infty)$ and the same argument in the proof of [19, Chapitre 7, Theorem 7.5, Theorem 7.7] one gets the regularity (5.1.4). We recall that $\mathbf{D}(A_p^\infty) = \cap_{n\in\mathbb{N}}\mathbf{D}(A_p^n)$.

Moreover, thanks to [29, Chapter 2, Theorem 4.6, page 101] we know that

$$\|A_p e^{-tA_p}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C_2(\Omega,p)}{t},$$

where $C_2(\Omega,p) = M_2\,\kappa_3(\Omega,p)$ for some constant $M_2 > 0$, which gives us estimate (5.1.6). Next, to prove estimate (5.1.7) we proceed in the same way as in the proof of the estimate (3.1.56) (see Chapter 3, Theorem 3.1.18 for the proof).

Finally, suppose that Ω is of class $C^{2,1}$, since the norm of $\boldsymbol{W}^{2,p}(\Omega)$ is equivalent to the graph norm of the stokes operator A_p one has estimate (5.1.8). $\square$

Estimates (5.1.5) and (5.1.7) allows to conclude the following corollary:

Corollary 5.1.2 (Weak Solutions for the Stokes Problem). *Let $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, $T < \infty$ and let $\boldsymbol{u}$ be the unique solution of Problem (5.1.1). Then $\boldsymbol{u}$ satisfies*

$$\forall\, 1 \leq q < 2, \qquad \boldsymbol{u} \in L^q(0,T;\, \boldsymbol{W}^{1,p}(\Omega)) \;\; \text{and} \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\, [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]').$$

Proof. Let $\boldsymbol{u}(t)$ be the unique solution of Problem (5.1.1). Thanks to Theorem 5.1.1 we know that $\boldsymbol{u}$ satisfies the estimates (5.1.5)-(5.1.8). Now thanks to Lemma 2.2.1 we know that

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{1,p}(\Omega)} \simeq \|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)}.$$

Thus one deduces directly that $\boldsymbol{u} \in L^q(0,T;\, \boldsymbol{W}^{1,p}(\Omega))$ for all $1 \leq q < 2$ and for all $0 < T < \infty$. Next, let us prove that $\frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\, [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]')$, set

$$\widetilde{\boldsymbol{u}}(t) = \boldsymbol{u}(t) - \sum_{j=1}^{J} \langle \boldsymbol{u}(t)\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j}\widetilde{\mathbf{grad}}\,q_j^\tau.$$

It is clear that $\boldsymbol{u}(t) \;=\; \widetilde{\boldsymbol{u}}(t) + \sum_{j=1}^{J}\langle \boldsymbol{u}(t)\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j}\widetilde{\mathbf{grad}}\,q_j^\tau$. Moreover thanks to [10, Proposition 4.3] we know that

$$\|\Delta\boldsymbol{u}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} \;=\; \|\Delta\widetilde{\boldsymbol{u}}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} \;\simeq\; \|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{W}^{1,p}(\Omega)} \;\leq\; \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}.$$

The last inequality comes from the fact (see Chapter 2, Lemma 2.2.8)

$$|\langle \boldsymbol{u}\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j}| \;\leq\; C(\Omega,p)\,\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

Thus $\frac{\partial \boldsymbol{u}}{\partial t} = \Delta\boldsymbol{u} \in L^q(0,T;\, [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]')$, for $T < \infty$ and $1 \leq q < 2$. $\square$

For the particular case $p = 2$ we have the following remark:

Remark 5.1.3. **(i)** Notice that in the Hilbertian case ($\boldsymbol{u}_0 \in \boldsymbol{L}^2_{\sigma,\tau}(\Omega)$), the property (5.1.5)-(5.1.8) are immediate. We will prove estimate (5.1.7). Observe that, thanks to Propositon 3.1.13 and Remark 3.1.14, on $\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$ we can express $\boldsymbol{u}(t)$ explicitly in the form

$$\boldsymbol{u}(t) \;=\; \sum_{j=1}^{J} \alpha_j \,\widetilde{\mathbf{grad}}\, q_j^\tau \;+\; \sum_{k=1}^{+\infty} \beta_k \, e^{-\lambda_k\, t}\, \boldsymbol{z}_k, \tag{5.1.9}$$

where

$$\alpha_j \;=\; \int_\Omega \boldsymbol{u}_0 \cdot \widetilde{\mathbf{grad}}\, \overline{q_j^\tau}\, \mathrm{d}\, x \qquad \text{and} \qquad \beta_k \;=\; \int_\Omega \boldsymbol{u}_0 \cdot \overline{\boldsymbol{z}_k}\, \mathrm{d}\, x.$$

As a result, using the fact that $A_2 \boldsymbol{z}_k = \lambda_k\, \boldsymbol{z}_k$ and the fact that

$$\int_\Omega |\mathbf{curl}\, \boldsymbol{z}_k|^2 \mathrm{d}\, x = \lambda_k \, \|\boldsymbol{z}_k\|^2_{\boldsymbol{L}^2(\Omega)} = \lambda_k$$

one has

$$\|\mathbf{curl}\, \boldsymbol{u}(t)\|^2_{\boldsymbol{L}^2(\Omega)} \;=\; \sum_{k=1}^{+\infty} \beta_k^2 \, e^{-2\lambda_k\, t} \lambda_k.$$

Finally, since

$$\|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2(\Omega)} = \sum_{j=1}^{J} \alpha_j^2 \;+\; \sum_{k=1}^{+\infty} \beta_k^2$$

estimate (5.1.7) follows directly. Similarly one gets directly estimates (5.1.5)-(5.1.8). We recall that $(\boldsymbol{z}_k)_k$ are eigenvectors for the Stokes operator associated to the eigenvalues $(\lambda_k)_k$ and they form with $(\widetilde{\mathbf{grad}}\, q_j^\tau)_{1\le j\le J}$ an orthonormal basis for $\boldsymbol{L}^2_{\sigma,\tau}(\Omega)$.
(ii) The solution $\boldsymbol{u}$ satisfies (see [7, Theorem 6.4])

$$\boldsymbol{u} \in L^2(0,T;\, \boldsymbol{H}^1(\Omega)) \;\; \text{and} \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^2(0,T;\, [\boldsymbol{H}^2_0(\mathrm{div},\Omega)]') \tag{5.1.10}$$

and

$$\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}\,\mathrm{t}}\, \|\boldsymbol{u}(t)\|^2_{\boldsymbol{L}^2(\Omega)} \;+\; \int_\Omega |\mathbf{curl}\, \boldsymbol{u}(t)|^2\, \mathrm{d}\, x \;=\; 0.$$

In other words for $p = 2$, Corollary 5.1.2 still holds true for $q = 2$ included.

We will now extend estimates (5.1.5)-(5.1.8) by giving the so called $L^p - L^q$ estimates

Theorem 5.1.4. *Let $1 < p \le q < \infty$, $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and let $\boldsymbol{u}(t)$ be the unique solution of Problem (5.1.1). The following estimates holds:*

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \;\le\; C\, t^{-3/2(1/p-1/q)} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}, \tag{5.1.11}$$

$$\|\mathbf{curl}\, \boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \;\le\; C\, t^{-1/2}\, t^{-3/2(1/p-1/q)} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \tag{5.1.12}$$

and

$$\forall\, m, n \in \mathbb{N}, \qquad \Big\| \frac{\partial^m}{\partial t^m} \Delta^n \boldsymbol{u}(t) \Big\|_{\boldsymbol{L}^q(\Omega)} \;\le\; C\, t^{-(m+n)}\, t^{-3/2(1/p-1/q)} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}. \tag{5.1.13}$$

Proof. The case $p = q$ follows directly from Theorem 5.1.1. Suppose that $p \neq q$, the proof is similar to the proof of [18, Corollary 4.6]. Let $s \in \mathbb{R}$ such that $\frac{3}{2}(\frac{1}{p} - \frac{1}{q}) < s < \frac{3}{2p}$ and set $\frac{1}{p_0} = \frac{1}{p} - \frac{2s}{3}$. It is clear that $p < q < p_0$. Let $\boldsymbol{u}(t)$ be the unique solution of Problem (5.1.1). Since for all $t > 0$, $\boldsymbol{u}(t) \in \mathbf{D}(A_p^\infty)$, then thanks to Corollary 4.1.11, $\boldsymbol{u}(t) \in \mathbf{D}(A_p^s) \hookrightarrow \boldsymbol{L}^{p_0}(\Omega)$. Now set $\alpha = \frac{1/p-1/q}{1/p-1/p_0} \in]0,1[$, we can easily verify that $\frac{1}{q} = \frac{\alpha}{p_0} + \frac{1-\alpha}{p}$. Thus $\boldsymbol{u}(t) \in \boldsymbol{L}^q(\Omega)$ and

$$
\begin{aligned}
\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} &\leq C\|\boldsymbol{u}(t)\|^{\alpha}_{\boldsymbol{L}^{p_0}(\Omega)}\|\boldsymbol{u}(t)\|^{1-\alpha}_{\boldsymbol{L}^p(\Omega)} \\
&\leq C\|A_p^s e^{-tA_p}\boldsymbol{u}_0\|^{\alpha}_{\boldsymbol{L}^p(\Omega)}\|e^{-tA_p}\boldsymbol{u}_0\|^{1-\alpha}_{\boldsymbol{L}^p(\Omega)} \\
&\leq C\,t^{-\alpha s}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}. \\
&= C\,t^{-3/2(1/p-1/q)}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}.
\end{aligned}
$$

Next, let $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega) \cap \boldsymbol{L}^q_{\sigma,\tau}(\Omega)$ then $\mathbf{curl}\,\boldsymbol{u}(t) \in \boldsymbol{L}^q(\Omega)$ and

$$
\begin{aligned}
\|\mathbf{curl}\,\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \leq C\,\|A_q^{1/2}\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} &= \|A_q^{1/2}e^{-tA_p/2}e^{-tA_p/2}\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\Omega)} \\
&\leq C\,t^{-1/2}\,\|e^{-tA_p/2}\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\Omega)} \\
&\leq C\,t^{-1/2}\,t^{-3/2(1/p-1/q)}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}.
\end{aligned}
$$

Now let $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, using the density of $\boldsymbol{L}^p_{\sigma,\rho}(\Omega) \cap \boldsymbol{L}^q_{\sigma,\tau}(\Omega)$ in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ we know that there exists a sequence $(\boldsymbol{u}_{0_m})_{m\geq 0}$ in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega) \cap \boldsymbol{L}^q_{\sigma,\tau}(\Omega)$ that converges to $\boldsymbol{u}_0$ in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. For all $m \in \mathbb{N}$ we set $\boldsymbol{u}_m(t) = e^{-tA_p}\boldsymbol{u}_{0_m}$, as a result the sequences $(\boldsymbol{u}_m(t))_{m\geq 0}$ and $(\mathbf{curl}\,\boldsymbol{u}_m(t))_{m\geq 0}$ converges to $\boldsymbol{u}(t)$ and $\mathbf{curl}\,\boldsymbol{u}(t)$ respectively in $\boldsymbol{L}^p(\Omega)$, where $u(t) = e^{-tA_p}\boldsymbol{u}_0$. On the other hand, for all $m, n \in \mathbb{N}$ one has

$$
\|\mathbf{curl}(\boldsymbol{u}_n(t) - \boldsymbol{u}_m(t))\|_{\boldsymbol{L}^q(\Omega)} \leq C\,t^{-1/2}\,t^{-3/2(1/p-1/q)}\|\boldsymbol{u}_{0_n} - \boldsymbol{u}_{0_m}\|_{\boldsymbol{L}^p(\Omega)}.
$$

Thus $(\mathbf{curl}\,\boldsymbol{u}_m(t))_{m\geq 0}$ is a Cauchy sequence in $\boldsymbol{L}^q_{\sigma,\tau}(\Omega)$ and converges to $\mathbf{curl}\,\boldsymbol{u}(t)$ in $\boldsymbol{L}^q(\Omega)$. This means that $\mathbf{curl}\,\boldsymbol{u}(t) \in \boldsymbol{L}^q(\Omega)$ and by passing to the limit as $m \to \infty$ one gets estimate (5.1.12).

Finally, thanks to Theorem 5.1.1 we know that for all $m, n \in \mathbb{N}$, $\frac{\partial^m}{\partial t^m}\Delta^n\boldsymbol{u} \in C^\infty((0,\infty),\,\mathbf{D}(A_p))$. Thus it belongs to $\boldsymbol{L}^q(\Omega)$ and

$$
\Big\|\frac{\partial^m}{\partial t^m}\Delta^n\boldsymbol{u}(t)\Big\|_{\boldsymbol{L}^q(\Omega)} = \|A_p^{(m+n)}e^{-tA_p}\boldsymbol{u}_0\|_{\boldsymbol{L}^q(\Omega)} \leq C\,t^{-(m+n)}\,t^{-3/2(1/p-1/q)}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}.
$$

$\square$

The following theorem studies the Problem (5.1.1) when the initial data $\boldsymbol{u}_0 \in \boldsymbol{X}_p$, (see 3.1.74 for the definition of $\boldsymbol{X}_p$). We will see that because $\boldsymbol{u}_0$ satisfies the compatibility conditions (3.1.25), estimates (5.1.5)-(5.1.8) still hold true with an exponential decay with respect to time.

Theorem 5.1.5. *Suppose that $\boldsymbol{u}_0 \in \boldsymbol{X}_p$ then Problem (5.1.1) has a unique solution $\boldsymbol{u}$ satisfying*

$$
\boldsymbol{u} \in C([0,\,+\infty[,\,\boldsymbol{X}_p) \cap C(]0,\,+\infty[,\,\mathbf{D}(A'_p)) \cap C^1(]0,\,+\infty[,\,\boldsymbol{X}_p), \tag{5.1.14}
$$

$$\boldsymbol{u} \in C^k(]0, +\infty[, \mathbf{D}(A_p'^{\ell})), \qquad \forall\, k,\, \ell \in \mathbb{N}. \tag{5.1.15}$$

Moreover there exists a constant $M > 0$ *such that for all* $0 < \mu < \lambda_1$*, the solution* $\boldsymbol{u}$ *satisfies the estimates:*

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq M\, \kappa_1(\Omega, p)\, e^{-\mu\, t}\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \tag{5.1.16}$$

and

$$\Big\|\frac{\partial \boldsymbol{u}(t)}{\partial t}\Big\|_{\boldsymbol{L}^p(\Omega)} \leq M\, \kappa_1(\Omega, p)\, \frac{e^{-\mu\, t}}{t}\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \tag{5.1.17}$$

In addition, if Ω *is of class* $C^{2,1}$ *the following estimates hold*

$$\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq M\, \kappa_1(\Omega, p)\, \frac{e^{-\mu\, t}}{\sqrt{t}}\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \tag{5.1.18}$$

and

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C(\Omega, p)\, \frac{e^{-\mu\, t}}{t}\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}. \tag{5.1.19}$$

We recall that λ_1 *is the first non zero eigenvalue of the Stokes operator defined above.*

Proof. As in the proof of Theorem 5.1.1, applying the semi-group theory to the operator A'_p one gets the existence and uniqueness of solution to the homogeneous Stokes Problem given by $\boldsymbol{u}(t) = e^{-tA'_p}\,\boldsymbol{u}_0$ and satisfying (5.1.14)-(5.1.15). We recall that the operator A'_p is the extension of the Stokes operator to $\boldsymbol{X}_p$ and it is defined by (3.1.77). The domain of the operator A'_p, $\mathbf{D}(A'_p)$ is defined by (3.1.76).

On the other hand, thanks to Theorem 3.1.37 and to [10] we know that

$$S(-A'_p) = \sup\{\mathrm{Re}\,\lambda \in \sigma(-A'_p)\} = -\lambda_1 < 0.$$

As a result thanks to [64, Chapitre 4, Theorem 4.3, page 118], there is a constant $M > 0$ such that for all $0 < \mu < \lambda_1$, $\|e^{-tA'_p}\|_{\mathcal{L}(\boldsymbol{X}_p)} \leq M\, \kappa_3(\Omega, p)\, e^{-\mu\, t}$, where $e^{-tA'_p}$ is the semi-group generated by the operator A'_p on $\boldsymbol{X}_p$. Thus one gets estimate (5.1.16). In addition, thanks to [64, Chapter 2, Theorem 6.13, page 76], we have

$$\|A'_p\, e^{-tA'_p}\| \leq M\, \kappa_1(\Omega, p)\, \frac{e^{-\mu\, t}}{t},$$

Thus one has estimate (5.1.17). Next, to prove estimate (5.1.18) we use the Gagliardo-Nirenberg inequality, in the same way as in the proof of the estimate (3.1.56) (see Chapter 3, Theorem 3.1.18, case $\langle \boldsymbol{u}\cdot\boldsymbol{n}\rangle_{\Sigma_j} = 0$, $1 \leq j \leq J$, for the proof). Finally, when Ω is of class $C^{2,1}$ we know that $\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{2,p}(\Omega)} \simeq \|\Delta \boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)}$ and estimate (5.1.19) follows directly. □

Remark 5.1.6. **(i)** Theorem 5.1.5 shows that in particular where the initial data $\boldsymbol{u}_0$ satisfies the compatibility conditions (3.1.25), it is the same for the unique solution $\boldsymbol{u}(t)$ of Problem (5.1.1) for all $t > 0$, which is not true if $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega) \setminus \boldsymbol{X}_p$. This comes from the fact that $\boldsymbol{u} \in C(]0, +\infty[\,, \mathbf{D}(A'_p))$, where $\mathbf{D}(A'_p)$ is given by (3.1.76). These compatibility conditions

give us the exponential decay of the semi-group, since they make the operator A'_p of bounded inverse (*i.e.* since $0 \in \rho(A'_p)$).
(ii) When the initial data $\boldsymbol{u}_0 \in \boldsymbol{X}_p$ the $L^p - L^q$ estimates (5.1.11-5.1.13) still hold true with an exponential decay in time. In other words there exist a constant $C > 0$ and $\delta > 0$ such that

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \leq C\, e^{-\delta\, t}\, t^{-3/2(1/p-1/q)} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)},$$

$$\|\mathbf{curl}\, \boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \leq C\, e^{-\delta\, t}\, t^{-1/2}\, t^{-3/2(1/p-1/q)} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}$$

and

$$\forall\, m, n \in \mathbb{N}, \qquad \Big\| \frac{\partial^m}{\partial t^m} \Delta^n \boldsymbol{u}(t) \Big\|_{\boldsymbol{L}^q(\Omega)} \leq C\, e^{-\delta\, t}\, t^{-(m+n)}\, t^{-3/2(1/p-1/q)} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}.$$

(iii) For $p = 2$, the solution $\boldsymbol{u}$ can be written explicitly in the form

$$\boldsymbol{u}(t) = \sum_{k=1}^{+\infty} \beta_k\, e^{-\lambda_k\, t}\, \boldsymbol{z}_k, \qquad \beta_k = \int_\Omega \boldsymbol{u}_0 \cdot \overline{\boldsymbol{z}_k}\, \mathrm{d}\, x$$

and the exponential decay with respect to time can be obtained directly. Moreover, contrary to the case $p \neq 2$ one has

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^2(\Omega)} \leq e^{-\lambda_1 t} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^2(\Omega)}. \tag{5.1.20}$$

In other words one has

$$\|e^{-tA'_2}\|_{\mathcal{L}(\boldsymbol{X}_2)} \leq e^{-\lambda_1 t}.$$

It is clear that estimate (5.1.20) is better than estimate (5.1.16). We recall that λ_1 is the first eigenvalue for the operator A'_p and it is equal to $\frac{1}{C_2(\Omega)}$ where $C_2(\Omega)$ is the constant of the Poincaré-type inequality (3.1.23).

In what follows, we study Problem (5.1.1) when the initial data $\boldsymbol{u}_0$ is more regular in order to get Strong Solution to Problem (5.1.1). We consider the case where $\boldsymbol{u}_0 \in \boldsymbol{V}^p_\tau(\Omega)$ (given by (3.1.2)). We recall that the space $\boldsymbol{V}^p_\tau(\Omega)$ is equal to the space $\boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega)$ (given by (3.1.3)) with equivalent norms.

