THÈSE

Pour l'obtention du grade de

DOCTEUR DE L'UNIVERSITÉ DE PAU ET DES PAYS DE L'ADOUR

Faculté des Sciences Fondamentales et Appliquées

DOMAINE DE RECHERCHE : MATHÉMATIQUES APPLIQUÉES - Option Analyse des EDP

Présentée par

M. Brahim Bougherara

Problèmes singuliers et bifurcation analytique

Directeur de thèse : **M. Jacques Giacomoni**

Soutenue le 11 septembre 2014

JURY

M. T. CAZENAVE	Directeur de Recherche CNRS à l'Université de Pierre et Marie Curie	Rapporteur
M. V.RĂDULESCU	Professeur à l'Université de Craiova	Rapporteur
M. C.A. STUART	Professeur à l'Ecole Polytechnique Fédérale de Lausanne	Rapporteur
M. A. FARINA	Professeur à l'Université de Picardie Jules Verne	Examinateur
M. M. AMARA	Professeur à l'U.P.P.A	Examinateur
M. G. CARBOU	Professeur à l'U.P.P.A	Examinateur
M. J. GIACOMONI	Professeur à l'U.P.P.A.	Directeur de thèse

Thèse préparée au sein du Laboratoire de Mathématiques Appliquées de Pau (L.M.A.P.),
dans l'école doctorale des sciences exactes et leurs applications de l' U.P.P.A.

Remerciements

Je tiens tout d'abord à remercier mon directeur de thèse Jacques Giacomoni, qui a accepté de m'encadrer pour ce travail de thèse. Merci à lui pour sa générosité : il m'a partagé ses connaissances et ses idées mathématiques, pour sa gentillesse et sa patience, il a consacré beaucoup de temps à discuter avec moi et à corriger mes textes mathématiques et non mathématiques. Je le remercie aussi pour ses qualités pédagogiques et sa bonne humeur constante. C'est un grand honneur d'avoir pu travailler avec lui.

Je souhaite remercier mes rapporteurs Thierry Cazenave, Charles Stuart et Vicentiu Radulescu, pour leur lecture rigoureuse du manuscrit de thèse et les nombreux conseils donnés pour l'amélioration de ce manuscrit. C'est un honneur pour moi qu'ils aient pris le temps de lire avec attention mon travail et de me faire part de leurs avis de spécialistes sur celui-ci. Je remercie aussi Alberto Farina et Gilles Carbou d'avoir participé à mon jury en tant qu'examinateurs. Merci également à Mohamed Amara d'avoir accepté d'être président de mon jury. Cela me fait honneur de présenter mes travaux de thèse devant cette commission d'examen.

Je tiens ensuite à remercier mes collaborateur avec qui j'ai travaillé durant cette thèse à savoir Peter Takàč et S. Prashanth, merci pour cette collaboration qui s'est révélée très enrichissante et stimulante. Merci encore à Peter pour son invitation à l'université de Rostock et pour la bonne réception, sans oublier à ce titre de remercier Merker Johen ainsi sa femme Katya pour leur générosité et gentillesse pendant ma séjour à Rostock. Je remercie aussi Jesús Hernández d'avoir commencé récemment une première collaboration avec moi.

Un grand merci à Paul Sauvy qui m'a aidé beaucoup pendant la thèse, il a répondu avec patience à toutes mes questions. Merci à lui pour de nombreuses discussions mathématiques. Je souhaite remercier Benoit qui a répondu avec patience à mes questions en topologie algébrique.

Je tiens à souligner combien il a été agréable de travailler au Laboratoire de Mathématiques et leurs Applications (LMAP, UMR 5142) à pau. C'est avec plaisir que je remercie tous les membres du département que j'ai eu la chance de croiser durant la thèse. Je remercie plus particulièrement Chérif Amrouche pour son aide et Guy Vallet pour son enseignement de qualité. Merci également à Marie Claire pour sa gentillesse et sa disponibilité pendant ces trois ans de la thèse. Un grand merci aux doctorants avec qui j'ai passé d'excellents moments : Ahmed, Hammou (merci Hammou pour ton aide

en Latex), Juan, Zina, Hind, Nikola, Oussama, Xung, Viatcheslav, Nestor et Thom, en espérant que leurs thèses se déroulent aussi bien que moi. Sans oublier de remercier les anciens qui sont déjà soutenu leurs thèses à savoir Cheik, Mohammed (Momo), Sweeta, Fabien, Loic, Hi Hai, Nelly et Carroline.

Je tiens à remercier bien sur ma famille en Algérie pour leur soutien à distance et aussi ma petite famille ici à Pau "Hakim, Yasmina et leur petite fille Ritège" pour leur générosité et leur soutien constant pendant la dernière année de thèse.

Table des matières

Notations **1**

Introduction générale **5**

I Bounded solutions to a quasilinear and singular parabolic equation with p-Laplacian **29**

1 Introduction . . . 29
2 Main results . . . 30
3 Logarithmic Sobolev inequalities . . . 31
4 Approximating problems for (P) . . . 33
4.1 Existence of positive solution u_ε . . . 34
4.1.1 Semi-discretization in time . . . 34
4.1.2 Energy estimates for $u_{\varepsilon,\Delta t}$ and $\tilde{u}_{\varepsilon,\Delta t}$ . . . 34
4.1.3 Passage to the limit and proof of theorem 4.1 . . . 38
4.2 Uniqueness of the positive solution u_ε . . . 40
5 L^∞-bounds for u_ε . . . 43
5.1 First estimate . . . 44
5.2 Second estimate . . . 48
6 Proofs of the main results . . . 56
6.1 Construction of subsolution to problem (P_ε) . . . 56
6.2 Proof of theorem 2.1 . . . 58
6.3 Proof of theorem 2.2 . . . 62

II Global bifurcation theory in the analytic framework **67**

1 Statement of the main result . . . 67
2 Analytic varieties theory . . . 69
2.1 $\mathbb{F}$-analytic varieties . . . 69
2.2 Weierstrass analytic varieties . . . 71
2.3 Germs of $\mathbb{C}$-analytic varieties and subspaces . . . 75
2.4 One-dimensional branches . . . 82
3 The Lyapunov-Schmidt reduction . . . 86
4 Proof of theorem 1.1 . . . 88

III Analytic global bifurcation for very singular problem and infinite turning points **93**
1 Introduction and preliminaries . . . 93
1.1 Introduction . . . 93
1.2 Definitions and main results . . . 96
1.3 Preliminaries . . . 99
2 The singular problem and the analytic bifurcation framework . . . 101
2.1 Analyticity of the solution operator . . . 101
2.2 Preliminary analysis of the linearised operator . . . 105
2.3 Local analytic path and minimal solutions . . . 109
2.3.1 Existence of minimal solution . . . 109
2.3.2 Invertibility of $\partial_2 F(\lambda, u_\lambda)$ and existence of a local path . . . 110
2.3.3 Local path containing set of all minimal solutions . . . 111
2.4 Global analytic path . . . 114
3 Infinite turning points in two dimensions via ODE analysis . . . 115
3.1 Singular solution and Oscillation criterion . . . 115
3.1.1 Transformation of the problem . . . 115
3.1.2 The singular solution . . . 117
3.1.3 Oscillation criterion . . . 119
3.2 Existence of infinitely many turning points in the bifurcation curve . . . 120

IV Some other applications of the global analytic bifurcation theorem **123**
1 Preliminaries . . . 123
2 The case when f(0) = 0 . . . 125
2.1 Superlinear nonlinearities . . . 125
2.2 Sublinear nonlinearities . . . 130
3 The case when $f(0) > 0$. . . . 132
3.1 Superlinear nonlinearities . . . 132
3.2 Sublinear nonlinearities . . . 134

Appendices **137**
A Polynomials with variable coefficients and analyticity . . . 137
A.1 Analyticity and Riemann Extension Theorem . . . 137
A.2 Polynomials with variable coefficients and analyticity . . . 137
B The critical groups in homology theory and Morse index . . . 141
B.1 Chain Complexes and homology . . . 141
B.2 Singular relative homology : Definitions and properties . . . 142
B.2.1 Construction of a singular chain complex . . . 142
B.2.2 Some properties . . . 143
B.3 Critical groups and Morse index . . . 144

Références bibliographiques **147**

Notations

Notations générales

$N \geq 2$ (ou d)	Entier naturel, dimension de l'espace de travail
$\mathbb{R}^N$	Espace euclidien muni de sa norme usuelle notée $\lvert \cdot \rvert$
Ω	Domaine borné et régulier.
$\partial\Omega$	Frontière de Ω
ν	Normale unitaire extérieure de Ω
$x = (x_1, \cdots, x_N)$	Elément de Ω
$d(x)$	Distance du point x à $\partial\Omega$ $(= d(x, \partial\Omega))$
$p > 1$ (ou q ou r)	Exposant de Lebesgue
$p' > 1$	Exposant conjugé de p vérifiant $\frac{1}{p} + \frac{1}{p'} = 1$
int(A)	Intérieur de l'ensemble A
$\operatorname{supp}_A f$	Support de de la fonction f sur A
∇v	Gradient de v défini par $\nabla v \stackrel{\text{def}}{=} \left(\frac{\partial v}{\partial x_1}, \cdots, \frac{\partial v}{\partial x_N}\right)$
$\Delta_p v$	p-Laplacien de v défini par $\Delta_p v \stackrel{\text{def}}{=} \operatorname{div}(\lvert\nabla v\rvert^{p-2}\nabla u)$
λ_1	Première valeur propre du p-Laplacien sur Ω
φ_1	Fonction propre strictement positive et L^p-renormalisée sur Ω, associée à λ_1
$\mathscr{C}(\overline{\Omega})$	Ensemble des fonctions continues sur $\overline{\Omega}$
$\mathscr{C}_0(\overline{\Omega})$	Ensemble des fonctions continues sur $\overline{\Omega}$ s'annulant sur $\partial\Omega$
$\mathscr{C}^{0,\alpha}(\overline{\Omega})$ (ou $\mathscr{C}^{\alpha}(\overline{\Omega})$)	Ensemble des fonctions de $\mathscr{C}(\overline{\Omega})$ α-Hölderiennes, avec $0 < \alpha < 1$; c'est à dire, $\mathscr{C}^{0,\alpha}(\overline{\Omega}) \stackrel{\text{def}}{=} \left\{v \in \mathscr{C}(\overline{\Omega}) \;\middle|\; \exists C > 0, \quad \forall x, y \in \overline{\Omega}, \quad \lvert v(x) - v(y)\rvert \leq C\lvert x - y\rvert^{\alpha}\right\}$
$\mathscr{C}^{1,\alpha}(\overline{\Omega})$	Ensemble des fonctions $\mathscr{C}^1(\overline{\Omega})$ α-Hölderiennes, avec $0 < \alpha < 1$; c'est à dire, $\mathscr{C}^{1,\alpha}(\overline{\Omega}) \stackrel{\text{def}}{=} \left\{v \in \mathscr{C}^1(\overline{\Omega}) \;\middle|\; \forall i \in \{1, \cdots, N\}, \quad \frac{\partial v}{\partial x_i} \in \mathscr{C}^{0,\alpha}(\overline{\Omega})\right\}$
$\mathcal{D}(\Omega)$	Ensemble des fonctions $\mathscr{C}^{\infty}$ à support compact sur Ω

$\mathcal{D}'(\Omega)$	Espace des distributions sur Ω
$\langle\cdot,\cdot\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)}$	Produit de dualité entre $\mathcal{D}'(\Omega)$ et $\mathcal{D}(\Omega)$
$L^p(\Omega)$, $L^\infty(\Omega)$	Espaces de Lebesgue standards sur Ω d'exposants p et ∞
$L^1_{\mathrm{loc}}(\Omega)$	Ensemble des fonctions localement intégrables sur Ω défini par $\mathrm{L}^1_{\mathrm{loc}}(\Omega) \overset{\mathrm{def}}{=} \{v \text{ mesurable sur } \Omega \mid \forall \Omega' \subset\subset \Omega, \quad v \in \mathrm{L}^1(\Omega')\}$
$W^{1,p}(\Omega)$	Espace de Sobolev standard sur Ω d'exposant p
$W^{1,p}_{\mathrm{loc}}(\Omega)$	Espace défini par $W^{1,p}_{\mathrm{loc}}(\Omega) \overset{\mathrm{def}}{=} \left\{v \in L^p_{\mathrm{loc}}(\Omega) \mid \forall i \in \{1,\cdots,N\}, \quad \frac{\partial v}{\partial x_i} \in L^p_{\mathrm{loc}}(\Omega)\right\}$
$W^{1,p}_0(\Omega)$	Adhérence de $\mathcal{D}(\Omega)$ dans $W^{1,p}(\Omega)$ pour la norme $\\|\cdot\\|_{W^{1,p}(\Omega)}$
$W^{-1,p'}(\Omega)$	Dual topologique de $W^{1,p}_0(\Omega)$
$\langle\cdot,\cdot\rangle_{W^{-1,p'}(\Omega)\times W^{1,p}_0(\Omega)}$	Produit de dualité entre $W^{-1,p'}(\Omega)$ et $W^{1,p}_0(\Omega)$
$\\|v\\|_{L^p(\Omega)}$	Norme de v sur $L^p(\Omega)$ définie par $\\|v\\|_{L^p(\Omega)} \overset{\mathrm{def}}{=} \left(\int_\Omega \vert v\vert^p\,dx\right)^{\frac{1}{p}}$
$\\|v\\|_{L^\infty(\Omega)}$	Norme de v dans $L^\infty(\Omega)$ définie par $\\|v\\|_{L^\infty(\Omega)} \overset{\mathrm{def}}{=} \operatorname*{ess\,sup}_{x\in\Omega} \vert v(x)\vert$
$\\|v\\|_{W^{1,p}(\Omega)}$	Norme de v sur $\mathrm{W}^{1,p}(\Omega)$ définie par $\\|v\\|_{W^{1,p}(\Omega)} \overset{\mathrm{def}}{=} \left(\\|v\\|_{L^p(\Omega)} + \\|\nabla v\\|_{L^p(\Omega)}\right)^{\frac{1}{p}}$
$\\|v\\|_{W^{1,p}_0(\Omega)}$	Norme de v sur $\mathrm{W}^{1,p}_0(\Omega)$ définie par $\\|v\\|_{W^{1,p}_0(\Omega)} \overset{\mathrm{def}}{=} \\|\nabla v\\|_{L^p(\Omega)}$, équivalente à $\\|v\\|_{W^{1,p}(\Omega)}$
$\\|f\\|_{W^{-1,p'}(\Omega)}$	Norme duale de f sur $W^{-1,p'}(\Omega)$

Chapitre I

$N \geq 2$	Dimension de l'espace de travail
$p > 1$	Paramètre relatif à Δ_p
Ω	Domaine borné et régulier dans $\mathbb{R}^N$
Q	Domaine d'étude défini par $Q \overset{\mathrm{def}}{=} (0,T)\times\Omega$
Γ	Frontière parabolique de Q définie par $\Gamma \overset{\mathrm{def}}{=} (0,T)\times\partial\Omega$
$\partial_t u$	Dérivée partielle de u par rapport au temps
r	Un réel assez grand
u_0	Condition initiale positive choisie dans $L^r(\Omega)$
δ	Exposant du terme singulier dans le second membre de (P), strictement positif
$\varepsilon > 0$	Paramètre du problème régularisé (P_ε)
$N >> 1$	Entier suffisamment grand
$0 = t_0 < \cdots < t_N = T$	Subdivision régulière de l'intervalle $[0,T]$
Δt	Pas de semi-discrétisation en temps égal à $\frac{T}{N}$
$\mathcal{D}(Q)$	Ensemble des fonctions $\mathscr{C}^\infty$ à support compact sur Q

$\mathcal{D}'(Q)$	Espace des distributions de Q
$\mathscr{C}\left([0,T], L^r(\Omega)\right)$	Ensemble des fonctions continues de $[0,T]$ dans $L^r(\Omega)$
$\mathscr{C}\left((a,b), W_0^{1,p}(\Omega)\right)$	Ensemble des fonctions continues de l'intervalle (a,b) dans $W_0^{1,p}(\Omega)$
$L^2(Q)$, $L^\infty(Q)$	Espaces de Lebesgue standard sur Q d'exposants 2 et ∞
$L^r\left(0,T;W_0^{1,p}(\Omega)\right)$	Espace de Sobolev à valeurs vectorielles avec $r=p$ ou $r=\infty$
$L^{p'}\left(0,T;\mathrm{W}^{-1,p'}(\Omega)\right)$	Dual topologique de $\mathrm{L}^p\left(0,T;W_0^{1,p}(\Omega)\right)$
$\langle\cdot,\cdot\rangle$	Produit de dualité entre $\mathrm{W}^{-1,p'}(\Omega)$ et $\mathrm{W}_0^{1,p}(\Omega)$
$\\|v\\|_{L^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)}$	Norme de v dans $\mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$ définie par $\\|v\\|_{\mathrm{L}^p\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)} \overset{\text{def}}{=} \left(\int_0^T \\|v(t,\cdot)\\|^p_{\mathrm{W}_0^{1,p}(\Omega)} ds\right)^{\frac{1}{p}}$
$\\|v\\|_{L^\infty\left(0,T;W_0^{1,p}(\Omega)\right)}$	Norme de v dans $L^\infty\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)$ définie par

$$\|v\|_{L^\infty\left(0,T;\mathrm{W}_0^{1,p}(\Omega)\right)} \overset{\text{def}}{=} \operatorname*{ess\,sup}_{t\in(0,T)} \|v(t,\cdot)\|_{W_0^{1,p}(\Omega)}$$

Chapitre II

$\mathcal{X}$, $\mathcal{Y}$	Espaces de Banach
$\mathbb{F}$	Le corps du complexes ou réels, ($\mathbb{C}$ ou $\mathbb{R}$)
$\mathcal{U}$	Ouvert de $\mathbb{R}\times\mathcal{X}$
n, m	Des entier naturels tels que $1\leq m\leq n$
δ	Réel positif assez petit
W (ou V)	Ouvert de $\mathbb{F}^m$ de la forme $\{(x_1,\cdots,x_m) : \lvert x_1\rvert,\cdots,\lvert x_m\rvert<\delta\}$
$F(\cdot,\cdot)$	Fonction analytique de $\mathcal{U}\subset\mathbb{F}\times\mathcal{X}$ dans $\mathcal{Y}$
$\partial_2F(\lambda,u)$	Dérivée de F en (λ,u) par rapport à la seconde variable u définie de $\mathcal{X}$ dans $\mathcal{Y}$
$\ker\partial_2F(\lambda,u)$	Le noyau de $\partial_2F(\lambda,u)$ définie par $\partial_2F(\lambda,u)=\{v\in\mathcal{Y} : \partial_2F(\lambda,u).v=0\}$
$\mathcal{S}$	Ensembles des solutions i.e. $\mathcal{S}=\{(\lambda,u)\in\mathcal{U} : F(\lambda,u)=0\}$
$\mathcal{R}$	Ensembles des solutions régulières i.e., $\mathcal{R}=\{(\lambda,u)\in\mathcal{S} : \ker\partial_2F(\lambda,u)=\{0\}\}$
$\mathcal{A}^+$	Branche de solutions de $\mathcal{R}$ locale et analytique émanant de $(0,0)$, i.e., $\mathcal{A}^+=\{(\lambda(s),u(s)) : (0,\epsilon)\mapsto(\lambda(s),u(s))$ est analytique et $(\lambda(0),u(0))=(0,0)\}\subset\mathcal{R}$.
$\mathcal{A}_n$	Composante connexe de $\mathcal{R}$. Pour $n=0$, $\mathcal{A}_0$ contient $\mathcal{A}^+$
$\mathcal{A}$	Ensemble de solutions connexe par arc
$\sim$ (ou $\sim_a$)	Relation d'équivalence en $a\in\mathbb{F}^n$ définie sur les ensembles de $\mathbb{F}^n$ par : $S\sim T$, $S,T\subset\mathbb{F}^n$ ssi $\exists\ O$ ouvert contenant a tel que $S\cap O=T\cap O$
$\gamma_a(S)$, $S\subset\mathbb{F}^n$	Le germe de S en a i.e., classe d'équivalence de S par la relation $\sim$
G	Ensemble fini des fonction analytique sur un voisinage U de a

$\mathrm{var}(U,G)$	Ensemble (variété) analytique définie par : $\mathrm{var}(U,G) = \{x \in U : g(x) = 0 \text{ pour tout } g \in G\}$
$\gamma_a(\mathrm{var}(U,G))$	Germe analytique
$\mathcal{V}_a(\mathbb{F}^n)$	Ensemble de tous les germes analytiques en a
D	Discriminant d'un polynôme (voir Définition A.2 dans l'Annexe A)
H	Ensemble de la forme : $\{h_{m+1}, \cdots, h_n : h_l$ est un polynômes de Weierstass en x_l, $l = m+1, \cdots, n\}$
$D(H)$	Produit de tous les discriminants des polynômes de H
$\mathrm{var}(W \times \mathbb{C}^{n-m}, H)$	Ensemble de Weierstrass (voir Définition 2.6)
B	Branche d'un ensemble de Weierstrass (voir définition 2.6)
$\mathcal{M}$	Sous-variété de $\mathbb{R}^n$

Chapitre III et Chapitre IV

δ	Exposant du terme singulier dans le second membre de (P_λ), strictement positif
$\lambda > 0$	Paramètre positif du problème (P_λ)
$F(\lambda, u)$	Opérateur-solution, associée au problèm (P_λ)
$\partial_2 F(\lambda, u)$	Opérateur linéarisé associé au problèm (P_λ)
$\mathcal{C}(\overline{\Omega})$	Espace des fonctions continues jusqu'au bord
$\mathcal{C}_\phi(\Omega)$	Espace à poids définie par : $\{u \in \mathcal{C}(\Omega) \mid \text{for some } C > 0, \lvert u(x)\rvert \le C\phi(x)\ \forall x \in \Omega\}$
$\mathcal{C}^+_\phi(\Omega)$	Le cône positif définie par : $\{u \in \mathcal{C}_\phi(\Omega) \mid \inf_{x\in\Omega} \frac{u(x)}{\phi(x)} > 0\}$
$\mathcal{S}$	Ensembles des solutions de (P_λ) i.e., $\mathcal{S} = \{(\lambda, u) \in \mathbb{R}_+ \times \mathcal{C}^+_\phi(\Omega) : F(\lambda, u) = 0\}$
$\mathcal{N}$	Ensembles des solutions régulier i.e., $\mathcal{N} = \{(\lambda, u) \in \mathcal{S} : \ker \partial_2 F(\lambda, u) = \{0\}\}$
$\mathcal{A}$	Ensemble connexe par arc de solutions de (P_λ)
$M(\lambda, u)$	Indice de Morse de l'opérateur $\partial_2 F(\cdot,\cdot)$ en (λ, u)
ϕ_1	Première fonction propre du Laplacien
λ_1	Première valeur propre du Laplacien
Λ_1	Première valeur propre de $\partial_2 F(\lambda, u_\lambda)$
$\underline{u}_\lambda$, $\overline{u}_\lambda$	Sous- et sur-solution de (P_λ) dans Ω
$H^s_0(\Omega)$	Espace de Hilbert standard représentant $W^{s,2}_0(\Omega)$
$\mathrm{H}^{-s}(\Omega)$	Dual topologique de $\mathrm{H}^s_0(\Omega)$ représentant $\mathrm{W}^{-s,2}(\Omega)$

Introduction générale

Cette thèse concerne l'étude de quelques problèmes paraboliques et elliptiques non linéaires singuliers. Précisément, les problèmes types que nous avons étudiés, posés sur un domaine borné et régulier Ω dans $\mathbb{R}^N$, avec $N \geq 2$ et avec conditions de Dirichlet homogènes présentent dans le second membre de l'équation un terme singulier de la forme $u^{-\delta}$, $\delta > 0$, qui tend donc vers l'infini au bord du domaine Ω. Cette singularité pose un certain nombre de difficultés qui ne permettent pas d'utiliser directement les méthodes classiques de l'analyse non-linéaire fondées entre autres sur des résultats de compacité (injections compactes de Sobolev, régularité elliptique et parabolique). Dans les démonstrations des principaux résultats des chapitres I et III qui portent sur l'existence, l'unicité ou la multiplicité, le comportement asymptotique et la régularité des solutions des problèmes considérés, nous montrons comment pallier ces difficultés. Ceci suppose d'adapter certaines techniques bien connues mais aussi d'introduire de nouvelles méthodes. Dans ce contexte, une étape importante est la détermination précise du comportement des solutions au bord de domaine qui permet d'adapter le principe de comparaison faible, d'utiliser la théorie globale et locale de la régularité elliptique et parabolique et d'appliquer dans un contexte nouveau la théorie globale de la bifurcation analytique.

La thèse se présente sous forme de deux parties indépendantes. Dans la première partie (qui correspond essentiellement au chapitre I), nous nous sommes intéressés à un problème quasilinéaire parabolique fortement singulier faisant intervenir l'opérateur p-Laplacien. Le second membre de l'équation se compose d'une non-linéarité f dépendant non linéairement de la solution et de son gradient d'une part et d'autre part d'un terme singulier de la forme $u^{-\delta}$, avec $\delta > 0$ quelconque. La donnée initiale u_0 en $t = 0$ est supposée positive ou nulle presque partout et prise dans $L^r(\Omega)$ avec $r \geq 2$ assez grand. Dans ce contexte, on démontre pour tout $\delta > 0$, l'existence locale dans $C([0, T]; L^r(\Omega))$ pour un certain $T > 0$ d'une solution faible strictement positive (à tout $t > 0$ dans l'intervalle d'existence $(0, T)$). Pour $\delta < 2 + \frac{1}{p-1}$, on démontre l'unicité de la solution en utilisant un principe de comparaison faible fondé sur la monotonie dans $W_0^{1,p}(\Omega)$ d'un opérateur non linéaire bien choisi. On démontre également dans ce cas des résultats de régularité dans l'espace d'énergie $W_0^{1,p}(\Omega)$. L'existence de solutions faibles dans le cas général repose sur des estimations a priori dans L^∞ obtenues via l'utilisation d'inégalités de type log-Sobolev combinées à des inégalités de Gagliardo-Nirenberg.

La deuxième partie (correspondant essentiellement aux chapitre II,III) porte sur l'étude de problèmes de bifurcation globale dans le cas analytique. Cette analyse vise à décrire finement les propriétés des branches de solutions d'une classe d'équations semilinéaires elliptiques en fonction d'un paramètre λ apparaissant dans ces équations. Nous nous intéresssons plus particulièrement à un problème posé dans un domaine borné de $\mathbb{R}^2$ et faisant intervenir une nonlinéarité qui se comporte comme une perturbation de $t \mapsto e^{at^2}$, avec $a > 0$, pour t grand et qui contient un terme de la forme $t^{-\delta}$. Le comportement à l'infini du type e^{u^2} correspond à la "croissance critique" en dimension 2 (d'après l'inégalité de Trudinger-Moser [93]) et induit un défaut de compacité dans $H_0^1(\Omega)$. Nous établissons pour une certaine classe de ces perturbations le comportement asymptotique des solutions positives via le diagramme de bifurcation en introduisant un paramètre λ dans la nonlinéarité. Précisément nous démontrons que la branche de solutions émanant de $\lambda = 0$ admet au moins un point de bifurcation asymptotique. Dans le cas où le domaine est un disque, nous établissons le résultat plus précis suivant : les solutions le long de la branche convergent vers une solution singulière (solution non bornée) lorsque la norme des solutions converge vers l'infini. Ce phénomène qui avait déja été observé par Joseph-Lundgren [82] pour des nonlinéarités du type e^t pour les dimensions entre 3 et 9 et par Dancer pour des problèmes surcritiques en dimension supérieure ou égale à 3 est complètement nouveau en dimension 2. Pour établir ces résultats, nous faisons appel à différents outils d'analyse nonlinéaire. Un outil important est la théorie analytique de la bifurcation globale qui a été introduite par Dancer [43] et généralisée par Buffoni, Dancer et Toland [30]. Cette théorie a été utilisée pour établir l'existence d'ondes solitaires à la surface de l'eau dans [30], voir aussi Plotnikov [102]. Tirant profit de cette théorie qui ne s'applique que pour des opérateurs analytiques, nous montrons l'existence d'arcs de solutions continues et analytiques par morceaux et nous pouvons décrire plus finement le comportement global de la branche de solutions obtenue. Un résultat important dans ce cadre est que la branche de solutions admet une infinité de "points de retournement" avant d'atteindre la solution singulière. Ces points de retournement correspondent à un changement de l'indice de Morse des solutions qui tend vers l'infini le long de la branche. Ceci montre en particulier la forte instabilité de ces solutions singulières qui ne peuvent admettre d'indice de Morse fini.

Il faut souligner ici que la théorie classique de la bifurcation globale introduite par Rabinowitz [104] fondée sur le degré topologique de Leray-Schauder ne permet pas en général de démontrer l'existence de tels arcs paramétrés de solutions pour lequels on peut identifier des points de bifurcation secondaire ou de points de retournement. En effet la théorie du degré topologique fournit des ensembles connexes de solutions mais qui ne correspondent pas nécessairement à des arcs paramétrés réguliers. Dans le chapitre II, nous donnons une présentation succincte de la théorie globale de la bifurcation analytique et nous reprenons, en détaillant quelques points clés, la construction de cette théorie faite dans Buffoni et Toland [32]. Le résultat important de ce chapitre est le théorème 5, qui établit l'existence et le comportement global d'arcs analytiques de solutions (λ, x) d'une équation non linéaire de la forme

$$F(\lambda, x) = 0, \ \lambda \in \mathbb{R}, \quad x \in \mathcal{X} \setminus \{0\} \tag{1}$$

où $F : \mathbb{R} \times \mathcal{X} \to \mathcal{Y}$ est analytique, $\mathcal{X}, \mathcal{Y}$ sont des espaces de Banach. Un outil clé dans ce chapitre est la réduction de Liapunov-Schmidt qui s'obtient à l'aide du théorème des fonctions implicites. La réduction de Liapunov-Schmidt permet localement de remplacer un problème en dimension infinie par une formulation équivalente en dimension finie, ce qui garantit entre autres la compacité locale. On introduit alors la notion de germes analytiques de dimension n en un point $a \in \mathbb{C}^n$ qui désigne l'ensemble des zéros communs contenus dans un voisinage de a, d'une collection finie des fonctions analytiques en a. L'étude des germes et en particulier les germes de dimension 1 permet de construire la continuation analytique de la branche au delà d'un point singulier de cette branche via le théorème de représentation de Puiseux. Soulignons qu'en un point singulier, le théorème des fonctions implicites ne peut être appliqué. Un outil préparatoire important à ce stade est le théorème de Weierstrass qui permet "localement" de remplacer une fonction analytique de n variable par un polynôme à une indéterminée dont les coefficients sont des fonctions analytiques de $n-1$ variables, et ceci sans changer (localement) l'ensemble des zéros de la fonction. Par une étude élaborée des polynômes de Weierstrass (via des outils comme le résultant de deux polynômes qui permet de savoir s'ils sont premiers entre eux et le discriminant d'un polynôme qui permet d'identifier les racines multiples de ce polynôme) et le théorème de structure 2.4 donnée au chapitre II, on peut facilement décrire l'ensemble de ses zéros (le germe) comme un réunion finie des branches (précisément des variétés connexes par arc de dimension $m < n$). Un autre outil important est le théorème de prolongement de Riemann qui permet d'étendre un certain nombre des propriétés démontrées pour cet ensemble sur tout un voisinage. Dans le cas de germes de dimension 1, le théorème de représentation de Puiseux permet de remplacer une branche par une courbe paramétrée analytique unique. Ceci implique qu'une branche de solutions d'un problème analytique réel de la forme (1) peut localement en chacun de ses points être paramétrée d'une manière analytique. Ceci implique par extension qu'un ensemble S de solutions d'un problème analytique réel (1) admet au voisinage de chacun de ses points d'accumulations une paramétrisation analytique. En conséquence une courbe analytique de solutions de S admettrait au voisinage de son extrémité une paramétrisation analytique ce qui donne lieu à un argument de continuation de la branche et permet d'obtenir une branche globale de solutions analytique par morceaux et de démontrer le résultat principal de ce chapitre. Les éléments de construction de la branche précédents et des arguments de compacité permet alors d'établir l'alternative sur le comportement global de la branche : soit la branche est bornée et forme une boucle (la branche paramétrée est alors périodique) soit la branche est non bornée ou rencontre le bord de l'ouvert où F est définie.

Dans le chapitre III, on applique cette théorie au problème semilinéaire elliptique critique et singulier considéré. La première étape consiste à démontrer dans un cadre plus général l'existence d'un arc paramétré analytique de solutions (λ, u) non borné de ce problème pour tout $\delta > 0$ en choisissant les espaces fonctionnels adaptés. Notons que pour des valeurs grandes de δ, les solutions ne sont pas assez régulières pour travailler dans le cadre variationnel de l'espace de Sobolev $H_0^1(\Omega)$. On utilise alors des espaces de Banach à poids ce qui assure l'analyticité de l'opérateur associé au problème, certaines propriétés de l'opérateur linéarisé et ainsi l'applicabilité de la théorie analytique de la bifurcation. On souligne ici le contraste entre le manque de régularité de la nonlinéarité du problème elliptique et la régularité de l'opérateur obtenu dans les espaces à poids. En dimension 2 lorsque le domaine Ω est la boule unité, on démontre que la branche obtenue admet une infinité de "points de retournement" et converge vers une certaine solution singulière. Pour cela, s'appuyant sur des résultats étendus de Gidas, Ni, Nirenberg [70] par Bal, Giacomoni [16] sur la symétrie radiale des solutions et en utilisant des transformations du type Emden-Fowler, on réduit le problème sous forme équivalente à une équation différentielle. On démontre que l'équation linéarisée en la solution singulière associée admet une infinité de zéros s'accumulant à l'origine en utilisant un critère classique d'oscillations donné par Fite [63] concernant le nombre de zéros d'une équation différentielle. Ensuite on en déduit que l'indice de Morse constant par morceaux le long de la branche de solutions est non borné, ce qui implique par un résultat de bifurcation l'existence des "points de retournement" de la branche.

Dans le chapitre IV que nous ne détaillons pas dans cette introduction, d'autres applications du théorème globale de la bifurcation analytique sont donnés. Précisément, nous présentons une typologie des diagrammes de bifurcation obtenus selon les classes de nonlinéarités considérées dans le problème semilinéaire elliptique.

Nous allons maintenant décrire avec plus de précision les principaux résultats obtenus dans les chapitres I à III et les idées principales utilisées dans leurs preuves.

Résultats principaux du Chapitre I : Ce chapitre porte sur l'étude d'un problème parabolique, quasi-linéaire et fortement singulier de la forme suivante :

$$(\mathrm{P}) \qquad \begin{cases} \partial_t u - \Delta_p u &= u^{-\delta} + f(x,u,\nabla u) & \text{in } Q_T \stackrel{\text{def}}{=} (0,T)\times\Omega, \\ u &= 0 & \text{on } \Gamma \stackrel{\text{def}}{=} (0,T)\times\partial\Omega, \\ u(0,.) &= u_0 \geq 0 & \text{in } \Omega. \end{cases}$$

Dans cette étude, Ω représente un ouvert borné et régulier de $\mathbb{R}^N$, $N \geq 2$, $\Delta_p u$ est l'opérateur p-Laplacien défini par $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ avec $2 \leq p < \infty$, l'exposant δ est supposé réel strictement positif, la donnée initiale u_0 vérifiant

$$u_0 \in L^r(\Omega) \quad \text{et} \quad u_0 \geq 0 \quad \text{p.p. dans } \Omega \tag{2}$$

où $r \geq 2$ est choisi suffisamment grand. La non-linéarité $f : \Omega \times \mathbb{R}_+ \times \mathbb{R}^N \longrightarrow \mathbb{R}$ est supposée bornée inférieurement et de Carathéodory i.e., pour tout $(s,\xi) \in \mathbb{R}_+ \times \mathbb{R}^N$, la fonction $f(\cdot,s,\xi) : \Omega \longrightarrow \mathbb{R}$ est Lebesgue-mesurable, et, presque pour tout $x \in \Omega$, la fonction $f(x,\cdot,\cdot) : \mathbb{R}_+ \times \mathbb{R}^N \longrightarrow \mathbb{R}$ est continue. Quant au comportement de f à l'infini on suppose l'hypothèse suivante :

$$f(x,s,\xi) \;\leq \left(as^{q-1}+b\right) + c|\xi|^{p-\frac{p}{q}} \text{ p.p. } x \in \Omega \text{ pour } s \geq 0, |\xi| \geq M. \tag{3}$$

où $a, c, M > 0$, $b \geq 0$ et $q \in [p, p^*)$ avec $p^* = \frac{pN}{N-p}$ si $p < N$ et $p^* = \infty$ si $p \geq N$ désigne l'exposant critique. Pour démontrer l'unicité de la solution (pour des valeur de δ petit), on suppose que f est localement Lipschitzienne par rapport à s uniformément en $x \in \Omega$ et $\xi \in \mathbb{R}^N$ et vérifie l'hypothèse suivante

$$\begin{aligned} &f(x,s,\xi) := F(x,s,\xi_p) \text{ où } \xi_p = |\xi|^{p-2}\xi \text{ et il existe } C > 0 \text{ tel que} \\ &\qquad |F(x,s,\xi) - F(x,s,\xi')| \leq C|\xi-\xi'| \quad \text{ pour tout } \xi,\xi' \in \mathbb{R}^N \end{aligned} \tag{4}$$

<u>**Exemple**</u> **:** Un exemple des fonctions f satisfaisant les hypothèses précédentes est le suivant :

$$(x, s, \xi) \mapsto b(x)s^q + c(x) + \mathcal{A}(x).|\xi|^{p-2}\xi$$

avec $q < p^* - 1$, $b, c \in L^\infty(\Omega)$ et $\mathcal{A} \in (L^\infty(\Omega))^N$.

Ce type de problèmes, pour $p = 2$, apparait dans plusieurs modèles physiques : dans l'étude des fluides non Newtoniens (en particulier les fluides pseudo-plastiques), dans le phénomène de couche limite pour des fluides visqueux (regarder [95], [53], [91]), dans les modèles de Langmuir-Hinshelwood cinétiques de réaction chimique de type catalyse ([11], [101]), dans les modèles cinétiques enzymatiques (regarder [17]), ainsi que dans la théorie de conduction de la chaleur pour des matériaux conducteurs ([83]) et dans les modes guidés d'un champ électromagnétique dans un milieu non-linéaire ([65]). Pour $p \neq 2$, tel problème apparait dans l'étude d'écoulements turbulents d'un gaz dans les milieux poreux (regarder [100]). Pour plus de détail, nous référons au monographe Hernández-Mancebo-Vega [80], l'ouvrage Ghergu-Radulescu [66] et à leurs bibliographies. Ces références donnent en particulier un bon état de l'art sur les problèmes elliptiques singuliers : existence, régularité des solutions, comportement au bord, unicité et multiplicité des solutions. Nous présentons les travaux réalisés importants dans ce cadre un peu plus loin. En ce qui concerne les problèmes paraboliques singuliers, la littérature existante est moins fournie.

Les problèmes paraboliques de la forme (P) ont fait l'objet de quelques travaux antérieurs. Dans M. Badra, K. Bal and J. Giacomoni [14], le problème (P) a été considéré dans le cas où $0 < \delta < 2 + \frac{1}{p-1}$. Dans ce travail, la non-linéarité f est supposée par ailleurs indépendante de ξ et vérifie la condition de sous-homogénéité suivante :

$$0 \leq \limsup_{s \to +\infty} \frac{f(x, s)}{s^{p-1}} < \lambda_1(\Omega)$$

(où $\lambda_1(\Omega)$ désigne la première valeur propre de $-\Delta_p$ dans Ω avec les conditions de Dirichlet homogènes sur le bord). Cette condition de sous-homogénéité (par rapport à l'homogénéité de l'opérateur p-Laplacien) garantit l'existence de solutions globales (i.e. définies pour tout temps). De plus, la donnée initiale u_0 est supposée dans $W_0^{1,p}(\Omega)$ et satisfait "la condition de cône" suivante :

$$\left\{ \begin{array}{rcll} c_1 d(x) & \leq u_0 \leq & c_2 d(x) & \text{if } \delta < 1, \\ c_1 d(x) \log^{\frac{1}{p}} \left(\frac{k}{d(x)} \right) & \leq u_0 \leq & c_2 \log^{\frac{1}{p}} \left(\frac{k}{d(x)} \right) & \text{if } \delta = 1, \\ c_1 d(x)^{\frac{p}{p-1+\delta}} & \leq u_0 \leq & c_2 \left(d(x)^{\frac{p}{p-1+\delta}} + d(x) \right) & \text{if } \delta > 1, \end{array} \right. \tag{5}$$

pour certains constantes $c_1, c_2 > 0$ et $k > 0$ assez grand, où $d(x) \stackrel{\text{def}}{=} d(x, \partial\Omega)$. Cette condition de cône qui via un principe de comparaison est stable le long du flot est utilisée pour contrôler la norme du terme singulier dans le dual d'un espace fonctionnel bien choisi. Dans ce travail, sont discutés à la fois l'existence et l'unicité de solutions faibles ainsi que le comportement en temps long des solutions (stabilisation et convergence vers une solution stationnaire). L'approche utilisée pour démontrer l'existence de solutions faibles est la méthode de semi-discrétisation en temps qui s'appuie sur l'étude précise du problème quasilinéaire elliptique associé. L'existence d'une solution faible et positive $u \in \mathscr{C}([0, T]; W_0^{1,p}(\Omega))$ est obtenu pour tout $T > 0$. En imposant quelques restrictions supplémentaires sur f et en utilisant un résultat de régularité de SIMON [110] dans les espaces de Besov et le principe de comparaison faible, les auteurs ont démontré la convergence asymptotique dans $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ vers l'unique solution de l'équation stationnaire associée. En considérant une condition plus faible sur la positivité de la donnée initiale u_0 (précisément, la stricte positivité sur tout compact de Ω), l'existence d'une solution faible est aussi démontrée pour $0 < \delta < 1$ dans [13].

D'autres travaux antérieurs se sont intéressés au cas particulier où $p = 2$. Dans TAKÁČ [115], un résultat de stabilisation dans $\mathscr{C}^1$ est établi pour une certain classe de problèmes paraboliques avec coefficients singuliers via l'utilisation judicieuse d'espaces de Sobolev à poids. Dans ce travail, en faisant appel à la théorie des semi-groupes analytiques telle qu'elle est développée dans HENRY [79], l'auteur démontre en premier lieu l'existence (locale) d'une solution du problème parabolique dans $C([0,T], C_0^1(\overline{\Omega}))$ pour un certain $0 < T \leq \infty$ et pour toute donnée initiale u_0 dans $\left[C_0^1(\overline{\Omega})\right]^+$, l'intérieur du cône positif de $C_0^1(\overline{\Omega})$. La preuve de ce résultat utilise de manière cruciale le fait que le domaine de l'opérateur m-accrétif correspondant est un sous-ensemble de $\left[C_0^1(\overline{\Omega})\right]^+$. Le comportement des solutions globales est aussi discuté en utilisant la théorie des systèmes dynamiques dans le contexte des espaces de dimension infinie (voir à ce sujet HENRY [79] section 5.3). Une étape importante pour prouver la convergence vers une solution stationnaire est de montrer la compacité pour des temps longs des trajectoires associées dans un espace fonctionnel adapté. Dans [115], cette compacité est démontrée via les espaces de Sobolev à poids $W^{2,p}(\Omega; d^\mu) \cap W_0^{1,p}(\Omega; d^\mu)$ (avec $d \stackrel{\text{def}}{=} d(\cdot, \partial\Omega)$ la distance au bord) et combinée à une extension d'une inégalité de type LOJASIEWICZ-SIMON qui assure la stabilisation dans $C^1(\overline{\Omega})$. Nous citons également le résultat de HERNÁNDEZ, MANCEBO et VEGA [80], où une classe similaire de problèmes paraboliques singuliers est étudiée. L'existence locale de solutions faibles est également obtenue via la théorie des semi-groupes analytiques, en particulier par l'utilisation du Théorème 3.3.3 dans [79] dans $\left[C_0^1(\overline{\Omega})\right]^+$. En démontrant des résultats de régularité sur le problème stationnaire linéarisé et via l'application du théorème [79, Theorem 5.1.1], les auteurs établissent la stabilité asymptotique (avec convergence exponentielle) des solutions stationnaires stables (i.e. dont la première valeur propre de l'opérateur linéarisé est strictement positive). Nous soulignons que le trait commun entre les deux travaux précédents est de considérer le domaine de l'opérateur comme un sous ensemble de $\left[C_0^1(\overline{\Omega})\right]^+$, ce qui donne un contrôle implicite du terme singulier près du bord. Ceci ne fonctionne pas pour des valeurs grandes du paramètre de la singularité (i.e. $\delta \geq 1$). En effet, dans ce cas, les solutions n'appartiennent pas à $C^1(\overline{\Omega})$. Une première alternative est de considérer l'approche de [14] basée sur l'invariance le long des trajectoires de cônes définis par des sous-solutions mais ceci suppose de restreindre l'ensemble des données initiales admissibles.

Dans cette thèse (chapitre I), nous avons apporté un certain nombre de résultats nouveaux concernant l'existence et l'unicité de solutions faibles de (P) dans le cas quasi-linéaire dégénéré ($p \geq 2$) sans restriction sur δ et sans supposer d'hypothèse de stricte positivité sur la donnée initiale ce qui n'avait pas été considéré auparavant. Au regard de la litterature existante, nous traitons par ailleurs des non-linéarités f plus générales dépendant du gradient de la solution (avec effet d'un terme de convection supplémentaire). Notre approche est de construire la solution faible comme limite de solutions approchées obtenues en appliquant la théorie des semi-groupes sur des problèmes perturbés où le terme singulier est régularisé. La construction de sous-solutions pour ces problèmes approximés permet de lever les contraintes sur les données initiales et d'obtenir un contrôle uniforme sur les termes approximant le terme singulier. Avant de détailler les résultats du chapitre I, nous référons enfin aux travaux qui ont traité le cas où le terme singulier est un terme d'absorption. Dans ce cas la validité d'un principe de comparaison n'est plus vérifiée. Dans l'article [47], DAVILA, MONTENEGRO ont étudié les solutions positives du problème d'évolution prescrit en $t = 0$ par une donnée initiale positive, représenté par l'équation singulière

$$u_t - \Delta u + \chi_{\{u>0\}} g(u) = 0,$$

avec des conditions homogènes de Dirichlet dans un domaine Ω borné de $\mathbb{R}^N$ et $g(u) = u^{-\beta} - f(u)$ (avec $0 < \beta < 1$) qui est singulier en $u = 0$ et $\chi_{\{u>0\}}$ étant la fonction caractéristique de l'ensemble $\{x \in \Omega :\ u(x) > 0\}$. Un résultat d'existence de solutions globales est démontré et dans l'hypothèse où le problème stationnaire

$$-\Delta u + \chi_{\{u>0\}}(u^{-\beta} - \lambda f(u)) = 0, \qquad \lambda > 0,$$

admet des solutions strictement positives presque partout dans Ω (ce qui se produit pour certaines classes de fonctions f). De plus, les solutions convergent vers une solution stationnaire quand $t \to +\infty$. Inversement, si les solutions du problème stationnaire s'annulent sur des sous-ensembles de mesure non nulle (ce qui se produit pour une certaine classe de fonctions f), alors les solutions du problème d'évolution s'annulent en temps fini (l'extinction en temps fini est alors observée). Enfin les auteurs donnent quelques résultats d'unicité et de comportement de solutions au voisinage du bord. Dans WINKLER [120], l'extinction en temps fini est aussi démontrée pour une classe différente de problèmes paraboliques avec terme d'absorption singulier tandis que dans WINKLER [121] et [119] des résultats de non-unicité et de contraction de support de la solution sont établis. Dans GIACOMONI-SAUVY-SHMAREV [68], des résultats d'extinction "totale" de la solution sont démontrés pour une équation quasilinéaire parabolique avec terme d'absorption singulier.

On donne maintenant les résultats principaux de ce chapitre.
On commence d'abord par définir la notion d'une solution faible de (P) :

Définition 1 *Une fonction positive* $u : (0,T) \times \Omega \longrightarrow \mathbb{R}_+$ *est une solution faible de* (P) *si*

- $u \in L^p(0,T; W^{1,p}_{loc}(\Omega)) \cap \mathscr{C}([0,T]; L^r(\Omega)) \cap \mathscr{C}(Q_T)$,
- $u > 0, \quad \forall (t,x) \in (0,T) \times \Omega$ *et* $\forall \phi \in \mathscr{C}^1_c(Q_T)$, *on a :*

$$-\int_{Q_T} u \partial_t \phi \mathrm{d}x\mathrm{d}t + \int_{Q_T} |\nabla u|^{p-1} \nabla u . \nabla \phi \mathrm{d}x\mathrm{d}t - \int_{Q_T} u^{-\delta} \phi \mathrm{d}x\mathrm{d}t - \int_{Q_T} f(x,u,\nabla u) \phi \mathrm{d}x\mathrm{d}t = 0,$$

- $u(0) := u(0,.) = u_0$ *p.p.*, $x \in \Omega$.

On impose les restrictions suivantes sur p, q et r :

$$2 \leq p \leq q < \min\{p^*, p(1 + \frac{r}{N})\}, \tag{6}$$

$$r \geq q, \tag{7}$$

et

$$r \geq \max(2, 1+\delta). \tag{8}$$

Alors, le théorème suivant établit un résultat d'existence d'une solution faible pour tout $\delta > 0$.

Théorème 1 *Supposons que (6), (7) et (8) sont satisfaites et que f est une fonction de Carathéodory satisfaisant* (3)*. On suppose que la donnée initiale u_0 vérifie* (2)*. Alors il existe $T_m > 0$ tel que pour tout $T < T_m$ le problème (P) admet une solution faible u strictement positive sur $(0,T)$ et telle que*

$$u^{\frac{r-2+p}{p}} \in L^p(0,T; W_0^{1,p}(\Omega)).$$

Le deuxième résultat principal donne l'unicité de la solution sous la restriction $0 < \delta < 2 + \frac{1}{p-1}$. Cette restriction est la condition nécessaire et suffisante pour obtenir les solutions du problème elliptique associé dans $W_0^{1,p}(\Omega)$. Le résultat établit également des propriétés de régularité de la solution.

Théorème 2 *Soit $\delta < 2 + \dfrac{1}{p-1}$. Supposons que les hypothèses du théorème 1 sont satisfaites. Alors, le problème* (P) *admet une solution faible u dans $L^p(0,T; W_0^{1,p}(\Omega))$. Si $s \mapsto f(x,s,\xi)$ est localement lipschitzienne uniformément en $x \in \Omega$ et $\xi \in \mathbb{R}^N$ et satisfait* (4)*, Alors la solution u est unique. De plus si* (3) *est remplacé par la condition plus restrictive suivante :*

$$f(x,s,\xi) \leq as^{q-1} + b + c|\xi|^{\alpha}, \ 0 \leq 2\alpha \leq p, \quad 2(q-1) \leq r \tag{9}$$

p.p. $x \in \Omega$ et pour $s \geq 0$, $|\xi|$ assez grand, alors on a

$$u \in \mathscr{C}((0,T); W_0^{1,p}(\Omega)), \quad \partial_t u \in L^2((\eta,T)\times\Omega), \ \forall \eta \in (0,T).$$

Idées de la preuve du théorème 1

La démonstration du théorème s'appuie sur l'étude du problème régularisé (P_ε) suivant

$$(\mathrm{P}_\varepsilon) \qquad \begin{cases} \partial_t u_\varepsilon - \Delta_p u_\varepsilon = (u_\varepsilon + \varepsilon)^{-\delta} + f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \ \text{ dans } Q_T = (0,T)\times\Omega, \\ u_\varepsilon = 0 \ \text{ sur } (0,T)\times\partial\Omega, \ u_\varepsilon > 0 \text{ in } Q_T, \\ u_\varepsilon(0,.) \overset{\text{def}}{=} u_{\varepsilon,0} = \min\{\frac{1}{\varepsilon}, u_0\} \geq 0 \text{ dans } \Omega, \end{cases}$$

où

$$f_\varepsilon(x,u,\nabla u) \overset{\text{def}}{=} \min\left\{\frac{1}{\varepsilon}, f(x,u,\nabla u)\right\}$$

L'existence et la régularité de la solution de ce problème sont établis par le théorème suivant :

Théorème 3 *Soit f est une fonction de Carathéodory et soit u_0 satisfaisant* (2)*. Alors le problème* (P_ε) *admet une solution faible et positive u_ε telle que*

$$u_\varepsilon \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q_T) \cap \mathscr{C}([0,T]; L^s(\Omega)), \ \forall T > 0, \ \forall s \in [1,\infty). \tag{10}$$

La preuve de ce théorème est basée sur la méthode de semi-discrétisation en temps. Des inégalités d'énergie permettent d'obtenir des estimations a priori sur les solutions discrètes approchées et puis par passage à la limite en faisant tendre le pas de discrétisation vers 0, on obtient l'existence de la solution dans les espaces fonctionnels requis.

Une fois démontrée l'existence de la solution u_ε pour le problème approché (P_ε), on passe à la limite $\epsilon \to 0^+$ pour montrer l'existence de la solution de (P).

Pour cela, on établit certaines estimations dans $L^\infty((0,T)\times\Omega)$. On étudie tout d'abord l'opérateur définie par $Au = -\Delta_p u - \frac{1}{u^\delta}$. On montre que cette opérateur engendre un semi-groupe continu de contraction sur $L^\infty(\Omega)$ qui correspond à la limite de semi-groupes engendrés par $A_\varepsilon u = -\Delta_p u - \frac{1}{(u+\varepsilon)^\delta}$. On peut alors démontrer les estimations a priori en utilisant la propriété d'ultra-contractivité (voir E.B. Davies [43] et [E.B. Davies, B. Simon [46], chap. 2, 59-81]) qui implique que pour tout $t > 0$, l'opérateur e^{At} transforme les ensemble bornés de $L^r(\Omega)$ en ensembles bornés de $L^\infty(\Omega)$ comme il est montré dans la proposition suivante :

Proposition 1 *Soient $\delta > 0$ et $p, q, r \geq 2$ tels que (6) et (8) sont satisfaits. Soit f une fonction Carathéodory satisfaisant (3). Supposons que $u_0 \in L^r(\Omega)$ et $u_0 \geq 0$ presque partout dans Ω. Alors il existe $T_m > 0$ tel que la solution u_ε de (P_ε) vérifie les estimations suivantes pour tout $T < T_m$:*

$$\exists C_1 > 0 : \quad \|u_\varepsilon\|_{L^r(\Omega)} \leq C_1 < \infty \text{ indépendamment de } \varepsilon > 0 \text{ et } \forall t \in [0,T] \tag{11}$$

et

$$\forall \eta \in (0,T),\ \exists C_\eta > 0 : \ \|u_\varepsilon\|_{L^\infty(\Omega)} \leq C_\eta < \infty \text{ indépendamment de } \varepsilon > 0 \text{ et } \forall t \in (\eta, T]. \tag{12}$$

L'outil principal pour démontrer ces estimations est l'utilisation des inégalités de type log-Sobolev combinées avec des inégalités de Gagliardo-Nirenberg, en multipliant l'équation de (P_ε) par le terme non-linéaire $u^{\varrho(t)}$, où pour un $S > 0$ fixé, $\varrho : [0,S) \to [2,\infty)$ est une fonction de classe $\mathscr{C}^1$ bien choisie de manière que $\varrho(t) \to \infty$ lorsque $t \to S^-$.

Les inégalités log-Sobolev ont été introduites par L. Gross [75] afin d'établir des estimations de L^p-L^q avec $1 < p < q < \infty$, en vue de démontrer les effets régularisants (propriété d'hypercontractivité) de processus de diffusion d'équations issus de la physique mathématique comme dans l'équation de Schrödinger. Cette technique a été aussi utilisée avec succès pour les équations paraboliques, quasi-linéaires par F. Cipriano, G. Grillo [38], M. Del Pino, J. Dolbeault, I. Gentil [49], P. Takáč [114], M. Bonforte, J.L. Vázquez [24], M. Bonforte, G. Grillo [23]. Une approche alternative pour montrer des estimations locales pour ce type d'équations est d'appliquer la technique des itérations de Moser comme dans E. DiBenedetto [57], G. Lieberman [87] et M.M. Porzio [103].

Afin de contrôler le terme singulier au bord de domaine, on construit une sous-solution donnée par l'expression suivante :

$$\underline{u}_\varepsilon(t,x) = \varphi_\varepsilon(t)\left[(\varphi_1 + \varepsilon^{\frac{p-1+\delta}{p}})^{\frac{p}{p-1+\delta}} - \varepsilon\right] \tag{13}$$

où $\varphi_\varepsilon(t) = \left[(1+\delta)\eta t + \varepsilon^{\delta+1}\right]^{\frac{1}{\delta+1}} - \varepsilon$ avec $\eta > 0$ assez petit.

Les estimations L^∞ données par la proposition précédente sont cruciales pour pouvoir appliquer le résultat de la régularité locale pour les problèmes paraboliques ([DiBenedetto-Freidman[57]], [Chen[123]]) et obtenir la compacité nécessaire pour démontrer l'existence d'une solution du problème (P) en passant à la limite $\epsilon \to 0^+$.

Idées de la preuve du théorème 2

La démonstration du théorème repose principalement sur le résultat suivant qui donne l'unicité, l'existence d'une fonction de Lyapunov et la régularité de la solution pour le problème approché (P_ε) et d'un passage à la limite quand $\epsilon \to 0^+$:

Théorème 4 *Sous les mêmes hypothèses du théorème* (3)*, et en supposant que f est localement lipschitzienne par rapport à s uniformément en $x \in \Omega$ et $\xi \in \mathbb{R}^N$ tel que* (4) *est satisfaite, alors la solution u_ε est unique et appartient à $\mathscr{C}\left((0,T); W_0^{1,p}(\Omega)\right)$. De plus, pour tout t_0, t tels que $0 < t_0 \leq t \leq T$ on a (pour $\delta \neq 1$)*

$$\begin{aligned}
&\int_{t_0}^{t}\int_\Omega (\partial_s u_\varepsilon)^2 \,\mathrm{d}x\mathrm{d}s + \frac{1}{p}\int_\Omega |\nabla u_\varepsilon(t)|^p \mathrm{d}x - \frac{1}{1-\delta}\int_\Omega (u_\varepsilon(t)+\varepsilon)^{1-\delta}\,\mathrm{d}x \\
&= \frac{1}{p}\int_\Omega |\nabla u_\varepsilon(t_0)|^p \mathrm{d}x - \frac{1}{1-\delta}\int_\Omega (u_\varepsilon(t_0)+\varepsilon)^{1-\delta}\,\mathrm{d}x \\
&+ \int_{t_0}^{t}\int_\Omega f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\,\partial_s u_\varepsilon \mathrm{d}x\mathrm{d}s,
\end{aligned} \tag{14}$$

(si $\delta = 1$, on remplace le terme de la forme $(1-\delta)^{-1}x^{1-\delta}$ dans expressions ci-dessus par le terme $\log(x)$). De plus, la suite $\{u_\varepsilon\}_{\varepsilon>0}$ est croissante.

Pour montrer ce résultat, on démontre un principe de comparaison faible pour l'équation ϵ-approchée en utilisant certaines propriétés liées à la monotonie de $v \to \Delta_p v - \frac{1}{(v+\epsilon)^\delta}$ et de la sous-homogénéité de $v \to f(\cdot,\cdot,\nabla v)$. Il en découle la monotonie de $\epsilon \to u_\epsilon$ et l'unicité de u_ϵ en utilisant un argument classique basé sur le lemme de Gronwall (voir section 3 du chapitre I). Cet argument d'unicité est également utilisé pour l'unicité de la solution de (P).

Résultats principaux du Chapitre II : Dans ce chapitre, nous nous somme intéressés à donner la preuve du théorème de bifurcation globale analytique dans la forme qui est utilisée dans le chapitre III. Une forme assez proche est donnée dans la monographie [Buffoni and Toland [32], Chapitre 9] et utilisée dans l'article [30] pour l'existence d'ondes solitaires à la surface de l'eau et pour démontrer la présence de bifurcations sous-harmoniques.

Soient $\mathcal{X}$, $\mathcal{Y}$ des espaces de Banach, $\mathcal{U} \subset \mathbb{R} \times \mathcal{X}$ un ouvert contenant $(0,0)$ dans son adhérence et $F : \mathcal{U} \to \mathcal{Y}$ une fonction analytique. Considérons le problème suivant :

$$F(\lambda, x) = 0, \quad (\lambda, x) \in \mathcal{U} \tag{15}$$

On définit aussi l'ensemble des solutions

$$\mathcal{S} = \{(\lambda, x) \in \mathcal{U} \, : \, F(\lambda, x) = 0\}$$

et l'ensemble des solutions non singulières

$$\mathcal{N} = \{(\lambda, x) \in \mathcal{S} \,:\, Ker(\partial_x F(\lambda, x)) = \{0\}\}\,.$$

En supposant l'hypothèse (G2) ci dessous, un point de $\mathcal{S}$ est un point où l'opérateur linéarisé $\partial_x F(\lambda, x)$ est inversible et donc pour lequel on peut appliquer le théorème des fonctions implicites.

Définition 2 *Un arc distingué est une composante connexe de* $\mathcal{N}$.

Supposons que

(G1) Toute partie bornée de $\mathcal{S}$ est relativement compacte dans $\mathbb{R} \times \mathcal{X}$.

(G2) $\partial_x F(\lambda, x)$ est un opérateur de Fredholm d'indice 0 pour tout $(\lambda, x) \in \mathcal{S}$.

(G3) Il existe une application analytique $(\lambda, u) \,:\, (0, \epsilon) \ni s \to (\lambda(s), u(s)) \in \mathcal{S}$ telle que $\partial_x F(\lambda(s), u(s))$ est inversible pour tout $s \in (0, \epsilon)$ et $\lim_{s \to 0^+} (\lambda(s), u(s)) = (0, 0)$.

Soit

$$\mathcal{A}^+ = \{(\lambda(s), u(s)) \,:\, s \in (0, \epsilon)\}\,.$$

Clairement, $\mathcal{A}^+ \subset \mathcal{S}$ et représente une partie de la branche globale maximale. Le résultat suivant donne une extension globale $\mathcal{A}$ de $\mathcal{A}^+$ ne contenant qu'un ensemble discret de points singuliers, ce qui est une des conséquences de l'analyticité de $\mathcal{A}$.

Théorème 5 *Sous les hypothèses (G1)-(G3). On a,* $[0, \epsilon) \ni s \mapsto (\lambda(s), u(s))$ *se prolonge en une application (notée de même)* $(\lambda, u) : (0, \infty) \to \mathcal{S}$ *telle que :*

(a) Soit $\mathcal{A} \stackrel{\text{def}}{=} \{(\lambda(s), u(s)) \,:\, s > 0\}$*. Alors,* $\mathcal{A} \cap \mathcal{N}$ *est une réunion au plus dénombrable d'arcs distingués* $\bigcup_{i=0}^{n} \mathcal{A}_i$, $n \leq \infty$.

(b) $\mathcal{A}^+ \subset \mathcal{A}_0$.

(c) $\{s > 0 \,:\, ker(\partial_x F(\lambda(s), u(s))) \neq \{0\}\}$ *est un ensemble discret.*

(d) Pour tout $s^* \in (0, \infty)$*, il existe une re-paramétrisation* $\rho^* \,:\, (-1, 1) \to \mathbb{R}$ *continue et injective telle que* $\rho^*(0) = s^*$ *et*

$$(-1, 1) \ni t \to (\lambda(\rho^*(t)), u(\rho^*(t))) \in \mathcal{A} \text{ est analytique.}$$

De plus, l'application $s \mapsto \lambda(s)$ *est injective dans le voisinage à droite de* $s = 0$ *et pour tout* $s^* > 0$ *il existe* $\epsilon^* > 0$ *tel que* λ *est injective d'une part sur* $[s^*, s^* + \epsilon^*]$ *et d'autre part sur* $[s^* - \epsilon^*, s^*]$.

(e) L'une de ces trois assertions est vérifiée.

(i) $\|(\lambda(s), u(s))\|_{\mathbb{R} \times \mathcal{X}} \to \infty$ *quand* $s \to \infty$.

(ii) La suite $\{(\lambda(s), u(s))\}$ *converge vers un point de la frontière de* $\mathcal{U}$ *lorsque* $s \to \infty$.

(iii) $\mathcal{A}$ *est une boucle fermée :*

$$\mathcal{A} = \{(\lambda(s), u(s)) \,:\, 0 \leq s \leq T, (\lambda(T), u(T)) = (0, 0) \text{ pour un certain } T > 0\}\,.$$

Dans ce cas, en choisissant le plus petit $T > 0$, on a

$$(\lambda(s+T), u(s+T)) = (\lambda(s), u(s)) \text{ pour tout } s \geq 0.$$

(f) Supposons que $\partial_x F(\lambda(s_1), u(s_1))$ est inversible pour certain $s_1 > 0$. Si $s_2 \neq s_1$ on a $(\lambda(s_1), u(s_1)) = (\lambda(s_2), u(s_2))$, alors (e)(iii) est satisfait et $|s_1 - s_2|$ est un entier multiple de T. En particulier, l'application $s \mapsto (\lambda(s), u(s))$ est injective sur $[0, T)$.

Soulignons que les assertions (e) et (f) du théorème 5 établissent les différentes alternatives du comportement global de la branche étendue. La preuve de ce théorème utilise des outils complexes comme la réduction de Lyapunov-Schmidt et l'analyse complexe à plusieurs variables, plus particulièrement la théorie des germes analytiques.

La réduction de Lyapunov-Schmidt donnée par le théorème ci-dessus est un outil puissant en théorie de la bifurcation qui permet de ramener de manière équivalente un problème posé dans un espace de Banach de dimension infinie à un problème en dimension finie (représentée par l'équation réduite) pour lequel il est plus facile d'obtenir de la compacité et de caractériser l'ensemble des zéros. La preuve de la réduction de Liapunov Schmidt est fondée sur le théorème des fonctions implicites.

Théorème 6 *Soient $\mathcal{X}$ et $\mathcal{Y}$ deux espaces de Banach et U un ouvert de $\mathbb{K} \times \mathcal{X}$ où $\mathbb{K}$ est le corps des complexes ou des réels. Soit $F \in \mathscr{C}^k(U, Y)$, $k \in \mathbb{N}^*$ tel que $F(\lambda_0, x_0) = 0$ où $(\lambda_0, x_0) \in U$, on suppose que la différentielle de F par rapport à la seconde variable $A := \partial_2 F(\lambda_0, x_0)$ est de Fredholm, $\ker(A) \neq \{0\}$. Soit $q \in \mathbb{N}^*$ la codimension de $R(A)$ le rang de A. Alors il existe deux ouverts $U_0 \subset U$ et $V \subset \mathbb{K} \times \ker(A)$, et deux applications $\psi \in \mathscr{C}^k(V, X)$ et $h \in \mathscr{C}^k(V, \mathbb{K}^q)$ tels que $(\lambda_0, x_0) \in U_0$, $(\lambda_0, 0) \in V$ et $\psi(\lambda_0, 0) = x_0$ avec*

$$F(\lambda, x) = 0 \text{ dans } U_0 \iff \begin{cases} h(\lambda, \xi) = 0 \text{ pour certain } (\lambda, \xi) \in V, \\ \psi(\lambda, \xi) = x. \end{cases} \tag{16}$$

Un des résultats importants de la théorie des germes analytiques est le théorème de la représentations de Puiseux (voir théorèmes 2.5, 2.2 du chapitre II). Ce résultat nous permet d'étendre la branche locale $\mathcal{A}^+$ donnée par l'hypothèse (G_3) au-delà du point singulier représentant l'extrémité de la branche maximale contenant $\mathcal{A}^+$ si cette branche maximale ne satisfait pas les alternatives (i)-(iii) de l'assertion (e) du théorème 5. Après avoir établi une analyse locale au voisinage de ce point à l'aide de la réduction de Lyapunov-Schmidt et des éléments de la théorie des germes analytiques dont un des résultats importants est le théorème de structure qui permet d'obtenir que seulement un nombre fini de branches de solutions traversent ce point singulier, on démontre que l'ensemble des solutions réels au voisinage de ce point singulier contient un arc de solutions qui admet une paramétrisation localement injective et analytique.

Résultats principaux du Chapitre III : Les travaux présentés dans ce chapitre ont fait l'objet d'une publication dans **Calculus of Variations and P.D.E.**
(version en ligne http ://link.springer.com/article/10.1007/s00526-014-0735-8).

Dans ce chapitre, on s'intéresse au problème elliptique semi-linéaire et singulier (P_λ) ci-dessous paramétré par λ. Pour obtenir une descripton précise du diagramme de bifurcation associé, on met en application la théorie globale de la bifurcation analytique présentée dans le chapitre II sur ce problème.

$$(\mathrm{P}_\lambda) \qquad \begin{cases} -\Delta u = \lambda(u^{-\delta} + f(u)) \quad \text{dans } \Omega, \\ u > 0 \quad \text{dans } \Omega, \ \ u|_{\partial\Omega} = 0, \end{cases}$$

où $\Omega \subset \mathbb{R}^N$ avec $N \geq 2$, est un ouvert borné et régulier et l'exposant δ est strictement positif. Nous considérons les solutions (λ, u) de (P_λ) dans l'espace fonctionnel $\mathbb{R}_+ \times \mathcal{X}$, où

$$\mathcal{X} := \mathcal{C}_\phi(\Omega) = \{u \in \mathcal{C}(\Omega) \mid \text{for some } C > 0, |u(x)| \leq C\phi(x) \ \forall x \in \Omega\},$$

muni de la norme $\|u\|_{\mathcal{C}_\phi(\Omega)} \stackrel{\text{def}}{=} \sup_{x\in\Omega} |\frac{u(x)}{\phi(x)}|$ est un espace de Banach à poids. Le poids ϕ associé est une fonction puissance de la distance $d(x)$. L'exposant correspondant dépend de δ et est relié au comportement des solutions de (P_λ) au bord de Ω. On suppose que la non-linéarité f satisfait :

(f_0) $f : [0, \infty) \to [0, \infty)$ est deux fois différentiable avec $f(0) = 0$.

On suppose par ailleurs les hypothèses supplémentaires suivantes :

(f_1) $f(t)$ est un produit fini de fonctions de la forme $g(t^p), p > 0$, où g est une fonction réelle analytique sur $\mathbb{R}$.

(f_2) $\liminf\limits_{t\to\infty} \dfrac{f(t)}{t} = \infty$.

Ces hypothèses assurent l'applicabilité de la théorie de la bifurcation analytique. Le comportement surlinéaire de f contenu dans (f_2) implique la nonexistence de solutions de (P_λ) pour des valeurs grandes de λ. L'hypothèse (f_3) suivante assure que la branche des solutions minimales est un arc paramétré continu et donc par les hypothèses précédentes analytique. Elle fait donc entièrement partie de la branche analytique de solutions de (P_λ) émanant de $(0, 0)$ si (f_3) est réalisée.

(f_3) l'application $t \mapsto t^{-\delta} + f(t)$ est convexe sur $(0, \infty)$.

