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Convergence du schéma Marker-and-Cell pour les équations de Navier-Stokes incompressible.

Jury

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Résumé

Le schéma Marker-And-Cell (MAC) est un schéma de discrétisation des équations aux dérivées partielles sur maillages cartésiens, très connu en mécanique des fluides. Nous nous intéressons ici à son analyse mathématique dans le cadre des écoulements incompressibles sur des maillages cartésiens non-uniformes en dimension 2 ou 3. Dans le schéma MAC, les inconnues discrètes sont les composantes normales de la vitesse aux faces du maillage, et la pression au centre des mailles dites "primales", sur lesquelles l'équation de continuité admet une discrétisation naturelle, la solution approchée est donc à divergence discrète nulle. Le bilan discret de chaque composante de la quantité de mouvement est écrit sur les mailles duales attachées à la même composante de la vitesse. Le terme de convection non linéaire est discrétisé de manière à être compatible avec une équation de continuité discrète sur les mailles duales, et coïncide avec la discrétisation habituellement utilisée en mécanique des fluides sur maillage uniforme. Dans un premier temps nous discrétisons les équations de Navier-Stokes pour un écoulement incompressible stationnaire; nous établissons des estimations a priori sur les suites de vitesses et pressions approchées qui permettent d'une part d'établir l'existence d'une solution au schéma numérique, et d'autre part d'obtenir la compacité de ces suites lorsque le pas d'espace tend vers 0. Nous montrons alors la convergence de ces suites (à une sous-suite près) vers une solution faible du problème continu lorsque le pas de discrétisation tend vers 0 pour des normes appropriées, ce qui nécessite une analyse fine du terme de convection non linéaire. Nous nous intéressons ensuite aux équations de Navier-Stokes en régime instationnaire avec une discrétisation en temps implicite. Nous démontrons là encore que le schéma préserve les propriétés de stabilité du problème continu (estimation $L^2(H^1)$ et $L^\infty(L^2)$ pour la vitesse), et obtenons ainsi l'existence d'une solution au schéma numérique. Puis, grâce à des techniques de compacité et en passant à la limite dans le schéma, nous démontrons qu'une suite de vitesses approchées (obtenue par une suite de discrétisations dont les pas d'espace et de temps tendent vers zéro) converge, à l'extraction d'une sous-suite près, vers une solution faible du problème continu. Si l'on se restreint au problème (linéaire) de Stokes, et en supposant de plus que la condition initiale de la vitesse est dans H^1, nous obtenons également une estimation sur la pression qui permet de montrer la convergence forte des pressions approchées.

Enfin nous étendons l'analyse aux écoulements incompressibles à masse volumique variable. Les inconnues de masse volumique sont situées dans les mailles primales (avec la pression) et l'équation de bilan de masse est discrétisée par un schéma volumes finis décentré amont, ce qui permet d'obtenir une estimation L^∞ sur la masse volumique; on démontre aussi les mêmes estimations *a priori* sur la vitesse volumique que dans le cas d'une masse volumique constante, ce qui permet d'établir l'existence d'une solution au schéma. Par des arguments de compacité et de passage à la limite, on montre alors la convergence du schéma.

Mots clés :

Equations de Navier-Stokes, Méthode de volume finis, Schéma MAC, Fluide incompressible, Ecoulements à masse volumique variable.

ABSTRACT

The Marker-And-Cell (MAC) scheme is a discretization scheme for partial derivative equations on Cartesian meshes, which is very well known in fluid mechanics. Here we are concerned with its mathematical analysis in the case of incompressible flows on two or three dimensional non-uniform Cartesian grids. In the MAC scheme, the discrete unknowns are the normal components of the velocity and the pressure at the center of the so-called "primal" cells, on which a natural discretization of the continuity equation is easy to obtain, the approximate solution is discrete divergence free. Each component of the momentum balance equation is discretized on the dual mesh associated to the same component of the velocity. In particular, the velocity convection operator is approximated so as to be compatible with a discrete continuity equation on the duals cells; this discretization coincides with the usual discretization on uniform meshes. We first discretize the steady-state incompressible Navier-Stokes equations. We show some *a priori* estimates that allow to show the existence of a solution to the scheme and some compactness and consistency results, in particular some new results for the velocity convection operator. By a passage to the limit on the scheme, we show that the approximate solutions obtained with the MAC scheme converge (up to a subsequence since no uniqueness result is known for the continuous problem) to a weak solution of the Navier-Stokes equations, thanks to a careful analysis of the nonlinear convection term. Then, we analyze the convergence of the unsteady-case Navier-Stokes equations. The algorithm is implicit in time. We first show that the scheme preserves the stability properties of the continuous problem ($L^2(H^1)$- and $L^\infty(L^2)$-estimates for the velocity), which yields, by a topological degree technique, the existence of a solution. Then, invoking compactness arguments and passing to the limit in the scheme, we prove that any sequence of solutions (obtained with a sequence of discretizations the space and time step of which tend to zero) converges up to the extraction of a subsequence to a weak solution of the continuous problem. If we restrict ourselves to the (linear) Stokes equations and assume that the initial velocity belongs to H^1, then we obtain estimates on the pressure and prove the convergence of the sequences of approximate pressures. Finally, we extend the analysis of the scheme to incompressible variable density flows. The density unknowns are located in the primal mesh (with the pressure) and the mass balance equation is discretized by an upstream finite volume scheme, which provides an L^∞ estimate on the density; we show the same it a priori estimates on the velocity as in the case of a constant density. We deduce the existence of a solution to the scheme. Again, by compactness arguments and passage to the limit in the scheme, we show the convergence of approximate solutions to an exact solution of the problem.

Key-words :

Navier-Stokes equations, Finite volume method, MAC scheme, Incompressible Fluid, Variable density flows.

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Chapter 1

INTRODUCTION GÉNÉRALE

La dynamique des fluides est un domaine actif de la recherche avec de nombreux problèmes non résolus ou partiellement résolus. La résolution d'un problème de dynamique des fluides demande normalement de calculer diverses propriétés des fluides comme la vitesse, la viscosité, la masse volumique, la pression et la température en tant que fonctions de l'espace et du temps. Un modèle fréquemment utilisé est celui donné par les équations de Navier-Stokes compressible ou incompressible. L'omniprésence de la mécanique des fluides dans la nature qui nous entoure, motive la nécessité de l'étude de ces équations. Au début des années trente du 20^e siècle, J.Leray a publié ses célèbres travaux [59], [60], [61], en particulier [61] sur les équations de Navier-Stokes incompressible. Il s'agit de valider mathématiquement la théorie de Navier-Stokes relative au mouvement d'un fluide visqueux. Le programme consiste à démontrer (ou à nier) l'existence et l'unicité globale de solutions pour le système de Navier-Stokes incompressible. Cet article indique la source d'inspiration, à savoir les deux articles de C.Oseen de 1911 et 1912 (voir [69] et [70]) et annonce des résultats révolutionnaires : l'existence de solutions globales dites turbulentes, c'est-à-dire très irrégulières. Ces résultats ont été le point de départ de très nombreux travaux de recherche dans le cadre des mathématiques actuelles, voir par exemple [63], [6]. Les solutions des équations de Navier-Stokes sont en général impossibles à calculer explicitement. Pour résoudre les problèmes intervenant en mécanique des fluides, on a donc recours à des approximations numériques. De nombreuses approches sont possibles pour résoudre numériquement les équations de Navier-Stokes. En ce qui concerne les schémas de discrétisation en espace, une méthode particulièrement populaire en mécanique des fluides numériques est le schéma dit “Marker-and-Cell”, qui, bien qu'il ait été introduit il y a 50 ans, n'a été que récemment analysé mathématiquement. L'objet de cette thèse est l'analyse mathématique du schéma “Marker-and-Cell” dans le cadre des écoulements incompressibles sur des maillages cartésiens non uniformes en dimension 2 ou 3.

1.1 Modélisation

Considérons un écoulement d'un fluide occupant un domaine $\Omega \subset \mathbb{R}^d, d = 2,3$ pendant un intervalle de temps $[0,T]$. Le fluide est caractérisé en tout point $\boldsymbol{x} \in \Omega$ et pour $t \in [0,T]$ par sa vitesse $\boldsymbol{u}(\boldsymbol{x},t)$, sa masse volumique $\rho(\boldsymbol{x},t)$ et sa pression $p(\boldsymbol{x},t)$. Les équations de Navier-Stokes sont obtenues en appliquant les lois de conservation à un volume élémentaire ω d'un fluide :

Conservation de la masse

$$\int_\omega \frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \boldsymbol{u}) = 0.$$

Conservation de la quantité de mouvement

$$\int_\omega \frac{\partial \rho \boldsymbol{u}}{\partial t} + \mathrm{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) - \int_{\partial\omega} \sigma \boldsymbol{n} = \int_\omega \rho \boldsymbol{f}$$

où $\partial\omega$ est le bord de ω, $\boldsymbol{n}$ la normale à $\partial\omega$ sortante de ω, σ est le tenseur des contraintes de l'écoulement et $\boldsymbol{f}$ est la masse volumique massique des forces appliquées au fluide.

Conservation de l'énergie

$$\int_\omega \frac{\partial \rho E}{\partial t} + \mathrm{div}(\rho E \boldsymbol{u}) - \int_{\partial\omega} (\phi - \sigma \boldsymbol{u}) \cdot \boldsymbol{n} = \int_\omega \rho \boldsymbol{f} \cdot \boldsymbol{u}$$

où ϕ est le flux de chaleur et E est l'énergie spécifique totale : $E = e + \frac{1}{2}|\boldsymbol{u}|^2$ où e est l'énergie spécifique interne et $|\cdot|$ désigne la norme euclidienne. En supposant que toutes ces fonctions de t et de $\boldsymbol{x}$ sont suffisamment régulières, on obtient en utilisant le théorème de divergence :

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \boldsymbol{u}) = 0 \\
&\frac{\partial \rho \boldsymbol{u}}{\partial t} + \mathrm{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) - \mathrm{div}\ \sigma = \rho \boldsymbol{f} \\
&\frac{\partial \rho E}{\partial t} + \mathrm{div}(\rho E \boldsymbol{u}) + \mathrm{div}(\phi - \sigma \boldsymbol{u}) = \rho \boldsymbol{f} \cdot \boldsymbol{u}
\end{aligned}$$

Supposons maintenant que le fluide est newtonien, c'est à dire qu'il existe deux réels λ et μ (appelés coefficients de Lamé), tels que :

$$\begin{aligned}
&\sigma = \tau - p\mathrm{Id} \\
&\tau = \lambda \mathrm{div}(\boldsymbol{u})\mathrm{Id} + 2\mu D(\boldsymbol{u})
\end{aligned}$$

où $D(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^t)$, p est la pression, τ le tenseur des contraintes visqueuses, Id la matrice identité et $D(\boldsymbol{u})$ est appelé le tenseur des déformations de l'écoulement. Enfin, en supposant que le fluide suit la loi de Fourier $\phi = -k\nabla T$ où k est la conductivité thermique, μ est constante et T la température, on obtient le système :

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{u}) = 0 \\
&\frac{\partial \rho \boldsymbol{u}}{\partial t} + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) - \mu \Delta \boldsymbol{u} + \nabla(p - (\lambda + \mu)\operatorname{div}(\boldsymbol{u})) = \rho \boldsymbol{f} \\
&\frac{\partial \rho E}{\partial t} + \operatorname{div}(\rho E \boldsymbol{u}) - \operatorname{div}(k \nabla T) - \operatorname{div}(\tau \boldsymbol{u}) + \operatorname{div}(p \boldsymbol{u}) = \rho \boldsymbol{f} \cdot \boldsymbol{u}
\end{aligned}
$$

que l'on doit compléter par une loi d'état pour relier toutes les grandeurs thermodynamiques, et des conditions aux limites et initiales. Dans toute la suite de cette thèse, nous allons nous intéresser à l'approximation numérique de modèles de fluides isothermes et incompressibles. Un fluide est dit incompressible si le volume occupé par toute partie du fluide se conserve dans son mouvement. Plus précisément, soit $y(\boldsymbol{x}, t)$ la position à l'instant t d'une particule située en $\boldsymbol{x}$ à l'instant 0, et soit $\omega(t) = \{\boldsymbol{y}(\boldsymbol{x}, t) : \boldsymbol{x} \in \omega(0)\}$ la région occupée à l'instant t par le fluide qui se trouvait dans la région $\omega(0)$ à l'instant 0. la vitesse $\boldsymbol{u}$ du fluide au point $\boldsymbol{y}(\boldsymbol{x}, t)$ à l'instant t est donnée par:

$$\boldsymbol{u}(\boldsymbol{y}(\boldsymbol{x}, t), t) = \frac{\partial \boldsymbol{y}}{\partial t}(\boldsymbol{x}, t).$$

On note $Y(t)$ l'application $\boldsymbol{x} \mapsto \boldsymbol{y}(\boldsymbol{x}, t)$; on a donc, $\omega(t) = Y(t)(\omega(0))$ et $Y(0) = \mathrm{Id}$. Si le volume $|\omega(t)|$ se conserve quand t varie, alors la divergence de la vitesse est nulle, ce qui s'énonce comme suit.

Theorem 1.1.1 (Condition d'incompressibilité [78, Théorème 1.1.]). *Soit Ω un ouvert de $\mathbb{R}^d$ et $T > 0$. Soit $Y \in C^1([0, T]; C^1(\Omega; \mathbb{R}^d))$ telle que, pour tout $t \in [0, T]$, $Y(t)$ soit une bijection de Ω sur son image $\Omega(t)$, d'inverse C^1, et pour tout boule ouverte $\omega(0) \subset \Omega$, on ait $|\omega(t)| = |\omega(0)|$. Alors*

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}(\boldsymbol{x}, t) = 0, \forall t \in [0, T] \quad et \quad \boldsymbol{x} \in \omega(t). \tag{1.1.1}$$

Dans le cas où le fluide est incompressible et isotherme, il n'y a plus d'équation d'énergie, et la pression p n'est plus définie par une loi d'état mais devient un multiplicateur de Lagrange associée à la contrainte de divergence nulle (1.1.1). Cette pression doit être considérée comme la fluctuation de la vraie pression thermodynamique autour de sa moyenne. Sous ces hypothèses, les équations de Navier-Stokes s'écrivent :

$$\partial_t \rho + \boldsymbol{u} \cdot \boldsymbol{\nabla} \rho = 0, \tag{1.1.2a}$$

$$\rho(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) - \mu \Delta \boldsymbol{u} + \boldsymbol{\nabla} p = \rho \boldsymbol{f}, \tag{1.1.2b}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0. \tag{1.1.2c}$$

On appelle souvent ces équations “les équations de Navier-Stokes pour un fluide incompressible non homogène” ou encore “les équations de Navier-Stokes incompressible à masse volumique variable”. Si maintenant le fluide est homogène, c.à.d. si sa masse volumique est indépendante de $\boldsymbol{x}$ l'équation de conservation de la masse se réduit à $\frac{\partial \rho}{\partial t} = 0$, et donc ρ est une constante. Le système (1.1.2) devient alors :

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \tag{1.1.3}$$

$$\rho(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) - \mu \Delta \boldsymbol{u} + \boldsymbol{\nabla} p = \rho \boldsymbol{f} \tag{1.1.4}$$

Pour obtenir les équations de Navier-Stokes incompressibles “standard”, il nous reste à adimensionner les équations (1.1.3) et (1.1.4). On fixe une échelle de temps t_0, une échelle d'espace l_0 et une taille caractéristique f_0 pour les forces appliquées à l'écoulement. On en déduit une vitesse caractéristique u_0 :

$$u_0 = \frac{l_0}{t_0}$$

Et on pose :

$$t^* = \frac{t}{t_0}, x^* = \frac{x}{l_0}, \boldsymbol{u}^* = \frac{\boldsymbol{u}}{u_0}, \boldsymbol{f}^* = \frac{\boldsymbol{f}}{f_0}, p^* = \frac{p}{\rho u_0^2}$$

On obtient alors en omettant les “*”

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

$$\rho(\frac{u_0}{t_0}\frac{\partial \boldsymbol{u}}{\partial t} + \frac{u_0^2}{l_0}(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) - \mu \frac{u_0}{l_0^2}\Delta \boldsymbol{u} + \frac{\rho u_0^2}{l_0}\boldsymbol{\nabla} p = \rho f_0 \boldsymbol{f}$$

i.e.

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \frac{\mu}{\rho u_0 l_0}\Delta \boldsymbol{u} + \boldsymbol{\nabla} p = \frac{f_0}{u_0/t_0}\boldsymbol{f}$$

Le système dépend des deux paramètres suivants :

- Le nombre de Reynolds :

$$\mathcal{R}e = \frac{\rho u_0 l_0}{\mu}$$

- Le nombre de Froude :

$$\mathcal{F}r = \frac{\rho u_0 / t_0}{f_0}$$

On obtient :

$$\begin{aligned} &\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \\ &\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \frac{1}{\mathcal{R}e}\Delta \boldsymbol{u} + \boldsymbol{\nabla} p = \frac{1}{\mathcal{F}r}\boldsymbol{f} \end{aligned}$$

En prenant $\mathcal{F}r = 1$, on obtient les équations de Navier-Stokes:

$$\begin{aligned} &\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \\ &\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \frac{1}{\mathcal{R}e}\Delta \boldsymbol{u} + \boldsymbol{\nabla} p = \boldsymbol{f} \end{aligned}$$

Si l'on impose une vitesse nulle au bord du domaine Ω d'écoulement, un fluide visqueux incompressible et homogène est donc décrit par les équations de Navier-Stokes suivantes :

- cas instationnaire

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \tag{1.1.5a}$$

$$\partial_t \boldsymbol{u} - \frac{1}{\mathcal{R}e}\Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \boldsymbol{\nabla} p = \boldsymbol{f}, \tag{1.1.5b}$$

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0, \tag{1.1.5c}$$

$$\boldsymbol{u} = 0, \quad \text{sur} \quad \partial\Omega. \tag{1.1.5d}$$

  où $\boldsymbol{u}_0$ est la condition initiale, donnée,

- cas stationnaire

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \tag{1.1.6a}$$

$$-\frac{1}{\mathcal{R}e}\Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \boldsymbol{\nabla} p = \boldsymbol{f}, \tag{1.1.6b}$$

$$\boldsymbol{u} = 0, \quad \text{sur} \quad \partial\Omega \tag{1.1.6c}$$

Dans le cas où le nombre de Reynolds $\mathcal{R}e$ est suffisamment petit, on peut négliger les effets non linéaires et les systèmes (1.1.5) et (1.1.6) se simplifient encore pour donner les équations de Stokes instationnaires

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1.1.7a}$$

$$\partial_t \boldsymbol{u} - \frac{1}{\mathcal{R}e}\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, \tag{1.1.7b}$$

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0, \tag{1.1.7c}$$

$$\boldsymbol{u} = 0, \quad \text{sur} \quad \partial\Omega. \tag{1.1.7d}$$

ou stationnaires

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1.1.8a}$$

$$-\frac{1}{\mathcal{R}e}\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, \tag{1.1.8b}$$

$$\boldsymbol{u} = 0, \quad \text{sur} \quad \partial\Omega \tag{1.1.8c}$$

1.2 Résultats d'existence pour les équations de Navier-Stokes

Nous rappelons ici les résultats d'existence pour les équations de Navier-Stokes incompressible. Comme nous avons dit au début de l'introduction, Jean Leray a introduit la notion de la solution faible (solution turbulente) des équations de Navier-Stokes incompressible. Il démontre l'existence et l'unicité d'une solution faible en dimension 2 pour un fluide occupant tout l'espace, et l'existence d'une solution faible en dimension 3. Il ne peut pas résoudre le problème de l'unicité en dimension 3, qui est toujours une question ouverte. Ces questions théoriques sont très bien décrites dans l'ouvrage [6]. Nous allons citer quelques théorèmes d'existence qu'on pourra trouver dans [77, 78] ou [6]. On définit les espaces $\mathcal{V}$, V, et H comme suit:

$$\mathcal{V} = \{\boldsymbol{v} \in (\mathcal{D}(\Omega)^d) : \nabla \cdot \boldsymbol{v} = 0\},$$
$$V \text{ la fermeture de } \mathcal{V} \text{ dans } (H^1(\Omega))^d,$$
$$H \text{ la fermeture de } \mathcal{V} \text{ dans } (L^2(\Omega))^d.$$

Cas de Navier-Stokes stationnaire

Theorem 1.2.1 (Théorème 6.4 et 6.5[78])**.** *On se donne* Ω *ouvert borné de* $\mathbb{R}^d$, $d \leq 4$ *et* $\boldsymbol{f} \in (H^{-1}(\Omega))^d$.

a)Il existe $\boldsymbol{u} \in V$, $p \in L^2_{loc}(\Omega)$, *telle que* $\boldsymbol{\nabla} p \in (H^{-1}(\Omega))^d$, *vérifiant*

$$-\frac{1}{\mathcal{R}e}\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{\nabla} p = \boldsymbol{f},$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \quad \boldsymbol{u} = 0 \, sur \, \partial\Omega.$$

b)Toute solution de la partie (a) vérifie

$$\frac{1}{\mathcal{R}e}\|\boldsymbol{u}\|_{H^1(\Omega)^d} \leq c(\Omega)\|\boldsymbol{f}\|_{H^{-1}(\Omega)^d}$$
$$\|\boldsymbol{\nabla} p\|_{H^{-1}(\Omega)^d} \leq c(\Omega)\|\boldsymbol{f}\|_{H^{-1}(\Omega)^d} + \big(\frac{c(\Omega)}{\frac{1}{\mathcal{R}e}}\|\boldsymbol{f}\|_{H^{-1}(\Omega)^d}\big)^2,$$

où $c(\Omega)$ *une constante ne dépendant que de* Ω.

Ce théorème nous donne l'existence (dans la partie (a)) d'une solution des équations de Navier-Stokes avec une vitesse nulle au bord, mais rien n'assure l'unicité. Il y a unicité des solutions assez petites et dans la partie (b) nous donne une estimation de toute solution.

Cas de Navier-Stokes instationnaire

Theorem 1.2.2 (Théorème 9.4[78]). *Soient* Ω *un ouvert borné de* $\mathbb{R}^d$, $d \leq 4$,
$\boldsymbol{f} \in L^2(0,T;(H^{-1}(\Omega))^d)$, *et* $\boldsymbol{u}_0 \in H$.
a)Il existe $\boldsymbol{u}$ *et* p *telles que :*
$\boldsymbol{u} \in L^2(0,T;V) \cap L^\infty(0,T;H) \cap C(0,T;L^2(\Omega)^d - faible)$,
$p \in W^{-1,\infty}(0,T,L^2_{loc}(\Omega))$, $\boldsymbol{\nabla} p \in (W^{-1,1}(]0,T[\times\Omega))^d$, *et*

$$\partial_t \boldsymbol{u} - \frac{1}{\mathcal{R}e}\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{\nabla} p = \boldsymbol{f},$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \boldsymbol{u}(\boldsymbol{x},0) = u_0.$$

b)Il existe un réel $c(\Omega)$ *et, pour chaque* u_0 *et* $\boldsymbol{f}$ *une solution qui vérifie de plus*

$$\sup_{0\leq t\leq T}\|\boldsymbol{u}\|_{L^2(\Omega)^d} \leq \|\boldsymbol{u}_0\|_{L^2(\Omega)^d} + \frac{c(\Omega)}{\sqrt{\frac{1}{\mathcal{R}e}}}\|\boldsymbol{f}\|_{L^2(0,T;H^{-1}(\Omega)^d)}.$$

Ce théorème nous donne l'existence (dans la partie (a)) d'une solution des équations de Navier-Stokes instationnaire, mais on a perdu l'unicité et la partie (b) nous donne une estimation sur la vitesse.

Cas de Navier-Stokes à masse volumique variable

l'existence d'une solution pour le Cas de Navier-Stokes non homogène a été démontrée par [77] voir aussi [5] pour d'autres comditions aux limites.

Theorem 1.2.3 (Théorème 9[77]). *Soient Ω un ouvert borné de $\mathbb{R}^d$, $d \leq 3$, avec une frontière lipschitzienne, $\boldsymbol{f} \in L^1(0,T;L^2(\Omega)^d)$, $u_0 \in H$, $\rho_0 \in L^\infty(\Omega)$ et $\rho_0 \geq 0$.*
a)Il existe

$$\boldsymbol{u} \in L^2(0,T;V), \rho \in L^\infty(\Omega\times]0,T[), p \in W^{1,\infty}(0,T,L^2(\Omega)).$$

telles que:

$$\begin{aligned}
&\inf_\Omega \rho_0 \leq \rho \leq \sup_\Omega \rho_0,\\
&\rho\boldsymbol{u} \in L^\infty(0,T;L^2(\Omega)^d) \cap L^2(0,T;W^{-1,3/2}(\Omega)^d),\\
&\rho \in C([0,T];W^{1,\infty}(\Omega)), \int_\Omega \rho\boldsymbol{u}\cdot\boldsymbol{v}\ \mathrm{d}\boldsymbol{x} \in C([0,T]), \quad \forall \boldsymbol{v} \in V,
\end{aligned}$$

vérifiant:

$$\begin{aligned}
&\frac{\partial\rho\boldsymbol{u}}{\delta t} + \boldsymbol{\nabla}\cdot(\boldsymbol{u}\rho\boldsymbol{u}) - \frac{1}{\mathfrak{Re}}\Delta\boldsymbol{u} = \rho\boldsymbol{f} - \boldsymbol{\nabla}p\\
&\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0,
\end{aligned}$$

et la conditions initiale faible

$$\begin{aligned}
&\rho(\boldsymbol{x},0) = \rho_0,\\
&\Big(\int_\Omega \rho\boldsymbol{u}\cdot\boldsymbol{v}\ \mathrm{d}\boldsymbol{x}\Big)(\boldsymbol{x},0) = \int_\Omega \rho_0\boldsymbol{u}_0\cdot\boldsymbol{v}\ \mathrm{d}\boldsymbol{x}, \quad \forall \boldsymbol{v} \in V.
\end{aligned}$$

Ce théorème nous donne l'existence de la solution des équations de Navier-Stokes à masse volumique variable en remplaçant la condition $\rho\boldsymbol{u}(\boldsymbol{x},0) = \rho_0\boldsymbol{u}_0$ par une condition faible. Cette condition ne peut pas être formulée de manière plus forte sans hypothèse de régularité supplémentaire, car il n'est pas possible de donner un sens à $\rho\boldsymbol{u}(\boldsymbol{x},0)$ ni à $\boldsymbol{u}(\boldsymbol{x},0)$. Notons que ce résultat d'existence a été généralisé par P.L. Lions au cas d'une viscosité dépendant de la masse volumique variable [63].

1.3 Approximation numérique des équations de Navier-Stokes

Les méthodes numériques utilisées pour l'approximation numérique des équations aux dérivées partielles sont nombreuses et variées: différences finies, éléments finis, volumes finis, Galerkine discontinu, méthodes spectrales... Les méthodes d'éléments finis, introduites dans les années 1950, ont suscité de nombreuses analyses mathématiques pour l'approximation d'équations aux dérivées partielles de divers types : voir par exemple les ouvrages [13, 79, 18] et leurs références, et plus particulièrement pour les équations de Navier-Stokes incompressibles [79]. L'analyse des méthodes d'élements finis pour la discrétisation des équations de Navier Stokes incompressibles a fait l'objet de plusieurs articles de Heywood et Rannacher [50, 51, 52, 53]. Les méthodes d'éléments finis mixtes ont elles aussi été largement étudiés, voir par exemple les ouvrages de référence [41, 3, 4], et l'article plus récent [54]. Les méthodes de volumes finis colocalisés , c. à. d. avec inconnues vitesse et pression situées dans les mailles, [65, 29, 57], sont aussi utilisées depuis longtemps pour les équations de Stokes et Navier-Stokes, même si leur analyse mathématique est relativement récente [28, 25, 27]. Leur inconvénient majeur est de ne pas être intrinsèquement stables, contrairement aux méthodes à maillage décalé. Encore plus récemment ont été introduites les méthodes de type Galerkine-discontinu ; celles-ci permettent l'approximation de solutions d'EDP peu régulières sur des maillages généraux, voir par exemple [16] pour une introduction générale et [15, 12] pour l'analyse dans le cadre des équations de Navier-Stokes incompressibles. Mais depuis des décennies, une des méthodes de discrétisation les plus utilisées dans la communauté de mécanique des fluides numérique est la méthode de discrétisation de dite "marker-and-cell" (MAC), développée par Harlow and Welsh au milieu des années 1960 [45]. Elle est une des méthodes les plus populaires [71, 72, 80] pour l'approximation des équations de Navier-Stokes, en raison de sa simplicité et de ses propriétés de stabilité remarquables. Même si elle est souvent présentée comme une méthode de différences finies [71], elle fait partie des méthodes de volumes finis, dans le sens où les équations que l'on discéretise sont les équations de bilan sur les mailles, et que ce sont les flux qui sont discrétisés par différences finies. La première analyse de l'estimation d'erreur est celle de [73] pour les équations de Stokes instationnaire sur un maillage carré. L'analyse mathématique du schéma dans le cas des équations de Stokes stationnaire est effectuée dans [67] pour un maillage rectangulaire uniforme avec une régularité H^2 sur la pression, en passant par une formulation vorticité-pression. Toujours dans le cas des équations de Stokes stationnaires, le schéma MAC a été reformulé de plusieurs manières possibles pour obtenir des estimations d'erreur : éléments finis mixtes [40, 44], Galerkin discontinu [56]. En utilisant les outils qui ont été développés pour la théorie des volumes finis [22, 23], une estimation d'erreur d'ordre 1 pour un maillage non uniforme a été obtenue dans [1], avec une convergence d'ordre 2 pour un maillage uniforme, sous la régularité usuelle (H^2 pour la vitesse, H^1 pour la pression). La convergence avec un second membre dans $H^{-1}(\Omega)$ (et sans hypothèse de régularité supplémentaire) a été démontrée dans [2]. Récemment, la superconvergence du schéma MAC pour les équations de Stokes stationnaires a été obtenue pour les maillages rectangulaires non uniformes [62], sous des hypothèses de régularité C^4 pour la vitesse et C^3 pour la pression et une hypothèse de convergence supplémentaire sur la pression. Les travaux sur l'analyse mathématique du schéma MAC pour les équations de Navier-Stokes sont beaucoup plus rares. Le premier résultat d'estimation d'erreur est obtenue par Nicolaides et Wu [68] pour un maillage rectangulaire uniforme, en utilisant une formulation vorticité pression. Plus récemment, la convergence du schéma MAC a été démontrée dans le cas stationnaire et instationnaire en dimension 2 ou 3 mais pour un schéma MAC modifié défini sur un maillage localement raffiné [8]. Pour les équations de Stokes sur un maillage uniforme, ce dernier schéma coïncide avec le schéma MAC original qui est classiquement utilisé dans les codes CFD. Toutefois, pour les équations de Navier-Stokes, le terme de convection non linéaire est discrétisé par éléments finis (voir e.g. [79]), qui ne coïncide pas avec la formulation classique du schéma MAC donnée par [71]. Cette discrétisation entraîne un stencil plus large. Des expériences numériques [9] semblent prouver qu'elle n'est pas aussi efficace que le schéma MAC classique. Au chapitre 2, nous effectuons l'analyse du schéma MAC classique pour les équations de Navier-Stokes stationnaire et instationnaires en variables primitives sur un maillage rectangulaire non uniforme en dimension deux ou trois, et, comme dans [8], sans aucune hypothèse de régularité sur la solution.

Dans le cas des écoulements non homogènes incompressibles, les analyses mathématiques de schémas de discrétisation sont encore plus rares : L'article [64] donne une analyse de convergence de la méthode de Galerkine discontinu avec une viscosité qui peut dépendre de la masse volumique. L'article [58] s'intéresse à l'analyse mathématique d'un schéma à mailles décalées, dans lequel l'approximation des termes de diffusion est effectué à l'aide des éléments finis de type Rannacher-Turek. Que ce soit pour Rannacher-Turek [58] ou pour le schéma MAC que nous étudions au chapitre 3, une caractéristique essentielle des schémas développés est que l'énergie cinétique (discrète) reste contrôlée : dans le même esprit que les travaux récents effectués pour la dicrétisation sur maillage décalé de l'opérateur de convection dans l'équation de quantité de mouvement, dans le cadre des équations de Navier-Stokes ou Euler compressible. Ces travaux ont été développés en particulier dans le but d'obtenir un schéma préservant l'équilibre de l'énergie cinétique [47, 39, 38]. D'une part la conservation de l'énergie cinétique permet d'obtenir des proprétés de stabilité du schéma; de fait, les essais numériques effectués à l'IRSN avec le code ISIS [55] montrent qu'on obtient ainsi des schémas particulièrement robustes. D'autre part, dans le cas des équations d'Euler, le fait de pouvoir obtenir un bilan d'énergie cinétique discrèt a été un point clé pour obtenir un schéma consistant en préservant l'ensemble convexe des états admissibles [49, 46]. La difficulté réside dans le fait que, comme dans le cas continu, l'obtention du bilan d'énergie cinétique s'obtient à partir du bilan de quantité de mouvement et du bilan de masse (utilisé deux fois) ; or le bilan de quantité de mouvement est écrit sur les cellules des maillages vitesse, tandis que d'une équation du bilan de masse soit convaincu sur les mêmes cellules (double), tandis que le bilan de masse est écrit sur les cellules du maillage pression. Comme dans [58], nous allons donc développer une procédure de calcul des termes de convection qui soit compatible avec un bilan de masse discret lui-même reconstruit sur les mailles vitesses à partir des bilans de masse "naturels" des mailles pression.

1.4 Plan des chapitres suivants

Dans le chapitre 2, nous introduisons le maillage MAC et les opérateurs discrets. Le schéma MAC est basé sur une grille rectangulaire (en 2D) ou paralllepipédique rectangle (en 3D) qu'on appelle "maillage primal" ou "maillage pression". Les inconnues discrètes sont les composantes normales de la vitesse aux faces du maillage, et la pression au centre des mailles. L'équation de continuité $\mathrm{div}\boldsymbol{u} = 0$ admet une discrétisation naturelle sur chaque maille pression. Le bilan discret de chaque composante de la quantité de mouvement est écrit sur les mailles duales attachées à la même composante de la vitesse. En particulier, le terme de convection non linéaire est discrétisé de manière à être compatible avec une équation de continuité discrète sur les mailles duales. Il coïncide avec la discrétisation habituellement utilisée en mécanique des fluides sur un maillage uniforme [71], contrairement au schéma de [8]. Les opérateurs de divergence discrète $\mathrm{div}_{\mathcal{M}}$ et du gradient discret $\boldsymbol{\nabla}_{\mathcal{E}}$ sont construits de manière à respecter une propriété de dualité qui est l'équivalent discret de la propriété

$$\int_\Omega \nabla q \cdot \boldsymbol{v} = -\int_\Omega q \; \mathrm{div}\boldsymbol{v}, \text{ si } \quad \boldsymbol{v}\cdot\boldsymbol{n} = 0, \text{ sur } \partial\Omega,$$

pour des champs scalaire q et vectoriel $\boldsymbol{v}$ suffisamment réguliers. Cette propriété de dualité est très importante pour l'étude de la stabilité du schéma et le passage à la limite dans le schéma. Le schéma MAC pour les équations de Navier-Stokes stationnaire s'écrit: Trouver $\boldsymbol{u}$ dans l'espace de vitesse discret $\mathbf{H}_{\mathcal{E},0}$ (muni de la norme H^1 discrète) et la pression dans l'espace de pression discret à moyenne nulle $L_{\mathcal{M},0}$ sur la maille primale qui vérifient

$$-\Delta_{\mathcal{E}}\boldsymbol{u} + \boldsymbol{C}_{\mathcal{E}}(\boldsymbol{u})\boldsymbol{u} + \boldsymbol{\nabla}_{\mathcal{E}}p = \boldsymbol{f}_{\mathcal{E}},$$

$$\mathrm{div}_{\mathcal{M}}\boldsymbol{u} = 0,$$

où $\Delta_{\mathcal{E}}$ est le Laplacien discret, $\boldsymbol{C}_{\mathcal{E}}(\boldsymbol{u})$ le terme de convection discret et $\boldsymbol{f}_{\mathcal{E}}$ est la valeur moyenne de $\boldsymbol{f}$ sur la maille duale.
Une formulation faible du schéma s'écrit :

$$\begin{aligned}
&\text{Trouver } (\boldsymbol{u}, p) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M},0} \text{ et, } \forall (\boldsymbol{v}, q) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M}},\\
&\quad \int_\Omega \nabla_{\mathcal{E}}\boldsymbol{u} : \nabla_{\mathcal{E}}\boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + b_{\mathcal{E}}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) - \int_\Omega p \,\mathrm{div}_{\mathcal{M}}(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathcal{P}_{\mathcal{E}}\boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x},\\
&\quad \int_\Omega \mathrm{div}_{\mathcal{M}}\boldsymbol{u}\, q \,\mathrm{d}\boldsymbol{x} = 0,
\end{aligned}$$

où $L_{\mathcal{M}}$ est l'espace de pression discret et $b_{\mathcal{E}}$ est la forme faible du terme non linéaire discret définie ainsi:

$$\begin{aligned}
\forall (\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \in {\mathbf{H}_{\mathcal{E},0}}^3, \qquad & b_{\mathcal{E}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \sum_{i=1}^{d} b_{\mathcal{E}}^{(i)}(\boldsymbol{u}, v_i, w_i),\\
\text{where for } i = 1,\ldots,d, \quad & b_{\mathcal{E}}^{(i)}(\boldsymbol{u}, v_i, w_i) = \int_\Omega C_{\mathcal{E}}^{(i)}(\boldsymbol{u}) v_i \; w_i \,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Pour obtenir les estimations a priori sur la vitesse et la pression on aura besoin d'étudier la forme faible du terme non linéaire discret $b_{\mathcal{E}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$. On démontre l'estimation suivante :

$$\forall (\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \in \boldsymbol{E}_{\mathcal{E}} \times {\mathbf{H}_{\mathcal{E},0}}^2, \qquad |b_{\mathcal{E}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \le C_{\eta_{\mathcal{M}}} \;\|\boldsymbol{u}\|_{1,\mathcal{E},0}\; \|\boldsymbol{v}\|_{1,\mathcal{E},0}\|\boldsymbol{w}\|_{1,\mathcal{E},0},$$

où $\boldsymbol{E}_{\mathcal{E}}$ est l'espace de vitesse discret à divergence discrète nulle et $C_{\eta_{\mathcal{M}}}$ ne dépendant que de la régularité du maillage (définie dans le chapitre 2). De plus, la forme trilinéaire $b_{\mathcal{E}}$ est nativement antisymétrique :

$$b_{\mathcal{E}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = -b_{\mathcal{E}}(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}), \quad \forall \boldsymbol{u} \in \boldsymbol{E}_{\mathcal{E}}$$

et donc en particulier,

$$b_{\mathcal{E}}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{u}) = 0, \quad \forall \boldsymbol{u} \in \boldsymbol{E}_{\mathcal{E}}.$$

En utilisant ce dernier résultat, on prend $\forall \boldsymbol{u} \in \boldsymbol{E}_{\mathcal{E}}$ comme une fonction test dans la forme faible du schéma et grâce à la dualité, on obtient une estimation sur la vitesse:

$$\|\boldsymbol{u}\|_{1,\mathcal{E},0} \leq \text{diam}(\Omega)\|\boldsymbol{f}\|_{L^2(\Omega)^d}.$$

Pour estimer la pression discrète on remarque d'abord que le schéma MAC est Inf-sup stable (comme cela a été remarqué dans l'article [76]) puis on prend l'interpolée de Fortin (défini dans le Lemme 2.3.3 au chapitre 2) d'une fonction régulière comme fonction test dans la forme faible du schéma, on obtient:

$$\|p\|_{L^2} \leq C_{\eta_{\mathcal{M}}}\|\boldsymbol{f}\|_{L^2(\Omega)^d}.$$

Ces estimations a priori sur les suites de vitesses et pressions approchées nous permettent d'une part d'établir l'existence d'une solution du schéma numérique en appliquant le théorème du degré topologique, et d'autre part de montrer la convergence de ces suites (à une sous suite près) vers une solution faible du problème continu lorsque le pas de discrétisation tend vers 0 pour des normes appropriées. Ce résultat est l'objet du théorème 2.3.8 dont la preuve nécessite une analyse fine du terme de convection non linéaire dont la consistance est donnée dans le Lemme 2.3.6. Dans le cas stationnaire on obtient une convergence forte de la vitesse discrète dans $L^2(\Omega)^d$ grâce à l'estimation sur la vitesse. De plus, l'estimation sur les translations de la vitesse [22, Theorem 14.2] donne la régularité de la limite dans $H_0^1(\Omega)^d$. Le fait que le maillage MAC est cartésien permet d'avoir une convergence forte du gradient discret de la vitesse, ce qui facilite le passage à la limite sur le terme de convection non linéaire. Grâce à une estimation L^2 sur la pression, on obtient la convergence faible (à sous-suite près) des suites de pression approchées. En prenant à nouveau l'interpolée de Fortin d'une fonction régulière comme fonction test dans la forme faible du schéma, on obtient que la suite des normes L^2 des pressions approchées tend vers la norme de la pression limite, ce qui donne donc la convergence forte de la pression. Nous étudions ensuite la discrétisation des équations de Navier-Stokes instationnaire par un schéma implicite en temps. Nous considérons pour cela une partition uniforme $0 = t_0 < t_1 < \cdots < t_N = T$ de l'intervalle de temps $(0, T)$ avec un pas temps $\delta t = t_{n+1} - t_n \ : t_n = n\ \delta t$ pour $n \in \{0, \cdots, N-1\}$. Le champ de vitesse $\boldsymbol{u}(\cdot, t_{n+1})$ au temps t_{n+1} est approchée par une fonction $\boldsymbol{u}^{(n+1)}$ constante par mailles duales en espace. De même, la pression $p(\boldsymbol{x}, t_{n+1})$ au temps t_{n+1} est approchées par une fonction $p^{(n+1)}$ constante par mailles primales en espace. Le schéma MAC implicite en temps pour les équations de Navier-Stokes instationnaire s'écrit:

Initialization

$$\boldsymbol{u}^{(0)} = \widetilde{\mathcal{P}}_{\mathcal{E}} \boldsymbol{u}_0$$

Étape $n \geq 0$. Résoudre pour $\boldsymbol{u}^{(n+1)}$ et $p^{(n+1)}$:

$$\boldsymbol{u}^{(n+1)} \in \mathbf{H}_{\mathcal{E},0},\ p^{(n+1)} \in L_{\mathcal{M},0},$$

$$\eth_t \boldsymbol{u}^{(n+1)} - \Delta_{\mathcal{E}} \boldsymbol{u}^{(n+1)} + \boldsymbol{C}_{\mathcal{E}}(\boldsymbol{u}^{(n+1)})\boldsymbol{u}^{(n+1)} + \boldsymbol{\nabla}_{\mathcal{E}}\ p^{(n+1)} = \boldsymbol{f}^{(n+1)},$$

$$\mathrm{div}_{\mathcal{M}} \boldsymbol{u}^{(n+1)} = 0,$$

où $\widetilde{\mathcal{P}}_{\mathcal{E}} \boldsymbol{u}_0$ est la valeur moyenne de la vitesse initiale sur la face σ ($\sigma \in \mathcal{E}^{(i)}$, $\mathcal{E}^{(i)}$ est l'ensemble des faces du maillage primal qui sont orthogonales à la i ème composante du vecteur de base canonique $\boldsymbol{e}_i$) et $\eth_t \boldsymbol{u}^{(n+1)}$ est la dérivée discrète en temps de la vitesse.

Nous démontrons que le schéma préserve les propriétés de stabilité du problème continu. On a une estimation $L^2(0,T;\mathbf{H}_{\mathcal{E},0})$ et $L^\infty(0,T;L^2(\Omega)^d)$ sur la vitesse discrète et une estimation $L^{4/3}(0,T;\boldsymbol{E}'_{\mathcal{E}})$ sur la dérivée discrète en temps de la vitesse très importante pour démontrer la compacité:

$$\|\eth_t \boldsymbol{u}\|_{L^{4/3}(0,T;\boldsymbol{E}'_{\mathcal{E}})} \leq C,$$

où $\|\eth_t \boldsymbol{u}\|_{L^{4/3}(0,T;\boldsymbol{E}'_{\mathcal{E}})} = \left(\int_0^T \|\eth_t \boldsymbol{u}\|_{\boldsymbol{E}'_{\mathcal{E}}}^{4/3}\ \mathrm{d}t\right)^{3/4}$ et $\|\boldsymbol{v}\|_{\boldsymbol{E}'_{\mathcal{E}}} = \max\{\left|\int_\Omega \boldsymbol{v}\cdot\boldsymbol{\varphi}\ \mathrm{d}\boldsymbol{x}\right|\ ;\ \boldsymbol{\varphi} \in \boldsymbol{E}_{\mathcal{E}}, \|\boldsymbol{\varphi}\|_{1,\mathcal{E},0} \leq 1\}$, et $C \geq 0$ ne dépend que de la vitesse initiale, Ω, $\eta_{\mathcal{M}}$ et $\boldsymbol{f}$. Nous démontrons ensuite la convergence du schéma. D'abord nous prouvons la compacité de suites de solutions approchées grâce au théorème d'Aubin-Simon discret que l'on rappelle dans l'annexe. On démontre la convergence dans $L^2(\Omega\times(0,T))$ et la régularité de la limite : $\bar{\boldsymbol{u}} \in L^2(0,T;\boldsymbol{E}(\Omega))$ et $\partial_t \bar{\boldsymbol{u}} \in L^{4/3}(0,T;\boldsymbol{E}'(\Omega))$, où $\boldsymbol{E}(\Omega)$ est l'espace des fonctions $H_0^1(\Omega)^d$ à divergence nulle et $\boldsymbol{E}'(\Omega)$ est son dual. Nous concluons alors la convergence de ces suites vers une solution faible du problème continu. Dans le cas des équations (linéaires) de Stokes instationnaire, nous pouvons obtenir une estimation sur la pression qui permet d'avoir la compacité des suite des pressions approchées. En effet, en utilisant le lemme de Necas, on peut construire par interpolation de Fortin une fonction $\boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0}$ telle que $\mathrm{div}_{\mathcal{M}} \boldsymbol{v} = p$ et dont la norme H^1 discrète soit contrôlée par la norme L^2 de la pression. On peut alors obtenir l'estimation

$$\|p\|_{L^2(0,T;L^2(\Omega))} \leq C,$$

où C ne dépend que de Ω, $\eta_{\mathcal{M}}$ et $\boldsymbol{f}$, à condition d'avoir supposé la régularité $H_0^1(\Omega)^d$ de la condition initiale de la vitesse, qui est nécessaire pour obtenir une estimation $L^2(0,T;\,L^2(\Omega)^d)$ sur la dérivée discrète en temps de la vitesse. Donc nous pouvons conclure que la pression approchée converge vers une solution faible des équations de Stokes avec une donnée H^1 sur la condition initiale de la vitesse.