Proposition 5.1.7 (Strong Solutions for the homogeneous Stokes Problem)**.** *Suppose that Ω is of class $C^{2,1}$ and let $\boldsymbol{u}_0 \in \boldsymbol{V}^p_\tau(\Omega)$ (given by (3.1.2)) and $T < \infty$. The unique solution $\boldsymbol{u}(t)$ of Problem (5.1.1) satisfies in particular*

$$\forall\, 1 \leq q < 2, \qquad \boldsymbol{u} \in L^q(0,T;\, \boldsymbol{W}^{2,p}(\Omega)) \;\text{ and }\; \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega)). \tag{5.1.21}$$

Moreover we have the estimates

$$\Big\| \frac{\partial \boldsymbol{u}}{\partial t} \Big\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C(\Omega, p)}{\sqrt{t}} \|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)} \tag{5.1.22}$$

and

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C(\Omega, p)\Big(1 + \frac{1}{\sqrt{t}}\Big) \|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)}. \tag{5.1.23}$$

Proof. Let $\boldsymbol{u}_0 \in \boldsymbol{V}^p_\tau(\Omega)$ and let $\boldsymbol{u}(t)$ be the unique solution of Problem (5.1.1). Set $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{u}(t)$. It is clear that $\boldsymbol{z}(t)$ is a solution of the problem

$$\begin{cases} \frac{\partial \boldsymbol{z}}{\partial t} - \Delta \boldsymbol{z} = \mathbf{curl}\,\boldsymbol{f}, \qquad \operatorname{div} \boldsymbol{z} = 0 & \text{in} \quad \Omega \times (0,T), \\ \boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{0}, & \text{on} \quad \Gamma \times (0,T), \\ \boldsymbol{z}(0) = \mathbf{curl}\,\boldsymbol{u}_0 & \text{in} \quad \Omega. \end{cases}$$

Using the analyticity of the operator $\mathcal{A}_p$, on $\boldsymbol{L}^p_\sigma(\Omega)$ (see Theorem 3.3.10) and proceeding in a similar way as in the proof of Theorem 5.1.1 we have

$$\|\mathbf{curl}\,\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C(\Omega,p)}{\sqrt{t}}\,\|\boldsymbol{z}_0\|_{\boldsymbol{L}^p(\Omega)}.$$

We recall that the operator $\mathcal{A}_p$ is defined by (3.3.7) and it is equal to the $-\Delta$ with normal boundary condition (see Proposition 3.3.1).

As a result

$$\Big\|\frac{\partial \boldsymbol{u}}{\partial t}\Big\|_{\boldsymbol{L}^p(\Omega)} = \|\Delta \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C(\Omega,p)}{\sqrt{t}}\,\|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)}.$$

Finally using the fact that (see Remark 3.1.5) $\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{2,p}(\Omega)} \simeq \|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} + \|\Delta \boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)}$, (5.1.21)-(5.1.23) follow directly. $\square$

Remark 5.1.8. As in Remark 5.1.3 (ii), for $p = 2$, the solution $\boldsymbol{u}$ satisfies (5.1.21) for $q = 2$ included.

Remark 5.1.9. When $\boldsymbol{u}_0 \in \boldsymbol{V}^p_\tau(\Omega)$ the following $L^p - L^q$ estimates hold

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \leq C\,t^{-1/2}\,t^{-3/2(1/p-1/q)}\|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)}. \tag{5.1.24}$$

The following theorem shows that by taking the initial data less regular one gets the very-weak solution to Problem (5.1.1).

Theorem 5.1.10 (Very weak solutions for the homogeneous Stokes Problem)**.** *Let $\boldsymbol{u}_0 \in [\boldsymbol{H}^{p'}_0(\operatorname{div},\Omega)]'_{\sigma,\tau}$, $T < \infty$ and let $\boldsymbol{u}$ be the unique solution of Problem (5.1.1). Then $\boldsymbol{u}$ satisfies*

$$\forall\, 1 \leq q < 2, \qquad \boldsymbol{u} \in L^q(0,T;\,\boldsymbol{L}^p(\Omega)) \ \text{ and } \quad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\,[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}). \tag{5.1.25}$$

Proof. Using the analyticity of the Stokes semi-group on $[\boldsymbol{H}^{p'}_0(\operatorname{div},\Omega)]'_{\sigma,\tau}$ (see Chapter 3, Theorem 3.1.28), we know that, when the initial data $\boldsymbol{u}_0 \in [\boldsymbol{H}^{p'}_0(\operatorname{div},\Omega)]'_{\sigma,\tau}$ the solution $\boldsymbol{u}(t) \in \boldsymbol{W}^{1,p}(\Omega)$ for all $t > 0$. As a result, using the interpolation inequality we have

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\,\|\boldsymbol{u}(t)\|^{1/2}_{\boldsymbol{W}^{1,p}(\Omega)}\|\boldsymbol{u}(t)\|^{1/2}_{\boldsymbol{W}^{-1,p}(\Omega)}. \tag{5.1.26}$$

On the other hand, thanks to Corollary 3.1.24 we know that

$$\begin{aligned} \|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{1,p}(\Omega)} &\simeq \|\boldsymbol{u}(t)\|_{[\boldsymbol{H}^{p'}_0(\operatorname{div},\Omega)]'} + \|\Delta \boldsymbol{u}(t)\|_{[\boldsymbol{H}^{p'}_0(\operatorname{div},\Omega)]'} \\ &\leq \big(1 + \frac{1}{t}\big)\,\|\boldsymbol{u}_0\|_{\boldsymbol{H}^{p'}_0(\operatorname{div},\Omega)]'}. \end{aligned} \tag{5.1.27}$$

Moreover, thanks to the continuous embeddings $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]' \hookrightarrow \boldsymbol{W}^{-1,p}(\Omega)$ and to the semi-group theory we have

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{-1,p}(\Omega)} \leq C(\Omega,p)\,\|\boldsymbol{u}(t)\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} \leq C(\Omega,p)\,\|\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}. \tag{5.1.28}$$

As a result, putting together (5.1.26), (5.1.27) and (5.1.28) one gets

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\left(1+\frac{1}{t}\right)^{1/2}\|\boldsymbol{u}_0\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}.$$

Thus, for every $T<\infty$ and for every $1\leq q<2$, $\boldsymbol{u}\in L^q(0,T;\,\boldsymbol{L}^p(\Omega))$.

It remains to prove that $\frac{\partial \boldsymbol{u}}{\partial t}\in L^q(0,T;\,[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau})$. We proceed in a similar way as in the proof of Corollary 5.1.2. We set

$$\widetilde{\boldsymbol{u}}(t)=\boldsymbol{u}(t)-\sum_{j=1}^{J}\langle\boldsymbol{u}(t)\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j}\widetilde{\mathbf{grad}}\,q_j^{\tau}.$$

It is clear that $\boldsymbol{u}(t)=\widetilde{\boldsymbol{u}}(t)+\sum_{j=1}^{J}\langle\boldsymbol{u}(t)\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j}\widetilde{\mathbf{grad}}\,q_j^{\tau}$. Moreover thanks to [10, Theorem 4.15] we know that

$$\|\Delta\boldsymbol{u}\|_{[\boldsymbol{T}^{p'}(\Omega)]'}=\|\Delta\widetilde{\boldsymbol{u}}\|_{[\boldsymbol{T}^{p'}(\Omega)]'}\simeq\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^p(\Omega)}\leq\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

The last inequality comes from the fact (see Chapter 2, Lemma 2.2.8)

$$|\langle\boldsymbol{u}\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j}|\leq C(\Omega,p)\,\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

Thus $\frac{\partial\boldsymbol{u}}{\partial t}=\Delta\boldsymbol{u}\in L^q(0,T;\,[\boldsymbol{T}^{p'}(\Omega)]')$ and the result is proved. □

Remark 5.1.11. **(i)** Applying the semi-group theory to the operator B_p (given by (3.1.65)) which is the extension of the Stokes operator to $[\boldsymbol{H}_0^{p'}(\mathrm{div}),\Omega)]'_{\sigma,\tau}$ one observe that when the given data $\boldsymbol{u}_0\in[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$ the Problem (5.1.1) has a unique solution $\boldsymbol{u}$ satisfying

$$\boldsymbol{u}\in C([0,+\infty[,\,[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau})\cap C(]0,+\infty[,\,\mathbf{D}(B_p))\cap C^1(]0,+\infty[,\,[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}),$$

$$\boldsymbol{u}\in C^k(]0,+\infty[,\,\mathbf{D}(B_p^{\ell})),\qquad\forall\,k\in\mathbb{N},\ \forall\,\ell\in\mathbb{N}^*.$$

We recall that the domain $\mathbf{D}(B_p)$ is given by (3.1.66).

Moreover we have the estimates

$$\|\boldsymbol{u}(t)\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}\leq C(\Omega,p)\,\|\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'},$$

$$\left\|\frac{\partial\boldsymbol{u}(t)}{\partial t}\right\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}\leq\frac{C(\Omega,p)}{t}\,\|\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}$$

and

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{1,p}(\Omega)}\leq C(\Omega,p)\,(1+\frac{1}{t})\|\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}.$$

(ii) Suppose that Ω is of class $C^{2,1}$, applying the semi-group theory to the operator C_p (given by (3.1.72)) which is the extension of the Stokes operator to $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ one observe that when the given data $\boldsymbol{u}_0 \in [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ the Problem (5.1.1) has a unique solution $\boldsymbol{u}$ satisfying

$$\boldsymbol{u} \in C([0, +\infty[, [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}) \cap C(]0, +\infty[, \mathbf{D}(C_p)) \cap C^1(]0, +\infty[, [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}),$$

$$\boldsymbol{u} \in C^k(]0, +\infty[, \mathbf{D}(C_p^\ell)), \qquad \forall\, k \in \mathbb{N},\ \forall\, \ell \in \mathbb{N}^*.$$

We recall that the domain of the operator C_p is given by (3.1.73).

Moreover we have the estimates

$$\|\boldsymbol{u}(t)\|_{[\boldsymbol{T}^{p'}(\Omega)]'} \leq C(\Omega, p)\, \|\boldsymbol{u}_0\|_{[\boldsymbol{T}^{p'}(\Omega)]'},$$

$$\Big\|\frac{\partial \boldsymbol{u}(t)}{\partial t}\Big\|_{[\boldsymbol{T}^{p'}(\Omega)]'} \leq \frac{C(\Omega, p)}{t}\, \|\boldsymbol{u}_0\|_{[\boldsymbol{T}^{p'}(\Omega)]'}.$$

and

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega, p)\, (1 + \frac{1}{t})\, \|\boldsymbol{u}_0\|_{[\boldsymbol{T}^{p'}(\Omega)]'}.$$

Remark 5.1.12 (Optimal initial value)**.** It is an important question what is the optimal (weakest possible) initial value to obtain a unique strong, weak or very weak solution to Problem (5.1.1).

(i) A unique solution $\boldsymbol{u}$ of Problem (5.1.1) is said to be a strong solution if it satisfies

$$1 < p, q < \infty,\ \ T \leq \infty,\ \ \boldsymbol{u} \in L^q(0, T;\, \boldsymbol{W}^{2,p}(\Omega)),\ \text{ and }\ \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0, T;\, \boldsymbol{L}^p(\Omega)).$$

The assumption $\boldsymbol{u}_0 \in \boldsymbol{V}^p_\tau(\Omega)$ is not optimal and may be replaced by the properties

$$\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega), \qquad \int_0^\infty \|A_p e^{-tA_p} \boldsymbol{u}_0\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\, t < \infty, \tag{5.1.29}$$

where $1 < p, q < \infty$ and e^{-tA_p} is the semi-group generated by the Stokes operator on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. With an initial value $\boldsymbol{u}_0$ satisfying (5.1.29) the unique solution $\boldsymbol{u}$ of Problem (5.1.1) satisfies (5.1.21) for all $1 < p, q < \infty$ and for all $T \leq \infty$ (see Proposition 5.1.7).

(ii) A unique solution $\boldsymbol{u}$ of Problem (5.1.1) is said to be a weak solution if it satisfies

$$1 < p, q < \infty,\ \ T \leq \infty,\ \ \boldsymbol{u} \in L^q(0, T;\, \boldsymbol{W}^{1,p}(\Omega)),\ \text{ and }\ \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0, T;\, [\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]'). \tag{5.1.30}$$

The optimal choice of the initial value $\boldsymbol{u}_0$ to obtain a unique weak solution to Problem (5.1.1) satisfying the maximal regularity (5.1.30) is

$$\boldsymbol{u}_0 \in [\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]'_{\sigma,\tau}, \qquad \int_0^\infty \|B_p e^{-tB_p} \boldsymbol{u}_0\|^q_{[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'} \mathrm{d}\, t < \infty, \tag{5.1.31}$$

where $1 < p, q < \infty$ and e^{-tB_p} is the semi-group generated by the Stokes operator on $[\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]'_{\sigma,\tau}$. Observe that for the weak solution, the choice of an initial value $\boldsymbol{u}_0$ satisfying (5.1.31) is better than $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ (see Corollary 5.1.2), since it allows us to obtain a unique solution satisfying (5.1.30) for all $1 < p, q < \infty$ and for $T = \infty$ included.

(iii) A unique solution $\boldsymbol{u}$ of Problem (5.1.1) is said to be a very weak solution if it satisfies

$$1 < p, q < \infty, \;\; T \leq \infty, \;\; \boldsymbol{u} \in L^q(0, T;\, \boldsymbol{L}^p(\Omega)), \;\; \text{and} \;\; \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0, T;\, [\boldsymbol{T}^{p'}(\Omega)]'). \tag{5.1.32}$$

The optimal choice of the initial value $\boldsymbol{u}_0$ to obtain a unique very weak solution to Problem (5.1.1) satisfying the maximal regularity (5.1.32) is

$$\boldsymbol{u}_0 \in [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}, \qquad \int_0^{\infty} \|C_p e^{-tC_p} \boldsymbol{u}_0\|^q_{[\boldsymbol{T}^{p'}(\Omega)]'} \mathrm{d}\, t < \infty, \tag{5.1.33}$$

where $1 < p, q < \infty$ and e^{-tC_p} is the semi-group generated by the Stokes operator on $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$. Notice that for the very weak solution, the choice of an initial value $\boldsymbol{u}_0$ satisfying (5.1.33) is better than $\boldsymbol{u}_0 \in [\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'_{\sigma,\tau}$ (see Theorem 5.1.10), since it allows us to obtain a unique solution satisfying (5.1.25) for all $1 < p, q < \infty$ and for $T = \infty$ included.

Thanks to Remark 3.1.42, the following remark shows that when the initial data satisfies a compatibility conditions similar to (3.1.25) the estimates in the Remark 5.1.11 still holds true with an exponential decay with respect to time.

Remark 5.1.13. **(i)** Since the operator $-B'_p$ generates a bounded analytic semi-group on $\boldsymbol{Y}_p$ (see Remark 3.1.42), when $\boldsymbol{u}_0 \in \boldsymbol{Y}_p$ the Problem (5.1.1) has a unique solution $\boldsymbol{u}$ satisfying

$$\boldsymbol{u} \in C([0, +\infty[, \boldsymbol{Y}_p) \cap C(]0, +\infty[, \mathbf{D}(B'_p)) \cap C^1(]0, +\infty[, \boldsymbol{Y}_p),$$
$$\boldsymbol{u} \in C^k(]0, +\infty[, \mathbf{D}(B'^{\ell}_p)), \qquad \forall\, k,\, \ell \in \mathbb{N}.$$

Moreover there exists a constant $C(\Omega, p)$ and a constant $\mu > 0$, the solution $\boldsymbol{u}$ satisfies the estimates:

$$\|\boldsymbol{u}(t)\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} \leq C(\Omega, p)\, e^{-\mu\, t}\, \|\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'},$$
$$\left\|\frac{\partial \boldsymbol{u}(t)}{\partial t}\right\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} \leq C(\Omega, p)\, \frac{e^{-\mu\, t}}{t}\, \|\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}$$

and

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(\Omega, p)\, \frac{e^{-\mu\, t}}{t}\, \|\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}.$$

(ii) Suppose that Ω is of Class $C^{2,1}$, since the operator $-C'_p$ generates a bounded analytic semi-group on $\boldsymbol{Z}_p$ (see Remark 3.1.42), when $\boldsymbol{u}_0 \in \boldsymbol{Z}_p$ the Problem (5.1.1) has a unique solution $\boldsymbol{u}$ satisfying

$$\boldsymbol{u} \in C([0, +\infty[, \boldsymbol{Z}_p) \cap C(]0, +\infty[, \mathbf{D}(C'_p)) \cap C^1(]0, +\infty[, \boldsymbol{Z}_p),$$
$$\boldsymbol{u} \in C^k(]0, +\infty[, \mathbf{D}(C'^{\ell}_p)), \qquad \forall\, k,\, \ell \in \mathbb{N}.$$

Moreover there exists a constant $C(\Omega, p)$ and a constant $\mu > 0$, the solution $\boldsymbol{u}$ satisfies the estimates:

$$\|\boldsymbol{u}(t)\|_{[\boldsymbol{T}^{p'}(\Omega)]'} \leq C(\Omega, p)\, e^{-\mu\, t}\, \|\boldsymbol{u}_0\|_{[\boldsymbol{T}^{p'}(\Omega)]'},$$

$$\Big\|\frac{\partial \boldsymbol{u}(t)}{\partial t}\Big\|_{[\boldsymbol{T}^{p'}(\Omega)]'} \leq C(\Omega,p)\,\frac{e^{-\mu\, t}}{t}\,\|\boldsymbol{u}_0\|_{[\boldsymbol{T}^{p'}(\Omega)]'}$$

and

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\,\frac{e^{-\mu\, t}}{t}\,\|\boldsymbol{u}_0\|_{[\boldsymbol{T}^{p'}(\Omega)]'}.$$

5.1.2 The inhomogeneous Stokes problem

We consider the abstract Cauchy-Problem on a complex Banach space X:

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{A}\,u(t) = f(t) \quad 0 \leq t \leq T \\ \quad u(0) = 0, \end{cases} \tag{5.1.34}$$

where $-\mathcal{A}$ is the infinitesimal generator of an analytic semi-group in X. In general, for $f \in L^p(0,T;\,X)$, the analyticity of $-\mathcal{A}$ is not enough to obtain a solution to Problem (5.1.34) satisfying

$$u \in W^{1,p}(0,T;\,X) \cap L^p(0,T;\,D(\mathcal{A})), \tag{5.1.35}$$

unless X is a Hilbert space (see [15, 72] for instance). Usually it is necessary to impose some further regularity condition on f such as Hölder continuity, (see [64] for instance). However, using the concept of ζ-convexity and a perturbation argument, [28, 43] have proved the existence of solution to Problem (5.1.34) satisfying (5.1.35), when the pure imaginary powers of $\mathcal{A}$ are bounded and satisfy estimate (2.3.4). Moreover, [43, Theorem 2.1] extends [28, Theorem 3.2] in two directions: First, the operator $\mathcal{A}$ may not have bounded inverse and second, the maximal interval of time is $T = \infty$. It is important to know that, Kato has proved [48, 49] a similar result in the case of a Hilbert space. More precisely, he has proved that the pure imaginary powers of a maximal accretive operator are bounded and satisfies an estimate of type (2.3.4). The property of ζ-convexity of a Banach space is stronger than the property of uniform convexity or reflexivity. More precisely, we have the following definition (see [65, Theorem 2] for example)

Definition 5.1.14. *A Banach space X is ζ-convex if there is a symmetric biconvex function ζ on $X \times X$ such that $\zeta(0,0) > 0$ and*

$$\forall\, x, y \in X,\;\; \|x\|_X \geq 1, \qquad \zeta(x,y) \leq \|x+y\|_X.$$

As an immediate consequence of Definition 5.1.14 we have:

Theorem 5.1.15. *Every Hilbert space is ζ-convex, ζ-convexity is preserved by Banach space's isomorphism and closed subspaces. Cartesian products and quotients of ζ-convex spaces are ζ-convex.*

On the other hand, [28, 43] used the following essential property of a ζ-convex Banach space (see [22, Theorem 3.3]):

Theorem 5.1.16. *A Banach space X is ζ-convex if and only if the truncated Hilbert transform*

$$(H_\varepsilon f)(t) \;=\; \frac{1}{\pi}\int_{|\tau|>\varepsilon}\frac{f(t-\tau)}{\tau}\,\mathrm{d}\tau, \qquad f\in L^s(\mathbb{R};\,X) \tag{5.1.36}$$

converges as $\varepsilon \to 0$ for almost all $t\in\mathbb{R}$. Moreover there is a constant $C=C(s,X)$ independent of f such that

$$\|Hf\|_{L^s(\mathbb{R},\,X)} \;\le\; C\,\|f\|_{L^s(\mathbb{R};\,X)}, \qquad 1<s<\infty,$$

where $(Hf)(t)=\lim_{\varepsilon\to 0}(H_\varepsilon f)(t)$.