Avant de donner les résultats principaux de ce chapitre, nous décrivons la littérature concernant les problèmes semilinéaire elliptiques singuliers. Un des tous premiers résultats obtenu dans le cadre des problèmes elliptiques singuliers, est Stuart [113], où l'existence d'une solution classique (dans $C^2(\Omega) \cap C(\overline{\Omega})$) est établi via l'utilisation d'un schéma itératif et le principe du maximum. Dans Crandall-Rabinowitz-Tartar [41] qu'on peut considérer à certains égards comme le travail pionnier et qui a motivé dans la suite une longue série de travaux pour cette classe de problèmes, une version plus générale du problème suivant est étudiée :

$$(\tilde{\mathrm{P}}) \begin{cases} Lu = \frac{\tilde{p}(x)}{u^\delta} \quad \text{dans } \Omega \\ u|_{\partial\Omega} = 0, \ u > 0 \text{ dans } \Omega \end{cases}$$

où Ω est un ouvert borné régulier ($\partial\Omega$ est au moins C^2) , L désigne un opérateur linéaire uniformément elliptique, $\tilde{p} \in C(\overline{\Omega})$ avec $0 < \tilde{p}(x)$ dans Ω. L'existence et l'unicité de solutions classiques ainsi que le comportement précis au bord du domaine de ces solutions sont établis. Dans le cas d'une non-linéarité plus générale, l'existence de solutions est obtenue en utilisant la théorie du degré topologique et un argument d'approximation par régularisation du terme singulier. Le cas où le potentiel $\tilde{p} > 0$ est singulier est analysé dans Gomes [72] moyennant des restrictions sur δ qui garantit l'existence de solutions dans $C^1(\overline{\Omega})$ via un argument de point fixe de Schauder sur la formulation intégrale équivalente. La compacité requise pour l'application du point fixe est assurée via des estimations près du bord de la fonction de Green et de son gradient. Dans Del Pino [50], une classe plus générale de potentiels (sans la condition de stricte positivité) est considérée. Dans ce cadre, l'auteur démontre l'existence de solutions à variation bornée. La régularité dans l'espace d'énergie dans le cas de non-linéarités plus singulières est traitée dans Lazer-McKenna [86] (avec le potentiel $\tilde{p} \in C^{0,\alpha}(\overline{\Omega})$ et strictement positif dans $\overline{\Omega}$). Les auteurs démontrent que la solution qui appartient à $C^{2,\delta}(\Omega) \cap C(\overline{\Omega})$ est dans $H_0^1(\Omega)$ si et seulement si $\delta < 3$ et dans $C^1(\overline{\Omega})$ si et seulement si $\delta < 1$. La régularité höldérienne des solutions (dans $C^{1,\alpha}(\overline{\Omega})$ et $C^{0,\alpha}(\overline{\Omega})$) est établie de manière plus générale dans Gui-Lin [76] par des estimations sur la représentation intégrale via la fonction de Green (en particulier les estimations sur l'expression faisant intervenir la partie singulière de la fonction de Green). Concernant la régularité dans les espaces de Sobolev, Díaz, Hernández, Rakotoson ont récemment montré que les solutions appartenaient à $W^{1,q}(\Omega)$ avec $q > 1$ dépendant de δ. La preuve de ce résultat fait appel à de récentes contributions de Díaz, Rakotoson (voir [54]) se situant dans la continuité de Brezis-Cazenave-Martel-Ramiandrisoa [27] sur l'existence (et unicité) de solutions très faibles pour des problèmes semi-linéaires elliptiques avec données dans les espaces à poids $L^q(\Omega,\, d^\beta(x, \partial\Omega))$ avec $0 < \beta < 1$. Lorsque le potentiel $\tilde{p}$ est dans un espace $L^m(\Omega)$ (avec $m \geq 1$), l'existence d'une solution u telle que $u^\gamma \in H_0^1(\Omega)$ est établie dans Boccardo-Orsina [22] en reprenant une approche similaire à celle de [41] et en utilisant des techniques (de troncature) de Stampacchia [111]. Des résultats de non-existence très intéressants sont également démontrés dans ce travail quand $\tilde{p}$ est une mesure de Radon (ex. une masse de Dirac). Dans Coclite-Palmieri [39], dans le cadre d'un problème perturbé de ($\tilde{\mathrm{P}}$) faisant apparaître un paramètre de bifurcation et un terme sur-linéaire, des résultats de non-existence sont démontrés pour des valeurs grandes du paramètre. Ceci suggère l'existence de solutions multiples pour des valeurs plus petites du paramètre. Par des méthodes variationnelles (minimisation sous la contrainte naturelle de Nehari), Yijing, Shaoping, Yiming [124] ont démontré dans le cadre de problèmes perturbés de ($\tilde{\mathrm{P}}$) par un terme surlinéaire de croissance sous critique la multiplicité de solutions. Ce résultat a été ensuite étendu dans le cas critique respectivement par Haitao dans [78] en utilisant la méthode de Perron qui correspond à la version varationnelle de la méthode de sur et sous solutions et par Hirano-Saccon-Shioji dans [81] en reprenant la méthode de Nehari. Le cas où la non-linéarité est de croissance exponentielle et sur-exponentielle (cas critique compris) à l'infini a été investi dans Adimurthi-Giacomoni [4]. Des résultats de non-existence dans le cas radial sont aussi démontrés dans ce travail en utilisant la méthode de tir. Nous reviendrons sur cette contribution dans la présentation du chapitre III.

Le cas des systèmes semi-linéaires elliptiques singuliers ont été plus récemment étudiés dans Hernández-Mancebo-Vega [80] et dans Ghergu [64]. Quand le système est coopératif, l'existence d'une solution est obtenue via un schéma itératif monotone (via le principe du maximum faible). Si le système n'est pas coopératif, l'existence de solution est établie par le théorème du point fixe de Schauder appliqué dans un sous-ensemble convexe de $C^0(\overline{\Omega})$ bien choisi. La compacité de l'application considérée fait appel aux résultats de régularité dans $C^{0,\alpha}(\overline{\Omega})$ établis dans Gui-Lin [76] .

L'existence de continua de solutions pour les problèmes elliptiques singuliers dans le cas de non-linéarités surlinéaires (en contraste avec les hypothèses de sous linéarité faites dans [41]) a été établi pour $\delta < 1$ dans Hernandez, Mancebo, Vega en utilisant l'analyse spectrale du problème linéarisé dans l'espace $C^1(\overline{\Omega})$ établie dans ce même travail. Mais cette approche ne fonctionne pas pour $\delta \geq 1$. Dans Dhanya, Giacomoni, Prashanth, Saoudi [52], l'existence de continua de solutions est étendue au cas $\delta < 3$ en utilisant les propriétés de l'opérateur nonlinéaire monotone $u \to -\Delta u - \frac{\lambda}{u^\delta}$ dans $H_0^1(\Omega)$, les propriétés de son inverse dans $C_0(\overline{\Omega})$ et les résultats de régularité de [76]. Dans ce chapitre, nous étendons ces résultats pour tout $\delta > 0$ en utilisant les résultats du chapitre II sur la bifurcation analytique. Cette approche est nouvelle pour les problèmes elliptiques singuliers. Elle présente l'avantage d'établir l'existence d'une branche globale connexe par arc en contraste avec les ensembles connexes de solutions obtenus avec la théorie du degré topologique. Elle ne s'applique cependant que dans le cadre d'opérateurs analytiques.

On donne maintenant l'énoncé de tous les résultats obtenus dans ce chapitre :
On commence par la définition suivante

Définition 3 *Soit $\phi \in \mathcal{C}(\Omega)$ avec $\phi > 0$ dans Ω. On définit l'espace de Banach suivant*

$$\mathcal{C}_\phi(\Omega) = \{u \in \mathcal{C}(\Omega) \mid \textit{for some } C > 0, |u(x)| \leq C\phi(x)\ \forall x \in \Omega\}$$

avec la norme $\|u\|_{\mathcal{C}_\phi(\Omega)} \stackrel{\text{def}}{=} \sup_{x\in\Omega} |\frac{u(x)}{\phi(x)}|$.

On définit ensuite le sous-ensemble ouvert et connexe de $\mathcal{C}_\phi(\Omega)$:

$$\mathcal{C}_\phi^+(\Omega) = \{u \in \mathcal{C}_\phi(\Omega) \mid \inf_{x\in\Omega} \frac{u(x)}{\phi(x)} > 0\}.$$

Enfin, on choisit la fonction poids ϕ en fonction de δ comme suit :

$$\phi = \phi_\delta \stackrel{\text{def}}{=} \begin{cases} \varphi_1 & 0 < \delta < 1, \\ \varphi_1(-\log \varphi_1)^{\frac{1}{2}} & \delta = 1, \\ \varphi_1^{\frac{2}{\delta+1}} & \delta > 1. \end{cases} \tag{17}$$

Considérons l'Opérateur-Solution associé à (P_λ)

$$F(\lambda, u) = u - \lambda(-\Delta)^{-1}(u^{-\delta} + f(u)),\ (\lambda, u) \in \mathbb{R}^+ \times \mathcal{C}^+_{\phi_\delta}(\Omega),\ \delta > 0. \tag{18}$$

On démontre dans un premier temps que si f satisfait l'hypothèse (f_1), l'opérateur F est analytique de $\mathbb{R}^+ \times \mathcal{C}^+_{\phi_\delta}(\Omega)$ dans $\mathcal{C}_{\phi_\delta}(\Omega)$. On montre aussi que les ensembles bornés et fermés de l'ensemble $\mathcal{S} = \{(\lambda, x) \in \mathcal{U} : F(\lambda, x) = 0\}$ sont compacts dans $\mathcal{C}_{\phi_\delta}(\Omega)$. On établit ensuite les propriétes de l'opérateur linéarisé. En particulier, on démontre qu'il est de Fredholm d'indice 0. En utilisant les résultats sur la bifurcation globale analytique, on démontre :

Théorème 7 *Soit f satisfaisant (f_0), (f_1) et (f_2). Alors, il existe $\Lambda \in (0, \infty)$ et un ensemble non borné $\mathcal{A} \subset (0, \Lambda] \times \mathcal{C}^+_{\phi_\delta}(\Omega)$ de solutions de (P_λ) qui est globalement paramétrisé par une application continue :*

$$(0,\infty) \ni s \to (\lambda(s), u(s)) \in \mathcal{A} \subset \mathcal{S}.$$

qui vérifie les propriétés suivantes sur la branche $\mathcal{A}$ *:*

(i) $(\lambda(s), u(s)) \to (0,0)$ *dans* $\mathbb{R} \times \mathcal{C}_{\phi_\delta}(\Omega)$ *lorsque* $s \to 0^+$.

(ii) Pour tout $s_0 > 0$ *petit, la portion de la branche* $\{(\lambda(s), u(s)) : 0 < s < s_0\}$ *coïncide avec la branche des solutions minimales de* (P_λ) *avec* $\lambda < \lambda(s_0)$. *De plus, ceci reste vrai dès que* $\partial_u F$ *est inversible le long de la branche des solutions minimales.*

(iii) $\|u(s)\|_{\mathcal{C}_{\phi_\delta}(\Omega)} \to \infty$ *quand* $s \to \infty$.

(iv) $\mathcal{A}$ *admet au moins un point de bifurcation asymptotique* $\Lambda_a \in [0, \Lambda]$. *i.e., il existe une suite* $\{s_n\}_{n\in\mathbb{N}} \subset (0,\infty)$, $\{(\lambda(s_n), u(s_n))\} \subset \mathcal{A}$ *telle que* $s_n \to \infty$, $\lambda(s_n) \to \Lambda_a$ *et* $\|u(s_n)\|_{\mathcal{C}_{\phi_\delta}(\Omega)} \to \infty$.

(v) $\{s \geq 0 : \partial_u F(\lambda(s), u(s))$ *n'est pas inversible* $\}$ *est un ensemble discret.*

(vi) ($\mathcal{A}$ *est une branche connexe par arc "analytique") En chacun de ses point,* $\mathcal{A}$ *admet une re-paramétrisation analytique local dans le sens suivant : Pour tout* $s^* \in (0,\infty)$ *il existe une application continue et injective* $\rho^* : (-1,1) \to \mathbb{R}$ *telle sue* $\rho^*(0) = s^*$ *et la reparamétrisation*

$$(-1,1) \ni t \to (\lambda(\rho^*(t)), u(\rho^*(t))) \in \mathcal{A} \text{ est analytique.}$$

En plus, l'application $s \mapsto \lambda(s)$ *est injective dans un voisinage à droite de* $s = 0$ *et pour tout* $s^* > 0$ *il existe* $\epsilon^* > 0$ *telle que* λ *est injective sur* $[s^*, s^* + \epsilon^*]$ *et sur* $[s^* - \epsilon^*, s^*]$.

En supposant l'hypothèse (f_3), on obtient que la branche des solutions minimales est entièrement incluse dans $\mathcal{A}$:

Corolaire 1 *Pour* $\delta > 0$ *et* f *vérifiant* $(f_0), (f_1), (f_2)$ *et* (f_3). *On a les les propriétés supplémentaires suivantes de la branche donnée par le théorème précédent.*

(i) La branche des solutions minimales tout entière $\{(\lambda, u_\lambda) : \lambda \in (0,\Lambda)\}$ *coïncide avec la portion de la branche de* $\mathcal{A}$ *qui contient* $(0,0)$ *dans son adhérence. En particulier, la branche des solutions minimales est est une courbe analytique dans* $\mathbb{R}^+ \times \mathcal{C}^+_{\phi_\delta}(\Omega)$.

(ii) Si $(\Lambda, u_\Lambda) \in \mathcal{A}$ *pour certain* $u_\Lambda \in \mathcal{C}^+_{\phi_\delta}(\Omega)$, *alors* $\mathcal{A}$ *admet un point de retournement vers la gauche en* (Λ, u_Λ).

Remarque 1 *Si* $\delta < 3$, *grâce à l'inégalité de Hardy, on obtient aussi que la branche* $\mathcal{A}$ *définit une courbe continue dans* $\mathbb{R}^+ \times H^1_0(\Omega)$.

Idées de la preuve du théorème 7 et du corollaire 1

Le premier point est d'identifier des espaces appropriés sur lesquels l'opérateur-solution F (donné par (18)) est bien défini et analytique. La présence d'un terme singulier dans le second membre nous amène à considérer des espaces de Banach avec poids. La fonction de poids ϕ_δ définie par (17) est une sous-solution (à une constante multiplicative près qui dépend de λ) du problème (P_λ). Ce choix nous permet d'identifier le comportement précis des solutions au bord du domaine et de définir la notion de solution très faible en utilisant les résultats de Dìaz, Rakotoson [54] et pour laquelle on montre la validité d'un principe de comparaison faible. Ceci permet alors de démontrer que l'image de F est bien dans $\mathcal{C}_{\phi_\delta}(\Omega)$. Ensuite, nous étudions certaines propriétés de l'opérateur linéarisé de F définie par :

$$\begin{cases} \partial_u F(\lambda, u) \ : \ \mathcal{C}_{\phi_\delta}(\Omega) \to \mathcal{C}_{\phi_\delta}(\Omega), \\ \partial_u F(\lambda, u)w = w + (-\Delta)^{-1}[a_\lambda(u)w] \\ \text{ où } a_\lambda(u) = \lambda(\delta u^{-1-\delta} - f'(u)). \end{cases} \tag{19}$$

On obtient ces propriétés en faisant appel à certains résultats sur l'existence et l'unicité des solutions faibles au sens donné dans [Díaz-Rakotoson, [54] et [55]] pour des problèmes semilinéaires elliptiques linéaires du second ordre avec un terme de droite dans $L^1(\cdot, d^\alpha)$ avec $\alpha \in (0, 1)$. L'existence de la branche locale $\mathcal{A}^+$ de solutions de (P_λ) émanant de $(0, 0)$ est démontrée via le théorème des fonctions implicite version analytique. En effet, on montre que le problème (P_λ) admet des solutions minimales pour des valeurs petites de λ et que l'opérateur linéarisé est inversible le long de la courbe formée par ces solutions. Ensuite on démontre l'existence de la branche globale en appliquant le théorème 5. Le comportement global de la branche est déduit du fait que pour des valeurs de λ petites, (P_λ) admet une solution unique et que par conséquent il existe une unique branche émanant de $(0, 0)$. Ceci écarte le scénario d'une branche qui forme une boucle fermée. De plus les solutions sont dans $\mathcal{C}^+_{\phi_\delta}(\Omega)$ et donc l'alternative (ii) de l'assertion (e) du théorème 5 ne se produit pas. La branche $\mathcal{A}$ est donc non bornée. De plus, par l'hypothèse de surlinéarité (f_2), il existe $\Lambda > 0$ tel que (P_λ) n'a pas des solutions pour $\lambda > \Lambda$. On en déduit alors l'existence d'au moins un point de bifurcation asymptotique.

En suppposant l'hypothèse de convexité (f_3), on démontre que le problème linéarisé (qui vérifie une propriété de monotonie par rapport à λ par (f_3)) le long de la branche minimale est inversible et par le théorème des fonctions implicites est donc analytique. On en déduit alors par unicité de la branche émanant de $(0, 0)$ que toute la branche minimale est dans $\mathcal{A}$, ce qui établit le corollaire 1. Dans Cazenave, Escobedo, Porzio [35], une discussion sur la régularité de la branche des solutions minimales est menée pour différentes nonlinéarités et le lien entre la régularité de la branche et la stricte positivité de la première valeur propre du problème linéarisé est établi.

On considère alors le cas particulier de la dimension 2 et où f a un comportement critique à l'infini au sens de Trudinger Moser. La définition ci dessous précise ce comportement critique pour toute dimension : soit

$$f^*(t) = \begin{cases} |t|^{\frac{N+2}{N-2}} \text{ si } N \geq 3, \\ e^{t^2} \text{ si } N = 2. \end{cases}$$

Définition 4 *(i) On dit que* $f : [0, \infty) \to [0, \infty)$ *a un comportement au plus critique si* $\limsup_{t\to\infty} \frac{f(t)}{f^*(t)} < \infty$ *quand* $N \geq 3$ *et* $\limsup_{t\to\infty} \frac{f(t)}{f^*(\alpha t)} < \infty$ *pour un certain* $\alpha > 0$ *quand* $N = 2$.
(ii) On dit que $h : [0, \infty) \to [0, \infty)$ *est une perturbation de* f^* *si* $\lim_{t\to\infty} \frac{h(t)}{f^*(t)} = 0$ *dans le cas* $N \geq 3$ *et* $\lim_{t\to\infty} \frac{h(t)}{f^*(\varepsilon t)} = 0$ *pour tout* $\varepsilon > 0$ *dans le cas* $N = 2$.
(iii) On dit que $f : [0, \infty) \to [0, \infty)$ *a un comportement critique si* $f(t) = f^*(t) + h(t)$ *dans le cas* $N \geq 3$ *et si* $f(t) = h(t)f^*(t)$ *dans le cas* $N = 2$ *avec* h *une perturbation de* f^*.

On va démontrer qu'en dimension 2 et pour une certaine classe de fonctions f avec un comportement critique et dans le cas où Ω est une boule de $\mathbb{R}^2$, l'existence de solutions singulières et la présence d'une infinité dénombrable de points de retournements dans la branche $\mathcal{A}$. Ces points de retournement correspondent à des points de discontinuités de l'indice de Morse, indice qui est constant par morceaux et tend vers l'infini le long de $\mathcal{A}$. On définit en premier lieu la notion de solution singulière de (P_λ) de la façon suivante :

Définition 5 *On dit que $u^* \in L^1_{loc}(\Omega)$ est une solution singulière de (P_λ) si $\inf_K u^* > 0$ pour tout compact $K \subset \Omega$, $f(u^*) \in L^1_{loc}(\Omega)$, u^* n'est pas borné sur Ω et u^* résout $-\Delta u^* = \lambda((u^*)^{-\delta} + f(u^*))$ dans Ω, au sens des distributions.*

On donne maintenant les résultats correspondants :

Théorème 8 *Fixons $0 < \delta < 1$. On pose $h(t) = t^{2+p_1} \prod_{i=2}^n e^{\alpha_i t^{p_i}}$, où $p_1 > 0, 1 < p_i < 2, i = 2, ..., n$ sont distincts et $\alpha_i < 0$. Étant donné $f(t) = h(t)e^{t^2}$, soit u^* une solution singulière à symétrie radiale de (P_λ). Alors, pour toute suite $\{u_i\}_i$ de (P_λ) convergeant vers u^* localement uniformément dans $B_1 \setminus \{0\}$, l'indice de Morse u_i tends vers l'infini lorsque $i \to \infty$.*

Remarque 2 *Pour $0 < \delta < 1$, l'analyse spectrale de l'opérateur linéarisé montre que l'indice de Morse de $\partial_u F$ est bien défini en toute solution faible. On pourra se référer à l'annexe B qui donne les définitions et propriétés importantes de l'indice de Morse.*

Théorème 9 *Soit $0 < \delta < 1$. On considère h sous la forme $h(t) = t^{2+p_1} \prod_{i=2}^n e^{\alpha_i t^{p_i}}$, où $p_1 > 0, 1 < p_i < 2, i = 2, ..., n$ sont distincts $\alpha_i < 0$. Soit $f(t) = h(t)e^{t^2}$. Alors la branche continue et analytique par morceaux $\mathcal{A}$ obtenue dans le Théorème 7 admet une infinité de points de retournements.*

Plus précisément, il existe une suite $\{(\lambda_i, u_i)\}_{i\in\mathbb{N}} \subset \mathcal{A}$ telle que $\|u_i\|_{\mathcal{C}_{\phi_\delta}(\Omega)}$ et l'indice de Morse de $\partial_u F(\lambda_i, u_i)$ sont strictement croissants et tendent vers l'infini quand $i \to \infty$. De plus, la branche $\mathcal{A}$ admet un point de retournement en (λ_i, u_i) ce qui implique que pour λ proche et différent de λ_i par valeur inférieure ou supérieure, il existe au moins deux solutions distinctes proches de u_i.

Pour établir ces deux résultats, on démontre le résultat suivant :

Lemme 1 *Soit $0 < \delta < 1$ et $h(t) = t^{2+p_1} \prod_{i=2}^n e^{\alpha_i t^{p_i}}$, où $p_1 > 0, 1 < p_i < 2, i = 2, ..., n$ sont distincts et $\alpha_i < 0$. Soit $f(t) = h(t)e^{t^2}$. Soit $\Omega = B_1(0)$ la boule ouverte unité de $\mathbb{R}^2$. Alors il existe $\lambda^* > 0$ et une solution radiale singulière $u^* \in L^\infty_{loc}(B_1(0) \setminus \{0\})$ qui est non bornée à l'origine et qui est solution de $-\Delta u^* = \lambda^*((u^*)^{-\delta} + f(u^*))$ dans $B_1(0)$ avec $u^* = 0$ sur $\partial B_1(0)$. De plus, $u^* \notin H^1_{loc}(B_1(0))$.*

Avant de donner quelques points clé de la preuve des théorèmes 9 et 8, on donne quelques éléments de l'état de l'art sur les problèmes elliptiques critiques en dimension 2 : Pour ces problèmes qui font intervenir des nonlinéarités qui sont des perturbations de $e^{\alpha u^2}$ avec $\alpha > 0$, on a une phénoménologie comparable à celle qu'on observe en dimension supérieure avec l'exposant critique $\frac{N+2}{N-2}$. Précisément, nous avons le résultat suivant du à Trudinger-Moser :

Théorème 10 *1. Soit $p < \infty$ et Ω un domaine borné de $\mathbb{R}^2$. Alors $u \in H^1_0(\Omega)$ implique $e^{u^2} \in L^p(\Omega)$. Par ailleurs, l'application $H^1_0(\Omega) \ni u \to e^{u^2} \in L^p(\Omega)$ est continue dans la topologie associée à la norme.*

2.

$$4\pi = \max\left\{c\,;\ \sup_{\|w\|\leq 1}\int_\Omega e^{c|w|^2} < +\infty\right\}. \tag{20}$$

Des améliorations de (20) ont été démontrées, voir par exemple ADIMURTHI-DRUET [3]. De manière similaire à ce qui se produit en dimension supérieure, $H_0^1(\Omega) \ni u \hookrightarrow e^{4\pi u^2} \in L^1(\Omega)$ n'est pas compact pour la topologie faible. Précisément, si on considère une suite $\{w_n\} \subset H_0^1(\Omega)$ telle que $\|w_n\| \leq 1$, on a par le théorème 10 $\sup_n \int_\Omega e^{4\pi w_n^2} < \infty$ mais pas a priori convergence de $e^{4\pi w_n^2}$ dans $L^1(\Omega)$. De ce fait, la question de l'existence de solutions pour

$$(P_1)\begin{cases} -\Delta u = f(x,u) \ \text{ dans } \ \Omega \subset \mathbb{R}^2 \\ u|_{\partial\Omega} = 0 \end{cases}$$

où $f(x,u)$ a un comportement critique (voir condition (H2) ci-après) est loin d'être évidente si on recherche les solutions de (P_1) comme points critiques de la fonctionnelle associée :

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F(x,u)dx$$

dans l'espace de Sobolev $H_0^1(\Omega)$ avec $F(x,t) := \int_0^t f(x,s)ds$. En particulier, la condition de Palais-Smale n'est pas automatiquement satisfaite. Une autre similitude avec le cas de la dimension supérieure est que l'exposant 2 est une limite de compacité. En effet, $H_0^1(\Omega) \ni u \hookrightarrow e^{4\pi u^\alpha} \in L^1(\Omega)$ est compacte pour la topologie faible si $\alpha < 2$. De même, la perte de compacité dans le cas $\alpha = 2$ fait apparaitre des phénomènes de concentration comme l'ont mis en évidence Adimurthi et Struwe pour des énergies petites (théorème 1.1 dans [7] généralisant [112] où le cas radial est considéré) et ensuite Druet (théorème 1 dans [59]) dans le cas général en appliquant une approche développée en dimension supérieure dans [60]. Ce phénomène de concentration est démontré à partir d'une suite de solutions $\{u_k\}$ de (P_1) mais contrairement au cas de la dimension $n \geq 3$ ne s'étend pas à une suite de Palais Smale quelconque comme l'ont démontré ADIMURTHI et PRASHANTH dans [5] : Les suites de Palais Smale peuvent présenter différents types de perte de compacité. En dimension supérieure (i.e. $n \geq 3$), la perte de compacité des suites de Palais Smale se traduit sous la forme d'un méchanisme universel, via le lemme de concentration-compacité ([89]). Pour les propriétés de compacité, de phénomènes de concentration dans ce cas, on peut consulter également (entre autres) [15], [29], [106] et[113].

Nous allons maintenant donner quelques résultats d'existence concernant (P_1) lorsque $f(x,\cdot) = h(x,\cdot)e^{b(\cdot)^2}$ $(b > 0)$ a une croissance critique et satisfait les conditions suivantes :

(A1) $f(x,0) = 0$, $f(x,t) > 0$ quand $t > 0$;

(A2) $\forall\, \epsilon > 0$, $\lim_{t\to\infty}\sup_{x\in\bar\Omega} h(x,t)e^{-\epsilon t^2} = 0$, $\lim_{t\to+\infty}\inf_{x\in\bar\Omega} h(x,t)e^{\epsilon t^2} = +\infty$;

(A3) Il existe $M > 0$ et $\sigma \in [0,1[$ tel que $F(x,t) \leq M(1 + f(x,t)t^\sigma)$.

Dans [2], ADIMURTHI prouve l'existence d'au moins une solution. Précisément, il démontre le résultat suivant :

Théorème 11 *Soit $f(x,t) = h(x,t)e^{bt^2}$. On suppose que f a une croissance critique et satisfait de plus*

(A4) $f'(x,t) := \frac{\partial}{\partial t}f(x,t) > \frac{f(x,t)}{t}$ pour $t > 0$.

Alors, si $\sup_{x\in\bar\Omega} f'(x,0) < \lambda_1(\Omega)$ *et si* $\limsup_{t\to+\infty} \inf_{x\in\bar\Omega} h(x,t)t = \infty$*, il existe une solution* u_0 *dans* $H_0^1(\Omega)$ *de* (P_1).

Le théorème 11 est démontré dans le cas plus général d'équations faisant intervenir l'opérateur n-Laplacien ($\Delta_n\cdot := -\nabla.(|\nabla\cdot|^{n-2}\nabla\cdot)$ qui coincide avec $-\Delta$ si $n=2$ et pour lequel une extension du théorème 10 est vérifiée :

Théorème 12 (Moser) *Soit* $n \geq 2$. *On a*

1) *Soit* $u \in W_0^{1,n}(\Omega)$, $p < \infty$, *alors* $e^{|u|^{\frac{n}{n-1}}} \in L^p(\Omega)$;

2) $(\frac{\alpha_n}{b})^{n-1} = \max\left\{c^n;\ \sup_{\|w\|\leq 1}\int_\Omega e^{b(c|w|)^{\frac{n}{n-1}}} < \infty\right\}$ *où* $\alpha_n = n\omega_n^{\frac{1}{n-1}}$ *avec* $\omega_n = |S^{n-1}|$.

La preuve est basée sur une méthode de minimisation sous contrainte due à Nehari ([98]). Donnons les éléments principaux de la preuve dans le cas $n=2$. En définissant

$$\frac{a(\Omega,f)^2}{2} = \inf\left\{E(u);\ \int_\Omega |\nabla u|^2\,dx = \int_\Omega f(x,u)u,\ u \not\equiv 0\right\}, \tag{21}$$

il démontre que le minimiseur de (21) est une solution de (P_1). Ceci nécessite de prouver que la condition de Palais Smale est satisfaite. Pour cela, il montre que $a(\Omega,f)^2 < (\frac{\alpha_2}{b})$ (qui correspond au premier niveau critique où la condition de Palais Smale échoue) en utilisant les fonctions de Moser $\{\phi_k\}$ obtenues par troncation et changement d'échelle de la solution fondamentale :

$$\phi_k(x) = \frac{1}{\sqrt{2\pi}}\begin{cases}(\log k)^{\frac{1}{2}} \text{ pour } |x| \leq \frac{1}{k}\\ \frac{\log(\frac{1}{r})}{(\log k)^{\frac{1}{2}}} \text{ pour } \frac{1}{k} \leq |x| \leq 1\\ 0 \text{ pour } |x| \geq 1,\end{cases}$$

comme fonctions tests et la condition $\limsup_{t\to+\infty} \inf_{x\in\bar\Omega} h(x,t)t = \infty$. La condition $\sup_{x\in\bar\Omega} f'(x,0) < \lambda_1(\Omega)$ assure que $0 < a(\Omega,f)^2$ et donc que u_0 est non triviale. Il n'est pas difficile de voir que cette condition est nécessaire. Le comportement à l'infini de h intervient donc de manière cruciale pour obtenir la compacité des suites de Palais Smale. La question qui se pose alors est de savoir si la condition $\limsup_{t\to+\infty} \inf_{x\in\bar\Omega} h(x,t)t = \infty$ est optimale. Dans [6], Adimurthi et Prashanth répondent positivement à cette question. Investissant le cas radial :

$$(P_\lambda)\begin{cases}-u''(r) - \frac{u'(r)}{r} = \lambda h(u)e^{u^2}(r), \quad r < 1,\\ u'(0) = 0\ \ , u(1) = 0 \text{ et } u > 0 \text{ pour } r < 1.\end{cases}$$

Ils démontrent que (P_λ) n'admet pas de solution pour $\lambda > 0$ petit si h appartient aux deux classes $\mathcal{H}_1$ et $\mathcal{H}_2$ suivantes :

$$\mathcal{H}_1 = \{h\ :\ h \text{ est asymptotique à l'infini à } e^{-s^\beta},\ 1 < \beta < 2\},$$

$$\begin{aligned}\mathcal{H}_2 &= \{h\,:\,h \text{ est asymptotique à l'infini à } e^{-s^\beta}, 0<\beta\le 1\}\\ &\quad \cup\{h\,:\,h \text{ décroit à l'infini polynomialement}\}.\end{aligned}$$

La preuve de la non existence de solutions pour λ petit et pour ces classes de fonctions h utilise une méthode de tir et la transformation d'Emden-Fowler comme dans [12]. Ce résultat de non existence combiné avec le résultat de [2] prouve que la condition $\limsup_{t\to+\infty} \inf_{x\in\bar\Omega} h(x,t)t=\infty$ trace la frontière entre existence et non existence de solutions pour le problème (P_1) une fois la condition $\sup_{x\in\bar\Omega} f'(x,0)<\lambda_1(\Omega)$ satisfaite.

Dans [8], le résultat de [2] (avec toujours $\sup_{x\in\bar\Omega} f'(x,0)<\lambda_1(\Omega)$) est étendu aux solutions changeant de signe en utilisant une méthode variationnelle de type linking, due à Bartolo-Benci-Fortunato (voir [19]). La compacité des suites de Palais Smale est cette fois ci assurée par la condition :

$$\limsup_{t\to+\infty} \inf_{x\in\bar\Omega} \frac{\log(h(x,t))}{t}=\infty.$$

Cette condition s'avère optimale puisque dans [9] est démontrée la non existence de solutions radiales changeant de signe dans un disque de rayon suffisamment petit et pour $f(x,t):=f(t)=te^{t^2+t^\beta}$ avec $0\le\beta\le 1$.

Dans [4], les auteurs étudient le problème perturbé singulier suivant :

$$(P)\begin{cases} -\Delta u=\lambda(\frac{p(x)}{u^\delta}+h(x,u)e^{4\pi u^2}) \text{ dans } \Omega\subset\mathbb{R}^2\\ u|_{\partial\Omega}=0\ ,\ u>0 \text{ dans } \Omega\end{cases}$$

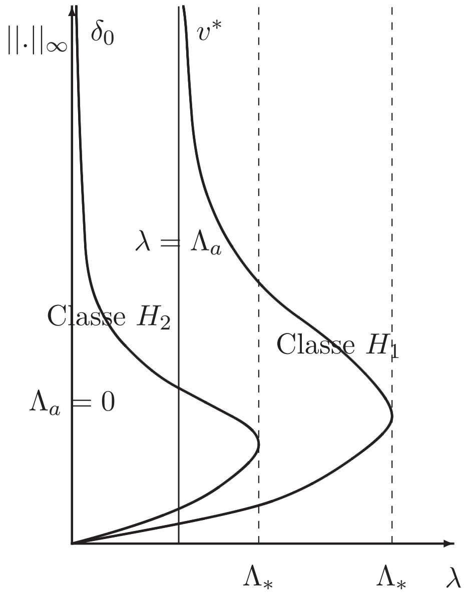
où $0<p\in L^\infty$, $\lambda>0$, $0<\delta<3$ et Ω un domaine borné régulier. La présence du terme sous linéaire $\frac{p(x)}{u^\delta}$ donne l'existence d'une solution minimale pour des valeurs petites de λ. La question que les auteurs se sont alors posée est de savoir sous quelle condition sur h à l'infini peut-on obtenir l'existence d'une seconde solution pour λ petit et si cette condition est optimale. En suivant une approche similaire à [6] (étude du problème variationnel dans le cas non radial et étude du cas radial par méthode de tir), ils démontrent que la condition sur la perturbation h qui sépare le cas d'unicité et de multiplicité pour les solutions pour $\lambda>0$ petit est donnée par :

$\exists\ \beta$ tel que $0\le\beta<1$ et $\lim_{t\to+\infty} h(x,t)te^{t^\beta}=+\infty$ uniformément sur $x\in\bar\Omega$.

Cette condition sépare les classes suivantes :

$\mathcal{H}_1=\{h\,:\,h$ est asymptotique à l'infini à e^{-s^β} $1<\beta<2\}$,
$\mathcal{H}_2=\{h$ est asymptotique à l'infini à e^{-s^β} $0<\beta\le 1\}$
$\cup\{h$ décroit polynomialement à l'infini$\}$.

Pour la classe $\mathcal{H}_1$, on a unicité de solutions de (P) dans le cas radial pour des valeurs de $\lambda>0$ petites tandis que pour la classe $\mathcal{H}_2$ on a multiplicité de solutions. Pour les diagrammes de bifurcation correspondants (voir figure ci jointe), cela signifie que dans le cas de la classe de $\mathcal{H}_2$, 0 est l'unique point de bifurcation asymptotique. Pour la classe $\mathcal{H}_1$, la branche de solutions admet au moins un point de bifurcation asymptotique Λ_a qui est nécessairement différent de 0 :



Bifurcation pour le cas sous linéaire et singulier

La question qui se pose alors est de déterminer le comportement des solutions au voisinage des points de bifurcation asymptotique pour les deux classes. Dans le cas de la classe $\mathcal{H}_2$, [5] dans le cas radial (voir aussi [52],[99] et [96]) et [7] dans le cas non radial ont démontré via un changement d'échelle bien choisi un phénomène de concentration (la fonction énergie s'accumulant au premier niveau d'énergie où la condition de Palais Smale n'est pas satisfaite). Un résultat récent et à paraître de Adimurthi-Giacomoni montre dans le cas d'une sous-classe $\mathcal{H}_2$ l'unicité de la branche asymptotique (i.e. unicité des solutions avec norme L^∞ grande). Ceci implique que les solutions "grandes" ont un indice de Morse égal à 1 et coincident avec les solutions obtenues par le Lemme du Col. Pour la classe $\mathcal{H}_1$, les théorèmes 9 et 8 établissent un tout autre phénomène : les solutions "grandes" convergent vers une solution singulière et l'indice de Morse tend vers l'infini.

Idées de la preuve des théorèmes 9 et 8

On applique la théorie globale de la bifurcation analytique avec la non-linéarité $f(t) = h(t)e^{t^2}$, où h est une perturbation de e^{t^2} de la classe $\mathcal{H}_1$ défini plus haut. Par ailleurs f vérifie les hypothèses (f_0), (f_1) et (f_2) du théorème 7. D'après le théorème 1.3 de [67], le problème (P_λ) admet une solution unique pour des valeurs de λ petites. Ceci implique que 0 ne peut être point de bifurcation asymptotique. La surlinéarité de f à l'infini implique l'existence d'au moins un point de bifurcation asymtotique qui est nécessairement différent de 0. En utilisant les résultats de [16], toutes les solutions de (P_λ) sont radiales symétriques. On peux alors étudier l'équation différentielle d'Emden-Fowler associée par méthode de tir comme dans [12]. En établissant les asymptotiques de certaines quantités caractéristiques de l'équation d'Emden-Fowler, on montre alors que toute suite de solutions (λ_i, u_i) de la branche $\mathcal{A}$ satisfaisant $\|u_i\|_{\mathcal{C}_{\phi_\delta}(\Omega)} \to \infty$ vérifie : il existe u^*, solution singulière de (P_λ) s'annulant au bord de B_1 et $\lambda^* > 0$ telle qu' à une sous suite près, $\lambda_i \to \lambda^*$ et $u_i \to u^*$ uniformément sur tout compact de $B_1 \setminus \{0\}$. On montre aussi que $u^* \in L^p(\Omega)$ pour tout $p \geq 1$ mais $u^* \notin H_0^1(\Omega)$.

On montre ensuite en utilisant un critère classique d'oscillation (voir [63]) que la solution du problème linéarisé en u^* admet une infinité de zéros s'accumulant à l'origine. On peut alors démontrer que l'indice de Morse des solutions u_i est non borné et donc non borné le long de la branche analytique $\mathcal{A}$. D'après un résultat classique de la théorie de la bifurcation (voir par exemple théorème II.7.3 dans KIELHÖFER [84]) ceci implique que la branche admet un point de retournement en chaque point de l'ensemble où l'indice de Morse change. Par conséquent, $\mathcal{A}$ admette une infinité dénombrable de points de retournements.

Cette propriété d'existence d'une infinité de "points de retournements", a été remarquée pour des non-linéarités sur-critiques en dimension supérieur par [DANCER, [43]] (regarder aussi [GUO-WEI, [77]]) en utilisant la théorie globale de la bifurcation analytique, et aussi pour des non-linéarités exponentielles en dimension supérieur par [JOSEPH-LUNDGREN, [82]] avec une approche différente basée sur l'étude de sytèmes dynamiques associés (voir chapitre IV). A travers une analyse rigoureuse du problème linéarisé, sont également démontrés dans ces travaux l'unicité du point de bifurcation asymptotique.

L'unicité du point de bifurcation constitue aussi une perspective de recherche intéressante pour notre problème. Pour établir un tel résultat, il faudrait parvenir à des propriétés fines sur les oscillations de la solution du problème linéarisé en u^*.

Chapitre I

Bounded solutions to a quasilinear and singular parabolic equation with p-Laplacian

Les résultats présentés dans ce chapitre font l'objet d'un travail soumis en collaboration avec Jacques Giacomoni et Peter Takáč.

1 Introduction

We investigate the following non-linear and singular parabolic equation :

$$\text{(P)} \qquad \begin{cases} \partial_t u - \Delta_p u = u^{-\delta} + f(x,u,\nabla u) & \text{in } Q_T \overset{\text{def}}{=} (0,T)\times\Omega, \\ u = 0 & \text{on } \Gamma = (0,T)\times\partial\Omega, \\ u(0,.) = u_0 \geq 0 & \text{in } \Omega. \end{cases}$$

where Ω stands for a regular bounded open subset of $\mathbb{R}^N$, $\Delta_p u \overset{\text{def}}{=} \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $2 \leq p < \infty$, $\delta > 0$ and $T > 0$, and the initial datum satisfies

$$u_0 \in L^r(\Omega) \quad \text{and} \quad u_0 \geq 0 \quad \text{a.e. in } \Omega \tag{I.1}$$

with $r \geq 2$ large enough. The function $f : \Omega \times \mathbb{R}_+ \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is bounded below and Carathéodory function, i.e., for every $(s,\xi) \in \mathbb{R}_+ \times \mathbb{R}^N$, the function $f(.,s,\xi) : \Omega \longrightarrow \mathbb{R}$ is Lebesgue-measurable, and, for almost all $x \in \Omega$, the function $f(x,.,.) : \mathbb{R}_+ \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is continuous. Concerning the asymptotic behaviour of f, we impose the following hypothesis :

$$f(x,s,\xi) \leq \left(as^{q-1} + b\right) + c|\xi|^{p-\frac{p}{q}} \text{ for a.a. } x \in \Omega \text{ for } s \geq 0, |\xi| \geq M. \tag{I.2}$$

where $a, c, M > 0$ and $b \geq 0$ and $q \in [p, p^*)$ where $p^* = \frac{pN}{N-p}$ if $p < N$ and $p^* = \infty$ if $p \geq N$. In order to show the uniqueness, we assume that f is locally Lipschitz with respect to s uniformly in $x \in \Omega$ and $\xi \in \mathbb{R}^N$ and satisfying the following hypothesis

$$\left.\begin{array}{c} f(x,s,\xi) := F(x,s,\xi_p) \text{ where } \xi_p = |\xi|^{p-2}\xi \text{ and there exists } C > 0 : \\ |F(x,s,\xi) - F(x,s,\xi')| \leq C|\xi - \xi'| \quad \text{for all } \xi, \xi' \in \mathbb{R}^N \end{array}\right\} \tag{I.3}$$

A prototype example of f satisfying the above conditions is $(x,s,\xi) \mapsto b(x)s^q + c(x) + \mathcal{A}(x).|\xi|^{p-2}\xi$ with $q < p^* - 1$, $b, c \in L^\infty(\Omega)$ and $\mathcal{A} \in (L^\infty(\Omega))^N$.

This problem has already been studied by M. Badra, K. Bal and J. Giacomoni in the paper [13]. In this work, the authors deal with f independent from ξ and satisfying the following subhomogeneous growth condition

$$0 \leq \limsup_{s\to+\infty} \frac{f(x,s)}{s^{p-1}} < \lambda_1(\Omega)$$

(where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta_p$ in Ω with zero Dirichlet boundary conditions) and $u_0 \in W_0^{1,p}(\Omega)$ satisfying a cone condition (implying a "strong positivity assumption" on u_0). In contrast, our results only require u_0 to be nonnegative and in $L^r(\Omega)$ with r large enough. We also point out that we get existence of weak solutions for any $\delta > 0$ (which is new for $\delta \geq 2 + \frac{1}{p-1}$) whereas in [13] $\delta < 2 + \frac{1}{p-1}$ is imposed.
In this chapter, we first prove a general existence result (see Theorem 2.1) for any $\delta > 0$. More precisely, we prove that there exists a positive weak solution $u \in \mathscr{C}([0,T];L^r(\Omega))$ such that $u^\gamma \in L^p(0,T;W_0^{1,p}(\Omega))$ for some $\gamma > 2$. If $\delta < 2 + \frac{1}{p-1}$, we prove additional properties of solutions (see Theorem 2.2) : the solution is unique and belongs to $L^p(0,T;W_0^{1,p}(\Omega)) \cap \mathscr{C}((0,T);W_0^{1,p}(\Omega))$.

To prove our main results, we study an auxiliary problem (see Problem (P_ϵ)) where the singular term is approximated by $\frac{1}{(u+\epsilon)^\delta}$. We obtain the existence of a weak solution by passing to the limit as $\epsilon \to 0^+$. We point out that the operator A, defined by $Au \stackrel{\text{def}}{=} -\Delta_p u - \frac{1}{u^\delta}$, generates a weak-star continuous semigroup of contractions on $L^\infty(\Omega)$ as a limit of semigroups generated by A_ϵ defined by $A_\epsilon u = -\Delta_p u - \frac{1}{(u+\epsilon)^\delta}$. Consequently, we are able to prove a priori bounds by using the smoothing property (ultracontractivity) that states that for every time $t > 0$, e^{At} maps bounded sets from $L^r(\Omega)$ into bounded sets in $L^\infty(\Omega)$. For that we use logarithmic Sobolev inequalities combined with inequalities of Gagliardo-Nirenberg type. This approach studied much earlier by L. Gross [75], was used for quasilinear parabolic equations by F. Cipriano, G. Grillo [38], M. Del Pino, J. Dolbeault, I. Gentil [49], P. Takáč [114], M. Bonforte, J.L. Vázquez [24], M. Bonforte, G. Grillo [23]. An alternative approach that can be used to derive the local estimates is to apply Moser's iteration technique as it is performed in E. DiBenedetto [57], G. Lieberman [87] and M.M. Porzio [103].

Such a problem with $p = 2$ arises in the study of non-Newtonian fluids (in particular *pseudoplastic* fluids), boundary-layer phenomena for viscous fluids (see [95], [53], [91]), in the Langmuir-Hinshelwood model of chemical hetereogeneous catalyst kinetics (see [11], [101]), in enzymatic kinetics models (see [17]), as well as in the theory of heat conduction in electrically conducting materials (see [83]) and in the study of guided modes of an electromagnetic field in nonlinear medium (see [65]). Problem (P_t) with $p \neq 2$ arises specifically in the study of turbulent flow of a gas in porous media (see [100]). We refer to the survey Hernández-Mancebo-Vega [80], the book Ghergu-Radulescu [66] and the bibliography therein for more details about the corresponding models.

2 Main results

We first define the notion of a positive weak solution to (P) as follows :

Definition 2.1 *A function $u : (0,T)\times\Omega \longrightarrow \mathbb{R}$ is a* positive weak solution *of (P) if*

– $u \in L^p(0,T;W^{1,p}_{loc}(\Omega)) \cap \mathscr{C}([0,T];L^r(\Omega)) \cap \mathscr{C}(Q_T)$,

– $u > 0 \quad \forall (t,x) \in (0,T)\times\Omega$ *and* $\forall \phi \in \mathscr{C}^1_c(Q_T)$, *we have :*

$$-\int_{Q_T} u\partial_t\phi \mathrm{d}x\mathrm{d}t + \int_{Q_T} |\nabla u|^{p-1}\nabla u.\nabla\phi \mathrm{d}x\mathrm{d}t - \int_{Q_T} u^{-\delta}\phi \mathrm{d}x\mathrm{d}t - \int_{Q_T} f(x,u,\nabla u)\phi \mathrm{d}x\mathrm{d}t = 0$$

– $u(0) := u(0,.) = u_0$ *a.e.* $x \in \Omega$.

We now state the main results. First, we assume the following restrictions on the numbers p, q and r :

$$2 \le p \le q < \min\{p^*, p(1+\frac{r}{N})\} \tag{I.4}$$

$$r \ge q \tag{I.5}$$

and

$$r \ge \max(2, 1+\delta). \tag{I.6}$$

Then we prove the following theorems :

Theorem 2.1 *Let (I.4), (I.5) and (I.6) hold. Assume that f is a bounded below Carathéodory function and satisfying* (I.2) *and assume that u_0 satisfies* (I.1)*. Then there exists $T_m > 0$ such that for any $T < T_m$, Problem (P) has at least a positive weak solution u such that*

$$u^{\frac{r-2+p}{p}} \in L^p(0,T;W^{1,p}_0(\Omega)).$$

Theorem 2.2 *Let $\delta < 2 + \dfrac{1}{p-1}$. Assume the same hypotheses of theorem 2.1. Then, problem (*P*) has a positive weak solution u in $L^p(0,T;W^{1,p}_0(\Omega))$. If f is locally Lipschitz with respect to s uniformly in $x \in \Omega$ and $\xi \in \mathbb{R}^N$ and satisfying* (I.3)*, then u is unique and in addition if instead of* (I.2) *f satisfies the following more restrictive condition*

$$f(x,s,\xi) \le as^{q-1} + b + c|\xi|^\alpha, \ 0 \le 2\alpha \le p, \quad 2(q-1) \le r \tag{I.7}$$

for a.e. $x \in \Omega$ and for $s \ge 0$, $|\xi|$ large, then

$$u \in \mathscr{C}((0,T);W^{1,p}_0(\Omega)), \quad \partial_t u \in L^2((\eta,T)\times\Omega), \ \forall \eta \in (0,T).$$

Remark 2.1 *A prototype example of f satisfying the above conditions* (I.3) *and* (I.7) *is* $(x,s,\xi) \mapsto b(x)s^{q_1} + c(x) + \dfrac{\mathcal{A}(x).|\xi|^{p-2}\xi}{1+|\xi|^{q_2}}$ *with* $2q_1 \le r$, $p - q_2 - 1 \le \dfrac{p}{2}$ *and* $b, c \in L^\infty(\Omega)$ *and* $\mathcal{A} \in (L^\infty(\Omega))^N$.

3 Logarithmic Sobolev inequalities

The logarithmic Sobolev inequalities are used in order to establish an "L^p-to-L^q" smoothing effect $(1 < p < q < \infty)$ of an analytic semigroup of bounded linear operators on $L^2(\Omega)$, modelling a diffusion

process in an arbitrary domain $\Omega \in \mathbb{R}^N$. This idea is due to L.Cross [75, p. 1066], the proof of theorem 1. In the paper of E.B. Davies and B. Simon [46], this idea was developed further : an equivalence relation was established between the "L^2-to-L^∞" smoothing effect of semigroup e^{-Ht} $(t \geq 0)$ on $L^2(\Omega)$, termed ultracontractivity, and the corresponding logarithmic Sobolev inequality for the (infinitesimal) generator $-H$ of this semigroup. To describe this phenomena in details below, we use the monograph by E. B. Davies [45, Chap. 2, pp. 59-81].
Let e^{-Ht} $(t \geq 0)$ be symmetric Markov C^0-semigroup on $L^2(\Omega)$ with the generator $-H$. The reader is referred to E. B. Davies [45, Chap. 1, pp. 21-25] for the definition and basic properties of a symmetric Markov semigroup. In particular, H is a positive definite, selfadjoint operator on $L^2(\Omega)$. Assume that the semigroup e^{-Ht} $(t \geq 0)$ is ultracontractive with

$$\|e^{-Ht}\|_{L^2 \to L^\infty} \leq e^{M(t)} \quad \text{for all } t > 0, \tag{I.8}$$

where $M : (0, \infty) \to \mathbb{R}$ is monotonically decreasing continuous function. Here, $\|e^{-Ht}\|_{L^2 \to L^\infty}$ denotes the norm of the bounded linear operator e^{-Ht} from $L^2(\Omega)$ to $L^\infty(\Omega)$. Let Q denote the quadratic form associated with the operator H,

$$Q(f) \stackrel{\text{def}}{=} \int_\Omega (Hf)\overline{f} \mathrm{d}x, \quad f \in \mathrm{dom}(H),$$

where $\mathrm{dom}(H)$ stands for the domain of H. The domain of Q is the Friedrichs energy space $\mathrm{Quad}(H)$. Then $0 \leq f \in \mathrm{Quad}(H) \cap L^1(\Omega) \cap L^\infty(\Omega)$ implies that $f^2 \log f \in L^1(\Omega)$ and the logarithmic Sobolev inequality

$$\int_\Omega f^2 \log f \mathrm{d}x \leq \eta Q(f) + M(\eta)\|f\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)} \log \|f\|_{L^2(\Omega)} \tag{I.9}$$

is valid for all $\eta > 0$. This result is taken from [45, Theorem 2.2.3, p. 64].

For each $t > 0$, e^{-Ht} is an integral operator on $L^2(\Omega)$,

$$(e^{-Ht}f)(x) = \int_\Omega K(x, y; t)f(y)\mathrm{d}x, \quad x \in \Omega,\ f \in L^2(\Omega), \tag{I.10}$$

with a non-negative kernel $K : \Omega \times \Omega \times (0, \infty) \to \mathbb{R}$, by [45, Lemma 2.1.2, p. 59]. Moreover, if $\Omega \subset \mathbb{R}^N$ has N-finite dimensional Lebesgue measure $|\Omega|_N$, then the kernel has the representation

$$0 \leq K(x, y; t) = \sum_{n=1}^{\infty} \exp(E_n t)\phi_n(x)\phi_n(y) \tag{I.11}$$

where the infinite series converges uniformly on $\Omega \times \Omega \times [\alpha, \infty)$ for any $\alpha > 0$; see [45, Theorem 2.1.4, p. 60]. Here, $E_1,\ E_2, \cdots,\ E_n, \cdots$ are the eigenvalues of H, repeated according to their multiplicity, with the associated eigenfunction ϕ_n $(n = 1, 2, \cdots)$.

For the Dirichlet Laplacian $H = -\Delta$ in $L^2(\Omega)$, we have

$$0 \leq K(x, y; t) \leq (4\pi t)^{-N/2} e^{-(x-y)^2/4t} \quad \text{for } x, y \in \Omega \text{ and } t > 0, \tag{I.12}$$

see E. B. Davies [45, Example 2.1.9, pp. 63]. consequently,

$$\|e^{-Ht}\|_{L^1\to L^\infty} \le (4\pi t)^{-N/2} \tag{I.13}$$

which yields, by [45, Lemma 2.2, p. 59],

$$\|e^{-Ht}\|_{L^2\to L^\infty} \le (8\pi t)^{-N/4} \quad \text{for all } t > 0. \tag{I.14}$$

Hence, inequality (I.8) holds with $M(t) = -(N/4)\log(8\pi t)$ and whenever $0 \le f \in W_0^{1,p}(\Omega)\cap L^1(\Omega)\cap L^\infty(\Omega)$, then also $f^2\log f \in L^1(\Omega)$ and the logarithmic Sobolev inequality (I.9) becomes

$$\int_\Omega f^2 \log f \mathrm{d}x \le \eta \int_\Omega |\nabla f|^2 \mathrm{d}x - \frac{N}{4}\log(8\pi\eta)\|f\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)}\log\|f\|_{L^2(\Omega)} \tag{I.15}$$

for all $\eta > 0$. It is an easy exercise to show that if inequality (I.15) holds for $\eta = 1$ then it holds for all $\eta > 0$ as well. Indeed, one may use the substitution $x = \eta^{1/2}y$ in $\mathbb{R}^N$ as the constants in this inequality are independent from the domain $\Omega \subset \mathbb{R}^N$.

Finally, given any $2 \le r < \infty$, from inequality (I.9) one can derive

$$\begin{aligned}\int_\Omega g^r \log g \mathrm{d}x \quad &\le \quad \eta\int_\Omega (Hg)g^{r-2}\mathrm{d}x + \frac{2}{r}M(\frac{2\eta}{r'})\|g\|^r_{L^r(\Omega)} \\ &\quad + \|g\|^r_{L^r(\Omega)}\log\|g\|_{L^r(\Omega)}.\end{aligned} \tag{I.16}$$

for every $g \in \mathcal{D}_+ \stackrel{\text{def}}{=} \bigcup_{t>0} e^{Ht}(L^1(\Omega)\cap L^\infty(\Omega))_+$; see the proof of lemma 2.2.6 in [45, p. 67]. As usual, $r' = \dfrac{r}{r-1}$ and $X_+ = \{f \in X : f \ge 0\}$ denotes the positive cone in an ordered Banach space X.

For the Dirichlet Laplacian $H = -\Delta$ in $L^2(\Omega)$, inequality (I.16) becomes

$$\begin{aligned}\int_\Omega g^r \log g \mathrm{d}x \quad &\le \quad \eta\int_\Omega |\nabla g|^2 g^{r-2}\mathrm{d}x - \frac{N}{2r}\log\left(\frac{16\pi\eta}{r}\right)\|g\|^r_{L^r(\Omega)} \\ &\quad + \|g\|^r_{L^r(\Omega)}\log\|g\|_{L^r(\Omega)}.\end{aligned} \tag{I.17}$$

for all $2 \le r < \infty$ and for every $g \in \mathcal{D}_+ = \bigcup_{t>0} e^{\Delta t}(L^1(\Omega)\cap L^\infty(\Omega))_+$. This inequality follows also directly from (I.15) by setting $f = g^{r/2}$ and replacing η by $2\eta/r$.

4 Approximating problems for (P)

Let $\varepsilon > 0$ and denote

$$f_\varepsilon(x,u,\nabla u) \stackrel{\text{def}}{=} \min\left\{\frac{1}{\varepsilon}, f(x,u,\nabla u)\right\}.$$

We assume that f is a Carathéodory function bounded below. Hence, f_ϵ is a Carathéodory function, as well, bounded below and above (by $\frac{1}{\varepsilon}$). These properties of f and f_ε will be used throughout this section without further notice.

In order to prove our results, we will work by approximation, regularizing the singular term $u^{-\delta}$ and studying the behavior of a sequence u_ε of solutions of the approximated problems (P_ε) below. We consider the following problem :

$$(\mathrm{P}_\varepsilon) \qquad \begin{cases} \partial_t u_\varepsilon - \Delta_p u_\varepsilon = (u_\varepsilon + \varepsilon)^{-\delta} + f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) & \text{in } Q_T = (0,T)\times\Omega, \\ u_\varepsilon = 0 \ \text{ on } (0,T)\times\partial\Omega, \ u_\varepsilon > 0 \text{ in } Q_T, \\ u_\varepsilon(0,.) := u_{\varepsilon,0} = \min\{\frac{1}{\varepsilon}, u_0\} \geq 0 \text{ in } \Omega. \end{cases}$$

4.1 Existence of positive solution u_ε

We prove in this subsection the following :

Theorem 4.1 *Let f be a bounded below Carathéodory function and u_0 satisfies* (I.1). *Then Problem* (P_ε) *admits at least one global positive weak solution u_ε such that*

$$u_\varepsilon \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q_T) \cap \mathscr{C}([0,T]; L^s(\Omega)), \ \forall T > 0, \ \forall s \in [1, \infty). \tag{I.18}$$

In order to prove this theorem, we use a semi-discretization in time approach (see J.L. Lions [87]).

4.1.1 Semi-discretization in time

Let $N >> 1$ and $\Delta t = \frac{T}{N}$, $t_n = n\Delta t$, $n = 0, ..., N$ and we consider the following alternative quasilinear elliptic problem :

$$(\mathrm{P_n}) \qquad \begin{cases} \frac{u_\varepsilon^n - u_\varepsilon^{n-1}}{\Delta t} - \Delta_p u_\varepsilon^n = (u_\varepsilon^n + \varepsilon)^{-\delta} + f_\varepsilon(x, u_\varepsilon^{n-1}, \nabla u_\varepsilon^{n-1}) & \text{in } \Omega, \\ u_\varepsilon^n = 0 \ \text{ on } \partial\Omega, \ u_\varepsilon^n > 0 \text{ on } \Omega, \quad for\ n \geq 2 \end{cases}$$

with $u_\varepsilon^0 \overset{\text{def}}{=} u_{\varepsilon,0} \in L^\infty(\Omega)$ and u_ε^1 satisfies the following problem :

$$(\mathrm{P_1}) \qquad \begin{cases} \frac{u_\varepsilon^1 - u_\varepsilon^0}{\Delta t} - \Delta_p u_\varepsilon^1 = (u_\varepsilon^1 + \varepsilon)^{-\delta} + f_\varepsilon(x, u_\varepsilon^0, 0) & \text{in } \Omega, \\ u_\varepsilon^1 = 0 \ \text{ on } \partial\Omega, \ u_\varepsilon^1 > 0 \text{ on } \Omega. \end{cases}$$

It is well known that $(\mathrm{P_n})$ has a positive solution $u_\varepsilon^n \in W_0^{1,p}(\Omega)$ for any $n \geq 1$. Hence we define functions $u_{\varepsilon,\Delta t}$, $\tilde{u}_{\varepsilon,\Delta t}$ for $n = 1, ..., N$ and $t \in (t_{n-1}, t_n]$, by

$$u_{\varepsilon,\Delta t}(t) = u_\varepsilon^n, \quad \tilde{u}_{\varepsilon,\Delta t}(t) = \frac{t - t_{n-1}}{\Delta t}(u_\varepsilon^n - u_\varepsilon^{n-1}) + u_\varepsilon^{n-1}, \text{ and } \tilde{u}_{\varepsilon,\Delta t}(0) := u_\varepsilon^0 = u_{\varepsilon,0},$$

so $\tilde{u}_{\varepsilon,\Delta t}$ and $u_{\varepsilon,\Delta t}$ satisfy

$$\partial_t \tilde{u}_{\varepsilon,\Delta t} - \Delta_p u_{\varepsilon,\Delta t} = \left(u_{\varepsilon,\Delta t} + \varepsilon\right)^{-\delta} + f_\varepsilon(x, u_{\varepsilon,\Delta t}(\cdot - \Delta t), \nabla u_{\varepsilon,\Delta t}(\cdot - \Delta t)) \quad \text{in } (\eta, T) \tag{I.19}$$

for any $\eta > 0$ such that $\Delta t < \eta$.