Dans le chapitre 3, nous étendons l'analyse aux écoulements incompressibles à masse volumique variable. La masse volumique $\rho(\boldsymbol{x}, t_{n+1})$ au temps t_{n+1} est approchée par une fonction $\rho^{(n+1)}$ constante par mailles primales en espace. Le schéma implicite semi–discrétisé en temps pour les équations de Navier-Stokes incompressible instationnaire inhomogène s'écrit:

$$\begin{aligned}
&\frac{1}{\delta t}(\rho^{(n+1)} - \rho^{(n)}) + \mathrm{div}(\rho^{(n+1)}\boldsymbol{u}^{(n+1)}) = 0,\\
&\frac{1}{\delta t}(\rho^{(n+1)}\boldsymbol{u}^{(n+1)} - \rho^{(n)}\boldsymbol{u}^{(n)}) + \mathrm{div}(\rho^{(n+1)}\boldsymbol{u}^{(n+1)} \otimes \boldsymbol{u}^{(n+1)}) - \Delta\boldsymbol{u}^{(n+1)} + \boldsymbol{\nabla} p^{(n+1)} = 0,\\
&\mathrm{div}\boldsymbol{u}^{(n+1)} = 0.
\end{aligned}$$

Comme nous l'avons mentionné plus haut, le terme de convection non linéaire est discrétisé de manière à être compatible avec une équation de continuité discrète sur les mailles duales ; l'équation de bilan de masse est discrétisée par un schéma volumes finis décentré amont, ce qui permet d'obtenir une estimation L^∞ sur la masse volumique: on obtient également une inégalité BV faible classique en hyperbolique, [7, 22], que l'on l'utilise pour démontrer la convergence du schéma. Puis, nous démontrons des estimations semblable à celle obtenues dans le cas d'un fluide homogène pour la vitesse discrète. En particulier, le bilan d'énergie cinétique discret donne des estimations sur la vitesse discrète $L^2(0,T;\mathbf{H}_{\mathcal{E},0})$ et $L^\infty(0,T;L^2(\Omega)^d)$, qui par un argument de degré topologique entraîne l'existence de la solution du schéma. On obtient une estimation $L^{4/3}(0,T;\boldsymbol{E}'_{\rho,\mathcal{E}})$ sur la dérivée discrète en temps

$$\|\eth_t\boldsymbol{u}\|_{L^{4/3}(0,T;\boldsymbol{E}'_{\rho,\mathcal{E}})} \leq C.$$

où $\|\eth_t\boldsymbol{u}\|_{L^{4/3}(0,T;\boldsymbol{E}'_{\rho,\mathcal{E}})} = \Big(\int_0^T \|\eth_t\boldsymbol{u}\|^{4/3}_{\boldsymbol{E}'_{\rho,\mathcal{E}}}\ \mathrm{d}t\Big)^{3/4}$ et $\|\boldsymbol{v}\|_{\boldsymbol{E}'_{\rho,\mathcal{E}}} = \max\ \Big\{\Big|\int_\Omega \rho\boldsymbol{v}\cdot\boldsymbol{w}\ \mathrm{d}\boldsymbol{x}\Big|\ ;\ \boldsymbol{w} \in \boldsymbol{E}_{\mathcal{E}}$ and $\|\boldsymbol{w}\|_{1,\mathcal{E},0} \leq 1\Big\}$, et $C \geq 0$ ne dépend que de la vitesse initiale, Ω, $\eta_{\mathcal{M}}$, $\rho_{\max}$ et $\boldsymbol{f}$. Nous démontrons la convergence du schéma, on prouve la compacité en utilisant [32, Proposition 4.47 and Theorem 4.53] que nous rappelons dans l'annexe dans la proposition 4.0.7. On démontre la compacité de suites de la vitesse dans $L^2(\Omega \times (0,T))$ grâce à l'estimation sur la translation :

$$\|\boldsymbol{u}_m(\cdot,\cdot+\tau) - \boldsymbol{u}_m\|_{L^2(0,T;L^2(\Omega))} \leq C(\tau^{1/2} + \delta t)$$

puis on prouve la régularité de la limite : $\bar{\boldsymbol{u}} \in L^2(0,T;\boldsymbol{E}(\Omega))$. La suite de masse volumique approchée converge vers $\bar{\rho}$ in $\in L^2(0,T;L^2(\Omega))$. Nous prouvons que toute suite de solutions discrètes (obtenue par une suite de discrétisations dont les pas d'espace et de temps tendent vers zéro) converge, à l'extraction d'une sous suite près, vers une solution faible du problème continu. Nous faisons le passage à la limite dans le schéma. Pour l'équation de la masse discrète; on utilise la convergence faible sur la masse volumique, la convergence forte sur la vitesse et l'inégalité BV faible pour estimer les termes des restes. Pour le passage à la limite dans l'équation de la quantité du mouvement, la seule difficulté est dans le terme de convection non linéaire. On introduit un autre terme $b_{\mathcal{M}}$ dans le lemme 3.5.1 dans le chapitre 2. Il est défini dans le maillage primal en fonction des inconnues w_K et $\hat{v}_\sigma$ qui sont déjà utilisées dans le chapitre 2 dans la démonstration de la consistance du terme non linéaire discret pour les équations de Navier-Stokes instationnaire. Dans le lemme 3.5.1 on contrôle l'erreur entre $b_{\mathcal{E}}$ et $b_{\mathcal{M}}$ puis on fait le passage à la limite sur le terme $b_{\mathcal{M}}$ en utilisant la convergence forte de la masse volumique.

1.5 Perspectives

Les schémas considérés dans cette thèse sont totalement implicites ; leur mise en œuvre implique la résolution d'un système entièrement non-linéaire couplé. Par conséquent, l'utilisation de ce schéma semble être difficile dans un contexte réel de calcul, principalement en raison du coût de calcul et du manque de robustesse. Dans ce qui suit, nous décrivons trois autres discrétisations en temps possibles, qui produisent des systèmes plus faciles à résoudre grâce à un découplage partiel des équations discrètes ; pour chacune d'entre elles, nous donnons les conditions dans lesquelles ces schémas satisfont les estimations de stabilité semblables à celles respectées par le schéma implicite, ce qui permet par la suite d'étendre le résultat de la convergence. Nous donnons les algorithmes de temps correspondant, en gardant la même discrétisation spatiale MAC, explicitée dans les chapitres suivants. La première discrétisation en temps est obtenue par un traitement explicite de la vitesse convective dans l'équation de bilan de masse:

$$\begin{aligned}
&\frac{1}{\delta t}(\rho^{(n+1)} - \rho^{(n)}) + \operatorname{div}(\rho^{(n+1)}\boldsymbol{u}^{(n)}) = 0,\\
&\frac{1}{\delta t}(\rho^{(n+1)}\boldsymbol{u}^{(n+1)} - \rho^{(n)}\boldsymbol{u}^{(n)}) + \operatorname{div}(\rho^{(n+1)}\boldsymbol{u}^{(n)} \otimes \boldsymbol{u}^{(n+1)}) - \Delta\boldsymbol{u}^{(n+1)} + \boldsymbol{\nabla} p^{(n+1)} = 0,\\
&\operatorname{div}\boldsymbol{u}^{(n+1)} = 0.
\end{aligned}$$

Ce schéma satisfait les mêmes estimations sur la masse volumique et la vitesse que pour le cas implicite, sans aucune restriction sur le pas de temps et l'analyse de convergence reste encore valable dans ce cas. L'avantage d'une telle discrétisation provient du découplage des équations de bilan de masse et de quantité de mouvement. dans un contexte de calcul, la difficulté est maintenant réduite à calculer la solution du système linéaire associé aux équations de Navier-Stokes linéarisées, par exemple par les méthodes de Newton ou quasi-Newton, les méthodes de type SIMPLE, les méthodes de lagrangien augmenté... Un autre schéma intéressant dans le cas d'une viscosité faible est le schéma dćouplé suivant, où la pression est traitée de façon implicite (ce qui est obligatoire pour des raisons de stabilité):

$$\frac{1}{\delta t}(\rho^{(n+1)} - \rho^{(n)}) + \mathrm{div}(\rho^{(n+1)}\boldsymbol{u}^{(n)}) = 0, \tag{1.5.1}$$

$$\frac{1}{\delta t}(\rho^{(n+1)}\boldsymbol{u}^{(n+1)} - \rho^{(n)}\boldsymbol{u}^{(n)}) + \mathrm{div}(\rho^{(n+1)}\boldsymbol{u}^{(n)} \otimes \boldsymbol{u}^{(n)}_{upw}) - \mu\Delta\boldsymbol{u}^{(n)} + \boldsymbol{\nabla} p^{(n+1)} = 0 \tag{1.5.2}$$

$$\mathrm{div}\boldsymbol{u}^{(n+1)} = 0. \tag{1.5.3}$$

Dans ce schéma, l'équation de la masse est résolue indépendamment de l'équation de la quantité de mouvement. et la solution est obtenue par une résolution d'un problème elliptique sur la pression p^n qui n'est pas difficile dans le contexte de calcul. En effet, en prenant la divergence discrète de (1.5.2)) et en utilisant (1.5.3), on obtient une relation de la forme $-\boldsymbol{\nabla} p^{(n+1)} = F^{(n)}$. La stabilité L^2 de ce schéma (*i.e.* l'équation d'énergie cinétique discrète) est vérifiée sous une condition de CFL sur le pas de temps de la forme $\delta t \leq c(h + \frac{h^2}{\mu})$ [49], à condition d'utiliser un choix upwind dans le terme de convection de l'équation de la quantité de mouvement (alors qu'on a utilisé un choix centré dans le cadre de cette thèse). L'analyse de convergence est semblable au cas du schéma implicite, avec une petite différence en raison de ce choix upwind. Un schéma très intéressant en termes de calcul est le schéma de correction pression suivant, inspiré des méthodes de type prédiction-correction [10, 43]. Il est possible de résoudre l'étape de correction (1.5.6)-(1.5.7) en résolvant un problème elliptique en pression.

$$\frac{1}{\delta t}(\rho^{(n+1)} - \rho^{(n)}) + \mathrm{div}(\rho^{(n+1)}\boldsymbol{u}^{(n)}) = 0. \tag{1.5.4}$$

Etape de prédiction :

$$\frac{1}{\delta t}(\rho^{(n+1)}\tilde{\boldsymbol{u}}^{(n+1)} - \rho^{(n)}\boldsymbol{u}^{(n)}) + \mathrm{div}(\rho^{(n+1)}\boldsymbol{u}^{(n)} \otimes \tilde{\boldsymbol{u}}^{(n+1)}) - \Delta\tilde{\boldsymbol{u}}^{(n+1)} + (\frac{\rho^{(n+1)}}{\rho^{(n)}})^{\frac{1}{2}}\boldsymbol{\nabla} p^{(n)} = 0 \tag{1.5.5}$$

Etape de correction (projection)

$$\frac{1}{\delta t}\rho^{(n+1)}(\boldsymbol{u}^{(n+1)} - \tilde{\boldsymbol{u}}^{(n+1)}) + \boldsymbol{\nabla} p^{(n+1)} - (\frac{\rho^{(n+1)}}{\rho^{(n)}})^{\frac{1}{2}}\boldsymbol{\nabla} p^{(n)} = 0 \tag{1.5.6}$$

$$\mathrm{div}\boldsymbol{u}^{(n+1)} = 0 \tag{1.5.7}$$

Ce schéma est inconditionnellement stable-L^2, *i.e.* sans aucune restriction sur le pas de temps. Plus précisément, grâce à la pondération du gradient par le quotient $(\rho^{(n+1)}/\rho^{(n)})^{\frac{1}{2}}$ dans (1.5.5), il est possible de prouver qu'au début et à la fin de cette étape, les vitesses $(\boldsymbol{u}^{(n+1)})_{0\leq n\leq N}$ sont contrôlées par la norme discrète $L^\infty(L^2)$, tandis que les vitesses intermédiaires $(\tilde{\boldsymbol{u}}^{(n+1)})_{0\leq n\leq N}$ sont contrôlées par la norme discrète $L^2(H^1_0)$ grâce au terme de diffusion (voir [46] pour un calcul similaire dans le cas des équations d'Euler compressible). Une conséquence de ces différentes estimations sur $\tilde{\boldsymbol{u}}$ et $\boldsymbol{u}$ est que le terme de convection $\mathrm{div}(\rho^{(n+1)}\boldsymbol{u}^{(n)} \otimes \tilde{\boldsymbol{u}}^{(n+1)})$ ne peut plus être contrôlé comme dans le cas implicite. La dérivée en temps de la vitesse $\partial_t\boldsymbol{u}^{(n+1)}$ est contrôlée par la norme discrète $L^1(W^{-1,p})$ où $p > 2$. Malheureusement, cela ne suffit pas pour appliquer le lemme d'Aubin-Simon discret (voir Annexe, Théorème 4.0.8) pour démontrer la compacité. La convergence de ce schéma est une question ouverte. On note cependant que dans le cas de masse volumique constante, les estimations d'erreur sont obtenues lorsqu'on suppose que la solution du problème continu est régulière (voir [74, 75, 43]). Enfin, notons que la convergence du schéma MAC a été prouvée pour les équations de Stokes stationnaire [20]. Des travaux sont en cours concernant la convergene du schéma implicite en temps pour les équations de Navier-Stokes stationnaire [36] et semi-stationnaires [34].

Chapter 2

CONVERGENCE OF THE MAC SCHEME FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

Abstract. This chapter is a submitted paper written in collaboration with T.Gallouët, R.Herbin and J-C. Latché [33]. Section 2.3 concerns the steady case is; already published in FVCA7 [48]. We prove in this paper the convergence of the Marker and cell (MAC) scheme for the discretization of the steady-state and unsteady-state incompressible Navier-Stokes equations in primitive variables on non-uniform Cartesian grids, without any regularity assumption on the solution. *A priori* estimates on solutions to the scheme are proven ; they yield the existence of discrete solutions and the compactness of sequences of solutions obtained with family of meshes the space step of which tends to zero. We then establish that the limit is a weak solution to the continuous problem.

2.1 Introduction

Let Ω be an open bounded domain of $\mathbb{R}^d$ with $d = 2$ or $d = 3$. We consider the steady-state incompressible Navier-Stokes equations, which read:

$$\mathrm{div}\bar{\boldsymbol{u}} = 0, \qquad \text{in } \Omega, \tag{2.1.1a}$$

$$-\Delta\bar{\boldsymbol{u}} + (\bar{\boldsymbol{u}} \cdot \nabla)\bar{\boldsymbol{u}} + \boldsymbol{\nabla}\bar{p} = \boldsymbol{f}, \qquad \text{in } \Omega, \tag{2.1.1b}$$

$$\bar{\boldsymbol{u}} = 0, \qquad \text{on } \partial\Omega. \tag{2.1.1c}$$

where $\bar{\boldsymbol{u}}$ stands for the (vector-valued) velocity of the flow, $\bar{p}$ for the pressure and $\boldsymbol{f}$ is a given field of $L^2(\Omega)^d$, and where for two given vector fields $\boldsymbol{v} = (v_1, \dots, v_d)$ and $\boldsymbol{w} = (w_1, \dots, w_d)$, the quantity $(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}$ is a vector field whose components are $((\boldsymbol{v}\cdot\nabla)\boldsymbol{w})_i = \sum_{k=1}^d v_k \partial_k w_i$, $i = 1, \dots, d$.

A weak formulation of Problem (2.1.1) reads:

$$\text{Find } (\bar{\boldsymbol{u}},\bar{p}) \in H_0^1(\Omega)^d \times L_0^2(\Omega) \ \text{ such that, } \ \forall (\boldsymbol{v},q) \in H_0^1(\Omega)^d \times L_0^2(\Omega),$$

$$\int_\Omega \nabla\bar{\boldsymbol{u}} : \nabla \boldsymbol{v} \ \mathrm{d}\boldsymbol{x} + \int_\Omega ((\bar{\boldsymbol{u}}\cdot\nabla)\boldsymbol{u})\cdot\boldsymbol{v} \ \mathrm{d}\boldsymbol{x} - \int_\Omega \bar{p}\,\mathrm{div}\boldsymbol{v} \ \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \ \mathrm{d}\boldsymbol{x}, \tag{2.1.2a}$$

$$\int_\Omega q \ \mathrm{div}\bar{\boldsymbol{u}} \ \mathrm{d}\boldsymbol{x} = 0, \tag{2.1.2b}$$

where $L_0^2(\Omega)$ stands for the subspace of $L^2(\Omega)$ of zero mean-valued functions.
We shall consider the transient Navier-Stokes equations:

$$\mathrm{div}\bar{\boldsymbol{u}} = 0 \qquad \text{in } \Omega\times(0,T), \tag{2.1.3a}$$

$$\partial_t\bar{\boldsymbol{u}} - \Delta\bar{\boldsymbol{u}} + (\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}} + \boldsymbol{\nabla}\bar{p} = \boldsymbol{f} \qquad \text{in } \ \Omega\times(0,T), \tag{2.1.3b}$$

$$\bar{\boldsymbol{u}} = 0 \qquad \text{on } \ \partial\Omega\times(0,T), \tag{2.1.3c}$$

$$\bar{\boldsymbol{u}}(\boldsymbol{x},0) = \boldsymbol{u}_0. \qquad \text{in } \Omega. \tag{2.1.3d}$$

This problem is posed for $(\boldsymbol{x},t)$ in $\Omega\times(0,T)$ where $T\in\mathbb{R}_+^*$ and Ω is an open bounded domain of $\mathbb{R}^d; d = 2$ or 3, $\bar{\boldsymbol{u}}$ stands for the (vector-valued) velocity of the flow, $\bar{p}$ for the pressure, $\boldsymbol{f}$ is a given vector field of $L^2(\Omega\times(0,T))^d$ and $\boldsymbol{u}_0 \in L^2(\Omega)$. Denoting by $\boldsymbol{E}(\Omega) = \{\boldsymbol{u}\in H_0^1(\Omega)^d \ ; \ \mathrm{div}\ \boldsymbol{u} = 0,\ a.e. \text{ in } \Omega\}$ the set of divergence free functions, we consider the following weak formulation of the transient problem (2.1.3) (see e.g. [6]).

$$\text{Find } \boldsymbol{u}\in L^2(0,T;\boldsymbol{E}(\Omega))\cap L^\infty(0,T;L^2(\Omega)^d) \ ; \text{ such that, } \forall \boldsymbol{v}\in L^2(0,T;\boldsymbol{E}(\Omega))\cap C_c^\infty(\Omega\times[0,T))$$
$$-\int_0^T\int_\Omega \bar{\boldsymbol{u}}(\boldsymbol{x},t)\cdot\partial_t\boldsymbol{v}(\boldsymbol{x},t)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t - \int_\Omega \boldsymbol{u}_0(\boldsymbol{x})\cdot\boldsymbol{v}(\boldsymbol{x},0)\ \mathrm{d}\boldsymbol{x} + \int_0^T\int_\Omega \nabla\bar{\boldsymbol{u}}(\boldsymbol{x},t) : \nabla\boldsymbol{v}(\boldsymbol{x},t)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t$$
$$+\int_0^T\int_\Omega ((\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}})(\boldsymbol{x},t)\cdot\boldsymbol{v}(\boldsymbol{x},t)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t = \int_0^T\int_\Omega \boldsymbol{f}(\boldsymbol{x},t)\cdot\boldsymbol{v}(\boldsymbol{x},t)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t. \tag{2.1.4}$$

The aim of this paper is to show, under minimal regularity assumptions on the solution, that sequences of approximate solutions obtained by the discretization of problem (2.1.1)(resp. (2.1.3)) by the Marker-And-cell (MAC) scheme converge to a solution of (2.1.2)(resp. (2.1.4)) as the mesh size tends to 0.
The Marker-And-Cell (MAC) scheme, introduced in the middle of the sixties [45], is one of the most popular methods [71, 80] for the approximation of the Navier-Stokes equations in the engineering framework, because of its simplicity, its efficiency and its remarkable mathematical properties. The first error analysis seems to be that of [73] in the case of the time-dependent Stokes equations on uniform square grids. The mathematical analysis of the scheme was per-

formed for the steady-state Stokes equations in [68] for uniform rectangular meshes with H^2 regularity assumption on the pressure. Error estimates for the MAC scheme applied to the Stokes equations have been obtained by viewing the MAC scheme as a mixed finite element method [40, 44] or a divergence conforming DG method [56]. Error estimates for rectangular meshes were also obtained for the related covolume method, see [11] and references therein. Using the tools that were developed for the finite volume theory [22, 23], an order 1 error estimate for non-uniform meshes was obtained in [1], with order 2 convergence for uniform meshes, under the usual regularity assumptions (H^2 for the velocities, H^1 for the pressure). It was recently shown in [62] that under higher regularity assumptions (C^4 for the velocities and C^3 for the pressure) and an additional convergence assumption on the pressure, superconvergence is obtained for non uniform meshes. Note also that the convergence of the MAC scheme for the Stokes equations with a right-hand-side in $H^{-1}(\Omega)$ was proven in [2]. Mathematical studies of the MAC scheme for the non linear Navier-Stokes equations are scarcer. A pioneering work was that of [68] for the steady-state Navier-Stokes equations and for uniform rectangular grids. More recently, a variant of the MAC scheme was defined on locally refined grids and the convergence proof was performed for both the steady-state and time dependent cases in two or three space dimensions [8]. For the Stokes equations on uniform grids, this latter scheme coincides with the usual MAC scheme that is classically used in CFD codes. However, for the Navier-Stokes equations, the nonlinear convection term is discretised in a manner which is similar to the finite element framework (see e.g. [79]), which no longer coincides with the usual MAC scheme, even on uniform grids. This discretization entails in a larger stencil, and numerical experiments [9] tend to show that is not as efficient as the classical MAC scheme. Our purpose here is to analyse the classical MAC scheme for the Navier-Stokes equations in primitive variables on a non-uniform rectangular mesh in two or three dimensions, and, as in [8], without regularity assumptions on the solutions. In section 2.2 we introduce the MAC space grid and the discrete operators. In particular, the velocity convection operator is approximated so as to be compatible with a discrete continuity equation on the duals cells ; this discretization coincides with the usual discretization on uniform meshes [71], contrary to the scheme of [8]. We introduce the MAC scheme for the steady state Navier-Stokes equations in Section 2.3. We give a weak formulation of the scheme. Velocity and pressure estimates are thus obtained, which lead to the compactness of sequences of approximate solutions. We then show that any prospective limit is a weak solution of the Navier-Stokes equations. In Section 2.4, we turn to the unsteady Navier-Stokes equations. An essential feature of the studied scheme is that the (discrete) kinetic energy remains controlled. We show the compactness of approximate sequences of solutions thanks to a discrete Aubin-Simon argument, and again conclude that any prospective limit of the approximate velocities is a weak solution of the Navier-Stokes equations thanks to a passage to the limit in the scheme. In the case of the unsteady Stokes equations, we are able to obtain some estimates which yield the compactness of sequences of approximate pressures; we are then able to conclude that the approximate pressure converges to a weak solution of the Stokes equations as the mesh size and time steps tend to 0.

2.2 Space discretization

We assume that the domain Ω is a union of rectangles ($d = 2$) or orthogonal parallelepipeds ($d = 3$), and, without loss of generality, we assume that the edges (or faces) of these rectangles (or parallelepipeds) are orthogonal to the canonical basis vectors, denoted by $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d)$.

Definition 2.2.1 (MAC grid)**.** A discretization of Ω with MAC grid, denoted by $\mathcal{D}$, is given by $\mathcal{D} = (\mathcal{M}, \mathcal{E})$, where:

- the pressure (or primal) grid denoted by $\mathcal{M}$, which consists of a union of possibly non uniform rectangles; a generic cell of this grid is denoted by K, and its mass center $\boldsymbol{x}_K$. A generic face (or edge in the two-dimensional case) of such a cell is denoted by $\sigma \in \mathcal{E}(K)$, and its mass center $\boldsymbol{x}_\sigma$, where $\mathcal{E}(K)$ denotes the set of all faces of K. The set of all faces of the mesh is denoted by $\mathcal{E}$; we have $\mathcal{E} = \mathcal{E}_{\rm int} \cup \mathcal{E}_{\rm ext}$, where $\mathcal{E}_{\rm int}$ (resp. $\mathcal{E}_{\rm ext}$) are the edges of $\mathcal{E}$ that lie in the interior (resp. on the boundary) of the domain. The set of faces that are orthogonal to the i^{th} unit vector $\boldsymbol{e}_i$ of the canonical basis of $\mathbb{R}^d$ is denoted by $\mathcal{E}^{(i)}$, for $i = 1, \ldots, d$. We then have $\mathcal{E}^{(i)} = \mathcal{E}^{(i)}_{\rm int} \cup \mathcal{E}^{(i)}_{\rm ext}$, where $\mathcal{E}^{(i)}_{\rm int}$ (resp. $\mathcal{E}^{(i)}_{\rm ext}$) are the edges of $\mathcal{E}^{(i)}$ that lie in the interior (resp. on the boundary) of the domain.
- For each $\sigma \in \mathcal{E}$, we write that $\sigma = K|L$ if $\sigma = \partial K \cap \partial L$ and we write that $\sigma = \overrightarrow{K|L}$ if, furthermore, $\sigma \in \mathcal{E}^{(i)}$ and $\overrightarrow{\boldsymbol{x}_K \boldsymbol{x}_L} \cdot \boldsymbol{e}_i > 0$ for some $i \in [\![1, d]\!]$. A dual cell D_σ associated to a face $\sigma \in \mathcal{E}$ is defined as follows:

  * if $\sigma = K|L \in \mathcal{E}_{\rm int}$ then $D_\sigma = D_{K,\sigma} \cup D_{L,\sigma}$, where $D_{K,\sigma}$ (resp. $D_{L,\sigma}$) is the half-part of K (resp. L) adjacent to σ (see Fig. 2.1 for the two-dimensional case) ;
  * if $\sigma \in \mathcal{E}_{\rm ext}$ is adjacent to the cell K, then $D_\sigma = D_{K,\sigma}$.

  A primal cell K will be denoted $K = [\overrightarrow{\sigma\sigma'}]$ if $\sigma, \sigma' \in \mathcal{E}^{(i)} \cap \mathcal{E}(K)$ for some $i = 1, \ldots, d$ are such that $(\boldsymbol{x}_{\sigma'} - \boldsymbol{x}_\sigma) \cdot \boldsymbol{e}_i > 0$. A dual face separating two duals cells D_σ and $D_{\sigma'}$ is denoted by $\epsilon = \sigma|\sigma'$ or $\epsilon = \overrightarrow{\sigma|\sigma'}$ when specifying its orientation: more precisely we write that $\epsilon = \overrightarrow{\sigma|\sigma'}$ if $\overrightarrow{\boldsymbol{x}_\sigma \boldsymbol{x}_{\sigma'}} \cdot \boldsymbol{e}_j > 0$ for some $j \in [\![1, d]\!]$. To any dual face ϵ, we associate a distance d_ϵ as sketched on Figure 2.1. For a dual face $\epsilon \subset \partial D_\sigma, \sigma \in \mathcal{E}^{(i)}$, $i \in [\![1, d]\!]$, the distance d_ϵ is defined by:

$$d_\epsilon = \begin{cases} d(\boldsymbol{x}_\sigma, \boldsymbol{x}_{\sigma'}) & \text{if } \epsilon = \overrightarrow{\sigma|\sigma'} \in \widetilde{\mathcal{E}}^{(i)}_{\text{int}}, \\ d(\boldsymbol{x}_\sigma, \epsilon) & \text{if } \epsilon \in \widetilde{\mathcal{E}}^{(i)}_{\text{ext}} \cap \tilde{\mathcal{E}}(D_\sigma) \end{cases} \tag{2.2.1}$$

where $d(\cdot,\cdot)$ denotes the Euclidean distance in $\mathbb{R}^d$, and the set $\widetilde{\mathcal{E}}^{(i)}$ of the faces of the i-th dual mesh (associated to the ith velocity component) is decomposed into the internal and boundary edges: $\widetilde{\mathcal{E}}^{(i)} = \widetilde{\mathcal{E}}^{(i)}_{\text{int}} \cup \widetilde{\mathcal{E}}^{(i)}_{\text{ext}}$.

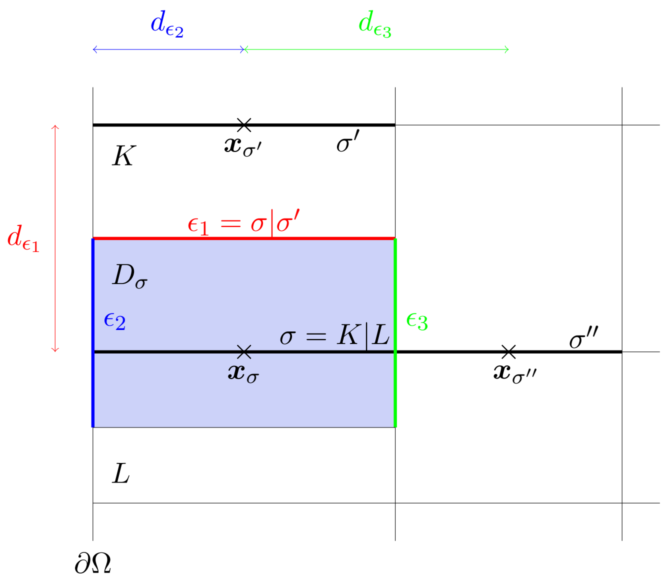


FIG. 2.1 – *Notations for control volumes and dual cells (for the second component of the velocity).*

We define the regularity of the mesh $\mathcal{M}$ by:

$$\eta_{\mathcal{M}} = \max\left\{\frac{|\sigma|}{|\sigma'|},\ \sigma \in \mathcal{E}^{(i)}, \forall \sigma' \in \mathcal{E}^{(j)}, i, j \in [\![1, d]\!], i \neq j\right\}, \tag{2.2.2}$$

where $|\cdot|$ stands for the $(d-1)$-dimensional measure of a subset of $\mathbb{R}^{d-1}$ (in the sequel, it is also be used to denote or d-dimensional measure of a subset $\mathbb{R}^d$). We also define the size of the mesh by

$$h_{\mathcal{M}} = \max\{\text{diam}(K), K \in \mathcal{M}\}.$$

The discrete velocity unknowns are associated to the velocity cells and are denoted by $(u_\sigma)_{\sigma \in \mathcal{E}^{(i)}}$, $i = 1, \ldots, d$, while the discrete pressure unknowns are associated to the primal cells and are are denoted by $(p_K)_{K \in \mathcal{M}}$.

Definition 2.2.2 (Discrete spaces)**.** Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a MAC grid in the sense of Definition 2.2.1. The discrete pressure space $L_\mathcal{M}$ is defined as the set of piecewise constant functions over each of the grid cells K of $\mathcal{M}$, and the discrete $i-th$ velocity space $H_\mathcal{E}^{(i)}$ as the set of piecewise constant functions over each of the grid cells D_σ , $\sigma \in \mathcal{E}^{(i)}$. We shall denote by $L_{\mathcal{M},0}$ the functions of $L_\mathcal{M}$ with zero mean value. As in the continuous case, the Dirichlet boundary conditions are (partly) incorporated into the definition of the velocity spaces, and, to this purpose, we introduce $H_{\mathcal{E},0}^{(i)} \subset H_\mathcal{E}^{(i)}, i = 1, \dots, d$, defined as follows:

$$H_{\mathcal{E},0}^{(i)} = \Big\{ u \in H_\mathcal{E}^{(i)},\ u(\boldsymbol{x}) = 0 \ \forall \boldsymbol{x} \in D_\sigma,\ \sigma \in \widetilde{\mathcal{E}}_{\text{ext}}^{(i)},\ i = 1, \dots, d \Big\}.$$

We then set $\mathbf{H}_{\mathcal{E},0} = \prod_{i=1}^d H_{\mathcal{E},0}^{(i)}$. Since we are dealing with piecewise constant functions, it is useful to introduce the characteristic functions $\chi_K, K \in \mathcal{M}$ and $\chi_{D_\sigma}, \sigma \in \mathcal{E}$ of the pressure and velocity cells, defined by

$$\chi_K(\boldsymbol{x}) = \begin{cases} 1 \text{ if } \boldsymbol{x} \in K, \\ 0 \text{ if } \boldsymbol{x} \notin K, \end{cases} \qquad \chi_{D_\sigma}(\boldsymbol{x}) = \begin{cases} 1 \text{ if } \boldsymbol{x} \in D_\sigma, \\ 0 \text{ if } \boldsymbol{x} \notin D_\sigma. \end{cases}$$

We can then write a function $\boldsymbol{u} \in \mathbf{H}_{\mathcal{E},0}$ as $\boldsymbol{u} = (u_1, \dots, u_d)$ with $u_i = \sum_{\sigma \in \mathcal{E}^{(i)}} u_\sigma \chi_{D_\sigma}$, $i \in [\![1, d]\!]$ and a function $p \in L_\mathcal{M}$ as $p = \sum_{K \in \mathcal{M}} p_K \chi_K$.

Let us now introduce the discrete operators which are used to write the numerical scheme.

Discrete divergence and gradient operators The discrete divergence operator $\text{div}_\mathcal{M}$ is defined by:

$$\text{div}_\mathcal{M} : \left| \begin{array}{l} \mathbf{H}_{\mathcal{E},0} \longrightarrow L_\mathcal{M} \\ \boldsymbol{u} \longmapsto \text{div}_\mathcal{M} \boldsymbol{u} = \displaystyle\sum_{K \in \mathcal{M}} \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| u_{K,\sigma}\ \chi_K, \end{array} \right. \tag{2.2.3}$$

$$\text{with } u_{K,\sigma} = u_\sigma \boldsymbol{n}_{K,\sigma} \cdot \boldsymbol{e}_i \text{ for } \sigma \in \mathcal{E}^{(i)} \cap \mathcal{E}(K), i = 1, \dots, d. \tag{2.2.4}$$

where $\boldsymbol{n}_{K,\sigma}$ denotes the unit normal vector to σ outward K. Note that we have the usual finite volume property of local conservativity of the flux through an interface $\sigma = K|L$ between the cells $K, L \in \mathcal{M}$, *i.e.*

$$u_{K,\sigma} = -u_{L,\sigma}, \quad \forall \sigma = K|L \in \mathcal{E}_{\text{int}}. \tag{2.2.5}$$

We can now define the discrete divergence free velocity space $\boldsymbol{E}_\mathcal{E}(\Omega) = \{\boldsymbol{u} \in \mathbf{H}_{\mathcal{E},0}\ ;\ \text{div}_\mathcal{M}\, \boldsymbol{u} = 0\}$.

The discrete divergence of $\boldsymbol{u} = (u_1, \ldots, u_d) \in \mathbf{H}_{\mathcal{E},0}$ may also be written as

$$\mathrm{div}_{\mathcal{M}}(\boldsymbol{u}) = \sum_{i=1}^{d} (\eth_i u_i)_K \chi_K, \tag{2.2.6}$$

where the discrete derivative $(\eth_i u_i)_K$ of u_i on K is defined by

$$(\eth_i u_i)_K = \frac{|\sigma|}{|K|}(u_{\sigma'} - u_\sigma) \text{ with } K = [\overrightarrow{\sigma\sigma'}], \sigma, \sigma' \in \mathcal{E}^{(i)}. \tag{2.2.7}$$

The discrete derivatives and divergence are consistent in the following sense:

Lemma 2.2.3 (Discrete derivative and divergence consistency)**.** *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a MAC grid, and let $\Pi_\mathcal{E}$ be an interpolator from $C_c^\infty(\Omega)^d$ to $\mathbf{H}_{\mathcal{E},0}$ such that, for any $\boldsymbol{\varphi} = (\varphi_1, \cdots, \varphi_d)^t \in (C_c^\infty(\Omega))^d$, there exists $C_{\boldsymbol{\varphi}} \geq 0$ depending only on $\boldsymbol{\varphi}$ such that*

$$\begin{aligned} &\Pi_\mathcal{E}\boldsymbol{\varphi} = \left(\Pi_\mathcal{E}^{(1)}\varphi_1, \cdots, \Pi_\mathcal{E}^{(d)}\varphi_d\right) \in H_{\mathcal{E},0}^{(1)} \times \cdots \times H_{\mathcal{E},0}^{(d)}, \text{ where} \\ &|\Pi_\mathcal{E}^{(i)}\varphi_i(\boldsymbol{x}) - \varphi_i(\boldsymbol{x}_\sigma)| \leq C_{\boldsymbol{\varphi}} h_\mathcal{M}^2 \ \forall \boldsymbol{x} \in D_\sigma, \ \forall \sigma \in \mathcal{E}^{(i)}, \ \forall i = 1, \cdots, d, \end{aligned} \tag{2.2.8}$$

Then there exists $C_{\boldsymbol{\varphi},\eta} \geq 0$, where η is the regularity of the mesh defined by (2.2.2)*, such that; $|\eth_i \Pi_\mathcal{E}^{(i)}\varphi_i(\boldsymbol{x}) - \partial_i\varphi_i(\boldsymbol{x})| \leq C_{\boldsymbol{\varphi},\eta} h_\mathcal{M}$ for a.e. $\boldsymbol{x} \in \Omega$. As a consequence, if $(\mathcal{D}_n)_{n\in\mathbb{N}} = (\mathcal{M}_n, \mathcal{E}_n)_{n\in\mathbb{N}}$ is a sequence of MAC grids such that $\eta_n \leq \eta$ for all n and $h_{\mathcal{M}_n} \to 0$ as $n \to +\infty$, then $\mathrm{div}_{\mathcal{M}_n}(\Pi_{\mathcal{E}_n}\boldsymbol{\varphi}) \to \mathrm{div}\boldsymbol{\varphi}$ uniformly as $n \to +\infty$.*

The gradient in the discrete momentum balance equation is built as the dual operator of the discrete divergence, and reads:

$$\boldsymbol{\nabla}_\mathcal{E} : \left| \begin{aligned} &L_\mathcal{M} \longrightarrow \mathbf{H}_{\mathcal{E},0} \\ &p \longmapsto \boldsymbol{\nabla}_\mathcal{E} p \\ &\boldsymbol{\nabla}_\mathcal{E} p(\boldsymbol{x}) = (\eth_1 p(\boldsymbol{x}), \ldots, \eth_d p(\boldsymbol{x}))^t, \end{aligned} \right. \tag{2.2.9}$$

where $\eth_i p \in H_{\mathcal{E},0}^{(i)}$ is the discrete derivative of p in the i-th direction, defined by:

$$\eth_i p(\boldsymbol{x}) = \frac{|\sigma|}{|D_\sigma|}\ (p_L - p_K) \quad \forall \boldsymbol{x} \in D_\sigma, \text{ for } \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\mathrm{int}}^{(i)}, \ i = 1, \ldots, d. \tag{2.2.10}$$

Note that in fact, the discrete gradient of a function of $L_\mathcal{M}$ should only be defined on the internal faces, and does not need to be defined on the external faces; we set it here in $\mathbf{H}_{\mathcal{E},0}$ (that is zero on the external faces) for the sake of simplicity. Again, the definition of the discrete derivatives of the pressure on the MAC grid is evidently consistent in the following sense:

Lemma 2.2.4 (Discrete gradient consistency)**.** *Let $\Pi_{\mathcal{M}}$ be an interpolator from $C_c^\infty(\Omega)$ to $L_{\mathcal{M}}$ such that, for any $\psi \in C_c^\infty(\Omega)$, there exists $C_\psi \geq 0$ depending only on ψ such that*

$$|\Pi_{\mathcal{M}}\psi(\boldsymbol{x}) - \psi(\boldsymbol{x}_K)| \leq C_\psi h_{\mathcal{M}}^2,\ \forall \boldsymbol{x} \in K,\ \forall K \in \mathcal{M}. \tag{2.2.11}$$

then there exists $C_{\psi,\eta} \geq 0$ depending only on ψ and η such that

$$|\eth_i \Pi_{\mathcal{M}}\psi(\boldsymbol{x}) - \partial_i\psi(\boldsymbol{x})| \leq C_{\psi,\eta} h_{\mathcal{M}}, \forall \sigma \in \mathcal{E}^{(i)},\ \forall i = 1, \ldots, d.$$

Let us then verify that the discrete gradient and divergence are dual.