We summarized above some useful properties of ζ-convex Banach spaces essential in our work. However, for further readings on ζ-convex Banach spaces we refer to [22, 65].

We will use the following theorem which is proved in [43, Theorem 2.1]:

Theorem 5.1.17. *Let X be a ζ-convex Banach space. Assume that $0<T\le\infty$, $1<p<\infty$ and that $\mathcal{A}\in\mathcal{E}_K^\theta(X)$ for some $K\ge 1$, $0\le\theta<\pi/2$ and $\mathcal{E}_K^\theta(X)$ as in Definition 2.3.6. Then for every $f\in L^p(0,T;\,X)$ there exists a unique solution $\boldsymbol{u}$ of the abstract Cauchy-Problem (5.1.34) satisfying the properties:*

$$u\in L^p(0,T_0;\,D(\mathcal{A})),\quad T_0\le T \;\text{ if } T<\infty \;\text{ and }\; T_0<T\;\text{ if } T=\infty,$$

$$\frac{\partial u}{\partial t}\in L^p(0,T;\,X)$$

and

$$\int_0^T \Big\|\frac{\partial u}{\partial t}\Big\|_X^p \,\mathrm{d}\,t \,+\, \int_0^T \|\mathcal{A}u(t)\|_X^p\,\mathrm{d}\,t \;\le\; C\int_0^T \|f(t)\|_X^p\,\mathrm{d}\,t$$

with $C=C(p,\theta,K,X)$ independent of f and T.

Applying now Theorem 5.1.17 to the Stokes operator with Navier-type boundary conditions yields strong-solution to Problem (5.0.1) with the boundary condition (3.1.1) and a global in time L^p-L^q estimates.

First we recall the following proposition which has been proved in [65, Proposition 3]:

Proposition 5.1.18. *Let Ω be an open domain in $\mathbb{R}^3$. The space $L^p(\Omega)$ is ζ-convex if and only if $1<p<\infty$.*

The following proposition shows the ζ-convexity of the dual spaces $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'$ and $[\boldsymbol{T}^{p'}(\Omega)]'$.

Proposition 5.1.19. *Let $1<p<\infty$, the dual spaces $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'$ and $[\boldsymbol{T}^{p'}(\Omega)]'$ are ζ-convex Banach spaces.*

Proof. We will prove the ζ-convexity of $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'$ because the ζ-convexity of $[\boldsymbol{T}^{p'}(\Omega)]'$ is similar. Let $\boldsymbol{f}\in L^s(\mathbb{R};\,[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]')$, then for almost all $t\in\mathbb{R}$, there exists $\boldsymbol{\psi}(t)\in\boldsymbol{L}^p(\Omega)$ and $\chi(t)\in L^p(\Omega)$ such that

$$\boldsymbol{f}(t) \;=\; \boldsymbol{\psi}(t)\,+\,\nabla\chi(t), \qquad \|\boldsymbol{f}(t)\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} \;=\; \max(\|\boldsymbol{\psi}(t)\|_{\boldsymbol{L}^p(\Omega)},\,\|\chi(t)\|_{L^p(\Omega)}).$$

Since $\boldsymbol{f} \in L^s(\mathbb{R};\, [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]')$, it is clear that $\boldsymbol{\psi} \in L^s(\mathbb{R};\, \boldsymbol{L}^p(\Omega))$ and $\chi \in L^s(\mathbb{R}; L^p(\Omega))$. On the other hand we can easily verify that

$$(H_\varepsilon \boldsymbol{f})(t) \,=\, (H_\varepsilon \boldsymbol{\psi})(t) \,+\, \nabla(H_\varepsilon \chi)(t).$$

Next, since $\boldsymbol{L}^p(\Omega)$ (respectively $L^p(\Omega)$) is ζ-convex then $(H_\varepsilon \boldsymbol{\psi})(t)$ (respectively $(H_\varepsilon \chi)(t)$) converges as $\varepsilon \to 0$ to $H\boldsymbol{\psi}(t)$ (respectively to $H\chi(t)$). Moreover we have the estimate

$$\|H\boldsymbol{\psi}(t)\|_{L^s(\mathbb{R};\, \boldsymbol{L}^p(\Omega))} \,\leq\, C(s,\Omega,p)\, \|\boldsymbol{\psi}\|_{L^s(\mathbb{R};\, \boldsymbol{L}^p(\Omega))}$$

and

$$\|H\chi(t)\|_{L^s(\mathbb{R};\, L^p(\Omega))} \,\leq\, C(s,\Omega,p)\, \|\psi\|_{L^s(\mathbb{R};\, L^p(\Omega))}$$

This means that $(H_\varepsilon \boldsymbol{f})(t)$ converges as $\varepsilon \to 0$ to $H\boldsymbol{f}(t) \,=\, H\boldsymbol{\psi}(t) \,+\, \nabla\, H\chi(t)$. Moreover we have the estimate

$$\|H\boldsymbol{f}(t)\|_{L^s(\mathbb{R};\, [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]')} \,\leq\, C(s,\Omega,p)\, \|\boldsymbol{f}\|_{L^s(\mathbb{R};\, [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]')},$$

which ends the proof. $\square$

Next, let us consider the Problem:

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} = \boldsymbol{f}, & \mathrm{div}\, \boldsymbol{u} = 0 \quad \text{in} \quad \Omega \times (0,T), \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Gamma \times (0,T), \\ & \boldsymbol{u}(0) = \mathbf{0} \quad \text{in} \quad \Omega, \end{cases} \tag{5.1.37}$$

where $\boldsymbol{f} \in L^q(0,T;\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$ and $1 < p,q < \infty$. The following theorem gives the maximal $L^p - L^q$ regularity for the inhomogeneous Problem (5.1.37).

Theorem 5.1.20. *Suppose that Ω is of class $C^{2,1}$ and let $1 < p,q < \infty$ and $0 < T \leq \infty$. Then for every $\boldsymbol{f} \in L^q(0,T;\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$ there exists a unique solution $\boldsymbol{u}$ of the Problem (5.1.37) satisfying*

$$\boldsymbol{u} \in L^q(0,T_0;\, \boldsymbol{W}^{2,p}(\Omega)), \quad T_0 \leq T \ \textit{if}\ T < \infty \ \text{ and } \ T_0 < T \ \textit{if}\ T = \infty, \tag{5.1.38}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega)) \tag{5.1.39}$$

and

$$\int_0^T \Big\|\frac{\partial \boldsymbol{u}}{\partial t}\Big\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\,t \,+\, \int_0^T \|\Delta \boldsymbol{u}(t)\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\,t \,\leq\, C(p,q,\Omega) \int_0^T \|\boldsymbol{f}(t)\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\,t. \tag{5.1.40}$$

Proof. Thanks to Theorem 5.1.15 and Proposition 5.1.18 we deduce that for all $1 < p < \infty$, $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ is ζ-convex. In addition, thanks to Theorem 4.1.6 we know that $A_p = -\Delta$ belongs $\mathcal{E}^{\theta_0}_K(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))$ with $0 < \theta_0 < \pi/2$. Thus, applying Theorem 5.1.17 to the operator A_p one gets the existence and uniqueness of a solution $\boldsymbol{u}$ to Problem (5.1.37) satisfying (5.1.38)-(5.1.40). $\square$

Remark 5.1.21. Similarly we can easily verify that $\boldsymbol{X}_p$ given by (3.1.74) is ζ-convex. Ten, for all $\boldsymbol{f} \in L^q(0,T;\, \boldsymbol{X}_p)$, Problem (5.1.37) has a unique solution $\boldsymbol{u}$ that satisfies (5.1.38)-(5.1.40). In this case the solution $\boldsymbol{u}$ satisfies in addition

$$\langle \boldsymbol{u}(t)\cdot\boldsymbol{n}\,,\,1\rangle_{\Sigma_j} \;=\; 0;\;\; 1\leq j\leq J, \qquad 0\leq t\leq\infty.$$

Now we consider the inhomogeneous Stokes problem (5.0.1) with the Navier-type boundary condition (3.1.1), where $\boldsymbol{f} \in L^q(0,T;\, \boldsymbol{L}^p(\Omega))$ and $\boldsymbol{u}_0 = \boldsymbol{0}$. The following theorem shows that the pressure can be decoupled from the problem using the weak Neumann Problem (2.2.7).

Theorem 5.1.22 (Strong Solutions for the inhomogeneous Stokes Problem)**.** *Suppose that Ω is of class $C^{2,1}$ and let $0 < T \leq \infty$, $1 < p, q < \infty$, $\boldsymbol{f} \in \boldsymbol{L}^q(0,T;\boldsymbol{L}^p(\Omega))$ and $\boldsymbol{u}_0 = \boldsymbol{0}$. The Problem (5.0.1) with (3.1.1) has a unique solution $(\boldsymbol{u},\pi)$ such that*

$$\boldsymbol{u} \in L^q(0,T_0;\, \boldsymbol{W}^{2,p}(\Omega)),\;\;\; T_0 \leq T \;\; \textit{if}\;\; T<\infty \;\;\text{ and }\;\; T_0 < T \;\;\textit{if}\;\; T=\infty, \tag{5.1.41}$$

$$\pi \in L^q(0,T;\, W^{1,p}(\Omega)/\mathbb{R}), \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\, \boldsymbol{L}^p(\Omega)) \tag{5.1.42}$$

and

$$\int_0^T \Big\|\frac{\partial \boldsymbol{u}}{\partial t}\Big\|^q_{\boldsymbol{L}^p(\Omega)}\,\mathrm{d}\,t \,+\, \int_0^T \|\Delta\boldsymbol{u}(t)\|^q_{\boldsymbol{L}^p(\Omega)}\,\mathrm{d}\,t \,+\, \int_0^T \|\pi(t)\|^q_{W^{1,p}(\Omega)/\mathbb{R}}\,\mathrm{d}\,t$$
$$\leq\; C(p,q,\Omega)\int_0^T \|\boldsymbol{f}(t)\|^q_{\boldsymbol{L}^p(\Omega)}\,\mathrm{d}\,t. \tag{5.1.43}$$

Proof. Let $\boldsymbol{f} \in \boldsymbol{L}^q(0,T;\, \boldsymbol{L}^p(\Omega))$, thanks to Lemma 2.2.9 we know that for almost all $0<t<T$, the problem

$$\mathrm{div}(\mathbf{grad}\,\pi(t) - \boldsymbol{f}(t)) = 0, \qquad \text{in}\,\Omega, \qquad (\mathbf{grad}\,\pi(t) - \boldsymbol{f}(t))\cdot\boldsymbol{n} = 0, \qquad \text{on}\,\Gamma,$$

has a unique solution $\pi(t) \in W^{1,p}(\Omega)/\mathbb{R}$ that satisfies the estimate

$$\text{for a.e. } t\in(0,T) \qquad \|\pi(t)\|_{W^{1,p}(\Omega)/\mathbb{R}} \;\leq\; C(\Omega)\|\boldsymbol{f}(t)\|_{\boldsymbol{L}^p(\Omega)}.$$

It is clear that $\pi \in L^q(0,T;\, W^{1,p}(\Omega)/\mathbb{R})$, $(\boldsymbol{f} - \mathbf{grad}\,\pi) \in L^q(0,T;\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$. As a result, thanks to Theorem 5.1.20, Problem (5.0.1) with (3.1.1) has a unique solution $(\boldsymbol{u},\pi)$ satisfying (5.1.41)-(5.1.43). $\square$

The following theorem shows that using estimate (4.1.23), the ζ-convexity of $[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'$, Lemma 2.2.9 (part (ii)) concerning the weak Neumann Problem and proceeding in a similar way as in the proof of Theorem 5.1.22 one gets the weak solutions for the inhomogeneous Stokes problem. We will skip the proof because it is similar to the proof of Theorem 5.1.22.

Theorem 5.1.23 (Weak Solutions for the inhomogeneous Stokes Problem). *Let $1 < p, q < \infty$, $\boldsymbol{u}_0 = 0$ and let $\boldsymbol{f} \in L^q(0,T;\, [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]')$, $0 < T \leq \infty$. The Problem (5.0.1) with (3.1.1) has a unique solution $(\boldsymbol{u}, \pi)$ satisfying*

$$\boldsymbol{u} \in L^q(0,T_0;\ \boldsymbol{W}^{1,p}(\Omega)), \quad T_0 \leq T \ \textit{if}\ T < \infty \ \text{ and } \ T_0 < T \ \textit{if}\ T = \infty,$$

$$\pi \in L^q(0,T;\ L^p(\Omega)/\mathbb{R}), \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T; \in [\boldsymbol{H}_0^{p'}(\mathrm{div}\Omega)]'_{\sigma,T})$$

and

$$\int_0^T \left\| \frac{\partial \boldsymbol{u}}{\partial t} \right\|^q_{[\boldsymbol{H}_0^{p'}(\mathrm{div}\Omega)]'} \mathrm{d}\,t + \int_0^T \|\Delta \boldsymbol{u}(t)\|^q_{[\boldsymbol{H}_0^{p'}(\mathrm{div}\Omega)]'} \mathrm{d}\,t + \int_0^T \|\pi(t)\|^q_{L^p(\Omega)/\mathbb{R}} \mathrm{d}\,t \\ \leq C(p,q,\Omega) \int_0^T \|\boldsymbol{f}(t)\|^q_{[\boldsymbol{H}_0^{p'}(\mathrm{div}\Omega)]'} \mathrm{d}\,t.$$

Similarly using the estimate (4.1.24), ζ-convexity of $[\boldsymbol{T}^{p'}(\Omega)]'$ and Lemma 2.2.9 (part (iii)), one has the very weak solution for the inhomogeneous Stokes problem:

Theorem 5.1.24 (Very weak solutions for the inhomogeneous Stokes Problem). *Suppose that Ω is of class $C^{2,1}$, let $0 < T \leq \infty$, $1 < p, q < \infty$, $\boldsymbol{u}_0 = 0$ and $\boldsymbol{f} \in L^q(0,T;\, [\boldsymbol{T}^{p'}(\Omega)]')$. Then the time dependent Stokes Problem (5.0.1) with the boundary condition (3.1.1) has a unique solution $(\boldsymbol{u}, \pi)$ satisfying*

$$\boldsymbol{u} \in L^q(0,T_0;\, \boldsymbol{L}^p(\Omega)), \quad T_0 \leq T \ \textit{if}\ T < \infty \ \text{ and } \ T_0 < T \ \textit{if}\ T = \infty,$$

$$\pi \in L^q(0,T;\ W^{-1,p}(\Omega)/\mathbb{R}), \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T; \in [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau})$$

and

$$\int_0^T \left\| \frac{\partial \boldsymbol{u}}{\partial t} \right\|^q_{[\boldsymbol{T}^{p'}(\Omega)]'} \mathrm{d}\,t + \int_0^T \|\Delta \boldsymbol{u}(t)\|^q_{[\boldsymbol{T}^{p'}(\Omega)]'} \mathrm{d}\,t + \int_0^T \|\pi(t)\|^q_{W^{-1,p}(\Omega)/\mathbb{R}} \mathrm{d}\,t \\ \leq C(p,q,\Omega) \int_0^T \|\boldsymbol{f}(t)\|^q_{[\boldsymbol{T}^{p'}(\Omega)]'} \mathrm{d}\,t.$$

5.2 Stokes problem with Navier slip boundary conditions

In this section we solve the time dependent Stokes Problem (5.0.1) with the boundary condition (3.2.1), using the semi-group theory. We recall that the Stokes operator $\mathbb{A}_p$ with Navier-slip boundary condition is defined in Chapter 3 by (3.2.5) and its domain $\mathbf{D}(\mathbb{A}_p)$ is defined by (3.2.4).

5.2.1 The homogeneous problem

Consider the two problems

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} + \mathbb{A}_p \boldsymbol{u} = \mathbf{0}, & \operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \Omega \times (0,T), \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0} \quad \text{on} \quad \Gamma \times (0,T), \\ & \boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in} \quad \Omega \end{cases} \tag{5.2.1}$$

and

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + \nabla \pi = \mathbf{0}, & \operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \Omega \times (0,T), \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0} \quad \text{on} \quad \Gamma \times (0,T), \\ & \boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in} \quad \Omega. \end{cases} \tag{5.2.2}$$

Notice that a function $\boldsymbol{u} \in C([0, +\infty[, \boldsymbol{L}^p_{\sigma,\tau}(\Omega)) \cap C(]0, +\infty[, \mathbf{D}(\mathbb{A}_p)) \cap C^1(]0, +\infty[, \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$ solve the Problem (5.2.1) if and only if there exists a function $\pi \in C(]0, \infty[\,; W^{1,p}(\Omega)/\mathbb{R})$ such that $(\boldsymbol{u}, \pi)$ solves (5.2.2). Indeed, let $\boldsymbol{u}$ be a solution of (5.2.1). Thus $\mathbb{A}_p \boldsymbol{u} = -P\Delta \boldsymbol{u} = -\frac{\partial \boldsymbol{u}}{\partial t}$. Now, since $(\boldsymbol{u}, \frac{\partial \boldsymbol{u}}{\partial t}) \in \mathbf{D}(\mathbb{A}_p) \times \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, then due to [8] there exists $\pi \in W^{1,p}(\Omega)/\mathbb{R}$ such that $\mathbb{A}_p \boldsymbol{u} = -\Delta \boldsymbol{u} + \nabla \pi = -\frac{\partial \boldsymbol{u}}{\partial t}$. Moreover we have the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)/\boldsymbol{\mathcal{T}}^p(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C(\Omega, p) \Big\| \frac{\partial \boldsymbol{u}}{\partial t} \Big\|_{\boldsymbol{L}^p(\Omega)},$$

where $\boldsymbol{\mathcal{T}}^p(\Omega)$ (given by (3.2.13)) is the Kernel of the Stokes operator with Navier-slip boundary condition. This means that that the mapping $\frac{\partial \boldsymbol{u}}{\partial t} \mapsto \pi$ is continuous from $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ to $W^{1,p}(\Omega)$. As a result, $\pi \in C(]0, \infty[\,; W^{1,p}(\Omega)/\mathbb{R})$ and $(\boldsymbol{u}, \pi)$ solves (5.2.2). Conversely, let $(\boldsymbol{u}, \pi)$ be a solution of (5.2.2). Applying the Helmholtz-projection P (defined by (3.1.7)) to the first equation of Problem (5.2.2), one gets directly that $\boldsymbol{u}$ solves (5.2.1).

For the homogeneous Stokes Problem (5.2.2), the analyticity of the semi-group give us a unique solutions satisfying all the regularity desired. Applying the semi-group theory to the Stokes operator on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, $[\boldsymbol{H}^{p'}_0(\operatorname{div}, \Omega)]'_{\sigma,\tau}$ and $(\boldsymbol{T}^{p'}(\Omega))'_{\sigma,\tau}$ respectively, one gets the strong, weak and very weak solutions to the homogeneous Stokes Problem (5.2.2). We skip the proof of the following theorem because it is similar to the proof of Theorem 5.1.1.