4.1.2 Energy estimates for $u_{\varepsilon,\Delta t}$ and $\tilde{u}_{\varepsilon,\Delta t}$

First, we derive the energy estimates to get a priori bounds of $u_{\varepsilon,\Delta t}$ and $\tilde{u}_{\varepsilon,\Delta t}$ independently of Δt. Multiplying by $\Delta t u_\varepsilon^n$, integrating on Ω and summing from $n = 1$ to m where $m \in \{1, ..., N\}$, we obtain

$$\sum_{n=1}^{m}\int_\Omega (u_\varepsilon^n-u_\varepsilon^{n-1})u_\varepsilon^n\mathrm{d}x+\Delta t\sum_{n=1}^{m}\|u_\varepsilon^n\|^p_{W_0^{1,p}(\Omega)}=\Delta t\sum_{n=1}^{m}\int_\Omega (u_\varepsilon^n+\varepsilon)^{-\delta}u_\varepsilon^n\mathrm{d}x$$
$$+\Delta t\sum_{n=2}^{m}\int_\Omega f_\varepsilon(x,u_\varepsilon^{n-1},\nabla u_\varepsilon^{n-1})u_\varepsilon^n\mathrm{d}x+\Delta t\int_\Omega f_\varepsilon(x,u_\varepsilon^0,0)u_\varepsilon^1\mathrm{d}x. \tag{I.20}$$

Next, we have

$$\begin{aligned}\sum_{n=1}^{m}\int_\Omega (u_\varepsilon^n-u_\varepsilon^{n-1})u_\varepsilon^n\mathrm{d}x &= \sum_{n=1}^{m}\int_\Omega \frac{1}{2}\left[(u_\varepsilon^n-u_\varepsilon^{n-1})^2+(u_\varepsilon^n)^2-(u_\varepsilon^{n-1})^2\right]\mathrm{d}x\\ &= \frac{1}{2}\sum_{n=1}^{m}\int_\Omega (u_\varepsilon^n-u_\varepsilon^{n-1})^2\mathrm{d}x+\frac{1}{2}\int_\Omega (u_\varepsilon^m)^2\mathrm{d}x-\frac{1}{2}\int_\Omega (u_\varepsilon^0)^2\mathrm{d}x\end{aligned}$$

and

$$\int_\Omega (u_\varepsilon^n+\varepsilon)^{-\delta}u_\varepsilon^n\mathrm{d}x\le C\int_\Omega (u_\varepsilon^n)^2\mathrm{d}x+C(\varepsilon)$$

where $C(\varepsilon)$ is a positive constant depending only on ε. From the boundedness of f_ε and by the Young inequality we have

$$\sum_{n=2}^{m}\int_\Omega f_\varepsilon(x,u_\varepsilon^{n-1},\nabla u_\varepsilon^{n-1})u_\varepsilon^n\mathrm{d}x+\int_\Omega f_\varepsilon(x,u_\varepsilon^0,0)u_\varepsilon^1\mathrm{d}x\le C+\sum_{n=1}^{m}\int_\Omega (u_\varepsilon^n)^2\mathrm{d}x.$$

Hence, we obtain

$$\begin{aligned}\frac{1}{2}\sum_{n=1}^{m}\int_\Omega (u_\varepsilon^n-u_\varepsilon^{n-1})^2\mathrm{d}x+\frac{1}{2}\int_\Omega (u_\varepsilon^m)^2\mathrm{d}x \quad &+ \quad \Delta t\sum_{n=1}^{m}\|u_\varepsilon^n\|^p_{W_0^{1,p}(\Omega)}\le\\ & 2\Delta t\sum_{n=1}^{m}\int_\Omega (u_\varepsilon^n)^2\mathrm{d}x+\tilde{C}(\varepsilon).\end{aligned}$$

Using the discrete Gronwall lemma (see [107] p. 43 for instance), it follows that $u_{\varepsilon,\Delta t}$, $\tilde{u}_{\varepsilon,\Delta t}$ are bounded in $L^\infty(0,T;L^2(\Omega))$ independently of Δt. Then,

$$\{u_{\varepsilon,\Delta t}\}\text{ is bounded in }L^p(0,T;W_0^{1,p}(\Omega)) \tag{I.21}$$

and

$$\{\tilde{u}_{\varepsilon,\Delta t}\}\text{ is bounded in }L^p(\eta,T;W_0^{1,p}(\Omega))\quad\forall\eta\in(0,T) \tag{I.22}$$

independently of Δt. Also $\{u_{\varepsilon,\Delta t}\},\{\tilde{u}_{\varepsilon,\Delta t}\}$ are bounded in $L^\infty(Q_T)$. Indeed, we consider the function v_ε solution to the following equation

$$\begin{cases} v'(t)=\varepsilon^{-\delta}+\frac{1}{\varepsilon}\\ v(0)=\frac{1}{\varepsilon}.\end{cases}$$

Let us denote $v_\varepsilon^n = v_\varepsilon(t_n)$ for $n \in \{0, ..., N\}$. Then we have

$$\begin{cases} \frac{v_\varepsilon^n - v_\varepsilon^{n-1}}{\Delta t} = \varepsilon^{-\delta} + \frac{1}{\varepsilon}, \\ v_\varepsilon^n > 0 \quad \text{on } \partial\Omega. \end{cases}$$

Hence, by the weak comparison principle for elliptic problems with the p-Laplacian (see TOLKSDORF [116], §3.3), v_ε^n is a supersolution to the discrete scheme ($\mathrm{P_n}$) and by the weak comparison principle one can show that for $n = 0, ..., N$: $u_\varepsilon^n \leq v_\varepsilon^n \leq C(T) < \infty$ independently of n. Now, let us derive the second energy estimate : Multiplying the equation in ($\mathrm{P_n}$) by $\frac{t_n + t_{n-1}}{2}(u_\varepsilon^n - u_\varepsilon^{n-1})$, integrating on Ω and summing for $n = 2$ to $m \in \{2, ..., N\}$, we have :

$$\begin{aligned} &\frac{\Delta t}{2}\sum_{n=2}^{m}(t_n + t_{n-1})\int_\Omega (\frac{u_\varepsilon^n - u_\varepsilon^{n-1}}{\Delta t})^2 \mathrm{d}x - \frac{1}{2}\sum_{n=2}^{m}(t_n + t_{n-1}) < \Delta_p u_\varepsilon^n, (u_\varepsilon^n - u_\varepsilon^{n-1}) > \\ &\quad = \frac{1}{2}\sum_{n=2}^{m}(t_n + t_{n-1})\int_\Omega (u_\varepsilon^n + \varepsilon)^{-\delta}(u_\varepsilon^n - u_\varepsilon^{n-1})\mathrm{d}x \\ &\quad\quad + \frac{1}{2}\sum_{n=2}^{m}(t_n + t_{n-1})\int_\Omega f_\varepsilon(x, u_\varepsilon^{n-1}, \nabla u_\varepsilon^{n-1})(u_\varepsilon^n - u_\varepsilon^{n-1})\mathrm{d}x. \end{aligned} \tag{I.23}$$

By convexity argument, we obtain :

$$\begin{aligned} - \quad & \frac{1}{2}\sum_{n=2}^{m}(t_n + t_{n-1}) < \Delta_p u_\varepsilon^n, (u_\varepsilon^n - u_\varepsilon^{n-1}) > \\ = \quad & \frac{1}{2}\sum_{n=2}^{m}(t_n + t_{n-1})\int_\Omega |\nabla u_\varepsilon^n|^{p-2}\nabla u_\varepsilon^n.(\nabla u_\varepsilon^n - \nabla u_\varepsilon^{n-1}) \\ \geq \quad & \sum_{n=2}^{m}\frac{1}{2p}(t_n + t_{n-1})\int_\Omega (|\nabla u_\varepsilon^n|^p - |\nabla u_\varepsilon^{n-1}|^p)\mathrm{d}x \\ = \quad & \frac{t_m}{p}\int_\Omega |\nabla u_\varepsilon^m|^p \mathrm{d}x - \frac{\Delta t}{p}\int_\Omega |\nabla u_\varepsilon^1|^p \mathrm{d}x - \frac{\Delta t}{2p}\sum_{n=2}^{m}\int_\Omega (|\nabla u_\varepsilon^n|^p + |\nabla u_\varepsilon^{n-1}|^p)\mathrm{d}x \\ \geq \quad & \frac{t_m}{p}\int_\Omega |\nabla u_\varepsilon^m|^p \mathrm{d}x - \frac{2}{p}\|\nabla u_{\varepsilon,\Delta t}\|^p_{L^p(0,t_m;W_0^{1,p}(\Omega))}. \end{aligned}$$

We estimate now the other terms in the right hand-side of (I.23). Again, using convexity arguments, we estimate $-\frac{1}{1-\delta}\int_\Omega (u_\varepsilon^n + \varepsilon)^{1-\delta}\, dx$ as follows (we do it only for $\delta \neq 1$) :

$$\frac{1}{1-\delta}\left[\int_\Omega \left(u_\varepsilon^{n-1} + \varepsilon\right)^{1-\delta}\mathrm{d}x - \int_\Omega (u_\varepsilon^n + \varepsilon)^{1-\delta}\,\mathrm{d}x\right] \leq -\int_\Omega \frac{u_\varepsilon^n - u_\varepsilon^{n-1}}{(u_\varepsilon^n + \varepsilon)^\delta}\mathrm{d}x.$$

Hence for $\delta \neq 1$ (if $\delta = 1$, we replace the terms of the form $(1-\delta)^{-1}x^{1-\delta}$ in the below expressions by term $\log(x)$),

$$
\begin{aligned}
&\frac{1}{2}\sum_{n=2}^{m}(t_n+t_{n-1})\int_\Omega (u_\varepsilon^n+\varepsilon)^{-\delta}(u_\varepsilon^n-u_\varepsilon^{n-1})\mathrm{d}x \\
&\leq \frac{1}{2(1-\delta)}\sum_{n=2}^{m}(t_n+t_{n-1})\left[\int_\Omega (u_\varepsilon^n+\varepsilon)^{1-\delta}\,\mathrm{d}x-\int_\Omega \left(u_\varepsilon^{n-1}+\varepsilon\right)^{1-\delta}\mathrm{d}x\right] \\
&= \frac{1}{2(1-\delta)}\left[t_m\int_\Omega (u_\varepsilon^m+\varepsilon)^{1-\delta}\mathrm{d}x-\Delta t\int_\Omega (u^1+\varepsilon)^{1-\delta}\mathrm{d}x\right] \\
&-\frac{\Delta t}{1-\delta}\sum_{n=2}^{m}\int_\Omega \left[(u_\varepsilon^n+\varepsilon)^{1-\delta}+(u_\varepsilon^{n-1}+\varepsilon)^{1-\delta}\right]\mathrm{d}x \\
&\leq \frac{t_m}{1-\delta}\int_\Omega (u_\varepsilon^m+\varepsilon)^{1-\delta}\mathrm{d}x+C\int_0^{t_m}\int_\Omega (u_{\varepsilon,\Delta t}+\varepsilon)^{1-\delta}\mathrm{d}x
\end{aligned}
$$

and by Young's inequality

$$
\begin{aligned}
&\frac{1}{2}\sum_{n=2}^{m}(t_n+t_{n-1})\int_\Omega f_\varepsilon(x,u_\varepsilon^{n-1},\nabla u_\varepsilon^{n-1})(u_\varepsilon^n-u_\varepsilon^{n-1})\mathrm{d}x \\
&\leq \Delta t\sum_{n=2}^{m}(t_n+t_{n-1})\left(\int_\Omega \left[f_\varepsilon(x,u_\varepsilon^{n-1},\nabla u_\varepsilon^{n-1})\right]^2\mathrm{d}x+\frac{1}{4}\int_\Omega\left(\frac{u_\varepsilon^n-u_\varepsilon^{n-1}}{\Delta t}\right)^2\mathrm{d}x\right) \\
&\leq 2T\int_0^{t_m}\int_\Omega \left[f_\varepsilon(x,u_{\varepsilon,\Delta t},\nabla u_{\varepsilon,\Delta t})\right]^2\mathrm{d}x\mathrm{d}s+\frac{\Delta t}{4}\sum_{n=2}^{m}(t_n+t_{n-1})\int_\Omega\left(\frac{u_\varepsilon^n-u_\varepsilon^{n-1}}{\Delta t}\right)^2\mathrm{d}x.
\end{aligned}
$$

Hence, we substitute these bounds in (I.23), we obtain for any $m \in \{2,...,N\}$

$$
\begin{aligned}
&\frac{1}{2}\|t^{\frac{1}{2}}\partial_t\tilde{u}_{\varepsilon,\Delta t}\|^2_{L^2(0,t_m;L^2(\Omega))}+\frac{t_m}{p}\int_\Omega|\nabla u_\varepsilon^m|^p\mathrm{d}x\leq\frac{1}{2}\int_\Omega(u_\varepsilon^1-u_\varepsilon^0)^2\mathrm{d}x \\
&+\frac{t_m}{1-\delta}\int_\Omega(u_\varepsilon^m+\varepsilon)^{1-\delta}\mathrm{d}x+C\int_0^{t_m}\int_\Omega(u_{\varepsilon,\Delta t}+\varepsilon)^{1-\delta}\mathrm{d}x \\
&+2T\int_0^{t_m}\int_\Omega\left[f_\varepsilon(x,u_{\varepsilon,\Delta t},\nabla u_{\varepsilon,\Delta t})\right]^2\mathrm{d}x\mathrm{d}s+\frac{2}{p}\|\nabla u_{\varepsilon,\Delta t}\|^p_{L^p(0,t_m;W_0^{1,p}(\Omega))}.
\end{aligned}
\tag{I.24}
$$

So, by using the first energy estimate (I.21), and taking into account that f_ε is bounded, we deduce that

$$
\sup_{t\in(0,T)} t\int_\Omega|\nabla u_{\varepsilon,\Delta t}|^p\mathrm{d}x\leq\max_{m\in\{1,\ldots,N\}}t_m\int_\Omega|\nabla u_\varepsilon^m|^p\mathrm{d}x\leq C_1
$$

and

$$
\|t^{\frac{1}{2}}\partial_t\tilde{u}_{\Delta t}\|_{L^2(0,T;L^2(\Omega))}=\max_{m\in\{2,\ldots,N\}}\|t^{\frac{1}{2}}\partial_t\tilde{u}_{\Delta t}\|_{L^2(0,t_m;L^2(\Omega))}\leq C_2
$$

where $C_1, C_2 > 0$ are independent of Δt. Hence, we have

$$
\left\{\begin{array}{lll}
u_{\varepsilon,\Delta t} & \text{is bounded in} & L^\infty(\eta,T;W_0^{1,p}(\Omega))\ \forall\eta\in(0,T), \\
t^{\frac{1}{2}}\partial_t\tilde{u}_{\varepsilon,\Delta t} & \text{is bounded in} & L^2(Q_T)
\end{array}\right.
\tag{I.25}
$$

uniformly in Δt. From (I.19), we easily get that

$$\partial_t \tilde{u}_{\varepsilon,\Delta t} \text{ is bounded in } L^{p'}(0,T;W^{-1,p'}(\Omega)). \tag{I.26}$$

4.1.3 Passage to the limit and proof of theorem 4.1

From the above estimates and up to a subsequence, we have, as $\Delta t \to 0$

$$u_{\varepsilon,\Delta t} \rightharpoonup u_\varepsilon \ in\ L^p(0,T;W_0^{1,p}(\Omega)), \tag{I.27}$$

$$\tilde{u}_{\varepsilon,\Delta t} \overset{*}{\rightharpoonup} \tilde{u}_\varepsilon \ in\ L^\infty(Q_T)\cap L^\infty(\eta,T;W_0^{1,p}(\Omega)),\ \forall \eta\in(0,T), \tag{I.28}$$

$$\partial_t\tilde{u}_{\varepsilon,\Delta t} \rightharpoonup \partial_t\tilde{u}_\varepsilon \ in\ L^2((\eta,T)\times\Omega)) \ \ \forall\eta\in(0,T), \tag{I.29}$$

and

$$\partial_t\tilde{u}_{\varepsilon,\Delta t} \rightharpoonup \partial_t\tilde{u}_\varepsilon \ in\ L^{p'}(0,T;W^{-1,p'}(\Omega)). \tag{I.30}$$

Therefore, for any $\eta\in(0,T)$ and for $N >> 1$ (large enough), there exists a unique $N'\le N$ such that $\eta\in(t_{N'},t_{N'+1}]$. Then,

$$\begin{aligned}\|u_{\varepsilon,\Delta t}-\tilde{u}_{\varepsilon,\Delta t}\|_{L^\infty(\eta,T;L^2(\Omega))} &\le 2\max_{n\in\{N',\dots,N\}}\|u_\varepsilon^n-u_\varepsilon^{n-1}\|_{L^2(\Omega)}\\ &\le 2\Delta t\sum_{n=N'}^{N}\|\frac{u_\varepsilon^n-u_\varepsilon^{n-1}}{\Delta t}\|_{L^2(\Omega)} = O((\Delta t)^{\frac12}).\end{aligned} \tag{I.31}$$

Hence we deduce that $u_\varepsilon=\tilde{u}_\varepsilon$ in Q_T. We also obtain by the first energy estimate (I.21) and (I.26) that $u_\varepsilon\in W^{1,p'}(0,T;W^{-1,p'}(\Omega))\cap L^p(0,T;W_0^{1,p}(\Omega))$. Thus, $u_\varepsilon\in\mathscr{C}([0,T];L^2(\Omega))$ (see V. Barbu [18], Lemma 4.1, p. 167). Since $u_\varepsilon\in L^\infty(Q_T)$, we deduce that

$$u_\varepsilon\in\mathscr{C}([0,T];L^s(\Omega)),\quad \forall s\in[2,\infty). \tag{I.32}$$

To obtain that, it suffices to apply the following interpolation inequality (see Brézis [26])

$$\|v\|_{L^s(\Omega)}\le\|v\|^\alpha_{L^\infty(\Omega)}\|v\|^{1-\alpha}_{L^2(\Omega)},\ \ \frac1s=\frac{\alpha}{\infty}+\frac{1-\alpha}{2},\quad v\in L^\infty(\Omega); \tag{I.33}$$

hence, $s=\frac{2}{1-\alpha}$. We apply the compactness result of Aubin-Simon (see [110]) to infer that

$$u_{\varepsilon,\Delta t}\underset{\Delta t\to0}{\longrightarrow}u_\varepsilon \quad \text{in } L^2(Q_T). \tag{I.34}$$

Therefore, by (I.31), we obtain also

$$\tilde{u}_{\varepsilon,\Delta t}\underset{\Delta t\to0}{\longrightarrow}u_\varepsilon \quad \text{in } L^2((\eta,T)\times\Omega). \tag{I.35}$$

It remains to show that u satisfies equation (P_ε) in the sense of distributions and $u(0) = u_0$. We multiply (I.19) by $u_{\varepsilon,\Delta t} - u_\varepsilon$ and integrate on $\tilde{Q}_T = (\eta, T) \times \Omega$. Then, by convexity arguments, we obtain

$$\begin{aligned}\int_{\tilde{Q}_T} \partial_t \tilde{u}_{\varepsilon,\Delta t}(u_{\varepsilon,\Delta t} - u_\varepsilon)\mathrm{d}x\mathrm{d}t \quad &+ \quad \frac{1}{p}\left(\int_{\tilde{Q}_T} |\nabla u_{\varepsilon,\Delta t}|^p \mathrm{d}x\mathrm{d}t - \int_{\tilde{Q}_T} |\nabla u_\varepsilon|^p \,\mathrm{d}x\mathrm{d}t\right) \\ &\leq \quad \int_{\tilde{Q}_T} (u_{\varepsilon,\Delta t} + \varepsilon)^{-\delta}(u_{\varepsilon,\Delta t} - u_\varepsilon)\mathrm{d}x\mathrm{d}t \\ &+ \quad \int_{\tilde{Q}_T} (u_{\varepsilon,\Delta t} - u_\varepsilon) f(x, u_{\varepsilon,\Delta t}(\cdot - \Delta t), \nabla u_{\varepsilon,\Delta t}(\cdot - \Delta t))\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Then, we get that

$$\overline{\lim_{\Delta t \to 0}} \int_{\tilde{Q}_T} |\nabla u_{\varepsilon,\Delta t}|^p \mathrm{d}x\mathrm{d}t - \int_{\tilde{Q}_T} |\nabla u_\varepsilon|^p \,\mathrm{d}x\mathrm{d}t \leq 0.$$

In the other hand, by (I.27) we have

$$\int_{\tilde{Q}_T} |\nabla u_\varepsilon|^p \mathrm{d}x\mathrm{d}t \leq \underline{\lim}_{\Delta t \to 0} \int_{\tilde{Q}_T} |\nabla u_{\varepsilon,\Delta t}|^p \,\mathrm{d}x\mathrm{d}t.$$

Thus we infer that $\lim_{\Delta t \to 0} \int_{\tilde{Q}_T} |\nabla u_{\varepsilon,\Delta t}|^p \mathrm{d}x\mathrm{d}t = \int_{\tilde{Q}_T} |\nabla u_\varepsilon|^p \,\mathrm{d}x\mathrm{d}t$. By the uniform convexity of the space $L^p(\eta, T; W_0^{1,p}(\Omega))$, we deduce that

$$u_{\varepsilon,\Delta t} \longrightarrow u_\varepsilon \quad in \quad L^p(\eta, T; W_0^{1,p}(\Omega)) \text{ as } \Delta t \to 0, \quad \forall \eta \in (0, T). \tag{I.36}$$

Therefore,

$$\nabla u_{\varepsilon,\Delta t} \longrightarrow \nabla u_\varepsilon \ \text{ in } \ L^p((\eta, T) \times \Omega) \text{ as } \Delta t \to 0. \tag{I.37}$$

Consequently, there is a sequence of time increments $(\Delta t)_n \to 0$ as $n \to \infty$, such that

$$\nabla u_{\varepsilon,(\Delta t)_n} \longrightarrow \nabla u_\varepsilon \text{ a.e. in } Q_T \text{ as } n \to \infty. \tag{I.38}$$

Then, from the compactness results above, it is easy to show that u satisfies (P_ε) in the sense of distributions. We now prove that $u_\varepsilon(0) = u_{\varepsilon,0}$. Since $u_\varepsilon \in \mathscr{C}([0, T], L^2(\Omega))$, it suffices to prove that $u_\varepsilon(t) \rightharpoonup u_{\varepsilon,0}$ in $L^2(\Omega)$ as $t \to 0$. Multiplying (I.19) by $\varphi \in \mathscr{D}(\Omega)$ and integrating on $(t_1, t_2) \times \Omega$ with $t_2 > t_1 > 0$, we get

$$\begin{aligned}&\int_\Omega \tilde{u}_{\varepsilon,\Delta t}(t_2)\varphi \mathrm{d}x - \int_\Omega \tilde{u}_{\varepsilon,\Delta t}(t_1)\varphi \mathrm{d}x + \int_{t_1}^{t_2} \int_\Omega |\nabla u_{\varepsilon,\Delta t}|^{p-2} \nabla u_{\varepsilon,\Delta t}.\nabla \varphi \mathrm{d}x\mathrm{d}s \\ &= \int_{t_1}^{t_2} \int_\Omega (u_{\varepsilon,\Delta t} + \varepsilon)^{-\delta} \varphi \mathrm{d}x\mathrm{d}t + \int_{t_1}^{t_2} \int_\Omega f_\varepsilon(x, u_{\varepsilon,\Delta t}(\cdot - \Delta t), \nabla u_{\varepsilon,\Delta t}(\cdot - \Delta t))\varphi \mathrm{d}x\mathrm{d}t.\end{aligned}$$

Using the continuity of $\tilde{u}_{\varepsilon,\Delta t}$ and passing to the limit as $t_1 \to 0$, we obtain :

$$\begin{aligned}&\int_\Omega \tilde{u}_{\varepsilon,\Delta t}(t_2)\varphi \mathrm{d}x - \int_\Omega u_{\varepsilon,0}\varphi \mathrm{d}x + \int_0^{t_2} \int_\Omega |\nabla u_{\varepsilon,\Delta t}|^{p-2} \nabla u_{\varepsilon,\Delta t}.\nabla \varphi \mathrm{d}x\mathrm{d}s \\ &= \int_0^{t_2} \int_\Omega (u_{\varepsilon,\Delta t} + \varepsilon)^{-\delta} \varphi \mathrm{d}x\mathrm{d}t + \int_0^{t_2} \int_\Omega f_\varepsilon(x, u_{\varepsilon,\Delta t}(\cdot - \Delta t), \nabla u_{\varepsilon,\Delta t}(\cdot - \Delta t))\varphi \mathrm{d}x\mathrm{d}t\end{aligned}$$

(after the extension $f_\varepsilon(x, u_{\varepsilon,\Delta t}(\cdot - \Delta t), \nabla u_{\varepsilon,\Delta t}(\cdot - \Delta t)) = f_\varepsilon(x, u_{\varepsilon,0}, 0)$ on $(0, \Delta t)$). From the energy estimates, the Lebesgue theorem and passing to the limit as $\Delta t \to 0$ and $t_2 \to 0$, we obtain that

$$\lim_{t\to 0} \int u_\varepsilon(t)\varphi \mathrm{d}x = \int_\Omega u_{\varepsilon,0}\varphi \mathrm{d}x.$$

Hence, by (I.32) we infer that $u_\varepsilon(0) = u_{\varepsilon,0}$.

4.2 Uniqueness of the positive solution u_ε

We now show the uniqueness of the solution to (P_ε).

Theorem 4.2 *Under the same hypotheses of Theorem* (4.1)*, if in addition, f is locally Lipschitz with respect to s uniformly in $x \in \Omega$ and $\xi \in \mathbb{R}^N$ and satisfying* (I.3)*, then u is unique and belongs to $\mathscr{C}\left((0,T); W_0^{1,p}(\Omega)\right)$ such that for all $t_0, t : 0 < t_0 \leq t \leq T$ we have (for $\delta \neq 1$)*

$$\begin{aligned}
&\int_{t_0}^{t}\int_\Omega (\partial_s u_\varepsilon)^2 \,\mathrm{d}x\mathrm{d}s + \frac{1}{p}\int_\Omega |\nabla u_\varepsilon(t)|^p \mathrm{d}x - \frac{1}{1-\delta}\int_\Omega (u_\varepsilon(t)+\varepsilon)^{1-\delta}\,\mathrm{d}x \\
&= \frac{1}{p}\int_\Omega |\nabla u_\varepsilon(t_0)|^p \mathrm{d}x - \frac{1}{1-\delta}\int_\Omega (u_\varepsilon(t_0)+\varepsilon)^{1-\delta}\,\mathrm{d}x \\
&+ \int_{t_0}^{t}\int_\Omega f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\,\partial_s u_\varepsilon \mathrm{d}x\mathrm{d}s,
\end{aligned} \tag{I.39}$$

(if $\delta = 1$, we replace the terms of the form $(1-\delta)^{-1}x^{1-\delta}$ in the above expressions by term $\log(x)$), in addition, the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ is nonincreasing.

Proof. Let u, v be two weak solutions to (P_ε), then we have

$$\partial_t(u-v) - (\Delta_p u - \Delta_p v) - \left((u+\varepsilon)^{-\delta} - (v+\varepsilon)^{-\delta}\right) = f_\varepsilon(x, u, \nabla u) - f_\varepsilon(x, v, \nabla v),$$

in the weak sense analogous to Definition 1 in Section 2. Multiplying by $(u-v)^+$ the above expression, integrating by parts, using the fact that f is Lipschitz in respect to u uniformly in x and ξ and from (I.3), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega (u-v)^{+2}\mathrm{d}x + \int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\nabla(u-v)^+\mathrm{d}x \\
&- \int_\Omega \left[(u+\varepsilon)^{-\delta}\mathrm{d}x - (v+\varepsilon)^{-\delta}\right](u-v)^+\,\mathrm{d}x \\
&= \int_\Omega \Big([f_\varepsilon(x, u\nabla u) - f_\varepsilon(x, v, \nabla u)] + [f_\varepsilon(x, v, \nabla u) - f_\varepsilon(x, v, \nabla v)]\Big)(u-v)^+\mathrm{d}x \\
&\leq C_1\int_\Omega (u-v)^{+2}\mathrm{d}x + C_2\int_\Omega ||\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v|(u-v)^+\mathrm{d}x.
\end{aligned} \tag{I.40}$$

In addition, we have

$$\begin{aligned}
|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v &= \Big[|\nabla(v+t(u-v))|^{p-2}\,\nabla(v+t(u-v))\Big]_0^1 \\
&= \mathbf{A}(u,v)\nabla(u-v)
\end{aligned}$$

where the matrix $\mathbf{A}(u,v) = (a_{i,j}(u,v))_{1\le i,j\le N}$ is given by

$$\begin{aligned} a_{i,j}(u,v) = &\int_0^1 |\nabla(v+s(u-v))|^{p-2} \\ &\times \left[\delta_{ij} + (p-2)\frac{\frac{\partial}{\partial x_i}(v+s(u-v))\frac{\partial}{\partial x_j}(v+s(u-v))}{|\nabla(v+s(u-v))|^2}\right] \mathrm{d}s. \end{aligned} \tag{I.41}$$

From Theorem 0.1 in Lieberman [87], $a_{i,j}(u,v) \in L^\infty(Q_T)\ \forall\, i,\, j \in (1,...,N)$. Furthermore, $\mathbf{A}$ is positive definite and symmetric. Hence there exists a constant $M > 0$ such that

$$0 \le |\mathbf{A}\xi|^2 \le\ M\ \mathbf{A}\xi.\xi \quad \forall\, \xi \in \mathbb{R}^n. \tag{I.42}$$

Then, from (I.40) and by the Young's Inequality, we obtain

$$\begin{aligned} &\frac{1}{2}\quad \frac{d}{dt}\int_\Omega (u-v)^{+2}\mathrm{d}x + \int_\Omega \mathbf{A}(u,v)\nabla(u-v).\nabla(u-v)^+\mathrm{d}x \\ \le\ &C_1\int_\Omega (u-v)^{+2}\mathrm{d}x + C_2\sqrt{M}\int_\Omega \left(\mathbf{A}(u,v)\nabla(u-v)^+.\nabla(u-v)^+\right)^{\frac{1}{2}}(u-v)^+\mathrm{d}x \\ \le\ &C_3\int_\Omega (u-v)^{+2}\mathrm{d}x + \frac{1}{2}\int_\Omega \mathbf{A}(u,v)\nabla(u-v)^+.\nabla(u-v)^+\mathrm{d}x. \end{aligned} \tag{I.43}$$

Then, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u-v)^{+2}\mathrm{d}x \le C_3\int_\Omega (u-v)^{+2}\mathrm{d}x.$$

Since we have also $\int_\Omega [(u-v)(0)]^{+2}\,\mathrm{d}x = 0$, by Gronwall's lemma we deduce $(u-v)^+ \equiv 0$ and then $u \le v$. By reversing the roles of u and v, we get the equality $u = v$.
By the same argument, one can show that

$$0 < \varepsilon_1 < \varepsilon_2 \Longrightarrow u_{\varepsilon_2} \le u_{\varepsilon_1} \tag{I.44}$$

which implies that the family $\{u_\varepsilon\}_{\varepsilon>0}$ is nonincreasing in $\varepsilon > 0$.

Let us now prove that $u_\varepsilon \in \mathscr{C}((0,T);W_0^{1,p}(\Omega))$. First, we observe that since $u_\varepsilon \in \mathscr{C}([0,T];L^r(\Omega))$ and $u_\varepsilon \in L^\infty(\eta,T;W_0^{1,p}(\Omega))$ for any $\eta \in (0,T)$, it follows that $u_\varepsilon : (0,T) \to W_0^{1,p}(\Omega)$ is weakly continuous and then

$$\|u_\varepsilon(t_0)\|_{W_0^{1,p}(\Omega)} \le \liminf_{t\to t_0} \|u_\varepsilon(t)\|_{W_0^{1,p}(\Omega)} \quad \forall t_0 \in (0,T). \tag{I.45}$$

Let $0 < t_0 < t \le T$, and let $m := m(N) \in \mathbb{N}^*$ such that $t = m\Delta t$ for a suitable $\Delta t > 0$. Making a semidiscretization in time on $(t_0,t)\times\Omega$ with the initial datum in the iterative schema $(\mathrm{P_n})$ $v^0 = u_\varepsilon(t_0) \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, we can easily adapt the above proof to show the existence and uniqueness of

$$v_\varepsilon \in L^p(t_0,T;W_0^{1,p}(\Omega)) \cap L^\infty((0,T)\times\Omega).$$

Furthermore, v_ε is a weak solution (in the sense of Definition 1.1 replacing 0 by t_0) to the problem :

$$\begin{cases} \partial_t v_\varepsilon - \Delta_p v_\varepsilon &= (v_\varepsilon+\varepsilon)^{-\delta} + f_\varepsilon(x, v_\varepsilon, \nabla v_\varepsilon) & \text{in } (t_0, T)\times\Omega, \\ v_\varepsilon &= 0 & \text{on } (t_0,T)\times\partial\Omega, \\ v_\varepsilon(0) &:= u_\varepsilon(t_0) & \text{in } \Omega. \end{cases}$$

We multiply the first equation of (P_n) by $(v_\varepsilon^n - v_\varepsilon^{n-1})$ (changing 0 by t_0, u_ε^0 by $u_\varepsilon(t_0)$) and we sum from 1 to m we obtain

$$\sum_{n=1}^m \Delta t \int_\Omega \left(\frac{v_\varepsilon^n - v_\varepsilon^{n-1}}{\Delta t}\right)^2 \mathrm{d}x + \sum_{n=1}^m \int_\Omega |\nabla v_\varepsilon^n|^{p-2}\nabla v_\varepsilon^n . \nabla\left(v_\varepsilon^n - v_\varepsilon^{n-1}\right)\mathrm{d}x =$$
$$\sum_{n=1}^m \int_\Omega \left[(v_\varepsilon^n+\varepsilon)^{-\delta}(v_\varepsilon^n - v_\varepsilon^{n-1}) + f_\varepsilon(x, v_\varepsilon^{n-1}, \nabla v_\varepsilon^{n-1})(v_\varepsilon^n - v_\varepsilon^{n-1})\right]\mathrm{d}x. \tag{I.46}$$

Using convexity arguments, we obtain :

$$\begin{aligned} &\int_{t_0}^{t_m}\int_\Omega (\partial_s \tilde{v}_{\varepsilon,\Delta t})^2 \mathrm{d}x\mathrm{d}s + \frac{1}{p}\int |\nabla v_{\varepsilon,\Delta t}(t)|^p \mathrm{d}x - \frac{1}{p}\int |\nabla u_\varepsilon(t_0)|^p \mathrm{d}x \\ &\le \frac{1}{1-\delta}\int_\Omega (v_{\varepsilon,\Delta t}(t)+\varepsilon)^{1-\delta}\mathrm{d}x - \frac{1}{1-\delta}\int_\Omega (u_\varepsilon(t_0)+\varepsilon)^{1-\delta}\mathrm{d}x \\ &+ \int_{t_0}^{t_m}\int_\Omega f_\varepsilon(x, v_{\varepsilon,\Delta t}(\cdot-\Delta t), \nabla v_{\varepsilon,\Delta t}(\cdot - \Delta t))\partial_s \tilde{v}_{\varepsilon,\Delta t}\mathrm{d}x\mathrm{d}s. \end{aligned}$$

(with the extension $f_\varepsilon(x, v_{\varepsilon,\Delta t}(\cdot-\Delta t), \nabla v_{\varepsilon,\Delta t}(\cdot-\Delta t)) = f_\varepsilon(x, v_{\varepsilon,0}, \nabla u_\varepsilon(t_0))$ on $(t_0, t_0+\Delta t)$). We pass now to the limit inf as $\Delta t \to 0$ (say $m \to \infty$), we have by (I.29)

$$\int_{t_0}^t \int_\Omega (\partial_s v_\varepsilon(s)^2 \mathrm{d}x\mathrm{d}s \le \varliminf_{\Delta t\to 0}\int_{t_0}^{t_m}\int_\Omega (\partial_s \tilde{v}_{\varepsilon,\Delta t})^2 \mathrm{d}x\mathrm{d}s.$$

Using (I.36), we obtain

$$\frac{1}{p}\int |\nabla v_\varepsilon(t)|^p \mathrm{d}x \le \varliminf_{\Delta t\to 0}\frac{1}{p}\int |\nabla v_{\varepsilon,\Delta t}(t)|^p \mathrm{d}x.$$

From (I.34) and by Lebesgue theorem we deduce that

$$\int_\Omega (v_{\varepsilon,\Delta t}(t)+\varepsilon)^{1-\delta}\mathrm{d}x \underset{\Delta t\to 0}{\longrightarrow} \int_\Omega (v_\varepsilon(t)+\varepsilon)^{1-\delta}\mathrm{d}x \tag{I.47}$$

and finally, from (I.34) and (I.38) it follows that $f_\varepsilon(x, v_{\varepsilon,\Delta t}, \nabla v_{\varepsilon,\Delta t})$ converges in $L^2(\Omega)$ to $f_\varepsilon(x, v_\varepsilon, \nabla v_\varepsilon)$. Hence by (I.29) we obtain that

$$\int_{t_0}^t\int_\Omega f_\varepsilon(x, \nabla v_{\varepsilon,\Delta t}(\cdot-\Delta), \nabla v_{\varepsilon,\Delta t}(\cdot-\Delta t))\partial_s v_{\varepsilon,\Delta t}\mathrm{d}x\mathrm{d}s \underset{\Delta t\to 0}{\longrightarrow} \int_{t_0}^t\int_\Omega f_\varepsilon(x, v_\varepsilon, \nabla v_\varepsilon)\partial_s v_\varepsilon \mathrm{d}x\mathrm{d}s. \tag{I.48}$$

Finally, we can infer that

$$\begin{aligned}&\int_{t_0}^t\int_\Omega(\partial_s v_\varepsilon)^2\mathrm{d}x+\frac{1}{p}\int|\nabla v_\varepsilon(t)|^p\mathrm{d}x-\frac{1}{p}\int|\nabla u_\varepsilon(t_0)|^p\mathrm{d}x\\&\le\frac{1}{1-\delta}\int_\Omega(v_\varepsilon(t)+\varepsilon)^{1-\delta}\,\mathrm{d}x-\frac{1}{1-\delta}\int_\Omega(u_\varepsilon(t_0)+\varepsilon)^{1-\delta}\,\mathrm{d}x\\&+\int_{t_0}^t\int_\Omega f_\varepsilon(x,v_\varepsilon,\nabla v_\varepsilon)\partial_s v_\varepsilon\mathrm{d}x\mathrm{d}s\end{aligned}\tag{I.49}$$

From the uniqueness of the solution proved above we have $v_\varepsilon=u_\varepsilon$ a.e. in $(t_0,t)\times\Omega$ and then u_ε satisfies also the above inequality. Hence, we get from (I.49) as $t\to t_0^+$

$$\overline{\lim_{t\to t_0^+}}\|u_\varepsilon(t)\|_{W_0^{1,p}(\Omega)}\le\|u_\varepsilon(t_0)\|_{W_0^{1,p}(\Omega)}.\tag{I.50}$$

From (I.45) and (I.50) it follows that $u_\varepsilon(t)\to u_\varepsilon(t_0)$ in $W_0^{1,p}(\Omega)$ as $t\to t_0^+$. Then, u_ε is right continuous on $(0,T]$. Let us now prove the left continuity. Let $h>0$ small enough and multiply the equation satisfied by u_ε, by $\tau_h(u_\varepsilon)=\frac{u_\varepsilon(s+h)-u_\varepsilon(s)}{h}$ and integrating over $(t_0,t)\times\Omega$, we get by convexity arguments that :

$$\begin{aligned}&\int_{t_0}^t\int_\Omega\partial_s u_\varepsilon\tau_h(u_\varepsilon)\mathrm{d}x\mathrm{d}s+\frac{1}{hp}\left(\int_t^{t+h}\int_\Omega|\nabla u_\varepsilon(s)|^p\mathrm{d}x\mathrm{d}s-\int_{t_0}^{t_0+h}\int_\Omega|\nabla u_\varepsilon(s)|^p\mathrm{d}x\mathrm{d}s\right)\\&\ge\frac{1}{(1-\delta)h}\left[\int_t^{t+h}\int_\Omega(u_\varepsilon+\varepsilon)^{1-\delta}\,\mathrm{d}x\mathrm{d}s-\int_{t_0}^{t_0+h}\int_\Omega(u_\varepsilon+\varepsilon)^{1-\delta}\,\mathrm{d}x\mathrm{d}s\right]\\&+\int_{t_0}^t\int_\Omega f_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\tau_h(u_\varepsilon)\mathrm{d}x\mathrm{d}s.\end{aligned}\tag{I.51}$$

We pass now to the limit as $h\to 0$, and taking into account that u_ε is right-continuous from $(0,T)$ to $W_0^{1,p}(\Omega)$, we get easily

$$\begin{aligned}&\int_{t_0}^t\int_\Omega(\partial_s u_\varepsilon)^2\mathrm{d}x+\frac{1}{p}\int|\nabla u_\varepsilon(t)|^p\mathrm{d}x-\frac{1}{p}\int|\nabla u_\varepsilon(t_0)|^p\mathrm{d}x\\&\ge\frac{1}{1-\delta}\int_\Omega(u_\varepsilon(t)+\varepsilon)^{1-\delta}\,\mathrm{d}x-\frac{1}{1-\delta}\int_\Omega(u_\varepsilon(t_0)+\varepsilon)^{1-\delta}\,\mathrm{d}x\\&+\int_{t_0}^t\int_\Omega f_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\partial_s u_\varepsilon\mathrm{d}x\mathrm{d}s\end{aligned}\tag{I.52}$$

which implies together with (I.49) that the above inequality is in fact an equality, Hence u_ε satisfies (I.39) and $u_\varepsilon\in\mathscr{C}((0,T);W_0^{1,p}(\Omega))$ and this conclude the proof of theorem 4.1.

5 L^∞-bounds for u_ε

In order to get a solution to Problem (P), we pass to a limit sequence $\{u_\varepsilon\}_\varepsilon$ of the approximate problems (P_ε) as $\varepsilon\to 0$. For that we need a priori L^∞-bounds for $\{u_\varepsilon\}$ independently of ε. The approach similar as in Takáč [114] that we use is based on estimates obtained by logarithmic Sobolev inequalities combined with inequalities of Gagliardo-Nirenberg type. First, we need the following technical identity which is essentially due to L.Gross [75, Lemma 1.1, p. 1065] ; see also F.Cipriani and G.Grillo [38, Lemma 3.2, p. 220] or E. B. Davies [45, Lemma 2.2.2, p. 64].

Lemma 5.1 *Let $r : [0,T) \longrightarrow [2,+\infty)$ be a continuously differentiable function, where $0 < T < \infty$. Assume that $g : [0,T) \longrightarrow L^2(\Omega)$ is continuous, i.e.,*
$g \in \mathscr{C}\left([0,T]; L^2(\Omega)\right)$, and satisfies

$$g \in L^p\left(0,T; W_0^{1,p}(\Omega)\right) \cap W^{1,p'}\left(0,T; W^{-1,p'}(\Omega)\right).$$

In addition, $g(t) \in L^\infty(\Omega)$ for every $t \in [0,T)$ and the norm $\|g(t)\|_{L^\infty(\Omega)}$ is bounded in $[0,T)$. Then the function $t \longmapsto \|g(t)\|_{L^{r(t)}(\Omega)}^{r(t)}$ is locally absolutely continuous in $[0,T)$ and satisfies

$$\begin{aligned}\frac{d}{dt}\|g(t)\|_{L^{r(t)}(\Omega)}^{r(t)} &= r(t)\int_\Omega |g(t)|^{r(t)-2} g(t) g'(t)\,\mathrm{d}x \\ &\quad + r'(t)\int_\Omega |g(t)|^{r(t)}\log|g(t)|\mathrm{d}x.\end{aligned} \tag{I.53}$$

5.1 First estimate

Proposition 5.1 *Let $\delta > 0$ and let $p,q,r \geq 2$, such that (I.4) and (I.6) hold. Let f satisfy (I.2). We assume that $u_0 \in L^r(\Omega)$. Then there exists $T_m > 0$ such that the solution u_ε of (P_ε) verifies the following bounds for any $T < T_m$:*

$$\exists C_1 > 0 : \quad \|u_\varepsilon\|_{L^r(\Omega)} \leq C_1 < \infty \text{ independently of } \varepsilon \text{ and } \forall t \in [0,T] \tag{I.54}$$

Proof. Let $\varrho > 0$ that we first consider fixed. Multiply the first equation of (P_ε) by $u_\varepsilon^{\varrho+1} \in L^p(0,T; W_0^{1,p}(\Omega))$ and integrate by parts on Ω, we obtain :

$$\begin{aligned}&\frac{1}{2+\varrho}\frac{d}{\mathrm{d}t}\int_\Omega |u_\varepsilon(t)|^{2+\varrho}\mathrm{d}x + (1+\varrho)\int_\Omega |\nabla u_\varepsilon(t)|^p |u_\varepsilon(t)|^\varrho \mathrm{d}x \\ &= \int_\Omega (u_\varepsilon(t)+\varepsilon)^{-\delta} u_\varepsilon(t)^{\varrho+1}\mathrm{d}x + \int_\Omega f(x,u_\varepsilon(t),\nabla u_\varepsilon)(t) u_\varepsilon(t)^{\varrho+1}\mathrm{d}x.\end{aligned} \tag{I.55}$$

From the growth condition (I.2) and Young's inequality, we estimate the last term in (I.55) as follows :

$$\begin{aligned}&\int_\Omega f_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon) u_\varepsilon^{\varrho+1}\mathrm{d}x \leq \int_\Omega \left(a u_\varepsilon^{q+\varrho} + b u_\varepsilon^{1+\varrho}\right)\mathrm{d}x + \int_\Omega c u_\varepsilon^{1+\varrho}|\nabla u_\varepsilon|^{p-\frac{p}{q}}\mathrm{d}x \\ &= \int_\Omega \left(a u_\varepsilon^{q+\varrho} + b u_\varepsilon^{1+\varrho}\right)\mathrm{d}x + \int_\Omega \left(c u_\varepsilon^{1+\frac{\varrho}{q}}\right)\left(|\nabla u_\varepsilon|^{\frac{p}{q'}} u_\varepsilon^{\frac{\varrho}{q'}}\right)\mathrm{d}x \\ &\leq \int_\Omega \left(a u_\varepsilon^{q+\varrho} + b u_\varepsilon^{1+\varrho}\right)\mathrm{d}x + c'\int_\Omega u_\varepsilon^{q+\varrho}\mathrm{d}x + \frac{1+\varrho}{4}\int_\Omega |\nabla u_\varepsilon|^p u_\varepsilon^\varrho \mathrm{d}x \\ &\leq \int_\Omega \left(a' u_\varepsilon^{q+\varrho} + b u_\varepsilon^{1+\varrho}\right)\mathrm{d}x + \frac{1+\varrho}{4}\int_\Omega |\nabla u_\varepsilon|^p u_\varepsilon^\varrho \mathrm{d}x\end{aligned} \tag{I.56}$$

with a suitable positive constant a'. We now substitute this bound in (I.55) to obtain

$$\begin{aligned}\frac{1}{2+\varrho}\frac{d}{\mathrm{d}t}\int_\Omega |u_\varepsilon(t)|^{2+\varrho}\mathrm{d}x + \frac{3(1+\varrho)}{4}\int_\Omega |\nabla u_\varepsilon(t)|^p|u_\varepsilon|^\varrho \mathrm{d}x \\ \leq \int_\Omega u_\varepsilon^{\varrho+1-\delta}\mathrm{d}x + \int_\Omega b u_\varepsilon^{1+\varrho}\mathrm{d}x + \int_\Omega a' u_\varepsilon^{q+\varrho}\mathrm{d}x.\end{aligned} \tag{I.57}$$

To estimate the first two terms in the right hand-side, we use the Hölder inequality as follows :

$$\int_\Omega u_\varepsilon^{\varrho+1-\delta}\mathrm{d}x + \int_\Omega b u_\varepsilon^{\varrho+1}\mathrm{d}x \leq C_1\left(\int_\Omega u_\varepsilon^{2+\varrho}dx\right)^{\frac{\varrho+1-\delta}{2+\varrho}}\mathrm{d}x + C_2\left(\int_\Omega u_\varepsilon^{2+\varrho}dx\right)^{\frac{\varrho+1}{2+\varrho}}\mathrm{d}x \tag{I.58}$$

for every $\varrho \geq \delta - 1$.
We now estimate the last integral $\int_\Omega a u_\varepsilon^{q+\varrho}$ on the right hand side of inequality (I.57).

The following lemma is an adaptation of Takáč [114, Lemma 5.1, p.338].

Lemma 5.2 *Let $p, q, r \in [2,\infty)$ verify* (I.4). *Then there exists a constant $C_3 > 0$ independent of ε such that the estimates*

$$a'\int_\Omega u_\varepsilon^{q+\varrho}\mathrm{d}x \leq \frac{1}{4}(1+\varrho)\int_\Omega |\nabla u_\varepsilon|^p u_\varepsilon^\varrho \mathrm{d}x + C_3\,\|u_\varepsilon\|_{L^{2+\varrho}(\Omega)}^{q+\varrho+(q-p)E(\varrho)} \tag{I.59}$$

hold for every $u_\varepsilon \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and fo all $\varrho \in [\varrho_0,\infty)$, where $\varrho_0 = r-2$ and

$$E(\varrho) = \frac{q-2}{p\left(1+\frac{2+\varrho}{N}\right)-q} \tag{I.60}$$

Proof. We use inequalities of Gagliardo-Nirenberg-type, see R. A. Adams and J. J. F. Fournier [1, Chap. 5]. We treat only the case $q > 2$, leaving a number of amendments for $q = 2$ to the reader. Recall that $2 \leq p \leq q \leq p^*$. There exists a constant $c_{p,q} > 0$ such that the following inequality holds, whenever $2 \leq R \leq p \leq Q \leq q$:

$$\|v\|_{L^Q(\Omega)} \leq c_{p,q}\|\nabla v\|_{L^p(\Omega)}^\theta \|v\|_{L^R(\Omega)}^{1-\theta} \quad \text{for all } v \in W_0^{1,p}(\Omega) \tag{I.61}$$

where

$$\theta \stackrel{\text{def}}{=} \left(\frac{1}{R}-\frac{1}{Q}\right)\left(\frac{1}{R}-\frac{1}{p}+\frac{1}{N}\right)^{-1}. \tag{I.62}$$

Notice that $0 \leq \theta < 1$ with ($\theta > 0$ if $R < Q$) and

$$\frac{1}{Q} = \theta\left(\frac{1}{p}-\frac{1}{N}\right)+\frac{1-\theta}{R}.$$

Now we write inequality (I.61) as

$$\int_\Omega |v|^Q\mathrm{d}x \leq c_{p,q}^Q\left(\int_\Omega |\nabla v|^p\mathrm{d}x\right)^{\theta Q/p}\left(\int_\Omega |v|^R\mathrm{d}x\right)^{(1-\theta)Q/R} \tag{I.63}$$

for all $v \in W_0^{1,p}(\Omega)$.

Next, let us take any $\varrho \in \mathbb{R}_+$ and $u_\varepsilon \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. We substitute

$$Q = \frac{q+\varrho}{1+\frac{\varrho}{p}} = p \cdot \frac{q+\varrho}{p+\varrho} \text{ and } R = \frac{2+\varrho}{1+\frac{\varrho}{p}} = p \cdot \frac{2+\varrho}{p+\varrho}$$

together with $v = |u_\varepsilon|^{\varrho/p} u_\varepsilon = |u_\varepsilon|^{1+\varrho/p}$ in inequality (I.63), thus arriving at

$$\begin{aligned} \int_\Omega |u_\varepsilon|^{q+\varrho} \mathrm{d}x \quad &\leq \quad c_{p,q}^{p(q+\varrho)/(p+\varrho)} \left(1 + \frac{\varrho}{p}\right)^{\theta p(q+\varrho)/(p+\varrho)} \\ &\times \left(\int_\Omega |\nabla u_\varepsilon|^p u_\varepsilon^\varrho \mathrm{d}x\right)^{\theta(q+\varrho)/(p+\varrho)} \|u_\varepsilon\|_{L^{2+\varrho}(\Omega)}^{(1-\theta)(q+\varrho)} \end{aligned} \tag{I.64}$$

where, by equation (I.62),

$$\theta = \frac{1}{q+\varrho} \cdot \frac{q-2}{1 - \frac{2+\varrho}{p+\varrho}(1-\frac{p}{N})} = \frac{1}{q+\varrho} \cdot \frac{q-2}{p(1+\frac{2+\varrho}{N}) - 2}. \tag{I.65}$$

Again, notice that $0 < \theta < 1$ ($\theta > 0$ because of $q > 2$) and

$$\frac{1}{q+\varrho} = \theta \cdot \frac{1 - \frac{p}{N}}{p+\varrho} + \frac{1-\theta}{2+\varrho}.$$

We set

$$P \stackrel{\text{def}}{=} \frac{p+\varrho}{\theta(q+\varrho)} = \frac{p(1+\frac{2+\varrho}{N}) - 2}{q-2} = 1 + \frac{p(1+\frac{2+\varrho}{N}) - 2}{q-2} = 1 + \frac{1}{E}. \tag{I.66}$$

Le us recall that

$$E \equiv E(\varrho) = \frac{q-2}{p\left(1+\frac{2+\varrho}{N}\right) - q} \leq E(\varrho_0) \stackrel{\text{def}}{=} E_0 < \infty \quad \forall \varrho \in [\varrho_0, \infty). \tag{I.67}$$

and $E > 0$ owing to $q > 2$. Then P satisfies

$$P \geq P_0 \stackrel{\text{def}}{=} 1 + \frac{p(1+\frac{2+\varrho_0}{N}) - 2}{q-2} = 1 + \frac{1}{E_0} > 1.$$

Its conjugate exponent is given by

$$P' = \frac{P}{P-1} = 1 + E \tag{I.68}$$

and satisfies

$$P' \leq P_0' = \frac{P_0}{p_0 - 1} = 1 + E_0 < \infty.$$

Below we need also the exponent

$$(1-\theta)(q+\varrho)P' = q + \varrho + \frac{(q-p)(q-2)}{p(1+\frac{2+\varrho}{N}) - q} = q + \varrho + (q-p)E \tag{I.69}$$

We estimate the right-hand side of (I.64) by Young's inequality using the pair of conjugate exponents (P, P'), for any $0 < \eta < \infty$,

$$\begin{aligned}\int_\Omega |u_\varepsilon|^{q+\varrho}\mathrm{d}x \le\ & c_{p,q}^{p(q+\varrho)/(p+\varrho)} \\ & \times \left[\frac{\eta}{P}\left(1+\frac{\varrho}{p}\right)^p \int_\Omega |\nabla u_\varepsilon|^p u_\varepsilon^\varrho \mathrm{d}x + \frac{\eta^{-P'/P}}{P'} \|u_\varepsilon\|_{L^{2+\varrho}(\Omega)}^{q+\varrho+(q-p)E} \right]. \end{aligned} \tag{I.70}$$

Here, formulas (I.66), (I.68) and (I.69) have been employed. We choose

$$\eta = \frac{(1+\varrho)P}{4ac_{p,q}^{p(q+\varrho)/(p+\varrho)}}\left(1+\frac{\varrho}{p}\right)^{-p} \tag{I.71}$$

and observe that

$$\begin{aligned} & ac_{p,q}^{p(q+\varrho)/(p+\varrho)}\frac{\eta^{-P'/P}}{P'} \\ & = \frac{1}{1+E}\left(ac_{p,q}^{p(q+\varrho)/(p+\varrho)}\right)^{1+E}\left(\frac{4}{(1+\varrho)(1+E^{-1})}\right)^E \left(1+\frac{\varrho}{p}\right)^{pE} \\ & \le \frac{1}{4}\left(4ac_{p,q}^{p(q+\varrho)/(p+\varrho)}\right)^{1+E}\left(1+\frac{\varrho}{p}\right)^{pE} \le C_4 < \infty \end{aligned} \tag{I.72}$$

with a constant C_4 independent from $\varrho \ge \varrho_0$. The last claim follows from

$$pE\left(1+\frac{\varrho}{p}\right) = \frac{(q-2)(p+\varrho)}{p(1+\frac{2+\varrho}{N})-q},$$

where

$$\frac{p+\varrho}{p(1+\frac{2+\varrho}{N})-q} \in \left[\frac{N}{p}, \frac{p+r-2}{p(1+\frac{r}{N})-q}\right] = \left[\frac{N}{p}, \frac{N}{p}\left(1+\frac{q-p+\frac{p}{N}(p-2)}{p(1+\frac{r}{N})-q}\right)\right]$$

combained with

$$\left(1+\frac{\varrho}{p}\right)^{1/(1+\frac{\varrho}{p})} \longrightarrow 1 \text{ as } \varrho \searrow 0 \text{ or } \varrho \nearrow \infty.$$

Finally, applying (I.71) and (I.72) to the right-hand side of (I.70) we arrive at (I.59) as desired (we point out that this result remain valid for any $\overline{\varrho} \ge \varrho$). $\square$

We substitute these bounds (I.58) and (I.59) in (I.57) to get

$$\begin{aligned}\frac{1}{2+\varrho}\frac{d}{dt} \parallel u_\varepsilon(t) \parallel_{L^{2+\varrho}(\Omega)}^{2+\varrho} & + \beta\int_\Omega |\nabla u_\varepsilon|^p u_\varepsilon^\varrho \mathrm{d}x \le C_1 \parallel u_\varepsilon(t) \parallel_{L^{2+\varrho}(\Omega)}^{\varrho+1-\delta} \\ & + C_2 \parallel u_\varepsilon(t) \parallel_{L^{2+\varrho}(\Omega)}^{\varrho+1} + C_3\|u_\varepsilon(t)\|_{L^{2+\varrho}(\Omega)}^{(2+\varrho)B(\varrho)} \end{aligned} \tag{I.73}$$

with $\beta = \frac{1}{4}(1+\varrho) > 0$ and $B(\varrho) = \frac{q+\varrho+(q-p)E(\varrho)}{2+\varrho} \ge 1$. Using the inequality : $\theta^{r_1} \le \theta^{r_2} + 1$ for any $\theta > 0$ and $r_1 \le r_2$, we deduce that there exist $C_4, C_5 > 0$ such that

$$\begin{aligned} C_1 \parallel u_\varepsilon(t) \parallel_{L^{2+\varrho}(\Omega)}^{\varrho+1-\delta} + C_2 \parallel u_\varepsilon(t) \parallel_{L^{2+\varrho}(\Omega)}^{\varrho+1} & + C_3\|u_\varepsilon(t)\|_{L^{2+\varrho}(\Omega)}^{(2+\varrho)B(\varrho)} \\ & \le C_4\|u_\varepsilon(t)\|_{L^{2+\varrho}(\Omega)}^{(2+\varrho)B(\varrho)} + C_5. \end{aligned} \tag{I.74}$$

Thus, (I.73) together with (I.74) yield

$$\frac{1}{2+\varrho}\frac{d}{dt}\parallel u_\varepsilon(t)\parallel_{L^{2+\varrho}(\Omega)}^{2+\varrho}+\beta\int_\Omega|\nabla u_\varepsilon|^p u_\varepsilon^\varrho \mathrm{d}x\leq C_4\|u_\varepsilon(t)\|_{L^{2+\varrho}(\Omega)}^{(2+\varrho)B(\varrho)}+C_5. \tag{I.75}$$

This implies the following differential inequality :

$$\frac{1}{2+\varrho}\frac{d}{dt}\parallel u_\varepsilon(t)\parallel_{L^{2+\varrho}(\Omega)}^{2+\varrho}\leq C_4\|u_\varepsilon(t)\|_{L^{2+\varrho}(\Omega)}^{(2+\varrho)B(\varrho)}+C_5. \tag{I.76}$$

By a classical comparison argument with sub- and super-solutions applied to (I.76), we have

$$\parallel u_\varepsilon(t)\parallel_{L^{2+\varrho}(\Omega)}^{2+\varrho}\leq U\left(t\right)\ \text{ for every } 0\leq t<T_m$$

where $U:[0,T_m)\longrightarrow\mathbb{R}_+$ is the unique solution of the following initial value problem

$$\begin{cases}\frac{1}{2+\varrho}\frac{d}{dt}U\left(t\right)=(C_4+C_5)U\left(t\right)^{B(\varrho)}\\ U\left(0\right)=\max\left\{\sup\limits_{\varepsilon>0}\parallel u_{\varepsilon,0}\parallel_{L^{2+\varrho}(\Omega)}^{2+\varrho},1\right\}=\max\left\{\parallel u_0\parallel_{L^r(\Omega)}^{r},1\right\}<\infty\end{cases}$$

with $[0,T_m)$ being the maximal time interval of existence for the above initial value problem ($0<T\leq\infty$), and then, the first estimate (I.54) holds. □

5.2 Second estimate

We show now a bound of the sequence (u_ε)in $L^\infty(\Omega)$ independently of $\varepsilon>0$ and $t\in[\eta,T)$ with $\eta>$ and T small enough. The estimate is given by the following proposition.

Proposition 5.2 *We assume the same hypotheses of proposition 5.1. Then there exists $T_m>0$ such that the solution u_ε of (P_ε) verifies the following bounds for any $T<T_m$:*

$$\forall\eta\in(0,T),\ \exists C_\eta>0:\ \|u_\varepsilon\|_{L^\infty(\Omega)}\leq C_\eta<\infty\ \textit{independently of } \varepsilon \textit{ and } \forall t\in(\eta,T]. \tag{I.77}$$

Proof. In order to prove this estimate, we take $\varrho:[0,T)\longrightarrow[2,\infty)$ continuously differentiable function with $\varrho'\left(t\right)>0$ for all $0\leq t<T$, and $\varrho\left(0\right)=\varrho_0\overset{\text{def}}{=}r-2$, this function will be specified more precisely later. We apply the lemma 5.1 with the function $g\left(t\right)$ replaced by the function $u_\varepsilon\left(t\right)$ and $r\left(t\right)=2+\varrho\left(t\right)$ to obtain

$$\begin{aligned}\frac{1}{2+\varrho\left(t\right)}\frac{d}{\mathrm{d}t}\|u_\varepsilon\left(t\right)\|_{L^{2+\varrho(t)}(\Omega)}^{2+\varrho(t)} &= \int_\Omega u_\varepsilon^{\varrho+1}\partial_t u_\varepsilon \mathrm{d}x\\ &\quad+\frac{\varrho'\left(t\right)}{2+\varrho\left(t\right)}\int_\Omega u_\varepsilon^{\varrho+2}\log u_\varepsilon\mathrm{d}x\\ &= -(\varrho+1)\int_\Omega|\nabla u_\varepsilon|^p|u_\varepsilon|^\varrho\mathrm{d}x+\int_\Omega(u_\varepsilon+\varepsilon)^{-\delta}u_\varepsilon^{\varrho+1}\mathrm{d}x\\ &\quad+\int_\Omega f(x,u_\varepsilon,\nabla u_\varepsilon)u_\varepsilon^{\varrho+1}\mathrm{d}x+\frac{\varrho'\left(t\right)}{2+\varrho}\int_\Omega u_\varepsilon^{\varrho+2}\log u_\varepsilon\mathrm{d}x\end{aligned}$$

We infer that

$$\frac{1}{2+\varrho(t)}\frac{d}{dt}\|u_\varepsilon(t)\|^{2+\varrho(t)}_{L^{2+\varrho(t)}(\Omega)}+(1+\varrho)\int_\Omega|\nabla u_\varepsilon|^p|u_\varepsilon|^\varrho \mathrm{d}x\leq$$
$$\int_\Omega u_\varepsilon^{\varrho+1-\delta}\mathrm{d}x+\int_\Omega f(x,u_\varepsilon,\nabla u_\varepsilon)u_\varepsilon^{\varrho+1}\mathrm{d}x+\frac{\varrho'(t)}{2+\varrho}\int_\Omega u_\varepsilon^{\varrho+2}\log u_\varepsilon \mathrm{d}x. \tag{I.78}$$

The first two summands on the right-hand side are estimated as above (see (I.56)-(I.74)), it remains to estimate the logarithmic term in the right-hand side of (I.78). The estimate is given by the following lemma which is also an adaptation of Takáč [114, lemma 6.1, 6.2, pp. 342].

Lemma 5.3 *If $p=2$, we have*

$$\begin{aligned}\frac{\varrho'}{2+\varrho}\int_\Omega u_\varepsilon^{\varrho(t)+2}\log u_\varepsilon\mathrm{d}x \quad &\leq \quad \frac{1}{4}(1+\varrho)\int_\Omega|\nabla u_\varepsilon|^2|u_\varepsilon|^\varrho\mathrm{d}x\\ &+\frac{N\varrho'}{2(2+\varrho)^2}\log\left(\frac{\varrho'}{4\pi(1+\varrho)}\right)\|u_\varepsilon\|^{2+\varrho}_{L^{2+\varrho}(\Omega)}\\ &+\frac{\varrho'}{2+\varrho}\|u_\varepsilon\|^{2+\varrho}_{L^{2+\varrho}(\Omega)}\log\|u_\varepsilon\|_{L^{2+\varrho}(\Omega)}\end{aligned}\tag{I.79}$$

for all $0\leq t<T$. If $p>2$, then there exists a constant $C_6>0$ such that

$$\begin{aligned}&\frac{\varrho'}{2+\varrho}\int_\Omega u_\varepsilon^{\varrho(t)+2}\log u_\varepsilon\mathrm{d}x\leq\frac{1}{4}(1+\varrho)\int_\Omega|\nabla u_\varepsilon|^p|u_\varepsilon|^\varrho\mathrm{d}x\\ &+C_6\left(\frac{\eta\varrho'}{(2+\varrho)(1+\varrho)^{2/p}}\right)^{p/(p-2)}\|u_\varepsilon\|^{\varrho}_{L^{2+\varrho}(\Omega)}\\ &+\varphi_\eta(\varrho)\varrho'\|u_\varepsilon\|^{2+\varrho}_{L^{2+\varrho}(\Omega)}+\frac{\varrho'}{2+\varrho}\|u_\varepsilon\|^{2+\varrho}_{L^{2+\varrho}(\Omega)}\log\|u_\varepsilon\|_{L^{2+\varrho}(\Omega)}\end{aligned}\tag{I.80}$$

for all $t\in[0,T)$ and $\eta>0$, where

$$\varphi_\eta(\varrho)\overset{def}{=}-\frac{N}{2(2+\varrho)^2}\log\frac{16\pi\eta}{2+\varrho}\quad \textit{for } \varrho\in\mathbb{R}_+. \tag{I.81}$$

Proof. Let us prove the case $p=2$. We use the logarithmic Sobolev inequality (I.17) to estimate

$$\begin{aligned}\int_\Omega u_\varepsilon^{2+\varrho}\log u_\varepsilon\mathrm{d}x \quad &\leq \quad \eta\int_\Omega|\nabla u_\varepsilon|^2u_\varepsilon^\varrho\mathrm{d}x-\frac{N}{2(2+\varrho)}\log\left(\frac{16\pi\eta}{2+\varrho}\right)\|u_\varepsilon\|^{2+\varrho}_{L^{2+\varrho}(\Omega)}\\ &+\|u_\varepsilon\|^{2+\varrho}_{L^{2+\varrho}(\Omega)}\log\|u_\varepsilon\|_{L^{2+\varrho}(\Omega)}\end{aligned}\tag{I.82}$$

for any $0<\eta<\infty$. Now we take

$$\eta=\frac{(2+\varrho)(1+\varrho)}{4\varrho'(t)}>0$$

and recalling that $\varrho'(t)>0$, use the expression

$$\frac{16\pi\eta}{2+\varrho}=\frac{4\pi(1+\varrho)}{\varrho'(t)}>0$$

to get inequality (I.79). We prove now the case $p > 2$. We consider $\eta : [0, T) \to (0, \infty)$ as a continuous function rather than a constant. Again, we begin with inequality (I.82). The first term on the right-hand side is estimated by Young's inequality,

$$|\nabla u_\varepsilon|^2 \leq \frac{2}{p}\kappa |\nabla u_\varepsilon|^p + \frac{p-2}{p}\kappa^{-2/(p-2)}$$

for any $0 < \kappa < \infty$, which entails

$$\int_\Omega |\nabla u_\varepsilon|^2 u_\varepsilon^\varrho \mathrm{d}x \leq \frac{2}{p}\kappa \int_\Omega |\nabla u_\varepsilon|^p \mathrm{d}x + \frac{p-2}{p}\kappa^{-2/(p-2)} \int_\Omega u_\varepsilon^\varrho \mathrm{d}x.$$

Setting

$$\kappa = \frac{p}{8\eta}\left(\frac{\varrho'}{(2+\varrho)(1+\varrho)}\right)^{-1}$$

we arrive at

$$\begin{aligned}
&\frac{\varrho'}{2+\varrho}\int_\Omega |\nabla u_\varepsilon|^2 u_\varepsilon^\varrho \mathrm{d}x \\
&\leq \frac{1}{4\eta}(1+\varrho)\int_\Omega |\nabla u_\varepsilon|^p u_\varepsilon^\varrho \mathrm{d}x \\
&\quad + \frac{p-2}{2}\cdot\frac{\varrho'}{2+\varrho}\left(\frac{p}{8\eta}\right)^{-2/(p-2)}\left(\frac{\varrho'}{(2+\varrho)(1+\varrho)}\right)^{p/(p-2)}\int_\Omega u_\varepsilon^\varrho \mathrm{d}x \\
&\leq \frac{1}{4\eta}(1+\varrho)\int_\Omega |\nabla u_\varepsilon|^p u_\varepsilon^\varrho \mathrm{d}x \\
&\quad + \frac{p-2}{2}\cdot\left(\frac{8\eta}{p}\right)^{2/(p-2)}\left(\frac{\varrho'}{(2+\varrho)(1+\varrho)^{2/p}}\right)^{p/(p-2)}\int_\Omega u_\varepsilon^\varrho \mathrm{d}x
\end{aligned} \tag{I.83}$$

The last integral is estimated as follows,

$$\begin{aligned}
\int_\Omega u_\varepsilon^\varrho \mathrm{d}x &\leq \left(\int_\Omega u_\varepsilon^{2+\varrho}\mathrm{d}x\right)^{\varrho/(2+\varrho)}\left(\int_\Omega \mathrm{d}x\right)^{2/(2+\varrho)} \\
&= |\Omega|^{2/(2+\varrho)}\|u_\varepsilon\|^\varrho_{L^{2+\varrho}(\Omega)}.
\end{aligned} \tag{I.84}$$

Next, from inequality (I.82),(I.83) and (I.84) we deduce

$$\begin{aligned}
&\frac{\varrho'}{2+\varrho}\int_\Omega u_\varepsilon^{2+\varrho}\log u_\varepsilon \mathrm{d}x \frac{1}{4}(1+\varrho)\int_\Omega |\nabla u_\varepsilon|^p u_\varepsilon^\varrho \mathrm{d}x \\
&\frac{p-2}{2}\cdot\left(\frac{8}{p}\right)^{2/(p-2)}\left(\frac{\eta\varrho'}{(2+\varrho)(1+\varrho)^{2/p}}\right)^{p/(p-2)} |\Omega|^{2/(2+\varrho)}\|u_\varepsilon\|^\varrho_{L^{2+\varrho}(\Omega)} \\
&-\frac{N}{2}\cdot\frac{\varrho'}{(2+\varrho)^2}\log\left(\frac{16\pi\eta}{2+\varrho}\right)\|u_\varepsilon\|^{2+\varrho}_{L^{2+\varrho}(\Omega)} + \frac{\varrho'}{(2+\varrho)}\|u_\varepsilon\|^{2+\varrho}_{L^{2+\varrho}(\Omega)}\log\|u_\varepsilon\|_{L^{2+\varrho}(\Omega)}
\end{aligned}$$

for any $0 < \eta < \infty$. Finally, we arrive at inequality (I.80) by taking the constant

$$C_3 = \frac{p-2}{2}\cdot\left(\frac{8}{p}\right)^{2/(p-2)}\max\{|\Omega|, 1\} > 0 \tag{I.85}$$

and the function $\varphi_\eta : \mathbb{R}_+ \to \mathbb{R}$ as defined in (I.81). $\square$

Let us now complete the proof of the proposition. We add the estimates in (I.74) and (I.80) and apply the result to the right hand-side of $(I.78)$, we get for $p > 2$ (leaving a few necessary modifications for $p = 2$ to the reader) :

$$\begin{aligned}\frac{1}{2+\varrho}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon(t)\|^{2+\varrho}_{L^{2+\varrho(t)}(\Omega)} &\le C_4\|u_\varepsilon\|^{q+\varrho+(q-p)E(\varrho)}_{L^{2+\varrho}(\Omega)} + C_5\\ &+C_6\left(\frac{\eta\varrho'}{(2+\varrho)(1+\varrho)^{2/p}}\right)^{p/(p-2)}\|u_\varepsilon\|^{\varrho}_{L^{2+\varrho}(\Omega)}\\ &+\varphi_\eta(\varrho)\,\varrho'\,\|u_\varepsilon\|^{2+\varrho}_{L^{2+\varrho}(\Omega)} + \frac{\varrho'}{2+\varrho}\|u_\varepsilon\|^{2+\varrho}_{L^{2+\varrho}(\Omega)}\log\|u_\varepsilon\|_{L^{2+\varrho}(\Omega)}\end{aligned} \tag{I.86}$$

for a.a. $t \in (0, T)$. Finally, using the identity

$$\begin{aligned}\frac{1}{2+\varrho}\frac{d}{dt}\|u_\varepsilon\|^{2+\varrho}_{L^{2+\varrho}(\Omega)} &= \|u_\varepsilon\|^{1+\varrho}_{L^{2+\varrho}(\Omega)}\cdot\frac{\mathrm{d}}{dt}\|u_\varepsilon\|_{L^{2+\varrho}(\Omega)}\\ &+\frac{\varrho'}{2+\varrho}\|u_\varepsilon\|^{2+\varrho}_{L^{2+\varrho}(\Omega)}\log\|u_\varepsilon\|_{L^{2+\varrho}(\Omega)}\end{aligned}$$

for a.a. $t \in (0, T)$, from inequality (I.86) we derive

$$\begin{aligned}\|u_\varepsilon\|^{1+\varrho}_{L^{2+\varrho}(\Omega)}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon\|_{L^{2+\varrho}(\Omega)} &\le C_4\|u_\varepsilon\|^{\varrho+Z(\varrho)}_{L^{2+\varrho}(\Omega)} + C_5\\ +C_6\left(\frac{\eta\varrho'}{(2+\varrho)(1+\varrho)^{2/p}}\right)^{p/(p-2)}&\|u_\varepsilon\|^{\varrho}_{L^{2+\varrho}(\Omega)} + \varphi_\eta(\varrho)\,\varrho'\,\|u_\varepsilon\|^{2+\varrho}_{L^{2+\varrho}(\Omega)}\end{aligned} \tag{I.87}$$

for a.a. $t \in (0, T)$. Here, we have denoted

$$Z \equiv Z(\varrho) \stackrel{\text{def}}{=} q + (q-p)E(\varrho) = q + \frac{(q-p)(q-2)}{p(1+\frac{2+\varrho}{N}) - q} \tag{I.88}$$

and

$$Z_0 \stackrel{\text{def}}{=} Z(\varrho_0) = q + \frac{(q-p)(q-2)}{p(1+\frac{r}{N}) - q}. \tag{I.89}$$

Notice that, we can check easily from (I.60) that for all $\varrho \ge \varrho_0$ one has $0 \le E(\varrho) \le E_0 = E(\varrho_0) < \infty$ which implies $q \le Z(\varrho) \le Z_0 < \infty$ as well. It is easy to see that inequality (I.87) entails

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon\|^{2}_{L^{2+\varrho}(\Omega)} &\le C_4\|u_\varepsilon\|^{Z(\varrho)}_{L^{2+\varrho}(\Omega)} + C_5\|u_\varepsilon\|^{-\varrho}_{L^{2+\varrho}(\Omega)}\\ +C_6\left(\frac{\eta\varrho'}{(2+\varrho)(1+\varrho)^{2/p}}\right)^{p/(p-2)} &+ \varphi_\eta(\varrho)\,\varrho'\,\|u_\varepsilon\|^{2}_{L^{2+\varrho}(\Omega)}\end{aligned} \tag{I.90}$$

for a.a. $t \in (0, T)$.

It remains to show that the norm $\nu(t) = \|u_\varepsilon(t,\cdot)\|_{L^{2+\varrho}(\Omega)}$ $(0 \le t < T)$ stays bounded for all $0 \le t < S$, where $S \in (0, T)$ is a suitable number and the upper bound on $\nu(t)$ $(0 \le t < S)$ depends solely on an upper bound on

$$U_0 \overset{\text{def}}{=} \max\{\sup_{\varepsilon>0} \|u_\varepsilon(0,\cdot)\|_{L^r(\Omega)}, 1\}.$$

Then also

$$\|u_\varepsilon(S,\cdot)\|_{L^\infty(\Omega)} \le \liminf_{t\to S^-} \|u_\varepsilon(t,\cdot)\|_{L^{2+\varrho(t)}(\Omega)} < \infty \tag{I.91}$$

as desired. In fact, we will be able to take S arbitrarily small, $0 < S < T$. Subsequently, applying estimate (I.54), we may replace U_0 by

$$\max\{\sup_{\substack{0\le t<T\\ \varepsilon>0}} \|u_\varepsilon(t,\cdot)\|_{L^r(\Omega)}, 1\}$$

and the initial condition for $u(t,\cdot)$ at $t = 0$ by that at $t = t_0$ for any $t_0 \in [0, T - S)$, thus obtaining $u(s,\cdot) \in L^\infty(\Omega)$ with $\|u_\varepsilon(s,\cdot)\|_{L^\infty(\Omega)}$ uniformly bounded for $S \le s < T$ and $\varepsilon > 0$.

Estimating $\nu(t)$ $(0 \le t < S)$ is based on a suitable choice of the functions $\eta(t)$ and $\varrho(t)$ in inequality (I.90). But first we weaken and simplify this inequality as follows : By an argument with sub- and supersolution applied to (I.90) for the square norm $\|u_\varepsilon(t,\cdot)\|^2_{L^{2+\varrho}(\Omega)}$, we have

$$\|u_\varepsilon(t,\cdot)\|^2_{L^{2+\varrho}(\Omega)} \le \tilde{U}(t) \quad \text{for every } 0 \le t < \min\{T, T_{\max}\}, \tag{I.92}$$

where $\tilde{U} : [0, T_{\max}) \to \mathbb{R}_+$ is the solution of the initial value problem

$$\begin{cases} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{U}(t) &=& (C_4 + C_5)\tilde{U}(t)^{Z(\varrho)/2} \\ && +C_6\left(\frac{\eta\varrho'}{(2+\varrho)(1+\varrho)^{2/p}}\right)^{p/(p-2)} \\ && +\varphi_{\eta(t)}\left(\varrho(t)\right)\varrho'(t)\tilde{U}(t) \quad \text{for } 0 \le t < T_{\max}, \\ \tilde{U}(0) &=& \tilde{U}_0 \overset{\text{def}}{=} \max\{\|u_0\|^2_{L^r(\Omega)}, 1\} \ge 1 \end{cases} \tag{I.93}$$

with $[0, T_{\max})$ being the maximal time interval of existence $(0 < T_{\max} \le \infty)$.

Since we are interested is S small enough only, we may replace T by $\min\{T, T_{\max}\}$ without loss of generality, i.e., we may assume $T \le T_{\max}$. Notice that, by our hypotheses $\varrho'(t) > 0$ and $0 < 8\pi\eta \le 1$, we have $\tilde{U}'(t) > 0$ for all $0 < t < T$. Thus, in the inequality (I.92), we allowed to use the function $U(t) = \tilde{U}(t)^{\frac{1}{2}}$ which yields

$$\|u(.,t\|_{L^{2+\varrho(t)}(\Omega)} \le U(t) \quad \text{for every } 0 \le t < T. \tag{I.94}$$

Furthermore, by (I.93), the function $U(t)$ satisfies

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}U(t) &=& (C_4 + C_5)U(t)^{Z(\varrho)-1} \\ && +C_6\left(\frac{\eta\varrho'}{(2+\varrho)(1+\varrho)^{2/p}}\right)^{p/(p-2)} U(t)^{-1} \\ && +\varphi_{\eta(t)}\left(\varrho(t)\right)\varrho'(t)U(t) \quad \text{for } 0 \le t < T_{\max}, \\ U(0) &=& U_0 \overset{\text{def}}{=} \max\{\|u_0\|_{L^r(\Omega)}, 1\} \ge 1 \end{cases} \tag{I.95}$$

As above, our hypotheses $\varrho'(t) > 0$ and $0 < 8\pi\eta \le 1$ guarantee $U'(t) > 0$ and $U(t) \ge 1$ for all $0 \le t < T$.

Next, we compute the solution of problem (I.95) by the change of variable $\varrho = \varrho(t) \ge \varrho_0$ in $U(t) = \hat{U}(\varrho(t))$ for all t small enough, say, $0 \le t < T'_{\max}$ provided $\eta \equiv \eta_\nu : [0,T) \to (0,\infty$ and $\varrho \equiv \varrho_\nu : [0,T) \to \mathbb{R}_+$ are chosen as follows, where $0 \le \nu \le \frac{1}{8\pi}$ is a constant : The function η is the superposition $\eta(t) \stackrel{\text{def}}{=} \hat{\eta}(\varrho(t), U(t))$ defined for every $0 \le t < T$, where

$$\hat{\eta}(\varrho, U) \stackrel{\text{def}}{=} \begin{cases} \nu U^{-2\frac{Z(\varrho)-p}{p}} & \text{if} \quad \varrho_0 \le \varrho < \infty \text{ and } U > 0, \\ \nu U^{-\frac{q-p}{p}} & \text{if} \quad \varrho = \infty \text{ and } U > 0 \end{cases} \tag{I.96}$$

Note that $Z(\varrho) \searrow q$ as $\varrho \nearrow$, from (I.88). Hence $\hat{\eta} : [\varrho_0, \infty] \times (0,\infty) \to (0,\infty)$ is continuous. The function ϱ is constructed in a more complicated manner depending on U. We take the pair of functions (ϱ, U) to be the unique solution of the system of coupled equations (I.95) and the following one,

$$\begin{cases} \frac{d}{dt}\varrho(t) = \frac{1}{\nu} U(t)^{Z(t)-2}(2+\varrho)(1+\varrho)^{2/p} & \text{for } 0 \le t < T, \\ \varrho(0) = \varrho_0 (= r - 2 \ge 0). \end{cases} \tag{I.97}$$

Similarly as above, $[0, T'_{\max})$ is the maximal time interval of existence ($0 \le T'_{\max} \le T$), i.e., if $T'_{\max} < T$ then

$$\varrho(t) \nearrow \infty \text{ as } t \searrow T'_{\max}.$$

We set formally $\varrho(t) = \infty$ for $T'_{\max} \le t < T$. Hence, the function $\eta : [0,T) \to (0,\infty)$ is continuous and satisfies $0 < 8\pi\eta \le 1$ for all $0 \le t < T$.