Lemma 2.2.5 (Discrete div $-\boldsymbol{\nabla}$ duality)**.** *Let $q \in L_{\mathcal{M}}$ and $\boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0}$ then we have:*

$$\int_\Omega q\ \mathrm{div}_{\mathcal{M}}\boldsymbol{v}\ \mathrm{d}\boldsymbol{x} + \int_\Omega \nabla_{\mathcal{E}} q \cdot \boldsymbol{v}\ \mathrm{d}\boldsymbol{x} = 0. \tag{2.2.12}$$

Proof. Let $q \in L_{\mathcal{M}}$ and $\boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0}$. By the definition (2.2.3) of the discrete divergence operator, we have:

$$\int_\Omega q\ \mathrm{div}_{\mathcal{M}}\boldsymbol{v}\ \mathrm{d}\boldsymbol{x} = \sum_{K\in\mathcal{M}} q_K \sum_{\sigma\in\mathcal{E}(K)} |\sigma| v_{K,\sigma}.$$

with $v_{K,\sigma} = v_\sigma \boldsymbol{n}_{K,\sigma} \cdot \boldsymbol{e}_i$ for $\sigma \in \mathcal{E}^{(i)} \cap \mathcal{E}(K), i = 1, \ldots, d$. Thanks to the conservativity (2.2.5) of the flux we get that:

$$\begin{aligned}\int_\Omega q\ \mathrm{div}_{\mathcal{M}}\boldsymbol{v}\ \mathrm{d}\boldsymbol{x} &= \sum_{\sigma\in\mathcal{E}_{\mathrm{int}},\sigma=K|L} (q_K|\sigma|v_{K,\sigma} + q_L|\sigma|v_{L,\sigma}) \\ &= \sum_{\sigma\in\mathcal{E}_{\mathrm{int}},\sigma=K|L} |\sigma|(q_K - q_L)v_{K,\sigma}.\end{aligned}$$

Therefore, by the definition (2.2.10) of the discrete derivative of q, we get:

$$\int_\Omega q\ \mathrm{div}_{\mathcal{M}}\boldsymbol{v}\ \mathrm{d}\boldsymbol{x} = -\sum_{i=1}^d \sum_{\sigma\in\mathcal{E}^{(i)},\sigma=K|L} |D_\sigma| v_\sigma \eth_i q = -\int_\Omega \boldsymbol{\nabla}_{\mathcal{E}} q \cdot \boldsymbol{v}\ \mathrm{d}\boldsymbol{x},$$

which concludes the proof. □

Discrete Laplace operator - For $i = 1\ldots, d$, we classically define the i^{th} component of the discrete Laplace operator by:

$$-\Delta_{\mathcal{E}}^{(i)} : \left| \begin{array}{l} H_{\mathcal{E},0}^{(i)} \longrightarrow H_{\mathcal{E},0}^{(i)} \\ u_i \longmapsto -\Delta_{\mathcal{E}} u_i = -\sum_{\sigma \in \mathcal{E}^{(i)}} (\Delta u)_\sigma \chi_{D_\sigma}, \text{ with } -(\Delta u)_\sigma = \frac{1}{|D_\sigma|} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} \phi_{\sigma,\epsilon}(u_i) \end{array} \right. \tag{2.2.13}$$

$\tilde{\mathcal{E}}(D_\sigma)$ denotes the faces of D_σ and

$$\phi_{\sigma,\epsilon}(u_i) = \begin{cases} \frac{|\epsilon|}{d_\epsilon}(u_\sigma - u_{\sigma'}), \text{ if } \epsilon = \overrightarrow{\sigma|\sigma'} \in \widetilde{\mathcal{E}}_{\text{int}}^{(i)}, \\ \frac{|\epsilon|}{d_\epsilon} u_\sigma, \text{ if } \epsilon \in \widetilde{\mathcal{E}}_{\text{ext}}^{(i)} \cap \tilde{\mathcal{E}}(D_\sigma) \end{cases}$$

where d_ϵ is defined by (2.2.1). Note that we have the usual finite volume property of local conservativity of the flux through an interface $\epsilon = \overrightarrow{\sigma\ |\sigma'}$:

$$\phi_{\sigma,\epsilon}(u_i) = -\phi_{\sigma',\epsilon}(u_i), \quad \forall \epsilon = \overrightarrow{\sigma|\sigma'} \in \widetilde{\mathcal{E}}_{\text{int}}^{(i)}. \tag{2.2.14}$$

Then the discrete Laplace operator of the full velocity vector is defined by

$$\begin{array}{rl} -\Delta_{\mathcal{E}} : & \mathbf{H}_{\mathcal{E},0} \longrightarrow \mathbf{H}_{\mathcal{E},0} \\ & \boldsymbol{u} \mapsto -\Delta_{\mathcal{E}} \boldsymbol{u} = (-\Delta_{\mathcal{E}}^{(1)} u_1, \ldots, -\Delta_{\mathcal{E}}^{(d)} u_d)^t. \end{array} \tag{2.2.15}$$

Discrete convection operator - Let us consider the momentum equation (2.1.1b) for the i^{th} component of the velocity, and integrate it on a cell D_σ, $\sigma \in \mathcal{E}^{(i)}$. By the Stokes formula we then need to discretise $\sum_{\epsilon \subset \partial D_\sigma} \int_\epsilon u_i \boldsymbol{u} \cdot \boldsymbol{n}_{\sigma,\epsilon} \, \mathrm{d}\gamma(\boldsymbol{x})$, where $\boldsymbol{n}_{\sigma,\epsilon}$ denotes the unit normal vector to ϵ outward D_σ and $\mathrm{d}\gamma(\boldsymbol{x})$ denotes the integration with respect to the $d-1$-dimensional Lebesgue measure. For $\epsilon = \sigma|\sigma'$, the convection flux $\int_\epsilon u_i \boldsymbol{u} \cdot \boldsymbol{n}_{\sigma,\epsilon} \, \mathrm{d}\gamma(\boldsymbol{x})$ is approximated by $|\epsilon| u_{\sigma,\epsilon} u_\epsilon$, where

$$u_\epsilon = (u_\sigma + u_{\sigma'})/2, \tag{2.2.16}$$

and $|\epsilon| u_{\sigma,\epsilon}$ is the numerical mass flux through ϵ outward D_σ; this flux must be chosen carefully to obtain the L^2 stability of the scheme. More precisely, we need that a discrete counterpart of the free divergence of $\boldsymbol{u}$ be satisfied also on the dual cells. We distinguish two cases (see figure 2.2):

- First case – The vector $\boldsymbol{e}_i$ is normal to ϵ, and ϵ is included in a primal cell K, with

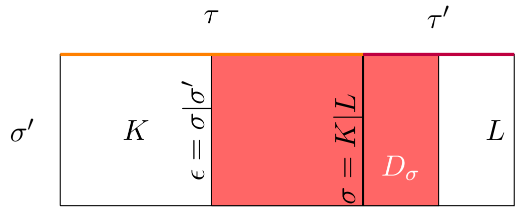


FIG. 2.2 – *dual fluxes*

$\mathcal{E}^{(i)}(K) = \{\sigma, \sigma'\}$. Then the mass flux through $\epsilon = \sigma|\sigma'$ is given by:

$$|\epsilon| u_{\sigma,\epsilon} = \frac{1}{2}\,(-|\sigma| u_{K,\sigma} + |\sigma'| u_{K,\sigma'}). \tag{2.2.17}$$

- Second case – The vector $\boldsymbol{e}_i$ is tangent to ϵ, and ϵ is the union of the halves of two primal faces τ and τ' such that $\sigma = K|L$ with $\tau \in \mathcal{E}(K)$ and $\tau' \in \mathcal{E}(L)$. The mass flux through ϵ is then given by:

$$|\epsilon| u_{\sigma,\epsilon} = \frac{1}{2}\,(|\tau| u_{K,\tau} + |\tau'| u_{L,\tau'}). \tag{2.2.18}$$

Note that with this definition, we again have the usual finite volume property of local conservativity of the flux through an interface $\overrightarrow{\sigma|\sigma'}$, *i.e.*

$$|\epsilon| u_{\sigma,\epsilon} = -|\epsilon| u_{\sigma',\epsilon} \tag{2.2.19}$$

together with the following discrete free divergence condition on the dual cells:

$$\sum_{\epsilon \in \mathcal{E}(D_\sigma)} |\epsilon| u_{\sigma,\epsilon} = \frac{1}{2} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| u_{K,\sigma} + \frac{1}{2} \sum_{\sigma \in \mathcal{E}(L)} |\sigma| u_{L,\sigma} = 0. \tag{2.2.20}$$

Note that we have also $u_{\sigma,\epsilon} = 0$ if $\epsilon \subset \partial\Omega$, which is consistent with the boundary conditions (2.1.1c).

We now define the i-th component $C_{\mathcal{E}}^{(i)}(\boldsymbol{u})$ of the non linear convection operator by:

$$C_{\mathcal{E}}^{(i)}(\boldsymbol{u}) : \left| \begin{array}{l} H_{\mathcal{E},0}^{(i)} \longrightarrow H_{\mathcal{E},0}^{(i)} \\ v \longmapsto C_{\mathcal{E}}^{(i)}(\boldsymbol{u}) v = \displaystyle\sum_{\sigma \in \widetilde{\mathcal{E}}_{\rm int}^{(i)}} \frac{1}{|D_\sigma|} \left(\sum_{\substack{\epsilon \in \tilde{\mathcal{E}}(D_\sigma) \\ \epsilon = \sigma|\sigma'}} |\epsilon| u_{\sigma,\epsilon} \frac{v_\sigma + v_{\sigma'}}{2} \right) \chi_{D_\sigma}, \end{array} \right. \tag{2.2.21}$$

and the full discrete convection operator $\boldsymbol{C}_{\mathcal{E}}(\boldsymbol{u}),\ \mathbf{H}_{\mathcal{E},0} \longrightarrow \mathbf{H}_{\mathcal{E},0}$ by

$$\boldsymbol{C}_{\mathcal{E}}(\boldsymbol{u})\boldsymbol{v} = (C_{\mathcal{E}}^{(1)}(\boldsymbol{u})v_1, \dots, C_{\mathcal{E}}^{(d)}(\boldsymbol{u})v_d)^t.$$

2.3 The steady case

With the notations introduced in the previous sections, the MAC scheme for the discretisation of Problem (2.1.1) on a MAC grid $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ reads:

$$\boldsymbol{u} \in \mathbf{H}_{\mathcal{E},0},\ p \in L_{\mathcal{M},0}, \tag{2.3.1a}$$

$$-\Delta_{\mathcal{E}}\boldsymbol{u} + \boldsymbol{C}_{\mathcal{E}}(\boldsymbol{u})\boldsymbol{u} + \boldsymbol{\nabla}_{\mathcal{E}}p = \mathcal{P}_{\mathcal{E}}\boldsymbol{f}, \tag{2.3.1b}$$

$$\mathrm{div}_{\mathcal{M}}\boldsymbol{u} = 0, \tag{2.3.1c}$$

where $L_{\mathcal{M},0} = \{q \in L_{\mathcal{M}} \ \ \int_\Omega q \ \mathrm{d}\boldsymbol{x} = 0\}$ and $\mathcal{P}_{\mathcal{E}}$ is the cell mean-value operator defined by

$$\begin{aligned} &\mathcal{P}_{\mathcal{E}}\boldsymbol{v} = \left(\mathcal{P}_{\mathcal{E}}^{(1)}v_1, \cdots, \mathcal{P}_{\mathcal{E}}^{(d)}v_d\right) \in H_{\mathcal{E},0}^{(1)} \times \cdots \times H_{\mathcal{E},0}^{(d)}, \text{ where for } i = 1, \dots d,\\ &\mathcal{P}_{\mathcal{E}}^{(i)} : \ \ L^1(\Omega) \longrightarrow H_{\mathcal{E},0}^{(i)}\\ &\qquad v_i \ \longmapsto \mathcal{P}_{\mathcal{E}}v_i \ ; \ i = 1, \cdots, d,\\ &\mathcal{P}_{\mathcal{E}}^{(i)}v_i = \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \left(\frac{1}{|D_\sigma|}\int_{D_\sigma} v_i(\boldsymbol{x}) \ \mathrm{d}\boldsymbol{x}\right)\chi_{D_\sigma}. \end{aligned} \tag{2.3.2}$$

2.3.1 Weak form of the scheme

We first recall the definition of the discrete H_0^1 inner product [22]; it is obtained by multiplying the discrete Laplace operator scalarly by a test function $\boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0}$ and integrating over the computational domain. A simple reordering of the sums (which may be seen as a discrete integration by parts) yields, thanks to the conservativity of the diffusion flux (2.2.14):

$$\begin{aligned} &\forall(\boldsymbol{u}, \boldsymbol{v}) \in {\mathbf{H}_{\mathcal{E},0}}^2, \qquad \int_\Omega -\Delta_{\mathcal{E}}\boldsymbol{u} \cdot \boldsymbol{v} \ \mathrm{d}\boldsymbol{x} = [\boldsymbol{u}, \boldsymbol{v}]_{1,\mathcal{E},0} = \sum_{i=1}^{d} [u_i, v_i]_{1,\mathcal{E}^{(i)},0},\\ &\text{with } [u_i, v_i]_{1,\mathcal{E}^{(i)},0} = \sum_{\substack{\epsilon \in \widetilde{\mathcal{E}}_{\mathrm{int}}^{(i)} \\ \epsilon = \overrightarrow{\sigma|\sigma'}}} \frac{|\epsilon|}{d_\epsilon} \ (u_\sigma - u_{\sigma'}) \ (v_\sigma - v_{\sigma'}) + \sum_{\substack{\epsilon \in \widetilde{\mathcal{E}}_{\mathrm{ext}}^{(i)} \\ \epsilon \subset \partial(D_\sigma)}} \frac{|\epsilon|}{d_\epsilon} \ u_\sigma \ v_\sigma. \end{aligned} \tag{2.3.3}$$

The bilinear forms $\left|\begin{array}{l} H^{(i)}_{\mathcal{E},0} \times H^{(i)}_{\mathcal{E},0} \to \mathbb{R} \\ (u,v) \mapsto [u_i, v_i]_{1,\mathcal{E}^{(i)},0} \end{array}\right.$ and $\left|\begin{array}{l} \mathbf{H}_{\mathcal{E},0} \times \mathbf{H}_{\mathcal{E},0} \to \mathbb{R} \\ (\boldsymbol{u},\boldsymbol{v}) \mapsto [\boldsymbol{u},\boldsymbol{v}]_{1,\mathcal{E},0} \end{array}\right.$ are inner products on $H^{(i)}_{\mathcal{E},0}$ and $\mathbf{H}_{\mathcal{E},0}$ respectively, which induce the following scalar and vector discrete H^1_0 norms:

$$\|u_i\|^2_{1,\mathcal{E}^{(i)},0} = [u_i, u_i]_{1,\mathcal{E}^{(i)},0} = \sum_{\substack{\epsilon \in \widetilde{\mathcal{E}}^{(i)}_{\text{int}} \\ \epsilon = \overrightarrow{\sigma|\sigma'}}} \frac{|\epsilon|}{d_\epsilon} (u_\sigma - u_{\sigma'})^2 + \sum_{\substack{\epsilon \in \widetilde{\mathcal{E}}^{(i)}_{\text{ext}} \\ \epsilon \subset \partial(D_\sigma)}} \frac{|\epsilon|}{d_\epsilon} u^2_\sigma \text{ for } i = 1, \dots, d, \tag{2.3.4a}$$

$$\|\boldsymbol{u}\|^2_{1,\mathcal{E},0} = [\boldsymbol{u}, \boldsymbol{u}]_{1,\mathcal{E},0} = \sum_{i=1}^{d} \|u_i\|^2_{1,\mathcal{E}^{(i)},0}. \tag{2.3.4b}$$

Since we are working on Cartesian grids, this inner product may be formulated as the L^2 inner

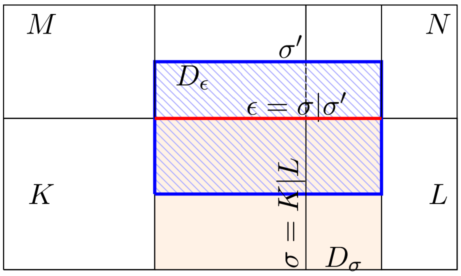


FIG. 2.3 – *Full grid for definition of the derivative of the velocity.*

product of discrete gradients. Indeed, consider the following discrete gradient of each velocity component u_i.

$$\nabla_{\mathcal{E}^{(i)}} u_i = (\eth_1 u_i, \dots, \eth_d u_i) \text{ with } \eth_j u_i = \sum_{\substack{\epsilon \in \widetilde{\mathcal{E}}^{(i)} \\ \epsilon \perp \boldsymbol{e}_j}} (\eth_j u_i)_{D_\epsilon} \, \chi_{D_\epsilon}, \tag{2.3.5}$$

where $(\eth_j u_i)_{D_\epsilon} = \dfrac{u_{\sigma'} - u_\sigma}{d_\epsilon}$ with $\epsilon = \overrightarrow{\sigma|\sigma'}$, and $D_\epsilon = \epsilon \times \boldsymbol{x}_\sigma \boldsymbol{x}_{\sigma'}$ (see Figure 2.3). This definition is compatible with the definition of the discrete derivative $(\eth_i u_i)_K$ given by (2.2.7), since, if $\epsilon \subset K$ then $D_\epsilon = K$. With this definition, it is easily seen that

$$\int_\Omega \nabla_{\mathcal{E}^{(i)}} u \cdot \nabla_{\mathcal{E}^{(i)}} v \, \mathrm{d}\boldsymbol{x} = [u, v]_{1,\mathcal{E}^{(i)},0}, \forall u, v \in H^{(i)}_{\mathcal{E},0}, \forall i = 1, \dots, d. \tag{2.3.6}$$

where $[u, v]_{1,\mathcal{E}^{(i)},0}$ is the discrete H^1_0 inner product defined by (2.3.3). We may then define

$$\nabla_{\mathcal{E}} \boldsymbol{u} = (\nabla_{\mathcal{E}^{(1)}} u_1, \dots, \nabla_{\mathcal{E}^{(d)}} u_d),$$

so that

$$\int_\Omega \nabla_\mathcal{E}\boldsymbol{u} : \nabla_\mathcal{E}\boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = [\boldsymbol{u},\boldsymbol{v}]_{1,\mathcal{E},0}.$$

With this formulation, the MAC scheme for the linear Stokes problem can be interpreted as a gradient scheme in the sense introduced in [24], see [30] and [19] for more details on the generalization of this formulation to other schemes. In the stationary case, we can show the (strong) convergence of this discrete gradient to the gradient of the exact velocity, and thus also show the strong convergence of the pressure, see section 2.4.4. The weak form $b_\mathcal{E}$ of the nonlinear convection operator is defined by:

$$\begin{aligned} &\forall(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \in {\mathbf{H}_{\mathcal{E},0}}^3, \qquad b_\mathcal{E}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \sum_{i=1}^d b_\mathcal{E}^{(i)}(\boldsymbol{u},v_i,w_i),\\ &\text{where for } i=1,\ldots,d, \quad b_\mathcal{E}^{(i)}(\boldsymbol{u},v_i,w_i) = \int_\Omega C_\mathcal{E}^{(i)}(\boldsymbol{u}) v_i \; w_i \, \mathrm{d}\boldsymbol{x}. \end{aligned} \tag{2.3.7}$$

We are now in position to introduce a weak formulation of the scheme, which reads:

Find $(\boldsymbol{u},p) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M},0}$ and, $\forall(\boldsymbol{v},q) \in \mathbf{H}_{\mathcal{E},0} \times L_\mathcal{M}$,

$$\int_\Omega \nabla_\mathcal{E}\boldsymbol{u} : \nabla_\mathcal{E}\boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + b_\mathcal{E}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) - \int_\Omega p \, \mathrm{div}_\mathcal{M}(\boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} = \int_\Omega \mathcal{P}_\mathcal{E}\boldsymbol{f}\cdot\boldsymbol{v} \, \mathrm{d}\boldsymbol{x}, \tag{2.3.8a}$$

$$\int_\Omega \mathrm{div}_\mathcal{M}\boldsymbol{u} \, q \, \mathrm{d}\boldsymbol{x} = 0, \tag{2.3.8b}$$

and which is equivalent to the MAC scheme (2.3.1).

2.3.2 Existence and stability

Lemma 2.3.1 (Estimate on $b_\mathcal{E}$)**.** *Let $\mathcal{D} = (\mathcal{M},\mathcal{E})$ be a MAC grid and let $b_\mathcal{E}$ be defined by* (2.3.7)*. There exists $C_{\eta_\mathcal{M}} > 0$, depending only on the regularity $\eta_\mathcal{M}$ of the mesh defined by* (2.2.2) *such that:*

$$\forall(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \in \boldsymbol{E}_\mathcal{E} \times {\mathbf{H}_{\mathcal{E},0}}^2, \qquad |b_\mathcal{E}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \le C_{\eta_\mathcal{M}} \; \|\boldsymbol{u}\|_{L^4(\Omega)^d} \|\boldsymbol{v}\|_{1,\mathcal{E},0} \; \|\boldsymbol{w}\|_{L^4(\Omega)^d} \tag{2.3.9}$$

and

$$\forall(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \in \boldsymbol{E}_\mathcal{E} \times {\mathbf{H}_{\mathcal{E},0}}^2, \qquad |b_\mathcal{E}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \le C_{\eta_\mathcal{M}} \; \|\boldsymbol{u}\|_{1,\mathcal{E},0} \; \|\boldsymbol{v}\|_{1,\mathcal{E},0} \|\boldsymbol{w}\|_{1,\mathcal{E},0}, \tag{2.3.10}$$

Proof. We closely follow the proof of the estimate in the continuous case, where the nonlinear term $b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \int_\Omega((\boldsymbol{u}\cdot\nabla)\boldsymbol{v})\cdot\boldsymbol{w} \, \mathrm{d}\boldsymbol{x}$ is estimated thanks to the Hölder inequality and the

Sobolev embedding: there exist $C_1 \geq 0$ and $C_2 \geq 0$ depending only on Ω such that

$$\begin{aligned}|b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| &\leq C_1 \|\boldsymbol{u}\|_{L^4(\Omega)^d} \|\nabla \boldsymbol{v}\|_{L^2(\Omega)^{d\times d}} \|\boldsymbol{w}\|_{L^4(\Omega)^d} \\ &\leq C_2 \|\nabla \boldsymbol{u}\|_{L^2(\Omega)^{d\times d}} \|\nabla \boldsymbol{v}\|_{L^2(\Omega)^{d\times d}} \|\nabla \boldsymbol{w}\|_{L^2(\Omega)^{d\times d}}.\end{aligned}$$

Let $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \in \boldsymbol{E}_{\mathcal{E}} \times {\mathbf{H}_{\mathcal{E},0}}^2$. Thanks to (2.2.20), we have:

$$b_{\mathcal{E}}^{(i)}(\boldsymbol{u}, v_i, w_i) = \sum_{\sigma \in \mathcal{E}^{(i)}} w_\sigma \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} |\epsilon| (v_\epsilon - v_\sigma) u_{\sigma,\epsilon}.$$

From the definition (2.2.16) of u_ϵ and with a discrete integration by parts, we get that:

$$b_{\mathcal{E}}^{(i)}(\boldsymbol{u}, v_i, w_i) = -\frac{1}{2} \sum_{\epsilon = \overrightarrow{\sigma|\sigma'} \in \widetilde{\mathcal{E}}_{\rm int}^{(i)}} (v_\sigma - v_{\sigma'}) |\epsilon| u_{\sigma,\epsilon} (w_{\sigma'} + w_\sigma)$$

From the definition (2.2.17)-(2.2.18) of $u_{\sigma,\epsilon}$ we have for $\epsilon = \overrightarrow{\sigma|\sigma'} \in \widetilde{\mathcal{E}}_{\rm int}^{(i)}$:

$$|u_{\sigma,\epsilon}| \leq \begin{cases} \frac{1}{2}(|u_\sigma| + |u_{\sigma'}|) & \text{if } \epsilon \perp \boldsymbol{e}_i, \\ \frac{1}{2}(|u_\tau| + |u_{\tau'}|) & \text{if } \epsilon \not\perp \boldsymbol{e}_i \text{ and } \epsilon \subset \tau \cup \tau', \end{cases}$$

where τ and τ' are the faces of $\mathcal{E}^{(j)}, j \neq i$ such that $\epsilon \subset \tau \cup \tau'$. Thus,

$$\begin{aligned} b_{\mathcal{E}}^{(i)}(\boldsymbol{u}, v_i, w_i) \leq & \sum_{\substack{\epsilon = \overrightarrow{\sigma|\sigma'} \in \widetilde{\mathcal{E}}_{\rm int}^{(i)} \\ \epsilon \perp \boldsymbol{e}_i}} |\epsilon| (|u_\sigma| + |u_{\sigma'}|) |v_\sigma - v_{\sigma'}| |w_\sigma + w_{\sigma'}| \\ & + \sum_{\substack{\epsilon = \overrightarrow{\sigma|\sigma'} \in \widetilde{\mathcal{E}}_{\rm int}^{(i)} \\ \epsilon \not\perp \boldsymbol{e}_i, \epsilon \subset \tau \cup \tau'}} |\epsilon| (|u_\tau| + |u_{\tau'}|) |v_\sigma - v_{\sigma'}| |w_\sigma + w_{\sigma'}|. \end{aligned}$$

Using Hölder's inequality, we get:

$$\begin{aligned}\sum_{\substack{\epsilon=\overrightarrow{\sigma|\sigma'}\in\widetilde{\mathcal{E}}^{(i)}_{\rm int}\\ \epsilon\perp\boldsymbol{e}_i}}|\epsilon||u_\sigma||v_\sigma-v_{\sigma'}||w_\sigma| &= \sum_{\substack{\epsilon=\overrightarrow{\sigma|\sigma'}\in\widetilde{\mathcal{E}}^{(i)}_{\rm int}\\ \epsilon\perp\boldsymbol{e}_i}}|\epsilon|^{\frac{1}{4}}d_\epsilon^{\frac{1}{4}}|u_\sigma|\frac{\sqrt{|\epsilon|}}{\sqrt{d_\epsilon}}|v_\sigma-v_{\sigma'}||\epsilon|^{\frac{1}{4}}d_\epsilon^{\frac{1}{4}}|w_\sigma|\\
&\le \Big(\sum_{\substack{\epsilon=\overrightarrow{\sigma|\sigma'}\in\widetilde{\mathcal{E}}^{(i)}_{\rm int}\\ \epsilon\perp\boldsymbol{e}_i}}|\epsilon|d_\epsilon|u_\sigma|^4\Big)^{\frac{1}{4}}\Big(\sum_{\substack{\epsilon=\overrightarrow{\sigma|\sigma'}\in\widetilde{\mathcal{E}}^{(i)}_{\rm int}\\ \epsilon\perp\boldsymbol{e}_i}}(v_\sigma-v_{\sigma'})^2\frac{|\epsilon|}{d_\epsilon}\Big)^{\frac{1}{2}}\\
&\qquad\Big(\sum_{\substack{\epsilon=\overrightarrow{\sigma|\sigma'}\in\widetilde{\mathcal{E}}^{(i)}_{\rm int}\\ \epsilon\perp\boldsymbol{e}_i}}|\epsilon|d_\epsilon|w_\sigma|^4\Big)^{\frac{1}{4}}\\
&\le \|u_i\|_{L^4(\Omega)}\|v_i\|_{1,\mathcal{E}^{(i)},0}\|w_i\|_{L^4(\Omega)}.\end{aligned}$$

Therefore, with similar computations for the terms involving $u_{\sigma'}$, u_τ,$u_{\tau'}$, $u_{\sigma'}$ and $w_{\sigma'}$, we get:

$$\begin{aligned}b^{(i)}_{\mathcal{E}}(\boldsymbol{u},v_i,w_i)\le C_{\mathcal{M}}\big[&\|u_i\|_{L^4(\Omega)}\|v_i\|_{1,\mathcal{E}^{(i)},0}\|w_i\|_{L^4(\Omega)}\\
&+\sum_{\substack{j\in[\![1,d]\!]\\ j\neq i}}\|u_j\|_{L^4(\Omega)}\|v_i\|_{1,\mathcal{E}^{(i)},0}\|w_i\|_{L^4(\Omega)}\big],\forall i\in[\![1,d]\!],\end{aligned}$$

where $C_{\mathcal{M}}$ only depends on $\eta_{\mathcal{M}}$ (2.2.2). We then deduce (2.3.9). By the discrete Sobolev inequality [22, Lemma 3.5], we also have

$$\begin{aligned}b^{(i)}_{\mathcal{E}}(\boldsymbol{u},v_i,w_i)\le C_{\eta_{\mathcal{M}}}\big[&\|u_i\|_{1,\mathcal{E}^{(i)},0}\|v_i\|_{1,\mathcal{E}^{(i)},0}\|w_i\|_{1,\mathcal{E}^{(i)},0}\\
&+\sum_{\substack{j\in[\![1,d]\!]\\ j\neq i}}\|u_j\|_{1,\mathcal{E}^{(i)},0}\|v_i\|_{1,\mathcal{E}^{(i)},0}\|w_i\|_{1,\mathcal{E}^{(i)},0}\big],\forall i\in[\![1,d]\!],\end{aligned}$$

from which we get (2.3.10). □

Lemma 2.3.2 ($b_{\mathcal{E}}$ is skew-symmetric)**.** *Let* $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})\in\boldsymbol{E}_{\mathcal{E}}\times\mathbf{H}_{\mathcal{E},0}\times\mathbf{H}_{\mathcal{E},0}$ *then ;*

$$b_{\mathcal{E}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=-b_{\mathcal{E}}(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}),\tag{2.3.11}$$

and therefore

$$\forall\boldsymbol{u}\in\boldsymbol{E}_{\mathcal{E}},\qquad b_{\mathcal{E}}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{u})=0.\tag{2.3.12}$$

Proof. The proof follows that of the continuous case, which is based on a integration by parts. Indeed

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=\int_\Omega(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}\,\mathrm{d}\boldsymbol{x}=-\int_\Omega(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}=-b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}).$$

Let $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{E}_{\mathcal{E}} \times \mathbf{H}_{\mathcal{E},0} \times \mathbf{H}_{\mathcal{E},0}$. By (2.2.20) we have:

$$b_{\mathcal{E}}^{(i)}(\boldsymbol{u}, v_i, w_i) = \sum_{\sigma\in\mathcal{E}^{(i)}} w_\sigma \sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)} |\epsilon|(v_\epsilon - v_\sigma)u_{\sigma,\epsilon}, \text{ for any } i \in [\![1,d]\!].$$

From the definition (2.2.16) of u_ϵ and with a discrete integration by parts, we get by conservativity of the flux (2.2.19) that:

$$\begin{aligned} b_{\mathcal{E}}^{(i)}(\boldsymbol{u}, v_i, w_i) &= -\frac{1}{2} \sum_{\epsilon=\overrightarrow{\sigma|\sigma'}\in\widetilde{\mathcal{E}}^{(i)}_{\text{int}}} |\epsilon|(v_\sigma - v_{\sigma'})u_{\sigma,\epsilon}(w_{\sigma'} + w_\sigma) \\ &= \frac{1}{2} \sum_{\epsilon=\overrightarrow{\sigma'|\sigma}\in\widetilde{\mathcal{E}}^{(i)}_{\text{int}}} |\epsilon|(v_\sigma + v_{\sigma'})u_{\sigma',\epsilon}(w_{\sigma'} - w_\sigma). \end{aligned}$$

which yields (2.3.11) thanks to another discrete integration by parts. □

In order to obtain an a priori estimate on the pressure, we introduce a so-called Fortin interpolation operator, which preserves the divergence. The following lemma is given in [35, Theorem 1, case $q = 2$], and we re-state here with our notations for the sake of clarity.

Lemma 2.3.3 (Fortin interpolation operator)**.** *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a MAC grid of Ω. For $\boldsymbol{v} \in \mathbf{H}_{\mathcal{E}},$, we define $\widetilde{\mathcal{P}}_{\mathcal{E}}\boldsymbol{v}$ by*

$$\begin{aligned} &\widetilde{\mathcal{P}}_{\mathcal{E}}\boldsymbol{v} = \left(\widetilde{\mathcal{P}}_{\mathcal{E}}^{(1)}v_1, \cdots, \widetilde{\mathcal{P}}_{\mathcal{E}}^{(1)}v_d\right) \in \mathbf{H}_{\mathcal{E}}, \text{ where for } i = 1, \dots d, \\ &\widetilde{\mathcal{P}}_{\mathcal{E}}^{(i)} : \quad H_0^1(\Omega) \longrightarrow H_{\mathcal{E},0}^{(i)} \\ &\qquad\quad v_i \longmapsto \widetilde{\mathcal{P}}_{\mathcal{E}} v_i \ ; \ i = 1, \cdots, d, \\ &\qquad\quad \widetilde{\mathcal{P}}_{\mathcal{E}}^{(i)} v_i(\boldsymbol{x}) = \frac{1}{|\sigma|} \int_\sigma v_i(\boldsymbol{x}) \, \mathrm{d}\gamma(\boldsymbol{x}), \ \forall \boldsymbol{x} \in D_\sigma, \ \sigma \in \mathcal{E}^{(i)}. \end{aligned} \tag{2.3.13}$$

For $q \in L^2(\Omega)$, we define $\mathcal{P}_{\mathcal{M}} q \in L_{\mathcal{M}}$ by:

$$\mathcal{P}_{\mathcal{M}} q(\boldsymbol{x}) = \frac{1}{|K|} \int_K q(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}. \tag{2.3.14}$$

Let $\eta_{\mathcal{M}} > 0$ be defined by (2.2.2). Let $\boldsymbol{\varphi} \in (H_0^1(\Omega))^d$, then

$$\mathrm{div}_{\mathcal{M}}(\widetilde{\mathcal{P}}_{\mathcal{E}}\boldsymbol{\varphi}) = \mathcal{P}_{\mathcal{M}}(\mathrm{div}\boldsymbol{\varphi}), \tag{2.3.15a}$$

$$\|\widetilde{\mathcal{P}}_{\mathcal{E}}\boldsymbol{\varphi}\|_{1,\mathcal{E},0} \le C_{\eta_{\mathcal{M}}} \|\nabla\boldsymbol{\varphi}\|_{(L^2(\Omega))^d}, \tag{2.3.15b}$$

where $C_{\eta_{\mathcal{M}}}$ depends only on $\eta_{\mathcal{M}}$ and Ω. In particular, if $\mathrm{div}\boldsymbol{\varphi} = 0$, then $\mathrm{div}_{\mathcal{M}}(\widetilde{\mathcal{P}}_{\mathcal{E}}\boldsymbol{\varphi}) = 0$.

Theorem 2.3.4 (Existence and estimates)**.** *There exists a solution to* (2.3.8)*, and there exists* $C_{\eta_{\mathcal{M}}} > 0$ *depending only on the regularity* $\eta_{\mathcal{M}}$ *of the mesh and* Ω*, such that any solution of* (2.3.8) *satisfies the following stability estimate:*

$$\|\boldsymbol{u}\|_{1,\mathcal{E},0} + \|p\|_{L^2(\Omega)} \leq C_{\eta_{\mathcal{M}}} \; \|\boldsymbol{f}\|_{L^2(\Omega)^d}. \tag{2.3.16}$$

Proof. Let us start by an *a priori* estimate on the approximate velocity. Assume that $(\boldsymbol{u}, p) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M},0}$ satisfies (2.3.1); taking $\boldsymbol{v} = \boldsymbol{u}$ in (2.3.8a) we get that:

$$\|\boldsymbol{u}\|^2_{1,\mathcal{E},0} = \int_\Omega p \; \mathrm{div}_{\mathcal{M}} \; \boldsymbol{u} \; \mathrm{d}\boldsymbol{x} - b_{\mathcal{E}}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}) + \int_\Omega \mathcal{P}_{\mathcal{E}} \boldsymbol{f} \cdot \boldsymbol{u} \; \mathrm{d}\boldsymbol{x}.$$

Since $\mathrm{div}_{\mathcal{M}}\boldsymbol{u} = 0$ and $b_{\mathcal{E}}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}) = 0$ by (2.3.11) this yields that

$$\|\boldsymbol{u}\|_{1,\mathcal{E},0} \leq \mathrm{diam}(\Omega) \|\boldsymbol{f}\|_{(L^2)^d}. \tag{2.3.17}$$

thanks to the fact that $\|\mathcal{P}_{\mathcal{E}} \boldsymbol{f}\|_{(L^2(\Omega))^d} \leq \|\boldsymbol{f}\|_{(L^2(\Omega))^d}$ and to the discrete Poincaré inequality [22, Lemma 9.1]. An *a priori* estimate on the pressure is obtained by remarking as in [76] that the MAC scheme is inf-sup stable. Indeed, since $p \in L^2_0(\Omega)$, there exists $\boldsymbol{\varphi} \in (H^1_0(\Omega))^d$ such that $\mathrm{div}\boldsymbol{\varphi} = p$ a.e. in Ω and

$$\|\boldsymbol{\varphi}\|_{(H^1_0(\Omega))^d} \leq c\|p\|_{L^2(\Omega)}, \tag{2.3.18}$$

where c depends only on Ω [66]. Taking $\boldsymbol{v} = \widetilde{\mathcal{P}}_{\mathcal{E}} \boldsymbol{\varphi}$ (defined by (2.3.13)) as test function in (2.3.8a), we obtain thanks to Lemma 2.3.3 that

$$[\boldsymbol{u}, \boldsymbol{v}]_{1,\mathcal{E},0} + b_{\mathcal{E}}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - \int_\Omega p^2 \; \mathrm{d}\boldsymbol{x} = \int_\Omega \mathcal{P}_{\mathcal{E}} \boldsymbol{f} \cdot \boldsymbol{v} \; \mathrm{d}\boldsymbol{x}.$$

Thanks to the estimate (2.3.10) on $b_{\mathcal{E}}$ and the Cauchy-Schwarz inequality we get:

$$\|p\|^2_{L^2(\Omega)} \leq \|\boldsymbol{u}\|_{1,\mathcal{E},0} \|\boldsymbol{v}\|_{1,\mathcal{E},0} + C_{\eta_{\mathcal{M}}} \|\boldsymbol{u}\|^2_{1,\mathcal{E},0} \|\boldsymbol{v}\|_{1,\mathcal{E},0} + \|\boldsymbol{f}\|_{L^2(\Omega)^d} \|\boldsymbol{v}\|_{L^2(\Omega)^d},$$

which yields

$$\|p\|_{L^2} \leq C_{\eta_{\mathcal{M}}} \|\boldsymbol{f}\|_{L^2(\Omega)^d}. \tag{2.3.19}$$

thanks to (2.3.15b), (2.3.18) and to the estimate (2.3.17). Let us now prove the existence of a solution to (2.3.8). Consider the continuous mapping

$$F : \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M},0} \times [0,1] \longrightarrow \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M},0}$$
$$(\boldsymbol{u}, p, \zeta) \mapsto F(\boldsymbol{u}, p, \zeta) = (\hat{\boldsymbol{u}}, \hat{p})$$

where $(\hat{\boldsymbol{u}}, \hat{p}) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M},0}$ is such that

$$\int_\Omega \hat{\boldsymbol{u}} \cdot \boldsymbol{v} = [\boldsymbol{u}, \boldsymbol{v}]_{1,\mathcal{E},0} + \zeta\ b_{\mathcal{E}}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - \int_\Omega p \ \mathrm{div}_{\mathcal{M}}\ \boldsymbol{v} - \int_\Omega \boldsymbol{f}_{\mathcal{E}} \cdot \boldsymbol{v},\ \forall \boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0} \tag{2.3.20a}$$
$$\int_\Omega \hat{p}\ q = \int_\Omega \mathrm{div}_{\mathcal{M}}\ \boldsymbol{u}\ q,\ \forall q \in L_{\mathcal{M}}. \tag{2.3.20b}$$

It is easily checked that F is indeed a one to one mapping, since the values of $\hat{u}^{(i)}; i = 1, \cdots, d$, and $\hat{p}$ are readily obtained by setting in this system $v_i = 1_{D_\sigma}$, $v_j = 0, j \neq i$ in (2.3.20a) and $q = 1_K$ in (2.3.20b). The mapping F is continuous; moreover, if $(\boldsymbol{u}, p) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M},0}$ is such that $F(\boldsymbol{u}, p, \zeta) = (0, 0)$, then for any $(\boldsymbol{v}, q) \in \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M}}$,

$$[\boldsymbol{u}, \boldsymbol{v}]_{1,\mathcal{E},0} + \zeta\ b_{\mathcal{E}}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - \int_\Omega p\, \mathrm{div}_{\mathcal{M}}(\boldsymbol{v})\ \mathrm{d}\boldsymbol{x} = \int_\Omega \mathcal{P}_{\mathcal{E}} \boldsymbol{f} \cdot \boldsymbol{v}\ \mathrm{d}\boldsymbol{x},$$
$$\int_\Omega \mathrm{div}_{\mathcal{M}}\ (\boldsymbol{u})\, q\ \mathrm{d}\boldsymbol{x} = 0.$$

The arguments used in the above estimates on possible solutions of (2.3.8) may be used in a similar way to show that $(\boldsymbol{u}, p)$ is bounded independently of ζ. Since $F(\boldsymbol{u}, p, 0) = 0$ is a bijective affine function by the stability of the linear Stokes problem (see [2]), the existence of at least one solution $(\boldsymbol{u}, p)$ to the equation $F(\boldsymbol{u}, p, 1) = 0$ which is exactly (2.3.8), follows by a topological degree argument which is recalled in the Appendix (Theorem 4.0.11, see also [14] for the theory, [21] for the first application to a nonlinear scheme and [26, Theorem 4.3] for an easy formulation of the result which can be used here). □

2.3.3 Convergence analysis

Lemma 2.3.5 (Full grid velocity interpolate)**.** *For a given MAC mesh* $(\mathcal{M}, \mathcal{E})$*, we define, for* $i, j = 1, \ldots, d$*, the i-th full grid velocity reconstruction operator by*

$$\mathcal{R}_{\mathcal{E}}^{(i,j)} :\ H_{\mathcal{E},0}^{(i)} \to H_{\mathcal{E},0}^{(j)}$$
$$v \mapsto \mathcal{R}_{\mathcal{E}}^{(i,j)} v = \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}^{(j)}} \widehat{v}_\sigma \chi_{D_\sigma}, \tag{2.3.22}$$

where

$$\widehat{v}_\sigma = v_\sigma \text{ if } \sigma \in \mathcal{E}^{(i)}, \quad \widehat{v}_\sigma = \frac{1}{\operatorname{card}(\mathcal{N}_\sigma)} \sum_{\sigma' \in \mathcal{N}_\sigma} v_{\sigma'} \text{ otherwise,} \tag{2.3.23}$$

$$\text{where, for any } \sigma \in \mathcal{E} \setminus \mathcal{E}^{(i)},\ \mathcal{N}_\sigma = \{\sigma' \in \mathcal{E}^{(i)}, \overline{D_\sigma} \cap \sigma' \neq \emptyset\}. \tag{2.3.24}$$

Then there exists $C \geq 0$, depending only on the regularity of the mesh defined by (2.2.2)*, such that, for any $v \in L^2(\Omega)$, and any $i, j = 1, \ldots, d$, $\|\mathcal{R}_\mathcal{E}^{(i,j)} v\|_{L^2(\Omega)} \leq C\|v\|_{L^2(\Omega)}$.*

Proof. Let us prove the bound on $\|\mathcal{R}_\mathcal{E}^{(i,j)}\|_{L^2(\Omega)}$ for $d = 2$, $i = 1$ and $j = 2$. Other cases are similar. Let $v \in H_{\mathcal{E},0}^{(i)}$. By definition of $\mathcal{R}_\mathcal{E}^{(i,j)} v$, retaining for each $\sigma \in \mathcal{E}_{\text{int}}$ the cells where v_σ is involved and noting that $\left[\frac{1}{4}(a+b+c+d)\right]^2 \leq a^2 + b^2 + c^2 + d^2$, we have:

$$\begin{aligned} \|\mathcal{R}_\mathcal{E}^{(i,j)} v\|^2_{L^2(\Omega)} &\leq \sum_{\substack{\sigma \in \mathcal{E}^{(i)}_{\text{int}} \\ \sigma = K|L}} v_\sigma^2 (|D_{\sigma_K^t}| + |D_{\sigma_K^b}| + |D_{\sigma_L^t}| + |D_{\sigma_L^b}|) \\ &\leq 4\eta^2 \sum_{\substack{\sigma \in \mathcal{E}^{(i)}_{\text{int}} \\ \sigma = K|L}} v_\sigma^2 |D_\sigma| \end{aligned}$$

where $D_{\sigma_K^t}$ (resp. $D_{\sigma_K^b}$ denotes the velocity cell associated to the top (resp. bottom) edge of K, with $\sigma = K|L$, see Figure 2.4.

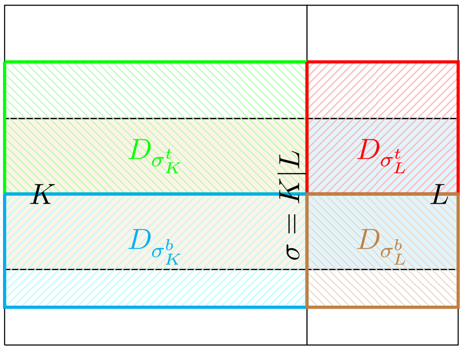


FIG. 2.4 – *Full grid velocity interpolate.*

□

Lemma 2.3.6 (Convergence of the full grid velocity interpolate). *Let $(\mathcal{M}_n, \mathcal{E}_n)_{n\in\mathbb{N}}$ be a sequence MAC meshes such that $h_{\mathcal{M}_n} \to 0$ as $n \to +\infty$, and $(\eta_{\mathcal{M}_n})_{n\in\mathbb{N}}$ remains bounded. Let $\bar{v} \in L^2(\Omega)$, and let $(v_n)_{n\in\mathbb{N}}$ be such that $v_n \in H^{(i)}_{\mathcal{E}_n,0}$ and v_n converges to $\bar{v}$ as $n \to +\infty$ in $L^2(\Omega)$. Let $i, j = 1, \ldots, d$ and $\mathcal{R}^{(i,j)}_{\mathcal{E}_n}$ be the full grid velocity reconstruction operator defined by* (2.3.22)*. Then $\mathcal{R}^{(i,j)}_{\mathcal{E}_n} v_n \to \bar{v}$ in $L^2(\Omega)$ as $n \to +\infty$.*

Proof. Let $\varphi \in C_c^\infty(\Omega)$. Denoting $\mathcal{R}_{\mathcal{E}_n}^{(i,j)}$ by $\mathcal{R}_n$ and $\mathcal{P}_{\mathcal{E}_n}^{(i)}$ by $\mathcal{P}_n$ for short (recall that $\mathcal{P}_{\mathcal{E}_n}^{(i)}$ is defined by (2.3.2)) we have:

$$\|\mathcal{R}_n v_n - \bar{v}\|_{L^2(\Omega)} \leq \|\mathcal{R}_n v_n - \mathcal{R}_n \circ \mathcal{P}_n \bar{v}\|_{L^2(\Omega)} + \|\mathcal{R}_n \circ \mathcal{P}_n \bar{v} - \mathcal{R}_n \circ \mathcal{P}_n \varphi\|_{L^2(\Omega)} + \|\mathcal{R}_n \circ \mathcal{P}_n \varphi - \varphi\|_{L^2(\Omega)} + \|\varphi - \bar{v}\|_{L^2(\Omega)}.$$

Since $\mathcal{R}_n v_n = \mathcal{R}_n \circ \mathcal{P}_n v_n$, and thanks to the fact that $\|\mathcal{R}_n\|_{L^2(\Omega)}$ is bounded (see Lemma 2.3.5) and that $\|\mathcal{P}_n\|_{L^2(\Omega)} \leq 1$, we get that there exists $C \geq 0$ such that

$$\|\mathcal{R}_n v_n - \bar{v}\|_{L^2(\Omega)} \leq C\|v_n - \bar{v}\|_{L^2(\Omega)} + C\|\bar{v} - \varphi\|_{L^2(\Omega)} + \|\mathcal{R}_n \circ \mathcal{P}_n \varphi - \varphi\|_{L^2(\Omega)} + \|\varphi - \bar{v}\|_{L^2(\Omega)}.$$

Let $\varepsilon > 0$. Let us choose $\varphi \in C_c^\infty(\Omega)$ so that $\|\varphi - \bar{v}\|_{L^2(\Omega)} \leq \frac{\varepsilon}{C+1}$. There exists n_1 such that $C\|v_n - \bar{v}\|_{L^2(\Omega)} \leq \varepsilon$, $\forall n \geq n_1$, and there exists n_2 such that $\|\mathcal{R}_n \circ \mathcal{P}_n \varphi - \varphi\|_{L^2(\Omega)} \leq \varepsilon$, $\forall n \geq n_2$. Therefore, for $n \geq \max(n_1, n_2)$, we get:

$$\|\mathcal{R}_n v_n - \bar{v}\|_{L^2(\Omega)} \leq 3\varepsilon,$$

which concludes the proof. □

Lemma 2.3.7 (Weak consistency of the nonlinear convection term)**.** *Let $(\mathcal{D}_n)_{n\in\mathbb{N}}$, with $\mathcal{D}_n = (\mathcal{M}_n, \mathcal{E}_n)$ be a sequence of meshes such that $h_{\mathcal{M}_n} = \max_{K\in\mathcal{M}_n} \operatorname{diam}(K) \to 0$ as $n \to +\infty$; assume that there exists $\eta > 0$ such that $\eta_{\mathcal{M}_n} \leq \eta$ for any $n \in \mathbb{N}$ (with $\eta_{\mathcal{M}_n}$ defined by* (2.2.2)*). Let $(\boldsymbol{v}_n)_{n\in\mathbb{N}}$ and $(\boldsymbol{w}_n)_{n\in\mathbb{N}}$ be two sequences of functions such that*

- *$\boldsymbol{v}_n \in \mathbf{H}_{\mathcal{E}_n,0}$ and $\boldsymbol{w}_n \in \mathbf{H}_{\mathcal{E}_n,0}$,*
- *the sequences $(\boldsymbol{v}_n)_{n\in\mathbb{N}}$ and $(\boldsymbol{w}_n)_{n\in\mathbb{N}}$ converge in $L^2(\Omega)^d$ to $\bar{\boldsymbol{v}} \in L^2(\Omega)^d$ and $\bar{\boldsymbol{w}} \in L^2(\Omega)^d$ respectively.*

Let $(\Pi_{\mathcal{E}_n})_{n\in\mathbb{N}}$ be a family of interpolators satisfying (2.2.8) *and let $\boldsymbol{\varphi} \in C_c^\infty(\Omega)^d$.*
Then $b_{\mathcal{E}}(\boldsymbol{v}_n, \boldsymbol{w}_n, \Pi_{\mathcal{E}_n}\boldsymbol{\varphi}) \to b(\bar{\boldsymbol{v}}, \bar{\boldsymbol{w}}, \boldsymbol{\varphi})$ as $n \to +\infty$.