Theorem 5.2.1. *Let $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, then problem (5.2.2) has a unique solution $(\boldsymbol{u}, \pi)$ satisfying*

$$\boldsymbol{u} \in C([0, +\infty[, \boldsymbol{L}^p_{\sigma,\tau}(\Omega)) \cap C(]0, +\infty[, \mathbf{D}(\mathbb{A}_p)) \cap C^1(]0, +\infty[, \boldsymbol{L}^p_{\sigma,\tau}(\Omega)),$$

$$\boldsymbol{u} \in C^k(]0, +\infty[, \mathbf{D}(\mathbb{A}^\ell_p)), \qquad \forall\, k \in \mathbb{N},\ \forall\, \ell \in \mathbb{N}^*$$

and

$$\pi \in C(]0, \infty[\,; W^{1,p}(\Omega)/\mathbb{R}).$$

We recall that $\mathbf{D}(\mathbb{A}_p)$ is given by (3.2.4). Moreover we have the estimates

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq\ C(\Omega, p)\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \tag{5.2.3}$$

and

$$\Big\| \frac{\partial \boldsymbol{u}(t)}{\partial t} \Big\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C(\Omega, p)}{t} \, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}.$$

In addition, if Ω is of class $C^{2,1}$ the following estimates hold

$$\|\mathbf{D}(\boldsymbol{u})\|_{\boldsymbol{L}^p(\Omega)} \ \leq \ \frac{C(\Omega, p)}{\sqrt{t}} \, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \tag{5.2.4}$$

and

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{2,p}(\Omega)} \ \leq \ C(\Omega, p)\,(1 + \frac{1}{t})\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}.$$

We also observe the following corollary

Corollary 5.2.2 (Weak Solutions for the Stokes Problem)**.** *Let $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, $T < \infty$ and let $(\boldsymbol{u}, \pi)$ be the unique solution of Problem (5.2.2). Then for all $1 \leq q < 2$, $(\boldsymbol{u}, \pi)$ satisfies*

$$\boldsymbol{u} \in L^q(0, T;\, \boldsymbol{W}^{1,p}(\Omega)), \qquad \pi \in L^q(0, T,\, L^p(\Omega)/\mathbb{R})$$

and

$$\frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0, T;\, [\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]').$$

Proof. Let $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and let $(\boldsymbol{u}, \pi)$ be the unique solution of Problem (5.2.2). Thanks to [8] we know that

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \simeq \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{D}(\boldsymbol{u})\|_{\boldsymbol{L}^p(\Omega)}.$$

As a result, using estimates (5.2.3) and (5.2.4) we deduce directly that $\boldsymbol{u} \in L^q(0, T;\, \boldsymbol{W}^{1,p}(\Omega))$ for all $1 \leq q < 2$ and for all $T < \infty$. Moreover, thanks to Chapter 3, Theorem 3.2.10 and Remark 3.2.12 we know that

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \simeq \|\boldsymbol{u}\|_{[\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]'} + \| - \Delta \boldsymbol{u} + \nabla \pi\|_{[\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]'}.$$

Since $-\Delta \boldsymbol{u} + \nabla \pi = -\frac{\partial \boldsymbol{u}}{\partial t}$, we deduce directly that $\frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0, T;\, [\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]')$, for all $1 \leq q < 2$ and for all $T < \infty$. Finally, thanks to [8] one has

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{\mathcal{T}}^p(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \leq C(\Omega, p) \Big\| \frac{\partial \boldsymbol{u}}{\partial t} \Big\|_{[\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]'}.$$

As a result we obtain directly that $\pi \in L^q(0, T;\, L^p(\Omega)/\mathbb{R})$, for all $1 \leq q < 2$ and for all $T < \infty$. $\square$

As in the previous section, the unique solution of Problem (5.2.2) satisfies the so called $L^p - L^q$ estimates. We skip the proof of the following Theorem, because it is similar to the proof of Theorem 5.1.4

Theorem 5.2.3. *Let $1 < p \leq q < \infty$, $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and let $\boldsymbol{u}(t)$ be the unique solution of Problem (5.2.2). The following estimates holds:*

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \leq C\, t^{-3/2(1/p-1/q)} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)},$$

$$\|\mathbf{D}(\boldsymbol{u}(t))\|_{\boldsymbol{L}^q(\Omega)} \leq C\, t^{-1/2}\, t^{-3/2(1/p-1/q)} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}$$

and

$$\forall\, m, n \in \mathbb{N}, \qquad \Big\| \frac{\partial^m}{\partial t^m} \mathbb{A}^n_p \boldsymbol{u}(t) \Big\|_{\boldsymbol{L}^q(\Omega)} \leq C\, t^{-(m+n)}\, t^{-3/2(1/p-1/q)} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}.$$

Now applying the semi-group theory to the operator $\mathbb{B}_p$ (defined by (3.2.33)) which is the extension of the Stokes operator $\mathbb{A}_p$ to $[\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]'_{\sigma,\tau}$ one has the following theorem : (we recall that the dual space $[\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]'_{\sigma,\tau}$ is given by (3.1.64)).

Theorem 5.2.4. *Suppose that the given data $\boldsymbol{u}_0 \in [\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]'_{\sigma,\tau}$ then the Problem (5.2.2) has a unique solution $(\boldsymbol{u}, \pi)$ satisfying*

$$\boldsymbol{u} \in C([0, +\infty[,\, [\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]'_{\sigma,\tau}) \cap C(]0, +\infty[,\, \mathbf{D}(\mathbb{B}_p)) \cap C^1(]0, +\infty[,\, [\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]'_{\sigma,\tau}),$$

$$\boldsymbol{u} \in C^k(]0, +\infty[,\, \mathbf{D}(\mathbb{B}^\ell_p)), \qquad \forall\, k \in \mathbb{N},\ \forall\, \ell \in \mathbb{N}^*$$

and

$$\pi \in C(]0, \infty[\,;\, L^p(\Omega)/\mathbb{R}).$$

We recall that $\mathbf{D}(\mathbb{B}_p)$ is defined by (3.2.32). Moreover we have the estimates

$$\|\boldsymbol{u}(t)\|_{[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'} \leq C(\Omega, p)\, \|\boldsymbol{u}_0\|_{[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'}, \tag{5.2.5}$$

$$\Big\| \frac{\partial \boldsymbol{u}(t)}{\partial t} \Big\|_{[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'} \leq \frac{C(\Omega, p)}{t}\, \|\boldsymbol{u}_0\|_{[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'}$$

and

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(\Omega, p)\, (1 + \frac{1}{t}) \|\boldsymbol{u}_0\|_{[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'}.$$

The following corollary shows that when the initial data is in $[\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]'_{\sigma,\tau}$ we have the very weak solution for the homogeneous Stokes problem (5.2.2).

Corollary 5.2.5 (Very weak solutions for the homogeneous Stokes Problem)**.** *Let $\boldsymbol{u}_0 \in [\boldsymbol{H}^{p'}_0(\mathrm{div}, \Omega)]'_{\sigma,\tau}$, $T < \infty$ and let $(\boldsymbol{u}, \pi)$ be the unique solution of Problem (5.2.2). Then for all $1 \leq q < 2$, $(\boldsymbol{u}, \pi)$ satisfies*

$$\boldsymbol{u} \in L^q(0, T;\, \boldsymbol{L}^p(\Omega)), \qquad \pi \in L^q(0, T;\, W^{-1,p}(\Omega)/\mathbb{R})$$

and

$$\frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0, T;\, [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}).$$

Proof. Since the Stokes operator with Navier-slip boundary conditions generates a bounded analytic semi-group on $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$, when the initial data $\boldsymbol{u}_0 \in [\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'_{\sigma,\tau}$ the homogeneous Stokes Problem (5.2.2) has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$. We proceed as in the proof of Theorem 5.1.10. First, thanks to Theorem 3.2.10, Remark 3.2.12 and Theorem 5.2.4 we know that

$$\begin{aligned}\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} &\simeq \|\boldsymbol{u}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'} + \|-\Delta\boldsymbol{u} + \nabla\pi\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}\\ &\leq (1+\frac{1}{t})\|\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}.\end{aligned}$$

Using estimate (5.2.5) and the continous embedding of $[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'$ in $\boldsymbol{W}^{-1,p}(\Omega)$ one has

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{-1,p}(\Omega)} \leq C(\Omega,p)\,\|\boldsymbol{u}_0\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'}.$$

As a result substituting in the interpolation inequality

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\|\boldsymbol{u}\|^{1/2}_{\boldsymbol{W}^{1,p}(\Omega)}\|\boldsymbol{u}\|^{1/2}_{\boldsymbol{W}^{-1,p}(\Omega)},$$

one gets directly that $\boldsymbol{u} \in L^q(0,T;\,\boldsymbol{L}^p(\Omega))$, for all $1 \leq q < 2$ and for all $T < \infty$.

To prove that $\frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\,[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau})$ we use the fact that (see Chapter 3 Theorem 3.2.13 and Remark 3.2.15)

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \simeq \|\boldsymbol{u}\|_{[\boldsymbol{T}^{p'}(\Omega)]'} + \|-\Delta\boldsymbol{u} + \nabla\pi\|_{[\boldsymbol{T}^{p'}(\Omega)]'}.$$

Since $-\Delta\boldsymbol{u} + \nabla\pi = -\frac{\partial \boldsymbol{u}}{\partial t}$ we deduce directly that $\frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\,[\boldsymbol{T}^{p'}(\Omega)]')$, for all $1 \leq q < 2$ and for all $T < \infty$. Finally thanks to [8] one has

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)/\boldsymbol{\mathcal{T}}^p(\Omega)} + \|\pi\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leq C(\Omega,p)\Big\|\frac{\partial \boldsymbol{u}}{\partial t}\Big\|_{[\boldsymbol{T}^{p'}(\Omega)]'}.$$

As a result we obtain directly that $\pi \in L^q(0,T;\,W^{-1,p}(\Omega)/\mathbb{R})$, for all $1 \leq q < 2$ and for all $T < \infty$.

□

In addition the analyticity of the semi-group generated by the Stokes operator on $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ (see (3.1.71) for the definition of $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$) allows us to obtain the following theorem. We recall that the extension of the Stokes operator with Navier slip boundary conditions to $[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ is defined by (3.2.38) and its domain $\mathbf{D}(\mathbb{C}_p)$ is defined by (3.2.37).

Theorem 5.2.6. *Suppose that the given data $\boldsymbol{u}_0 \in [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}$ the Problem (5.2.2) has a unique solution $(\boldsymbol{u},\pi)$ satisfying*

$$\boldsymbol{u} \in C([0,\,+\infty[,\,[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}) \cap C(]0,\,+\infty[,\,\mathbf{D}(\mathbb{C}_p)) \cap C^1(]0,\,+\infty[,\,[\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,\tau}),$$

$$\boldsymbol{u} \in C^k(]0,\,+\infty[,\,\mathbf{D}(\mathbb{C}_p^{\ell})), \qquad \forall\, k \in \mathbb{N},\ \forall\, \ell \in \mathbb{N}^*$$

and

$$\pi \in C(]0,\infty[\,;\, W^{-1,p}(\Omega)/\mathbb{R}).$$

Moreover we have the estimates

$$\|\boldsymbol{u}(t)\|_{[\boldsymbol{T}^{p'}(\Omega)]'} \leq C(\Omega,p)\,\|\boldsymbol{u}_0\|_{[\boldsymbol{T}^{p'}(\Omega)]'},$$

$$\Big\|\frac{\partial \boldsymbol{u}(t)}{\partial t}\Big\|_{[\boldsymbol{T}^{p'}(\Omega)]'} \leq \frac{C(\Omega,p)}{t}\,\|\boldsymbol{u}_0\|_{[\boldsymbol{T}^{p'}(\Omega)]'}.$$

and

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\,(1+\frac{1}{t})\,\|\boldsymbol{u}_0\|_{[\boldsymbol{T}^{p'}(\Omega)]'}.$$

When the initial data are more regular we have the strong solution for the homogeneous Stokes problem (5.2.2).

Proposition 5.2.7 (Strong Solutions for the homogeneous Stokes Problem)**.** *Suppose that* Ω *is of class* $C^{2,1}$ *and let* $\boldsymbol{u}_0 \in \boldsymbol{V}^p_\tau(\Omega)$ *(given by* (3.1.2)*) and* $T<\infty$*. The unique solution* $(\boldsymbol{u},\pi)$ *of Problem (5.2.2) satisfies for all* $1 \leq q < 2$

$$\boldsymbol{u} \in L^q(0,T;\, \boldsymbol{W}^{2,p}(\Omega)), \qquad \pi \in L^q(0,T;\, W^{1,p}(\Omega)/\mathbb{R})$$

and

$$\frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega)).$$

Proof. First we recall that thanks to Chapter 4, Theorem 4.2.3 we know that

$$\mathbf{D}(\mathbb{A}_p^{1/2}) = \boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega) = \boldsymbol{V}^p_\tau(\Omega),$$

where $\boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega)$ is given by (3.1.3). Now let $\boldsymbol{u}_0 \in \boldsymbol{V}^p_\tau(\Omega)$ and let $\boldsymbol{u}(t) = e^{-t\mathbb{A}_p}\boldsymbol{u}_0$ be the unique solution of Problem (5.2.2) with $e^{-t\mathbb{A}_p}$ is the semi-group generated by the Stokes operator $\mathbb{A}_p$ on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. As a result using Proposition 2.3.12 (property (2.3.6)) one has

$$\begin{aligned}
\|\mathbb{A}_p\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} = \|\mathbb{A}_p^{1/2}\mathbb{A}_p^{1/2}e^{-t\mathbb{A}_p}\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} &= \|\mathbb{A}_p^{1/2}e^{-t\mathbb{A}_p}\mathbb{A}_p^{1/2}\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}\\
&\leq \frac{c}{\sqrt{t}}\|\mathbb{A}_p^{1/2}\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}\\
&\leq \frac{C}{\sqrt{t}}\|\boldsymbol{u}_0\|_{\boldsymbol{W}^{1,p}(\Omega)}
\end{aligned}$$

Since $\frac{\partial \boldsymbol{u}}{\partial t} = \mathbb{A}_p\boldsymbol{u}$ we deduce directly that $\frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\boldsymbol{L}^p_{\sigma,\tau}(\Omega))$, for all $1 \leq q < 2$ and for all $T<\infty$.

Next using Theorem 5.2.1 we know that the solution $\boldsymbol{u}(t) \in \boldsymbol{W}^{2,p}(\Omega)$ for all $t>0$. In addition, thanks to Chapter 3, Theorem 3.2.4 and Remark 3.2.9 we know that

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{2,p}(\Omega)} \simeq \|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbb{A}_p\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)}.$$

As a result $\boldsymbol{u} \in L^q(0,T;\, \boldsymbol{W}^{2,p}(\Omega))$ for all $1 \leq q < 2$ and for all $T<\infty$.

It remains to prove that $\pi \in L^q(0,T;\, W^{1,p}(\Omega))$, for all $1 \leq q < 2$ and for all $T < \infty$. Indeed since $-\Delta \boldsymbol{u} + \nabla\pi = -\frac{\partial \boldsymbol{u}}{\partial t}$ one has thanks to [8]

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{2,p}(\Omega)/\boldsymbol{\mathcal{T}}^p(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq \left\|\frac{\partial \boldsymbol{u}}{\partial t}\right\|_{\boldsymbol{L}^p(\Omega)}.$$

This ends the proof. $\square$

5.2.2 The inhomogeneous problem

In this subsection we consider the inhomogeneous Stokes problem

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{f}, & \operatorname{div}\boldsymbol{u} = 0 \quad \text{in} \quad \Omega\times(0,T), \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, & [\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0} \quad \text{on} \quad \Gamma\times(0,T), \\ & \boldsymbol{u}(0) = \mathbf{0} \quad \text{in} \quad \Omega. \end{cases} \tag{5.2.6}$$

As it is stated above for a function $\boldsymbol{f} \in L^q(0,T;\boldsymbol{L}^p_{\sigma,\tau}(\Omega))$, the analyticity of the Stokes semi-group is not enough to obtain a unique solution $(\boldsymbol{u},\pi)$ to Problem (5.2.6) satisfying the maximal $L^p - L^q$ regularity : For $1 < p, q < \infty$ and $T \leq \infty$,

$$\boldsymbol{u} \in L^q(0,T;\, \mathbf{D}(\mathbb{A}_p)), \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\, L^p(\Omega))$$

and

$$\pi \in L^q(0,T;\, W^{1,p}(\Omega)/\mathbb{R}).$$

Usually we need to impose further regularity assumptions on $\boldsymbol{f}$ (such that $\boldsymbol{f}$ is locally Hölder continuous).

For the inhomogeneous Stokes Problem (5.2.6) we have the maximal $L^p - L^q$ regularity for the problem. This is guaranteed by the boundedness of the pure imaginary powers of the Stokes operator $\mathbb{A}_p$ (see Chapter 4, Theorem 4.2.1). Applying now Theorem 5.1.17 to the operator $\mathbb{A}_p$ we obtain directly the following theorem

Theorem 5.2.8 (Strong Solutions for the inhomogeneous Stokes Problem)**.** *Suppose that Ω is of class $C^{2,1}$ and let $0 < T \leq \infty$, $1 < p, q < \infty$, $\boldsymbol{f} \in \boldsymbol{L}^q(0,T;\boldsymbol{L}^p_{\sigma,\tau}(\Omega))$ and $\boldsymbol{u}_0 = \mathbf{0}$. The Problem (5.2.6) has a unique solution $(\boldsymbol{u},\pi)$ such that*

$$\boldsymbol{u} \in L^q(0,T_0;\, \boldsymbol{W}^{2,p}(\Omega)), \quad T_0 \leq T \ \textit{if}\ T<\infty \ \text{ and } \ T_0 < T \ \textit{if}\ T = \infty,$$

$$\pi \in L^q(0,T;\, W^{1,p}(\Omega)/\mathbb{R}), \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\, \boldsymbol{L}^p(\Omega))$$

and

$$\int_0^T \left\|\frac{\partial \boldsymbol{u}}{\partial t}\right\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\,t + \int_0^T \|\mathbb{A}_p\boldsymbol{u}(t)\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\,t + \int_0^T \|\pi(t)\|^q_{W^{1,p}(\Omega)/\mathbb{R}} \mathrm{d}\,t$$
$$\leq C(p,q,\Omega) \int_0^T \|\boldsymbol{f}(t)\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\,t.$$

The boundedness of the pure imaginary powers of the operators $\mathbb{B}_p$ and $\mathbb{C}_p$ respectively (see Chapter 4 Remark 4.2.2) allows us to obtain weak and very weak solutions to Problem (5.2.6).