Naturally, we wish to show that there exists a number $0 < \hat{\nu} \le \frac{1}{8\pi}$ such that indeed $T'_{\max} < T$ whenever $0 < \nu \le \hat{\nu}$. This will guarantee (I.91),i.e.,

$$\begin{aligned} \|u_\varepsilon(S,\cdot)\|_{L^\infty(\Omega)} &\le \liminf_{t\to S^-} \|u_\varepsilon(t,\cdot)\|_{L^{2+\varrho(t)}(\Omega)} \\ &\le \liminf_{t\to S^-} U(t) = U(S) < \infty, \end{aligned} \tag{I.98}$$

with $S = T'_{\max} < T$ whenever $0 < \nu < \hat{\nu}$.

Lemma 5.4 *Let $0 < \nu \le \frac{1}{8\pi}$ be arbitrary and set*

$$C = C_4 + C_5 + C_6 > 0.$$

Given the choice of $\eta(t) = \hat{\eta}(\varrho(t), U(t))$ for every $0 \le t < T$, the pair of functions $(\varrho, U) : [0,T) \to \mathbb{R}_+ \times (0,\infty)$ satisfies equations (I.95) *and* (I.97) *for all $0 \le t < T'_{\max} (\le T)$ if and only if $U(t) = \hat{U}(\varrho(t))$ holds for all $0 \le t < T'_{\max}$, where $\hat{U} : [\varrho_0, \infty) \to (0,\infty)$ is the unique solution of the initial value problem*

$$\begin{cases} \frac{d}{d\varrho}\hat{U}(\varrho) = \left[\dfrac{C\nu}{(2+\varrho)(1+\varrho)^{2/p}} + \varphi_\nu(\varrho)\right]\hat{U}(\varrho) \\ \qquad\qquad + \dfrac{N(Z(\varrho)-p)}{p(2+\varrho)^2}\hat{U}(\varrho).\log\hat{U}(\varrho) \quad for\ \varrho_0 \le \varrho < \infty, \\ \hat{U}(\varrho_0) = U_0 = \max\{\|u_0\|_{L^r(\Omega)}, 1\} \ge 1 \end{cases} \tag{I.99}$$

and ϱ is the unique solution of

$$\begin{cases} \frac{d}{d\varrho}\varrho(t) = \frac{1}{\nu}\hat{U}(\varrho)^{Z(\varrho)-2}(2+\varrho)(1+\varrho)^{2/p}\ for\ 0 \le t < T'_{\max}, \\ \varrho(0) = \varrho_0 = r-2 \ge 0. \end{cases} \tag{I.100}$$

Remark 5.1 *It is easy to see that the problem* (I.99) *is equivalent to the following in-homogeneous linear problem for unknown function $V(\varrho) = \log\hat{U}(\varrho)$:*

$$\begin{cases} \frac{d}{d\varrho}V(\varrho) = \dfrac{C\nu}{(2+\varrho)(1+\varrho)^{2/p}} + \varphi_\nu(\varrho) \\ \qquad\qquad + \dfrac{N(Z(\varrho)-p)}{p(2+\varrho)^2}V(\varrho) \quad for\ \varrho_0 \le \varrho < \infty, \\ V(\varrho_0) = V_0 = \max\{\log\|u_0\|_{L^r(\Omega)}, 0\} \ge 0. \end{cases} \tag{I.101}$$

Let us now prove the lemma

Proof. In order to apply our choices, equations (I.96) and (I.97), to problem (I.95), we first calculate the expression

$$\frac{\eta(t)\varrho'(t)}{(2+\varrho)(1+\varrho)^{2/p}} = U(t)^{(p-2)Z(\varrho(t))/p} \quad \text{for } 0 \le t < T'_{\max}$$

and then insert it into (I.95), thus arriving at

$$\begin{cases} \frac{d}{dt}U(t) = \left[C + \frac{1}{\nu}(2+\varrho)(1+\varrho)^{2/p}\varphi_{\eta(t)}\left(\varrho(t)\right)\right]U(t)^{Z(\varrho(t))-1} \\ \qquad \text{for } 0 \le t < T'_{\max}, \\ U(\varrho_0) = U_0 = \max\{\|u_0\|_{L^r(\Omega)}, 1\} \ge 1. \end{cases} \tag{I.102}$$

By the standard theory for system of ordinary differential equations, the system of equations (I.97) and (I.102) for the pair of unknown functions (ϱ, U) of time t, with $\eta(t) = \hat{\eta}(\varrho(t), U(t))$ for every $0 \le t < T$, possesses a unique solution on a maximal interval $0 \le t < T'_{\max}$ for some $T'_{\max} \in (0, T]$. We apply (I.96) and (I.97) again, to get

$$\varphi_{\eta(t)}(\varrho(t)) = \varphi_\nu(\varrho(t)) + \frac{N(Z(\varrho(t)) - p}{p(2+\varrho(t))^{2/p}}\log U(t),$$

$$U(t)^{Z(\varrho(t))-2} = \frac{\nu\varrho'(t)}{(2+\varrho)(1+\varrho)^{1/p}},$$

respectively, and consequently, problem (I.102) becomes

$$\begin{cases} \frac{1}{\varrho'(t)}\frac{d}{dt}U(t) = \left[\dfrac{C\nu}{(2+\varrho)(1+\varrho)^{2/p}} + \varphi_\nu(\varrho(t))\right] U(t) \\ \qquad + \dfrac{N(Z(\varrho(t))-p)}{p(2+\varrho(t))^2}\, U(t)\ \log U(t) \\ \qquad \text{for } 0 \le t < T'_{\max}, \\ U(0) = U_0 = \max\{\|u_0\|_{L^r(\Omega)}, 1\} \ge 1. \end{cases} \tag{I.103}$$

As the function $\varrho(t)$ satisfies $\varrho'(t) > 0$ for all $t \in [0, T'_{\max})$, we are allowed to replace the time variable t in problem (I.103) above by ϱ, thus turning it into the new independent variable for an equivalent initial value problem, namely, problem (I.99). Finally, problem (I.100) is obtained from (I.97). □

We complete now the proof of our proposition 5.2. We show that

$$(\varrho(t), U(t)) \nearrow (\infty, U_\infty) \quad \text{as } t \to T'_{\max}, \tag{I.104}$$

where $1 \le U_\infty < \infty$ and $0 < T'_{\max} < T$, whenever $\nu > 0$ is small enough, say, $0 < \nu < \hat{\nu}$ for some $0 < \hat{\nu} \le \frac{1}{8\pi}$. This claim can be derived from equations (I.99) and (I.100) as follows.

Problem (I.99) is equivalent the in-homogeneous linear problem (I.101) with the coefficients

$$A(\varrho) \stackrel{\text{def}}{=} \frac{N(Z(\varrho(t))-p)}{p(2+\varrho(t))^2} \text{ and } B(\varrho) \stackrel{\text{def}}{=} \frac{C\nu}{(2+\varrho)(1+\varrho)^{2/p}} + \varphi_\nu(\varrho)$$

defined for $\varrho_0 \le \varrho < \infty$. Recall that the expressions $\varphi_\nu(\varrho)$ and $Z(\varrho)$ have been defined in (I.81) and (I.88), respectively. As we take only $0 < \nu \le \frac{1}{8\pi}$, we have $A(\varrho) > 0$ and $B(\varrho) > 0$ together with

$$\int_{\varrho_0}^\infty A(\varrho)\mathrm{d}\varrho < \infty \text{ and } \int_{\varrho_0}^\infty \mathrm{B}(\varrho)\mathrm{d}\varrho < \infty.$$

It follows that

$$\begin{aligned} 0 \le V(\varrho) \quad &\le \quad V_0 \exp\left(\int_{\varrho_0}^\varrho A(\sigma)\mathrm{d}\sigma\right) + \int_{\varrho_0}^\varrho \exp\left(\int_\tau^\varrho A(\sigma)\mathrm{d}\sigma\right) B(\tau)\mathrm{d}\tau \\ &\le \quad \left(V_0 + \int_{\varrho_0}^\infty B(\tau)\mathrm{d}\tau\right) \exp\left(\int_{\varrho_0}^\infty A(\sigma)\mathrm{d}\sigma\right) \equiv \hat{C}_\nu(V_0) < \infty \end{aligned} \tag{I.105}$$

for all $\varrho_0 \le \varrho < \infty$. The constant $\hat{C}_\nu(V_0) > 0$ depends solely upon N, $|\Omega|_N$, p, q, r, ν, and the upper bound $U_0 = \exp(V_0) \ge 1$ on the initial norm

$$\|u_0\|_{L^r(\Omega)} \le U_0.$$

So we have verified $U_\infty \le \exp\left(\hat{C}_\nu(V_0)\right) < \infty$ as desired in (I.104).

It remains to show that $\varrho(t) \nearrow \infty$ as $t \nearrow T'_{\max}$. From problem (I.97) with $U(t) \geq U_\infty \geq 1$ for $0 \leq t < T'_{\max}$, we deduce

$$\begin{cases} \frac{d}{dt}\varrho(t) \geq \frac{1}{\nu}(2+\varrho)(1+\varrho)^{2/p} & \text{for } 0 \leq t < T'_{\max}, \\ \qquad \varrho(o) = 0. \end{cases} \tag{I.106}$$

Now let $\tilde{\varrho} \equiv \tilde{\varrho}_\nu : [0, \tilde{T}_{\max}) \to \mathbb{R}_+$ be the solution of the corresponding initial value problem

$$\begin{cases} \frac{d}{dt}\tilde{\varrho}(t) = \frac{1}{\nu}(2+\tilde{\varrho})(1+\tilde{\varrho})^{2/p} & \text{for } 0 \leq t < \tilde{T}_{\max}, \\ \qquad \tilde{\varrho}(o) = 0. \end{cases} \tag{I.107}$$

where $[0, \tilde{T}_{\max}) \to \mathbb{R}_+$ is the maximal time interval of existence ($0 < \tilde{T}_{\max} \equiv \tilde{T}_{\max,\nu} \leq \infty$.) It is easy to see that $\tilde{T}_{\max,\nu} = \nu \tilde{T}_{\max,1} < \infty$ whenever $0 < \nu \leq 1/8\pi$. The number $\tilde{T}_{\max,1}$ depends solely upon p. We compare equations (I.106) and (I.107) to conclude that $T'_{\max} \leq \tilde{T}_{\max,\nu} = \nu \tilde{T}_{\max,1}$ whenever $0 < \nu \leq 1/8\pi$. thus, fixing

$$\hat{\nu} = \min\{T/\tilde{T}_{\max,1}, 1/8\pi\}$$

we can achieve $T'_{\max} < T$ whenever $0 < \nu < \hat{\nu}$. Moreover, we get $T'_{\max} \to 0^+$ as $\nu \to 0^+$. This proves (I.98) with $S = T'_{\max} < T$ arbitrarily small, $0 < S < T$. Notice that the initial condition U_0 for $U(t)$ at $t = 0$, starting from problem (I.95) is independent of $\varepsilon > 0$. This choice makes the function $U(t)$ of $t \in [0, T)$ independent from ε, as well.
The conclusion of proposition 5.2 follows immediately from (I.98).

Notice that, applying the estimate (I.54) obtained by 5.1, we may replace U_0 by

$$U_0 \overset{\text{def}}{=} \max\{\sup_{\varepsilon>0, 0\leq t<T} \|u_\varepsilon(\cdot,t)\|_{L^r(\Omega)},\ 1\}, \quad 1 \leq U_0 \leq \max\{c, 1\},$$

and the initial condition for $u_\varepsilon(\cdot, t)$ at $t = 0$ by that at $t = t_0$ for any $t_0 \in [0, T - S)$, thus obtaining $u_\varepsilon(\cdot, t) \in L^\infty(\Omega)$ with

$$\|u_\varepsilon(\cdot, s)\|_{L^\infty(\Omega)} \leq U(S) < \infty \quad \text{whenever } S \leq s < T.$$

Recall that S may be chosen arbitrarily small, but the function $U : [0, T) \to (0, \infty)$ will depend on this choice at every time $t \in [0, T)$.

6 Proofs of the main results

6.1 Construction of subsolution to problem (P_ε)

In order to control the singular term near to the boundary of Ω, we need to construct an appropriate subsolution for (P_ε).

We set $\varphi_\varepsilon(t) = \left[(1+\delta)\eta t + \varepsilon^{\delta+1}\right]^{\frac{1}{\delta+1}} - \varepsilon$ with $\eta > 0$ small enough and define the function $\underline{u}_\varepsilon$ as follows

$$\underline{u}_\varepsilon(t, x) = \varphi_\varepsilon(t)\left[(\varphi_1 + \varepsilon^{\frac{p-1+\delta}{p}})^{\frac{p}{p-1+\delta}} - \varepsilon\right] \tag{I.108}$$

where φ_1 is a positive eigenfunction associated with the principal eigenvalue λ_1 of $-\Delta_p$ defined by

$$-\Delta_p\varphi_1 = \lambda_1\varphi^{p-1} \quad \text{ in } \Omega; \quad \varphi_1 = 0 \quad \text{ on } \partial\Omega, \tag{I.109}$$

$\varphi_1 \in W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega})$ and $\varphi_1 > 0$ in Ω. From the strong maximum principle (see J.L. Vázquez [118]), φ_1 satisfies

$$\frac{\partial\varphi_1}{\partial n} := \nabla\varphi_1.\overrightarrow{n} < 0 \text{ on } \partial\Omega. \tag{I.110}$$

and consequently

$$\int_\Omega \varphi_1^\alpha \mathrm{d}x < +\infty \quad \forall\ \alpha > -1. \tag{I.111}$$

Let $C > 0$ and let us prove that for η small enough independently of ε and for all $\varepsilon > 0$ small enough, $\underline{u}_\varepsilon$ is subsolution of the problem

$$(P_s)\begin{cases} \partial_t u - \Delta_p u = (u+\varepsilon)^{-\delta} - C \ \text{ in } Q_T, \\ u = 0 \ \text{ on } (0,T)\times\partial\Omega,\ u > 0 \ \text{ in } Q_T, \\ u(0,x) = u_0 \text{ in } \Omega. \end{cases}$$

Setting $\alpha = \frac{p}{p-1+\delta}$, we have :

$$\begin{aligned} \partial_t\underline{u}_\varepsilon - \Delta_p\underline{u}_\varepsilon &= \eta\,(\varphi_\varepsilon(t)+\varepsilon)^{-\delta}\left[(\varphi_1+\varepsilon^{\frac{1}{\alpha}})^\alpha - \varepsilon\right] \\ &\quad +\lambda_1\alpha^{p-1}\varphi_\varepsilon(t)^{p-1}(\varphi_1+\varepsilon^{\frac{1}{\alpha}})^{(\alpha-1)(p-1)}\varphi_1^{p-1} \\ -\alpha^{p-1}(\alpha-1)(p-1)|\nabla\varphi_1|^p\varphi_\varepsilon(t)^{p-1}(\varphi_1+\varepsilon^{\frac{1}{\alpha}})^{\overbrace{\alpha p - p - \alpha}^{-\alpha\delta}} \end{aligned} \tag{I.112}$$

and we have

$$(\underline{u}_\varepsilon+\varepsilon)^{-\delta} = \left[\varphi_\varepsilon(t)(\varphi_1+\varepsilon^{\frac{1}{\alpha}})^\alpha + (1-\varphi_\varepsilon(t))\varepsilon\right]^{-\delta}$$

The first and third terms in the right hand-side of the equation in (I.112) are dominating in respect with the second one for η and ε small enough. Since $\varphi_1 \in C^1(\overline{\Omega})$, there exists $c_1 > 0$ such that $|\alpha^{p-1}(\alpha-1)(p-1)|\nabla\varphi_1|^p \ \leq c_1$. Then, it suffices to show that

$$\begin{aligned} &\eta\,(\varphi_\varepsilon(t)+\varepsilon)^{-\delta}\left[(\varphi_1+\varepsilon^{\frac{1}{\alpha}})^\alpha - \varepsilon\right] + c_1\varphi_\varepsilon(t)^{p-1}(\varphi_1+\varepsilon^{\frac{1}{\alpha}})^{-\alpha\delta} \\ &\leq [\varphi_\varepsilon(t)(\varphi_1+\varepsilon^{\frac{1}{\alpha}})^\alpha + (1-\varphi_\varepsilon)\varepsilon]^{-\delta} - C. \end{aligned}$$

To this aim, we distinguish the following two cases :

Case 1 : $\varphi_\varepsilon(t)(\varphi_1+\varepsilon^{\frac{1}{\alpha}})^\alpha \geq (1-\varphi_\varepsilon)\varepsilon$. Then

$$(\underline{u}_\varepsilon+\varepsilon)^{-\delta} \geq 2^{-\delta}\varphi_\varepsilon(t)^{-\delta}(\varphi_1+\varepsilon^{\frac{1}{\alpha}})^{-\alpha\delta}. \tag{I.113}$$

Hence

$$\begin{aligned}
&\eta\,(\varphi_\varepsilon(t)+\varepsilon)^{-\delta}\left[(\varphi_1+\varepsilon^{\frac{1}{\alpha}})^{\alpha}-\varepsilon\right]+c_1\varphi_\varepsilon(t)^{p-1}(\varphi_1+\varepsilon^{\frac{1}{\alpha}})^{-\alpha\delta}\\
&\leq \eta(\varphi_\varepsilon(t))^{-\delta}(\varphi_1+\varepsilon^{\frac{1}{\alpha}})^{\alpha}+c_1\varphi_\varepsilon(t)^{p-1}(\varphi_1+\varepsilon^{\frac{1}{\alpha}})^{-\alpha\delta}\\
&\leq (\underline{u}_\varepsilon+\varepsilon)^{-\delta}-C
\end{aligned}$$

for η small enough independently of ε.

Case 2 : $\varphi_\varepsilon(t)(\varphi_1+\varepsilon^{\frac{1}{\alpha}})^{\alpha}\leq(1-\varphi_\varepsilon)\varepsilon$. Then

$$(\underline{u}_\varepsilon+\varepsilon)^{-\delta}\geq 2^{-\delta}(1-\varphi_\varepsilon)^{-\delta}\varepsilon^{-\delta}\geq 2^{-\delta}\varepsilon^{-\delta}.$$

In the other hand, we have the following estimate :

$$\begin{aligned}
\eta\,(\varphi_\varepsilon(t)+\varepsilon)^{-\delta}\left[(\varphi_1+\varepsilon^{\frac{1}{\alpha}})^{\alpha}-\varepsilon\right] &+ c_1\varphi_\varepsilon(t)^{p-1}(\varphi_1+\varepsilon^{\frac{1}{\alpha}})^{-\alpha\delta}\\
&\leq (c_2\eta+c_1\varphi_\varepsilon(t)^{p-1})\varepsilon^{-\delta}
\end{aligned}$$

for a suitable positive constant c_2 independent of ε. Furthermore, for η small enough, we have $c_2\eta+c_1\varphi_\varepsilon(t)^{p-1}<2^{-\delta}$. It follows that $\underline{u}_\varepsilon$ is a subsolution to (P_s) for η small enough. Then, we have by the weak comparison principle :

$$u_\varepsilon\geq\underline{u}_\varepsilon\quad\text{a.e. in }(0,T)\times\Omega. \tag{I.114}$$

Indeed, taking C large enough such that $-C\leq f(x,s,\xi)$ a.e. in $\Omega\times\mathbb{R}_+\times\mathbb{R}^n$, then we have

$$\partial_t(\underline{u}_\varepsilon-u_\varepsilon)-(\Delta_p\underline{u}_\varepsilon-\Delta_p u_\varepsilon)-\left[(\underline{u}_\varepsilon+\varepsilon)^{-\delta}-(u_\varepsilon+\varepsilon)^{-\delta}\right]\leq 0.$$

Multiplying by $(\underline{u}_\varepsilon-u_\varepsilon)$ and integrating on $\Omega^+=\{\underline{u}_\varepsilon\geq u_\varepsilon\}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega^+}(\underline{u}_\varepsilon-u_\varepsilon)^2\mathrm{d}x\leq-\int_{\Omega^+}\left(|\nabla\underline{u}_\varepsilon|^{p-2}\nabla\underline{u}_\varepsilon-|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\right).\nabla(\underline{u}_\varepsilon-u_\varepsilon)\mathrm{d}x\\
&+\int_{\Omega^+}\left[(\underline{u}_\varepsilon+\varepsilon)^{-\delta}-(u_\varepsilon+\varepsilon)^{-\delta}\right](\underline{u}_\varepsilon-u_\varepsilon)\mathrm{d}x\\
&\leq\int_{\Omega^+}\left[(\underline{u}_\varepsilon+\varepsilon)^{-\delta}-(u_\varepsilon+\varepsilon)^{-\delta}\right](\underline{u}_\varepsilon-u_\varepsilon)\mathrm{d}x\leq 0
\end{aligned}$$

which implies (I.114).

6.2 Proof of theorem 2.1

First, let us denote that in the case where $\delta\geq 2+\frac{2}{p-1}$, the second energy estimate can not hold. To overcome this difficulty, we use the local regularity result in X. Chen [123], see also [57]. For that, we show the following preliminary result :

Lemma 6.1 *Let K be a compact subset of Ω, then the sequence u_ε is bounded in $L^p(0,T;W^{1,p}(K))$ uniformly in ε.*

Proof. It suffices to show that for any $\varphi \in \mathscr{D}(\Omega)$ such that $\varphi \geq 0$ and for a suitable constant C (depending on φ but not on ε) :

$$\int_0^T \int_\Omega |\nabla u_\varepsilon|^p \varphi^p \mathrm{d}x\mathrm{d}t \leq C < \infty.$$

To this aim, we multiply by $u_\varepsilon \varphi^p$, and integrate by parts on $Q_T = (0,T) \times \Omega$ to obtain

$$\begin{aligned} & \frac{1}{2}\int_\Omega u_\varepsilon(T)^2 \varphi^p \mathrm{d}x - \frac{1}{2}\int_\Omega u_{\varepsilon,0}^2 \varphi^p \mathrm{d}x \\ & + \int_{Q_T} |\nabla u_\varepsilon|^p \varphi^p \mathrm{d}x\mathrm{d}t + p\int_{Q_T} \left(|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \varphi^{p-1}\right).(\nabla \varphi u_\varepsilon)\,\mathrm{d}x\mathrm{d}t \\ & = \int_{Q_T} (u_\varepsilon + \varepsilon)^{-\delta} u_\varepsilon \varphi^p \mathrm{d}x\mathrm{d}t + \int_{Q_T} f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon \varphi^p \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{I.115}$$

By Young's inequality, we have

$$\begin{aligned} p\int_{Q_T} \left(|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \varphi^{p-1}\right).(\nabla \varphi u_\varepsilon)\,\mathrm{d}x\mathrm{d}t & \leq \frac{1}{4}\int_{Q_T} |\nabla u_\varepsilon|^p \varphi^p \mathrm{d}x\mathrm{d}t \\ & + (4(p-1))^{p-1}\int_{Q_T} |\nabla \varphi|^p u_\varepsilon^p \mathrm{d}x\mathrm{d}t. \end{aligned}$$

Using the growth condition (I.2) and by Young's inequality, it follows :

$$\begin{aligned} \int_{Q_T} f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon \varphi^p \mathrm{d}x\mathrm{d}t & \leq \int_{Q_T} \left(a u_\varepsilon^q + b u_\varepsilon + c|\nabla u_\varepsilon|^{p(1-1/q)} u_\varepsilon\right) \varphi^p \mathrm{d}x\mathrm{d}t \\ & \leq \int_{Q_T} \left(a' u_\varepsilon^q + b u_\varepsilon\right) \varphi^p \mathrm{d}x\mathrm{d}t + \frac{1}{4}\int_{Q_T} |\nabla u_\varepsilon|^p \varphi^p \mathrm{d}x\mathrm{d}t. \end{aligned}$$

Then, by the estimate (I.54) and (I.5), there exists $C_1 > 0$ independent of ε such that

$$\int_0^T \int_\Omega f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon \varphi^p \mathrm{d}x\mathrm{d}t \leq C_1 + \frac{1}{4}\int_0^T \int_\Omega |\nabla u_\varepsilon|^p \varphi^p \mathrm{d}x\mathrm{d}t. \tag{I.116}$$

The singular term is estimated as follows (for $\delta \geq 1$) :

$$\int_{Q_T} (u_\varepsilon + \varepsilon)^{-\delta} u_\varepsilon \varphi^p \mathrm{d}x\mathrm{d}t \leq \int_{Q_T} \underline{u}_\varepsilon^{1-\delta} \varphi^p \mathrm{d}x\mathrm{d}t \leq C_2 < \infty \tag{I.117}$$

where C_2 is independent of ε (if $\delta < 1$ we use (I.54)). So, gathering (I.115), (I.116) and (I.117), we obtain

$$\begin{aligned} \frac{1}{2}\int_\Omega u_\varepsilon(T)^2 \varphi^p \mathrm{d}x + \frac{1}{2}\int_{Q_T} |\nabla u_\varepsilon|^p \varphi^p \mathrm{d}x\mathrm{d}t & \leq \frac{1}{2}\int_\Omega u_{\varepsilon,0}^2 \varphi^p \mathrm{d}x + C_1 + C_2 \\ & \leq \frac{1}{2}\int_\Omega u_0^2 \varphi^p \mathrm{d}x + C_1 + C_2. \end{aligned} \tag{I.118}$$

Hence, the lemma is proved.

Proof of Theorem 2.1 : We first prove that u_ε converge as $\varepsilon \to 0$, to a function u satisfying (P) in the sense of distributions. From X. Chen [123, Theorem 1.1] (which extends the former result in [DiBenedetto, [56]]), and thanks to (I.77) and (I.114), we get :

$$\{\nabla u_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } \mathscr{C}^{0,\alpha}(\tilde{Q}), \quad \text{with } 0<\alpha<1 \tag{I.119}$$

for any compact set $\tilde{Q}$ of Q_T. From the compactness of $\mathscr{C}^{0,\alpha}(\tilde{Q})$ in $\mathscr{C}(\tilde{Q})$, we infer that up to a subsequence

$$u_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u \text{ and } \nabla u_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \nabla u, \quad in\ \mathscr{C}_{loc}(Q_T). \tag{I.120}$$

Hence, we can show that the limit u satisfies the equation in (P) in the sense of distributions. Indeed, thanks to (I.120), we have for any $\varphi \in \mathscr{C}_c^1(Q_T)$

$$\int_{Q_T} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon.\nabla\varphi \mathrm{d}x\mathrm{d}t \underset{\varepsilon\to 0}{\longrightarrow} \int_{Q_T} |\nabla u|^{p-2}\nabla u.\nabla\varphi \mathrm{d}x\mathrm{d}t$$

and using the boundedness of u_ε in $L^r(Q_T)$ given by (I.54), we obtain

$$-\int_{Q_T} u_\varepsilon\partial_t\varphi \mathrm{d}x\mathrm{d}t \underset{\varepsilon\to 0}{\longrightarrow} -\int_{Q_T} u\partial_t\varphi \mathrm{d}x\mathrm{d}t.$$

In the other hand we have :

$$(u_\varepsilon+\varepsilon)^{-\delta}\varphi + f_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\varphi \quad\leq\quad \underline{u}_\varepsilon^{-\delta}\varphi + (au_\varepsilon^{q-1}+b+c|\nabla u_\varepsilon|^{p-p/q})\varphi \leq C<\infty$$

where $C>0$ is independent of ε. Using the Lebesgue theorem, we get as $\varepsilon\to 0$:

$$\int_{Q_T} ((u_\varepsilon+\varepsilon)^{-\delta} + f_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon))\varphi \mathrm{d}x\mathrm{d}t \to \int_{Q_T} (u^{-\delta}+f(x,u,\nabla u))\varphi \mathrm{d}x\mathrm{d}t.$$

Let us now prove that $u^{\frac{\varrho+p}{p}} \in L^p(0,T;W_0^{1,p}(\Omega))$ with $\varrho = r-2$. For that, we integrate the inequality (I.75) from 0 to T and using the estimate $(I.54)$, we obtain :

$$\int_0^T\int_\Omega |\nabla u_\varepsilon^{\frac{\varrho+p}{p}}|^p dx\mathrm{d}t = \int_0^T\int_\Omega |\nabla u_\varepsilon|^p u_\varepsilon^\varrho dx\mathrm{d}t \leq C<\infty. \tag{I.121}$$

Then $u_\varepsilon^{\frac{\varrho+p}{p}}$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$, which implies that $u_\varepsilon^{\frac{\varrho+p}{p}} \underset{\varepsilon\to 0}{\longrightarrow} w$ weakly in $L^p(0,T;W_0^{1,p}(\Omega))$ and by (I.120) we deduce that $u^{\frac{\varrho+p}{p}} = w \in L^p(0,T;W_0^{1,p}(\Omega))$. Now we prove that $u\in\mathscr{C}([0,T);L^r(\Omega))$ and $u(0)=u_0$. We show first that $u\in\mathscr{C}([\eta,T);L^r(\Omega))$ for any $\eta\in(0,T)$. Let $t_0>0$. From (I.119) and (I.120), we have $u(t)\longrightarrow u(t_0)$ in Ω as $t\to t_0$. Then from (I.77) and by applying the Lebesgue theorem, we deduce that $u(t)\longrightarrow u(t_0)$ in $L^r(\Omega)$ as $t\to t_0$. We show now that $\lim_{t\to 0} u(t) = u_0$ in $L^r(\Omega)$. Integrating on $(0,t)\times\Omega$ the inequality (I.75), we obtain :

$$\begin{aligned}\frac{1}{2+\varrho}\int_\Omega u_\varepsilon(t)^{2+\varrho}\mathrm{d}x \quad &- \quad \frac{1}{2+\varrho}\int_\Omega u_{\varepsilon,0}^{2+\varrho}\mathrm{d}x + \beta\int_0^t\int_\Omega |\nabla u_\varepsilon|^p u_\varepsilon^\varrho \mathrm{d}x\mathrm{d}s \\ &\leq \int_0^t \left(C_4\|u_\varepsilon(s)\|_{L^{2+\varrho}(\Omega)}^{(2+\varrho)B(\varrho)} + C_5\right)\mathrm{d}s.\end{aligned} \tag{I.122}$$

Furthermore, from(I.121) it follows that

$$\int_0^t\int_\Omega |\nabla u|^p u^\varrho \mathrm{d}x\mathrm{d}s \le \varliminf_{\varepsilon\to 0}\int_0^t\int_\Omega |\nabla u_\varepsilon|^p u_\varepsilon^\varrho \mathrm{d}x\mathrm{d}s.$$

From (I.54), (I.77) and (I.120) and by the Lebesgue theorem, we get that

$$\int_\Omega u_\varepsilon(t)^{2+\varrho}\mathrm{d}x \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega u(t)^{2+\varrho}\mathrm{d}x,$$

$$\int_\Omega u_{\varepsilon,0}^{2+\varrho}\mathrm{d}x \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega u_0^{2+\varrho}\mathrm{d}x$$

and

$$\int_0^t \|u_\varepsilon\|_{L^{2+\varrho}(\Omega)}^{(2+\varrho)B(\varrho)}\mathrm{d}s \underset{\varepsilon\to 0}{\longrightarrow} \int_0^t \|u\|_{L^{2+\varrho}(\Omega)}^{(2+\varrho)B(\varrho)}\mathrm{d}s.$$

Hence, passing to the limit inf as $\varepsilon\downarrow 0^+$ in (I.122), we get

$$\begin{aligned}
&\frac{1}{2+\varrho}\int_\Omega u(t)^{2+\varrho}\mathrm{d}x - \frac{1}{2+\varrho}\int_\Omega u_0^{2+\varrho}\mathrm{d}x + \beta\int_0^t\int_\Omega |\nabla u|^p u^\varrho \mathrm{d}x\mathrm{d}s\\
&\le \varliminf_{\varepsilon\to 0}\left[\frac{1}{2+\varrho}\int_\Omega u_\varepsilon(t)^{2+\varrho}\mathrm{d}x - \frac{1}{2+\varrho}\int_\Omega u_0^{2+\varrho}\mathrm{d}x + \beta\int_{\varepsilon,0}^t\int_\Omega |\nabla u_\varepsilon|^p u_\varepsilon^\varrho \mathrm{d}x\mathrm{d}s\right]\\
&\le \varliminf_{\varepsilon\to 0}\int_0^t\left(C_4\|u_\varepsilon\|_{L^{2+\varrho}(\Omega)}^{(2+\varrho)B(\varrho)}\mathrm{d}s + C_5\right)\mathrm{d}s = \int_0^t\left(C_4\|u\|_{L^{2+\varrho}(\Omega)}^{(2+\varrho)B(\varrho)}\mathrm{d}s + C_5\right)\mathrm{d}s
\end{aligned}$$

from which we obtain as $t\to 0$

$$\varlimsup_{t\to 0}\int_\Omega u(t)^{2+\varrho}\mathrm{d}x \le \int_\Omega u_0^{2+\varrho}\mathrm{d}x. \tag{I.123}$$

We now prove that $u(t)$ converges weakly to u_0 in $L^r(\Omega)$. Let $0<t<T$ and $\varphi\in\mathscr{D}(\Omega)$. Multiplying the equation in (P_ε) by φ, and integrating by parts in $(0,t)\times\Omega$, we obtain

$$\begin{aligned}
&\int_\Omega u_\varepsilon(t)\varphi\mathrm{d}x - \int_\Omega u_{\varepsilon,0}\varphi\mathrm{d}x + \int_0^t\int_\Omega |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon.\nabla\varphi\mathrm{d}x\mathrm{d}s\\
&= \int_0^t\int_\Omega (u_\varepsilon+\varepsilon)^{-\delta}\varphi\mathrm{d}x\mathrm{d}s + \int_0^t\int_\Omega f_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\varphi\mathrm{d}x\mathrm{d}s.
\end{aligned}\tag{I.124}$$

By (I.114) we have easily

$$\int_0^t\int_\Omega (u_\varepsilon+\varepsilon)^{-\delta}\varphi\mathrm{d}x\mathrm{d}s \underset{\varepsilon\longrightarrow 0}{\longrightarrow} \int_0^t\int_\Omega u^{-\delta}\varphi\mathrm{d}x\mathrm{d}s$$

From Lemma 6.1 and (I.119), it follows that

$$\lim_{\varepsilon\to 0}\int_0^t\int_\Omega |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon.\nabla\varphi\mathrm{d}x\mathrm{d}s = \int_0^t\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla\varphi\mathrm{d}x\mathrm{d}s$$

and

$$\lim_{\varepsilon\to 0}\int_0^t\int_\Omega \left||\nabla u_\varepsilon|^{p-p/q} - |\nabla u|^{p-p/q}\right|\varphi\mathrm{d}x\mathrm{d}s = 0.$$

Hence, from BREZIS [26, theorem 4.9], and by (I.77), we get that up to an extraction of a subsequence of the family $\{u_\varepsilon\}$, there exists $g \in L^1(Q_T)$ such that

$$|f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\varphi| \leq (a u_\varepsilon^{q-1} + b + c|\nabla u_\varepsilon|^{p-p/q})\varphi \leq g.$$

Hence, by the Lebesgue theorem, we deduce that

$$\lim_{\varepsilon\to 0}\int_0^t\int_\Omega f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\varphi \mathrm{d}x\mathrm{d}t = \int_0^t\int_\Omega f(x, u, \nabla u)\varphi \mathrm{d}x\mathrm{d}t.$$

Then we can pass to the limit in (I.124) as $\varepsilon \to 0$ and $t \to 0$, we obtain that

$$\lim_{t\to 0}\int u(t)\varphi \mathrm{d}x = \int_\Omega u_0\varphi \quad \forall\varphi \in \mathscr{D}(\Omega).$$

This implies that $u(t) \underset{t\to 0}{\rightharpoonup} u_0$ in $L^r(\Omega)$. Hence, recalling (I.123), by the uniform convexity of $L^r(\Omega)$, we deduce that

$$u: \ [0, T) \longrightarrow L^r(\Omega) \text{ is continuous and } u(0) = u_0.$$

6.3 Proof of theorem 2.2

First, we prove some energy estimates :

First energy estimate : Let $0 \leq t \leq T$. Multiplying the first equation of (P_ε) by u_ε and integrating by parts on $(0, t) \times \Omega$, we obtain :

$$\frac{1}{2}\int_\Omega u_\varepsilon(t)^2\mathrm{d}x - \frac{1}{2}\int_\Omega u_0^2\mathrm{d}x + \int_0^t\int_\Omega |\nabla u_\varepsilon|^p\mathrm{d}x\mathrm{d}s = \int_0^t\int_\Omega (u_\varepsilon+\varepsilon)^{-\delta}u_\varepsilon\mathrm{d}x\mathrm{d}s$$
$$+\int_0^t\int_\Omega f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)u_\varepsilon\mathrm{d}x\mathrm{d}s. \tag{I.125}$$

We employ the growth condition $(I.2)$ and the Young's inequality to obtain

$$\begin{aligned}
&\int_0^t\int_\Omega f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)u_\varepsilon\mathrm{d}x\mathrm{d}s\\
&\leq a\int_0^t\int_\Omega u_\varepsilon^q\mathrm{d}x\mathrm{d}s + b\int_0^t\int_\Omega u_\varepsilon\mathrm{d}x\mathrm{d}s + c\int_0^t\int_\Omega |\nabla u_\varepsilon|^{p(1-1/q)}u_\varepsilon\mathrm{d}x\mathrm{d}s\\
&\leq \int_0^t\int_\Omega \left(a' u_\varepsilon^q + b u_\varepsilon\right)\mathrm{d}x\mathrm{d}s + \frac{1}{2}\int_0^t\int_\Omega |\nabla u_\varepsilon|^p\mathrm{d}x\mathrm{d}s.
\end{aligned} \tag{I.126}$$

From $(I.5)$ and $(I.54)$, we get easily that

$$\int_0^t\int_\Omega f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)u_\varepsilon\mathrm{d}x\mathrm{d}s \leq \tilde{C} + \frac{1}{2}\int_0^t\int_\Omega |\nabla u_\varepsilon|^p\mathrm{d}x\mathrm{d}s$$

where $\tilde{C}$ is independent of ε. Furthermore, if $\delta \leq 1$, we have by (I.54)

$$\int_0^t\int_\Omega (u_\varepsilon+\varepsilon)^{-\delta}u_\varepsilon\mathrm{d}x\mathrm{d}s \leq \int_0^t\int_\Omega u_\varepsilon^{1-\delta}\mathrm{d}x\mathrm{d}s \leq C. \tag{I.127}$$

Otherwise, if $\delta > 1$, then by using (I.114), we obtain

$$\begin{aligned}\int_0^t \int_\Omega (u_\varepsilon + \varepsilon)^{-\delta} u_\varepsilon \mathrm{d}x\mathrm{d}s \leq \int_0^t \int_\Omega u_\varepsilon^{1-\delta} \mathrm{d}x\mathrm{d}s &\leq \int_0^t \int_\Omega \underline{u_\varepsilon}^{1-\delta} \mathrm{d}x\mathrm{d}s \\ &\leq \int_0^t \int_\Omega \varphi_\varepsilon^{1-\delta} \varphi_1^{\alpha(1-\delta)} \mathrm{d}x\mathrm{d}s \qquad \text{(I.128)}\end{aligned}$$

and we have

$$\int_0^t \int_\Omega \varphi_\varepsilon^{1-\delta} \varphi_1^{\alpha(1-\delta)} \mathrm{d}x\mathrm{d}s = \int_0^t [\varphi_\varepsilon(t)]^{1-\delta} \mathrm{d}s \int_\Omega \varphi_1^{\alpha(1-\delta)} \mathrm{d}x < C < \infty \qquad \text{(I.129)}$$

independently of ε for $\delta < 2 + \frac{1}{p-1}$ (see (I.111)).
Finally, going back to equation (I.125) and using (I.126)–(I.129), we obtain that :

$$\{u_\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^p(0,T;W_0^{1,p}(\Omega)) \qquad \text{(I.130)}$$

independently of ε. This implies that $u_\varepsilon \rightharpoonup u$ in $L^p(0,T;W_0^{1,p}(\Omega))$ as $\varepsilon \to 0$. The uniqueness of u follows from the weak comparison principle as in the proof of Theorem 4.1. We now give the proof for the second assertion of Theorem 2.2, for this purpose we show :
Second energy estimate : Passing to the limit inf in (I.24) as $\Delta t \to 0$ and using (I.36), we obtain that the derivative $\partial_t u_\varepsilon$ satisfies

$$\begin{aligned}\int_0^t \int_\Omega s|\partial_s u_\varepsilon|^2 \mathrm{d}x\mathrm{d}s + \frac{t}{p}\int_\Omega |\nabla u_\varepsilon(t)|^p \mathrm{d}x &\leq C_1 \int_0^t \int_\Omega [f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)]^2 \mathrm{d}x\mathrm{d}s \\ &+ C_2 \int_0^t \int_\Omega |\nabla u_\varepsilon|^p \mathrm{d}x\mathrm{d}s + C_3 \qquad \text{(I.131)}\end{aligned}$$

where C_1, C_2 and C_3 are independent of ε. From the growth condition (I.7), we estimate the first term of the right-hand side in (I.131) as follows :

$$\int_0^t \int_\Omega [f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)]^2 \mathrm{d}x\mathrm{d}s \leq C \int_0^t \int_\Omega \left(u_\varepsilon^{2(q-1)} + 1 + |\nabla u_\varepsilon|^{2\alpha}\right) \mathrm{d}x\mathrm{d}s. \qquad \text{(I.132)}$$

Substituting (I.132) in (I.131) and using (I.54) and (I.130), we infer that

$$u_\varepsilon \text{ is bounded in } L^\infty(\eta, T; W_0^{1,p}(\Omega)) \quad \forall \eta \in (0,T) \qquad \text{(I.133)}$$

and

$$\partial_t u_\varepsilon \text{ is bounded in } L^2((\eta,T)\times\Omega) \quad \forall \eta \in (0,T) \qquad \text{(I.134)}$$

independently of ε. Let us now prove that $u \in \mathscr{C}((0,T]; W_0^{1,p}(\Omega))$. First, we claim that

$$u_\varepsilon \longrightarrow u \text{ in } L^p(\eta, T; W_0^{1,p}(\Omega)) \text{ as } \varepsilon \to 0, \quad \forall \eta \in (0,T). \qquad \text{(I.135)}$$

Indeed, from (I.133) and (I.134) and the compactness result of Aubin-Simon (see [110]) we deduce that up to a subsequence, we have as $\varepsilon \to 0$

$$u_\varepsilon \longrightarrow u \text{ in } L^2((\eta, T) \times \Omega), \quad \forall \eta \in (0, T). \tag{I.136}$$

Multiplying the first equation of (P_ε) by $u_\varepsilon - u$ and integrating on $\tilde{Q}_T = (\eta, T) \times \Omega$, we obtain by convexity arguments that (we do it only for $\delta \neq 1$)

$$\begin{aligned}
&\int_{\tilde{Q}_T} \partial_t u_\varepsilon (u_\varepsilon - u)\mathrm{d}x\mathrm{d}t + \frac{1}{p}\int_{\tilde{Q}_T} |\nabla u_\varepsilon|^p \mathrm{d}x\mathrm{d}t - \frac{1}{p}\int_{\tilde{Q}_T} |\nabla u|^p \mathrm{d}x\mathrm{d}t \\
&\leq \frac{1}{1-\delta}\int_{\tilde{Q}_T} \left[(u_\varepsilon + \varepsilon)^{1-\delta} - (u+\varepsilon)^{1-\delta}\right] \mathrm{d}x\mathrm{d}t \\
&+ \int_{\tilde{Q}_T} f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)(u_\varepsilon - u)\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{I.137}$$

From (I.134) and (I.136), we have :

$$\int_{\tilde{Q}_T} \partial_t u_\varepsilon (u - u_\varepsilon)\mathrm{d}x\mathrm{d}t \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

Since $\delta < 2 + \frac{1}{p-1}$ and using (I.114), we have

$$\int_{\tilde{Q}_T} (u_\varepsilon + \varepsilon)^{1-\delta}\mathrm{d}x\mathrm{d}t \underset{\varepsilon \to 0}{\longrightarrow} \int_{\tilde{Q}_T} u^{1-\delta}\mathrm{d}x\mathrm{d}t. \tag{I.138}$$

We observe by (I.7) that $f(x, u, \nabla u) \in L^2(\Omega)$ and since

$$f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} f(x, u, \nabla u) \text{ in } \mathscr{D}'(Q_T) \text{ and pointwise in } Q_T,$$

it follows that $f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)$ converge weakly to $f(x, u, \nabla u)$ in $L^2(\Omega)$ and the last term in the right hand side in (I.137) converges to 0 as $\varepsilon \to 0$. Hence we get from (I.137) that

$$\overline{\lim_{\varepsilon \to 0}} \int_{\tilde{Q}_T} |\nabla u_\varepsilon|^p \mathrm{d}x\mathrm{d}t \leq \int_{\tilde{Q}_T} |\nabla u|^p \mathrm{d}x\mathrm{d}t.$$

Thus, (I.135) follows from the above inequality and (I.130).
Again since $u \in \mathscr{C}([0, T]; L^r(\Omega))$ and $u \in L^\infty(\eta, T; W_0^{1,p}(\Omega))$ it follows that $u : t \in (0, T) \longrightarrow W_0^{1,p}(\Omega)$ is weakly continuous and satisfies

$$\|u(t_0)\|_{W_0^{1,p}(\Omega)} \leq \liminf_{t \to t_0} \|u(t)\|_{W_0^{1,p}(\Omega)} \quad \forall\, t_0 \in (0, T). \tag{I.139}$$

Let $t_0 > 0$ and v_ε the solution of (P_ε) with the initial data $v_\varepsilon(t_0) = u(t_0) \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$. Similarly as in the proof of theorem 4.1, we derive the above energy estimates for v_ε in $(t_0, T) \times \Omega$. Then, from the uniqueness of the solution for the problem (P), we deduce that

$$v_\varepsilon \longrightarrow u \text{ in } L^p(t_0, T; W_0^{1,p}(\Omega)) \text{ as } \varepsilon \to 0. \tag{I.140}$$

On the other hand, from (I.39), we deduce that v_ε satisfies (for $\delta \neq 1$)

$$\begin{aligned}
&\int_{t_0}^{t}\int_\Omega (\partial_s v_\varepsilon)^2 \,\mathrm{d}x\mathrm{d}s + \frac{1}{p}\int_\Omega |\nabla v_\varepsilon(t)|^p \mathrm{d}x - \frac{1}{1-\delta}\int_\Omega (v_\varepsilon(t)+\varepsilon)^{1-\delta}\,\mathrm{d}x \\
&= \frac{1}{p}\int_\Omega |\nabla u(t_0)|^p \mathrm{d}x - \frac{1}{1-\delta}\int_\Omega (u(t_0)+\varepsilon)^{1-\delta}\,\mathrm{d}x \\
&+ \int_{t_0}^{t}\int_\Omega f_\varepsilon(x, v_\varepsilon, \nabla v_\varepsilon)\,\partial_s v_\varepsilon \mathrm{d}x\mathrm{d}s.
\end{aligned} \tag{I.141}$$

From (I.140) and (I.7) we have $f_\varepsilon(x, v_\varepsilon, \nabla v_\varepsilon) \to f(x, u, \nabla u)$ in $L^2((t_0, T)\times\Omega)$ as $\varepsilon \to 0$. This implies together with (I.134) that

$$\int_{t_0}^{t}\int_\Omega f_\varepsilon(x, v_\varepsilon, \nabla v_\varepsilon)\partial_s v_\varepsilon \mathrm{d}x\mathrm{d}s \underset{\varepsilon\to 0}{\longrightarrow} \int_{t_0}^{t}\int_\Omega f(x, u, \nabla u)\partial_s u \mathrm{d}x\mathrm{d}s.$$

Hence, (I.141) yields

$$\begin{aligned}
&\int_{t_0}^{t}\int_\Omega (\partial_s u)^2 \,\mathrm{d}x\mathrm{d}s + \frac{1}{p}\int_\Omega |\nabla u(t)|^p \mathrm{d}x - \frac{1}{1-\delta}\int_\Omega [u(t)]^{1-\delta}\mathrm{d}x \\
&\leq \frac{1}{p}\int_\Omega |\nabla u(t_0)|^p \mathrm{d}x - \frac{1}{1-\delta}\int_\Omega [u(t_0)]^{1-\delta}\,\mathrm{d}x + \int_{t_0}^{t}\int_\Omega f(x, u, \nabla u)\,\partial_s u \mathrm{d}x\mathrm{d}s.
\end{aligned}$$

Hence to get the continuity of $u : (0, T) \longrightarrow W_0^{1,p}(\Omega)$, we argue by the same way of the proof of Theorem 4.1. This completes the proof of Theorem 2.2.

Chapitre II

Global bifurcation theory in the analytic framework

1 Statement of the main result

We give in this section some results concerning global analytic bifurcation theory. For that we use some material from [BUFFONI AND TOLAND [32], Chapter 9] . Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces over $\mathbb{R}$, $\mathcal{U} \subset \mathbb{R} \times \mathcal{X}$ an open set containing $(0,0)$ in its closure and $F : \mathcal{U} \to \mathcal{Y}$ be an $\mathbb{R}$-analytic function. Define the solution set of $F(\lambda, x) = 0$

$$\mathcal{S} = \{(\lambda, x) \in \mathcal{U} \,:\, F(\lambda, x) = 0\}$$

and the non-singular solutions set

$$\mathcal{R} = \{(\lambda, x) \in \mathcal{S} \,:\, \ker(\partial_2 F(\lambda, x)) = \{0\}\}\,.$$

Suppose that

(G0) $(\lambda, 0) \in \mathcal{U}$ and $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$.

(G1) Bounded closed subsets of $\mathcal{S}$ are compact in $\mathbb{R} \times \mathcal{X}$.

(G2) $\partial_2 F(\lambda, x)$ is a Fredholm operator of index zero for all $(\lambda, x) \in \mathcal{S}$

(G3) There exists an analytic function $(\lambda, u) \,:\, (0, \epsilon) \to \mathcal{S}$ such that
$F(\lambda(s), u(s)) = 0$ for all $s \in (0, \epsilon)$ and $\lim_{s \to 0} (\lambda(s), u(s)) = (0, 0)$

and

$$\mathcal{A}^+ = \{(\lambda(s), u(s)) \,:\, s \in (0, \epsilon)\} \subset \mathcal{R}.$$

Obviously, $\mathcal{A}^+ \subset \mathcal{S}$. The following result gives a global extension of the function (λ, u) from $(0, \epsilon)$ to $(0, \infty)$ in the $\mathbb{R}$-analytic case.

Theorem 1.1 *Suppose (G0)-(G3) hold. Then there exists a continuous curve $\mathcal{A}$ which extends $\mathcal{A}^+$ as follows :*

(a) (λ, u) can be extended as a continuous map (still called) $(\lambda, u) : (0, \infty) \to \mathcal{S}$.

(b) $\mathcal{A}^+ \subset \mathcal{A} \overset{\text{def}}{=} \{(\lambda(s), u(s)) : s \geq 0\} \subset \mathcal{S}$.

(c) $\{s \geq 0 \,:\, \partial_2 F(\lambda(s), u(s))$ is not invertible $\}$ is a discrete set.

(d) *At each of its points $\mathcal{A}$ has a local analytic re-parametrization in the following sense : For each $s^* \in (0,\infty)$ there exists a continuous, injective map $\rho^* : (-1,1) \to \mathbb{R}$ such that $\rho^*(0) = s^*$ and the re-parametrisation*

$$(-1,1) \ni t \to (\lambda(\rho^*(t)), u(\rho^*(t))) \in \mathcal{A} \text{ is analytic.}$$

Furthermore, the map $s \mapsto \lambda(s)$ is injective in a right neighbourhood of $s = 0$ and for each $s^* > 0$ there exists $\epsilon^* > 0$ such that λ is injective on $[s^*, s^* + \epsilon^*]$ and on $[s^* - \epsilon^*, s^*]$.

(e) *One of the following occurs.*

(i) $\|(\lambda(s), u(s))\|_{\mathbb{R}\times\mathcal{X}} \to \infty$ *as* $s \to \infty$.

(ii) *the sequence $\{(\lambda(s), u(s))\}$ approaches the boundary of $\mathcal{U}$ as $s \to \infty$.*

(iii) *$\mathcal{A}$ is the closed loop :*

$$\mathcal{A} = \{(\lambda(s), u(s)) : 0 \leq s \leq T, (\lambda(T), u(T)) = (0,0) \text{ for some } T > 0\}.$$

In this case, choosing the smallest such $T > 0$ we have

$$(\lambda(s+T), u(s+T)) = (\lambda(s), u(s)) \text{ for all } s \geq 0.$$

(f) *Suppose $\partial_2 F(\lambda(s_1), u(s_1))$ is invertible for some $s_1 > 0$. If for some $s_2 \neq s_1$, we have $(\lambda(s_1), u(s_1)) = (\lambda(s_2), u(s_2))$ then (e)(iii) occurs and $|s_1 - s_2|$ is an integer multiple of T. In particular, the map $s \mapsto (\lambda(s), u(s))$ is injective on $[0,T)$.*

Remark 1.1 *(1) There is no claim that $\mathcal{A}$ is a maximal connected subset of $\mathcal{S}$. Other curves or manifolds in $\mathcal{S}$ may intersect $\mathcal{A}$.*

(2) $\mathcal{A}$ may self-intersect in the sense that while $s \mapsto (\lambda(s), u(s))$ is locally injective, it need not be globally injective. For example, in part (e)(iii) of the theorem it is clearly not globally injective.

(3) In part (d), it can happen that the parametrisation has zero derivative, in which case $\{(\lambda(s), u(s)) : |s - s^*| < \delta^*\} \subset \mathcal{A}$ may not be smooth curve even though it has a local analytic parametrisation at every point. Of course, for δ^* sufficiently small, the two segments of the set $\{(\lambda(s), u(s)) : 0 < |s - s^*| < \delta^*\}$, with $(\lambda(s^*), u(s^*))$ deleted, are smooth and can be parametrised by λ.

(4) Alternative (e)(i) is much stronger than the claim that $\mathcal{A}$ is unbounded in $\mathbb{R} \times \mathcal{X}$.

(5) In several cases, the existence of the local analytic path $\mathcal{A}^+$ is obtained by the implicit function theorem in the analytic framework (see for instance the application of Theorem 1.1 in the next chapter). In case of bifurcation from the first eigenvalue, the existence of $\mathcal{A}^+$ is a consequence of the well known Crandall-Rabinowitz result [40]. More precisely, if $F : \mathbb{F} \times \mathcal{X} \to \mathcal{Y}$ is of class $\mathcal{C}^k$, $k \geq 2$, and that $F(\lambda, 0) = 0 \in \mathcal{Y}$ for all $\lambda \in \mathbb{F}$ and if the following conditions hold

$$\partial_2 F(\lambda_0, 0) \text{ is a Fredholm operator of index zero};$$

$$\ker(\partial_2 F(\lambda_0, 0)) = \text{span } \{\xi_0\} \text{ for some } x_0 \in X \setminus \{0\};$$

and the transversality condition

$$\partial^2_{1,2}F(\lambda_0,0)(1,\xi_0) \notin R(\partial_2 F(0,0)).$$

Then there exist $\varepsilon > 0$ *and* $\mathcal{C}^{k-1}$ *map* $(\Lambda,\chi) : (-\epsilon,\epsilon) \to \mathbb{F}\times X$ *such that* $(\Lambda(0),\chi(0)) = (\lambda_0,\xi_0)$ *and* $F(\Lambda(s), s\chi(s)) = 0$ *for all* $0 \le |s| < \epsilon$*, and there exists an open* $U_0 \subset \mathbb{F}\times X$ *such that*

$$\{(\lambda,x)\in U_0 : F(\lambda,x)=0,\ x\neq 0\} = \{(\Lambda(s), s\chi(s)) : 0<|s|<\epsilon\}$$

This identity means that there is unique branch emanating from $(\lambda_0,0)$*. In addition, if* F *is analytic then* Λ *and* χ *are analytic. For the proof of this results we refer to [40, Theorem 1.7] and one uses the analytic version of the Implicit Function Theorem to get the analyticity of the bifurcating curve.*

The proof is based on the Lyapunov-Schmidt reduction (see Theorem 3.1) and the analytic varieties theory, which are the purpose of the following subsections.

2 Analytic varieties theory

We give here some results concerning analytic varieties that we will use further in order to prove Theorem 1.1. We refer to Lojasiewicz [90] for a further description of the theory of analytic varieties. The consequence of these results is that under suitable conditions, an analytic real problem of the form

$$F(\lambda,x)=0, \qquad \text{where } \lambda \text{ is a real parameter,}$$

admits a bifurcate branch of solutions which can be locally analytically parametrised in a neighbourhood of each point. This property is deduced by Puiseux representation in the real case, which gives locally (in particular in a neighbourhood of any singular point) an injective and analytic parametrization of one-dimensional analytic varieties.

2.1 $\mathbb{F}$-analytic varieties

Here, $\mathbb{F}$ denotes $\mathbb{R}$ or $\mathbb{C}$. A function $f : U\subset\mathbb{C}^n \to \mathbb{C}$ is said to be real-on-real if $f(U\cap\mathbb{R}^n)\subset\mathbb{R}$.

Definition 2.1 *Let* $a\in\mathbb{F}^n$*,* $n\ge 1$*. Two sets* S *and* T *of* $\mathbb{F}^n$ *are said to be equivalent at* a *if there is and open and non empty neighbourhood* O *of* a *in* $\mathbb{F}^n$*, such that* $O\cap S = O\cap T$*. We note that this is an equivalence relation on* $2^{\mathbb{F}^n}$*. The corresponding equivalence class, denoted by* $\gamma_a(S)$ *for* $S\subset\mathbb{F}^n$*, is called the germ of* S *at* a *and if* $T\in\gamma_a(S)$*, we say that* T *is a representative of* $\gamma_a(S)$*.*

Remark 2.1 *The finite unions, intersections, inclusion and complements of germs of sets at* a *are defined by the same operations on representatives. (It is easy to check that these are well defined operations on germs, and independent of the chosen representatives)*

Definition 2.2 *Let U a non-empty open set of $\mathbb{F}^n$ and G denotes a finite collection of $\mathbb{F}$-analytic functions $g : U \to \mathbb{F}$. The set*

$$\operatorname{var}(U, G) = \{x \in U : g(x) = 0, \text{ for all } g \in G\}.$$

is called the $\mathbb{F}$-analytic variety generated by G on U. If $U \subset \mathbb{C}^n$ and the elements of G are real-on-real, we say that $\operatorname{var}(U, G)$ is real-on-real. The germ at a of an analytic variety is referred to as an $\mathbb{F}$-analytic germ. We denote by $\mathcal{V}_a(\mathbb{F})$, the set of all $\mathbb{F}$-analytic germs at $a \in \mathbb{F}$.

Definition 2.3 *Let $\alpha \in \mathcal{V}_a(\mathbb{F})$ such that $\alpha = \gamma_0(\operatorname{var}(U, G))$.*

(1) A point $x \in \operatorname{var}(U, G)$ is said to be m-regular if there exists a neighbourhood O of x in $\mathbb{F}^n$ such that $O \cap \operatorname{var}(U, G)$ is an $\mathbb{F}$-analytic manifold of dimension m.

(2) The dimension, $\dim_{\mathbb{F}} \alpha$, is the largest integer m such that every representative of α contains an m-regular point (the point a itself need not be m-regular). If no such integer exists, we say that $\dim_{\mathbb{F}} \alpha = -1$.

Lemma 2.1 *Let $\mathcal{M} \subset \mathbb{F}^n$ is an $\mathbb{F}$-analytic manifold and let $a \in \mathcal{M}$. Then*

(1) $\gamma_a(\mathcal{M}) \in \mathcal{V}_a(\mathbb{F}^n)$.

(2) If $a \in U \cap \mathcal{M}$ and $\operatorname{var}(U, G)$ is an $\mathbb{C}$-analytic variety, there is an open neighbourhood W of a in $\mathcal{M}$ such that $W \setminus \operatorname{var}(W, G)$ is either empty or dense in W.

To prove the above lemma, we need the following well-know result about analytic functions :

Lemma 2.2 *Suppose that X and Y are Banach spaces, that $U \subset X$ is an open connected set and that $F : U \to Y$ is $\mathbb{F}$-analytic. Supppose also that there is a non empty open set $W \subset U$ on which F is identically 0. Then F is identically zero on U.*

Proof. (1) Without loss of generality, suppose that $a = 0 \in \mathcal{M}$ and let $Z_1 = \operatorname{range} df[0]$ (according to the notations in the definition of m-dimensional manifold A.1 in the Annex A), then $\mathbb{F}^n = Z_1 \oplus Z_2$, and write

$$f(x) = f_1(x) + f_2(x), \ f_i(x) \in Z_i, \ i = 1, 2, \ x \in U_0 \subset \mathbb{F}^m$$

where U_0 is a neighbourhood of $0 \in \mathbb{F}^m$. By hypothesis, $df[0]$ has rank m and $df_1[0] : \mathbb{F}^m \to Z_1$ is a bijection. By the implicit inverse theorem, U_0 can be chosen so that $f_1 : U_0 \to W_0$ is a bijection with an analytic inverse for some neighbourhood W_0 of $f_1(0) = 0 \in Z_1$. Now the set $\{f(x) : x \in U_0\}$ is representative of $\gamma_0(\mathcal{M})$ and we have

$$\begin{aligned} \{f(x) : x \in U_0\} &= \{(f_(x), f_2(x)) : x \in U_0\} \\ &= \{(y, f_2 \circ f_1^{-1}(y)) : y \in W_0\} \\ &= \{(y, z) \in W_0 \times Z_2 : z - f_2 \circ f_1^{-1}(y) = 0\}. \end{aligned}$$

Since $f_2 \circ f_1^{-1}$ is analytic this show that $\gamma_0(\mathcal{M}) \in \mathcal{V}_0(\mathbb{F}^n)$.

(2) Let now $\operatorname{var}(U, G)$ an $\mathbb{F}$-analytic variety in $\mathbb{F}^n$, let $0 \in \mathcal{M}$ and let U_0 as above. Let $B \subset U_0$ a ball centred at $0 \in \mathbb{F}^m$ and let $W = f(B)$. Then $0 \in W$, which is a relatively open subset of $\mathcal{M}$.

Suppose that $W \setminus \mathrm{var}(U, G)$ is not dense in W. Then there is an open set $\hat{W} \subset W$ such that $\hat{W} \subset \mathrm{var}(U, G)$. Let $\hat{B} = f^{-1}(\hat{W})$. Then $g \circ f = 0$ on $\hat{B}$ for all $g \in G$. Since $g \circ f$ is analytic and appealing Lemma 2.2 , the set of its zeros has an empty interior, then $g \circ f = 0$ on B for all $g \in G$. Hence $W \subset \mathrm{var}(U, G)$, in other words, $W \setminus \mathrm{var}(U, G) = W \setminus \mathrm{var}(W, G)$ is empty. This proves the result.
□

Lemma 2.3 *Let* $\mathrm{var}(U, G)$ *an* $\mathbb{F}$*-analytic variety in* $\mathbb{F}^n$ *and* $\mathcal{M} \subset U$ *a connected* $\mathbb{F}$*-analytic manifold such that* $\mathcal{M} \cap \mathrm{var}(U, G)$ *has non-empty interior relative to* $\mathcal{M}$*. Then* $\mathcal{M} \subset \mathrm{var}(U, G)$*.*

Proof. Let $\overset{\circ}{N}$ denote the relative interior in $\mathcal{M}$ of $N = \mathcal{M} \cap \mathrm{var}(U, G)$. Note that $\overset{\circ}{N}$ is open in $\mathcal{M}$ by definition and it is non-empty by hypothesis. Suppose that x belongs to the boundary of $\overset{\circ}{N}$ in $\mathcal{M}$. By Lemma 2.1, there is an open neighbourhood W of x in $\mathcal{M}$ such that $W \setminus \mathrm{var}(U, G)$ is either empty or dense in W. Now $W \cap \overset{\circ}{N} \neq 0$ since x is on the boundary of $\overset{\circ}{N}$, and since $\overset{\circ}{N} \subset \mathcal{M} \cap \mathrm{var}(U, G)$ is open, $W \setminus \mathrm{var}(U, G)$ is not dense in W. Hence it is empty, which implies that $x \in \overset{\circ}{N}$. Thus $\overset{\circ}{N}$ is closed in $\mathcal{M}$. By connectedness $\overset{\circ}{N} = \mathcal{M}$ and $\mathcal{M} \subset \mathrm{var}(U, G)$. □

Definition 2.4 *A germ* $\alpha \in \mathcal{V}_a(\mathbb{F}^n)$ *is said to be irreducible if* $\alpha = \alpha_1 \cup \alpha_2$ *for some germs* $\alpha_1, \alpha_2 \in \mathcal{V}_a(\mathbb{F}^n)$ *then* $\alpha = \alpha_1$ *or* $\alpha = \alpha_2$*.*

For example, $\emptyset$ and $\{a\}$, with $a \in \mathbb{F}^n$, are irreducible.

2.2 Weierstrass analytic varieties

The Weierstrass Theorem A.2 given in the Annex A, implies that the set of zeros of an analytic function f in a neighbourhood of $0 \in \mathbb{F}^n$ such that $f(0) = 0$, coincides with the zero level set of a polynomial in x_n, of the form

$$x_n^q + \sum_{k=0}^{q-1} a_k(x_1, \cdots, x_{n-1}) x_n^k \text{ with } a_k(0, \cdots, 0) = 0$$

where the coefficients a_k are analytic in a neighbourhood of $0 \in \mathbb{F}^{n-1}$ and the roots of the polynomial are simple. We need to study further this kind of level set and in this purpose, we present below some useful results about Weierstrass analytic varieties defined in definition 2.6.

Let $W = \{(x_1, \cdots, x_m) \in \mathbb{C}^m : |x_1|, \cdots, |x_m| < \delta\}$ for some $\delta > 0$ and $m \in \mathbb{N}$. (in the case $m = 0$, $W = \{0\}$). In this paragraph we consider $\mathbb{F} = \mathbb{C}$.

Definition 2.5 *A Weirstrass polynomial in* $z \in \mathbb{C}$ *defined on* W *is a polynomial of the form*

$$z^p + \sum_{k=0}^{p-1} a_k(x_1, \cdots, x_m) z^k, \quad p \in \mathbb{N}^* \tag{II.1}$$

such that $a_k : W \to \mathbb{C}$, $k \in \{0, \cdots, p-1\}$, *are an analytic functions which vanish at* $0 \in \mathbb{C}^m$*, and such that the associated discriminant* $D(a_0, \cdots, a_{p-1}, 1) \not\equiv 0$ *(see definition A.2 in the Annex A). We note that when* $m = 0$*, Weierstrass polynomials are of the form* z^p*,* $p \geq 1$*.*

Remark 2.2 *Let $P(z) = z^p + \sum_{k=0}^{p-1} a_k(x_1, \cdots, x_m)z^k$ is a polynomial defined on W such that the analytic functions a_k, $k \in \{0, \cdots, p\}$, vanish at 0 and such that its discriminant is identically zero on W. Then from the simplification Theorem A.3 (Annex A), there is a Weierstrass polynomial h which have the same roots as P (in particular its discriminant is not identically zero on W). Hence the hypothesis $D(a_0, \cdots, a_{p-1}, 1) \not\equiv 0$ in the above definition is not necessary and without loss of generality, we can always assume it.*

Suppose now that $0 \leq m < n$. Let $H = \{h_{m+1}, \cdots, h_n\}$ a family of $n - m$ functions, such that each $h_l \in H$, $l \in \{m+1, \cdots, n\}$, is a Weierstrass polynomial in $z = x_l \in \mathbb{C}$ and with coefficients defined on W, and we write

$$h_l \equiv h_l(x_1, \cdots, x_m;\ x_l) = x_l^{p_l} + \sum_{k=0}^{p_l-1} a_k^l(x_1, \cdots, x_m)x_l^k,\ p_l \in N^*,\ l \in \{m+1, \cdots, n\}.$$

Remark 2.3 *For all $l \in \{m+1, \cdots, n\}$, the polynomial $h_l \in H$ is a analytic function independent of x_k, $\forall k \in \{m+1, \cdots, n\} \setminus \{l\}$. In our study we consider h_l as an analytic function defined on $W \times \mathbb{C}^{n-m}$.*

So we give the following definition :

Definition 2.6 *Let $m, n \in \mathbb{N}$ such that $m < n$ and let H a family of $n - m$ Weierstrass-polynomials as above.*

a) A Weierstrass analytic variety is a set of $\mathbb{C}^n$ of the form

$$\mathrm{var}(W \times \mathbb{C}^{n-m}, H),$$

b) Its discriminant $D = D(H) : W \to \mathbb{C}$ is the product of all the discriminants of the polynomials of H.

c) The branches of a Weierstrass analytic variety is $\mathrm{var}(W \times \mathbb{C}^{n-m}, H)$ are the connected components (i.e. maximal connected set) of the set

$$\mathrm{var}(W \times \mathbb{C}^{n-m}, H) \setminus \left(\mathrm{var}(W, D) \times \mathbb{C}^{n-m}\right). \tag{II.2}$$

Remark 2.4 *The discriminants of the polynomials h_l, $l \in \{m+1, \cdots, n\}$ and then the discriminant of a Weierstrass analytic variety $D(H)$, are analytic functions on W. By abuse of notation, we consider sometimes $D(H)$ as an analytic function on $W \times \mathbb{C}^{n-m}$ even if it is independent of $(x_{m+1}, \cdots, x_n) \in \mathbb{C}^{n-m}$. Then the set given in* (II.2) *can be written as follows :*

$$\mathrm{var}(W \times \mathbb{C}^{n-m}, H) \setminus \mathrm{var}(W \times \mathbb{C}^{n-m}, H \cup D). \tag{II.3}$$

Remark 2.5 *Let $x = (x_1, \cdots, x_m) \in W$ such that $D(x) \neq 0$, then there are at most K points $\xi_j(x) = (x_{m+1}^j, \cdots, x_n^j) \in \mathbb{C}^{n-m}$, $1 \leq j \leq K$, where K is the product of the degrees of h_l, and x_l^j is simple root of h_l, for all $l \in \{m+1, \cdots, n\}$. Hence by the analytic implicit function theorem, each*

$\xi_j(x)$ depends locally and analytically on $x = (x_1, \cdots, x_m) \in W$. Thus each branch is a connected $\mathbb{C}$-analytic manifold of dimension m and it projects onto a part of $W \setminus \mathrm{var}(W, \{D\})$ which is connected by Proposition A.1 in Annex A, and this part is open and closed (by the continuity of roots with respect to $(x_1, \cdots, x_m) \in W$ (see Proposition A.2 in Annex A). Hence the branch projects onto the whole set $W \setminus \mathrm{var}(W, \{D\})$.

Example 2.1 *In the previous definition we take $n = 2$, $m = 1$, $V = \mathbb{C}$ and $H = \{h_2\}$ where $h_2(x_1; Z) = Z^2 - x_1,\ x_1 \in \mathbb{C}$. Then the Weierstrass analytic variety is given by*

$$E := \mathrm{var}(V \times \mathbb{C}, H) = \{(x_1, x_2) \in \mathbb{C}^2 : x_1 = x_2^2\}.$$

Not that $D(H)$ vanishes only at 0 then $\mathrm{var}(V, D(H)) = \{0\}$. Here E has exactly one branch, $B = \{(x_1, x_2) \in \mathbb{C}^2 : x_1 = x_2^2,\ x_1 \neq 0\}$. This illustrate both that a branch is connected, but not in general simply connected.