Proof. Let $i \in [\![1, d]\!]$. We have: $b_{\mathcal{E}_n}(\boldsymbol{v}_n, \boldsymbol{w}_n, \Pi_{\mathcal{E}_n}\boldsymbol{\varphi}) = \sum_{i=1}^d b_{\mathcal{E}}^{(i)}(\boldsymbol{v}, w_i, \Pi_{\mathcal{E}}^{(i)}\varphi_i)$, where we have omitted the sub- and superscripts n for the sake of clarity in the right hand side of the equality, with:

$$b_{\mathcal{E}}^{(i)}(\boldsymbol{v}, w_i, \Pi_{\mathcal{E}}^{(i)}\varphi_i) = \sum_{\sigma\in\mathcal{E}^{(i)}} \varphi_{i,\sigma} \sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)} |\epsilon| v_{\sigma,\epsilon} w_\epsilon = S_1 + S_2,$$

where $\varphi_{i,\sigma} = \varphi_i(\boldsymbol{x}_\sigma)$, with

$$S_1 = \sum_{\substack{\epsilon=\overrightarrow{\sigma|\sigma'}\in\widetilde{\mathcal{E}}^{(i)}_{\mathrm{int}}\\ \epsilon\perp \boldsymbol{e}_i,\epsilon\subset K}} |\epsilon| \frac{v_\sigma + v_{\sigma'}}{2}\frac{w_\sigma + w_{\sigma'}}{2}\varphi_{i,\sigma}, \quad S_2 = \sum_{\substack{\epsilon=\overrightarrow{\sigma|\sigma'}\in\widetilde{\mathcal{E}}^{(i)}_{\mathrm{int}}\\ \epsilon\not\perp \boldsymbol{e}_i,\epsilon\subset \tau\cup\tau'}} |\epsilon| \frac{v_\tau + v_{\tau'}}{2}\frac{w_\sigma + w_{\sigma'}}{2}\varphi_{i,\sigma},$$

where τ and τ' are the faces of $\mathcal{E}^{(j)}, j \neq i$ such that $\epsilon \subset \tau \cup \tau'$.

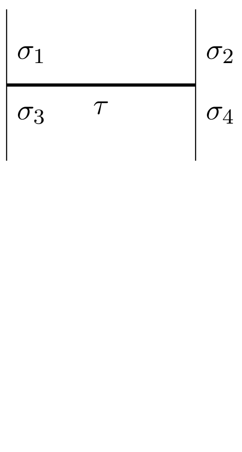
For $K \in \mathcal{M}$ and $\sigma, \sigma' \in \mathcal{E}(K) \cap \mathcal{E}^{(i)}$ we denote by $\tilde{v}_{K,i}$ the mean value $\frac{1}{2}(v_\sigma + v_{\sigma'})$. Reordering over the edges, we get that

$$\begin{aligned} S_1 &= \sum_{\substack{K\in\mathcal{M}\\ K=\overrightarrow{[\sigma\sigma']}}} |\sigma| \tilde{v}_{K,i}\tilde{w}_{K,i}(\varphi_{i,\sigma} - \varphi_{i,\sigma'}) \\ &= \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}}\\ \sigma=K|L}} |(D_{K,\sigma}\tilde{v}_{K,i} + D_{L,\sigma}\tilde{v}_{L,i})\tilde{w}_{K,i}\eth_i\Pi^{(i)}_{\mathcal{E}}\varphi_i \\ &\to -\int_\Omega \bar{v}_i\ \bar{w}_i\ \partial_i\varphi_i\ \mathrm{d}\boldsymbol{x} \text{ as } n \to +\infty \end{aligned}$$

thanks to the fact that $\tilde{v}_{n,i} \to \bar{v}_i$ and $\tilde{w}_{n,i} \to \bar{w}_i$ in $L^2(\Omega)$, and thanks to Lemma 2.2.3. Now $S_2 = \sum_{\substack{j\in[1,d]\\ j\neq i}} S_{2,j}$ with

$$S_{2,j} = \sum_{\tau\in\mathcal{E}^{(j)}_{\mathrm{int}}} |\tau|\frac{v_\tau}{4}[\sum_{k=1}^{4} (w_{\sigma_3} + w_{\sigma_1})\,\varphi_{i,\sigma_1} + (w_{\sigma_4} + w_{\sigma_2})\,\varphi_{i,\sigma_2} - (w_{\sigma_1} + w_{\sigma_3})\,\varphi_{i,\sigma_3} - (w_{\sigma_2} + w_{\sigma_4})\,\varphi_{i,\sigma_4}]$$

where $(\sigma_k)_{k=1,\ldots,4}$ are the four neighbouring faces (or edges) of τ belonging to $\mathcal{E}^{(i)}$, *i.e.* such that $\bar{\tau} \cap \bar{\sigma}_k \neq \emptyset$, see the following figure:



Thus,

$$S_{2,j} = \sum_{\tau \in \mathcal{E}_{\rm int}^{(j)}} |\tau| \frac{v_\tau}{4} \left[(w_{\sigma_3} + w_{\sigma_1}) (\varphi_{i,\sigma_1} - \varphi_{i,\sigma_3}) + (w_{\sigma_4} + w_{\sigma_2}) (\varphi_{i,\sigma_2} - \varphi_{i,\sigma_4}) \right]$$

$$= - \sum_{\tau \in \mathcal{E}^{(j)}} |D_\tau| v_\tau \hat{w}_\tau \partial_j \varphi_i(\boldsymbol{x}_\tau) + R$$

where $|R| \leq C_{\boldsymbol{\varphi},\eta} \|v_{n,i}\|_{L^2(\Omega)} \|w_{n,j}\|_{L^2(\Omega)} h_n$, with $C_{\boldsymbol{\varphi},\eta} \geq 0$ depending only on $\boldsymbol{\varphi}$ and η. Hence by Lemma 2.3.6,

$$S_{2,j} \to - \int_\Omega \bar{v}_i \ \bar{w}_j \ \partial_j \varphi_i \ {\rm d}\boldsymbol{x} \text{ as } n \to +\infty,$$

which concludes the proof. □

Theorem 2.3.8 (Convergence of the scheme). *Let $(\mathcal{D}_n)_{n\in\mathbb{N}}$, with $\mathcal{D}_n = (\mathcal{M}_n, \mathcal{E}_n)$ be a sequence of meshes such that $h_{\mathcal{M}_n} = \max_{K\in\mathcal{M}_n} {\rm diam}(K) \to 0$ as $n \to +\infty$; assume that there exists $\eta > 0$ such that $\eta_{\mathcal{M}_n} \leq \eta$ for any $n \in \mathbb{N}$ (with $\eta_{\mathcal{M}_n}$ defined by (2.2.2)). Let $(\boldsymbol{u}_n, p_n)$ be a solution to the MAC scheme (2.3.1) or its weak form (2.3.8), for $\mathcal{D} = \mathcal{D}_n$. Then there exists $\bar{\boldsymbol{u}} \in H_0^1(\Omega)^d$ and $\bar{p} \in L^2(\Omega)$ such that, up to a subsequence:*

- *the sequence $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ converges to $\bar{\boldsymbol{u}}$ in $L^2(\Omega)^d$,*
- *the sequence $(\nabla_n \boldsymbol{u}_n)_{n\in\mathbb{N}}$ converges to $\nabla\bar{\boldsymbol{u}}$ in $L^2(\Omega)^{d\times d}$,*
- *the sequence $(p_n)_{n\in\mathbb{N}}$ converges to $\bar{p}$ in $L^2(\Omega)$,*
- *$(\bar{\boldsymbol{u}}, \bar{p})$ is a solution to the weak formulation (2.1.2).*

Proof. Thanks to the estimate (2.3.17) on the velocity, we can apply the classical translate estimate [22, Theorem 14.2] and the estimates on the translates [22, Theorem 14.1] to obtain the existence of a subsequence of approximate solutions $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ which converges to some $\bar{\boldsymbol{u}} \in L^2(\Omega)^d$. From the estimates on the translates, we also get the regularity of the limit, that is $\bar{\boldsymbol{u}} \in H_0^1(\Omega)^d$. The estimate (2.3.19) on the pressure then yields the weak convergence of a subsequence of $(p_n)_{n\in\mathbb{N}}$ to some $\bar{p}$ in $L^2(\Omega)$. Let us then pass to the limit in the scheme in order to prove its (weak) consistency.

Passing to the limit in the mass balance equation: Let $\psi \in C_c^\infty(\Omega)$, taking $\psi_n = \Pi_{\mathcal{M}_n}\psi$ the interpolate of ψ satisfying (2.2.11) as test function in (2.3.8b) and using (2.2.12), we get that:

$$0 = \int_\Omega {\rm div}_{\mathcal{M}_n} \boldsymbol{u}_n \psi_n \ {\rm d}\boldsymbol{x} = - \int_\Omega \nabla_{\mathcal{M}_n} \ \psi_n \cdot \boldsymbol{u}_n \ {\rm d}\boldsymbol{x} = - \sum_{i=1}^d \int_\Omega u_n^{(i)} \eth_i \psi_n \ {\rm d}\boldsymbol{x}.$$

Therefore, thanks to Lemma 2.2.4,

$$0 = \lim_{n\to+\infty} \int_\Omega \mathrm{div}_{\mathcal{M}_n} \boldsymbol{u}_n \psi_n \ \mathrm{d}\boldsymbol{x} = -\sum_{i=1}^{d} \int_\Omega \bar{u}^{(i)} \partial_i \psi \ \mathrm{d}\boldsymbol{x} = -\int_\Omega \bar{\boldsymbol{u}} \cdot \nabla \psi \ \mathrm{d}\boldsymbol{x} = \int_\Omega \mathrm{div} \bar{\boldsymbol{u}} \ \psi \ \mathrm{d}\boldsymbol{x}.$$

so that $\bar{\boldsymbol{u}}$ satisfies (2.3.8b).

Passing to the limit in the momentum balance equation: Let $\boldsymbol{\varphi} = (\varphi_1, \cdots, \varphi_d)^t \in$

$(C_c^\infty(\Omega))^d$, and take $\boldsymbol{\varphi}_n = \Pi_{\mathcal{E}_n} \boldsymbol{\varphi} = (\varphi_{n,1}, \cdots, \varphi_{n,d})^t \in \boldsymbol{H}_{\mathcal{E}_n,0}$ as test function in (2.3.8a); where $\Pi_{\mathcal{E}_n} \boldsymbol{\varphi}$ is an interpolate of $\boldsymbol{\varphi}$ satisfying (2.2.8) this yields:

$$\int_\Omega \nabla_{\mathcal{E}_n} \boldsymbol{u}_n : \nabla_{\mathcal{E}_n} \boldsymbol{\varphi}_n \ \mathrm{d}\boldsymbol{x} + b_{\mathcal{E}}(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{\varphi}_n) - \int_\Omega p_n \ \mathrm{div}_{\mathcal{M}_n} \boldsymbol{\varphi}_n \ \mathrm{d}\boldsymbol{x} = \int_\Omega \mathcal{P}_{\mathcal{E}_n} \boldsymbol{f} \cdot \boldsymbol{\varphi}_n \ \mathrm{d}\boldsymbol{x}. \quad (2.3.25)$$

Thanks to the L^2 convergence of $\boldsymbol{u}_n$ to $\bar{\boldsymbol{u}}$, to the weak L^2 convergence of p_n to $\bar{p}$ and to the uniform convergence of $\mathcal{P}_{\mathcal{E}_n} \boldsymbol{f}$ to $\boldsymbol{f}$ and of $\mathrm{div}_{\mathcal{M}_n} \boldsymbol{\varphi}_n$ to $\mathrm{div} \boldsymbol{\varphi}$ (see Lemma 2.2.3) as $n \to +\infty$, we have

$$\int_\Omega \mathcal{P}_{\mathcal{E}_n} \boldsymbol{f} \cdot \boldsymbol{\varphi}_n \ \mathrm{d}\boldsymbol{x} \to \int_\Omega \boldsymbol{f} \cdot \bar{\boldsymbol{\varphi}} \ \mathrm{d}\boldsymbol{x} \text{ and } \int_\Omega p_n \ \mathrm{div}_{\mathcal{M}_n} \boldsymbol{\varphi}_n \ \mathrm{d}\boldsymbol{x} \to \int_\Omega \bar{p} \ \mathrm{div} \ \bar{\boldsymbol{\varphi}} \ \mathrm{d}\boldsymbol{x} \text{ as } n \to \infty$$

From [22, Proof of Theorem 9.1], we get that

$$\int_\Omega \nabla_{\mathcal{E}_n} u_{n,i} : \nabla_{\mathcal{E}_n} \varphi_{n,i} \ \mathrm{d}\boldsymbol{x} = [u_{n,i}, \varphi_{n,i}]_{1,\mathcal{E}_n^{(i)},0} \to -\int_\Omega \bar{u}_i \Delta \varphi_i \ \mathrm{d}\boldsymbol{x} \text{ as } n \to +\infty.$$

and therefore

$$\int_\Omega \nabla_{\mathcal{E}_n} \boldsymbol{u}_n : \nabla_{\mathcal{E}_n} \boldsymbol{\varphi}_n \ \mathrm{d}\boldsymbol{x} \to -\sum_{i=1}^{d} \int_\Omega \bar{u}_i \Delta \varphi_i \ \mathrm{d}\boldsymbol{x} = \int_\Omega \nabla \bar{\boldsymbol{u}} : \nabla \boldsymbol{\varphi} \ \mathrm{d}\boldsymbol{x} \text{ as } n \to +\infty.$$

By Lemma 2.3.7, we have

$$\lim_{n\to+\infty} b_{\mathcal{E}_n}(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{\varphi}_n) = \int_\Omega (\bar{\boldsymbol{u}} \cdot \nabla) \bar{\boldsymbol{u}} \cdot \boldsymbol{\varphi} \ \mathrm{d}\boldsymbol{x}. \quad (2.3.26)$$

Passing to the limit as $n \to +\infty$ in (2.3.25) thus yields that $\bar{\boldsymbol{u}}$ and $\bar{p}$ satisfy (2.1.2). Let us now prove the strong convergence of $\nabla_{\mathcal{E}_n} \boldsymbol{u}_n$ to $\nabla \bar{\boldsymbol{u}}$ in $L^2(\Omega)$. The sequence $(\boldsymbol{\nabla}_{\mathcal{E}_n} \boldsymbol{u}_n)_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)^{d\times d}$ and therefore, there exists $\xi \in L^2(\Omega)^{d\times d}$ and a subsequence still denoted by $(\boldsymbol{\nabla}_{\mathcal{E}_n} \boldsymbol{u}_n)_{n\in\mathbb{N}}$ converging to ξ weakly in $L^2(\Omega)^{d\times d}$. Since $\int_\Omega \boldsymbol{\nabla}_{\mathcal{E}_n} \boldsymbol{u}_n \ \boldsymbol{\varphi}_n \ \mathrm{d}\boldsymbol{x} = \int_\Omega \mathrm{div}_n \boldsymbol{\varphi}_n \ \boldsymbol{u}_n \ \mathrm{d}\boldsymbol{x}$, the uniqueness of the limit in the sense of distributions implies that $\nabla \bar{\boldsymbol{u}} = \xi$. Taking $\boldsymbol{\varphi}_n = \boldsymbol{u}_n$

in (2.3.25) this yields:

$$\int \nabla_{\mathcal{E}_n} \boldsymbol{u}_n : \nabla_{\mathcal{E}_n} \boldsymbol{u}_n \, \mathrm{d}\boldsymbol{x} = \int_\Omega \mathcal{P}_{\mathcal{E}_n} \boldsymbol{f} \cdot \boldsymbol{u}_n \, \mathrm{d}\boldsymbol{x}.$$

Passing to the limit as $n \to \infty$ we get that:

$$\|\boldsymbol{\nabla}_{\mathcal{E}_n} \boldsymbol{u}_n\|^2_{L^2(\Omega)^{d\times d}} = \|\boldsymbol{u}_n\|^2_{1,\mathcal{E}_n,0} \to \int_\Omega \boldsymbol{f} \cdot \bar{\boldsymbol{u}} \, \mathrm{d}\boldsymbol{x} = \|\boldsymbol{\nabla} \bar{\boldsymbol{u}}\|^2_{L^2(\Omega)^{d\times d}},$$

which implies the strong convergence of the discrete gradient of the velocity.

Let us finally prove the strong convergence of the pressure. Let $\boldsymbol{\varphi}_n \in (H^1_0(\Omega))^d$ be such that $\mathrm{div}\boldsymbol{\varphi}_n = p_n$ a.e. in Ω and

$$\|\boldsymbol{\varphi}_n\|_{H^1_0(\Omega)^d} \leq c\|p_n\|_{L^2(\Omega)},$$

where c depends only on Ω. Let $\boldsymbol{\psi}_n = \widetilde{\mathcal{P}}_{\mathcal{E}_n} \boldsymbol{\varphi}_n$; thanks to Lemma 2.3.3, we have $\|\boldsymbol{\psi}_n\|_{1,\mathcal{E}_n,0} \leq c\, C_{\eta_n} \|p_n\|_{L^2(\Omega)}$, and since p_n is piecewise constant, we get that $\mathrm{div}_{\mathcal{M}_n} \boldsymbol{\psi}_n = p_n$, Therefore, taking $\boldsymbol{\psi}_n = \widetilde{\mathcal{P}}_{\mathcal{E}_n} \boldsymbol{\varphi}_n$ as test function in (2.3.8a), we obtain:

$$\int_\Omega p_n^2 \, \mathrm{d}\boldsymbol{x} = \int_\Omega \nabla_{\mathcal{E}_n} \boldsymbol{u}_n : \nabla_{\mathcal{E}_n} \boldsymbol{\psi}_n \, \mathrm{d}\boldsymbol{x} + b_\mathcal{E}(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{\psi}_n) - \int_\Omega \mathcal{P}_{\mathcal{E}_n} \boldsymbol{f} \cdot \boldsymbol{\psi}_n \, \mathrm{d}\boldsymbol{x},$$

and $\|\boldsymbol{\psi}_n\|_{1,\mathcal{E},0} \leq c\, C_{\eta_n} \|p_n\|_{L^2(\Omega)}$.

From the bound on $\|\boldsymbol{\psi}_n\|_{1,\mathcal{E},0}$ we know that $\boldsymbol{\psi}_n$ converges to some $\boldsymbol{\psi} \in H^1_0(\Omega)^d$ in $L^2(\Omega)$ and thanks to (2.3.6) that $\nabla_{\mathcal{E}_n} \boldsymbol{\psi}_n \to \nabla \boldsymbol{\psi}$ weakly in $(L^2(\Omega)^{d\times d})$ as $n \to +\infty$. Passing to the limit as $n \to \infty$ we get that

$$\|p_n\|^2_{L^2(\Omega)} \to \int_\Omega \nabla \bar{\boldsymbol{u}} : \nabla \boldsymbol{\psi} \, \mathrm{d}\boldsymbol{x} + b(\bar{\boldsymbol{u}}, \bar{\boldsymbol{u}}, \boldsymbol{\psi}) - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\psi} \, \mathrm{d}\boldsymbol{x}.$$

Since $(\bar{\boldsymbol{u}}, \bar{p})$ satisfies (2.1.2), this implies that $\|p_n\|_{L^2(\Omega)} \to \|\bar{p}\|_{L^2(\Omega)}$, which in turn yields that $p_n \to \bar{p}$ in $L^2(\Omega)$ as $n \to +\infty$. □

2.4 Unsteady case

2.4.1 Time discretization

Let us now turn to the time discretization of the problem (2.1.3); we consider a MAC grid $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ of Ω in the sense of Definition 2.2.1, and a partition $0 = t_0 < t_1 < \cdots < t_N = T$ of the time interval $(0, T)$, and, for the sake of simplicity, a constant time step $\delta t = t_{n+1} - t_n$; hence $t_n = n\delta t$ for $n \in \{0, \cdots, N-1\}$. Let $\{u_\sigma^{(n+1)}, \sigma \in \mathcal{E}^{(i)}, n \in \{0, \cdots, N-1\}\}$ and

$\{p_K^{(n+1)}, K \in \mathcal{M}, n \in \{0, \cdots, N-1\}\})$ be the sets of discrete velocity and pressure unknowns; we define the corresponding piecewise constant functions $\boldsymbol{u} = (u_1, \ldots, u_d)$ and p. For the velocities, these constant functions are of the form:

$$u_i = \sum_{n=0}^{N-1} \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} u_\sigma^{(n+1)} \chi_{D_\sigma} \chi_{]t_n, t_{n+1}]},$$

where $\chi_{]t_n,t_{n+1}]}$ is the characteristic function of the interval $]t_n, t_{n+1}]$. We denote by $X_{i,\mathcal{E},\delta t}$ the set of such piecewise constant functions on time intervals and dual cells, and we set $\boldsymbol{X}_{\mathcal{E},\delta t} = \prod_{i=1}^d X_{i,\mathcal{E},\delta t}$. For the pressure, the constant functions are of the form:

$$p = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} p_K^{(n+1)} \chi_K \chi_{]t_n, t_{n+1}]},$$

and we denote by $Y_{\mathcal{M},\delta t}$ the space of such piecewise constant functions. We look for an approximation $(\boldsymbol{u}, p) \in \boldsymbol{X}_{\mathcal{E},\delta t} \times Y_{\mathcal{M},\delta t}$ of $(\bar{\boldsymbol{u}}, \bar{p})$ solution of the problem (2.1.3). For $\sigma \in \mathcal{E}^{(i)}$, $i \in \{1, \cdots, d\}$ the value $u_\sigma^{(n+1)}$ is an expected approximation of $u_i(\boldsymbol{x}, t_{n+1})$, for $\boldsymbol{x} \in D_\sigma$, and the value $p_K^{(n+1)}$ is an expected approximation of $p(\boldsymbol{x}, t_{n+1})$ for $\boldsymbol{x} \in K$. For a given $\boldsymbol{u} \in \boldsymbol{X}_{\mathcal{E},\delta t}$ associated to the set of discrete velocity unknowns $\{u_\sigma^{(n+1)}, \sigma \in \mathcal{E}^{(i)}, n \in \{0, \cdots, N-1\}\}$, and for $n \in \{0, \cdots, N-1\}$, we denote by $u_i^{(n)} \in H_{\mathcal{E},0}^{(i)}$ the piecewise constant function defined by $u_i^{(n)}(\boldsymbol{x}) = u_\sigma^{(n)}$ for $\boldsymbol{x} \in D_\sigma, \sigma \in \mathcal{E}^{(i)}$, and set $\boldsymbol{u}^{(n)} = (u_1^{(n)}, \ldots, u_d^{(n)})^t \in \mathbf{H}_\mathcal{E}$. Setting

$$\boldsymbol{u}(\cdot, 0) = \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} u_\sigma^{(0)} \chi_{D_\sigma} = \mathcal{P}_\mathcal{E} \boldsymbol{u}_0,$$

we define the discrete time derivative $\eth_t \boldsymbol{u} \in \boldsymbol{X}_{\mathcal{E},\delta t}$ by

$$\eth_t \boldsymbol{u} = \sum_{n=0}^{N-1} \frac{1}{\delta t} (\boldsymbol{u}^{(n+1)} - \boldsymbol{u}^{(n)}) \chi_{]t_n, t_{n+1}]}$$

Denoting by $\boldsymbol{u}^{(n)} = \boldsymbol{u}(\cdot, t_n)$ and $p^{(n)} = p(\cdot, t_n)$, the time-implicit MAC scheme for the transient Navier-Stokes reads:

Initialization

$$\boldsymbol{u}^{(0)} = \widetilde{\mathcal{P}}_{\mathcal{E}} \boldsymbol{u}_0 \tag{2.4.1a}$$

Step $n \geq 0$. Solve for $\boldsymbol{u}^{(n+1)}$ and $p^{(n+1)}$:

$$\boldsymbol{u}^{(n+1)} \in \mathbf{H}_{\mathcal{E},0}, \ p^{(n+1)} \in L_{\mathcal{M},0}, \tag{2.4.1b}$$

$$\eth_t \boldsymbol{u}^{(n+1)} - \Delta_{\mathcal{E}} \boldsymbol{u}^{(n+1)} + \boldsymbol{C}_{\mathcal{E}}(\boldsymbol{u}^{(n+1)})\boldsymbol{u}^{(n+1)} + \boldsymbol{\nabla}_{\mathcal{E}}\ p^{(n+1)} = \boldsymbol{f}_{\mathcal{E}}^{(n+1)}, \tag{2.4.1c}$$

$$\mathrm{div}_{\mathcal{M}} \boldsymbol{u}^{(n+1)} = 0, \tag{2.4.1d}$$

where for all $n \in \{0, \ldots, N-1\}$, $\boldsymbol{f}_{\mathcal{E}}^{(n+1)} = \mathcal{P}_{\mathcal{E}} \boldsymbol{f}(\cdot, t^{(n+1)})$ (recall that $\mathcal{P}_{\mathcal{E}}$ is the mean value operator defined by (2.3.2)). A weak formulation of Step n of the scheme (2.4.1) reads:

$$\text{Find } \boldsymbol{u}^{(n+1)} \in \boldsymbol{E}_{\mathcal{E}}\ ;\ n \in \{0, \cdots, N-1\}, \text{ such that, for any } \boldsymbol{v} \in \boldsymbol{E}_{\mathcal{E}},$$
$$\int_\Omega \eth_t \boldsymbol{u}^{(n+1)} \cdot \boldsymbol{v}\ \mathrm{d}\boldsymbol{x} + \int_\Omega \nabla \boldsymbol{u}^{(n+1)} \cdot \nabla \boldsymbol{v}\ \mathrm{d}\boldsymbol{x} + b_{\mathcal{E}}(\boldsymbol{u}^{(n+1)}, \boldsymbol{u}^{(n+1)}, \boldsymbol{v}) = \int_\Omega \boldsymbol{f}_{\mathcal{E}}^{(n+1)} \cdot \boldsymbol{v}\ \mathrm{d}\boldsymbol{x}. \tag{2.4.2}$$

2.4.2 Existence and estimates on the approximation solution

Lemma 2.4.1 (Existence and first estimates on the velocity)**.** *There exists at least a solution* $\boldsymbol{u} \in \boldsymbol{X}_{\mathcal{M},\delta t}$ *satisfying* (2.4.1)*. Furthermore, there exists* $C > 0$ *depending only on* $\boldsymbol{u}_0$ *and* $\boldsymbol{f}$ *such that any function* $\boldsymbol{u} \in \boldsymbol{X}_{\mathcal{M},\delta t}$ *satisfying* (2.4.1) *satisfies:*

$$\|\boldsymbol{u}\|_{L^2(0,T;\mathbf{H}_{\mathcal{E},0})} \leq C, \tag{2.4.3}$$

$$\|\boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega)^d)} \leq C, \tag{2.4.4}$$

where $\|\boldsymbol{u}\|^2_{L^2(0,T;\mathbf{H}_{\mathcal{E},0})} = \sum_{n=0}^{N-1} \delta t \|\boldsymbol{u}^{(n+1)}\|^2_{1,\mathcal{E},0}$, $\|\boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega)^d)} = \max\{\|\boldsymbol{u}^{(n+1)}\|_{L^2(\Omega)^d},\ n \in [\![0, N-1]\!]\}$, *and* $\boldsymbol{u}^{(n)} = \boldsymbol{u}(\cdot, t_n)$*.*

Proof. We prove the a priori estimates (2.4.3) and (2.4.4). The existence of a solution then follows by a topological degree argument as for the stationary case.

Let $M \leq N-1$; taking $\boldsymbol{v} = \boldsymbol{u}^{(n+1)}$ in (2.4.2), multiplying by δt and summing the result over $n \in \{0, \cdots, M\}$, we obtain thanks to Lemma 2.3.2 and to the Cauchy-Schwarz inequality:

$$\sum_{n=0}^{M} \sum_{\sigma \in \mathcal{E}^{(i)}} |D_\sigma| u_\sigma^{(n+1)} (u_\sigma^{(n+1)} - u_\sigma^{(n)}) + \sum_{n=0}^{M} \delta t \|u_i^{(n+1)}\|^2_{1,\mathcal{E}^{(i)},0} \leq \sum_{n=0}^{M} \delta t \|f_i(.,t_{n+1})\|_{L^2(\Omega)} \|u_i(.,t_{n+1})\|_{L^2(\Omega)}.$$

Using the fact that for all $a, b \in \mathbb{R}$, $a(a-b) = \frac{1}{2}(a-b)^2 + \frac{1}{2}a^2 - \frac{1}{2}b^2$ for the first term of the left hand-side and the discrete Poincaré and Young inequalities for the right and side, we get that

$$\|u_i^{(M+1)}\|^2_{L^2(\Omega)} + \sum_{n=0}^{M} \delta t \|u_i^{(n+1)}\|^2_{1,\mathcal{E}^{(i)},0} \leq \|u_i^{(0)}\|^2_{L^2(\Omega)} + C_P^2 \|f^{(i)}\|^2_{L^2(0,T;L^2(\Omega))},$$

where $C_P > 0$ depends only on Ω. On one hand, this inequality yields the L^∞ estimate (2.4.4); on the other hand, taking $M = N-1$ and summing for $i = 1, \ldots, d$, we get the L^2 estimate (2.4.3).

□

Next we turn to an estimate on the discrete time derivative. To this end, we introduce the following discrete dual norms on $\mathbf{H}_{\mathcal{E},0}$ and $\boldsymbol{X}_{\mathcal{E},\delta t}$.

$$\begin{aligned} &\boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0} \mapsto \|\boldsymbol{v}\|_{\boldsymbol{E}'_\mathcal{E}} = \max\{\left|\int_\Omega \boldsymbol{v} \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x}\right| \; ; \; \boldsymbol{\varphi} \in \boldsymbol{E}_\mathcal{E} \text{ and } \|\boldsymbol{\varphi}\|_{1,\mathcal{E},0} \leq 1\}, \\ &\boldsymbol{v} \in \boldsymbol{X}_{\mathcal{E},\delta t} \mapsto \|\boldsymbol{v}\|_{L^{4/3}(0,T;\boldsymbol{E}'_\mathcal{E})} = \left(\sum_{n=0}^{N-1} \delta t \|\boldsymbol{v}^{n+1)}\|^{4/3}_{\boldsymbol{E}'_\mathcal{E}} \right)^{3/4}. \end{aligned} \tag{2.4.5}$$

Lemma 2.4.2 (Estimate on the dual norm of the discrete time derivative)**.** *Let* $\boldsymbol{u} \in \boldsymbol{X}_{\mathcal{E},\delta t}$ *be a solution to* (2.4.1)*. Then there exists* $C > 0$ *depending only on* $\boldsymbol{u}_0$, Ω, $\eta_\mathcal{M}$ *and* $\boldsymbol{f}$ *such that:*

$$\|\eth_t \boldsymbol{u}\|_{L^{4/3}(0,T;\boldsymbol{E}'_\mathcal{E})} \leq C.$$

Proof. If $\boldsymbol{u} \in \boldsymbol{X}_{\mathcal{E},\delta t}$ is a solution to (2.4.1) then $\boldsymbol{u}^{(n+1)} = \boldsymbol{u}(\cdot, t_{n+1}) \in \boldsymbol{E}_\mathcal{E}$ is a solution to (2.4.2); taking $\boldsymbol{v} \in \boldsymbol{E}_\mathcal{E}$ such that $\|\boldsymbol{v}\|_{1,\mathcal{E},0} \leq 1$ as test function in (2.4.2) we have $\forall n \in \{0, \cdots, N-1\}$:

$$\int_\Omega \eth_t \boldsymbol{u}^{(n+1)} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \int_\Omega \nabla \boldsymbol{u}^{(n+1)} \cdot \nabla \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \; b_\mathcal{E}(\boldsymbol{u}^{(n+1)}, \boldsymbol{u}^{(n+1)}, \boldsymbol{v}) = \int_\Omega \boldsymbol{f}_\mathcal{E}^{(n+1)} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}.$$

By Lemma 2.3.2 and thanks to the estimate (2.3.9) we have

$$|b_\mathcal{E}(\boldsymbol{u}^{(n+1)}, \boldsymbol{u}^{(n+1)}, \boldsymbol{v})| \leq C_{\eta_\mathcal{M}} \|\boldsymbol{u}^{(n+1)}\|^2_{L^4(\Omega))^d}.$$

Using the Cauchy-Schwarz inequality, we note that

$$\|\boldsymbol{u}^{(n+1)}\|_{L^4(\Omega)^d}^4 = \int_\Omega |\boldsymbol{u}^{(n+1)}||\boldsymbol{u}^{(n+1)}|^3 \, \mathrm{d}\boldsymbol{x} \le \|\boldsymbol{u}^{(n+1)}\|_{L^2(\Omega)^d}\|\boldsymbol{u}^{(n+1)}\|_{L^6(\Omega)^d}^3.$$

Therefore, thanks to the estimate (2.4.4) of Lemma 2.4.1 and to the discrete Poincaré inequality, there exists $\widetilde{C}_{\eta_\mathcal{M}} > 0$ depending only on Ω and on the regularity of the mesh, such that

$$\int_\Omega \eth_t \boldsymbol{u}^{(n+1)} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} \le \widetilde{C}_{\eta_\mathcal{M}} (\|_{(L^6(\Omega))^d}^{3/2} + \|\boldsymbol{u}^{(n+1)}\|_{1,\mathcal{E},0} + \|\boldsymbol{f}_\mathcal{E}^{(n+1)}\|_{(L^2(\Omega))^d}),$$

Hence

$$\begin{aligned}\|\eth_t \boldsymbol{u}^{(n+1)}\|_{\boldsymbol{E}'_\mathcal{E}}^{4/3} &\le 9\widetilde{C}_{\eta_\mathcal{M}}^{\frac{4}{3}} \left(\|\boldsymbol{u}^{(n+1)}\|_{L^6(\Omega)^d}^2 + \|\boldsymbol{u}^{(n+1)}\|_{1,\mathcal{E},0}^{4/3} + \|\boldsymbol{f}_\mathcal{E}^{(n+1)}\|_{L^2(\Omega)^d}^{4/3}\right) \\ &\le 9\widetilde{C}_{\eta_\mathcal{M}}^{\frac{4}{3}} \left(\|\boldsymbol{u}^{(n+1)}\|_{L^6(\Omega)^d}^2 + \|\boldsymbol{u}^{(n+1)}\|_{1,\mathcal{E},0}^{2} + \|\boldsymbol{f}_\mathcal{E}^{(n+1)}\|_{L^2(\Omega)^d}^{2} + 2\right).\end{aligned}$$

Multiplying this latter inequality by δt and summing for $n = 0, \dots, N-1$, we get

$$\|\boldsymbol{u}\|_{L^{4/3}(0,T;\boldsymbol{E}'_\mathcal{E})}^{\frac{4}{3}} \le 9\widetilde{C}_{\eta_\mathcal{M}}^{\frac{4}{3}} \left(\|\boldsymbol{u}\|_{L^2(0,T,L^6(\Omega)^d)}^2 + \|\boldsymbol{u}\|_{L^2(0,T,\mathbf{H}_{\mathcal{E},0})}^2 + \|\boldsymbol{f}\|_{L^2(0,T,L^2(\Omega)^d)}^2 + 2T\right).$$

We conclude by the discrete Sobolev inequality [22, Lemma 3.5] and thanks to the $L^2(0,T;\mathbf{H}_{\mathcal{E},0})$ estimate on $\boldsymbol{u}$ given in (2.4.3). □

2.4.3 Convergence analysis

Theorem 2.4.3 (Convergence of the scheme)**.** *Let $(\delta t_m)_{m\in\mathbb{N}}$ and $(\mathcal{D}_m)_{m\in\mathbb{N}} = (\mathcal{M}_m, \mathcal{E}_m)_{m\in\mathbb{N}}$ be a sequence of time steps and MAC grids (in the sense of Definition 2.2.1) such that $\delta t_m \to 0$ and $h_{\mathcal{M}_m} \to 0$ as $m \to +\infty$; assume that there exists $\eta > 0$ such that $\eta_{\mathcal{M}_m} \le \eta$ for any $m \in \mathbb{N}$ (with $\eta_{\mathcal{M}_m}$ defined by (2.2.2)). Let $\boldsymbol{u}_m$ be a solution to (2.4.2) for $\delta t = \delta t_m$ and $\mathcal{D} = \mathcal{D}_m$. Then there exists $\bar{\mathbf{u}} \in L^2(0,T;\boldsymbol{E}(\Omega))$ such that, up to a subsequence:*

- *the sequence $(\boldsymbol{u}_m)_{m\in\mathbb{N}}$ converges to $\bar{\mathbf{u}}$ in $L^{4/3}(0,T;L^2(\Omega)^d)$,*
- *$\bar{\mathbf{u}}$ is a solution to the weak formulation (2.1.4).*
- *$\partial_t \bar{\mathbf{u}} \in L^{4/3}(0,T;E'(\Omega))$.*

  If $d = 2$, the solution is unique and therefore the whole sequence converges and $\partial_t \bar{\mathbf{u}} \in L^2(0,T;E'(\Omega))$ see e.g. [6, pages 389-391].

Proof. We proceed in four steps.

First step: compactness in $L^{4/3}(0,T;L^2(\Omega)^d)$.

The first step consists in applying the discrete Aubin-Simon theorem 4.0.8 in order obtain the existence of subsequence of $(\boldsymbol{u}_m)_{m\in\mathbb{N}}$ which converges to $\bar{\boldsymbol{u}}$ in $L^{4/3}((0,T);L^2(\Omega)^d)$. In our setting, we apply Theorem 4.0.8 with $p=\frac{4}{3}$; the Banach space B of is $L^2(\Omega)^d$, and the spaces X_m and Y_m consist in the space $\mathbf{H}_{\mathcal{E}_m,0}$ endowed with the norms defined respectively in (2.3.4) and (2.4.5). By [22, Theorem 14.2] and the Kolmogorov compactness theorem (see e.g. [22, Theorem 14.1]) we obtain that $(X_m,Y_m)_{m\in\mathbb{N}}$ is compactly embedded in B in the sense of Definition 4.0.5. Let us then show that the sequence $(X_m,Y_m)_{m\in\mathbb{N}}$ is compact-continuous in $L^2(\Omega)^d$ in the sense of Definition 4.0.6. Let $\boldsymbol{v}_m\in\mathbf{H}_{\mathcal{E}_m,0}$ such that $(\|\boldsymbol{v}_m\|_{1,\mathcal{E}_m,0})_{m\in\mathbb{N}}$ is bounded and $\|\boldsymbol{v}_m\|_{\boldsymbol{E}'_m}\to 0$ as $m\to+\infty$. Assume that $\boldsymbol{v}_m\to\boldsymbol{v}$ in $(L^2(\Omega))^d$; by definition (2.4.5) of the dual norm, we have

$$\int_\Omega \boldsymbol{v}_m\cdot\boldsymbol{v}_m\,\mathrm{d}\boldsymbol{x}\le\|\boldsymbol{v}_m\|_{1,\mathcal{E}_m,0}\|\boldsymbol{v}_m\|_{\boldsymbol{E}'_m}.$$

Passing to the limit in this inequality as $m\to\infty$, we get that $\boldsymbol{v}=0$, so that the sequence $(X_m,Y_m)_{m\in\mathbb{N}}$ is compact-continuous in $L^2(\Omega)^d$. We now check the three assumptions (H1), (H2) and (H3) of Theorem 4.0.8: By Lemma 2.4.1, the sequence $\|\boldsymbol{u}_m\|_{L^1(0,T;\mathbf{H}_{\mathcal{E},0})}$ is bounded, and thanks to the discrete Poincaré inequality, the sequence $(\boldsymbol{u}_m)_{m\in\mathbb{N}}$ is also bounded in $L^{4/3}(0,T;(L^2(\Omega)^d))$; furthermore, the sequence $\|\eth_t\boldsymbol{u}_m\|_{L^{4/3}(0,T;\boldsymbol{E}'_{\mathcal{E}})}$ is bounded by Lemma 2.4.2. Hence, Theorem 4.0.8 applies and there exists $\bar{\boldsymbol{u}}\in L^{4/3}(0,T;L^2(\Omega)^d)$ such that, up to a subsequence,

$$\boldsymbol{u}_m\to\bar{\boldsymbol{u}}\text{ in }L^{4/3}\left(0,T;L^2(\Omega)^d\right)\text{ as }m\to+\infty.$$

Step 2: Convergence in $L^2(\Omega\times(0,T))$.
By Lemma 2.4.1, the sequence $(\boldsymbol{u}_m)_{m\in\mathbb{N}}$ is bounded in $L^\infty(0,T,L^2(\Omega)^d)$, and therefore, there exists $\hat{\boldsymbol{u}}\in L^\infty(0,T;L^2(\Omega)^d)$ and a subsequence $(\boldsymbol{u}_{\phi(m)})_{m\in\mathbb{N}}$ converging to $\hat{\boldsymbol{u}}$ $\star$-weakly in $L^\infty(0,T;L^2(\Omega)^d)$. Since $\boldsymbol{u}_{\phi(m)}\to\bar{\boldsymbol{u}}$ in $L^{4/3}(0,T;L^2(\Omega)^d)$, the uniqueness of the limit in the sense of distributions implies that $\bar{\boldsymbol{u}}=\hat{\boldsymbol{u}}$ so that $\bar{\boldsymbol{u}}\in L^\infty(0,T;L^2(\Omega)^d)$. By a classical interpolation result on $L^p(0,T)$ spaces, we have

$$\|\bar{\boldsymbol{u}}-\boldsymbol{u}_m\|_{L^2(0,T;L^2(\Omega)^d)}\le\|\bar{\boldsymbol{u}}-\boldsymbol{u}_m\|^{2/3}_{L^{4/3}(0,T;L^2(\Omega)^d)}\|\bar{\boldsymbol{u}}-\boldsymbol{u}_m\|^{1/3}_{L^\infty(0,T;L^2(\Omega)^d)},$$

which implies that $\boldsymbol{u}_m$ converges towards $\bar{\boldsymbol{u}}$ in $L^2(0,T;L^2(\Omega)^d)$ as m tends to infinity. **Step 3:**

Weak consistency of the scheme
The notion of weak consistency that we use here is the Lax-Wendroff notion: we show that if a sequence of approximate solutions of the scheme converges to some limit, then this limit is a weak solution to the original problem. Let us then show that $\bar{\boldsymbol{u}}$ satisfies (2.1.4). Let

$\boldsymbol{\varphi} \in C_c^\infty(\Omega \times [0,T))^d$, such that $\operatorname{div}\boldsymbol{\varphi} = 0$. By Lemma 2.3.3, we have $\operatorname{div}_{\mathcal{M}_m} \widetilde{\mathcal{P}}_{\mathcal{E}_m} \boldsymbol{\varphi}(\cdot, t_n) = 0$, and so we can take $\boldsymbol{\varphi}_m^{(n)} = \widetilde{\mathcal{P}}_{\mathcal{E}_m} \boldsymbol{\varphi}(\cdot, t_n) \in \boldsymbol{E}_{\mathcal{E}}$ as test function in (2.4.2) ; multiplying by δt_m and summing for $n = \{0, \ldots, N_m - 1\}$ (with $N_m \delta t_m = T$), we then get:

$$\sum_{n=0}^{N_m-1} \delta t_m \Big(\int_\Omega \eth_t \boldsymbol{u}_m^{(n+1)} \cdot \boldsymbol{\varphi}_m^{(n)} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_\Omega \nabla_{\mathcal{E}_m} \boldsymbol{u}_m^{(n+1)} : \nabla_{\mathcal{E}_m} \boldsymbol{\varphi}_m^{(n)} \,\mathrm{d}\boldsymbol{x} + b_{\mathcal{E}_m}(\boldsymbol{u}_m^{(n+1)}, \boldsymbol{u}_m^{(n+1)}, \boldsymbol{\varphi}_m^{(n)}) \\ - \int_\Omega \mathcal{P}_{\mathcal{E}_m} \boldsymbol{f}^{(n+1)} \cdot \boldsymbol{\varphi}_m^{(n)} \,\mathrm{d}\boldsymbol{x} \Big) = 0.$$

The first term of the left handside reads $T_{1m} = \sum_{i=1}^d T_{1m,i}$ with

$$\begin{aligned} T_{1m,i} &= \sum_{n=0}^{N_m-1} \sum_{\sigma \in \mathcal{E}^{(i)}} |D_\sigma| \; (u_{m,\sigma}^{(n+1)} - u_{m,\sigma}^{(n)}) \varphi_{m,\sigma}^{(n)} \\ &= -\sum_{n=0}^{N_m-1} \delta t \sum_{\sigma \in \mathcal{E}^{(i)}} |D_\sigma| u_{m,\sigma}^{(n+1)} \frac{\varphi_{m,\sigma}^{(n+1)} - \varphi_{m,\sigma}^{(n)}}{\delta t} - \sum_{\sigma \in \mathcal{E}^{(i)}} |D_\sigma| u_{m,\sigma}^{(0)} \; \varphi_{m,\sigma}^{(0)} \\ &= -\int_0^T \int_\Omega u_{m,i}(\boldsymbol{x},t) \eth_t \varphi_{m,i}(\boldsymbol{x},t) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_\Omega \mathcal{P}_{\mathcal{E}_m}^{(i)} u_{0,i}(\boldsymbol{x}) \varphi_m^{(0)}(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

We know that $u_{m,i} \to \overline{u}_i$ in $L^2(0,T;L^2(\Omega))$ as $m \to +\infty$. By definition, the discrete partial derivative $\eth_t \varphi_{m,i}$ converges uniformly to $\partial \varphi_i$ as $m \to +\infty$. Moreover, $\mathcal{P}_{\mathcal{E}_m} u_{0,i}$ converges to $(\bar{u}_{0,i})$ in $L^q(\Omega)$ for all q in $[1,2]$, and $\varphi_{m,\sigma}^{(0)}$ converges to $\bar{\varphi}_i(\cdot,0)$ in $L^q(\Omega)$ for all q in $[1,\infty]$. Hence

$$T_{1m} \to -\int_0^T \int_\Omega \bar{\boldsymbol{u}}(\boldsymbol{x},t) \cdot \partial_t \boldsymbol{\varphi}(\boldsymbol{x},t) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_\Omega \bar{\boldsymbol{u}}_0(\boldsymbol{x}) \cdot \boldsymbol{\varphi}(\boldsymbol{x},0) \,\mathrm{d}\boldsymbol{x} \text{ as } m \to \infty. \tag{2.4.6}$$

Let us then study the second term of the left hand side. We have

$$\int_\Omega \nabla_{\mathcal{E}_m} \boldsymbol{u}_m^{(n+1)} : \nabla_{\mathcal{E}_m} \boldsymbol{\varphi}_m^{(n)} \,\mathrm{d}\boldsymbol{x} = \int_\Omega \nabla_{\mathcal{E}_m} \boldsymbol{u}_m^{(n+1)} : \nabla_{\mathcal{E}_m} \boldsymbol{\varphi}_m^{(n+1)} \,\mathrm{d}\boldsymbol{x} \\ + \int_\Omega \nabla_{\mathcal{E}_m} \boldsymbol{u}_m^{(n+1)} : \nabla_{\mathcal{E}_m} (\boldsymbol{\varphi}_m^{(n)} - \boldsymbol{\varphi}_m^{(n+1)}) \,\mathrm{d}\boldsymbol{x}.$$

As in the stationary case, we get that

$$\sum_{n=0}^{N_m-1} \delta t_m \int_\Omega \nabla_{\mathcal{E}_m} \boldsymbol{u}_m^{(n+1)} : \nabla_{\mathcal{E}_m} \boldsymbol{\varphi}_m^{(n+1)} \,\mathrm{d}\boldsymbol{x} \to \int_0^T \int_\Omega \nabla \bar{\boldsymbol{u}} \cdot \nabla \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \text{ as } m \to +\infty.$$

Moreover, thanks to the regularity of $\boldsymbol{\varphi}$,

$$\int_\Omega \nabla_{\mathcal{E}_m} \boldsymbol{u}_m^{(n+1)} : \nabla_{\mathcal{E}_m} (\boldsymbol{\varphi}_m^{(n+1)} - \boldsymbol{\varphi}_m^{(n)}) \,\mathrm{d}\boldsymbol{x} \le \delta t_m C_\varphi \|\boldsymbol{u}_m^{(n+1)}\|_{1,\mathcal{E},0}$$

where C_φ depends only on $\boldsymbol{\varphi}$. We thus get that

$$\sum_{n=0}^{N_m-1} \delta t_m \int_\Omega \nabla_{\mathcal{E}_m} \boldsymbol{u}_m^{(n+1)} : \nabla_{\mathcal{E}_m} (\boldsymbol{\varphi}_m^{(n+1)} - \boldsymbol{\varphi}_m^{(n)}) \, \mathrm{d}\boldsymbol{x} \to 0 \text{ as } m \to +\infty.$$

Similarly, we have

$$\int_\Omega \mathcal{P}_{\mathcal{E}_m} \boldsymbol{f}^{(n+1)} \cdot (\boldsymbol{\varphi}_m^{(n)} - \boldsymbol{\varphi}_m^{(n+1)}) \, \mathrm{d}\boldsymbol{x} \leq \delta t C_{\boldsymbol{\varphi}} \|\boldsymbol{f}(\cdot, t_{n+1})\|_{L^2(\Omega)} \to 0 \text{ as } m \to +\infty,$$

so that

$$\sum_{n=0}^{N_m-1} \delta t_m \int_\Omega \mathcal{P}_{\mathcal{E}_m} \boldsymbol{f}^{(n+1)} \cdot \boldsymbol{\varphi}_m^{(n)} \, \mathrm{d}\boldsymbol{x} \to \int_0^T \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \text{ as } m \to +\infty.$$

The convection term is dealt with by remarking that an easy adaptation of Lemma 2.3.7 to the time-dependent framework implies that

$$\sum_{m=0}^{N-1} \delta t_m b_{\mathcal{E}}(\boldsymbol{u}_m^{(n+1)}, \boldsymbol{u}_m^{(n+1)}, \boldsymbol{\varphi}_m^{(n)}) \to \int_0^T b(\bar{\boldsymbol{u}}, \bar{\boldsymbol{u}}, \boldsymbol{\varphi}) \, \mathrm{d}t \text{ as } n \to +\infty.$$

Therefore, $\bar{u}$ is indeed a solution of (2.1.4).