Theorem 5.2.9 (Weak Solutions for the inhomogeneous Stokes Problem)**.** *Let* $1 < p, q < \infty$, $\boldsymbol{u}_0 = 0$ *and let* $\boldsymbol{f} \in L^q(0, T; [\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'_{\sigma,T})$, $0 < T \leq \infty$. *The Problem (5.2.6) has a unique solution* $(\boldsymbol{u}, \pi)$ *satisfying*

$$\boldsymbol{u} \in L^q(0, T_0;\ \boldsymbol{W}^{1,p}(\Omega)), \quad T_0 \leq T \ \textit{if}\ T < \infty \ \text{ and } \ T_0 < T \ \textit{if}\ T = \infty,$$

$$\pi \in L^q(0, T;\ L^p(\Omega)/\mathbb{R}), \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0, T; \in [\boldsymbol{H}_0^{p'}(\mathrm{div}\Omega)]'_{\sigma,T})$$

and

$$\int_0^T \left\| \frac{\partial \boldsymbol{u}}{\partial t} \right\|^q_{[\boldsymbol{H}_0^{p'}(\mathrm{div}\Omega)]'} \mathrm{d}\,t + \int_0^T \|\mathbb{B}_p \boldsymbol{u}(t)\|^q_{[\boldsymbol{H}_0^{p'}(\mathrm{div}\Omega)]'} \mathrm{d}\,t + \int_0^T \|\pi(t)\|^q_{L^p(\Omega)/\mathbb{R}} \mathrm{d}\,t \leq C(p, q, \Omega) \int_0^T \|\boldsymbol{f}(t)\|^q_{[\boldsymbol{H}_0^{p'}(\mathrm{div}\Omega)]'} \mathrm{d}\,t.$$

Theorem 5.2.10 (Very weak solutions for the inhomogeneous Stokes Problem)**.** *Suppose that* Ω *is of class* $C^{2,1}$, *let* $0 < T \leq \infty$, $1 < p, q < \infty$, $\boldsymbol{u}_0 = 0$ *and* $\boldsymbol{f} \in L^q(0, T; [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,T})$. *Then the evolutionary Stokes Problem (5.2.6) has a unique solution* $(\boldsymbol{u}, \pi)$ *satisfying*

$$\boldsymbol{u} \in L^q(0, T_0; \boldsymbol{L}^p(\Omega)), \quad T_0 \leq T \ \textit{if}\ T < \infty \ \text{ and } \ T_0 < T \ \textit{if}\ T = \infty,$$

$$\pi \in L^q(0, T;\ W^{-1,p}(\Omega)/\mathbb{R}), \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0, T; \in [\boldsymbol{T}^{p'}(\Omega)]'_{\sigma,T})$$

and

$$\int_0^T \left\| \frac{\partial \boldsymbol{u}}{\partial t} \right\|^q_{[\boldsymbol{T}^{p'}(\Omega)]'} \mathrm{d}\,t + \int_0^T \|\mathbb{C}_p \boldsymbol{u}(t)\|^q_{[\boldsymbol{T}^{p'}(\Omega)]'} \mathrm{d}\,t + \int_0^T \|\pi(t)\|^q_{W^{-1,p}(\Omega)/\mathbb{R}} \mathrm{d}\,t \leq C(p, q, \Omega) \int_0^T \|\boldsymbol{f}(t)\|^q_{[\boldsymbol{T}^{p'}(\Omega)]'} \mathrm{d}\,t.$$

5.3 Stokes problem with normal and pressure boundary conditions

In this section we solve the time dependent Stokes Problem (5.0.1) with the boundary condition (3.3.1) using the semi-group theory. As described in Chapter 3, Proposition 3.3.1, the Stokes operator with normal and pressure boundary condition coincides with the $-\Delta$ operator. In addition due to boundary conditions (3.3.1) the pressure is decoupled from the solution of the velocity components. Indeed taking the divergence of the first equation in (5.0.1), the pressure is a solution of a Dirichlet problem

$$\Delta \pi = \mathrm{div} \boldsymbol{f} \qquad \text{in } \ \Omega \times (0, T), \qquad \pi = 0 \qquad \text{on } \ \Gamma \times (0, T).$$

Assuming that $\operatorname{div} \boldsymbol{f} = 0$ in Ω, the pressure π is equal to zero and we are reduced to study the following heat problem:

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega \times (0,T), \\ \qquad\qquad\qquad\quad \boldsymbol{u} \times \boldsymbol{n} = 0 & \text{on} \quad \Gamma \times (0,T), \\ \qquad\qquad\qquad\quad \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in} \qquad \Omega. \end{cases} \tag{5.3.1}$$

As in the case of the heat equation the following theorem shows that the solution of Problem (5.3.1) with $\boldsymbol{f} = \boldsymbol{0}$ (*i.e.* the homogeneous problem) is regular for $t > 0$. Moreover it allows us to describe the decay in time of this solutions. The proof of the following theorem is similar to the proof of Theorem 5.1.1 and can be obtained directly using the analyticity of the Stokes semi-group with normal and pressure boundary conditions on $\boldsymbol{L}^p_\sigma(\Omega)$ (proved in Chapter 3, Theorem 3.3.10). We recall that the Stokes operator with normal and pressure boundary conditions is the operator $\mathcal{A}_p$ defined in Chapter 3 by (3.3.7) and its domain is given by (3.3.6).

Theorem 5.3.1. *We suppose that* $\boldsymbol{u}_0 \in \boldsymbol{L}^p_\sigma(\Omega)$, *and* $\boldsymbol{f} = \boldsymbol{0}$. *Then Problem* (5.3.1) *has a unique solution* $\boldsymbol{u}(t)$ *satisfying*

$$\boldsymbol{u} \in C([0, +\infty[, \boldsymbol{L}^p_\sigma(\Omega)) \cap C(]0, +\infty[, \mathbf{D}(\mathcal{A}_p)) \cap C^1(]0, +\infty[, \boldsymbol{L}^p_\sigma(\Omega)),$$

$$\boldsymbol{u} \in C^k(]0, +\infty[, \mathbf{D}(\mathcal{A}^\ell_p)), \qquad \forall\, k \in \mathbb{N},\ \forall\, \ell \in \mathbb{N}^*.$$

Moreover we have the estimates

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq C_1(\Omega, p)\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \tag{5.3.2}$$

and

$$\Big\|\frac{\partial \boldsymbol{u}(t)}{\partial t}\Big\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C_2(\Omega, p)}{t}\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}.$$

In addition, if Ω *is of class* $C^{2,1}$ *the following estimates hold*

$$\|\mathbf{curl}\, \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C_3(\Omega, p)}{\sqrt{t}}\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \tag{5.3.3}$$

and

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C_4(\Omega, p)\, (1 + \frac{1}{t})\, \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}.$$

Next, as in the case of the Stokes operator with Navier or Navier-type boundary conditions, we can extend estimates (5.3.2)-(5.3.3) to the so called $L^p - L^q$ estimates. The proof is similar to the proof of Theorem 5.1.4 and it is done using Sobolev embedding for the domains of fractional powers of the Stokes operator (proved Chapter 4, Corollary 4.3.5) and an interpolation inequality, (see Theorem 5.1.4 for a similar proof).

Theorem 5.3.2. *Let $1 < p \leq q < \infty$, $\boldsymbol{u}_0 \in \boldsymbol{L}^p_\sigma(\Omega)$ and suppose that $\boldsymbol{f} = \boldsymbol{0}$. Let $\boldsymbol{u}(t)$ be the unique solution of Problem* (5.3.1)*, then $\boldsymbol{u}(t) \in \boldsymbol{L}^q(\Omega)$ for all $1 < p \leq q < \infty$. Moreover, the following estimates:*

$$\begin{aligned}\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} &\leq C\,t^{-3/2(1/p-1/q)}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)},\\ \|\mathbf{curl}\,\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} &\leq C\,t^{-1/2}\,t^{-3/2(1/p-1/q)}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}\end{aligned}$$

and

$$\forall\, m,n \in \mathbb{N}, \qquad \Big\|\frac{\partial^m}{\partial t^m}\Delta^n \boldsymbol{u}(t)\Big\|_{\boldsymbol{L}^q(\Omega)} \leq C\,t^{-(m+n)}\,t^{-3/2(1/p-1/q)}\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}.$$

Now, we consider a special initial data that leads to an exponential decay of the solution $\boldsymbol{u}$ with respect to time. More precisely, when $\boldsymbol{u}_0$ belongs to $\boldsymbol{\mathcal{X}}_p$ (see (3.3.54) for the definition of $\boldsymbol{\mathcal{X}}_p$). We skip the proof of the following theorem which is based on the well-known properties of analytic semi-groups when the infinitesimal generator is of bounded inverse (see Theorem 5.1.5 for a similar proof).

Theorem 5.3.3. *Suppose that $\boldsymbol{u}_0 \in \boldsymbol{\mathcal{X}}_p$ and $\boldsymbol{f} = \boldsymbol{0}$. Then Problem* (5.3.1) *has a unique solution $\boldsymbol{u}$ satisfying*

$$\boldsymbol{u} \in C([0,\,+\infty[,\,\boldsymbol{\mathcal{X}}_p) \cap C(]0,\,+\infty[,\,\mathbf{D}(\mathcal{A}'_p)) \cap C^1(]0,\,+\infty[,\,\boldsymbol{\mathcal{X}}_p),$$

$$\boldsymbol{u} \in C^k(]0,\,+\infty[,\,\mathbf{D}(\mathcal{A}'^{\ell}_p)), \qquad \forall\, k,\,\ell \in \mathbb{N}.$$

Moreover there exist constant $M, \mu > 0$ such that, the solution $\boldsymbol{u}$ satisfies the estimates:

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\,e^{-\mu\,t}\,\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}$$

and

$$\Big\|\frac{\partial \boldsymbol{u}(t)}{\partial t}\Big\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\,\frac{e^{-\mu\,t}}{t}\,\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}$$

In addition, if Ω is of class $C^{2,1}$ the following estimates hold

$$\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\,\frac{e^{-\mu\,t}}{\sqrt{t}}\,\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}$$

and

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C(\Omega,p)\,\frac{e^{-\mu\,t}}{t}\,\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}.$$

Remark 5.3.4. (i) We recall that the operator $\mathcal{A}'_p$ defined by (3.3.56) with domain $\mathbf{D}(\mathcal{A}'_p)$ (given by (3.3.55)) is the restriction of the Stokes operator $\mathcal{A}_p$ to the space $\boldsymbol{\mathcal{X}}_p$. Moreover, the operator $\mathcal{A}'_p$ is invertible with bounded inverse.

(ii) Theorem 5.3.3 shows that in particular where the initial data $\boldsymbol{u}_0$ satisfies the compatibility conditions (3.3.23), it is the same for the unique solution $\boldsymbol{u}(t)$ of Problem (5.3.1) for all $t > 0$, which is not true if $\boldsymbol{u}_0 \in \boldsymbol{L}^p_\sigma(\Omega) \setminus \boldsymbol{\mathcal{X}}_p$. This comes from the fact that $\boldsymbol{u} \in C(]0, +\infty[\,,\,\mathbf{D}(\mathcal{A}'_p))$, where $\mathbf{D}(\mathcal{A}'_p)$ is given by (3.3.55). These compatibility conditions give us the exponential decay of the semi-group, since they make the operator $\mathcal{A}'_p$ of bounded inverse (*i.e.* $0 \in \rho(\mathcal{A}'_p)$).

As in the case of the Stokes operator with Navier-type boundary conditions, in order to apply Theorem 5.1.17 and to get the maximal $L^p - L^q$ regularity for the inhomogeneous Stokes problem with normal and pressure boundary conditions we prove the ζ-convexity of $[\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'$.

Proposition 5.3.5. *Let $1 < p < \infty$, the dual space $[\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'$ is a ζ-convex Banach space.*

Proof. We proceed as in the proof of Proposition 5.1.19. Let $\boldsymbol{f} \in L^s(\mathbb{R};\, [\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]')$, then for almost all $t \in \mathbb{R}$, there exists $\boldsymbol{\psi}(t) \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{\xi}(t) \in \boldsymbol{L}^p(\Omega)$ such that

$$\boldsymbol{f}(t) \,=\, \boldsymbol{\psi}(t) \,+\, \mathbf{curl}\boldsymbol{\xi}(t) \qquad \|\boldsymbol{f}(t)\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'} \,=\, \max(\|\boldsymbol{\psi}(t)\|_{\boldsymbol{L}^p(\Omega)},\, \|\boldsymbol{\xi}(t)\|_{\boldsymbol{L}^p(\Omega)}).$$

Since $\boldsymbol{f} \in L^s(\mathbb{R};\, [\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]')$, it is clear that $\boldsymbol{\psi} \in L^s(\mathbb{R};\, \boldsymbol{L}^p(\Omega))$ and $\boldsymbol{\xi} \in L^s(\mathbb{R}; \boldsymbol{L}^p(\Omega))$. On the other hand we can easily verify that

$$(H_\varepsilon \boldsymbol{f})(t) \,=\, (H_\varepsilon \boldsymbol{\psi})(t) \,+\, \mathbf{curl}(H_\varepsilon \boldsymbol{\xi})(t).$$

We recall that $(H_\varepsilon \boldsymbol{f})(t)$ is the truncated Hilbert transform of $\boldsymbol{f}$ given by (5.1.36).

Next since $\boldsymbol{L}^p(\Omega)$ is ζ-convex (see Proposition 5.1.18), then $(H_\varepsilon \boldsymbol{\psi})(t)$ (respectively $(H_\varepsilon \boldsymbol{\xi})(t)$) converges as $\varepsilon \to 0$ to $H\boldsymbol{\psi}(t)$ (respectively to $H\boldsymbol{\xi}(t)$). Moreover we have the estimate

$$\|H\boldsymbol{\psi}(t)\|_{L^s(\mathbb{R};\, \boldsymbol{L}^p(\Omega))} \,\leq\, C(s, \Omega, p)\, \|\boldsymbol{\psi}\|_{L^s(\mathbb{R};\, \boldsymbol{L}^p(\Omega))}$$

and

$$\|H\boldsymbol{\xi}(t)\|_{L^s(\mathbb{R};\, \boldsymbol{L}^p(\Omega))} \,\leq\, C(s, \Omega, p)\, \|\boldsymbol{\xi}\|_{L^s(\mathbb{R};\, \boldsymbol{L}^p(\Omega))}$$

This means that $(H_\varepsilon \boldsymbol{f})(t)$ converges as $\varepsilon \to 0$ to $H\boldsymbol{f}(t) \,=\, H\boldsymbol{\psi}(t) \,+\, \mathbf{curl}\, H\boldsymbol{\xi}(t)$. Moreover we have the estimate

$$\|H\boldsymbol{f}(t)\|_{L^s(\mathbb{R};\, [\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]')} \,\leq\, C(s, \Omega, p)\, \|\boldsymbol{f}\|_{L^s(\mathbb{R};\, [\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]')},$$

which ends the proof. $\square$

The following theorem gives weak solutions for problem (5.3.1).

Theorem 5.3.6. *Let $1 < p < \infty$, $\boldsymbol{u}_0 \in \boldsymbol{L}^p_\sigma(\Omega)$ and let $\boldsymbol{f} \in L^q(0, T;\, [\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'_\sigma)$ with $1 < q < 2$ and $0 < T \leq \infty$. Then, the Problem (5.3.1) has a unique solution $\boldsymbol{u}$ that satisfies:*

$$\boldsymbol{u} \in L^q(0, T_0;\; \boldsymbol{W}^{1,p}(\Omega)), \quad T_0 \leq T \ \textit{ if } T < \infty \ \text{ and } \ T_0 < T \ \textit{ if } T = \infty, \tag{5.3.4}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0, T;\, [\boldsymbol{H}_0^{p'}(\mathbf{curl}\Omega)]'_\sigma) \tag{5.3.5}$$

and

$$\int_0^T \Big\|\frac{\partial \boldsymbol{u}}{\partial t}\Big\|^q_{[\boldsymbol{H}_0^{p'}(\mathbf{curl}\Omega)]'} \mathrm{d}\, t + \int_0^T \|\Delta \boldsymbol{u}(t)\|^q_{[\boldsymbol{H}_0^{p'}(\mathbf{curl}\Omega)]'} \mathrm{d}\, t \,\leq\, C\Big(\int_0^T \|\boldsymbol{f}(t)\|^q_{[\boldsymbol{H}_0^{p'}(\mathbf{curl}\Omega)]'} \mathrm{d}\, t + \|\boldsymbol{u}_0\|^q_{\boldsymbol{L}^p(\Omega)}\Big), \tag{5.3.6}$$

with some constant C independent of T.

Proof. The proof is done in two steps:

Step 1: *homogeneous problem* ($\boldsymbol{f} = \boldsymbol{0}$).

Thanks to Lemma 2.2.1, we know that

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{1,p}(\Omega)} \simeq \|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)}.$$

Since $\boldsymbol{u}$ satisfies (5.3.2)-(5.3.3), the solution $\boldsymbol{u}$ clearly belongs to $\in L^q(0,T;\ \boldsymbol{W}^{1,p}(\Omega))$ for all $1 < q < 2$ and $0 < T < \infty$. To prove that $\frac{\partial \boldsymbol{u}}{\partial t} = \Delta \boldsymbol{u} \in L^q(0,T;\ [\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]')$, we set

$$\widetilde{\boldsymbol{u}}(t) = \boldsymbol{u}(t) - \sum_{i=1}^{I} \langle \boldsymbol{u}(t) \cdot \boldsymbol{n}\,,\,1\rangle_{\Gamma_i} \mathbf{grad}\, q_i^N.$$

Thanks to [11, Corollary 4.4], we have

$$\|\Delta \boldsymbol{u}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'} = \|\Delta \widetilde{\boldsymbol{u}}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'} \simeq \|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}.$$

Thus one has (5.3.5) for all $1 \leq q < 2$ and for all $T < \infty$.
Step 2: *Inhomogeneous Problem* ($\boldsymbol{u}_0 = \boldsymbol{0}$).

Thanks to Proposition 5.3.5, we know that the dual space $[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'_\sigma$ is a ζ-*convex* Banach space. Next, due to Theorem 4.3.1 we know that the pure imaginary powers of the Stokes operator with normal and pressure boundary conditions are bounded on $[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'_\sigma$ and satisfy estimate (4.3.2). This means that the operator $\mathcal{B}_p$ belongs to $\mathcal{E}_K^{\theta_0}([\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'_\sigma)$ with $0 < \theta_0 < \pi/2$. As a result applying Theorem 5.1.17 we obtain a unique weak solution for the Problem (5.3.1) satisfying the regularity (5.3.4)-(5.3.5) and estimate (5.3.6).

We recall that the operator $\mathcal{B}_p$ is the extension of the Stokes operator to the dual space $[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'_\sigma$ and it is defined in Chapter 3 by (3.3.49)-(3.3.50) and its domain is given by (3.3.51). □

As described above, when the external force $\boldsymbol{f}$ is not a divergence free function, the pressure doesn't vanish in Ω and can be decoupled from the problem using a Dirichlet problem. The following theorem gives weak solution for the Stokes problem (5.0.1) with the boundary condition (3.3.1).

Corollary 5.3.7. *Let* $1 < p < \infty$, $\boldsymbol{u}_0 \in \boldsymbol{L}^p_\sigma(\Omega)$ *and let* $\boldsymbol{f} \in L^q(0,T;\ [\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]')$ *with* $1 < q < 2$ *and* $0 < T \leq \infty$. *The Stokes Problem* (5.0.1) *with normal and pressure boundary condition* (3.3.1) *has a unique solution* $(\boldsymbol{u},\,\pi)$ *satisfying*

$$\boldsymbol{u} \in L^q(0,T_0;\ \boldsymbol{W}^{1,p}(\Omega)),\quad T_0 \leq T\ \textit{ if } T < \infty \ \text{ and } \ T_0 < T\ \textit{ if } T = \infty, \tag{5.3.7}$$

$$\pi \in L^q(0,T;\ W_0^{1,p}(\Omega)), \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\ [\boldsymbol{H}_0^{p'}(\mathbf{curl}\Omega)]') \tag{5.3.8}$$

and

$$\int_0^T \Big\|\frac{\partial \boldsymbol{u}}{\partial t}\Big\|^q_{[\boldsymbol{H}_0^{p'}(\mathbf{curl}\Omega)]'}\,\mathrm{d}\,t + \int_0^T \|\Delta \boldsymbol{u}(t)\|^q_{[\boldsymbol{H}_0^{p'}(\mathbf{curl}\Omega)]'}\,\mathrm{d}\,t + \int_0^T \|\pi(t)\|^q_{W_0^{1,p}(\Omega)}\,\mathrm{d}\,t$$
$$\leq C(p,q,\Omega)\Big(\int_0^T \|\boldsymbol{f}(t)\|^q_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'}\,\mathrm{d}\,t + \|\boldsymbol{u}_0\|^q_{\boldsymbol{L}^q(\Omega)}\Big). \quad (5.3.9)$$

Proof. The result for the pressure is derived by taking the divergence of the first equation in (5.0.1). Indeed π is solution of the following Dirichlet problem:

$$\Delta\,\pi = \operatorname{div}\boldsymbol{f} \quad \text{in}\,\Omega \qquad \text{and} \qquad \pi = 0 \quad \text{on}\,\Gamma. \quad (5.3.10)$$

Since $\boldsymbol{f}(t) \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'$ for all $0 < t < T$, then $\operatorname{div}\boldsymbol{f} \in W^{-1,p}(\Omega)$ and by standard L^p regularity, there exists a unique $\pi \in W_0^{1,p}(\Omega)$ solution of the Dirichlet Problem (5.3.10) satisfying for a.e. $t \in (0,T)$:

$$\|\pi(t)\|_{W_0^{1,p}(\Omega)} \leq C(\Omega)\|\boldsymbol{f}(t)\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'}.$$

Next, applying Theorem 5.3.6 with a right hand side $(\boldsymbol{f} - \nabla\pi) \in L^q(0,T;\,[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'_\sigma)$, The Stokes problem (5.0.1) with the boundary condition (3.3.1) has a unique solution $(\boldsymbol{u},\pi)$ satisfying (5.3.7)-(5.3.9). $\square$

We observe the following remark

Remark 5.3.8. **(i)** If $\boldsymbol{u}_0 = \mathbf{0}$, then the solution $(\boldsymbol{u},\,\pi)$ of the Stokes problem (5.0.1)-(3.3.1) given in Corollary 5.3.7 satisfies the maximal regularity (5.3.7)-(5.3.8) for $1 < p,q < \infty$. This follows directly from Theorem 5.1.17.