If $h_2(x_1; Z) = Z^2 + x_1^2$ then $E = \{(z, \pm iz) \in \mathbb{C}^2 : z \in \mathbb{C}\}$ which has two branches $B_\pm = \{(z, \pm iz) \in \mathbb{C}^2 : z \in \mathbb{C},\ z \neq 0\}$ and neither of them is closed under complex conjugation even though E is real-on-real. The following lemma gives an information about this issue.

This lemma will be also used to show some results concerning the real case, in particular steps 2 and 3 in the proof of le theorem 1.1.

Lemma 2.4 *Let B a branch of a Weierstrass analytic variety $\mathrm{var}(W \times \mathbb{C}^{n-m}, H)$. Then, if $\mathrm{var}(W \times \mathbb{C}^{n-m}, H)$ is real-on-real and $B \cap \mathbb{R}^n \neq 0$ then*

$$B^* = \{\overline{z} = (\overline{z}_1, \cdots, \overline{z}_n) \in \mathbb{C}^n, z \in B\} = B$$

Proof. Since every function of H has Taylor expansion at 0 with coefficients real, and B is a maximal connected set in $\mathrm{var}(W \times \mathbb{C}^{n-m}, H) \setminus \mathrm{var}(W \times \mathbb{C}^{n-m}, H \cup D)$, then the set B^* is also maximal in the same sense (because $h(\overline{z}) = \overline{h(z)},\ \forall h \in H$) and by hypothesis $B \cap B^* \neq \emptyset$. Hence $B = B^*$. $\square$

We give now some lemmas in order to characterize the structure of the branches, in particular, the properties of their closure.

Lemma 2.5 *Let B a branch of a Weierstrass analytic variety $E = \mathrm{var}(W \times \mathbb{C}^{n-m}, H)$, with discriminant $\mathcal{D} = D(H)$. Let $\overline{B}$ the relative closure of B in $W \times \mathbb{C}^{n-m}$. Then,*

$$\overline{B} = \mathrm{var}(W \times \mathbb{C}^{n-m}, G)$$

for some finite collection G of analytic functions $g : W \times \mathbb{C}^{n-m} \to \mathbb{C}$.
In addition, if H is real-on-real and $B \cap \mathbb{R}^n \neq 0$, then G is real-on-real and $\overline{B}$ is real-on-real $\mathbb{C}$-analytic variety and $\overline{B} \cap \mathbb{R}^n$ is an $\mathbb{R}$-analytic variety with $\dim \gamma_0(\overline{B} \cap \mathbb{R}^n) = m$.

Proof. Let $x' = (x_1, \cdots, x_m) \in W \setminus \mathrm{var}(W, \{\mathcal{D}\})$, then we can write B as follows

$$[\{x'\} \times \mathbb{C}^{n-m}] \cap B = \{(x', \xi_j(x')) : \xi_j(x') = (\xi_{j,m+1}(x'), \cdots \xi_{j,n}(x')) \in \mathbb{C}^{n-m},\ 1 \leq j \leq K\}$$

where $K \in \mathbb{N}^*$. By the remark 2.5, the dependence of ξ_j on x' is locally $\mathbb{C}$-analytic ($W \setminus \mathrm{var}(W, \{\mathcal{D}\})$ is not in general connected). Therefore, by the connectedness of B, K is independent of $x' \in W \setminus \mathrm{var}(W, \{\mathcal{D}\})$. Let $\xi \in \mathbb{C}^{n-m}$. We remark that $x = (x', \xi) \in B$ if and only if

$$\prod_{j=1}^{K} \langle \varrho, \xi - \xi_j(x') \rangle = 0$$

for all $\varrho \in \mathbb{C}^{n-m}$. Hence, writing this product as follows

$$\prod_{j=1}^{K} \langle \varrho, \xi - \xi_j(x') \rangle = \sum_{\{\sigma \in \mathbb{N}^{n-m} \setminus \{0\} : |\sigma| = K\}} \varrho^{\sigma} \tilde{g}_{\sigma}(x', \xi). \tag{II.4}$$

Note that, for $x' = (x_1, \cdots, x_m) \in W \setminus \mathrm{var}(W, \{\mathcal{D}\})$,

$$x = (x_1, \cdots, x_n) \in \mathbb{C}^n \Longleftrightarrow \tilde{g}_{\sigma}(x_1, \cdots, x_n) = 0 \quad \forall \sigma \in \{s \in \mathbb{N}^* : |s| = K\}.$$

Therefore, $\tilde{g}_{\sigma}$ is bounded on $[W \times \mathbb{C}^{n-m}] \times \mathrm{var}(W, \{\mathcal{D}\})$, then by the Riemann extension theorem (Theorem A.1 in the Annex A), $\tilde{g}_{\sigma}$ can be extended as an analytic function g_{σ} on all of the $W \times \mathbb{C}^{n-m}$. To completes the proof of the first part, we set $G = \{g_{\sigma} : \sigma \in \mathbb{N}^*,\ |\sigma| = K\}$.

Now, suppose that H is real-on-real and $B \cap \mathbb{R}^n \neq 0$. From the implicit function theorem, $B \cap \mathbb{R}^n$ is an $\mathbb{R}$-analytic manifold of dimension m. By the lemma 2.4, for each $j \in \{1, \cdots, r\}$ and for all $x' \in W \setminus \mathrm{var}(W, \{\mathcal{D}\})$, $\xi_j(x') = \overline{\xi_k(\overline{x'})}$ for some k. Therefore the left side of (II.4) is real when ϱ, x' and ξ are real vectors. Therefore $\tilde{g}_{\sigma}$ is real when x' and ξ are real vectors. This shows that G is real-on-real, then $\gamma_0(\overline{B})$ is real-on-real $\mathbb{C}$-analytic variety and $\gamma_0(\overline{B} \cap \mathbb{R}^n)$ is real-on-real $\mathbb{R}$-analytic variety with $\dim_{\mathbb{R}} B \cap \mathbb{R}^n = m$. □

Remark 2.6 *For $m = n-1$, $\overline{B} = \mathrm{var}(W \times \mathbb{C}, G)$ is a Weierstrass analytic variety on W. Indeed, if $m = n-1$ then from (II.4), G has only one element g given by*

$$g(x', \xi) = \prod_{j=1}^{K} (\xi - \xi_j(x')), \text{ for all } (x', \xi) \in W \times \mathbb{C}$$

which is a polynomial in $x_n = \xi$ with coefficients analytic on W and its principal coefficient is 1, all the others vanish at $0 \in W$ and its discriminant is not identically 0.

Example 2.2 *Let $n = 3$, $m = 1$, $W = \mathbb{C}$ and $E = \mathrm{var}(W \times \mathbb{C}^2, \{h, k\})$ where*

$$h(x, y) = y^2 - x^3, \qquad k(x, z) = z^2 - x^3.$$

Then it is easy to show that E has two branches $B_{\pm}$ such that

$$B_{\pm} = \mathrm{var}(W \times \mathbb{C}^2, \{h, k, l^{\pm}\}), \quad \text{where } l^{\pm}(x, y, z) = yz \pm x^3,\ (x, y, z) \in \mathbb{C}^3. \tag{II.5}$$

However $\overline{B}_{\pm}$ are not Weierstrass analytic varieties. Suppose that this is false and that $\overline{B}_{-}$ is a Weierstrass analytic variety defined by

$$y^p + \sum_{k=0}^{p-1} a_k(x) y^k = 0, \quad z^p + \sum_{k=0}^{q-1} b_l(x) z^l = 0, \tag{II.6}$$

where the discriminant of the polynomials is non-zero almost everywhere. Therefore, for almost all x in a neighbourhood of 0 *in* $\mathbb{C}$, *there are exactly pq solutions of* (II.6)*. However, for the same x there are two points (x, y, z) on $\overline{B}_{-}$. Hence $pq = 2$. Suppose $p = 1$ and $q = 2$. Then the system*

$$y = -a_0(x) \text{ and } z^2 = -b_1(x)z - b_0(x)$$

is equivalent to (2.2) *with a minus sign. But this is false since* (2.2) *does not determine y as an analytic function of x in a neighbourhood of* 0*. A similar contradiction is reached for $\overline{B}_{+}$.*

2.3 Germs of $\mathbb{C}$-analytic varieties and subspaces

The purpose of this subsection is to give a description of one-dimensional germs and the way they emerge from higher dimensional germs as it is given in theorems 2.4 and 2.5. These properties will be used in the proof of Theorem 1.1.

Let $\alpha \in \mathcal{V}_0(\mathbb{C}^n)$ a germ of an analytic variety. If $n = 1$ then $\alpha \in \{\emptyset, \{0\}, \gamma_0(\mathbb{C})\}$. If $\{0\} \subset \alpha$ but $\alpha \notin \{\{0\}, \gamma_0(\mathbb{C}^n)\}$ then $n \geq 2$. Moreover, since $\alpha \neq \gamma_0(\mathbb{C}^n)$, there is a non-trivial subspace $T \subsetneqq \mathbb{C}^n$ such that

$$\gamma_0(T) \cap \alpha = \{0\}. \tag{II.7}$$

Indeed, suppose that $\text{var}(U, G)$ is a representative of α. Then there is at least one $\mathbb{C}$-analytic function $g \in G$, such that $g \not\equiv 0$ on U. Hence there is $b \in \mathbb{C}^n$, such that $g \not\equiv 0$ on the subspace $Y = \{tb, t \in \mathbb{C}\}$. Since g restricted to $Y \cap U$ is a $\mathbb{C}$-analytic function of one complex variable, its zeros are isolated. This show that the complex linear space Y satisfies $\alpha \cap \gamma_0(Y) = \{0\}$. Let us denote that if $T = \mathbb{C}^n$ then $\alpha = \{0\}$. Hence the result.
In addition, if α is real-on-real (Definition 2.2), then $\alpha \cap \gamma_0(\mathbb{R}^n) \neq \gamma_0(\mathbb{R}^n)$. Indeed, we argue by contradiction, suppose that $\alpha \cap \gamma_0(\mathbb{R}^n) = \gamma_0(\mathbb{R}^n)$, and let $g \in G$ is real-on-real. Then $g \equiv 0$ in a neighbourhood of $0 \in \mathbb{R}^n$. It follows due to the Taylor expansion that $d^{(k)}g(0) = 0$, for all $k \in \mathbb{N}$. Hence due to the analyticity of g, we can use again the Taylor expansion to deduce that $g \equiv 0$ in a neighbourhood of $0 \in \mathbb{C}^n$, and this contradicts with hypothesis. Thus, if $\alpha \neq \gamma_0(\mathbb{C}^n)$ then $\alpha \cap \gamma_0(\mathbb{R}^n) \neq \gamma_0(\mathbb{R}^n)$ and hence there exists a real subspace $\hat{T} = \{tb : t \in \mathbb{R}\}$ with $b \in \mathbb{R}^n$ such that $\gamma_0(\hat{T}) \cap \alpha = \{0\}$. Moreover $\gamma_0(T) \cap \alpha = \{0\}$ on the complex space $T = \{zb : z \in \mathbb{C}\}$.

Definition 2.7 *A complex linear subspace of $T \subset \mathbb{C}^n$ is said to be a complexified subspace of $\mathbb{C}^n$ if it has a real basis. Equivalently T is a complexified subspace if*

$$T^* = \{\overline{z} = (\overline{z}_1, \cdots, \overline{z}_n) \in \mathbb{C}^n : z \in T\} = T$$

Remark 2.7 *Any complexified subspace Z_1 of $\mathbb{C}^n$ has (in fact many) complementary complexified* *subspace Z_2 of $\mathbb{C}^n$ such that $\mathbb{C}^n = Z_1 \oplus Z_2$. This ensure that the choice of basis, in the last part of the next lemma, is possible.*

Lemma 2.6 *Let $\alpha \in \mathcal{V}_0(\mathbb{C}^n)$, $n \geq 2$ and T a linear subspace of $\mathbb{C}^n$ such that* (II.7) *holds. Choose a basis of $\mathbb{C}^n$ such that*

$$T = \{(0, \cdots, 0, z_{m+1}, \cdots, z_n) \in \mathbb{C}^n : (z_{m+1}, \cdots, z_n) \in \mathbb{C}^{n-m}\} \tag{II.8}$$

Then $\gamma_0(P(E))$ is the germ of an analytic variety in $\mathbb{C}^m$, where E is a representative of α and $P(E)$ the projection onto $\mathbb{C}^m$ of E, i.e.

$$P(E) = \{(z_1, \cdots, z_m) \in \mathbb{C}^m,\ \exists (z_{m+1}, \cdots, z_n) \in \mathbb{C}^{n-m} : (z_1, \cdots, z_n) \in E\}. \tag{II.9}$$

If T is a complexified subspace, α is real-on-real, and choosing a real basis of $\mathbb{C}^n$ such that (II.8) *holds, then $\gamma_0(P(E))$ is real-on-real.*

Proof. The cases $m = 0$ ($\alpha = \gamma_0(\mathbb{C}^n),\ T = \{0\}$) and $m = n$ ($\alpha = \{0\},\ T = \mathbb{C}^n$) are trivial. Suppose that $0 < m < n$. In the coordinates (II.8), let $\alpha = \gamma_0(E)$ such that $E = \mathrm{var}(V \times B_\delta(\mathbb{C}), \{f_1, \cdots, f_k\}\}$ where $k \in N^*$, $V = \{(x_1, \cdots, x_{n-1}) \in \mathbb{C}^{n-1} : |x_1|, \cdots, |x_{n-1}| < \delta\}$ and $B_\delta(\mathbb{C}) = \{x_n \in \mathbb{C} : |x_n| < \delta\}$ for small positive δ. Since

$$\{(0, \cdots, 0, z_n) \in \mathbb{C}^n\} \subset T \text{ and } \gamma_0(T) \cap \alpha = \{0\}$$

$f_j(0, \cdots, 0, \cdot) \not\equiv 0$ on $B_\delta(\mathbb{C})$ for some $j \in \{1, \cdots, k\}$. Relabeling $\{f_1, \cdots, f_k\}$ if necessary, we suppose that $j = 1$. By the assertion (2) of Weierstrass Preparation Theorem A.2 in Annex A, there is no loss of generality in supposing f_1 is given by a polynomial of the form :

$$h_1(x_1, \cdots, x_{n-1};\ x_n) = x_n^p + \sum_{\nu=0}^{p-1} a_\nu(x_1 \cdots, x_{n-1}) x_n^\nu \text{ with } a_\nu(0) = 0 \tag{II.10}$$

where $p \geq 1$ is a degree of h. Then by the assertion (1) of Weierstrass preparation theorem A.2, we may suppose without loss of generality that each of the other f_j, in the definition of α, has the form

$$f_j(x_1, \cdots, x_n) = h_j(x_1, \cdots, x_{n-1};\ x_n), \quad j \in \{2, \cdots, k\}. \tag{II.11}$$

where h_j is a polynomial in x_n defined on V of degree at most $p-1$. Thus the family $\{f_1, \cdots, f_k\}$ of analytic functions satisfies the hypotheses of the projection lemma A.1 (see Annex A). Therefore the projection of $E = \mathrm{var}(V \times B_\delta(\mathbb{C}), \{f_1, \cdots, f_k\}\}$ (a representative of α) onto $\mathbb{C}^{n-1} \times \{0\}$ is an analytic variety in $\mathbb{C}^{n-1}$. Let $\beta \in \mathcal{V}_0(\mathbb{C}^{n-1})$ denotes its germ. If $m = n-1$ then the proof is complete. If $m < n-1$,

$$\hat{T} = \{(z_1, \cdots, z_{n-1}) \in \mathbb{C}^{n-1} : z_1 = \cdots = z_m = 0)\}$$

Since $\alpha \cap \gamma_0(T) = \{0\}$, $\beta \cap \gamma_0(\hat{T}) = \{0\}$ in $\mathbb{C}^{n-1}$. We can repeat the argument $n - m$ times to prove the first part of the lemma.

In the case when T is a complexified subspace of $\mathbb{C}^n$ and α is real-on-real, the projection lemma A.1 at each step gives a real-on-real analytic variety. This completes the proof. $\square$

Definition 2.8 *Let T a linear space of $\mathbb{C}^n$ satisfies* (II.7)*, Then T is said to be maximal with respect to $\alpha \in \mathcal{V}_0(\mathbb{C}^n)$ if*

$$\alpha \cap \gamma_0(Y) \neq \{0\} \textit{ for any subspace } Y \subset \mathbb{C}^n \textit{ such that } T \subsetneqq Y.$$

Lemma 2.7 *Suppose $n \geq 2$. Let T a non trivial subspace of $\mathbb{C}^n$ satisfying* (II.7) *which is maximal with respect to $\alpha \in \mathcal{V}_0(\mathbb{C}^n)$ and let $m = n - \dim T \in \{1, \cdots, n-1\}$ and choose the coordinates* (II.8)*. Then $\gamma_0(P(E)) = \gamma_0(\mathbb{C}^m)$.*

Proof. We have seen in the previous lemma that $P(E)$ is an analytic variety of $\mathbb{C}^m$ at 0. Suppose that its germ at 0 is $\beta \neq \gamma_0(\mathbb{C}^m)$. Then there exists a non-trivial subspace $L \subset \mathbb{C}^m$ such that $\beta \cap \gamma_0(L) = \{0\}$. Let $Y = L \times \mathbb{C}^{n-m}$. Then $T \subsetneqq Y$ and $\gamma_0(Y) \cap \alpha = \{0\}$. which violates the maximality of T. $\square$

We give now the same result concerning the real-on-real germs.

Lemma 2.8 *Suppose that $\alpha \in \mathcal{V}_0(\mathbb{C}^n)$ is real-on-real, $n \geq 2$, and T is a complexified subspace of $\mathbb{C}^n$ (see definition 2.7) such that $\alpha \cap \gamma_0(T) = \{0\}$. Suppose that T is maximal in the sense that if $\tilde{T}$ is a complexified subspace then*

$$(T \subset \tilde{T} \neq T) \Longrightarrow (\alpha \cap \gamma_0(\tilde{T}) \neq \{0\}) \tag{II.12}$$

Then, with respect to the real basis such that

$$T = \{(z_1, \cdots, z_n) \in \mathbb{C}^n : z_1 = \cdots = z_n = 0\}$$

we have $\gamma_0(P(E)) = \gamma_0(\mathbb{C}^m)$.

Proof. We have in the lemma 2.6 that in this case $P(E)$ is real-on-real analytic variety in $\mathbb{C}^m$. Suppose that its germ $\gamma_0(P(E)) \neq \gamma_0(\mathbb{C}^m)$. Then, as we have seen at the beginning of this subsection, there exists a non-trivial complexified subspace L of $\mathbb{C}^m$ such that $\gamma_0(P(E)) \cap \gamma_0(L) = \{0\}$. Hence $\tilde{T} = L \times \mathbb{C}^{n-m}$ is a complexified subspace of $\mathbb{C}^n$ satisfies $\alpha \cap \gamma_0(\tilde{T}) = \{0\}$, which violates the maximality of T. This proves the lemma. $\square$

Example 2.3 *Let $n = 4$, and let $g, h : \mathbb{C}^4 \to C$ defined by*

$$g(w, x, y, z) = y^2 + z^2, \quad h(w, x, y, z) = x, \quad (w, x, y, z) \in \mathbb{C}^4.$$

Then $\alpha = \gamma_0(E)$ is real-on-real, where

$$E = \mathrm{var}(\mathbb{C}^4, \{g, h\}) = \{(w, 0, y, \pm iy) : w, y \in \mathbb{C}\}$$

and we have

$$E \cap \mathbb{R}^4 = \{(w,0,0,0) : w \in \mathbb{R}\}.$$

The subspace $T = \{0\} \times \mathbb{C}^3$ is a complexified subspace satisfies $\gamma_0(T) \cap E \neq \{0\}$, then it doesn't satisfy Definition 2.8 and (II.12)*. It is easy to show that the complexified subspaces $\{0\} \times \mathbb{C}^2 \times \{0\}$ and $\{0\} \times \mathbb{C} \times \{0\} \times \mathbb{C}$ are maximal in the sense of Definition 2.8 and* (II.12)*.*

Theorem 2.1 (Representation theorem) *Let $\alpha \in \mathcal{V}_0(\mathbb{C}^n)\backslash\{\{0\}, \gamma_0(\mathbb{C}^n)\}$, $\{0\} \subset \alpha$ and $n \geq 2$. After a change of the basis of $\mathbb{C}^n$ as in* (II.8)*, there exists a Weierstrass analytic variety* $\mathrm{var}(W \times \mathbb{C}^{n-m}, H)$*, $m \in \{1, \cdots, n-1\}$, such that*

$$\alpha \subset \gamma_0(\mathrm{var}(W \times \mathbb{C}^{n-m}, H)).$$

Proof. Let T a subspace of $\mathbb{C}^n$ such that (II.7) holds. We choose the coordinates (II.8). Hence as in the proof of the lemma 2.6, and by using the simplification theorem A.3 (see the annex A), we can reduce the problem to the case when $\alpha = \gamma_0(E)$, with $E = \mathrm{var}(V \times \mathbb{C}, G)$, where $G = \{f_1, \cdots, f_k\}$, $k \in \mathbb{N}^*$ such that

$$f_1(x_1, \cdots, x_n) = x_n^p + \sum_{\nu=0}^{p-1} a_\nu(x_1 \cdots, x_{n-1}) x_n^\nu \text{ with } a_\nu(0) = 0$$

with $a_\nu(0) = 0$, $0 \leq \nu \leq p-1$, $D(a_0, \cdots, a_{p-1}, 1) \not\equiv 0$ on V, and f_j, $j \geq 2$, is a polynomial in the same variable $x_n \in \mathbb{C}$, with coefficients that are analytic functions of the other variables on V. Note that f_1 is a Weierstrass polynomial (Defintion 2.5). If $k = 1$, then $m = n-1$ and the theorem holds with $W = V$ and $H = \{f_1\}$. In fact, we have $\alpha = \gamma_0(\mathrm{var}(W \times \mathbb{C}, H)$ in this case. This also covers the case $n = 2$. Suppose now that $n \geq 3$ and $k \geq 2$. We argue by induction on n. Let $\mathcal{A} = \mathrm{var}(V, K)$ be the projection of $E = \mathrm{var}(V \times \mathbb{C}, G)$ into $\mathbb{C}^{n-1}$ (see the projection lemma A.1 in the annex A). Then since $n \geq m+2$, we have $\{0\} \subsetneqq \gamma_0(\mathcal{A}) \neq \gamma_0(\mathbb{C}^{n-1})$ and we have $\gamma_0(\hat{T}) \cap \gamma_0(\mathcal{A}) = \{0\}$ in $\mathbb{C}^{n-1}$ where

$$\hat{T} = \{(z_1, \cdots, z_{n-1}) \in \mathbb{C}^{n-1} : z_1 = \cdots = z_m = 0)\}.$$

Hence by the inductive hypothesis, we suppose that the theorem holds for all $\hat{n} \in \mathbb{N}$ such that $2 \leq \hat{n} \leq n-1$. Then let $\mathrm{var}(W \times \mathbb{C}^{n-m-1}, \hat{H})$ be a Weierstrass analytic variety such that $\gamma_0(\mathcal{A}) \subset \gamma_0(\mathrm{var}(W \times \mathbb{C}^{n-m-1}, \hat{H}))$ where

$$W = \{(x_1, \cdots, x_m) \in \mathbb{C}^m : |x_1|, \cdots, |x_m| < \delta\} \text{ with } \delta > 0 \text{ small enough}$$

$\hat{H} = \{\hat{h}_l, m+1 \leq l \leq n-1\}$, $\hat{\mathcal{D}} : W \to \mathbb{C}$ is its discriminant which is not identically zero. We now construct the Weierstrass analytic variety satisfying the statement of theorem 2.1. For $(x_1, \cdots, x_m) \in W \setminus \mathrm{var}(W, \{\mathcal{D}\})$ and $x_n \in \mathbb{C}$, we set

$$\tilde{h}_n(x_1, \cdots,\ x_n) = \prod_{\substack{(\xi_{m+1}, \cdots, \xi_{n-1}) \\ \in \Upsilon(x_1, \cdots, x_m)}} f_1(x_1, \cdots, x_m, \xi_{m+1}, \cdots, \xi_{n-1}, x_n)$$

where

$$\Upsilon(x_1, \cdots, x_m) = \{(\xi_{m+1}, \cdots, \xi_{n-1}) \in \mathbb{C}^{n-m-1} : \hat{h}_l(x_1, \cdots, x_m;\ \xi_l) = 0,\ m+1 \leq l \leq n-1\}.$$

Since $\hat{\mathcal{D}} \not\equiv 0$, the dependence of ξ_l on $(x_1, \cdots, x_m) \in W \setminus \mathrm{var}(W, \{\hat{\mathcal{D}}\})$ is locally analytic by implicit function theorem. (Recall that the set $W \setminus \mathrm{var}(W, \{\hat{\mathcal{D}}\})$ is an open, dense, connected subset of W (see Proposition A.1 in the Annex A)). By choosing a smaller value of δ in the definition of W if necessary, we see that the coefficients of $\tilde{h}_n$ are bounded and hence, by the Riemann extension theorem A.1 in the annex A, $\tilde{h}_n$ can be extended as $\mathbb{C}$-analytic function to all of W. Note that in $\tilde{h}_n$, the coefficient of the highest power of x_n is 1 and all the other coefficients vanish at $0 \in W$. After simplification (Remark 2.2), $\tilde{h}_n$ becomes a Weierstrass polynomial defined on W. Hence we set $H = \hat{H} \cup \{\hat{h}_n\}$ and then we can easily see that the germ α satisfies

$$\alpha \subset \gamma_0(\mathrm{var}(W \times \mathbb{C}^{n-m}, H)).$$

This completes the proof. □

Theorem 2.2 *Let α a germ as in the preceding theorem. Let T be a maximal subspace of $\mathbb{C}^n$ satisfying* (II.7) *(see Definition 2.8) and choose the coordinates* (II.8). *Hence $\gamma_0(B) \subset \alpha$ for some branch B of the Weierstrass analytic variety* $\mathrm{var}(W \times \mathbb{C}^{n-m}, H)$ *obtained in Theorem 2.1.*

Proof. Similarly as in the proof of Theorem 2.1, let $E = \mathrm{var}(W \times \mathbb{C}^{n-m}, G)$ be a representative of α in the new coordinates. Let $\mathcal{D} : W \to \mathbb{C}$ the discriminant of a Weierstrass analytic variety $\mathrm{var}(W \times \mathbb{C}^{n-m}, H)$. Note that $\mathcal{D} \not\equiv 0$ on W and then $W \setminus \mathrm{var}(W, \{\mathcal{D}\})$ is an open, dense, connected subset of W (see Proposition A.1 in the annex A). Let U be a non-empty open, connected subset of $W \setminus \mathrm{var}(W, \{\mathcal{D}\})$. Then we have

$$[U \times \mathbb{C}^{n-m}] \cap [\mathrm{var}(W \times \mathbb{C}^{n-m}, H)] = \{(x, \xi_j(x)) \in \mathbb{C}^n : 1 \leq j \leq r\}$$

where $r \in \mathbb{N}^*$ is the product of degrees of polynomial of H, and $\xi_j : U \to \mathbb{C}^{n-m}$, $j \in \{1, \cdots, r\}$, are analytic mappings (see Remark 2.5). For $j \in \{1, \cdots, k\}$, we set

$$U_j = \{x \in U : (x, \xi_j(x)) \notin E = \mathrm{var}(W \times \mathbb{C}^{n-m}, G)\}$$

Note that U_j can be written as follows :

$$U_j = U \setminus \mathrm{var}(U, \{\tilde{g}_1, \cdots, \tilde{g}_k\})$$

where $\tilde{g}_\nu : U \ni x \mapsto g_\nu(x, \xi_j(x))$, $g_\nu \in G$ for all $\nu \in \{1, \cdots, k\}$. Hence by Proposition A.1, either U_j is empty or is an open, dense subset of U. Otherwise, we claim that the intersection :

$$\bigcap_{j=1}^{r} U_j = \emptyset.$$

Indeed, let $x \in \bigcap_{j=1}^{r} U_j$, hence

$$(x, \xi_j(x)) \notin E, \quad \forall j \in \{1, \cdots, r\}.$$

From Lemma 2.7, we have $\gamma_0(P(E)) = \gamma_0(\mathbb{C}^m)$, then there exists $\zeta \in \mathbb{C}^{n-m}$ such that

$$(x, \zeta) \in E \subset \mathrm{var}(W \times \mathbb{C}^{n-m}, H) \text{ and } \zeta \neq \xi_j(x),\ \forall\, j \in \{1, \cdots, r\}.$$

Thus $x \in \mathrm{var}(W, \{\mathcal{D}\})$ which contradicts the fact that $x \in U$. Hence, there is some j such that $U_j = \emptyset$ and then the non-empty set $O = \{(x, \xi_j(x)),\ x \in U\} \subset E$. Let B the branch which contains O. Then by Lemma 2.3, we deduce that $B \subset E$ and hence the result. $\square$

Theorem 2.3 *Let α, $\mathrm{var}(W \times \mathbb{C}^{n-m}, H)$, $1 \leq m \leq n-1$, given by Theorem 2.1. Let $\mathcal{D}$ the discriminant of a Weierstrass analytic variety $\mathrm{var}(W \times \mathbb{C}^{n-m}, H)$, and choose m such that*

$$n - m = \max\{\dim T : T \textit{ is a linear subspace of } \mathbb{C}^n,\ \gamma_0(T) \cap \alpha = \{0\}\}.$$

Then, $\dim_{\mathbb{C}} \alpha = m$, $\mathrm{var}(W \times \mathbb{C}^{n-m}, H)$ has no manifold of dimension larger that m and $\mathrm{var}(W \times \mathbb{C}^{n-m}, H \cup \{\mathcal{D}\})$ has no manifold of dimension equal to (or greater that) m.

Proof. Let B the branch given by Theorem 2.2. Then $m = \dim_{\mathbb{C}} B \leq \dim_{\mathbb{C}} \alpha$. If $\mathcal{D}$ is nowhere zero, or zero only at $0 \in W$, then it is easy to see that $E = \mathrm{var}(W \times \mathbb{C}^{n-m}, H)$ contains no manifolds of dimension larger than m. Suppose now that $\{0\} \subset \gamma_0(\mathrm{var}(W, \{\mathcal{D}\})) \notin \{\{0\}, \gamma_0(\mathbb{C}^m)\}$ and let $\mathcal{M}$ be a manifold of dimension strictly larger than m. which is a subset of E. Then, since $\dim B = m$, $B \cap \mathcal{M} = \emptyset$ for all branches B of E. In other words, $\mathcal{M} \subset \mathrm{var}(W \times \mathbb{C}^{n-m}, H \cup \{\mathcal{D}\})$.
Let $L = \{(0, \cdots, 0, z_m) : z_m \in \mathbb{C}\}$ a subspace of $\mathbb{C}^m$. Then we have

$$\gamma_0(\mathrm{var}(W, \{\mathcal{D}\})) \cap \gamma_0(L) = \{0\}$$

and hence

$$\gamma_0(\mathrm{var}(W \times \mathbb{C}^{n-m}, H \cup \{\mathcal{D}\})) \cap \gamma_0(Y) = \{0\}$$

where $Y = L \times \mathbb{C}^{n-m}$. Therefore, The representation Theorem 2.1 yields the existence of a Weierstrass analytic variety $\mathrm{var}(\tilde{W} \times \mathbb{C}^{n-\tilde{m}}, \tilde{H})$, $\tilde{m} < m$, such that

$$\mathcal{M} \subset \mathrm{var}(W \times \mathbb{C}^{n-m}, H \cup \{\mathcal{D}\}) \subset \mathrm{var}(\tilde{W} \times \mathbb{C}^{n-\tilde{m}}, \tilde{H}).$$

Repeat finitely often we find that this holds with $\tilde{m} = 1$, which is impossible. Hence $\mathrm{var}(W \times \mathbb{C}^{n-m}, H \cup \{\mathcal{D}\})$ contains no manifold of dimension larger than m and $\mathrm{var}(W \times \mathbb{C}^{n-m}, H \cup \{\mathcal{D}\}))$ contains no manifold of dimension m or more and then $\dim_{\mathbb{C}} \alpha = m$. $\square$

Lemma 2.9 *Let B be a branch of a Weierstrass analytic variety $\mathrm{var}(W \times \mathbb{C}^{n-m}, H)$ and suppose $\alpha \in \mathcal{V}_0(\mathbb{C}^n)$ is such that*

$$\gamma_0(\overline{B}) \neq \alpha \subset \gamma_0(\overline{B}).$$

Then $\dim_{\mathbb{C}} \alpha < m = \dim_{\mathbb{C}} B$.

Proof. Let $m' = \dim_{\mathbb{C}} \alpha$, and suppose that $E = \mathrm{var}(U, G) \subset \overline{B} \cap (W \times \mathbb{C}^{n-m})$ such that $\gamma_0(E) = \alpha$, where U is an open set with $B \cup \{0\} \subset U$. Let $D(H)$ denote the discriminant of H on W and suppose that $m' \geq m$. We will infer that $\gamma_0(\overline{B}) = \alpha$, a contradiction which will prove the lemma.

Define an analytic manifold $\mathcal{M} \subset E$ as consisting of all m'-regular points of E. If $\mathcal{M} \subset \mathrm{var}(W \times \mathbb{C}^{n-m}, H \cup \{D(H)\}$, it follows by the previous theorem that $m' < m$. Since this is false by assumption, $\mathcal{M} \cap B \neq \emptyset$ and $m' = \dim_{\mathbb{C}} B = m$. Therefore there exists a point $z \in \mathcal{M} \cap B$ which has a neighbourhood O_z in B which is a subset of $\mathcal{M}$. From the lemma 2.3 and the fact that B is a connected manifold, it follows that $B \subset E$. Hence $\alpha = \gamma_0(\overline{B})$. This contradiction proves the lemma. □

Theorem 2.4 (Structure theorem.) *Let* $\alpha \in \mathcal{V}_0(\mathbb{C}^n)$ *with* $n \geq 2$ *and* $\{0\} \subsetneqq \alpha \neq \gamma_0(\mathbb{C}^n)$. *Then there exist sets* $B_1, \cdots, B_N$, *where each* B_j, $1 \leq j \leq N$, *after a linear change of coordinates (depending on* j*), is a branch of a Weierstrass analytic variety (depending on* j*), such that*

(1) $\alpha = \gamma_0(B_1 \cup \cdots, B_N \cup \{0\})$ *;*

(2) $\dim_{\mathbb{C}} \alpha = \max_{1 \leq j \leq N} \{\dim_{\mathbb{C}} B_j\}$ *;*

(3) If $\mathcal{M} \subset \mathbb{C}^n$, *is a connected manifold, of dimension* $l \in \{1, \cdots, N\}$, *of* l*-regular points of a representative of* α *and* $\gamma_0(\mathcal{M}) \neq \emptyset$, *then there exists* $j \in \{1, \cdots, N\}$ *such that* $\gamma_0(\mathcal{M}) \in \gamma_0(\overline{B}_j)$ *and* $\dim_{\mathbb{C}} B_j = l$.

(4) $\alpha \cap \gamma_0(\mathbb{R}^n) = \gamma_0(\tilde{B}_1, \cdots, \tilde{B}_K \cup \{0\})$ *where* $\tilde{B}_j$ *denotes those branches which intersect with* $\mathbb{R}^n$ *non-trivially ;*

(5) $\dim_{\mathbb{R}}(\alpha \cap \mathbb{R}^n) = \max_{1 \leq j \leq K} \dim_{\mathbb{R}}(\tilde{B}_j \cap \mathbb{R}^n)$.

Proof. We argue by induction on $m = \dim_{\mathbb{C}} \alpha$. Suppose that $m = 1$, then it suffices to show

$$\alpha = \gamma_0 \left(\bigcup_{\gamma_0(B) \subset \alpha} B \cup \{0\} \right)$$

where B is a branch of the Weierstrass analytic variety $\mathrm{var}(W \times \mathbb{C}^{n-1}, H)$ obtained in the representation theorem 2.1. (Note that by Theorem 2.2, there is at least one branch B with germ contained in α). Indeed, when $m = 1$, the discriminant $\mathcal{D}$ of $\mathrm{var}(W \times \mathbb{C}^{n-1}, H)$ is an analytic function defined on $W \subset \mathbb{C}$. Hence its zeros are isolated and then, there is no loss of generality in assuming that the discriminant $\mathcal{D}$ is non-zero on $W \setminus \{0\}$ and that $\overline{B} = B \cup \{0\}$ for all branches B. Now, recall that $\alpha \subset \gamma_0(\mathrm{var}(W \times \mathbb{C}^{n-1}, H))$, and let B a branch of $\mathrm{var}(W \times \mathbb{C}^{n-1}, H)$ such that $\alpha \cap \gamma_0(B) \neq \{0\}$ and $\gamma_0(B) \not\subset \alpha$. Then $\gamma_0(\overline{B}) \not\subset \alpha$ and recall that $\gamma_0(\overline{B})$ is a germ of an analytic variety (lemma 2.5) and then $\alpha \cap \gamma_0(\overline{B})$ is also an analytic germ in $\mathbb{C}^n$ with

$$\gamma_0(\overline{B}) \neq \alpha \cap \gamma_0(\overline{B}) \subset \gamma_0(\overline{B}).$$

Therefore, by Lemma 2.9, $\dim_{\mathbb{C}} \Big(\alpha \cap \gamma_0(\overline{B}) \Big) = 0$. Hence $\alpha \cap \gamma_0(\overline{B}) = \{0\}$. Since this is false we conclude that $\gamma_0(\overline{B}) \subset \alpha$ for every branch B of $\mathrm{var}(W \times \mathbb{C}^{n-1}, H)$ with $\{0\} \subsetneqq \alpha \cap \gamma_0(\overline{B})$. Then

$$\alpha = \gamma_0 \left(\bigcup_{\gamma_0(B) \subset \alpha} B \cup \{0\} \right)$$

and the theorem follows in the case $m = 1$. Suppose now $m \geq 2$ and make the induction hypothesis that the results (1)-(3) of the theorem hold for all smaller values of m.
We have according to the representation theorem 2.1,

$$\alpha \subset \gamma_0(\mathrm{var}(W \times \mathbb{C}^{n-m}, H)) = \gamma_0 \left(\bigcup_B \overline{B} \right)$$

where the union is over all the branches B of $\mathrm{var}(W \times \mathbb{C}^{n-m}, H)$. In addition to the branches B with $\gamma_0(B) \subset \alpha$, consider branches $\tilde{B}$ of $\mathrm{var}(W \times \mathbb{C}^{n-m}, H)$ such that

$$\emptyset \neq \gamma_0(\overline{\tilde{B}}) \cap \alpha \neq \gamma_0(\overline{\tilde{B}})$$

Since, by Lemma 2.9, the germ $\gamma_0(\overline{\tilde{B}}) \cap \alpha$ has dimension strictly smaller than m, we can apply the inductive hypothesis to each of these branches to complete the proof of (1) and (2) in the theorem. Let us prove (3). Note that $\dim_{\mathbb{C}} \mathcal{M} \leq m = \dim_{\mathbb{C}} \alpha$ and that we may suppose $\mathcal{M} \subset W \times \mathbb{C}^{n-m}$. If

$$\gamma_0(\mathcal{M}) \not\subset \gamma_0(\mathrm{var}(W \times \mathbb{C}^{n-m}, H \cup \{\mathcal{D}\})),$$

then $\mathcal{M} \cap B \neq \emptyset$ for at least one branch of $\mathrm{var}(W \times \mathbb{C}^{n-m}, H)$. For $z \in \mathcal{M} \cap B$, $\mathcal{M}$ and B coincide in a neighbourhood of z and $\dim_{\mathbb{C}} \mathcal{M} = \dim_{\mathbb{C}} B$. Recall that by Lemma 2.5 $\gamma_0(\overline{B})$ is an analytic variety, then it follows from Lemma 2.3 that $\gamma_0(\mathcal{M}) \subset \gamma_0(\overline{B})$. Now if

$$\mathcal{M} \not\subset \gamma_0(\mathrm{var}(W \times \mathbb{C}^{n-m}, H \cup \{\mathcal{D}\})),$$

we apply the inductive hypothesis to obtain the result.

Parts (4) and (5) follows from Lemma 2.5 □

Corollary 2.1 *In the previous theorem, suppose in addition that α is real-on-real. Then $\gamma_0(\overline{B})$ is real-on-real for each branch B emerging in Theorem 2.4(1) with $B_j \cap \mathbb{R}^n \neq \emptyset$.*

Proof. If α is real-on-real, we can check easily that the Weierstrass analytic variety $\mathrm{var}(\tilde{W} \times \mathbb{C}^{n-\tilde{m}}, \tilde{H})$ constructed in the representation Theorem 2.1 is real-on-real. If the maximal complexified spaces are used as in Lemma 2.8, then the corollary follows from the Lemma 2.5. □

2.4 One-dimensional branches

Theorem 2.5 (Puiseux' Representation) *Suppose $m = 1$, $1 \leq n \in \mathbb{N}$ and B is a branch of a Weierstrass analytic variety $E = \mathrm{var}(W \times \times \mathbb{C}^{n-1}, H)$ where W is chosen so that the discriminant*

$\mathcal{D}(H)$ is non-zero on $W \setminus \{0\}$. Then there exists $K \in \mathbb{N}$, $\delta > 0$ and a $\mathbb{C}$-analytic function

$$\psi : \{z \in \mathbb{C} : |z|^K < \delta\} \to \mathbb{C}^{n-1}$$

such that the mapping $z \mapsto (z^K, \psi(z))$ is injective, $\psi(0) = 0$ and

$$\{0\} \cup B = \overline{B} \cap (W \times \mathbb{C}^{n-1}) = \{(z^K, \psi(z)) : |z|^K < \delta\}.$$

Proof. Let $H = \{h_2, \cdots, h_n\}$ where $h_k \equiv h_k(z_1;\ z_k)$ is a Weierstrass polynomial of degree p_k, say, $2 \leq k \leq n$. If the discriminant $\mathcal{D}(H)$ is not zero at $z_1 = 0$ (the case when all the polynomials h_k, $2 \leq k \leq n$, are of degree 1) then for all $2 \leq k \leq n$, $h_k(z_1;\ z_k) = x_k - \psi_k(z_1)$, where ψ_k is an analytic function on W which vanishes at 0. In this case $B = \mathrm{var}(W \times \times \mathbb{C}^{n-1}, H)$ and the theorem holds with $K = 1$ and

$$\psi(z_1) = (\psi_2(z_1), \cdots, \psi_n(z_1)), z_1 \in W.$$

Now, suppose that $\mathcal{D}(H)$ is zero at 0. Note that for $z_1 \in W \setminus \{0\}$, each of the polynomials $h_k(z_1;\ z_k)$ has only simple roots. Let $\hat{W}$ denote the half-plane in $\mathbb{C}$ defined by

$$\hat{W} = \{z \in \mathbb{C} : z = \rho + i\theta,\ -\infty < \rho < \log \delta,\ \theta \in \mathbb{R}\}$$

where $\delta > 0$ is given in the definition of W, and let

$$\hat{h}_k(z, z_k) = h_k(e^z;\ z_k),\ z \in \hat{W},\ z_k \in \mathbb{C}.$$

Let

$$\hat{H} = \{\hat{h}_2, \cdots, \hat{h}_n\} \text{ and } \hat{E} = \mathrm{var}(\hat{W} \times \mathbb{C}^{n-1}, \hat{H}).$$

It is clear that B is a branch of E if and only if $\hat{B}$ is a branch of $\hat{W}$, where

$$B = \{(e^z, \xi) : (z, \xi) \in \hat{B}\},\quad \xi \in \mathbb{C}^{n-1}.$$

Since $\mathcal{D}(H)$ is nowhere zero on $W \times \{0\}$, $\mathcal{D}(\hat{H})$ is no where zero on $\hat{W}$ and every point of $\hat{E}$ is 1-regular. By the implicit function theorem, we can therefore write

$$\left(\{z\} \times \mathbb{C}^{n-1}\right) \cap \hat{E} = \{(z, \xi_j(z)) : 1 \leq j \leq p\}$$

where $p = \prod_{k=1}^{n} p_k$. By the analytic implicit function theorem, each ξ_j is defined locally on $\hat{W}$ as a $\mathbb{C}$-analytic function with values in $\mathbb{C}^{n-1}$ and since $\hat{W}$ is connected, it can be defined as an analytic function on $\hat{W}$. Thus $\hat{E}$ is the union of the disjoint graphs of the functions $\xi_j : \hat{W} \to \mathbb{C}^{n-1},\ 1 \leq j \leq p$. Recall that for $z \in \hat{W}$, each component of ξ_j is a simple root of some polynomial $h_k(e^{z_1};\ z_k)$, $1 \leq j \leq n$. Therefore the set-valued map

$$z \mapsto \{(e^z, \xi_j(z)) : 1 \leq j \leq p\}$$

is $2\pi i$-periodic on $\hat{W}$. Moreover if, for some $\hat{z} \in \hat{W}$ and some $m \in \mathbb{Z}$,

$$\xi_{j_1}(\hat{z}) = \xi_{j_2}(\hat{z} + 2\pi i m)$$

then by the analytic implicit function theorem and analytic continuation, the previous equality holds for every $\hat{z} \in \hat{W}$. Therefore, for each $j \in \{1, \cdots, p\}$, the mapping

$$z \mapsto (e^z, \xi_j(z)) \in E, \quad z \in \hat{W} \tag{II.13}$$

is periodic with period $2\pi i k_j$ and is injective on the set $W_j = \{z = \rho + \theta \in \hat{W} : 0 < \theta \leq 2\pi k_j\}$, for some $k_j \in \{1, \cdots, p\}$. It is easy to see that its image on W_j is both open closed in E and hence is a branch of E.
For a given branch B. choose j such that the image of (II.13) on W_j coincides with B. We have seen that an injective parametrisation of B is given by

$$B = \{(e^z, \xi_j(z)) : z \in W_j\}.$$

Since $z \mapsto \xi_j(k_j z)$ has period (not necessarily minimal) $2\pi i$, we can define an analytic function $\tilde{\psi} : \{z \in \mathbb{C} : 0 < |z| < \delta^{1/k_j}\} \to \mathbb{C}$ by

$$\tilde{\psi}(z) = \xi_j(k_j \log z)$$

where it does not matter which determination of complex log is chosen. Thus

$$\xi_j(k_j z) = \tilde{\psi}(e^z), \quad k_j z \in \hat{W}.$$

This gives a new injective parametrisation of B. namely

$$B = \{(z^{k_j}, \tilde{\psi}(z)) : 0 < |z| < \delta^{1/k_j}\},$$

where $\tilde{\psi}$ is analytic and $\lim_{z \to 0} \tilde{\psi}(z) = 0$. The Riemann extension theorem (Theorem A.1) means that $\tilde{\psi}$ has an analytic extension ψ defined on the ball $\{z : |z| < \delta^{1/k_j}\}$, with $\psi(0) = 0$ and let $K = k_j$ to complete the proof. $\square$

Corollary 2.2 (Intersection with $\mathbb{R}^n$) *In Theorem 2.5, suppose in addition that $\gamma_0(B \cap \mathbb{R}^n) \notin \{\emptyset, \{0\}\}$. Then there exists $k \in \mathbb{N}^*$ with $0 \leq k \leq 2K - 2$, such that*

$$B \cap \mathbb{R}^n = \{((-1)^k r^K, \psi(r \exp(k\pi i/K)) : -\delta^{1/K} < r < \delta^{1/K}\}, \tag{II.14}$$

and this parametrisation is injective.

Proof. Since $\gamma_0(B \cap \mathbb{R}^n) \notin \{\emptyset, \{0\}\}$, it follows from the previous theorem the existence of a sequence $\{z_j : j \in \mathbb{N}\} \subset \mathbb{C}$ with $z_j \to 0$ as $j \to \infty$, such that $z_j^K \in \mathbb{R}$ and $\psi(z_j) \in \mathbb{R}^{n-1}$ for all $j \in \mathbb{N}$. Therefore, without loss of generality (In fact up to an extraction of a subsequence) we may assume, for some

$k \in \{0, 1, \cdots, 2K-1\}$, that $z_j = |z_j| \exp(k\pi i/K)$ and

$$\psi(|z_j| \exp(k\pi i/K)) \in \mathbb{R}^{n-1} \text{ for all } j \in \mathbb{N}.$$

Since ψ is $\mathbb{C}$-analytic function of one complex variable, we can infer that

$$\psi(r \exp(k\pi i/K)) \in \mathbb{R}^{n-1}, \quad \forall r \in \mathbb{R}, \text{ with } -\delta^{1/K} < r < \delta^{1/K}.$$

Suppose that there is $l \in \{0, 1, \cdots, 2K-1\}$ different from k and a sequence $\rho_j > 0$ such that $\psi(\rho_j \exp(l\pi i/K))$ is real and $\rho_j \to 0$ as $j \to \infty$, then by the preceding argument, we may assume that $\psi(r \exp(l\pi i/K))$ is real for all r with $-\delta^{1/K} < r < \delta^{1/K}$. We will now show that $l - k \in K\mathbb{Z}$. Suppose that this is false. For $p \in \mathbb{N}$,

$$\frac{d^p\psi}{dr^p}(r \exp(k\pi i/K))\big|_{r=0} = \exp(p\pi i(k-l)/K) \frac{d^p\psi}{dr^p}(r \exp(l\pi i/K))\big|_{r=0}$$

and the derivatives are real. Therefore, for all p with $(k-l)p \notin K\mathbb{Z}$, we have

$$\frac{d^p\psi}{dr^p}(0) = 0$$

Let $p_0 \in \mathbb{N}$ the generator of the ideal $\{p \in \mathbb{Z} : (l-k)p \in K\mathbb{Z}\}$. Then the power series expansion of $\psi(z)$ at $z = 0$ involves only powers of z^{p_0} and it follows that

$$\psi(z_1) = \psi(z_2)$$

for all $z_1, z_2 \in \mathbb{C}$ such that $z_1^{p_0} = z_2^{p_0}$ and since p_0 divides K, $z_1^K = z_2^K$. Therefore if $z_1^{p_0} = z_2^{p_0}$,

$$(z_1^K, \psi(z_1)) = (z_2^K, \psi(z_2)), \quad \forall z_1, z_2 \in \mathbb{C}.$$

Now the injectivity in the theorem above give that $p_0 = 1$ and $k - 1 \in K\mathbb{Z}$. This completes the proof. $\square$

Example 2.4 *Consider the collection H of three Weierstrass polynomials*

$$Z^2 - z_1, \quad Z^3 - z_1^2, \quad Z^4 - z_1^3,$$

and let W be the disc of radius 2 with centre 0 in $\mathbb{C}$. Then the corresponding Weierstrass analytic variety,

$$\operatorname{var}(W \times \mathbb{C}^3, H) = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : |z_1| < 2, \ z_2^2 - z_1 = z_3^3 - z_1^2 = z_4^4 - z_1^3 = 0\}$$

has two branches. To see, we take for example the branch B_1 which contains the point $(1,1,1,1) \in \mathbb{C}^4$, contains also the closed Jordan curve

$$\Gamma_1 = \{(e^{it}, e^{it/2}, e^{2it/3}, e^{3it/4}) : t \in [0, 24\pi]\}.$$

Γ_1 *projects onto the unit circle in* W *and contains* 12 *points above* $1 \in C$*, namely the points*

$$\{(e^{2k\pi i}, e^{k\pi i}, e^{4k\pi i/3}, e^{3k\pi i/2}) : k \in \mathbb{N} \cap [0, 12]\}.$$

Similarly, the branch B_2 *containing* $(1, -1, 1, 1)$ *contains the closed Jordan curve*

$$\Gamma_1 = \{(e^{it}, e^{i(\pi+t/2)}, e^{2it/3}, e^{3it/4}) : t \in [0, 24\pi]\}.$$

Γ_2 *also has the following* 12 *points above* $1 \in \mathbb{C}$

$$\{(e^{2k\pi i}, e^{(k+1)\pi i}, e^{4k\pi i/3}, e^{3k\pi i/2}) : k \in \mathbb{N} \cap [0, 12]\}$$

and projects onto the unit circle in W*. We show now that* $\Gamma_1 \cap \Gamma_2 \neq \emptyset$*. It suffices to show that the following system has no solution :*

$$\begin{aligned} t = s \bmod 2\pi, \quad \pi + t/2 = s/2 \bmod 2\pi, \\ 2t/3 = 2s/3 \text{ mod, } 2\pi, \quad 3t/4 = 3s/4, \bmod 2\pi \end{aligned}$$

These equations imply that

$$2k = 4l - 2 = 3m = 8n/3 \textit{ for some } k, l, m, n \in \mathbb{Z},$$

which have no solutions. Therefore $\Gamma_1 \cap \Gamma_2 \neq \emptyset$*. Since there are at most* $2 \times 3 \times 4 = 24$ *points of* $\text{var}(W \times \mathbb{C}^3, H)$ *above* $1 \in W$*, there are at most two branches of this variety. Thus the variety has exactly two branches and* $K = 12$ *in Theorem 2.5. Moreover condidates for the function* ψ *(remember it is not unique) corresponding to these branches, are*

$$\psi_1(z) = (z^6, z^8, z^9) \textit{ and } \psi_2(z) = (z^6, z^8, -iz^9, \quad \textit{for all } |z|^{12} < 2.$$

Moreover

$$B_1 \cap \mathbb{R}^n = \{(r^{12}, r^6, r^8, r^9) : r \in (-2^{1/12}, 2^{1/12})\}$$

$$B_2 \cap \mathbb{R}^n = \{(r^{12}, -r^6, r^8, r^9) : r \in (-2^{1/12}, 2^{1/12})\},$$

which correspond to $k = 0$ *and* $k = 6$*,respectively, in Corollary 2.2.*

3 The Lyapunov-Schmidt reduction

The Lyapunov-Schmidt procedure is a method for reducing, locally in a neighbourhood of a known solution, an infinite-dimensional equation to an equivalent one involving an equation in finite dimensions.

Theorem 3.1 *Let* X *and* Y *are Banach spaces and* U *an open set of* $\mathbb{K} \times X$ *where* $\mathbb{K}$ *denotes the real or complex field. Let* $F \in \mathscr{C}^k(U, Y)$ *for some* $k \in \mathbb{N}^*$ *such that* $F(\lambda_0, x_0) = 0$ *where* $(\lambda_0, x_0) \in U$*, we suppose also that the partial derivative* $A := \partial_2 F(\lambda_0, x_0)$ *is a Fredholm operator and* q *is the* *co-dimension of* $R(A)$ *the range of* A, $\ker(A) \neq \{0\}$. *Then there exist two open sets,* $U_0 \subset U$ *and* $V \subset \mathbb{K} \times \ker(A)$, *and two mappings* $\psi \in \mathscr{C}^k(V, X)$ *and* $h \in \mathscr{C}^k(V, \mathbb{K}^q)$ *such that* $(\lambda_0, x_0) \in U_0$, $(\lambda_0, 0) \in V$ *and* $\psi(\lambda_0, 0) = x_0$ *with*

$$F(\lambda, x) = 0 \text{ in } U_0 \Longleftrightarrow \begin{cases} h(\lambda, \xi) = 0 & \text{for some } (\lambda, \xi) \in V \\ \psi(\lambda, \xi) = x \end{cases} \tag{II.15}$$

Remark 3.1 *In the event that* F *is* $\mathbb{K}$*-analytic, it will be clear from the proof and the analytic implicit function theorem that* h *and* ψ *will also be* $\mathbb{K}$*-analytic.*

Proof. Since A is a Fredholm operator there exists two closed subspaces $W \subset X$ and $Z \subset Y$ such that

$$X = \ker(A) \oplus W \text{ and } Y = Z \oplus R(A)$$

with $\dim Z = q$. Let $P : Y \to Y$ the projection on Z defined by

$$\forall y \in Y = Z \oplus R(A),\ P(y) = z \text{ and } (I - P)y = w \text{ where } (z, w) \in Z \times R(A). \tag{II.16}$$

The operator P is bounded (see §2.4 in the chapter 2 of [26]). For $\lambda \in \mathbb{K}$, $\xi \in \ker(A)$ and $\eta \in W$ such that $(\lambda, x_0 + \xi + \eta) \in U$, let

$$G(\lambda, \xi, \eta) := (I - P)F(\lambda, x_0 + \xi + \eta). \tag{II.17}$$

The mapping G is well defined on some open set of $\mathbb{K} \times \ker(A) \times W$. Note that $G(\lambda_0, 0, 0) = 0$ and

$$\partial_3 G(\lambda_0, 0, 0)\eta = (I - p)A\eta = A\eta, \quad \forall \eta \in W.$$

Hence $\partial_3 G(\lambda_0, 0, 0) = A_{|W} \in \mathcal{L}(W, R(A))$ and by the implicit function theorem, there exist open sets $U_0 \subset U$, $V \in \mathbb{K} \times \ker(A)$, and a mapping $\phi \in \mathscr{C}^k(V, W)$ such that $(\lambda_0, x_0) \in U_0$, $(\lambda_0, 0) \in V$, $\phi(\lambda_0, 0) = 0$ and such that

$$\big((\lambda, x_0 + \xi + \eta) \in U_0 : G(\lambda, \xi, \eta) = 0\big) \Longleftrightarrow \big((\lambda, \xi) \in V \text{ and } \eta = \phi(\lambda, \xi)\big). \tag{II.18}$$

Hence, it suffices to put

$$\psi(\lambda, \xi) = x_0 + \xi + \phi(\lambda, \xi) \text{ and } h(\lambda, \xi) = PF(\lambda, \psi(\lambda, \xi)) \in Z. \tag{II.19}$$

Then for all $(\lambda, \xi) \in V$,

$$h(\lambda, \xi) = 0 \text{ if and only if } PF(\lambda, \psi(\lambda, \xi)) = 0$$

and we have from (II.18) $(I - P)F(\lambda, \psi(\lambda, \xi)) = G(\lambda, \xi, \eta) = 0$ and hence

$$h(\lambda, \xi) = 0 \text{ if and only if } F(\lambda, \psi(\lambda, \xi)) = 0.$$

Finally choose a basis for the q-dimensional space Z and thereby identify Z with $\mathbb{K}^q$. $\square$

Corollary 3.1 *Let the hypotheses of Theorem 3.1 and let U_0, V h and ψ be given by its conclusion. Then we can choose U_0 and V small enough such that*

$$\forall (\lambda,\xi)\in V, \quad \dim\ \ker(\partial_2 F(\lambda,\psi(\lambda,\xi)) = \dim\ \ker(\partial_2 h(\lambda,\xi)). \tag{II.20}$$

Proof. From (II.18) and (II.17), we have

$$(I-P)\partial_2 F(\lambda, x_0+\xi+\phi(\lambda,\xi)) \equiv 0, \quad \forall(\lambda,\xi)\in V. \tag{II.21}$$

By derivation this identity with respect to ξ, it follows that

$$(I-P)\partial_2 F(\lambda, x_0+\xi+\phi(\lambda,\xi))(v+\partial_2\phi(\lambda,\xi)) = 0, \quad \forall v\in \ker(A) \tag{II.22}$$

Therefore, if $\partial_2 F(\lambda, x_0+\xi+\phi(\lambda,\xi))(v+w)=0$ with $v\in\ker(A)$ and $w\in W$ then

$$(I-P)\partial_2 F(\lambda, x_0+\xi+\phi(\lambda,\xi))(w-\partial_2\phi(\lambda,\xi)v) = 0. \tag{II.23}$$

Since $(I-P)A : W \to R(A)$ is bijective, $(I-P)\partial_2 F(\lambda, x_0+\xi+\phi(\lambda,\xi))_{|W}$ is also a bijection when the sets U_0 and V have been chosen with sufficiently small diameters. Therefore $w=\partial_2\phi(\lambda,\xi)v$ and so

$$\begin{aligned}
\forall (v,w)\in \ker(A)\times W: \quad & \partial_2 F(\lambda, x_0+\xi+\phi(\lambda,\xi))(v+w)=0\\
\iff \quad & P\partial_2 F(\lambda, x_0+\xi+\phi(\lambda,\xi))(v+w)=0 \text{ and } w=\partial_2\phi(\lambda,\xi)v\\
\iff \quad & \partial_2 h(\lambda,\xi)=0.
\end{aligned}$$

The corollary follows. □

4 Proof of theorem 1.1

The proof is organized below in a few steps.

Step 1. First, we give the following definition of distinguished arcs.

Definition 4.1 *A distinguished arc is a maximal connected subset of $\mathcal{R}$.*

We recall that $\mathcal{R}$ is the non-singular solutions set

$$\mathcal{R} = \{(\lambda,x)\in\mathcal{S} : \ker(\partial_2 F(\lambda,x))=0\}.$$

and we have supposed that $\mathcal{A}^+ \subset \mathcal{R}$. Hypothesis (G2) and the implicit function theorem ensure that a distinguished arc is the graph of an $\mathbb{R}-$ analytic function of λ. More precisely, if $\mathcal{I}$ is a distinguished arc then exists a (possibly infinite) open interval I and an $\mathbb{R}$- analytic function $g: I\to X$ such that

$$\{(\lambda, g(\lambda) : \lambda\in I\} = \mathcal{I}.$$

Step 2. (Liapunov-Schmidt reduction)
We consider the singular points $(\lambda_*, x_*) \in \mathbb{R} \times X$ defined as follows

$$F(\lambda_*, x_*) = 0 \text{ and } \ker(\partial_2 F(\lambda_*, x_*)) \neq \{0\}.$$

To study the structure of $\mathcal{S}$ in a neighbourhood of point $(\lambda_*, x_*) \in \mathcal{S}\backslash\mathcal{R}$, we use the Lyapunov-Schmidt reduction. Theorem 3.1 yields the existence of
a neighbourhood V of (λ_*, x_*) in $\mathbb{R} \times \ker(\partial_2 F(\lambda_*, x_*))$,
$\mathbb{R}$-analytic maps $\psi : V \to X$, $h : V \to \mathbb{R}^q$ ($q = \dim\ker(\partial_2 F(\lambda_*, x_*)) \geq 1$) with the following properties

- $\psi(\lambda_*, 0) = x_*$,
- for all $(\lambda, \xi) \in V : h(\lambda, \xi) = 0 \Longleftrightarrow F(\lambda, \psi(\lambda, \xi)) = 0$,
- if $F(\lambda, x) = 0$ such that $\|(\lambda, x) - (\lambda_*, x_*)\|$ is sufficiently small, then there exists $\xi \in \ker(\partial_2 F(\lambda_*, x_*))$ such that $(\lambda, \xi) \in V$ and $x = \psi(\lambda, \xi)$
- from the corollary 3.1, dim $\ker(\partial_2 F(\lambda, \psi(\lambda, \xi)) = \dim \ \ker(\partial_2 h(\lambda, \xi)), \ (\lambda, \xi) \in V$.

Let A, $\mathcal{M}$ the sets

$$\begin{aligned} A &= \{(\lambda, \xi) \in V : h(\lambda, \xi) = 0\} = \operatorname{var}(V, \{h\}), \\ \mathcal{M} &= \{(\lambda, \xi) \in V : h(\lambda, \xi) = 0, \ \ker(\partial_2 h(\lambda, \xi)) = \{0\}\} \\ &= \{(\lambda, \xi) \in V : (\lambda, \psi(\lambda, \xi) \in \mathcal{R}\}. \end{aligned}$$

The corollary 3.1 implies that the elements of $\mathcal{M}$ are 1-regular points of A. Let $\{M_j, j \in J\}$ denote the non-empty connected components of $\mathcal{M}$ which have the property that $\gamma_{(\lambda_*,0)}(M_j) \neq \emptyset$.
Since h is an $\mathbb{R}$- analytic function on the $(q+1)$- dimensional real vector space V, the s component of $h(\lambda, \xi)$ are real function defined locally in a neighbourhood of $(\lambda_*, 0) \in V$ by Taylor series, the n^{th} term of which is a sum of terms of the form

$$h^*_{k_1, \cdots, k_{q+1}} x_1^{k_1} \cdots x_{q+1}^{k_{q+1}}$$

Here $(x_1, \cdots, x_{q+1}) \in \mathbb{R}^{q+1}$ are the coefficients of $(\lambda_*, 0) - (\lambda, \xi)$ in some linear cordinate system. Replacing $(x_1, \cdots, x_{q+1}) \in \mathbb{R}^{q+1}$ with $(z_1, \cdots, z_{q+1}) \in \mathbb{C}^{q+1}$ leads to a real to real $\mathbb{C}$-analytic extension h^c of h in a complex neighbourhood V^c od $(\lambda, 0)$ and a corresponding $\mathbb{C}$-analytic variety. Let

$$\begin{aligned} A^c &= \{(\lambda, \xi) \in V^c : h^c(\lambda, \xi) = 0\} = \operatorname{var}(V^c, \{h^c\}), \\ \mathcal{M}^c &= \{(\lambda, \xi) \in V^c : h^c(\lambda, \xi) = 0, \ \ker(\partial_2 h^c(\lambda, \xi)) = \{0\}\} \\ &= \{(\lambda, \xi) \in V^c : (\lambda, \psi(\lambda, \xi) \in \mathcal{R}\} \end{aligned}$$

and let $\{M_j^c, j \in J^c\}$ be the non-empty connected components of $\mathcal{M}^c$ with $\gamma_{(\lambda_*,0)}(\mathbb{R}^{q+1} \cap M_j^c) \neq \emptyset$. Note that for each $j \in J$ there exists $\hat{j} \in J^c$ such that $M_j \subset M^c_{\hat{j}}$.
Step 3. (Application of the Structure Theorem 2.4)
Theorem 2.4 on the structure of complex analytic varieties, when applied to $\mathcal{A}^c$ gives, for each $j \in J^c$,

the existence of a real-on-real branch B_j with

$$\gamma_{(\lambda_*,0)}(M_j^c) \subset \gamma_{(\lambda_*,0)}(\overline{B}_j), \quad \dim B_j = 1 \text{ and } B_j \subset A^c.$$

By making the neighbourhood V^c smaller if necessary, we may suppose that $B_j \setminus \{(\lambda_*, 0)\} \subset M_j^c$. by Theorem 2.4 there are finitely many branches and hence finitely many M_j^c and M_j. By Theorem 2.5 each of these one-dimensional branches B_j admits a $\mathbb{C}$-analytic parametrisation in a neighbourhood of $(\lambda_*, 0)$.

We now return to the setting of $\mathbb{R}^n$. From corollary 2.2, we obtain that $\overline{\mathcal{M}}$, in a neighbourhood of $(\lambda_*, 0)$, is the union of a finite number of curves which pass through $(\lambda_*, 0) \in V$, intersect one another only at $(\lambda_*, 0)$ and are given by parametrisation (II.13). Thus, in our previous notation, each M_j, $j \in J$, is paired, in a unique way with another $M_{\tilde{j}}$, $\tilde{j} \in J$, so that their union with the point $(\lambda_*, 0)$ form one of these curves in V.

Step 4. (Routes of length N and existence of a maximal route)

First, we give the following definition

Definition 4.2 *A route of length $N \in \mathbb{N}^* \cup \{\infty\}$ is a set $\{\mathcal{A}_n : 0 \le n \le N-1\}$ of distinguished arcs and a set $\{(\lambda_n, x_n) : 0 \le n \le N-1\} \in \mathbb{R} \times X$ such that*

(a) $(\lambda_0, x_0) = (0, 0)$ *;*

(b) $\mathcal{A}_0$ *contains* $\mathcal{A}^+ = \{(\lambda(s), u(s)) \,:\, s \in (0, \epsilon)\}$ *;*

(c) If $N > 1$, then for all $0 \le n \le N-1$,

$$(\lambda_{n+1}, x_{n+1}) \in \partial\mathcal{A}_n \cup \partial\mathcal{A}_{n+1} \setminus \{(\lambda_n, x_n)\}$$

and there exists an injective $\mathbb{R}$-analytic map $\rho : (-1, 1) \to \mathcal{A}_n \cup \mathcal{A}_{n+1} \cup \{(\lambda_n, x_n)\}$ with $\rho(0) = (\lambda_{n+1}, x_{n+1})$, (the analyticity of ρ implies that $\mathcal{A}_{n+1}$ is uniquely determined by $\mathcal{A}_n$ and vice versa.)

(d) The mapping $n \mapsto \mathcal{A}_n$ is injective.

Next, let us show the existence of a maximal route. The existence of a route of length 1 is obvious and we have $(0, 0) \in \partial\mathcal{A}_0$. Parts (c) and (d) of the definition of a route imply that $\mathcal{A}_n \neq \mathcal{A}_{n+1}$ and $\mathcal{A}_{n+1}$ is uniquely determined by $\mathcal{A}_n$. Therefore, if

$$\{\mathcal{A}_n^j : 0 \le n \le N_j - 1\},\ \{(\lambda_n^j, x_n^j) : 0 \le n \le N_j - 1\},\ j \in \{1, 2\}$$

are two routes with $N_1 \le N_2$, it follows that

$$\mathcal{A}_n^1 = \mathcal{A}_n^2 \text{ and } (\lambda_n^1, x_n^1) = (\lambda_n^2, x_n^2).$$

Hence, under the hypothesis of Theorem 1.1 there exists a maximal route of length $N \in \mathbb{N}^* \cup \{\infty\}$ which we denote by

$$\{\mathcal{A}_n : 0 \le n \le N-1\}.$$

Step 5. (Parametrisation of a maximal route) Due to the definitions 4.1 and 4.2, we can define a continuous parametrisation

$$[0, N[\ni s \mapsto (\lambda(s), u(s)) \in \mathbb{R} \times X$$

of

$$\mathcal{A} \stackrel{\text{def}}{=} \bigcup_{n=0}^{N-1} [\{(\lambda_n, x_n)\} \cup \mathcal{A}_n]$$

such that $(\lambda(n), u(n)) = (\lambda_n, x_n)$ and $(\lambda, u)_{|]n,n+1[}$ is a parametrization of $\mathcal{A}_n$ for all $n \in \{0, 1, \cdots, N-1\}$.
We claim that if $\liminf_{s\to N} \|(\lambda(s), u(s))\| < \infty$, then $N < \infty$ and there is a singular point $(\lambda_*, x_*) \in \partial\mathcal{A}_{N-1}$ such that $F(\lambda_*, x_*) = 0$.
Indeed, if $\liminf_{s\to N} \|(\lambda(s), u(s))\| < \infty$, then there exists a sequence $\{s_k,\ k \in \mathbb{N}\}$ such that $\lim_{k\to\infty} s_k = N$ and $\{(\lambda(s_k), x(s_k)),\}$ is bounded in $\mathbb{R} \times X$. Hence, from the compactness hypothesis of Theorem 1.1, $\{(\lambda(s_k), x(s_k)),\ k \in \mathbb{N}\}$ is relatively compact. Without loss of generality, we may suppose that it converges to $(\lambda_*, x_*) \in \mathcal{S}$. If $N = \infty$, then every neighbourhood of (λ_*, x_*) intersects infinitely many distinct distinguished arcs and according to the local analysis below, if $(\lambda_*, x_*) \in U$ then in a neighbourhood of this point, the solution set is a $\mathbb{R}$- analytic variety which intersects with a finite number of distinguished arcs. This shows that the sequence $(\lambda(s_k), x(s_k))$ (up to a subsequence) is contained in some distinguished arc. Then the condition (d) of definition 4.2 implies that for s large enough, $(\lambda(s), x(s))$ belongs to this arc. Hence N is finite and we have $(\lambda(s), x(s)) \to (\lambda_*, x_*)$ as $s \to N$.

Hence, if $N = \infty$, then either $\lim_{s\to N} \|(\lambda(s), u(s))\| = \infty$ or $\lim\limits_{s\to\infty} \operatorname{dist}((\lambda(s), u(s)), \partial U) = 0$. Therefore one of the two first alternatives in (e) of Theorem 1.1 holds.