Step 4: Regularity of the limit

Thanks to [22, theorems 14.1 and 14.2] the sequence of normed vector spaces $(\mathbf{H}_{\mathcal{E}_m,0}, \|\cdot\|_{1,\mathcal{E}_m,0})_{m\in\mathbb{N}}$ is $L^2(\Omega)^d$-limit-included in $H_0^1(\Omega)^d$ in the sense of Definition 4.0.9. We have $\boldsymbol{u}_m \to \bar{\boldsymbol{u}}$ in $L^2(0,T,L^2(\Omega))$ as $m \to \infty$ and $(\|\boldsymbol{u}_m\|_{L^2(0,T;\mathbf{H}_{\mathcal{E}_m,0})})_{m\in\mathbb{N}}$ is bounded thanks to Lemma 2.4.1. Therefore Theorem 4.0.10 applies and $\bar{\boldsymbol{u}} \in L^2(0,T;\boldsymbol{E}(\Omega))$.

Let us finally show that $\partial_t \bar{\boldsymbol{u}} \in L^{4/3}(0,T;E'(\Omega))$. Let $\boldsymbol{\varphi} \in C_c^\infty(\Omega\times(0,T))$ such that $\mathrm{div}\boldsymbol{\varphi} = 0$. Let $\boldsymbol{\varphi}_m \in \boldsymbol{E}_{\mathcal{E}_m}$ be defined by

$$\boldsymbol{\varphi}_m(\cdot,t) = \frac{1}{\delta t} \int_{t_n}^{t_{n+1}} \widetilde{\mathcal{P}}_{\mathcal{E}} \boldsymbol{\varphi}(\cdot,t) \, \mathrm{d}t \text{ for } t \in [t_n, t_{n+1}[.$$

Thanks to Lemma 2.4.2, there exists $C \geq 0$ depending only on $\boldsymbol{u}_0$, Ω, η and $\boldsymbol{f}$ such that:

$$\int_0^T \int_\Omega \eth_t \boldsymbol{u}_m \cdot \boldsymbol{\varphi}_m \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \leq C \|\boldsymbol{\varphi}_m\|_{L^4(0,T;\mathbf{H}_{\mathcal{E},0})}.$$

By Lemma 2.3.3, there exists C_2 depending only on η and Ω, such that $\|\boldsymbol{\varphi}_m\|_{L^4(0,T;\mathbf{H}_{\mathcal{E},0})} \leq C_2 \|\boldsymbol{\varphi}\|_{L^4(0,T;\boldsymbol{E}(\Omega))}$, where $\boldsymbol{E}(\Omega)$ is endowed with the H_0^1 norm. Hence passing to the limit as

$m \to +\infty$ in a similar way as for T_{1m} in Step 1, we get that

$$\int_0^T \int_\Omega \boldsymbol{u} \cdot \partial_t \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} \le C C_2 \|\boldsymbol{\varphi}\|_{L^4(0,T;\boldsymbol{E}(\Omega))}.$$

We then get that $\partial_t \bar{\boldsymbol{u}} \in L^{4/3}(0,T;\boldsymbol{E}'(\Omega))$ by density. □

2.4.4 Case of the unsteady Stokes equations

In the case of the unsteady Stokes equations, that is Problem (2.1.3) where we omit the non-linear convection term in (2.1.3b), stronger estimates can be obtained, which entail the weak convergence of the pressure. We assume in this section that $\boldsymbol{u}_0 \in H^1(\Omega)^d$ and that $\mathrm{div}\boldsymbol{u}_0 = 0$, and consider the following weak formulation of the unsteady Stokes problem:

$$\begin{aligned} &\text{Find } (\bar{\boldsymbol{u}}, \bar{p}) \in L^2(0,T;\boldsymbol{E}(\Omega)) \times L^2(0,T;L^2(\Omega)) \text{ such that } \forall \boldsymbol{\varphi} \in C_c^\infty([0,T[\times\Omega)^d \\ &-\int_0^T \int_\Omega \bar{\boldsymbol{u}}(\boldsymbol{x},t) \cdot \partial_t \boldsymbol{\varphi}(\boldsymbol{x},t) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t - \int_\Omega \boldsymbol{u}_0(\boldsymbol{x}) \cdot \boldsymbol{\varphi}(\boldsymbol{x},0) \, \mathrm{d}\boldsymbol{x} + \int_0^T \int_\Omega \nabla \bar{\boldsymbol{u}}(\boldsymbol{x},t) : \nabla \boldsymbol{\varphi}(\boldsymbol{x},t) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\ &\qquad - \int_0^T \int_\Omega \bar{p} \, \mathrm{div}\boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t = \int_0^T \int_\Omega \boldsymbol{f}(\boldsymbol{x},t) \cdot \boldsymbol{\varphi}(\boldsymbol{x},t) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t. \end{aligned} \tag{2.4.7}$$

Note that this formulation does not use divergence free test functions as in (2.1.4). Indeed, in the case of the Stokes equations, we are able to show the (weak) convergence of the pressure and we thus consider a formulation in which the pressure is present. Note that the two formulations are in fact equivalent.

The scheme We look for an approximation $(\boldsymbol{u}, p) \in \boldsymbol{X}_{\mathcal{E},\delta t} \times Y_{\mathcal{M},\delta t}$ of $(\boldsymbol{u}, p)$ solution to the problem (2.4.7); we consider the time-implicit MAC scheme which reads:

Initialization

$$\boldsymbol{u}^{(0)} = \widetilde{\mathcal{P}}_\mathcal{E} \boldsymbol{u}_0 \tag{2.4.8a}$$

Step $n \ge 0$. Solve for $\boldsymbol{u}^{(n+1)}$ and $p^{(n+1)}$:

$$\boldsymbol{u}^{(n+1)} \in \mathbf{H}_{\mathcal{E},0}, \; p^{(n+1)} \in L_\mathcal{M}, \; \int_\Omega p^{(n+1)} \, \mathrm{d}\boldsymbol{x} = 0, \tag{2.4.8b}$$

$$\eth_t \boldsymbol{u}^{(n+1)} - \Delta_\mathcal{E} \boldsymbol{u}^{(n+1)} + \boldsymbol{\nabla}_\mathcal{E} \, p^{(n+1)} = \boldsymbol{f}_\mathcal{E}^{(n+1)}, \tag{2.4.8c}$$

$$\mathrm{div}_\mathcal{M} \boldsymbol{u}^{(n+1)} = 0, \tag{2.4.8d}$$

Note that the choice of the discretization of the initial condition in (2.4.8a), together with the assumption $\text{div}\boldsymbol{u}_0 = 0$ implies that $\text{div}_{\mathcal{M}}\ \boldsymbol{u}^{(0)} = 0$; this fact is important for the obtention of the estimates. A weak formulation of (2.4.8b)–(2.4.8d) reads:

$$\text{Find } (\boldsymbol{u}^{(n+1)}, p^{(n+1)}) \in \boldsymbol{E}_{\mathcal{E}} \times L_{\mathcal{M}}\ ; \int_\Omega p^{(n+1)}\ \mathrm{d}\boldsymbol{x} = 0, \text{ and } \forall \boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0}, \tag{2.4.9}$$

$$\int_\Omega \eth_t \boldsymbol{u}^{(n+1)}\ \boldsymbol{v}\ \mathrm{d}\boldsymbol{x} + \int_\Omega \nabla_{\mathcal{E}} \boldsymbol{u}^{(n+1)} : \nabla_{\mathcal{E}} \boldsymbol{v}\ \mathrm{d}\boldsymbol{x} - \int_\Omega p^{(n+1)} \text{div}_{\mathcal{M}}\ \boldsymbol{v}\ \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}_{\mathcal{E}}^{(n+1)} \cdot \boldsymbol{v}\ \mathrm{d}\boldsymbol{x}. \tag{2.4.10}$$

The estimates of Lemma 2.4.1 on the approximate solutions obtained in the case of the Navier-Stokes equations are of course still valid. However we get stronger estimates on $\eth_t \boldsymbol{u}$ and on p, as we proceed to show.

Lemma 2.4.4 (Estimates on the discrete time derivative)**.** *Let $\boldsymbol{u}^{(n+1)} \in \mathbf{H}_{\mathcal{E},0}$ be a solution to* (2.4.8)*; then there exists $C > 0$ depending only on $\boldsymbol{u}_0$, Ω, $\eta_{\mathcal{M}}$ and $\boldsymbol{f}$ such that:*

$$\|\eth_t\ \boldsymbol{u}\|_{L^2(0,T;L^2(\Omega)^d)} \le C, \tag{2.4.11}$$

$$\|\eth_t\ \boldsymbol{u}\|_{L^\infty(0,T;\mathbf{H}_{\mathcal{E},0})} \le C. \tag{2.4.12}$$

Proof. Let $\boldsymbol{u}^{(n+1)} \in \boldsymbol{E}_{\mathcal{E}}$ be a solution to (2.4.8b)-(2.4.8d). Taking $\boldsymbol{v} = \eth_t \boldsymbol{u}^{(n+1)}$ in (2.4.10) we get:

$$\int_\Omega (\eth_t \boldsymbol{u}^{(n+1)})^2\ \mathrm{d}\boldsymbol{x} + \int_\Omega \nabla_{\mathcal{E}} \boldsymbol{u}^{(n+1)} : \nabla_{\mathcal{E}}(\eth_t \boldsymbol{u}^{(n+1)})\ \mathrm{d}\boldsymbol{x}$$
$$- \int_\Omega p^{(n+1)}\ \text{div}_{\mathcal{M}}(\eth_t \boldsymbol{u}^{(n+1)})\ \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}_{\mathcal{E}}^{(n+1)} \cdot \eth_t \boldsymbol{u}^{(n+1)}\ \mathrm{d}\boldsymbol{x}. \tag{2.4.13}$$

By linearity of the discrete time derivative discrete divergence operators, and thanks to (2.4.8d), we get that $\text{div}_{\mathcal{M}}(\eth_t \boldsymbol{u}^{(n+1)}) = \eth_t(\text{div}_{\mathcal{M}} \boldsymbol{u}^{(n+1)}) = 0$. Multiplying (2.4.13) by δt and summing the result over $n \in \{0, \cdots, M\}$; $M \le N-1$ we obtain $T_1 + T_2 = T_3$ where

$$T_1 = \sum_{n=0}^{M} \delta t \int_\Omega (\eth_t \boldsymbol{u}^{(n+1)})^2\ \mathrm{d}\boldsymbol{x},\ T_2 = \sum_{n=0}^{M} \delta t \int_\Omega \nabla_{\mathcal{E}} \boldsymbol{u}^{(n+1)} : \eth_t(\nabla_{\mathcal{E}} \boldsymbol{u}^{(n+1)})\ \mathrm{d}\boldsymbol{x}, \text{ and}$$

$$T_3 = \sum_{n=0}^{M} \delta t \int_\Omega \boldsymbol{f}_{\mathcal{E}}^{(n+1)} \cdot \eth_t \boldsymbol{u}^{(n+1)}\ \mathrm{d}\boldsymbol{x}.$$

We have:

$$T_2 = \sum_{n=0}^{M} (\frac{1}{2} \|\boldsymbol{u}^{(n+1)}\|_{1\mathcal{E},0}^2 - \frac{1}{2}\|\boldsymbol{u}^{(n)}\|_{1,\mathcal{E},0}^2 + \frac{1}{2}\|\boldsymbol{u}^{(n+1)} - \boldsymbol{u}^{(n)}\|_{1,\mathcal{E},0}^2) \ge \frac{1}{2}\|\boldsymbol{u}^{M+1}\|_{1,\mathcal{E},0}^2 - \frac{1}{2}\|\boldsymbol{u}_0\|_{1,\mathcal{E},0}^2.$$

By the Cauchy-Schwarz and the Young inequalities we obtain:

$$T_3 \leq \sum_{n=0}^{M} \delta t \Big(\int_\Omega |\boldsymbol{f}(.,t_{n+1})|^2 \, \mathrm{d}\boldsymbol{x} \Big)^{\frac{1}{2}} \Big(\int_\Omega (\eth_t \boldsymbol{u}^{(n+1)})^2 \, \mathrm{d}\boldsymbol{x} \Big)^{\frac{1}{2}}$$
$$\leq \frac{1}{2} \|\boldsymbol{f}\|^2_{L^2(0,T;L^2(\Omega)^d)} + \frac{1}{2} \sum_{n=0}^{M} \delta t \int_\Omega (\eth_t \boldsymbol{u}^{(n+1)})^2 \, \mathrm{d}\boldsymbol{x}.$$

Gathering the above inequalities, we get that:

$$\sum_{n=0}^{M} \delta t \int_\Omega (\eth_t \boldsymbol{u}^{(n+1)})^2 \, \mathrm{d}\boldsymbol{x} + \|\boldsymbol{u}^{M+1}\|^2_{1,\mathcal{E},0} \leq \|\boldsymbol{f}\|^2_{L^2(0,T;L^2(\Omega)^d)} + \|\boldsymbol{u}_0\|^2_{1,\mathcal{E},0}. \tag{2.4.14}$$

This in turn yields the $L^\infty(L^2)$ estimate (2.4.12) and the $L^2(L^2)$ estimate (2.4.11) (taking $M = N-1$) on the discrete derivative, with $C = \sqrt{2}(\|\boldsymbol{f}\|_{L^2(0,T;L^2(\Omega)^d)} + \|\boldsymbol{u}_0\|_{(H^1)^d})$.

□

Lemma 2.4.5 (Estimate on the pressure). *Let* $(\boldsymbol{u}, p) \in \boldsymbol{X}_{\mathcal{M},\delta t} \times Y_{\mathcal{M},\delta t}$ *be a solution to* (2.4.8). *There exists* $C \geq 0$ *depending only on* Ω, $\eta_\mathcal{M}$ *and* $\boldsymbol{f}$ *such that:*

$$\|p\|_{L^2(0,T;L^2(\Omega))} \leq C. \tag{2.4.15}$$

Proof. With the same arguments as in the proof of the pressure estimate in Proposition 2.3.4, we choose $\boldsymbol{v} = \widetilde{\mathcal{P}}_\mathcal{E} \boldsymbol{\varphi}$ as test function in (2.4.10), where $\boldsymbol{\varphi} \in H_0^1(\Omega)^d$ is such that $\mathrm{div}\boldsymbol{\varphi} = p^{(n+1)}$ and $\|\nabla \boldsymbol{\varphi}\|_{L^2(\Omega)^{d\times d}} \leq c \|p^{(n+1)}\|_{L^2(\Omega)}$, with c depending only on Ω. Thanks to (2.3.15a) we then obtain:

$$\int_\Omega \eth_t \boldsymbol{u}^{(n+1)} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \int_\Omega \nabla_\mathcal{E} \boldsymbol{u}^{(n+1)} : \nabla_\mathcal{E} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} - \|p^{(n+1)}\|^2_{L^2(\Omega)} = \int_\Omega \boldsymbol{f}_\mathcal{E}^{(n+1)} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x},$$

Thanks to the Cauchy-Schwarz and Poincaré inequalities and to the estimate (2.3.15a) we then get that there exists $C_{\eta_\mathcal{M}}$ depending on Ω and on the regularity of the mesh such that

$$\|p^{(n+1)}\|^2_{L^2(\Omega)} \leq C_{\eta_\mathcal{M}} \left(\|\eth_t \boldsymbol{u}^{(n+1)}\|^2_{(L^2(\Omega))^d} + \|\boldsymbol{u}^{(n+1)}\|^2_{1,\mathcal{E},0} + \|\boldsymbol{f}_\mathcal{E}^{(n+1)}\|^2_{L^2(\Omega)^d} \right).$$

Summing (2.4.4) over $n \in \{0, \cdots, N-1\}$ and multiplying by δt yields the result thanks to (2.4.3), and (2.4.11).

□

Theorem 2.4.6 (Convergence of the scheme). *Let* $(\delta t)_m \in (0,T)$ *and* $(\mathcal{D}_m)_{m\in\mathbb{N}}$ *be a sequence* *of meshes such that* $(\delta t)_m \to 0$ *and* $\max_{K\in\mathcal{M}_m} \operatorname{diam}(K) \to 0$ *as* $m \to +\infty$ *; assume that there exists* $\eta > 0$ *such that* $\eta_{\mathcal{M}_m} \leq \eta$ *for any* $m \in \mathbb{N}$ *(with* $\eta_{\mathcal{M}_m}$ *defined by* (2.2.2)*). Let* $(\boldsymbol{u}_m, p_m)$ *be a solution to* (2.4.8) *for* $(\delta t)_m = \delta t$ *and* $\mathcal{D} = \mathcal{D}_m$*. Then there exists* $(\bar{\boldsymbol{u}}, \bar{p}) \in L^2(0,T;\boldsymbol{E}(\Omega)) \times L^2(0,T;L^2(\Omega))$ *such that, up to a subsequence:*

- *the sequence* $(\boldsymbol{u}_m)_{m\in\mathbb{N}}$ *converges to* $\bar{\boldsymbol{u}}$ *in* $L^2(0,T;L^2(\Omega)^d)$*,*
- *the sequence* $(p_m)_{m\in\mathbb{N}}$ *weakly converges to* $\bar{p}$ *in* $\in L^2(0,T;L^2(\Omega))$*,*
- $(\bar{\boldsymbol{u}}, \bar{p})$ *is a solution to the weak formulation* (2.4.7)*.*

Proof. The convergence of the sequence of discrete solutions of the velocity follow from the Theorem 2.4.3 and the convergence of the sequence of discrete solutions of the pressure in $L^2(0,T;L^2(\Omega))$ follow from the estimate (2.4.15). Let us then show that $(\bar{\boldsymbol{u}}, \bar{p})$ satisfies (2.4.7). Let $\boldsymbol{\varphi} \in C_c^\infty(\Omega \times [0,T))^d$. Taking $\boldsymbol{\varphi}_m^{(n)} = \widetilde{\mathcal{P}}_{\mathcal{E}_m}\boldsymbol{\varphi}(\cdot, t_n) \in \mathbf{H}_{\mathcal{E}_m,0}$ as test function in (2.4.10), multiplying by δt_m and summing for $n = \{0, \ldots, N_m - 1\}$ (with $N_m \delta t_m = T$), we obtain:

$$\sum_{n=0}^{N_m-1} \delta t_m \Big(\int_\Omega \eth_t \boldsymbol{u}_m^{(n+1)} \cdot \boldsymbol{\varphi}_m^{(n)} \, \mathrm{d}\boldsymbol{x} + \int_\Omega \nabla_{\mathcal{E}_m} \boldsymbol{u}_m^{(n+1)} : \nabla_{\mathcal{E}_m} \boldsymbol{\varphi}_m^{(n)} \, \mathrm{d}\boldsymbol{x} - \int_\Omega p_m^{(n+1)} \mathrm{div}_{\mathcal{M}_m} \boldsymbol{\varphi}_m^{(n)} \, \mathrm{d}\boldsymbol{x} - \int_\Omega \mathcal{P}_{\mathcal{E}_m} \boldsymbol{f}^{(n+1)} \cdot \boldsymbol{\varphi}_m^{(n)} \, \mathrm{d}\boldsymbol{x} \Big) = 0.$$

Let us deal with the pressure term, (all other terms of the equation can be dealt with as in the proof of Theorem 2.4.3). We have:

$$\int_\Omega p_m^{(n+1)} \mathrm{div}_{\mathcal{M}_m} \, \boldsymbol{\varphi}_m^{(n)} \, \mathrm{d}\boldsymbol{x} = \int_\Omega p_m^{(n+1)} \mathrm{div}_{\mathcal{M}_m} \, \boldsymbol{\varphi}_m^{(n+1)} \, \mathrm{d}\boldsymbol{x} + \int_\Omega p_m^{(n+1)} \mathrm{div}_{\mathcal{M}_m} \, (\boldsymbol{\varphi}_m^{(n)} - \boldsymbol{\varphi}_m^{(n+1)}) \, \mathrm{d}\boldsymbol{x}.$$

By Lemma 2.3.3 and thanks to the regularity of $\boldsymbol{\varphi}$,

$$\begin{aligned} \int_\Omega p_m^{(n+1)} \mathrm{div}_{\mathcal{M}_m} \, (\boldsymbol{\varphi}^{(n)} - \boldsymbol{\varphi}^{(n+1)}) \, \mathrm{d}\boldsymbol{x} &\leq \|p_m^{(n+1)}\|_{L^2(\Omega)} \|\mathrm{div}\, (\boldsymbol{\varphi}^{(n)} - \boldsymbol{\varphi}^{(n+1)})\|_{L^2(\Omega)} \\ &\leq \delta t_m \, C_{\boldsymbol{\varphi}} \|p_m^{(n+1)}\|_{L^2(\Omega)} \\ &\to 0 \text{ as } m \to +\infty. \end{aligned}$$

We proved in the stationary case (see the proof of Theorem 2.3.8) that

$$\int_\Omega p_m^{(n+1)} \mathrm{div}_{\mathcal{M}_m} \, \boldsymbol{\varphi}_m^{(n+1)} \, \mathrm{d}\boldsymbol{x} \to \int_\Omega \bar{p} \, \mathrm{div} \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} \text{ as } m \to \infty,$$

and this concludes the proof that $(\bar{u}, \bar{p})$ is indeed a solution of (2.4.7). □

Chapter 3

CONVERGENCE OF THE MAC SCHEME FOR VARIABLE DENSITY INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

Abstract. This chapter is written in collaboration with T.Gallouët, R.Herbin and J-C. Latché and it will be submitted soon. We prove in this paper the convergence of the Marker and Cell (MAC) scheme for time-dependent variable density Navier-Stokes equations. The algorithm is implicit in time, and the space approximation is based on non uniform MAC grids. The mass conservation equation and momentum conservation equations are discretized in such a way that the kinetic energy remains controlled. We first show that the scheme preserves the stability properties of the continuous problem (L^∞-estimate for the density, $L^\infty(L^2)$- and $L^2(H^1)$-estimates for the velocity), which yields, by a topological degree technique, the existence of a solution. Then, invoking compactness arguments and passing to the limit in the scheme, we prove that any sequence of solutions (obtained with a sequence of discretizations the space and time step of which tend to zero) converges up to the extraction of a subsequence to a weak solution of the continuous problem.

3.1 Introduction

The Marker-And-Cell (MAC) scheme, introduced in the middle of the sixties [45], is one of the most popular methods [71, 80] for the approximation of the compressible and incompressible Navier-Stokes equations in the engineering framework, because of its simplicity, and its efficiency. Moreover, it progressively appeared in the litterature that this scheme enjoys remarkable mathematical properties. The mathematical analysis of the scheme for the incompressible Navier-Stokes equations on non-uniform meshes was recently performed [8, 48, 33]. The scheme was also proved to be convergent for the compressible Stokes equations [20] and stable for the compressible Euler [46] and Navier-Stokes [42] equations at all Mach number. The continuous problem addressed in this paper reads, in its strong from:

$$\partial_t \bar\rho + \operatorname{div}(\bar\rho \bar{\boldsymbol u}) = 0, \tag{3.1.1a}$$

$$\partial_t \bar\rho\ \bar{\boldsymbol u} + \operatorname{div}(\bar\rho\ \bar{\boldsymbol u} \otimes \bar{\boldsymbol u}) - \Delta \bar{\boldsymbol u} + \boldsymbol\nabla \bar p = \boldsymbol f, \tag{3.1.1b}$$

$$\operatorname{div}\ \bar{\boldsymbol u} = 0, \tag{3.1.1c}$$

This problem is posed for $(\boldsymbol x, t)$ in $\Omega \times (0,T)$ where $T \in \mathbb{R}^+$ and Ω is an open bounded connected subset of $\mathbb{R}^d$, with $d \in \{2,3\}$; we suppose that Ω may be covered by a structured grid, so Ω is a finite union of rectangles if $d = 2$ and of rectangular parallelepipeds if $d = 3$. We assume that the source term $\boldsymbol f$ belongs to $L^2(0,T;L^2(\Omega)^d)$. The variables $\bar\rho$, $\bar{\boldsymbol u}$ and $\bar p$ are respectively the density, the velocity and the pressure of the flow. The three above equations respectively express the mass conservation, the momentum balance and the incompressibility of the flow. This system is supplemented with initial and boundary conditions:

$$\bar{\boldsymbol u}|_{\partial\Omega} = 0, \qquad \bar{\boldsymbol u}|_{t=0} = \boldsymbol u_0, \qquad \bar\rho|_{t=0} = \rho_0. \tag{3.1.2}$$

We assume that the initial data satisfies the following properties:

$$\rho_0 \text{ belongs to } L^\infty(\Omega)\ \text{; we denote } \rho_{\min} = \operatorname*{ess\,inf}_{\boldsymbol x \in \Omega}\ \rho_0(\boldsymbol x),\ \rho_{\max} = \operatorname*{ess\,sup}_{\boldsymbol x \in \Omega}\ \rho_0(\boldsymbol x),$$

$$\text{and we suppose that } \rho_{\min} > 0, \tag{3.1.3a}$$

$$\boldsymbol u_0 \in L^2(\Omega)^d. \tag{3.1.3b}$$

A well-known consequence of equations (3.1.1a) and (3.1.1c), is the following maximum principle:

$$\rho_{\min} \le \bar\rho(\boldsymbol x, t) \le \rho_{\max}, \qquad \text{for } a.e.\ (\boldsymbol x, t) \in \Omega \times (0,T), \tag{3.1.4}$$

which shows that the natural regularity for $\bar\rho$ is $L^\infty(\Omega \times (0,T))$. For the velocity $\bar{\boldsymbol u}$, a classical calculation allows to derive natural estimates for the solutions. Taking the scalar product of (3.1.1b) by $\bar{\boldsymbol u}$ and using twice the mass conservation equation (3.1.1a) yields the kinetic energy equation:

$$\partial_t(\frac{1}{2}\bar\rho|\bar{\boldsymbol u}|^2) + \operatorname{div}(\frac{1}{2}\bar\rho|\bar{\boldsymbol u}|^2\bar{\boldsymbol u}) - \Delta\bar{\boldsymbol u}\cdot\bar{\boldsymbol u} + \nabla\bar p \cdot \bar{\boldsymbol u} = 0. \tag{3.1.5}$$

Integrating over Ω, one gets, since div $\bar{\boldsymbol u} = 0$ and $\bar{\boldsymbol u}_{|\partial\Omega} = 0$, that, for all $t \in (0,T)$:

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\bar\rho(\boldsymbol x,t)|\bar{\boldsymbol u}(\boldsymbol x,t)|^2\ \mathrm{d}\boldsymbol x + \int_\Omega \nabla\bar{\boldsymbol u}(\boldsymbol x,t) : \nabla\bar{\boldsymbol u}(\boldsymbol x,t)\ \mathrm{d}\boldsymbol x = 0.$$

Integrating over time yields, once again for all $t \in (0,T)$:

$$\int_\Omega \frac{1}{2}\bar{\rho}(\boldsymbol{x},t)|\bar{\boldsymbol{u}}(\boldsymbol{x},t)|^2 \,\mathrm{d}\boldsymbol{x} + \int_0^t \int_\Omega |\nabla \bar{\boldsymbol{u}}(\boldsymbol{x},t)|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_\Omega \frac{1}{2}\rho_0(\boldsymbol{x})|\boldsymbol{u}_0(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x},\ \forall t \in (0,T).$$

This shows that the natural regularity for $\bar{\boldsymbol{u}}$ is $L^\infty((0,T);L^2(\Omega)) \cap L^2((0,T);H_0^1(\Omega)^d)$. This leads to define the weak solutions to problem (3.1.1) as follows:

Definition 3.1.1. Let $\rho_0 \in L^\infty(\Omega)$ such that $\rho_0 > 0$ for a.e. $\boldsymbol{x} \in \Omega$, and let $\boldsymbol{u}_0 \in L^2(\Omega)^d$. A pair $(\bar{\rho}, \bar{\boldsymbol{u}})$ is a weak solution of problem (3.1.1) if it satisfies the following properties:

- $\bar{\rho} \in \{\rho \in L^\infty(\Omega \times (0,T)),\ \rho > 0 \ a.e. \text{ in } \Omega \times (0,T)\}$.
- $\bar{\boldsymbol{u}} \in \{\boldsymbol{u} \in L^\infty(0,T;L^2(\Omega)^d) \cap L^2(0,T;H_0^1(\Omega)^d),\ \mathrm{div}\ \boldsymbol{u} = 0 \ a.e. \text{ in } \Omega \times (0,T)\}$.
- For all φ in $C_c^\infty(\Omega \times [0,T))$,

$$-\int_0^T \int_\Omega \bar{\rho}(\boldsymbol{x},t)(\partial_t \varphi(\boldsymbol{x},t) + \bar{\boldsymbol{u}}(\boldsymbol{x},t) \cdot \nabla \varphi(\boldsymbol{x},t)) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_\Omega \rho_0(\boldsymbol{x})\varphi(\boldsymbol{x},0)\,\mathrm{d}\boldsymbol{x}. \qquad (3.1.6)$$

- For all $\boldsymbol{v}$ in $\{\boldsymbol{w} \in C_c^\infty(\Omega \times [0,T))^d, \mathrm{div}\ \boldsymbol{w} = 0\}$,

$$\int_0^T \int_\Omega \big[-\bar{\rho}(\boldsymbol{x},t)\bar{\boldsymbol{u}}(\boldsymbol{x},t) \cdot \partial_t \boldsymbol{v}(\boldsymbol{x},t) - (\bar{\rho}(\boldsymbol{x},t)\bar{\boldsymbol{u}}(\boldsymbol{x},t) \otimes \bar{\boldsymbol{u}}(\boldsymbol{x},t)) : \nabla \boldsymbol{v}(\boldsymbol{x},t)$$
$$+ \nabla \bar{\boldsymbol{u}}(\boldsymbol{x},t) : \nabla \boldsymbol{v}(\boldsymbol{x},t)\big] \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_\Omega \rho_0(\boldsymbol{x})\boldsymbol{u}_0(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x},0)\,\mathrm{d}\boldsymbol{x} + \int_0^T \int_\Omega \boldsymbol{f}(\boldsymbol{x},t) \cdot \boldsymbol{v}(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t. \qquad (3.1.7)$$

The existence of such a weak solution was proven in [77] and a convergence result for the discontinuous Galerkin approximation of the problem was proven in [64]. We show in the sequel that under minimal regularity assumptions on the solution, the sequences of approximate solutions obtained by the discretization of problem (3.1.1) by the MAC scheme converge (up to subsequence) to a weak solution of (3.1.1) as the mesh size tends to 0 (which, by the way, yields another proof of the existence of weak solutions). An essential feature of the scheme is that the (discrete) kinetic energy remains controlled. The paper is organized as follows. We recall in Section 3.2 the main features of the MAC discretization and apply it to the equations (3.1.1) in order to obtain the discrete problem. As in [33], the velocity convection operator is approximated so as to be compatible with a discrete continuity equation on the duals cells. Velocity and density estimates are thus obtained, which lead to the compactness of sequences of approximate solutions. We then show that the prospective limit is a weak solution of the variable density incompressible Navier-Stokes equations, in the sense of Definition 3.1.1.

3.2 The MAC discretization

Most of the tools used here were already introduced in [33] but we need to recall them here for the sake of clarity. We recall that the domain Ω is assumed to be a union of rectangles ($d = 2$) or orthogonal parallelepipeds ($d = 3$), and, without loss of generality, we assume that the edges (or faces) of these rectangles (or parallelepipeds) are orthogonal to the canonical basis vectors, denoted by $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_d)$.

Definition 3.2.1 (MAC grid)**.** A discretization of Ω with MAC grid, denoted by $\mathcal{D}$, is given by $\mathcal{D} = (\mathcal{M}, \mathcal{E})$, where:

- The pressure (or primal) grid, denoted by $\mathcal{M}$, consists of a union of possibly non uniform rectangles or rectangular parallelepipeds, the faces of which are thus, by assumption on Ω, orthogonal to one of the canonical basis vectors of $\mathbb{R}^d$. A generic cell of this grid is denoted by K, and its mass center $\boldsymbol{x}_K$. A generic face (or edge in the two-dimensional case) of such a cell is denoted by $\sigma \in \mathcal{E}(K)$, and its mass center $\boldsymbol{x}_\sigma$, where $\mathcal{E}(K)$ denotes the set of all faces of K. The set of all faces of the mesh is denoted by $\mathcal{E}$; we have $\mathcal{E} = \mathcal{E}_{\text{int}} \cup \mathcal{E}_{\text{ext}}$, where $\mathcal{E}_{\text{int}}$ (resp. $\mathcal{E}_{\text{ext}}$) are the edges of $\mathcal{E}$ that lie in the interior (resp. on the boundary) of the domain. The set of faces that are orthogonal to the i^{th} unit vector $\boldsymbol{e}_i$ of the canonical basis of $\mathbb{R}^d$ is denoted by $\mathcal{E}^{(i)}$, for $i = 1, \dots, d$. We then have $\mathcal{E}^{(i)} = \mathcal{E}^{(i)}_{\text{int}} \cup \mathcal{E}^{(i)}_{\text{ext}}$, where $\mathcal{E}^{(i)}_{\text{int}}$ (resp. $\mathcal{E}^{(i)}_{\text{ext}}$) are the edges of $\mathcal{E}^{(i)}$ that lie in the interior (resp. on the boundary) of the domain.

- For each $\sigma \in \mathcal{E}$, we write that $\sigma = K|L$ if $\sigma = \partial K \cap \partial L$ and we write that $\sigma = \overrightarrow{K|L}$ if, furthermore, $\sigma \in \mathcal{E}^{(i)}$ and $\overrightarrow{\boldsymbol{x}_K\boldsymbol{x}_L} \cdot \boldsymbol{e}_i > 0$ for some $i \in [\![1, d]\!]$. A dual cell D_σ associated to a face $\sigma \in \mathcal{E}$ is defined as follows:

  * if $\sigma = K|L \in \mathcal{E}_{\text{int}}$ then $D_\sigma = D_{K,\sigma} \cup D_{L,\sigma}$, where $D_{K,\sigma}$ (resp. $D_{L,\sigma}$) is the half-part of K (resp. L) adjacent to σ (see Fig. 2.1 for the two-dimensional case) ;
  * if $\sigma \in \mathcal{E}_{\text{ext}}$ is adjacent to the cell K, then $D_\sigma = D_{K,\sigma}$.

  A primal cell K will be denoted $K = [\overrightarrow{\sigma\sigma'}]$ if $\sigma, \sigma' \in \mathcal{E}^{(i)} \cap \mathcal{E}(K)$ for some $i = 1, \dots, d$ are such that $(\boldsymbol{x}_{\sigma'} - \boldsymbol{x}_\sigma) \cdot \boldsymbol{e}_i > 0$. A dual face separating two duals cells D_σ and $D_{\sigma'}$ is denoted by $\epsilon = \sigma|\sigma'$ or $\epsilon = \overrightarrow{\sigma|\sigma'}$ when specifying its orientation: more precisely we write that $\epsilon = \overrightarrow{\sigma|\sigma'}$ if $\overrightarrow{\boldsymbol{x}_\sigma\boldsymbol{x}_{\sigma'}} \cdot \boldsymbol{e}_j > 0$ for some $j \in [\![1, d]\!]$. To any dual face ϵ, we associate a distance d_ϵ as sketched on Figure 2.1. For a dual face $\epsilon \subset \partial D_\sigma, \sigma \in \mathcal{E}^{(i)}$, $i \in [\![1, d]\!]$, the distance d_ϵ is defined by:

$$d_\epsilon = \begin{cases} d(\boldsymbol{x}_\sigma, \boldsymbol{x}_{\sigma'}) & \text{if } \epsilon = \overrightarrow{\sigma|\sigma'} \in \widetilde{\mathcal{E}}^{(i)}_{\text{int}}, \\ d(\boldsymbol{x}_\sigma, \epsilon) & \text{if } \epsilon \in \widetilde{\mathcal{E}}^{(i)}_{\text{ext}} \cap \tilde{\mathcal{E}}(D_\sigma) \end{cases} \tag{3.2.1}$$

where $d(\cdot,\cdot)$ denotes the Euclidean distance in $\mathbb{R}^d$, and the set $\widetilde{\mathcal{E}}^{(i)}$ of the faces of the i-th dual mesh (associated to the ith velocity component) is decomposed into the internal and boundary edges: $\widetilde{\mathcal{E}}^{(i)} = \widetilde{\mathcal{E}}^{(i)}_{\text{int}} \cup \widetilde{\mathcal{E}}^{(i)}_{\text{ext}}$.

We define the regularity of the mesh $\mathcal{M}$ by:

$$\eta_{\mathcal{M}} = \max \left\{ \frac{|\sigma|}{|\sigma'|},\ \sigma \in \mathcal{E}^{(i)}, \forall \sigma' \in \mathcal{E}^{(j)}, i, j \in [\![1, d]\!], i \neq j \right\}, \tag{3.2.2}$$

where $|\cdot|$ stands for the $(d-1)$-dimensional measure of a subset of $\mathbb{R}^{d-1}$ (in the sequel, it is also be used to denote or d-dimensional measure of a subset of $\mathbb{R}^d$). We also define the space step by

$$h_{\mathcal{M}} = \max\{\text{diam}(K), K \in \mathcal{M}\}.$$

The discrete velocity unknowns are associated to the velocity cells and are denoted by $(u_\sigma)_{\sigma \in \mathcal{E}^{(i)}}$, $i = 1, \ldots, d$, while the discrete pressure unknowns are associated to the primal cells and are denoted by $(p_K)_{K \in \mathcal{M}}$.

Definition 3.2.2 (Discrete spaces)**.** Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a MAC grid in the sense of Definition 3.2.1. The discrete pressure space $L_{\mathcal{M}}$ is defined as the set of piecewise constant functions over each of the grid cells K of $\mathcal{M}$; the approximation space for the i^{th} velocity component, denoted by $H^{(i)}_{\mathcal{E}}$, is defined as the set of piecewise constant functions over each of the grid cells D_σ, $\sigma \in \mathcal{E}^{(i)}$. We denote by $L_{\mathcal{M},0}$ the functions of $L_{\mathcal{M}}$ with zero mean value. As in the continuous case, the Dirichlet boundary conditions are (partly) incorporated in the definition of the velocity spaces, and, to this purpose, we introduce $H^{(i)}_{\mathcal{E},0} \subset H^{(i)}_{\mathcal{E}}, i = 1, \ldots, d$, defined as follows:

$$H^{(i)}_{\mathcal{E},0} = \left\{ u \in H^{(i)}_{\mathcal{E}},\ u(\boldsymbol{x}) = 0\ \forall \boldsymbol{x} \in D_\sigma,\ \sigma \in \widetilde{\mathcal{E}}^{(i)}_{\text{ext}} \right\}.$$

We then set $\mathbf{H}_{\mathcal{E},0} = \prod_{i=1}^d H^{(i)}_{\mathcal{E},0}$. Since we are dealing with piecewise constant functions, it is useful to introduce the characteristic functions $\chi_K, K \in \mathcal{M}$ and $\chi_{D_\sigma}, \sigma \in \mathcal{E}$ of the pressure and velocity cells, defined by

$$\chi_K(\boldsymbol{x}) = \begin{cases} 1 \text{ if } \boldsymbol{x} \in K, \\ 0 \text{ if } \boldsymbol{x} \notin K, \end{cases} \qquad \chi_{D_\sigma}(\boldsymbol{x}) = \begin{cases} 1 \text{ if } \boldsymbol{x} \in D_\sigma, \\ 0 \text{ if } \boldsymbol{x} \notin D_\sigma. \end{cases}$$

We can then write a function $\boldsymbol{u} \in \mathbf{H}_{\mathcal{E},0}$ as $\boldsymbol{u} = (u_1, \dots, u_d)$ with $u_i = \sum_{\sigma \in \mathcal{E}^{(i)}} u_\sigma \chi_{D_\sigma}$, $i \in [\![1, d]\!]$ and a function $p \in L_\mathcal{M}$ as $p = \displaystyle\sum_{K \in \mathcal{M}} p_K \chi_K$.