(ii) If we consider the homogeneous Stokes problem ($\boldsymbol{f} = \mathbf{0}$), the assumption $\boldsymbol{u}_0 \in \boldsymbol{L}^p_\sigma(\Omega)$ is not optimal to obtain maximal regularity (5.3.7)-(5.3.8) for $1 < p,q < \infty$. Indeed, the maximal $L^p - L^q$ regularity is obtained when the initial value $\boldsymbol{u}_0$ satisfies

$$\boldsymbol{u}_0 \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'_\sigma, \qquad \int_0^\infty \|\mathcal{B}_p e^{-t\mathcal{B}_p}\boldsymbol{u}_0\|^q_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'}\mathrm{d}\,t < \infty, \quad (5.3.11)$$

where $1 < p,q < \infty$ and $e^{-t\mathcal{B}_p}$ is the semi-group generated by the Stokes operator on $[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'_\sigma$.

When the initial data is more regular we obtain the strong solution for the Stokes problem with normal and pressure boundary condition.

Theorem 5.3.9. *Suppose that Ω is of class $C^{2,1}$, $\boldsymbol{f} \in L^q(0,T;\,\boldsymbol{L}^p_\sigma(\Omega))$ and $\boldsymbol{u}_0 \in \boldsymbol{V}^p_N(\Omega)$ (given by (3.3.4)) with $1 \leq q < 2$ and $T < \infty$. The unique solution $\boldsymbol{u}(t)$ of Problem (5.3.1) satisfies in particular*

$$\boldsymbol{u} \in L^q(0,T;\,\boldsymbol{W}^{2,p}(\Omega)) \;\; \text{and} \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0,T;\,\boldsymbol{L}^p_\sigma(\Omega)). \quad (5.3.12)$$

Moreover we have the estimate

$$\int_0^T \Big\|\frac{\partial \boldsymbol{u}}{\partial t}\Big\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\,t + \int_0^T \|\Delta \boldsymbol{u}(t)\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\,t \\ \leq C(p,q,\Omega)\Big(\int_0^T \|\boldsymbol{f}(t)\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\,t + \|\boldsymbol{u}_0\|^q_{\boldsymbol{W}^{1,p}(\Omega)}\Big). \quad (5.3.13)$$

Proof. The inhomogeneous case ($\boldsymbol{u}_0 = \boldsymbol{0}$), is obtained by applying Theorem 5.1.17 for the Stokes operator $\mathcal{A}_p$ (defined by (3.3.7)). This is true, because the pure imaginary powers of the Stokes operator $\mathcal{A}_p$ are bounded and satisfies estimate (4.3.1) and since the space $\boldsymbol{L}^p_\sigma(\Omega)$ is ζ-*convex* (being a closed subspace of a $\zeta-$convex Banach space).

Now, for the homogeneous problem ($\boldsymbol{f} = \boldsymbol{0}$), we proceed in the same way as in the proof of Proposition 5.2.7. This is guaranteed since $\boldsymbol{u}_0 \in \boldsymbol{V}^p_N(\Omega) = \mathbf{D}(\mathcal{A}_p^{1/2})$. □

Remark 5.3.10. **(i)** As described in Remark 5.3.8, for the homogeneous problem, the assumption $\boldsymbol{u}_0 \in \boldsymbol{V}^p_N(\Omega)$ is not optimal and may be replaced by the properties

$$\boldsymbol{u}_0 \in \boldsymbol{L}^p_\sigma(\Omega), \qquad \int_0^\infty \|\mathcal{A}_p e^{-t\mathcal{A}_p}\boldsymbol{u}_0\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\,t < \infty, \quad (5.3.14)$$

where $1 < p, q < \infty$ and $e^{-t\mathcal{A}_p}$ is the semi-group generated by the Stokes operator on $\boldsymbol{L}^p_\sigma(\Omega)$. With an initial value $\boldsymbol{u}_0$ satisfying (5.3.14) the unique solution $\boldsymbol{u}$ of Problem (5.3.1) satisfies (5.3.12) for all $1 < p, q < \infty$ and for all $T \leq \infty$.
(ii) In the case where $\boldsymbol{u}_0 = \boldsymbol{0}$ and $\boldsymbol{f} \in L^q(0, T;\, \boldsymbol{L}^p_\sigma(\Omega))$, for all $1 < p, q < \infty$ and $T \leq \infty$ the unique solution $\boldsymbol{u}$ of Problem (5.3.1) satisfies the maximal $L^p - L^q$ regularity (5.3.12). This is obtained directly by applying Theorem 5.1.17 to the Stokes operator $\mathcal{A}_p$.

Corollary 5.3.11. *Suppose that* Ω *is of class* $C^{2,1}$ *and let* $0 < T < \infty$, $1 < p < \infty$, $1 \leq q < 2$, $\boldsymbol{f} \in \boldsymbol{L}^q(0, T; \boldsymbol{L}^p(\Omega))$ *and* $\boldsymbol{u}_0 \in \boldsymbol{V}^p_N(\Omega)$. *The solution* $(\boldsymbol{u}, \pi)$ *of the Stokes problem* (5.0.1)-(3.3.1) *satisfies:*

$$\boldsymbol{u} \in L^q(0, T_0;\, \boldsymbol{W}^{2,p}(\Omega)), \quad T_0 \leq T \;\; \textit{if}\;\; T < \infty \;\; \text{and} \;\; T_0 < T \;\; \textit{if}\;\; T = \infty, \quad (5.3.15)$$

$$\pi \in L^q(0, T;\, W^{1,p}_0(\Omega)), \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0, T;\, \boldsymbol{L}^p(\Omega)) \quad (5.3.16)$$

and

$$\int_0^T \Big\|\frac{\partial \boldsymbol{u}}{\partial t}\Big\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\,t + \int_0^T \|\Delta \boldsymbol{u}(t)\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\,t + \int_0^T \|\pi(t)\|^q_{W^{1,p}(\Omega)} \mathrm{d}\,t \\ \leq C(p,q,\Omega)\Big(\int_0^T \|\boldsymbol{f}(t)\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\,t + \|\boldsymbol{u}_0\|^q_{\boldsymbol{W}^{1,p}(\Omega)}\Big). \quad (5.3.17)$$

Proof. To proof is similar to the proof of Corollary 5.3.7. We apply Theorem 5.3.9 with the right hand side $\boldsymbol{f} - \nabla\,\pi$ which belongs to $L^q(0,\, T;\, \boldsymbol{L}^p_\sigma(\Omega))$. □

Remark 5.3.12. Similarly to the case of weak solutions, the maximal regularity (5.3.15)-(5.3.17) holds for any $1 < p, q < \infty$ if $\boldsymbol{u}_0 = \boldsymbol{0}$. But for the case $\boldsymbol{f} = \boldsymbol{0}$, the assumption $\boldsymbol{u}_0 \in \boldsymbol{V}^p_N(\Omega)$ is not optimal to obtain the maximal regularity and may be replaced by:

$$\boldsymbol{u}_0 \in \boldsymbol{L}^p_\sigma(\Omega), \qquad \int_0^\infty \|\mathcal{A}_p e^{-t\mathcal{A}_p} \boldsymbol{u}_0\|^q_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\, t < \infty, \tag{5.3.18}$$

where $e^{-t\mathcal{A}_p}$ is the semi-group generated by the Stokes operator on $\boldsymbol{L}^p_\sigma(\Omega)$.

Chapter 6

Applications to the Navier-Stokes Problem

There is an extensive literature on the solvability of the initial value problem for the Navier-Stokes equation in L^2- spaces. Hopf proved the existence of a global weak solution, using the Faedo-Galerkin approximation and an energy inequality. However the uniqueness and the global regularity of Hopf's solution are still open problem for $n \geq 3$. Fujita and Kato [51, 35] use the semi-group theory and the fractional powers of a non-negative operator to study the non-stationary Navier Stokes Problem with Dirichlet boundary condition in the Hilbert space L^2. Furthermore, for $n = 3$ they proved the existence of a unique global strong solution if the initial value has square-summable half derivative. Later on, Giga and Miyakawa [41] prove the existence of a unique local in time strong solution without assuming that the initial velocity is regular. To establish their result, they develop an L^r theory generalising the L^2 theory of Kata and Fujita [51, 35]. In [40], Giga constructs a unique local in time mild solution for a class of semi-linear parabolic equation. He also show that his result includes the semi-linear heat equation and the Navier-Stokes system with Dirichlet boundary condition.

In this chapter we consider the Navier-Stokes Problem

$$\left\{\begin{array}{llll} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \boldsymbol{0}, & \operatorname{div}\boldsymbol{u} = 0 & \text{in} & \Omega\times(0,T), \\ & \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in} & \Omega, \end{array}\right. \tag{6.0.1}$$

with the boundary conditions considered in the previous chapters respectively. We will show that some informations on the linear Problem (3.1.1) can be used to obtain local mild and classical solutions to Problem (6.0.1). For this reason we will use the semi-group theory developped in the previous chapters for the Stokes operator. The proofs will be done for the Navier-Stokes problem with Navier-type boundary conditions (3.1.1) but it can be applied also for the Navier-slip boundary conditions (3.2.1) and the boundary condition involving the pressure (3.3.1).

First, using the $L^p - L^q$ estimates (1.0.22) for the homogeneous Stokes problem and proceeding as Giga in [40] we will prove the existence of a local in time mild solution for the homogeneous Navier-Stokes problem with Navier-type boundary conditions.

Next, using the fractional powers of the Stokes operator we will estimate the non-linear term $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$. Then, proceeding as Giga and Miyakawa [41] we prove that the solution $\boldsymbol{u} \in \mathbf{D}(A_p)$ for all $t \in (0, T_*]$ for certain $T_* < T$.

6.1 Navier-Stokes Problem with Navier-type boundary conditions

Consider the abstract semilinear parabolic equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + \mathcal{A}\boldsymbol{u} = \boldsymbol{F}\boldsymbol{u} \qquad u(0) = \boldsymbol{a}, \qquad \text{in} \quad \Omega \times]0, T[, \tag{6.1.1}$$

where Ω is an arbitrary domain of $\mathbb{R}^3$. Giga considers Problem (6.1.1) in L^p-type vector valued function spaces, $1 < p < \infty$, $\boldsymbol{F}\boldsymbol{u}$ represents the non-linear term and $-\mathcal{A}$ is the infinitesimal generator of a strongly continuous semi-group $e^{-t\mathcal{A}}$ in some closed subspace $\boldsymbol{E}^p$ of $\boldsymbol{L}^p(\Omega)$ equipped with the norm of $\boldsymbol{L}^p(\Omega)$. He constructs in [40] a unique local in time mild solution in $L^q(0, T_*, \boldsymbol{L}^p(\Omega))$ for the Problem (6.1.1). He also proves that the constructed solution is global in time for small initial data. His analysis is based on the regularisation property of the linear part $e^{-t\mathcal{A}}$ and some assumptions on the non-linear term. Giga solves the Problem (6.1.1) via the corresponding integral equation

$$\boldsymbol{u}(t) = e^{-t\mathcal{A}}\boldsymbol{a} + \int_0^t e^{-(t-s)\mathcal{A}}\boldsymbol{F}\boldsymbol{u}(s)\,\mathrm{d}\,s. \tag{6.1.2}$$

To prove his theorem, Giga gives some assumptions on the operator $\mathcal{A}$ and the nonlinear term $\boldsymbol{F}\boldsymbol{u}$.

Having the Navier-Stokes problem (6.1.7) in mind, we consider in the sequel a particular case of [40]. We assume that :

(i) There exists a continuous projection P from $L^p(\Omega)$ to $\boldsymbol{E}^p$ for all $1 < p < \infty$ such that the restriction of P on $\mathcal{D}(\Omega)$ is independent of p and $\mathcal{D}(\Omega) \cap \boldsymbol{E}^p$ is dense in $\boldsymbol{E}^p$.

(ii) For a fixed $0 < T < \infty$ the following estimate holds

$$(\mathbf{A}) \quad \|e^{-t\mathcal{A}}\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \leq M\, t^{-\frac{3}{2}(\frac{1}{s} - \frac{1}{p})}\|\boldsymbol{f}\|_{\boldsymbol{L}^r(\Omega)}, \qquad \boldsymbol{f} \in \boldsymbol{E}^r, \quad 0 < t < T, \tag{6.1.3}$$

with $p \geq r > 1$ and the constant $M = M(p, r, T)$ depends only on p, r and T.

(iii) The non-linear $\boldsymbol{F}\boldsymbol{u}$ can be written in the form

$$\boldsymbol{F}\boldsymbol{u} = \boldsymbol{L}\,\boldsymbol{G}\boldsymbol{u}, \tag{6.1.4}$$

where $\boldsymbol{L}$ is the linear part and $\boldsymbol{G}$ is the non-linear part.

(iv) We suppose also that the following estimate holds

$$(\mathbf{N1}) \quad \|e^{-t\mathcal{A}}\boldsymbol{L}\,\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \leq N_1 t^{-1/2}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \qquad \boldsymbol{f} \in \boldsymbol{E}^p, \quad 0 < t < T, \tag{6.1.5}$$

where the constant $N_1 = N_1(p, T)$ depends only on p and T.

(v) The operator $\boldsymbol{G}$ satisfies the following estimate

$$(\mathbf{N2}) \quad \|\boldsymbol{G}\boldsymbol{v} - \boldsymbol{G}\boldsymbol{w}\|_{\boldsymbol{L}^s(\Omega)} \leq N_2\|\boldsymbol{v} - \boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)}(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)}), \qquad \boldsymbol{G}\mathbf{0} = \mathbf{0}, \tag{6.1.6}$$

for all $\boldsymbol{v}$, $\boldsymbol{w} \in \boldsymbol{E}^p$, for $s = \frac{p}{2} > 1$ and $N_2 = N_2(p)$ depends only on p, $1 < p < \infty$.

We know state the result of Giga [40, Theorem 1, Theorem 2] who proves the existence and uniqueness of mild solutions of (6.1.2) assuming ($\mathbf{A}$), ($\mathbf{N1}$) and ($\mathbf{N2}$). In what follows BC denotes the space of bounded and continuous functions and C denotes positive constant whose value may change from one line to the next.

Theorem 6.1.1 (Giga's abstract existence and uniqueness theorem)**.** *Let $\boldsymbol{u}_0 \in \boldsymbol{E}^p$, $p \geq 3$. Then there is $T_0 > 0$ and a unique mild solution of (6.1.2) on $[0, T_0)$ such that*

$$\boldsymbol{u} \in BC([0, T_0);\, \boldsymbol{E}^p) \cap L^q(0, T_0;\, \boldsymbol{E}^r)$$

with

$$q > p,\ r > p, \qquad \frac{2}{q} + \frac{3}{r} = \frac{3}{p}.$$

Moreover there is a positive constant ε such that if $\|\boldsymbol{u}_0\|_{\boldsymbol{E}^p} \leq \varepsilon$, then T_0 can be taken as infinity for $p = 3$.

Now we want to apply Giga's abstract existence and uniqueness result to the Navier-Stokes problem with Navier-type boundary conditions (3.1.1) to get the existence and uniqueness of a local in time mild solution. Consider the problem

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla\pi = \mathbf{0}, & \operatorname{div}\boldsymbol{u} = 0 \quad \text{in} \quad \Omega \times (0, T), \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, & \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Gamma \times (0, T), \\ & \boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in} \quad \Omega, \end{cases} \tag{6.1.7}$$

where $(\boldsymbol{u}\cdot\nabla) = \sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j}$ and Ω is a bounded domain of $\mathbb{R}^3$ of class $C^{2,1}$. For simplicity the external force is assumed to be zero.

Applying the Helmholtz projection P defined by (3.1.7) to the first equation of system (6.1.7), we get

$$\frac{\partial \boldsymbol{u}}{\partial t} + A_p\boldsymbol{u} = -P(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega). \tag{6.1.8}$$

We recall that the operator A_p is the Stokes operator with Navier-type boundary conditions defined in Chapter 3 by (3.1.5). Let us verify assumptions (**A**), (**N1**) and (**N2**) described above for the Stokes operator A_p, in our case $\boldsymbol{E}^p = \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. First, observe that the assumption (**A**) is the $L^p - L^q$ estimate (5.1.11), proved in Chapter 5, Theorem 5.1.4. Thus assumption (**A**) holds. We next verify the assumptions for the non-linear term

$$\boldsymbol{F}\boldsymbol{u} = -P(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}. \tag{6.1.9}$$

Since $\operatorname{div}\boldsymbol{u} = 0$ in Ω, we can easily verify that

$$\forall\ 1 \le i \le 3, \qquad (\boldsymbol{u}\cdot\nabla\boldsymbol{u})_i = \sum_{j=1}^{3} \frac{\partial(u_j u_i)}{\partial x_j}.$$

As in [40] let $(g_{ij})_{1\le i,j\le 3}$ be a matrix and for all $1 \le i \le 3$ we set $\boldsymbol{g}_i = (g_{ij})_{1\le j\le 3}$. We define $\boldsymbol{L}$ by

$$\boldsymbol{L}\,\boldsymbol{g}_i = P\operatorname{div}\boldsymbol{g}_i. \tag{6.1.10}$$

We recall that P is the Helmholtz projection defined above. The non-linear term $\boldsymbol{F}\boldsymbol{u}$ is expressed by $\boldsymbol{L}\,\boldsymbol{G}\boldsymbol{u}$, where $(\boldsymbol{G}\boldsymbol{u})(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x}))$ and

$$\boldsymbol{g}(\boldsymbol{u}):\quad \mathbb{R}^3 \longmapsto \mathbb{R}^9, \qquad (\boldsymbol{g}(\boldsymbol{u}))_{ij} = -u_i u_j, \quad 1 \le i,j \le 3.$$

It is easy to see that for all $\boldsymbol{y}$, $\boldsymbol{z} \in \mathbb{R}^3$ $\boldsymbol{g}$ satisfies

$$|\boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{g}(\boldsymbol{z})| \le N_2\,|\boldsymbol{y}-\boldsymbol{z}|(|\boldsymbol{y}| + |\boldsymbol{z}|), \qquad \boldsymbol{g}(\boldsymbol{0}) = \boldsymbol{0},$$

with $|\cdot|$ denotes the norm on $\mathbb{R}^k$, $k \in \{3, 9\}$. Thus $\boldsymbol{G}$ satisfies (**N2**).
It remains to verify the assumption (**N1**). To this end we prove the following lemmas and propositions

Proposition 6.1.2. *Consider the Helmholtz projection $P : \boldsymbol{L}^p(\Omega) \longmapsto \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ defined in (3.1.7). The adjoint P^* of P is equal to the continuous embedding $I : \boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega) \longmapsto \boldsymbol{L}^{p'}(\Omega)$.*

Proof. First we recall that for all $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $P\boldsymbol{f} = \boldsymbol{f} - \operatorname{\mathbf{grad}}\pi$ in Ω, where π is the unique solution of Problem (2.2.7). We recall also that for all $1 < p < \infty$ $(\boldsymbol{L}^p(\Omega))' \simeq \boldsymbol{L}^{p'}(\Omega)$ and $(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))' \simeq \boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega)$. Let $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{v} \in \boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega)$ we have

$$\langle P\boldsymbol{u}\,,\,\boldsymbol{v}\rangle_{\boldsymbol{L}^p(\Omega)\times\boldsymbol{L}^{p'}(\Omega)} = \int_\Omega (\boldsymbol{u} - \operatorname{\mathbf{grad}}\pi)\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x$$

with $\pi \in \boldsymbol{W}^{1,p}(\Omega)/\mathbb{R}$ is the unique solution of the problem:

$$\operatorname{div}(\operatorname{\mathbf{grad}}\pi - \boldsymbol{u}) = 0 \qquad \text{in } \Omega, \qquad (\operatorname{\mathbf{grad}}\pi - \boldsymbol{u})\cdot\boldsymbol{n} = 0 \qquad \text{on } \Gamma.$$