If $N < \infty$ and $\liminf_{s\to N} \|(\lambda(s), u(s))\| = \infty$, then by similar arguments as above, we can deduce that $\lim s \to N\|(\lambda(s), u(s))\| < \infty$. Hence it is the straightforward to map $[0, N)$ to $[0, \infty)$ to obtain a parametrisation of $\mathcal{A}$ satisfying parts (a)–(d) of Theorem 1.1.

Finally, if $N < \infty$ and $\liminf_{s\to N} \|(\lambda(s), u(s))\| < \infty$, then (λ_{N-1}, x_{N-1}) and (λ_*, x_*) are the ends of $\mathcal{A}_{N-1}$, and the unique continuation of $\mathcal{A}_{N-1}$ at (λ_*, x_*), as a distinguished arc $\mathcal{A}_*$ distinct from $\mathcal{A}_{N-1}$, is ensured by corollary 2.2. Then we can prove that $\mathcal{A}_* = \mathcal{A}_0$. Indeed, from the maximality of the route $\{\mathcal{A}_n : 0 \le n \le N-1\}$, it follows that $\mathcal{A}_*$ coincides with some distinguished arc $\mathcal{A}_m$ where

$$m \in \{0, \cdots, N-2\} \tag{II.24}$$

Suppose that $m \neq 0$. As $\mathcal{A}_* = \mathcal{A}_m$ is a continuation of $\mathcal{A}_{N-1}$ then

$$\mathcal{A}_{N-1} \in \{\mathcal{A}_{m-1}, \mathcal{A}_{m+1}\}.$$

If $\mathcal{A}_{N-1} = \mathcal{A}_{m-1}$ then from assertion (d) in Definition 4.2, we obtain that $N-1 = m-1$ which contradicts (II.24). Hence $\mathcal{A}_{N-1} = \mathcal{A}_{m+1}$, then $m = N-2$ and $\mathcal{A}_m = \mathcal{A}_{N-2}$, it follows that $(\lambda_*, x_*) = (\lambda_{N-2}, x_{N-2})$ and then we deduce that $\mathcal{A}_{N-1} = \mathcal{A}_{N-3}$ which contradicts assertion (d) in Definition 4.2. Hence $m = 0$ and then $(\lambda_*, x_*) \in \{(\lambda_0, x_0), (\lambda_1, x_1)\}$. If $(\lambda_*, x_*) = (\lambda_1, x_1)$, then $\mathcal{A}_{N-1} = \mathcal{A}_1$, by (d) we obtain $N = 2$ which is not possible together with $(\lambda_*, x_*) = (\lambda_1, x_1)$. Then $(\lambda_*, x_*) = (\lambda_0, x_0)$ and (iii) in (e) holds. To complete the proof, it suffices to extend $s \mapsto (\lambda(s), x(s))$ on $[0, \infty)$ as follows :

$$\forall s \in [0, N), \quad \forall k \in \mathbb{N} : (\lambda(s + kN), x(s + kN)) = (\lambda(s), x(s)).$$

The part f) of Theorem 1.1 follows from the implicit function theorem applied at $(\lambda(s_1), u(s_1))$. Indeed, since $s_1 \neq s_2$, we have $(\lambda(s_1), u(s_1)) \in \mathcal{A}_{n_1}$ and $(\lambda(s_2), u(s_2)) \in \mathcal{A}_{n_2}$, where $0 \leq n_1, n_2 < N$. Since $\partial_2 F(\lambda, u)$ is invertible at $(\lambda(s_1), u(s_1))$, from the implicit function theorem it follows that $\mathcal{A}_{n_1}$ and $\mathcal{A}_{n_2}$ coincide in a neighbourhood of the point where they intersect. Moreover, we know that $\partial_2 F(\lambda, u)$ is invertible along of $\mathcal{A}_{n_1}$ and $\mathcal{A}_{n_1}$. Then by the same argument it follows that $\mathcal{A}_{n_1} = \mathcal{A}_{n_2}$. Thus from the definition of the maximal root, (e)(iii) occurs and $|s_1 - s_2|$ is an integer multiple of T.

This observation leads to the conclusion that $(\lambda, u) : [0, \infty) \to \mathcal{S}$ is locally injective which completes the proof of Theorem 1.1.

Chapitre III

Analytic global bifurcation for very singular problem and infinite turning points

Nous présentons ici les résultats issus de BOUGHERARA-GIACOMONI-PRASHANTH qui ont fait l'objet d'une publication [25], dans la revue **Calculus of Variations and P.D.E.**
(version en ligne http ://link.springer.com/article/10.1007/s00526-014-0735-8).

1 Introduction and preliminaries

1.1 Introduction

Let $\Omega \subset R^N, N \geq 2$, be a bounded domain with smooth boundary. We consider the following singular semilinear elliptic problem with the bifurcation parameter $\lambda > 0$:

$$(P_\lambda) \qquad \begin{cases} -\Delta u = \lambda(u^{-\delta} + f(u)) \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \ \ u|_{\partial\Omega} = 0. \end{cases}$$

Here, the singular exponent δ can be any positive number. Throughout this paper, we assume without further mention the following basic condition on the nonlinearity f :

(f_0) $f : [0, \infty) \to [0, \infty)$ is a twice continuously differentiable map with $f(0) = 0$.

In various places, we need to assume that additionally some of the following three conditions are satisfied by f :

(f_1) $f(t)$ is a finite product of functions of the form $g(t^p), p > 0$, where g is a real entire function on $\mathbb{R}$.

(f_2) $\liminf\limits_{t\to\infty} \dfrac{f(t)}{t} = \infty$.

(f_3) the map $t \mapsto t^{-\delta} + f(t)$ is convex on $(0, \infty)$.

Remark 1.1 *It is easy to see that* $f(t) = t^{2+p_1} \prod_{i=2}^n (1 + t^{p_i}),\ p_i > 0,$ *satisfies conditions* $(f_0) - (f_2)$*. Similarly,* $f(t) = t^{2+p_1} \prod_{i=2}^n e^{\alpha_i t^{p_i}}$ *where* $p_i > 0$ *are distinct and* $\alpha_j > 0$ *for* j *such that* $p_j = \max\{p_2, p_3, \cdots, p_n\}$*, also satisfies these conditions.*

Remark 1.2 *If* g *is a* C^2 *function on* $(0, \infty)$*, convex on* (a, ∞) *with* $inf_{(0,a)} g'' \geq -M$ *for some positive constants* a, M*, then* $f_\nu(t) \stackrel{\text{def}}{=} t^{-\delta} + \nu g(t)$ *satisfies* (f_3) *for all small* $\nu > 0$.

Define

$$f^*(t) = \begin{cases} |t|^{\frac{N+2}{N-2}} \text{ if } N \geq 3, \\ e^{t^2} \text{ if } N = 2. \end{cases}$$

We make the following definition concerning functions of "critical growth" depending on the dimension N :

Definition 1.1 *(i) We say a map $f : [0, \infty) \to [0, \infty)$ has "atmost critical" growth if*

$$\limsup_{t\to\infty} \frac{f(t)}{f^*(t)} < \infty \textit{ when } N \geq 3 \textit{ and } \limsup_{t\to\infty} \frac{f(t)}{f^*(\alpha t)} < \infty \textit{ for some } \alpha > 0 \textit{ when } N = 2.$$

(ii) We say a map $h : [0, \infty) \to [0, \infty)$ is a perturbation of f^* if $\lim_{t\to\infty} \frac{h(t)}{f^*(t)} = 0$ when $N \geq 3$ and $\lim_{t\to\infty} \frac{h(t)}{f^*(\varepsilon t)} = 0$ for any $\varepsilon > 0$ when $N = 2$.

Definition 1.2 *We say that $u \in L^1_{loc}(\Omega)$ is a weak solution to $-\Delta u = \lambda(u^{-\delta} + f(u))$ if $ess.inf_K\ u > 0$ for any compact set $K \subset \Omega$, $f(u) \in L^1_{loc}(\Omega)$ and for any $\phi \in C_0^\infty(\Omega)$, u satisfies :*

$$-\int_\Omega u\Delta\phi = \lambda \int_\Omega (u^{-\delta} + f(u))\phi.$$

Our approach in this paper will be to regard the problem (P_λ) as a bifurcation problem, with the bifurcation parameter being λ, posed in $\mathbb{R}^+ \times \mathcal{C}_{\phi_\delta}(\Omega)$ where $\mathcal{C}_{\phi_\delta}(\Omega)$ is a closed subspace of $\mathcal{C}_0(\overline{\Omega})$ weighted by a suitable power of the distance to the boundary function (see §1.2). It is easy to see that any $u \in \mathcal{C}_0(\overline{\Omega})$ that weakly solves (P_λ) is infact twice continuously differentiable in Ω. Therefore, we consider the following set of all classical solutions

$$\mathcal{S} = \{u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}_0(\overline{\Omega}), u > 0 \text{ solves } (P_\lambda)\}. \tag{III.1}$$

We remark that, as we allow any $\delta > 0$ as the singular exponent, working in the standard variational setting of Sobolev spaces is ruled out. See [52] for a bifurcation analysis of the problem (P_λ) in such a Sobolev space setting with the restriction $0 < \delta < 3$. We also refer to the work of [Haitao,[78]] where variational methods are used for $0 < \delta < 1$ and to [Hirano-Saccon-Shioji, [81]] where ideas related to non-differentiable functionals are used to treat any $\delta > 0$. In this work, we would like to develop a bifurcational framework and present a global bifurcation picture for the set of solutions $\mathcal{S}$. This is usually done using the framework of [Crandall-Rabinowitz, [40]]. But there is also a theory of bifurcation in the context of analytic operators that has been initiated by Dancer (see [44],[42]) from the observation that connectedness in the topological theory can be replaced with path-connectedness if the operators are real-analytic. See [Buffoni, Dancer, Toland, [30] and [31]] for further development of these ideas. In this work we choose to employ this analytic framework to study $\mathcal{S}$.

There are mainly two reasons for this choice. With this framework, we have the additional advantage of obtaining an analytic globally path connected branch of solutions for any $\delta > 0$. Indeed, the full branch of minimal solutions can be shown to lie in this analytic path (see lemma 2.8). Secondly, the global analytic bifurcation theory allows us to show, for certain choices of the nonlinearity f of "critical growth" (see definition 1.1) in a ball in two dimensions, that we do have infinite number of "turning points" along this unbounded analytic path of solutions. Here, by a "turning point" we mean a point on the path at which the path bends back (in the sense of Definition 1.8). We show that this is a consequence of the Morse index of solutions becoming unbounded along this path (see Theorem 1.5). To our knowledge, this is the only instance of such phenomenon in two dimensions.

This property of infinite number of "turning points" was observed for supercritical nonlinearities in higher dimension by [Dancer, [43]] (see also [Guo-Wei, [77]]) using the analytic global bifurcation theory and for exponential nonlinearities in higher dimensions by [Joseph-Lundgren, [82]] with a different approach based on a clever study of suitable dynamical systems. We refer also to the work of [61] in this direction. In their study of water waves (motivated by some ideas of [Plotnikov [102]]), the analytic bifurcation framework has been used by [Buffoni, Dancer, Toland, [30],[31]] to show the presence of sub-harmonic bifurcations.

We would like to point out that in the topological version of global bifurcation theory due to [Rabinowitz [104]], it is difficult, and in general not always possible, to obtain the existence of such a path of solutions upon which secondary bifurcation points can be identified.

We now describe the contents of the two main sections of the paper. In section 2, we put the solution operator $F(\lambda, u)$ corresponding to the problem (P_λ) in the framework of analytic bifurcation theory. A crucial point is the identification of the subspace $\mathcal{X}$ of $\mathcal{C}(\Omega)$ between which the operator is analytic. This space turn out to be those of continuous functions on Ω weighted by an appropriate power of the distance to the boundary function (see proposition 2.2). We remark that if the nonlinearity itself is real analytic, it is not difficult to prove that the solution operator is analytic with the choice $\mathcal{X} = \mathcal{C}_0(\overline{\Omega})$ (see proposition 2.1). It is useful to refer to the discussion in [Dancer,[44]] and [Takáč [115]]) in this context.

The functional framework having been identified, in lemma 2.7, we show the existence of analytic path of solutions to (P_λ) for small $\lambda > 0$ using the analytic version of implicit function theorem. We make use of some recent results about existence and uniqueness of very weak solutions in [Díaz-Rakotoson, [54] and [55]] in this process. The results of section §2 extend, to any $\delta > 0$ in the analytic context, the results obtained in [[52]] where the theory of global bifurcation of [Rabinowitz [104]] is applied for $\delta < 3$ in the C^2- category.

The existence of a global analytic path of solutions in Theorem 1.1 is then obtained by applying Theorem 1.1 in the "positive cone" generated in $\mathcal{X}$ by the respective power of the distance function to the boundary.

In section §3, we focus on the case when Ω is the unit ball B_1 in two dimensions and $0 < \delta < 1$. We apply the global bifurcation result contained in Theorem 1.1 to a set of nonlinearities of the form $f(t) = h(t)e^{t^2}$. Here h will be a perturbation of e^{t^2} in the sense of definition 1.1, which decays to zero as $t \to \infty$. Such class of nonlinearities f will correspond to the ones identified in theorem 1.3 of [67] that ensure (P_λ) posed on B_1 has exactly one solution for all small λ. With these choices of Ω and f we show (see Theorem 1.3) that the Morse index of the solutions becomes unbounded on a discrete set of points along this global unbounded analytic path. Hence this path admits a “turning point" at each point of this discrete set.

To establish this result, a crucial fact is the existence of a radial singular solution to (P_λ) posed on B_1 for the choice of f as explained above (see lemma 3.3). We then do an asymptotic analysis of the linearisation (about the singular solution) of the ODE associated to (P_λ) and recalling an oscillation criteria for ODEs, we prove that solutions of such linearised ODEs oscillate infinitely often near the origin. As we can show that any unbounded sequence of solutions along the global analytic path converges to such a singular solution, it follows that the Morse index of such solutions tends to infinity (see Theorem 1.2). We remark that this result which only concerns the class of functions f similar to those satisfying conditions in theorem 1.3 of [67] is in contrast with the case where f satisfies conditions similar to those of theorem 1.4 in the same work. In the latter case, one can prove existence of large solutions of mountain pass type which implies that the Morse index of such solutions is 1 (see also [[4]]). We think that it is also the case for all large solutions belonging to the global analytic path and we will analyse this particular situation in a forthcoming paper.

1.2 Definitions and main results

Definition 1.3 *Given $\phi \in \mathcal{C}(\Omega)$ with $\phi > 0$ in Ω, define the Banach space*

$$\mathcal{C}_\phi(\Omega) = \{u \in \mathcal{C}(\Omega) \mid \textit{for some } C > 0, |u(x)| \leq C\phi(x)\ \forall x \in \Omega\}$$

with the norm : $\|u\|_{\mathcal{C}_\phi(\Omega)} \stackrel{\text{def}}{=} \sup_{x\in\Omega} |\frac{u(x)}{\phi(x)}|$.

Definition 1.4 *Define the following open convex subset of $\mathcal{C}_\phi(\Omega)$:*

$$\mathcal{C}^+_\phi(\Omega) = \{u \in \mathcal{C}_\phi(\Omega) \mid \inf_{x\in\Omega} \frac{u(x)}{\phi(x)} > 0\}.$$

Let φ_1 be the first positive eigenfunction for $-\Delta$ in $H^1_0(\Omega)$. We normalise as $\|\varphi_1\|_{L^\infty(\Omega)} = 1$. We define ϕ_δ as follows :

$$\phi_\delta = \begin{cases} \varphi_1 & 0 < \delta < 1, \\ \varphi_1(-\log\varphi_1)^{\frac{1}{2}} & \delta = 1, \\ \varphi_1^{\frac{2}{\delta+1}} & \delta > 1. \end{cases} \tag{III.2}$$

Definition 1.5 *Let $\mathcal{U}$ be an open subset of a real Banach space $\mathcal{E}$, $\mathcal{G}$ is a real Banach space and $F : \mathcal{U} \to \mathcal{G}$. We say F is real analytic on $\mathcal{U}$ if for each $x \in \mathcal{U}$ there is an $\varepsilon > 0$ and continuous k-homogeneous polynomials $P_k : \mathcal{E} \to \mathcal{G}$ such that $F(x+h) = \sum_{k=0}^\infty P_k(h)$ if $\|h\| < \varepsilon$.*

Consider the following solution operator associated to (P_λ) :

$$F(\lambda, u) = u - \lambda(-\Delta)^{-1}(u^{-\delta} + f(u)),\ (\lambda, u) \in \mathbb{R}^+ \times \mathcal{C}^+_{\phi_\delta}(\Omega),\ \delta > 0. \tag{III.3}$$

We give below the statements of all the results that we will prove in this paper. The result below (proved in sub-section 2.4) shows the existence of a global analytic path (in the sense of statement (vi) below) of solutions.

Theorem 1.1 *Let f satisfy conditions (f_1) and (f_2). Then, there exists $\Lambda \in (0, \infty)$ and an unbounded set $\mathcal{A} \subset (0, \Lambda] \times \mathcal{C}^+_{\phi_\delta}(\Omega)$ of solutions to (P_λ) which is globally parametrised by a continuous map :*

$$(0, \infty) \ni s \to (\lambda(s), u(s)) \in \mathcal{A} \subset \mathcal{S}.$$

Moreover, the following properties hold along this path $\mathcal{A}$:

(i) $(\lambda(s), u(s)) \to (0, 0)$ in $\mathbb{R} \times \mathcal{C}_{\phi_\delta}(\Omega)$ as $s \to 0^+$.

(ii) For some $s_0 > 0$ small, the portion of the path $\{(\lambda(s), u(s)) : 0 < s < s_0\}$ coincides with the minimal solution branch to (P_λ), $\lambda < \lambda(s_0)$. Infact, this remains true as long as $\partial_u F$ is invertible along the minimal solutions branch.

(iii) $\|u(s)\|_{\mathcal{C}_{\phi_\delta}(\Omega)} \to \infty$ as $s \to \infty$.

(iv) $\mathcal{A}$ has at least one asymptotic bifurcation point $\Lambda_a \in [0, \Lambda]$. That is, there exist sequences $\{s_n\}_{n\in\mathbb{N}} \subset (0, \infty)$, $\{(\lambda(s_n), u(s_n))\} \subset \mathcal{A}$ such that $s_n \to \infty$, $\lambda(s_n) \to \Lambda_a$ and $\|u(s_n)\|_{\mathcal{C}_{\phi_\delta}(\Omega)} \to \infty$.

(v) $\{s \geq 0 : \partial_u F(\lambda(s), u(s))$ is not invertible $\}$ is a discrete set.

(vi) ($\mathcal{A}$ is an "analytic" path) At each of its points $\mathcal{A}$ has a local analytic re-parameterization in the following sense : For each $s^* \in (0, \infty)$ there exists a continuous, injective map $\rho^* : (-1, 1) \to \mathbb{R}$ such that $\rho^*(0) = s^*$ and the re-parametrisation

$$(-1, 1) \ni t \to (\lambda(\rho^*(t)), u(\rho^*(t))) \in \mathcal{A} \text{ is analytic.}$$

Furthermore, the map $s \mapsto \lambda(s)$ is injective in a right neighborhood of $s = 0$ and for each $s^* > 0$ there exists $\epsilon^* > 0$ such that λ is injective on $[s^*, s^* + \epsilon^*]$ and on $[s^* - \epsilon^*, s^*]$.

Remark 1.3 *(i) We note that though $\mathcal{A}$ has a global continuous parametrisation, it has only a local analytic parametrisation. It is easy to see that if $\partial_u F(\lambda_0, u_0)$ is invertible for some $(\lambda_0, u_0) \in \mathcal{A}$, then by the analytic implicit function theorem λ can be taken as the analytic parametrising variable in a neighborhood of λ_0.*

(ii) Infact, $\mathcal{A}$ has a nice structure made of such analytic arcs (parametrised by λ) and a countable collection of singular points. See statement (a) of theorem 1.1.

Remark 1.4 *(a) In Theorem 1.1, the nonlinearity under consideration is singular at zero. Instead, if we consider (P_λ) with the nonlinearity $t^\beta + f(t), 0 < \beta < 1$, then again theorem 1.1 holds good with the choice of the solution space $\mathbb{R}^+ \times \mathcal{C}^+_{\varphi_1}(\Omega)$ (or $\mathbb{R}^+ \times C^{1,+}_0(\overline{\Omega})$ where $C^{1,+}_0(\overline{\Omega})$ is the interior of the positive cone of $C^1_0(\overline{\Omega})$).*

*(b) If we take $t + f(t)$ as the nonlinearity in (P_λ), as in Chapter 8 and 9 of [32], we can show that theorem 1.1, except for assertions (i) and (ii), holds with the choice of the function space $\mathbb{R}^+ \times \mathcal{C}^+_{\varphi_1}(\Omega)$. Infact, $\mathcal{A}$ emanates from $(\lambda_1(\Omega), 0)$ and satisfies additionally properties (a) and (b) of Theorem 1.1. The proof of this bifurcation from the simple eigenvalue is given in Theorem 8.3.1 in [32] and uses the approach in [*Crandall-Rabinowitz*, [40]]. We will not consider this case in the present paper.*

In the case f satisfies condition (f_3) as well, we can give additional information about the analytic path $\mathcal{A}$:

Corollary 1.1 *Let $\delta > 0$, f satisfy conditions $(f_1), (f_2)$ and (f_3). Then, in addition to assertions in theorem 1.1 the following properties hold.*

(i) The entire branch of minimal solutions $\{(\lambda, u_\lambda) : \lambda \in (0, \Lambda)\}$ coincides with a path-connected portion of $\mathcal{A}$ containing $(0, 0)$ in its closure. In particular, the minimal solution branch is an analytic curve in $\mathbb{R}^+ \times \mathcal{C}^+_{\phi_\delta}(\Omega)$.

(ii) If $(\Lambda, u_\Lambda) \in \mathcal{A}$ for some $u_\Lambda \in \mathcal{C}^+_{\phi_\delta}(\Omega)$, then $\mathcal{A}$ bends to the left of $\{\lambda = \Lambda\}$ at the point (Λ, u_Λ) .

Remark 1.5 *(a) If $N = 2$, $0 < \delta < 3$ and f has "atmost critical" growth in the sense of definition 1.1, by application of Trudinger-Moser imbedding [93], we can show that $(\Lambda, u_\Lambda) \in \mathcal{A}$ where u_Λ is the extremal solution obtained as the limit of minimal solutions. Thus, the assertion (ii) above implies the assertion 4) in Theorem 1.1 in [52] regarding the bending of the bifurcation curve to the left of Λ at u_Λ. In addition, the corresponding C^2 curve so obtained there is analytic. When $N \geq 3$ and f is again "atmost critical", using standard elliptic regularity (for example, see Lemma A.6, [69]), we can conclude the same. Thus, in these "atmost critical" growth situations, the minimal solution branch forms a bounded portion of the unbounded global solution branch. For many works related to such extremal solutions found at the end of the minimal solution branch and its regularity, we refer to [33] and the references given there.*

(b) If $\delta < 3$, from the Hardy inequality we obtain also that $\mathcal{A}$ defines a continuous curve in $\mathbb{R}^+ \times H^1_0(\Omega)$. Additionally, if f is "atmost critical", it can be seen that $\mathcal{A}$ is unbounded in $\mathbb{R}^+ \times H^1_0(\Omega)$ iff it is unbounded in $\mathbb{R}^+ \times \mathcal{C}_{\phi_\delta}(\Omega)$.

We now apply the analytic bifurcation theory to (P_λ) posed in the unit ball $B_1 \subset \mathbb{R}^2$. From [16], all solutions to (P_λ) on B_1 are radially symmetric. We first define the notion of singular solutions to (P_λ) :

Definition 1.6 *We call $u^* \in L^1_{loc}(\Omega)$ a singular solution to (P_λ) if $\inf_K u^* > 0$ for any compact set $K \subset \Omega$, $f(u^*) \in L^1_{loc}(\Omega)$, u^* is unbounded on Ω and u^* solves, in the sense of distributions, $-\Delta u^* = \lambda((u^*)^{-\delta} + f(u^*))$ in Ω.*

Remark 1.6 *It is not necessary in the above definition that u^* vanishes at the boundary.*

Remark 1.7 *If $0 < \delta < 1$, thanks to lemmas 3.5 and 3.6, the Morse Index of $\partial_u F$ at any weak solution is well defined.*

We state the following result about the singular solutions to (P_λ) which is of independent interest :

Theorem 1.2 *Fix $0 < \delta < 1$. Take $h(t) = t^{2+p_1} \prod_{i=2}^n e^{\alpha_i t^{p_i}}$, where $p_1 > 0, 1 < p_i < 2, i = 2, ..., n$ are distinct and $\alpha_i < 0$. Let $f(t) = h(t)e^{t^2}$. Let u^* be a radial singular solution to (P_λ) on B_1. Then, for any sequence of solutions $\{u_i\}_i$ of (P_λ) converging to u^* locally uniformly in $B_1 \setminus \{0\}$, the Morse index of u_i tends to infinity as $i \to \infty$. In particular, the Morse index of u^* is $+\infty$.*

Finally, the following result concerning infinite "turning points" rests on the existence of such a singular solution :

Theorem 1.3 *Let $\Omega = B_1 \subset \mathbb{R}^2$ and $0 < \delta < 1$. Take $h(t) = t^{2+p_1} \prod_{i=2}^n e^{\alpha_i t^{p_i}}$, where $p_1 > 0, 1 < p_i < 2, i = 2, ..., n$ are distinct and $\alpha_i < 0$. Let $f(t) = h(t)e^{t^2}$. Then the continuous analytic path $\mathcal{A}$ obtained in Theorem 1.1 admits infinitely many "turning points".*

More precisely, there exists a sequence $\{(\lambda_i, u_i)\}_{i\in\mathbb{N}} \subset \mathcal{A}$ such that $\|u_i\|_{\mathcal{C}_{\phi_\delta}(\Omega)}$ and the Morse index of u_i tend to infinity as $i \to \infty$ and the path $\mathcal{A}$ locally bends back at (λ_i, u_i).

We will prove the above two theorems in sub-section 3.2.

Remark 1.8 *Theorem 1.3 can be shown to hold as well when the singular term is replaced by a term as in remark 1.4(a) and (b).*

1.3 Preliminaries

We recall first some results concerning global analytic bifurcation theory from [Buffoni and Toland [32], Chapter 9] . Let $\mathcal{X}$, $\mathcal{Y}$ be real Banach spaces, $\mathcal{U} \subset \mathbb{R} \times \mathcal{X}$ an open set containing $(0, 0)$ in its closure and $F : \mathcal{U} \to \mathcal{Y}$ be an $\mathbb{R}$-analytic function. Define the solution set

$$\mathcal{S} = \{(\lambda, x) \in \mathcal{U} \,:\, F(\lambda, x) = 0\}$$

and the non-singular solution set

$$\mathcal{N} = \{(\lambda, x) \in \mathcal{S} \,:\, Ker(\partial_x F(\lambda, x)) = \{0\}\} .$$

Definition 1.7 *A distinguished arc is a maximal connected subset of $\mathcal{N}$.*

Suppose that

(G1) Bounded closed subsets of $\mathcal{S}$ are compact in $\mathbb{R} \times \mathcal{X}$.

(G2) $\partial_x F(\lambda, x)$ is a Fredholm operator of index zero for all $(\lambda, x) \in \mathcal{S}$.

(G3) There exists an analytic function $(\lambda, u) : (0, \epsilon) \to \mathcal{S}$ such that $\partial_x F(\lambda(s), u(s))$ is invertible for all $s \in (0, \epsilon)$ and $\lim_{s\to 0^+} (\lambda(s), u(s)) = (0, 0)$.

Let

$$\mathcal{A}^+ = \{(\lambda(s), u(s)) \,:\, s \in (0, \epsilon)\} .$$

Obviously, $\mathcal{A}^+ \subset \mathcal{S}$. The following result gives a global extension of the function (λ, u) from $(0, \epsilon)$ to $(0, \infty)$ in the $\mathbb{R}$-analytic case.

Theorem 1.4 *Suppose (G1)-(G3) hold. Then, (λ, u) can be extended as a continuous map (still called) $(\lambda, u) : (0, \infty) \to \mathcal{S}$ with the following properties :*

(a) Let $\mathcal{A} \overset{\text{def}}{=} \{(\lambda(s), u(s)) : s > 0\}$. Then, $\mathcal{A} \cap \mathcal{N}$ is an atmost countable union of distinct distinguished arcs $\bigcup_{i=0}^n \mathcal{A}_i$, $n \leq \infty$.

(b) $\mathcal{A}^+ \subset \mathcal{A}_0$.

(c) $\{s > 0 \,:\, ker(\partial_x F(\lambda(s), u(s))) \neq \{0\}\}$ is a discrete set.

(d) At each of its points $\mathcal{A}$ has a local analytic re-parameterization in the following sense : For each $s^* \in (0,\infty)$ there exists a continuous, injective map $\rho^* : (-1,1) \to \mathbb{R}$ such that $\rho^*(0) = s^*$ and the re-parametrisation

$$(-1,1) \ni t \to (\lambda(\rho^*(t)), u(\rho^*(t))) \in \mathcal{A} \text{ is analytic.}$$

Furthermore, the map $s \mapsto \lambda(s)$ is injective in a right neighborhood of $s = 0$ and for each $s^* > 0$ there exists $\epsilon^* > 0$ such that λ is injective on $[s^*, s^* + \epsilon^*]$ and on $[s^* - \epsilon^*, s^*]$.

(e) One of the following occurs.

(i) $\|(\lambda(s), u(s))\|_{\mathbb{R}\times\mathfrak{X}} \to \infty$ as $s \to \infty$.

(ii) the sequence $\{(\lambda(s), u(s))\}$ approaches the boundary of $\mathfrak{U}$ as $s \to \infty$.

(iii) $\mathcal{A}$ is the closed loop :

$$\mathcal{A} = \{(\lambda(s), u(s)) : 0 \le s \le T, (\lambda(T), u(T)) = (0,0) \text{ for some } T > 0\}.$$

In this case, choosing the smallest such $T > 0$ we have

$$(\lambda(s+T), u(s+T)) = (\lambda(s), u(s)) \text{ for all } s \ge 0.$$

(f) Suppose $\partial_x F(\lambda(s_1), u(s_1))$ is invertible for some $s_1 > 0$. If for some $s_2 \neq s_1$, we have $(\lambda(s_1), u(s_1)) = (\lambda(s_2), u(s_2))$ then (e)(iii) occurs and $|s_1 - s_2|$ is an integer multiple of T. In particular, the map $s \mapsto (\lambda(s), u(s))$ is injective on $[0, T)$.

Remark 1.9 *We remark that theorem 9.1.1 in [32] deals with "bifurcation from the first eigenvalue" type of situation. The conditions $(G1)-(G3)$ assumed there are required only to ensure that the starting analytic path corresponding to $\mathcal{A}^+$ is available for global extension. In our case, we make this as an assumption $(G3)$ above. Hence the proof given in [32] holds good in our case as well.*

We make the following definitions of a "turning point" and a "potential operator" :

Definition 1.8 *("turning point") Let E be a real Banach space. Given a map $L : \mathbb{R}\times E \to E$ consider its set of zeroes $Z_L \overset{\text{def}}{=} \{(\lambda, x) : L(\lambda, x) = 0\}$. We call $(\lambda_0, x_0) \in Z_L$ a "turning point" in Z_L if there exist two sequences $\{(\lambda_k, \hat{x}_k)\}_{k\in\mathbb{N}}$, $\{(\lambda_k, \tilde{x}_k)\}_{k\in\mathbb{N}} \subset Z_L$ which converge to (λ_0, x_0) and $\hat{x}_k \neq \tilde{x}_k$ for all $k \in \mathbb{N}$.*

Definition 1.9 *("potential operator") Let E be a real Banach space with an inner-product $\langle\cdot,\cdot\rangle$ that is continuous on $E \times E$. Let $U \subset E$ be an open set and $G : U \to E$ be a continuous map. We say that G is a potential operator if there exists a continuously differentiable map $g : U \to \mathbb{R}$ such that*

$$\nabla g(x)h = \langle G(x), h\rangle, \quad \text{for all } x \in U, h \in E.$$

In the context of problems with a "potential operator" structure, we state the following theorem from section 11.3 of [32] which links the change of Morse index with the existence of "turning points" :

Theorem 1.5 *Let E a real Banach space, H a real Hilbert space and $\mathcal{U}$ an open set in $\mathbb{R} \times E$. We suppose that E is continuously imbedded and dense in H. Let $G : \mathcal{U} \to H$ be a real analytic map. We assume that there exists a continuously differentiable "potential" map $g : \mathcal{U} \to \mathbb{R}$ such that $< G(\lambda, u), \phi >= g_u(\lambda, u)\phi$ for any $u \in \mathcal{U}, \phi \in H$, i.e. G is a potential operator. In addition, we suppose that*

i) $\forall(\lambda, u) \in \mathcal{U}$, $\partial_u G(\lambda, u) : E \to H$ is a Fredholm operator with 0 index;

(ii) $M(\lambda, u)$, the Morse index, is well defined for $\forall(\lambda, u) \in \mathcal{U}$ such that $G(\lambda, u) = 0$;

iii) For any compact set of solutions to $G(\lambda, u) = 0$ in $\mathcal{U}$, the spectrum of $\partial_u G(\lambda, u)$ is uniformly bounded below.

Let $\mathcal{L} = \{(\lambda(s), u(s)) \in \mathcal{U} : s \in (-\varepsilon, \varepsilon)\}$ be a curve of solutions to $G(\lambda, u) = 0$ such that 0 does not belong to the spectrum of $\partial_u G(\lambda(s), u(s))$ for $s \in (-\varepsilon, \varepsilon)\backslash\{0\}$ and such that

$$\lim_{s \to 0^-} M(\lambda(s), u(s)) \neq \lim_{s \to 0^+} M(\lambda(s), u(s)).$$

Then $(\lambda(0), u(0))$ is a "turning point".

The above result is due to [H. Kielhöfer, [85]] and independently to [Chow-Lauterbach, [37]] (see also Theorem II.7.3. p. 236 in [84]).

2 The singular problem and the analytic bifurcation framework

2.1 Analyticity of the solution operator

In this section we use the definition 1.5 of an analytic map between Banach spaces.

Proposition 2.1 *Let $g(t) : \mathbb{R} \to \mathbb{R}$ be an entire function with $g(0) = 0$. Define $M_k(a) = max_{[-a,a]} g^{(k)}$, $k = 1, 2, 3, ...$ Assume that for any $a \geq 0$, there exists $\mu > 0$ such that the series $\sum_{k=0}^{\infty} \frac{M_k(a)}{k!} \mu^k$ converges. Then, for any $\phi \in \mathcal{C}_0(\overline{\Omega}), \phi > 0$ in Ω, we have $\mathcal{C}_\phi(\Omega) \ni u \mapsto g(u) \in \mathcal{C}_\phi(\Omega)$ is an analytic map. Furthermore, if $\inf_{[0,\infty)} g' > 0$, then g maps $\mathcal{C}_\phi^+(\Omega)$ into itself.*

Proof. We expand $g(t + \theta) = \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{k!} \theta^k$ where θ is chosen so that $t + \theta$ lies in the interval of convergence around t. We then have for $u, h \in \mathcal{C}_\phi(\Omega)$, with $\|h\|_{\mathcal{C}_\phi(\Omega)} < \varepsilon$, $\varepsilon > 0$ small,

$$g(u + h) = \sum_{k=0}^{\infty} \frac{g^{(k)}(u)}{k!} h^k. \tag{III.4}$$

Let $\max_\Omega |u| \leq C\|\phi\|_{L^\infty(\Omega)}$. Then

$$\frac{|g^{(k)}(u)|}{k!} h^k \phi^{-1} \leq \frac{M_k(C\|\phi\|_{L^\infty(\Omega)})}{k!} \|\phi\|_{L^\infty(\Omega)}^{k-1} \varepsilon^k.$$

By choosing ε small enough, we obtain that the series in (III.4) is dominated by the convergent series whose terms are given by the right hand side of the last inequality, and hence converges uniformly in $\mathcal{C}_\phi(\Omega)$. If $\inf_{[0,\infty)} g' > 0$ then by mean-value theorem it follows that g maps $\mathcal{C}_\phi^+(\Omega)$ into itself. $\square$

Remark 2.1 *We see that the conclusion g is an analytic self-map on $\mathcal{C}_\phi(\Omega)$ in proposition 2.1 holds good if the radius of convergence of g at $t = 0$ is infinite.*

Recall (see equation (III.3)) the solution operator defined as :

$$F(\lambda, u) = u - \lambda(-\Delta)^{-1}(u^{-\delta} + f(u)),\ (\lambda, u) \in \mathbb{R}^+ \times \mathcal{C}^+_{\phi_\delta}(\Omega), \delta > 0.$$

Proposition 2.2 *Let f satisfy the condition (f_1). Then the map F takes $\mathbb{R} \times \mathcal{C}^+_{\phi_\delta}(\Omega)$ into $\mathcal{C}_{\phi_\delta}(\Omega)$ and is analytic.*

Proof. *Step 1 : The map $\mathcal{C}^+_{\phi_\delta}(\Omega) \ni u \longmapsto u^{-\delta} + f(u) \in \mathcal{C}^+_{\phi_\delta^{-\delta}}(\Omega)$ is analytic.*
Since $u \in \mathcal{C}^+_{\phi_\delta}(\Omega)$, we can find positive constants C_1, C_2 such that

$$0 < C_1 \leq \frac{u(x)}{\phi_\delta(x)} \leq C_2 < \infty \quad \forall x \in \Omega \ \ (C_1, C_2 \text{ depend on } u). \tag{III.5}$$

It then follows that $u^{-\delta} \in \mathcal{C}^+_{\phi_\delta^{-\delta}}(\Omega)$ whenever $u \in \mathcal{C}^+_{\phi_\delta}(\Omega)$. We expand for real numbers t, h with $t > 0$ and $t + h > 0$:

$$(t + h)^{-\delta} = \sum_{k=0}^{\infty} C^\delta_k t^{-\delta-k} h^k \tag{III.6}$$

where

$$C^\delta_k = \frac{\prod_{j=0}^{k-1}(-\delta - j)}{k!},\ \ k = 1, 2, ... \tag{III.7}$$

For a fixed $t > 0$, we note that the series in (III.6) converges uniformly on compact subsets of $(-t, t)$.

Let $0 < \varepsilon < C_1$. We now choose $h \in \mathcal{C}_{\phi_\delta}(\Omega)$ such that $\|h\|_{\mathcal{C}_{\phi_\delta}(\Omega)} \leq \varepsilon$. That is,

$$\sup_{x\in\Omega} \frac{|h(x)|}{\phi_\delta(x)} \leq \varepsilon. \tag{III.8}$$

From (III.5), (III.8) we obtain that

$$(u \pm h)(x) > 0 \quad \text{in} \ \ \Omega.$$

Therefore, from the expansions (III.6) and (III.7) we get

$$[(u + h)(x)]^{-\delta} = \sum_{k=0}^{\infty} C^\delta_k (u(x))^{-\delta} \left(\frac{h(x)}{u(x)}\right)^k,\ \ x \in \Omega. \tag{III.9}$$

Using the pointwise estimates in (III.5) and (III.8) we obtain the following estimate for each term in (III.9)

$$C^\delta_k (u(x))^{-\delta} \left(\frac{h(x)}{u(x)}\right)^k \leq \{\varepsilon^k C_1^{-\delta-k} C^\delta_k\}(\phi_\delta(x))^{-\delta},\ \ x \in \Omega.$$

Hence,
$$\left|C_k^\delta (u(x))^{-\delta}\left(\frac{h(x)}{u(x)}\right)^k\right|(\phi_\delta(x))^\delta \le C_1^{-\delta}C_k^\delta\left(\frac{\varepsilon}{C_1}\right)^k,\quad x\in\Omega. \tag{III.10}$$

We note that the series $\sum_{k=0}^\infty C_k^\delta\, x^k$ converges uniformly on compact sets of $[0,1)$. Therefore, since we chose $\varepsilon < C_1$, from (III.10) we obtain that the series

$$\sum_{k=0}^\infty C_k^\delta (u(x))^{-k-\delta}h(x)^k \text{ converges uniformly in } \mathcal{C}_{\phi_\delta^{-\delta}}(\Omega) \text{ for } \|h\|_{\mathcal{C}_{\phi_\delta}(\Omega)} \text{ small.}$$

This shows the analyticity of the map $u^{-\delta}$. By similar arguments as above we can show that the map $\mathcal{C}^+_{\phi_\delta}(\Omega) \ni u \longmapsto u^p \in \mathcal{C}^+_{\phi_\delta^p}(\Omega)$ is analytic for any $p>0$. Since f satisfies the condition (f_1), using lemma 2.1 we obtain that the map $\mathcal{C}^+_{\phi_\delta}(\Omega) \ni u \longmapsto f(u) \in \mathcal{C}_{\phi_\delta^p}(\Omega)\cap\{u\ge 0\}$ is analytic. We note that the inclusion $\mathcal{C}_{\phi_\delta^p}(\Omega) \hookrightarrow \mathcal{C}_{\phi_\delta^{-\delta}}(\Omega)$ is analytic and hence we obtain the analyticity of the map $f(u)$ between $\mathcal{C}^+_{\phi_\delta}(\Omega)$ and $\mathcal{C}_{\phi_\delta^{-\delta}}(\Omega)$. We also have that $f(u)\ge 0$ in Ω for all $u\in\mathcal{C}^+_{\phi_\delta}(\Omega)$ which means that $u^{-\delta}+f(u)\in\mathcal{C}^+_{\phi_\delta^{-\delta}}(\Omega)$. This finishes step 1.

Step 2 : The map $\mathcal{C}_{\phi_\delta^{-\delta}}(\Omega) \ni u \longmapsto (-\Delta)^{-1}u \in \mathcal{C}_{\phi_\delta}(\Omega)$ is a linear continuous map (and hence analytic). Furthermore, this map takes $\mathcal{C}^+_{\phi_\delta^{-\delta}}(\Omega)$ into $\mathcal{C}^+_{\phi_\delta}(\Omega)$.

First, we show that Δ^{-1} is well defined on $\mathcal{C}_{\phi_\delta^{-\delta}}(\Omega)$. That is, we want to show that the following problem has a unique solution in the sense of distributions :

$$w\in L^1_{\text{loc}}(\Omega)\quad \begin{cases} -\Delta w = u \;\; in \;\; \Omega, \quad u\in\mathcal{C}_{\phi_\delta^{-\delta}}(\Omega),\\ w=0 \text{ on } \partial\Omega.\end{cases} \tag{III.11}$$

Later, we show that $w\in\mathcal{C}^+_{\phi_\delta}(\Omega)$ if $u\in\mathcal{C}^+_{\phi_\delta^{-\delta}}(\Omega)$.

If $\delta<1$ it is easy to check that $u\in L^p(\Omega)$ for some $p>1$. Hence, by standard elliptic theory, we can obtain the unique solution w to (III.11) in $W^{2,p}(\Omega)$. In particular, w satisfies the PDE in (III.11) pointwise a.e.in Ω.

If $\delta\ge 1$, the main idea is to use the fact

$$u\in\mathcal{C}_{\phi_\delta^{-\delta}}(\Omega) \Longrightarrow u\in H^{-s}(\Omega) \text{ for some } s>0.$$

To see this, we use the following H^s-version of the Hardy' inequality (see [Triebel, [117]] or [Grisvard, [74]]), Theorem 1.4.4.4. :

$$\|d(x,\partial\Omega)^{-s}\xi\|_{L^2(\Omega)} \le C\|\xi\|_{H_0^s(\Omega)}\ \forall\xi\in H_0^s(\Omega),\ s\in[0,2]\backslash\{\frac{1}{2},\frac{3}{2}\}. \tag{III.12}$$

For any $\delta>0$, let
$$s_0(\delta)=\frac{1}{2}\,\frac{3\delta-1}{\delta+1}. \tag{III.13}$$

Note that

$$\lim_{\delta\to\infty} s_0(\delta) = \frac{3}{2},\ s_0(\delta) \text{ is strictly increasing for } \delta > 0, \text{ and hence } s_0(\delta) > \frac{1}{2} \text{ for } \delta > 1. \tag{III.14}$$

We also check easily that

$$d(x,\partial\Omega)^{s-\frac{2\delta}{\delta+1}} \in L^2(\Omega) \ \ if \ \ s > s_0(\delta). \tag{III.15}$$

Now, given $\delta \geq 1$, fix $s \in (s_0(\delta), \frac{3}{2})$. Then, using (III.12), (III.14) and (III.15), we get for any $u \in \mathcal{C}_{\phi_\delta^{-\delta}}(\Omega)$, $\xi \in H_0^s(\Omega)$,

$$\begin{aligned} |\textstyle\int_\Omega u\xi\,| &= |\textstyle\int_\Omega (ud(x,\partial\Omega)^s)(\xi d(x,\partial\Omega)^{-s})| \\ &\leq C\textstyle\int_\Omega (d(x,\partial\Omega)^{s-\frac{2\delta}{\delta+1}})(d(x,\partial\Omega)^{-s}|\xi|) \\ &\leq C\|d(x,\partial\Omega)^{s-\frac{2\delta}{\delta+1}}\|_{L^2(\Omega)}\|d(x,\partial\Omega)^{-s}\xi\|_{L^2(\Omega)} \\ &\leq C\|\xi\|_{H_0^s(\Omega)}. \end{aligned} \tag{III.16}$$

From (III.16), we immediately obtain that

$$u \in \mathcal{C}_{\phi_\delta^{-\delta}}(\Omega) \Longrightarrow u \in H^{-s}(\Omega) \ \ \forall s \in (s_0(\delta), 3/2). \tag{III.17}$$

Thanks to (III.17) and standard elliptic theory on $H_0^s(\Omega)$ spaces, we can solve (III.11) uniquely for $w \in H_0^{2-s}(\Omega)$ $\forall s \in (s_0(\delta), 3/2)$. By Sobolev imbedding we then obtain that $w \in L^p(\Omega)$ for some $p > 1$.

Therefore, for any $\delta > 0$, we can ensure that the unique solution $w \in L^p(\Omega)$ for some $p > 1$. Thus, w solves (III.11) in the sense of distributions. Clearly, since $u \in \mathcal{C}_{\phi_\delta^{-\delta}}(\Omega)$, from elliptic regularity we obtain that

$$w \in W^{2,p}_{\text{loc}}(\Omega),\ 1 \leq p < \infty \text{ (in particular } w \in \mathcal{C}(\Omega)). \tag{III.18}$$

It follows then, that

$$-\Delta w = u \ \text{ pointwise a.e. in } \Omega. \tag{III.19}$$

Let v_δ solve

$$-\Delta v_\delta = v_\delta^{-\delta},\ v_\delta > 0 \text{ in } \Omega,\ v_\delta = 0 \text{ on } \partial\Omega. \tag{III.20}$$

From theorem 2.2 in [41] we obtain that $v_\delta \in \mathcal{C}^+_{\phi_\delta}(\Omega)$ for all $\delta > 0$.

Thus, from (III.19), (III.20) and noting that $|u(x)| \leq \|u\|_{\mathcal{C}_{\phi_\delta^{-\delta}}(\Omega)}\phi_\delta^{-\delta}(x),\ x \in \Omega$, for the choice $\theta_u = \|v_\delta\|^\delta_{\mathcal{C}_{\phi_\delta}(\Omega)}\|u\|_{\mathcal{C}_{\phi_\delta^{-\delta}}(\Omega)}$, we get

$$\begin{cases} -\Delta(\theta_u v_\delta \pm w) \geq 0 \ \text{ distribution sense in } \Omega, \\ \theta_u v_\delta \pm w = 0 \ \text{ on } \partial\Omega. \end{cases}$$

From the weak comparison principle we obtain

$$|w(x)| \leq \|v_\delta\|^{\delta+1}_{\mathcal{C}_{\phi_\delta}(\Omega)}\|u\|_{\mathcal{C}^{-\delta}_{\phi_\delta}(\Omega)}\phi_\delta(x) \ \text{ in } \Omega. \tag{III.21}$$

In particular, from (III.21) we get the continuity of the map $(-\Delta)^{-1}:\mathcal{C}_{\phi_\delta^{-\delta}}(\Omega)\to\mathcal{C}_{\phi_\delta}(\Omega)$. If $u\in\mathcal{C}^+_{\phi_\delta^{-\delta}}(\Omega)$ then we can check for $\theta>0$ small that $-\Delta(w-\theta v_\delta)\geq 0$ and hence by maximum principle, $w\in\mathcal{C}^+_{\phi_\delta}(\Omega)$. This proves step 2. The proof of the proposition follows by combining steps 1 and 2. $\square$

2.2 Preliminary analysis of the linearised operator

We recall the solution operator F we studied in the last section :

$$\begin{cases} F(\lambda,u):\mathbb{R}^+\times\mathcal{C}^+_{\phi_\delta}(\Omega)\longrightarrow\mathcal{C}_{\phi_\delta}(\Omega)\\ F(\lambda,u)=u-\lambda(-\Delta)^{-1}[u^{-\delta}+f(u)],\ \delta>0.\end{cases}$$

If f satisfies condition (f_1), from proposition 2.2 we know that F is an analytic map. The linearisation of F with respect to the second variable at $(\lambda,u)\in\mathbb{R}^+\times\mathcal{C}^+_{\phi_\delta}(\Omega)$ is given as :

$$\begin{cases}\partial_u F(\lambda,u)\ :\ \mathcal{C}_{\phi_\delta}(\Omega)\to\mathcal{C}_{\phi_\delta}(\Omega),\\ \partial_u F(\lambda,u)w=w+(-\Delta)^{-1}[a_\lambda(u)w]\\ \text{where } a_\lambda(u)=\lambda(\delta u^{-1-\delta}-f'(u)).\end{cases}\tag{III.22}$$

Remark 2.2 *In the above definition, we note that $a_\lambda(u)w\in\mathcal{C}_{\phi_\delta^{-\delta}}(\Omega)$ and hence $(-\Delta)^{-1}[a_\lambda(u)w]\in\mathcal{C}_{\phi_\delta}(\Omega)$ as explained in Step 2 of lemma 2.2.*

In this section we put the operator F and the function spaces in the framework of analytic global bifurcation theory as given in [32]. This will enable us to prove the existence of an analytic global path of solutions to F in $\mathbb{R}^+\times\mathcal{C}^+_{\phi_\delta}(\Omega)$. We prove some lemmas below for this purpose.

Consider the following problem where h is a non-negative bounded function on Ω that is locally Hölder continuous :

$$(P)\qquad\begin{cases}-\Delta w+\lambda k w=\lambda w^{-\delta}+h,\ w>0,\ \text{ in }\ \Omega,\ (\lambda,\delta>0),\\ w=0\text{ on }\partial\Omega,\\ k\geq 0\text{ a constant.}\end{cases}$$

Lemma 2.1 *The problem (P) admits a unique distributional solution $w\in W_0^{1,q}(\Omega)\cap\mathcal{C}_0(\overline{\Omega})$ for some $q>1$. Assume further that there exists $\overline{\phi}\in\mathcal{C}^+_{\phi_\delta}(\Omega)$ which is a super solution for (P). Then we additionally have $w\in[c\phi_\delta,\overline{\phi}]$ for all $c>0$ small enough. In particular, $w\in\mathcal{C}^+_{\phi_\delta}(\Omega)$.*

Proof. Given $k>0,h\in L^\infty(\Omega),h\geq 0$, we consider the regularised problem for $\varepsilon>0$:

$$(P_\varepsilon)\qquad\begin{cases}-\Delta w+\lambda k w=\lambda(w^++\varepsilon)^{-\delta}+h\text{ in }\Omega,\\ w=0\text{ on }\partial\Omega.\end{cases}$$

We note that for a suitably small $c>0$, the function $\underline{\phi}^\varepsilon=(c^{\frac{1+\delta}{2}}\varphi_1+\varepsilon^{\frac{1+\delta}{2}})^{\frac{2}{1+\delta}}-\varepsilon$ is a subsolution for (P_ε) $\forall\varepsilon>0$. The unique positive solution of $-\Delta\bar{w}_\varepsilon+\lambda k\bar{w}_\varepsilon=\lambda\varepsilon^{-\delta}+\|h\|_{L^\infty(\Omega)},\bar{w}_\varepsilon\in H^1_0(\Omega)$, is a super-solution. By comparison principle, it is easy to see that $\underline{\phi}^\varepsilon<\bar{w}_\varepsilon$. Then, using standard arguments we obtain a solution $w_\varepsilon \in [\underline{\phi}^\varepsilon, \bar{w}_\varepsilon]$ of (P_ε) and uniquely so by the non-increasing nature of the right hand side in (P_ε). That w_ε is Hölder continuous on $\overline{\Omega}$ follows by elliptic regularity. By maximum principle we conclude as well that $w_\varepsilon > 0$ in Ω. We have the following monotonicity properties of w_ε with respect to $\varepsilon > 0$, namely :

$$w_{\varepsilon_1} \leq w_{\varepsilon_2} \text{ in } \Omega \text{ if } \varepsilon_2 \leq \varepsilon_1, \tag{III.23}$$

$$w_{\varepsilon_2} + \varepsilon_2 \leq w_{\varepsilon_1} + \varepsilon_1 \text{ in } \Omega \text{ if } \varepsilon_2 \leq \varepsilon_1. \tag{III.24}$$

(III.23) follows from (III.24) by the maximum principle. To show (III.24), we note that $v_\varepsilon = w_\varepsilon + \varepsilon$ solves

$$\begin{cases} -\Delta v_\varepsilon + \lambda k v_\varepsilon = \lambda v_\varepsilon^{-\delta} h + \lambda k \varepsilon \text{ in } \Omega, \\ v_\varepsilon = \varepsilon \text{ on } \partial\Omega. \end{cases}$$

By considering the set $\{v_{\varepsilon_2} > v_{\varepsilon_1}\}$ and using the comparison principle we get

$$v_{\varepsilon_2} \leq v_{\varepsilon_1} \text{ in } \Omega \quad \text{if } \varepsilon_2 \leq \varepsilon_1$$

which shows (III.24). Thus $\{w_\varepsilon\}$ is a Cauchy sequence in $\mathcal{C}_0(\overline{\Omega})$ from (III.24) and hence from the fact $w_\varepsilon \in [\underline{\phi}^\varepsilon, \bar{w}_\varepsilon]$, we get

$$w_\varepsilon \longrightarrow w \text{ (say)} \quad \text{in } \mathcal{C}_0(\overline{\Omega}) \text{ and } w \geq c\phi_\delta. \tag{III.25}$$

Let $\psi \in \mathcal{C}_c^2(\Omega)$ and multiply (P_ε) with ψ and integrate by parts to get,

$$-\int_\Omega w_\varepsilon \Delta\psi + \int_\Omega \lambda k w_\varepsilon \psi = \int_\Omega \lambda (w_\varepsilon + \varepsilon)^{-\delta} \psi + \int_\Omega h\psi, \quad \psi \in \mathcal{C}_c^2(\Omega). \tag{III.26}$$

Using the monotonicity of $w_\varepsilon + \varepsilon$ with respect to ε to pass to the limit $\varepsilon \to 0$ in (III.26), we get that w is a distributional solution of (P).

Since $h \in L^\infty(\Omega)$, from (III.25) we obtain that $h + \lambda w^{-\delta} \in L^1(\Omega, d(x,\partial\Omega)^\alpha)$ for some $\alpha < 1$. Then, by [Díaz-Rakotoson, Theorems 3 and 4, [54]], we get $w \in W_0^{1,q}(\Omega)$ for some $q > 1$. Thus, (P) admits a solution with the desired properties. Suppose (P) admits two distinct solutions w_1, w_2. Thanks to elliptic regularity, $w_1, w_2 \in \mathcal{C}^2(\Omega) \cap \mathcal{C}_0(\overline{\Omega})$. By considering the set $A \overset{\text{def}}{=} \{w_1 < w_2\} \subset\subset \Omega$ and applying the classical maximum principle in A if it is non-empty, we obtain a contradiction. Hence A is an empty set. This proves uniqueness for (P). Suppose now (P) has a super solution $\overline{\phi} \in \mathcal{C}_{\phi_\delta}^+(\Omega)$. Clearly, a super solution of (P) is also a super solution of (P_ε). We can check that $\underline{\phi}^\varepsilon \leq \phi_\delta$ for all $\varepsilon > 0$ if $c > 0$ is chosen small enough. Therefore, for such c, we get $\underline{\phi}^\varepsilon \leq \overline{\phi}$ for all ε. Hence, we can repeat the above arguments by replacing $\bar{w}_\varepsilon$ by $\overline{\phi}$ to get the solution $w \in \mathcal{C}_{\phi_\delta}^+(\Omega)$. $\square$

Remark 2.3 *Since a sub solution of $(P_\varepsilon), \varepsilon > 0$, is also a sub solution of (P) we can check that if $c > 0$ is chosen small (independent of ε), $\underline{\phi}^\varepsilon = (c^{\frac{1+\delta}{2}}\varphi_1 + \varepsilon^{\frac{1+\delta}{2}})^{\frac{2}{1+\delta}} - \varepsilon$ is a sub solution of (P) for all $\varepsilon > 0$ small.*

Lemma 2.2 *Let $a \in \mathcal{C}(\Omega)$ be such that $0 \leq a(x) \leq a_1 d(x,\partial\Omega)^{-2}$ for some positive constant a_1. Then,* *given* $z \in \mathcal{C}_{\phi_\delta}(\Omega)$ *there exists a unique distributional solution* $v \in \mathcal{C}_{\phi_\delta}(\Omega)$ *solving*

$$-\Delta v + av = az \text{ in } \Omega. \tag{III.27}$$

Further, the estimate $\|v\|_{\mathcal{C}_{\phi_\delta}(\Omega)} \leq C\|az\|_{\mathcal{C}_{\phi_\delta^{-\delta}}(\Omega)}$ *holds for some constant* $C > 0$ *independent of* z.

Proof. We note that $az \in \mathcal{C}_{\phi_\delta^{-\delta}}(\Omega) \subset L^1(\Omega, d(x, \partial\Omega)^\alpha)$ for some $\alpha \in (0,1)$. Hence by Theorem 2.1 in [55] the equation (III.27) admits a unique distributional solution

$$v \in L^1(\Omega) \cap W^{1,q}(\Omega, d(\cdot, \partial\Omega)) \text{ for some } q > 1.$$

We only need to show that $v \in \mathcal{C}_{\phi_\delta}(\Omega)$. The obstacle to showing this is the regularity of a which could have strong singularity near $\partial\Omega$, and this prevents the application of comparison principle. To overcome this, we perform the following regularisation :

Let $h_n \in \mathcal{C}_0^\infty(\Omega)$ be such that $h_n \to az$ locally uniformly in Ω. Let

$$\tilde{h}_n = \min\{2\|az\|_{\mathcal{C}_{\phi_\delta^{-\delta}}(\Omega)}\phi_\delta^{-\delta}, \max\{-2\|az\|_{\mathcal{C}_{\phi_\delta^{-\delta}}(\Omega)}\phi_\delta^{-\delta}, h_n\}\}.$$

Then, $\tilde{h}_n$ is a minimum/maximum of locally Hölder continuous functions and since it has compact support it is Hölder continuous on $\overline{\Omega}$. Further, given a compact set $K \subset \Omega$, we can check $\tilde{h}_n = h_n$ for all large n. Hence

$$\sup_n \|\tilde{h}_n\|_{\mathcal{C}_{\phi_\delta^{-\delta}}(\Omega)} \leq 2\|az\|_{\mathcal{C}_{\phi_\delta^{-\delta}}(\Omega)}, \ \tilde{h}_n \to az \text{ in } \mathcal{C}_{\mathrm{loc}}(\Omega), \ \tilde{h}_n \in \mathcal{C}_0^{0,\alpha}(\Omega), \ \alpha \text{ depends on } n!$$

Further, we choose a sequence $\{a_n\}$ of Hölder continuous functions on $\overline{\Omega}$, $a_n \to a$ in $\mathcal{C}_{loc}(\Omega)$ such that $0 \leq a_n(x) \leq a(x)$. Let v_n solve :

$$\begin{cases} -\Delta v_n + a_n v_n = \tilde{h}_n \text{ in } \Omega, \\ v_n = 0 \text{ on } \partial\Omega. \end{cases} \tag{III.28}$$

Clearly, by classical theory, $v_n \in \mathcal{C}^{2,\alpha}(\Omega) \cap \mathcal{C}_0(\overline{\Omega})$, where $\alpha = \alpha(n)$ depends on n. Let $\psi \in \mathcal{C}^+_{\phi_\delta}(\Omega)$ denote the unique solution (see [41]) of the problem $-\Delta\psi = \psi^{-\delta}$ in Ω, $\psi = 0$ on $\partial\Omega$.

We note the following inequality satisfied by the supersolution $M\psi$ for any $M > 0$ and a fixed $c > 0$,

$$-\Delta M\psi + a_n M\psi \geq M\psi^{-\delta} \geq cM\phi_\delta^{-\delta}. \tag{III.29}$$

We now choose M such that $cM = 2\|az\|_{\mathcal{C}_{\phi_\delta^{-\delta}}(\Omega)}$ so that

$$|\tilde{h}_n| \leq 2\|az\|_{\mathcal{C}_{\phi_\delta^{-\delta}}(\Omega)}\phi_\delta^{-\delta} = cM\phi_\delta^{-\delta}. \tag{III.30}$$

From (III.28) -(III.30) and classical comparison test, we get

$$|v_n| \leq C\|az\|_{\mathcal{C}_{\phi_\delta^{-\delta}}(\Omega)}\phi_\delta, \quad C \text{ independent of } z. \tag{III.31}$$

Recall , $\tilde{h}_n \to az,\ a_n \to a$ in $\mathcal{C}_{\mathrm{loc}}(\Omega)$. We note that $\{v_n\}$ is bounded in $W^{2,p}_{\mathrm{loc}}(\Omega)$ and let $v_n \to \tilde{v}$ in $\mathcal{C}_{\mathrm{loc}}(\Omega)$. Then, in the sense of distributions

$$\begin{cases} -\Delta\tilde{v} + a\tilde{v} = az \text{ in } \Omega, \\ \tilde{v} = 0 \text{ on } \partial\Omega. \end{cases}$$

By the uniqueness result given in theorem 2.1 in [55], we get $\tilde{v} = v$ and hence from (III.31) we get $\|v\|_{\mathcal{C}_{\phi_\delta}(\Omega)} \leq C\|az\|_{\mathcal{C}_{\phi_\delta^{-\delta}}(\Omega)}$. □

Corollary 2.1 *Let $h \in \mathcal{C}_{\phi_\delta^{-\delta}}(\Omega), a \in \mathcal{C}(\Omega)$ and $0 \leq a \leq a_1 d^{-2}$ for some constant $a_1 > 0$. If $v \in \mathcal{C}_{\phi_\delta}(\Omega) \cap \mathcal{C}^2(\Omega)$ is a classical solution of $-\Delta v + av = h$ in Ω, then we have $\|v\|_{\mathcal{C}_{\phi_\delta}(\Omega)} \leq C\|h\|_{\mathcal{C}_{\phi_\delta^{-\delta}}(\Omega)}$ for some constant $C > 0$ independent of h.*

Lemma 2.3 *The map $\partial_u F(\lambda, u)$ is Fredholm with index 0 for all $(\lambda, u) \in \mathbb{R}^+ \times \mathcal{C}^+_{\phi_\delta}(\Omega)$ and $\delta > 0$.*

Proof. We can write $\partial_u F(\lambda, u) = I + A_\lambda + B_\lambda$ where

$$A_\lambda w = (-\Delta)^{-1}[\delta\lambda u^{-1-\delta} w],$$

and

$$B_\lambda w = (-\Delta)^{-1}[-\lambda f'(u) w].$$

By appealing to lemma 2.2 with $a = \delta\lambda u^{-1-\delta}$ it is easy to see that $I + A_\lambda$ is invertible on $\mathcal{C}_{\phi_\delta}(\Omega)$. By classical theory, B_λ is compact on $\mathcal{C}_{\phi_\delta}(\Omega)$. It then follows that $\partial_u F(\lambda, u)$ is Fredholm with index 0. We finally describe the the spectrum of the linearised operator $\partial_u F(\lambda, u_\lambda)$ in $\mathcal{C}_{\varphi_1}(\Omega)$ for $\delta < 1$. Given an operator or a space X, we denote its complexification by X^*. □

Lemma 2.4 *For $\delta < 1$, the linearised operator $\partial_u F(\lambda, u_\lambda)$ is a compact perturbation of the identity.*

Proof. For convenience, denote $L_\lambda \stackrel{\mathrm{def}}{=} \partial_u F(\lambda, u_\lambda) = I + A_\lambda + B_\lambda$ (see lemma 2.3 for definition). Clearly, B_λ is a compact operator on $\mathcal{C}_{\phi_\delta}(\Omega)$ for any $\delta > 0$. For $\delta < 1$, we recall (see (III.2)) that $\mathcal{C}_{\phi_\delta}(\Omega) \stackrel{\mathrm{def}}{=} \mathcal{C}_{\varphi_1}(\Omega)$. It remains to show that $A_\lambda \stackrel{\mathrm{def}}{=} (-\Delta)^{-1}(\lambda\delta u_\lambda^{-1-\delta}\cdot)$ is compact on $\mathcal{C}_{\varphi_1}(\Omega)$. Let $\{\psi_n\} \subset \mathcal{C}_{\varphi_1}(\Omega)$ be a bounded sequence. Letting $\xi_n = A_\lambda(\psi_n)$, we see that ξ_n solves $-\Delta\xi_n = \lambda\delta u_\lambda^{-1-\delta}\psi_n,\ \ \xi_n \in \mathcal{C}_{\varphi_1}(\Omega)$. Since $u_\lambda \in \mathcal{C}^+_{\varphi_1}(\Omega)$, we see that $\{-\Delta\xi_n\}$ is bounded in $\mathcal{C}_{\varphi_1^{-\delta}}(\Omega)$. Then, from theorem 1.1(i) in [76], we obtain that $\{\xi_n\}$ is a bounded sequence in $\mathcal{C}^{1,\theta}(\overline{\Omega}) \cap \mathcal{C}_0(\overline{\Omega})$ for some $\theta \in (0,1)$. The compactness of A_λ now follows. □

Lemma 2.5 *If $\delta < 1$, the spectrum of the linearised operator $\partial_u F(\lambda, u_\lambda)$ on $\mathcal{C}_{\varphi_1}(\Omega)$ consists of only countable compact set of real numbers whose members other than 1 are isolated eigenvalues.*

Proof. We again denote $L_\lambda \stackrel{\mathrm{def}}{=} \partial_u F(\lambda, u_\lambda) = I + A_\lambda + B_\lambda$ as before. That the spectrum is a countable compact set whose members other that 1 are isolated eigenvalues follows from the spectral theory of compact operators and lemma 2.4. To show it is real we proceed as follows. Let $\mu = \mu_1 + i\mu_2 \in \mathbb{C}$ be an eigenvalue of L_λ. That is, $(\mu I^* - (I + A_\lambda + B_\lambda)^*)\psi = 0$ for some non-zero $\psi = \psi_1 + i\psi_2 \in \mathcal{C}^*_{\varphi_1}(\Omega)$. Writing out this equation in terms of real and imaginary parts, we obtain the following system (recall from (III.22) that $a_\lambda(u_\lambda) \stackrel{\text{def}}{=} \lambda(\delta u_\lambda^{-1-\delta} - f'(u_\lambda)))$,

$$\begin{aligned} -(1-\mu_1)\Delta\psi_1 - \mu_2\Delta\psi_2 + a_\lambda(u_\lambda)\psi_1 &= 0, \\ -(1-\mu_1)\Delta\psi_2 + \mu_2\Delta\psi_1 + a_\lambda(u_\lambda)\psi_2 &= 0. \end{aligned} \tag{III.32}$$

Let us assume for the moment that f' is nondecreasing. Taking $(\mu_1, \mu_2) = (1, 0)$ in the last system implies that supports of ψ_1, ψ_2 are contained in the set $\{\delta u_\lambda^{-\delta-1} = f'(u_\lambda)\}$. This set is of measure zero in Ω if f' is a non-decreasing function. Hence $\psi_1 \equiv \psi_2 \equiv 0$. This means 1 is not an eigenvalue of $\partial_u F(\lambda, u_\lambda)$ in this case.

We can rewrite the above system in matrix form for $\mu_1 \neq 1, \mu_2 \neq 0$ as,

$$\begin{bmatrix} -\Delta\psi_1 \\ -\Delta\psi_2 \end{bmatrix} = -\frac{1}{2} a_\lambda(u_\lambda) \begin{bmatrix} \frac{1}{1-\mu_1} & \frac{1}{1-\mu_1} \\ \frac{1}{\mu_2} & -\frac{1}{\mu_2} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}.$$

We note that the right hand side in the above equation belongs to $\mathcal{C}_{\varphi_1^{-\delta}}(\Omega) \times \mathcal{C}_{\varphi_1^{-\delta}}(\Omega)$. Since $\delta < 1$, this means that the right hand side is in $H^{-1}(\Omega) \times H^{-1}(\Omega)$. Now, arguing as in step 2 of proposition 2.2, we obtain that infact $\psi_1, \psi_2 \in H^1_0(\Omega)$ and solve the system (III.32) in the H^1-weak sense. If $\mu_2 = 0$ and $\mu_1 \neq 1$, then the system in (III.32) is uncoupled and we again get as before that $\psi_1, \psi_2 \in H^1_0(\Omega)$ solve (III.32) in the H^1-weak sense. If $\mu_1 = 1$ and $\mu_2 \neq 0$ again from (III.32) we get the same conclusion. Now multiplying the first equation in (III.32) by ψ_2 and the second by ψ_1 and integrating by parts, we get that $\mu_2 = 0$. Since any spectral value other than 1 is an eigenvalue, the lemma follows. $\square$

2.3 Local analytic path and minimal solutions

2.3.1 Existence of minimal solution

Let $M_0 > 0$ be chosen so that $\delta t^{-\delta-1} - f'(t) \geq 0$ in $(0, M_0)$. We first show the existence of minimal solutions for all small $\lambda > 0$.