Let us now introduce the discrete operators which are used to write the numerical scheme.

Discrete divergence and gradient operators - Let $\boldsymbol{n}_{K,\sigma}$ stand for the unit normal vector to σ outward K, and $u_{K,\sigma}$ be defined as $u_{K,\sigma} = u_\sigma\, \boldsymbol{n}_{K,\sigma} \cdot \boldsymbol{e}_i$ for any face $\sigma = K|L \in \mathcal{E}^{(i)}$, $i = 1, \dots, d$. We are now in position to define the discrete divergence operator $\mathrm{div}_\mathcal{M}$:

$$\mathrm{div}_\mathcal{M} : \left| \begin{array}{l} L_\mathcal{M} \times \mathbf{H}_{\mathcal{E},0} \longrightarrow L_\mathcal{M} \\ (\rho, \boldsymbol{u}) \mapsto \mathrm{div}_\mathcal{M}(\rho \boldsymbol{u}) = \displaystyle\sum_{K \in \mathcal{M}} \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}\ \chi_K, \end{array} \right. \tag{3.2.3}$$

$$\text{with } F_{K,\sigma} = |\sigma|\, \rho_\sigma u_{K,\sigma} \text{ for } K \in \mathcal{M},\ \sigma \in \mathcal{E}(K), \text{ and } \rho_\sigma = \begin{cases} \rho_K \text{ if } \ u_{K,\sigma} \geq 0, \\ \rho_L \text{ otherwise.} \end{cases} \tag{3.2.4}$$

Note that the above definition is compatible with the classical divergence of the velocity:

$$\mathrm{div}_\mathcal{M} \boldsymbol{u} = \sum_{K \in \mathcal{M}} \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma|\, u_{K,\sigma}\ \chi_K. \tag{3.2.5}$$

The discrete divergence-free velocity space is denoted by $\boldsymbol{E}_{\mathcal{E}(\Omega)} = \{\boldsymbol{u} \in \mathbf{H}_{\mathcal{E},0}\ ;\ \mathrm{div}_\mathcal{M}\ \boldsymbol{u} = 0\}$. The discrete divergence of $\boldsymbol{u} = (u_1, \dots, u_d) \in \mathbf{H}_{\mathcal{E},0}$ may also be written as

$$\mathrm{div}_\mathcal{M}(\boldsymbol{u}) = \sum_{i=1}^{d} (\eth_i u_i)_K \chi_K, \tag{3.2.6}$$

where the discrete derivative $(\eth_i u_i)_K$ of u_i on the cell $K \in \mathcal{M}$ is defined by

$$(\eth_i u_i)_K = \frac{u_{\sigma'} - u_\sigma}{d(\boldsymbol{x}_{\sigma'}, \boldsymbol{x}_\sigma)},\ \sigma, \sigma' \in \mathcal{E}^{(i)} \cap \mathcal{E}(K); (\boldsymbol{x}_{\sigma'} - \boldsymbol{x}_\sigma) \cdot \boldsymbol{e}_i > 0. \tag{3.2.7}$$

The gradient in the discrete momentum balance equation is built as the dual operator of the discrete divergence, and reads:

$$\nabla_{\mathcal{E}} : \left| \begin{array}{l} L_{\mathcal{M}} \longrightarrow \mathbf{H}_{\mathcal{E},0} \\ p \longmapsto \nabla_{\mathcal{E}} p(\boldsymbol{x}) = (\eth_1 p(\boldsymbol{x}), \dots, \eth_d p(\boldsymbol{x}))^t \end{array} \right. \tag{3.2.8}$$

where $\eth_i p \in H^{(i)}_{\mathcal{E},0}$ is the discrete derivative of p in the i-th direction, defined by:

$$\eth_i p = \sum_{\sigma \in \mathcal{E}^{(i)}} (\eth p)_\sigma \ \chi_{D_\sigma}, \quad \text{with, for } \sigma = \overrightarrow{K|L} \in \mathcal{E}^{(i)}_{\text{int}},\ (\eth p)_\sigma = \frac{|\sigma|}{|D_\sigma|}\ (p_L - p_K). \tag{3.2.9}$$

The discrete divergence defined by (3.2.5) and the discrete pressure gradient (3.2.8) are dual in the following sense [33, Lemma 2.5]:

$$\forall q \in L_{\mathcal{M}},\ \forall \boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0}, \quad \int_\Omega q \ \mathrm{div}_{\mathcal{M}} \boldsymbol{v} \ \mathrm{d}\boldsymbol{x} + \int_\Omega \nabla_{\mathcal{E}} q \cdot \boldsymbol{v} \ \mathrm{d}\boldsymbol{x} = 0. \tag{3.2.10}$$

The discrete derivatives on the MAC grid are consistent in the sense that the discrete derivative of the interpolate of a smooth function tends to the derivative of the smooth function as the mesh size tends to 0, see [33, Lemma 2.3 and Lemma 2.4].

Discrete Laplace operator - For $i = 1 \dots, d$, we classically define the i^{th} component of the discrete Laplace operator by:

$$-\Delta^{(i)}_{\mathcal{E}} : \left| \begin{array}{l} H^{(i)}_{\mathcal{E},0} \longrightarrow H^{(i)}_{\mathcal{E},0} \\ u_i \longmapsto -\Delta_{\mathcal{E}} u_i = -\displaystyle\sum_{\sigma \in \mathcal{E}^{(i)}} (\Delta u)_\sigma \chi_{D_\sigma}, \text{ with } -(\Delta u)_\sigma = \frac{1}{|D_\sigma|} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} \phi_{\sigma,\epsilon}(u_i). \end{array} \right.$$

In this relation, $\tilde{\mathcal{E}}(D_\sigma)$ denotes the faces of D_σ and

$$\phi_{\sigma,\epsilon}(u_i) = \begin{cases} \dfrac{|\epsilon|}{d_\epsilon}(u_\sigma - u_{\sigma'}), \text{ if } \ \epsilon = \sigma|\sigma' \in \widetilde{\mathcal{E}}^{(i)}_{\text{int}}, \\[2ex] \dfrac{|\epsilon|}{d_\epsilon} u_\sigma, \text{ if } \epsilon \in \widetilde{\mathcal{E}}^{(i)}_{\text{ext}} \cap \tilde{\mathcal{E}}(D_\sigma), \end{cases}$$

where d_ϵ is defined by (3.2.1). Note that we have the usual finite volume property of local conservativity of the flux through an interface: $\phi_{\sigma,\epsilon}(u_i) = -\phi_{\sigma',\epsilon}(u_i),\ \forall \epsilon = \sigma|\sigma' \in \widetilde{\mathcal{E}}^{(i)}_{\text{int}}$. Then the discrete Laplace operator $-\Delta_{\mathcal{E}},\ \mathbf{H}_{\mathcal{E},0} \longrightarrow \mathbf{H}_{\mathcal{E},0}$, is given by:

$$-\Delta_{\mathcal{E}} \boldsymbol{u} = (-\Delta^{(1)}_{\mathcal{E}} u_1, \dots, -\Delta^{(d)}_{\mathcal{E}} u_d)^t.$$

The discrete velocity convection operator - As in [47, 46, 42], the discretisation of the nonlinear convection term is performed so as to ensure a discrete kinetic energy inequality. Integrating the the i^{th} component of the momentum balance equation (3.1.1b) on a cell D_σ, $\sigma \in \mathcal{E}^{(i)}$, we see that we need to discretize the term

$$\sum_{\epsilon \subset \partial D_\sigma} \int_\epsilon \rho u_i \boldsymbol{u} \cdot \boldsymbol{n}_{\sigma,\epsilon} \, \mathrm{d}\gamma(\boldsymbol{x}),$$

where $\boldsymbol{n}_{\sigma,\epsilon}$ denotes the unit normal vector to ϵ outward D_σ and $\mathrm{d}\gamma(\boldsymbol{x})$ denotes the integration with respect to the $d-1$-dimensional Lebesgue measure. For $\epsilon = \sigma|\sigma'$, the convection flux $\int_\epsilon \rho u_i \boldsymbol{u} \cdot \boldsymbol{n}_{\sigma,\epsilon} \, \mathrm{d}\gamma(\boldsymbol{x})$ is approximated by $F_{\sigma,\epsilon} u_\epsilon$, where ,

$$u_\epsilon = (u_\sigma + u_{\sigma'})/2, \tag{3.2.11}$$

and $F_{\sigma,\epsilon}$ is the numerical mass flux through ϵ outward D_σ which is defined, as in [47, 46, 42] so as to ensure that a local discrete mass balance holds on each cell D_σ of the dual mesh. This is crucial in order to recover a discrete kinetic energy balance by working (as in the continuous case) on the momentum equation which is indeed discretized on the dual cells. We distinguish two cases:

- First case – The vector $\boldsymbol{e}_i$ is normal to ϵ, and ϵ is included in a primal cell K, with $\mathcal{E}^{(i)}(K) = \{\sigma, \sigma'\}$. Then the mass flux through $\epsilon = \sigma|\sigma'$ is given by:

$$F_{\sigma,\epsilon} = \frac{1}{2} \, (-F_{K,\sigma} + F_{K,\sigma'}). \tag{3.2.12}$$

  (Recall that $F_{K,\sigma}$ is the numerical flux through σ outward K, given by (3.2.4).)

- Second case – The vector $\boldsymbol{e}_i$ is tangent to ϵ, and ϵ is the union of the halves of two primal faces τ and τ' such that $\sigma = K|L$ with $\tau \in \mathcal{E}(K)$ and $\tau' \in \mathcal{E}(L)$. The mass flux through ϵ is then given by:

$$F_{\sigma,\epsilon} = \frac{1}{2} \, (F_{K,\tau} + F_{L,\tau'}). \tag{3.2.13}$$

We now define the i-th component $C^{(i)}_{\mathcal{E}}(\rho \boldsymbol{u})$ of the non linear convection operator by:

$$C^{(i)}_{\mathcal{E}}(\rho \boldsymbol{u}) : \quad \left| \begin{array}{l} H^{(i)}_{\mathcal{E},0} \longrightarrow H^{(i)}_{\mathcal{E},0} \\ v \longmapsto C^{(i)}_{\mathcal{E}}(\rho \boldsymbol{u}) v = \displaystyle\sum_{\sigma \in \widetilde{\mathcal{E}}^{(i)}_{\mathrm{int}}} \frac{1}{|D_\sigma|} \sum_{\substack{\epsilon \in \tilde{\mathcal{E}}(D_\sigma) \\ \epsilon = \sigma|\sigma'}} F_{\sigma,\epsilon} \, \frac{v_\sigma + v_{\sigma'}}{2} \, \chi_{D_\sigma}. \end{array} \right.$$

and the full discrete convection operator $\boldsymbol{C}_{\mathcal{E}}(\rho\boldsymbol{u}),\ \mathbf{H}_{\mathcal{E},0} \longrightarrow \mathbf{H}_{\mathcal{E},0}$ by

$$\boldsymbol{C}_{\mathcal{E}}(\rho\boldsymbol{u})\boldsymbol{v} = (C_{\mathcal{E}}^{(1)}(\rho\boldsymbol{u})v_1, \ldots, C_{\mathcal{E}}^{(d)}(\rho\boldsymbol{u})v_d)^t.$$

3.3 The scheme

3.3.1 Definition of the scheme (or strong formulation)

We consider a MAC grid $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ of Ω in the sense of Definition 3.2.1, and a partition $0 = t_0 < t_1 < \cdots < t_N = T$ of the time interval $(0, T)$, and, for the sake of simplicity, a constant time step $\delta t = t_{n+1} - t_n$; hence $t_n = n\delta t$ for $n \in \{0, \cdots, N\}$. Let $\{u_\sigma^{(n+1)}, \sigma \in \mathcal{E}^{(i)}, n \in \{0, \cdots, N-1\}\}$, $\{p_K^{(n+1)}, K \in \mathcal{M}, n \in \{0, \cdots, N-1\}$, and $\{\rho_K^{(n+1)}, K \in \mathcal{M}, n \in \{0, \cdots, N-1\}$ be the sets of discrete velocity, pressure and density unknowns; we define the corresponding piecewise constant functions $\boldsymbol{u} = (u_1, \ldots, u_d)$, p and ρ. For the velocities, these piecewise constant functions are of the form:

$$u_i = \sum_{n=0}^{N-1} \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} u_\sigma^{(n+1)} \chi_{D_\sigma} \chi_{]t_n, t_{n+1}]},$$

where $\chi_{]t_n,t_{n+1}]}$ is the characteristic function of the interval $]t_n, t_{n+1}]$. We denote by $X_{i,\mathcal{E},\delta t}$ the set of such piecewise constant functions on time intervals and dual cells, and we set $\boldsymbol{X}_{\mathcal{E},\delta t} = \prod_{i=1}^d X_{i,\mathcal{E},\delta t}$. The pressure and density the discrete functions are defined by:

$$p = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} p_K^{(n+1)} \chi_K \chi_{]t_n,t_{n+1}]}, \qquad \rho = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} \rho_K^{(n+1)} \chi_K \chi_{]t_n,t_{n+1}]},$$

and we denote by $Y_{\mathcal{M},\delta t}$ the space of such piecewise constant functions. We look for an approximation $(\boldsymbol{u}, \rho, p) \in \boldsymbol{X}_{\mathcal{E},\delta t} \times Y_{\mathcal{M},\delta t}^2$ of $(\boldsymbol{u}, \rho, p)$ solution of the problem (3.1.1). For $\sigma \in \mathcal{E}^{(i)}$, $i \in \{1, \cdots, d\}$ the value $u_\sigma^{(n+1)}$ is an expected approximation of $u_i(\boldsymbol{x}, t_{n+1})$, for $\boldsymbol{x} \in D_\sigma$, and the values $p_K^{(n+1)}$ and $\rho_K^{(n+1)}$ an expected approximation of $p(\boldsymbol{x}, t_{n+1})$ and $\rho(\boldsymbol{x}, t_{n+1})$ for $\boldsymbol{x} \in K$. We define the discrete derivative $\eth_t \rho \in Y_{\mathcal{M},\delta t}$ of the approximate density ρ by

$$\eth_t \rho(\boldsymbol{x}, t) = \sum_{n=0}^{N-1} \eth_t \rho^{(n+1)}(\boldsymbol{x}) \chi_{]t_n,t_{n+1}]}(t) \text{ for } t \in]t_n, t_{n+1}] \text{ and } n \in \{0, \cdots, N-1\},$$

$$\text{where } \eth_t \rho^{(n+1)} = \sum_{K \in \mathcal{M}} \frac{\rho_K^{(n+1)} - \rho_K^{(n)}}{\delta t} \chi_K \in L_{\mathcal{M}}.$$

Next, for a given approximate density $\rho \in L_{\mathcal{M}}$ and a given approximate velocity $\boldsymbol{u} \in \boldsymbol{X}_{\mathcal{E},\delta t}$, we define an approximate momentum function $\rho\boldsymbol{u} \in \boldsymbol{X}_{\mathcal{E},\delta t}$ by

$$(\rho\boldsymbol{u})(\cdot,t) = (\rho\boldsymbol{u})^{(n+1)} = ((\rho u_1)^{(n+1)}, \dots, (\rho u_d)^{(n+1)})^t \text{ for } t \in]t_n, t_{n+1}] \text{ and } n \in \{0, \cdots, N-1\},$$

where, for $n \in \{0, \cdots, N-1\}$

$$(\rho u_i)^{(n+1)} = \sum_{\sigma \in \mathcal{E}^{(i)}} \rho_{D_\sigma}^{(n+1)} u_\sigma^{(n+1)} \chi_{D_\sigma} \in H_{\mathcal{E},0}^{(i)} \text{ for } i = 1, \dots, d,$$

with

$$|D_\sigma|\, \rho_{D_\sigma}^{(n+1)} = |D_{K,\sigma}|\, \rho_K^{(n+1)} + |D_{L,\sigma}|\, \rho_L^{(n+1)} \text{ for any } \sigma = K|L \in \mathcal{E}_{\text{int}}^{(i)}. \tag{3.3.1}$$

We may then define the discrete derivative $\eth_t(\rho\boldsymbol{u}) \in \boldsymbol{X}_{\mathcal{E},\delta t}$ of such an approximate momentum by

$$\eth_t(\rho\boldsymbol{u})(\cdot,t) = (\eth_t(\rho u_1)^{(n+1)}, \dots, \eth_t(\rho u_d)^{(n+1)})^t \text{ for } t \in]t_n, t_{n+1}] \text{ and } n \in \{0, \cdots, N-1\}$$

$$\text{where } \eth_t(\rho u_i)^{(n+1)} = \sum_{\sigma \in \mathcal{E}^{(i)}} \frac{\rho_{D_\sigma}^{(n+1)} u_\sigma^{(n+1)} - \rho_{D_\sigma}^{(n)} u_\sigma^{(n)}}{\delta t} \chi_{D_\sigma} \in H_{\mathcal{E},0}^{(i)}$$

$$\text{for } i = 1, \dots, d, \text{ and } n \in \{0, \cdots, N-1\},$$

Definition 3.3.1 (Cell and face mean value interpolators)**.** Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a MAC grid on Ω, and let $\boldsymbol{v} \in (H_0^1(\Omega))^d$. Let $\mathcal{P}_{\mathcal{E}}\boldsymbol{v}$ and $\widetilde{\mathcal{P}}_{\mathcal{E}}\boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0}$ be the cell and edge mean-value interpolates of the velocity field $\boldsymbol{v}$, defined by $\mathcal{P}_{\mathcal{E}}\boldsymbol{v} = (\mathcal{P}_{\mathcal{E}}^{(1)} v_1, \cdots, \mathcal{P}_{\mathcal{E}}^{(d)} v_d)$ and $\widetilde{\mathcal{P}}_{\mathcal{E}}\boldsymbol{v} = (\widetilde{\mathcal{P}}_{\mathcal{E}}^{(1)} v_1, \cdots, \widetilde{\mathcal{P}}_{\mathcal{E}}^{(1)} v_d)$ respectively, where, for $i = 1, \dots d$,

$$\begin{aligned} \mathcal{P}_{\mathcal{E}}^{(i)} : \ H_0^1(\Omega) &\longrightarrow H_{\mathcal{E},0}^{(i)} \\ v_i &\longmapsto \mathcal{P}_{\mathcal{E}} v_i = \sum_{\sigma \in \mathcal{E}^{(i)}} (\mathcal{P}_{\mathcal{E}} v_i)_\sigma \, \chi_{D_\sigma} \end{aligned} \qquad \begin{aligned} \widetilde{\mathcal{P}}_{\mathcal{E}}^{(i)} : \ H_0^1(\Omega) &\longrightarrow H_{\mathcal{E},0}^{(i)} \\ v_i &\longmapsto \widetilde{\mathcal{P}}_{\mathcal{E}} v_i = \\ &\sum_{\sigma \in \mathcal{E}^{(i)}} (\widetilde{\mathcal{P}}_{\mathcal{E}} v_i)_\sigma \, \chi_{D_\sigma} \end{aligned}$$

with, for $\sigma \in \mathcal{E}^{(i)}$,

$$(\mathcal{P}_{\mathcal{E}}^{(i)} v_i)_\sigma = \frac{1}{|D_\sigma|} \int_{D_\sigma} v_i(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \quad \text{and} \quad (\widetilde{\mathcal{P}}_{\mathcal{E}}^{(i)} v_i)_\sigma = \frac{1}{|\sigma|} \int_\sigma v_i(\boldsymbol{x}) \, \mathrm{d}\gamma(\boldsymbol{x}).$$

For $q \in L^2(\Omega)$, we define $\mathcal{P}_{\mathcal{M}} q \in L_{\mathcal{M}}$ by:

$$\mathcal{P}_{\mathcal{M}} q(\boldsymbol{x}) = \frac{1}{|K|} \int_K q(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}.$$

Let $\boldsymbol{u} \in \boldsymbol{X}_{\mathcal{E},\delta t}$ and $(\rho, p) \in Y^2_{\mathcal{E},\delta t}$; setting $\boldsymbol{u}^{(n)} = \boldsymbol{u}(\cdot, t_n)$, $\rho^{(n)} = \rho(\cdot, t_n)$ and $p^{(n)} = p(\cdot, t_n)$, the time-implicit MAC scheme for the variable density Navier-Stokes equations reads:

Initialization

$$\boldsymbol{u}^{(0)} = \widetilde{\mathcal{P}}_{\mathcal{E}} \boldsymbol{u}_0, \quad \rho^{(0)} = \mathcal{P}_{\mathcal{M}} \rho_0 \tag{3.3.2a}$$

Step $n+1$, **for** $n \in \{0, \cdots, N-1\}$ – Solve for $\boldsymbol{u}^{(n+1)} \in \mathbf{H}_{\mathcal{E},0}, \rho^{(n+1)} \in L_{\mathcal{M}}$ and $p^{(n+1)} \in L_{\mathcal{M},0}$:

$$\eth_t \rho^{(n+1)} + \mathrm{div}_{\mathcal{M}}(\rho^{(n+1)} \boldsymbol{u}^{(n+1)}) = 0 \tag{3.3.2b}$$

$$\eth_t(\rho \boldsymbol{u})^{(n+1)} + \boldsymbol{C}_{\mathcal{E}}(\rho^{(n+1)} \boldsymbol{u}^{(n+1)}) \, \boldsymbol{u}^{(n+1)} - \Delta_{\mathcal{E}} \boldsymbol{u}^{(n+1)} + \nabla_{\mathcal{E}} \, p^{(n+1)} = \boldsymbol{f}_{\mathcal{E}}^{(n+1)}, \tag{3.3.2c}$$

$$\mathrm{div}_{\mathcal{M}} \boldsymbol{u}^{(n+1)} = 0, \tag{3.3.2d}$$

where for all $n \in \{0, \ldots, N-1\}$, $\boldsymbol{f}_{\mathcal{E}}^{(n+1)} = \mathcal{P}_{\mathcal{E}} \boldsymbol{f}(\cdot, t_{n+1})$ and we recall that $L_{\mathcal{M},0} = \{q \in L_{\mathcal{M}}, \ \int_\Omega q \, \mathrm{d}\boldsymbol{x} = 0\}$. Note that, thanks to the assumptions on ρ_0 and its approximation $\rho^{(0)}$, we have $\rho_{\min} \leq \rho_K^{(0)} \leq \rho_{\max}$, for any $K \in \mathcal{M}$.

Lemma 3.3.2 (Mass balance on the dual cells)**.** *If* $(\rho, \boldsymbol{u}) \in L_{\mathcal{M}} \times \boldsymbol{X}_{\mathcal{E},\delta t}$ *satisfies the mass balance equation* (3.3.2b)*, then the following mass balance on the dual cells holds:*

$$\frac{|D_\sigma|}{\delta t} (\rho_{D_\sigma}^{(n+1)} - \rho_{D_\sigma}^{(n)}) + \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}^{(n+1)} = 0,$$

where ρ_{D_σ} *is defined by* (3.3.1) *and* $F_{\sigma,\epsilon}$ *by* (3.2.12)-(3.2.13)*.*

Proof. Thanks to definition of the dual flux (3.2.12)-(3.2.13), we have for $\sigma = K|L$:

$$\sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}^{(n+1)} = \frac{1}{2} \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}^{(n+1)} + \frac{1}{2} \sum_{\sigma \in \mathcal{E}(L)} F_{L,\sigma}^{(n+1)}.$$

Using (3.3.1) and (3.3.2b) we get that:

$$\frac{|D_\sigma|}{\delta t} (\rho_{D_\sigma}^{(n+1)} - \rho_{D_\sigma}^{(n)}) + \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}^{(n+1)} = 0.$$

□

3.3.2 Weak formulation of the scheme

We recall that (see e.g. [33])

$$\forall(\boldsymbol{u},\boldsymbol{v}) \in {\mathbf{H}_{\mathcal{E},0}}^2, \qquad \int_\Omega -\Delta_{\mathcal{E}}\boldsymbol{u}\cdot\boldsymbol{v}\ \mathrm{d}\boldsymbol{x} = [\boldsymbol{u},\boldsymbol{v}]_{1,\mathcal{E},0} = \sum_{i=1}^d [u_i,v_i]_{1,\mathcal{E}^{(i)},0},$$
$$\text{with } [u_i,v_i]_{1,\mathcal{E}^{(i)},0} = \sum_{\substack{\epsilon\in\widetilde{\mathcal{E}}^{(i)}_{\mathrm{int}}\\ \epsilon=\sigma|\sigma'}} \frac{|\epsilon|}{d_\epsilon}\ (u_\sigma - u_{\sigma'})\ (v_\sigma - v_{\sigma'}) + \sum_{\substack{\epsilon\in\widetilde{\mathcal{E}}^{(i)}_{\mathrm{ext}}\\ \epsilon\subset\partial(D_\sigma)}} \frac{|\epsilon|}{d_\epsilon}\ u_\sigma\ v_\sigma. \tag{3.3.3}$$

The bilinear forms $\left|\begin{array}{l} H^{(i)}_{\mathcal{E},0}\times H^{(i)}_{\mathcal{E},0}\to\mathbb{R} \\ (u,v)\mapsto [u_i,v_i]_{1,\mathcal{E}^{(i)},0}\end{array}\right.$ and $\left|\begin{array}{l} \mathbf{H}_{\mathcal{E},0}\times \mathbf{H}_{\mathcal{E},0}\to\mathbb{R} \\ (\boldsymbol{u},\boldsymbol{v})\mapsto [\boldsymbol{u},\boldsymbol{v}]_{1,\mathcal{E},0}\end{array}\right.$ are inner products on $H^{(i)}_{\mathcal{E},0}$ and $\mathbf{H}_{\mathcal{E},0}$ respectively, which induce the following discrete H^1_0 norms:

$$\|u_i\|^2_{1,\mathcal{E}^{(i)},0} = [u_i,u_i]_{1,\mathcal{E}^{(i)},0} = \sum_{\substack{\epsilon\in\widetilde{\mathcal{E}}^{(i)}_{\mathrm{int}}\\ \epsilon=\sigma|\sigma'}} \frac{|\epsilon|}{d_\epsilon}\ (u_\sigma - u_{\sigma'})^2 + \sum_{\substack{\epsilon\in\widetilde{\mathcal{E}}^{(i)}_{\mathrm{ext}}\\ \epsilon\subset\partial(D_\sigma)}} \frac{|\epsilon|}{d_\epsilon}\ u_\sigma^2, \quad \text{for } i=1,\ldots,d, \tag{3.3.4a}$$

$$\|\boldsymbol{u}\|^2_{1,\mathcal{E},0} = [\boldsymbol{u},\boldsymbol{u}]_{1,\mathcal{E},0} = \sum_{i=1}^d \|u_i\|^2_{1,\mathcal{E}^{(i)},0}. \tag{3.3.4b}$$

This inner product may be formulated as the L^2 inner product of discrete gradients. Indeed, for $i=1,\ldots,d$, consider the following discrete gradient of the i^{th} velocity component u_i:

$$\nabla_{\mathcal{E}^{(i)}} u_i = (\eth_1 u_i,\ldots,\eth_d u_i) \text{ with } \eth_j u_i = \sum_{\substack{\epsilon\in\widetilde{\mathcal{E}}^{(i)}\\ \epsilon\perp\boldsymbol{e}_j}} (\eth_j u_i)_{D_\epsilon}\ \chi_{D_\epsilon}, \tag{3.3.5}$$

where $(\eth_j u_i)_{D_\epsilon} = (u_{\sigma'} - u_\sigma)/d_\epsilon$ with $\epsilon = \overrightarrow{\sigma|\sigma'}$, and $D_\epsilon = \epsilon\times\boldsymbol{x}_\sigma\boldsymbol{x}_{\sigma'}$ (see Figure 2.3). This definition is compatible with the definition of the discrete derivative $(\eth_i u_i)_K$ given by (3.2.7), since, if $\epsilon\subset K$ then $D_\epsilon = K$. With this definition, it is easily seen that

$$\int_\Omega \nabla_{\mathcal{E}^{(i)}} u\cdot\nabla_{\mathcal{E}^{(i)}} v\ \mathrm{d}\boldsymbol{x} = [u,v]_{1,\mathcal{E}^{(i)},0}, \quad \forall u,v\in H^{(i)}_{\mathcal{E},0}, \tag{3.3.6}$$

where $[u,v]_{1,\mathcal{E}^{(i)},0}$ is the discrete H^1_0 inner product defined by (3.3.3). We may then define

$$\nabla_{\mathcal{E}}\boldsymbol{u} = (\nabla_{\mathcal{E}^{(1)}} u_1,\ldots,\nabla_{\mathcal{E}^{(d)}} u_d),$$

so that

$$\int_\Omega \nabla_\mathcal{E} \boldsymbol{u} : \nabla_\mathcal{E} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = [\boldsymbol{u}, \boldsymbol{v}]_{1,\mathcal{E},0}.$$

Let us now define the weak form of the nonlinear convection operator, which we denote by $b_\mathcal{E}$:

$$\forall (\rho, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in L_\mathcal{M} \times {\mathbf{H}_{\mathcal{E},0}}^3, \quad b_\mathcal{E}(\rho\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \sum_{i=1}^{d} b_\mathcal{E}^{(i)}(\rho\boldsymbol{u}, v_i, w_i)$$
$$= \sum_{i=1}^{d} \int_\Omega \Big(C_\mathcal{E}^{(i)}(\rho\boldsymbol{u})\, v_i \Big) \; w_i \, \mathrm{d}\boldsymbol{x}. \tag{3.3.7}$$

With these notations, a weak formulation of the scheme reads:

For $n \in \{0, \cdots, N-1\}$, find $(\rho^{(n+1)}, \boldsymbol{u}^{(n+1)}) \in L_\mathcal{M} \times \boldsymbol{E}_\mathcal{E}$ such that, for any $(q, \boldsymbol{v}) \in L_\mathcal{M} \times \boldsymbol{E}_\mathcal{E}$,

$$\int_\Omega \Big(\eth_t \rho^{(n+1)} + \mathrm{div}_\mathcal{M}(\rho^{(n+1)} \boldsymbol{u}^{(n+1)}) \Big) q \, \mathrm{d}\boldsymbol{x} = 0, \tag{3.3.8}$$

$$\int_\Omega \eth_t (\rho\boldsymbol{u})^{(n+1)} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + b_\mathcal{E}(\rho^{(n+1)}\boldsymbol{u}^{(n+1)}, \boldsymbol{u}^{(n+1)}, \boldsymbol{v}) + \int_\Omega \nabla_\mathcal{E} \boldsymbol{u}^{(n+1)} : \nabla_\mathcal{E} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}$$
$$= \int_\Omega \boldsymbol{f}_\mathcal{E}^{(n+1)} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}. \tag{3.3.9}$$

Another weak formulation, featuring the pressure, reads:

For $n \in \{0, \cdots, N-1\}$, find $(\rho^{(n+1)}, \boldsymbol{u}^{(n+1)}, p^{(n+1)}) \in L_\mathcal{M} \times \boldsymbol{E}_\mathcal{E} \times L_{\mathcal{M},0}$ such that, for any $(q, \boldsymbol{v}) \in L_\mathcal{M} \times \mathbf{H}_\mathcal{E}$, Equation (3.3.8) holds, and

$$\int_\Omega \eth_t (\rho\boldsymbol{u})^{(n+1)} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + b_\mathcal{E}(\rho^{(n+1)}\boldsymbol{u}^{(n+1)}, \boldsymbol{u}^{(n+1)}, \boldsymbol{v}) + \int_\Omega \nabla_\mathcal{E} \boldsymbol{u}^{(n+1)} : \nabla_\mathcal{E} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}$$
$$+ \int_\Omega \nabla_\mathcal{E} p^{(n+1)} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}_\mathcal{E}^{(n+1)} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}. \tag{3.3.10}$$

This formulation is strictly equivalent to (*i.e.* yields the same algebraic equations as) the "strong" form of the scheme, given by (3.3.2).

3.4 Existence and estimates on the approximation solution

The following uniform estimate is a classical consequence of the upwind choice in the mass equation and of the fact that the velocity is divergence-free.

Lemma 3.4.1 (Estimate on the density)**.** *Let $\mathcal{D}$ be a MAC discretization in the sense of*

Definition 3.2.1. For $n \in \{1, \cdots, N-1\}$*, assume* $\rho^{(n)} \in L_{\mathcal{M}}$ *is such that* $0 < \rho_{\min} \le \rho^{(n)} \le \rho_{\max}$*. If* $\rho^{(n+1)} \in L_{\mathcal{M}}$ *and* $\boldsymbol{u}^{(n+1)} \in \mathbf{H}_{\mathcal{E}}$ *satisfy the discrete mass balance* (3.3.2b) *and the divergence constraint* (3.3.2d)*, then*

$$\rho_{\min} \le \rho^{(n+1)} \le \rho_{\max}. \tag{3.4.1}$$

The following result gives a classical weak BV inequality (see [7] for the seminal paper and its use in the linear case and [22, Chapter V] in the general nonlinear case) which will be useful for the convergence proof (see also [46, Theorem 3] for the one-dimensional case).

Lemma 3.4.2 (Weak BV estimate for the density)**.** *Any solution to the scheme* (3.3.2) *satisfies the following equality, for all* $K \in \mathcal{M}$ *and* $0 \le n \le N-1$*:*

$$\frac{|K|}{2\delta t}\big[(\rho_K^{(n+1)})^2 - (\rho_K^{(n)})^2\big] + \frac{1}{2}\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\,(\rho_\sigma^{(n+1)})^2\, u_{K,\sigma}^{(n+1)} + \mathcal{R}_K^{n+1} = 0, \tag{3.4.2}$$

where

$$\mathcal{R}_K^{n+1} = \frac{|K|}{2\delta t}\big(\rho_K^{(n+1)} - \rho_K^{(n)}\big)^2 - \frac{1}{2}\sum_{\sigma\in\mathcal{E}(K)} |\sigma|\,\big(\rho_\sigma^{(n+1)} - \rho_K^{(n+1)}\big)^2\; u_{K,\sigma}^{(n+1)}. \tag{3.4.3}$$

As a consequence, we get that

$$\frac{1}{2}\sum_{K\in\mathcal{M}} |K|(\rho_K^{(n+1)})^2 + \frac{\delta t}{2}\sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=K|L}} |\sigma|\,(\rho_L^{(n+1)} - \rho_K^{(n+1)})^2\; |u_{K,\sigma}^{(n+1)}| + \mathcal{R}_\rho^{(n+1)} = \frac{1}{2}\sum_{K\in\mathcal{M}} |K|(\rho_K^{(n)})^2, \tag{3.4.4}$$

where $\mathcal{R}_\rho^{(n+1)} = \frac{1}{2}\sum_{K\in\mathcal{M}} |K|(\rho_K^{(n+1)} - \rho_K^{(n)})^2 \ge 0$*. Thus, the following weak* BV *estimate holds:*

$$\sum_{n=1}^{N} \delta t \sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=K|L}} |\sigma|\,\big(\rho_L^{(n)} - \rho_K^{(n)}\big)^2\; |u_{K,\sigma}^{(n)}| \le C, \tag{3.4.5}$$

where $C \ge 0$ *depends only on the* L^2*-norm of this initial data for the density* ρ_0*.*

Proof. Multiply the discrete mass balance equation (3.3.2b) by $|K|\rho_K^{(n+1)}$. In the discrete time derivative term, use the identity $2(a^2 - ab) = (a^2 - b^2) + (a-b)^2$ with $a = \rho_K^{(n+1)}$ and $b = \rho_K^{(n)}$. In the discrete convection term, use the identity $2ab = a^2 + b^2 - (a-b)^2$ with $a = \rho_K^{(n+1)}$ and $b = \rho_\sigma^{n+1}$. Furthermore, note that

$$\sum_{\sigma\in\mathcal{E}(K)} |\sigma|(\rho_K^{(n+1)})^2 u_{K,\sigma}^{(n+1)} = (\rho_K^{(n+1)})^2(\mathrm{div}_{\mathcal{M}}\boldsymbol{u})_K = 0.$$

Relation (3.4.4) is obtained by summing (3.4.2) over the cells of the mesh. As usual, the convective term (*i.e.* the second term in (3.4.2)) vanishes by conservativity; the second term in (3.4.4) is obtained by summing over the cells the second term of the remainder (3.4.3), and using the definition of the upwind approximation of the density at the face. Finally, the weak BV estimate (3.4.5) is obtained by summing (3.4.4) over n. □

We then give an estimate on the velocity which is a discrete equivalent of the kinetic energy balance. Recall that in the continuous setting, the kinetic energy balance is formally obtained by multiplying the i-th component of the momentum balance equation (3.1.1b) by the i-th component u_i of $\boldsymbol{u}$. Using the mass balance equation (3.1.1a) twice, this yields

$$\partial_t(\frac{1}{2}\rho u_i^2) + \operatorname{div}(\frac{1}{2}\rho u_i^2\ \boldsymbol{u}) + \partial_i p\ u_i = f_i u_i,$$

and thus, summing over the components:

$$\partial_t(\rho E_k) + \operatorname{div}(\rho E_k\ \boldsymbol{u}) + \nabla p \cdot \boldsymbol{u} = \boldsymbol{f} \cdot \boldsymbol{u}, \text{ with } E_k = \frac{1}{2}|\boldsymbol{u}|^2.$$

In the discrete setting, this multiplication must be performed on the dual mesh, since the velocity unknowns are defined at the faces and the momentum balance equations are written on the corresponding dual cells. Thanks to the choice of the fluxes on the faces of the dual mesh, which ensures that a discrete mass balance equation holds on the dual grid cells, it is proven in [46, Lemma 3.1] that a discrete equivalent of the above formal computation can be performed to yield a discrete kinetic energy balance.

Lemma 3.4.3 (Discrete kinetic energy balance)**.** *Any solution to the scheme* (3.3.2) *satisfies the following equality, for* $1 \le i \le d$*, for all* $\sigma \in \mathcal{E}^{(i)}$ *and* $0 \le n \le N-1$*:*

$$\frac{1}{2\delta t}(\rho_{D_\sigma}^{(n+1)}(u_\sigma^{(n+1)})^2 - \rho_{D_\sigma}^{(n)}(u_\sigma^{(n)})^2) + \frac{1}{2|D_\sigma|}\sum_{\substack{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)\\ \epsilon=\sigma|\sigma'}} F_{\sigma,\epsilon}^{(n+1)} u_\sigma^{(n+1)} u_{\sigma'}^{(n+1)} - (\Delta u)_\sigma^{(n+1)} u_\sigma^{(n+1)} \\ + (\eth p)_\sigma^{(n+1)} u_\sigma^{(n+1)} - f_\sigma^{(n+1)} u_\sigma^{(n+1)} = -R_\sigma^{n+1}, \quad (3.4.6)$$

with $R_\sigma^{n+1} = \dfrac{1}{2\delta t}\rho_{D_\sigma}^{(n)}\big(u_\sigma^{(n+1)} - u_\sigma^{(n)}\big)^2.$

Thanks to this identity, we are now in position to state the following uniform estimates for the velocity.

Lemma 3.4.4 (Estimates on the velocity)**.** *There exists $C > 0$ depending only on $\boldsymbol{u}_0$, ρ_0 and $\boldsymbol{f}$ such that, for any function $\boldsymbol{u} \in \boldsymbol{X}_{\mathcal{E},\delta t}$ satisfying* (3.3.2)*, the following estimates hold:*

$$\|\boldsymbol{u}\|_{L^2(0,T;\mathbf{H}_{\mathcal{E},0})} = \sum_{n=0}^{N-1} \delta t \ \|\boldsymbol{u}^{(n+1)}\|^2_{1,\mathcal{E},0} \leq C, \tag{3.4.7}$$

$$\|\boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega)^d)} = \max_{0\leq n\leq N-1} \|\boldsymbol{u}^{(n+1)}\|_{L^2(\Omega)^d} \leq C. \tag{3.4.8}$$

Proof. Let us sum the discrete kinetic balance equation (3.4.6) over the faces $\sigma \in \mathcal{E}^{(i)}$ and sum for $i = 1, \ldots, d$ and for $n = 0, \ldots, M$ with $M \leq N-1$. Thanks to the duality of the discrete gradient and divergence operators (see (3.2.10)), using (3.3.2d), noting that the convection term vanishes by conservativity of the numerical flux and that the density remains positive, we get:

$$\sum_{i=1}^{d} \sum_{\sigma\in\mathcal{E}^{(i)}} |D_\sigma| \frac{1}{2} \rho_{D_\sigma}^{(M+1)} (u_\sigma^{(M+1)})^2 - \sum_{i=1}^{d} \sum_{\sigma\in\mathcal{E}^{(i)}} |D_\sigma| \frac{1}{2} \rho_{D_\sigma}^{(0)} (u_\sigma^{(0)})^2 + \delta t \sum_{n=0}^{M} \|\boldsymbol{u}^{(n+1)}\|^2_{1,\mathcal{E},0} \\ - \sum_{n=0}^{M} \sum_{i=1}^{d} \sum_{\sigma\in\mathcal{E}^{(i)}} \delta t |D_\sigma| f_\sigma^{(n+1)} u_\sigma^{(n+1)} \leq 0.$$

By Lemma 3.4.1 and thanks to the Cauchy-Schwarz, discrete Poincaré and Young inequalities, we then get the existence of $C > 0$ depending only on Ω such that

$$\sum_{n=0}^{M} \delta t \ \|\boldsymbol{u}^{(n+1)}\|^2_{1,\mathcal{E},0} + \rho_{\min} \|\boldsymbol{u}^{(M+1)}\|^2_{L^2(\Omega)} \leq \rho_{\max} \|\boldsymbol{u}_0\|^2_{L^2(\Omega)} + C \, \|\boldsymbol{f}\|^2_{L^2(0,T;L^2(\Omega)^d)}.$$

On one hand, this inequality yields the $L^\infty(L^2)$ estimate (3.4.8) ; on the other hand, taking $M = N-1$ we get the $L^2(\mathbf{H}_{\mathcal{E},0})$ estimate (3.4.7). □

Lemma 3.4.5 (Estimate on $b_\mathcal{E}$)**.** *There exists $C_{\eta_\mathcal{M}} > 0$, depending only on the regularity $\eta_\mathcal{M}$ of the mesh defined by* (2.2.2) *such that, $\forall (\rho, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in L_\mathcal{M} \times \boldsymbol{E}_\mathcal{E} \times {\mathbf{H}_{\mathcal{E},0}}^2$,*

$$|b_\mathcal{E}(\rho\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leq C_{\eta_\mathcal{M}} \, \|\rho\|_{L^\infty(\Omega)} \, \|\boldsymbol{u}\|_{L^4(\Omega)^d} \, \|\boldsymbol{v}\|_{1,\mathcal{E},0} \, \|\boldsymbol{w}\|_{L^4(\Omega)^d} \tag{3.4.9}$$

$$\leq C_{\eta_\mathcal{M}} \, \|\rho\|_{L^\infty(\Omega)} \, \|\boldsymbol{u}\|_{1,\mathcal{E},0} \, \|\boldsymbol{v}\|_{1,\mathcal{E},0} \, \|\boldsymbol{w}\|_{1,\mathcal{E},0}. \tag{3.4.10}$$

These estimates can be proven by an easy adaptation of the proof of [33, Lemma 3.1].

Next we turn to an estimate the discrete derivative $\eth_t \boldsymbol{u}$ of the velocity for a certain norm depending on ρ; this estimate is used later to obtain the compactness of the sequence of approximate velocities. To this end, we denote by $L_{\mathcal{M},b}$ the set $\{\rho \in L_{\mathcal{M}}; \rho_{\min} \leq \rho \leq \rho_{\max} \text{ a.e.}\}$, and for a given $\rho \in L_{\mathcal{M},b}$, we introduce the following discrete norms on $\mathbf{H}_{\mathcal{E},0}$ and $\boldsymbol{X}_{\mathcal{E},\delta t}$:

$$\begin{aligned}
&\text{for } \boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0}, \quad \|\boldsymbol{v}\|_{\boldsymbol{E}'_{\rho,\mathcal{E}}} = \max \Big\{ \Big| \int_\Omega \rho \boldsymbol{v} \cdot \boldsymbol{w} \, \mathrm{d}\boldsymbol{x} \Big| \; ; \; \boldsymbol{w} \in \boldsymbol{E}_{\mathcal{E}} \text{ and } \|\boldsymbol{w}\|_{1,\mathcal{E},0} \leq 1 \Big\}, \\
&\text{for } \boldsymbol{v} \in \boldsymbol{X}_{\mathcal{E},\delta t}, \quad \|\boldsymbol{v}\|_{L^{4/3}(0,T;\boldsymbol{E}'_{\rho,\mathcal{E}})} = \left(\sum_{n=0}^{N-1} \delta t \|\boldsymbol{v}^{(n+1)}\|^{4/3}_{\boldsymbol{E}'_{\mathcal{E}}} \right)^{3/4}.
\end{aligned} \tag{3.4.11}$$

Note that the notation $\| \cdot \|_{\boldsymbol{E}'_{\rho,\mathcal{E}}}$ is somewhat formal since this is not a dual norm. However, if ρ is constant, it is in fact the dual norm to the norm in $E_{\mathcal{E}}$ that we used for the homgeneneous incompressible case [33], which reads $\|\boldsymbol{v}\|_{\boldsymbol{E}'_{\mathcal{E}}} = \max \Big\{ \Big| \int_\Omega \boldsymbol{v} \cdot \boldsymbol{w} \, \mathrm{d}\boldsymbol{x} \Big| \; ; \; \boldsymbol{w} \in \boldsymbol{E}_{\mathcal{E}} \text{ and } \|\boldsymbol{w}\|_{1,\mathcal{E},0} \leq 1 \Big\},$.