As a result,

$$\langle P\boldsymbol{u}\,,\,\boldsymbol{v}\rangle_{\boldsymbol{L}^p(\Omega)\times\boldsymbol{L}^{p'}(\Omega)} = \int_\Omega \boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}\,x + \int_\Omega \pi\operatorname{div}\overline{\boldsymbol{v}}\,\mathrm{d}\,x - \langle \pi\,,\,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma = 0,$$

where $\langle\cdot\,,\,\cdot\rangle_\Gamma = \langle\cdot\,,\,\cdot\rangle_{\boldsymbol{W}^{1/p',p}(\Gamma)\times\boldsymbol{W}^{-1/p',p'}(\Gamma)}$. This means that

$$\langle P\boldsymbol{u}\,,\,\boldsymbol{v}\rangle_{\boldsymbol{L}^p(\Omega)\times\boldsymbol{L}^{p'}(\Omega)} = \langle\boldsymbol{u}\,,\,\boldsymbol{v}\rangle_{\boldsymbol{L}^p(\Omega)\times\boldsymbol{L}^{p'}(\Omega)} = \langle\boldsymbol{u}\,,\,P^*\boldsymbol{v}\rangle_{\boldsymbol{L}^p(\Omega)\times\boldsymbol{L}^{p'}(\Omega)}.$$

□

Lemma 6.1.3. *For all $1\leq j\leq 3$, the operator $(I+A_p)^{-1/2}P\frac{\partial}{\partial x_j}$ is a linear bounded operator from $\boldsymbol{L}^p(\Omega)$ to $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, for all $1<p<\infty$.*

Proof. The proof is similar to the proof of [41, Lemma 2.1]. First observe that the operator

$$\frac{\partial}{\partial x_j}\,I\,(I+A_p)^{-1/2}\;:\quad \boldsymbol{L}^p_{\sigma,\tau}(\Omega)\longmapsto \mathbf{D}((I+A_p)^{1/2})\longmapsto \boldsymbol{W}^{1,p}(\Omega)\longmapsto \boldsymbol{L}^p(\Omega)$$

is continuous for each p, $1<p<\infty$, where I denotes the continuous embedding of $\mathbf{D}((I+A_p)^{1/2}) = \boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega)$ in $\boldsymbol{W}^{1,p}(\Omega)$. As a result the adjoint operator $\Big[\frac{\partial}{\partial x_j}\,I\,(I+A_p)^{-1/2}\Big]^*$ is continuous from $\boldsymbol{L}^{p'}(\Omega)$ to $\boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega)$. Let us prove that

$$\Big[\frac{\partial}{\partial x_j}\,I\,(I+A_p)^{-1/2}\Big]^* = (I+A_{p'})^{-1/2}P\frac{\partial}{\partial x_j}. \tag{6.1.11}$$

First thanks to Remark 2.3.14 in Chapter 2, we know that the adjoint operator of $(I+A_p)^{1/2}$ is equal to $(I+A_{p'})^{1/2}$ thus the adjoint operator of $(I+A_p)^{-1/2}$ is equal to $(I+A_{p'})^{-1/2}$. Now let $\boldsymbol{u}\in\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and let $\boldsymbol{v}\in\boldsymbol{\mathcal{D}}(\Omega)$, one has

$$\begin{aligned}\Big\langle\frac{\partial}{\partial x_j}\,I\,(I+A_p)^{-1/2}\boldsymbol{u}\,,\,\boldsymbol{v}\Big\rangle_{\boldsymbol{L}^p(\Omega)\times\boldsymbol{L}^{p'}(\Omega)} &= -\Big\langle I\,(I+A_p)^{-1/2}\boldsymbol{u}\,,\,\frac{\partial\boldsymbol{v}}{\partial x_j}\Big\rangle_{\boldsymbol{L}^p(\Omega)\times\boldsymbol{L}^{p'}(\Omega)}\\ &= -\Big\langle (I+A_p)^{-1/2}\boldsymbol{u}\,,\,P\,\frac{\partial\boldsymbol{v}}{\partial x_j}\Big\rangle_{\boldsymbol{L}^p(\Omega)\times\boldsymbol{L}^{p'}(\Omega)} \qquad (6.1.12)\\ &= -\Big\langle \boldsymbol{u}\,,\,(I+A_{p'})^{-1/2}P\,\frac{\partial\boldsymbol{v}}{\partial x_j}\Big\rangle_{\boldsymbol{L}^p(\Omega)\times\boldsymbol{L}^{p'}(\Omega)}.\end{aligned}$$

The equality (6.1.12) comes from the fact that the adjoint of the Helmoltz projection P is equal to I (see Proposition 6.1.2). As a result for all $\boldsymbol{v}\in\boldsymbol{\mathcal{D}}(\Omega)$ one has

$$\Big[\frac{\partial}{\partial x_j}\,I\,(I+A_p)^{-1/2}\Big]^*\boldsymbol{v} = (I+A_{p'})^{-1/2}P\frac{\partial\boldsymbol{v}}{\partial x_j}.$$

Since $\Big[\frac{\partial}{\partial x_j}\,I\,(I+A_p)^{-1/2}\Big]^*$ is continuous from $\boldsymbol{L}^{p'}(\Omega)$ to $\boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega)$, then for all $\boldsymbol{v}\in\boldsymbol{\mathcal{D}}(\Omega)$ one has

$$\Big\|(I+A_{p'})^{-1/2}P\frac{\partial\boldsymbol{v}}{\partial x_j}\Big\|_{\boldsymbol{L}^{p'}(\Omega)} = \Big\|\Big[\frac{\partial}{\partial x_j}\,I\,(I+A_p)^{-1/2}\Big]^*\boldsymbol{v}\Big\|_{\boldsymbol{L}^{p'}(\Omega)} \leq C\|\boldsymbol{v}\|_{\boldsymbol{L}^{p'}(\Omega)}.$$

Thus the operator $(I+A_{p'})^{-1/2}P\frac{\partial}{\partial x_j}\;:\;\boldsymbol{\mathcal{D}}(\Omega)\subset\boldsymbol{L}^{p'}(\Omega)\longmapsto\boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega)$ is continuous for the norm of $\boldsymbol{L}^{p'}(\Omega)$. As a result using the density of $\boldsymbol{\mathcal{D}}(\Omega)$, the operator $(I+A_{p'})^{-1/2}P\frac{\partial}{\partial x_j}$ can be extended to a linear continuous operator from $\boldsymbol{L}^{p'}(\Omega)$ to $\boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega)$. Moreover (6.1.11) holds. □

As a consequence of Lemma 6.1.3 we have the following corollary

Corollary 6.1.4. *Let $\boldsymbol{L}$ be the operator defined in* (6.1.10). *The following estimate holds*

$$\forall\, \boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \qquad \|e^{-tA_p}\boldsymbol{L}\,\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C(p,T)}{t^{1/2}}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{6.1.13}$$

where e^{-tA_p} is the semi-group generated by the Stokes operator with Navier-type boundary conditions on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and $C(p,T)$ is a constant depending on p and T.

Proof. First we recall that the Stokes operator with Navier-type boundary conditions generates a bounded analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ for all $< p < \infty$, (see Chapter 3, Theorem 3.1.19). We also know that for a fixed $\lambda > 0$ one has

$$\forall\, t > 0, \qquad e^{-tA_p} = e^{\lambda t}e^{-t(\lambda I + A_p)},$$

where $e^{-t(\lambda I + A_p)}$ is the analytic semi-group generated by the operator $-(\lambda\, I + A_p)$ on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. Now, let $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ one has

$$\begin{aligned}
\|e^{-tA_p}\boldsymbol{L}\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} &= e^t\|e^{-t(I+A_p)}\boldsymbol{L}\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \\
&= e^t\,\|(I + A_p)^{1/2}\,e^{-t(I+A_p)}\,(I + A_p)^{-1/2}\,\boldsymbol{L}\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \\
&\leq \frac{Ce^T}{t^{1/2}}\,\|(I + A_p)^{-1/2}\,\boldsymbol{L}\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \\
&\leq \frac{Ce^T}{t^{1/2}}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.
\end{aligned}$$

The last inequality comes from the fact that the operator $(I + A_p)^{-1/2}\boldsymbol{L}$ is a bounded operator from $\boldsymbol{L}^p(\Omega)$ into $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ which is a consequence of Lemma 6.1.3. ☐

Remark 6.1.5. Corollary 6.1.4 means the Stokes operator with Navier-type boundary conditions satisfies the assumption (**N1**)

We thus have checked all assumptions that guarantee the existence and uniqueness of local in time mild solution for the Navier-Stokes Problem (6.1.8). As a result applying Theorem6.1.1 to the Stokes operator A_p with $\boldsymbol{E}^p = \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ we have the following theorem :

Theorem 6.1.6 (Existence and uniqueness)**.** *Let $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, $p \geq 3$. There is a $T_0 > 0$ and a unique mild solution of* (6.1.8) *on* $[0, T_0)$ *such that*

$$\boldsymbol{u} \in BC([0, T_0);\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega)) \cap L^q(0, T_0;\, \boldsymbol{L}^r_{\sigma,\tau}(\Omega))$$

with

$$q > p,\ \ r > p, \qquad \frac{2}{q} + \frac{3}{r} = \frac{3}{p}.$$

Moreover there is a positive constant ε such that if $\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p_{\sigma,\tau}(\Omega)} \leq \varepsilon$ then T_0 can be taken as infinity for $p = 3$.

Next we want to prove that the mild solution obtained above is a classical solution. For this reason we will proceed as in [41]. We start by the following lemma.

Lemma 6.1.7. *Let $0 \le \delta \le \frac{1}{2} + \frac{3}{2}(1 - \frac{1}{p})$ and $1 < p < \infty$. Then*

$$\forall\, \boldsymbol{u},\ \boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega), \qquad \|(I + A_p)^{-\delta} P(\boldsymbol{u} \cdot \nabla)\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \le M\, \|(I + A_p)^\theta \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \|(I + A_p)^\rho \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}, \tag{6.1.14}$$

where the constant $M = M(\delta, \theta, \rho, p)$ provided that

$$\delta + \theta + \rho \ge 1/2 + 3/2p, \qquad \theta > 0, \quad \rho > 0, \qquad \rho + \delta > 1/2.$$

Remark 6.1.8. By density of $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ in $\mathbf{D}((I + A_p)^\alpha)$ for all $0 \le \alpha \le 1$ one has estimate (6.1.14) for all $\boldsymbol{u} \in \mathbf{D}((I + A_p)^\theta)$ and for all $\boldsymbol{v} \in \mathbf{D}((I + A_p)^\rho)$.

Proof of Lemma 6.1.7. Assume that $0 \le \varepsilon \le \frac{3}{2}(1 - \frac{1}{p})$. Thanks to Chapter 4, Corollary 4.1.11 we know that the operator

$$(\lambda I + A_{p'})^{-\varepsilon} \,:\, \boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega) \longmapsto \mathbf{D}((I + A_{p'})^\varepsilon) \hookrightarrow \boldsymbol{L}^{s'}_{\sigma,\tau}(\Omega)$$

is a bounded linear operator with

$$\frac{1}{s'} = \frac{1}{p'} - \frac{2\varepsilon}{3}.$$

By duality this implies that the operator

$$(I + A_p)^{-\varepsilon} \,:\, \boldsymbol{L}^{s}_{\sigma,\tau}(\Omega) \longrightarrow \boldsymbol{L}^{p}_{\sigma,\tau}(\Omega)$$

extends uniquely to a bounded linear operator from $\boldsymbol{L}^{s}_{\sigma,\tau}(\Omega)$ to $\boldsymbol{L}^{p}_{\sigma,\tau}(\Omega)$ with

$$\frac{1}{s} = \frac{1}{p} + \frac{2\varepsilon}{3}. \tag{6.1.15}$$

(i) First consider the case $\delta \ge 1/2$ and take $\varepsilon = \delta - \frac{1}{2}$ and observe that with such ε, the operator $(I + A_p)^{-\varepsilon}$ is a bounded linear operator from $\boldsymbol{L}^{s}_{\sigma,\tau}(\Omega)$ to $\boldsymbol{L}^{p}_{\sigma,\tau}(\Omega)$, where s is given by (6.1.15). Using Lemma 6.1.3 one has

$$\|(I + A_p)^{-\delta} P(\boldsymbol{u} \cdot \nabla)\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} = \Big\| \sum_{j=1}^{3} (I + A_p)^{-\varepsilon - 1/2} P \frac{\partial(u_j \boldsymbol{v})}{\partial x_j} \Big\|_{\boldsymbol{L}^p(\Omega)} \le C \,\|\, |\boldsymbol{u}| \cdot |\boldsymbol{v}| \,\|_{\boldsymbol{L}^s(\Omega)}. \tag{6.1.16}$$

We recall that since $\operatorname{div} \boldsymbol{u} = 0$ in Ω we have

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{v} = \sum_{j=1}^{3} \frac{\partial(u_j \boldsymbol{v})}{\partial x_j}.$$

By assumption we can take r_1 and r_2 such that

$$\frac{1}{r_1} \ge \frac{1}{p} - \frac{2\theta}{3}, \qquad \frac{1}{r_2} \ge \frac{1}{p} - \frac{2\rho}{3}, \qquad \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{s}, \qquad 1 < r_1,\, r_2 < \infty. \tag{6.1.17}$$

As a result Hölder inequality and (6.1.17) yield

$$\|\,|\boldsymbol{u}|\cdot|\boldsymbol{v}|\,\|_{\boldsymbol{L}^s(\Omega)} \le \|\boldsymbol{u}\|_{\boldsymbol{L}^{r_1}(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{L}^{r_2}(\Omega)} \le C\,\|(I+A_p)^{\theta}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\|(I+A_p)^{\rho}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}. \quad (6.1.18)$$

Finally putting together (6.1.16) and (6.1.18) we obtain the required result.

(ii) The case $0 \le \delta \le 1/2$ is obtain in the same way as in the proof of [41, Lemma 2.2]. □

In the particular case where $p > 3$ we have the following proposition

Proposition 6.1.9. *Let $p > 3$ then for all $\boldsymbol{u}$, $\boldsymbol{v} \in \mathbf{D}(A_p^{1/2})$ one has*

$$\|P(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \le C\,\|(I+A_p)^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\|(I+A_p)^{1/2}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}. \quad (6.1.19)$$

Proof. First, since for $p > 3$, $\boldsymbol{W}^{1,p}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$ one has

$$\begin{aligned}
\|P(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} &\le C\,\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(\Omega)}\|\nabla\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}\\
&\le C\,\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}\\
&\le C\,\|(I+A_p)^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\|(I+A_p)^{1/2}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}
\end{aligned}$$

The last inequality comes from the fact that $\mathbf{D}(A_p^{1/2}) = \boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega)$ and for all $\boldsymbol{u} \in \mathbf{D}(A_p^{1/2})$ the norm $\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}$ is equivalent to the norm $\|(I+A_p)^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$. □

Consider now the non-linear term $\boldsymbol{F}\boldsymbol{u}$ defined by (6.1.9), we have the following proposition

Proposition 6.1.10. *Let δ, θ, ρ be as in Lemma 6.1.7 and let $1 < p < \infty$. For all $\boldsymbol{u}$, $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega)$ one has*

$$\begin{aligned}
\|(I+A_p)^{-\delta}(\boldsymbol{F}\boldsymbol{u}-\boldsymbol{F}\boldsymbol{v})\|_{\boldsymbol{L}^p(\Omega)} \le C\,\|(I+A_p)^{\theta}(\boldsymbol{u}-\boldsymbol{v})\|_{\boldsymbol{L}^p(\Omega)}\|(I+A_p)^{\rho}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} +{}&\\
\|(I+A_p)^{\theta}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}\|(I+A_p)^{\rho}(\boldsymbol{u}-\boldsymbol{v})\|_{\boldsymbol{L}^p(\Omega)}.& \quad (6.1.20)
\end{aligned}$$

Moreover for $p > 3$ we have

$$\begin{aligned}
\|\boldsymbol{F}\boldsymbol{u}-\boldsymbol{F}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \le C\,\|(I+A_p)^{1/2}(\boldsymbol{u}-\boldsymbol{v})\|_{\boldsymbol{L}^p(\Omega)}\|(I+A_p)^{1/2}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} +{}&\\
\|(I+A_p)^{1/2}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}\|(I+A_p)^{1/2}(\boldsymbol{u}-\boldsymbol{v})\|_{\boldsymbol{L}^p(\Omega)}.& \quad (6.1.21)
\end{aligned}$$

Proof. Just observe that

$$\boldsymbol{F}\boldsymbol{u}-\boldsymbol{F}\boldsymbol{v} \,=\, P(\boldsymbol{u}-\boldsymbol{v})\cdot\nabla\boldsymbol{u} \,+\, P\boldsymbol{v}\cdot\nabla(\boldsymbol{u}-\boldsymbol{v}).$$

As a result, estimates (6.1.20) and (6.1.21) follow directly from Lemma 6.1.7 and Proposition 6.1.9. □

Before we state our theorem we recall the following lemma which is proved by Fujita and Kato in [35, Lemma 2.12]

Lemma 6.1.11. *Let $-\mathcal{A}$ be the infinitesimal generator of a bounded analytic semi-group on a Banach space X and let $e^{-t\mathcal{A}}$ be the semi-group generated by $\mathcal{A}$. Consider the function*

$$v(t) = \int_0^t e^{-(t-s)\mathcal{A}} f(s)\,\mathrm{d}\,s, \qquad 0 \le t \le T, \quad T > 0 \tag{6.1.22}$$

with $f \in C((0,T];\, X)$ is assumed to satisfy

$$\sup_{0<s\le t} s^{\lambda} \|f(s)\|_X \le M(t) < \infty, \qquad (0 < t \le T), \tag{6.1.23}$$

for some constant $0 \le \lambda < 1$ and a real valued function M.

If $0 \le \alpha < 1$, then $\mathcal{A}^{\alpha} v(t)$ exists for each $t \in (0,T]$ and satisfy the inequality

$$\|\mathcal{A}^{\alpha} v(t)\|_X \;\le\; t^{1-\alpha-\lambda}\, M(t)\, B(1-\alpha\,,\, 1-\lambda),$$

where $B(\cdot\,,\,\cdot)$ represents the beta function

$$B(x,y) \;=\; \int_0^1 s^{x-1}(1-s)^{y-1}\,\mathrm{d}\,s.$$

Moreover, $\mathcal{A}^{\alpha} v \in C^{\vartheta}((0,T],\, X)$ for any $0 < \vartheta < 1-\alpha$. In particular, we have $\mathcal{A}^{\alpha} v \in C^{\vartheta}([0,T];\, X)$ with $\mathcal{A}^{\alpha} v(0) = 0$ if $0 < \vartheta \le 1-\alpha-\lambda$. We recall that $C^{\vartheta}((0,T];\, X)$ is the set of Hölder continuous function with exponent ϑ.

Consider again the function $v(t)$ given by (6.1.22), the following lemma is proved by Fujita and Kato in [35, Lemma 2.14]

Lemma 6.1.12. *Under the same assumptions of Lemma 6.1.11, when the function f is Hölder continuous with some exponent $0 < \vartheta < 1$ on $[0,T]$, the function v is also Hölder continuous with exponent $1+\vartheta$ on $(0,T]$ and $\mathcal{A}v$ is Hölder continuous with exponent ϑ on $(0,T]$. Furthermore*

$$\frac{\partial v}{\partial t} + \mathcal{A}\,v \;=\; f.$$

The following theorem shows that the solution $\boldsymbol{u}(t)$ of Theorem 6.1.6 is in $\mathbf{D}(A_p^{\alpha})$ for all $t \in (0,T_*]$ and for all $0 < \alpha < 1-\delta$, where δ satisfies the assumptions of Lemma 6.1.7.

Theorem 6.1.13. *Let δ be as in Lemma 6.1.7 be fixed and let $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, $p \ge 3$. There exists a maximal interval of time $T_* \in (0,T)$ such that the unique solution $\boldsymbol{u}(t)$ of Problem (6.1.8) is in $C((0,T_*];\, \mathbf{D}(A_p^{\alpha}))$ for all $0 < \alpha < 1-\delta$. Moreover the solution $\boldsymbol{u}(t)$ satisfies*

$$\|(I + A_p)^{\alpha} \boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \le K_{\alpha} t^{-\alpha}, \tag{6.1.24}$$

for some constant $K_{\alpha} > 0$.