Lemma 2.6 *We can find $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ there exists a minimal solution $u_\lambda \in \mathcal{C}^+_{\phi_\delta}(\Omega)$ solving (P_λ). Furthermore, $\|u_\lambda\|_{\mathcal{C}(\overline{\Omega})} \to 0$ as $\lambda \to 0^+$. If we choose λ_0 further small so that $\sup\limits_{0<\lambda<\lambda_0} \|u_\lambda\|_{\mathcal{C}(\overline{\Omega})} < M_0$, then u_λ is the only solution in the domain $(0, \lambda_0) \times \{u \in \mathcal{C}^+_0(\overline{\Omega}) : \|u\|_{\mathcal{C}(\overline{\Omega})} < M_0\}$.*

Proof. We can check that $\underline{\phi}_\lambda = c\lambda^{\frac{1}{1+\delta}}\phi_\delta$ is a subsolution for the problem (P_λ) for all $\lambda > 0$ if $c > 0$ is chosen small enough. Consider the purely singular problem :

$$(S_\lambda) \qquad \begin{cases} -\Delta\psi = \lambda\psi^{-\delta}, \ \psi > 0, \ \text{ in } \Omega, \\ \psi = 0 \text{ on } \partial\Omega. \end{cases}$$

Then, from [41] it is well known that (S_λ) admits a unique solution $\psi_\lambda \in \mathcal{C}^+_{\phi_\delta}(\Omega)$ for all $\lambda > 0$. Let $\xi_\lambda \in H^1_0(\Omega)$ denote the unique positive solution of $-\Delta\xi = \lambda$ in Ω. Then, it can be checked that for some $\lambda_0 > 0$, $\overline{\phi}_\lambda = \psi_\lambda + \xi_\lambda$ is a super solution for (P_λ) for all $0 < \lambda < \lambda_0$. We can check that $\overline{\phi}_\lambda \in \mathcal{C}^+_{\phi_\delta}(\Omega)$ as well. From the estimates in theorem 2.2 and 2.5 in [41], we get that

$$\liminf_{\lambda\to 0^+} \inf_{\overline{\Omega}} \frac{\overline{\phi}_\lambda}{\lambda^{\frac{1}{1+\delta}}\phi_\delta} > 0. \tag{III.33}$$

Hence, by choosing $c > 0$ smaller if necessary, we can ensure $\underline{\phi}_\lambda < \overline{\phi}_\lambda$ in Ω for all $\lambda \in (0, \lambda_0)$. We now choose a constant $k > 0$ so that the function $f(t) + kt$ is non-decreasing on $[0, \|\overline{\phi}_\lambda\|_{L^\infty(\Omega)}]$. Define the following monotone iteration scheme for all $\lambda \in (0, \lambda_0)$:

$$(P_n) \qquad \begin{cases} u_0 = \underline{\phi}_\lambda. \text{ Given } \ u_n \in [\underline{\phi}_\lambda, \overline{\phi}_\lambda], \text{ solve} \\ -\Delta u_{n+1} - \lambda u_{n+1}^{-\delta} + \lambda k u_{n+1} = \lambda(f(u_n) + ku_n) \ \text{ in } \ \Omega, \\ \qquad\qquad\qquad\qquad u_{n+1} = 0 \ \text{ on } \ \partial\Omega, \\ u_{n+1} \in [\underline{\phi}_\lambda, \overline{\phi}_\lambda], \ u_{n+1} \in W^{1,q}_0(\Omega) \text{ for some } q > 1. \end{cases}$$

It is easy to see that $\underline{\phi}_\lambda, \overline{\phi}_\lambda$ are respectively sub and super solutions of the PDE in (P_n). Hence from lemma 2.1 we see that the iteration scheme in (P_n) is well defined. We also obtain the monotonicity of the iterates viz., $u_{n+1} \geq u_n$ from the monotonicity of the operator $-\Delta(\cdot) - \lambda(\cdot)^{-\delta} + \lambda k(\cdot)$. We then obtain the minimal solution u_λ for all $\lambda \in (0, \lambda_0)$ in a standard way. Clearly $\|u_\lambda\|_{\mathcal{C}_{\phi_\delta}(\Omega)} \to 0$ as $\lambda \to 0^+$. Suppose $\tilde{u}_\lambda$ is another solution with $\lambda < \lambda_0, \|u_\lambda\|_{\mathcal{C}(\overline{\Omega})} < M_0$. Then $w_\lambda \overset{\text{def}}{=} u_\lambda - \tilde{u}_\lambda$ solves

$$-\Delta w_\lambda + \lambda[\delta\xi_\lambda^{-\delta-1} - f'(\xi_\lambda)]w_\lambda = 0$$

where ξ_λ lies between u_λ and $\tilde{u}_\lambda$. This makes $\delta\xi_\lambda^{-\delta-1} - f'(\xi_\lambda) \geq 0$ and hence $w_\lambda \equiv 0$. □

2.3.2 Invertibility of $\partial_2 F(\lambda, u_\lambda)$ and existence of a local path

Definition 2.1 *Let* $\Lambda = \sup\{\lambda > 0 : (P_\lambda)$ *has a solution*$\}$.

Remark 2.4 *Using the above lemma, it follows that* $\Lambda > 0$. *Using the method of monotone iteration it can also be seen that* (P_λ) *has a minimal solution* u_λ *for all* $\lambda \in (0, \Lambda)$. *Suppose, additionally* f *satisfies the "superlinear" condition* (f_2). *Then, it can be shown that* $\Lambda < \infty$.

Definition 2.2 $\mathcal{M} = \{(\lambda, u_\lambda) : 0 < \lambda < \Lambda, u_\lambda$ *is the minimal solution of* $(P_\lambda)\} \subset \mathcal{S}$.

As a first step, we show that a portion of the minimal solution branch $\mathcal{M}$ containing the solutions with small norm can be parametrised by an analytic curve.

Lemma 2.7 *Let* $\delta > 0$ *be fixed and* f *satisfy condition* (f_1). *For* $\lambda_0 > 0$ *chosen with the restriction as given in lemma 2.6, the portion of the minimal solution branch* $\{(\lambda, u_\lambda) \in \mathcal{M} : \lambda \in (0, \lambda_0)\}$ *lies on a path parametrised by an analytic map.*

Proof. Recall $a_\lambda(u_\lambda) \stackrel{\text{def}}{=} \lambda[\delta u_\lambda^{-1-\delta} - f'(u_\lambda)]$. We show that $\partial_u F(\lambda, u_\lambda)$ is invertible if $\lambda \in (0, \lambda_0)$. Thanks to lemma 2.3, it is enough to show that $\partial_u F(\lambda, u_\lambda)$ is onto $\mathcal{C}_{\phi_\delta}(\Omega)$ for all such λ. By the choice of $\lambda_0 > 0$ it is clear that for all $\lambda \in (0, \lambda_0)$ we have $a_\lambda(u_\lambda) \geq 0$. The "onto"ness of $\partial_u F(\lambda, u_\lambda)$ follows now by applying lemma 2.2 with $a = a_\lambda(u_\lambda)$. The lemma follows by applying the analytic implicit function theorem 4.5.4 in [32] at (λ, u_λ), for $\lambda \in (0, \lambda_0)$. □

2.3.3 Local path containing set of all minimal solutions

We show now that the linearised operator is still invertible along of minimal solutions if we suppose that f satisfies (f_3). Consequently, we can parametrise the set of minimal solutions by applying the implicit function theorem.

Proposition 2.3 *Let $u \in \mathcal{C}^+_{\phi_\delta}(\Omega) \cap \mathcal{C}^2(\Omega), \lambda > 0$. If $\partial_u F(\lambda, u)\varphi = 0$ for some $\varphi \in \mathcal{C}^2(\Omega) \cap \mathcal{C}_{\phi_\delta}(\Omega)$, then infact $\varphi \in H^1_0(\Omega) \cap \mathcal{C}_{\varphi_1}(\Omega)$ and is a H^1-weak solution of $-\Delta\varphi + a_\lambda(u)\varphi = 0$. Conversely, if for some $\theta \in \mathbb{R}$, the function $\varphi \in H^1_0(\Omega)$ is a non-negative H^1-weak solution of $-\Delta\varphi + a_\lambda(u)\varphi = \theta\varphi$ (obtained by constrained minimisation for instance), then $\varphi \in \mathcal{C}_{\varphi_1}(\Omega) \cap \mathcal{C}^2(\Omega)$.*

Proof. Let $\partial_u F(\lambda, u)\varphi = 0$ for some non-trivial $\varphi \in \mathcal{C}^2(\Omega) \cap \mathcal{C}_{\phi_\delta}(\Omega)$. Define the following minimisation problem

$$\inf_{\psi \in H^1_0(\Omega)} \int_\Omega |\nabla\psi|^2 \mathrm{d}x + \int_\Omega \lambda\delta u^{-\delta-1}\psi^2 \mathrm{d}x - \int_\Omega \lambda f'(u)\varphi\psi \mathrm{d}x.$$

It is easy to see that the functional being minimised is coercive and weakly lower semicontinous on $H^1_0(\Omega)$ and hence has a minimiser $\psi_0 \in H^1_0(\Omega)$. We note that $u^{-1-\delta} \sim d^{-2}$ near $\partial\Omega$. Recalling Hardy's inequality, we can check that ψ_0 is a non-trivial H^1-weak solution of

$$-\Delta\psi_0 + \lambda\delta u^{-\delta-1}\psi_0 = \lambda f'(u)\varphi.$$

It follows, by standard elliptic regularity, that $\psi_0 \in \mathcal{C}^2(\Omega)$. Let

$$M = \sup_{\overline{\Omega}} \lambda|f'(u)\varphi|.$$

Comparing in H^1-weak sense with the equation satisfied by $-\Delta\xi = M$ in Ω, $\xi \in H^1_0(\Omega)$, we obtain that $\psi_0 \in \mathcal{C}_{\varphi_1}(\Omega)$. This means that $\varphi - \psi_0 \in \mathcal{C}^2(\Omega) \cap \mathcal{C}_0(\overline{\Omega})$ is a classical solution of

$$-\Delta(\varphi - \psi_0) + \lambda\delta u^{-\delta-1}(\varphi - \psi_0) = 0.$$

This implies, by the maximum principle, $\varphi \equiv \psi_0$. Therefore, $\varphi \in H^1_0(\Omega) \cap \mathcal{C}_{\varphi_1}(\Omega)$ and is a H^1-weak solution of $-\Delta\varphi + a_\lambda(u)\varphi = 0$. Conversely, let $-\Delta\varphi + a_\lambda(u)\varphi = \theta\varphi$ for a non-negative $\varphi \in H^1_0(\Omega)$. Letting the operator $L = -\Delta - f'(u) - \theta$, $f^i = g = 0$ in theorem 8.15 of [71] we get that $\varphi \in L^\infty(\Omega)$. A comparison with the solution ξ of $-\Delta\xi = 1, \xi \in H^1_0(\Omega)$ implies that $\varphi \in \mathcal{C}_{\varphi_1}(\Omega)$. That $\varphi \in \mathcal{C}^2(\Omega)$ follows from classical elliptic regularity. □

Lemma 2.8 *If f satisfies condition (f_3), then $\partial_u F(\lambda, u_\lambda)$ is invertible for any $(\lambda, u_\lambda) \in \mathcal{M}$. Consequently, if f satisfies additionally the condition (f_1), the full set of minimal solutions $\mathcal{M}$ is parametrised by an analytic map.*

Proof. Let $\epsilon > 0$. Consider the problem

$$(P_\lambda^\epsilon) \qquad \begin{cases} -\Delta w = \lambda[(w+\epsilon)^{-\delta} + f(w)] & \text{in } \Omega, \\ w > 0 \quad \text{in } \Omega, \ \ w|_{\partial\Omega} = 0. \end{cases}$$

We detail the argument for $\delta > 1$. The case $\delta \leq 1$ is similar but simpler. Let $\rho_\epsilon = (c^{\frac{\delta+1}{2}}\varphi_1 + \epsilon^{\frac{\delta+1}{2}})^{\frac{2}{\delta+1}} - \epsilon$. Given $\lambda > 0$, we can find $c = c_\lambda > 0$ small enough so that ρ_ϵ is a sub-solution for (P_λ^ϵ). We note that u_λ is a super-solution to this problem. Further restricting c_λ if necessary, we can ensure that $\rho_\epsilon \leq u_\lambda$ as well as $\rho_\epsilon \leq \phi_\delta$. Hence the method of monotone iteration can be applied now to obtain a minimal solution w_λ^ϵ for (P_λ^ϵ). We note that $\rho_\epsilon < w_\lambda^\epsilon < u_\lambda$ for all ϵ. Define for $\lambda \in (0, \Lambda)$,

$$\Lambda_1(\lambda) = \inf_{\phi \in H_0^1(\Omega), \int_\Omega \phi^2 = 1} \int_\Omega |\nabla \phi|^2 \mathrm{d}x + \int_\Omega a_\lambda(u_\lambda)\phi^2 \mathrm{d}x, \tag{III.34}$$

$$\Lambda_1^\epsilon(\lambda) = \inf_{\phi \in H_0^1(\Omega), \int_\Omega \phi^2 = 1} \int_\Omega |\nabla \phi|^2 + \lambda\delta(w_\lambda^\epsilon + \epsilon)^{-1-\delta}\phi^2 \mathrm{d}x - \int_\Omega \lambda f'(w_\lambda^\epsilon)\phi^2 \mathrm{d}x. \tag{III.35}$$

We now claim that $\Lambda_1^\epsilon(\lambda) \geq 0$. Suppose, $\Lambda_1^\epsilon(\lambda) < 0$ for some $\epsilon > 0$, $\lambda \in (0, \Lambda)$. Let $\varphi_\lambda^\epsilon \in H_0^1(\Omega)$ be the non-negative minimiser for (III.35). Since we assumed $\Lambda_1^\epsilon(\lambda) < 0$, we can check that (here we need f is $\mathcal{C}^2$) for $\mu > 0$ small enough the function $\tilde{w}_\lambda^\epsilon \overset{\text{def}}{=} w_\lambda^\epsilon - \mu\varphi_\lambda^\epsilon$ is a super-solution to (P_λ^ϵ) which lies above ρ_ϵ. Hence, again by monotone iteration procedure, we obtain a solution to (P_λ^ϵ) which lies below w_λ^ϵ, contradicting its minimality. Hence, $\Lambda_1^\epsilon(\lambda) \geq 0$ for all ϵ. Since $w_\lambda^\epsilon \leq u_\lambda$, by elliptic regularity we obtain a function w_λ such that $w_\lambda^\epsilon \to w_\lambda$ in $\mathcal{C}_{loc}(\Omega)$. Clearly, w_λ solves (P_λ) and by minimality of u_λ we obtain $w_\lambda \equiv u_\lambda$. We note that $w_\lambda^\epsilon + \epsilon \geq \rho_\epsilon + \epsilon \geq c_\lambda \phi_\delta$. Let φ_λ be the non-negative minimiser for (III.34). By Hardy's inequality,

$$\int_\Omega (w_\lambda^\epsilon + \epsilon)^{-1-\delta}\varphi_\lambda^2 \leq c_\lambda^{-1-\delta} \int_\Omega d^{-2}\varphi_\lambda^2 < \infty.$$

Therefore, by dominated convergence theorem,

$$\begin{aligned} \Lambda_1(\lambda) = & \int_\Omega \{|\nabla\varphi_\lambda|^2 + \lambda\delta u_\lambda^{-1-\delta}\varphi_\lambda^2\}\mathrm{d}x - \int_\Omega \lambda f'(u_\lambda)\varphi_\lambda^2 \mathrm{d}x \\ = & \int_\Omega |\nabla\varphi_\lambda|^2 + \lambda\delta(w_\lambda^\epsilon + \epsilon)^{-1-\delta}\varphi_\lambda^2 \mathrm{d}x - \int_\Omega \lambda f'(w_\lambda^\epsilon)\varphi_\lambda^2 \mathrm{d}x + o_\epsilon(1) \\ \geq & \Lambda_1^\epsilon(\lambda) + o_\epsilon(1). \end{aligned}$$

Hence, we conclude $\Lambda_1(\lambda) \geq 0$ for any $\lambda \in (0, \Lambda)$.

Suppose $\Lambda_1(\lambda_0) = 0$ for some $0 < \lambda_0 < \Lambda$. Then, we have

$$\int_\Omega |\nabla\varphi_{\lambda_0}|^2 \mathrm{d}x + \int_\Omega a_{\lambda_0}(u_{\lambda_0})\varphi_{\lambda_0}^2 \mathrm{d}x = 0.$$

The above equality implies that $\int_\Omega a_{\lambda_0}(u_{\lambda_0})\varphi_{\lambda_0}^2 \mathrm{d}x < 0$. Let $0 < \lambda < \lambda_0$. Since u_λ, u_{λ_0} are minimal solutions they are ordered as $u_\lambda < u_{\lambda_0}$. Hence, using (f_3), we get for any $\lambda < \lambda_0$ that

$$\int_\Omega |\nabla \phi_{\lambda_0}|^2 \mathrm{d}x + \int_\Omega a_\lambda(u_\lambda)\phi_{\lambda_0}^2 \mathrm{d}x < 0.$$

Therefore, it follows that $\Lambda_1(\lambda) < 0$ for $\lambda < \lambda_0$ which contradicts the established fact above. This finishes the proof that $\Lambda_1(\lambda) > 0$ for all $0 < \lambda < \Lambda$. Suppose now for some $\lambda \in (0, \Lambda)$, the linearised operator $\partial_u F(\lambda, u_\lambda)$ is not invertible. Then we can find $\varphi \in \mathcal{C}^2(\Omega) \cap \mathcal{C}_{\phi_\delta}(\Omega)$, $\int_\Omega \varphi^2 = 1$ solving $-\Delta\varphi + a_\lambda(u_\lambda)\varphi = 0$. From proposition 2.3, we obtain that $\varphi \in H_0^1(\Omega)$ is a H^1-weak solution of $-\Delta\varphi + a_\lambda(u_\lambda)\varphi = 0$. This means that $\Lambda_1(\lambda) = 0$, a contradiction. Therefore, we can apply the analytic implicit function theorem 4.5.4 in [32] at any $(\lambda, u_\lambda), \lambda \in (0, \Lambda)$, to prove the lemma. □

Remark 2.5 *In the concave case when the nonlinearity is $t^\alpha + f(t)$, $0 < \alpha < 1$, we can not conclude $\Lambda_1(\lambda) > 0\ \forall \lambda \in (0, \Lambda)$. In this context, refer to the paper [*Cazenave, Escobedo and Pozio*, [35]] for counter examples. Hence, the analytic path obtained by using implicit function theorem coincides with the minimal solution branch $\{(\lambda, u_\lambda)\}$ as long as $\Lambda_1(\lambda) > 0$, but not for all $\lambda \in (0, \Lambda)$.*

Lemma 2.9 *Closed and bounded subsets of the set of solutions $\mathcal{S} = \{(\lambda, u) \in \mathbb{R}^+ \times \mathcal{C}^+_{\phi_\delta}(\Omega) : F(\lambda, u) = 0\}$ are compact.*

Proof. We note that if $(\lambda, u) \in \mathcal{S}$, then u solves (P_λ). We claim that

$$\inf_{\{(\lambda,u)\in\mathcal{S}\}} \inf_\Omega \left(\frac{u}{\phi_\delta}\right) \geq c\lambda^{\frac{1}{\delta+1}}, \quad \text{for some constant } c > 0. \tag{III.36}$$

To see this, we note that if $(\lambda, u) \in \mathcal{S}$, then $u \geq u_\lambda$ where u_λ is the minimal solution to (P_λ). It can be seen that there exists a constant $c > 0$ such that $c\lambda^{\frac{1}{\delta+1}}\phi_\delta$ is a subsolution of (P_λ) for all $\lambda > 0$. Hence $u_\lambda \geq c\lambda^{\frac{1}{\delta+1}}\phi_\delta$ and the claim follows. Let $\mathcal{B} \subset \mathcal{S}$ be a closed and bounded set. Then, we can find a constant $M > 0$ such that $\lambda + \|u\|_{\mathcal{C}_{\phi_\delta}(\Omega)} \leq M$ for all $(\lambda, u) \in \mathcal{B}$. Therefore, from (III.36),

$$|\Delta u| \leq c^{-\delta}\lambda^{\frac{1}{\delta+1}}\phi_\delta^{-\delta} + \lambda \sup_{[0,M]} f \leq c^{-\delta}M^{\frac{1}{\delta+1}}\phi_\delta^{-\delta} + M \sup_{[0,M]} f \ \text{ for all } u \in \mathcal{B}. \tag{III.37}$$

Thanks to the above estimate on Δu, we can appeal to proposition 3.4 in [76] to obtain that

$$\sup_{\mathcal{B}} \|u\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega})} < \infty \ \text{ for some } \alpha \in (0, 1).$$

Let $\{(\lambda_n, u_n)\} \subset \mathcal{B}$. We will show that $\{(\lambda_n, u_n)\}$ has a convergent subsequence in $\mathbb{R}^+ \times \mathcal{C}_{\phi_\delta}(\Omega)$. From the Hölder estimate above, upto a subsequence, we have that $(\lambda_n, u_n) \to (\lambda_0, u_0)$ (say) in $\mathcal{C}(\overline{\Omega})$. If $\lambda_0 = 0$, then from the first inequality in (III.37) we get that $\Delta u_n \to 0$ locally uniformly in Ω. It then follows that u_0 is harmonic in Ω and $u_0 = 0$ on $\partial\Omega$. Hence $u_0 \equiv 0$. Using lemma 2.6 we get that $u_n \equiv u_{\lambda_n}$ (the minimal solution) and the equation (III.33) there implies that $(\lambda_n, u_n) \to (0, 0)$ in $\mathbb{R}^+ \times \mathcal{C}_{\phi_\delta}(\Omega)$. This gives a contradiction as $\mathcal{B}$ is closed and being a subset of $\mathcal{S}$, $(0, 0)$ doesnot belong to $\mathcal{B}$. Hence, necessarily, $\lambda_0 > 0$. Then, using (III.37) and the bound for u_n in $\mathcal{C}_{\phi_\delta}(\Omega)$, we get $-\Delta u_0 \in \mathcal{C}^+_{\phi_\delta^{-\delta}}(\Omega)$. From step 2 of proposition 2.2 we get that $u_0 \in \mathcal{C}^+_{\phi_\delta}(\Omega)$. Let $w_n = u_0 - u_n$. Then, again using (III.36), we see that w_n solves an equation of the form

$$\begin{aligned} -\Delta w_n + \lambda_0 \delta \xi_n^{-\delta-1} w_n &= (\lambda_0 - \lambda_n) u_n^{-\delta} + (\lambda_0 - \lambda_n) f(u_n) + \lambda_0 (f(u_{\lambda_0}) - f(u_n)) \\ &= o_n(1) \phi_\delta^{-\delta} \text{ where } \xi_n \text{ lies between } u_n \text{ and } u_0. \end{aligned}$$

Noting that $\xi_n^{-\delta-1} \sim d^{-2}$ near $\partial\Omega$, from corollary 2.1 we get that $u_n \to u_0$ in $\mathcal{C}_{\phi_\delta}(\Omega)$. □

2.4 Global analytic path

We are now ready to give the **Proof of Theorem 1.1.** Choose $\mathcal{U} = \mathbb{R}^+ \times \mathcal{X} = \mathbb{R}^+ \times \mathcal{C}_{\phi_\delta}(\Omega)$ and the "positive" cone $\mathcal{K} = \mathcal{C}^+_{\phi_\delta}(\Omega)$. Note that $\mathcal{K}$ is open in $\mathcal{X}$. Condition $(G1)$ in Theorem 1.1 holds because of Lemma 2.9. Conditions $(G2)-(G3)$ are satisfied thanks to Proposition 2.2, Lemma 2.3 and Lemma 2.7. Indeed, in view of lemma 2.7, we may take $\mathcal{A}^+$ to be the portion of minimal solution branch given by $\{(\lambda, u_\lambda) \in \mathcal{M} : 0 < \lambda < \lambda_0\}$. We fix an analytic parametrisation $\mathcal{A}^+ = \{(\lambda(s), u(s)) : 0 < s < s_0\}$ for some $s_0 > 0$.

Applying now Theorem 1.1, we can extend the analytic map $(\lambda(s), u(s))$ for all $s > 0$ to get a global analytic and continuous path $\mathcal{A}$ of solutions to (P_λ) containing $\mathcal{A}^+$ and satisfying assertions $(a)-(f)$ of Theorem 1.1. This shows assertion (v). From the very construction of $\mathcal{A}$ from $\mathcal{A}^+$ assertion (i) holds. Conclusion (ii) is a consequence of lemmas 2.7 and 2.8. Note that $\mathcal{A}$ contains the full minimal solutions branch if $\partial_u F(\lambda, u_\lambda)$ is invertible for all $\lambda \in (0, \Lambda)$. Thanks to (III.36), we see that $u(s) \in \mathcal{K}$ for all $s > 0$. The case $e(ii)$ in Theorem 1.1 can be ruled out as follows. If $(\lambda(s_n), u(s_n))$ converges as $s_n \to \infty$ to a boundary point $(0, u_0)$ of $\mathcal{U} = \mathbb{R}^+ \times \mathcal{C}_{\phi_\delta}(\Omega)$, then $\Delta u(s_n) \to 0$ in $\mathcal{C}_{loc}(\Omega)$. That is, $\Delta u_0 = 0$ and hence $u_0 \equiv 0$. From lemma 2.6 we get that $u(s_n)$ is the minimal solution for all large s_n. We note that, from the analytic implicit function theorem, the distinguished arc $\mathcal{A}_0$ containing minimal solutions starting from $(0,0)$ is isolated from other solutions. Therefore, we obtain that the distinguished arc corresponding to all large s must coincide with $\mathcal{A}_0$. This gives a contradiction, as distinguished arcs are distinct from (a) of theorem 1.1. We can rule out $e(iii)$ (i.e., $\mathcal{A}$ forms a closed loop containing $(0,0)$) by the same argument. Hence we have only the case $e(i)$ occurring. Since (P_λ) has no solution for $\lambda > \Lambda$, we obtain that $\|u(s)\|_{\mathcal{C}_{\phi_\delta}(\Omega)} \to \infty$ as $s \to \infty$. This shows (iii). Assertion (iv) is a consequence of (iii). Assertions (v) and (vi) are exactly the assertions (c) and (d) in theorem 1.1.

Assuming that f satisfies conditions $(f_1), (f_2)$ and (f_3), we now give the

Proof of Corollary 1.1. From Lemma 2.8, the minimal solutions branch belongs to $\mathcal{A}$ and hence (i) of corollary 1.1 holds. The bending of the analytic path $\mathcal{A}$ to the left of Λ at the point (Λ, u_Λ) follows from assertion (d) of theorem 1.1 regarding the injectivity of the $\lambda(s)$ map.

3 Infinite turning points in two dimensions via ODE analysis

Throughout this section we take $\Omega = B_1(0) \subset \mathbb{R}^2$ and assume f has the form given in statement of theorem 1.3.

3.1 Singular solution and Oscillation criterion

3.1.1 Transformation of the problem

We make use of O.D.E. techniques like shooting argument to study in detail the linearisation of (P_{λ^*}) at the singular solution u^*. To this end, we perform below some transformations that will put (P_λ) into the equivalent form given by the classical Emden-Fowler equations. Define $\tilde{f}(t) \stackrel{\text{def}}{=} t^{-\delta} + f(t),\ t > 0$. First, it can be shown that any positive solution of (P_λ) in $B_1(0)$ is radial (see [16]). Next, we note that if u is a radial solution to (P_λ) then $z(r) \stackrel{\text{def}}{=} u(x), r = |x|$, solves the following O.D.E. version of (P_λ) :

$$(P_\lambda^{rad}) \qquad \left\{ \begin{array}{c} \left. \begin{array}{rcl} -(rz')' & = & \lambda r \tilde{f}(z) \\ z & > & 0 \end{array} \right\} \text{ in } (0,1), \\ z'(0) = z(1) = 0. \end{array} \right.$$

Letting now $R = \lambda^{\frac{1}{2}}$, (P_λ^{rad}) can be rewritten as the following O.D.E. boundary value problem :

$$(P_R^{rad}) \qquad \left\{ \begin{array}{c} \left. \begin{array}{rcl} -(rz')' & = & r \tilde{f}(z) \\ z & > & 0 \end{array} \right\} \text{ in } (0,R), \\ z'(0) = z(R) = 0. \end{array} \right.$$

We finally make the following singular Emden-Fowler transformation :

$$y(t) = z(r), \text{ where } r = 2e^{-\frac{t}{2}}, \ \ t \in (2\log(2/R), \infty).$$

Then, it can be checked that (P_R^{rad}) is equivalent to the following problem with $T = 2\log(\frac{2}{R})$:

$$\begin{array}{c} \left. \begin{array}{rcl} -y'' & = & e^{-t}\tilde{f}(y) \\ y & > & 0 \end{array} \right\} \text{ in } (T,\infty), \\ y(T) = y'(\infty) = 0. \end{array}$$

For our purpose, instead of the above boundary value problem, it will be more convenient to consider the following initial-value problem depending upon a parameter $\gamma > 0$:

$$(P_\gamma) \qquad \left\{ \begin{array}{c} -y'' = e^{-t}\tilde{f}(y), \\ y(\infty) = \gamma, y'(\infty) = 0. \end{array} \right.$$

Any solution of the above problem will be denoted as $y(\cdot\,;\gamma)$. Since $\tilde{f}(t) > 0$ for $t > 0$, it follows from (P_γ) that y is a strictly concave function as long as it is positive. Therefore, there exists $T_0(\gamma) > -\infty$ such that $y(T_0(\gamma);\gamma) = 0$ and $y(t;\gamma) > 0$ for all $t > T_0(\gamma)$. $T_0(\gamma)$ thus defined, is clearly the first zero of the solution y of (P_γ) as we move left from infinity. Let $s_0 > 0$ be any real number such that

$g \stackrel{\text{def}}{=} \log \tilde{f}$ is convex and increasing for all $t > s_0$. For an integer $m \geq 0$ and any $\gamma > m + s_0$ define $t_m(\gamma)$ to be the point at which $y(t_m(\gamma); \gamma) = m + s_0$. We first prove the following lemmas.

Lemma 3.1 *$T_0(\gamma)$ is a continuous function of γ in $(0, \infty)$.*

Proof. Le $\gamma_* > 0$ be fixed and we show that T_0 is continuous at γ_* in $(0, \infty)$. Clearly from continuous dependence of solutions on the initial data for regular ODE, we obtain that for any $\varepsilon > 0$, we can find $\delta > 0$, such that

$$|u(t, \gamma) - y(t, \gamma_*)| \leq \varepsilon, \text{ for all } t \geq T_0(\gamma_*) + \varepsilon \text{ and } |\gamma - \gamma_*| \leq \delta.$$

Suppose that T_0 is discontinuous at γ_*. Then without loss of generality, for some subsequence $\gamma_k \to \gamma_*$, we will have $\limsup_{k\to\infty} T_0(\gamma_k) < T_0(\gamma_0)$. This give a contradiction in view of the continuous statement above and the fact that each $y(., \gamma_k)$ is a concave function. This proves the lemma. $\square$

The following lemma is an adaptation of [67, Proposition 4.2] for the more general N-Laplacian operator

Lemma 3.2 *Let $\tilde{f}(t) = t^{-\delta} + h(t)e^{t^2}$, with $h(t) = t^{2+p_1} \prod_{i=2}^{n} e^{\alpha_i t^{p_i}}$. Then* $\limsup_{\gamma\to\infty} T_0(\gamma) < \infty$.

Proof. Recall that $g := \log \tilde{f}$ is convex on (s_0, ∞). We define the following function

$$\theta_\gamma(t) = \frac{g^{\frac{1}{2}}(y(t, \gamma))}{t} \text{ for } t \geq T_0(\gamma)$$

Note that

$$\theta'_\gamma(t) = \frac{1}{2} y'(t, \gamma) g'(y(t, \gamma)) g^{-\frac{1}{2}}(y(t, \gamma)) t^{-1} - g^{\frac{1}{2}}(y(t, \gamma)) t^{-2}.$$

Suppose for a subsequence $\gamma_k \to \infty$ that $T_0(\gamma_k) \to \infty$. Note that $\tilde{f}$ satisfies the assumption (**A1**) – (**A5**) of [67]. Then we can use the lemma 4.1 of [67], we have

$$\theta_{\gamma_k}(t_0) \leq 0 \text{ for } \gamma_k \text{ large,}$$

(recall that $t_0 := t_0(\gamma_k) > T_0(\gamma_k)$ satisfies $y(t_0, \gamma_k) = s_0$). Hence, fix $s_0 > 0$ large enough. It is easy to show that, there exists a constant $c > 0$ such that

$$2\frac{g(s)}{g'(s)} \leq s\left(1 - \frac{1}{s^2}\right) \text{ for all } s > s_0 \tag{III.38}$$

Fix $\varepsilon > 0$ small enough such that $s_0 - \varepsilon > s_0\left(1 - \dfrac{1}{s_0^2}\right)$ and we choose $t_0^\varepsilon := t_0^\varepsilon(\gamma_k) < t_0(\gamma_k)$ such that $y(t_0^\varepsilon, \gamma_k) = \varepsilon$.

Since $y := y(., \gamma_k)$ solves the problem (P_γ) with $\gamma = \gamma_k$, we have

$$y(t_0) - y(t_0^\varepsilon) = \int_{t_0^\varepsilon}^{t_0} \left(y'(t_0) + \int_s^{t_0} e^{-\tau} f(y(\tau)) \mathrm{d}\tau\right) \mathrm{ds}.$$

We can find a constant $C_\varepsilon > 0$ independent of γ_k, such that for any $\tau \in [t_0^\varepsilon, t_0]$, $f(y(\tau) \le C_\varepsilon$. Using (III.38) in the above inequality, we get

$$\begin{aligned} s_0 - \varepsilon &\le y(t_0)(t_0 - t_0^\varepsilon) + C_\varepsilon \int_{t_0^\varepsilon}^{t_0} e^{-\tau} \mathrm{d}\tau \\ &\le y(t_0)t_0 + C_\varepsilon e^{t_0^\varepsilon} \\ &\le s_0 \left(1 - \frac{1}{s_0^2}\right) + C_\varepsilon e^{-T_0(\gamma_k)} \end{aligned}$$

The above inequality gives a contradiction if $T_0(\gamma_k) \to \infty$ as $k \to \infty$. This completes the proof. □

3.1.2 The singular solution

From the corresponding O.D.E., we note that any radially symmetric solution u to (P_λ) is also radially decreasing.

Lemma 3.3 *Take $0 < \delta < 1$ and $h(t) = t^{2+p_1} \prod_{i=2}^n e^{\alpha_i t^{p_i}}$, where $p_1 > 0, 1 < p_i < 2, i = 2, ..., n$ are distinct and $\alpha_i < 0$. Let $f(t) = h(t)e^{t^2}$. Let $\Omega = B_1(0)$ the open unit ball in $\mathbb{R}^2$. Then there exists $\lambda^* > 0$ and a positive radial singular solution $u^* \in L^\infty_{loc}(B_1(0) \setminus \{0\})$ that blows up at the origin solving $-\Delta u^* = \lambda^*((u^*)^{-\delta} + f(u^*))$ in $B_1(0)$ with $u^* = 0$ on $\partial B_1(0)$. Furthermore, $u^* \notin H^1_{loc}(B_1(0))$.*

Proof. We divide the proof into several steps.

Step 1 : Given a sequence $\gamma_n \to \infty$ as $n \to \infty$, there exists a sequence $\{R_n\}$ of positive numbers with $\liminf_{n\to\infty} R_n > 0$ *and a sequence of radially decreasing solutions $\{u_n\}$ to $(P^{rad}_{R_n})$ with $u_n(0) = \gamma_n$.*

Given $\{\gamma_n\}$ as in the statement, let $y_n = y(\cdot, \gamma_n)$ denote the solution of (P_{γ_n}). It can easily be checked that $\tilde{f}(t) = t^{-\delta} + h(t)e^{t^2}$ satisfies the assumptions $(\mathbf{A1}) - (\mathbf{A5})$ of [67]. Now, by Proposition 4.2 of [67] , $T_* \overset{\text{def}}{=} \limsup_{n\to\infty} T_0(\gamma_n) < \infty$. Up to a subsequence of $\{\gamma_n\}$, we can assume that $T_* = \lim_{n\to\infty} T_0(\gamma_n)$. From remark 2.4 it follows that $T_* > -\infty$. Let $R_n = 2e^{-\frac{T_0(\gamma_n)}{2}}$ and $u_n(x) = y_n(2\log(2/|x|)), |x| \le R_n$. Then, u_n solves $(P^{rad}_{R_n})$ with $u_n(0) = \gamma_n$ and $\liminf_{n\to\infty} R_n > 0$.

Step 2 : Let $\{\gamma_n\}, \{y_n\}$ be as in Step 1. Extend y_n to $[T_* - 1, T_0(\gamma_n))$ by zero. Then the extended sequence (still denoted as $\{y_n\}$) is uniformly bounded on compact subsets of $[T_* - 1, \infty)$.

We define the following energy function associated to (P_{γ_n}) :

$$E_n(t) = y_n'(t) - \frac{1}{2}(y_n'(t))^2 g'(y_n(t)) - e^{g(y_n(t))-t},\ t > T_0(\gamma_n).$$

Hence, $E_n'(t) = -\frac{1}{2}(y_n')^3 g''(y_n) \le 0\ \forall t \ge t_0(\gamma_n)$ since y_n is strictly increasing and g is convex for this range of t. Since $\lim_{t\to\infty} E_n(t) = 0$ we obtain that E_n is a non-negative function on $(t_0(\gamma_n), \infty)$. This immediately implies that

$$y_n'(t)g'(y_n(t)) < 2,\ \forall t \ge t_0(\gamma_n). \tag{III.39}$$

Now, integrating the ODE in (P_{γ_n}) we have,

$$\int_{t_0(\gamma_n)}^\infty \tilde{f}(y_n(t))e^{-t}dt = y_n'(t_0(\gamma_n)).$$

From (III.39) and recalling that $y_n(t_0(\gamma_n)) = s_0$, we get,

$$\sup_n \int_{t_0(\gamma_n)}^{\infty} \tilde{f}(y_n(t))e^{-t}dt < \infty. \tag{III.40}$$

Let $[a, b] \subset [T_* - 1, \infty)$. Define $A = \{n : t_0(\gamma_n) > b\}, B = \{n : t_0(\gamma_n) \leq b\}$. We note that $\sup_n y_n(b) < \infty$. Otherwise, a subsequence of $\{y_n(b)\}$ tends to ∞ and hence by monotonicity of y_n, this subsequence converges uniformly to ∞ in $[b, b+1]$, violating (III.40). Again by monotonicity of y_n, we have $\sup_{[a,b]} y_n \leq y_n(b)$. Therefore, $\sup_{n\in A} \sup_{[a,b]} y_n(t) \leq \sup_{n\in A} y_n(b) \leq \sup_n y_n(t_0(\gamma_n)) = s_0$. Similarly, $\sup_{n\in B} \sup_{[a,b]} y_n(t) \leq \sup_{n\in B} y_n(b) < \infty$. This finishes Step 2.

Step 3 : Constructing the singular solution.

Recall that for an integer $m \geq 0$ and any n such that $\gamma_n > m + s_0$ we defined $t_m(\gamma_n)$ to be the point at which $y_n(t_m(\gamma_n)) = m + s_0$.

We claim that $S_m \overset{\text{def}}{=} \limsup_{\gamma_n > m+s_0} t_m(\gamma_n) < \infty$. To see this, for $\gamma_n > m + s_0$, define $z_n(t) = y_n(t) - m - s_0$. Then $-z_n'' = e^{-t}\tilde{f}(z_n + m + s_0) \overset{\text{def}}{=} e^{-t}\overline{f}(z)$, and $t_m(\gamma_n)$ is the first zero of $z_n(t)$ as t decreases from infinity. It can also be checked that $\overline{f}$ and $\overline{g} = \log \overline{f}$ satisfies assumptions $(\mathbf{A1}) - (\mathbf{A5})$ of [67]. Now, again by Proposition 4.2 of [67] we get $S_m < \infty$.

From step 2 and the fact that y_n solves the ODE we obtain that $\{y_n\}$ and $\{y_n'\}$ have uniformly convergent subsequences in any compact sub-interval of (T_*, ∞). By using a diagonalisation process we can obtain a subsequence of $\{y_n\}$, which we will denote by $\{y_n\}$ again, and a positive, continuous and nondecreasing function y^* on (T_*, ∞) such that $y_n \to y^*$ locally uniformly in (T_*, ∞). Furthermore, $y_n' \to (y^*)'$ pointwise in (T_*, ∞) (see theorem 7.17 in [108]). By restricting to a further subsequence of $\{\gamma_n\}$ so that $\lim_{n\to\infty} t_m(\gamma_n) = S_m$, we obtain

$$y^*(S_m) = \lim_{n\to\infty} y_n(S_m) = \lim_{n\to\infty} y_n(t_m(\gamma_n)) = m + s_0.$$

Since y^* is nondecreasing, we obtain that $y^*(t) \to \infty$ as $t \to \infty$.

Integrating the ODE satisfied by y_n once, we get,

$$y_n'(t) - y_n'(s) = \int_s^t \tilde{f}(y_n(\rho))e^{-\rho}d\rho, \;\; T_* < s < t < \infty.$$

Using the convergence of y_n, y_n', we can pass to the limit as $n \to \infty$ on either side of the above equation to obtain that y^* also satisfies the same integral equation. That is, y^* solves the equation $-(y^*)'' = e^{-t}\tilde{f}(y^*)$ in (T_*, ∞).

Now,

$$\int_{T_0(\gamma_n)}^{\infty} \tilde{f}(y_n)e^{-t}dt \leq C + \int_{T_0(\gamma_n)}^{t_0(\gamma_n)} y_n^{-\delta}e^{-t}dt + \int_{t_0(\gamma_n)}^{\infty} f(y_n)e^{-t}dt.$$

Thanks to the boundary estimate in (III.36) we have, $y_n(t) \geq k(1 - e^{-\frac{(t-T_0(\gamma_n))}{2}})$ for all $t > T_0(\gamma_n)$ and $n \geq 1$, for some $k > 0$. Hence, the first integral on the right in the last inequality is uniformly bounded for all n. That the second integral in this inequality is uniformly bounded with respect to n

follows from (III.40). Hence we obtain that

$$\sup_n \int_{T_0(\gamma_n)}^{\infty} \tilde{f}(y_n)e^{-t}dt < \infty. \tag{III.41}$$

We now come to the value for y^* at T_*. Since y^* is non decreasing, y^* has a right limit at T_*. Integrating the ODE satisfied by y_n between $t \in [T_0(\gamma_n), t_0(\gamma_n))$ and $t_0(\gamma_n)$ and using (III.39) and (III.41), we deduce that y_n' is uniformly bounded on $[T_0(\gamma_n), T_* + 1]$. Consequently, the extended sequence y_n is uniformly bounded in the Lipshitz norm on $[T_* - 1, T_* + 1]$. Then, by Ascoli-Arzela theorem, we get that

$$y^*(T_*) = \lim_{n\to\infty} y_n(T_*) = 0. \tag{III.42}$$

Again from (III.41) and Fatou's Lemma, $\int_{T_*}^{\infty} \tilde{f}(y^*)e^{-t}dt < \infty$. Thus, to summarise, y^* solves the differential equation $-(y^*)'' = e^{-t}\tilde{f}(y^*)$ in (T_*, ∞) with $y^*(T_*) = 0$ and $\int_{T_*}^{\infty} \tilde{f}(y^*(t))e^{-t}dt < \infty$. Defining $R_* = 2e^{-\frac{T_*}{2}}$, going back to our original variable $x \in B_{R_*}$ and defining $z^*(x) = y^*(2\log(\frac{2}{|x|}))$ we obtain that z^* solves the following problem :

$$\begin{cases} \left.\begin{array}{rcl} -\Delta z^* & = & \tilde{f}(z^*) \\ z^* & > & 0 \end{array}\right\} \text{ in } B_{R_*} \setminus \{0\}, \\ z^* = 0 \text{ on } \partial B_{R_*}, \ \lim_{|x|\to 0} z^*(x) = \infty, \quad \tilde{f}(z^*) \in L^1(B_{R_*}). \end{cases}$$

It is clear that $\liminf_{t\to\infty} \tilde{f}(t)t^{-1} > 0$ and z^* is continuous in $B_{R_*} \setminus \{0\}$. Thus by the result of Brezis-Lions [28], in fact z^* solves the problem $-\Delta z^* = \tilde{f}(z^*) + \alpha\delta_0$ in the sense of distributions in B_{R_*} for some $\alpha \geq 0$. Since $\tilde{f}(t)$ behaves like $e^{t^2 - t^{\beta}}$ for all large $t > 0$ and some $\beta \in (1, 2)$, $\tilde{f}$ is a super-exponential type nonlinearity. Hence from Theorem 2.1 of [51] we obtain that $\alpha = 0$. Letting now $u^*(x) = z^*(R_* x), x \in B_1(0)$, we get that u^* is the required singular solution with $\lambda_* = R_*^2$.

If $u^* \in H^1_{loc}(\Omega)$, by Trudinger-Moser imbedding [93], we obtain that $u^* \in L^p_{loc}(\Omega)$ for all $p \geq 1$ and hence is bounded, a contradiction. □

Remark 3.1 *Let f, Ω be as in lemma 3.3. An inspection of the proof of this lemma shows that for any unbounded (in $L^\infty(\Omega)$) sequence $\{u_n\}$ of solutions to (P_{λ_n}), necessarily* $\liminf_{n\to\infty} \lambda_n > 0$ *and we can find a singular solution of (P_{λ_*}) for any λ_* that is a subsequential limit of such a $\{\lambda_n\}$.*

3.1.3 Oscillation criterion

Let y^*, u^* denote the singular solutions as constructed above. The linearised problem at the singular solution u^* obtained after performing the Emden-Fowler transformations is :

$$-w'' = \{e^{g(y^*(t))-t}g'(y^*(t))\}w \quad \text{in } (T_*, \infty) \tag{III.43}$$

Lemma 3.4 *Let $h(t) = t^{2+p_1}\prod_{i=2}^{n} e^{\alpha_i t^{p_i}}$, where $p_1 > 0, 1 < p_i < 2, i = 2, ..., n$ are distinct and $\alpha_i < 0$. Let $f(t) = h(t)e^{t^2}$. Then, every solution w of* (III.43) *oscillates infinitely often in any neighbourhood of ∞.*

Proof. From the construction of singular solution in lemma 3.3, we get a sequence $\{\gamma_n\}_{n\geq 1}$ such that $\gamma_n \to \infty$ as $n \to \infty$ and the corresponding sequence of solutions $y(t;\gamma_n) \to y^*(t)$ and $y'(t;\gamma_n) \to (y^*)'(t)$ as $n \to \infty$ uniformly on compact subsets of (T_*, ∞). We defined $S_0 = \limsup_{n\to\infty} t_0(\gamma_n)$.

In order to show the lemma, we use the oscillation criterion (see [Fite, [63]] and [Wintner, [122]]) :

$$\lim_{\tau\to\infty} \int_{S_0+1}^{\tau} e^{g(y^*(t))-t} g'(y^*(t))dt = \infty. \tag{III.44}$$

Suppose

$$\int_{S_0+1}^{\infty} e^{g(y^*(t))-t} g'(y^*)(t)dt < \infty. \tag{III.45}$$

Upon integrating by parts we have, for any $S_0 + 1 < \tau_1 < \tau_2 < \infty$,

$$\begin{aligned}\int_{\tau_1}^{\tau_2} e^{g(y^*(t))-t} g'(y^*)(t)dt &= -\int_{\tau_1}^{\tau_2} (y^*)'' g'(y^*)(t)dt \\ &= -[(y^*)'g'(y^*)]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} [(y^*)']^2 g''(y^*)(t)dt.\end{aligned} \tag{III.46}$$

From (III.39), we obtain that $(y^*)'(t)g'(y^*)(t)$ is a bounded function for all $t > S_0 + 1$. We also note that $g''(t) \geq c > 0$ for all $t > S_0 + 1$ and some $c > 0$. Therefore from (III.45) and (III.46) we obtain that $\int_{S_0+1}^{\infty} [(y^*)']^2 dt < \infty$ and $\int_{S_0+1}^{\infty} (y^*)^2 e^{-t} dt < \infty$. This gives a contradiction to the fact that u^* is not a H^1 function in a neighbourhood of the origin. Hence the oscillation criterion (III.44) holds.

3.2 Existence of infinitely many turning points in the bifurcation curve

Throughout this sub-section we take $\Omega = B_1(0)$, the unit ball in $\mathbb{R}^2$, $\delta < 1$ and f satisfying the assumptions in theorem 1.3.

Lemma 3.5 *Fix $\lambda > 0$. Then, $F(\lambda, u)$ is a potential operator from $\mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}^+_{\phi_1}(\Omega)$ to $\mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}_0(\overline{\Omega})$ (see definition 1.9).*

Proof. We let $E = \mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}_0(\overline{\Omega}), U = E \cap \mathcal{C}^+_{\phi_1}(\Omega)$ and take the inner-product $\langle\cdot,\cdot\rangle_{H^1_0(\Omega)}$ on E. Then $F(\lambda,\cdot) : U \to E$ is a continuous map (this follows from the estimate in theorem 1.1(i) in [76]). Define $g_\lambda : U \to \mathbb{R}$ as

$$g_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \lambda \int_\Omega \frac{1}{1-\delta}|u|^{1-\delta} - \lambda \int_\Omega \int_0^u f(t)dt,\ u \in H^1_0(\Omega).$$

It is easy to see that g_λ is continuously differentiable on U and $\nabla g_\lambda(u)h = \langle F(\lambda,u), h\rangle_{H^1_0(\Omega)}$ for all $u \in U, h \in H^1_0(\Omega)$.

Lemma 3.6 *Let $u \in \mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}^+_{\phi_1}(\Omega)$. Then $\partial_u F(\lambda, u)$ is a compact perturbation of the identity from $\mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}_0(\overline{\Omega})$ to itself and its spectrum on this space coincides with that on $\mathcal{C}_{\phi_1}(\Omega)$ given in lemma 2.5.*

Proof. It is enough to show that $A_\lambda \stackrel{\text{def}}{=} (-\Delta)^{-1}(\lambda\delta u_\lambda^{-1-\delta}\cdot)$ is a compact operator on $\mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}_0(\overline{\Omega})$. Let $\{\psi_n\} \subset \mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}_0(\overline{\Omega})$ be a bounded sequence. Letting $\xi_n = A_\lambda(\psi_n)$, we see that ξ_n solves

$-\Delta\xi_n = \lambda\delta u_\lambda^{-1-\delta}\psi_n, \quad \xi_n \in \mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}_0(\overline{\Omega})$. Since $u_\lambda \in \mathcal{C}^+_{\varphi_1}(\Omega)$, we see that $\{-\Delta\xi_n\}$ is bounded in $\mathcal{C}_{\varphi_1^{-\delta}}(\Omega)$. Then, from theorem 1.1(i) in [76], we obtain that $\{\xi_n\}$ is a bounded sequence in $\mathcal{C}^{1,\theta}(\overline{\Omega})$ for some $\theta \in (0,1)$. The compactness of A_λ now follows. Thus, any spectral value other than 1 is an eigenvalue and an argument similar to that in lemma 2.5 shows that these eigenvalues are real. Since $\mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}_0(\overline{\Omega}) \subset \mathcal{C}_{\varphi_1}(\Omega)$, eigenfunctions of $\partial_u F(\lambda, u)$ on $\mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}_0(\overline{\Omega})$ are also its eigenfunctions on $\mathcal{C}_{\varphi_1}(\Omega)$. Conversely, any eigenfunction of $\partial_u F(\lambda, u)$ on $\mathcal{C}_{\varphi_1}(\Omega)$ is in the space $\mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}_0(\overline{\Omega})$ thanks to theorem 1.1(i) of [76] again.

Lemma 3.7 *The Morse index of the linearised operator $\partial_u F(\lambda, u)$ is unbounded along the analytic branch $\mathcal{A}$ and it is $+\infty$ at u^*.*

Proof. From the regularity result in theorem 1.1(i) [76] we obtain that any solution of (P_λ) is in $\mathcal{C}^1(\overline{\Omega})$. Therefore, $\mathcal{A} \subset \mathbb{R} \times [\mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}^+_{\phi_1}(\Omega)]$. In view of lemmas 3.5 and 3.6, we can define the Morse index of $\partial_u F(\lambda, u)$ to be the number of its negative eigenvalues (counting multiplicity). From the norm unboundedness of $\mathcal{A}$ and lemma 3.3 we obtain a sequence $\lambda_i \to \lambda^*$ (see remark 3.1) such that the corresponding sequence of solutions $\{u_{\lambda_i}\}$ of (P_{λ_i}) have $\|u_{\lambda_i}\|_{\mathcal{C}_{\phi_1}(\Omega)} \to \infty$ and converge in $\mathcal{C}^1_{loc}(B_1(0) \setminus \{0\})$ to a radial singular solution u^* of (P_{λ_*}) as $i \to \infty$. We now show that the Morse index of $\partial_u F(\lambda_i, u_{\lambda_i})$ tends to infinity in a similar way as in [43]. Indeed, from the Lemma 3.4, we deduce that for any integer $M >> 1$, there is $n^* = n^*(M)$ such that for $n \geq n^*$, the graph of y_n intersects with that of y^* at least M times in the interval $(\max(T_*, T_0(\gamma_n)), \infty)$, (where y_n, y^*, T_* and $T_0(\gamma_n)$ are defined as in the proof of the Lemma 3.3). Hence we see that $y_n - y^*$ has at least M zeros in $(\max(T_*, T_0(\gamma_n)), \infty)$ and thus there are at least $\dfrac{[M]}{2} - 1$ intervals I_i $(i = 1, \cdots, \dfrac{[M]}{2} - 1)$ on which $y_n - y^* > 0$. Frome the convexity of g near infinity, we can see that the function w_n^i defined by $w_n^i \stackrel{\text{def}}{=} y_n - y^*$ on I_i and $w_n^i = 0$ otherwise, satisfies :

$$-(w_n^i)'' < \{e^{g(y^*(t))-t} g'(y^*(t))\} w_n^i \quad \text{in } I_i, \quad w_n^i > 0 \text{ in } I_i, \text{ and } w_n^i = 0 \text{ on } \partial I_i.$$

Multiplying w_n^i on both sides of the above inequality and integrating it on the annular domain I_i, then we obtain

$$\int_{I_i} \left[(w_n^i)'\right]^2 \mathrm{d}t - \int_{I_i} \{e^{g(y^*(t))-t} g'(y^*(t))\}(w_n^i)^2 \mathrm{d}t < 0.$$

Since $w_n^i \in H_0^1(I_*)$ where $I_* = (T_*, +\infty)$,

$$\int_{I_*} w_n^i w_n^j \mathrm{d}t = 0, \quad i \neq j$$

and from a basic argument of decomposition in an orthogonal basis, we deduce that the Morse index of y^* is greater than $\dfrac{[M]}{2} - 1$ and the arbitrariness of M implies that the Morse index of y^* is $+\infty$. Then the Morse index of u^* is also $+\infty$. By the same way we can show that the Morse index of $\partial_u F(\lambda, u)$ is unbounded along the sequence (λ_i, u_i).

From Theorem 1.1, we recall that we can construct an analytic path of solutions $\mathcal{A} = \{(\lambda(s), u(s)) : s \in (0, \infty)\}$ to (P_λ) bifurcating from $(0,0)$ satisfying

$$\|u(s)\|_{C_{\phi_\delta}(\Omega)} \to \infty \text{ as } s \to \infty.$$

We further recall that along the path $\mathcal{A}$ the linearised operator $\partial_u F(\lambda, u)$ is Fredholm of index zero. It is invertible except on a set of isolated points $\Sigma \stackrel{\text{def}}{=} \{(\lambda(s_i), u(s_i)) : i \in \mathbb{N}\}$. Consequently, the implicit function theorem can be applied at any $s \neq s_i$ to obtain that locally near such an s the solution set $\mathcal{A}$ can be described as a curve parametrised by $\lambda(s)$. We now show that the curve $\mathcal{A}$ "turns back" at infinitely many points of Σ. In particular, this implies that the set Σ is infinite.

Proof of theorem 1.2 : The proof is an immediate consequence of the Lemma 3.7.

Proof of theorem 1.3 : From lemma 3.7 we can find an infinite sequence $\{(\lambda_i, u_i)\} \subset \mathcal{A}$ where the Morse index of $\partial_u F(\lambda, u)$ changes. Necessarily $\partial_u F(\lambda_i, u_i)$ is not invertible i.e., (λ_i, u_i) is in Σ. We note that, again by the C^1-estimate in [76], it follows that the operator norm of $\partial_u F(\lambda, u)$ in the space $\mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}_0(\overline{\Omega})$ is uniformly bounded for any (λ, u) chosen in a compact (in $\mathbb{R} \times [\mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}_0(\overline{\Omega})]$) subset of $\mathcal{A}$. Then, appealing to theorem 1.5 with $H = H_0^1(\Omega)$, we get that (λ_i, u_i) is a "turning point" for $\mathcal{A}$ in the sense of definition 1.8.

Remark 3.2 *In the above proof, another way to show that $(\lambda_i, u_i) = (\lambda(s_i), u(s_i))$ is a "turning point" is to use the invariance of critical groups by homotopy (where the critical groups are defined in the Annex B). Indeed, arguing by contradiction, if (λ_i, u_i) is not a turning point, the path $\mathcal{A}$ locally near (λ_i, u_i) is a curve parametrised by λ. Then the critical groups of these solutions must be locally independent of λ by homotopy invariance of the critical groups. From the formula for the critical groups at a non-degenerate point given by Theorem B.1 in the Annex B, it follows that the Morse index of the linearised operator $\partial_2 F(\lambda, u)$ must be constant in a deleted neighborhood of (λ_i, u_i) which contradicts with our choice of (λ_i, u_i).*

Acknowledgement : J. Giacomoni and S.Prashanth were funded by IFCAM (Indo-French Centre for Applied Mathematics) under the project "Singular phenomena in reaction diffusion equations and in conservation laws" .

Chapitre IV

Some other applications of the global analytic bifurcation theorem

1 Preliminaries

In this section, we consider the following bifurcation problem with the parameter λ :

$$(\mathrm{P}_\lambda) \qquad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u_{|\partial\Omega} = 0, \quad u > 0 \text{ on } \Omega, \end{cases} \tag{IV.1}$$

where Ω is a regular bounded domain in $\mathbb{R}^N$, and $f : \mathbb{R} \to \mathbb{R}$ is some given non linearity. In order to apply the analytic global bifurcation theory, we assume that the mapping

$$\begin{aligned} U \subset \mathcal{X} &\longrightarrow \mathcal{Y} \\ u &\longmapsto f(u) \end{aligned}$$

is analytic, where $\mathcal{X}$, $\mathcal{Y}$ are Banach spaces and U is an open subset of $\mathcal{X}$. In this chapter, we describe the picture of bifurcation diagrams associated with (P_λ) considering various asymptotic behaviors of the nonlinearity f. Thanks to the theory of analytic bifurcation described in the chapter II, we show in each situation the existence of unbounded analytic path of solutions to (P_λ) and give its global behaviour.

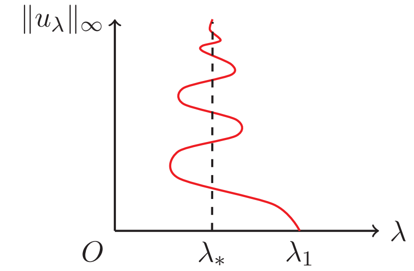
We mention that in P. L. Lions [88] has investigated such kind of bifurcation problems by using the topological bifurcation theory based on the Leray Schauder degree (see [104], [40]). We point out that the Leray Schauder degree provide the existence of connected set of solutions but not path-connected sets. In order to get a more precise description of the set of solutions to (P_λ), we apply the results from Chapter II. In many situations we are able to identify secondary bifurcation points in the global analytic branch. In the general case, it is sometimes possible to apply the implicit function theorem and then prove that a portion of the connected set given by topological degree is a smooth curve in $\mathbb{R}\times\mathcal{X}$ but this approach often fails to prove the existence of a secondary bifurcation point on the global bifurcation branch. Nevertherless the result proved in [40] (bifurcation from the first eigenvalue) can be alternatively used in some situations in this case.

The global bifurcation theory in the analytic framework has been already used for problem (P_λ)

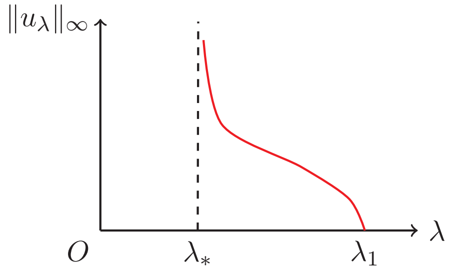
with nonlinearities f which are asymptotically like a supercritical power at infinity and Ω a ball. In E. N. Dancer [43], it is shown that the branch of positive solutions undergo infinitely many bifurcations in this case. In Z. Guo and J. Wei [77], the following problem is studied :

$$\begin{cases} -\Delta u = u^p + \lambda u & \text{in } \Omega = B_1 \\ u_{|\partial\Omega} = 0, \quad u > 0 \text{ on } \Omega. \end{cases} \tag{IV.2}$$

According to the classical bifurcation theory [40] and [105], $(\lambda_1, 0)$ is a bifurcation point from which emanates an unbounded branch $\mathcal{A}$ of solutions (λ, u). F. Merle and L.A. Peletier [92] has previously proved that there exists a unique value $\lambda_* > 0$ such that a singular radial solution u_* exists for this problem. In the paper [77], the global structure of the branch $\mathcal{A}$ is further studied. The authors show for $N \geq 3$ and for $(N+2)/(N-2) < p < p_c$, where the Joseph-Lundgren exponent p_c (see [82]) is given by $p_c = \frac{(N-2)^2+8\sqrt{N-1}}{(N-2)(N-10)}$ if $N \geq 11$ and $p_c = \infty$ otherwise, that for all $k \in \mathbb{N}$ there exist k bounded radial solution for any λ sufficiently close to λ_*. In particular there are infinitely many classical solutions for $\lambda = \lambda_*$. This result is proved also by Dolbeault and Flores [58], using geometric dynamical system method and contrats with the case $N \geq 12$ and $p \geq p_c^2$ where the Morse index of solutions is 1 and where the solution is unique for $\lambda \in (\lambda_*, \lambda_1)$ (see [77]). In [77], they prove the nonexistence of regular solutions for $\lambda < \lambda_*$ in case of $p \geq p_c^2$ and $N \geq 12$. Precisely, we have the following bifurcation diagrams.

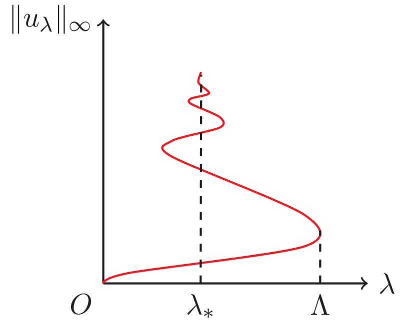


$N \geq 3$, $(N+2)/(N-2) < p < p_c$.

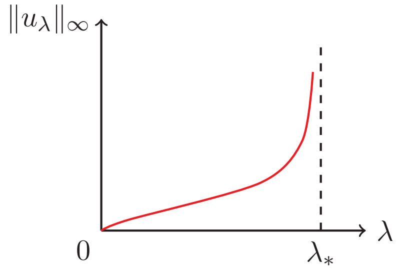


$N \geq 12, \quad p \geq p_c^2$.

Let us denote that in this case, λ_* is unique and we believe that the first situation occurs also in our study in §3 of the chapter III. Joseph and Lundgren in [82] have dealt with the case where $f(u) = e^u$, $3 \leq n \leq 9$, and Ω is radially symmetric in (P_λ). They prove the existence of radially symmetric solutions in reducing (P_λ) to an ordinary differential equation. Clever phase plane techniques are used to determine the precise description of the solution set. The associated bifurcation diagram is as follows :



In particular, they prove the uniqueness of the singular solution and the oscillation of the global branch around $\lambda = \lambda_*$. For higher dimensions ($n \geq 10$), they prove that the following picture of global bifurcation occurs :



The authors prove that in this case the extremal solution (i.e. the minimal solution for $\lambda = \lambda_*$) is singular and then the branch can not bend back. For discussion about the boundedness of the extremal solution, we refer to X. Cabré [33] and X. Cabré and Y. Martel [34]. In the next sections, we consider different behaviour of f near 0 and ∞. We prove the existence and give the global behaviour of continuous and piecewise analytic arcs of solutions.

2 The case when f(0) = 0

2.1 Superlinear nonlinearities

Let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz and continuous function such that $f(0) = 0$. By superlinear behavior, we mean the following condition hold :

$$\liminf_{t\to\infty} \frac{f(t)}{t} > \lambda_1 \tag{IV.3}$$

where λ_1 is the first eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary conditions. We assume the following growth conditions :

$$\lim_{t\to\infty} \frac{f(t)}{t^\sigma} = 0, \quad \text{with } \sigma = \frac{N+2}{N-2} \text{ if } N \geq 3, \text{ and } \sigma < \infty \text{ if } N = 1, 2, \tag{IV.4}$$

and

$$\lim_{t\to\infty} \frac{f(t)}{t} = +\infty. \tag{IV.5}$$

The assumption below ensures that the solutions of (P_λ) are uniformly bounded in $L^\infty(\Omega)$ by the results from D. G. DeFigueiredo, P. L. Lions and R. D. Nussbaum [48].

$$\limsup_{t\to\infty} \frac{tf(f) - \theta F(t)}{t^2 f(t)^{2/N}} \leq 0, \quad \text{with } \theta \in (0, \frac{2N}{N-2}),\ N \geq 3, \tag{IV.6}$$

where $F(t) = \int_0^t f(s)\mathrm{d}s$. If $N = 2$, we point out that this last condition is unnecessary.

We define now the functional space :

$$\mathcal{C}_{\phi_1}(\Omega) = \{u \in \mathcal{C}(\Omega) : \exists c > 0, |u(x)| \leq c\phi_1(x)\},$$

This space is a Banach space with the norm

$$\|u\|_{\mathcal{C}_{\phi_1}(\Omega)} = \sup_{x\in\Omega} \frac{u(x)}{\phi_1(x)}.$$

We consider the positive cone $U \stackrel{\text{def}}{=} \mathcal{C}^+_{\phi_1}(\Omega)$ where

$$\mathcal{C}^+_{\phi_1}(\Omega) = \{u \in \mathcal{C}_{\phi_1}(\Omega) \mid \inf_{x\in\Omega} \frac{u(x)}{\phi_1(x)} > 0\}.$$

As in proposition 2.1 in Chapter III, in order to apply the analytic global bifurcation theory, we assume in the entire chapter the following assumption

$$\left.\begin{array}{l} f(t) \text{ is a finite product of functions of the form } g(t^p),\ p>0, \\ \text{where } g \text{ is a real entire function on } \mathbb{R}. \end{array}\right\} \tag{IV.7}$$

Remark 2.1 *If f satisfies* (IV.7)*, then it is easy to show that the operator*

$$\begin{array}{rcl} f : U & \longrightarrow & \mathcal{C}_{\phi_1}(\Omega) \\ u & \longmapsto & f(u) \end{array}$$

is analytic (see proposition 2.1 in Chapter III).

We give now the following theorem concerning the superlinear case :

Theorem 2.1 *Let us assume that Ω is convex and f satisfies* (IV.5), (IV.4), (IV.7) *and* (IV.6)*. We also suppose that f is differentiable in $\mathbb{R}^+$ with $f'(0) \neq 0$ and*

$$f(t) > 0 \quad \textit{for all } t > 0. \tag{IV.8}$$

Then there exists unbounded analytic connected-path $\mathcal{A}$ of solutions (λ, u_λ) to (P_λ) emanating from $(\frac{\lambda_1}{f'(0)}, 0)$ towards the asymptotic bifurcation point $(0,0)$, that is

$$\mathcal{A} = \{(\lambda(s), u(s)),\ s \in [0,\infty) : (\lambda(s), u(s)) \textit{ is solution de } (\mathrm{P}_\lambda)\}$$

where

$$(\lambda, u) : [0,\infty) \longrightarrow \mathbb{R}_+ \times \mathcal{C}_{\phi_1}(\Omega)$$

is analytic such that $(\lambda(0), u(0)) = (\frac{\lambda_1}{f'(0)}, 0)$ and $\lambda(s) \to 0,\ \|u(s)\|_{L^\infty(\Omega)} \to \infty$ as $s \to \infty$.

Remark 2.2 *Instead of the hypothesis f is differentiable in $\mathbb{R}_+$ in the above theorem, if we assume that f is only differentiable in $\mathbb{R}^*_+$ and $\lim_{t\to 0^+} \frac{f(t)}{t} = +\infty$ and keeping hypotheses, then we get a similar result but the branch emanates in this case from $(0,0)$ (see Remark 2.5 in Chapter III).*

Proof. We define the solution-operator to (P_λ)

$$\begin{array}{rccl} F: & \mathbb{R}_+ \times \mathcal{C}^+_{\phi_1}(\Omega) & \longrightarrow & \mathcal{C}_{\phi_1}(\Omega) \\ & (\lambda, u) & \longmapsto & u - \lambda(-\Delta)^{-1}(f(u)) \end{array} \tag{IV.9}$$

Then, as in Chapter III, one can show that F is analytic. The linearised operator $\partial_2 F(\lambda, u)$ is defined for all $(\lambda, u) \in \mathbb{R}_+ \times \mathcal{C}^+_{\phi_1}(\Omega)$ as follows

$$\begin{array}{rccl} \partial_2 F(\lambda, u): & \mathcal{C}_{\phi_1}(\Omega) & \longrightarrow & \mathcal{C}_{\phi_1}(\Omega) \\ & \varphi & \longmapsto & \varphi - \lambda(-\Delta)^{-1}(f'(u)\varphi) \end{array} \tag{IV.10}$$

Note that $\partial_2 F(\lambda, u) = id + B_\lambda$, where

$$\begin{array}{rccl} B_\lambda: & \mathcal{C}_{\phi_1}(\Omega) & \longrightarrow & \mathcal{C}_{\phi_1}(\Omega) \\ & \varphi & \longmapsto & -\lambda(-\Delta)^{-1}(f'(u)\varphi) \end{array}$$

which is compact by the elliptic regularity theory which provides $C^{1,\theta}(\overline{\Omega})$ regularity with $0 < \theta < 1$. Hence $\partial_2 F(\lambda, u)$ is a Fredholm operator of index 0.

Again, using similar arguments as in Chapter III, we can show that bounded closed sets of solutions of (P_λ) are compact in $\mathbb{R} \times \mathcal{C}_{\phi_1}(\Omega)$.

Let us define

$$\lambda_* = \sup\{\lambda \in \mathbb{R}_+ : (\mathrm{P}_\lambda) \text{ has a solution }\}.$$

We have to show that $\lambda_* < \infty$. This is an easy consequence of (IV.5) and (IV.8). Indeed these assumptions imply that there exists $\alpha > 0$ such that $f(t) \geq \alpha t$ for all $t \geq 0$. Now multiplying (P_λ) by the first positive renormalized eigenfunction of $-\Delta$ with boundary Dirichlet conditions, ϕ_1, and integrating by parts, we get

$$\lambda_1 \int_\Omega u\phi_1 \mathrm{dx} = \lambda \int_\Omega f(u)\phi_1 \mathrm{dx} \geq \alpha\lambda \int_\Omega u\phi_1 \mathrm{dx}.$$

Therefore, $\lambda \leq \frac{\lambda_1}{\alpha}$ and thus $\lambda_* < \infty$.