With these notations, we may state the following result.

Lemma 3.4.6 (Estimate on the discrete time derivative of the velocity)**.** *Let* $\boldsymbol{u} \in \boldsymbol{X}_{\mathcal{E},\delta t}$ *be a solution to* (3.3.2)*. Then there exists* $C > 0$ *depending only on* $\boldsymbol{u}_0$*,* Ω*,* $\eta_{\mathcal{M}}$ *and* $\boldsymbol{f}$ *such that:*

$$\|\eth_t \boldsymbol{u}\|_{L^{4/3}(0,T;\boldsymbol{E}'_{\rho,\mathcal{E}})} \leq C.$$

Proof. Let $(\rho, \boldsymbol{u}) \in L_{\mathcal{M}} \times \boldsymbol{X}_{\mathcal{E},\delta t}$ satisfy (3.3.2). Then using Lemma (3.3.2), we have

$$\begin{aligned}
\rho^{(n)}_{D_\sigma} \frac{1}{\delta t}(u^{(n+1)}_\sigma - u^{(n)}_\sigma) &= \frac{1}{\delta t}(\rho^{(n+1)}_{D_\sigma} u^{(n+1)}_\sigma - \rho^{(n)}_{D_\sigma} u^{(n)}_\sigma) - u^{(n+1)}_\sigma \frac{1}{\delta t}(\rho^{(n+1)}_{D_\sigma} - \rho^{(n)}_{D_\sigma}) \\
&= -\frac{1}{|D_\sigma|} \sum_{\substack{\epsilon \in \tilde{\mathcal{E}}(D_\sigma) \\ \epsilon = \sigma|\sigma'}} F^{(n+1)}_{\sigma,\epsilon} \; \frac{u^{(n+1)}_\sigma + u^{(n+1)}_{\sigma'}}{2} + (\Delta u)^{(n+1)}_\sigma - (\eth \; p)^{(n+1)}_\sigma \\
&\qquad + f^{(n+1)}_\sigma + \frac{1}{|D_\sigma|} \sum_{\substack{\epsilon \in \tilde{\mathcal{E}}(D_\sigma) \\ \epsilon = \sigma|\sigma'}} F^{(n+1)}_{\sigma,\epsilon} \; u^{(n+1)}_\sigma.
\end{aligned}$$

Let $\boldsymbol{v} \in \boldsymbol{E}_{\mathcal{E}}$ such that $\|\boldsymbol{v}\|_{1,\mathcal{E},0} \leq 1$; multiplying the above equality by $|D_\sigma| v_\sigma$, summing the result over $\sigma \in \mathcal{E}^{(i)}$ and $i \in \{1, \cdots, d\}$, and using the discrete duality property (3.2.10), we obtain

$$\begin{aligned}
\int_\Omega \rho^{(n)} \eth_t \boldsymbol{u}^{(n+1)} \; \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} &= \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}^{(i)}} \sum_{\substack{\epsilon \in \tilde{\mathcal{E}}(D_\sigma) \\ \epsilon = \sigma|\sigma'}} F^{(n+1)}_{\sigma,\epsilon} \; (u^{(n+1)}_\sigma - u^{(n+1)}_\epsilon) v_\sigma \\
&\qquad - \int_\Omega \nabla_{\mathcal{E}} \boldsymbol{u}^{(n+1)} \cdot \nabla_{\mathcal{E}} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{f}^{(n+1)}_{\mathcal{E}} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Noting that

$$(u_\sigma^{(n+1)} - u_\epsilon^{(n+1)})v_\sigma = u_\sigma^{(n+1)} v_\epsilon - \frac{1}{2}(u_\sigma^{(n+1)} v_{\sigma'} + u_{\sigma'}^{(n+1)} v_\sigma),$$

we get that

$$\sum_{i=1}^{d} \sum_{\sigma\in\mathcal{E}^{(i)}} \sum_{\substack{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)\\ \epsilon=\sigma|\sigma'}} F_{\sigma,\epsilon}^{(n+1)}\ (u_\sigma^{(n+1)} - u_\epsilon^{(n+1)})v_\sigma = -b_{\mathcal{E}}(\rho^{(n+1)}\boldsymbol{u}^{(n+1)}, \boldsymbol{v}, \boldsymbol{u}^{(n+1)}) + R$$

$$\text{with } R = -\frac{1}{2}\sum_{i=1}^{d} \sum_{\sigma\in\mathcal{E}^{(i)}} \sum_{\substack{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)\\ \epsilon=\sigma|\sigma'}} F_{\sigma,\epsilon}^{(n+1)}\ (u_\sigma^{(n+1)} v_{\sigma'} + u_{\sigma'}^{(n+1)} v_\sigma) = 0$$

thanks to the conservativity of the dual fluxes. We thus obtain that

$$\int_\Omega \rho^{(n)} \eth_t \boldsymbol{u}^{(n+1)}\ \cdot \boldsymbol{v}\ \mathrm{d}\boldsymbol{x} = -b_{\mathcal{E}}(\rho^{(n+1)}\boldsymbol{u}^{(n+1)}, \boldsymbol{v}, \boldsymbol{u}^{(n+1)}) - \int_\Omega \nabla_{\mathcal{E}} \boldsymbol{u}^{(n+1)} \cdot \nabla_{\mathcal{E}} \boldsymbol{v}\ \mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{f}_{\mathcal{E}}^{(n+1)} \cdot \boldsymbol{v}\ \mathrm{d}\boldsymbol{x}.$$

Thanks to (3.4.9) we obtain :

$$|b_{\mathcal{E}}(\rho^{(n+1)}\boldsymbol{u}^{(n+1)}, \boldsymbol{v}, \boldsymbol{u}^{(n+1)})| \le C_{\eta_{\mathcal{M}}} \|\rho\|_{L^\infty(\Omega)} \|\boldsymbol{u}^{(n+1)}\|_{L^4(\Omega))^d}^2.$$

Using the Cauchy-Schwarz inequality, we note that

$$\|\boldsymbol{u}^{(n+1)}\|_{L^4(\Omega)^d}^4 = \int_\Omega |\boldsymbol{u}^{(n+1)}||\boldsymbol{u}^{(n+1)}|^3\ \mathrm{d}\boldsymbol{x} \le \|\boldsymbol{u}^{(n+1)}\|_{L^2(\Omega)^d} \|\boldsymbol{u}^{(n+1)}\|_{L^6(\Omega)^d}^3.$$

Therefore, thanks to the estimate (3.4.8) of Lemma 3.4.4, thanks to Lemma 3.4.1 and to the discrete Poincaré inequality, there exists $\widetilde{C}_{\eta_{\mathcal{M}}} > 0$ depending only on Ω, $\rho_{\max}$ and on the regularity of the mesh, such that

$$\int_\Omega \rho^{(n)} \eth_t \boldsymbol{u}^{(n+1)} \cdot \boldsymbol{v}\ \mathrm{d}\boldsymbol{x} \le \widetilde{C}_{\eta_{\mathcal{M}}} (\|\boldsymbol{u}^{(n+1)}\|_{(L^6(\Omega))^d}^{3/2} + \|\boldsymbol{u}^{(n+1)}\|_{1,\mathcal{E},0} + \|\boldsymbol{f}_{\mathcal{E}}^{(n+1)}\|_{(L^2(\Omega))^d}).$$

Hence there exists $C > 0$ depending only on Ω, $\rho_{\max}$ and on the regularity of the mesh such that

$$\begin{aligned}\|\eth_t \boldsymbol{u}^{(n+1)}\|_{\boldsymbol{E}'_{\rho,\mathcal{E}}}^{4/3} &\le C\left(\|\boldsymbol{u}^{(n+1)}\|_{L^6(\Omega)^d}^2 + \|\boldsymbol{u}^{(n+1)}\|_{1,\mathcal{E},0}^{4/3} + \|\boldsymbol{f}_{\mathcal{E}}^{(n+1)}\|_{L^2(\Omega)^d}^{4/3}\right)\\ &\le C\left(\|\boldsymbol{u}^{(n+1)}\|_{L^6(\Omega)^d}^2 + \|\boldsymbol{u}^{(n+1)}\|_{1,\mathcal{E},0}^2 + \|\boldsymbol{f}_{\mathcal{E}}^{(n+1)}\|_{L^2(\Omega)^d}^2 + 2\right).\end{aligned}$$

Multiplying this latter inequality by δt and summing for $n = 0, \dots, N-1$, we conclude the proof of the lemma thanks to the discrete Sobolev inequality [22, Lemma 3.5] and thanks to the $L^2(0,T;\mathbf{H}_{\mathcal{E},0})$ estimate on $\boldsymbol{u}$ given by (3.4.7). □

Let us now give a (non-uniform) bound on the pressure that will be useful to prove the existence of a solution to the scheme (3.3.2).

Lemma 3.4.7 (Bound on the pressure)**.** *Let $(\rho^{(n)}, \boldsymbol{u}^{(n)}, p^{(n)}) \in L_{\mathcal{M}} \times \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M},0}$ be given and assume that $(\rho^{(n+1)}, \boldsymbol{u}^{(n+1)}, p^{(n+1)}) \in L_{\mathcal{M}} \times \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M},0}$ satisfies (3.3.2). Then there exists $C_{\delta t} > 0$ depending only on ρ_0, δt, $\eta_{\mathcal{M}}$, $\boldsymbol{f}$, and Ω such that:*

$$\|p^{(n+1)}\|_{L^2(\Omega)} \leq C_{\delta t}. \tag{3.4.12}$$

Proof. Let $(\rho^{(n+1)}, \boldsymbol{u}^{(n+1)}, p^{(n+1)})$ be a solution to (3.3.2); we choose $\boldsymbol{v} = \widetilde{\mathcal{P}}_{\mathcal{E}}\boldsymbol{\varphi}$ as test function in (3.3.10), where $\boldsymbol{\varphi} \in H_0^1(\Omega)^d$ is such that $\operatorname{div}\boldsymbol{\varphi} = p^{(n+1)}$ and $\|\nabla\boldsymbol{\varphi}\|_{L^2(\Omega)^{d\times d}} \leq c\|p^{(n+1)}\|_{L^2(\Omega)}$, with c depending only on Ω. By Lemma 2.3.3, we then obtain:

$$\|p^{(n+1)}\|^2_{L^2(\Omega)} = T_1 + T_2 + T_3 + T_4, \qquad \text{with } T_1 = \int_\Omega \eth_t(\rho\boldsymbol{u})^{(n+1)} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x},$$
$$T_2 = b_{\mathcal{E}}((\rho\boldsymbol{u})^{(n+1)}, \boldsymbol{u}^{(n+1)}, \boldsymbol{v}),\ T_3 = \int_\Omega \nabla_{\mathcal{E}}\boldsymbol{u}^{(n+1)} : \nabla_{\mathcal{E}}\boldsymbol{v}\, \mathrm{d}\boldsymbol{x} \text{ and } T_4 = -\int_\Omega \boldsymbol{f}_{\mathcal{E}}^{(n+1)} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}.$$

By the Cauchy-Schwarz inequality and thanks to the L^∞ bound (3.4.1) on ρ we get that

$$|T_1| \leq \frac{\rho_{\max}}{\delta t} \sum_{i=1}^{d} \int_\Omega (|u_i^{(n+1)}| + |u_i^{(n)}|)|v_i|\, \mathrm{d}\boldsymbol{x} \leq \frac{\rho_{\max}}{\delta t} \|\boldsymbol{u}\|_{L^\infty(0,T,(L^2(\Omega)^d)}\|\boldsymbol{v}\|_{L^2(\Omega)^d}.$$

Using the fact that $\widetilde{\mathcal{P}}_{\mathcal{E}}$ is a Fortin interpolator (see Lemma 2.3.3), since $\|\nabla\boldsymbol{\varphi}\|_{L^2(\Omega)^{d\times d}} \leq c\|p^{(n+1)}\|_{L^2(\Omega)}$, we get by the Poincaré inequality and by the $L^\infty(L^2)$ estimate (3.4.8) that there exists $C_1 \geq 0$ depending only on ρ_0, δt, $\eta_{\mathcal{M}}$, $\boldsymbol{f}$, and Ω such that $|T_1| \leq C_1\|p^{(n+1)}\|_{L^2(\Omega)}$. Moreover, thanks to (3.4.10) we obtain:

$$\begin{aligned} |b_{\mathcal{E}}((\rho\boldsymbol{u})^{(n+1)}, \boldsymbol{v}, \boldsymbol{u}^{(n+1)})| &\leq C_{\eta_{\mathcal{M}}}\|\rho^{(n+1)}\|_{L^\infty}\|\boldsymbol{u}^{(n+1)}\|^2_{1,\mathcal{E},0}\|\boldsymbol{v}\|_{1,\mathcal{E},0} \\ &\leq C_{\eta_{\mathcal{M}}}\rho_{\max}\|\boldsymbol{u}^{(n+1)}\|^2_{1,\mathcal{E},0}\|\nabla\boldsymbol{\varphi}\|_{L^2(\Omega)^{d\times d}}. \end{aligned}$$

Therefore, thanks to the estimate (3.4.7), there exists $C_2 \geq 0$ depending only on ρ_0, δt, $\eta_{\mathcal{M}}$, $\boldsymbol{f}$, and Ω such that $|T_2| \leq C_2 \|p^{(n+1)}\|_{L^2(\Omega)}$. Finally,

$$\begin{aligned} T_3 &\leq \|\boldsymbol{u}^{(n+1)}\|_{1,\mathcal{E},0} \|\boldsymbol{v}\|_{1,\mathcal{E},0} \\ &\leq C_{\eta_{\mathcal{M}}} \ \|\boldsymbol{u}^{(n+1)}\|_{1,\mathcal{E},0} \|p^{(n+1)}\|_{L^2(\Omega)} \end{aligned}$$

and $T_4 \leq \|\boldsymbol{f}_{\mathcal{E}}^{(n+1)}\|_{L^2(\Omega)^d} \|\boldsymbol{v}\|_{L^2(\Omega)^d} \ \mathrm{d}\boldsymbol{x} \leq C_{\eta_{\mathcal{M}}} \|\boldsymbol{f}_{\mathcal{E}}^{(n+1)}\|_{L^2(\Omega)^d} \|p^{(n+1)}\|_{L^2(\Omega)}$, which concludes the proof. □

The existence of a solution to the scheme (3.3.2) is obtained by a topological degree argument. Its proof is based on an abstract theorem which we recall in the Appendix : see Theorem 4.0.11.

Theorem 3.4.8 (Existence of a solution)**.** *For a given* $n \in \{1, \cdots, N-1\}$, *let us assume that the density* $\rho^{(n)}$ *is such that* $0 < \rho_{\min} \leq \rho_K^{(n)} \leq \rho_{\max}$ *for all* $K \in \mathcal{M}$. *Then the non-linear system* (3.3.2) *admits at least one solution* $(\rho^{(n+1)}, \boldsymbol{u}^{(n+1)}, p^{(n+1)})$ *in* $L_{\mathcal{M}} \times \mathbf{H}_{\mathcal{E},0} \times L_{\mathcal{M},0}$, *and any possible solution satisfies the estimates* (3.4.1), (3.4.7) *and* (3.4.12).

Proof. This proof is based on a topological degree argument, see Theorem 4.0.11 in the Appendix. Let $N_{\mathcal{M}} = \mathrm{card}(\mathcal{M})$ and $N_{\mathcal{E}} = \mathrm{card}(\mathcal{E}_{\mathrm{int}})$; we identify $L_{\mathcal{M}}$ with $\mathbb{R}^{N_{\mathcal{M}}}$ and $\mathbf{H}_{\mathcal{E},0}$ with $\mathbb{R}^{N_{\mathcal{E}}}$. Let $V = \mathbb{R}^{N_{\mathcal{M}}} \times \mathbb{R}^{N_{\mathcal{E}}} \times \mathbb{R}^{N_{\mathcal{M}}}$ and let us introduce the function $F : V \times [0,1] \to V$ defined by:

$$F(\rho, \boldsymbol{u}, p, \lambda) = \left| \begin{array}{ll} \dfrac{1}{\delta t}(\rho_K - \rho_K^{(n)}) + \lambda \dfrac{1}{|K|} \displaystyle\sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}, & K \in \mathcal{M} \\ \dfrac{1}{\delta t}(\rho_{D_\sigma} u_\sigma - \rho_{D_\sigma}^{(n)} u_\sigma^{(n)}) + \lambda \dfrac{1}{|D_\sigma|} \displaystyle\sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon} u_\epsilon - (\Delta u)_\sigma + (\eth p)_\sigma - f_\sigma, & \sigma \in \mathcal{E}_{\mathrm{int}} \\ -\dfrac{1}{|K|} \displaystyle\sum_{\sigma \in \mathcal{E}(K)} |\sigma| u_{K,\sigma} + \dfrac{1}{|K|} \displaystyle\sum_{L \in \mathcal{M}} |L|\, p_L, & K \in \mathcal{M}. \end{array} \right.$$

The function F is continuous from $V \times [0,1]$ to V and the problem $F(\rho, \boldsymbol{u}, p, 1) = 0$ is equivalent to the system (3.3.2). Moreover, an easy verification shows that the problem $F(\rho, \boldsymbol{u}, p, \lambda) = 0$ for λ in $[0,1]$, satisfies the same estimates as stated in (3.4.1), (3.4.7) and (3.4.12) uniformly in λ. Hence, defining

$$\mathcal{O} = \{(\rho, \boldsymbol{u}, p) \in V \text{ s.t. } \frac{\rho_{\min}}{2} < \rho < 2\ \rho_{\max},\ \|\boldsymbol{u}\|_{1,\mathcal{E},0} < C \text{ and } \|p\|_{L^2(\Omega)} \leq C\},$$

with C (strictly) greater than the right-hand sides of (3.4.7) and (3.4.12), the second hypothesis of Theorem 4.0.11 is also satisfied. Therefore, in order to prove the existence of at least one solution to the scheme (3.3.2), it remains to show that the topological degree of $F(\rho, \boldsymbol{u}, p, 0)$

with respect to 0_V and $\mathcal{O}$ is non-zero. The function $G : (\rho, \boldsymbol{u}, p) \to F(\rho, \boldsymbol{u}, p, 0)$ is clearly differentiable on $\mathcal{O}$, and its Jacobian matrix is given by:

$$\text{Jac } G(\rho, \boldsymbol{u}, p) = \left(\begin{array}{c|c} \frac{1}{\delta t} Id_{\mathbb{R}^{N_\mathcal{M} \times N_\mathcal{M}}} & 0 \\ \hline A & \\ & S(\boldsymbol{u}, p) \\ 0 & \end{array} \right)$$

where A is some matrix in $\mathbb{R}^{N_\mathcal{E} \times N_\mathcal{M}}$ and $S(\boldsymbol{u}, p) \in \mathbb{R}^{(N_\mathcal{E}+N_\mathcal{M})\times(N_\mathcal{E}+N_\mathcal{M})}$ is the Jacobian matrix associatedto the MAC discretization of the following Stokes problem: Find $(\boldsymbol{u}, p)$ such that $\int_\Omega p(\boldsymbol{x}) = 0$ and

$$\begin{aligned} &\frac{1}{\delta t}\rho(\boldsymbol{x})\ \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, && \text{in } \Omega, \\ &\text{div } \boldsymbol{u} = 0, && \text{in } \Omega, \\ &\boldsymbol{u} = 0, && \text{on } \partial\Omega \end{aligned}$$

where $\rho(\boldsymbol{x}) > 0$ is a given (since the derivation is performed with respect to $\boldsymbol{u}$ and p with a fixed density) non-negative (by the maximum principle) function approximated by a given constant ρ_{D_σ} on each dual cell D_σ. Hence $S(\boldsymbol{u}, p)$ is invertible and so is Jac $G(\rho, \boldsymbol{u}, p)$. This implies that the topological degree of $F(\rho, \boldsymbol{u}, p, 0)$ is non-zero and by Theorem 4.0.11, there exists at least one solution $(\rho, \boldsymbol{u}, p)$ to the equation $F(\rho, \boldsymbol{u}, p, 1) = 0$, *i.e.* to the scheme (3.3.2). □

3.5 Convergence of the scheme

In order to prove the convergence of the scheme, we introduce an alternate convection operator $b_\mathcal{M}$, defined on the pressure grid and easier to manipulate in the proofs. It relies on the reconstruction of each velocity component on all edges (or faces in 3D) of the mesh. For a given MAC

mesh $(\mathcal{M}, \mathcal{E})$, we define, for $i, j = 1, \dots, d$, the i-th full grid velocity reconstruction operator by

$$\begin{aligned} \mathcal{R}_\mathcal{E}^{(i,j)} :\quad & H_{\mathcal{E},0}^{(i)} \to L^2(\Omega) \\ & v \mapsto \mathcal{R}_\mathcal{E}^{(i,j)} v = \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(j)}} \widehat{v}_\sigma \chi_{D_\sigma}, \end{aligned}$$

where

$$\widehat{v}_\sigma = v_\sigma \text{ if } \sigma \in \mathcal{E}^{(i)}, \quad \widehat{v}_\sigma = \frac{1}{\operatorname{card}(\mathcal{N}_\sigma)} \sum_{\sigma' \in \mathcal{N}_\sigma} v_{\sigma'} \text{ otherwise,} \tag{3.5.1}$$

with, for any $\sigma \in \mathcal{E} \setminus \mathcal{E}^{(i)}$, $\mathcal{N}_\sigma = \{\sigma' \in \mathcal{E}^{(i)}, \overline{D_\sigma} \cap \sigma' \neq \emptyset\}$. We recall that by [33, Lemma 3.5], there exists $C \geq 0$, depending only on the regularity of the mesh defined by (3.2.2), such that, for any $v \in L^2(\Omega)$, $\|\mathcal{R}_\mathcal{E}^{(i,j)} v\|_{L^2(\Omega)} \leq C\|v\|_{L^2(\Omega)}$. Moreover, by [33, Lemma 3.6] if $(\mathcal{M}_n, \mathcal{E}_n)_{n\in\mathbb{N}}$ be a sequence MAC meshes such that $h_{\mathcal{M}_n} \to 0$ as $n \to +\infty$, and $(\eta_{\mathcal{M}_n})_{n\in\mathbb{N}}$ remains bounded, and $\bar{v} \in L^2(\Omega)$ and $(v_n)_{n\in\mathbb{N}}$ are such that $v_n \in H^{(i)}_{\mathcal{E}_n,0}$ and v_n converges to $\bar{v}$ as $n \to +\infty$ in $L^2(\Omega)$, then $\mathcal{R}_{\mathcal{E}_n}^{(i,j)} v_n \to \bar{v}$ in $L^2(\Omega)$ as $n \to +\infty$.

Lemma 3.5.1 (Alternate convection term)**.** *Let $\boldsymbol{u} \in \mathbf{H}_{\mathcal{E},0}$ and $\rho \in L_\mathcal{M}$, let $i \in \{1,\dots,d\}$ and let $v,\ w \in H^{(i)}_{\mathcal{E},0}$. For $\sigma \in \mathcal{E}$, let $\widehat{v}_\sigma$ be defined by* (3.5.1)*, and for $K \in \mathcal{M}$, let $w_K = \frac{1}{2}\sum_{\sigma\in\mathcal{E}^{(i)}(K)} w_\sigma$ where $\mathcal{E}^{(i)}(K)$ is the set of faces of K which are included in $\mathcal{E}^{(i)}$. Let $b^{(i)}_\mathcal{M} : L_\mathcal{M} \times \mathbf{H}_{\mathcal{E},0} \times H^{(i)}_{\mathcal{E},0} \times H^{(i)}_{\mathcal{E},0} \to \mathbb{R}$ be defined by:*

$$b^{(i)}_\mathcal{M}(\rho, \boldsymbol{u}, v, w) = \sum_{K\in\mathcal{M}} w_K \sum_{\sigma\in\mathcal{E}(K)} F_{K,\sigma}\ \widehat{v}_\sigma, \tag{3.5.2}$$

Then there exists $C \geq 0$ depending only on the regularity $\eta_\mathcal{M}$ defined by (3.2.2) *such that:*

$$|\beta^{(i)}(\rho\boldsymbol{u}, v, w)| = |b^{(i)}_\mathcal{E}(\rho\boldsymbol{u}, v, w) - b^{(i)}_\mathcal{M}(\rho\boldsymbol{u}, v, w)| \leq C\ h\ \|\rho\|_{L^\infty(\Omega)} \|\boldsymbol{u}\|_{1,\mathcal{E},0}\ \|v\|_{1,\mathcal{E}^{(i)},0}\ \|w\|_{1,\mathcal{E}^{(i)},\infty}, \tag{3.5.3}$$

with

$$\|w\|_{1,\mathcal{E}^{(i)},\infty} = \max_{\epsilon\in\widetilde{\mathcal{E}}^{(i)}} \frac{1}{d_\epsilon} |[w_\epsilon]| \ \textit{with}\ [w_\epsilon] = w_\sigma - w_{\sigma'} \ \textit{for}\ \epsilon = \sigma|\sigma'.$$

Proof. By definition (see (3.3.7)), we have

$$b^{(i)}_\mathcal{E}(\rho, \boldsymbol{u}, v, w) = \sum_{\sigma\in\mathcal{E}^{(i)}} w_\sigma \sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon} v_\epsilon,$$

where v_ϵ is defined by (3.2.11) and $F_{\sigma,\epsilon}$ by (3.2.12)-(3.2.13). The sum of fluxes on the faces of a velocity cell may be dispatched to the two intersected pressure cells, as shown in Figure 3.1; hence, we may rewrite $b^{(i)}_\mathcal{E}(\rho, \boldsymbol{u}, v, w)$ as a sum over the primal cells, and using (3.2.12)-(3.2.13), we get:

$$b^{(i)}_\mathcal{E}(\rho, \boldsymbol{u}, v, w) = \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}^{(i)}(K)} w_\sigma \Big[F_{\sigma,\epsilon_K} v_{\epsilon_K} + \sum_{\substack{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)\\ \epsilon\not\perp\boldsymbol{e}_i}} \frac{F_{K,\tau_\epsilon}}{2}\ v_\epsilon\Big],$$

where:

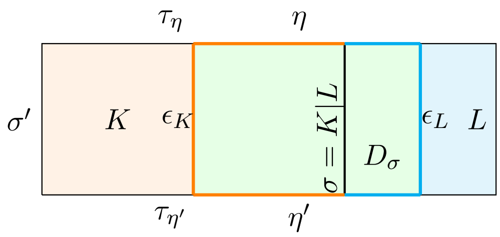


FIG. 3.1 – *Fluxes: from a dual cell to two pressure cells.*

- $\epsilon_K \in \widetilde{\mathcal{E}}^{(i)}$ is the unique face (or edge in 2D) of the dual mesh which is included in K and normal to $\boldsymbol{e}_i$

- $\tau_\epsilon \in \mathcal{E}(K)$ is the face (or edge) which belongs to $\mathcal{E}^{(j)}$ for some $j \neq i$ and intersects the dual face ϵ which is parallel to $\boldsymbol{e}_i$. (They are the faces τ_η and $\tau_{\eta'} \in \mathcal{E}^{(2)}$ in Figure 3.1).

By conservativity, adding $F_{K,\sigma}\ v_\sigma$ in the internal sum does not change the value of the sum, and therefore

$$b_{\mathcal{E}}^{(i)}(\rho, \boldsymbol{u}, v, w) = \sum_{K\in\mathcal{M}} \ \sum_{\sigma\in\mathcal{E}^{(i)}(K)} w_\sigma \Big[F_{K,\sigma}\ v_\sigma + F_{\sigma,\epsilon_K}\ v_{\epsilon_K} + \sum_{\substack{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)\\ \epsilon\not\perp \boldsymbol{e}_i}} \frac{F_{K,\tau_\epsilon}}{2}\ v_\epsilon\Big]. \tag{3.5.4}$$

Thanks to the definition of $\hat{v}_\sigma$ and F_{K,τ_ϵ}, we have:

$$\sum_{\sigma\in\mathcal{E}^{(i)}(K)} \Big[F_{K,\sigma}\ v_\sigma + \sum_{\substack{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)\\ \epsilon\not\perp \boldsymbol{e}_i}} \frac{F_{K,\tau_\epsilon}}{2}\ v_\epsilon\Big] = \sum_{\sigma\in\mathcal{E}(K)} F_{K,\sigma}\ \hat{v}_\sigma,$$

so that

$$\mathcal{R}^{(i)}(\rho, \boldsymbol{u}, v, w) = \sum_{K\in\mathcal{M}} \ \sum_{\sigma\in\mathcal{E}^{(i)}(K)} (w_\sigma - w_K) \Big[F_{K,\sigma}\ v_\sigma + F_{\sigma,\epsilon_K}\ v_{\epsilon_K} + \sum_{\substack{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)\\ \epsilon\not\perp \boldsymbol{e}_i}} \frac{F_{K,\tau_\epsilon}}{2}\ v_\epsilon\Big],$$

Using (3.2.12), we remark that a discrete mass balance is satisfied over the half-cells $D_{K,\sigma}$, in the sense that:

$$F_{K,\sigma} + F_{\sigma,\epsilon_K} + \sum_{\substack{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)\\ \epsilon\not\perp \boldsymbol{e}_i}} \frac{F_{K,\tau}}{2} = \frac{1}{2} \sum_{\tau\in\mathcal{E}(K)} F_{K,\tau}.$$

Therefore,

$$\begin{aligned}
&\mathcal{R}^{(i)}(\rho,\boldsymbol{u},v,w) = \mathcal{R}^{(i,1)}(\rho,\boldsymbol{u},v,w) + \mathcal{R}^{(i,2)}(\rho,\boldsymbol{u},v,w) \text{ with}\\
&\mathcal{R}^{(i,1)}(\rho,\boldsymbol{u},v,w) = \frac{1}{2}\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}^{(i)}(K)} (w_\sigma - w_K)\ v_\sigma \sum_{\sigma'\in\mathcal{E}(K)} F_{K,\sigma'}\\
&\mathcal{R}^{(i,2)}(\rho,\boldsymbol{u},v,w) = \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}^{(i)}(K)} (w_\sigma - w_K)\ \Big[F_{\sigma,\epsilon_K}(v_{\sigma_K} - v_\sigma) + \sum_{\substack{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)\\ \epsilon\not\perp \boldsymbol{e}_i,\epsilon=\overrightarrow{\sigma|\sigma'}}} \frac{1}{2}F_{K,\tau_\epsilon}(v_{\sigma'} - v_\sigma)\Big].
\end{aligned}$$

Let us denote by $[v]_K^{(i)}$ the jump $v_\sigma - v_{\sigma'}$ where $\sigma,\sigma'\in\mathcal{E}(K)\cap\mathcal{E}^{(i)}$ are oriented as in Figure 3.1. We have

$$\begin{aligned}
|\mathcal{R}^{(i,1)}(\rho,\boldsymbol{u},v,w)| &\le \frac{1}{2}\sum_{K\in\mathcal{M}} |[w]_K^{(i)}||[v]_K^{(i)}| \sum_{\sigma'\in\mathcal{E}(K)} |F_{K,\sigma'}|\\
&\le C_\eta \frac{1}{2} h\|\rho\|_{L^\infty}\|w\|_{1,\mathcal{E}^{(i)},\infty}\|v\|_{1,\mathcal{E}^{(i)},0}\|\boldsymbol{u}\|_{(L^2(\Omega))^2},
\end{aligned}$$

where C_η depends only on the regularity of the mesh. The treatment of the term $\mathcal{R}^{(i,2)}(\rho,\boldsymbol{u},v,w)$ is then similar to that of the term $\mathcal{R}^{(i)}(\rho,\boldsymbol{u},v,w)$ in the proof of [33, Lemma 2.21]. We give it here for the sake of completeness:

$$\begin{aligned}
\mathcal{R}^{(i,2)}(\rho,\boldsymbol{u},v,w) &= \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}^{(i)}(K)} (w_\sigma - w_K)\ \Big[F_{\sigma,\epsilon_K}(v_{\epsilon_K} - v_\sigma) + \sum_{\substack{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)\\ \epsilon\not\perp \boldsymbol{e}_i}} \frac{F_{K,\tau_\epsilon}(\rho,\boldsymbol{u})}{2}(v_\epsilon - v_\sigma)\Big],\\
&= \frac{1}{2}\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}^{(i)}(K)} (w_\sigma - w_K)\ \Big[F_{\sigma,\epsilon_K}(v_{\sigma_K} - v_\sigma)\\
&\qquad + \sum_{\substack{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)\\ \epsilon\not\perp \boldsymbol{e}_i,\epsilon=\overrightarrow{\sigma|\sigma'}}} \frac{1}{2}F_{K,\tau_\epsilon}(\rho,\boldsymbol{u})(v_{\sigma'} - v_\sigma)\Big], \quad (3.5.5)
\end{aligned}$$

where σ_K denotes the face or edge of the primal mesh that is opposite to σ in the cell K. Hence,

$$\begin{aligned}|\mathcal{R}^{(i,2)}(\rho,\boldsymbol{u},v,w)| &\leq h\|w\|_{1,\mathcal{E}^{(i)},\infty}\|\rho\|_{L^\infty}\Big(\sum_{\substack{\epsilon\in\widetilde{\mathcal{E}}^{(i)},\epsilon\perp\boldsymbol{e}_i\\ \epsilon=\sigma|\sigma'}}|\epsilon|(|u_\sigma|+|u_{\sigma'}|)|v_\sigma-v_{\sigma'}| \\ &\qquad+\sum_{\substack{\epsilon\in\widetilde{\mathcal{E}}^{(i)}_{\rm int},\epsilon\not\perp\boldsymbol{e}_i\\ \epsilon=\sigma|\sigma'\subset\tau\cup\tau'}}|\epsilon|(|u_\tau|+|u_{\tau'}|)\ |v_\sigma-v_{\sigma'}|)\Big) \\ &\leq h\|w\|_{1,\mathcal{E}^{(i)},\infty}\|\rho\|_{L^\infty}\Big(\sum_{\substack{\epsilon\in\widetilde{\mathcal{E}}^{(i)},\epsilon\perp\boldsymbol{e}_i\\ \epsilon=\sigma|\sigma'}}|\epsilon|d_\epsilon(|u_\sigma|+|u_{\sigma'}|)\frac{|v_\sigma-v_{\sigma'}|}{d_\epsilon} \\ &\qquad+\sum_{\substack{\epsilon\in\widetilde{\mathcal{E}}^{(i)}_{\rm int},\epsilon\not\perp\boldsymbol{e}_i\\ \epsilon=\sigma|\sigma'\subset\tau\cup\tau'}}d_\epsilon|\epsilon|(|u_\tau|+|u_{\tau'}|)\ \frac{|v_\sigma-v_{\sigma'}|}{d_\epsilon})\Big).\end{aligned}$$

By the Cauchy-Schwarz inequality, we get that

$$\begin{aligned}|\mathcal{R}^{(i,2)}(\rho,\boldsymbol{u},v,w))|^2 &\leq h^2\|w\|^2_{1,\mathcal{E}^{(i)},\infty}\|\rho\|^2_{L^\infty}\|v\|^2_{1,\mathcal{E}^{(i)},0} \\ &\Big(\sum_{\substack{\epsilon\in\widetilde{\mathcal{E}}^{(i)},\epsilon\perp\boldsymbol{e}_i\\ \epsilon=\sigma|\sigma'}}d_\epsilon|\epsilon|(|u_\sigma|+|u_{\sigma'}|)^2+\sum_{\substack{\epsilon\in\widetilde{\mathcal{E}}^{(i)}_{\rm int},\epsilon\not\perp\boldsymbol{e}_i\\ \epsilon=\sigma|\sigma'\subset\tau\cup\tau'}}d_\epsilon|\epsilon|(|u_\tau|+|u_{\tau'}|)^2\ \Big).\end{aligned}$$

There exists C depending only on the regularity of the mesh such that:

$$\sum_{\substack{\epsilon\in\widetilde{\mathcal{E}}^{(i)},\epsilon\perp\boldsymbol{e}_i\\ \epsilon=\sigma|\sigma'}}d_\epsilon|\epsilon|(|u_\sigma|+|u_{\sigma'}|)^2+\sum_{\substack{\epsilon\in\widetilde{\mathcal{E}}^{(i)}_{\rm int},\epsilon\not\perp\boldsymbol{e}_i\\ \epsilon=\sigma|\sigma'\subset\tau\cup\tau'}}d_\epsilon|\epsilon|(|u_\tau|+|u_{\tau'}|)^2\leq C^2\sum_{\sigma\in\mathcal{E}}|D_\sigma||u_\sigma|^2,$$

so that

$$|\mathcal{R}^{(i,2)}(\rho,\boldsymbol{u},v,w))|\leq Ch\|\rho\|_{L^\infty}\|w\|_{1,\mathcal{E}^{(i)},\infty}\|v\|_{1,\mathcal{E}^{(i)},0}\|\boldsymbol{u}\|_{(L^2(\Omega))^d}.$$

This concludes the proof of (3.5.3). □

Lemma 3.5.2 (Estimate on the time translates of the velocity)**.** *Let $\boldsymbol{u}\in X_{\mathcal{E},\delta t}$ and $\rho\in Y_{\mathcal{M},\delta t}$ and let $\tau>0$ then*

$$\int_0^{T-\tau}\int_\Omega\rho(t)|\boldsymbol{u}(t+\tau)-u(t)|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\leq C_{\eta_\mathcal{M},T}\|\rho\|_{L^\infty(\Omega\times(0,T))}(\|\boldsymbol{u}\|^3_{L^2(0,T;\mathbf{H}_{\mathcal{E},0})}+1)\sqrt{\tau+\delta t}\quad(3.5.6)$$

where $C_{\eta_\mathcal{M},T}>0$ only depends on T and on the regularity of the mesh $\eta_\mathcal{M}$.

Proof. As in [64] we use the Lions compactness argument and follow the proof of the continuous case, see e.g. [6, pages 444-452]. Let $\boldsymbol{w} = \boldsymbol{u}(\cdot, \cdot + \tau) - \boldsymbol{u}$. We wish to estimate the term

$$A = \int_0^{T-\tau} \int_\Omega (\rho(t)\boldsymbol{u}(t+\tau) - \rho(t)\boldsymbol{u}(t)) \cdot \boldsymbol{w}(t) \ \mathrm{d}t = \int_0^{T-\tau} (A_1(t) + A_2(t)) \ \mathrm{d}t, \ \text{with}$$
$$A_1(t) = \int_\Omega (\rho(t+\tau)\boldsymbol{u}(t+\tau) - \rho(t)\boldsymbol{u}(t)) \cdot \boldsymbol{w}(t) \ \mathrm{d}t,$$
$$A_2(t) = \int_\Omega (\rho(t) - \rho(t+\tau))\, \boldsymbol{u}(t+\tau) \cdot \boldsymbol{w}(t) \ \mathrm{d}t.$$

Let us first remark that

$$\begin{aligned}(\rho(t+\tau)\boldsymbol{u}(t+\tau) - \rho(t)\boldsymbol{u}(t)) \cdot \boldsymbol{w}(t) &= \Big[\sum_{n; t_n \in (t, t+\tau)} (\rho^{(n+1)}\boldsymbol{u}^{(n+1)} - \rho^{(n)}\boldsymbol{u}^{(n)}) \Big] \cdot \boldsymbol{w}(t) \\ &= \sum_{n=1}^{N-1} (\rho^{(n+1)}\boldsymbol{u}^{(n+1)} - \rho^{(n)}\boldsymbol{u}^{(n)}) \cdot \boldsymbol{w}(t) \chi_{(t,t+\tau)}(t_n)\end{aligned}$$

so that, thanks to the momentum equation (3.3.2c), we have $A_1(t) = A_{11}(t) + A_{12}(t) + A_{13}(t) + A_{14}(t)$ with

$$A_{11}(t) = \sum_{n=1}^{N-1} \delta t \ \int_\Omega \boldsymbol{f}^{(n+1)} \cdot \boldsymbol{w}(t) \ \chi_{(t,t+\tau)}(t_n),$$
$$A_{12}(t) = -\sum_{n=1}^{N-1} \delta t \ \int_\Omega \nabla_{\mathcal{E}} p^{(n+1)} \cdot \boldsymbol{w}(t) \ \chi_{(t,t+\tau)}(t_n),$$
$$A_{13}(t) = -\sum_{n=1}^{N-1} \delta t \ \int_\Omega \Delta_{\mathcal{E}} \boldsymbol{u}^{(n+1)} \cdot \boldsymbol{w}(t) \ \chi_{(t,t+\tau)}(t_n),$$
$$A_{14}(t) = \sum_{n=1}^{N-1} \delta t \ \int_\Omega \boldsymbol{C}_{\mathcal{E}}(\rho^{(n+1)}\boldsymbol{u}^{(n+1)})\boldsymbol{u}^{(n+1)} \cdot \boldsymbol{w}(t) \ \chi_{(t,t+\tau)}(t_n).$$

It is easily seen that

$$\int_0^{T-\tau} A_{11}(t) \le C\sqrt{\tau} \|\boldsymbol{f}\|_{L^2(0,T;L^2(\Omega))} \|\boldsymbol{u}\|_{L^2(0,T;L^2(\Omega))}$$

Moreover, thanks to the discrete duality property and the fact that $\boldsymbol{u}$ is discrete-divergence-free, we have $A_{12}(t) = 0$. Next we write that

$$\int_\Omega \Delta_{\mathcal{E}} \boldsymbol{u}^{(n+1)} \cdot \boldsymbol{w}(t) = -\int_\Omega \nabla_{\mathcal{E}} \boldsymbol{u}^{(n+1)} : \nabla_{\mathcal{E}} \boldsymbol{w} \ \mathrm{d}\boldsymbol{x} \le \|\boldsymbol{u}^{(n+1)}\|_{1,\mathcal{E},0}^2 \| \|\boldsymbol{w}\|_{1,\mathcal{E},0},$$

and therefore,

$$\int_0^{T-\tau} A_{13}(t) \leq (5\tau + 4\delta t)\|\boldsymbol{u}\|^2_{L^2(0,T;\mathbf{H}_{\mathcal{E},0})}$$

Let us now turn to the term $A_{14}(t)$. We have

$$A_{14}(t) \leq \|\rho\|_{L^\infty(\Omega\times(0,T))} \sum_{n=1}^{N-1} \delta t \|\boldsymbol{u}^{(n+1)}\|^2_{L^4(\Omega)} \|\boldsymbol{w}(t)\|_{1,\mathcal{E},0} \chi_{(t,t+\tau)}(t_n).$$

Therefore, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\int_0^{T-\tau} A_{14}(t)\ \mathrm{d}t &\leq \|\rho\|_{L^\infty(\Omega\times(0,T))} \sum_{n=1}^{N-1} \delta t \|\boldsymbol{u}^{(n+1)}\|^2_{L^4(\Omega)} \int_0^{T-\tau} \|\boldsymbol{w}(t)\|_{1,\mathcal{E},0} \chi_{(t_n-\tau,t_n)}(t)\ \mathrm{d}t. \\ &\leq \|\rho\|_{L^\infty(\Omega\times(0,T))} \sqrt{\tau} \|\boldsymbol{u}\|^2_{L^2(0,T;L^4(\Omega))} \|\boldsymbol{u}\|_{L^2(0,T;\mathbf{H}_{\mathcal{E},0})}.\end{aligned}$$

Let us now turn to the term A_2. As for A_1, we write that

$$\begin{aligned}(\rho(t+\tau) - \rho(t))\boldsymbol{u}(t+\tau)\cdot\boldsymbol{w}(t) &= \sum_{n;t_n\in(t,t+\tau)} (\rho^{(n+1)} - \rho^{(n)})\boldsymbol{u}(t+\tau)\cdot\boldsymbol{w}(t) \\ &= \sum_{n=1}^{N-1} (\rho^{(n+1)} - \rho^{(n)})\boldsymbol{u}(t+\tau)\cdot\boldsymbol{w}(t)\chi_{(t,t+\tau)}(t_n)\end{aligned}$$

Thanks to the discrete mass equation (3.3.2b), we have:

$$A_2(t) = \sum_{n=1}^{N-1} \delta t \int_\Omega \mathrm{div}_{\mathcal{M}}(\rho^{(n+1)}\boldsymbol{u}^{(n+1)})\boldsymbol{u}(t+\tau)\cdot\boldsymbol{w}(t)\chi_{(t,t+\tau)}(t_n) = \sum_{n=1}^{N-1} \delta t I_{n,t},$$

with $I_{n,t} = \int_\Omega \mathrm{div}_{\mathcal{M}}(\rho^{(n+1)}\boldsymbol{u}^{(n+1)})\boldsymbol{u}(t+\tau)\cdot\boldsymbol{w}(t)\ \mathrm{d}\boldsymbol{x}$. For t and n fixed, we now obtain a bound for $I_{n,t}$. We skip the index $(n+1)$ and we set $\boldsymbol{v} = \boldsymbol{u}(t+\tau)$. By definition of the discrete divergence operator, we have:

$$\begin{aligned}I_{n,t} &= \sum_{K\in\mathcal{M}} \frac{1}{|K|} \left(\sum_{\sigma\in\mathcal{E}_K} |\sigma|\rho_\sigma u_{K,\sigma}\right) \left(\sum_{\sigma\in\mathcal{E}_K} |D_{K,\sigma}| v_{K,\sigma} w_{K,\sigma}\right) \\ &= \frac{1}{2} \sum_{K\in\mathcal{M}} \left(\sum_{\sigma\in\mathcal{E}_K} |\sigma|\rho_\sigma u_{K,\sigma}\right) \left(\sum_{\sigma\in\mathcal{E}_K} v_{K,\sigma} w_{K,\sigma}\right).\end{aligned}$$

Reordering this sum, we obtain, with $\psi_\sigma = v_\sigma w_\sigma$

$$|I_{n,t}| \leq \sum_{\substack{\sigma\in\mathcal{E}_{\rm ext}\\ \sigma=K|L}} |\sigma||\rho_\sigma||u_\sigma|(\sum_{j=1}^{d} T_{\sigma,j}),$$

with, if $\sigma \in \mathcal{E}_i$,

$$T_{\sigma,i} = \frac{1}{2}(|\psi_\sigma - \psi_{\sigma_K}| + |\psi_\sigma - \psi_{\sigma_L}|),\ \sigma_K \in \mathcal{E}_K \cap \mathcal{E}_i,\ \sigma_L \in \mathcal{E}_L \cap \mathcal{E}_i$$

and for $j \neq i$,

$$T_{\sigma,j} = \frac{1}{2} \sum_{\substack{\sigma_K\in\mathcal{E}_j\cap\mathcal{E}_K,\\ \sigma_L\in\mathcal{E}_j\cap\mathcal{E}_L,\\ \sigma_K\sigma_L//e_i}} |\psi_{\sigma_K} - \psi_{\sigma_L}|.$$

We now remark that for all σ_1, σ_2 we have $\psi_{\sigma_1} - \psi_{\sigma_2} = v_{\sigma_1}(w_{\sigma_1} - w_{\sigma_2}) + w_{\sigma_2}(v_{\sigma_1} - v_{\sigma_2})$. Using the upper bound of ρ, the regularity of the mesh and Hölder Inequality (noting that $(1/2 + 1/6 + 1/6) \leq 1$), we obtain, for some C only depending on the regularity of the mesh and the upper bound of ρ,

$$\begin{aligned} I_{n,t} &\leq C\|\boldsymbol{u}^{n+1}\|_{L^6(\Omega)}(\|\boldsymbol{u}(t+\tau)\|_{L^6(\Omega)}\|\boldsymbol{w}(t), \|_{1,\mathcal{E},0} + \|\boldsymbol{w}(t)\|_{L^6(\Omega)}\|\boldsymbol{u}(t+\tau)\|_{1,\mathcal{E},0} \\ &\leq C\|\boldsymbol{u}^{n+1}\|_{L^6(\Omega)}(\|\boldsymbol{u}(t+\tau)\|^2_{L^6(\omega)} + \|\boldsymbol{w}(t)\|^2_{1,\mathcal{E},0} + \|\boldsymbol{w}(t)\|^2_{L^6(\Omega)} + \|\boldsymbol{u}(t+\tau)\|^2_{1,\mathcal{E},0}. \end{aligned}$$

The Cauchy-Schwarz Inequality gives

$$\begin{aligned} \sum_{n=1}^{N-1} \delta t\|\boldsymbol{u}^{n+1}\|_{L^6(\Omega)}\chi_{(t,t+\tau)}(t_n) &\leq (\sum_{n=1}^{N-1} \delta t\|\boldsymbol{u}^{n+1}\|^2_{L^6(\Omega)})^{1/2}(\sum_{n=1}^{N-1} \delta t\chi_{(t,t+\tau)}(t_n))^{1/2} \\ &\leq \|\boldsymbol{u}\|_{L^2(]0,T[,L^6(\Omega))}(\tau+\delta t)^{1/2}. \end{aligned}$$

Then,

$$A_2(t) \leq C\|\boldsymbol{u}\|_{L^2(]0,T[,L^6(\Omega))}(\tau+\delta)^{1/2}(\|\boldsymbol{u}(t+\tau)\|^2_{L^6(\omega)} + \|\boldsymbol{w}(t)\|^2_{1,\mathcal{E},0} + \|\boldsymbol{w}(t)\|^2_{L^6(\omega)} + \|\boldsymbol{u}(t+\tau)\|^2_{1,\mathcal{E},0}).$$

Since the $L^2(0,T;L^6(\Omega))$-norm of $\boldsymbol{u}$ is bounded by its $L^2(0,T;\boldsymbol{H}_{\mathcal{E},0})$, we obtain the desired inequality.