Proof. First, we note that, thanks to Theorem 6.1.6, there exists a $T_0 > 0$ such that the unique solution $\boldsymbol{u}(t)$ of Problem (6.1.8) obtained in Theorem 6.1.6 is in $BC([0,T_0);\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$. Moreover for all $0 \le t \le T_0$, $\boldsymbol{u}(t)$ is given by

$$\boldsymbol{u}(t) = \boldsymbol{u}_0(t) + \boldsymbol{S}\boldsymbol{u}(t) \tag{6.1.25}$$

with

$$\boldsymbol{u}_0(t) = e^{-tA_p}\boldsymbol{u}_0 \qquad \text{and} \qquad \boldsymbol{S}\boldsymbol{u}(t) = \int_0^t e^{-(t-s)A_p}\boldsymbol{F}\boldsymbol{u}(s)\,\mathrm{d}\,s, \tag{6.1.26}$$

where $\boldsymbol{F}\boldsymbol{u} = -P(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$. In addition, thanks to [40, Theorem 1], we know that by construction there exists a sequence $(\boldsymbol{u}_m(t))_{m\ge 0}$ such that $(\boldsymbol{u}_m)_m$ converges to $\boldsymbol{u}$ in $BC([0,T_0];\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$ and $\boldsymbol{u}_m(t)$ is defined recursively by

$$\boldsymbol{u}_0(t) = e^{-tA_p}\boldsymbol{u}_0 \qquad \text{and} \quad \forall\, m \ge 1, \qquad \boldsymbol{u}_{m+1}(t) = \boldsymbol{u}_0(t) + \boldsymbol{S}\boldsymbol{u}_m(t). \tag{6.1.27}$$

Now, let δ be as in Lemma 6.1.7 and $0 < \alpha < 1-\delta$. Since e^{-tA_p} is a bounded analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, then $\boldsymbol{u}_0(t) \in \mathbf{D}(A_p) \hookrightarrow \mathbf{D}(A_p^\alpha)$ and

$$\|(I+A_p)^\alpha \boldsymbol{u}_0(t)\|_{\boldsymbol{L}^p(\Omega)} = \|(I+A_p)^\alpha e^{-tA_p}\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} = e^t\|(I+A_p)^\alpha e^{-t(I+A_p)}\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \le K_{\alpha 0}t^{-\alpha}, \tag{6.1.28}$$

with

$$K_{\alpha 0} = \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \sup_{0<t\le T_0} e^t t^\alpha \|(I+A_p)^\alpha e^{-t(I+A_p)}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))}. \tag{6.1.29}$$

The factor e^t in (6.1.29) is irrelevant since our existence is local in time.

Suppose that for some $m \ge 1$, $\boldsymbol{u}_m(t) \in \mathbf{D}(A_p^\alpha)$ for all $0 < t \le T_0$ and satisfies

$$\|(I+A_p)^\alpha \boldsymbol{u}_m(t)\|_{\boldsymbol{L}^p(\Omega)} \le K_{\alpha m}t^{-\alpha}, \qquad \forall\ \alpha,\ 0 < \alpha < 1-\delta \tag{6.1.30}$$

for some constant $K_{\alpha m} > 0$ and let us prove that $\boldsymbol{u}_{m+1}(t) \in \mathbf{D}(A_p^\alpha)$ and satisfies

$$\|(I+A_p)^\alpha \boldsymbol{u}_{m+1}(t)\|_{\boldsymbol{L}^p(\Omega)} \le K_{\alpha m+1}t^{-\alpha}, \qquad 0 < \alpha < 1-\delta,$$

for some constant $K_{\alpha m+1} > 0$. We shall estimate $\|(I+A_p)^\alpha \boldsymbol{u}_{m+1}(t)\|_{\boldsymbol{L}^p(\Omega)}$ by using the explicit formula (6.1.27). Observe that

$$(I+A_p)^\alpha \boldsymbol{u}_{m+1}(t) = (I+A_p)^\alpha \boldsymbol{u}_0(t) + (I+A_p)^\alpha \boldsymbol{S}\boldsymbol{u}_m(t), \tag{6.1.31}$$

where $\boldsymbol{S}\boldsymbol{u}(t)$ is given by (6.1.26).

$$\begin{aligned}
\|(I+A_p)^\alpha \boldsymbol{S}\boldsymbol{u}_m(t)\|_{\boldsymbol{L}^p(\Omega)} &\le \int_0^t \|(I+A_p)^\alpha e^{-(t-s)A_p}\boldsymbol{F}\boldsymbol{u}_m(s)\|_{\boldsymbol{L}^p(\Omega)}\,\mathrm{d}\,s\\
&\le e^T\int_0^t \|(I+A_p)^\alpha e^{-(t-s)(I+A_p)}\boldsymbol{F}\boldsymbol{u}_m(s)\|_{\boldsymbol{L}^p(\Omega)}\,\mathrm{d}\,s\\
&\le e^T\int_0^t \|(I+A_p)^{\alpha+\delta} e^{-(t-s)(I+A_p)}(I+A_p)^{-\delta}\boldsymbol{F}\boldsymbol{u}_m(s)\|_{\boldsymbol{L}^p(\Omega)}\,\mathrm{d}\,s\\
&\le C_{\alpha+\delta}\, e^T\int_0^t (t-s)^{-\alpha-\delta}\|(I+A_p)^{-\delta}\boldsymbol{F}\boldsymbol{u}_m(s)\|_{\boldsymbol{L}^p(\Omega)}\,\mathrm{d}\,s.
\end{aligned} \tag{6.1.32}$$

As in the proof of [41, Theorem 2.3] to estimate the term $\|(I+A_p)^{-\delta}\boldsymbol{F}\boldsymbol{u}_m(s)\|_{\boldsymbol{L}^p(\Omega)}$ we choose $\theta>0$ and $\rho>0$ such that

$$\theta+\rho+\delta=1, \qquad 0<\theta<1-\delta, \qquad 1/2<\delta+\rho<1.$$

We can easily verify that θ, ρ and δ satisfy the assumptions of Lemma 6.1.7. Thus using Lemma 6.1.7 and (6.1.30) one has

$$\begin{aligned}\|(I+A_p)^{-\delta}\boldsymbol{F}\boldsymbol{u}_m(s)\|_{\boldsymbol{L}^p(\Omega)} &\leq M\|(I+A_p)^{\theta}\boldsymbol{u}_m(s)\|_{\boldsymbol{L}^p(\Omega)}\|(I+A_p)^{\rho}\boldsymbol{u}_m(s)\|_{\boldsymbol{L}^p(\Omega)}\\ &\leq M\,K_{\theta m}K_{\rho m}\,s^{\delta-1}.\end{aligned} \tag{6.1.33}$$

Now putting together (6.1.32) and (6.1.33) one has

$$\|(I+A_p)^{\alpha}\boldsymbol{S}\boldsymbol{u}_m(t)\|_{\boldsymbol{L}^p(\Omega)} \leq C_{\alpha+\delta}\,M\,K_{\theta m}K_{\rho m}\,e^{T}\int_0^t(t-s)^{-\alpha-\delta}\,s^{\delta-1}\,\mathrm{d}\,s. \tag{6.1.34}$$

Putting together (6.1.31), (6.1.28) and (6.1.34) and using Lemma 6.1.11 one has

$$\|(I+A_p)^{\alpha}\boldsymbol{u}_{m+1}(t)\|_{\boldsymbol{L}^p(\Omega)}\leq K_{\alpha m+1}t^{-\alpha} \tag{6.1.35}$$

with $K_{\alpha m+1}$ defined recursively by

$$K_{\alpha m+1}=K_{\alpha 0}\;+\;M\,e^{T}\,C_{\alpha+\delta}B(1-\delta-\alpha,\delta)K_{\theta m}K_{\rho m} \tag{6.1.36}$$

and $B(\cdot\,,\,\cdot)$ denotes the beta function. Thus $\boldsymbol{u}_m(t)$ is well defined for each $m\geq 0$ as an element of $C((0,T_0];\,\mathbf{D}(A_p^{\alpha}))$ for all $0<\alpha<1-\delta$, moreover $\boldsymbol{u}_m(t)$ satisfies (6.1.35) with $K_{\alpha m}$ defined recursively by (6.1.29) and (6.1.36).

As in the proof of [41, Theorem 2.3] we can show that if

$$K_0\;<\;\frac{1}{4C_1MB_1}, \tag{6.1.37}$$

with $C_1=\max(C_{\theta+\delta}\,,\,C_{\rho+\delta})$ and $B_1=\max(B(1-\delta-\theta\,,\,\delta)\,,\,B(1-\delta-\rho\,,\,\delta))$, then

$$\|(I+A_p)^{\alpha}\boldsymbol{u}_{m+1}(t)\|_{\boldsymbol{L}^p(\Omega)}\leq K_{\alpha}t^{-\alpha}, \tag{6.1.38}$$

with a constant K_α independent of m. As a result, for all $0<t<T_0$ the sequence $(\boldsymbol{u}_m(t))_{m\geq 0}$ is bounded in $\mathbf{D}(A_p^{\alpha})$ and thus it converges weakly in $\mathbf{D}(A_p^{\alpha})$ to a function denoted by $\boldsymbol{v}(t)$ and $(I+A_p)^{\alpha}\boldsymbol{u}_m(t)$ converges weakly to $(I+A_p)^{\alpha}\boldsymbol{v}(t)$ in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. In the other hand $\boldsymbol{u}_m(t)$ converges to $\boldsymbol{u}(t)$ in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ thus $\boldsymbol{u}(t)=\boldsymbol{v}(t)$ and $\boldsymbol{u}(t)\in\mathbf{D}(A_p^{\alpha})$ for all $0<t<T_0$. As stated in the proof of [41, Theorem 2.3], if $T>0$ is chosen sufficiently small then $K_{\alpha 0}$, $0<\alpha<1-\delta$ becomes small and K_0 satisfies (6.1.37). This shows the existence of $T_*>0$ such that $\boldsymbol{u}\in C((0,T_*];\,\mathbf{D}(A_p^{\alpha}))$.

Finally observe that since $(I+A_p)^{\alpha}\boldsymbol{u}_m(t)$ converges weakly to $(I+A_p)^{\alpha}\boldsymbol{u}(t)$ in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ then

$$\|(I+A_p)^{\alpha}\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)}\;\leq\;\liminf_m\|(I+A_p)^{\alpha}\boldsymbol{u}_m(t)\|_{\boldsymbol{L}^p(\Omega)}\;\leq\;K_{\alpha}t^{-\alpha}.$$

Thus one has estimate (6.1.24). $\square$

The next step is to prove that the solution $\boldsymbol{u}$ of Problem (6.1.8) is in $C((0,T_*];\mathbf{D}(A_p))$. Since the Stokes operator generates a bounded analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ then $\boldsymbol{u}_0(t)$ defined in (6.1.26) is in $\mathbf{D}(A_p)$ for all $t>0$. It remains to prove that $\boldsymbol{Su}(t)$ defined in (6.1.26) is in $\mathbf{D}(A_p)$ for all $0<t\le T_*$. The proof is done in three steps. First we prove that $(I+A_p)^\alpha\boldsymbol{u},\ 0<\alpha<1-\delta$ is Hölder continuous on every interval $[\varepsilon,\,T_*]$. This gives us that the non-linear term $\boldsymbol{Fu}$ is also Hölder continuous on every interval $[\varepsilon,\,T_*]$ and thus $\boldsymbol{u}\in\mathbf{D}(A_p)$ for all $0<t\le T_*$.

Proposition 6.1.14. *Let $0\le\delta<1$ be as in Lemma 6.1.7, $0<\alpha<1-\delta$, let $\boldsymbol{u}_0\in\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, $p\ge 3$ and let $\boldsymbol{u}(t)$ be the unique solution of Problem* (6.1.8). *Then $(I+A_p)^\alpha\boldsymbol{u}$ is Hölder continuous on every interval $[\varepsilon,T_*]$, $(0<\varepsilon<T_*)$.*

Proof. First we recall that for all $0<t\le T_*$

$$(I+A_p)^\alpha\boldsymbol{u}(t)=(I+A_p)^\alpha\boldsymbol{u}_0(t)+(I+A_p)^\alpha\boldsymbol{Su}(t)$$

with $\boldsymbol{u}_0(t)$ and $\boldsymbol{Su}(t)$ are defined in (6.1.26). Since the operators A_p and $I+A_p$ generates bounded analytic semi-groups on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and since $e^{-tA_p}=e^te^{-t(I+A_p)}$ then for all $\boldsymbol{u}_0\in\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, $(I+A_p)^\alpha\boldsymbol{u}_0(t)$ is Hölder continuous on every interval $[\varepsilon,T_*]$, $0<\varepsilon<T_*$, (see Chapter 2 Proposition 2.3.19).

Let us prove the Hölder continuity of $(I+A_p)^\alpha\boldsymbol{Su}(t)$. observe that

$$\begin{aligned}&(I+A_p)^\alpha\boldsymbol{Su}(t+h)-(I+A_p)^\alpha\boldsymbol{Su}(t)=\\&\quad\int_0^t(I+A_p)^\alpha\big[e^{t+h-s}e^{-(t+h-s)(I+A_p)}\boldsymbol{Fu}(s)-e^{t-s}e^{-(t-s)(I+A_p)}\boldsymbol{Fu}(s)\big]\,\mathrm{d}\,s+\\&\qquad\qquad\int_t^{t+h}e^{t+h-s}(I+A_p)^\alpha e^{-(t+h-s)(I+A_p)}\boldsymbol{Fu}(s)\,\mathrm{d}\,s.\end{aligned}$$

As a result,

$$\|(I+A_p)^\alpha\boldsymbol{Su}(t+h)-(I+A_p)^\alpha\boldsymbol{Su}(t)\|_{\boldsymbol{L}^p(\Omega)}\le I_1+I_2$$

with

$$I_1=e^T\int_0^t\|(I+A_p)^\alpha e^{-(t-s)(I+A_p)}(e^{-h(I+A_p)}-I)\boldsymbol{Fu}(s)\|_{\boldsymbol{L}^p(\Omega)}\,\mathrm{d}\,s \tag{6.1.39}$$

and

$$I_2=e^T\int_t^{t+h}\|(I+A_p)^\alpha e^{-(t+h-s)(I+A_p)}\boldsymbol{Fu}(s)\|_{\boldsymbol{L}^p(\Omega)}\,\mathrm{d}\,s. \tag{6.1.40}$$

We recall that the factor e^T in I_1 and I_2 is irrelevant since our existence is local in time. Now as in the proof of [41, Proposition 2.4], let $0<\mu<1-\delta-\alpha$ then

$$\begin{aligned}I_1&=e^T\int_0^t\|(I+A_p)^{\alpha+\delta+\mu}e^{-(t-s)(I+A_p)}(e^{-h(I+A_p)}-I)(I+A_p)^{-\mu-\delta}\boldsymbol{Fu}(s)\|_{\boldsymbol{L}^p(\Omega)}\,\mathrm{d}\,s\\&\le C_\mu\|(e^{-h(I+A_p)}-I)(I+A_p)^{-\mu}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))}\int_0^t(t-s)^{-\alpha-\delta-\mu}s^{\delta-1}\,\mathrm{d}\,s.\end{aligned} \tag{6.1.41}$$

The last inequality comes from the fact that $\|(I+A_p)^{-\delta}\boldsymbol{F}\boldsymbol{u}(s)\|_{\boldsymbol{L}^p(\Omega)} \leq C s^{-\delta-1}$ which is a consequence of estimate (6.1.24) and Lemma 6.1.7. Now using Lemma 2.3.16 in Chapter 2 one has

$$\|(e^{-h(I+A_p)} - I)(I+A_p)^{-\mu}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \leq \frac{C}{\mu} h^{\mu}.$$

Substituting in (6.1.41) one has as in the proof of [41, Proposition 2.4]

$$I_1 \leq C\, h^{\mu}, \tag{6.1.42}$$

with some constant C depending on ε and μ.
Next consider the integral I_2 given by (6.1.40) one has

$$\begin{aligned} I_2 &\leq e^T \int_t^{t+h} \|(I+A_p)^{\alpha+\delta} e^{-(t+h-s)(I+A_p)}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \|(I+A_p)^{-\delta}\boldsymbol{F}\boldsymbol{u}(s)\|_{\boldsymbol{L}^p(\Omega)}\,\mathrm{d}\,s \\ &\leq C_\varepsilon \int_t^{t+h} (t+h-s)^{-\alpha-\delta}\,\mathrm{d}\,s \leq \frac{C_\varepsilon}{1-\delta-\alpha} h^{1-\delta-\alpha} \\ &\leq C h^{\mu} \end{aligned} \tag{6.1.43}$$

with

$$C_\varepsilon = \sup_{\varepsilon \leq t \leq T_*} \|(I+A_p)^{-\delta}\boldsymbol{F}\boldsymbol{u}(s)\|_{\boldsymbol{L}^p(\Omega)}.$$

Finally putting together (6.1.42) and (6.1.43) one gets directly the Hölder continuity of $(I+A_p)^{\alpha}\boldsymbol{S}\boldsymbol{u}$ on $(0, T_*]$. □

Now we can prove the Hölder continuity of $\boldsymbol{F}\boldsymbol{u}$ given by (6.1.9).

Proposition 6.1.15. *Under the same assumptions of Proposition 6.1.14, let $\boldsymbol{u}$ be the unique solution of Problem* (6.1.8)*. Then $\boldsymbol{F}\boldsymbol{u}$ is Hölder continuous on every interval* $[\varepsilon, T_*]$*,* $0 < \varepsilon < T_*$.

Proof. Let $\boldsymbol{u}(t)$ be the unique solution of Problem (6.1.8). Thanks to Theorem 6.1.13 we know that $\boldsymbol{u} \in C((0, T_*]; \mathbf{D}(A_p^{\alpha}))$ for all $0 < \alpha < 1-\delta$, where δ is as in Lemma 6.1.7. Under a suitable choice of δ we can show that $\boldsymbol{u} \in C((0, T_*]; \mathbf{D}(A_p^{1/2}))$. Now, using Proposition 6.1.4, estimate (6.1.21) one has

$$\begin{aligned} \|\boldsymbol{F}\boldsymbol{u}(t+h) - \boldsymbol{F}\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq\ & C\,\|(I+A_p)^{1/2}(\boldsymbol{u}(t+h) - \boldsymbol{u}(t))\|_{\boldsymbol{L}^p(\Omega)}\|(I+A_p)^{1/2}\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} + \\ & \|(I+A_p)^{1/2}\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)}\|(I+A_p)^{1/2}(\boldsymbol{u}(t+h) - \boldsymbol{u}(t))\|_{\boldsymbol{L}^p(\Omega)}. \end{aligned}$$

Next, using the fact $(I+A_p)^{1/2}\boldsymbol{u}$ is Hölder continuous on every interval $[\varepsilon, T_*]$ (see Proposition 6.1.14), there exists $0 < \mu < 1-\delta-1/2$ such that

$$\|\boldsymbol{F}\boldsymbol{u}(t+h) - \boldsymbol{F}\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq C\, h^{\mu}$$

and the result is proved. □

Theorem 6.1.16. *Let* $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, $p \geq 3$ *and let* $\boldsymbol{u}(t)$ *be the unique solution of Problem* (6.1.8)*, then*

$$\boldsymbol{u} \in C((0, T_*], \mathbf{D}(A_p)) \cap C^1((0, T_*];\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega)).$$

Proof. First we recall that the solution $\boldsymbol{u}$ is given explicitly by (6.1.25). We recall also that since e^{-tA_p} is an analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ then $\boldsymbol{u}_0(t) \in \mathbf{D}(A_p)$ for all $t > 0$. It suffices to verify that $\boldsymbol{S}\boldsymbol{u}(t) \in \mathbf{D}(A_p)$ for all $t \in (0,\, T_*]$ which is a consequence of Proposition 6.1.15 and Theorem 2.3.20 in Chapter 2. Moreover, thanks to Lemma 6.1.12 one has $\boldsymbol{u} \in C^1((0, T];\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$ this ends the proof. $\square$

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