Without loss of generality, we can assume that $f'(0) = 1$. Then, it is well known that $A = \partial_2 F(\lambda_1, 0) = id - \lambda_1(-\Delta)^{-1}$ satisfies

$$\ker A = span\{\phi_1\}, \qquad \text{and } \dim\ker A = \mathrm{codim}\mathrm{R}(A)$$

where $\mathrm{R}(A)$ denotes the range space of the operator A. In addition the transversality condition is satisfied, i.e. $\partial^2_{12} F(\lambda_1, 0)\phi_1 \frac{\phi_1}{\lambda_1} \notin R(A)$. Hence we can apply the results of Crandall-Rabinowitz on the bifurcation from a simple eigenvalue see [40, Theorem 1.7] and the specific form [32, Theorem 8.3.1] in the analytic case. Then, we deduce the existence of unique local and nontrivial analytic branch $\mathcal{A}^+$ which emanates from $(\frac{\lambda_1}{f'(0)}, 0) = (\lambda_1, 0)$. Now we are ready to apply the global bifurcation Theorem 1.1 given in Chapter II (see also Remark 1.1 in the same chapter), it follows the existence of a branch

$\mathcal{A}$ extending the local branch $\mathcal{A}^+$ where

$$\mathcal{A} = \{(\lambda(s), u(s)), s \in [0, \infty),\ (\lambda(0), u(0)) = (\lambda_1, 0)\}$$

and

$$(\lambda, u) : [0, \infty) \longrightarrow \mathbb{R}_+ \times \mathcal{C}_{\phi_1}(\Omega) \text{ is piecewise analytic.}$$

We show now that $\mathcal{A}$ is unbounded. According to the global bifurcation theorem, if $\mathcal{A}$ is bounded then either it converges to a boundary point, say $(0, u_0) \in \mathbb{R}_+ \times \mathcal{C}_{\phi_1}(\Omega)$, or $\mathcal{A}$ is a loop. This latter case is impossible since the bifurcating branch from $(\frac{\lambda_1}{f'(0)}, 0)$ is unique. Then, only the first alternative can occur. Hence, there exists a sequence $(\lambda_n, u_n) = (\lambda(s_n), u(s_n)) \in \mathbb{R}_+ \times \mathcal{C}_{\phi_1}(\Omega)$ such that

$$\lambda_n \to 0, \text{ and } u_n \to u_0 \text{ in } \mathcal{C}_{\phi_1}(\Omega) \text{ as } s_n \to \infty.$$

Then we have $-\Delta u_n = \lambda_n f(u_n) \to 0$ in $\mathcal{C}(K)$ as $s_n \to \infty$ for any compact $K \subset \Omega$. Hence we infer that $-\Delta u_0 = 0$ (with homogeneous Dirichlet boundary conditions) and then $u_0 = 0$. We have to show now that this is a contradiction. By the assumptions $f(0) = 0$ and f differentiable, we have

$$\exists c > 0, \exists t_0 > 0, \text{ for all } t \in [0, t_0) : f(t) \le ct.$$

Let n large enough such that $\|u_n\|_{L^\infty(\Omega)} \le t_0$ and $\lambda_n < \frac{\lambda_1}{c}$ are small enough, we obtain after multiplying (P_λ) by ϕ_1 and integrating by parts

$$\lambda_1 \int_\Omega u_n \phi_1 \mathrm{dx} = \lambda \int_\Omega f(u_n)\phi_1 \mathrm{dx} \le c\lambda_n \int_\Omega u_n \phi_1 \mathrm{dx}$$

which contradicts $\lambda_n \to 0$. Then the branch $\mathcal{A}$ is unbounded. Le's now show that 0 is the unique asymptotic bifurcation point. Due to a priori estimates in $L^\infty(\Omega)$ given by DeFigueiredo, Lions, Nussbaum [48] and in view of assumptions (IV.5),(IV.6), we deduce that for all $0 < \alpha < \beta$, there exists $C > 0$ such that, if (λ, u) is solution of (P_λ) with $\lambda \in [\alpha, \beta]$, then $\|u\|_{L^\infty(\Omega)} \le C$. Then 0 and only 0 is an asymptotic bifurcation point. This completes the proof.

□

Remark 2.3 *The convexity of the domain and the assumption* (IV.6) *are technical assumptions which are used to get uniform a priori bounds for the solutions of* (P_λ) *(see [88]). Of course the theorem still holds under other assumptions which imply uniform a priori bounds, as in [48].*

We give now a condition which implies that the branch bifurcates at $(\frac{\lambda_1}{f'(0)}, 0)$ to the right :

Corollary 2.1 *Under the assumptions of the above theorem 2.1 (a), if $f(t) < f'(0)t$, for $t > 0$ small, then $\lambda_* > \frac{\lambda_1}{f'(0)}$.*

Proof. Without loss of generality we suppose that $f'(0) = 1$. Let $t_1 > 0$ such that

$$f(t) < t \quad \text{for all } 0 < t < t_0.$$

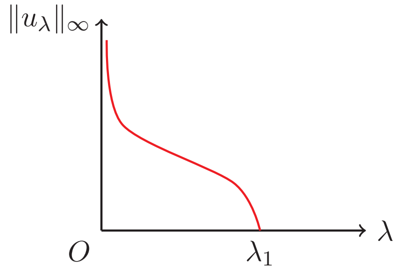
Let us recall that the bifurcating branch $\mathcal{A}^+$ given by Crandall-Rabinowitz theorem is unique and emanates from $(\lambda_1, 0)$. Then locally near $(\lambda_1, 0)$ the only admissible solutions (λ, u) of (P_λ) are for $\lambda > \lambda_1$. This proves that $\lambda_* > \lambda_1$. □

Remark 2.4 *(1) Under the same assumptions of the above corollary, the branch $\mathcal{A}$ has at least one "turning point" at some (λ_0, u_0) with $\lambda_0 \in (\frac{\lambda_1}{f'(0)}, \lambda_*]$, then from the path-connectedness of the global branch $\mathcal{A}$, we deduce that there are at least two distinct ordered solution of (P_λ) for all $\lambda \in [\frac{\lambda_1}{f'(0)}, \lambda_*)$.*

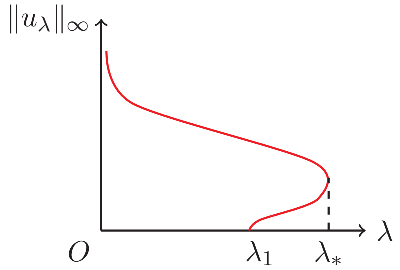
(2) In the case where $\lim\limits_{t\to 0^+} \dfrac{f(t)}{t} = +\infty$ (see remark 2.2), we have at least two ordered solutions for all $\lambda \in (0, \lambda_*)$.

We summarize these results by giving the corresponding bifurcation diagrams. The curves below represent the maximum norm of u in respect to λ, whenever (λ, u) is solution of (P_λ).

– Case 1. $f'(0) = 1$, and $f(t) > t$ for $t > 0$ small.
  Example. $f(t) = t + t^p$ $(1 < p < (N+2)/(N-2))$.

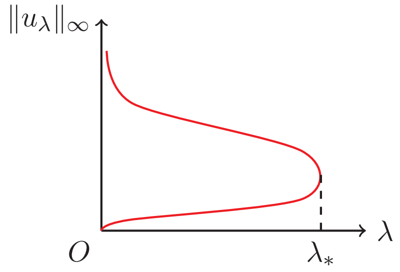


– Case 2. $f'(0) = 1$, and $f(t) < t$ for $t > 0$ small.
  Example. $f(t) = t(1 - \sin t) + t^p$ $(2 < p < (N+2)/(N-2))$.



– Case 3. $\lim_{t\to 0^+} \dfrac{f(t)}{t} = +\infty$.

Example. $f(t) = \sqrt{t} + t^p$ $(1 < p < (N+2)/(N-2))$.



2.2 Sublinear nonlinearities

We still assume that f is locally Lipschitz continuous function from $\mathbb{R}^+$ to $\mathbb{R}$ and that $f(0) = 0$. In addition f is assumed to be sublinear, that is

$$\limsup_{t\to\infty} \frac{f(t)}{t} < \lambda_1. \tag{IV.11}$$

We give the following existence result (see for example H. Amann [10], or H. Berestycki and P. L. Lions [21]) :

Theorem 2.2 *In addition to* (IV.11)*, let us assume that f satisfies :*

$$\liminf_{t\to 0^+} \frac{f(t)}{t} > \lambda_1. \tag{IV.12}$$

Then the problem

$$(\mathrm{P}) \qquad \begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u_{|\partial\Omega} = 0, \quad u > 0 & \text{on } \Omega, \end{cases} \tag{IV.13}$$

admits a maximal solution.

Since f is continuous and locally Lipschitz, there exists a constant $k > 0$ such that $t \mapsto f(t) + kt$ is nondecreasing for $t \in [0, C]$, for some constant $C > 0$ large. Then the existence of a maximal solution is obtained by constructing a monotone decreasing iteration scheme $\{u_n\}$ starting with $u_0 = C$.

Remark 2.5 *If f satisfies* (IV.12) *and*

$$f(\beta) = 0, \qquad \text{for some } \beta > 0, \tag{IV.14}$$

then the problem (P) *admits a maximum solution among all solution less than β. Indeed, by the maximum principle, every solution of* (P) *with f replaced by $\tilde{f} = f(t \wedge \beta)$, is less than β. Hence it is also solution to* (P)*. Let us remark that $\tilde{f}$ satisfies obviously* (IV.11)*.*

Remark 2.6 *In the case when f satisfies* (IV.11), (IV.12) *and*

$$t \longmapsto f(t)/t \text{ is decreasing for } t > 0, \tag{IV.15}$$

then it is well known (see H. Berestycki *[20] for an elegant proof) that* (P) *has a unique positive solution (the uniqueness follows in fact from the monotonicity of the operator* $-\Delta - \frac{f(u)}{u}$*, see also* Junping Shi *and* Miaoxin Yao *[109, Lemma 2.3]). In particular if* f *is is strictly concave then* (IV.15) *is satisfied.*

Now, we turn to the parametrized version (P_λ).

Theorem 2.3 *We assume that* f *is differentiable near* 0 *with* $f'(0) = 1$ *(without loss of generality) and we assume the hypotheses* (IV.7) *and*

$$\liminf_{t\to\infty} \frac{f(t)}{t} = 0 \ (\textit{resp. } f(\beta) = 0 \textit{ for some } \beta > 0), \tag{IV.16}$$

and

$$f(t) > 0 \textit{ for all } t > 0 \ (\textit{ resp. for all } \beta > t > 0). \tag{IV.17}$$

Let λ_* *be the infimum of all* $\lambda > 0$ *such that there exists a solution of* (P_λ) *(resp. less than* β*). Then we have :* $0 < \lambda_* \le \lambda_1$*, and there exists an unbounded set* $\mathcal{A} \subset [\lambda_*, \infty) \times \mathcal{C}_{\phi_1}(\Omega)$ *of solutions to* (P_λ) *which is globally parametrized by a continuous map :*

$$(0, \infty) \ni s \longmapsto (\lambda(s), u(s)) \in \mathcal{A}$$

such that $(\lambda(s), u(s)) \to (\lambda_1, 0)$ *as* $s \to 0$ *and*

$$\|(\lambda(s), u(s))\|_\infty \longrightarrow \infty \textit{ as } s \to \infty.$$

If we assume that f *is differentiable in* $\mathbb{R}^{+*}$ *only and* $\lim\limits_{t\to 0^+} f(t)/t = \infty$*, then* $\lambda_* = 0$ *and the branch* $\mathcal{A}$ *emanates from* $(0, 0)$*.*

Remark 2.7 *If* $f'(0) = 0$*, it is easy to prove that there is no bifurcation point and the global branch can not emanate from some* $(\tilde{\lambda}, 0)$ *with* $\tilde{\lambda} \in \mathbb{R}^+$*.*

Let us give now a condition which implies $\lambda_* < \lambda_1$.

Corollary 2.2 *Under the assumption of theorem 2.3 with* $f'(0) = 1$ *and if the following condition*

$$f(t) > t, \textit{ for } t > 0 \textit{ and sufficiently small}$$

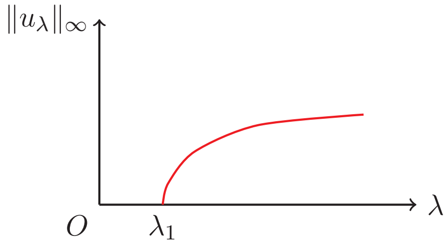
then $\lambda_* < \lambda_1$*.*

Remark 2.8 *If* $f(t) < t$*, for* $t > 0$ *small, then locally near* $(\lambda_1, 0)$ *the solutions* (λ, u) *to* (P_λ) *exist only for* $\lambda > \lambda_1$*.*

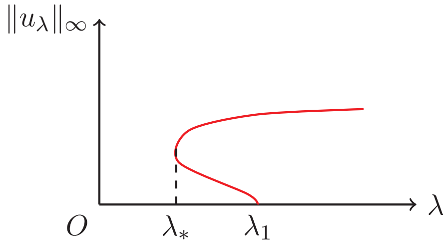
The proofs of Theorem 2.3 and Corollary 2.2 are similar to those of Theorem 2.1 and Corollary 2.1. Therefore we skip them. We point out that the existence of maximal solutions which provide uniform bounds of solutions (for λ bounded) implies the nonexistence of asymptotic bifurcation points. Under conditions of Corollary 2.2, we can prove the existence of minimal solutions for $\lambda \in (\lambda_*, \lambda_1)$ by lower and super solution technique and that a portion of the minimal solution branch belongs to $\mathcal{A}$.

We summarize again the results of this section with the associated bifurcation diagrams.

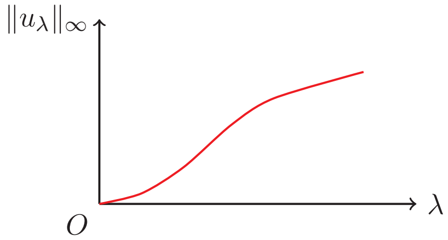
- Case 1. $f'(0) = 1$, and $f(t) < t$ for $t > 0$ small.
  Example. $f(t) = t - t^p$ $(1 < p < \infty)$. Let us denote her that f is concave, then (P_λ) has a unique solution for every $\lambda \geq \lambda_1$ and by the maximum principle, the solutions are increasing with respect to λ (see Remark 2.6).



- Case 2. $f'(0) = 1$, and $f(t) > t$ for $t > 0$ small.
  Example. $f(t) = t + \alpha t^2 - \beta t^3,\ \alpha, \beta > 0$.



- Case 3. $\lim_{t\to 0^+} \dfrac{f(t)}{t} = +\infty$.
  Example. $f(t) = t^p$ $(0 < p < 1)$.



3 The case when $f(0) > 0$.

3.1 Superlinear nonlinearities

In this section, we still assume that f is locally Lipschitz continuous function from $\mathbb{R}^+$ to $\mathbb{R}$ and we assume that $f(0) > 0$ and we will restrict our attention to the case when f satisfies (IV.8) Our result about the existence of a global connected-path is the following

Theorem 3.1 *Let us suppose that f is differentiable on $(0,\infty)$ and satisfies*

$$\lim_{t\to 0} f'(t)\ t < \infty, \tag{IV.18}$$

that the hypotheses (IV.7),(IV.8), (IV.5), (IV.4), (IV.6) *hold, and that Ω is convex. Then there exist λ_* such that (P_λ) has no solution for $\lambda > \lambda_*$, and an unbounded connected-path $\mathcal{A}$ emanating from $(0,0)$ such that*

(i) There exists $\lambda_0 \in (0,\lambda_*)$ such that for all $\lambda \in (0,\lambda_0)$, there exist at least two distinct positive solutions $\underline{u}_\lambda, u_\lambda \in \mathcal{A}$, where $\underline{u}_\lambda$ is a minimal solution,

(ii) $\|u_\lambda\|_\infty \to \infty$ when $\lambda \to 0^+$.

Remark 3.1 *(1) Again assumption* (IV.6)*(and the convexity of Ω) is technical, it is used only in (ii) in order to ensure some uniform a priori estimates and the theorem still holds with any assumption (replacing* (IV.6)*) providing a priori bounds of the solutions of (P_λ) (alternative assumptions providing uniform bounds can be found in [48]).*

(2) If we assume in addition that f is strictly convex on $\mathbb{R}_+$, then the connected-path $\mathcal{A}$ satisfies

i) $(\lambda_*, u_{\lambda_*}) \in \mathcal{A}$ is a "turning point" of $\mathcal{A}$;

ii) for all $\lambda \in (0,\lambda_*)$, there exist two distinct and orderd positive solutions $\underline{u}_\lambda, u_\lambda \in \mathcal{A}$ such that $\underline{u}_\lambda$ is the minimal solution. In this case the the derivative $\partial_u F(\lambda, u_\lambda)$ is invertible along the curve of minimal solutions for $\lambda \in (0,\lambda_*)$.

Remark 3.2 *The example $f(t) = \alpha + \beta t$ with $\alpha, \beta > 0$ shows that assumption* (IV.5) *is crucial in order to get*

i) a solution for $\lambda = \lambda_*$,

ii) at least two solutions for $0 < \lambda < \lambda_*$.

Indeed in this case it is well known that (P_λ) has a solution if and only if $\lambda < \lambda_* = \lambda_1/\beta$.

Proof of Theorem 3.1 : The proof is similar the to that of Theorem 2.1. The crucial point here is to show the existence of a local analytic branch of solutions emanating from $(0,0)$. For that, it suffices to prove that the linearised operator $\partial_u F(\lambda,u) = id - \lambda\left[(-\Delta)^{-1} f'(u)\cdot\right]$ is invertible for λ small enough, and then we can apply the implicit function theorem along the minimal branch for small $\lambda > 0$. Indeed, we have by hypothesis (IV.18), $f'(u) \in C_{\phi_1^{-1}}(\Omega)$, for all $u \in C^+_{\phi_1}(\Omega)$. Hence by variational method, we can show that the following problem

$$\begin{cases} -\Delta w - \lambda f'(u) w = g \text{ in } \Omega, \\ w = 0 \text{ on } \partial\Omega, \end{cases}$$

where $g \in \mathcal{C}_{\phi_1}(\Omega)$, has one and only one positive solution $w \in H^1_0(\Omega) \cap \mathcal{C}^+_{\phi_1}(\Omega)$, for λ small enough. For the uniqueness it is enough to remark that the first eigenvalue μ_1 of the operator $-\Delta - \lambda f'(u)$ is positive. Indeed μ_1 is defined by

$$\mu_1 = \inf_{\substack{\varphi \in H^1_0(\Omega) \\ \int_\Omega \varphi^2 = 1}} \left(\int_\Omega |\nabla\varphi|^2 \mathrm{d}x - \lambda \int f'(u)\varphi^2 \mathrm{d}x \right)$$

Using Young inequality and the fact that $f'(u) \in C_{\phi_1^{-1}}(\Omega)$, we have

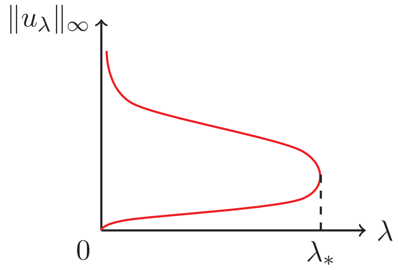
$$\lambda \int f'(u)\varphi^2 \mathrm{d}x \leq \frac{1}{2}\lambda \left(\int (f'(u)\phi_1)^2\varphi^2 \mathrm{d}x + \int_\Omega (\frac{\varphi}{\phi_1})^2 \mathrm{d}x \right)$$

We have $f'(u)\varphi \in L^\infty(\Omega)$, then $\int (f'(u)\phi_1)^2\varphi^2 \mathrm{d}x \leq C_1 \int_\Omega |\nabla\varphi|^2 \mathrm{d}x$ and by Hardy inequality, we have $\int_\Omega (\frac{\varphi}{\phi_1})^2 \mathrm{d}x \leq C_2 \int_\Omega |\nabla\varphi|^2 \mathrm{d}x$. Hence by choosing λ small enough, we obtain that

$$\mu_1 \geq C_3 \inf_{\substack{\varphi \in H_0^1(\Omega) \\ \int_\Omega \varphi^2 = 1}} \int_\Omega |\nabla\varphi|^2 \mathrm{d}x > 0.$$

This concludes the proof.

We present the result of this section through a bifurcation diagram representing the set of solutions to (P_λ) (taking into account the previous remark), (e.g., $f(t) = \alpha + \beta t^p$, $p \geq (N+2)/(N-2)$. here f is convex).



3.2 Sublinear nonlinearities

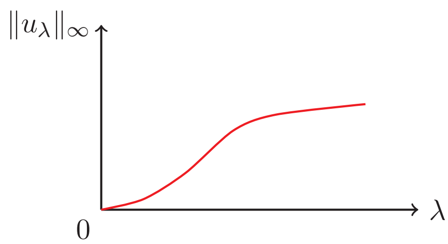
The main result is given by the following theorem. The proof of this result uses similar arguments as in above subsections. Therefore, we skip it.

Theorem 3.2 *Let us assume that f satisfies* (IV.7)*, either* (IV.8) *together with*

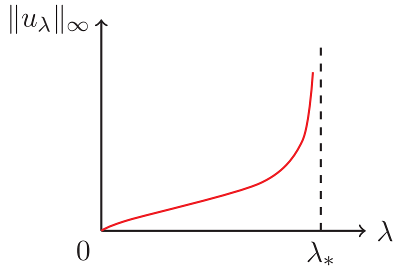
$$\lim_{t \to \infty} f(t)t^{-1} = 0$$

or that there exists $\beta > 0$ such that $f(\beta) = 0$. Then there is an unbounded connected-path $\mathcal{A}$ emanating from $(0,0)$. In addition, if f is concave, then for all $\lambda > 0$ there exists a unique solution u_λ.

The following diagram summarizes the situation given in the above theorem.
Example. $f(t) = \alpha + \beta t^\theta$, $\alpha, \beta > 0$, $\theta \in (0,1)$.



Remark 3.3 *If we assume instead of* (IV.16), $\limsup_{t\to\infty} \frac{f(t)}{t} = K < \infty$, *then there exists a connected-path* $\mathcal{A}$ *emanating from* $(0, 0)$ *and has asymptotic bifurcation point in for* $\lambda_* = \lambda_1/K$.



Appendices

A Polynomials with variable coefficients and Analyticity

A.1 Analyticity and Riemann Extension Theorem

Definition A.1 (Definition of a manifold) *A set $\mathcal{M} \subset \mathbb{F}$ is called an m-dimensional $\mathbb{F}$-analytic manifold if, for all points $a \in \mathcal{M}$, there is an open neighbourhood U_a of $0 \in \mathbb{F}^n$ and an analytic function $f : U_a \to \mathcal{M}$ such that $f(0) = a$, $df[0]$ is a finite-dimensional linear transformation of rank m, and f maps open sets of U into relatively open sets of $\mathcal{M}$. onto*

Theorem A.1 *(***Riemann extension theorem***) Let $U \subset \mathbb{C}^n$ a non-empty connected open set and let $\operatorname{var}(U, G) \neq U$ an analytic variety. Then if $\tilde{h}$ is an analytic function on $U \setminus \operatorname{var}(U, G)$ such that*

$$\sup\{|\tilde{h}(x)| : x \in U \setminus \operatorname{var}(U, G)\} < \infty$$

then there is an analytic function h such that $h \equiv \tilde{h}$ on $U \setminus \operatorname{var}(U, G)$ (h is called the extension of $\tilde{h}$).

To prove the above theorem, we use the following preliminary result of independent interest :

Proposition A.1 *Suppose that $U \subset \mathbb{F}^n$ is open, connected and $g_k : U \to \mathbb{F}$ is $\mathbb{F}$-analytic, $1 \leq k \leq m$. Let $E = \{x \in U \,:\, g_k(x) = 0 \in \mathbb{F},\, 1 \leq k \leq m\}$.*

(a) If $E \neq U$, then $U \backslash E$ is open, dense in U.

(b) If, in addition, $\mathbb{F} = \mathbb{C}$, then $U \backslash E$ is connected.

For the proofs of the above results, we refer to Federer [62] and Dieudonné [53].

A.2 Polynomials with variable coefficients and analyticity

Let us denote that the theory of polynomials with variable constants is to be found in Mumford [94] and Narasimhan [97].

Proposition A.2 (Continuous dependence of roots) *Let $p \in \mathbb{N}^*$.*

(1) Let $\hat{z} \in \mathbb{C}$ is a simple root of a polynomial

$$Z^p + \hat{a}_{p-1} Z^{p-1} + \cdots + \hat{a}_0$$

with constant complex coefficients. Then there is a $\mathbb{C}$*-analytic function* f*, defined on a neighbourhood of* $(\hat{a}_{p-1}, \cdots, \hat{a}_0)$*, such that* $z = f(a_0, \cdots, a_{p-1})$ *is a simple root of the polynomial*

$$Z^p + a_{p-1}Z^{p-1} + \cdots + a_0$$

and $\hat{z} = f(\hat{a}_{p-1}, \cdots, \hat{a}_0)$*. In addition, if* $\hat{a}_{p-1}, \cdots, \hat{a}_0, \hat{z} \in \mathbb{R}$*, then*

$$a_0, \cdots, a_{p-1} \in \mathbb{R} \Longrightarrow z = f(a_0, \cdots, a_{p-1}) \in \mathbb{R}.$$

(2) Suppose now that $\hat{z}$ *is of multiplicity* $q \geq 1$ *of the polynomial in part (1). Then for every* $\varepsilon > 0$ *small enough, there exist* $\delta > 0$ *such that the polynomial*

$$Z^p + a_{p-1}Z^{p-1} + \cdots + a_0$$

has exactly q *complex roots, counted according to their multiplicity, in the set*

$$\{z \in \mathbb{C} : |z - \hat{z}| < \varepsilon\}$$

provided that $|\hat{a}_{p-1} - a_{p-1}|, \cdots, |\hat{a}_0 - a_0| < \delta$

(3) For all $\varepsilon > 0$*, there exists* $\delta > 0$ *such that*

$$|a_0|, \cdots, |a_{p-1}| < \delta \Longrightarrow |z| < \varepsilon$$

where z *is a root of the polynomial*

$$Z^p + a_{p-1}Z^{p-1} + \cdots + a_0$$

with complex coefficients.

Theorem A.2 (Weierstrass Preparation Theorem) *Let* $U \subset \mathbb{F}^n$ *an open set contains* 0*,* $f : U \to \mathbb{F}$ *is analytic,* $f(0) = 0$ *and for* $(0, \cdots, x_n) \in U$*,*

$$f(0, \cdots, 0, x_n) = x_n^q v(x_n) \text{ where } v(0) \neq 0 \text{ and } q \geq 1$$

where v *is analytic in a neighbourhood of* $0 \in \mathbb{F}^n$*.*

(1) For all analytic function ϕ *in a neighbourhood of* $0 \in \mathbb{F}^n$ *such that* $\phi(0) = 0$*, there are* $b_1, \cdots, b_{n-1}$ *analytic on a neighbourhood of* $0 \in \mathbb{F}^{n-1}$ *and* a *analytic on a neighbourhood of* $0 \in \mathbb{F}^n$ *such that*

$$\phi(x_1, \cdots, x_n) = a(x_1, \cdots, x_n)f(x_1, \cdots, 0, x_n) + \sum_{k=0}^{q-1} b_k(x_1, \cdots, x_{n-1})x_n^k$$

on a neighbourhood of $0 \in \mathbb{F}^n$*.*

(2) There are analytic functions h*,* $a_0, \cdots, a_{q-1}$ *in neighbourhood of* $0 \in \mathbb{F}^{n-1}$ *such that* $a_0(0) =$

$\cdots = a_{q-1}(0) = 0$ *and an analytic functions* h *in neighbourhood of* $0 \in \mathbb{F}^n$ *such that* $h(0) \neq 0$ *and*

$$f(x_1, \cdots, x_n) = h(x_1, \cdots, x_n)\left(x_n^q + \sum_{k=0}^{q} a_k(x_1, \cdots, x_{n-1})x_n^k\right).$$

(3) the functions a, u, a_k, b_k *are uniquely determined if the neighbourhoods are open connected.*

(4) If $U \subset \mathbb{C}^n$ *and* f *is real-on-real, then* h *and* a_k *are real-on-real.*

For the proof we refer to R.Narasimhan [97].

Discriminant of polynomial. Consider two polynomials of the complex variable z, with constant coefficients in $\mathbb{C}$, given by

$$A(z) = a_p z^p + \cdots + a_1 z + a_0, \ p \geq 1, \ a_p \neq 0,$$

$$B(z) = b_p z^q + \cdots + b_1 z + a_0, \ q \geq 1, \ a_q \neq 0.$$

Then A and B are said to be co-prime if the constant polynomial "1" of degree 0 is their greatest common divisor. An elementary criterion says that A and B are co-prime if and only if there exist two polynomial U and V such that

$$\begin{cases} A(z)U(z) + B(z)V(z) = 0, \\ U(z =\not\equiv 0, \ V(z) \not\equiv 0, \\ \deg(U(z)) < q, \quad \deg(V(z)) < p. \end{cases} \tag{.1}$$

If

$$U(z) = c_{q-1}z^{q-1} + \cdots + c_1 z + c_0, \ V(z) = d_{p-1}z^{p-1} + \cdots + d_1 z + d_0,$$

then (.1) is equivalent to

$$\begin{bmatrix}
a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\
a_1 & a_0 & \cdots & 0 & b_1 & b_0 & \cdots & 0 \\
\vdots & a_1 & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \cdots & a_0 & b_{q-1} & b_{q-2} & b_{q-3} & \cdots & 0 \\
\vdots & \vdots & \cdots & a_1 & b_q & b_{q-1} & b_{q-2} & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
a_{p-1} & a_{p-2} & \cdots & a_{p-q} & \vdots & \vdots & \vdots & \cdots & b_0 \\
a_p & a_{p-1} & \cdots & a_{p-q+1} & \vdots & \vdots & \vdots & \cdots & b_1 \\
0 & a_p & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & & b_{q-1} \\
0 & 0 & \cdots & a_{p-1} & 0 & 0 & 0 & \cdots & b_q
\end{bmatrix} \times \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{q-1} \\ d_0 \\ d_1 \\ \vdots \\ d_{p-1} \end{bmatrix} = 0.$$

In the above system we have supposed $p > q$. The above complex $(p+q) \times (p+q)$ matrix is called the resultant matrix of A and B. Its determinant $\mathcal{D}(a_0, \cdots, a_p, b_0, \cdots, b_q)$, is called **the resultant** of A

and B. The resultant is a polynomial in the coefficients $a_0, \cdots, a_p, b_0, \cdots, b_q$ which is zero if and only if A and B are not co-prime.

Definition A.2 (Discriminant of polynomial) *Consider the polynomial*

$$A(z) = a_p z^p + \cdots + a_1 z + a_0, \; p \geq 1, \; a_p \neq 0,$$

the coefficients of which are complex. The discriminant $\mathcal{D}(a_0, \cdots, a_p)$ *of* A *is the resultant of* A *and* A'*, where* A' *denotes the polynomial obtained by differentiating* A *with respect to* z*.*

It is clear that if A has a multiple root $z_0 \in \mathbb{C}$, then $z - z_0$ is a common factor of A and A'. Then the discriminant of A vanishes exactly when A has at least one multiple root.

Theorem A.3 (Simplification of polynomials by Euclid's Algorithm)
Let $A(z_1, \cdots, z_m;\; z)$ *be a polynomial defined as follows*

$$A(z_1, \cdots, z_m;\; z) = z^p + a_{p-1}(z_1, \cdots, z_m) z^{p-1} + \cdots + a_0(z_1, \cdots, z_m)$$

where $p \geq 1$, $a_{p-1}, \cdots, a_0$ *are* $\mathbb{C}$*-analytic function on*

$$V = \{(z_1, \cdots, z_m) \in \mathbb{C}^m : |z_1|, \cdots, |z_m| < \delta\}, \; \delta > 0.$$

Suppose that its discriminant $D(a_0, \cdots, a_{p-1}, 1) \equiv 0$ *on* V*. Then there exists another polynomial*

$$E(z_1, \cdots, z_m;\; z) = z^q + e_{q-1}(z_1, \cdots, z_m) z^{p-1} + \cdots + e_0(z_1, \cdots, z_m)$$

of degree q*, and the coefficients* $e_{p-1}, \cdots, e_0$ *are* $\mathbb{C}$*-analytic function on* V*, which has the same roots as* $A(z_1, \cdots, z_m;\; z)$*, possibly with smaller multiplicity, and* $D(e_0, \cdots, e_{p-1}, 1) \not\equiv 0$ *on* V*. (In particular,* $E(z_1, \cdots, z_m;\; z)$ *has no multiple roots for* $(z_1, \cdots, z_m)$ *in an open dense connected subset* W *of* V*). If* $A(z_1, \cdots, z_m;\; z)$ *is real-on-real then so is* E*. (*E *is called the simplification of* A*).*

Lemma A.1 (Projection lemma) *Let* $A_1(z_1, \cdots, z_m;\; z)$ *be a polynomial of degree* p*, defined on* V *as in the previous theorem and let* $A_j(z_1, \cdots, z_m;\; z)$ *be a polynomial in* z *of degree at most* $p - 1$*,* $2 \leq j \leq k$*. Let*

$$\mathcal{A} = \{(z_1, \cdots, z_m) \in V : \exists z \in \mathbb{C}, \; A_j(z_1, \cdots, z_m;\; z) = 0 \text{ for all } j \in \{1, \cdots, k\}\}.$$

Then there exists a finite family $\{g_i : i \in I\}$ *of analytic functions on* V *such that*

$$\mathcal{A} = \{(z_1, \cdots, z_m) \in V : \; g_i(z_1, \cdots, z_m) = 0 \text{ for all } i \in I\}.$$

If the polynomials A_j *are real-on-real, then the functions in* $\mathcal{A}$ *are real-on-real.*

The reader is referred to D. Mumford [94, sect. 4.11] and also to B. Buffoni and J. F. Toland [32, Chapter 6] for the proof.

B The critical groups in Homology theory and Morse index

In this section, we study the relationship between these Morse type numbers and the deformation property of the underlying manifold via the deformation property of homology (The main result of this section is the theorem B.1). First, we begin by giving some definitions and properties of the homology. For more illustrations, we refer to K. C.Chang [36] and M. J. Greenberg and J. R.Harper [73].

B.1 Chain Complexes and homology

Definition B.1 *Let R be Ring and let $(M,+)$ be an Abelian group. Then M is called a left R-module if there exists a scalar multiplication : $\mu : R\times M \to M$ denoted by $\mu(r,m) = rm$ for all $r \in R$ and $m \in M$, such that for all $r, r_1, r_2 \in M$ and all $m, m_1, m_2 \in M$,*

$$r(m_1 + m_2) = rm_1 + rm_2, \quad (r_1 + r_2)m = r_1m + r_2m, \quad r_1(r_2m) = (r_1r_2)m, \quad 1m = m.$$

"Module" will always mean left module unless stated otherwise.

Let us denote that if R is a field, then M is a vector space over R.

Example B.1 *1- Any ring R is a module over itself.*
2- Every Abelian group $(A,+)$ is a $\mathbb{Z}$-module.

Definition B.2 *1- Any subset of M that is a left R-module under the operations induced from M, is called a submodule of M.*

2- The submodule generated by $X \subset M$ denoted by $\langle X\rangle$, is the smallest submodule of M which contains X. It is not difficult to show that

$$\langle X\rangle = \sum_{x\in X} Rx.$$

3- The module M is called a free module if there exists a subset $X \subset M$ such that each element $m \in M$ can be expressed uniquely as a finite sum, $m = \sum_{i=1}^{n} a_ix_i$, with $a_1, \cdots, a_n \in R$ and $x_1, \cdots, x_n \in X$.

Definition B.3 *Let R be a ring. A chain complex over R is a sequence $\{C_q, \partial_q\}_{q\in\mathbb{Z}}$ denoted by (C_*, ∂), of R-modules C_q and homomorphisms $\partial_q : C_q \to C_{q-1}$ such that $\partial_q\partial_{q+1} = 0$,*

$$\cdots C_{q+1} \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} C_{q-1} \cdots$$

with $C_q = 0$ if $q < 0$. The homomorphism ∂_q is called a boundary operator.

Let us denote $Z_q(C_*, \partial) = \ker(\partial_q : C_q \to C_{q-1})$ and $B_q(C_*, \partial) = \mathrm{Im}(\partial_{q+1} : \mathrm{C_{q+1}} \to \mathrm{C_q})$. Since $\partial_q\partial_{q+1} = 0$, we have $B_q(C_*, \partial) \subset Z_q(C_*, \partial)$.

Definition B.4 *We call*

$$H_q(C_*, R) \stackrel{\text{def}}{=} Z_q(C_*, \partial)/B_q(C_*, \partial)$$

the q-homology module.

B.2 Singular relative homology : Definitions and properties

We take a countably infinite product $\mathbb{R}^\infty$ of copies of $\mathbb{R}$, and consider the vectors

$$\begin{aligned} e_0 &= (0,0,\cdots,0,\cdots) \\ e_1 &= (1,0,\cdots,0,\cdots) \\ \cdots & \quad\quad \cdots \\ e_i &= (0,0,\cdots,\underset{i^{th}}{1},\cdots) \\ \cdots & \quad\quad \cdots \end{aligned}$$

We identify $\mathbb{R}^n$ with the subspace having all components after the n-th equal to 0.

B.2.1 Construction of a singular chain complex

We define the standard q-simplex Δ_q

$$\Delta_q = \left\{ \sum_{i=0}^{q} \lambda_i e_i,\ \lambda_i \geq 0,\ \sum_{i=0}^{q} \lambda_i = 1 \right\}.$$

Thus Δ_0 is a point, Δ_1 is the unit interval, Δ_2 is a triangle (including its interior), etc.

Given a topological space X, a singular q-simplex in X is a continuous map $\Delta_q \to X$. Thus for $q = 0$ it can be identified with a point in X ; for $q = 1$ it is a path in X, etc. Let R be a ring. We denote by $S_q(X;R)$ the free R-module generated by the set of the singular q-simplex in X,

$$S_q(X;R) = \left\{ \sum_{i\in I} r_i\sigma_i \mid \sigma_i : \Delta_q \to X \text{ is continuous}, r_i \in R,\ I \text{ is finte} \right\}.$$

We define the boundary operator by the homomorphism $\partial_q : S_q(X;R) \to S_{q-1}(X;R)$

$$\partial_q\sigma = \sum_{i=0}^{q} (-1)^i \sigma \circ \varepsilon_q^i$$

where the map $\varepsilon_q^i : \Delta_{q-1} \to \Delta_q$ is defined by

$$(t_0,\cdots,t_{q-1}) \mapsto (t_0,\cdots,\underset{i^{th}}{0},\cdots,t_{q-1}).$$

Let us prove show that $\partial_q \circ \partial_{q+1} = 0$. For $\sigma \in S_{q+1}(X;R)$

$$\begin{aligned} (\partial_q \circ \partial_{q+1})(\sigma) &= \sum_{i=0}^{q+1}\sum_{j=0}^{q} (-1)^{i+j} \sigma \circ \varepsilon_{q+1}^i \circ \varepsilon_q^j \\ &= \sum_{i>j} (-1)^{i+j} \sigma \circ \varepsilon_{q+1}^i \circ \varepsilon_q^j + \sum_{i\leq j} (-1)^{i+j} \sigma \circ \varepsilon_{q+1}^i \circ \varepsilon_q^j \end{aligned}$$

replacing j by $j-1$ in the second term in this sum, we obtain

$$(\partial_q \circ \partial_{q+1})(\sigma) \sum_{i>j} (-1)^{i+j} \sigma \circ \varepsilon^i_{q+1} \circ \varepsilon^j_q - \sum_{i<j} (-1)^{i+j} \sigma \circ \varepsilon^i_{q+1} \circ \varepsilon^{j-1}_q .$$

Using this inequality $\varepsilon^i_{q+1} \circ \varepsilon^j_q = \varepsilon^j_{q+1} \circ \varepsilon^{i-1}_q$ and replacing i by j, we obtain that $(\partial_q \circ \partial_{q+1})(\sigma) = 0$.

Hence the sequence $(S_*, \partial) := \{S_q(X;R), \partial_q\}$ defines a chain complex over R.

Definition B.5 $(S_*(X;R), \partial)$ *is called a singular chain complex and*

$$H_q(X,R) \stackrel{\text{def}}{=} H_q(S_*, \partial)$$

is called the singular q-homology module.

Given a pair of topological spaces (X,A), with $A \subset X$ (such pair will be called topological pair). The boundary operator ∂ induces a homomorphism $\overline{\partial}$ which makes the following diagram commutative

$$\begin{tikzcd}
S_q(X;R) \arrow[r] \arrow[d, "\partial_q"'] & S_q(X;R)/S_q(A;R) \arrow[d, "\overline{\partial}"'] \\
S_{q-1}(X;R) \arrow[r] & S_{q-1}(X;R)/S_{q-1}(A;R)
\end{tikzcd}$$

and we have clearly $\overline{\partial}_q \circ \overline{\partial}_{q+1} = 0$. Then we define $Z_q(X,A;\partial) = \ker \overline{\partial}_q$ and $B_q(X,A;\partial) = \mathrm{Im}\overline{\partial}_{\mathrm{q+1}}$.

Definition B.6 *We call*

$$H_q(X,A;R) \stackrel{\text{def}}{=} Z_q(C_*, \overline{\partial})/B_q(C_*, \overline{\partial})$$

the singular relative q-homology module.

B.2.2 Some properties

Given two topological pairs (X,Y) and (X',Y'), we say that a map $f:(X,Y) \to (X',Y')$ is continuous if $f: X \to X'$ is continuous and $f(Y) \subset Y$. Two continuous maps $f,g:(X,Y) \to (X',Y')$ are called homotopic and we write $f \cong g$, if $\exists\ F:[0,1] \times (X,Y) \to (X',Y')$, which is continuous and satisfies

$$F(0,.) = f,\ \ F(1,.) = g,$$

and

$$F:[0,1] \times Y \to Y'.$$

The topological pairs (X,Y) and (X',Y') are called homotopically equivalent if there exist a continuous maps

$$\phi : (X,Y) \to (X',Y')$$

$$\psi : (X',Y') \to (X,Y)$$

satisfying

$$\psi \circ \phi = \mathrm{id}_{(\mathrm{X,Y})}, \quad \phi \circ \psi \cong \mathrm{id}_{(\mathrm{X',Y'})}.$$

We say (X',Y') is a deformation retract of (X,Y) if $X' \subset X$, $Y' \subset Y$, and if $\exists\ \eta : [0,1] \times X \to X$ satisfying

$$\eta(0,\cdot) = \mathrm{id_X},\ \eta(1,\mathrm{X}) \subset \mathrm{X'},\ \eta(1,\mathrm{Y}) \subset \mathrm{Y'},$$
$$\eta(t,Y) \subset Y \text{ and } \eta(t,\cdot)|_{X'} = \mathrm{id_{X'}},\ \forall \mathrm{t} \in [0,1].$$

Then we have this important property
Homotopy invariance : If (X,Y) and (X',Y') are homotopically equivalent, then

$$H_q(X,Y,R) \cong H_q(X',Y',R),\ \forall q.$$

If(X',Y') is a deformation retract of (X,Y), then

$$H_q(X,Y,R) \cong H_q(X',Y',R),\ \forall q.$$

Another basic property is the following
Excision property : If $U \subset X$ satisfies $\overline{U} \subset \mathrm{int(Y)}$, then

$$H_q(X \setminus U, Y \setminus U, R) \cong H_q(X,Y,R)$$

We need the following lemma for the proof of the below theorem.

Lemma B.1 *We have*
$$H_q(B^m, S^{m-1}; R) \cong \begin{cases} R & \text{if } q = m \\ 0 & \text{otherwise.} \end{cases}$$

For the proof we refer to M. J. Greenberg and J. R.Harper [73, Example 14.4, p.76].

B.3 Critical groups and Morse index

Definition B.7 *Let p be an isolated critical point of f, and $c = f(p)$. We call*

$$C_p(f,p) \stackrel{\text{def}}{=} H_q(U_p \cap f_c, U_p \cap (f_c \setminus \{p\}, R)$$

the q^{th} critical group of f at p, where U_p is a neighborhood of p contains only p as critical point, $H_q(U_p \cap f_c, U_p \cap (f_c \setminus \{p\}, R)$ is the singular relative homology.

According to the excision property of singular homology, the critical groups are well-defined (they do not depend on a special choice of the neighborhood U_p).

Definition B.8 *Let $f \in C^2(M,\mathbb{R})$, with M a Riemann manifold and p a critical point of f.*
(1) p is called non-degenerate if $d^2 f(p)$ has a bounded inverse.
(2) Since $A = d^2 f(p)$ is a self-adjoint operator which possesses a resolution of identity, we call the dimension of the negative space corresponding to the spectral decomposing, the Morse index of p, denoted by $\mathrm{ind(f,p)}$ *(it can be ∞).*

We give the following Morse Lemma :

Lemma B.2 *Suppose that $f \in \mathscr{C}^2(M, \mathbb{R})$ and that p is a non-degenerate critical point. Then there exists a neighborhood U_p of p and a local diffeomorphism $\psi : U_p \to T_p(M)$ with $\psi(p) = \theta$, such that*

$$f \circ \psi^{-1}(\xi) = f(p) + \frac{1}{2}\langle d^2 f(p)\xi, \xi\rangle, \ \forall \xi \in \psi(U_p),$$

where $\langle , \rangle$ is the inner product of the Hilbert space H, on which the Riemannian manifold M is modeled.

Now we are in a position to compute the critical groups of a nondegenerate critical point via its Morse index (See [36, Theorem 4.1, p.34]).

Theorem B.1 *Suppose that $f \in C^2(M, \mathbb{R})$ and that p is a non-degenerate critical point of f with Morse index j. Then*

$$C_q(f, p) = \begin{cases} R & q = j, \\ 0 & q \neq j. \end{cases}$$

Proof. According to the Morse lemma, we may restrict ourselves to a special case where f is a quadratic function on the Hilbert space $H : f(x) = \frac{1}{2}\langle Ax, x\rangle$, where A is a bounded, invertible, self-adjoint operator. Let $P_\pm$ be the orthogonal projection onto the positive (negative) subspace $H_\pm$ with respect to the spectral decomposition of A. We have

$$f(x) = \frac{1}{2}\left(\|(AP_+)^{1/2}x\|^2 - \|((-AP_-)^{1/2}x\|^2\right).$$

On an ϵ-ball B_ϵ centred at 0, it is easily seen that

$$B_\epsilon \cap f_0 = \left\{x \in H | \|y_+\| \le \|y_-\|, \ y_\pm = (\pm AP_\pm)^{1/2}x\right\}.$$

We define a deformation

$$\eta(t, x) = y_- + ty_+ \quad \forall (t, x) \in [0, 1] \times (B_\epsilon \cap f_0).$$

It is a strong deformation retract from $(B_\epsilon \cap f_0, B_\epsilon \cap (f_0 \setminus \{0\}))$ to $(H_- \cap B_\epsilon, H_- \setminus \{0\} \cap B_\epsilon)$, and the two pairs $(H_- \cap B_\epsilon, H_- \setminus \{0\} \cap B_\epsilon)$ and (B^j, S^{j-1}) are homotopically equivalent. Thus if j is finite, then

$$\begin{aligned} C_q(f, p) &\cong H_q(B_\epsilon \cap f_0, B_\epsilon \cap (f_0 \setminus \{0\}), R) \\ &\cong H_q(H_- \cap B_\epsilon, (H_- \setminus \{0\}) \cap B_\epsilon, R) \\ &\cong H_q(B^j, S^{j-1}, R) \cong \begin{cases} R & j = q \\ 0 & q \neq j. \end{cases} \end{aligned}$$

For $j = \infty$, since S^∞ is contractible (hompotpic to a point), then we always have

$$C_q(f, p) \cong 0.$$

This complete the proof. $\square$

Références bibliographiques

[1] R. A. Adams and J. J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[2] Adimurthi. Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the n-Laplacian. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 17(3) :393–413, 1990.

[3] Adimurthi and O. Druet. Blow-up analysis in dimension 2 and a sharp form of Trudinger-Moser inequality. *Comm. Partial Differential Equations*, 29(1-2) :295–322, 2004.

[4] Adimurthi and J. Giacomoni. Multiplicity of positive solutions for a singular and critical elliptic problem in $\mathbb{R}^2$. *Commun. Contemp. Math.*, 8(5) :621–656, 2006.

[5] Adimurthi and S. Prashanth. Failure of Palais-Smale condition and blow-up analysis for the critical exponent problem in $\mathbb{R}^2$. *Proc. Indian Acad. Sci. Math. Sci.*, 107(3) :283–317, 1997.

[6] Adimurthi and S. Prashanth. Critical exponent problem in $\mathbb{R}^2$-border-line between existence and non-existence of positive solutions for Dirichlet problem. *Adv. Differential Equations*, 5(1-3) :67–95, 2000.

[7] Adimurthi and M. Struwe. Global compactness properties of semilinear elliptic equations with critical exponential growth. *J. Funct. Anal.*, 175(1) :125–167, 2000.

[8] Adimurthi and S. L. Yadava. Multiplicity results for semilinear elliptic equations in a bounded domain of $\mathbb{R}^2$ involving critical exponents. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 17(4) :481–504, 1990.

[9] Adimurthi and S.L. Yadava. Existence of nodal solutions of elliptic equations with critical growth in $\mathbb{R}^2$. *Trans. A.M.S.*, 332(1) :449–458., 1992.

[10] H. Amann. "Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces" (SIAM Rev. **18** (1976), no. 4, 620–709). *SIAM Rev.*, 19(4) :vii, 1977.

[11] R. Aris. *Mathematical modelling techniques*, volume 24 of *Research Notes in Mathematics*. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1979.

[12] F. V. Atkinson and L. A. Peletier. Ground states and Dirichlet problems for $-\Delta u = f(u)$ in $\mathbb{R}^2$. *Arch. Rational Mech. Anal.*, 96(2) :147–165, 1986.

[13] M. Badra, K. Bal, and J. Giacomoni. Some results about a quasilinear singular parabolic equation. *Differ. Equ. Appl.*, 3(4) :609–627, 2011.

[14] M. Badra, K. Bal, and J. Giacomoni. A singular parabolic equation : existence, stabilization. *J. Differential Equations*, 252(9) :5042–5075, 2012.

[15] A. Bahri and J.-M. Coron. On a nonlinear elliptic equation involving the critical Sobolev exponent : the effect of the topology of the domain. *Comm. Pure Appl. Math.*, 41(3) :253–294, 1988.

[16] K. Bal and J. Giacomoni. A remark about symmetry of solutions to singular equations and applications. In *Eleventh International Conference Zaragoza-Pau on Applied Mathematics and Statistics*, volume 37 of *Monogr. Mat. García Galdeano*, pages 25–35. Prensas Univ. Zaragoza, Zaragoza, 2012.

[17] H. T. Banks. *Modeling and control in the biomedical sciences*. Springer-Verlag, Berlin, 1975. Lecture Notes in Biomathematics, Vol. 6.

[18] V. Barbu. *Nonlinear semigroups and differential equations in Banach spaces*. Editura Academiei Republicii Socialiste România, Bucharest ; Noordhoff International Publishing, Leiden, 1976. Translated from the Romanian.

[19] P. Bartolo, V. Benci, and D. Fortunato. Abstract critical point theorems and applications to some nonlinear problems with "strong" resonance at infinity. *Nonlinear Anal.*, 7(9) :981–1012, 1983.

[20] H. Berestycki. Le nombre de solutions de certains problèmes semi-linéaires elliptiques. *J. Funct. Anal.*, 40(1) :1–29, 1981.

[21] H. Berestycki and P.-L. Lions. Some applications of the method of super and subsolutions. In *Bifurcation and nonlinear eigenvalue problems (Proc., Session, Univ. Paris XIII, Villetaneuse, 1978)*, volume 782 of *Lecture Notes in Math.*, pages 16–41. Springer, Berlin, 1980.

[22] L. Boccardo and L. Orsina. Semilinear elliptic equations with singular nonlinearities. *Calc. Var. Partial Differential Equations*, 37(3-4) :363–380, 2010.

[23] M. Bonforte and G. Grillo. Super and ultracontractive bounds for doubly nonlinear evolution equations. *Rev. Mat. Iberoam.*, 22(1) :111–129, 2006.

[24] M. Bonforte and J. L. Vázquez. Positivity, local smoothing, and Harnack inequalities for very fast diffusion equations. *Adv. Math.*, 223(2) :529–578, 2010.

[25] B. Bougherara, J. Giacomoni, and S. Prashanth. Analytic global bifurcation and infinite turning points for very singular problems. *J. Calculus of variations and PDEs*.

[26] H. Brézis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[27] H. Brézis, T. Cazenave, Y. Martel, and A. Ramiandrisoa. Blow up for $u_t - \Delta u = g(u)$ revisited. *Adv. Differential Equations*, 1(1) :73–90, 1996.

[28] H. Brézis and P. L. Lions. A note on isolated singularities for linear elliptic equations. In *Mathematical analysis and applications, Part A*, volume 7 of *Adv. in Math. Suppl. Stud.*, pages 263–266. Academic Press, New York-London, 1981.

[29] H. Brézis and L. Nirenberg. Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math.*, 36(4) :437–477, 1983.

[30] B. Buffoni, E. N. Dancer, and J. F. Toland. The regularity and local bifurcation of steady periodic water waves. *Arch. Ration. Mech. Anal.*, 152(3) :207–240, 2000.

[31] B. Buffoni, E. N. Dancer, and J. F. Toland. The sub-harmonic bifurcation of Stokes waves. *Arch. Ration. Mech. Anal.*, 152(3) :241–271, 2000.

[32] B. Buffoni and J. Toland. *Analytic theory of global bifurcation*. Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2003. An introduction.

[33] X. Cabré. Regularity of radial extremal solutions of semilinear elliptic equations. *Bol. Soc. Esp. Mat. Apl. SeMa*, (34) :92–98, 2006.

[34] X. Cabré and Y. Martel. Weak eigenfunctions for the linearization of extremal elliptic problems. *J. Funct. Anal.*, 156(1) :30–56, 1998.

[35] T. Cazenave, M. Escobedo, and M. A. Pozio. Some stability properties for minimal solutions of $-\Delta u = \lambda g(u)$. *Port. Math. (N.S.)*, 59(4) :373–391, 2002.

[36] K. C. Chang. *Infinite-dimensional Morse theory and multiple solution problems.* Progress in Nonlinear Differential Equations and their Applications, 6. Birkhäuser Boston, Inc., Boston, MA, 1993.

[37] S.-N. Chow and R. Lauterbach. A bifurcation theorem for critical points of variational problems. *Nonlinear Anal.*, 12(1) :51–61, 1988.

[38] F. Cipriani and G. Grillo. Uniform bounds for solutions to quasilinear parabolic equations. *J. Differential Equations*, 177(1) :209–234, 2001.

[39] M. M. Coclite and G. Palmieri. On a singular nonlinear Dirichlet problem. *Comm. Partial Differential Equations*, 14(10) :1315–1327, 1989.

[40] M. G. Crandall and P. H. Rabinowitz. Bifurcation from simple eigenvalues. *J. Functional Analysis*, 8 :321–340, 1971.

[41] M. G. Crandall, P. H. Rabinowitz, and L. Tartar. On a Dirichlet problem with a singular nonlinearity. *Comm. Partial Differential Equations*, 2(2) :193–222, 1977.

[42] E. N. Dancer. Bifurcation theory for analytic operators. *Proc. London Math. Soc. (3)*, 26 :359–384, 1973.

[43] E. N. Dancer. Infinitely many turning points for some supercritical problems. *Ann. Mat. Pura Appl. (4)*, 178 :225–233, 2000.

[44] E. N. Dancer. Finite Morse index solutions of exponential problems. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 25(1) :173–179, 2008.

[45] E. B. Davies. *Heat kernels and spectral theory*, volume 92 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1990.

[46] E. B. Davies and B. Simon. Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians. *J. Funct. Anal.*, 59(2) :335–395, 1984.

[47] J. Dávila and M. Montenegro. Existence and asymptotic behavior for a singular parabolic equation. *Trans. Amer. Math. Soc.*, 357(5) :1801–1828, 2005.

[48] D. G. DeFigueiredo, P. L. Lions, and Nussbaum. A priori estimates and existence results for positive solutions of semilinear elliptic equations. *J. Math. Pures Appl.*

[49] M. Del Pino, J. Dolbeault, and I. Gentil. Nonlinear diffusions, hypercontractivity and the optimal L^p-Euclidean logarithmic Sobolev inequality. *J. Math. Anal. Appl.*, 293(2) :375–388, 2004.

[50] M. A. Del Pino. A global estimate for the gradient in a singular elliptic boundary value problem. *Proc. Roy. Soc. Edinburgh Sect. A*, 122(3-4) :341–352, 1992.

[51] R. Dhanya, J. Giacomoni, and S. Prashanth. Isolated singularities for the exponential type semilinear elliptic equation in $\mathbb{R}^2$. *Proc. Amer. Math. Soc.*, 137(12) :4099–4107, 2009.

[52] R. Dhanya, J. Giacomoni, S. Prashanth, and K. Saoudi. Global bifurcation and local multiplicity results for elliptic equations with singular nonlinearity of super exponential growth in $\mathbb{R}^2$. *Adv. Differential Equations*, 17(3-4) :369–400, 2012.

[53] J. I. Díaz. *Nonlinear partial differential equations and free boundaries*, volume I of *Research Notes in Mathematics 106.* Pitman (Advanced Publishing Program), Boston, MA, 1985.

[54] J. I. Díaz and J. M. Rakotoson. On the differentiability of very weak solutions with right-hand side data integrable with respect to the distance to the boundary. *J. Funct. Anal.*, 257(3) :807–831, 2009.

[55] J. I. Díaz and J. M. Rakotoson. On very weak solutions of semi-linear elliptic equations in the framework of weighted spaces with respect to the distance to the boundary. *Discrete Contin. Dyn. Syst.*, 27(3) :1037–1058, 2010.

[56] E. DiBenedetto. $C^{1,\alpha}$ local regularity of weak solutions of degenerate elliptic equations. *Nonlinear Anal.*, **7**-(8) : 827–850, 1983.

[57] E. DiBenedetto and A. Freidman. Hölder estimates for non-linear degenerate parabolic equations. *J. Reine Angew. Math.*

[58] J. Dolbeault and I. Flores. Geometry of phase space and solutions of semilinear elliptic equations in a ball. *Trans. Amer. Math. Soc.*, 359(9) :4073–4087 (electronic), 2007.

[59] O. Druet. Multibumps analysis in dimension 2 : quantification of blow-up levels. *Duke Math. J.*, 132(2) :217–269, 2006.

[60] O. Druet, E. Hebey, and F. Robert. *Blow-up theory for elliptic PDEs in Riemannian geometry*, volume 45 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 2004.

[61] A. Farina. Stable solutions of $-\Delta u = e^u$ on $\mathbb{R}^N$. *C. R. Math. Acad. Sci. Paris*, 345(2) :63–66, 2007.

[62] H. Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[63] W. B. Fite. Concerning the zeros of the solutions of certain differential equations. *Trans. Amer. Math. Soc.*, 19(4) :341–352, 1918.

[64] M. Ghergu. Lane-Emden systems with negative exponents. *J. Funct. Anal.*, 258(10) :3295–3318, 2010.

[65] M. Ghergu and V. Rădulescu. Multi-parameter bifurcation and asymptotics for the singular Lane-Emden-Fowler equation with a convection term. *Proc. Roy. Soc. Edinburgh Sect. A*, 135(1) :61–83, 2005.

[66] M. Ghergu and V. Rădulescu. *Singular elliptic problems : bifurcation and asymptotic analysis*, volume 37 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, Oxford, 2008.

[67] J. Giacomoni, S. Prashanth, and K. Sreenadh. Uniqueness and multiplicity results for N-Laplace equation with critical and singular nonlinearity in a ball. *Asymptot. Anal.*, 61(3-4) :195–227, 2009.

[68] J. Giacomoni, P. Sauvy, and S. Shmarev. Complete quenching for a quasilinear parabolic equation. *J. Math. Anal. Appl.*, 410(2) :607–624, 2014.

[69] J. Giacomoni, I. Schindler, and P. Takáč. Sobolev versus Hölder local minimizers and existence of multiple solutions for a singular quasilinear equation. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 6(1) :117–158, 2007.

[70] B. Gidas, W. M. Ni, and L. Nirenberg. Symmetry and related properties via the maximum principle. *Comm. Math. Phys.*, 68(3) :209–243, 1979.

[71] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[72] S. M. Gomes. On a singular nonlinear elliptic problem. *SIAM J. Math. Anal.*, 17(6) :1359–1369, 1986.

[73] M. J. Greenberg and J. R. Harper. *Algebraic topology*, volume 58 of *Mathematics Lecture Note Series*. Benjamin/Cummings Publishing Co., Inc., Advanced Book Program, Reading, Mass., 1981. A first course.

[74] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[75] L. Gross. Logarithmic Sobolev inequalities. *Amer. J. Math.*, 97(4) :1061–1083, 1975.

[76] C. Gui and F.-H. Lin. Regularity of an elliptic problem with a singular nonlinearity. *Proc. Roy. Soc. Edinburgh Sect. A*, 123(6) :1021–1029, 1993.

[77] Z. Guo and J. Wei. Global solution branch and Morse index estimates of a semilinear elliptic equation with super-critical exponent. *Trans. Amer. Math. Soc.*, 363(9) :4777–4799, 2011.

[78] Y. Haitao. Multiplicity and asymptotic behavior of positive solutions for a singular semilinear elliptic problem. *J. Differential Equations*, 189(2) :487–512, 2003.

[79] D. Henry. *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1981.

[80] J. Hernández, F. J. Mancebo, and J. Vega. On the linearization of some singular, nonlinear elliptic problems and applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 19(6) :777–813, 2002.

[81] N. Hirano, C. Saccon, and N. Shioji. Brezis-Nirenberg type theorems and multiplicity of positive solutions for a singular elliptic problem. *J. Differential Equations*, 245(8) :1997–2037, 2008.

[82] D. D. Joseph and T. S. Lundgren. Quasilinear Dirichlet problems driven by positive sources. *Arch. Rational Mech. Anal.*, 49 :241–269, 1972/73.

[83] H. B. Keller and D. S. Cohen. Some positone problems suggested by nonlinear heat generation. *J. Math. Mech.*, 16 :1361–1376, 1967.

[84] H. Kielhöfer. *Bifurcation Theory an introduction with applications to partial differential equations*. 156.

[85] H. Kielhöfer. A bifurcation theorem for potential operators. *J. Funct. Anal.*, 77(1) :1–8, 1988.

[86] A. C. Lazer and P. J. McKenna. On a singular nonlinear elliptic boundary-value problem. *Proc. Amer. Math. Soc.*, 111(3) :721–730, 1991.

[87] G. M. Lieberman. Boundary and initial regularity for solutions of degenerate parabolic equations. *Nonlinear Anal.*, 20(5) :551–569, 1993.

[88] P.-L. Lions. On the existence of positive solutions of semilinear elliptic equations. *SIAM Rev.*, 24(4) :441–467, 1982.

[89] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. II. *Rev. Mat. Iberoamericana*, 1(2) :45–121, 1985.

[90] S. Lojasiewicz. *Introduction to complex analytic geometry*. Birkhäuser Verlag, Basel, 1991. Translated from the Polish by Maciej Klimek.

[91] C. D. Luning and W. L. Perry. Positive solutions of negative exponent generalized Emden-Fowler boundary value problems. *SIAM J. Math. Anal.*, 12(6) :874–879, 1981.

[92] F. Merle and L. A. Peletier. Positive solutions of elliptic equations involving supercritical growth. *Proc. Roy. Soc. Edinburgh Sect. A*, 118(1-2) :49–62, 1991.

[93] J. Moser. A sharp form of an inequality by N. Trudinger. *Indiana Univ. Math. J.*, 20 :1077–1092, 1970/71.

[94] D. Mumford. *Algebraicheskaya geometriya. I. Komplektsnye proektivnye mnogoobraziya*. "Mir", Moscow ; distributed by Imported Publications, Chicago, Ill., 1979. Translated from the English by Ju. I. Manin.

[95] A. Nachman and A. Callegari. A nonlinear singular boundary value problem in the theory of pseudoplastic fluids. *SIAM J. Appl. Math.*, 38(2) :275–281, 1980.

[96] K. Nagasaki and T. Suzuki. Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially dominated nonlinearities. *Asymptotic Anal.*, 3(2) :173–188, 1990.

[97] R. Narasimhan. *Introduction to the theory of analytic spaces.* Lecture Notes in Mathematics, No. 25. Springer-Verlag, Berlin-New York, 1966.

[98] Z. Nehari. On a class of nonlinear second-order differential equations. *Trans. Amer. Math. Soc.*, 95 :101–123, 1960.

[99] T. Ogawa and T. Suzuki. Nonlinear elliptic equations with critical growth related to the Trudinger inequality. *Asymptotic Anal.*, 12(1) :25–40, 1996.

[100] D. O'Regan. Some general existence principles and results for $(\phi(y'))' = qf(t, y, y'),\ 0 < t < 1$. *SIAM J. Math. Anal.*, 24(3) :648–668, 1993.

[101] W. L. Perry. A monotone iterative technique for solution of pth order ($p < 0$) reaction-diffusion problems in permeable catalysis. *J. Comput. Chem.*, 5(4) :353–357, 1984.

[102] P. I. Plotnikov. Nonuniqueness of solutions of a problem on solitary waves, and bifurcations of critical points of smooth functionals. *Izv. Akad. Nauk SSSR Ser. Mat.*, 55(2) :339–366, 1991.

[103] M. M. Porzio. $L^{\infty}_{\mathrm{loc}}$-estimates for degenerate and singular parabolic equations. *Nonlinear Anal.*, 17(11) :1093–1107, 1991.

[104] P. H. Rabinowitz. Some global results for nonlinear eigenvalue problems. *J. Functional Analysis*, 7 :487–513, 1971.

[105] P. H. Rabinowitz. Variational methods for nonlinear elliptic eigenvalue problems. *Indiana Univ. Math. J.*, 23 :729–754, 1973/74.

[106] O. Rey. Le rôle de la fonction de Green dans une équation elliptique non linéaire avec l'exposant critique de Sobolev. *C. R. Acad. Sci. Paris Sér. I Math.*, 305(13) :591–594, 1987.

[107] T. Roubíček. *Nonlinear partial differential equations with applications*, volume 153 of *International Series of Numerical Mathematics*. Birkhäuser/Springer Basel AG, Basel, second edition, 2013.

[108] W. Rudin. *Principles of mathematical analysis.* McGraw-Hill Book Co., New York-Auckland-Düsseldorf, third edition, 1976. International Series in Pure and Applied Mathematics.

[109] J. Shi and M. Yao. On a singular nonlinear semilinear elliptic problem. *Proc. Roy. Soc. Edinburgh Sect. A*, 128(6) :1389–1401, 1998.

[110] J. Simon. Compact sets in the space $L^p(0, T; B)$. *Ann. Mat. Pura Appl. (4)*, **146** : 65–96, 1987.

[111] G. Stampacchia. Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. *Ann. Inst. Fourier (Grenoble)*, 15(fasc. 1) :189–258, 1965.

[112] M. Struwe. Critical points of embeddings of $H_0^{1,n}$ into Orlicz spaces. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 5(5) :425–464, 1988.

[113] C. A. Stuart. Existence and approximation of solutions of non-linear elliptic equations. *Math. Z.*, 147(1) :53–63, 1976.

[114] P. Takáč. L^{∞}-bounds for weak solutions of an evolutionary equation with the p-laplacian. In *Proceedings of the 2004 International Conference on "Function Spaces, Differential Operators and Nonlinear Analysis" (FSDONA) in honor of Alois Kufner*, May 28 – June 2, 2004, Brno-Milovy, Czech Republic.

[115] P. Takáč. Stabilization of positive solutions for analytic gradient-like systems. *Discrete Contin. Dynam. Systems*, 6(4) :947–973, 2000.

[116] P. Tolksdorf. On the Dirichlet problem for quasilinear equations in domains with conical boundary points. *Comm. Partial Differential Equations*, 8(7) :773–817, 1983.

[117] H. Triebel. *Interpolation theory, function spaces, differential operators*. Johann Ambrosius Barth, Heidelberg, second edition, 1995.

[118] J. L. Vázquez. A strong maximum principle for some quasilinear elliptic equations. *Appl. Math. Optim.*, 12(3) :191–202, 1984.

[119] M. Winkler. Blow-up in a degenerate parabolic equation. *Indiana Univ. Math. J.*, 53(5) :1415–1442, 2004.

[120] M. Winkler. Instantaneous shrinking of the support in degenerate parabolic equations with strong absorption. *Adv. Differential Equations*, 9(5-6) :625–643, 2004.

[121] M. Winkler. Infinite-time quenching in a fast diffusion equation with strong absorption. *NoDEA Nonlinear Differential Equations Appl.*, 16(1) :41–61, 2009.

[122] A. Wintner. On the non-existence of conjugate points. *Amer. J. Math.*, 73 :368–380, 1951.

[123] C. Xinfu. Hölder estimate for the solution of degenerate parabolic equations. *Acta Math. Appl. Sinica*.

[124] S. Yijing, W. Shaoping, and L. Yiming. Combined effects of singular and superlinear nonlinearities in some singular boundary value problems. *J. Differential Equations*, 176(2) :511–531, 2001.

Résumé : Cette thèse s'inscrit dans le domaine mathématique de l'analyse des équations aux dérivées partielles non-linéaires. Précisément, nous nous sommes intéressés à une classe de problèmes elliptiques et paraboliques avec coefficients singuliers. Ce manque de régularité pose un certain nombre de difficultés qui ne permettent pas d'utiliser directement les méthodes classiques de l'analyse non-linéaire fondées entre autres sur des résultats de compacité. Dans les démonstrations des principaux résultats, nous montrons comment pallier ces difficultés. Ceci suppose d'adapter certaines techniques bien connues mais aussi d'introduire de nouvelles méthodes. Dans ce contexte, une étape importante est l'estimation fine du comportement des solutions qui permet d'adapter le principe de comparaison faible, d'utiliser la régularité elliptique et parabolique et d'appliquer dans un nouveau contexte la théorie globale de la bifurcation analytique. La thèse se présente sous forme de deux parties indépendantes.

– Dans la première partie (chapitre I de la thèse), nous avons étudié un problème quasi-linéaire parabolique fortement singulier faisant intervenir l'opérateur p-Laplacien. On a démontré l'existence local et la régularité de solutions faibles. Ce résultat repose sur des estimations a priori dans L^∞ obtenues via l'utilisation d'inégalités de type log-Sobolev combinées à des inégalités de Gagliardo-Nirenberg. On démontre l'unicité de la solution pour un intervalle de valeurs du paramètre de la singularité en utilisant un principe de comparaison faible fondé sur la monotonie dans $W_0^{1,p}(\Omega)$ d'un opérateur non linéaire adéquat.
– Dans la deuxième partie (correspondant aux Chapitres II, III et IV de la thèse), nous sommes intéressé à l'étude de problèmes de bifurcation globale. On a établi pour ces problèmes l'existence de continuas non bornés de solutions qui admettent localement une paramétrisation analytique. Pour établir ces résultats, nous faisons appel à différents outils d'analyse non linéaire. Un outil important est la théorie analytique de la bifurcation globale qui a été introduite par DANCER (voir Chapitre II de la thèse). Pour un problème semilinéaire elliptique avec croissance critique en dimension 2, on montre que les solutions le long de la branche convergent vers une solution singulière (solution non bornée) lorsque la norme des solutions converge vers l'infini. Par ailleurs nous montrons que la branche admet une infinité dénombrable de "points de retournement" correspondant à un changement de l'indice de Morse des solutions qui tend vers l'infini le long de la branche.

Mots clés : opérateur p-Laplacien, principe de comparaison, problèmes singuliers, problèmes elliptiques/paraboliques, théorie globale de la bifurcation analytique.

Abstract : This thesis is concerned with the mathematical study of nonlinear partial differential equations. Precisely, we have investigated a class of nonlinear elliptic and parabolic problems with singular coefficients. This lack of regularity involves some difficulties which prevent the straighforward application of classical methods of nonlinear analysis based on compactness results. In the proofs of the main results, we show how to overcome these difficulties. Precisely we adapt some well known techniques together with the use of new methods. In this framework, an important step is to estimate accurately the solutions in order to apply the weak comparison principle, to use the regularity theory of parabolic and elliptic equations and to develop in a new context the analytic theory of global bifurcation. The thesis presents two independent parts.

– In the first part (corresponding to Chapter I), we are interested by a nonlinear and singular parabolic equation involving the p-Laplacian operator. We established for this problem that for any non-negative initial datum chosen in a certain $L^r(\Omega)$, there exists a local positive weak solution. For that we use some L^∞ estimates based on logarithmic Sobolev inequalities to get ultracontractivity of the associated semi-group. Additionaly, for a range of values of the singular coefficient, we prove the uniqueness of the solution and further regularity results.
– In the second part (corresponding to Chapters II, III and IV of the thesis), we are concerned with the study of global bifurcation problems involving singular nonlinearities. We establish the existence of a piecewise analytic global path of solutions to these problems. For that we use crucially the analytic bifurcation theory introduced by DANCER (described in Chapter II of the thesis). In the frame of a class of semilinear elliptic problems involving a critical nonlinearity in two dimensions, we further prove that the piecewise analytic path of solutions admits asymptotically a singular solution (i.e. an unbounded solution), whose Morse index is infinite. As a consequence, this path admits a countable infinitely many "turning points" where the Morse index is increasing.

Keywords : p-Laplacian operator, comparaison principle, singular problems, elliptic/parabolic problems, analytic theory of global bifurcation.