$\square$

Theorem 3.5.3 (Convergence of the scheme)**.** *Let* $(\delta t_m)_{m\in\mathbb{N}}$ *and* $(\mathcal{D}_m)_{m\in\mathbb{N}} = (\mathcal{M}_m, \mathcal{E}_m)_{m\in\mathbb{N}}$ *be* *a sequence of time steps and MAC grids (in the sense of Definition 2.2.1) such that* $\delta t_m \to 0$ *and* $h_{\mathcal{M}_m} \to 0$ *as* $m \to +\infty$ *; assume that there exists* $\eta > 0$ *such that* $\eta_{\mathcal{M}_m} \leq \eta$ *for any* $m \in \mathbb{N}$ *(with* $\eta_{\mathcal{M}_m}$ *defined by* (2.2.2)*). Let* $(\rho_m, \boldsymbol{u}_m)$ *be a solution to* (3.3.2) *for* $\delta t = \delta t_m$ *and* $\mathcal{D} = \mathcal{D}_m$*. Then there exists* $\bar\rho$ *with* $\rho_{\min} \leq \bar\rho \leq \rho_{\max}$ *and* $\bar{\boldsymbol{u}} \in L^2(0,T;\boldsymbol{E}(\Omega))$ *such that, up to a subsequence:*

- *the sequence* $(\boldsymbol{u}_m)_{m\in\mathbb{N}}$ *converges to* $\bar{\boldsymbol{u}}$ *in* $L^2(0,T;L^2(\Omega)^d)$,
- *the sequence* $(\rho_m)_{m\in\mathbb{N}}$ *converges to* $\bar\rho$ *in* $\in L^2(0,T;L^2(\Omega))$,
- $(\bar\rho,\bar{\boldsymbol{u}})$ *is a solution to the weak formulation* (3.1.6) *and* (3.1.7).

Proof. We proceed in six steps.

First step: Weak convergence for ρ.

By (3.4.1), there exists a subsequence of $(\rho_m)_{m\in\mathbb{N}}$, still denoted $(\rho_m)_{m\in\mathbb{N}}$, which converges star-weakly to some function $\bar\rho$ in $L^\infty(\Omega\times(0,T))$, *i.e.*:

$$\lim_{m\to\infty}\int_0^T\int_\Omega \rho_m(\boldsymbol{x},t)\varphi(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_0^T\int_\Omega \bar\rho(\boldsymbol{x},t)\varphi(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,\quad \forall\varphi\in L^1(\Omega\times(0,T)). \tag{3.5.7}$$

Furthermore, an easy consequence of (3.4.1) and (3.5.7) is the non-negativity of the integrals

$$\int_0^T\int_\Omega(\bar\rho(\boldsymbol{x},t)-\rho_{\min})\chi_A(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \text{ and } \int_0^T\int_\Omega(\rho_{\max}-\bar\rho(\boldsymbol{x},t))\chi_A(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

for any borelian set A of $\Omega\times(0,T)$, which is equivalent to $\rho_{\min}\leq\bar\rho(\boldsymbol{x},t)\leq\rho_{\max}$ a.e. in $\Omega\times(0,T)$.

Step 2: compactness in $L^2(0,T;L^2(\Omega)^d)$.

The second step consists in applying [32, Proposition 4.47 and Theorem 4.53] which we recall in the appendix in Proposition 4.0.7 in order obtain the existence of subsequence of $(\boldsymbol{u}_m)_{m\in\mathbb{N}}$ which converges to $\bar{\boldsymbol{u}}$ in $L^2(0,T;L^2(\Omega)^d)$. In our setting, we apply Proposition 4.0.7 with $p=2$; the Banach space B of is $L^2(\Omega)^d$, and the spaces X_m consist in the space $\mathbf{H}_{\mathcal{E}_m,0}$ endowed with the norm defined in (3.3.4). By [22, Theorem 14.2] and the Fréchet-Kolmogorov compactness theorem (see e.g. [22, Theorem 14.1]) we obtain that $(X_m)_{m\in\mathbb{N}}$ is compactly embedded in B in the sense of Definition 4.0.5. We now check the three assumptions (1), (2) and (3) of Proposition 4.0.7: By Lemma 3.4.4, the sequence $\|\boldsymbol{u}_m\|_{L^1(0,T;\mathbf{H}_{\mathcal{E},0})}$ is bounded, and thanks to the discrete Poincaré inequality, we also have that is bounded in $L^2(0,T;(L^2(\Omega)^d))$; furthermore, Thanks to the estimates (3.4.7) and to Lemma 3.5.2, there exists $C>0$ independent of m such that

$$\|\boldsymbol{u}_m(\cdot,\cdot+\tau)-\boldsymbol{u}_m\|_{L^2(0,T;L^2(\Omega))}\leq C(\tau^{1/2}+\delta t)$$

Hence, Proposition 4.0.7 applies and there exists $\bar{\boldsymbol{u}} \in L^2(0,T;(L^2)^d)$ such that, up to a subsequence,

$$\boldsymbol{u}_m \to \bar{\boldsymbol{u}} \text{ in } L^2\left(0,T;L^2(\Omega)^d\right) \text{ as } m \to +\infty.$$

Step 3 : Passing to the limit in the mass balance equation Let us show that $(\bar{\rho}, \bar{\boldsymbol{u}})$ satisfies (3.1.6). Let $\psi \in C_c^\infty(\Omega \times [0,T))$, taking $\psi_m^{(n)} = \mathcal{P}_{\mathcal{M}_m}\psi(\cdot,t_n) \in L_{\mathcal{M}}$ as test function in (3.3.8), multiplying by δt_m and summing for $n = \{0,\ldots,N_m-1\}$ (with $N_m\delta t_m = T$), we get:

$$\sum_{n=0}^{N_m-1} \delta t_m \int_\Omega \eth_t \rho_{\mathcal{M}_m}^{(n+1)}\ \psi_m^{(n)}\ \mathrm{d}\boldsymbol{x} + \sum_{n=0}^{N_m-1} \delta t_m \int_\Omega \mathrm{div}_{\mathcal{M}_m}(\rho\boldsymbol{u})_m^{(n+1)}\ \psi_m^{(n)}\ \mathrm{d}\boldsymbol{x} = T_{1,m} + T_{2,m} = 0,$$

with

$$T_{1,m} = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{M}} |K|(\rho_K^{(n+1)} - \rho_K^{(n)})\psi_K^{(n)} \text{ and } T_{2,m} = \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} \psi_K^{(n)} \sum_{\sigma\in\mathcal{E}(K)} F_{K,\sigma}^{(n+1)},$$

where we have dropped the subscript m for short. Performing a discrete integration by parts in $T_{1,m}$, we get:

$$T_{1,m} = -\sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} |K|\rho_K^{(n+1)} \frac{(\psi_K^{(n+1)} - \psi_K^{(n)})}{\delta t} - \sum_{K\in\mathcal{M}} |K|\rho_K^{(0)}\psi_K^{(0)},$$

so that

$$T_{1,m} = -\int_0^T \int_\Omega \rho_m(\boldsymbol{x},t)\eth_t\psi_m(\boldsymbol{x},t)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t - \int_\Omega \rho_m^{(0)}(\boldsymbol{x})\psi_m(\boldsymbol{x},0)\ \mathrm{d}\boldsymbol{x},$$

The boundedness of ρ_0 and the definition of the initial conditions for the scheme ensures that the sequence $(\rho_m^{(0)})_{m\in\mathbb{N}}$ converges to ρ_0 in $L^q(\Omega)$ for all $q \in [1,\infty)$. Since, the sequence of discrete solutions of the interpolate time derivatives $\eth_t\psi_m$ converges uniformly towards $\eth_t\psi$, we thus obtain:

$$\lim_{m\to\infty} T_{1,m} = -\int_0^T \int_\Omega \bar{\rho}(\boldsymbol{x},t)\partial_t\psi(\boldsymbol{x},t)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t - \int_\Omega \rho_0(\boldsymbol{x})\psi(\boldsymbol{x},0)\ \mathrm{d}\boldsymbol{x}.$$

Using the expression of the mass flux $F_{K,\sigma}$ and reordering the sum in $T_{2,m}$ we get:

$$T_{2,m} = -\sum_{i=1}^{d}\sum_{n=0}^{N-1} \delta t \sum_{\sigma=K|L\in\mathcal{E}_{\mathrm{int}}^{(i)}} |D_\sigma|\rho_\sigma^{(n+1)} u_{K,\sigma}^{(n+1)} \frac{|\sigma|}{|D_\sigma|}(\psi_L^{(n)} - \psi_K^{(n)}),$$

We decompose the sum in two terms, $T_{2,m} = \mathcal{T}_{2,m} + \mathcal{R}_{2,m}$ with

$$\mathcal{T}_{2,m} = -\sum_{i=1}^{d}\sum_{n=0}^{N-1} \delta t \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}^{(i)}} (|D_{K,\sigma}|\rho_K^{(n+1)} + |D_{L,\sigma}|\rho_L^{(n+1)}) u_{K,\sigma}^{(n+1)} \frac{|\sigma|}{|D_\sigma|}(\psi_L^{(n)} - \psi_K^{(n)})$$
$$= -\int_0^T \int_\Omega \rho_m(\boldsymbol{x},t)\boldsymbol{u}_m(\boldsymbol{x},t)\cdot\nabla_{\mathcal{E}}\psi_m(\boldsymbol{x},t)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t,$$

and

$$\mathcal{R}_{2,m} = -\sum_{i=1}^{d}\sum_{n=0}^{N-1} \delta t \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}^{(i)}} \big[|D_\sigma|\rho_\sigma^{(n+1)} - |D_{K,\sigma}|\rho_K^{(n+1)} - |D_{L,\sigma}|\rho_L^{(n+1)}\big] u_{K,\sigma}^{(n+1)} \frac{|\sigma|}{|D_\sigma|}(\psi_L^{(n)} - \psi_K^{(n)}).$$

Since ρ_m converges to $\bar\rho$ weakly in $L^r(\Omega\times(0,T))$ for $r\in[1,+\infty)$, that $\nabla_{\mathcal{E}}\psi_m$ converges uniformly to $\nabla\psi$, and that $\boldsymbol{u}_m$ converges to $\bar{\boldsymbol{u}}$ in $L^2(\Omega\times(0,T))^d$, we have Therefore

$$\lim_{m\to 0} \mathcal{T}_{2,m} = -\int_0^T\int_\Omega \bar\rho(\boldsymbol{x},t)\bar{\boldsymbol{u}}(\boldsymbol{x},t)\cdot\nabla\psi(\boldsymbol{x},t)\ \mathrm{d}\boldsymbol{x}\ \mathrm{d}t.$$

Thanks to the upwind choice (3.2.4) for ρ_σ, the remainder term $\mathcal{R}_{2,m}$ can be bounded as follows:

$$|\mathcal{R}_{2,m}| \le \sum_{n=0}^{N-1}\delta t\sum_{i=1}^{d}\sum_{\substack{\sigma\in\mathcal{E}_{\rm int}^{(i)}\\ \sigma=K|L}} |\sigma||\rho_L^{(n+1)} - \rho_K^{(n+1)}||u_\sigma^{(n+1)}||\psi_L^{(n)} - \psi_K^{(n)}|$$

Therefore, thanks to the Cauchy-Schwarz inequality and the weak BV estimate (3.4.5), we have:

$$|\mathcal{R}_{2,m}| \le \Big(\sum_{n=0}^{N-1}\delta t\sum_{i=1}^{d}\sum_{\substack{\sigma\in\mathcal{E}_{\rm int}^{(i)}\\ \sigma=K|L}} |\sigma||\rho_L^{(n+1)} - \rho_K^{(n+1)}|^2|u_\sigma^{(n+1)}|\Big)^{\frac12}$$
$$\Big(\sum_{n=0}^{N-1}\delta t\sum_{i=1}^{d}\sum_{\substack{\sigma\in\mathcal{E}_{\rm int}^{(i)}\\ \sigma=K|L}} |\sigma||\psi_L^{(n)} - \psi_K^{(n)}|^2|u_\sigma^{(n+1)}|\Big)^{\frac12}$$
$$\le \sqrt{C}\Big(\sum_{n=0}^{N-1}\delta t\sum_{i=1}^{d}\sum_{\substack{\sigma\in\mathcal{E}_{\rm int}^{(i)}\\ \sigma=K|L}} |\sigma||\psi_L^{(n)} - \psi_K^{(n)}|^2|u_\sigma^{(n+1)}|\Big)^{\frac12}.$$

By regularity of ψ, we then have

$$
\begin{aligned}
|\mathcal{R}_{2,m}| &\leq \sqrt{C} C_{\varphi} \sqrt{h_m} \Big(\sum_{n=0}^{N-1} \delta t \sum_{i=1}^{d} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}}^{(i)} \\ \sigma = K|L}} |D_\sigma| |u_\sigma^{(n+1)}| \Big)^{\frac{1}{2}} \\
&\leq \sqrt{C} C_{\varphi} T |\Omega| \sqrt{h_m} \|\boldsymbol{u}_m\|_{L^2(\Omega \times (0,T)}.
\end{aligned}
$$

Therefore, $|\mathcal{R}_{2,m}| \to 0$ as $h_m \to 0$.

Step 4 : Strong convergence of ρ. Let us now show that the density converges strongly in L^2. In fact, the proof is the exact same as that of [58, Proposition 8.6] and we give it here for the sake of completeness.

Since $(\rho_m)_{m \in \mathbb{N}}$ is bounded in $L^\infty(\Omega \times (0,T))$, it is sufficient, by interpolation, to prove the strong convergence of ρ_m towards $\bar{\rho}$ in $L^2(\Omega \times (0,T))$. From the L^∞ weak star convergence of ρ_m, we also get that ρ_m converges weakly in $L^2(\Omega \times (0,T))$, and therefore

$$
\|\bar{\rho}\|_{L^2(\Omega \times (0,T))} \leq \lim inf_{m \in \mathbb{N}} \|\rho_m\|_{L^2(\Omega \times (0,T))}.
$$

By (3.4.4), we have for all n in $\{0, \cdots, N-1\}$:

$$
\sum_{K \in \mathcal{M}} |K| (\rho_K^{(n+1)})^2 \leq \sum_{K \in \mathcal{M}} |K| (\rho_K^{(0)})^2 \leq \|\rho_0\|_{L^2(\Omega)}^2,
$$

which yields $\|\rho_m(.,t)\|_{L^2(\Omega)}^2 \leq \|\rho_0(.,t)\|_{L^2(\Omega)}^2$ for all $t \in (0,T)$ and $m \in \mathbb{N}$. Moreover, since $\bar{\rho}$ is a solution to (3.1.6), one has $\bar{\rho} \in C^0(0,T; L^2(\Omega))$ and $\|\bar{\rho}(.,t)\|_{L^2(\Omega)} = \|\rho_0\|_{L^2(\Omega)}$ for all $t \in (0,T)$[17]. Therefore, we have $\|\rho_m(.,t)\|_{L^2(\Omega)}^2 \leq \|\bar{\rho}(.,t)\|_{L^2(\Omega)}^2$ for all $t \in [0,T)$ and all $m \in \mathbb{N}$. Integrating this last inequality for $t \in [0,T)$, we obtain $\|\rho_m\|_{L^2(\Omega \times (0,T))}^2 \leq \|\bar{\rho}\|_{L^2(\Omega \times (0,T))}^2$ for all $m \in \mathbb{N}$, and passing to the limit as m goes to infinity yields:

$$
\limsup_{n \to \infty} \|\rho_m\|_{L^2(\Omega \times (0,T))} \leq \|\bar{\rho}\|_{L^2(\Omega \times (0,T))}.
$$

This proves that $\lim_{m \to \infty} \|\rho_m\|_{L^2(\Omega \times (0,T))} = \|\bar{\rho}\|_{L^2(\Omega \times (0,T))}$ which in turn yields that ρ_m converges strongly to $\bar{\rho}$ in $L^2(\Omega \times (0,T))$.

Step 5 : Passing to the limit in the momentum balance equation Let us then show that $(\bar{\rho}, \bar{\boldsymbol{u}})$ satisfies (3.1.7). Let $\boldsymbol{\varphi} \in C_c^\infty(\Omega \times [0,T))^d$, such that $\mathrm{div}\boldsymbol{\varphi} = 0$. By Lemma 2.3.3, we have $\mathrm{div}_{\mathcal{M}_m} \widetilde{\mathcal{P}}_{\mathcal{E}_m} \boldsymbol{\varphi}(\cdot, t_n) = 0$, and so we can take $\boldsymbol{\varphi}_m^{(n)} = \widetilde{\mathcal{P}}_{\mathcal{E}_m} \boldsymbol{\varphi}(\cdot, t_n) \in \boldsymbol{E}_{\mathcal{E}}$ as test function in (3.3.9) ; multiplying by δt_m and summing for $n = \{0, \ldots, N_m - 1\}$ (with $N_m \delta t_m = T$), we then get:

$$\sum_{n=0}^{N_m-1} \delta t_m \Big[\int_\Omega \eth_t(\rho \boldsymbol{u})_m^{(n+1)} \cdot \boldsymbol{\varphi}_m^{(n)} \,\mathrm{d}\boldsymbol{x} + b_{\mathcal{E}}((\rho\boldsymbol{u})_m^{(n+1)}, \boldsymbol{u}_m^{(n+1)}, \boldsymbol{\varphi}_m^{(n)}) + \int_\Omega \nabla_{\mathcal{E}} \boldsymbol{u}_m^{(n+1)} : \nabla_{\mathcal{E}} \boldsymbol{\varphi}_m^{(n)} \,\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{f}_{\mathcal{E}_m}^{(n+1)} \cdot \boldsymbol{\varphi}_m^{(n)} \,\mathrm{d}\boldsymbol{x}\Big] = 0,$$

which is equivalent to $\displaystyle\sum_{i=1}^{d} \big[T_{1,i}^{(m)} + T_{2,i}^{(m)} + T_{3,i}^{(m)} + T_{4,i}^{(m)}\big] = 0$ with

$$T_{1,i}^{(m)} = \sum_{n=0}^{N-1} \sum_{\sigma\in\mathcal{E}^{(i)}} \int_\Omega \eth_t(\rho u_i)_m^{(n+1)} \varphi_{m,i}^{(n)} \,\mathrm{d}\boldsymbol{x}, \qquad T_{2,i}^{(m)} = \sum_{n=0}^{N-1} \delta t\; b_{\mathcal{E}}^{(i)}((\rho\boldsymbol{u})_m^{(n+1)}, \boldsymbol{u}_m^{(n+1)}, \boldsymbol{\varphi}_m^{(n)}),$$

$$T_{3,i}^{(m)} = \sum_{n=0}^{N-1} \delta t [u_{i,m}^{(n+1)}, \varphi_{i,m}^{(n)}]_{1,\mathcal{E}^{(i)},0}, \qquad T_{4,i}^{(m)} = \sum_{n=0}^{N-1} \delta t \int_\Omega f_{i,\mathcal{E}_m}^{(n+1)} \varphi_{i,m}^{(n)} \,\mathrm{d}\boldsymbol{x}.$$

where we have dropped the subscript m for short.

Let us first pass to the limit in the time derivative term $T_{1,i}^{(m)}$; we have

$$\begin{aligned} T_{1,i}^{(m)} &= \sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}^{(i)}} |D_\sigma|\, \frac{\rho_{D_\sigma}^{(n+1)} u_\sigma^{(n+1)} - \rho_{D_\sigma}^{(n)} u_\sigma^{(n)}}{\delta t}\, \varphi_\sigma^{(n)} \\ &= -\sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}^{(i)}} |D_\sigma| \rho_{D_\sigma}^{(n+1)} u_\sigma^{(n+1)} \frac{(\varphi_\sigma^{(n+1)} - \varphi_\sigma^{(n)})}{\delta t} - \sum_{\sigma\in\mathcal{E}^{(i)}} |D_\sigma| \rho_{D_\sigma}^{(0)} u_\sigma^{(0)}\, \varphi_\sigma^{(0)}. \end{aligned}$$

Thanks to the definition of the quantity ρ_{D_σ} we have:

$$\begin{aligned} T_{1,i}^{(m)} &= -\sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}^{(i)}} (|D_{K,\sigma}|\rho_K^{(n+1)} + |D_{L,\sigma}|\rho_L^{(n+1)}) u_\sigma^{(n+1)} \frac{(\varphi_\sigma^{(n+1)} - \varphi_\sigma^{(n)})}{\delta t} \\ &\qquad - \sum_{\sigma\in\mathcal{E}^{(i)}} (|D_{K,\sigma}|\rho_K^{(0)} + |D_{L,\sigma}|\rho_L^{(0)}) u_\sigma^{(0)}\, \varphi_\sigma^{(0)} \\ &= -\int_0^T \int_\Omega \rho_m(\boldsymbol{x},t) u_{i,m}(\boldsymbol{x},t) \eth_t \varphi_{i,m}(\boldsymbol{x},t) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_\Omega \rho_m^{(0)}(\boldsymbol{x}) u_{i,m}^{(0)}(\boldsymbol{x}) \varphi_{i,m}^{(0)}(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

The sequence $(\rho_m)_{m\in\mathbb{N}}$ converges (up to a subsequence) to $\bar\rho$ weakly in $L^2(\Omega\times(0,T))$ as $m \to +\infty$, and the sequence $(u_{i,m})_{m\in\mathbb{N}}$ converges (up to a subsequence) to $\bar u_i$ in $L^2(\Omega\times(0,T))$; since $\eth_t\varphi_{i,m}$ converges uniformly towards $\partial_t\varphi_i$, we may thus pass to the limit in the first integral. In addition, from the initialization of the scheme and the assumed regularity of the initial data (i.e. $\rho_0 \in L^\infty(\Omega)$ and $u_0 \in L^2(\Omega)$, $\rho_m^{(0)}$ converges to ρ_0 in $L^q(\Omega)$ for all q in $[1,\infty)$ and $(u_{i,m}^{(0)})$ converges to $u_{i,0}$ in $L^q(\Omega)$ for all q in $[1,2]$; since $\varphi_{i,m}^{(0)}$ converges $\varphi_i(.,0)$ uniformly, we may also pass to the limit in the second integral. Hence,

$$\sum_{i=1}^d T_{1,i}^{(m)} \to -\int_0^T \int_\Omega \bar\rho(\boldsymbol{x},t)\bar{\boldsymbol{u}}(\boldsymbol{x},t) \ \cdot \partial_t \boldsymbol{\varphi}(\boldsymbol{x},t) \ \mathrm{d}\boldsymbol{x} \ \mathrm{d}t - \int_\Omega \rho_0(\boldsymbol{x})\boldsymbol{u}_0(\boldsymbol{x}) \cdot \boldsymbol{\varphi}(\boldsymbol{x},0) \ \mathrm{d}\boldsymbol{x}, \text{ as } m \to \infty.$$

By Lemma 3.5.1 and thanks to the estimates 3.4.8,3.5.3, we have

$$|b_{\mathcal{E}}^{(i)}((\rho \boldsymbol{u})_m^{(n+1)}, \boldsymbol{u}_m^{(n+1)}, \boldsymbol{\varphi}_m^{(n)}) - b_{\mathcal{M}}^{(i)}((\rho \boldsymbol{u})_m^{(n+1)}, \boldsymbol{u}_m^{(n+1)}, \boldsymbol{\varphi}_m^{(n)})| \to 0 \text{ as } n \to +\infty,$$

and therefore $\lim_{m\to+\infty} T_{2,i}^{(m)} = \lim_{m\to+\infty} \widetilde{T}_{2,i}^{(m)}$ with

$$\begin{aligned}\widetilde{T}_{2,i}^{(m)} &= \sum_{n=0}^{N-1} \delta t \ b_{\mathcal{M}}^{(i)}(\rho \boldsymbol{u})_m^{(n+1)}, u_{i,m}^{(n+1)}, \varphi_{i,m}^{(n)}) \\ &= \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} \varphi_{K,i}^{(n)} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \rho_\sigma^{(n+1)} \ u_{K,\sigma}^{(n+1)} \ (\hat u_\sigma^{(i)})^{(n+1)}\end{aligned}$$

where we recall that $\varphi_{K,i}^{(n)} = \frac{1}{2}\left(\varphi_\sigma^{(n)} + \varphi_{\sigma'}^{(n)}\right)$ with $\sigma,\sigma' \in \mathcal{E}(K) \cap \mathcal{E}^{(i)}$. By a change of summation, we thus get that

$$\begin{aligned}\widetilde{T}_{2,i}^{(m)} &= \sum_{j=1}^d \sum_{n=0}^{N-1} \delta t \sum_{\sigma=K|L\in\mathcal{E}_{\mathrm{int}}^{(j)}} |\sigma| \rho_\sigma^{(n+1)} u_{K,\sigma}^{(n+1)} \ (\hat u_\sigma^{(i)})^{(n+1)} \ (\varphi_{L,i}^{(n)} - \varphi_{K,i}^{(n)}) \\ &= -\int_0^T \int_\Omega \widetilde\rho_{m,i} \hat u_{m,i} u_{m,j} \eth_j \varphi_{m,i} \ \mathrm{d}\boldsymbol{x} \ \mathrm{d}t,\end{aligned}$$

where $\widetilde\rho_{m,i} = \sum_{n=0}^{N-1} \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}^{(i)}} \rho_\sigma^{(n+1)} \chi_{[t_n,t_{n+1})} \chi_{D_\sigma}$ and $\varphi_{m,i} = \sum_n \varphi_i^{(n)} \chi_{[t_n,t_{n+1})}$. By Step 5, ρ_m tends to $\bar\rho$ in $L^2((0,T)\times\Omega)$. Thanks to the upwind choice, the values $\rho_{m,i}$ on the D_σ are equal to $(\rho_m)_K$ or $(\rho_m)_L$, so that $\rho_{m,i}$ also tends to $\bar\rho$ in $L^2((0,T)\times\Omega)$ as $m \to +\infty$, and therefore almost everywhere up to a subsequence. By Lemma 3.4.1, $\rho_{m,i}$ is bounded in $L^\infty((0,T)\times\Omega)$. By Step 3, $u_{m,j}$ converges (up to a subsequence) to $\bar u_j$ in $L^2((0,T)\times\Omega)$. Thanks to [33, Lemma 3.6], $\hat u_{m,j}$ also converges to $\bar u_j$ in $L^2((0,T)\times\Omega)$. Moreover, $\eth_j \varphi_{m,i}$ converges uniformly to $\partial_j \varphi_i$. Thus, by the dominated convergence theorem, up to a subsequence, $\widetilde\rho_{m,i} \hat u_{m,i} u_{m,j} \eth_j \varphi_{m,i}$ tends to $\bar{\rho}\bar{u}_j\bar{u}_i\partial_j\varphi_i$ in $L^1((0,T)\times\Omega)$ as $m\to+\infty$, so that

$$\lim_{m\to+\infty} T_{2,i}^{(m)} = \int_0^T \int_\Omega \bar{\rho}\bar{u}_j\bar{u}_i\partial_j\varphi_i \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

which concludes the proof of convergence of the non linear convection term.

The diffusion term reads

$$[\boldsymbol{u}_m^{(n+1)}, \boldsymbol{\varphi}_m^{(n)}]_{1,\mathcal{E},0} = [\boldsymbol{u}_m^{(n+1)}, \boldsymbol{\varphi}_m^{(n+1)}]_{1,\mathcal{E},0} + [\boldsymbol{u}_m^{(n+1)}, \boldsymbol{\varphi}_m^{(n)} - \boldsymbol{\varphi}_m^{(n+1)}]_{1,\mathcal{E},0}.$$

By an easy adaptation of the proof of [22, Theorem 4.2], we get that

$$\sum_{n=0}^{N_m-1} \delta t_m [\boldsymbol{u}_m^{(n+1)}, \boldsymbol{\varphi}_m^{(n+1)}]_{1,\mathcal{E},0} \to \int_0^T \int_\Omega \nabla\bar{\boldsymbol{u}}\cdot\nabla\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \text{ as } m\to+\infty.$$

Moreover, thanks to the regularity of $\boldsymbol{\varphi}$,

$$[\boldsymbol{u}_m^{(n+1)}, \boldsymbol{\varphi}_m^{(n+1)} - \boldsymbol{\varphi}_m^{(n)}]_{1,\mathcal{E},0} \le \delta t_m C_\varphi \|\boldsymbol{u}_m^{(n+1)}\|_{1,\mathcal{E},0}$$

where C_φ depends only on $\boldsymbol{\varphi}$. We thus get that

$$\sum_{n=0}^{N_m-1} \delta t_m [\boldsymbol{u}_m^{(n+1)}, \boldsymbol{\varphi}_m^{(n+1)} - \boldsymbol{\varphi}_m^{(n)}]_{1,\mathcal{E},0} \to 0 \text{ as } m\to+\infty.$$

Finally, we have

$$\int_\Omega \boldsymbol{f}_{\mathcal{E}_m}^{(n+1)}\cdot(\boldsymbol{\varphi}_m^{(n)} - \boldsymbol{\varphi}_m^{(n+1)})\,\mathrm{d}\boldsymbol{x} \le \delta t C_{\boldsymbol{\varphi}} \|\boldsymbol{f}(\cdot,t_{n+1})\|_{L^2(\Omega)} \to 0 \text{ as } m\to+\infty,$$

so that

$$\sum_{n=0}^{N_m-1} \delta t_m \int_\Omega \boldsymbol{f}_{\mathcal{E}_m}^{(n+1)}\cdot\boldsymbol{\varphi}_m^{(n)}\,\mathrm{d}\boldsymbol{x} \to \int_0^T \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \text{ as } m\to+\infty.$$

Step 6: Regularity of the limit.

Thanks to [22, Theorem 14.2] the sequence of normed vector spaces $(\mathbf{H}_{\mathcal{E}_m,0}, \|\cdot\|_{1,\mathcal{E}_m,0})_{m\in\mathbb{N}}$ is $L^2(\Omega)^d$-limit-included in $H_0^1(\Omega)^d$ in the sense of Definition 4.0.9. We have $\boldsymbol{u}_m \to \bar{\boldsymbol{u}}$ in $L^2(0,T,L^2(\Omega))$ as $m\to\infty$ and $(\|\boldsymbol{u}_m\|_{L^2(0,T;\mathbf{H}_{\mathcal{E}_m,0})})_{m\in\mathbb{N}}$ is bounded thanks to Lemma 3.4.4. Therefore Theorem 4.0.10 applies and $\bar{\boldsymbol{u}}\in L^2(0,T;H_0^1(\Omega)^d)$. We then remark that $\mathrm{div}\bar{\boldsymbol{u}}=0$ adapting the proof that $\mathrm{div}\bar{\boldsymbol{u}}$ of the stationary case (see the proof of [33, Theorem 3.8]), we get that $\bar{\boldsymbol{u}}\in L^2(0,T;\boldsymbol{E}(\Omega))$.

□

Chapter 4

APPENDIX: FUNCTIONAL ANALYSIS

We first give a consequence of the Kolmogorov theorem which can be found in [22, Theorem 3.10] (we give it for the sake of completeness).

Theorem 4.0.4 (Compactness of a bounded sequence and regularity of the limit)**.** *Let Ω be an open bounded set of $\mathbb{R}^d$ with a Lipschitz continuous boundary, $d \geq 1$, and $(u_n)_{n\in\mathbb{N}}$ a bounded sequence of $L^2(\Omega)$. For $n \in \mathbb{N}$, one defines $\widetilde{u}_n$ by $\widetilde{u}_n = u_n$ a.e. on Ω and $\widetilde{u}_n = 0$ a.e. on $\mathbb{R}^d \setminus \Omega$. Assume that their exist $C \in \mathbb{R}$ and $\{h_n, n \in \mathbb{N}\} \subset \mathbb{R}_+$ such that $h_n \to 0$ as $n \to \infty$ and*

$$\|\widetilde{u}_n(\cdot + \xi) - \widetilde{u}_n\|^2_{L^2(\mathbb{R}^d)} \leq C\,|\xi|\,(|\eta| + h_n), \quad \forall \xi \in \mathbb{R}^d.$$

Then $(u_n)_{n\in\mathbb{N}}$ is relatively compact in $L^2(\Omega)$. Furthermore, if $u_n \to u$ then $u \in H^1_0(\Omega)$.

Definition 4.0.5 (Compactly embedded sequence of spaces)**.** Let B be a Banach space; a sequence $(X_m)_{m\in\mathbb{N}}$ of Banach spaces included in B is compactly embedded in B if any sequence $(u_m)_{m\in\mathbb{N}}$ satisfying:

- $u_m \in X_m$ ($\forall m \in \mathbb{N}$),
- the sequence $(\|u_m\|_{X_m})_{m\in\mathbb{N}}$ is bounded,

is relatively compact in B.

Definition 4.0.6 (Compact-continuous sequence of spaces)**.** Let B be a Banach space, and let $(X_m)_{m\in\mathbb{N}}$ and $(Y_m)_{m\in\mathbb{N}}$ be sequences of Banach spaces such that $X_m \subset B$ for $m \in \mathbb{N}$. The sequence $(X_m, Y_m)_{m\in\mathbb{N}}$ is compact-continuous in B if the following conditions are satified:

- The sequence $(X_m)_{m\in\mathbb{N}}$ is compactly embedded in B (see Definition 4.0.5),
- $X_m \subset Y_m$ (for all $m \in \mathbb{N}$),

- if the sequence $(u_m)_{m\in\mathbb{N}}$ is such that $u_m \in X_m$ (for all $m \in \mathbb{N}$), $(\|u_m\|_{X_m})_{m\in\mathbb{N}}$ is bounded and $\|u_m\|_{Y_m} \to 0$ as $m \to +\infty$, then any subsequence of $(u_m)_{m\in\mathbb{N}}$ converging in B converges to 0 (in B).

Proposition 4.0.7 (Time compactness with a sequence of subspaces)**.** *$1 \le p < +\infty$, $T > 0$. Let B be a Banach space and $(X_m)_{n\in\mathbb{N}}$ be a sequence of Banach spaces compactly embedded in B (see Definition 4.0.6). Let $(u_m)_{m\in\mathbb{N}}$ be a sequence of $L^p((0,T),B)$ satisfying the following conditions*

1. *The sequence $(u_m)_{m\in\mathbb{N}}$ is bounded in $L^p((0,T),B)$.*
2. *The sequence $(\|u_m\|_{L^1((0,T),X_m)})_{m\in\mathbb{N}}$ is bounded.*
3. *There exists a nondecreasing function from $(0,T)$ to $\mathbb{R}_+$ such that $\lim_{h\to 0^+}\eta(h) = 0$ and, for all $h \in (0,T)$ and $m \in \mathbb{N}$,*

$$\int_0^{T-h} \|u_m(t+h) - u_m(t)\|_B^p dt \le \eta(h).$$

Then, the sequence $(u_m)_{m\in\mathbb{N}}$ is relatively compact in $L^p((0,T),B)$.

Proof. In order to apply Kolmogorov's theorem (see e.g. [32, Theorem 4.38], we only have to prove that for all $\varphi \in C_c^\infty(\mathbb{R},\mathbb{R})$, the sequence $\{\int_0^T u_m\varphi dt,\ m \in \mathbb{N}\}$ is relatively compact in B.

Let $\varphi \in C_c^\infty(\mathbb{R},\mathbb{R})$. For $m \in \mathbb{N}$, one has, with $\|\varphi\|_u = \max_{t\in\mathbb{R}}|\varphi(t)|$,

$$\|\int_0^T u_m\varphi dt\|_{X_m} \le \|\varphi\|_u \|u_m\|_{L^1((0,T),X_m)}.$$

Then, since the sequence $(\|u_m\|_{L^1((0,T),X_m)})_{m\in\mathbb{N}}$ is bounded, the sequence $\{\|\int_0^T u_m\varphi dt\|_{X_m},\ m \in \mathbb{N}\}$ is also bounded. Therefore the sequence $\{\int_0^T u_m\varphi dt,\ m \in \mathbb{N}\}$ in relatively compact in B. □

The following theorem is proved [8] and is a generalization of a previous work carried out in [37].

Theorem 4.0.8 (Aubin-Simon Theorem with a sequence of subspaces and a discrete derivative.)**.** *Let $1 \le p < \infty$, let B be a Banach space, and let $(X_m)_{m\in\mathbb{N}}$ and $(Y_m)_{m\in\mathbb{N}}$ be sequences of Banach spaces such that $X_m \subset B$ for $m \in \mathbb{N}$. We assume that the sequence $(X_m, Y_m)_{m\in\mathbb{N}}$ is compact-continuous in B. Let $T > 0$ and $(u^{(m)})_{m\in\mathbb{N}}$ be a sequence of $L^p(0,T;B)$ satisfying the following conditions:*

- *(H1) the sequence* $(u^{(m)})_{m\in\mathbb{N}}$ *is bounded in* $L^p(0,T;B)$.
- *(H2) the sequence* $(\|u^{(m)}\|_{L^1(0,T;X_m)})_{m\in\mathbb{N}}$ *is bounded.*
- *(H3) the sequence* $(\|\eth_t u^{(m)}\|_{L^p(0,T;Y_m)})_{m\in\mathbb{N}}$ *is bounded.*

Then there exists $u \in L^p(0,T;B)$ *such that, up to a subsequence,* $u^{(m)} \to u$ *in* $L^p(0,T;B)$.

Definition 4.0.9 (B-limit-included)**.** Let B be a Banach space, $(X_m)_{m\in\mathbb{N}}$ be a sequence of Banach spaces included in B and X be a Banach space included in B. The sequence $(X_m)_{m\in\mathbb{N}}$ is B-limit-included in X if there exists $C \in \mathbb{R}$ such that if u is the limit in B of a subsequence of a sequence $(u_m)_{m\in\mathbb{N}}$ verifying $u_m \in X_m$ and $\|u_m\|_{X_m} \le 1$, then $u \in X$ and $\|u\|_X \le C$.

Theorem 4.0.10 (Regularity of the limit)**.** *Let* $1 \le p < \infty$ *and* $T > 0$. *Let* B *be a Banach space,* $(X_m)_{m\in\mathbb{N}}$ *be a sequence of Banach spaces included in* B *and* B*-limit-included in* X *(where* X *is a Banach space included in* B*). Let* $T > 0$ *and, for* $m \in \mathbb{N}$, *Let* $u_m \in L^p(0,T;X_m)$. *We assume that the sequence* $(\|u_m\|_{L^p(0,T;X_m)})_{m\in\mathbb{N}}$ *is bounded and that* $u_m \to u$ *a.e. as* $m \to \infty$. *Then* $u \in L^p(0,T;X)$.

Proof. Since $u_n \to u$ in $L^p((0,T),B)$ as $n \to +\infty$, we can assume, up to subsequence, that $u_n \to u$ in B a.e..

Then, since the the sequence $(X_n)_{n\in\mathbb{N}}$ is B-limit-included in X, we obtain, with C given by Definition 4.0.9,

$$\|u\|_X \le C \liminf_{n\to+\infty} \|u_n\|_{X_n} \text{ a.e..}$$

Using now Fatou's lemma, we have

$$\int_0^T \|u(t)\|_X^p dt \le C^p \int_0^T \liminf_{n\to+\infty} \|u_n(t)\|_{X_n}^p dt \le C^p \liminf_{n\to+\infty} \int_0^T \|u_n(t)\|_{X_n}^p dt.$$

Then, since $(\|u_n\|_{L^p((0,T),X_n)})_{n\in\mathbb{N}}$ is bounded, we conclude that $u \in L^p((0,T),X)$. □

The following theorem follows from standard arguments of the topological degree theory (see [14] for an overview of the theory and e.g. [21, 31] for other uses in the same objective as here, namely the proof of existence of a solution to a numerical scheme).

Theorem 4.0.11. *Let* N *and* M *be two positive integers and* $V = \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^N$. *Let* $b \in V$ *and* $f(.)$ *and* $F(.,.)$ *be two continuous functions respectively from* V *and* $V \times [0,1]$ *to* V *satisfying:*

1. $F(.,1) = f(1)$;

2. *$\forall \alpha \in [0,1]$, if an element v of $\overline{\mathcal{O}}$ (the closure of $\mathcal{O}$) is such that $F(v,\alpha) = b$, then $v \in \mathcal{O}$, where $\mathcal{O}$ is defined as follows:*

$$\mathcal{O} = \{(x,y,z) \in V \ \textit{s.t.}\ C_0 < x < C_1 \ \textit{and}\ \|y\|_M < C_2 \ \textit{and}\ \|z\|_N < C_3\}$$

   where, for any real number c and vector x, the notation $x > c$ means that each component of x is larger than c ; C_0 , C_1 , C_2 and C_3 are positive constants and $\|y\|_M$ and $\|z\|_N$ are two norms defined on $\mathbb{R}^M$ and $\mathbb{R}^N$ respectively,

3. *the topological degree of $F(.,0)$ with respect to b and $\mathcal{O}$ is equal to $d_0 \neq 0$.*
   Then the topological degree of $F(.,1)$ with respect to b and O is also equal to $d_0 = 0$; consequently, there exists at least one solution $v \in O$ to the equation $f(v) = b$.

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